ABSTRACT

RAHMAN, SYED MIZANUR. Finite Element Analysis and Related Numerical Schemes for Ratcheting Simulation. (Under the direction of Dr. Tasnim Hassan).

Towards developing a robust finite element simulation model, this dissertation determines the state-of-the-art of simulation of ratcheting responses of structures. With this objective, the study evaluated seven constitutive models for their ratcheting simulation capabilities for straight pipe and elbow pipe components. Both local (strain ratcheting) and global (e.g., load-deflection, ovalization) responses of these two piping components were considered in the evaluation. The models evaluated are Bilinear, Multilinear and Chaboche, modified Chaboche, Ohno-Wang, modified Ohno-Wang, and Abdel Karim-Ohno. The latter four models are currently not available in ANSYS, hence was implemented into it for this study. In search of the best numerical scheme for implementation of cyclic plasticity models into finite element programs, Euler and Runge-Kutta (both explicit and implicit) type numerical schemes for solving the nonlinear incremental plasticity equations and approximating consistency condition, and return type algorithms for updating back stresses, were evaluated. This numerical scheme evaluation was conducted at the materials level with respect to the simulations of stable hysteresis loop, uniaxial ratcheting and multiaxial ratcheting responses. Implicit radial return scheme was demonstrated to be the best numerical scheme for implementing cyclic plasticity models, in terms of stability and accuracy for large loading increments. Automated parameter determination tools based on genetic algorithm search technique were developed for determining the model parameters of the four advanced constitutive models. In addition, for evaluating constitutive models at the materials level, the strain
driven radial return algorithm was extended, such that both stress and strain increments can be prescribed simultaneously.

The study demonstrated that the existing cyclic plasticity models are not capable of simulating straight and elbow pipe ratcheting responses satisfactorily when the model parameters are determined from material level responses (stable hysteresis loop, and uniaxial and/or biaxial ratcheting responses). For improving the simulation capability of the existing models, this study proposed a semi-inverse approach for refinement of model parameters using both the material level and structural local responses simultaneously. This proposed approach is validated for modified Chaboche model with respect to the straight pipe responses. Finally, limitations of the existing models are identified and recommendations are made for developing a robust model for structural ratcheting simulations.
FINITE ELEMENT ANALYSIS AND RELATED NUMERICAL SCHEMES FOR RATCHETING SIMULATION

by

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A dissertation submitted to the Graduate Faculty of North Carolina State University in partial fulfillment of the requirements for the Degree of Doctor of Philosophy

CIVIL ENGINEERING

Raleigh

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[Signatures]
To my parents
BIOGRAPHY

Syed Mizanur Rahman was born on November 24, 1974 in Dhaka, Bangladesh. He received his Bachelor’s degree in Civil Engineering from Bangladesh University of Engineering and Technology (BUET), Dhaka, Bangladesh in December, 1998 and he also completed his M. Sc. degree in Structural Engineering from the same university. He worked as a Lecturer in Structural Engineering Division in BUET from December, 1998 to August, 2001. After that he enrolled in Department of Civil Engineering at North Carolina State University as Doctoral student in August, 2001 under the supervision of Dr. Tasnim Hassan. Currently, he is working as a structural engineer in nuclear industry at Charlotte, North Carolina.
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CHAPTER ONE
INTRODUCTION

1.1 Introduction
A major concern in nuclear power plants and chemical industries is unexpected fatigue failures (Gosselin, 1994; EPRI, 1992). Fatigue crack in structures is initiated locally under repeated reversal of loading during earthquakes, extreme weather, and mechanical and thermal service conditions. Under cyclic loading, various parts of the structure may experience stress reversals exceeding the elastic limit of the materials used. Stress cycles in the inelastic range may occur at discontinuities even for structures designed to be within the elastic limit. Under such inelastic load reversals, fatigue failure may involve progressive accumulation of deformation or strain (known as ratcheting) within plastically deformed zones. It is demonstrated that the presence of ratcheting during fatigue may reduce the crack initiation life and thus the fatigue life of components (Lu, 2002).

In case of many fatigue failures experienced by industries it was not possible to determine the root causes, when they were not related to defects or unexpected loading (Virginia Power, 1993; EPRI, 1992). It is not known if any of these failures were influenced by ratcheting failure mechanisms. Ratcheting fatigue failure mechanisms are not understood well. The simulation models available currently can not simulate structural ratcheting responses. A research program at NC State is making effort to understand ratcheting failure mechanism and develop simulation model for ratcheting fatigue. A set of ratcheting data were developed towards this effort (Corona, 1996; Modlin and Hassan, 2002). A number of cyclic plasticity models were evaluated and a modified model was proposed for simulating ratcheting responses at the materials level (Bari and Hassan, 2000, 2002). This study implemented these models into finite element programs for evaluating these models against the structural ratcheting responses. Various numerical challenges were overcome during the model implementation process. These and the current state-of-the-art of ratcheting simulation models are presented in this dissertation. First, a brief overview of the ratcheting phenomena and currently available ratcheting simulation models is given below.

Ratcheting is defined as the accumulation of deformation or strain in structures under the action of cyclic loading. Ratcheting is demonstrated with an experimental result in Fig. 1.1.
(from Hassan and Kyriakides, 1994). In this experiment, a straight pipe of stainless steel (SS304) is subjected to steady internal pressure and symmetric axial strain-controlled loading (Fig. 1.1a,b). The resulting axial stress-strain response of the material is shown in Fig. 1.1c. It is observed that the stress amplitude of the hysteresis loops initially increases with cycles and stabilized after a couple of cycles. When the circumferential strain is plotted against the axial strain responses, as shown in Fig. 1.1d, it shows accumulation of circumferential strain with cycles. This circumferential strain accumulation is induced by the presence of the internal pressure. This ratcheting phenomenon is not well understood under complex stress histories in structures.

Fig. 1.1 Ratcheting phenomenon under cyclic loading, (a) straight pipe under internal pressure and symmetric axial strain-controlled loading, (b) loading path, (c) axial stress-strain response, and (d) circumferential strain-axial strain response.

This dissertation studied the ratcheting responses and their simulations for two piping components, straight pipe and elbow pipe. Ratcheting failure in piping components under cyclic loading can be caused either by fatigue crack initiation or progressive collapse. In order to develop rational design methods for these failure mechanisms, it is essential to simulate the ratcheting responses in piping components. Simulation of ratcheting damage accumulation under cyclic loading remains to be a major challenge. Addressing this challenge requires experimental data and robust analysis methods. Moreover, in a structural
analysis under cyclic load, it is required to simulate both global deformation and local strain accumulation simultaneously.

Finite element programs like ANSYS and ABAQUS are commonly used for structural response simulation under cyclic loading. Hassan et. al. (1998) showed that existing finite element packages like ANSYS (1997) fails to simulate ratcheting of straight pipe components. This shortcoming of finite element analysis in simulating ratcheting was demonstrated to be related to the deficiency of the constitutive models. Hassan et. al. (1998) also demonstrated that through incorporating an improved constitutive model into ANSYS (1997), its structural ratcheting simulation can be improved. During that time, finite element packages like ANSYS5.1 included only the Bilinear (Prager, 1956) and Multilinear (Besseling, 1958) models. Since then significant advancement is made through incorporating advanced constitutive model into ANSYS and ABAQUS. Incorporation of the non-linear kinematic hardening model by Chaboche (1986, 1991) into ANSYS and ABAQUS was a major leap towards improved simulation of stress-strain and ratcheting responses of structures. In the recent years, a number of advanced cyclic plasticity models have been proposed by Chaboche (1994), Ohno and Wang (1993), Jiang and Sehitoglu (1996), Abdel Karim and Ohno (2000), Bari and Hassan (2002), Chen and Jiao (2003), Vincent et al., (2004), Taleb and Cailletaud (2005), and many others for improving ratcheting simulation under multiaxial loading.


This dissertation will evaluate the advanced cyclic plasticity models developed by Chaboche (1986), Ohno and Wang (1993), Abdel Karim and Ohno (2000), modified Chaboche (Bari
and Hassan, 2002), and modified Ohno-Wang (Chen and Jiao, 2003) in simulating ratcheting responses of straight and elbow piping components. These advanced models were implemented into ANSYS (2005) for this study. In addition the simplified models like Bilinear (Prager, 1956) and Multilinear (Besseling, 1958) models, which are available in ANSYS and are usually used by practicing engineers for design and analysis, will also be evaluated. The straight pipe ratcheting responses used in this study was developed by Prof. E. Corona at the University of Notre Dame (Corona 1996). The elbow pipe ratcheting response data was developed at North Carolina State University by Modlin and Hassan (2002).

1.2 Constitutive Models

Constitutive model defines the material stress-strain relationship in finite element analysis. The rate independent plasticity models considered in this study has the following common features:

i. von-Mises yield criterion (yield surface):
   \[ f(\sigma - \alpha) = \frac{3}{2} (s - a) : (s - a) = \sigma_0 \]  
   (1.1)

ii. Strain decomposition:
   \[ d\varepsilon = d\varepsilon^p + d\varepsilon^p \]  
   (1.2)

Hook’s law:
   \[ d\varepsilon^p = \frac{1 + \nu}{E} d\sigma - \frac{\nu}{E} tr(d\sigma)I \]  
   (1.3)

Flow rule:
   \[ d\varepsilon^p = \frac{1}{H} \left( \frac{\partial f}{\partial \sigma} : d\sigma \right) \frac{\partial f}{\partial \sigma} \]  
   (1.4)

where \( \sigma \) is the stress tensor, \( \varepsilon^p \) the plastic strain tensor, \( s \) the deviatoric stress tensor, \( \alpha \) the current center of the yield surface in total stress space, \( a \) the current center of the yield surface in deviatoric stress space, \( \sigma_0 \) the size of the yield surface, \( \nu \) the poisson’s ratio, \( E \) the elastic modulus and \( H \) is the plastic modulus.

iii. With the von-Mises yield criterion, the most important feature of a plasticity model in simulating ratcheting responses is the kinematic hardening rule, which determines the back stress increment:
   \[ da = f(\sigma, \varepsilon^p, a, d\sigma, d\varepsilon^p, ....) \]  
   (1.5)
A kinematic hardening rule dictates the translation of the yield surface during plastic loading and thereby influencing non-linear stress-strain and ratcheting responses.

Under cyclic loads, materials show cyclic hardening or cyclic softening or combination of the two. This feature is represented through isotropic hardening rule. Experimental study indicates that cyclic hardening and softening tends to cease after certain number of cycles and the size of the yield surface stabilizes (Morrow, 1965; Jhansale, 1975; Tuegel, 1987; Ishikawa and Sasaki, 1988). The change in size of the yield surface is expressed through the isotropic hardening rule. However, ratcheting keeps on occurring with cycles even after the material stabilizes. Thus, kinematic hardening is considered to be the primary reason for ratcheting response whereas isotropic hardening mostly influences the change in the rate of ratcheting during the initial cycles (see Fig. 1.1).

Currently available isotropic hardening models can’t simulate the cyclic hardening and softening under stress-controlled loading history (Hassan and Kyriakides, 1994 a,b). Hence, to evaluate the basic ratcheting aspects of the models considered, this study excludes the isotropic hardening features from the constitutive models. This is a logical assumption for Alloy 4130, the material of the straight pipes, because this material does not show much cyclic hardening or softening under strain-controlled cycles. The elbow pipe material, which is stainless steel (SS) 304L, on the other hand is a cyclically hardening material. Hence, for elbow pipes the assumption of cyclically stabilized material may introduce errors in the ratcheting rate simulations during initial cycles when cyclic hardening is prominent. Once a robust constitutive model for simulating structural ratcheting responses is identified, the future study can be extended towards searching of a robust isotropic hardening model for improving the accuracy of the ratcheting simulation.

The models studied in the dissertation are briefly presented below:

1.2.1 The Bilinear model

This simple linear form of kinematic hardening rule was first proposed by Prager (1956) as,

\[ da = Cd\varepsilon^p \]  

(1.6)

With this plasticity model material stress-strain is represented by a linear elastic and a linear plastic part. For uniaxial loading, this kinematic hardening rule dictates the linear movement of the yield surface in stress-plastic strain space as shown in Fig. 1.2. The simplicity
(linearity) of the model allows fast calculation for finite element simulation. Also, the model has the advantage of having only one plasticity parameter $C$ which can be easily determined from a uniaxial stress-strain response. Bari and Hassan (2000) demonstrated that the Bilinear plasticity model (Prager, 1956) fails to produce ratcheting under cyclic loading and yields shake down phenomena. As this model is widely used, its performance in simulating ratcheting responses of piping components will be evaluated in this study.

![Uniaxial stress-strain response for Bilinear model.](image)

1.2.2 The Multilinear model

Besseling (1958) proposed the multilinear kinematic model which is also known as sublayer or overlay model (Owen et.al., 1974). In this model, the uniaxial stress-strain response is represented by several linear segments, as shown in Fig. 1.3. In case of uniaxial loading stress, $\sigma$ is calculated for strain, $\varepsilon$ (for $\varepsilon > \varepsilon_1$) as (Owen et.al., 1974)

$$
\sigma = E\left[t_1\varepsilon_1 + t_2\varepsilon_2 + t_3\varepsilon_3 + t_4\varepsilon\right] 
$$

(1.7)

where $t_1 = \frac{E - E_{T1}}{E}$, $t_2 = \frac{E_{T1} - E_{T2}}{E}$, $t_3 = \frac{E_{T2} - E_{T3}}{E}$, $t_4 = \frac{E_{T3}}{E}$

This model can produce smooth shape of stress-strain response when a large number of linear segments are used. ANSYS allows automatic calculation of plasticity model parameters from the stress-strain points. Easy parameter determination for this plasticity model made it popular for analysis with ANSYS. This model, however, fails to produce ratcheting under uniaxial loading and underpredicts ratcheting under multiaxial loading (Bari and Hassan, 2000).
1.2.3 Chaboche model

The non-linear kinematic hardening rule was first proposed by Armstrong-Frederick (1966) with a ‘recall’ term (2\textsuperscript{nd} term in Eq. 1.8) which keeps memory of the strain path history. The kinematic rule was proposed in the form,

\[
d a = \frac{2}{3} C d \varepsilon^p - \gamma a dp
\]  

(1.8)

where, \( dp = \left| d \varepsilon^p \right| = \left[ \frac{2}{3} d \varepsilon^p \cdot d \varepsilon^p \right]^{1/2} \)

It can be shown that for uniaxial case, Eq.(1.8) reduces to (Chaboche, 1986) the form

\[
\alpha_s = \frac{C}{\gamma} \left[ 1 - \exp\left( -\gamma (\varepsilon_s^p) \right) \right]
\]  

(1.9)

A qualitative plot of Eq. 1.9 or the translation of the yield surface is shown in Fig. 1.4a (\( \alpha_s \) trace). For a cyclically stable material, the uniaxial stress path is obtained by simply adding to \( \alpha_s \) the yield surface size, \( \sigma_0 \). Although, this model can produce the nonlinear part of the stress-strain response reasonably well for small strain ranges, it stabilizes to a constant
stress at large strain range. Consequently, the model overpredicts the rate of ratcheting under uniaxial loading (Bari and Hassan, 2000).

![Diagram](image)

Fig. 1.4 Uniaxial stress-plastic strain response representation by (a) Armstrong-Frederick model (b) Chaboche model

Chaboche et al. (1979, 1986) demonstrated that when three or more Armstrong-Frederick type rules are superimposed as

\[
d a = \sum_{i=1}^{m} d a_i = \sum_{i=1}^{m} \left( \frac{2}{3} C_i d \varepsilon_p - \gamma_i a_i d \rho \right)
\]

the resulting kinematic hardening rule simulates the non-linear stress-strain response well. If each of the superimposed kinematic hardening rule are integrated for uniaxial case and plotted, the \( \alpha_i \) (for i = 1, 2, 3) traces obtained are shown in Fig. 1.4b. The trace of \( \alpha_x = \alpha_1 + \alpha_2 + \alpha_3 \) and \( \sigma_x \) are also shown in same figure. The improvement of the stress-strain response simulation by the Chaboche (1979) is achieved due to the fact that each of the superimposed kinematic hardening rules represents a specific segment of the stress-strain curve. For example, \( \alpha_1 \) represent the initial nonlinear part, \( \alpha_2 \) the knee part and \( \alpha_3 \) the constant slope part. Because of the constant slope yielded by the decomposed kinematic rule \( \alpha_3 \), the uniaxial ratcheting simulation is improved compare to the Armstrong Frederick (1966) model.
For further improvement of the ratcheting simulation, one of the later versions of the Chaboche model (1991) proposed the kinematic hardening rule with four decomposed rules as follows:

\[
\frac{da}{dt} = \sum_{i=1}^{m} da_i 
\]

(1.11)

\[
da_i = \frac{2}{3} C_i d \epsilon^p - \gamma_i a_i dp 
\text{ for } i = 1, 2, 3
\]

(1.11a)

\[
da_4 = \frac{2}{3} C_4 d \epsilon^p - \gamma_4 a_4 dp \left(1 - \frac{a_4}{f(a_4)} \right) 
\text{ for } i = 4
\]

(1.11b)

This model has nine parameters \( a_1, \ldots, a_4, \gamma_1, \ldots, \gamma_4 \), where \( a_4 \) is called the threshold term. Although, with the four decomposed rules and the threshold term the Chaboche model can simulate the stress-strain and uniaxial ratcheting responses well, it still overpredicts multiaxial ratcheting simulation (Bari and Hassan, 2000).

1.2.4 Modified Chaboche model

Bari and Hassan (2002) modified the Chaboche (1991) kinematic rule, to improve the model’s multiaxial ratcheting simulation, as follows:

\[
da = \sum_{i=1}^{4} da_i 
\]

(1.12)

\[
da_i = \frac{2}{3} C_i d \epsilon^p - \gamma_i a_i \delta^+ + (1 - \delta^+) (a_i : n) dp 
\text{ for } i = 1, 2, 3
\]

(1.12a)

\[
da_4 = \frac{2}{3} C_4 d \epsilon^p - \gamma_4 a_4 \delta^+ + (1 - \delta^+) (a_4 : n) \left(1 - \frac{a_4}{f(a_4)} \right) dp
\]

(1.12b)

The multiaxial parameter \( \delta^+ \) in the kinematic rule does not influence the uniaxial loading response simulation, but plays a key role in improving multiaxial ratcheting simulation (Bari and Hassan, 2002).

1.2.5 Ohno-Wang model

Ohno-Wang (1993) proposed the kinematic hardening rule as

\[
da = \sum_{i=1}^{M} da_i 
\]

(1.13)
where, $C_i-C_M$, $\gamma_i-\gamma_M$ and $m$ are model parameters and $M$ is the number of linear segments required for multilinear representation of the uniaxial stress-strain curve.

Similar to the Chaboche model, each kinematic hardening rule in Ohno-Wang model represents a specific part of the uniaxial stress strain curve. The second term in the Ohno-Wang kinematic hardening rule improves the ratcheting simulation for both uniaxial and multiaxial loading cycles (Bari and Hassan, 2002).

### 1.2.6 Modified Ohno-Wang model

Chen and Jiao (2003) demonstrated that the addition of the Bari and Hassan (2002) multiaxial parameter $\delta^*$ to the Ohno-Wang (1993) model can further improve the multiaxial ratcheting simulation. The modification of the model is proposed as

$$
d a = \sum_{i=1}^{M} d a_i
$$

$$
d a_i = \frac{2}{3} C_i d e^\varepsilon - \gamma_i a_i \left[ d e^\varepsilon : \frac{a_i}{f(\alpha_i)} \left( \frac{f(\alpha_i)}{C_i/\gamma_i} \right)^m \right]
$$

(1.14a)

and $d \delta^* = \beta (\delta^*_{\text{sat}} - \delta^*) dp$, with $\delta^* = \delta^*_0$ at $p = 0$

where $\delta^*_0$ and $\delta^*_{\text{sat}}$ is the initial and end values of $\delta^*$.

The additional multiaxial model parameters are $\beta, \delta^*_{\text{sat}}, \delta^*_0$ over the Ohno-Wang model parameters.

### 1.2.7 Abdel Karim-Ohno model


$$
d a = \sum_{i=1}^{M} d a_i
$$

$$
d a_i = \frac{2}{3} C_i d e^\varepsilon - \mu \gamma_i a_i dp - \gamma_i a_i H \left\{ \frac{3}{2} a_i : a_i - \left( \frac{C_i}{\gamma_i} \right)^2 \right\} \{d \lambda_i\}
$$

(1.15a)
where, \( d \lambda_i = d \epsilon^p : \frac{a_i}{C_i / \gamma_i} - \mu dp \) and \( C_i - C_M, \gamma_i - \gamma_M \) and \( \mu \) are model parameters.

Although this model improved the multiaxial ratcheting simulation to some extent but it over predicts uniaxial ratcheting simulation (Bari and Hassan, 2002).

1.3 Experimental results

This dissertation evaluates the above seven plasticity models against material and structural experimental responses as discussed in the following.

![Fig. 1.5 Loading histories and paths](image)

(a) (b) (c) (d)

Fig. 1.5 Loading histories and paths (a) uniaxial stress history (b) axial strain cycle with constant internal pressure (c) biaxial bow tie cycle (d) biaxial reverse bow tie cycle.

1.3.1 Material responses

The material data used in this dissertation were developed by Hassan and Kyriakides (1992), Hassan et.al. (1992) and Cororna et al. (1996). These data include uniaxial to complex biaxial ratcheting responses of cyclically stabilized carbon steels. The loading histories considered are shown in Fig. 1.5. Under the loading history shown in Fig. 1.5a is the uniaxial stress cycle between two fixed stress limits with a mean stress. The response from this loading history shown in Fig. 1.6a, where it is observed that stress-strain loop gradually shifts and thus accumulates axial strain with cycles. The plot of axial strain at positive stress peak versus the number of cycle demonstrates the rate of axial strain ratcheting as shown in Fig. 1.6b. For the loading history shown in Fig. 1.5b, circumferential strain ratchets with cycles due to the presence of steady circumferential stress as shown in Fig. 1.6c and 1.6d. The loading history shown in Fig. 1.5c and 1.5d are termed as bow-tie and reverse bow-tie ratcheting history. Readers are referred to Corona et al. (1996) for details and ratcheting responses of those two loading histories. These ratcheting responses are presented in Chapter 2 and 3, while discussing the simulations.
1.3.2 Structural responses

1.3.2.1 Straight pipe under internal pressure and cyclic bending

Two straight pipe ratcheting experiments were conducted at the University of Notre Dame by Prof. E. Corona (1996) for this study. In these experiments, straight pipes were subjected to steady internal pressure and cyclic bending. The schematic of the pipe specimen, loading and loading path are shown in Fig. 1.7. Detail discussion of the experimental setup are presented in chapter 5. In these experiments, the pipes were first subjected to internal pressure followed by cyclic rotation at both ends while keeping the internal pressure constant. Two experiments were conducted, each with the same internal pressure, but different amplitudes of end rotation. The pipe material undergoes multiaxial cyclic stresses in the plastic range which induces permanent strain and deformation. As the pipe bends, the circular cross-section of the pipe ovalizes. In these experiments, the strains were measured at top and side of the pipe at the midspan (see Fig. 1.7a). In addition, the ovalization of the pipe at the midspan was also measured. The results of these tests are discussed in detail in chapter 5.
1.3.2.2 Elbow pipe under cyclic bending and internal pressure

This set of elbow tests were conducted by Modlin and Hassan (2002). Two inches diameter short-radius elbow were welded to ten inches long straight pipes at each end of the elbow in fabricating elbow specimens as shown in Fig. 1.8a. The elbow specimens were subjected to steady internal pressure and cyclic opening-closing cycles (in Fig. 1.8). The elbow data set developed includes three monotonic tests, one opening and two closing and both without any internal pressure, and seven displacement or force-controlled opening-closing tests, with or without internal pressure. During the experiment, axial and circumferential strains were recorded at extrados, intrados and flank. In addition, the diameter changes (ovalization) at the midsection were measured.
Research objectives and methodology
The primary objective of this study was to develop a robust finite element code by implementing in it advanced cyclic plasticity models for simulating ratcheting and cyclic responses of piping components. The numerical tool developed was validated against a broad set of ratcheting and cyclic responses at the materials level, as well as, the structural level. Efficient numerical schemes were identified for implementing constitutive models into the finite element code. As the advanced cyclic plasticity models considered involve a large number of model parameters, to facilitate the use of these models by the industrial engineers an automated parameter optimization scheme, using the genetic algorithm technique was developed.

Brief overview and organization
This study evaluated several advanced cyclic plasticity models against a broad set of material level, straight pipe and elbow pipe ratcheting and cyclic responses. Currently, only the
Bilinear, Multilinear and Chaboche models are available in ANSYS. But ANSYS has an open architecture which allows user to customize ANSYS through implementing plasticity subroutines. The study implemented the modified Chaboche (Bari and Hassan, 2002), Ohno-Wang (1993), modified Ohno-Wang (Chen and Jiao, 2003) and Abdel Karim-Ohno (2000) models into ANSYS. Each of the advanced plasticity models have large number of parameters. These parameters are determined from three different material level experimental responses: uniaxial strain-controlled response (stable hysteresis loop), uniaxial ratcheting response and biaxial ratcheting response. The strengths of an advanced plasticity model might be undermined if the model parameters are not optimized for these responses. Manual optimization of a large number of parameters through simultaneous simulation of three experimental responses are tedious and error prone. Manual parameter determination for a model also requires extensive experience with the model. The heuristic search algorithm e.g. genetic algorithm, can automate the optimization of model parameters. In general, genetic algorithm optimizes the model parameters from a large range of parameters, which was found not efficient and it did not always converge. This study developed a scheme to include the physical significance of the parameters into the genetic algorithm search and thus enhanced the efficiency of the parameter optimization scheme. A specialized stepped optimization approach for the advanced cyclic plasticity models considered was developed to further enhance the efficiency of the parameter optimization scheme. The development of the new optimization scheme for these advanced plasticity models and its advantages compared to other parameter optimization schemes are discussed in Chapter Two.

As mentioned earlier, the modified Chaboche (Bari and Hassan, 2002), Ohno-Wang (1993), modified Ohno-Wang (Chen and Jiao, 2003) and Abdel Karim-Ohno (2000) models are implemented into ANSYS through its customization option. These models are highly nonlinear in nature. In finite element computation, the plasticity subroutines determine the stress increments for the prescribed strain increments. Structural analysis under cyclic loading are usually performed with large load increments, whereas the plasticity model (based on incremental plasticity theory) requires that the strain increment needs to be as small as possible. When the load increment is large, the plasticity model need to upgrade stresses for large strain increments, for which the numerical integration may suffer from oscillation and lack of stability due to the non-linear nature of the models. Reducing the
increment size to overcome this problem increases computational cost especially for large number of cycles. In search of a robust numerical scheme for plasticity model implementation, existing explicit, semi-implicit and implicit type of Euler and Runge-Kutta numerical schemes are studied with uniaxial and multiaxial ratcheting simulation. These methods can be implemented through either conventional or radial return approach. The implicit type radial return algorithm was observed to fulfill the requirements of accuracy and stability. With the radial return approach, it is possible to prescribe a large load increment without losing the inherent characteristics of the cyclic plasticity models. The study on the numerical integration and their various advantages and disadvantages related to the plasticity models considered are addressed in Chapter Three.

As mentioned earlier in finite element analysis, strain increments are input in the plasticity subroutine for stress increment calculation. Hence the radial return approach was developed only for strain increments and for plane stress case to work with shell elements. This approach is not suitable for validation of cyclic plasticity models without using a finite element code. Validation of the plasticity models using a finite element code requires more solution time and also introduces numerical error from the iteration of global convergence. So, this study developed a new algorithm called “stress return algorithm” for plasticity calculation for any combination of prescribed stresses and strains. The formulation and algorithm of stress return algorithm are given in Chapter Three.

The quadratic convergence of structural computations is dependent on the accurate representation of the tangent modulus which has to be consistent with the numerical integration schemes adapted for stress upgrading at an integration point. Chapter Four presents the numerical aspects of the formulation of consistent tangent modulus. A brief methodology for implementing cyclic plasticity subroutines into ANSYS (2005) is also presented in Chapter Four.

This study evaluates advanced cyclic plasticity models through customized ANSYS against the pipe and elbow experimental responses. The simulations started with the material parameters, which were determined through a parameter optimization algorithm. The simulations are compared against the local strain ratcheting responses at critical locations, as well as, global ovalization ratcheting responses. Ratcheting simulations of straight pipe responses and simulation improvement methodologies are discussed in Chapter Five.
Chapter Six presents elbow responses of a series of tests under monotonic and cyclic loading. The elbow is subjected to displacement or force controlled cyclic opening and closing loads along with or without internal pressure. The elbow responses under such loading are simulated with seven plasticity models. Simulated ovalization and strain ratcheting responses at critical locations are compared with experiments to evaluate the performance of the models. Simulation of the elbow pipe responses are presented and discussed in Chapter Six. Finally, Chapter Seven summarized the outcome of this study and future directions.
CHAPTER TWO
AUTOMATED PARAMETER DETERMINATION OF ADVANCED CONSTITUTIVE MODEL

2.1 Introduction

Advance constitutive models are developed to simulate material nonlinearities under cyclic loading. Metallic materials may accumulate strain or deformation under cycle loads in the inelastic range. This phenomenon is known as ratcheting and is an important phenomenon to be considered in structural analysis and design in a rational manner. Chaboche (1994), Ohno and Wang (1993), Abdel Karim and Ohno (2000), Bari and Hassan (2002) and many others developed improved constitutive models for representing cyclic nonlinear responses including ratcheting. Bari and Hassan (2002) demonstrated the performance of many of these modified models in simulating a wide set of uniaxial and biaxial ratcheting responses developed by Hassan and Kyriakides (1992), Hassan et al. (1992) and Corona, et al.(1996). These modified models however have large number of parameters, many of which do not have well defined physical meaning. Hence, most of these parameters are determined through a trial-and-error curve fitting approach using several experimental responses at the material level. In addition, the arbitrariness and multiaxial complexity (non-proportionality) of cyclic loading make such advanced models highly nonlinear and multimodal in functional and parameter space. Hence, optimization of advanced constitutive model parameters through manual operation is tedious and might be erroneous. Manual parameter determination for an advanced plasticity model also requires an in-depth knowledge of the model and experience with its parameter determination. These are few of the primary reasons why advanced cyclic plasticity models are not widely used for structural analysis and design. These problems could be overcome through developing an automated parameter optimization system using heuristic search technique (e.g. genetic algorithm).

In general, probabilistic search technique is used for constitutive model identification and it’s parameter determination. Probabilistic search technique employs highly exploitative search in function and parameter space using random number generator as a tool. In literature, genetic programming and neural network is used for constitutive model identification (Goldberg, 1989, Ishikawa and Sasaki, 1988, Feng and Yang, 2001, Furukawa et al., 2002, Gray et. al., 1998 and many). Genetic algorithm is commonly employed for identification of
model parameters (Gendy and Saleeb, 2000; Furukawa et al., 2002; Nanakorn and Meesomklin, 2001, Pal et. al., 1996 and others). GA is a systematic stochastic search procedure that uses random search as a tool to guide a highly exploitative search in parameter space. GA randomly generates an array of initial parameter sets and subsequently generates better sets of parameters using the initial sets through a systematic search technique. This iterative search technique gradually evolves towards better and better parameter sets and finally yields the best parameter set for given fitness criteria. This chapter discusses the development of such an automatic parameter determination scheme for modified Chaboche model developed by Bari and Hassan (2002). A new stepped GA optimization approach, which is found to be more efficient over the conventional GA approach in terms of fitness quality and optimization time is developed. Advantages of the stepped approach are presented. The stepped optimization approach is also used to develop optimization algorithm for Ohno-Wang (1993) and Abdel Karim-Ohno (2000) plasticity models. Methodology to implement stepped optimization approach is presented.

2.2 Constitutive Model

The rate-independent constitutive model considered in this study is composed of von-Mises yield criterion:

\[
 f(\sigma - \bar{\sigma}) = \frac{3}{2}(s - \bar{\sigma}):(s - \bar{\sigma})^{\frac{1}{2}} = \sigma_0
\]  

(2.1)

Flow rule :

\[
 d\varepsilon^p = \frac{1}{H} \left( \frac{\partial f}{\partial \sigma} : d\sigma \right) \frac{\partial f}{\partial \sigma}
\]  

(2.2)

The modified Chaboche kinematic hardening rule:

\[
 d\alpha = \sum_{i=1}^{4} d\alpha_i
\]  

(2.3)

\[
 d\alpha_i = \frac{2}{3} C_i d\varepsilon^p - \gamma_i (a_i \delta^p + (1 - \delta^p)(a_i : n)n) dp \quad \text{For } i = 1 \sim 3
\]  

(2.3a)

\[
 d\alpha_4 = \frac{2}{3} C_4 d\varepsilon^p - \gamma_4 (a_4 \delta^p + (1 - \delta^p)(a_4 : n)n) \left(1 - \frac{\bar{a}_4}{f(a_4)}\right) dp
\]  

(2.3b)

where \( \sigma \) is the stress tensor, \( \varepsilon^p \) is the plastic strain tensor, \( s \) is the deviatoric stress tensor, \( \alpha \) is the current center of the yield surface, \( a \) is the current center of the yield surface in
deviatoric space, $\sigma_0$ is the size of the yield surface, $H$ is the plastic modulus and
\[ dp = \left| d\varepsilon^p \right| = \left[ \frac{2}{\sqrt{3}} d\varepsilon^p \cdot d\varepsilon^p \right]^{\frac{1}{2}}. \]
The kinematic hardening rule includes four decomposed hardening rules (Eq. 2.3) as originally proposed by Chaboche, et al. (1979). In the kinematic hardening rule, $C$, $\gamma$, $\delta$, and $\delta'$ are model parameters. Rate-independent plasticity theory considered in this study assumes that the yield surface size and shape remain unchanged (cyclically stable material), for which the kinematic hardening rule describes the translation (evolution) of the yield surface. As in classical plasticity theory, the yield surface is given by the von-Mises yield criterion, and the flow rule determines the magnitude and direction of plastic strain increments.

Fig. 2.1 Hysteresis curve used for parameter determination

2.3 Manual procedure and experimental responses for parameter determination

Bari and Hassan (2000,2002) presented a systematic step by step approach for determining the parameters of modified Chaboche model. In this approach, parameters ($C_1, C_2, C_3, C_4, \gamma_1, \gamma_2, \gamma_4, a_4$) are determined through simulating a hysteresis curve (upper or lower curve) as shown in Fig. 2.1. Parameters $\gamma_3$ is determined by simulating a uniaxial ratcheting rate response as shown in Fig. 2.2b, which is obtained by plotting the maximum strain in each cycle from a uniaxial ratcheting response (Fig. 2.2a) as a function of the number of cycle. This ratcheting rate response is also used to simultaneously optimize the parameter $a_4$. Finally, the multiaxial parameter $\delta'$ is determined from the biaxial ratcheting rate response shown in Fig. 2.3b, which is obtained by plotting the maximum circumferential strain in each cycle from a biaxial ratcheting response (Fig. 2.3a) as a function of the number of cycle. The
readers are referred to Hassan and Kyriakides (1992a) and Hassan et.al. (1992b) for details of these experimental responses. For loading portion of the stabilized, symmetric hysteresis curve (Fig. 2.4), each of the decomposed hardening rules, except the third rule, starts at $-\frac{C_i}{\gamma_i}$ the starting plastic strain $-\varepsilon_L^p$ and reaches the value $C_i/\gamma_i$ at or prior to the final plastic strain $\varepsilon_L^p$. The third linear hardening rule (when, $\gamma_3 = 0$) passes through the origin. Guidelines that facilitate the parameter determination are, $C_1$ should be a very large value to match the plastic modulus at yielding and corresponding $\gamma_1$ also should be large enough to stabilize the first hardening rule immediately, $C_3$ is determined from the slope of the linear segment of the hysteresis curve at a high strain range, and finally, $C_2, C_4, \gamma_2$ and $\gamma_4$ are evaluated by trials to produce a good representation of the knee part of the experimental stable hysteresis curve, which also satisfy the relationship for higher strain values

$$\left(\frac{C_1}{\gamma_1} + \frac{C_2}{\gamma_2} + \frac{C_4}{\gamma_4} + \bar{a}_4\right) + \sigma_0^+ = \sigma_x^+ - \frac{C_3}{2}\left\{\varepsilon_x^p - (-\varepsilon_L^p)\right\}$$

(2.4)

Fig. 2.2. Uniaxial ratcheting test used for parameter determination. (a) axial stress-strain response, (b) axial strain ratcheting rate response.
Fig. 2.3 Biaxial ratcheting response used for parameter determination. (a) axial-circumferential strain response, (b) circumferential strain ratcheting rate response.

While determining the hysteresis curve parameters ($C_1$, $C_2$, $C_3$, $C_4$, $\gamma_1$, $\gamma_2$, $\gamma_4$, $\tilde{a}_4$), the value of $\gamma_3$, which mainly influence the uniaxial ratcheting response, is set to zero. While determining $\gamma_3$ in the subsequent step, all hysteresis curve parameters are kept constant, except $\tilde{a}_4$ as it also influences the uniaxial ratcheting response. However, any change in $\tilde{a}_4$ may degrade the hysteresis curve simulation, which may require reevaluation of the hysteresis curve parameters. Hence during $\gamma_3$ determination, several trial-and-error iterations between the hysteresis curve and the uniaxial ratcheting response are required, which makes this manual determination approach tedious and time consuming. The determination of multiaxial parameter $\delta'$ is simple and straightforward, as this parameter does not have any influence on uniaxial responses, but influences only the biaxial ratcheting response.

2.4 Automated parameter determination using genetic algorithm

As discussed above, the modified Chaboche model has nine interdependent model parameters, which are determined using three sets of experimental responses. Hence, parameter determination requires several trial-and-error iterations for obtaining a reasonable set of parameter values and thus the manual parameter value determination becomes tedious and time consuming. Determining a good set of parameter values manually also requires detailed knowledge of the constitutive model and extensive parameter determination experience. This study evaluates the use of heuristic search technique, e.g., genetic algorithm, towards automating the parameter value determination of advanced models and thus overcome the constraints involved in manual parameters determination.
Genetic algorithm (GA) is a robust stochastic search procedure that explores for solutions from a broad and promising region. The flow chart for conventional GA is shown in Fig. 2.5, where it is seen that the first step involves random generation of sets of real numbers as possible parameter values (population generation). Each parameter set is known as an individual in the population. For each individual set has a fitness value, which is determined based on how well it simulates observed responses. Fitness value is determined based on the root mean square deviation of distance between the experimental ($Y_{exp}$) and model simulation ($Y_m$) responses as follows:

Deviation measure,

$$f_{dist} = \sqrt{\frac{\sum_{i=1}^{N} (Y_{exp,i} - Y_{m,i})^2}{N}}$$  \(2.5\)

Normalized value,

$$\bar{f}_{dist} = \frac{f_{dist}}{\sqrt{\frac{\sum_{i=1}^{N} Y_{exp,i}^2}{N}}}$$  \(2.6\)
In GA parameter optimization, the least square distance between the experimental data and model simulations becomes the objective function to be minimized. In hysteresis curve simulation, it is necessary to simulate closely the end stress and plastic modulus ($C_3$) at higher strain range along with minimizing the overall least square distance. This additional requirement can be achieved by introducing constraints in optimization scheme for hysteresis loop and applying through penalty on fitness functions based on the match of the end stress and plastic modulus. When the parameter set violates the constraint, the fitness value calculated based on distance measure is penalized. In this case, linear penalty is added (e.g. linearly increasing the fitness value) to fitness value based on the violation of the constraint. Thus, those parameter sets that simulate the hysteresis curve well but could not match the end plastic modulus well will have higher fitness function (penalty is added) than those that satisfies both the criteria. As the objective in this optimization scheme is to minimize the fitness function value, parameter sets violating the constraint are less likely to be accepted as optimized parameter set. Similarly, during ratcheting simulations it is imperative to minimize the distance function as well as matching the stabilized ratcheting rate at higher number of cycles while simulating the ratcheting response. Final objective for ratcheting simulation is to satisfy both criteria: minimizing the distance function and matching stabilized ratcheting rate. The deviation measures in each objective are normalized with respect to their experimental observations as shown in Eq. 2.6. Then in optimization scheme, multiple objectives are satisfied by minimizing the maximum normalized deviation.
Once the fitness values for each individual in the population are calculated, the individual with the best fitness value is selected and saved as an elite individual. In the following steps, GA attempts to acquire improved sets of individuals through using the Darwinian principle of “survival of the fittest”. This requires operations of selection, crossover, mutation, and elitism (Fig. 2.5) to produce a better offspring population from the current population. The basic idea involves in these operations is that two better parents might produce a better offspring. In selection operation, binary tournament selection strategy is used, which picks the better individual from randomly chosen two individuals from the population. The process is continued until all parameter set individuals has gone through the selection operation for once and thus the better half of the population is selected. The binary selection process is repeated on the same population so that each selected individual undergoes selection criteria twice. Thus, the selection operation assures that the individuals with better fitness values would be copied in the next generation. The pictorial presentation of the binary section strategy is shown in Fig. 2.6 with the example a population of eight individual parameter set. Subsequently, crossover operator is employed to selected population. Randomly, two individuals are selected and their parameters exchange their information in order to generate two offspring in between two parents. In this case, arithmetic crossover is done to generate offspring from two parents using the arithmetic equation shown in Fig. 2.7. These equations generate two new parameter set in between parent parameter sets. With some random probability, parameters are also generated outside these two parents but within the parameter range. The operations of the arithmetic crossover are shown in Fig. 2.7 with example. The selected (better) population obtained from binary selection strategy is used for arithmetic crossover to produce a new offspring set. In mutation, from the new offspring population, offspring individuals are randomly picked and parameters are altered. In this process, all offspring individuals are arranged as long strings and randomly different parameters on the string are mutated (altered) within the range for making the search global and thus getting rid of local optimum. Mutation rate of 10% is selected in this study. Higher mutation rate gives better convergence initially but may not have convergence effect at the end, whereas smaller percentage of mutation works better for convergence to ultimate value from beginning to end.
After the generation of offspring population through selection, crossover and mutation, the parent individuals are discarded. The fitness values of the newly generated offspring obtained after selection, crossover and mutation operations are determined. The elitism operation is
followed for picking the individual with the best fitness value. During this operation, the fitness value of the elite individual from the current generation is compared to that of the elite from the previous generation and the better one is saved as elite. When the elite set of the previous generation is better than the elite set of the current generation, the elite set from previous generation replaces the worst individual of the current generation. This process ensures the survival of the best parameter set through generations. The elite set is scrutinized against the termination criteria to be determined if an acceptable individual (optimized parameter set) is achieved yet. If not, the search continues with selection, crossover, and mutation for generating better offspring following the steps in Fig. 2.5.

The conventional GA procedure described above was found inefficient for determining an optimized parameter sets for the modified Chaboche model. The time of convergence was large and the optimized best individual was not always a reasonable set of parameters. However, the study determined that if the initial random parameter generation includes the physical meaning of the parameters, the GA search technique converges quickly to a very good set of parameters. This modified procedure is discussed below.

![Fig. 2.8 Schematic of the mutation operation](image-url)
Fig. 2.9 Four segments of the hysteresis curve each of which is predominantly represented by one of the four kinematic hardening rules.

2.5 Initial parameter generation based on the physical meaning of parameters

As discussed above, the modified Chaboche kinematic hardening rule is superposition of four rules (Eq. 2.3). For a uniaxial, cyclically stable, symmetric hysteresis curve in the plastic strain and total stress space, these four rules add up to determine the total back stress, $\alpha_x$, as shown in Fig. 2.4. Each of the four decomposed kinematic hardening rules, except the third rule, starts with a specific slope ($C_i$) and this slope gradually approaches to zero with increasing plastic strain (Fig. 2.4). The third rule is linear and passes through zero when, $\gamma_3 = 0$, and is slightly nonlinear if $\gamma_3$ is non-zero. Thus, the hysteresis curve can be divided into four segments ($\Delta \varepsilon_1^p, \Delta \varepsilon_2^p, \Delta \varepsilon_3^p$ and $\Delta \varepsilon_4^p$), each of which is predominantly represented by one of the hardening rules as shown in Fig. 2.9. The first kinematic hardening rule predominantly represents the first segment by starting with a very large slope and stabilizing very quickly. The fourth segment is almost linear with a small slope and is represented by the third rule. The transition knee part of the hysteresis curve is divided in two segments, the second segment is represented predominantly by the fourth hardening rule and third segment by the second hardening rule. So, the initial slope of the first segment is a good approximation for $C_1$, that of the second segment for $C_4$, the third segment for $C_2$, and the fourth segment for $C_3$. The plastic strain ranges for these four segments are determined for seven different materials through extensive trial-and-error run of the modified Chaboche model program. The calculated range of these four segments is shown in Table 2.1. For a given segment, any value within the above determined range is found to be a good initial approximation for...
determining \( C_i \). In this study, the ranges used are 1% for the first segment, 15% for the second, 20% for the third, and the remainder for the fourth segment (Table 2.1). It will be shown later that updating these ranges at an intermediate stage increases the convergence rate in parameter determination.

<table>
<thead>
<tr>
<th>Segments</th>
<th>Range (% of ( 2\varepsilon^p ))</th>
<th>Used range (% of ( 2\varepsilon^p ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \varepsilon_1^p )</td>
<td>1-2%</td>
<td>1%</td>
</tr>
<tr>
<td>( \Delta \varepsilon_2^p )</td>
<td>15-20%</td>
<td>15%</td>
</tr>
<tr>
<td>( \Delta \varepsilon_3^p )</td>
<td>20-25%</td>
<td>20%</td>
</tr>
<tr>
<td>( \Delta \varepsilon_4^p )</td>
<td>Rest</td>
<td>Rest</td>
</tr>
</tbody>
</table>

In parameter determination, it is required to select a hysteresis curve with a plastic strain range \( \pm \varepsilon^p \) such that all hardening rules except the third rule stabilizes within this plastic strain range. Then it can be shown that (Bari and Hassan, 2002)

\[
2 \left( \frac{C_1}{\gamma_1} \right) + 2 \left( \frac{C_2}{\gamma_2} + \bar{a}_4 \right) + 2 \left( \frac{C_4}{\gamma_4} \right) = (\sigma^+ - \sigma^-) - 2\sigma_0 - 2C_p\varepsilon^p \tag{2.7}
\]

In Eq. 2.7, \( \sigma^+ \) is the positive peak and \( \sigma^- \) is the negative peak of the hysteresis curve. Hence, the right hand side of Eq. 2.7 can be determined. As a result, if the back stress contributions from each of the three rules in the left hand side of Eq. 2.7 can be determined, then initial values of \( \gamma_1, \gamma_2 \) and \( \gamma_4 \) could be estimated. Through extensive trial-and-error run of the modified Chaboche model program, these contributions are calculated for different materials and shown in Table 2.2. Contributions of 10%, 50% and 40% for the three hardening rules for estimating initial \( \gamma_1, \gamma_2 \), and \( \gamma_4 \) values are found to yield reasonable parameters as is demonstrated later. The initial estimates of \( C' \)'s and \( \gamma' \)'s are allowed to evolve or adapt with generation, which is found to improve the convergence speed.
After initial estimations of parameters, ranges used for each of the parameters for random generation of population are as follows: ±5% for $C_1$ and $C_3$ because these parameters can be estimated more accurately than others. The parameters $C_2$, $C_4$, $\gamma_1$, $\gamma_2$, and $\gamma_4$ are varied ±10% of the initial estimate. The value of $\gamma_3$ is generated within the range of 0-30 and $\overline{a}_4$ is
generated within the range $\frac{1}{4}$ to $\frac{1}{2}$ of yield stress. These ranges were verified to work well for seven different materials. The parameter $\delta'$ is varied between 0 and 1. In this study, this modified GA scheme is called simultaneous GA, as this scheme determines the parameters through simultaneous optimization of simulations for the three experimental responses. This scheme determines good parameter sets for the modified Chaboche model quite efficiently as will be presented later. The simultaneous GA scheme is shown in Fig. 2.10.

Another modified approach is developed in this study in which optimization of simulations for three different responses is performed in three steps. In first step, GA optimizes the hysteresis curve parameters ($C_1, C_2, C_3, C_4, \gamma_1, \gamma_2, \gamma_4, a_4$) using the initial estimates discussed in the last section. In this step, $\gamma_3$ is kept as zero. In second step, the hysteresis curve parameters are varied within a narrow range of $\pm 3\%$, but $\gamma_3$ and $a_4$ are varied as discussed in the last section. In this step, the optimization of the hysteresis curve and uniaxial ratcheting rate simulations are carried out simultaneously. Finally, in the third step, the biaxial ratcheting parameter $\delta'$ is varied between 0 and 1, whereas all other parameters determined earlier are varied within a narrow range of $\pm 3\%$. In this final step, simulations for the stabilized hysteresis curve, uniaxial ratcheting and biaxial ratcheting rates are optimized simultaneously. However, the parameters are varied within a narrow range than in the last scheme and hence are more efficient than the simultaneous GA scheme. This modified parameter determination scheme is called the stepped GA in this study. The stepped GA scheme is shown in Fig. 2.11.

### 2.6 Validation of the GA parameter determination schemes

The GA parameter determination schemes are verified using artificial responses developed with a known set of parameters. Thus it would be possible to evaluate the accuracy of the GA parameters through comparing these parameters against the known parameters. The artificial responses are generated using the modified Chaboche model program and the parameters shown in Table 2.3. The artificial responses include a hysteresis curve, and uniaxial and biaxial ratcheting rates as shown in Fig. 2.12 and Fig. 2.13. The stabilized hysteresis curve is obtained with axial strain amplitude ($\varepsilon_{\text{ax}}$) of 1%. The uniaxial ratcheting rate response is obtained with mean stress ($\sigma_{\text{xm}}$) of 6.5 ksi and amplitude stress ($\sigma_{\text{xa}}$) of 32.12 ksi. The biaxial ratcheting rate response is obtained with a steady hoop stress ($\sigma_{\text{0m}}$) of 9.65 ksi and axial
strain amplitude ($\varepsilon_{xc}$) of 0.50%. Elastic material properties used are modulus of elasticity ($E$) = 27,000 ksi, Poisson’s ratio, ($\nu$) = 0.30 and size of the yield surface, $\sigma_0 = 20.0$ ksi.

![Flow chart for stepped genetic algorithm](image)

**Fig. 2.11. Flow chart for stepped genetic algorithm**

<table>
<thead>
<tr>
<th>Parameter type</th>
<th>Parameter values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{1-4}$</td>
<td>50000.0, 3000.0, 400.0, 12000.0</td>
</tr>
<tr>
<td>$\gamma_{1-4}$</td>
<td>20000.0, 300.0, 10.0, 400.0</td>
</tr>
<tr>
<td>$\bar{d}_4$, $\delta'$</td>
<td>4.0, 0.25</td>
</tr>
</tbody>
</table>

Table 2.3

Parameters used for developing the artificial responses

The GA parameters determined using the general, simultaneous and stepped GA schemes are shown in Table 2.4. General GA optimized parameters are observed to show large variation compared to actual parameters. Only the fitness values are shown in Table 2.4. Comparison of the simultaneous and stepped GA parameter responses and the artificial responses used for
parameter determination are shown in Fig. 2.12 and Fig. 2.13. Even though the parameters by the two schemes are different, the difference in visual fitness of the responses to the artificial responses is undetectable. However, the fitness values and computation times in Table 2.4 shows that the stepped GA simulates all three responses better and yield faster convergence.

### Table 2.4

<table>
<thead>
<tr>
<th>Type</th>
<th>Parameter values</th>
<th>Simultaneous Optimization</th>
<th>Stepped Optimization</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>General GA Optimization</td>
<td>Simultaneous Optimization</td>
<td>Stepped Optimization</td>
</tr>
<tr>
<td>$C_{1-4}$</td>
<td>27697.0, 3627.0, 413.0, 11523.0, 45469.0, 2601.0, 13484.0</td>
<td>50281.0, 3012.0, 401.0, 12010.0</td>
<td></td>
</tr>
<tr>
<td>$\gamma_{1-4}$</td>
<td>12128.0, 342.0, 9.05, 3409.0, 20997.0, 264.0, 9.67, 3840.0,</td>
<td>20818.0, 302.0, 9.93, 4050.0</td>
<td></td>
</tr>
<tr>
<td>$\bar{a}_4, \delta$</td>
<td>2.87, 0.23, 4.23, 0.231, 4.05, 0.251</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Optimization</td>
<td>6984.9, 3840.2, 685.53</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time (sec)</td>
<td>0.0097, 0.0088, 0.0021</td>
<td>0.0303, 0.0199, 0.0047</td>
<td></td>
</tr>
<tr>
<td>Stable Loop</td>
<td>0.0303, 0.0199, 0.0047</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ratcheting</td>
<td>0.0303, 0.0199, 0.0047</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fitness, $f_{stab}$</td>
<td>0.0303, 0.0199, 0.0047</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Uniaxial</td>
<td>0.0303, 0.0199, 0.0047</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ratcheting</td>
<td>0.0303, 0.0199, 0.0047</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fitness, $f_{uni}$</td>
<td>0.0303, 0.0199, 0.0047</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Biaxial</td>
<td>0.0303, 0.0199, 0.0047</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ratcheting</td>
<td>0.0303, 0.0199, 0.0047</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fitness, $f_{bi}$</td>
<td>0.0303, 0.0199, 0.0047</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### 2.7 Application to real response data

The stepped GA optimization scheme is employed for the modified Chaboche (Bari and Hassan, 2002) plasticity model parameter determination from the real cyclic test responses. These real test responses include a hysteresis curve, uniaxial ratcheting and biaxial ratcheting response of carbon steel CS1026 as shown in Fig. 2.1-2.3. The stabilized hysteresis curve was obtained from a uniaxial strain controlled cyclic test with axial strain amplitude of 1.0% (Fig. 2.1). The uniaxial ratcheting rate response is obtained from stress controlled uniaxial cyclic test with mean stress ($\sigma_{xm}$) of 6.5 ksi and amplitude stress ($\sigma_{xa}$) of 32.12 ksi. The biaxial ratcheting rate response for CS1026 material is obtained from cyclic test with a steady hoop stress ($\sigma_{hm}$) of 9.65 ksi and axial strain amplitude ($\varepsilon_{xc}$) of 0.50%. The elastic material properties determined from the hysteresis curve are modulus of elasticity, ($E$) = 26,300 ksi, Poisson’s ratio, ($\nu$) = 0.302 and size of the yield surface, $\sigma_0$ = 18.8 ksi.
Fig. 2.12. Comparison of the exact results and simulation obtained by stepped GA optimized parameters for (a) loading part of hysteresis curve (b) uniaxial ratcheting response and (c) biaxial ratcheting response.

Fig. 2.13. Comparison of the exact results and simulation obtained by simultaneous GA optimized parameters for (a) loading part of hysteresis curve (b) uniaxial ratcheting response and (c) biaxial ratcheting response.

The stepped GA optimized parameters are given in Table 2.5, which also lists the manually determined parameters (Bari and Hassan, 2002). Fitness with the manually calculated and GA parameters for stress-strain, uniaxial ratcheting and biaxial ratcheting simulation are shown in Fig. 2.14 and Fig. 2.15. GA optimized parameters simulated the hysteresis curve better than manual parameters. Uniaxial ratcheting simulations from both set of parameters are compared in Fig. 2.14b and Fig. 2.15b, where it is observed that two simulations are similar. However, improvement in biaxial ratcheting simulation with GA parameters is
observed (Figs. 2.14c and 2.15c). Overall, the GA optimization scheme calculates better parameters than manual approach for modified Chaboche model.

Table 2.5
Constitutive model parameters determined by using stepped GA algorithm for real response

<table>
<thead>
<tr>
<th>Type</th>
<th>Manual Calculation (Bari &amp; Hassan, 2002)</th>
<th>GA optimization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{1-4}$</td>
<td>60000.0, 3228.0, 455.0, 15000.0</td>
<td>53024.0, 4685.0, 437.14, 16904.0</td>
</tr>
<tr>
<td>$\gamma_{1-4}$</td>
<td>20000.0, 400.0, 11.0, 5000.0</td>
<td>21397.0, 451.0, 7.85, 8241.0</td>
</tr>
<tr>
<td>$\bar{\bar{a}}_{4}, \delta'$</td>
<td>5.0, 0.18</td>
<td>3.81, 0.28</td>
</tr>
<tr>
<td>Optimization Time (sec)</td>
<td>-------</td>
<td>1283.5</td>
</tr>
<tr>
<td>Stable Loop Fitness, $f_{stb}$</td>
<td>0.017</td>
<td>0.0099</td>
</tr>
<tr>
<td>Uniaxial Ratcheting Fitness, $f_{uni}$</td>
<td>0.126</td>
<td>0.119</td>
</tr>
<tr>
<td>Biaxial Ratcheting Fitness, $f_{bi}$</td>
<td>0.144</td>
<td>0.068</td>
</tr>
<tr>
<td>Stable Loop Simulation</td>
<td>0.024</td>
<td>0.018</td>
</tr>
<tr>
<td>First Loop Simulation</td>
<td>0.173</td>
<td>0.131</td>
</tr>
<tr>
<td>Bow-tie Ratcheting Fitness, $f_{bow}$</td>
<td>0.27</td>
<td>0.29</td>
</tr>
</tbody>
</table>

The GA and traditional manual parameter sets are further evaluated against three more sets of cyclic responses obtained from CS1026 material. These are stabilized hysteresis loop from uniaxial strain controlled test, first hysteresis loop from stress controlled (ratcheting) tests and bow-tie ratcheting rate response. The GA parameters simulate the whole hysteresis loop better (Figs. 2.16a and 2.17a). During parameter determination only the loading part is considered. Hence, the loading part simulation matched better than the unloading part in both simulations. The improvement in first loop simulation with GA optimized parameters is apparent when Figs. 2.16b and 2.17b are compared. Even though ratcheting predicted by first loop (Figs. 2.16b and 2.17b) is not significantly modified, overall simulation of the first loop is better with GA parameters than manually determined parameters. The bow tie ratcheting
simulations with GA parameters is slightly inferior to manually determined parameters (see Figs. 2.16c and 2.17c).

Fig. 2.14. Comparison of the experimental results and simulation obtained by manual optimized parameters for modified Chaboche model for (a) loading part of hysteresis curve (b) uniaxial ratcheting response and (c) biaxial ratcheting response.

Fig. 2.15. Comparison of the experimental results and simulation obtained by stepped GA optimized parameters for modified Chaboche model for (a) loading part of hysteresis curve (b) uniaxial ratcheting response and (c) biaxial ratcheting response.
Fig. 2.16. Comparison of the experimental results and simulation obtained by manual optimized parameters for (a) hysteresis curve (b) first hysteresis loop of uniaxial ratcheting response and (c) bow-tie ratcheting response.

Fig. 2.17. Comparison of the experimental results and simulation obtained by Stepped GA optimized parameters for (a) hysteresis curve (b) first hysteresis loop of uniaxial ratcheting response and (c) bow-tie ratcheting response.

2.8 Optimization strategy for Ohno-Wang and Abdel Karim-Ohno models

The stepped optimization strategy is used for Ohno-Wang (1993) and Abdel Karim-Ohno (2000) plasticity models. The Ohno-Wang kinematic hardening rule (1993) is given by

\[ d \alpha = \sum_{i=1}^{M} d a_i \]  

\[ d a_i = \frac{2}{3} C_i \varepsilon^p - \gamma_i a_i \left( d \varepsilon^p : \frac{a_i}{f(a_i)} \right) \left( \frac{f(a_i)}{C_i / \gamma_i} \right)^{m_i} \]  

for \( i = 1 \sim M \).
where, \( C_1, C_M, \gamma_1, \gamma_M \) and \( m_i \) to \( m_M \) are model parameters. The power \( m_i \) is assumed to be same (= \( \gamma \)) for all hardening rule.

Ohno-Wang model parameters are determined from uniaxial hysteresis curve and uniaxial ratcheting experiment. The loading part of the stable hysteresis curve is first divided into \( M \) number of linear segments (Fig. 2.18a). Each linear segment represented by a decomposed kinematic hardening rule. As the center of the yield surface initially moves fast with small change in plastic strain, initial three segments are taken as 2%, 6% and 8% of the plastic strain range. The remaining plastic range is equally divided to the other segments. The parameters are determined from the following relationship (Jiang-Sehitoglu, 1996)

\[
C_i = \frac{\sigma_x^i - \sigma_x^{i-1}}{\varepsilon_p^i - \varepsilon_p^{i-1}} - \frac{\sigma_x^{i+1} - \sigma_x^i}{\varepsilon_p^{i+1} - \varepsilon_p^i} \quad \text{for } i = 2, (M - 1) \tag{2.9a}
\]

\[
\gamma_i = \frac{2}{\varepsilon_p^i + \varepsilon_p^0} \quad \text{for } i = 1, (M - 1) \tag{2.9b}
\]

The parameters \( C_1 \) and \( C_M \) is determined from

\[
C_1 = \frac{\sigma_x^1 - \sigma_x^0}{\varepsilon_p^1 - \varepsilon_p^0} \quad \tag{2.10a}
\]

\[
C_M = \frac{\sigma_x^M - \sigma_x^{M-1}}{\varepsilon_p^M - \varepsilon_p^{M-1}} \quad \tag{2.10b}
\]
The parameters $\gamma_M$ and $m_i (= m)$ are the ratcheting parameters.

The stepped GA optimization strategy developed in this study is used for parameter optimization for Ohno-Wang Model. In first step, parameters $C_1$ to $C_M$ and $\gamma_1$ to $\gamma_{M-1}$ are optimized through best representation of the loading and unloading part of hysteresis loop. The additional requirement of matching stress values at higher strain range is achieved through the constraint equation, as done for modified Chaboche model. In the first step, initial range for the parameters $C_2$ to $C_{M-1}$ and $\gamma_1$ to $\gamma_{M-1}$ are varied in range $\pm 10\%$ of the initial estimate. The parameter $C_M$ is determined from the end slope for loading part of the hysteresis curve for higher strain range (Eq. 2.10b). As $C_M$ can be measured precisely, it is varied in $\pm 5\%$ range. In this step, $\gamma_M$ is assumed a small value (e.g.10) and $m$ is kept equal to zero. The upper and lower range for $C_1$ is taken equal to

$$C_{1,\max} = \frac{\sigma_0^1 - \sigma_0^0}{\varepsilon_1^p - \varepsilon_0^p}$$

$$C_{1,\min} = \frac{\sigma_0^1 - \sigma_0^0}{\varepsilon_1^p - \varepsilon_0^p} - \frac{\sigma_2^1 - \sigma_1^1}{\varepsilon_2^p - \varepsilon_1^p}$$

In second step, $\gamma_M$ and $m$ are determined through primarily optimizing the simulations of uniaxial ratcheting. However, as $\gamma_M$ and $m$ may change the hysteresis curve simulation a little, $C_i$ to $C_M$ and $\gamma_1$ to $\gamma_{M-1}$ are further optimized by varying within $\pm 3\%$ range of the last determined values. The parameter $m$ is varied in between 0~1.0. Initial range (upper and lower value) for $\gamma_M$ in the optimization scheme are taken from Eq. 2.12 (Bari and Hassan, 2002).

$$\gamma_{M,\min} = \frac{2}{0.10 \ast \varepsilon_{L,2}^p + \varepsilon_0^p} \quad \text{and} \quad \gamma_{M,\max} = \frac{2}{0.025 \ast \varepsilon_{L,2}^p + \varepsilon_0^p}$$

The stepped GA optimization strategy for Ohno-Wang model was verified using the real test data of CS1026 simulation. This set of data was manually simulated by Bari and Hassan (2002) with twelve kinematic hardening rules. The GA optimized parameters are compared with manually determined parameters in Table 2.6. The simulations are compared in Fig. 2.19 and Fig. 2.20. The manually optimized and GA optimized parameters (Table 2.6) are further evaluated with bow tie ratcheting response simulation (shown in Fig. 2.21).
Table 2.6
Ohno-Wang model parameters for CS1026 material

<table>
<thead>
<tr>
<th>Type</th>
<th>Manually Determined Parameters (Bari &amp; Hassan, 2002)</th>
<th>Optimized Parameters (Stepped GA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No of decomposed rules</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>C</td>
<td>31940, 36213, 2520, 376, 11021, 4550, 3474, 2196, 857, 247, 98, 200</td>
<td>25898, 14113, 2227, 988, 274, 1136, 154, 80, 29, 17, 14, 371</td>
</tr>
<tr>
<td>γ</td>
<td>45202, 13944, 7728, 4955, 3692, 2135, 1230, 584, 295, 119, 50, 20</td>
<td>45202, 13944, 7728, 4955, 3692, 2135, 1230, 584, 295, 119, 50, 20</td>
</tr>
<tr>
<td>m</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>Stable loop fitness</td>
<td>0.032</td>
<td>0.029</td>
</tr>
<tr>
<td>Uniaxial ratcheting fitness</td>
<td>0.1642</td>
<td>0.0508</td>
</tr>
<tr>
<td>Bowtie ratcheting fitness</td>
<td>0.5467</td>
<td>0.4413</td>
</tr>
</tbody>
</table>

Fig. 2.19. Comparison of the experimental results and simulation obtained by manual optimized parameters for Ohno-Wang model for (a) hysteresis curve (b) uniaxial ratcheting response

Fig. 2.20. Comparison of the experimental results and simulation obtained by stepped GA optimized parameters for Ohno-Wang Model (a) hysteresis curve (b) uniaxial ratcheting response.
The stepped GA optimization strategy is further evaluated in optimizing the parameters for the same test data, but with seven and five decomposed kinematic hardening rules for the Ohno-Wang (1993) model. These parameters along with the simulation fitness values are shown in Table 2.7. Results in Table 2.7 indicate that with the GA optimization scheme it is possible to minimize the number of decomposed kinematic hardening rules in Ohno-Wang model while determining a good set of parameters. Reducing the number of kinematic hardening rules also decreases the finite element computation time with Ohno-Wang model.

![Graphs showing comparison of manual vs. stepped GA optimized parameters](image_url)

**Fig.2.21** Comparison of Bow tie ratcheting simulation with Ohno-Wang model for (a) manually optimized parameters (b) stepped GA optimized parameters.

<table>
<thead>
<tr>
<th>Type</th>
<th>Optimized Parameters (Stepped GA)</th>
<th>Optimized Parameters (Stepped GA)</th>
<th>Optimized Parameters (Stepped GA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No of decomposed Rules</td>
<td>12</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>$C$</td>
<td>25898, 14113, 2227, 988, 274, 1136, 154, 80, 29, 17, 14, 371 11875, 2178, 770, 275, 361, 462, 331, 341, 300, 208, 298, 29, 0.508</td>
<td>48522, 4159, 922, 246, 74, 25, 486 5464, 737, 288, 277, 195, 110, 48, 0.37</td>
<td>65181, 5065, 693, 230, 518 7985, 669, 439, 200, 36, 0.18</td>
</tr>
<tr>
<td>$\gamma$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stable Loop Fitness</td>
<td>0.029</td>
<td>0.0243</td>
<td>0.026</td>
</tr>
<tr>
<td>Uniaxial Ratcheting Fitness</td>
<td>0.0508</td>
<td>0.1204</td>
<td>0.054</td>
</tr>
</tbody>
</table>

**Table 2.7**

Ohno-Wang model parameters with different number of kinematic hardening rules for CS1026 material.
The kinematic hardening rules for Abdel Karim-Ohno (2000) model is given by

\[ d\mathbf{a} = \sum_{i=1}^{M} d\mathbf{a}_i \]  

(2.13)

\[ da_i = 2/3 C_i d\varepsilon^p - \mu_i \gamma_i a_i dp - \gamma_i a_i H\left(\frac{1}{2} a_i : a_i - \left(\frac{C_i}{\gamma_i}\right)^2\right) d\lambda_i \]  

(2.13a)

where \( d\lambda_i = d\varepsilon^p : \frac{a_i}{C_i / \gamma_i} - \mu_i dp \) for \( i = 1\sim M \),

and \( H \) stands for Heaviside step function. The parameter \( \mu_1 \) to \( \mu_M \) is assumed to be same (= \( \mu \)) for all hardening rule.

In the first step of parameter optimization with stepped GA for Abdel Karim-Ohno (2000) model, \( C_1 \) to \( C_M \) and \( \gamma_1 \) to \( \gamma_{M-1} \) are optimized through simulating the loading and unloading part of the hysteresis loop. Initial estimate for \( C_1 \) to \( C_M \) and \( \gamma_1 \) to \( \gamma_{M-1} \) parameters are obtained through dividing the loading part of the hysteresis loop into \( M \) segments as done with Ohno-Wang model (1993). Here, \( \gamma_M \) is assumed a small value (e.g.10). The ranges for variation of \( C_1 \) to \( C_M \) and \( \gamma_1 \) to \( \gamma_{M-1} \) are set \( \pm 10\% \) of the initial estimates. Similar to Ohno-Wang model, \( C_M \) is varied in \( \pm 5\% \) range only. In first optimization step, \( \mu \) are set equal to 1.0. In this step, \( C_1 \) to \( C_M \) and \( \gamma_1 \) to \( \gamma_{M-1} \) are determined from the best fit of loading and unloading part of the hysteresis loop. In second optimization step, \( \gamma_M \) and \( \mu \) are determined through primarily optimizing the simulations of uniaxial or biaxial ratcheting response. As the parameter \( \mu \) may change the simulation of hysteresis loop, \( C_1 \) to \( C_M \) and \( \gamma_1 \) to \( \gamma_{M-1} \) are further optimized in this step. These parameters \( C_1 \) to \( C_M \) and \( \gamma_1 \) to \( \gamma_{M-1} \) varied within \( \pm 3\% \) range and the initial range for \( \gamma_m \) are set according Eq. 2.12. The parameters \( \mu \) are varied in the range of 0~1.0. In this step, all the parameters are optimized through simulating hysteresis loop and uniaxial or biaxial ratcheting response simultaneously.
Table 2.8
Abdel Karim-Ohno model parameters for CS1026 material

<table>
<thead>
<tr>
<th>Type</th>
<th>Manually Determined Parameters (Bari &amp; Hassan, 2002)</th>
<th>Optimized Parameters (Stepped GA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No of Rules</td>
<td>12</td>
<td>5</td>
</tr>
<tr>
<td>C</td>
<td>31940,36214,2520,376,11021,4551,3475,2196,857,247,98,200</td>
<td>42206,4845,931,1236,378,45202,13944,7728,4955,3692,2135,1230,584,295,119,50,20</td>
</tr>
<tr>
<td>γ</td>
<td>0.31940,36214,2520,376,11021,4551,3475,2196,857,247,98,200</td>
<td>0.45202,13944,7728,4955,3692,2135,1230,584,295,119,50,20</td>
</tr>
<tr>
<td>µ</td>
<td>0.07</td>
<td>0.07</td>
</tr>
<tr>
<td>Stable Loop Fitness</td>
<td>0.0229</td>
<td>0.0269</td>
</tr>
<tr>
<td>Biaxial Ratcheting Fitness</td>
<td>0.1905</td>
<td>0.228</td>
</tr>
<tr>
<td>Bowtie ratcheting fitness</td>
<td>0.756</td>
<td>0.741</td>
</tr>
</tbody>
</table>

Fig. 2.22. Comparison of the experimental results and simulation obtained by manual optimized parameters for Abdel Karim-Ohno model for (a) hysteresis curve (b) biaxial ratcheting response.

Fig. 2.23. Comparison of the experimental results and simulation obtained by stepped GA optimized parameters for Abdel Karim-Ohno Model (a) hysteresis curve (b) biaxial ratcheting response.
Bari and Hassan (2002) determined the Abdel Karim-Ohno (2000) model parameters for CS1026 with 12 decomposed kinematic hardening rules from hysteresis loop and biaxial ratcheting simulation. The stepped optimization scheme can determine a set of parameters that simulate same hysteresis loop and biaxial ratcheting response of CS1026 with reasonable fitness values with five kinematic hardening rules for the Abdel-Karim-Ohno Model. The model parameters from the optimization scheme are compared with the manually determined parameters (Bari and Hassan, 2002) in Table 2.8. The simulations with these parameters are shown in Fig. 2.22 and Fig. 2.23, which demonstrates that the stepped GA can determine a good set of parameters for Abdel Karim-Ohno model. The two parameter sets shown in Table 2.8 are further evaluated with bow tie ratcheting response simulation which are shown in Fig. 2.24.

![Fig.2.24 Comparison of Bow tie ratcheting simulation with Abdel Karim-Ohno model for (a) manually optimized parameters (b) stepped GA optimized parameters.](image)

### 2.8 Conclusion

In this study, GA search technique is used to automate the determination of advanced constitutive model parameters. This study compared the parameters determined from manual approach and GA based search approach. Manual parameter determination is tedious, time consuming and requires pre-experience in parameter determination for advanced constitutive models. Automation of parameter search using GA can remove the need for experience in parameter determination and determine better parameters than manual approach. Conventional GA without any initial estimate of the parameters from their physical meaning is found not efficient and accurate in determining model parameters. In parameter search, whenever parameters are restricted to vary only within a narrow range about the initial
estimate based on physical significance of parameters, the efficiency of GA increases significantly. The efficiency of the GA based optimization scheme in determining the advanced constitutive model parameters increase significantly, when GA search is performed in three steps (stepped GA) rather than simultaneous search of all parameters.
CHAPTER THREE
NUMERICAL SCHEMES FOR IMPLEMENTING ADVANCED CYCLIC PLASTICITY MODELS

3.1 Introduction

Cyclic plasticity models are based on incremental relationship for capturing history dependence behavior. As discussed in Chapter Two, these plasticity relationships are given by

Strain decomposition: \( d \varepsilon = d \varepsilon^e + d \varepsilon^p \)  
(3.1)

Hook’s law: \( d \varepsilon^e = \frac{1 + \nu}{E} d \sigma - \frac{\nu}{E} \text{tr}(d \sigma) I \)  
(3.2)

Flow-rule: \( d \varepsilon^p = \frac{1}{H} \left( \frac{\delta f}{\delta \sigma} : d \sigma \right) \left( \frac{\delta f}{\delta \sigma} \right) \)  
(3.3)

von-Mises criteria: \( f(\sigma - \alpha) = \left[ \frac{3}{2}(\kappa - \alpha):(\kappa - \alpha) \right]^{1/2} = \sigma_0 \)  
(3.4)

And kinematic hardening rule (e.g. for Modified Chaboche model)

\[ da = \sum_{i=1}^{4} da_i \]  
(3.5)

\[ da_i = \frac{2}{3} C_i d \varepsilon^p - \gamma_i (a_i : \delta^i) + (1 - \delta^i)(a_i : n)n \] \( dp \), for \( i = 1, 2, 3 \) and  
(3.5a)

\[ da_4 = \frac{2}{3} C_4 d \varepsilon^p - \gamma_4 (a_4 : \delta^i) + (1 - \delta^i)(a_4 : n)n \left( 1 - \frac{a_4}{f(a_4)} \right) dp \]  
(3.5b)

where \( \sigma \) is the stress tensor, \( \varepsilon \) is the plastic strain tensor, \( \varepsilon^p \) is the plastic strain tensor, \( s \) is the deviatoric stress tensor, \( \alpha \) is the current center of the yield surface, \( a \) is the current center of the yield surface in deviatoric space, \( n \) is the normal to yield surface, \( \sigma_0 \) is the size of the yield surface, \( H \) is the plastic modulus and \( dp = |d \varepsilon^p| = \left[ \frac{2}{3} d \varepsilon^p : d \varepsilon^p \right]^{1/2} \). In the kinematic hardening model, \( C \)'s, \( \gamma \)'s, \( a_4 \) and \( \delta^i \) are model parameters of the modified Chaboche model (Bari and Hassan, 2002). In addition, plastic modulus \( H \) in Eq. 3.3 is calculated using the approximate consistency condition (Bari and Hassan, 2001), and is given by the relationship
\[ H = \sum_{i=1}^{4} H_i \]  

where, for modified Chaboche model

\[ H_i = C_i - \gamma_i \left( a_i : \frac{\partial f}{\partial \sigma} \right) \]  

for \( i = 1,2,3 \)  

\[ H_4 = C_4 - \gamma_4 \left( a_4 : \frac{\partial f}{\partial \sigma} \right) \left( 1 - \frac{\bar{a}_4}{f(\bar{a}_4)} \right) \]  

When these equations are written for uniaxial loading, it reduces to

\[ H = \sum_{i=1}^{4} H_i \]  

where, \( H_i = C_i - \gamma_i \alpha_i \)  

for \( i = 1,2,3 \)  

\[ H_4 = C_4 \]  

when \( \alpha_{4,x} \leq \bar{a}_4 \)  

\[ H_4 = C_4 - \gamma_4 \alpha_4 \]  

when \( \alpha_{4,x} > \bar{a}_4 \)  

After integration, the closed form solution of the stress-strain relationship (Eq. 3.3) for uniaxial monotonic loading is given by

\[ \sigma_x = \sum_{i=1}^{4} \alpha_i + \sigma_0 \]  

\[ \alpha_i = \frac{C_i}{\gamma_i} \left[ 1 - \exp(-\gamma_i \varepsilon_x^p) \right] \]  

for \( i = 1,2,3 \)  

\[ \alpha_4 = C_4 \varepsilon_x^p \]  

\[ \alpha_4 = \bar{a}_4 + \frac{C_4}{\gamma_4} \left[ 1 - \exp(-\gamma_4 (\varepsilon_x^p - \varepsilon_{xa}^p)) \right] \]  

\[ \alpha_4 > \bar{a}_4 \]  

where \( \varepsilon_{xa}^p = \bar{a}_4 / C_4 \)

A sketch of stress-strain relationship derived above for uniaxial monotonic loading in the nonlinear range is shown in Fig. 3.1. These closed form solutions (Eq. 3.7-3.8) only can be derived for uniaxial loading. For multiaxial loading, even for monotonic case, calculations are history dependent must be carried out incrementally. For capturing the history
dependence, the solution of the plasticity equations requires iterative scheme for each load increment. Moreover, as plasticity formulation are developed for differential increments (Eq. 3.1-3.6) and accurate result can only be obtained through numerical calculation using infinitesimal load increments. In reality, finite size of increment $\Delta \varepsilon^p$ is used; stress increment $\Delta \sigma^p$ is calculated through linear approximation over this increment size as shown in Fig. 3.1. Such linear approximation over the increment size introduces error in numerical calculation (see Fig. 3.1). The load increments must be sufficiently small to keep these errors within the acceptable range. This requirement of small load increments is impractical for structural analysis because of high computational cost. Hence, numerical schemes were developed for implementing plasticity models such that large load increments can be prescribed for structural analysis.

![Diagram showing finite increment calculation in representing the stress-strain response under uniaxial loading](image)

**Fig. 3.1** Finite increment calculation in representing the stress-strain response under uniaxial loading

Several numerical schemes of explicit, semi-implicit and implicit (Euler and Runge-Kutta types) type have been suggested to solve the complex, nonlinear, incremental plasticity equations at each integration point. Many attempts in representing the non-linear material response were made with Euler methods (Hunsaker et.al., 1976; Kreig and Key, 1976; Rice and Tracey, 1973; Ortiz and Popov, 1985 and many others). These incremental plasticity equations are also solved with Runge-Kutta methods (Buttner and Simeon, 2002). These methods basically solved the plasticity equations (Eq. 3.1 to 3.5) are using mainly two
In one approach, plasticity solutions use the first order approximation of von-Mises function \( df = 0 \), known as the consistency condition) with explicit calculation of plastic modulus, \( H \) (Chaboche, 1990). In this approach, solution algorithm uses the approximate consistency condition along with kinematic hardening rule for upgrading the stress and state variables for the increment step through linear shooting based on the plastic modulus. In plastic modulus calculation, computation for the incremental equations are performed using the Euler and Runge-Kutta numerical methods. This approach is referred to as the classical approach in this dissertation. The other solution approach uses two steps: elastic prediction and plastic correction (Wilkins, 1964). These are return type solution scheme for plasticity problems. Here, solution is first predicted assuming the strain increment as elastic increment and then necessary correction is performed to return to the yield surface. This return step can also be calculated with Euler and Runge-Kutta methods which also employ the kinematic hardening rule. Wilkins (1964) first proposed the radial return algorithm for the plasticity problem.

In solving plasticity problems, Euler and Runge-Kutta type numerical methods have been compared (Kreig and Kreig, 1977, Simo and Taylor, 1984, Dodds, R.H., 1987, Wang et. al., 2000, Sawyer et. al, 2001, Buttner and Simeon, 2002). Most of the studies verified the numerical techniques for monotonic loading only at the material and structural levels. These numerical schemes are not verified yet against cyclic loading either at material or structural levels. A main reason for this lagging behind is that the advanced cyclic plasticity models are mostly proposed after 80’s, whereas most numerical studies preceded that era. Demand on cyclic plasticity analyses is gradually increasing. Advanced cyclic plasticity models like Chaboche (1991), Bari and Hassan (2002), Ohno-Wang (1993), Abdel Karim-Ohno (2000) can simulate nonlinear and complex material responses. However the performance of these models with various numerical implementation schemes is not known. The importance of studying the numerical implementation lies in the fact that small error in each incremental step due to approximation in numerical scheme accumulates and might introduce significant error after large number of cycle.

This study is devoted to compare various numerical schemes for multiaxial cyclic plasticity. While structural analysis is performed in finite element methods, numerical calculation is carried out in two steps. In first step, local calculation is carried out at each integration points
of the element. In second step, global solution is carried out for nodal force calculations through Newton-Raphson iteration method. This chapter evaluates the local numerical schemes and suggests the best numerical scheme for implementing advanced cyclic plasticity models into finite element analysis codes.

This study examines several numerical schemes for implementing the Chaboche type advanced constitutive models (Bari-Hassan, 2002; Chaboche, 1989, 1991; Ohno–Wang, 1993; Abdel Karim–Ohno, 2000 and Chen-Jiao, 2003) for structural response simulation. These plasticity models are highly nonlinear which restricts the use of these models for large load increment. Numerical schemes suffer from lack of stability and accuracy when calculations with these models are performed with large strain increments. This study proposed a technique for the Chaboche type plasticity models in order to avoid numerical problems with large strain increments. Another aspect of the return type schemes is that these are proposed as strain driven approach in return type algorithm. These schemes calculate the stress increments for prescribed strain increments in finite element analysis. However, for evaluating plasticity models without using finite element analysis needs a numerical technique that can solve the plasticity equation when stress increments are also prescribed. Hence, this chapter proposed a new method called stress return algorithm for prescribing stress components in combination with the strain components cyclic plasticity calculations without the use of finite element analysis.

In subsequent sections, at first the implementation techniques for Euler and Runge-Kutta type solution schemes in classical and return approaches are described with respect to the modified Chaboche model. Description of numerical difficulties and techniques of mitigating these problems are presented. The last section presented the accuracy and stability of numerical schemes with respect to the closed form solutions and experimental responses of uniaxial and multiaxial ratcheting. The method of implicit elastic predictor-plastic corrector (radial return) algorithm is found to be optimum for cyclic plasticity. The methodology to implement radial return scheme for Ohno-Wang (1993), Abdel Karim-Ohno (2000) and modified Ohno-Wang model (2003) is presented in Appendix A.

3.2 Numerical schemes

As mentioned in the introduction, there are two types of solution scheme for plasticity calculations: classical approach and return mapping algorithm. In the subsequent
subsections, basic steps of classical plasticity calculation and return mapping solution scheme will be presented for the modified Chaboche constitutive model.

3.2.1 Classical approach

In the elastic range, stress increment for given strain increment is obtained using the Hook’s Law (Eq. 3.2). In the inelastic range, classical approach approximates the nonlinear stress-strain path through chordal approximations for a finite increment step. This linear approximation is based on plastic modulus calculation. This linear approximation certainly introduces discretization error (Doods, 1987) and this type of calculation scheme uses the information of the previous step only.

At the beginning of load increment step, stress increment calculation involves assuming the entire strain increment as elastic. The trial elastic stress increment is calculated as

\[ d\sigma_{n+1}^T = D^e : d\varepsilon_{n+1} \]  

and the trial total stress as

\[ \sigma_{n+1}^T = \sigma_n + d\sigma_{n+1}^T \]  

The yield condition is checked using the yield criterion (Eq. 3.4)

\[ f_{n+1}^T = \frac{3}{2} \left( \tilde{s}_{n+1}^T - a_n^T \right) \left( \tilde{s}_{n+1}^T - a_n^T \right) - \sigma_0^2 \]  

where \( \tilde{s}_{n+1}^T \) are the deviatoric components of \( \sigma_{n+1}^T \). If \( f_{n+1}^T \leq 0 \), \( \sigma_{n+1}^T \) is accepted as updated stress \( \sigma_{n+1} \), that is the strain increment \( d\varepsilon_{n+1} \) is an elastic increment.

If \( f_{n+1}^T > 0 \), for classical calculation method, the elastic part of the strain increment is calculated using the consistency condition

\[ f_{n+1}^T = \frac{3}{2} \left( \tilde{s}_n + kd\tilde{s}_{n+1}^T - a_n^T \right) \left( \tilde{s}_n + kd\tilde{s}_{n+1}^T - a_n^T \right) - \sigma_0^2 = 0 \]  

where \( k \) represents the part of elastic trial stress increment, that brings the total stress on the yield surface. That is, the elastic and elastic-plastic parts of the strain increment are \( kd\varepsilon_{n+1} \) and \( (1-k)d\varepsilon_{n+1} \), respectively.

The total stress and strains are updated for the elastic part of the increment as

\[ \sigma_{n+1} = \sigma_n + kd\sigma_{n+1}^T \]  

\[ \varepsilon_{n+1} = \varepsilon_n + kd\varepsilon_{n+1} \]
The calculation for the remaining elastic-plastic increment, \((1-k)d\varepsilon_{n+1}\) is performed according to the plasticity model. Note that when \(\sigma_n\) lies on the yield surface, \(k\) from Eq.3.11 will become zero, thus the total increment \(d\varepsilon_{n+1}\) becomes elastic-plastic increment.

Plasticity calculation in the classical approach is based on calculating plastic modulus and linear approximation for the increment. In this approach, for coupled models like the modified Chaboche constitutive model, the plastic modulus calculation is coupled with the kinematic hardening rule through the consistency condition, which requires that the total stress always lies on the yield surface, i.e.

\[ f(\sigma_{n+1} - \sigma_{n+1}) = \sigma_0 \]  

(3.13)

First order differentiation of Eq. 3.13 in gives

\[ \frac{\partial f}{\partial \sigma_{n+1}} = \frac{3}{2} \frac{(\sigma_{n+1} - \sigma_{n+1})}{\sigma_0} \]  

(3.14)

Note that

\[ \frac{\partial f}{\partial \sigma_{n+1}} : \frac{\partial f}{\partial \sigma_{n+1}} = \frac{3}{2} \]  

(3.15a)

\[ \eta_{n+1} : \frac{\partial f}{\partial \sigma_{n+1}} = \sqrt{\frac{3}{2}} \]  

(3.15b)

The flow rule (Eq. 3.3) can be rewritten as

\[ d\varepsilon_{n+1} = d\lambda \frac{\partial f}{\partial \sigma_{n+1}} \]  

(3.16)

where, the plastic multiplier, \(d\lambda = \frac{1}{H} \left( \frac{\partial f}{\partial \sigma_{n+1}} : d\sigma_{n+1} \right) \)

(3.16a)

Using Eq. 3.15a and Eq. 3.16, and \(dp = \left[ \frac{2}{3} d\varepsilon^p : d\varepsilon^p \right]^{\frac{1}{2}}\), it can be shown that

\[ dp = d\lambda \]  

(3.17)

First order approximation of the consistency condition (Eq. 3.13) can be written as

\[ df = 0 \]  

(3.18a)

or,

\[ \frac{\partial f}{\partial \sigma_{n+1}} : d\sigma_{n+1} = \frac{\partial f}{\partial \sigma_{n+1}} : d\sigma_{n+1} \]  

(3.18b)

Eq. 3.18b and Eq. 3.5 and Eq. 3.14 yields
\[
\frac{\partial f}{\partial \sigma_{n+1}} : d\sigma_{n+1} = \left[ \sum_{i=1}^{3} 2 \left( \frac{\partial f}{\partial \sigma_{n+1}} : \frac{\partial f}{\partial \sigma_{n+1}} \right) - \sum_{i=1}^{3} \gamma \left( \frac{\partial f}{\partial \sigma_{n+1}} : \left( 1 - \delta \right) a_{n+1,i} : n_{n+1} \left( n_{n+1} : \frac{\partial f}{\partial \sigma_{n+1}} \right) \right) \right] d\lambda
\]

Eq. 3.16a and Eq. 3.19 yields the plastic modulus similar to Chaboche et al. (1986),

\[
H = \frac{1}{d\lambda} \left\{ \frac{\partial f}{\partial \sigma_{n+1}} : d\sigma_{n+1} \right\} = \sum_{i=1}^{4} H_{i}
\]

where

\[
H_{i} = \frac{2}{3} C \left( \frac{\partial f}{\partial \sigma_{n+1}} : \frac{\partial f}{\partial \sigma_{n+1}} \right) - \gamma \left( \frac{\partial f}{\partial \sigma_{n+1}} + (1 - \delta) a_{n+1,i} : n_{n+1} \left( n_{n+1} : \frac{\partial f}{\partial \sigma_{n+1}} \right) \right) \quad \text{for } i=1,2,3
\]

\[
H_{4} = \frac{2}{3} C \left( \frac{\partial f}{\partial \sigma_{n+1}} : \frac{\partial f}{\partial \sigma_{n+1}} \right) - \gamma \left( \frac{\partial f}{\partial \sigma_{n+1}} + (1 - \delta) a_{n+1,4} : n_{n+1} \left( n_{n+1} : \frac{\partial f}{\partial \sigma_{n+1}} \right) \right) \left( 1 - \frac{\bar{a}_{4}}{f(a_{n+1,4})} \right)
\]

Applying von-Mises function and its derivative from Eq. 3.4 and Eq. 3.14, to Eq. 3.15a, plastic modulus can be calculated as in Eq. 3.21 which is independent of multiaxial parameter \( \delta \)

\[
H = \sum_{i=1}^{4} H_{i}
\]

where, \( H_{i} = C_{i} - \gamma_{i} \left( a_{n+1,i} : \frac{\partial f}{\partial \sigma_{n+1}} \right) \) for \( i=1,2,3 \) \n
\[
H_{4} = C_{4} - \gamma_{4} \left( a_{n+1,4} : \frac{\partial f}{\partial \sigma_{n+1}} \right) \left( 1 - \frac{\bar{a}_{4}}{f(a_{n+1,4})} \right)
\]

Eq. 3.3 and Eq. 3.14 yields

\[
d\varepsilon_{n+1}^{p} = \frac{9}{4H} \left\{ \left( s_{n+1} - a_{n+1} \right) \sigma_{0} / \sigma_{0} \right\} d\sigma_{n+1}
\]

Eq. 3.22 can be rewritten as

\[
d\varepsilon_{n+1}^{p} = \frac{9}{4H\sigma_{0}^{2}} \left( s_{n+1} - a_{n+1} \right) \otimes \left( s_{n+1} - a_{n+1} \right) d\sigma_{n+1}
\]

The Hooks law (Eq. 3.2) can be expressed as

\[
d\varepsilon_{n+1}^{c} = C d\sigma_{n+1}
\]
Plugging in the values of \( d\varepsilon^e_{n+1} \) and \( d\varepsilon^p_{n+1} \) in strain decomposition assumption Eq. 3.1 gives the elastic-plastic stress-strain relationship

\[
d\varepsilon^e_{n+1} = \left[ C + \frac{9}{4H_o^2} \left( \left( \varepsilon^e_{n+1} - a_{n+1} \right) \otimes \left( \varepsilon^p_{n+1} - a_{n+1} \right) \right) \right] d\sigma_{n+1} \tag{3.25}
\]

Stress increment for a given strain increment can be calculated using Eq. 3.25. Equation 3.25 can also be used for problems at the material level where stress and strain components are needed to be prescribed in any combination.

**3.2.1.1 Solution scheme**

The plasticity problem involving Eq. 3.25 can be described as initial value problems with incremental equations, where explicit integration is not always possible. The \( d\varepsilon - d\sigma \) relationship in Eq. 3.25 yields stress increment for an infinitesimal strain increment. However, finite element calculations are performed with finite size of increment, \( \Delta\varepsilon \). The numerical schemes linearly approximate the non-linear stress-strain path for \( \Delta\varepsilon \) based on the plastic modulus. Thus, accuracy of the numerical calculation is largely dependent on the plastic modulus calculation. In modified Chaboche model, plastic modulus calculation is coupled with kinematic hardening rule through the consistency condition. The expression for plastic modulus (Eq. 3.21), \( H \) is derived using consistency condition (Eq. 3.18) and kinematic hardening rule (Eq. 3.5). Here, Eq. 3.21 shows that plastic modulus calculation is a function of yield surface position \( (a_{\sigma_{n+1}}) \) and normal direction of the yield surface \( \left( \frac{\partial f}{\partial \sigma_{\sigma_{n+1}}} \right) \) at the new stress point. This can be expressed in a generalized form

\[
H_{\sigma_{n+1}} = g_1(a_{\sigma_{n+1}}, n_{\sigma_{n+1}}) \quad \text{where} \quad n_{\sigma_{n+1}} = \frac{2}{3} \frac{\partial f}{\partial \sigma_{\sigma_{n+1}}} \tag{3.26}
\]

For uniaxial loading, normal direction remains fixed irrespective of the position on the increment. Thus the plastic modulus calculation is dependent on position of the yield surface only. For multiaxial loading, plastic modulus is a function of both the normal direction and position of the yield surface. For latter case, normal direction changes with yield surface translation. The normal directions for uniaxial and multiaxial loading cases is demonstrated in Fig. 3.2 (Bari and Hassan, 2001). Figure 3.2a demonstrates that normal direction remains unchanged with translation of the yield surface under uniaxial loading, whereas for multiaxial case (Fig. 3.2b), normal direction changes with translation of the yield surface. In
plastic modulus calculation, \( a_{n+1} \) and \( \frac{\partial f}{\partial \sigma_{n+1}} \) are initially unknown. Different numerical schemes like Euler and Runge-Kutta methods are employed which assumes different values for \( a_{n+1} \) and \( \frac{\partial f}{\partial \sigma_{n+1}} \) for plastic modulus calculation. Euler and Runge-Kutta methods are approximation of Taylor’s series which are used for approximating non-linear response.

![Fig. 3.2 Yield surface translation and change in normal direction, (a) uniaxial loading and (b) multiaxial loading.](image)

**3.2.1.2 Euler methods**

Ortiz and Popov (1985) described the three Euler type numerical schemes in a generalized pattern. The explicit, semi-implicit and implicit Euler solution schemes are evaluated for plasticity calculation in this study. The general equation for plastic modulus calculation in Euler method can be expressed as

\[
H_{n+1} = \left(1 - \phi\right) g_1(a_n, \sigma_n) + \phi g_1(a^{(k)}_{n+1}, \sigma^{(k)}_{n+1})
\]

where \( k \) represents the number of iteration

**Explicit Euler**

Explicit Euler scheme (Butcher, 1987) uses the initial values only for integration (\( \phi = 0 \)). This is the first order approximation to Taylor Series. Explicit or Forward Euler calculates the plastic modulus based on the initial values. In explicit method, plastic modulus calculation and kinematic hardening rules are expressed as
\[ H_{n+1} = g_i(a_n, n_n) \]  
(3.28)

\[ \Delta a_{n+1} = \sum_{j=1}^{4} (\frac{2}{3} C_i \Delta \varepsilon_{n+1}^p - \gamma_i \{a_{n,j}, \delta^* + (1-\delta^*)(a_{n,j} : n_n) x_{n,j} \Delta p \}) \]  
(3.29)

where \( x_{n,i} = 1 \) for \( i = 1, 2, 3 \);

\[ x_{n,4} = \left( 1 - \frac{a_4}{f(a_{n,4})} \right) \]

Thus, new point is successively calculated based on the previous point in Explicit Euler scheme, hence also referred to as Forward Euler scheme. The schematic of the calculation is shown in Fig. 3.3 for a uniaxial monotonic response where stress increment \( \Delta \sigma \) is upgraded for the strain increment \( \Delta \varepsilon^p \) using the plastic modulus calculated based on the initial point only.

Fig. 3.3 Depiction of the classical forward Euler scheme

Fig. 3.4 Depiction of the classical modified Euler scheme

**Semi-implicit Euler**

Semi-implicit Euler method (Butcher, 1987) uses both initial and end values for plasticity calculation. This method is the second order approximation of Taylor series and calculates the plastic modulus using Eq. 3.27 for \( \phi = 0.5 \).

\[ H_{n+1} = 0.5 \ast \left( g_i(a_n, n_n) + g_i(a_{n+1}, n_{n+1}) \right) \]  
(3.30)
This approximation of plastic modulus is demonstrated in Fig. 3.4. This scheme is a two stage calculation process. In first step, plastic modulus \( g_1(a_n, n_n) \) is approximated from the initial values and then end values and plastic modulus \( g_1(a_n, n_n) \) are calculated using this plastic modulus. In second step, the average plastic modulus \( H_{n+1} \) is calculated based on the plastic modulus at initial and new points using Eq. 3.30 is used to obtain final stress and state variables. Then, stress increment \( \Delta \sigma \) is upgraded for the strain increment \( \Delta \varepsilon^p \) using the average plastic modulus as shown in Fig. 3.4.

Another semi-implicit Euler scheme proposed by Heun (Mathews, 1992) uses third order approximation of Taylor series. Heun’s method is similar to the semi-implicit Euler method presented above, involves iteration using

\[
H_{n+1}^{k+1} = 0.5 \left( g_1(a_n, n_n) + g_1(a_{n+1}^k, n_{n+1}^k) \right)
\]

where \( k \) is the number of iteration

Convergence of the plastic modulus is checked with the following criteria

\[
L_{chn} = \left| 1 - \frac{H_{n+1}^{(k+1)}}{H_{n+1}^{(k)}} \right| < \text{Convergence limit} \quad (3.31b)
\]

**Backward Euler**

The Backward Euler Scheme (Butcher, 1987) is an implicit solution technique which updates the increments based on iteratively converged end values. In backward Euler scheme, plastic modulus calculation and back stress increments are calculated using Eqs. 3.32 and 3.33. The solution is repeated until plastic modulus converges. For uniaxial monotonic loading, this iteration scheme is depicted in Fig. 3.5

\[
H_{n+1}^{k+1} = g_1(a_{n+1}^k, n_{n+1}^k)
\]

\[
\Delta a_{n+1} = \sum_{i=1}^4 \left( \frac{2}{3} C_i \Delta \varepsilon^p_{n+1} - \gamma_i (a_{n+1}, \Delta \varepsilon^p_{n+1} + \left( 1 - \delta_i \right) (a_{n+1}, n_{n+1}) \cdot \varepsilon_{n+1, i} \cdot \Delta p_{n+1} ) \right)
\]

where \( x_{n+1, i} = 1 \) for \( i = 1, 2, 3 \);

\[
x_{n+1, 4} = \left( 1 - \frac{\bar{a}_4}{f(a_{n+1, 4})} \right)
\]

Convergence of the plastic modulus is checked using (Eq. 3.31b).
3.2.1.3 Runge-Kutta methods

Explicit and implicit Runge-Kutta methods (Butcher, 1987) are higher order approximation of Taylor’s series.

**Explicit Runge-Kutta method**

Explicit Runge-Kutta method (Butcher, 1987) is the fourth order approximation to Taylor’s series. This method uses weighted average of the initial, two intermediate and end points for plastic modulus calculation, as follows

$$H_{n+1} = \left( g_1(a_{n+1}^0, n_{n+1}^0) + 2.0 \ast g_1(a_{n+1}^1, n_{n+1}^1) + 2.0 \ast g_1(a_{n+1}^2, n_{n+1}^2) + g_1(a_{n+1}^3, n_{n+1}^3) \right) / 6.0 \quad (3.34)$$

where superscripts 0 and 3 indicate the initial and end points; superscripts 1 and 2 indicate the two intermediate points of the increment and \( H_n^i = g_1(a_n^i, n_n^i) \); \( a_{n+1}^{i+1} = g_2(a_n^0, n_n^0, H_n^i) \); \( n_{n+1}^{i+1} = g_3(a_n^0, n_n^0, H_n^i) \). The approximation with explicit Runge-Kutta method is depicted in Fig. 3.6a.

**Implicit Runge-Kutta method**

The implicit Runge-Kutta scheme (Butcher, 1987) involves calculating the plastic modulus at two intermediate points, and approximates the plastic modulus using

$$H_{n+1}^{k+1} = 0.5 \ast \left( g_1^{k+1} + g_1^{k,2} \right) \quad (3.35)$$
Where \( w_1 = 0.5 \times (0.5 - 1/\sqrt{3}) \); \( w_2 = 0.5 \times (0.5 + 1/\sqrt{3}) \)

and superscripts 1 and 2 corresponds to two intermediate locations which are determined by \( w_1 \) and \( w_2 \) multiplied by the stress or strain increment.

The convergence of the iteration scheme is checked using Eq. 3.31b.

![Fig. 3.6 Depiction of the classical Runge-Kutta scheme (a) explicit Runge-Kutta (b) implicit Runge-Kutta.](image)

3.2.2 Return mapping algorithm

This section presents the return mapping algorithms with respect to modified Chaboche model. Implicit type return mapping method is known as radial return method (Wilkins, 1964) is commonly employed for solving the incremental equations of plasticity model (Kobayashi and Ohno, 2002; Sawyer et al., 2000, Wang et al., 2001). The return mapping algorithm is also known as elastic predictor-plastic corrector algorithm (Kreig and Kreig, 1977). In elastic prediction, stress is calculated elastically for given strain increment. Then in plastic correction, final stress increment is obtained by projecting the elastically predicted stresses on the yield surface along the direction radial to the yield surface. This returning direction to the yield surface can be selected from any of the initial to updated position of the yield surface. Based on the selected radial direction the return mapping scheme can be explicit, semi-implicit and implicit algorithm. Explicit return mapping scheme utilizes the normal direction for the initial position of the yield surface. Implicit type return mapping scheme utilizes the normal direction from the updated yield surface. Semi-implicit scheme uses some combination of the initial and updated positions of the yield surface. In subsequent
sections, the general methodology to implement return type algorithm with modified Chaboche plasticity model is discussed. Solution schemes with Euler and Runge-Kutta methods are presented.

### 3.2.2.1 Discretization of the constitutive relations

The rate-independent plasticity model with modified Chaboche kinematic hardening rule is discretized for the load increment \( n \) to \( n+1 \) with the following set of equations

\[
d\sigma_{n+1} = D^e : \left( d\varepsilon_{n+1}^e - d\varepsilon_{n+1}^p \right)
\]

\[
\sigma_{n+1} = \sigma_n + D^e : \left( d\varepsilon_{n+1}^e - d\varepsilon_{n+1}^p \right)
\]

\[
f_{e+1} = \frac{3}{2} \left( \varepsilon_{n+1}^e - a_{n+1} \right) : \left( \varepsilon_{n+1}^e - a_{n+1} \right) - \sigma_0^2
\]

\[
\varepsilon_{n+1}^p = \varepsilon_n^p + d\varepsilon_{n+1}^p
\]

\[
d\varepsilon_{n+1}^p = \sqrt{\frac{3}{2}} dp_{n+1} \{ (1-\phi)n_n + \phi n_{n+1} \}
\]

\[
n_n = \sqrt{\frac{3}{2}} \frac{s_n - a_n}{\sigma_0}
\]

\[
n_{n+1} = \sqrt{\frac{3}{2}} \frac{s_{n+1} - a_{n+1}}{\sigma_0}
\]

\[
a_{n+1} = \sum_{i=1}^{4} a_{n+1,i} = \sum_{i=1}^{4} a_{n,i} + \sum_{i=1}^{4} da_{n+1,i}
\]

\[
d a_{n+1} = \sum_{i=1}^{4} \frac{3}{2} C_i d\varepsilon_{n+1}^p - (1-\phi) \sum_{i=1}^{4} \gamma_i (a_{n,i} \delta^i + (1-\delta^i)(a_{n,i} : n_{n+1}) dp_{n+1} x_{n,i} +
\]

\[-\phi \sum_{i=1}^{4} \gamma_i (a_{n+1,i} \delta^i + (1-\delta^i)(a_{n+1,i} : n_{n+1}) dp_{n+1} x_{n+1,i}
\]

where, \( x_i = 1 \) for \( i = 1,2,3 \).

\[
x_i = \left( 1 - \frac{a_4}{f(a_i)} \right) \text{ for } i=4.
\]

\[
n_{n+1} : n_{n+1} = 1
\]

\[
d n_{n+1} : n_{n+1} = 0
\]

The value of \( \phi \) above determines if the numerical schemes used is explicit (\( \phi=0 \)), implicit (\( \phi = 1 \)) and semi-implicit (0<\( \phi <1 \)).

Eq. 3.41 is slightly rearranged to rewrite
\[ a_{n+1,j} = \left[ \left[ 1 + \phi \gamma \frac{dp_{n+1}x_{n+1,j}}{\partial} \right] - \frac{3}{2} \gamma \phi x_{n+1,j} \left( \frac{1 - \delta}{\sigma_0^2} \right) (\xi_{n+1} - a_{n+1}) \otimes (\xi_{n+1} - a_{n+1}) \right]^{-1} \]

\[ a_{n+1,j} = C \left( \phi (\xi_{n+1} - a_{n+1}) + (1 - \phi) (\xi_{n+1} - a_{n+1}) \right) - \left( \frac{1 - \delta}{\sigma_0} \right) \left( \xi_{n+1} - a_{n+1} \right) \otimes \left( \xi_{n+1} - a_{n+1} \right) \left( \xi_{n+1} - a_{n+1} \right) \]

(3.45)

3.2.2.2 Nonlinear scalar equation

The discretization presented above can be reduced to nonlinear scalar equations for determining unknown \( d\sigma_{n+1} \) which satisfies the discretized Eqs. 3.36 to 3.42 where all the variables are known at the beginning of the increment step. This reduction is performed in two parts, elastic prediction and plastic correction.

Elastic prediction

At the beginning of the increment, stress prediction involves calculating the trial elastic stress increment using

\[ d\sigma_{n+1}^T = D^e : d\varepsilon_{n+1} \]

\[ \sigma_{n+1}^T = \sigma_n + d\sigma_{n+1}^T \]

(3.9a)

(3.9b)

The yield condition is checked for the trial stress value using Eq. 3.4

\[ f_{n+1}^T = \frac{3}{2} \left( \xi_{n+1}^T - \varepsilon_n \right) : \left( \xi_{n+1}^T - \varepsilon_n \right) - \frac{\sigma_0^2}{2} \]

(3.10)

where \( \xi_{n+1}^T \) is the deviatoric component of \( \sigma_{n+1}^T \). If \( f_{n+1}^T \leq 0 \), \( \sigma_{n+1}^T \) is accepted as updated stress \( \sigma_{n+1} \).

If \( f_{n+1}^T > 0 \), the strain increment \( d\varepsilon_{n+1} \) contains an elastic and plastic part, when updated stress increment is calculated through satisfying the yield condition, \( f_{n+1}^T = 0 \). This step is known as plastic correction as discussed below.

Plastic correction

Actual stress can be calculated using the plastic correction \( D^p : d\varepsilon_{n+1}^p \) on the trial elastic stress.

61
\[ d\sigma_{n+1} = d\sigma_{n+1}^T - D^e : d\varepsilon_{n+1}^p \]  
\[ \sigma_{n+1} = \sigma_{n+1}^T - D^e : d\varepsilon_{n+1}^p \]  
(3.46a)  
(3.46b)

Noting that, for elastic isotropy and inelastic incompressibility,

\[ D^e : d\varepsilon_{n+1}^p = 2Gd\varepsilon_{n+1}^p, \]  
where \( G = \frac{E}{2(1+v)} \).

Eq. 3.46 can be converted into deviatoric form

\[ \tilde{s}_{n+1} = \tilde{s}_{n+1}^T - 2Gd\varepsilon_{n+1}^p \]  
(3.47)

Subtracting Eq. 3.41 from Eq. 3.47, the following can be obtained

\[ \tilde{s}_{n+1} - a_{n+1} = \tilde{s}_{n+1}^T - 2Gd\varepsilon_{n+1}^p - \sum_{i=1}^{4} a_{n+1,i} \]  
(3.48)

Putting the values of the \( a_{n+1} \) from Eq. 3.42 into Eq. 3.48

\[ \tilde{s}_{n+1} - a_{n+1} = \tilde{s}_{n+1}^T - 2Gd\varepsilon_{n+1}^p - \sum_{i=1}^{4} a_{n+1,i} - \frac{1}{2} \sum_{i=1}^{4} \gamma_i \left[ \delta a_{n+1,i} + (1-\delta)(a_{n+1,i} : n_{n+1}) n_{n+1} \right] x_{n+1,i} dp_{n+1} \]  
\[ + \phi \sum_{i=1}^{4} \gamma_i \left[ \delta a_{n+1,i} + (1-\delta)(a_{n+1,i} : n_{n+1}) n_{n+1} \right] x_{n+1,i} dp_{n+1} \]  
(3.49)

Rearranging the terms and using Eq. 3.39 and Eq. 3.40 to eliminate \( d\varepsilon_{n+1}^p \) from Eq. 3.49

\[ \tilde{s}_{n+1} - a_{n+1} = \tilde{s}_{n+1}^T - \sum_{i=1}^{4} a_{n+1,i} - (3G + \sum_{i=1}^{4} C_i) \frac{dp_{n+1}}{\sigma_0} \left\{ (1-\phi)(\tilde{s}_{n} - a_{n}) + \phi(\tilde{s}_{n+1} - a_{n+1}) \right\} \]  
\[ + (1-\phi) \sum_{i=1}^{4} \left[ \frac{3(1-\delta)}{2} \gamma_i \left( a_{n+1,i} : (s_{n+1} - a_{n+1}) \right) x_{n+1,i} dp_{n+1} \right] \]  
\[ + \phi \sum_{i=1}^{4} \left[ \frac{3(1-\delta)}{2} \gamma_i \left( a_{n+1,i} : (s_{n+1} - a_{n+1}) \right) x_{n+1,i} dp_{n+1} \right] \]  
(3.50)

The value of \((s_{n+1} - a_{n+1})\) obtained from Eq. 3.50 will be used in von-Mises function in Eq. 3.4 to solve for \( dp_{n+1} \). In Eq. 3.50, the term \((s_{n+1} - a_{n+1})\) cannot be separated explicitly for the modified term \((a_{n+1} : n_{n+1})\) for subsequent calculation of scalar variable, \( dp_{n+1} \). The above equation for the modified Chaboche model is more complex compared to similar
equations derived for other plasticity models like Armstrong–Frederick, Chaboche, Ohno-Wang model (Kobayashi and Ohno, 2002; Sawyer et. al., 2000, Doghri, 1995).

With rearrangement of the Eq. 3.50, it can be expressed in nonlinear form

\[
q_{n+1} = \frac{1}{\sigma_0} \sum_{i=1}^{N} \frac{\partial_p}{\partial_{\phi}} \left( \sigma_i \right) \frac{\partial_{\phi}}{\partial_{\Delta\phi}} \left( \sigma_i \right)
\]

where, \( q_{n+1} = s_{n+1} - a_{n+1} \) and \( q_{n} = s_{n} - a_{n} \) \( \quad (3.51) \)

To calculate the scalar parameter \( dp_{n+1} \) for determining \( d\sigma_{n+1} \) as end result, Eq. 3.51 can be substituted in yield function for \( f_{n+1} = 0 \) gives

\[
\frac{3}{2} \left( \sum_{i=1}^{N} C_i \right) \frac{dp_{n+1}}{\partial_{\phi}} \left( \sigma_i \right) \frac{\partial_{\phi}}{\partial_{\Delta\phi}} \left( \sigma_i \right) \\ \ \text{for semi-implicit scheme}
\]

\[
\left( \sum_{i=1}^{N} C_i \right) \left( \frac{dp_{n+1}}{\partial_{\phi}} \right) \left( \sigma_i \right) \left( \frac{\partial_{\phi}}{\partial_{\Delta\phi}} \right) \left( \sigma_i \right) \\ \text{for semi-implicit scheme}
\]

\[
= \left( \sigma_0 + \phi(3G + \sum_{i=1}^{N} C_i) dp_{n+1} \right) \quad (3.52)
\]

The yield function presented in Eq. 3.52 is a generalized nonlinear function of kinematic scalar variable \( dp_{n+1} \) for semi-implicit scheme. Calculating the kinematic variable for the increment step, all the variables can be updated from the known values at step \( n \) to \( n+1 \) using discretized Eq. 3.36-3.42. Solution for kinematic variable and state variable upgrading are different depending on the numerical scheme type explicit, semi-implicit and implicit used. Eq. 3.52 is derived as a strain driven approach for use with finite element where stress increments are calculated for prescribed strain increments to nonlinear plasticity problems.

The problems with simultaneous strain and stress prescribed case cannot be solved without finite element. This following section demonstrates the new formulations of “stress return algorithm” to implement with the return type algorithm for solving problems with stress prescribed cases. This formulation ideas is the extension of the idea of Doghri (1995) for plane stress element formulation.
3.2.2.3 Stress return mapping

When stresses are prescribed along with strain components, the problem reduced to calculate the unknown strain and stress components. A new algorithm based on return type scheme is proposed for such calculation. This methodology is referred as stress return mapping algorithm. The methodology presented in this section, will be described keeping similarity with discretization and algorithm for the strain driven approach presented in previous section. When one or more stress components are prescribed along with strain components, at first it is assumed that increment is elastic and unknown stress components are calculated from Eq. 3.9. The calculated stresses are checked against the von-Mises function as shown in Eq. 3.10. When $f_{\pi+1}^T > 0$, it is necessary to go for elastic-plastic calculation. In this case, strain driven formulations for return algorithm are still valid but need to be modified for given normal and shear stress components.

3.2.2.3.1 Normal stress components

When prescribed increments are in the elastic-plastic range, Eq. 3.36 is still valid and actual strain components along the given stress directions can be mathematically expressed using Eq. 3.39

\[
\begin{align*}
\delta e_1 &= \frac{d\sigma_1}{K + \frac{4}{3}G} - \frac{K - \frac{2}{3}G}{K + \frac{4}{3}G} (d\epsilon_1 + d\epsilon_2) + \frac{2G}{K + \frac{4}{3}G} \sqrt{\frac{3}{2}} dp\{(1 - \phi)n_{\pi+1,1} + \phi n_{\pi+1,1}\} \\
\delta e_2 &= \frac{d\sigma_2}{K + \frac{4}{3}G} - \frac{K - \frac{2}{3}G}{K + \frac{4}{3}G} (d\epsilon_1 + d\epsilon_3) + \frac{2G}{K + \frac{4}{3}G} \sqrt{\frac{3}{2}} dp\{(1 - \phi)n_{\pi+1,2} + \phi n_{\pi+1,2}\} \\
\delta e_3 &= \frac{d\sigma_3}{K + \frac{4}{3}G} - \frac{K - \frac{2}{3}G}{K + \frac{4}{3}G} (d\epsilon_2 + d\epsilon_3) + \frac{2G}{K + \frac{4}{3}G} \sqrt{\frac{3}{2}} dp\{(1 - \phi)n_{\pi+1,3} + \phi n_{\pi+1,3}\}
\end{align*}
\] (3.53a, 3.53b, 3.53c)

where 1, 2, 3 denotes the normal stress directions and

\[
K = \frac{E}{3(1 - 2\nu)}
\]

Now it is necessary to calculate trial elastic strain increments along the prescribed stress components. When a single normal stress component is prescribed, trial elastic strain increment is calculated from

\[
\delta e_1^T = \frac{d\sigma_1}{K + \frac{4}{3}G} - \frac{K - \frac{2}{3}G}{K + \frac{4}{3}G} (d\epsilon_2 + d\epsilon_3)
\] (3.54)
When two normal stress increments are prescribed, trial elastic strain increments can be calculated as

\[
d\varepsilon^T_1 = \frac{d\sigma_1}{K + G/3} - \frac{K - G}{K + G/3} (d\varepsilon^T_2 + d\varepsilon_3)
\]

(3.55a)

\[
d\varepsilon^T_2 = \frac{d\sigma_2}{K + G/3} - \frac{K - G}{K + G/3} (d\varepsilon^T_1 + d\varepsilon_3)
\]

(3.55b)

When three normal stress increments are prescribed, trial elastic strain increments can be calculated as

\[
d\varepsilon^T_1 = \frac{d\sigma_1}{K + G/3} - \frac{K - G}{K + G/3} (d\varepsilon^T_2 + d\varepsilon^T_3)
\]

(3.56a)

\[
d\varepsilon^T_2 = \frac{d\sigma_2}{K + G/3} - \frac{K - G}{K + G/3} (d\varepsilon^T_1 + d\varepsilon^T_3)
\]

(3.56b)

\[
d\varepsilon^T_3 = \frac{d\sigma_3}{K + G/3} - \frac{K - G}{K + G/3} (d\varepsilon^T_1 + d\varepsilon^T_2)
\]

(3.56c)

Now, Eq. 3.9 should be corrected for plastic strain increments and given stresses. Plastic strains for three axial directions are \( (d\varepsilon_1 - d\varepsilon^T_1) \), \( (d\varepsilon_2 - d\varepsilon^T_2) \) and \( (d\varepsilon_3 - d\varepsilon^T_3) \) respectively.

When a single normal stress component is prescribed Eq. 3.53 and Eq. 3.54 gives directly

\[
(d\varepsilon_1 - d\varepsilon^T_1) = \frac{\sqrt{3}}{2} k_1 dp \{ (1 - \phi) n_{n1,1} + \phi n_{n+1,1} \};
\]

where \( k_1 = 2G/(K + G/3) \)

(3.57)

When multiple normal stresses are prescribed, \( (d\varepsilon_1 - d\varepsilon^T_1) \), \( (d\varepsilon_2 - d\varepsilon^T_2) \) and \( (d\varepsilon_3 - d\varepsilon^T_3) \) cannot be obtained directly. In such case, subtracting Eq. 3.56 from Eq. 3.53 gives

\[
(d\varepsilon_1 - d\varepsilon^T_1) = -k_2 (d\varepsilon_2 - d\varepsilon^T_2) - k_2 (d\varepsilon_3 - d\varepsilon^T_3) + k_1 \sqrt{3} dp \{ (1 - \phi) n_{n+1,1} + \phi n_{n+1,1} \}
\]

(3.57a)

\[
(d\varepsilon_2 - d\varepsilon^T_2) = -k_2 (d\varepsilon_1 - d\varepsilon^T_1) - k_2 (d\varepsilon_3 - d\varepsilon^T_3) + k_1 \sqrt{3} dp \{ (1 - \phi) n_{n+1,2} + \phi n_{n+1,2} \}
\]

(3.57b)

\[
(d\varepsilon_3 - d\varepsilon^T_3) = -k_2 (d\varepsilon_1 - d\varepsilon^T_1) - k_2 (d\varepsilon_2 - d\varepsilon^T_2) + k_1 \sqrt{3} dp \{ (1 - \phi) n_{n+1,3} + \phi n_{n+1,3} \}
\]

(3.57c)
Where \( k_1 = 2G / (K + \frac{1}{2} G) \); \( k_2 = (K - \frac{1}{2} G) / (K + \frac{1}{2} G) \);

Solution of Eq.(3.57) gives
\[
(de_1 - de_1^T) = de_1^p; \quad (de_2 - de_2^T) = de_2^p; \quad (de_3 - de_3^T) = de_3^p
\] (3.58)

Basically, when three stress increments are prescribed Eq. 3.56 calculates the actual elastic strain increments. Thus, \((de_1 - de_1^T)\) simply calculates the plastic strain increments as found in Eq. 3.58.

Similarly, when two normal stress components are prescribed, Eq. 3.53 and Eq. 3.55 gives
\[
(de_1 - de_1^T) = \sqrt{\frac{3}{2}} \frac{k_1}{1 - k_2^2} \left( (1 - \phi)(n_{n,1} - k_z n_{n,2}) + \phi(n_{n+1,1} - k_z n_{n+1,2}) \right);
\] (3.59a)

\[
(de_2 - de_2^T) = \sqrt{\frac{3}{2}} \frac{k_1}{1 - k_2^2} \left( (1 - \phi)(n_{n,1} - k_z n_{n,2}) + \phi(n_{n+1,1} - k_z n_{n+1,2}) \right);
\] (3.59b)

### 3.2.2.3.2 Shear stress components

When shear stresses are prescribed, mathematically actual strain increment in given stress direction can be expressed from Eq. 3.36
\[
d\varepsilon_i = \frac{d\sigma_i}{2G}\;
\quad \text{where } i = 4, 5, 6 \text{ denotes the shear stress directions}
\] (3.60)

When shear stress increments are prescribed, trial elastic strain increment is calculated as
\[
d\varepsilon_i^T = \frac{d\sigma_i}{2G}\quad \text{where } i = 4, 5, 6
\] (3.61)

Subtracting Eq. 3.61 from Eq. 3.60 gives
\[
d\varepsilon_i - d\varepsilon_i^T = d\varepsilon_i^p = \sqrt{\frac{3}{2}} dp_n \left( (1 - \phi)n_{n,i} + \phi n_{n+1,i} \right); \quad \text{where } i = 4, 5, 6
\] (3.62)

### 3.2.2.3.3 Modification for stress prescribed cases

For each given stress, the basic Eq. 3.46 need to be modified. When stress components are prescribed in return algorithm, with modification terms Eq. 3.46 becomes as shown below
\[
d\sigma_{n+1} = d\sigma_{n+1}^T - D^T : d\varepsilon_{n+1}^p + \sum_{i=1}^{3}(K + 2G\delta_i)(d\varepsilon_{n+1,i} - d\varepsilon_{n+1,i}^T) + \sum_{i=4}^{6}2G\delta_i(d\varepsilon_{n+1,i} - d\varepsilon_{n+1,i}^T)
\] (3.63)

where, \( \delta_{ij} = \delta_{ji} = \frac{1}{2} \delta_{ij} \) for \( i = 1, 2, 3 \)
\[
\delta_{ij} = \delta_{ji} \quad \text{for } i = 4, 5, 6
\]

and \( \delta_{ij} = 1 \) when \( i = j \); \( \delta_{ij} = 0 \) when \( i \neq j \)
In Eq. 3.63, the third and fourth term represents the correction terms for given normal stress and shear stresses. Taking the deviatoric part of the Eq. 3.63 and subtracting Eq. 3.41
\[
d s_{n+1} - d a_{n+1} = d s_{n+1}^T - D : d e_{n+1}^p + \sum_{i=1}^3 (K_1 + 2G_0) (d e_{n+1,i} - d e_{n+1,i}^T) + \sum_{i=1}^6 2G_0 (d e_{n+1,i} - d e_{n+1,i}^T) - 4 d a_{n+1} \\
\text{(3.64)}
\]
Putting the values of \(d e_{n+1}^p\) and \(d a_{n+1}^p\) from Eq. 3.39 and Eq. 3.42 in Eq. 3.64 and rearranging we get
\[
q_{n+1} = \left[ s_{n+1}^p - a_n + (1 - \phi) (3G + \sum_{i=1}^3 C_i) \frac{d p_{n+1}}{\sigma_0} q_n + (1 - \phi) \sum_{i=1}^3 \gamma_i (\delta^i a_{n,i} + \frac{1 - \phi}{\sigma_0} (a_{n,i} : q_n) q_{n+1}) x_{n,i} d p_{n+1} + \right. \\
+ \phi \sum_{i=1}^4 \gamma_i (\delta^i a_{n,i} + \frac{1 - \phi}{\sigma_0} (a_{n,i} : q_n) q_{n+1}) x_{n,i} d p_{n+1} + d p_{n+1} Corr_1 + d p_{n+1} Corr_2 \right] \left[ (\sigma_0 / (\sigma_0 + (3G + \sum_{i=1}^3 C_i) d p_{n+1}) \right] \\
\text{(3.65)}
\]
Where \(Corr_1\) and \(Corr_2\) corresponds to correction for normal and shear stress components respectively.

**Normal stress correction (Corr1)**

When single normal stress component is given, the correction term corresponds to
\[
Corr_1 = \frac{3G}{\sigma_0} k_1 \left[ (1 - \phi) (s_{n+1,m} - a_{n+1,m}) + \phi (s_{n+1,m} - a_{n+1,m}) \right] \delta_m \\
\text{where } m \text{ is prescribed stress direction, 1 or 2 or 3.} \\
\text{(3.66a)}
\]

When two normal stress components are given the correction term becomes
\[
Corr_1 = (1 - \phi) \frac{3G}{\sigma_0} k_1 \left[ (s_{n+1,l} - a_{n+1,l}) - k_2 (s_{n+1,m} - a_{n+1,m}) \right] \delta_l + \left[ (s_{n+1,m} - a_{n+1,m}) - k_2 (s_{n+1,l} - a_{n+1,l}) \right] \delta_m \\
+ \phi \frac{3G}{\sigma_0} k_1 \left[ (s_{n+1,l} - a_{n+1,l}) - k_2 (s_{n+1,m} - a_{n+1,m}) \right] \delta_l + \left[ (s_{n+1,m} - a_{n+1,m}) - k_2 (s_{n+1,l} - a_{n+1,l}) \right] \delta_m \\
\text{(3.66b)}
\]

where \(l, m\) are prescribed stress directions, 1, 2 or 2, 3 or 1, 3.

When three normal stress components are given the correction term becomes
\[
Corr_1 = \frac{3G}{\sigma_0} \sum_{i=1}^3 \delta_m \left[ (1 - \phi) (s_{n+1,m} - a_{n+1,m}) + \phi (s_{n+1,m} - a_{n+1,m}) \right] \\
\text{(3.66c)}
\]
Shear stress correction \((\text{Corr2})\)

When one direct shear stress components are given in \(m\) direction, the correction term becomes

\[
\text{Corr2} = -\frac{6G}{\sigma_0} \left[ \delta_m \left( (1-\phi)(s_{n,m} - a_{n,m}) + \phi(s_{n+1,m} - a_{n+1,m}) \right) \right]
\]  

(3.67)

where \(m\) is prescribed stress direction, 4 or 5 or 6.

The correction terms are directly additive when more than one shear stress components are given.

Putting the value of \(\text{Corr1}\) and \(\text{Corr2}\) from Eq. 3.66 and Eq. 3.67 in Eq. 3.65, \(q_{n+1}\) can be obtained for prescribed stresses. Using Eq. 3.65 in yield function, Eq. 3.10 provides a nonlinear function of kinematic scalar variable, \(dp_{n+1}\). The nonlinear scalar equation of \(dp_{n+1}\) can be solved by the method of successive substitution. The converged value for \(dp_{n+1}\) allows to calculate \(q_{n+1}\) using Eq. 3.65. State variables are upgraded using Eq. 3.41. Plastic strain increment \(d\varepsilon_{\text{pl}}\) for the time step can be calculated from Eq. 3.39. Elastic part for the given strain directions can be calculated from strain decomposition rule. Unknown stress increments and elastic strain part for the given stress directions can be calculated from Hook’s law. The unknown total strain components can be obtained by adding elastic and plastic components.

3.2.2.4 Solving the nonlinear equation

The nonlinear equation presented in Eqs. 3.39, 3.42 and 3.52 are based on infinitesimal increment size of \(d\varepsilon\) and \(d\sigma\). Plasticity calculations are performed with finite size of \(\Delta\varepsilon\) and \(\Delta\sigma\). The \(\phi\) value determines the solution scheme as explicit, semi-implicit and implicit algorithm. The complexity for the solution of the nonlinear equation for scalar variable is dependent on \(\phi\) value.

Explicit return scheme

In explicit return scheme, nonlinear equation is obtained for \(\phi = 0\), thus Eq. 3.39, Eq. 3.42 and Eq. 3.52 reduce to

\[
\Delta\varepsilon_{\text{pl},n+1} = \sqrt{2} \Delta p_{n+1} \eta_{n}
\]  

(3.68)
\[ \Delta a_{n+1} = \sum_{i=1}^{4} \frac{2}{3} C_i \Delta \varepsilon^p_{n+1} - \sum_{i=1}^{4} \gamma_i \left( a_{n,i} \delta' + (1-\delta')(a_{n,i} : n_n) n_n \right) \Delta p_{n+1} x_{n,i} \]  

(3.69)

where, \( x_{n,i} = 1 \), for \( i = 1,2,3 \); \( x_{n,4} = \left( 1 - \frac{a_{4}}{f(a_{n,4})} \right) \).

\[ \frac{3}{2} \left( s_{n+1}^T - a_n + (3G + \sum_{i=1}^{4} C_i) \frac{\Delta p_{n+1}}{\sigma_0} + \sum_{i=1}^{4} \gamma_i \left( \delta a_{n,i} + \frac{3}{2} \left( 1-\delta' \right) (a_{n,i} : q_n) q_n \right) x_{n,i} \Delta p_{n+1} \right) = \sigma_0^2 \]  

(3.70)

Schematically, explicit return scheme can be presented as shown in Fig. 3.7, where it is observed that the return to the yield surface (plastic corrector) is along \( n_n \) normal to the initial yield surface. As this scheme utilizes initial yield surface position and corresponding normal direction, this scheme is known as forward Euler scheme in plasticity literature (Dodds, 1987).

Eqs. 3.70 is a nonlinear quadratic equation of scalar variable \( \Delta p_{n+1} \) in which only unknown is \( \Delta p_{n+1} \). Thus, the solution is straightforward and single step process. Direct solution is possible in this case. Solution for Eq. 3.70 yields two roots for \( \Delta p_{n+1} \). The smaller root gives the optimum solution and gives the minimum distance from the yield surface to the predicted elastic stress. The other root gives the solution for distance for intersection on other side of the yield surface, hence not applicable. Plastic strain and state variable increment can be calculated from Eq. 3.68 and Eq. 3.69. Stress increment can be calculated from Eq. 3.46a.
Semi-implicit return scheme

In semi-implicit return scheme, nonlinear equation is obtained for $0 < \phi < 1$. In modified Euler scheme (Kreig and Kreig, 1977), $\phi = 0.5$, thus Eqs. 3.39, 3.42 and 3.52 reduce to

$$\Delta \mathbf{E}_p^{n+1} = \sqrt{2} \Delta p_{n+1} \left( 0.5 \mathbf{n}_n + 0.5 \mathbf{n}_{n+1} \right)$$

$$\Delta \mathbf{a}_{n+1} = \frac{4}{3} C \Delta \mathbf{E}_p^{n+1} - 0.5 \sum_{j=1}^{4} \gamma_j \left( \mathbf{a}_{n,j} : \mathbf{n}_n \right) \Delta p_{n+1} \mathbf{x}_{n,j}$$

$$- 0.5 \sum_{j=1}^{4} \gamma_j \left( \mathbf{a}_{n+1,j} : \mathbf{n}_{n+1} \right) \Delta p_{n+1} \mathbf{x}_{n+1,j}$$

Thus, this numerical approach utilizes the return direction partially along the normal to initial position of the yield surface and partially along the normal to the updated position of the yield surface. Schematically, this approach can be presented as in Fig. 3.8.
In Eq. 3.73, $q_{n+1}$, $n_{n+1}$, and position of the center of the yield surface, $a_{n+1}$, are functions of the scalar variable $\Delta p_{n+1}$. Thus, Eq. 3.73 is implicit type of equation. The solution can be obtained with successive substitution algorithm. In modified Euler scheme, the solution is a two step process. In first step, $q_{n+1}$ and $a_{n+1}$ is taken as $q_n$ and $a_n$ and Eq. 3.73 is solved for $\Delta p_{n+1}\textsuperscript{(0)}$. Thus, $q_{n+1}\textsuperscript{(1)}$ and $a_{n+1}\textsuperscript{(1)}$ are updated using Eq.3.51 and Eq.3.45 for $\Delta p_{n+1}\textsuperscript{(0)}$ taking $\phi = 0$. In second step, $\Delta p_{n+1}\textsuperscript{(1)}$ is solved using the value of $q_{n+1}\textsuperscript{(1)}$ and $a_{n+1}\textsuperscript{(1)}$ from Eq. 3.73. Here, $\Delta p_{n+1}\textsuperscript{(1)}$ is accepted as final solution. Plastic strain increment and movement of the center of the yield surface are calculated through Eq. 3.71 and Eq. 3.72.

This modified Euler scheme can be further extended to the idea of Heun’s Method. In Heun’s return approach, the second step is repeated. In each iteration, the position and normal direction of the yield surface updated for a new value of $\Delta p_{n+1}\textsuperscript{(k)}$. The process is repeated until $\Delta p_{n+1}\textsuperscript{(k)}$ converges. This method is an iterative approach. Convergence of the iteration is measured with convergence of scalar variable $\Delta p_{n+1}$ and checked with following convergence criteria

$$L_{Rhm} = \left|1.0 - \frac{\Delta p_{n+1}\textsuperscript{(k+1)}}{\Delta p_{n+1}\textsuperscript{(k)}}\right| < 1.0 \times 10^{-8} \quad (3.74)$$

**Radial return algorithm**

The return algorithm (Wilkins, 1964) converts to fully implicit scheme for $\phi = 1$. In this approach, the constitutive equations are discretized in an incremental form:

$$\Delta \varepsilon_{n+1}^p = \sqrt{\frac{3}{2}} \Delta p_{n+1} \frac{n_{n+1}}{n_{n+1}} \quad (3.75)$$

$$\Delta a_{n+1} = \sum_{i=1}^{4} \frac{2}{3} C_i \Delta \varepsilon_{n+1}^p - \sum_{i=1}^{4} \gamma_i \left(a_{n+1,i} \delta' + (1 - \delta')(a_{n+1,i} \cdot n_{n+1})n_{n+1}\right) \Delta p_{n+1} x_{n+1,i} \quad (3.76)$$

where, $x_{n+1,i} = 1$, for $i = 1, 2, 3$; $x_{n+1,4} = \left(1 - \frac{a_4}{f(a_{n+1,4})}\right)$

This approach utilizes the normal direction of the updated yield surface as the return direction. Schematically, radial return scheme can be represented as in Fig. 3.9. For $\phi = 1$, Eq. 3.45, Eq. 3.51 and Eq. 3.52 become
\[
\begin{align*}
\mathbf{a}_{n+1,i} &= \left[ 1 + \frac{\gamma_i \Delta \mathbf{p}_{n+1} \mathbf{X}_{n+1,i} \mathbf{\delta}^2}{2} \right] I + \frac{3}{2} \gamma_i \Delta \mathbf{p}_{n+1} \mathbf{X}_{n+1,i} \frac{(1-\delta)}{\sigma_0^2} q_{n+1} \otimes q_{n+1} \right]^{-1} \left[ a_{n,j} + C_i \frac{\Delta \mathbf{p}_{n+1}}{\sigma_0} q_{n+1} \right] \\
q_{n+1} &= \left( \mathbf{s}_{n+1} - \mathbf{a}_n + \sum_{i=1}^{4} \gamma_i (\mathbf{\delta} \mathbf{a}_{n+1,i} + \frac{3}{2} \frac{1-\delta}{\sigma_0} (\mathbf{a}_{n+1,i} \otimes q_{n+1}) \mathbf{x}_i \Delta \mathbf{p}_{n+1} \right) \left( \frac{\sigma_0}{(\sigma_0 + 3G + \sum_{i=1}^{4} C_i) \Delta \mathbf{p}_{n+1}} \right) \\
\frac{3}{2} \left( \mathbf{s}_{n+1} - \mathbf{a}_n + \sum_{i=1}^{4} \gamma_i (\mathbf{\delta} \mathbf{a}_{n+1,i} + \frac{3}{2} \frac{1-\delta}{\sigma_0} (\mathbf{a}_{n+1,i} \otimes q_{n+1}) \mathbf{x}_i \Delta \mathbf{p}_{n+1} \right) \left( \mathbf{s}_{n+1} - \mathbf{a}_n + \sum_{i=1}^{4} \gamma_i (\mathbf{\delta} \mathbf{a}_{n+1,i} + \frac{3}{2} \frac{1-\delta}{\sigma_0} (\mathbf{a}_{n+1,i} \otimes q_{n+1}) \mathbf{x}_i \Delta \mathbf{p}_{n+1} \right) \\
&= \left( \frac{\sigma_0}{(\sigma_0 + 3G + \sum_{i=1}^{4} C_i) \Delta \mathbf{p}_{n+1}} \right)^2
\end{align*}
\]  

Thus, four equations from Eq. 3.77, Eq. 3.78, Eq. 3.79 make a system of nonlinear equation which can be solved in an iterative manner by Newton-Raphson method. The nonlinear system has the unknowns \( \Delta \mathbf{p}_{n+1}, q_{n+1}, \mathbf{a}_{n+1,i} \). In such a situation, Chaboche and Cailletaud (1996) described the need for a solution for system of equations with \( 6(N+1)+1 \) equations for constitutive model with kinematic hardening rules. Here, \( N \) represents the number of decomposed kinematic hardening rules in the constitutive model. The set of equations presented in this case thus require to solve 31 equations in modified Newton’s method. In solution with iterative approach (for implicit case), inversion of a matrix of dimension 31 will be too time consuming. Doghri (1993) proposed an explicit updating scheme for implicit radial return scheme to solve plasticity model with multiple Armstrong-Frederick type kinematic hardening rule without any approximation. Doghri (1995) extended the solution technique by Euler backward approach to plane stress problems. Doghri’s method also requires similar type of computational work in each step as demonstrated by Chaboche and Cailletaud (1996). The method proposed by Kobayshi and Ohno (2002) suggests that the system can be rearranged to proceed for solution in iterative manner with conversion to a nonlinear scalar equation similar to Eq. 3.52 and thus solve in an iterative manner. Similar type of nonlinear scalar equation can be formed for the rate independent plasticity models with Bilinear (Prager, 1956), Armstrong-Frederick (1966), Ohno-Wang (1993), Chaboche kinematic hardening (1991) rules. For the aforementioned models, the order of the similar nonlinear scalar equation is less than Eq. 3.79. For a linear type kinematic hardening model (Prager,1956) it possible to calculate \( \Delta \mathbf{p}_{n+1} \) in a single step without iteration (Kobayashi and...
Ohno, 2002). For any other non-linear kinematic hardening rule the solution approach should be iterative. Similar equations for Chaboche and Ohno-Wang kinematic hardening rules can be efficiently solved by successive iteration (Kobayashi and Ohno, 2002).

The additional complexity for the modified Chaboche model in Eq. 3.79 over the Armstrong-Frederick (1966) and Ohno-Wang (1993) models in solving the nonlinear equation comes from the term \((a_{n+1} : n_{n+1})p_{n+1}\). It is imperative to set a solution scheme for models with the modified Chaboche model. In Eq. 3.79, \(q_{n+1}^{k+1}, a_{n+1}^{k+1}\) and \(x_{n+1,i}^{k+1}\) are function of \(\Delta p_{n+1}\).

The Eq. 3.76 can be reset in the following form for iterative solution

\[
q_{n+1}^{k+1} = \frac{\sigma_0}{2} \left[ \sum_{i=1}^{3} \gamma_i \left( a_{n+1,i}^{k+1} : a_n + \sum_{i=1}^{4} \gamma_i \left( \delta a_{n+1,i} : a_n + \frac{1}{2} \frac{1}{\sigma_0} \left( \frac{1}{\sigma_0} \right) (a_{n+1,i} : q_{n+1}^{k+1}) q_{n+1}^{k+1} \right) x_{n+1,i}^{k+1} \Delta p_{n+1} \right] \right]
\]

\[
\sigma_0 + (3G + \sum_{i=1}^{4} C_i) \Delta p_{n+1}
\]

where, \(k\) denotes the iteration number for an increment step (3.80)

Substituting the values from Eq.3.80 to the yield function gives the nonlinear scalar equation

\[
\frac{3}{2} q_{n+1}^{k+1} : q_{n+1}^{k+1} = \sigma_0^2
\]

The solution will be obtained through the Newton-Raphson iteration scheme using Eq. 3.80 for implicit radial return algorithm. The substitution process can be started assuming \(a_{n+1}^{(0)} = a_n, q_{n+1}^{(0)} = \delta_n - a_n\) Eq. 3.81 which yields a nonlinear equation of second order with
kinematic scalar variable $\Delta p_{n+1}^{(l)}$. Now, $\Delta p_{n+1}^{(l)}$ can be determined from Eq. 3.81. Once $\Delta p_{n+1}^{(l)}$ is obtained, it is necessary to upgrade $q_{n+1}^{(k)}$ from Eq. 3.80 (where $k=1$). From the updated $q_{n+1}^{(k)}$ and $\Delta p_{n+1}^{(k)}$, the kinematic state variables are updated using Eq. 3.77. For given $q_{n+1}^{(k)}$ and $\Delta p_{n+1}^{(k)}$, upgrading of first three kinematic variables is single step process through

$$a_{n+1,i}^k = \left\{ 1 + \gamma_i \Delta p_{n+1} x_{n+1,i} \sigma \right\} I_2 + \frac{3}{2} \gamma_i \Delta p_{n+1} x_{n+1,i} \left( 1 - \delta_i \right) q_{n+1}^k \otimes q_{n+1}^k \right\}^{-1} \left[ a_{n+1,i} + C_i \frac{\Delta p_{n+1}}{\sigma_0} q_{n+1}^k \right]$$

where $x_{n+1,i} = 1$, for $i = 1,2,3$. (3.82)

For the fourth hardening rule, with given $q_{n+1}^{(k)}$ and $\Delta p_{n+1}^{(k)}$, Eq. 3.77. is still an implicit nonlinear equation as $x_{n+1,4} = \left\{ 1 - \bar{a}_4 \right\} \left( \frac{3}{2} a_{n+1,4} : a_{n+1,4} \right)^{\frac{1}{2}}$ is again a function of $a_{n+1,4}$. The one step upgrading similar to using Eq. 3.82 will have convergence difficulty with large strain increments. Therefore, at each iteration step upgrading of all the kinematic variables will be done through modified Newton’s method. In upgrading of state variables with the modified Newton’s scheme, the first, second and third rule will give Eq. 3.82. For the fourth kinematic rule, the process becomes iterative with modified Newton’s scheme. Using modified Newton’s scheme, it is possible to set an explicit upgrading scheme for the state variables of fourth hardening rule. The fourth rule can be expressed alternately as a function for a given $q_{n+1}^{(k)}$ and $\Delta p_{n+1}^{(k)}$.

$$\psi = a_{n+1,4} - \left[ a_{n+4} + C_4 \Delta p_{n+1} q_{n+1} / \sigma_0 \right] + L a_{n+1,4} \left( 1 - \bar{a}_4 \right) \left( \frac{3}{2} a_{n+1,4} : a_{n+1,4} \right)^{\frac{1}{2}} \right)$$

where $L = \gamma_i \Delta p_{n+1} \left[ \delta_i I + \frac{3}{2} \left( 1 - \delta_i \right) q_{n+1}^k \otimes q_{n+1}^k \right]$ (3.83)

For given $q_{n+1}^{(k)}$ and $\Delta p_{n+1}^{(k)}$, the function $\psi$ has one variable $a_{n+1,4}$. Now this function will be explicitly corrected in iterative manner to upgrade the state variable. For upgrading with modified Newton’s scheme,

$$\frac{\partial \psi}{\partial a_4} = I + L \left( 1 - \bar{a}_4 \right) \left( \frac{3}{2} a_{n+1,4} : a_{n+1,4} \right)^{\frac{1}{2}} + \frac{\bar{a}_4}{\left( \frac{3}{2} a_{n+1,4} : a_{n+1,4} \right)^{\frac{1}{2}}} L a_{n+1,4} \otimes a_{n+1,4}$$
The amount of correction, $C_{a_4}$, required can be determined explicitly from

$$C_{a_4} = -\frac{\partial \psi}{\partial a_{n+1,4}}$$

(3.85)

The updated value of the $a_{n+1,4}$ can be calculated from

$$a_{n+1,4}^{i+1} = a_{n+1,4}^i + C_{a_4}$$

(3.86)

where $i$ denotes the iteration number for fourth hardening rule.

As this approach is an iterative scheme, the process is repeated until $f(a_{n+1,4})$ converges.

The convergence criteria is set with

$$1 - \frac{f(a_{n+1,4}^{k+1})}{f(a_{n+1,4}^k)} < 1.0e - 6$$

(3.87)

Once the kinematic variables $a_{n+1,i}$ are upgraded, solution process involving Eq. 3.80 to Eq. 3.87 is repeated until $\Delta F_{n+1}^{(k)}$ converged. Convergence of $\Delta F_{n+1}^{(k)}$ is measured with the following criteria

$$L_{RB} = \left| 1.0 - \frac{\Delta F_{n+1}^{(k+1)}}{\Delta F_{n+1}^{(k)}} \right| < 1.0e - 7$$

(3.88)

The stress and state variable increment for the converged $\Delta F_{n+1}^{(k)}$ are the final solution for the radial return scheme for the increment.

**Explicit Runge-Kutta method**

Similar to return type modified Euler method, the idea of explicit Runge-Kutta method also can be used to select return direction on the yield surface. This method calculates the normal direction of the yield surface in two intermediate locations along with start and end point of the increments. The return direction in plastic strain increment calculation is taken as aggregate of the normal directions of four locations as shown in Eq. 3.89

$$\Delta e_{n+1}^p = \sqrt{\frac{3}{2}} \Delta F_{n+1} * \frac{1}{6} \left( n_{n+1}^0 + 2.0 * n_{n+1}^1 + 2.0 * n_{n+1}^2 + n_{n+1}^3 \right)$$

(3.89)

where superscript 1 and 2 demonstrates two intermediate locations of the increment.
3.3 Numerical difficulties

Efficient implementation of numerical schemes demands numerical calculation to be performed on large strain or stress increments for the plasticity model. The modified kinematic hardening rule is superposition of decomposed nonlinear incremental equations. First and fourth hardening rules considered in this modified Chaboche model are highly nonlinear in nature. The state variables representing these two rules start with an initial high slope ($C_i$), stabilize quickly and become asymptotic to horizontal at value $C_i/\gamma_i$. Figure 3.10a shows the closed form solution of the kinematic variable for the first kinematic rule. The rate of stabilization from initial steep part to horizontal asymptote is dependent on $\gamma_i$. Numerical simulation of such asymptotic curve with the present methods produces oscillation as shown in Fig. 3.10a. Such simulation demand very small increment size for numerical calculation.

At higher strain range, oscillation observed even with small increment size as the kinematic state variable becomes closer to stabilizing value (Fig. 3.10b). Appearance of such oscillation in numerical approximation is not explicitly reported in plasticity literature. Liu and Al-Bermani (1995) and R.H Dodds (1987) observed oscillation in finite element simulation with large increments. Chaboche and Cailletaud (1996) also reported saw-tooth oscillation for semi-implicit scheme. For each numerical integration scheme, there is a certain size of strain increment (termed as threshold increment size), below which numerical solution is stable, and for larger value numerical solution becomes unstable. This instability in finite element analysis causes divergence in solving global systems of nodal force equilibriums among the elements.

Oscillation in integration can appear for two phases. At first, oscillation appears for larger increment size. For the first kinematic hardening rule, it has very high initial slope and quickly stabilizes. When the increment size is such big (greater than $1/\gamma_i$), the numerical result crosses the limiting value (stabilizing value, $C_i/\gamma_i$). In the next increment, the kinematic hardening rule tries to come back to the stabilized value, as a result differs from actual slope, crosses the boundary value, thus the oscillation is produced (Fig. 3.10). If the integration is carried out a number of times at the same integration point, then it stabilizes in subsequent steps. This oscillation reappears due to lack of consistency at higher strain range. At the end of each step of plasticity calculation, it is necessary to satisfy the consistency
condition. The upgraded stresses at the end of each time step will not always stay on the yield surface. This deviation accumulates and causes oscillation at the end when the integration process is repeated. The oscillation appears to be fast when increment size is large (Fig. 3.10b). Chaboche and Cailletaud (1996) observed such oscillation in modified Euler rule even for a simple non-linear kinematic hardening model. The authors identified that the reason of this problem is the violation of the consistency at the end of the increment.

When the oscillation is observed at the beginning, adaptive increment size keep the increment sufficiently small, makes the solution stable. Since constitutive model will be implemented with cyclic loading, which requires repetitive calculation, solution with small increment size may become time consuming and expensive. Among the numerical methods Forward Euler integration scheme demands the smallest increment size for stable solution from nonlinear plasticity models. To overcome this numerical difficulty, numerical algorithms of higher order can be used. Hence, modified Euler and backward Euler method were implemented for classical plasticity numerical scheme. However, it was observed that semi-implicit and implicit Euler methods show oscillation at higher strain range. Another alternative can be higher order numerical algorithm than Backward Euler scheme. Methods of explicit and implicit Runge-Kutta (RK) type are also implemented for numerical approximation. Although RK methods allow higher strain increment than Euler methods, but oscillation observed oscillation still was observed in the simulation at higher strain range. However, a remedy to the problem was achieved as follows. Each kinematic hardening rule partly dictates the movement of the center of yield surface in deviatoric stress space in a manner that its components satisfies

\[ f(a_{n+1},j) = \left[ \frac{1}{2} a_{n+1,j} : a_{n+1,j} \right] \leq \left( \frac{C_i}{\gamma_i} \right)^2 \]  

(3.90a)

where \( C_i / \gamma_i \) is the stabilization value.

During upgrading of the state variable for the center of the yield surface using Eq.3.42, Eq.3.90a is checked. When the condition in Eq. 3.90a is violated for any decomposed rule, the numerical correction is applied for that rule to stay on the asymptotic value.

\[ f^c(a_{n+1},j) = \left[ \frac{1}{2} (a_{n,j} + k_c d a_{n+1,j}) : (a_{n,j} + k_c d a_{n+1,j}) \right] = \left( \frac{C_i}{\gamma_i} \right)^2 \]  

(3.90b)

when \( f(a_{n,j} + d a_{n+1,j}) > \left( \frac{C_i}{\gamma_i} \right)^2 \)
where $k_c$ is the correction factor to be determined and $k_c d a_{n+1,i}$ is the corrected actual movement of the yield surface. This correction approximates the curve as shown in Fig. 3.10a as corrected solution and checked against a closed form solution. This numerical fixing does not introduce any additional error in the simulation and helps to remove the oscillation and divergence in the numerical scheme. This correction is applicable with any numerical schemes for upgrading the center of the yield surface. This approach allows the modified Chaboche model to prescribe larger strain increments. This correction method will be referred as the oscillation correction.

![Graph](image)

Fig. 3.10 Presence of oscillation in uniaxial stress-strain response simulation by forward Euler method in (a) state variable, a calculation (b) stable hysteresis loop simulation with strain increment of 0.013%.

As mentioned before, another source of oscillation comes from violating the consistency condition at the end of the increment due to first order approximation of the consistency condition. The magnitude of inconsistency is different for different numerical schemes and increases for higher increment sizes. To solve the problem, stress scaling is an alternative solution. In numerical calculation, at the end of the solution the magnitude of stress increment is adjusted to satisfy the consistency condition. But for higher increment size, such adjustment damages the quality of the solution. Hence, Forward Euler scheme urges for smallest increment size for stress-strain simulation. The threshold increment size is dependent on parameter values. For the example in Fig. 3.10b, when the strain increment was 0.013% the oscillation appears at the higher strain range (Fig. 3.10b). With the proposed numerical correction and stress scaling, the numerical scheme can be applied with ten times higher strain increment size (0.13%) without loss of accuracy (Fig. 3.11a).
3.4 Numerical examples

In previous sections, numerical methodologies to implement the modified Chaboche
kinematic hardening rule are presented. This section presents the convergence and accuracy
of the local iteration scheme with numerical examples. The Plasticity model with modified
Chaboche model (Bari and Hassan, 2002) can be analytically solved for proportional loading.
The modified Chaboche model included the multiaxial ratcheting parameter $\delta'$. For
proportional loading, the modified Chaboche model reduces to the original Chaboche (1994)
model due to the elimination of $\delta'$ term. Uniaxial loading is proportional loading. Closed
form solution is obtained for uniaxial loading. For multiaxial cyclic loading, analytical
solution is not possible; hence the results are compared with experimental results. The
objective of this section is to find a numerical scheme which retains the characteristics of the
plasticity model when large strain or stress increments are prescribed.

3.4.1 Uniaxial cyclic loading

Closed form solution for loading part of uniaxial stable hysteresis curve is given by Bari and
Hassan (2000). The closed form solution can be extended to uniaxial cyclic loading for
calculation of uniaxial ratcheting. For virgin monotonic uniaxial loading closed form solution
can be given by

$$\sigma_x = \sum_{i=1}^{4} \alpha_i + \sigma_0$$  \hspace{1cm} (3.91)

$$\alpha_{i,x} = \frac{C_i}{\gamma_i} \left[ 1 - \exp \left( -\gamma_i (\varepsilon_{x_i}^p) \right) \right]$$  \hspace{1cm} \text{for } i = 1,2,3 \hspace{1cm} (3.91a)

$$\alpha_{i,x} = C_i \varepsilon_{x_i}^p$$  \hspace{1cm} \text{for } i = 4 \text{ and } \varepsilon_{x_4}^p \leq \varepsilon_{x_4}^p \hspace{1cm} (3.91b)

$$\alpha_{i,x} = a_4 + \frac{C_4}{\gamma_i} \left[ 1 - \exp \left( -\gamma_i (\varepsilon_{x_i}^p - \varepsilon_{x_4}^p) \right) \right]$$  \hspace{1cm} \text{for } i = 4 \text{ and } \varepsilon_{x_4}^p > \varepsilon_{x_4}^p \hspace{1cm} (3.91c)

where \( \varepsilon_{x_4}^p = a_4 / C_4 \)

For uniaxial cyclic loading and unloading

$$\sigma_x = \sum_{i=1}^{4} \alpha_i \pm \sigma_0$$  \hspace{1cm} (3.92)

$$\alpha_{i,x} = \frac{C_i}{\gamma_i} \left[ \pm 1 \mp \left( 1 - \frac{\gamma_i}{C_i} \alpha_{i,x} \right) \exp \left\{ \mp \gamma_i (\varepsilon_{x_i}^p - (\mp) \varepsilon_{x_i}^p) \right\} \right]$$  \hspace{1cm} \text{for } i = 1,2,3 \hspace{1cm} (3.92a)
\[\alpha_{ix} = C_{i} \{\varepsilon_{x}^{p} - (\mp \varepsilon_{L}^{p})\} + \alpha_{i,L}\]

for \(i = 4\) and \(\varepsilon_{x}^{p} \leq \varepsilon_{xa}^{p}\) \hspace{1cm} (3.92b)

\[\alpha_{ix} = \pm a_{4} + \frac{C_{i}}{\gamma_{i}} \left[ \pm 1 \exp \left( \pm \gamma_{i} \left( \varepsilon_{x}^{p} - \varepsilon_{xa}^{p} \right) \right) \right]\]

for \(i = 4\) and \(\varepsilon_{x}^{p} > \varepsilon_{xa}^{p}\) \hspace{1cm} (3.92c)

where \(\varepsilon_{xa}^{p} = (\pm a_{4} \mp \alpha_{4,L}) / C_{4} \pm (\mp \varepsilon_{L}^{p})\)

\(\pm \alpha_{i,L}\) is the end value from previous loading path at plastic strain limit \(\mp \varepsilon_{L}^{p}\).

The closed form solution presented in Eq. 3.91 and 3.92 has been used to obtain a uniaxial stable hysteresis loop and uniaxial ratcheting response for cyclic loading. The parameters used for hysteresis loop and uniaxial ratcheting simulation for the modified Chaboche kinematic hardening rule are shown in Table 3.1 (Bari and Hassan, 2002). In this section, performance of each numerical scheme is first compared with closed form solution of uniaxial strain controlled cyclic loop along with monotonic part. The closed form solution is obtained by strain-controlled loading in three steps, monotonic loading up to 1.3% strain, then unloading up to -1.2% and reloading again to 1.3% strain (shown in Fig. 3.11 as closed form solution) with parameters shown in Table 3.1. The large strain limits are chosen, so that a stable cyclic loop is obtained. It was observed that the efficiency of the numerical schemes is dependent on increment size. This closed form solution of cyclic loop is compared with the simulation of each numerical scheme for five different types of increments which ranges from small to large strain increments. Number of sub steps and its size is used for cyclic loop simulation in each case are given in Table 3.2. Cyclic loops are simulated with each numerical scheme for increments case1 to case6, only the critical results are presented for the discussion.

This section also compares the performance of the numerical schemes for closed-form solution of uniaxial ratcheting. This uniaxial ratcheting comparison is a measure of numerical scheme’s performance under cyclic loading. Uniaxial ratcheting for closed form results is calculated for stress cycle with mean \(\sigma_{xm} = 6.45\) ksi and amplitude \(\sigma_{xa} = 31.8\) ksi with the parameter set shown in Table 3.1 close to experimental loading range (Bari and Hassan, 2002) for the parameter set (CS1026 material) shown in Table 3.1. The strain accumulation at positive peak stress in each cycle is captured up to 45 cycles. Different size of stress increments as shown in Table 3.3 to obtain five different simulations. The subsequent sub sections also compares the various numerical schemes presented in previous section with
closed form ratcheting solution for its accuracy, adaptability with large increments and convergence with closed form solutions.

Table 3.1
Material parameters for CS1026

<table>
<thead>
<tr>
<th>Elastic parameters</th>
<th>$E = 26,300$ ksi; $\nu = 0.302$; $\sigma_0 = 18.80$ ksi</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plastic Parameters</td>
<td>$C_1-4 = 60,000; 3228; 455; 15,000$</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{1-4} = 20,000; 400; 11; 5000$</td>
</tr>
<tr>
<td></td>
<td>$\bar{a}_4 = 5.0, \delta' = 0.18$</td>
</tr>
</tbody>
</table>

Table 3.2
Size and number of increments used for uniaxial strain-controlled stress-strain response simulation

<table>
<thead>
<tr>
<th>Case</th>
<th>Strain increment size (%)</th>
<th>Number of incremental steps</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Monotonic loading</td>
<td>Unloading/Reloading</td>
</tr>
<tr>
<td>Case1</td>
<td>0.013</td>
<td>0.012</td>
</tr>
<tr>
<td>Case2</td>
<td>0.026</td>
<td>0.024</td>
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<tr>
<td>Case3</td>
<td>0.052</td>
<td>0.049</td>
</tr>
<tr>
<td>Case4</td>
<td>0.13</td>
<td>0.122</td>
</tr>
<tr>
<td>Case5</td>
<td>0.26</td>
<td>0.244</td>
</tr>
<tr>
<td>Case6</td>
<td>0.65</td>
<td>0.63</td>
</tr>
</tbody>
</table>
### Table 3.3
Size and number of stress increments used for uniaxial stress controlled ratcheting response simulation

<table>
<thead>
<tr>
<th>Case</th>
<th>Stress increment size (ksi)</th>
<th>Number of incremental steps</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Monotonic loading</td>
<td>Unloading/Reloading</td>
</tr>
<tr>
<td>Case 1</td>
<td>0.19</td>
<td>0.16</td>
</tr>
<tr>
<td>Case 2</td>
<td>0.38</td>
<td>0.32</td>
</tr>
<tr>
<td>Case 3</td>
<td>0.76</td>
<td>0.64</td>
</tr>
<tr>
<td>Case 4</td>
<td>1.53</td>
<td>1.27</td>
</tr>
<tr>
<td>Case 5</td>
<td>3.8</td>
<td>3.2</td>
</tr>
</tbody>
</table>

#### 3.4.1.1 Forward Euler method

Performance of the forward Euler numerical scheme is checked with closed form solutions. Forward Euler scheme is simplest and quick explicit method, as it does not require any iteration. In this study, forward Euler scheme is implemented with both classical and return approach. In uniaxial cyclic loop simulation, classical forward Euler observed to have a tendency of slight over prediction even for small increments. This over prediction is observed prominent where the curvature is high (Fig. 3.11a). In coupled constitutive models, classical forward Euler method uses the initial position of the yield surface for plastic modulus calculation. The plastic modulus is usually higher than the actual, as the dynamic recovery term (second term of Eq. 3.5a-b) in the kinematic hardening rule is lower than actual. Thus, forward Euler scheme usually over predicts for stress in strain prescribed cases or vice versa. Higher the increment step size, higher the plastic modulus than the actual for the time step. In classical approach, forward Euler scheme is found to produce small deviation from the von-Mises condition at the end of the solution for each step. It is necessary to go for stress scaling at the end of the solution to avoid violation of the consistency condition. Doods (1987) also reported similar inconsistency in this approach (tangent stiffness method) and suggested stress scaling. The forward Euler scheme requires the smallest increment size (threshold size) for stable numerical calculation without oscillation. Cyclic loop simulation shows oscillation for increments higher than threshold increment size as shown in Fig. 3.10b. More deviation observed with increasing increment size. When the corrections based on stabilized value
along with stress scaling is applied for each increment, simulation by this method improve significantly. Figure 3.11 demonstrates the improved performance of forward Euler scheme with the proposed corrections for large strain increment 0.13%. The corrections removed the oscillation and improved the accuracy of simulation.

Forward Euler scheme in return algorithm also observed to produce oscillation for increments greater than threshold size. Similar state variable correction and stress scaling is necessary for improving the simulation. This approach shows very good stress strain hysteresis curve simulation similar to the classical forward Euler scheme (compare Fig. 3.11a and Fig. 3.11b).

In uniaxial ratcheting simulation (Fig. 3.12), stress increment is prescribed and the material undergoes cycling loading with in two fixed two stress limits. Forward Euler scheme underpredicts ratcheting strains due to the over prediction of plastic modulus. Thus, uniaxial ratcheting shows underprediction than the actual ratcheting. As the step size is increased, underprediction of uniaxial ratcheting increased as shown in Fig. 3.12. Both classical and return type forward Euler schemes performed about the same in simulating uniaxial ratcheting (Fig. 3.12).

![Fig. 3.11 Uniaxial stable hysteresis loop simulation for with strain increment of 0.13% (a) Classical Forward Euler (b) Return type Forward Euler.](image-url)
3.4.1.2 Modified Euler Method

Modified Euler scheme was implemented in both classical and return algorithm. As the plastic modulus calculation takes the average of the initial and end plastic modulus, this scheme can more closely approximate the stable loop than the forward Euler scheme for smaller increments. Similar to forward Euler scheme simulation worsen with increased increment sizes. However, modified Euler scheme also has a threshold increment size which is larger than forward Euler scheme for the same set of model parameters. The simulation is good with classical modified Euler for strain increment 0.13% when oscillation correction and stress scaling is applied (Fig. 3.13a).

Modified Euler method proceeds with average plastic modulus calculation. The plastic modulus calculated in modified Euler scheme is smaller than the forward Euler scheme, then predicts smaller stress increment than forward Euler schemes for a given strain increment. Thus, with modified Euler scheme the prediction of hysteresis loop and uniaxial ratcheting either have similar or lower accuracy compare to the forward Euler scheme (Compare Fig. 3.11 to 3.13 and Fig. 3.12 to 3.14). However, with this method, the simulation got worse for larger increments. For simulation with larger strain increments this approach need larger stress scaling which deteriorates the quality of the simulation.

3.4.1.3 Heun’s Method

Heun’s method is an iterative approach. This approach tends to overpredict the hysteresis loop simulation as shown in Fig. 3.15a. For smaller increment, this approach converges very
fast and converged with in eight to ten iterations (e.g. Fig. 3.15b for load increment of case3). For larger increments this, solution algorithm produces non-converging conjugate solution (Fig. 3.15b for case4). In such a situation, successive iteration yields a pair of plastic modulus, $H$ which means use of one $H$ value yields the other $H$ value in iteration. In such case, iterative approach yields poor convergence rate. For non-converged solution, the stress strain response might show oscillation or degrades simulation. The degradation in simulation by Heun’s approach is clearly visible from Fig. 3.15a.

Fig. 3.13 Uniaxial stable hysteresis loop simulation with strain increments of 0.13% by (a) Classical modified Euler (b) Return type modified Euler

Fig. 3.14 Uniaxial ratcheting simulation by (a) classical modified Euler scheme and (b) return modified Euler scheme.

Heun’s return algorithm performs better than classical Huen’s algorithm as shown in Fig. 3.16a and 3.17b. For small strain increment it does not need any oscillation correction. The trend of overprediction is improved in this case. In most cases, convergence of the algorithm
obtained within six to eight iterations for small to moderate increment sizes (*case1* to *case3*). This approach observes convergence difficulty for larger increments (Fig. 3.16b). For larger increment (greater than 0.05% strain), it violates the consistency condition which requires stress scaling. For larger increments this method also suffers from oscillation. Again, the convergence rate is slow in classical Heun’s approach compared to return type Heun’s approach for the same size of increment (compare Fig. 3.15b and Fig. 3.16b for *step2* of *case4*).

![Figure 3.15](a) Uniaxial stable hysteresis loop simulation by Classical Heun’s method with increments of 0.13% (b) convergence of the local iteration scheme

![Figure 3.16](a) Uniaxial stable hysteresis loop simulation by Return type Heun’s method with strain increments of 0.13% (b) convergence of the local iteration scheme.
Despite all these problems, in return algorithm with Heun’s method gives slightly better simulation for uniaxial ratcheting than modified Euler method for small increments (Fig. 3.17). Heun’s approach is an iterative approach, which requires much more solution time than modified Euler scheme.

Fig. 3.17 Uniaxial ratcheting simulation by (a) classical Heun’s method and (b) return type Heun’s method.

Fig. 3.18 Uniaxial stress-strain simulation by Classical backward Euler scheme with strain increments 0.13%.

Fig. 3.19 Uniaxial ratcheting simulation by backward Euler scheme

Wang et. al (2000) described implementation techniques for return type modified Euler schemes. This method failed to yield good simulation for multiple Armstrong-Frederick type models even with small increments. The author discretized Eq. 3.42 as

\[ da_{n+1,i,j} = a_{n,i,j} + \frac{2}{3} C_i d\varepsilon^{p} - \gamma_1 \left( \frac{1}{2} a_{n,i,j} + \frac{1}{2} a_{n+1,i,j} \right) dp \]  

(3.93a)
\[ d\varepsilon^p = \sqrt{\sum d\varepsilon(n_{n+1})}/\sigma_0 \] (3.93b)

where \( n_{n+1} \) is the normal direction at the end of the increment.

In the first term of Eq. 3.93a, the author used the normal direction from the updated point only whereas in the dynamic recovery term is taken as average of the initial and updated position of the yield surface. This inconsistency becomes prominent for multiple back stress model and the author failed for modified Euler scheme. In backward type scheme, Eq. 3.93a is converted by the author as

\[ da_{n+1,i} = a_{n,i} + \frac{\dot{\gamma}}{C_i} d\varepsilon^p - \gamma_i a_{n+1,i} dp \] (3.93c)

Eq. 3.93c maintains the consistency between the normal direction and recovery term and thus improved the simulation.

### 3.4.1.4 Classical Backward Euler Method

In classical Euler type algorithms, backward Euler performs the best. For small and large increments, it can simulate the closed form solution of hysteresis loop well (Fig. 3.18a). Basically, classical algorithm gives chordal approximation which updates for increment through linear shooting based on plastic modulus calculated at the updated point. For linear shooting it needs stress scaling and oscillation correction for state variables. This method experiences similar convergence and simulation quality problem as Heun’s method and modified Euler method when the increment steps are large. It gives the best accuracy among the Euler methods for large increments. The backward Euler simulation for strain increments of case4 (shown in Fig. 3.18) clearly demonstrates supremacy in simulation than other classical approaches. However, in uniaxial ratcheting simulation, this approach has the tendency to overpredict than closed form solution. As this method uses the updated position of the yield surface (for normal direction and magnitude of kinematic variable) in plastic modulus calculation, in uniaxial case this modulus in general yields lower value. For this reason, in uniaxial ratcheting, it has the tendency to overpredict. But, Fig. 3.19 shows underestimation of ratcheting for larger increment sizes instead of overprediction due to nonconverged solution for larger increments.
Fig. 3.20 Uniaxial stable hysteresis loop simulation for increments with 0.052% by (a) Classical explicit Runge-Kutta scheme (b) Return type explicit Runge-Kutta scheme.

Fig. 3.21 Uniaxial ratcheting simulation by (a) Classical explicit Runge-Kutta scheme (b) Return type explicit Runge-Kutta scheme case 4 and 5 diverges.

### 3.4.1.5 Explicit Runge-Kutta Method

Classical explicit Runge-Kutta scheme can simulate strain driven cases for moderate size increments (0.052%) as shown in Fig. 3.20a. It has been observed that this method failed to simulate for larger strain increments (0.13%). This approach also needs the stress scaling and oscillation correction for state variables. This method has the advantage of explicit calculation, does not require any iteration which makes it fast in computation. Return type explicit Runge-Kutta scheme gives similar type of accuracy in simulation like classical Runge-Kutta method (Fig. 3.20b). For uniaxial ratcheting simulation, the simulation with explicit Runge-Kutta scheme gradually deteriorates as increment size increases. Return type
explicit Runge-Kutta scheme yields better results than classical explicit Runge-Kutta approach (compare Fig. 3.21a and 3.21b for case 2). With increasing step size (case 4 and case 5), return type explicit scheme fails to produce ratcheting rate due to instability in calculation (Fig. 3.21).

### 3.4.1.6 Implicit Runge-Kutta Method

This iterative scheme observed convergence difficulty for large strain increments (0.052%) which results in oscillation and poor simulation (Fig. 3.22a). Poor performance of this approach is due to averaging of plastic modulus of two intermediate points. For smaller increment the simulation improves (Fig. 3.22b). For uniaxial ratcheting, simulation with this method has the problem of convergence difficulty for large stress increment, deviates the simulation from the closed form ratcheting simulation (Fig. 3.23).

![Fig. 3.22](image1.png)

**Fig. 3.22** Uniaxial stable hysteresis loop simulation by classical implicit Runge-Kutta method for strain increments of (a) 0.052% and (b) 0.026%.

![Fig. 3.23](image2.png)

**Fig. 3.23** Uniaxial ratcheting simulation by classical implicit Runge-Kutta method.
3.4.1.7 Radial Return Method

Implicit type radial return algorithm is also implemented in this study. The simulations obtained through iterative radial return scheme are compared with the closed form solution. Comparison is presented in two steps, accuracy of the solution and convergence of the algorithm. The closed form results for the cyclic hysteresis loop are compared with simulations with three different strain increments of 0.13%, 0.26% and 0.65% (Fig. 3.24). It has been observed that for large strain increments (0.26% and 0.65%) radial return method is very close to exact solutions, calculates the values close to exact ones at end points and performs comparatively better than any other numerical schemes (Fig. 3.24b). However, for very high increment size, the fitness of the simulation deviates as shown in Fig. 3.24b as the simulation becomes chordal approximation. Similar observation for large increments for the plasticity models with Armstrong-Fredrick type hardening rule is also reported by Kobayashi and Ohno (2002). As this model takes the normal direction and position of the yield surface at the end of the increment step, it has a tendency to represent lower plastic modulus for the calculation. Such deviation becomes prominent when the increment is large as seen in simulation for strain increments of 0.65% (Fig. 3.24b). Another strong feature of this method is that it doesn’t show any oscillation in the simulation.

Fig. 3.24 Uniaxial hysteresis loop simulation by radial return method for strain increments of (a) 0.13% and (b) 0.26% and 0.65%.

As radial return is an iterative scheme it is important to check the convergence of the scheme. In this approach method of successive substitution has been implemented to calculate the nonlinear scalar variable $\Delta p$. At each incremental step, $\Delta p$ is calculated from updated normal direction and updated position of the yield surface and then state variables
are updated for the calculated $\Delta p$. Kobayashi and Ohno (2002) implemented the state variable upgrading scheme for the Armstrong-Frederick type Kinematic hardening rule as

$$a_{n+1,i} = a_{n,i} + \frac{1}{1 + \gamma, \Delta p^{(i)}} C_i \Delta \varepsilon^p$$

(3.94)

When such results are compared with closed form solution, under prediction becomes prominent with increasing $\Delta p$, although this provides a smooth shape of the kinematic hardening rule for decomposed nonlinear kinematic hardening rule with Armstrong-Frederick type rules. The upgrading scheme of the state variable shown according to Eq. 3.94 for the converged value of $\Delta p$ becomes implicit when $\Delta p^{(i)}$ is a function of $a_{n+1,i}$, which requires iteration. Single step upgrading using Eq. 3.94 observes convergence difficulty in the simulation under large increments for the Chaboche type plasticity model with the threshold term in fourth kinematic hardening rule. The method of successive substitution can be implemented for upgrading state variables but it also showed poor convergence rate for large increments in this case. At some steps, successive substitution yields conjugate solution for $\Delta p$ which slows the convergence rate. Kobayashi and Ohno (2002) suggested to implement Aitkin’s accelerating process to improve the convergence rate. Aitkin’s $\Delta^2$ process was also been implemented in this study but this did not change the convergence rate significantly. In such a case, when the solution is not converged, oscillation is observed in the simulation. Whereas the upgrading scheme following modified Newton’s scheme (Eq. 3.80 to 3.87) allows implementing the scheme for larger increments without any convergence difficulty in this case.

Thus the radial return scheme presented in previous section involves two types of iteration. At first, it is necessary to check the convergence of the nonlinear scalar variable $\Delta p^{(k+1)}_{n+1}$. The solution for $\Delta p^{(k+1)}_{n+1}$ in radial return scheme usually converges within six iterations for small to moderate increment sizes. Whereas for large increments it has been observed to continue twelve iterations when strain increment has a pure elastic part. The solution converged with in six iterations for increments having elastic-plastic part. Figure 3.25 demonstrates the convergence of couple of steps for strain increments of 0.26%. In each iteration, solution obtained for $\Delta p^{(k+1)}_{n+1}$ is used for state variable upgrading and the fourth rule needs iteration.
for state variable upgrading as demonstrated with Eq. 3.77. Using the modified Newton’s scheme, it has been observed that the solution converged within four iterations.

![Graph showing convergence](image)

Fig. 3.25 Quadratic convergence of radial return algorithm for large strain increments of 0.26%.

Even though this radial return approach is iterative, it yields accurate calculations with larger increments with accuracy and also provides stability in the numerical solution. Additionally, this approach yields converged solution within six iterations. The final solution is independent of any numerical correction like oscillation correction and stress scaling due to the use of exact consistency condition.

Uniaxial ratcheting responses obtained from this radial return scheme give the closest approximation while compared with closed form solution. This method with larger increments is observed to produce over prediction of ratcheting rate. This over prediction increases consistently with increasing increment sizes shown in Fig. 3.26. Studying the uniaxial monotonic, uniaxial hysteresis loop and uniaxial ratcheting simulations this radial return method seems to be best method for accuracy and stability for all increment sizes.
3.4.2 Multiaxial cyclic loading

The objective of this study is to evaluate the performance of numerical scheme and determine best scheme for multiaxial ratcheting simulation. Ratcheting simulation is influenced by numerical scheme as different schemes consider different position of the yield surface for plastic modulus and normal direction calculation. It was demonstrated that normal direction changes with change in position of the yield surface under multiaxial loading in Fig. 3.2a. Hence, the influence of a numerical schemes in simulating multiaxial ratcheting can be significant which was not the case in simulating uniaxial ratcheting. This section compares the performance of various numerical schemes for multiaxial ratcheting simulation. It is not possible to obtain an analytical solution for multiaxial cyclic loading like the uniaxial case. Hence, the performance comparisons for different schemes will be made based on experimental results under multiaxial cyclic loading.

This study compares the numerical schemes for four types of ratcheting for CS1026 material. Experimental results include uniaxial ratcheting, biaxial ratcheting, bowtie and reverse bowtie ratcheting for stabilized material. The loading histories used are shown in Fig. 1.5. Uniaxial ratcheting involves stress cycling between two fixed stress limits with a mean stress, $\sigma_{xm} = 6.52$ ksi and amplitude $\sigma_{xa} = 32.12$ ksi (Fig. 1.5a). Biaxial ratcheting involves axial strain cycling with an amplitude, $\varepsilon_{xc} = 0.5$ % and constant circumferential stress, $\sigma_{\theta m} = 9.65$ ksi (Fig. 1.5b). In bowtie ratcheting both the circumferential stress and axial strain cycles (Fig. 1.5c, $\varepsilon_{xc} = 0.5$ %, $\sigma_{\theta m} = 9.65$ ksi, $\sigma_{\theta a} = 2.36$ ksi). In reverse bow tie ratcheting, stress loading direction is opposite to bow tie ratcheting but the loading parameters were kept the same (Fig. 1.5d, $\varepsilon_{xc} = 0.5$ %, $\sigma_{\theta m} = 9.65$ ksi, $\sigma_{\theta a} = 2.36$ ksi).
The modified Chaboche model parameters were determined from uniaxial stable loop, uniaxial ratcheting and biaxial ratcheting responses for CS1026 material shown in Fig. 2.1 to Fig. 2.3 (Bari and Hassan, 2002). As a numerical schemes influence the parameter determination depending on the size of increments used, the following method was used for parameter determination. Stable loop parameters were determined using the closed form simulation; thus it is independent of any numerical scheme used. For uniaxial ratcheting, it was observed from previous section that all numerical schemes provide good simulation for small increments. All the numerical schemes also produced good simulations for biaxial ratcheting when small increments were used. Thus, parameters determined with very small increments will be representative for all numerical schemes. At first parameters are determined for the set of results using the optimization algorithm developed in Chapter Two. In this approach, parameter determination is made with semi-implicit (Heun’s approach) return algorithm with very small increments. The discretization used for uniaxial and biaxial loading in parameter determination is given in Table 3.6 (as Uniθ and Biθ). Parameters are shown in Table 3.4. Fitness is measured as root mean square (RMS) deviation in this case. Uniaxial and biaxial ratcheting fitness with this set of parameters are shown in Table 3.5. The fitness for uniaxial ratcheting varies in between 0.272 to 0.274 for all numerical schemes. Biaxial ratcheting fitness varies in between 0.292 to 0.294 for this set of parameter. The fitness is almost same for all numerical schemes. These parameters are used for larger increment to study the influence of numerical schemes in ratcheting simulation.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Magnitude of parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$~$C_4$</td>
<td>53441.00, 5103.00, 443.00, 19527.00</td>
</tr>
<tr>
<td>$\gamma_1$~$\gamma_4$</td>
<td>26056.00, 469.00, 8.65, 4693.00</td>
</tr>
<tr>
<td>$a_4$</td>
<td>1.58</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.26</td>
</tr>
</tbody>
</table>
Table 3.5
Uniaxial and biaxial ratcheting fitness for small discretization

<table>
<thead>
<tr>
<th>Numerical Scheme</th>
<th>Fitness (RMS)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Uniaxial Ratcheting</td>
<td>Biaxial Ratcheting</td>
</tr>
<tr>
<td>Forward Euler</td>
<td>0.272</td>
<td>0.293</td>
<td></td>
</tr>
<tr>
<td>Modified Euler</td>
<td>0.274</td>
<td>0.294</td>
<td></td>
</tr>
<tr>
<td>Heun’s Method</td>
<td>0.274</td>
<td>0.294</td>
<td></td>
</tr>
<tr>
<td>Radial Return</td>
<td>0.274</td>
<td>0.293</td>
<td></td>
</tr>
<tr>
<td>Explicit Runge-Kutta</td>
<td>0.273</td>
<td>0.292</td>
<td></td>
</tr>
</tbody>
</table>

At this stage deviation of simulations from the experimental results comes from the model responses and error due to numerical calculation. When the model and its parameters are fixed, different numerical schemes can only modify the error from the numerical source. The numerical error is minimized when the increment size is small. When the increment size increases, the magnitude of error from numerical sources increased. This section studies the effect of increment size on different numerical schemes in multiaxial ratcheting simulation. The increment sizes used in ratcheting simulation are given in Table 3.6a-d.

Table 3.6a
Load increments used in uniaxial ratcheting simulation (Fig. 1.5a)

<table>
<thead>
<tr>
<th>Increment Size</th>
<th>Monotonic $\Delta \sigma_x$ (ksi)</th>
<th>Cyclic $\Delta \sigma_x$ (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uni 0</td>
<td>0.0966</td>
<td>0.0803</td>
</tr>
<tr>
<td>Uni 1</td>
<td>0.1932</td>
<td>0.1606</td>
</tr>
<tr>
<td>Uni 2</td>
<td>0.3864</td>
<td>0.3212</td>
</tr>
<tr>
<td>Uni 3</td>
<td>0.7728</td>
<td>0.6424</td>
</tr>
<tr>
<td>Uni 4</td>
<td>1.5456</td>
<td>1.2848</td>
</tr>
</tbody>
</table>
Table 3.6b
Load increments used in biaxial ratcheting simulation (Fig. 1.5b)

<table>
<thead>
<tr>
<th>Increment Size</th>
<th>Monotonic $\Delta \sigma_0$ (ksi)</th>
<th>Cyclic $\Delta \varepsilon_x$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Bi 0$</td>
<td>0.0965</td>
<td>0.0025</td>
</tr>
<tr>
<td>$Bi 1$</td>
<td>0.193</td>
<td>0.005</td>
</tr>
<tr>
<td>$Bi 2$</td>
<td>0.386</td>
<td>0.010</td>
</tr>
<tr>
<td>$Bi 3$</td>
<td>0.643</td>
<td>0.020</td>
</tr>
<tr>
<td>$Bi 4$</td>
<td>1.206</td>
<td>0.040</td>
</tr>
<tr>
<td>$Bi 5$</td>
<td>1.206</td>
<td>0.100</td>
</tr>
</tbody>
</table>

Table 3.6c
Load increments used in bowtie ratcheting simulation (Fig. 1.5c)

<table>
<thead>
<tr>
<th>Bow 0</th>
<th>Monotonic $\Delta \sigma_0$ (ksi)</th>
<th>Cyclic $\Delta \sigma_x$ (ksi)</th>
<th>$\Delta \varepsilon_x$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0965</td>
<td>0.012,0.047*</td>
<td>0.0025</td>
</tr>
<tr>
<td></td>
<td>0.193</td>
<td>0.024,0.094</td>
<td>0.005</td>
</tr>
<tr>
<td></td>
<td>0.386</td>
<td>0.047,0.189</td>
<td>0.010</td>
</tr>
<tr>
<td></td>
<td>0.644</td>
<td>0.095,0.315</td>
<td>0.020</td>
</tr>
<tr>
<td></td>
<td>1.206</td>
<td>0.236, 0.591</td>
<td>0.050</td>
</tr>
<tr>
<td></td>
<td>1.206</td>
<td>0.473,1.182</td>
<td>0.100</td>
</tr>
</tbody>
</table>

*0.012 and 0.047 are the stress increments for inclined and vertical paths respectively in Fig. 1.5c
Table 3.6d
Load increments used in reverse-bowtie ratcheting simulation (Fig. 1.5d)

<table>
<thead>
<tr>
<th></th>
<th>Monotonic</th>
<th>Cyclic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta \sigma_0$ (ksi)</td>
<td>$\Delta \sigma_x$ (ksi)</td>
</tr>
<tr>
<td>$R_{bow\ 0}$</td>
<td>0.0965</td>
<td>0.012, 0.047*</td>
</tr>
<tr>
<td>$R_{bow\ 1}$</td>
<td>0.193</td>
<td>0.024, 0.094</td>
</tr>
<tr>
<td>$R_{bow\ 2}$</td>
<td>0.386</td>
<td>0.047, 0.189</td>
</tr>
<tr>
<td>$R_{bow\ 3}$</td>
<td>0.644</td>
<td>0.095, 0.315</td>
</tr>
<tr>
<td>$R_{bow\ 4}$</td>
<td>1.206</td>
<td>0.236, 0.591</td>
</tr>
<tr>
<td>$R_{bow\ 5}$</td>
<td>1.206</td>
<td>0.473, 1.182</td>
</tr>
</tbody>
</table>

*0.012 and 0.047 are the stress increments for inclined and vertical paths respectively in Fig. 1.5d

3.4.2.1 Forward Euler scheme

In uniaxial ratcheting, the parameters with smallest increment size overpredict the experimental ratcheting simulation by both the classical and return type algorithm with forward Euler method. With increasing increment size (from $Uni0$ to $Uni4$) the forward Euler is observed to simulate smaller ratcheting rate (Fig. 3.27a-b). Similarly, in biaxial ratcheting, forward Euler is observed to simulate smaller ratcheting rate with increasing increment sizes (Fig. 3.27c-d). In uniaxial case, normal direction of the yield surface remains fixed and influenced by the yield surface position. For multiaxial ratcheting, the simulation is influenced by position and normal direction of the yield surface. Forward Euler scheme makes the calculation based on normal direction and position of the yield surface at the beginning of the increment (initial position). When the increment size is small, the actual position and normal direction of the yield surface does not vary much from the initial. With increasing increment sizes these deviations become more prominent. For this reason, biaxial ratcheting simulation deviates more with increased increment size compared to uniaxial ratcheting simulation.

In bow tie and reverse bowtie loading, both stress and strain increments are applied simultaneously. Both classical and return algorithm has been observed to produce markedly different ratcheting rate with increased increment size (Fig. 3.27e-h). The change in ratcheting rate is more in these cases compared to the biaxial ratcheting case.
Fig. 3.27 Multiaxial ratcheting simulation with modified Chaboche model by classical and return type forward Euler scheme for (a), (b) Uniaxial ratcheting simulation (c), (d) Biaxial ratcheting simulation (e), (f) bow-tie ratcheting simulation and (g), (h) reverse bow tie ratcheting simulation
3.4.2.2 Modified Euler scheme

The performance of return type modified Euler in ratcheting simulation is more consistent compared to classical modified Euler scheme. In uniaxial ratcheting, classical scheme is observed to produce different ratcheting history with increasing increment size whereas return type modified algorithm have minor variation in simulation (Fig. 3.28a-b). In biaxial ratcheting simulation, classical Euler scheme deviates significantly for higher increment sizes (Fig. 3.28c). Such deviation is not observed for return type modified Euler scheme (Fig. 3.28d). Classical modified Euler showed marked change in ratcheting rate in bow tie and reverse bow tie case (Figs. 3.28e and 3.28g). In return algorithm for bow tie and reverse bow tie history, ratcheting rate variation with change in increment size is also noticeable (Figs. 3.28f and 3.28h). In general, return type modified Euler scheme simulates ratcheting rate for small to moderate increments consistently.

3.4.2.3 Heun’s method

The iterative Classical Heun’s approach produces change in uniaxial ratcheting rate for larger increment sizes (Fig. 3.29a). For biaxial ratcheting simulation, similar behavior is observed (Fig. 3.29c). For larger increments this approach also faces convergence difficulty and affected simulation quality for bow tie and reverse bowtie history (Fig. 3.29g and Fig. 3.29h). The increase in deviation in large increment is due to convergence difficulty observed in this method. The return type Heun’s approach is consistent in simulating uniaxial, biaxial, bowtie and reverse bowtie ratcheting rates (Fig. 3.29b, Fig. 3.29d, Fig. 3.29f and Fig. 3.29h). This return modified and Heun’s scheme shows more consistent simulation than forward Euler scheme with increasing increment sizes as this method takes the average of the normal directions and position of the yield surface from the initial and end position. Consistency in the simulation indicates that this normal direction is more representative than the initial normal direction in all cases.
Fig. 3.28 Multiaxial ratcheting simulation with modified Chaboche model by classical and return type modified Euler scheme for (a), (b) Uniaxial ratcheting simulation (c),(d) Biaxial ratcheting simulation (e),(f) bow-tie ratcheting simulation and (g), (h) reverse bow tie ratcheting simulation
3.4.2.4 Backward Euler scheme
Classical backward scheme can well represent uniaxial ratcheting even in large increments (Fig. 3.30a). Biaxial ratcheting is however severely affected with increasing increment size (Fig. 3.30b-d). Under multiaxial loading, this method fails to produce reasonable results for moderate size increments due to convergence difficulty. In multiaxial loading with large increments, this iterative approach fails converge plastic modulus calculation.

3.4.2.5 Runge-Kutta schemes
For multiaxial histories, explicit Runge-Kutta showed marked deviation in simulation with increased increment sizes (Fig. 3.31). Implicit Runge-Kutta scheme also shows convergence problem as seen in Fig. 3.32.

3.4.2.6 Radial return algorithm
The performance of the radial return algorithm is excellent in ratcheting simulation as observed in Fig. 3.33. This scheme is further tested for other larger increments for biaxial, bowtie and reverse bowtie ratcheting cases and observed that statement is still valid (Fig. 3.33). The deviation in ratcheting simulation with increasing increment size is unnoticeable in most cases except for uniaxial ratcheting.
Fig. 3.29 Multiaxial ratcheting simulation with modified Chaboche model by classical and return type Heun’s method for (a), (b) Uniaxial ratcheting simulation (c),(d) Biaxial ratcheting simulation (e),(f) bow-tie ratcheting simulation and (g), (h) reverse bow tie ratcheting simulation
3.4.2.6 Summary

In previous section, various numerical schemes of implementing plasticity models are compared in terms of accuracy, stability and efficiency of simulating responses. In search of a robust numerical algorithm, numerical schemes in modified Chaboche model evaluated against the uniaxial closed form solutions and multiaxial experimental results. Kreig and Kreig (1977) studied tangent stiffness method, modified Euler method and radial return method for plasticity problems with isotropic hardening subjected to monotonic loading and observed radial return method to have reasonable accuracy in monotonic loading. Dodds (1987) studied forward Euler, modified Euler and radial return scheme for rate independent plasticity with isotropic hardening or linear mixed-isotropic–kinematic hardening. The authors identified that elastic predictor–radial return algorithm satisfy the requirements for stable and efficient numerical algorithm when numerical schemes are compared under uniaxial monotonic loading. When the above studies were made, the kinematic hardening models were not well developed. This study extended the comparison of numerical schemes for uniaxial and multiaxial ratcheting responses.
Fig. 3.31 Multiaxial ratcheting simulation with modified Chaboche model by classical and return type explicit Runge-Kutta method for (a), (b) Uniaxial ratcheting simulation (c),(d) Biaxial ratcheting simulation (e),(f) bow-tie ratcheting simulation and (g), (h) reverse bow tie ratcheting simulation
Fig. 3.32 Multiaxial ratcheting simulation with modified Chaboche model by classical implicit Runge-Kutta method for (a) Uniaxial ratcheting simulation (b) Biaxial ratcheting simulation (c) bow-tie ratcheting simulation and (d) reverse bow tie ratcheting simulation

Fig. 3.33 Multiaxial ratcheting simulation with modified Chaboche model by radial return algorithm for (a) Uniaxial ratcheting simulation (b) Biaxial ratcheting simulation (c) bow-tie ratcheting simulation and (d) reverse bow tie ratcheting simulation
In general, return type algorithm performs better than classical plasticity calculation due to its stability in numerical simulation. For finite size of increment, return type algorithm will be a better choice than classical approach in implementation of plasticity models. Forward Euler scheme produces largest deviation from the consistency condition for the same size of increments among all numerical schemes studied. This approach needs state variable correction and stress scaling for implementation with large increments. The ratcheting simulation from this method deviates for biaxial, bowtie and reverse bow tie ratcheting response as the strain increment is increased. Return type Modified Euler and Heun’s scheme performs better than forward Euler scheme in simulating uniaxial and multiaxial ratcheting responses. These methods show deviation in multiaxial ratcheting simulation with increasing increment size. These methods need state variable correction and stress scaling at the end of each step, and thus affected the quality of the simulation. Explicit Runge-Kutta performs poorly in stress-strain cyclic loop simulation with large increments. Implicit Runge-Kutta method suffers from convergence difficulty with large increments.

The iterative radial return approach simulates the stress-strain response well even with large increments. This method maintains the persistent quality of the multiaxial ratcheting simulations with large increments. This method converges fast and can handle large strain increments with greater stability and reasonable accuracy. The numerical schemes except radial return method have been observed to become unstable for large increments. Radial return method is the best approach for plasticity calculation in terms of accuracy, time efficiency and convergence. As this method always takes the normal direction and position of the yield surface at the end point and also satisfies the consistency condition at the end points, the calculations are always consistent. It does not require any oscillation correction and stress scaling at the end of solution.

3.5 Conclusion

Numerical schemes in classical and return approach for plasticity computation are discussed in a general form. Discretization technique to each of the numerical scheme also explained. A special algorithm named “Stress return algorithm” is developed to solve the problem of simultaneous stress and strain prescription with return type algorithm. Numerical difficulties for these numerical schemes and their remedies are addressed in this chapter. Evaluation of
all the numerical schemes are made in three steps; uniaxial monotonic and hysteresis loop simulation, uniaxial ratcheting and multiaxial ratcheting simulations. The closed form solutions are presented for monotonic and hysteresis loop and uniaxial ratcheting simulation. The numerical schemes are compared for accuracy and stability for simulating the above responses with various sizes of increments. Evaluations are also made for multiaxial ratcheting simulation using experimental responses for various size of increments. The radial return method is found to be robust in simulating uniaxial stress-strain, and uniaxial and multiaxial ratcheting responses. The radial return method is recommended for implementing plasticity models.
CHAPTER FOUR
NUMERICAL ASPECTS OF IMPLEMENTING CYCLIC PLASTICITY MODEL INTO FINITE ELEMENT PROGRAMS

4.1 Introduction

Finite element method is widely used for analysis of the elastic-plastic response of structures. Finite element method employs constitutive model to capture the material stress strain response at each integration point. Structural response is the integrated output from all integration points. During structural analysis through finite element method, local and global level numerical integrations are carried out simultaneously. Local integration performs numerical calculations using constitutive equations for finite increments of loading at integration points in each element. Global integration solves for the nodal force equilibrium equations in iterative manner. The accuracy the structural solution is dependent on the accuracy of calculation at the integration points. When the local numerical scheme is iterative, solution time is dependent on the efficiency of the solution scheme. Solution time of the global iterative solution in nodal force is dependent on the consistency between the stiffness matrix and local numerical scheme adapted. The structural stiffness matrix in turn depends on the tangent modulus calculated for each integration point. Thus, efficient global solution is dependent on consistency between the local numerical scheme and tangent modulus at each integration point.

The accuracy and efficiency of the structural solution with a plasticity model is dependent on numerical techniques of implementing the plasticity model. Continuing demand of cyclic structural analysis for large number of cycles urges the need for solution in large load steps. It is imperative to employ an efficient numerical solution technique to maintain accuracy, stability and numerical efficiency for large load steps. The comparative study of numerical schemes in Chapter Three demonstrates that radial return algorithm satisfies conditions of accuracy, stability and efficiency with large strain increments. This study implemented the constitutive models in finite element program ANSYS with radial return method as local numerical scheme. The strain driven numerical scheme presented in Chapter Three for radial return method is used to calculate stresses for six prescribed strain components. This approach is used for solid elements. In structural analysis, shell elements are commonly used for pipe model generation. This strain driven approach needs to be modified for plane stress...
Case. This Chapter presents the formulations for plane stress case keeping similarity with the strain driven radial return algorithm presented in previous sections. In structural simulation, due to geometry of the structure or point of application of load, different parts of the structure may experience different state of deformation. To account for this different state of deformation in an efficient manner, implementation of local numerical scheme demands numerical solution with sub increment method. A methodology to implement with sub-increment is also presented in this chapter.

Solution time of nodal force equilibrium equations is dependent on the convergence rate of the iterative solution scheme. The quadratic convergence of this global solution scheme is dependent on the consistency between the tangent matrix and local numerical scheme. This chapter presents the formulations for consistent tangent modulus for radial return numerical scheme to implement with modified Chaboche (Bari and Hassan, 2002) plasticity model. The consistent tangent matrix formulations for Ohno-Wang (1993), Abdel Karim-Ohno (2000), and modified Ohno-Wang model (Chen and Jiao, 2003) are presented in Appendix B. The modified Chaboche model is implemented in ANSYS with radial return method. A methodology to implement plasticity model in ANSYS is discussed briefly with reference to ANSYS manual. At the end of this chapter, the implementation is verified with two numerical examples.

4.2 Radial return method for plane stress

Elasto-plastic analysis with shell elements is important for analysis of piping components. Special plane stress algorithm is necessary to implement plasticity model with shell elements in finite element packages. In strain driven implicit code, for plane stress case, out of plane strain component is an additional kinematic unknown. Additional condition for plane stress case is that the corresponding stress component is zero. For elastic increment, the unknown strain component can be readily solved for the known corresponding stress component. For strain increments having elastic and plastic parts, the plastic increment has to be calculated iteratively. Additional iteration for the plane stress case increases the computation time. Iterative calculation for plastic strain increment assumes inelastic incompressibility of the material. In this section, special solution scheme is presented for the plane stress case (Doghri, 1995) for cyclic plasticity models with the modified Chaboche (Bari and Hassan,
2002) kinematic hardening rule. The scheme will be presented keeping similarity with the strain driven part of the radial return method presented in previous Chapter (Section 3.2.2).

The objective of the plane stress problem can be described as calculating the stress increment \( \sigma_{n+1}^{T} \) and \( \varepsilon_{n+1,3} \) for a strain increment \( \varepsilon_{n+1} \) which satisfies the discretized Eqs. 3.36-3.42 and \( \sigma_{n+1,3,1} = 0 \). The algorithm still can be distinguished into two parts, elastic prediction and plastic correction.

In elastic prediction, trial elastic strain increment for the out of plane component \( \varepsilon_{n+1,3}^{T} \) can be calculated making the \( \sigma_{n+1,3} = 0 \) in Eq. 3.36 assuming the entire strain increment is elastic which results in

\[
d\varepsilon_{n+1,3}^{T} = -\frac{(K - \frac{2}{3} G)}{(K + \frac{4}{3} G)} (d\varepsilon_{n+1,1} + d\varepsilon_{n+1,2})
\]  \hspace{1cm} (4.1)

Trial elastic stress \( \sigma_{n+1}^{T} \), calculated from \( \varepsilon_{n+1}^{T} \) with \( \varepsilon_{n+1,3}^{T} \), is checked for the yield condition \( f_{n+1}^{T} \leq 0 \) and \( \sigma_{n+1}^{T} \) is accepted as updated stress \( \sigma_{n+1} \) when it satisfies the yield condition. Out of plane strain component is calculated using Eq. 4.1.

When the yield condition is not satisfied for elastically calculated stresses, \( f_{n+1}^{T} > 0 \), the unknown components of \( \sigma_{n+1} \) and \( \varepsilon_{n+1,3} \) is calculated for satisfying the consistency condition where \( d\varepsilon_{n+1,3} \) and \( d\varepsilon_{n+1,3}^{p} \) is the actual total and actual plastic strain increment. Eq. 3.36 is still valid

\[
d\sigma_{n+1} = D^{e} : (d\varepsilon_{n+1} - d\varepsilon_{n+1}^{p})
\]  \hspace{1cm} (4.2)

Eq. 4.2 is corrected for plastic strain in plane stress case,

\[
d\sigma_{n+1} = d\sigma_{n+1}^{T} - D^{e} : d\varepsilon_{n+1}^{p} + (K1 + 2G\delta_{3})(d\varepsilon_{n+1,3} - d\varepsilon_{n+1,3}^{T})
\]  \hspace{1cm} (4.3)

where \( \delta_{3} = \delta_{i3} \delta_{j3} - \frac{1}{3} \delta_{ij} \).

In Eq. 4.3 for the third stress component increment,

\[
d\sigma_{n+1,3}^{T} = (K - \frac{2}{3} G)(d\varepsilon_{n+1,1} + d\varepsilon_{n+1,2}) + (K + \frac{4}{3} G)d\varepsilon_{n+1,3}^{p} - 2Gd\varepsilon_{n+1,3}^{p} = 0
\]  \hspace{1cm} (4.4)

Therefore,

\[
d\varepsilon_{n+1,3} = \frac{2G}{(K + \frac{4}{3} G)} \sqrt{\frac{3}{2}} dp_{n+1}n_{n+1,3} - \frac{(K - \frac{2}{3} G)}{(K + \frac{4}{3} G)} (d\varepsilon_{n+1,1} + d\varepsilon_{n+1,2})
\]  \hspace{1cm} (4.5)

Subtracting Eq. 4.1 from Eq. 4.5,

\[
d\varepsilon_{n+1,3} - d\varepsilon_{n+1,3}^{T} = \frac{2G}{(K + \frac{4}{3} G)} \sqrt{\frac{3}{2}} dp_{n+1}n_{n+1,3}
\]  \hspace{1cm} (4.6a)
\[
d\varepsilon_{n+1,3} - d\varepsilon_{n+1,3}^T = \frac{2G}{(K + \frac{4}{3} G)} dp_{n+1} \frac{3}{2} \sigma_0 q_{n+1,3}
\]

(4.6b)

For plane stress case, the current stress can be computed as

\[
\sigma_{n+1} = \sigma_n + d\sigma_{n+1} = \sigma_n + d\sigma_{n+1}^T - (K' + 2G\delta_3)(d\varepsilon_{n+1,3} - d\varepsilon_{n+1,3}^T)
\]

(4.7)

Taking the deviatoric part of Eq. 4.7 and subtracting Eq. 3.41 (for \(\phi = 1\))

\[
s_{n+1} - a_{n+1} = s_n + d\tilde{s}_{n+1} - 2Gd\varepsilon_{p}^n + 2G\tilde{\sigma}(d\varepsilon_{n+1,3} - d\varepsilon_{n+1,3}^T) - \sum_{i=1}^{4} a_{n+1,i}
\]

(4.8)

Putting the values of \(d\varepsilon_{n+1,3} - d\varepsilon_{n+1,3}^T\) and \(a_{n+1}\) from Eq.4.6 and Eq.3.42 and rearranging we get

\[
q_{n+1} = \left[ \sigma_0 \left( s_{n+1} - a_0 + \sum_{i=1}^{4} \left( \delta a_{n+1,i} + \frac{3}{2} \frac{(1-\delta)}{\sigma_0} (a_{n+1,i} q_{n+1,i}) \right) dp_{n+1} + \frac{6G^2}{\sigma_0 (K + \frac{4}{3} G)} dp_{n+1} q_{n+1,3} \delta_3 \right) \right] \left[ \sigma_0 + (3G + \sum_{i=1}^{4} C) dp_{n+1} \right]
\]

(4.9)

The term \(\frac{6G^2}{\sigma_0 (K + \frac{4}{3} G)} q_{n+1,3} \delta_3\) in Eq. 4.9, could be obtained from Eq. 3.66a (for \(m = 3\)) when one normal stress component is known.

Using Eq. 4.9 in von-Mises function of Eq. 3.4, it is possible to find a nonlinear scalar equation of kinematic scalar term \(dp_{n+1}\). This nonlinear scalar equation can be solved by method of successive substitution presented in Chapter Three. This algorithm is used as local numerical scheme in customized ANSYS to implement modified Chaboche model for shell elements.

### 4.3 Strain paths for local numerical scheme

The local numerical scheme can be implemented in finite element method (for any type of elements) for given strain increments in two ways, path dependent strategy and path independent strategy. In each strategy, local numerical schemes calculate stresses for prescribed strain increments. In path dependent strategy, plasticity equations are solved for the residual strain increment prescribed for that iteration in a step. In path independent strategy, solution is carried out over the entire strain increment for the step. Path dependent strategy has two drawbacks. During integration of the scheme, residual strain direction can be strikingly different from the strain increment for the step. This approach also has the
probability of spurious unloading especially when stresses are overestimated in previous iteration (Dodds, 1987). As the path independent strategy is free from these two drawbacks, the constitutive models in this study are implemented with path independent strategy. During structural analysis due to geometry of the structure, a small load increment might produce a large strain increment or vice versa. Radial return scheme can handle large strain increment with good accuracy as observed in Chapter Three. However, it introduces small errors when increments are large. To eliminate this error, local iterations can be performed through sub-increment method. When the substep number is kept constant, the problem of large strain increment can still be present. Depending on the size of the step, number of substeps need to be adaptable to overcome this problem. In this study, number of sub steps is made adaptable with respect to a constant equivalent strain increment. This ensures the local that numerical calculation will be within allowable size of increment for acceptable accuracy. Such approach allows optimization of the solution time in a way that larger strain increments need large number of sub steps and smaller increments can be done with minimum number of sub-increments. As the number of steps is adaptable with a given strain increment, this sub-increment method also optimizes the solution time for required level of accuracy. But the solution time is increased with implementation of sub increment scheme. The radial return numerical scheme is implemented in ANSYS for solid elements. To solve the problem with shell elements, this study implemented plane stress algorithm also. Both the solution approach is used as subroutine in ANSYS for structural solution.

4.4 Consistent tangent modulus

After convergence of the local iteration for the numerical scheme, it is important to calculate the consistent tangent modulus to preserve the quadratic convergence rate at global level of finite element calculation. Numerical method is said to be consistent with the tangent modulus when prescribed stress increment for given strain increment from tangent modulus is consistent with stress calculated by local numerical scheme. Such tangent matrix is termed as consistent tangent modulus for the numerical scheme adapted. This study used four kinematic hardening rules in the plasticity model. The consistent tangent modulus is first prescribed in a general form for the modified Chaboche model with radial return algorithm. Expressions for Ohno-Wang, Abdel Karim-Ohno and modified Ohno-Wang model are given in Appendix B.
The discretized Eq. 3.36 to Eq. 3.41 gives the nonlinear stress-strain relationship for the interval \( n \) to \( n+1 \) for modified Chaboche (Bari and Hassan, 2002) which calculates the stress increment \( \Delta \sigma_{n+1} \) and \( \Delta \varepsilon_{n+1}^p \) for a given finite strain increment, \( \Delta \varepsilon_{n+1} \).

By differentiating Eq. 3.36, Eq. 3.39- Eq. 3.40a,

\[
\begin{align*}
\text{d} \sigma_{n+1} &= D' : (\text{d} \varepsilon_{n+1} - \text{d} \varepsilon_{n+1}^p) \\
\text{d} \varepsilon_{n+1}^p &= \sqrt{\frac{3}{2}} (d \Delta p_{n+1} n_{n+1} + \Delta p_{n+1} d n_{n+1}) \\
\text{d} n_{n+1} &= \sqrt{\frac{3}{2}} \left( \frac{d s_{n+1} - d a_{n+1}}{\sigma_0} \right)
\end{align*}
\]  

(4.10)

(4.11)

(4.12)

From deviatoric component of Eq. 4.10,

\[
\text{d} s_{n+1} = 2G(I_{\mathrm{e}d} : \text{d} \varepsilon_{n+1} - \text{d} \varepsilon_{n+1}^p) \\
\text{where } I_{\mathrm{e}d} = I - \frac{1}{3}(\mathbf{1} \otimes \mathbf{1})
\]  

(4.13)

By combining Eq. 3.43, Eq. 4.11 and Eq. 4.12

\[
\text{d} \Delta p_{n+1} = \sqrt{\frac{2}{3}} (\bar{n}_{n+1} : \text{d} \varepsilon_{n+1}^p)
\]  

(4.14)

Using Eq. 4.11- Eq. 4.14,

\[
\begin{align*}
\text{d} \varepsilon_{n+1}^p &= (\bar{n}_{n+1} \otimes \bar{n}_{n+1}) \text{d} \varepsilon_{n+1}^p + \frac{3}{2} \frac{\Delta p_{n+1}}{\sigma_0} 2G(I_{\mathrm{e}d} : \text{d} \varepsilon_{n+1} - \text{d} \varepsilon_{n+1}^p) - \frac{3}{2} \frac{\Delta p_{n+1}}{\sigma_0} \sum_{i=1}^{4} C_i d \varepsilon_{n+1}^p \\
&+ \frac{3}{2} \frac{\Delta p_{n+1}}{\sigma_0} (1-\delta') \sum_{i=1}^{4} (a_{n+1,i} \otimes \bar{n}_{n+1}) x_{n+1,i} d \varepsilon_{n+1}^p + \sqrt{\frac{3}{2}} \frac{\Delta p_{n+1}}{\sigma_0} \delta' \sum_{i=1}^{4} \gamma_i x_{n+1,i} (a_{n+1,i} \otimes \bar{n}_{n+1}) d \varepsilon_{n+1}^p
\end{align*}
\]  

(4.15)

Rearranging the terms of Eq. 4.15

\[
\begin{align*}
2G(I_{\mathrm{e}d} : \text{d} \varepsilon_{n+1}) &= -\frac{2\sigma_0}{3\Delta p_{n+1}} \text{d} \varepsilon_{n+1}^p + \frac{2}{3} \sum_{j=1}^{4} C_j \text{d} \varepsilon_{n+1}^p + 2Gd e_{n+1}^p - (1-\delta') \sum_{i=1}^{4} x_i (a_{n+1,i} \otimes \bar{n}_{n+1}) \text{d} \varepsilon_{n+1}^p \\
&- \frac{2\sigma_0}{3\Delta p_{n+1}} (\bar{n}_{n+1} \otimes \bar{n}_{n+1}) \text{d} \varepsilon_{n+1}^p - \sqrt{\frac{2}{3}} \delta' \sum_{i=1}^{4} \gamma_i k_i (a_{n+1,i} \otimes \bar{n}_{n+1}) \text{d} \varepsilon_{n+1}^p
\end{align*}
\]  

(4.16)

From Eq. 4.16
\[ 2G(I_d : \varepsilon_n^{e+1}) = L_{n+1} : \varepsilon_n^{p} \]  \hspace{1cm} (4.17)

where

\[ L_{n+1} = \left[ \frac{2\sigma_0}{3\Delta p_{n+1}} + \frac{2}{3} \sum_{i=1}^{4} C_i + 2G(1-\delta) \sum_{i,j=1}^{4} k_i (a_{n+1}^{p} : \varepsilon_{n+1}^{u}) \right] - \frac{\delta}{3} \sum_{i,j} k_i (a_{n+1}^{p} : \varepsilon_{n+1}^{u}) - \frac{2\sigma_0}{3\Delta p_{n+1}} (a_{n+1}^{u} : a_{n+1}^{u}) \]

From Eq. 4.10 and Eq. 4.17 expression for the consistent tangent modulus can be derived

\[ \frac{d\sigma_{n+1}}{d\varepsilon_{n+1}} = D^e - 4G^2 (L_{n+1} : I_d) \]  \hspace{1cm} (4.18)

The expression presented in Eq. 4.18 represents consistent tangent modulus for solid elements.

For plane stress algorithm, the expression in Eq. 4.18 need to be modified for plane stress case. The following formula can be used to get the consistent tangent modulus for plane stress (Sawer et. al., 2001)

\[ J_{i,j} = J_{i,j} - \frac{J_{i,3} J_{3,j}}{J_{3,3}} \]  \hspace{1cm} (4.19)

where \( J \) denotes the tangent modulus obtained using Eq. 4.18.

4.5 Implementation with ANSYS

4.5.1 ANSYS architecture for numerical calculation (ANSYS 9.0)

ANSYS has an open architecture which allows to link the users own constitutive models as a subroutine. With this option ANSYS allows user defined stress-strain relationship where mechanical behavior of material is necessary. Such subroutine is called to update stress, strain, state variables and consistent tangent modulus at each integration point during structural solution. The structural solution performed with such customized ANSYS took advantage of the experience of the related fields which is incorporated in ANSYS. The option named “USERMAT” is used for linking the plasticity routine. Choice of such an option restricted the use of elements in ANSYS. In such a situation, only \( 18X \) elements are available for structural model generation in ANSYS. In structural solution with customized ANSYS, it initially calls the subroutine USERMAT. Depending on the number of direct and shear stress components this subroutine calls other subrouines USERMAT3D, USERMATPS, USERMATBM and USERMAT1D for solid, plane stress, beam and one dimensional elements.
respectively. In these subroutines, input and output variables and their format are fixed. The input argument “prop” can be used to specify material elastic and plastic parameters for the plasticity model. The number of material parameters can be given outside the ANSYS subroutine as input in material properties definition. The subroutine takes input as initial stress, strain, plastic strain, state variables and strain increment as input at the beginning of time step. The state variables used to keep track of the kinematic variables associated with each constitutive model for each integration point.

The subroutine in customized ANSYS performs two tasks in all iteration: upgrading variables and calculating tangent modulus. The local numerical scheme for the plasticity models updates stresses, plastic strains and state variables for the trial strain increments given by the finite element program. ANSYS architecture works with path independent strategy for local upgrading of variables for the given strain increment. In all iterations for a step, the subroutine works with initial converged stresses, strains and state variables from the previous step and numerically calculates variables over the trial strain increments. When the variables are updated, the tangent modulus is calculated at the updated point. This tangent modulus is used for calculation of stiffness matrix for global iteration for the structure. This stiffness matrix finally calculates the residual strain increment at each integration point for the residual force in nodal force equilibrium iterations. In ANSYS for the next iteration, this subroutine starts with converged stress, strain, state variables and consistent tangent modulus for the previous time step and the updated stress, strain, state variables and consistent tangent modulus from the previous iteration is lost. The trial strain increments are updated which is composed of previous trial strain increment and current residual strain increment. In new iteration, the subroutine updates the variables for the total trial strain increment as done in path independent strategy. The calculation of tangent modulus in all iteration is used to calculate the trial residual strain increment only. This tangent modulus correlates the global iteration with local numerical scheme. The global iteration converges faster when the predicted residual strain increment calculation by the tangent modulus matches with the stress increment calculation with local numerical scheme. In first iteration of a time step, ANSYS initially passes zero strain increment along with stress, strain, plastic strain and state variables (converged value of previous time step) to form the structural stiffness matrix where local integration is avoided and only the consistent tangent modulus is calculated at
the beginning of the time step based on input quantities. This stiffness matrix formed in first iteration forms the trial strain increment for the total force. In subsequent iterations, stiffness matrix calculates the residual strain increments for the residual force in the global iteration. The implementation of plasticity model in customized ANSYS should be consistent with its inbuilt architecture for numerical schemes.

4.5.2 Implementation with ANSYS as subroutine

This study developed subroutines to implement with ANSYS for four constitutive models: plasticity model with modified Chaboche (Bari and Hassan, 2002), Ohno-Wang (1993), Abdel Karim-Ohno (2000) and modified Ohno-Wang (Chen and Jiao, 2003) kinematic hardening rules. The subroutines are developed for solid and shell elements. USERMAT3D and USERMATPS are the subroutines for three dimensional and plane stress analyses respectively. The subroutine updates stress, strain, plastic strain, state variables for the strain increment. These subroutines also give the consistent tangent modulus based on the updated solution of the integration point.

The developed subroutines take stress, strain and state variable at the beginning of the increment step as the input. The strain increment given as input for the step is trial strain increment obtained from finite element analysis for the load increment in that step. In this study, the state variables are also used to store the elastic strain and plastic strain for further improvement of the model. The kinematic variables for each decomposed rules which keep track of the location of the center of the yield surface are stored in state variables. Equivalent plastic strain increment is also stored as a state variable. The input quantities stress, strain, plastic strain and state variables come from the converged value at the beginning of increment step. The numerical calculation is performed for the total trial strain increment for the increment step during iteration as done in path independent strategy. The integration scheme updates the stress, strain, plastic strain and state variables as the final value at the end all iteration in a step. In implementation of plasticity models with ANSYS, first iteration calculates the consistent tangent modulus. This initial consistent tangent modulus forms the structural stiffness matrix to calculate the trial strain increment to start the numerical calculation. The subroutines are implemented with adaptive number of sub increments for local numerical calculation. It has been observed in Chapter Three that implicit type radial return scheme satisfies the requirement of stability and accuracy for large increment. The
consistent tangent modulus developed in previous section is used with local numerical scheme to get the quadratic convergence rate in global solution. The expression of *consistent tangent modulus* in Eq.4.19 uses scalar variable $\Delta p_{n+1}$ and variables at the updated point. This $\Delta p_{n+1}$ is calculated for the entire strain increment which is obtained through summing up $\Delta p_{n+1}$ calculated at each *sub increment* in increment step. *Consistent tangent modulus* is dependent on strain increment and is a function $\Delta p_{n+1}$ for the radial return scheme in a step.

In implementation with solid element (*USERMAT3D*) the strain driven radial return algorithm is implemented directly where six strain components are given to update six stress components. The expression for consistent tangent modulus in Eq. 4.18 can be used directly in these cases. For shell element (*USERMATPS*), it is necessary to implement the plane stress algorithm and the consistent tangent modulus needs to be modified as shown in Eq. 4.19. This study implemented plasticity models in ANSYS with radial return numerical scheme for local numerical calculation. An adaptive sub-increment approach is used based on fixed sub-increment size. It is also possible to implement plasticity models with forward Euler numerical scheme with sufficiently small sub increments for local calculation. Accuracy and stability can still be available in this case; solution time will increase due to large number of sub-increments. This increase in solution time is offset in some extent by explicit calculation compared to implicit radial return scheme. The tangent modulus can still be calculated at the updated point which is used for residual strain increment calculation. To get better convergence in global iteration with this forward Euler numerical scheme, next iteration is necessary to calculate based on residual trial increment only. In other words forward Euler scheme should be implemented with path dependent strategy. As mentioned before, ANSYS architecture works with path independent strategy for local calculation. ANSYS architecture by default calculates the variables for the total strain increment, not for the residual strain increment. For each step in a iteration, the subroutine works with initial converged stress and strains from the previous step and upgrades variables over the total trial strain increments. Thus the tangent modulus calculated at updated point becomes inconsistent for forward Euler numerical scheme. But, this tangent modulus is consistent tangent modulus for forward Euler scheme when used with path dependent strategy. When the local integration scheme is
forward Euler type, tangent modulus is calculated at updated stress points and residual trial strain increments are calculated, the system becomes inconsistent when the local integration attempts to integrate over the total trial strain increment in ANSYS. In such a case the convergence rate is very low for the global integration scheme and occasionally fails to converge. According to ANSYS architecture, when the solution scheme is called, it comes with the stress, strain and state variables at the beginning of the increment step and updated stress, strain and state variables in previous iteration are lost. It is not possible to convert the ANSYS calculation in a way so that numerical calculation always works on the residual strain increments. Thus, explicit type scheme is not adaptable in ANSYS architecture as the tangent modulus becomes inconsistent in this case.

4.6 Numerical examples

This section validates the implementation of constitutive models in customized ANSYS with numerical examples. This validation is performed in two steps. First, the accuracy and convergence of radial return scheme is checked with closed form solution of uniaxial material tests. Next, the convergence of the global force equilibrium is checked with pipe specimen under cyclic moment and internal pressure. Material level comparison is made for USERMAT3D and USERMATPS for three dimensional and plane stress problems. Convergence of the global force equilibrium is checked for the subroutine USERMATPS through geometric model of pipe specimen with shell elements.

4.6.1 Material test: uniaxial experiment

The accuracy of three dimensional radial return schemes for the modified Chaboche model is verified with 3D eight node iso-parametric element (Solid185) and plane stress element (Shell181). The 3D element has unit length, width and thickness. The element is subjected to displacement controlled cyclic loading to reproduce hysteresis loop. For this purpose, the element is subjected to monotonic loading up to 0.013 in, unloading up to -0.012 in and reloading up to 0.013 in. Closed form solution is calculated with the set of parameters shown in Table 4.1 for the plasticity model with modified Chaboche hardening rule. The simulation with Solid185 and Shell181 element is compared for a cyclic loading path along with monotonic path. The displacement path is simulated with incremental loading for the increments used in Table 4.2. Two different loading paths with large increments are used for uniaxial test simulation.
Fig. 4.1 (a) Uniaxial stress-strain simulation for a single SOLID185 element with customized ANSYS (b) local convergence at integration point for different steps of case1.

![Graph](image1)

Fig. 4.2 (a) Uniaxial stress-strain simulation for a single Shell181 element with customized ANSYS (b) local convergence at integration point for different steps of case1.

![Graph](image2)

Table 4.1

<table>
<thead>
<tr>
<th>Material parameters for comparison</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic parameters</td>
</tr>
<tr>
<td>$E = 26,300$ ksi ; $v = 0.302$ ; $\sigma_0 = 18.80$ ksi</td>
</tr>
<tr>
<td>Plastic Parameters</td>
</tr>
<tr>
<td>$C_{1-d} = 60,000$; 3228; 455; 15,000 $</td>
</tr>
<tr>
<td>$\gamma_{1-d} = 20,000$; 400; 11; 5,000 $</td>
</tr>
<tr>
<td>$\delta_4 = 5.0$, $\delta' = 0.18$</td>
</tr>
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</table>
Table 4.2

<table>
<thead>
<tr>
<th>Case</th>
<th>Displacement increment</th>
<th>Number of incremental steps</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Monotonic loading (in)</td>
<td>Unloading/Reloading (in)</td>
</tr>
<tr>
<td>Monotonic loading</td>
<td>Unloading/Reloading</td>
<td></td>
</tr>
<tr>
<td>Case 1</td>
<td>0.013</td>
<td>0.0122</td>
</tr>
<tr>
<td>Case 2</td>
<td>0.022</td>
<td>0.0204</td>
</tr>
</tbody>
</table>

The simulation obtained for two large increments is shown in Fig. 4.1a along with the closed form solution. It has been observed that for large increments the simulation gives correct results to the points where stresses are calculated. This accuracy is due to sub-increment method adapted in implementation. Although the increment is large in simulation, simulation is carried out with 0.01% strain increment due to sub-increment method adapted in the implementation of the plasticity models. In all cases, convergence of the local iteration scheme is found within 6 iterations except the first step (Fig. 4.1b). First step has elastic and plastic part. The stress strain curvature is high in first step, which is the reason for higher number of iterations. In plane stress case, simulation is also done for these two cases (Fig. 4.2). Plane stress algorithm also simulated accurately to those points where stresses are calculated (Fig. 4.2a). Local convergence is observed within 6 iterations except for first step (Fig. 4.2b).

4.6.2 Structural simulation: pipe subjected to cyclic moment and internal pressure

This section checks the consistency of the derived tangent modulus (Eq. 4.19-4.20) used with radial return algorithm for the plasticity model with modified Chaboche kinematic hardening rule. In this case, the thin pipe shown in Fig. 4.3 (L = 28”, d_m = 1.218”, t = 0.0359”) is subjected to cyclic moment (M) and internal pressure (p_s). The loading history of the pipe is given in Fig. 4.3b where M_a = 1 kip-in and steady internal pressure, p_s= 1.6 ksi. First internal pressure is applied in five steps. Subsequently, monotonic moment of 1 kip-in is applied in 10 steps and unloading and reloading is done in 20 steps. Material parameters are used as shown in Table 4.1. The moment rotation response is shown in Fig. 4.4a. The convergence of the residual moment along with iterations are shown in Fig. 4.4b for different steps. In this case, ANSYS default convergence criteria are used which is 0.1% for force and moments. It
has been observed that all the iterations completed with in six iterations as observed in Fig.4.1b.

Fig. 4.3 Pipe subjected to constant internal pressure ($p_s$) and cyclic bending moment ($M_a$).

Fig. 4.4 Experimental response of pipe under internal pressure and cyclic moment (a) Moment rotation response (b) Moment convergence with iteration in different loading steps.

4.7 Conclusion
This study addressed different numerical aspects of structural ratcheting simulation under cyclic loads. Local numerical algorithm dictates the accuracy of the solution. As this accuracy is influenced by size of the increment, this study suggests implementing radial return scheme with sub-increments. When such algorithm is to be implemented with ANSYS, user should implement with path independent strategy. The convergence of the global iteration scheme is dependent on the consistency between the stress integration algorithm and prediction of stresses through tangent modulus. Such a consistent tangent modulus is derived for modified Chaboche model. This consistent tangent modulus gives
quadratic convergence for the solution. Similar to modified Chaboche model, consistent tangent modulus is also derived for Ohno-Wang, Abdel Karim-Ohno and modified Ohno-Wang model and presented in Appendix B. These four models will be further used for structural analysis.
CHAPTER FIVE
RATCHETING SIMULATION OF STRAIGHT PIPE UNDER CYCLIC BENDING AND STEADY INTERNAL PRESSURE

5.1 Introduction
This chapter evaluates seven constitutive models for structural ratcheting simulation of straight pipe under cyclic bending and internal pressure. The plasticity models considered are bilinear (Prager, 1956), multilinear (Besseling, 1958) and the nonlinear kinematic hardening models by Chaboche (1986), which are currently available in the commercial finite element program ANSYS9.0 (2005), as well as the advanced constitutive models, such as, modified Chaboche model (Bari and Hassan, 2002), Ohno-Wang (1993) model and its improved versions proposed by Chen and Jiao (2003) and Abdel Karim-Ohno (2000) model. Straight pipe ratcheting data were developed at University of Notre Dame (Corona, 1996). Two pipe tests were conducted on straight pipe under cyclic bending and steady internal pressure. Material tests for determining constitutive model parameters were also conducted from the same batch of material.

The objective of the study is to develop a systematic numerical approach for structural ratcheting simulation. Towards developing a generalized approach, this study determines the model parameters from material experiments only and attempts to simulate the structural ratcheting using these parameters. The study simulates the structural responses using the commercial finite element code ANSYS. The plasticity models with bilinear, multilinear and Chaboche kinematic hardening rule are currently available in the ANSYS code. The advanced plasticity models with modified Chaboche (2002), Ohno-Wang (1993), Abdel Karim-Ohno (2000) and modified Ohno-Wang (2002) kinematic hardening rules were customized with ANSYS9.0 through the option of user programmable features.

Few studies so far addressed the simulation of piping components under cyclic bending and internal pressure (Corona and Kyriakides, 1991; Kulkarni et. al. 2003, 2004; Degrassi et. al., 2003). Hassan et. al. (1998) demonstrated that finite element analysis fails to simulate ratcheting damage accumulation mainly due to the deficiency of constitutive models. This study also showed that through incorporating improved constitutive models into finite element programs their structural ratcheting simulation can be improved. Kulkarni and his
co-workers (2003, 2004) simulated the structural responses for pipe under steady internal pressure and cyclic bending. In this study, only circumferential strain ratcheting was studied at top of the pipe where Chaboche model with three decomposed rule is used to simulate the structural response. In this simulation, model parameters were determined from material experiments and simulation was observed to represent the recorded circumferential strain ratcheting response well. The author demonstrated that Chaboche model failed to simulate the shakedown tendency of the strain ratcheting response. Degrassi et. al. (2003) performed nonlinear time history analysis on piping system and simulated the results through plasticity model with bilinear, multilinear and Chaboche kinematic hardening rules. The authors showed that the best simulation can be obtained with the Chaboche plasticity model in ANSYS. In this simulation, some of the model parameters were determined from elbow response, not material response. The elbow was a part of the piping system and might influence the overall response of piping system. From literature, it is observed that pipe ratcheting behavior under cyclic bending and internal pressure is not well understood. This urges the need for developing a numerical tool to predict the ratcheting response under cyclic bending and internal pressure. This lagging behind in ratcheting research is due to lack of experimental data and incapability of the finite element model in ratcheting simulation. To understand the ratcheting behavior under cyclic bending and internal pressure, this study developed experimental data. As the failure for finite element simulation is identified as the incapability of plasticity models incorporated in ANSYS in ratcheting simulation, this study incorporated four advanced constitutive models (Bari and Hassan, 2002; Ohno-Wang, 1993; Chen and Jiao, 2003 and Abdel Karim-Ohno, 2000) in ANSYS9.0. This study will evaluate these plasticity models through finite element simulation of experimental data and develop a numerical tool for structural ratcheting simulation.

5.2 Experiments

5.2.1 Experimental setup

The straight pipe test under cyclic bending and steady internal pressure was conducted at the University of Notre Dame (Corona, 1996). The test was conducted with a specially designed bending device originally developed by Kyriakides and his co-workers (1989, 1991). The test device is a four point bending machine which is capable of applying cyclic bending. Schematic of the machine is shown in Fig. 5.1 (Corona and Kyriakides, 1991). The pipe
ends are welded to solid rod at ends. Part of the solid rod is extended into the tube. The pure bending moment is applied through four rollers on each side with a sprocket assembly which is mounted on two heavy beams. Sprockets are connected to hydraulic cylinders and load cells through strand chains running over sprockets. Rotation of the sprocket is obtained through contracting and releasing the hydraulic cylinder. Rotation of the sprocket in turn rotates ends of the pipe specimen. The system of roller supported sprockets allows these to move in the horizontal direction. The pipe specimen was connected to a pump to pressurize and maintain the steady internal pressure during the test. First, the pipe is subjected to a steady internal pressure. Following, a symmetric cyclic rotation is applied at both ends. The reader is referred Corona and Kyriakides (1991) for details of the test setup.

![Diagram of the pure bending device](image)

**Fig. 5.1 Schematic of the pure bending device**

### 5.2.2 Experiments

The experiments were carried out on pipe of alloy steel 4130. The test specimen was thin walled pipe with thickness 0.911 mm and outside diameter of 31.85 mm and length 711 mm. The ratio of diameter to thickness is 35. In two experiments, the pipe specimen was subjected to loading as shown in Fig. 5.2a and the loading path is shown in Fig. 5.2b. The loading prescribed to the two tests is shown in Table 5.1. In two tests, the amplitude of the rotation varied but the internal pressure remained the same. To determine the model parameters, material experiments were conducted on the same batch of pipe used in the structural tests. Material tests include strain-controlled cyclic test, uniaxial ratcheting and biaxial ratcheting
tests. Readers are referred to Hassan and Kyriakides (1992) and Hassan et. al. (1992) for details on the material experiments.

![Schematic of the pipe specimen subjected to internal pressure and cyclic rotation](image1)

**Fig. 5.2** (a) Schematic of the pipe specimen subjected to internal pressure and cyclic rotation. (b) Loading path: cyclic bending at steady internal pressure. (c) Ovalization of the cross section under bending.

<table>
<thead>
<tr>
<th>Pipe Specimen</th>
<th>Rotation amplitude (radian)</th>
<th>Internal pressure (MPa)</th>
<th>No of cycles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pipe1</td>
<td>0.0924</td>
<td>11.03</td>
<td>80</td>
</tr>
<tr>
<td>Pipe2</td>
<td>0.1930</td>
<td>11.03</td>
<td>40</td>
</tr>
</tbody>
</table>

### 5.2.3 Experimental results

Cyclic bending of tubes induces progressive accumulation of ovalization of the tube cross section (see Fig. 5.2c) and strain ratcheting. Both, in plane and out of plane ovalization were recorded at the mid-pipe length with ovalization transducers (Vaze and Corona, 1998). Axial
strain and circumferential strain responses were recorded at the top and side of the pipe using strain gages as shown in Fig. 5.2a. A computer-based data acquisition system was used to record data during the experiment. The readers are referred to Vaze and Corona (1998) for details of the data acquisition system.

As mentioned before, the pipe experiments were rotation-controlled cyclic bending test under steady internal pressure. The moment-rotation response of pipe2 specimen is shown in Fig. 5.3a, where the hysteresis loops show that the loading in the pipe were prescribed in the inelastic range. The moments at maximum and minimum rotation are plotted against the number of cycles in Fig. 5.3b. This figure demonstrates that the responses of pipe are almost stable after cyclic softening in the initial couple of cycles. The ovalization along in-plane (ΔD_y) and out of plane (ΔD_x) directions are shown in Fig. 5.3c and 5.3d. The plot of the ovalization at positive and negative peak rotation against the number of cycles is shown in Fig. 5.3e and 5.3f. Figure 5.3c to 5.3f demonstrates that pipe gradually accumulates ovalization both in-plane and out-of-plane direction with cycles. The variation of moments with cycles can be best compared with simulation results through plotting the mean and amplitude of moment’s peaks as shown in Fig. 5.4a and 5.4b. These two figures demonstrates that the moment responses from both experiments are symmetric (zero mean value) and the responses are almost stable. The mean of the ovalization at rotation peaks in each cycle is obtained through the algebraic average of the ovalization. Figures 5.4c and 5.4d shows mean ovalization ratcheting response as function of the number of cycles from the two experiments. The amplitude of the ovalization response is obtained from the difference between the ovalization at the rotation peaks divided by 2. Figures 5.4e and 5.4f shows the plot of amplitude of ovalization along in-plane and out-of-plane bending directions for both experiments. Both pipe1 and pipe2 experiments were conducted for the same internal pressure, but the amplitude of rotation were different. For pipe1 specimen (θ_c=0.09 radian in Fig. 5.4c) exhibits small accumulation rate of in-plane ovalization and it stabilizes to a constant ratcheting rate after about 40 cycles. Out-of plane ovalization for this experiment gradually reduces with cycles as shown in Fig. 5.4d. For Pipe2 experiment, rotation amplitude prescribed was twice compared to the pipe1 experiment. Hence, the mean of ovalization plot (Fig. 5.4c) shows higher accumulation rate for pipe2 than that in pipe1 and it showed steady ratcheting rate for in-plane and out of plane ovalization after couple of cycles.
(Fig. 5.4c and 5.4d). In both tests, the amplitude of in-plane and out-of-plane ovalization in each cycle remained unchanged with cycles (Fig. 5.4e and 5.4f).

The axial and circumferential strain responses for the pipe2 experiment are shown in Fig. 5.5a to 5.5d. At the top of the pipe, axial strain responses show almost no ratcheting (symmetric strain response) whereas the circumferential strain ratcheting response shows significant ratcheting. At the side of the pipe (mean neutral axis), no ratcheting is demonstrated by either axial or circumferential strains as expected. The plot of the peaks of axial and circumferential strains at top and side of the pipe against the number of cycles are shown in Fig. 5.6a to 5.6d. Again, only the circumferential strain at the top is showing ratcheting response (Fig.5.6b). Comparison of the strain responses in pipe1 and pipe2 are shown by plotting mean and amplitude of strains against the number of cycles in Fig.5.7 and 5.8, which will be used for simulation later. The plot for mean of strain peaks against the number of cycles (Fig. 5.7a and 5.7c) verify the comment made about the axial strain ratcheting above. Figure 5.7b shows that the mean of circumferential strain ratchets for both experiments at the top of the pipe. Figure 5.7b shows the effect of the rotation on the rate of circumferential strain ratcheting. The amplitude of axial and circumferential strain cycles at top of the pipe remained unchanged with cycles (Fig. 5.8a-b). The amplitude of axial strain at side of the pipe increases with a small rate with cycles (Fig. 5.8c) whereas the amplitude of circumferential strain remains steady in most cases (Fig. 5.8d).

5.2.4 Material experiments for model parameter determination

Three material experiments were conducted at North Carolina State University from the same batch of pipe used in cyclic bending tests to determine the model parameters. The stable hysteresis response of alloy steel 4130 from a strain-controlled cyclic test is shown in Fig. 5.9a. The uniaxial ratcheting response is obtained from a stress controlled ratcheting test with $\sigma_{xa} = 510$ MPa, $\sigma_{xm} = 64$ MPa and is shown in Fig. 5.9b. The biaxial ratcheting response is obtained from symmetric strain-controlled test and steady internal pressure ($\varepsilon_{xc} = \pm0.4\%$, $\sigma_\theta = 187$ MPa) and is shown in Fig. 5.9c.
Fig. 5.3 Responses to symmetric rotation-bending and steady internal pressure of steel alloy pipe 4130 tube (a) moment-rotation, (b) Positive and negative moment peaks in each cycle versus the no of cycle, (c) in-plane ovalization-rotation response, (d) out-of-plane ovalization-rotation response, (e) in-plane ovalization and (f) out-of-plane ovalization at positive and negative moment peaks versus the number of cycle.
Fig. 5.4 Responses to symmetric rotation-bending and steady internal pressure of steel alloy pipe 4130 tube (a) mean of peak moments versus the no of cycle (b) amplitude of peak moments versus the no of cycle (c) mean of in-plane peak ovalization versus the no of cycle and (d) mean of out-of-plane peak ovalization versus the no of cycle (e) amplitude of in-plane ovalization peaks versus the no of cycles and (f) amplitude of out of plane ovalization versus the no of cycles
Fig. 5.5 Responses to symmetric rotation-bending and steady internal pressure of steel alloy pipe 4130 tube (a) axial strain at top versus rotation (b) hoop strain at top versus rotation (c) Axial strain at side versus rotation and (d) hoop strain at side versus rotation.

Fig. 5.6 Responses to symmetric rotation-bending and steady internal pressure of steel alloy pipe 4130 tube (a) axial strain peaks at each cycle at top versus the no of cycle (b) hoop strain peaks at each cycle at top versus the no of cycle (c) Axial strain peaks at each cycle at side versus the no of cycle and (d) hoop strain peaks at each cycle at side versus the no of cycle.
Fig. 5.7 Responses to symmetric rotation-bending and steady internal pressure of steel alloy pipe 4130 tube (a) mean of axial strain peaks at each cycle at top versus the no of cycle (b) mean of hoop strain peaks at each cycle at top versus the no of cycle (c) mean of axial strain peaks at each cycle at side versus the no of cycle and (d) mean of hoop strain peaks at each cycle at side versus the no of cycle.

5.3 Plasticity models
This study simulates the structural ovalization and strain responses through using the finite element package ANSYS for the plasticity models: bilinear (Prager, 1956), multilinear (Besseling, 1958) and Chaboche (1986) kinematic hardening model and using customized ANSYS for the advanced plasticity models with modified Chaboche (2002), Ohno Wang (1993), Abdel Karim-Ohno (2000) and modified Ohno-Wang (2003). The model parameters are determined from the material responses discussed in last section (Fig. 5.9). A genetic algorithm based optimization tool discussed in Chapter Two was used for parameter determination for all the models except bilinear and multilinear models for which parameters are determined manually.
Fig. 5.8 Responses to symmetric rotation-bending and steady internal pressure of steel alloy pipe 4130 tube (a) amplitude of axial strain peaks at each cycle at side versus the no of cycle (b) amplitude of hoop strain peaks at each cycle at side versus the no of cycle (c) amplitude of axial strain peaks at each cycle at side versus the no of cycle and (d) amplitude of hoop strain peaks at each cycle at side versus the no of cycle.

Fig. 5.9 Material response for the steel alloy pipe 4130 tube for parameter determination (a) Stable hysteresis loop (b) axial strains at positive stress peaks of uniaxial cycles from a stress controlled test (c) circumferential strain peaks from axial strain cyclic experiment.
5.4 Finite element model

It is well known that the accuracy of structural simulation is dependent on the load increment size and finite element mesh used. This section compares simulations from different finite element mesh of the pipe structure to determine the optimized mesh for finite element simulation. At first simulation was performed with very small elements. As very fine mesh with large number of elements increases the solution time considerably, especially for cyclic analysis it is necessary to optimize the mesh size for the pipe experiments. For cyclic analysis, this study accepts the mesh size which yielded the strain value maximum 1% different than that obtained for a very fine mesh.

The pipe (Fig. 5.2) was modeled using Shell181 elements in ANSYS. With the customized option of implementing constitutive model in ANSYS the element choices are limited to I8X series elements. Hence, the choice of element for modeling pipe structure was limited to Shell181 only. This Shell181 element is a four node layered element with six degrees of freedom at each node: translations along X, Y, Z directions and rotations about X, Y, Z directions. SHELL181 is well-suited for linear, large rotation, and large strain nonlinear applications. This element can take account of the change in thickness in stiffness calculations during nonlinear analysis. So, the Shell181 element is capable of fulfilling the requirement of large rotational analyses. Details of the element features are given in ANSYS manual 9.0.

The pipe shown in Fig.5.2a is subjected to internal pressure and rotation at ends. The structure is doubly symmetric in geometry and loading. That is, the pipe is symmetric about X-Y plane at the mid-pipe section and also symmetric to Y-Z plane (Fig.5.2a). The load is also symmetric (Fig.5.2b). Hence, only one quarter of the structure was modeled for analysis. Stress and strains were calculated at the location of strain gages through averaging the results from adjacent elements. As the stress and strain data are extracted from the element results at midspan, size of the element at the target location might influence the result. Therefore, the pipe mesh is refined at the midspan although the radius of curvature is same throughout the length of the pipe.

For mesh optimization, the structure is subjected to cyclic loading history for twenty cycles as shown in Fig.5.2a (p = 11.03 MPa, θc = 0.09 radian). The plasticity model used in mesh
study is Chaboche model (1986) in ANSYS. The Chaboche model parameters used in this study are

\[
\sigma_0 = 262.0 \text{ MPa}, \quad E = 183000 \text{ MPa}, \quad \nu = 0.302 \\
C_{1-4} = 837130, 111700, 22060, 217080 \text{ MPa} \\
\gamma_{1-4} = 43481, \quad 552, \quad 0.5, \quad 3789
\]

Eleven different mesh arrangements are considered for mesh study and the number of elements along different directions is given in Table 5.2. The mesh arrangement for \textit{mesh1} has a very fine mesh with large number of elements. As mentioned earlier accumulated circumferential strain from each mesh arrangement are compared with the results for the finest mesh arrangement (\textit{mesh1}) as shown in Table 5.3. Comparison of the results obtained for \textit{mesh1}, \textit{mesh2} and \textit{mesh3} arrangements concludes that reduction of number of elements at the end of the pipe does not influence the solution quality. Comparison of \textit{mesh4}, \textit{mesh5} and \textit{mesh6} reveals that element number can be reduced to certain number along the circumference without influencing the solution quality. Comparison of the \textit{mesh2}, \textit{mesh4} and \textit{mesh9} provides the idea of element number reduction along the axis at the middle of the pipe without loosing the solution accuracy. Comparison of the eleven meshes concludes that \textit{mesh11} has small axial and circumferential strain variation around 0.43% compare to the first mesh. \textit{Mesh11} has 732 elements only, which is less than one third of the elements of \textit{mesh1} arrangement, but yield acceptable result. Thus, \textit{mesh11} was used in this study. The mesh arrangement is shown Fig. 5.10.

<table>
<thead>
<tr>
<th>Table 5.2</th>
<th>Description of the considered finite element mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No of elements used</td>
</tr>
<tr>
<td></td>
<td>Along circumference (div1)</td>
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<tr>
<td>\textit{mesh 1}</td>
<td>32</td>
</tr>
<tr>
<td>\textit{mesh 2}</td>
<td>32</td>
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<td>\textit{mesh 3}</td>
<td>32</td>
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<td>\textit{mesh 9}</td>
<td>32</td>
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<tr>
<td>\textit{mesh 10}</td>
<td>32</td>
</tr>
<tr>
<td>\textit{mesh 11}</td>
<td>24</td>
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</tbody>
</table>
Table 5.3
Axial and circumferential strain at top of the pipe after 20 cycles

<table>
<thead>
<tr>
<th></th>
<th>After 2 Cycle</th>
<th>Deviation (%)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Axial Strain (%)</td>
<td>Circumferential Strain (%)</td>
<td>Axial Strain (%)</td>
<td>Circumferential Strain (%)</td>
</tr>
<tr>
<td>mesh 1</td>
<td>-0.392165</td>
<td>0.885973</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>mesh 2</td>
<td>-0.392156</td>
<td>0.885973</td>
<td>-0.002</td>
<td>0.000</td>
</tr>
<tr>
<td>mesh 3</td>
<td>-0.392218</td>
<td>0.886023</td>
<td>0.014</td>
<td>0.006</td>
</tr>
<tr>
<td>mesh 4</td>
<td>-0.392140</td>
<td>0.885963</td>
<td>-0.006</td>
<td>-0.001</td>
</tr>
<tr>
<td>mesh 5</td>
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<td>0.884807</td>
<td>-0.157</td>
<td>-0.132</td>
</tr>
<tr>
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</tr>
<tr>
<td>mesh 7</td>
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<td>0.885119</td>
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</tr>
<tr>
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<td>0.876790</td>
<td>-0.724</td>
<td>-1.036</td>
</tr>
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<td>mesh 9</td>
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<td>0.885970</td>
<td>-0.003</td>
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<tr>
<td>mesh 10</td>
<td>-0.388127</td>
<td>0.879145</td>
<td>-1.030</td>
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<tr>
<td>mesh 11</td>
<td>-0.390751</td>
<td>0.882220</td>
<td>-0.361</td>
<td>-0.424</td>
</tr>
</tbody>
</table>

Fig. 5.10 Schematic of the pipe mesh

5.5 Structural Simulation


5.5.1 Simulation with ANSYS

5.5.1.1 Bilinear Model

The parameters of the bilinear model (Prager, 1956) are determined by approximating the hysteresis stress-strain loop as shown in Fig. 5.11a. The model parameters include elastic modulus, $E = 183000$ MPa, yield stress, $\sigma_0 = 550$ MPa and plastic modulus, $E_p = 18620$ MPa.
The simulation for pipe1 and pipe2 experiments is compared with experimental results in Fig. 5.12 and 5.13. The moment curvature response shown in Fig. 5.12a and 5.12b showed overprediction of yield moment due to high yield stress parameter but underprediction of moments at higher curvature in both experiments. The simulated mean of in-plane and out-of-plane ovalization peaks underpredicted ovalization rate with shakedown phenomena for higher number of cycles (Fig.5.12c and 5.12e). The simulated amplitude of ovalization remained unchanged and matched well with experiments (Fig.5.12d and 5.12f).

![Graph](image)

**Fig. 5.11 Simulation of the material responses with the optimized parameters (a) hysteresis loop (b) uniaxial ratcheting (c) biaxial ratcheting.**

In the strain response simulation of pipe1 and pipe2 experiments, the mean axial strain peaks showed very small ratcheting rate similar to experimental responses (shown in Fig.5.13a). The simulated amplitudes of axial strain cycles also have very good correlations with both experiments (Fig.5.13b). The circumferential strain simulation at top of the pipe showed shakedown phenomena for higher number of cycles as seen in Fig. 5.13c. Structural simulation with bilinear model failed to produce the steady ratcheting rate at higher cycles for both pipe1 and pipe2 experiments. The simulated mean and amplitude of axial and
circumferential strains at side of the pipe are very small and mostly matched well with experiments (Fig. 5.13e-h) except small amplitude increases in Fig. 5.13f. In simulating the stress-strain response, the bilinear model cannot distinguish between loading and unloading curve in hysteresis loop simulations and cannot simulate uniaxial ratcheting as shown in 5.11b (Bari and Hassan, 2000). For biaxial loading history shown in Fig. 5.9c, this model shows shakedown phenomena with initial overprediction (Fig. 5.11c). The main deficiency of the structural simulations with the bilinear plasticity model is the shakedown phenomena as seen in Fig. 5.11b and 5.11c, Fig. 5.12c and 5.12e and Fig. 5.13a and 5.13c. The other disadvantage of this model is that the structural simulations cannot be improved by adjusting the model parameters.

### 5.5.1.1 Multilinear Model

This plasticity model with multilinear kinematic hardening rule (Besseling, 1958) approximates the stress-strain hysteresis loop very well as shown in Fig. 5.11a. All the parameters are determined from uniaxial hysteresis loop only. In the multilinear plasticity model, it is necessary to provide the uniaxial monotonic stress-strain curve for ANSYS simulation. For this purpose, the loading part of the cyclic hysteresis loop is converted to equivalent monotonic stress-strain response using Masing’s Method (1926). For an equivalent monotonic curve, at first the hysteresis loop is transferred to zero stress-strain point and then half of the stress and strain values are taken as the points of the equivalent monotonic stress-strain response. The simulation for the hysteresis loop is shown in Fig. 5.11a. In this simulation, model parameters describe the stress-strain relationship up to 0.75% strain. For strains greater than 0.75%, this model suffers from numerical integration problem. However, if the curve is extended up to 2% strain following the curvature of the stress-strain hysteresis loop, this numerical difficulty can be removed. The structural simulations for pipe1 and pipe2 experiments are shown in Fig. 5.12 and 5.13. The moment-rotation response produced a smooth shape but the simulation underpredicted both experiment responses (Fig. 5.12a-b). The ovalization and strain simulations for pipe1 experiment shows similar performance like bilinear kinematic hardening rule with shakedown phenomena (Fig. 5.12 and 5.13). The simulation of in-plane pipe ovalization for pipe2 experiment does not show any shakedown phenomena (Fig. 5.12c). Like the bilinear model, this model fails to distinguish the hysteresis curve difference between loading and
unloading responses and simulate shakedown of strain ratcheting as shown in Fig. 5.11b and 5.11c (Bari and Hassan, 2000). The shakedown phenomena are not observed in pipe2 ratcheting simulations of ovalization and strains (Fig. 5.11b and 5.11c).

Fig. 5.12 Structural responses simulation for straight pipes with the with bilinear, multilinear and Chaboche models (a) Moment-rotation response for pipe1 (b) Moment-rotation response for pipe2 (c) mean of in-plane ovalization, (d) amplitude of in-plane ovalization (e) mean of out-of-plane ovalization and (f) amplitude of out-of-plane ovalization.
Fig. 5.13 Structural response simulation for straight pipes with the bilinear, multilinear and Chaboche models (a) mean of axial strain at top (b) amplitude of axial strain at top, (c) mean of circumferential strain at top and (d) amplitude of circumferential strain at top, (e) mean of axial strain at side and (f) amplitude of axial strain at side and (g) mean of circumferential strain at side and (h) amplitude of circumferential strain at side.
5.5.1.3 Chaboche model

The model parameters for the Chaboche plasticity model (Chaboche, 1986) kinematic hardening rules are determined simulating hysteresis loop and uniaxial ratcheting response (Fig. 5.11a and 5.11b). The automated parameter optimization system presented in Chapter Two is used to determine the model. The parameters used in the simulation with Chaboche model are:

$$\sigma_0 = 262.0 \text{ MPa}, \ E = 183000 \text{ MPa}, \ \nu = 0.302$$
$$C_{1-4} = 837130, 111700, 22060, 217080 \text{ MPa}$$
$$\gamma_{1-4} = 43481, \ 552, \ 0.5, \ 3789$$

The simulations with Chaboche plasticity model for pipe1 and pipe2 experiments are shown in Fig.5.12 and 5.13. The moment rotation simulation with this model underpredicts the experimental responses (Fig. 5.12a-b). This model simulated the in-plane ovalization ratcheting along with the stabilized ratcheting rate for pipe1 experiment well but failed to simulate in-plane ovalization of pipe2 experiment (Fig. 5.12c). The simulation for out-of-plane diameter change shows increase in ovalization accumulation, whereas it decreased for pipe1 experiment (Fig. 5.12e). The simulated amplitudes of ovalization cycles however matched well with experiments (Fig. 5.12d and 5.12f). The simulation of the axial strain mean at top of the pipe remained unchanged with cycles and represent well with the experimental response (Fig.5.13a). The amplitude of the axial strain at top of the pipe has good correlation with both experimental responses (Fig.5.13b). The circumferential strain ratcheting simulation for pipe1 experiment showed initial overprediction, but it matches well the stabilized ratcheting rate. This model failed to simulate the steady circumferential strain ratcheting rate for pipe2 experiments (Fig.5.13c). The amplitude of the circumferential strain cycles matched well with experiments (Fig.5.13d). The mean and amplitude of the axial and circumferential strains at side of the pipe remained unchanged and represent experimental responses with acceptable accuracy (Fig. 5.13e-h).
From the structural simulation of *pipe1* and *pipe2* with bilinear (Prager, 1956), multilinear (Besseling, 1958) and Chaboche (1986) models we find that none of the models can simulate either the ovalization ratcheting or strain ratcheting well. Of the three models, the multilinear model performed the best in simulating ratcheting responses. It was a little surprise to see the poor performance of the Chaboche model. However, all three models simulated the ovalization and strain amplitudes well. The moment curvature responses are underpredicted, but this underprediction may not be the reason of poor ratcheting simulation. Hence, in order to find the state of the art of model in simulating structural ratcheting advanced plasticity models will be scrutinized in the following section.
Fig. 5.15 Structural response simulation for straight pipes with the Chaboche and modified Chaboche models (a) Moment-rotation response for pipe1 (b) Moment-rotation response for pipe2 (c) mean of in-plane ovalization, (d) amplitude of in-plane ovalization (e) mean of out-of-plane ovalization and (f) amplitude of out-of-plane ovalization (g) mean of circumferential strain at top and (h) amplitude of circumferential strain at top of pipe.
5.5.2 Simulation with customized ANSYS

5.5.2.1 Modified Chaboche model

The modified Chaboche (Bari and Hassan, 2002) model parameters below are determined using the hysteresis loop, uniaxial and biaxial ratcheting responses and using the optimization algorithm developed for parameter determination.

\[ \sigma_0 = 262.0 \text{ MPa}, \quad E = 183000 \text{ MPa}, \quad \nu = 0.302 \]

\[ C_{1-4} = 761240, 108580, 23732, 205797 \text{ MPa} \]

\[ \gamma_{1-4} = 34201, 551, 0.95, 4875 \]

\[ \bar{a}_4 = 15.4, \quad \delta' = 0.67, \]

The simulation for the material responses with the optimized model parameters are shown in Fig. 5.14. The structural simulations for pipe experiments are presented in Fig. 5.15. Figure 5.15 shows that the simulation from the Chaboche and modified Chaboche are almost the same for two pipe experiments. The additional parameter in the modified Chaboche model \( \bar{a}_4 \) and \( \delta' \), do not improve the structural simulation of the model compare to the Chaboche model.

5.5.2.2 Ohno-Wang Model

The Ohno-Wang model (1993) parameter set below were determined through optimizing the against the uniaxial hysteresis loop and uniaxial ratcheting responses.

\[ \sigma_0 = 262.0 \text{ MPa}, \quad E = 183000 \text{ MPa}, \quad \nu = 0.302 \]

\[ C_{1-5} = 311320, 161000, 76440, 59407, 24077 \text{ MPa} \]

\[ \gamma_{1-5} = 18946, 3565, 827, 508, 11.3 \]

\[ m_{1-5} = 0.60. \]

The simulations are shown in Fig. 5.16. Note that only five decomposed rules were used for simulating the material responses. Similar underprediction in the moment rotation responses as in the Chaboche and modified Chaboche models are observed (Fig. 5.17a and 5.17b). This model can simulate the biaxial ratcheting response vary well (Fig. 5.16c), even though material parameters were determined from hysteresis loop and uniaxial ratcheting responses only. The model, however, simulates the ovalization and strain responses similar to the Chaboche and modified Chaboche models (compare Fig. 5.15 and Fig. 5.17).

The Ohno-Wang model requires a large number of kinematic hardening rules (linear segments) to simulate the hysteresis loop well (Bari and Hassan, 2000). To optimize the
number of rules necessary for the Ohno-Wang model, parameters for 8 and 12 rules are also determined as shown in Table 5.3. Fig. 5.18 shows the material responses simulation from Ohno-Wang plasticity model with 5, 8 and 12 rules. The structural simulations for pipe2 experiment through Ohno-Wang model with 5, 8 and 12 rules are shown in Fig. 5.19. Not much improvement in simulation is observed as the number of rules are increased. However, when four rules were used the simulation deteriorates (not shown here). Hence, five kinematic rules were used in the simulations with Ohno-Wang model.

Fig. 5.16 Simulation of the material responses with the optimized parameters (a) hysteresis loop (b) uniaxial ratcheting (c) biaxial ratcheting.

<table>
<thead>
<tr>
<th>Number of decomposed rules, M</th>
<th>$C_{I-M}$ (MPa)</th>
<th>$\gamma_{I-M}$</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>314900, 176180, 50080, 50440, 2597, 2832, 1082, 3306</td>
<td>25133, 3768, 1304, 549, 527, 423, 391, 5.84</td>
<td>0.40</td>
</tr>
<tr>
<td>12</td>
<td>837300, 169780, 39970, 31020, 10645, 35412, 490, 20480, 3102, 4300, 5650, 22240</td>
<td>68487, 3335, 2803, 535, 585, 544, 594, 570, 566, 598, 563, 6.8</td>
<td>0.48</td>
</tr>
</tbody>
</table>
Fig. 5.17 Structural response simulation for straight pipes with the Ohno-Wang and modified Ohno-Wang models (a) Moment-rotation response for pipe1 (b) Moment-rotation response for pipe2 (c) mean of in-plane ovalization, (d) amplitude of in-plane ovalization (e) mean of out-of-plane ovalization and (f) amplitude of out-of-plane ovalization (g) mean of circumferential strain at top and (h) amplitude of circumferential strain at top of pipe.
Fig. 5.18 Simulation of the material responses with the optimized parameters (a) hysteresis loop (b) uniaxial ratcheting (c) biaxial ratcheting.

Fig. 5.19 Comparison of simulations for straight pipe from Ohno-Wang model with 5,8,12 decomposed rules (a) Moment-rotation response for pipe2 (b) mean of in-plane ovalization, (c) mean of out-of-plane ovalization and (d) mean of circumferential strain at top of pipe.
5.5.2.3 Modified Ohno-Wang Model

The parameters for the modified Ohno-Wang model (Chen and Jiao, 2003) are determined from uniaxial hysteresis loop, uniaxial and biaxial ratcheting experiments. These parameters are

\[ \sigma_0 = 262.0 \text{ MPa}, \quad E = 183000 \text{ MPa}, \quad \nu = 0.302 \]
\[ C_{1-5} = 314900, 166650, 76170, 59170, 23850 \text{ MPa} \]
\[ \gamma_{1-5} = 20133, \quad 3609, \quad 823, \quad 504, \quad 12 \]
\[ m_{1-5} = 0.59, \quad \delta_0' = 0.75, \quad \delta_{m'} = 1.00, \quad \beta = 0.23 \]

The material simulations with the determined parameters are shown in Fig. 5.16. Simulation with the modified Ohno-Wang model for the pipe experiments are shown in Fig.5.17. Except the small shift in the ovalization and strain ratcheting simulations, the Ohno-Wang and modified Ohno-Wang simulations are similar. However, the Ohno-Wang model simulate shakedown of circumferential strain ratcheting, which is improved by the modified Ohno-Wang model due to incorporation of multiaxial parameter, \( \delta' \) (shown in Fig.5.17g). But the circumferential strain ratcheting rate still remained under predicted for pipe2 experiment.

5.5.2.4 Abdel Karim-Ohno model

The model parameters for the Abdel Karim-Ohno model (2002) are determined through hysteresis loop and either uniaxial or biaxial ratcheting responses. The material parameters optimized through simulating hysteresis loop and uniaxial ratcheting responses (see in Fig.5.20) with five kinematic hardening rules are

\[ \sigma_0 = 262.0 \text{ MPa}, \quad E = 183000 \text{ MPa}, \quad \nu = 0.302 \]
\[ C_{1-5} = 342610, 208880, 60230, 52800, 22670 \text{ MPa} \]
\[ \gamma_{1-5} = 28292, \quad 3289, \quad 575, \quad 549, \quad 36.4, \]
\[ \mu_{1-5} = 0.80. \]

The simulations of pipe experiments with this parameter set are shown in Fig. 5.21. The simulation performance of this model is similar to Chaboche, modified Chaboche, Ohno-Wang and modified Ohno-Wang. The model ratcheting parameters are optimized from biaxial ratcheting response (\( \mu_{1-5} = 0.01 \)) as shown in Fig. 5.20, keeping all other parameters unchanged. However, the structural simulations overall do not change much (Fig.5.21).
Fig. 5.20 Simulation of the material responses with the optimized parameters (a) hysteresis loop (b) uniaxial ratcheting (c) biaxial ratcheting.

5.6 Observations from the structural simulations

Seven cyclic plasticity models are evaluated in terms of structural response simulation. In these simulations, the parameters of the models were determined from material responses only. These models failed to simulate structural responses in two aspects. First, the models underestimates moment-rotation response and secondly, these cannot simulate the ovalization and strain ratcheting responses for pipe2 experiment. This failure can be due to either determining an in-appropriate set of parameters from material responses or incapability of the constitutive model used in the simulation. The purpose of this section is to investigate the influence of the first reason. This study will not study the reasons of incapability of the plasticity models, rather will focus in determining a set of parameters that can simulate the material and structural responses with a reasonable accuracy. For this purpose a semi-inverse approach of model parameter determination has been used. This is not direct inverse approach. In this method, instead of using global and local structural responses in parameter determination, local strain responses recorded at critical locations of the structure are used for parameter refinement.
Fig. 5.21 Structural response simulation with the Abdel Karim-Ohno model (a) Moment-rotation response for pipe 1 (b) Moment-rotation response for pipe 2 (c) mean of in-plane ovalization, (d) amplitude of in-plane ovalization (e) mean of out-of-plane ovalization and (f) amplitude of out-of-plane ovalization (g) mean of circumferential strain at top and (h) amplitude of circumferential strain at top of pipe.
Fig. 5.22 Simulated axial strain and circumferential stress histories for (a) pipe1 and (b) pipe2.

Fig. 5.22 (c) Simulation of the circumferential strain ratcheting rates at top of the pipe in pipe1 and pipe2 experiments (d) Simulation of the circumferential strain response at top of pipe2 experiment with modified Chaboche model and structurally simulated stress history and biaxial stress history.

Fig. 5.23 Simulation of circumferential strain peaks with cycles at top of the pipe for both experiments with modified Chaboche model parameters

From the previous structural simulations in Fig. 5.15, it was observed that when the circumferential strain ratcheting (local response) at top or bottom of the pipe matched well, the ovalization ratcheting (global response) matched well in structural simulations. It was
determined that material points at top and bottom of the pipe1 and pipe2 simulations undergoes loading histories as shown in Fig.5.22a and 5.22b. The top/bottom material points of the pipe is subjected to almost symmetrical axial-strain cycle of amplitude of 0.425% and 0.88% in pipe1 and pipe2 experiments respectively along with the circumferential stress. These axial strain amplitudes simulated are close to the measured values (Fig. 5.13b). The simulated circumferential strain variations are shown in Fig.5.22. The measured circumferential strains at those points can be considered as ratcheting response for these loading histories. Plot of the peak of circumferential strains against the number of cycles gives the ratcheting rates at top for pipe1 and pipe2 specimen as shown in Fig. 5.22c.

First ratcheting simulations are performed with modified Chaboche model and material parameters determined earlier for stress-strain history shown in Fig.5.22b and for a biaxial history with steady circumferential stress of 186 MPa (average of the history in Fig.5.22b) and axial strain amplitude of 0.88%. From these two simulations with modified Chaboche model parameters (shown in Fig.5.22d), it is observed that the difference in the two loading histories has very small influence in material ratcheting rate simulation. This comparison indicates that the biaxial history ratcheting response should determine reasonable multiaxial parameter $\delta'$ of the modified Chaboche model. To further evaluate the idealization of the stress-history, the circumferential strain response was simulated for the pipe1 experiment. The results are shown in Fig.5.23, along with the simulated results of pipe2. The figure shows that the idealized simulations compare well to the structural simulation shown in Fig. 5.15e, in terms of stable ratcheting rates. Hence, for simplicity the biaxial stress history will be used in the semi-inverse method of parameter determination, instead of the actual simulated history shown in Fig.5.22.

The model parameters of the modified Chaboche model were determined using material experimental responses obtained under biaxial history. It is noted that the axial strain amplitude, $\varepsilon_{xc} = 0.4\%$, which is same as the axial–strain amplitude at top of pipe1 experiment. Hence, the modified Chaboche model with this parameter set simulated well for pipe1 experiment. Therefore, the failure of pipe2 simulations can be attributed to the fact that the material response used is not representative of the structural responses. The objective of this section is to find a new set of parameters that can simulate the material hysteresis loop,
uniaxial and biaxial ratcheting histories as well as pipe1 and pipe2 circumferential ratcheting histories at the top of the pipe.

The modified Chaboche model parameter refinement will be performed using the GA optimization algorithm and based on the optimized simulation of strain controlled hysteresis loop, uniaxial and biaxial strain ratcheting responses, circumferential strain responses from pipe1 and pipe2 experiments simultaneously. From several searches using GA optimization algorithm, a common set of parameters could not determined for simulating all these responses reasonably well. Therefore, a sensitivity study of the parameters was conducted to understand the influence of each of the parameter in simulating the responses mentioned above.

### 5.7 Sensitivity of the parameters

In this sensitivity study, each of the modified Chaboche model parameters is varied individually to observe its influence on the simulation of hysteresis loop, uniaxial and biaxial ratcheting, and pipe1 and pipe2 circumferential strain ratcheting. In this model, $C_1$ is determined directly from the initial slope of $\sigma_x - \varepsilon_x^p$ curve. Hence, this parameter is not varied for the sensitivity study. Here, $\gamma_1$ is changed from 34201 to 100000 and 4000 whereas all other parameters were kept the same. Figure 5.24 demonstrates the simulated responses. As $\gamma_1$ is increased, the contribution from first hardening rule $C_1/\gamma_1$ is decreased from 22.3 to 7.58, therefore hysteresis loop shape slightly underpredicted (shown in Fig.5.24a). This increase in $\gamma_1$ increased the uniaxial and biaxial ratcheting rates by a small amount (Fig.5.24b-d). When $\gamma_1$ reduced to 4000, the contribution from first hardening rule increased from 22.3 to 191.7. Therefore, the hysteresis loop overpredicts significantly (Fig.5.24a). As the hysteresis loop becomes stiffer, the initial uniaxial and biaxial ratcheting rates are slightly decreased but rates at higher cycles are not changed much compare to simulations from the material parameter (Fig.5.24b-d). From Fig. 5.24, it is observed that $\gamma_1$ influence the hysteresis loop simulation significantly, but the ratcheting simulations are influenced slightly.

As the objective of this study is to improve the ratcheting responses keeping the hysteresis loop simulation same, this parameter should not be varied much from the material response. Therefore, a small range of $\pm 3\%$ variation is selected for $\gamma_1$ to vary from the initial parameters for further refinement of the parameters using GA optimization.
The influence of the parameters of the second hardening rule ($C_2$ and $\gamma_2$) is shown in Fig.5.25 and 5.26. Figure 5.25 demonstrates simulations with $C_2 = 206840$ or $C_2 = 20684$ whereas all other parameters are same as material response determined parameters. Figure 5.25a demonstrates that increase or decrease in $C_2$ stiffen or soften the hysteresis loop respectively. Increase in $C_2$ slightly reduces the biaxial ratcheting rates or vice versa (Fig.5.25c-d). On the other hand, uniaxial ratcheting rates are significantly influenced by the decrease of $C_2$ (Fig.5.25b). When the parameter $\gamma_2$ is increased ($\gamma_2 = 1500$) or decreased ($\gamma_2 = 250$), the hysteresis loop becomes softer or stiffer similarly (Fig.5.26a). Increase or decrease of $\gamma_2$ has some degree of influence, but in an opposite order on the uniaxial and biaxial ratcheting simulation (Fig.5.26b-d). The above results indicate that for refining $C_2$ and $\gamma_2$, these can be searched within a narrow range ($\pm 3\%$) about the material parameters.

Among the parameters of the third hardening rule (parameters $C_3$ and $\gamma_3$), $C_3$ is measured from the stabilized slope of the hysteresis loop, hence should not be changed. Bari and Hassan (2002) demonstrated that $\gamma_3$ influences uniaxial ratcheting significantly but does not influence the hysteresis loop simulation. $\gamma_3$ usually is a small value. In determining its influence, $\gamma_3$ was increased from 0.95 to 12.00 and 4.00, and significant increase in the uniaxial and biaxial ratcheting rate simulations are observed (Fig.5.27b-d). But simulation of hysteresis loop remained unchanged (Fig.5.27a). Hence, $\gamma_3$ parameter has potential in improving ratcheting simulation without changing the hysteresis loop simulation. This parameter should be searched over a large range about the initial value of the material parameter.

The parameters for the fourth hardening rule are $C_4$, $\gamma_4$ and $\bar{a}_4$. Change in $C_4$ from 205797 to 507100 or 34475 and keeping all other material parameters same, causes small deviation of hysteresis loop simulation (Fig.5.28a), however has small to no influence on ratcheting simulations (Fig.5.28b-d). The influence of $\gamma_4$ was studied by changing it form 4875 to 15000 or 1500. The influence of $\gamma_4$ on the hysteresis loop and ratcheting simulations are also small as observed in Fig.5.29. When the threshold parameter $\bar{a}_4$ is varied from 15.40 to 82.7 or 41.30, the hysteresis loop becomes stiffer compared to material parameter hysteresis
loop (Fig.5.30a) and ratcheting rates reduced by small amount (Fig.5.30b-c). Since, change in parameters of the fourth hardening rule ($C_4$, $\gamma_4$ and $\bar{a}_4$), modifies the hysteresis loop to some extent but ratcheting rates are not influenced much, these parameters should be searched within a narrow range. Bari and Hassan (2002) proposed $\delta$ as a multiaxial parameter in modifying Chaboche model (Chaboche, 1991). Change in $\delta$ from 0.67 to 0.20 or 0.95, keeping all other material parameters same, keeps the hysteresis loop and uniaxial ratcheting simulation unchanged (Fig.5.31a-b), but modifies slightly the biaxial ratcheting simulation (Fig.5.31c-d).

From the above parameter sensitivity study, it is observed that change in $\gamma_3$ does not influence the material hysteresis loop simulation much, but significantly changes the uniaxial and biaxial ratcheting rate simulation. In the material parameter determination, the experimental ratcheting rates used were low, consequently, the uniaxial ratcheting parameter, $\gamma_3$ determined was small 0.95. As the $\gamma_3$ parameter was small ($\gamma_3 = 0.95$), the simulated biaxial ratcheting rate was also small. In the parameter set of Bari and Hassan (2000, 2002), $\gamma_3$ has a much higher value for modified Chaboche model, $\gamma_3 = 11$. Thus, the influence of $\delta$ on the biaxial ratcheting simulation was significant. Hence, an additional parametric study of $\delta$ was performed with $\gamma_3 = 4$, and $\delta$ with 0.20, 0.67 and 0.95, while keeping all other parameters are same. The influence of $\delta$ with high $\gamma_3$ on the simulation is shown in Fig.5.32. This figure demonstrates that the hysteresis loop simulation is unchanged (Fig.5.32a) but uniaxial ratcheting rate is overpredicted because of higher $\gamma_3$ (Fig.5.32b), and biaxial ratcheting is influenced significantly by $\delta$ (Fig.5.32d) when $\gamma_3$ is increased from the material determined value. These simulation results in Fig.5.32 indicates that the searching of $\gamma_3$ and $\delta$ parameters over a large range about the material parameters has the potential in improving the structural ratcheting simulation.
Fig. 5.24 Influence of $\gamma_1$ parameter in simulation of (a) hysteresis loop (b) uniaxial ratcheting response (c) biaxial ratcheting response (d) pipe1 and pipe2 circumferential strain ratcheting responses.

Fig. 5.25 Influence of $C_2$ parameter in simulation of (a) hysteresis loop (b) uniaxial ratcheting response (c) biaxial ratcheting response (d) pipe1 and pipe2 circumferential strain ratcheting responses.
Fig. 5.26 Influence of $\gamma_2$ parameter in simulation of (a) hysteresis loop (b) uniaxial ratcheting response (c) biaxial ratcheting response (d) pipe1 and pipe2 circumferential strain ratcheting responses.

Fig. 5.27 Influence of $\gamma_3$ parameter in simulation of (a) hysteresis loop (b) uniaxial ratcheting response (c) biaxial ratcheting response (d) pipe1 and pipe2 circumferential strain ratcheting responses.
Fig. 5.28 Influence of $C_4$ parameter in simulation of (a) hysteresis loop (b) uniaxial ratcheting response (c) biaxial ratcheting response (d) pipe1 and pipe2 circumferential strain ratcheting responses.

Fig. 5.29 Influence of $\gamma_4$ parameter in simulation of (a) hysteresis loop (b) uniaxial ratcheting response (c) biaxial ratcheting response (d) pipe1 and pipe2 circumferential strain ratcheting responses.
Fig. 5.30 Influence of $a_4$ parameter in simulation of (a) hysteresis loop (b) uniaxial ratcheting response (c) biaxial ratcheting response (d) pipe1 and pipe2 circumferential strain ratcheting responses.

Fig. 5.31 Influence of $\delta$ parameter in simulation of (a) hysteresis loop (b) uniaxial ratcheting response (c) biaxial ratcheting response (d) pipe1 and pipe2 circumferential strain ratcheting responses.
As mentioned earlier, the objective in this section is to refine the model parameters for improving the ratcheting simulation without affecting the hysteresis loop simulation keeping the material hysteresis loop simulation unchanged. Parameters $\gamma_1, C_2, \gamma', C_4, \gamma'$ and $a_4$ influence the hysteresis loop simulation significantly without changing the ratcheting simulation much. Therefore, these parameters should be searched within a small range ($\pm 3\%$). Again, $\gamma_3$ and $\delta'$, which influence the ratcheting simulations are searched as follows: $\gamma_3$ is varied from 0 to 12 and $\delta'$ is varied from 0.5 to 1.0. The GA optimization scheme determined a new set of parameter for the modified Chaboche model:

$\sigma_0 = 262.0$ MPa, $E = 183000$ MPa, $\nu = 0.302$

$C_{1-4} = 758150, 109860, 23289, 208710$ MPa

$\gamma_{1-4} = 33655, \quad 544, \quad 5.89, \quad 4872$

$\overline{a_4} = 12.50, \delta' = 0.78$.

The simulations for the responses used for the parameter determination as shown in Fig. 5.33. From this figure, it is observed that uniaxial ratcheting history is overpredicted, where as the material biaxial ratcheting and the material pipe ratcheting simulations are improved.
This modified set of parameters was used for the structural response simulation of pipe1 and pipe2. The simulated results are plotted and compared to the experimental results in Fig. 5.34. The moment-rotation response is still underpredicted similar to the material parameter simulations as shown in Fig. 5.34a and 5.34b. However, the modified parameter set predicted pipe2 ovalization and strain ratcheting responses very well, and pipe1 ratcheting responses are little overpredicted.

It is shown earlier in Fig.5.33 that this set of parameter could not simulate the material uniaxial ratcheting response well. It is noted here that the material uniaxial ratcheting response was obtained from a stress-controlled cyclic experiment but hysteresis loop and biaxial ratcheting responses were obtained from strain-controlled cyclic tests. The difference between the hysteresis curves from stress and strain-controlled experiments are observed when the 1st and 218th reloading curves from uniaxial ratcheting experiment is compared with the 1st and stable reloading curve from the uniaxial strain-controlled tests as shown in Fig.5.35a. To have a better comparison of the curves, these are shifted to start from the same initial point (shown in Fig.5.35b). It is been observed that the reloading curves from stress
controlled experiment is stiffer than the curves from the strain-controlled experiments. This explains the overpredicted simulation of the material uniaxial ratcheting response. To examine if the uniaxial ratcheting simulation can be improved, a composite curve was developed combining the initial part of the 218th stress controlled and reloading end part of the stable hysteresis curve as shown in Fig.5.35c. The composed curve shown in Fig.5.35c matches well with the 1st hysteresis curve from the strain controlled experiment. This composite curve is used for further parameter refinement of the model parameters. Thus determined parameters are,

\[ \sigma_0 = 262.0 \text{ MPa}, \ E = 183000 \text{ MPa}, \ v = 0.302 \]
\[ C_{1-4} = 714040, \ 88248, \ 22340, \ 162530 \text{ MPa} \]
\[ \gamma_{1-4} = 7868, \ 526, \ 6.12, 20387 \]
\[ \tilde{a}_4 = 69.2, \ \delta' = 0.81, \]

The simulations of the experimental responses used in parameter determination are shown in Fig.5.36. As expected, the hysteresis loop are slightly overpredicted, but the material uniaxial ratcheting simulation is improved with affecting the biaxial ratcheting simulations. The structural simulations with these parameters are shown in Fig.5.37. The moment-rotation response simulations for pipe experiments (shown in Fig.5.37) are improved slightly whereas the simulations of ovalization and strain ratcheting are not changed much.

The above study suggests a new semi-inverse approach for parameter determination, which is to determine the model parameters using the material hysteresis curve and local ratcheting responses at critical locations in structures. This parameter determination method requires identifying the stress-strain history at critical locations of the structure. In cases like pipe structure, the load history can be simplified for accelerating the optimization process. This loading history can be identified using finite element analysis for the structure with material response determined parameters.
Fig. 5.34 Structural response simulation through refined parameter set 1 and material parameters with modified Chaboche model: (a) Moment-rotation response for pipe 1 (b) Moment-rotation response for pipe 2 (c) mean of in-plane ovalization, (d) amplitude of in-plane ovalization (e) mean of out-of-plane ovalization and (f) amplitude of out-of-plane ovalization (g) mean of circumferential strain at top and (h) amplitude of circumferential strain at top of pipe.
Fig. 5.35 Comparison of hysteresis curves obtained from strain controlled and stress-controlled experiments (a) actual curves (b) shifted curves and (c) composite curve.

Fig. 5.36 Comparison refined parameter set 2 and material parameters simulation for (a) hysteresis loop (b) uniaxial ratcheting response (c) biaxial ratcheting response (d) pipe1 and pipe2 circumferential strain ratcheting responses.
Fig. 5.37 Structural response simulation through *refined parameter set 2 and material parameters* with modified Chaboche model (a) Moment-rotation response for pipe 1 (b) Moment-rotation response for pipe 2 (c) mean of in-plane ovalization, (d) amplitude of in-plane ovalization (e) mean of out-of-plane ovalization and (f) amplitude of out-of-plane ovalization (g) mean of circumferential strain at top and (h) amplitude of circumferential strain at top of pipe.
5.8 Conclusion

This chapter presents experimental ratcheting responses of straight pipes subjected to cyclic bending and internal pressure conducted by Corona (1996) for this project. Attempt was made to simulate these responses with ANSYS using bilinear, multilinear and Chaboche plasticity models. Parameters of these models were determined using the material responses. None of these three models could simulate the cyclic and ratcheting (ovalization and strain) responses. Hence, advanced cyclic plasticity models, such as, modified Chaboche, Ohno-Wang, Abdel Karim-Ohno model and modified Ohno-Wang were implemented in ANSYS for simulation of the pipe responses. Again, it was observed that, when the model parameters are determined from material responses it cannot simulate the piping responses well.

At this point, effort was made towards understanding the deficiency of the models through extensive study of the modified Chaboche model. A sensitivity study of modified Chaboche model parameters was performed with a view to improve the structural ratcheting simulation. It was shown that when the ratcheting parameters are determined using localized strain response in pipe, these parameters can be refined for improving the structural response simulation. This suggests that the material responses used for parameter determination of modified Chaboche model may not be representative of the structural responses. This study suggested a new semi-inverse approach for parameter determination of advanced cyclic plasticity models. This method uses both material responses (hysteresis loop and uniaxial ratcheting) and localized structural responses (circumferential strain response) from structures for parameter determination. This proposed parameter determination strategy will be further verified with structural simulations for elbow experiments in Chapter 6.
6.1 Introduction

Elbow components are widely used in nuclear power plants and chemical industries. Piping systems are observed to experience premature failures due to fatigue cracks or plastic collapse due to the accumulation of deformation or strain in elbow components (EPRI, 1994). These results indicate that fatigue failures may occur much earlier than the estimated life, and may result in unscheduled plant downtime. Elbow components in a piping system may be subjected to various combinations of internal pressure, bend and twisting cyclic loading. Due to geometry of the elbow structure, the stress distribution in the elbow pipes are much more complex than the straight pipes. The behavior of the elbow becomes more complex due to coupling between pressure and bending responses. Double curvature in elbow components induces stress riser, thus plastic zones form in the elbow resulting in permanent deformations or strains. These strains or deformations are accumulated locally under cyclic loading which induces these fatigue crack or plastic collapse failure. The low cycle ratcheting fatigue failure mechanism of elbow components are not well understood.

The objective of this chapter is to understand the low cycle fatigue failure phenomena of piping components under internal pressure and cyclic bending. Few studies in the literature addressed for the elbow behavior under in-plane bending and internal pressure (Shalaby and Younan, 1998; Degrassi et. al., 2003; Ayob et. al., 2003; Balan and Redektop, 2005; Robertson et. al., 2005). Most of the studies are monotonic experiments and few of them presented finite element simulations for monotonic loading. Degrassi et. al. (2003) performed non-linear time history finite element analysis of piping system with the object of simulating ratcheting responses under seismic excitation. They simulated the seismic responses of the piping systems using the bilinear, multilinear and Chaboche models in ANSYS 5.6. The study was performed in two steps. In first step, the Chaboche model parameters were determined from uniaxial strain controlled hysteresis loop and ratcheting response using the parameter determination method proposed by Bari and Hassan (2002). Degrassi et. al. (2003) determined bilinear and multilinear model parameters from hysteresis loop only. However,
one of the Chaboche model parameters, $\gamma_3$, was determined using the elbow ratcheting responses through finite element analysis. The authors (Degrassi et al., 2003) demonstrated that bilinear and multilinear models are incapable of producing ratcheting responses, whereas Chaboche model can simulate the cyclic ratcheting response of elbow through adjusting ratcheting parameters based on elbow ratcheting response. In the second step, these determined parameters were used for finite element simulation of deformation and strain ratcheting responses obtained from seismic experiments. They demonstrated that bilinear and multilinear model showed shake down phenomena in strain ratcheting simulation. They demonstrated that the Chaboche model performed best in strain ratcheting simulation and recommended the use of Chaboche model for ratcheting simulation. Balan and Redektop (2005) simulated the response of elbow specimen under cyclic bending and internal pressure with bilinear plasticity model in finite element Code ADINA. In this simulation, the author demonstrated the shakedown phenomena observed in simulation with bilinear model for hoop strain ratcheting at flank. More rigorous studies are needed to understand the behavior of elbow components under cyclic loading.

The concept of the behavior of the elbow components can be well understood through a systematic set of test results of elbow piping component and detailed finite element analysis. A series of elbow experiments were conducted at North Carolina State University (Modlin and Hassan, 2000). The study of the experimental responses depicts that internal pressure enhances the ratcheting rate. This Chapter will evaluate the existing plasticity models against these ratcheting responses of elbow pipe under cyclic bending and internal pressure. The plasticity models considered for detailed finite element analysis are bilinear (Prager, 1956), multilinear (Besseling, 1958) and Chaboche (1986) which are currently available in ANSYS and also the advanced models by Bari and Hassan (2002), Ohno-Wang (1993), modified Ohno-Wang (2003) and Abdel Karim-Ohno (2000) which are customized with ANSYS. The performances of the plasticity models are compared in order to identify the best model for structural ratcheting simulation.

6.2 Experiments

Modlin and Hassan (2002) conducted a series of elbow experiments by applying steady internal pressure and cyclic opening-closing bending. These results are presented first in order to demonstrate the ratcheting responses of the elbow components.
6.2.1 Elbow specimens

The elbow experiments were conducted in stainless steel (SS) 304L elbow specimens. Each elbow specimen was constructed of two-inch diameter, schedule 10, 90-degree, short radius (mean bend radius 80 mm) elbow piping component, each end of which is butt welded to a 254 mm long straight pipes as shown in Fig.6.1a. Straight pipe segments had lengths about 5 times the outside diameter of the elbow so that end constraints do not influence the elbow responses. A lug was attached to each end that sealed the elbow, allowing for internal pressure to be applied. The lug also provided a pinned-end connection at each end of the elbow as shown in Fig.6.1b. Each elbow specimen’s geometry was thoroughly measured at different sections (A to F) and different points (1 to 12) in each section as shown in Fig.6.2. Average thickness and diameter, and pipe and lug lengths of 10 elbow specimens are shown Table 6.1 with reference to Fig.6.2. Detail dimensions of the elbow specimens measured are presented in Table C.1 to C.10 in Appendix C with reference to specimen schematic shown in Fig.6.2. Geometry (width and height) of the butt welds between the pipe and elbow is measured at eight locations and are shown in Appendix C in Table C.11. SS308 weld sticks are used for welding.

![Image](image-url)

Fig. 6.1 (a) Picture of the elbow test setup (b) Idealization of the test.
Fig. 6.2 Elbow specimen diameter and thickness measurement (a) sections (A to F) measured (b) points (1 to 12) on each section measured.

Table 6.1
Geometry of elbow specimens

<table>
<thead>
<tr>
<th>Specimen No.</th>
<th>Average thickness, t (mm)</th>
<th>Average diameter, d (mm)</th>
<th>W (mm)</th>
<th>X (mm)</th>
<th>Y (mm)</th>
<th>Z (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pipe</td>
<td>Elbow</td>
<td>Pipe</td>
<td>Elbow</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Elbow 1</td>
<td>2.718</td>
<td>3.302</td>
<td>57.66</td>
<td>57.10</td>
<td>87.7</td>
<td>256.5</td>
</tr>
<tr>
<td>Elbow 2</td>
<td>2.845</td>
<td>3.251</td>
<td>57.51</td>
<td>57.10</td>
<td>88.1</td>
<td>256.5</td>
</tr>
<tr>
<td>Elbow 3</td>
<td>2.794</td>
<td>3.327</td>
<td>57.25</td>
<td>56.85</td>
<td>88.3</td>
<td>256.5</td>
</tr>
<tr>
<td>Elbow 4</td>
<td>3.099</td>
<td>3.683</td>
<td>57.40</td>
<td>56.90</td>
<td>88.1</td>
<td>256.5</td>
</tr>
<tr>
<td>Elbow 5</td>
<td>3.150</td>
<td>3.937</td>
<td>57.30</td>
<td>57.25</td>
<td>87.7</td>
<td>254.0</td>
</tr>
<tr>
<td>Elbow 6</td>
<td>3.073</td>
<td>3.759</td>
<td>57.91</td>
<td>57.25</td>
<td>88.6</td>
<td>254.0</td>
</tr>
<tr>
<td>Elbow 7</td>
<td>2.794</td>
<td>3.531</td>
<td>57.61</td>
<td>57.00</td>
<td>88.1</td>
<td>256.5</td>
</tr>
<tr>
<td>Elbow 8</td>
<td>3.150</td>
<td>3.835</td>
<td>57.30</td>
<td>56.85</td>
<td>86.7</td>
<td>254.0</td>
</tr>
<tr>
<td>Elbow 9</td>
<td>3.073</td>
<td>3.759</td>
<td>57.61</td>
<td>56.95</td>
<td>88.1</td>
<td>256.5</td>
</tr>
<tr>
<td>Elbow 10</td>
<td>3.073</td>
<td>3.734</td>
<td>57.51</td>
<td>56.79</td>
<td>88.3</td>
<td>254.0</td>
</tr>
</tbody>
</table>

6.2.2 Experimental setup
The test setup is shown in Fig. 6.1a. In experiments that included internal pressure, it was applied via pressurized hydraulic oil through the upper lugs using pneumatic pump. At top cross arm of the loading frame held the load-cell (shown in the picture, Fig. 6.1a), which is pin connected to the upper lug. The lower lug was pin connected to the actuator rod that applied displacement or force controlled loading to the elbow specimen. For tests with pressure, a steady pressure was maintained through the use of an accumulator.
6.2.3 Data acquisition

A computer based data acquisition system was used to record elbow ovalization and strains at the mid-section of the elbow. Elbow ovalizations were measured using LVDTs (Linear Variable Differential Transformers) through a device shown in Fig. 6.3. Figure 6.3a shows the elbow with the extrados at the bottom with a LVDT device setup. LVDT’s were mounted at both flanks, and the intrados and extrados. Thus, the changes in diameters were measured across the flanks and across the intrados and extrados. The change in diameter was determined by summing the output of the two LVDT’s aligned in each direction.

![Fig. 6.3 (a) Picture from above of LVDT layout (b) Schematic of the LVDT apparatus](image)

Strains are measured at pipe intrados (concave side of the elbow), extrados (convex side of the elbow), both flanks (at sides) and 45° position at midway between flank and extrados (see Fig.6.4). The strain gages used were all biaxial strain gages, and thus both axial and circumferential strains are measured at each of the points.

A uniaxial strain gage was also mounted on the outside of the straight pipe, at a distance eight inches away from the center of the pin connection. This strain gage provided data for checking elastic relationship between load displacement-strain, thus determined accuracy of the acquired data from the experiments.
6.2.3 Prescribed loading in the experiments

The elbow specimens were tested under both displacement-controlled and force-controlled opening-closing cyclic loading, with or without internal pressure. The loading histories followed the experiments shown in Fig. 6.5. Parameters of the displacement-controlled and force-controlled loading histories are shown in Table 6.2 and 6.3, respectively. The displacement controlled experiments included monotonic and cyclic experiments. Three monotonic experiments, two closing (Elbow1 and Elbow3) and one opening (Elbow2) are shown in Table 6.2. Two displacement-controlled cyclic experiments were conducted without internal pressure (Elbow4 and Elbow5). Two more displacement-controlled cyclic experiments were conducted in the presence of internal pressure (Elbow6 and Elbow7). These experiments can be divided in two sets; first set includes elbow4 and elbow5, the experiments without internal pressure but with different displacement amplitude ($\delta_c$); second set includes elbow4, elbow6 and elbow7, the experiments with different internal pressure, but same displacement amplitude.

Three force-controlled tests were conducted, Elbow8 and elbow9, with different internal pressure, but almost the same force-controlled loading parameters. Elbow10 experiment was intended to be duplication of elbow9, but could not be achieved.
Fig. 6.5  (a) Schematic of the load application (all dimensions are mm) (b) Displacement controlled-load history (c) Force-controlled load history

Table 6.2

Loading parameters in the displacement controlled experiments (Fig. 6.5b)

<table>
<thead>
<tr>
<th>Specimens</th>
<th>Displacement amplitude, δc (mm)</th>
<th>Steady Internal Pressure, p (MPa)</th>
<th>No of Cycles, N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elbow1</td>
<td>Monotonic closing</td>
<td>0.0</td>
<td>---</td>
</tr>
<tr>
<td>Elbow2</td>
<td>Monotonic opening</td>
<td>0.0</td>
<td>---</td>
</tr>
<tr>
<td>Elbow3</td>
<td>Monotonic closing</td>
<td>0.0</td>
<td>---</td>
</tr>
<tr>
<td>Elbow4</td>
<td>±11.80</td>
<td>0.0</td>
<td>672</td>
</tr>
<tr>
<td>Elbow5</td>
<td>±18.80</td>
<td>0.0</td>
<td>200</td>
</tr>
<tr>
<td>Elbow6</td>
<td>±11.80</td>
<td>11.03</td>
<td>853</td>
</tr>
<tr>
<td>Elbow7</td>
<td>±11.80</td>
<td>20.70</td>
<td>475</td>
</tr>
</tbody>
</table>
Table 6.3

<table>
<thead>
<tr>
<th>Force, Mean, $F_m$ (kN)</th>
<th>Amplitude, $F_a$ (kN)</th>
<th>Steady Internal Pressure, $p$ (MPa)</th>
<th>No of Cycles, $N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elbow8</td>
<td>0.19540 ±5.5174</td>
<td>0.0</td>
<td>1201</td>
</tr>
<tr>
<td>Elbow9</td>
<td>0.16005 ±5.1866</td>
<td>10.70</td>
<td>2120</td>
</tr>
<tr>
<td>Elbow10</td>
<td>0.08545 ±4.9195</td>
<td>10.34</td>
<td>7280</td>
</tr>
</tbody>
</table>

6.3 Experimental results

6.3.1 Monotonic responses

Elbow monotonic experiments were conducted by prescribing displacement-controlled loading either in the closing or opening directions. Figure 6.6a and 6.6b show the load displacement response for elbow1 and elbow2 experiments. For monotonic closing (elbow1), the force required for closing gradually increased until reaches the ultimate value, following which the force gradually reduces (Fig.6.6a), whereas the force increases gradually for monotonic opening experiment (Fig.6.6b). The in-plane (intrados to extrados) and out-of-plane (flank to flank) ovalization responses under monotonic closing (elbow1) and opening (elbow2) displacements are shown in Fig.6.7a and 6.7b. The axial and circumferential strain responses at elbow flank, intrados and extrados for elbow1 and elbow2 experiments are shown in Figs.6.7c to 6.7h. All ovalization and strain responses either monotonically increase or decreases. The load deformation ovalization, axial and circumferential strain responses at elbow flank, intrados and extrados for elbow3 experiment are shown and compared to elbow2 opening experiment in Figs. 6.8 and 6.9.

Fig. 6.6 Monotonic load-displacement response from (a) closing (elbow1) (b) opening (elbow2) experiments
Fig. 6.7 Responses from monotonic experiments for elbow1 and elbow2 specimens; (a) In-plane (intrados to extrados) ovalization (b) out-of-plane (flank to flank) ovalization (c) axial strain at flank (d) circumferential strain at flank (e) axial strain at intrados (f) circumferential strain at intrados (g) axial strain at extrados (h) circumferential strain at extrados.
6.3.2 Displacement-controlled elbow experiments

Four elbow experiments (Elbow4, Elbow5, Elbow6 and Elbow7) were conducted by prescribing cyclic displacement-controlled loading with or without internal pressure as shown in Table 6.2. The responses from these experiments will be presented in two groups: (1) Elbow4, Elbow6 and Elbow7 experiments with the same displacement amplitude but different levels of internal pressure, (2) Elbow4 and Elbow5 experiments with different displacement amplitude and no internal pressure. First, one of the elbow (Elbow7) experimental responses will be presented elaborately.

Elbow7 experiment was conducted with internal pressure of 20.70 MPa and cyclic displacement amplitude of 11.80 mm. The load-displacement response for the Elbow7 experiment is shown in Fig. 6.10a. This figure shows 1st cycle including the monotonic load-deformation response and every 50th cycles. Figure 6.10a demonstrates that the load deformation cycles stabilizes after a couple of cycles. The in-plane (intrados and extrados) and out-of-plane (flank and flank) ovalizations as function of displacements are shown in Fig.6.10b and 6.10c for every 50th cycle starting with the 1st cycle. The ovalization-displacement response gradually shifts upward indicating ovalization. The responses of axial and circumferential strains as a function of displacements for every 50th cycles at both flanks, intrados, and extrados are shown in Fig.6.11. Axial and circumferential strain responses gradually shows ratcheting with cycles. Note that the strain responses at the two flanks are not duplicating closely as expected (Figs. 6.11a and 6.11c).
Fig. 6.9 Responses from monotonic experiments for elbow2 and elbow3 specimens; (a) In-plane (intrados to extrados) ovalization (b) out-of-plane (flank to flank) ovalization (c) axial strain at flank (d) circumferential strain at flank (e) axial strain at intrados (f) circumferential strain at intrados (g) axial strain at extrados (h) circumferential strain at extrados.
The forces and ovalizations recorded at maximum and minimum displacements for *elbow7* experiment are plotted against the number of cycles in Fig. 6.12. Figure 6.12a shows that the force response is almost symmetric, but Fig.6.12b and 6.12c shows ratcheting of in-plane and out-of-plane ovalization. It is noted to the readers that the fluctuation in the peak response is due to time interval data ovalization. Consequently, the actual force peaks were not acquired. However, the acquired data still demonstrates that the force response is almost stable. The peak strain responses at maximum and minimum displacements for *elbow7* experiment at two flanks, extrados, and intrados are plotted in Fig. 6.13. Fig.6.11b and 6.11c show that the ovalization ratcheting rate is higher along the out-of-plane direction compared to along the in-plane direction. From this figure, it is observed that the circumferential strain at flanks ratchet the most, about 4-6% in 25 cycles.

**Elbow response- effect of internal pressure**

*Elbow4, elbow6* and *elbow7* experiments are compared to demonstrate the influence of internal pressure on elbow fatigue responses. The 1\textsuperscript{st} cycle of the load-displacement responses along with the monotonic part for *elbow4, elbow6* and *elbow7* experiments are shown in Fig. 6.14. Effect of internal pressure is not distinctively observed from the hysteresis loop comparison. The mean and amplitude of force, ovalization and strain responses are shown in Fig.6.15 and 6.16. Figures 6.15a and 6.15b shows the variation of mean (P\textsubscript{m}) and amplitude (P\textsubscript{a}) of forces with cycles respectively. The mean of force response are gradually decreasing with the increasing internal pressure as observed in Fig.6.15a. The amplitude of forces responses, on the other hand is increasing with the increase in internal pressure (Fig.6.15b). However, in any of the three experiments, either the mean or the amplitude of force is not changing much with cycles. The mean and amplitude of ovalization responses are shown in Fig.6.15c-f. Figs. 6.15c and 6.15d demonstrates that the internal pressure increase the ovalization ratcheting rates. However, the internal pressure does not change the amplitude of ovalization (Fig. 6.15e and Fig.6.15f). Also, for all experiments, the amplitude of ovalizations remained unchanged with cycles.

The mean and amplitude of axial and circumferential strain responses at both flanks, intrados, extrados and midway between extrados and flank are plotted against the number of cycles in Fig.6.16. The mean of axial strains are observed to have small to negligible amount of ratcheting at all locations for all elbow experiments (Fig. 6.16). The amplitude of axial
strains also changes small to negligible at allocations for all elbow experiments. Level of internal pressure or the number of cycles do not have much influence on the axial strain response (Fig. 6.16). However, the circumferential strain response is influenced by the level of internal pressure. Circumferential strain ratcheting rate at the flanks are influenced significantly by the level of internal pressure (Fig. 6.16b and 6.16f). The mean of circumferential strains at intrados and extrados are observed to be influenced by the level of pressure (Fig. 6.16j and 6.16n). The amplitude of circumferential strain are not influenced by internal pressure or the number of cycles (Fig. 6.16l and 6.16p). For the midway location between the extrados and flank, no conclusion on the influence of internal pressure can be drawn due to lack of data caused by strain gage failure (Fig. 6.16q-6.16t). It is interesting to note that even under zero internal pressure, circumferential strain ratchets at flank, extrados and intrados under displacement controlled cycles.

Fig. 6.10 Responses of elbow experiment (a) Load-displacement response; (b) in-plane (intrados-extrados) and (c) out-of-plane ovalization (flank-flank) responses.
Fig. 6.11 Axial and circumferential strain responses at two flanks, intrados and extrados for elbow experiment.
Effect of displacement amplitude on elbow responses is studied by comparing responses of elbow4 and elbow5 experiments which were conducted with zero internal pressure. The 1st cycle of load-displacement responses for these two experiments are shown in Fig. 6.14. The mean and amplitude of forces are plotted against the number of cycles in Fig.6.17a and 6.17b. The mean and amplitude of elbow ovalization and strain responses are compared in Figs.6.17c to 6.17f and 6.18a to 6.18t respectively. The displacement amplitude at zero internal pressure seems to influence the responses of out-of-plane ovalization ratcheting (Fig.6.16d), in-plane and out-of-plane ovalization amplitudes (Fig. 6.16e and 6.16f), circumferential strain ratcheting at both flanks (Fig.6.17b and 6.17f) only. Mean and amplitude force, in-plane ovalization, axial strain mean and amplitude at all locations and circumferential strain mean and amplitude at all locations except the circumferential strain means of flank seems not to be influenced by the displacement amplitude. In some case, conclusion cannot be drawn due to lack of data due to strain gage failure.
Fig. 6.13 Axial and circumferential strain responses at maximum and minimum displacements as a function of number of cycles for elbow experiment (a)-(d) at flanks, (e)-(f) at intrados and (g)-(h) at extrados.
Fig. 6.14 First cycle load-displacement response from (a) elbow4, (b) elbow5, (c) elbow6 and (d) elbow7 displacement controlled experiments.

Elbow5 experiment is done without internal pressure; only cyclic vertical displacements are applied as done in elbow4 experiment. The amplitude of applied displacement in elbow5 experiment is higher (1.5 times approximately) than elbow4 experiment. The mean of in-plane ovalization for elbow5 responses are observed to have negligible ratcheting responses whereas the out-of-plane ovalization shows ratcheting due to increase in displacement amplitude (Fig.6.17c to 6.17d). The amplitude increases with increase in amplitude of applied displacement for elbow5 but remained unchanged with cycles (Fig.6.17e to 6.17f). The mean of axial strain peaks at flanks, intrados and extrados demonstrate negligible ratcheting rate (Fig.6.18). Maximum circumferential strain ratcheting rate is observed at flank for the elbow5 experiment (Fig.6.18b and 6.18f). Amplitude of axial and hoop strains increased with increase in applied displacements but remained unchanged with cycles (Fig. 6.18).
Fig. 6.15 Mean and amplitude of force, in-plane ovalization and out-of-plane ovalization responses against the number of cycles for \textit{elbow4}, \textit{elbow6} and \textit{elbow7} experiments.
Fig. 6.16 Mean and amplitude of axial and circumferential strain responses against the number of cycles at both flanks from elbow4, elbow6 and elbow7 experiments.
Fig. 6.16 (continued) Mean and amplitude of axial and circumferential strain responses against the number of cycles at intrados and extrados from *elbow*4, *elbow*6 and *elbow*7 experiments.
6.3.3 Force-controlled elbow experiments

Two force-controlled experiments were conducted as shown in Table 6.3. In both experiments, the intention was to prescribe symmetrical force cycle, but small mean force was obtained as shown in Table 6.3. First, one of the force-controlled elbow responses will be presented elaborately. *Elbow*9 experiment was conducted with internal pressure of 10.70 MPa and force amplitude of 5.18 kN. The load-displacement response for the *elbow*9 experiment is shown in Fig.6.19a where 1st and every 100th cycles are shown. Figure 6.19a demonstrates that the load-displacement loops gradually shifts with cycles in the negative displacement direction, demonstrating ratcheting of displacement. The in-plane and out-of-plane diameter changes (ovalizations) with displacements are shown in Fig.6.19b and 6.19c for every 100th cycle starting with the 1st cycle, demonstrating ratcheting of in-plane and out-of-plane ovalization. The variation of axial and circumferential strains with displacements for every 100th cycles at both flanks, intrados, extrados, and midway between flank and extrados are shown in Fig.6.20a to 6.20j. Each of these responses demonstrating ratcheting of strains with cycles. Displacements at maximum and minimum peak forces as a function of the
number of cycles in Fig.6.21a shows the amplitude and ratcheting rate of displacement response. Amplitude and ratcheting rates of ovalization responses along the in-plane and out-of-plane directions for \textit{elbow9} force-controlled experiment are shown in Fig.6.21b and 6.21c. Amplitudes of axial and circumferential strain responses against the number of cycles at flank, intrados, extrados and midway between extrados and flanks are shown in Fig.6.22a to 6.22j.

![Graphs](image)

**Fig. 6.17** Mean and amplitude of force, in-plane ovalization and out-of-plane ovalization responses against the number of cycles from \textit{elbow4} and \textit{elbow5} experiments.
Fig. 6.18 Mean and amplitude of axial and circumferential strain responses against the number of cycles at both flanks for elbow4 and elbow5 experiments.
Fig. 6.18 (continued) Mean and amplitude of axial and circumferential strain responses against the number of cycles at intrados and extrados from elbow4 and elbow5 experiments.
Fig. 6.18 (continued) Mean and amplitude of axial and circumferential strain responses against the number of cycles at 45° midway between flank and extrados from *elbow4* and *elbow5* experiments.

Fig. 6.19 (a) Load displacement response for *elbow9* experiment; (b) in-plane and (c) out-of-plane ovalization responses against the applied displacement for *elbow9* experiment.
Fig. 6.20 Axial and circumferential strain responses from elbow9 force-controlled experiment (a)-(d) at both flanks, (e)-(f) intrados, (g)-(h) extrados, (i)-(j) midway between extrados and flank as a function of displacement.
The 1st cycle of load displacement loops for elbow8 and elbow9 experiments are shown in Fig. 6.23. Difference in hysteresis loop induced by the internal pressure is not seen clearly from Fig.6.23. Mean and amplitude of displacement, and in-plane and out-of-plane ovalizations are shown in Fig.6.24 to demonstrate the influence of internal pressure on the responses of force controlled elbow experiments. The mean and amplitude of axial and circumferential strains from elbow8 and elbow9 at both flanks, intrados, extrados and midway between extrados and flank are plotted against the number of cycles in Fig.6.25.

Elbow10 experiment was conducted of 10.34 MPa and force-controlled cycle of amplitude 4.92 kN and mean 0.08 kN. Responses from elbow10 are compared to those from elbow9 experiment in Figs.6.26 and 6.27.
Fig. 6.22 Axial and circumferential strain peaks from elbow9 experiment plotted against the number of cycles at (a)-(d) both flank, (e)-(f) intrados and (g)-(h) extrados, (i)-(j) midway between extrados-flank.
Fig. 6.23 First cycle of load-displacement response from (a) elbow8 (b) elbow9 and (c) elbow10 force controlled experiments.

Fig. 6.24 Mean and amplitude of force, in-plane ovalization and out-of-plane ovalization responses against the number of cycles from elbow8 and elbow9 experiments.
Fig. 6.25 Mean and amplitude of axial and hoop strain responses against the number of cycles at flank for elbow8 and elbow9 experiments.
Fig. 6.25 (continued) Mean and amplitude of axial and circumferential strain responses against the number of cycles at intrados and extrados from elbow8 and elbow9 experiments.
6.3.4 Material experiments

To determine model parameters, monotonic and cyclic tests were conducted on SS304L straight pipes obtained from the same batch of the pipes used for elbow specimen fabrication. The monotonic uniaxial stress-strain response is shown in Fig. 6.28a. The stress-strain response of the uniaxial ratcheting experiment is shown in 6.28b. The plot of axial strain peaks against the number of cycles from the uniaxial ratcheting experiment is shown in Fig. 6.28c. The biaxial ratcheting test shows circumferential strain accumulation as shown in Fig. 6.28d. The plot of circumferential strain peaks against the number of cycles is shown in Fig. 6.28e. The material parameter determination for cyclic experiments requires stable hysteresis loop from uniaxial strain-controlled test. As the hysteresis loop was not available in this case, a cyclic curve was composed through combining the reloading part of the stress-strain curve in uniaxial ratcheting experiment (Fig. 6.28b) and the end slope of the monotonic response (Fig. 6.28a). This composition for obtaining a hysteresis loop for model parameter determination is shown in Fig. 6.29.
Fig. 6.26 Mean and amplitude of displacements, in-plane ovalization and out-of plane ovalization responses against the number of cycles for elbow9 and elbow10 experiments.
Fig. 6.27 Mean and amplitude of axial and circumferential strain responses against the number of cycles at flanks from *elbow9* and *elbow10* experiments.
Fig. 6.27 (continued) Mean and amplitude of axial and circumferential strain responses against the number of cycles at intrados and extrados from *elbow9* and *elbow10* experiments.
6.4 Finite element model

6.4.1 Geometric model

The elbow specimen (Fig. 6.2b) and loading prescribed is doubly symmetric. The specimen was pin-connected at the top and bottom ends. In the experiment setup, the top end remained stationary while the bottom end translates with the actuator rod to prescribe the intended displacement and force-controlled loading. Using the double symmetry of the structure, as well as considering the symmetric loading condition, only one quarter of the specimen was modeled for finite element simulations. Consequently only half of the displacement was prescribed for displacement-controlled experimental simulations. The boundary conditions of the model came from the double symmetry of specimen and loading in addition, one point (extrados) on the symmetric cross-section was pinned in all three directions. The elbow and pipe considered in this study has thickness to diameter ratio of 14 and 18, which can be considered as thin shell structure. Hence, the elbow is modeled with *Shell181* elements in ANSYS9.0 (2005).
Fig. 6.28 Material responses of SS304L for model parameter determination (a) uniaxial stress-strain curve (b) uniaxial ratcheting (c) uniaxial ratcheting rate (d) circumferential strain ratcheting for biaxial loading history (e) biaxial ratcheting rate.

Fig. 6.29 Composed hysteresis curve from monotonic stress-strain curve and uniaxial ratcheting hysteresis curve for model parameter determination.

In elbow specimen, the straight pipe was welded at the ends of the elbow. The weld thickness and width was not same for all specimens. An average width and height of the
weld is used for modeling of the weld. Tan and Matzen (2002) showed that the best correlation of the experiment with finite element analysis is observed when the weld bead is modeled as half part of the elbow bend and half part of the straight pipe. The authors also suggested reducing actual weld width to 85% to take account of nonrectangular weld width. This study modeled the weld bead as half on the elbow and half on the straight pipe. Rather than reducing the width of the weld bead to 85%, this study modeled the actual weld shape. As no data are available for weld material properties, these were approximated by the SS304L properties presented earlier.

The straight pipe end of the elbow specimen was sealed with shell elements with high modulus of rigidity. The lug end was modeled as a rigid plate with high modulus of rigidity shell elements (Fig. 6.30a). The loading were prescribed at the lug end shown in Fig. 6.5a. The quarter of the elbow specimen was modeled with exact thickness for the pipe and elbow. The thickness of the elbow and the pipe is not uniform; the assumption of exact thickness is still an approximation of the structure. The pipe dimensions usually were circular in shape, but 90° elbow dimensions usually were elliptical as also observed from the measured diameters in Tables C1 to C10, Appendix C.

Fig. 6.30 Two of the Finite element meshes in Table 6.4 used for mesh convergence study (a) mesh1 and (b) mesh7
6.4.2 Mesh convergence study

A mesh convergence study was performed to select an optimum mesh for the analysis of elbow under cyclic bending and steady internal pressure. The elbow specimen and loading history of elbow7 experiment was selected for the mesh study. In this loading history, structure is subjected to cyclic displacement-controlled loading of amplitude 11.80 mm and internal pressure of 20.70 MPa. The analysis is performed with Chaboche plasticity model (1986) and the parameters used are as follows

\[ \sigma_0 = 179.0 \text{ MPa}, \quad E = 193740 \text{ MPa}, \quad \nu = 0.302 \]

\[ C_{1-4} = 147490, \quad 27006, \quad 4702, \quad 113597 \text{ MPa} \]

\[ \gamma_{1-4} = 2549, \quad 867, \quad 1.43, \quad 3706 \]

<table>
<thead>
<tr>
<th>Mesh type</th>
<th>No of elements used</th>
<th>Rigid end plate elements at the end plate</th>
<th>Total no. of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Along circumference</td>
<td>Along elbow bend</td>
<td>Along pipe length</td>
<td>No of Elements</td>
</tr>
<tr>
<td>Mesh 1</td>
<td>20</td>
<td>10</td>
<td>48</td>
</tr>
<tr>
<td>Mesh 2</td>
<td>16</td>
<td>8</td>
<td>36</td>
</tr>
<tr>
<td>Mesh 3</td>
<td>12</td>
<td>6</td>
<td>27</td>
</tr>
<tr>
<td>Mesh 4</td>
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<td>4</td>
<td>18</td>
</tr>
<tr>
<td>Mesh 5</td>
<td>16</td>
<td>8</td>
<td>18</td>
</tr>
<tr>
<td>Mesh 6</td>
<td>16</td>
<td>8</td>
<td>24</td>
</tr>
<tr>
<td>Mesh 7</td>
<td>12</td>
<td>6</td>
<td>18</td>
</tr>
</tbody>
</table>

First, the elbow responses of elbow7 were calculated with a very fine mesh, mesh1 in Table 6.4. For fine mesh, shell thickness to element edge ratio is kept close to 1.0 as minimum value (Fig.6.30a). The number of elements along the circumference, elbow and pipe length is shown in Table 6.4. The total number of elements required to model the quarter of the elbow specimen was 1319 for this fine mesh. Results with this mesh for upto 200 cycles will be used as reference for the mesh study. Elbow analysis responses from four meshes (mesh1 to mesh4 in Table6.4) are compared in Figs.6.31a and 6.31b for circumferential strain ratcheting at flank and intrados. The comparison of mesh1 to mesh4 responses show that simulation accuracy decreases for larger size elements along the circumference and elbow bends. Mesh2 yielded result with acceptable accuracy. Next reduction of elements along the straight pipe was examined through mesh5 and mesh6 (Table 6.4). Results in Fig.6.31c and 6.31d shows that the reduction in elements in pipe through gradual increase in length of the
elements along the pipe length away from the bend has very small influence on ratcheting rate calculation (Fig.6.32c and 6.32d). Further reduction of element in elbow was examined through mesh7 and results of mesh1 and mesh7 are compared in Fig. 6.31e and 6.31f.

Fig.6.31 Circumferential strain ratcheting simulation for mesh optimization study at flank and intrados.

Comparison of the results from seven different meshes (mesh1 to mesh7) show that mesh6 with 624 elements is the best choice for finite element model elbow response calculation. As this study performed cyclic elbow analysis for 600 cycles, solution time were needed to be optimized. Although mesh7 produces more deviation from mesh1 compared to mesh6 (Fig.6.31c and 6.31f), but mesh7 has only 352 elements close to half of the number of
elements in *mesh6*, thus cuts the solution time to approximately one fourth as general rule due to reduction in number of degrees of freedom. Hence, the initial model evaluation study for the elbow simulations was performed with *mesh7*. Once the best plasticity model is selected, final results will be obtained with *mesh6*. The mesh arrangement for *mesh7* is shown in Fig.6.30b.

**6.5 Structural simulation for displacement-controlled tests**

This study simulated the displacement-controlled experiments with ANSYS 9.0 (2005) with its bilinear, multilinear and Chaboche model. Simulations with modified Chaboche (2002), Ohno-Wang (1993), modified Ohno-Wang (2003) and Abdel Karim-Ohno (2000) model were performed through customizing these models into ANSYS9.0. The performances of these plasticity models are presented in subsequent subsections through simulating the experimental results of *elbow4, elbow6 and elbow7*.

**6.5.1 Bilinear and Multilinear models**

The model parameters for the bilinear (Prager,1956) and multilinear (Besseling,1958) plasticity model are determined through approximating the composed stress-strain response only (Fig.6.32a). The bilinear model parameters include elastic modulus, \( E = 193740 \text{ MPa} \); poison’s ratio, \( \nu = 0.302 \); yield stress, \( \sigma_0 = 297 \text{ MPa} \) and plastic modulus, \( E_p = 4480.0 \text{ MPa} \). Multilinear kinematic hardening rule parameters are determined from twelve stress-strain points on the curve. The simulation of the hysteresis curve and uniaxial and biaxial ratcheting responses with the determined parameter sets are shown in Fig.6.32. The load-displacement responses for all three elbow experiments from ANSYS9.0 with bilinear and multilinear models are shown in Fig.6.33. Bilinear model shows over prediction for the load-displacement response as shown in Fig. 6.33 and this is mainly due to the higher yield stress used in bilinear model for material stress-strain curve representation as shown in Fig. 6.32a. Multilinear model showed less over prediction than bilinear model in simulating the load-displacement responses (Fig.6.33), because this model can closely approximate the stress-strain responses (Fig.6.32a). Mean and amplitude of force responses are plotted against the number of cycles in Fig. 6.34a and 6.34b. The mean force responses decreased with increase in internal pressure and mean of force response stabilizes after couple of cycles which matched the trend of experiment (Fig.6.34a). The amplitude of force response simulations are little overpredicted as shown in Fig.6.34b.
Fig. 6.32 Model parameter determination and material response simulations with bilinear and multilinear models (a) hysteresis loop representation and (b) uniaxial ratcheting simulation and (c) biaxial ratcheting simulation.

Fig. 6.33 Load-displacement response simulation with bilinear and multilinear plasticity models for (a) elbow4 (b) elbow6 and (c) elbow7 experiments.

The mean and amplitude of ovalization response simulations with bilinear and multilinear model for elbow4, elbow6 and elbow7 experiments are compared with experimental results in
Fig. 6.34c to 6.34f. Both the bilinear and multilinear models show underprediction of ovalization ratcheting and shakedown phenomena. However, Fig.6.34e and 6.34f demonstrates that the simulated amplitudes of ovalization cycles match well with experimental responses. The amplitude of ovalization simulations remained unchanged for \textit{elbow4, elbow6} and \textit{elbow7} experiments with increasing internal pressure which is also observed in experiments (Fig. 6.34e and Fig. 6.34f). The mean and amplitude of axial and circumferential strain responses at both flanks, intrados and extrados for \textit{elbow4, elbow6} and \textit{elbow7} experiments are compared with simulation obtained with bilinear and multilinear plasticity models in Fig. 6.35. The mean of axial strain ratcheting simulation at flanks showed small over-prediction with shakedown phenomena as shown in Fig.6.35a and Fig.6.35e. The amplitudes of axial strain cycle simulations with bilinear and multilinear models matched well with experiments (Fig.6.35c and 6.35g). The simulated amplitude remained unchanged with cycles and remained same for \textit{elbow4, elbow6} and \textit{elbow7} experiments (Fig.6.35c and Fig.6.35g). Presence of internal pressure does not change much the axial strain amplitudes as observed in \textit{elbow4, elbow6} and \textit{elbow7} experiments. The circumferential strain ratcheting simulations for both flanks underpredict experimental responses and show shakedown phenomena which are not shown by experimental responses (Fig. 6.35b and Fig.6.35f). The amplitudes of circumferential strain ratcheting simulations are same in \textit{elbow4, elbow6} and \textit{elbow7} experiments and remained unchanged with cycles, as also observed in experiments (Fig. 6.35d and Fig. 6.35h).

At intrados and extrados, mean of axial strain responses remained unchanged with cycles for all experiments, so does the simulation except one which shows negative ratcheting rate (Fig. 6.35i). Axial strain amplitude simulations at intrados matched well with experiments and remained unchanged with cycles (Fig.6.35k). Circumferential strain simulation at intrados overpredict the experimental response (Fig.6.35j). In experiments, ratcheting rate gradually decreases to a small stabilized rate or shakedown whereas the simulations show steady ratcheting rate with both bilinear and multilinear models. The amplitude of circumferential strain at intrados showed overprediction (Fig.6.36l). Axial and circumferential strain simulations at extrados for \textit{elbow4, elbow6} and \textit{elbow7} experiments underpredict the experimental responses (Fig.6.35m and 6.35n) mainly because that the negative ratcheting (opposite to experimental trend) responses are simulated for all three experiments. The
amplitudes of axial and circumferential strain cycles matched well with experiments (Fig.6.35o and 6.35p). The axial and circumferential strain simulations at 45° midway between flank and extrados showed overprediction compared to experimental responses in elbow4, elbow6 and elbow7 experiments (Fig.6.35q and 6.36r). The amplitude of axial and circumferential strains at midway between extrados and flank matched well with experimental responses (Fig.6.35s and 6.35t).

Fig.6.34 Ovalization ratcheting simulation with bilinear and multilinear plasticity model for elbow4, elbow6 and elbow7 experiments; (a) mean of reaction forces (b) amplitude of reaction forces; (c) Mean of in-plane and (d) mean of out-of-plane ovalization; (e) amplitude of in-plane and (f) amplitude of out-of-plane ovalization ratcheting simulations.
Fig. 6.35 Mean and amplitude of axial and circumferential strain ratcheting simulation with bilinear and multilinear plasticity model for elbow4, elbow6 and elbow7 experiments at flanks.
Fig. 6.35 (continued) Mean and amplitude of axial and circumferential strain ratcheting simulation with bilinear and multilinear plasticity model for elbow4, elbow6 and elbow7 experiments at intrados and extrados.
In general, both the bilinear and multilinear model predicted the amplitude of force ovalization and strain responses well for all elbow experiments. Similar to experimental responses, the simulated amplitude of ovalization and strain cycles remained unchanged with increase in internal pressure in both bilinear and multilinear plasticity model. Both the bilinear and multilinear models are observed to produce shakedown in ratcheting simulation, where the experiments show continued ratcheting with a steady rate. The shakedown simulation is the deficiency of the kinematic hardening rule of bilinear and multilinear models. As observed in Fig.6.32 that the models are incapable of simulating uniaxial and biaxial strain ratcheting. The simulation of negative ratcheting at extrados (Fig.6.35i), opposite to the experimental ratcheting is a concern.

### 6.5.2 Chaboche and Modified Chaboche models

The model parameters for Chaboche plasticity model (1986) are determined through simulating the composed hysteresis curve and uniaxial ratcheting response (Fig.6.36a and 6.36b). The modified Chaboche model (Bari and Hassan, 2002) parameters are determined through simulating composed hysteresis curve, uniaxial ratcheting and biaxial ratcheting.
responses (Fig. 6.36a to 6.36c). The parameter optimization algorithm developed in Chapter Two was used to optimize the model parameters for these two models.

The optimized parameters of Chaboche (1986) model are:

\[
\begin{align*}
\sigma_0 &= 179.0 \text{ MPa}, \\
E &= 193740 \text{ MPa}, \\
\nu &= 0.302 \\
C_{i-4} &= 147490, 27006, 4702, 113597 \text{ MPa} \\
\gamma_{1-4} &= 2549, 867, 1.43, 3706
\end{align*}
\]

The parameters for modified Chaboche (Bari and Hassan, 2002) model are:

\[
\begin{align*}
\sigma_0 &= 179.0 \text{ MPa}, \\
E &= 193740 \text{ MPa}, \\
\nu &= 0.302 \\
C_{i-4} &= 162880, 26862, 4626, 134570 \text{ MPa} \\
\gamma_{1-4} &= 3404, 874, 1.42, 5013 \\
\bar{a}_4 &= 13.86, \delta' = 0.302,
\end{align*}
\]

![Graphs](a) (b) (c)

Fig. 6.36 Model parameter determination and material response simulations with Chaboche and modified Chaboche models (a) hysteresis loop representation and (b) uniaxial ratcheting simulation and (c) biaxial ratcheting simulation.
Fig. 6.37 Load-displacement response simulation with Chaboche and modified Chaboche models for (a) elbow4 (b) elbow6 and (c) elbow7 experiments.

The load-displacement simulation of the first loop deformation responses (first loop) for elbow4, elbow6 and elbow7 experiment using ANSYS 9.0 with Chaboche and modified Chaboche model are shown in Fig. 6.37. These simulations with Chaboche and modified Chaboche are similar are similar to the simulations by multilinear plasticity model. Variation of mean and amplitude of forces with cycles for these three experiments is shown in Fig.6.38a and 6.38b. The mean of the forces in the simulation has been observed to decrease with increase in internal pressure and also stabilized after couple of cycles (Fig.6.38a). The simulation for mean and amplitude of in-plane and out-of-plane ovalization for elbow4, elbow6 and elbow7 experiments are plotted and compared with experimental results in Fig. 6.38c-f. Chaboche and modified Chaboche models show improvement in simulating ovalization responses compare to the bilinear and multilinear model. The trend of shakedown in ratcheting observed with bilinear and multilinear plasticity model is improved in Chaboche and modified Chaboche model (compare Fig. 6.34 and Fig. 6.38). The Chaboche and modified Chaboche models overpredicted the stabilized ovalization ratcheting rates (Fig.6.38c and 6.38d). The amplitude of in-plane and out-of-plane ovalization simulations matched well with experiments and showed similar type of prediction observed for bilinear and multilinear plasticity model (Fig.6.38e and 6.38f).
The mean and amplitude of axial and circumferential strain cycles at flanks, intrados, extrados and midway between extrados-flank are plotted against the number of cycles in Fig. 6.39. Simulations of these responses with Chaboche and modified Chaboche model improved significantly compared to bilinear and multilinear models (Fig.6.39). Amplitudes of axial and circumferential strains matched well with simulations from both plasticity models (Fig. 6.39). However, the simulations show negative ratcheting rates for axial strains at intrados (Fig.6.39i). The circumferential strain ratcheting simulations are overpredicted by these two plasticity models (Fig. 6.39j). Amplitude of axial and circumferential strain cycles at intrados matched well (Fig.6.39k and 6.39l). The strain ratcheting simulations at extrados are not satisfactory (Fig.6.39m to 6.39n) as negative ratcheting rates are (opposite to experimental responses) are predicted. Although the amplitude of axial and circumferential strain cycles matched well with experiments, the ratcheting rates show overprediction for strain ratcheting simulations at midway between extrados and flank (Fig.6.39q to 6.39t).

In general, the modified Chaboche model shows less overprediction than Chaboche model for all three experiments. This improved performance of these models over bilinear and multilinear models might be related to improved multiaxial ratcheting simulation by earlier two models. The trend of shakedown phenomena observed in bilinear and multilinear model is improved in Chaboche and modified Chaboche model (shown in Fig.6.36b and 6.36c). The amplitudes of ovalization and strain cycle simulations with Chaboche and modified Chaboche model are similar as observed in bilinear and multilinear plasticity models. Hence, in the following, the evaluation of finite element simulations with other plasticity models performed with respect to the ratcheting responses only.
Fig. 6.38 Ovalization ratcheting simulation with Chaboche and modified Chaboche plasticity model for elbow4, elbow6 and elbow7 experiments; (a) mean of reaction forces (b) amplitude of reaction forces; (c) Mean of in-plane and (d) mean of out-of-plane (e) amplitude of in-plane and (f) amplitude of out-of-plane ovalization ratcheting simulations.
Fig. 6.39 Mean and amplitude of axial and circumferential strain ratcheting simulation with Chaboche and modified Chaboche plasticity model for \textit{elbow}4, \textit{elbow}6 and \textit{elbow}7 experiments at flanks.
Fig. 6.39 (continued) Mean and amplitude of axial and circumferential strain ratcheting simulation with Chaboche and modified Chaboche plasticity model for elbow4, elbow6 and elbow7 experiments at intrados and extrados.
6.5.3 Ohno-Wang and modified Ohno-Wang model

The parameters for the Ohno-Wang plasticity model (1993) are determined from composed hysteresis curve and uniaxial ratcheting experiments using the optimization tool developed in Chapter Two. The parameters for the modified Ohno-Wang model (Chen and Jiao, 2003) are determined from uniaxial hysteresis loop, uniaxial and biaxial ratcheting responses. The material simulations of material responses used for parameter determination with Ohno-Wang and modified Ohno-Wang model, using five kinematic hardening rules are shown in Fig. 6.40.

The parameters of Ohno-Wang model are

\[ \sigma_0 = 179.0 \text{ MPa}, \quad E = 193740 \text{ MPa}, \quad \nu = 0.302 \]

\[ C_{1-5} = 296020, 40914, 1000, 738, 3890 \]

\[ \gamma_{1-5} = 3870, 986, 518, 187, 35 \]

\[ m_{1-5} = 0.304. \]

The parameters for modified Ohno-Wang model are

\[ \sigma_0 = 179.0 \text{ MPa}, \quad E = 193740 \text{ MPa}, \quad \nu = 0.302 \]

\[ C_{1-5} = 271272, 45430, 903, 641, 4536 \]

\[ \gamma_{1-5} = 3563, 1137, 493, 268, 8.2 \]

\[ m_{1-5} = 0.34, \quad \delta_{0}^{'} = 0.71, \quad \delta_{u}^{'} = 1.00, \quad \beta = 0.37 \]
Fig. 6.40 Model parameter determination and material response simulations with Ohno-Wang and modified Ohno-Wang models (a) hysteresis loop representation and (b) uniaxial ratcheting simulation and (c) biaxial ratcheting simulation.

Fig. 6.41 Load-displacement response simulation with Ohno-Wang and modified Ohno-Wang plasticity models for (a) elbow4 (b) elbow6 and (c) elbow7 experiments.
The simulation for first loop, force, ovalization and strains are shown and compared with the experimental responses in Fig. 6.41, 6.42 and 6.43 respectively. In general, Ohno-Wang and modified Ohno-Wang models show similar performance in simulation compare to Chaboche and modified Chaboche model. However, Ohno-Wang and modified Ohno-Wang simulate the both in plane and out-of-plane ovalization ratcheting rates more closely than the modified Chaboche model (compare Fig. 6.38 to 6.42). However, negative ratcheting (opposite to the experimental responses) is also simulated at intrados and extrados by both Ohno-Wang and modified Ohno-Wang models similar to the modified Chaboche model (see Figs.6.39 and 6.43).
Fig. 6.43 Mean of axial and circumferential strain ratcheting simulation with Ohno-Wang and modified Ohno-Wang plasticity model for elbow4, elbow6 and elbow7 experiments at flanks, intrados, midway between extrados-flank and extrados.
6.5.4 Abdel Karim-Ohno model

Abdel Karim-Ohno (2000) model parameters are determined from uniaxial hysteresis loop and either uniaxial or biaxial ratcheting responses using the optimization tool developed in Chapter Two. It is found that the parameters determined from uniaxial ratcheting responses can simulate the biaxial ratcheting response similar to Ohno-Wang model (see Fig.6.44), therefore, only one set of ratcheting parameters are obtained for elbow response simulations. The parameters determined for Abdel Karim-Ohno model are

\[ \sigma_0 = 179.0 \text{ MPa}, \quad E = 193740 \text{ MPa}, \quad \nu = 0.302 \]

\[ C_{1-5} = 277650, \ 39430, \ 951, \ 689, \ 4385 \text{ MPa} \]

\[ \gamma_{1-5} = 3508, \ 1067, \ 554, \ 190, \ 0.9 \]

\[ \mu_{1-5} = 0.71 \]

Fig. 6.44 Model parameter determination and material response simulations with Ohno-Wang and Abdel Karim-Ohno models (a) hysteresis loop representation and (b) uniaxial ratcheting simulation and (c) biaxial ratcheting simulation.
Fig. 6.45 Load-displacement response simulation with Ohno-Wang and Abdel Karim-Ohno plasticity models for (a) elbow4, (b) elbow6, and (b) elbow7 experiments.

For comparison, the simulations with Abdel Karim-Ohno model are compared with Ohno-Wang simulations and experimental responses in Fig. 6.45, 6.46 and 6.47. Abdel Karim-Ohno and Ohno-Wang simulations for all the responses compared in these figures are similar, with Ohno-Wang performed better in most cases. The deficiency of simulating the negative ratcheting (opposite to experimental response) at intrados and extrados is shown by the Abdel Karim-Ohno model too.
Fig. 6.46 Ovalization ratcheting simulation with Ohno-Wang and Abdel Karim-Ohno plasticity model for elbow4, elbow6 and elbow7 experiments; (a) mean of reaction force (b) amplitude of reaction force (c) Mean of in-plane and (d) mean of out-of-plane (e) amplitude of in-plane and (f) amplitude of out-of-plane ovalization ratcheting simulations.
Fig. 6.47 Mean of axial and circumferential strain ratcheting simulation with Ohno-Wang and Abdel Karim-Ohno plasticity model for elbow4, elbow6 and elbow7 experiments at flanks, intrados, midway between extrados-flank and extrados.
6.6 **Comparison of solution time**

Solution time required for simulating a given problem with bilinear, multilinear and Chaboche models available in ANSYS9.0 are compared. As the other plasticity models like modified Chaboche (2002), Ohno-Wang (1993), modified Ohno-Wang (2003) and Abdel Karim-Ohno (2000) model are incorporated in ANSYS9.0 as external subroutine, those usually takes extra time in invoking of external routine, hence solution time of these advanced models is not compared here. The solution time compared by the three models considered was for simulating *elbow7* experiment upto 475 cycles, where each loading and unloading displacements were prescribed with 25 steps. The comparison is made in a WINDOWS 32.0-BIT machine with Intel Pentium 4 CPU, 2.8 GHz Processor, 1024 MB RAM and the solution time is given in Table 6.5.

<table>
<thead>
<tr>
<th>Plasticity model type</th>
<th>Solution Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bilinear model</td>
<td>21892</td>
</tr>
<tr>
<td>Multilinear Model</td>
<td>32066</td>
</tr>
<tr>
<td>Chaboche Model</td>
<td>56411</td>
</tr>
</tbody>
</table>

6.7 **Conclusion**

This study presented the elbow responses of three monotonic and seven cyclic experiments. Comparison of the *elbow4*, *elbow6* and *elbow7* experimental responses demonstrate that increase of internal pressure increase the ratcheting rate of ovalization and strain but the amplitude of ovalization and strain cycles remains unchanged with increase in internal pressure. The amplitude of ovalization and strain cycles enlarges with increase in amplitude of prescribed displacement when *elbow4* and *elbow5* results are compared. With the increase in prescribed displacement, mean ovalization and strain ratcheting rates are also increased. The bilinear and multilinear model cannot simulate ratcheting rate reasonable and also show shakedown after a number of cycles. The nonlinear models like Chaboche (1986), modified Chaboche (Bari and Hassan, 2002), Ohno-Wang (1993), modified Ohno-Wang (Chen and Jiao, 2003) and Abdel Karim-Ohno (2000) can simulate ratcheting rates of ovalization and
strain at flank reasonably, with small overprediction. A major deficiency of all the models studied is that they all simulate negative ratcheting at intrados or extrados or at both locations, when experimental ratcheting occurred in the positive direction. So, the current state of elbow ratcheting simulation is not satisfactory. So the improvement of the ratcheting parameters from stress-strain history at critical locations as shown in straight pipe ratcheting simulation in Chapter Five might show better performance in ratcheting simulation.
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APPENDICES
APPENDIX A
NUMERICAL SCHEMES FOR PLASTICITY MODELS

A.1 Numerical schemes

The radial return algorithm presented in Section 3.2.2 is for modified Chaboche plasticity model. This numerical approach can be used with Ohno-Wang (1993), Abdel Karim-Ohno (2000) and modified Ohno-Wang (Chen and Jiao, 2003) plasticity models too. However, the modified Chaboche plasticity model is different from Ohno-Wang, modified Ohno-Wang and Abdel-Karim-Ohno plasticity models is the kinematic hardening rule only. The implementation technique presented in Chapter three is still valid for these three advanced models. The necessary formulations for Ohno-Wang, modified Ohno-Wang and Abdel-Karim-Ohno plasticity model is presented in subsequent sections.

A.1.1 Ohno-Wang plasticity model

This Ohno-Wang (1993) plasticity model uses the kinematic hardening rule as

\[
\begin{align*}
\dot{\alpha}_i &= \frac{2}{3} C_i d\varepsilon^p - \gamma_i a_i \left\{ d\varepsilon^p : \frac{a_i}{f(\alpha_i)} \right\} \left( \frac{f(\alpha_i)}{C_i/\gamma_i} \right)^m \tag{A.1a}
\end{align*}
\]

where, \( C_i \), \( C_M \), \( \gamma_i \), \( \gamma_M \) and \( m \) are model parameters.

Equation (A.1a) can be rewritten as

\[
\begin{align*}
\dot{\alpha}_i &= \frac{2}{3} C_i d\varepsilon^p - \gamma_i a_i \left\{ \frac{3}{2} \frac{S_{n+1} - a_{n+1}}{\sigma_0} : \frac{a_i}{f(\alpha_i)} \right\} \left( \frac{f(\alpha_i)}{C_i/\gamma_i} \right)^m \tag{A.1b} d\sigma_{n+1}
\end{align*}
\]

where, \( C_i \), \( C_M \), \( \gamma_i \), \( \gamma_M \) and \( m \) are model parameters.

The equation (3.48) shown in Chapter Three is still valid. When the kinematic hardening rule Ohno-Wang kinematic hardening rule is used, Eq 3.49 is formed with Eq.A1.b and Eq.3.48 and as follows for \( \phi = 1.0 \)
\[ s_{n+1} - a_{n+1} = s_n^T - 2G \sum_{i=1}^M a_{ni} - \sum_{i=1}^M C_i \dot{e}_{n+1}^i + \sum_{i=1}^M c_i a_i \left( \frac{3( s_{n+1} - a_{n+1})}{2 \sigma_0} : \frac{a_i}{f(\alpha_i)} \right) \left( \frac{f(\alpha_i)}{C_i / \gamma_i} \right)^w dp_{n+1} \]  

(A.2)

Rearranging the terms we can get

\[ s_{n+1} - a_{n+1} = s_n^T - \sum_{i=1}^M a_{ni} - (3G + \sum_{i=1}^M C_i) \frac{dp_{n+1}}{\sigma_0} ( s_{n+1} - a_{n+1} ) + \sum_{i=1}^M c_i a_i \left( \frac{3( s_{n+1} - a_{n+1})}{2 \sigma_0} : \frac{a_i}{f(\alpha_i)} \right) \left( \frac{f(\alpha_i)}{C_i / \gamma_i} \right)^w dp_{n+1} \]  

(A.3)

The equation (A.3) becomes with Ohno-Wang kinematic hardening rule

\[ q_{n+1} = \left( \frac{s_{n+1} - a_n}{2} + \sum_{i=1}^M c_i a_i \right) \left( \frac{f(\alpha_i)}{C_i / \gamma_i} \right)^w dp_{n+1} \left( \sigma_0 + (3G + \sum_{i=1}^M C_i) dp_{n+1} \right) \]  

(A.4)

Putting the values of \( q_{n+1} \) in Eq. (3.36)

\[ \frac{3}{2} \left( \frac{s_{n+1} - a_n}{2} + \sum_{i=1}^M c_i a_i \right) \left( \frac{f(\alpha_i)}{C_i / \gamma_i} \right)^w dp_{n+1} \left( \sigma_0 + (3G + \sum_{i=1}^M C_i) dp_{n+1} \right) = \sigma_0 + (3G + \sum_{i=1}^M C_i) dp_{n+1} \]  

(A.5)

Equation (A.5) is quadratic equation of \( dp_{n+1} \) and the final solution can be obtained as shown in section 3.2.2 in Chapter Three.

### A.1.2 Abdel Karim-Ohno plasticity model

The Abdel Karim-Ohno (2000) kinematic hardening rule is

\[ d a_i = \sum_{i=1}^M d a_i \]  

(A.6)

\[ d a_i = \frac{2}{3} C_i d \varepsilon_{n+1}^i - \mu \gamma_i a_i dp_{n+1} - \gamma_i a_i H \left( \frac{3}{2} a_i : a_i - \frac{C_i}{\gamma_i} \right)^2 \]  

(A.7a)

where, \( d \lambda_i = d \varepsilon_{n+1}^i : \frac{a_i}{C_i / \gamma_i} - \mu dp_{n+1} \)  

(A.7b)

and \( C_i - C_M, \gamma_i - \gamma_M \) and \( \mu \) are model parameters.

Equation (A.7b) becomes

\[ d \lambda_i = \left( \frac{3}{2} \frac{(s_{n+1} - a_{n+1})}{\sigma_0} : \frac{a_i}{C_i / \gamma_i} - \mu \right) dp_{n+1} = d \lambda_i dp_{n+1} \]  

(A.7c)
Equation (3.48) is still valid. Combining equation Eq. A.7b and Eq.3.48 and as follows for \( \phi = 1.0 \)

\[
\sum_{i=1}^{M} \gamma_i a_i dp_{n+1} + \sum_{j=1}^{M} \gamma_j a_j + \sum_{i=1}^{M} \gamma_i a_i H \left[ \frac{3}{2} a_i : a_i - \left( \frac{C_i}{\gamma_i} \right)^2 \right] \left\{ d\gamma_i \right\} \]  
\( (A.8) \)

Rearranging the terms

\[
\sum_{i=1}^{M} \gamma_i a_i dp_{n+1} + \sum_{j=1}^{M} \gamma_j a_j + \sum_{i=1}^{M} \gamma_i a_i H \left[ \frac{3}{2} a_i : a_i - \left( \frac{C_i}{\gamma_i} \right)^2 \right] \left\{ d\gamma_i \right\} \]  
\( (A.9) \)

The equation (A.9) becomes

\[
q_{\gamma_{n+1}} = \left\{ \sum_{i=1}^{M} \gamma_i a_i dp_{n+1} + \sum_{j=1}^{M} \gamma_j a_j + \sum_{i=1}^{M} \gamma_i a_i H \left[ \frac{3}{2} a_i : a_i - \left( \frac{C_i}{\gamma_i} \right)^2 \right] \left\{ d\gamma_i \right\} \right\} \sigma_0 / (\sigma_0 + (3G + \sum_{i=1}^{M} C_d dp_{n+1}))  
\( (A.10) \)

Putting the values of \( q_{\gamma_{n+1}} \) in Eq. (3.36)

\[
\frac{3}{2} \left\{ \sum_{i=1}^{M} \gamma_i a_i dp_{n+1} + \sum_{j=1}^{M} \gamma_j a_j + \sum_{i=1}^{M} \gamma_i a_i H \left[ \frac{3}{2} a_i : a_i - \left( \frac{C_i}{\gamma_i} \right)^2 \right] \left\{ d\gamma_i \right\} \right\} \left\{ \sum_{i=1}^{M} \gamma_i a_i dp_{n+1} + \sum_{j=1}^{M} \gamma_j a_j + \sum_{i=1}^{M} \gamma_i a_i H \left[ \frac{3}{2} a_i : a_i - \left( \frac{C_i}{\gamma_i} \right)^2 \right] \left\{ d\gamma_i \right\} \right\} \]  
\( (A.11) \)

Equation A.11 is a quadratic equation of \( dp_{n+1} \) similar to Eq.A.5 and the final solution can be obtained as shown in section 3.2.2 in Chapter Three.

**A.1.3 Modified Ohno-Wang plasticity model**

Chen and Jiao (2003) modified the Ohno-Wang kinematic hardening rule as

\[
d\alpha = \sum_{i=1}^{M} d\alpha_i \]  
\( (A.12) \)

\[
d\alpha_i = \frac{2}{3} C_i d\varepsilon^r - \gamma_i (\delta^* a_i + (1-\delta^*) (a_i : n_i) n_i) \left( d\varepsilon^r : \frac{a_i}{f(\alpha_i)} \right) \left( \frac{f(\alpha_i)}{C_i / \gamma_i} \right)^m \]  
\( (A.12a) \)

where \( d\delta^* = \beta (\delta^*_{sat} - \delta^*) dp \) and \( \delta^*_{0} \) is the starting value of \( \delta^* \).

Equation (A.12a) becomes
\[ da_i = \frac{2}{3} C_i d \varepsilon^p - \gamma_i \left( \delta a_i + (1-\delta)(a_i : n_i)n_i \right) \left[ \frac{3}{2} \frac{s_{n+1} - a_{n+1}}{\sigma_0} \cdot \frac{a_i}{f(\alpha_i)} \right] \left( \frac{f(\alpha_i)}{C_i / \gamma_i} \right)^m dp_{n+1} \]  
(A.12c)

where, \( C_i, C_M, \gamma_i, \gamma_M \) and \( m \) are model parameters.

Again, Equation 3.48 is still valid. Combining equation Eq. A.12 and Eq. 3.48 and as follows for \( \phi = 1.0 \)

\[ s_{n+1} - a_{n+1} = s_{n+1}^T - 2Gd \varepsilon^p - \sum_{i=1}^{M} a_{n,j} - \sum_{i=1}^{M} \frac{2}{3} C_i d \varepsilon^p 
\]

\[ + \sum_{i=1}^{M} \gamma_i \left( \delta a_i + (1-\delta)(a_i : n_i)n_i \right) \left[ \frac{3}{2} \frac{s_{n+1} - a_{n+1}}{\sigma_0} \cdot \frac{a_i}{f(\alpha_i)} \right] \left( \frac{f(\alpha_i)}{C_i / \gamma_i} \right)^m dp_{n+1} \]  
(A.13)

Rearranging the terms we can get

\[ s_{n+1} - a_{n+1} = s_{n+1}^T - \sum_{i=1}^{M} a_{n,j} - (3G + \sum_{i=1}^{M} C_i) \frac{dp_{n+1}}{\sigma_0} (s_{n+1} - a_{n+1}) 
\]

\[ + \sum_{i=1}^{M} \gamma_i \left( \delta a_i + (1-\delta)(a_i : n_i)n_i \right) \left[ \frac{3}{2} \frac{s_{n+1} - a_{n+1}}{\sigma_0} \cdot \frac{a_i}{f(\alpha_i)} \right] \left( \frac{f(\alpha_i)}{C_i / \gamma_i} \right)^m dp_{n+1} \]  
(A.14)

The Eq. 3.3 becomes with Ohno-Wang kinematic hardening rule

\[ q_{n+1} = \left( s_{n+1}^T - a_n + \sum_{i=1}^{M} \gamma_i \left( \delta a_i + (1-\delta)(a_i : n_i)n_i \right) \left[ \frac{3}{2} \frac{s_{n+1} - a_{n+1}}{\sigma_0} \cdot \frac{a_i}{f(\alpha_i)} \right] \left( \frac{f(\alpha_i)}{C_i / \gamma_i} \right)^m dp_{n+1} \right) \left( \sigma_0 / (\sigma_0 + G + \sum_{i=1}^{M} C_i) dp_{n+1} \right) \]  
(A.15)

Putting the values of \( q_{n+1} \) in Eq. (3.36)

\[ \left( \frac{3}{2} \right) \left( s_{n+1}^T - a_n + \sum_{i=1}^{M} \gamma_i \left( \delta a_i + (1-\delta)(a_i : n_i)n_i \right) \left[ \frac{3}{2} \frac{s_{n+1} - a_{n+1}}{\sigma_0} \cdot \frac{a_i}{f(\alpha_i)} \right] \left( \frac{f(\alpha_i)}{C_i / \gamma_i} \right)^m dp_{n+1} \right) \]:

\[ \left( s_{n+1}^T - a_n + \sum_{i=1}^{M} \gamma_i \left( \delta a_i + (1-\delta)(a_i : n_i)n_i \right) \left[ \frac{3}{2} \frac{s_{n+1} - a_{n+1}}{\sigma_0} \cdot \frac{a_i}{f(\alpha_i)} \right] \left( \frac{f(\alpha_i)}{C_i / \gamma_i} \right)^m dp_{n+1} \right) = \left( \sigma_0 + (3G + \sum_{i=1}^{M} C_i) dp_{n+1} \right) \]  
(A.16)

Equation A.16 is quadratic equation of \( dp_{n+1} \) and the final solution can be obtained as shown in section 3.2.2 in Chapter Three.
APPENDIX B
CONSISTENT TANGENT MODULUS FOR PLASTICITY MODELS

B.1 Consistent tangent modulus
The Consistent tangent modulus for the Ohno-Wang (1993), Abdel-Karim-Ohno (2000) and modified Ohno-Wang (Chen and Jiao, 2003) plasticity models can be determined similar to modified Chaboche model (Bari and Hassan, 2002). The final expression for the consistent tangent modulus in Eq.4.17 is still valid with change in $L_{n+1}$.  

B.1.1 Consistent tangent modulus for Ohno-Wang model
The consistent tangent modulus for the Ohno-Wang (1993) plasticity model can be expressed as

$$\frac{d\sigma_{n+1}}{d\varepsilon_{n+1}} = D^e - 4G^2 (L_{n+1} : I_d)$$

where

$$L_{n+1} = \left[ \frac{2\sigma_0}{3\Delta p_{n+1}} + \frac{2}{3} \sum_{i=1}^{M} C_i + 2G \right] I - \frac{2\sigma_0}{3\Delta p_{n+1}} (n_{n+1} \otimes n_{n+1}) - \sum_{i=1}^{M} \frac{2}{3} \gamma_i \langle d\lambda_i \rangle \left( \frac{f(\alpha_i)}{C_i} \right)^w (a_i \otimes n_{n+1})$$

B.1.2 Consistent tangent modulus for Abdel Karim-Ohno model
Consistent tangent modulus for the Abdel Karim-Ohno (2000) plasticity model can be expressed similarly

$$\frac{d\sigma_{n+1}}{d\varepsilon_{n+1}} = D^e - 4G^2 (L_{n+1} : I_d)$$

where

$$L_{n+1} = \left[ \frac{2\sigma_0}{3\Delta p_{n+1}} + \frac{2}{3} \sum_{i=1}^{M} C_i + 2G \right] I - \frac{2\sigma_0}{3\Delta p_{n+1}} (n_{n+1} \otimes n_{n+1}) - \sum_{i=1}^{M} \frac{2}{3} \gamma_i \langle d\lambda_i \rangle \left( \mu_i \Delta p_{n+1} n_{n+1} + H_i \langle d\lambda_i \rangle (a_{n+1} \otimes n_{n+1}) \right)$$
### B.1.3 Consistent tangent modulus for modified Ohno-Wang model

Consistent tangent modulus for the modified Ohno-Wang (2003) plasticity model can be expressed similarly

\[
\frac{d\sigma_{n+1}}{d\varepsilon_{n+1}} = D^e - 4G^2 (L_{n+1} : I_d) \tag{B.2}
\]

where

\[
L_{n+1} = \left[ \frac{2\sigma_0}{3\lambda_p} + \frac{2}{3} \sum_{i=1}^{M} C_i + 2G \right] - \frac{2\lambda_p}{3\lambda_p} \left( n_i + \sum_{i=1}^{M} q_{a_{0i}} \otimes n_{a_{0i}} \right) - \sum_{i=1}^{M} \gamma_i \left( \delta^i + (1-\delta^i)q_{a_{0i}} \otimes q_{a_{0i}} \right) q_{a_{0i}} \otimes n_{a_{0i}} \left( \frac{f(a_i)}{C_i} \right)^n
\]
C.1 Elbow specimen dimensions

The elbow measurements are made using the following two keys as a reference. Letters A-K represent location along the elbows’ length, W-Z are dimensions in mm, and numbers 1-12 represent locations around the elbows’ circumference. In Fig C.2, the 1 position corresponds to the extrados, 2 the intrados, and 3 and 4 are the flank positions:

Fig. C.1 – Elbow measurement diagram

Fig C.2 – Elbow measurement diagram

The following tables list wall thickness at the locations specified, and the diameter across the intrados-extrados, and flank-flank.
### Table C.1
Monotonic opening test - Specimen *Elbow1*

<table>
<thead>
<tr>
<th>Wall thickness (mm)</th>
<th>Diameters (mm)</th>
<th>Diameters (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1  2  3  4  5  6  7  8  9  10  11  12</td>
<td>D$_{1,2}$</td>
<td>D$_{3,4}$</td>
</tr>
<tr>
<td>A</td>
<td>2.59  2.82  2.72  2.69</td>
<td>60.58</td>
</tr>
<tr>
<td>B</td>
<td>2.54  2.87  2.72  2.72</td>
<td>60.07</td>
</tr>
<tr>
<td>C</td>
<td>2.67  2.84  2.77  2.72</td>
<td>60.04</td>
</tr>
<tr>
<td>D</td>
<td>2.69  2.77  2.84  2.69</td>
<td>60.60</td>
</tr>
<tr>
<td>E</td>
<td>2.69  2.77  2.77  2.77</td>
<td>60.68</td>
</tr>
<tr>
<td>F</td>
<td>2.62  2.72  2.79  2.77</td>
<td>60.25</td>
</tr>
<tr>
<td>G</td>
<td>3.15  3.63  3.48  3.43</td>
<td>60.45</td>
</tr>
<tr>
<td>H</td>
<td>3.05  3.66  3.38  3.43</td>
<td>60.35</td>
</tr>
<tr>
<td>I</td>
<td>3.05  3.66  3.43  3.45</td>
<td>60.07</td>
</tr>
<tr>
<td>J</td>
<td>3.02  3.66  3.40  3.40</td>
<td>59.97</td>
</tr>
<tr>
<td>K</td>
<td>3.02  3.68  3.45  3.38</td>
<td>60.45</td>
</tr>
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</table>

### Table C.2
Monotonic closing test - Specimen *Elbow2*

<table>
<thead>
<tr>
<th>Wall thickness (mm)</th>
<th>Diameters (mm)</th>
<th>Diameters (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1  2  3  4  5  6  7  8  9  10  11  12</td>
<td>D$_{1,2}$</td>
<td>D$_{3,4}$</td>
</tr>
<tr>
<td>A</td>
<td>2.87  2.74  2.79  2.77</td>
<td>60.33</td>
</tr>
<tr>
<td>B</td>
<td>2.87  2.77  2.87  2.82</td>
<td>60.20</td>
</tr>
<tr>
<td>C</td>
<td>2.90  2.74  2.87  2.82</td>
<td>60.71</td>
</tr>
<tr>
<td>D</td>
<td>2.95  2.82  2.77  2.84</td>
<td>60.71</td>
</tr>
<tr>
<td>E</td>
<td>2.97  2.72  2.77  2.90</td>
<td>60.27</td>
</tr>
<tr>
<td>F</td>
<td>2.95  2.79  2.74  2.92</td>
<td>60.33</td>
</tr>
<tr>
<td>G</td>
<td>3.02  3.40  3.51  3.51</td>
<td>60.63</td>
</tr>
<tr>
<td>H</td>
<td>2.95  3.23  3.25  3.45</td>
<td>60.20</td>
</tr>
<tr>
<td>I</td>
<td>2.82  3.38  3.28  3.20</td>
<td>59.94</td>
</tr>
<tr>
<td>J</td>
<td>2.92  3.35  3.30  3.23</td>
<td>60.38</td>
</tr>
<tr>
<td>K</td>
<td>3.07  3.38  3.40  3.30</td>
<td>60.68</td>
</tr>
</tbody>
</table>

### Table C.3
Monotonic opening test - Specimen *Elbow3*

<table>
<thead>
<tr>
<th>Wall thickness (mm)</th>
<th>Diameters (mm)</th>
<th>Diameters (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1  2  3  4  5  6  7  8  9  10  11  12</td>
<td>D$_{1,2}$</td>
<td>D$_{3,4}$</td>
</tr>
<tr>
<td>A</td>
<td>2.92  2.59  2.69  2.84</td>
<td>60.40</td>
</tr>
<tr>
<td>B</td>
<td>2.95  2.64  2.74  2.87</td>
<td>60.30</td>
</tr>
<tr>
<td>C</td>
<td>2.92  2.67  2.72  2.82</td>
<td>60.35</td>
</tr>
<tr>
<td>D</td>
<td>2.69  2.87  2.72  2.92</td>
<td>60.30</td>
</tr>
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### Table C.9
Force-controlled cyclic test with internal pressure - Specimen Elbow9

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Table C.10
Force-controlled cyclic test with internal pressure - Specimen *Elbow10*

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C.2 Weld dimensions

The elbow weld measurements are made using the following Fig.C3 as a reference. Diameters with weld and without welds are measured and difference is calculated as the height of the weld as shown in Table C11.

![Elbow weld measurement diagram](image_url)

**Fig C.3** – Elbow weld measurement diagram
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