ABSTRACT

CAPALDI, MINDY BETH. Developing a New L_∞ Algebra Using Symmetric Brace Algebras. (Under the direction of Thomas Lada.)

Exploring the connections between L_∞ algebras and symmetric brace algebras is a relatively recent area of study. L_∞ algebras are the skew symmetric analog of A_∞ algebras. The latter structures allow for a transfer of associativity from a DG-algebra to a homotopic DG-module. Symmetric braces can greatly simplify the calculations involved in proving an L_∞ structure. We will investigate how to find a new L_∞ algebra structure when given an existing L_∞ structure and a collection of maps $\{f_i\}$ with relations that coincide with the definition of an L_∞ morphism, all within the context of symmetric brace algebra notation.

Developing a New L_∞ Algebra Using Symmetric Brace Algebras

by
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DEDICATION

This dissertation is dedicated to my wonderful husband, Alex Capaldi.

BIOGRAPHY

Mindy Beth Capaldi was born in Lexington, KY on May 30, 1984. She graduated high school in 1998 from Frankfort High School and went on to attend Georgetown College in Georgetown, KY. Mindy completed her undergraduate career with a Bachelor of Science degree in Mathematics, a Bachelor of Arts in History, and a minor in English. Clearly, she enjoyed school, so Mindy decided to continue on to graduate school at North Carolina State University where she would study mathematics. In 2008, after passing her preliminary written examinations in Topology, Linear/Lie Algebra, and Abstract Algebra, she received her Master of Science degree. Her interest in topology, in no small part due to the excellent teaching of Dr. Tom Lada, led to a focus on that subject that continues today.

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Chapter 1

Introduction

1.1 Background

T. Lada and M. Markl have explored some ideas of transferring homotopy structures. Consider a chain complex (V, d_V) equipped with an L_∞ structure and another chain complex (W, d_W) with chain maps $f : (V, d_V) \to (W, d_W), g : (W, d_W) \to (V, d_V)$ such that the composition gf is chain homotopic to the identity $I_V : V \to V$, via a chain-homotopy h. Symmetric braces may be used to transfer the homotopy structure on V to the space W [8]. In relation to other areas of mathematics and physics, R. Fulp, T. Lada, and J. Stasheff showed in [4] that data describing particles of high spin can be rewritten in the context of L_∞ algebra structures. T. Lada and M. Markl went on to demonstrate the more manageable interpretation of these results in terms of symmetric brace algebras.

1.2 L_∞ Algebras

Before defining L_∞ algebras it is helpful to explain their predecessor, A_∞ algebras. The latter were a construction by J. Stasheff in the 1960s. A_∞ algebras developed from trying to transfer associativity from a differential graded algebra, A, to a DG-module, B, that is homotopic to A. The algebra structure that B inherits from A will not in general be associative, but only homotopy associative up to higher dimensional homotopies.

An L_∞ algebra structure is the generalization of a graded Lie algebra in which the Jacobi expression is only homotopic to zero, rather than equal to zero, and this structure is the skew symmetric analog of an A_∞ algebra.

1.2.1 Definition of an L_∞ Algebra

An L_∞ structure on a graded vector space V is a collection of skew symmetric linear maps $l_n : V^{\otimes n} \to V$ of degree $n-2$ that satisfy the relations

$$\sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(v_{\sigma(1)}, ..., v_{\sigma(i)}) v_{\sigma(i+1)}, ..., v_{\sigma(n)}) = 0$$

where $\chi(\sigma)$ is the antisymmetric Koszul sign and σ is taken over all $(i, n-i)$ unshuffles [1].

This is the chain complex version, whereas the cochain complex version would require that the maps have degree $2-n$.

The antisymmetric Koszul sign is defined as $\chi(\sigma) = sgn(\sigma) \cdot \epsilon(\sigma; v_1, ..., v_n)$, or the sign of the permutation times the sign that arises from the degrees of the permuted elements.

Example: $(v_1, v_2) = -(-1)^{|v_1||v_2|}(v_2, v_1)$, where $|v_1|$ =degree of v_1 and $|v_2|$ = degree of v_2.

1.2.2 Examples For Small n

The following cases demonstrate the definition, including the signs involved, for $n = 1, 2, 3$.

$n = 1 : (-1)^{1(1-1)} l_1 l_1(v_1) = l_1 l_1(v_1) = 0$

This should automatically follow from the fact that l_1 is the differential in our understood chain complex.

$$n = 2 : ((-1)^{2(1-1)} l_1 l_2 + (-1)^{1(2-1)} l_2 l_1)(v_1 \otimes v_2)$$
$$= l_1 \left(l_2(v_1 \otimes v_2)\right) - l_2 \left(l_1(v_1) \otimes v_2 - (-1)^{|v_1||v_2|} l_1(v_2) \otimes v_1\right) = 0$$

$$n = 3 : ((-1)^{3(1-1)} l_1 l_3 + (-1)^{2(2-2)} l_2 l_2 + (-1)^{1(3-1)} l_3 l_1)(v_1 \otimes v_2 \otimes v_3)$$
$$= l_3 \left(l_1(v_1) \otimes v_2 \otimes v_3 - (-1)^{|v_1||v_2|} l_1(v_2) \otimes v_1 \otimes v_3 + (-1)^{|v_3|(|v_1|+|v_2|)} l_1(v_3) \otimes v_1 \otimes v_2\right)$$
$$+ l_2 \left(l_2(v_1 \otimes v_2) \otimes v_3 - (-1)^{|v_2||v_3|} l_2(v_1 \otimes v_3) \otimes v_2 + (-1)^{|v_1|(|v_2|+|v_3|)} l_2(v_2 \otimes v_3) \otimes v_1\right)$$
$$+ l_1(l_3(v_1 \otimes v_2 \otimes v_3) = 0$$

From the n=2 case, we see that l_2 is a chain map. Also, the line expanding $l_2 l_2$ from the n=3 case is equivalent to the Jacobi expression, which is therefore chain homotopic to zero via chain homotopy l_3.

It is useful to look at the L_∞ algebra definition without actually evaluating the maps over $(v_1, ..., v_n)$. In this case, we do not have to include the Koszul sign. The following is a table of this interpretation of the definition for small n.

1.3 Symmetric Brace Algebras

It is advantageous to analyze L_∞ algebra structures using symmetric brace algebras. Many signs and unshuffles can be absorbed into the symmetric brace notation. First,

Table 1.1: L_n Algebras

$n = 1$	$l_1 l_1 = 0$
$n = 2$	$l_1 l_2 - l_2 l_1 = 0$
$n = 3$	$l_1 l_3 + l_2 l_2 + l_3 l_1 = 0$
$n = 4$	$l_1 l_4 - l_2 l_3 + l_3 l_2 - l_4 l_1 = 0$
$n = 5$	$l_1 l_5 + l_2 l_4 + l_3 l_3 + l_4 l_2 + l_5 l_1 = 0$

let's look at the symmetric brace algebra definition and some small examples.

1.3.1 Definition of a Symmetric Brace Algebra

A symmetric brace algebra is a graded vector space B together with a collection of degree 0 multilinear braces $x \langle x_1, ..., x_n \rangle$ that are graded symmetric in $x_1, ..., x_n$ and satisfy the identities

$$x \langle \rangle = x$$

and

$$x \langle x_1, ..., x_m \rangle \langle y_1, ..., y_n \rangle =$$

$$\sum \epsilon \cdot x \left\langle x_1 \left\langle y_{i_1^1}, ..., y_{i_{t_1}^1} \right\rangle, x_2 \left\langle y_{i_1^2}, ..., y_{i_{t_2}^2} \right\rangle, ..., x_m \left\langle y_{i_1^m}, ..., y_{i_{t_m}^m} \right\rangle, y_{i_1^{m+1}}, ..., y_{i_{t_{m+1}}^{m+1}} \right\rangle$$

where the sum is taken over all unshuffle sequences

$$i_1^1 < ... < i_{t_1}^1, ..., i_1^{m+1} < ... < i_{t_{m+1}}^{m+1}$$

of $1, ..., n$ and where ϵ is the Koszul sign of the permutation

$$(x_1, ..., x_m, y_1, ..., y_n) \mapsto (x_1, y_{i_1^1}, ..., y_{i_{t_1}^1}, x_2, y_{i_1^2}, ..., y_{i_{t_2}^2}, ..., x_m, y_{i_1^m}, ..., y_{i_{t_m}^m}, y_{i_1^{m+1}}, ..., y_{i_{t_{m+1}}^{m+1}})$$

of elements of B [8].

1.3.2 Examples of the Relations

- $x \langle x_1 \rangle \langle y \rangle = x \langle x_1 \langle y \rangle \rangle + x \langle x_1, y \rangle$

- $x \langle x_1, x_2 \rangle \langle y \rangle = (-1)^{|x_2||y|} x \langle x_1 \langle y \rangle, x_2 \rangle + x \langle x_1, x_2 \langle y \rangle \rangle + x \langle x_1, x_2, y \rangle$

The order of x_1 and x_2 never changes.

- $x \langle x_1 \rangle \langle y_i, y_j \rangle = x \langle x_1 \langle y_i \rangle, y_j \rangle + (-1)^{|y_i||y_j|} x \langle x_1 \langle y_j \rangle, y_i \rangle + x \langle x_1 \langle y_i, y_j \rangle \rangle + x \langle x_1, y_i, y_j \rangle$

If $i = j$, then $x \langle x_1 \langle y_i \rangle, y_j \rangle = x \langle x_1 \langle y_j \rangle, y_i \rangle$ and there is a repeated term. This case ends up being rather important, because in our application we will not want both terms.

1.4 Objectives

Suppose we are given an L_∞ algebra structure on (V, d_V) and a collection of maps $\{f_i\} \in B_0(V)$ with relations that coincide with the definition of an L_∞ morphism. From these assumptions, we can achieve a new L_∞ algebra structure on (V, d_V). This process includes many calculations, a lot of induction, and numerous uses of combinatorics. In order to proceed, we must know some details about the signs involved (see Chapter 2). It will also be beneficial to understand the connections with L_∞ morphisms and the techniques of unshuffling used there (see Chapter 3). To reinforce comprehension of the main theorem, small general and concrete examples are included (see Chapter 6).

Chapter 2

Connecting L_∞ Algebras and Symmetric Braces

2.1 Fundamental Example

T. Lada and M. Markl provide a useful example of a symmetric brace algebra, as seen in [8]. It is given by the graded vector space composed of antisymmetric maps,

$$B_s(V) = \oplus_{p+k-1=s} Hom(V^{\otimes k}, V)_p^{as}$$

where V is a graded vector space and $Hom(V^{\otimes k}, V)_p^{as}$ denotes the space of k-multilinear maps of degree p that are antisymmetric in the sense that

$$f(v_1, ..., v_i, v_{i+1}, ..., v_k) = -(-1)^{|v_i||v_{i+1}|} f(v_1, ..., v_{i+1}, v_i, ..., v_k),$$

for any $v_1, ..., v_i, v_{i+1}, ..., v_k \in V$ and $1 \leq i \leq k-1$.

This is basically the antisymmetric Koszul sign at work. Keeping with the notation used in the respective definitions, the $\{l_n\}$ maps that compose an L_∞ algebra structure and the $\{f_i\}$ maps that make up a symmetric brace algebra both use the antisymmetric Koszul sign. The braces themselves only use the Koszul sign, however.

Within this symmetric brace algebra example there is also a definition in [8] of how to actually evaluate these braces on elements from $V^{\otimes k}$.

2.1.1 Definition of How to Evaluate Symmetric Braces

Given the graded vector space B_s, suppose we have a collection of maps $f \in Hom(V^{\otimes k}, V)_p^{as} \subset B_{p-k+1}(V)$ and $g_i \in Hom(V^{\otimes a_i}, V)_{q_i}^{as} \subset B_{q_i - a_i + 1}(V), 1 \leq i \leq n$. Define the symmetric brace $f\langle g_1, ..., g_n\rangle \in Hom(V^{\otimes r}, V)_{p+q_1+\cdots q_n}^{as}$,where $r := a_1 + ... + a_n + k - n$ by

$$f\langle g_1, ..., g_n\rangle (v_1, ..., v_r) := \sum (-1)^{\delta} \chi \cdot f(g_1 \otimes \cdots \otimes g_n \otimes 1^{\otimes k-n})(v_{i_1}, ..., v_{i_r})$$

with the summation taken over all unshuffles

$$i_1 < \cdots < i_{a_1}, i_{a_1+1} < \cdots < i_{a_1+a_2}, \cdots, i_{a_1+...+a_k+1} < \cdots < i_r$$

of elements of V, where χ is the antisymmetric Koszul sign of the permutation

$$(v_1, .., v_r) \rightarrow (v_{i_1}, ..., v_{i_r})$$

and

$$\delta = (k-1)q_1 + (k-2+a_1)q_2 + ... + (k-n+a_1+...+a_{n-1})q_n + \sum_{1 \leq i < j \leq n} a_i a_j + (n-1)a_1$$
$$+(n-2)a_2 + ... + a_{n-1}$$

Delta is needed to assure that we have skew-symmetry. This sign will be examined in more detail later. In evaluating symmetric braces, we lose some of the simplifying benefits of the notation, but this is what makes symmetric brace notation comparable to that of L_∞ algebras.

2.2 Application to L_∞ Algebras

An important and useful application of symmetric braces occurs in proving an L_∞ algebra structure ([8], Exercise 6). Suppose we are given maps $l_k \in Hom(V^{\otimes k}, V)^{as}_{k-2}$ and we let $l = l_1 + l_2 + \cdots$. Then an L_∞ algebra structure on V is given by the symmetric brace relation $l\langle l\rangle = 0$. This is a greatly simplified equation compared to the L_∞ algebra definition. Why does it work?

Example: Let $l = l_1 + l_2 + l_3$.

Then $l\langle l\rangle(x,y,z) = (l_1 + l_2 + l_3)\langle l_1 + l_2 + l_3\rangle(x,y,z)$

$= (l_1\langle l_1 + l_2 + l_3\rangle + l_2\langle l_1 + l_2 + l_3\rangle + l_3\langle l_1 + l_2 + l_3\rangle)(x,y,z)$

$= (l_1\langle l_1\rangle + l_1\langle l_2\rangle + l_1\langle l_3\rangle + l_2\langle l_1\rangle + l_2\langle l_2\rangle + l_2\langle l_3\rangle + l_3\langle l_1\rangle + l_3\langle l_2\rangle + l_3\langle l_3\rangle)(x,y,z)$

But $l_1^2 = 0$ since l_1 is a differential. Also, $l_2\langle l_3\rangle, l_3\langle l_2\rangle$, and $l_3\langle l_3\rangle$ are all 0 since they are acting on only the three terms (x,y,z). Two sets of brace combinations remain. One is $l_1\langle l_2\rangle + l_2\langle l_1\rangle = 0$, which is equivalent to L_2 (refer to Table 1.1) since

$$l_1\langle l_2\rangle + l_2\langle l_1\rangle = (-1)^{\delta} l_1 l_2 + (-1)^{\delta} l_2 l_1 = (-1)^{(1-1)0} l_1 l_2 + (-1)^{(2-1)(-1)} l_2 l_1 = l_1 l_2 - l_2 l_1$$

and similarly

$$l_1\langle l_3\rangle + l_2\langle l_2\rangle + l_3\langle l_1\rangle = (-1)^{(1-1)3} l_1 l_3 + (-1)^{(2-1)2} l_2 l_2 + (-1)^{(3-1)1} l_3 l_1 = l_1 l_3 + l_2 l_2 + l_3 l_1$$

which is equivalent to L_3.

2.3 L_∞ Morphisms

My goal is to get a new L_∞ algebra structure. To do so, I need to introduce a collection of maps $\{l'_n\}$. Eventually, I will show that these maps compose the new structure.

2.3.1 Definition of an L_∞ Morphism

For $f_i \in Hom(V^{\otimes i}, V)^{as}_{i-1}$, $l_i \in Hom(V^{\otimes i}, V)^{as}_{i-2}$, $l'_i \in Hom(V^{\otimes i}, V)^{as}_{i-2}$ and $(v_1, ..., v_n) \in V^{\otimes n}$, the collection $\{f_i\}_{i\geq 1}$ is an L_∞ morphism if

$$\sum_{i+j=n+1} (-1)^{j(i-1)} \sum_{\sigma \in S_{j,i-1}} \chi(\sigma) f_i(l_j(v_{\sigma(1)}, \ldots, v_{\sigma(j)}), v_{\sigma(j+1)}, \ldots, v_{\sigma(n)}) =$$

$$\sum_{j=1}^{n} \sum_{\substack{r_1+\cdots+r_j=n \\ r_1\leq\cdots\leq r_j}} (-1)^{\frac{j(j-1)}{2}+\sum_{i=1}^{j-1} r_i(j-i)}$$

$$\sum_{\sigma \in S^{<}_{r_1,\ldots,r_j}} \chi(\sigma)\beta(\sigma) l'_j(f_{r_1}(v_{\sigma(1)}, \ldots, v_{\sigma(r_1)}), \cdots, f_{r_j}(v_{\sigma(n-r_j+1)}, \ldots, v_{\sigma(n)}))$$

where $S_{j,i-1}$ denotes the set of $j, i-1$ unshuffles and $S^{<}_{r_1,\ldots,r_j}$ the set of $r_1, ..., r_j$-unshuffles satisfying $\sigma(r_1 + \cdots + r_{i-1} + 1) < \sigma(r_1 + \cdots + r_i + 1)$ if $r_i = r_{i+1}$. Also, $\chi(\sigma)$ is the antisymmetric Koszul sign for the permutation σ and $\beta(\sigma)$ is the sign from permuting "v"s and "f"s.
For example, $\beta(\sigma) l'_2 \langle f_1, f_3\rangle (v_1, v_2, v_3, v_4) = (-1)^{|v_1||f_3|} l_2 \langle f_1(v_1), f_3(v_2, v_3, v_4)\rangle$.

This is similar to the definition given in [3] with a few changes since I have already evaluated the maps on the elements. Beta is a result of the alteration. Here the elements $v_{\sigma(i)}$ have already been unshuffled and put in place to be evaluated.

$S^{<}_{r_1,...,r_j}$ puts a condition on the unshuffles of $r_1, ...r_j$ that has not appeared in other definitions. To summarize the condition, if $\sigma(i)$ and $\sigma(j)$ denote the subscripts of the first terms in two sets of the unshuffle, $\sigma(i) < \sigma(j)$ indicates that after unshuffling this ordering of the subscripts is increasing.

Example: (2,2) unshuffles of 1234 with this restriction are (12)(34), (13),(24), and (14)(23), leaving out the unshuffle (34)(12) since $3 > 1$ and unshuffles (24)(13) and (23)(14) since $2 > 1$.

2.4 Delta

I want to return to the delta sign used in Definition 2.1.1 for evaluating symmetric braces.

$\delta = (k-1)q_1 + (k-2+a_1)q_2 + (k-3+a_1+a_2)q_3 + ... + (a_1+a_2+...+a_{n-1})q_n + \sum_{1\leq i<j\leq n} a_i a_j + (n-1)a_1 + (n-2)a_2 + (n-3)a_3 + ... + a_{n-1}$

This sign actually matches the signs given in the definition of an L_∞ morphism.

2.4.1 Lemma

If $l \in Hom(V^{\otimes k}, V)^{as}_p$, $f_i \in Hom(V^{\otimes a_i}, V)^{as}_{q_i}$, $1 \leq i \leq n$ and
$\delta = (k-1)q_1 + (k-2+a_1)q_2 + (k-3+a_1+a_2)q_3 + ... + (a_1+a_2+...+a_{n-1})q_n + \sum_{1\leq i<j\leq n} a_i a_j + (n-1)a_1 + (n-2)a_2 + (n-3)a_3 + ... + a_{n-1}$, then

$$(-1)^{\delta} l_n(f_{a_1}, ..., f_{a_n}) = (-1)^{\frac{n(n-1)}{2} + \sum_{i=1}^{n-1} a_i(n-i)} l_n(f_{a_1}, ..., f_{a_n})$$

and

$$(-1)^{\delta} f_i l_j = (-1)^{j(i-1)} f_i l_j$$

Proof: The proof will be split into two sections, one for each of above equations.

Part 1: Show $\delta = \frac{n(n-1)}{2} + \sum_{i=1}^{n-1} a_i(n-i)$

Part one of the proof is only for terms $l_n(f_{a_1}, ..., f_{a_n})$, which can be found in both Definition 2.1.1 and 2.3.1. For the morphisms involved we know that $k = n$ and $q_i = a_i - 1$. Plugging that information into the original formula for δ, we get

$$\begin{aligned}
\delta &= (n-1)(a_1-1) + (n-2+a_1)(a_2-1) + (n-3+a_1+a_2)(a_3-1) + ... \\
&\quad +(a_1+a_2+...+a_{n-1})(a_n-1) + \sum_{1\le i<j\le n} a_i a_j + (n-1)a_1 + (n-2)a_2 + (n-3)a_3 + \\
&\quad ... + a_{n-1} \\
&= (n-1)a_1 - (n-1) + (n-2)a_2 + a_1 a_2 - (n-2+a_1) + (n-3)a_3 + \\
&\quad (a_1+a_2)a_3 - (n-3+a_1+a_2) + ... + (a_1+a_2+...+a_{n-1})a_n \\
&\quad -(a_1+a_2+...+a_{n-1}) + \sum_{1\le i<j\le n} a_i a_j + (n-1)a_1 + (n-2)a_2 + (n-3)a_3 + ... + a_{n-1} \\
&= -(n-1) - (n-2+a_1) - (n-3+a_1+a_2) - ... - (a_1+a_2+...+a_{n-1}) \\
&\quad +[a_1 a_2 + (a_1+a_2)a_3 + ... + (a_1+a_2+...+a_{n-1})a_n] \\
&\quad + \sum_{1\le i<j\le n} a_i a_j + 2(n-1)a_1 + 2(n-2)a_2 + 2(n-3)a_3 + ... + 2a_{n-1}
\end{aligned}$$

All of the terms that are doubled can be ignored since they don't affect the sign change. Also, the middle terms that I bracketed above end up being the same as $\sum_{1\le i<j\le n} a_i a_j$, so that too is doubled and can be left off.

$$
\begin{aligned}
&= -(n-1)-(n-2+a_1)-(n-3+a_1+a_2)-...-(a_1+a_2+...+a_{n-1})+\sum_{1\le i<j\le n} a_i a_j \\
&\quad +\sum_{1\le i<j\le n} a_i a_j \\
&= -(n-1)-(n-2+a_1)-(n-3+a_1+a_2)-...-(a_1+a_2+...+a_{n-1}) \\
&= [-(n-1)-(n-2)-(n-3)-...-1]+[-a_1-(a_1+a_2)-...-(a_1+a_2+...+a_{n-1})] \\
&= -\frac{n(n-1)}{2}-[(n-1)a_1-(n-2)a_2-...-2a_{n-2}-a_{n-1}] \\
&= -\frac{n(n-1)}{2}-\sum_{i=1}^{n-1} a_i(n-i) \\
&= -(\frac{n(n-1)}{2}+\sum_{i=1}^{n-1} a_i(n-i))
\end{aligned}
$$

This final line is exactly what I needed since the negative doesn't matter.

Part 2: Show $\delta = j(i-1)$ for $(-1)^{\delta} f_i l_j$

For $f \in Hom(V^{\otimes k}, V)_{k-1}$ and $l \in Hom(V^{\otimes a_i}, V)_{a_i-2}$, the δ sign for $\sum f(l)$ can be greatly simplified from

$\delta = (k-1)q_1 + (k-2+a_1)q_2 + (k-3+a_1+a_2)q_3 + ... + (a_1+a_2+...+a_{n-1})q_n + \sum_{1\le i<j\le n} a_i a_j + (n-1)a_1 + (n-2)a_2 + (n-3)a_3 + ... + a_{n-1}$

to $\delta = (k-1)q_1$

This is due to the fact that there is only one l_j, which that reduces δ to $(k-1)q_1$ immediately. But $q_1 = a_1 - 2$, so $\delta = (k-1)(a_1-2) = ka_1 - 2k - a_1 + 2 = a_1(k-1)$. Since we are dealing with signs, we can leave off the terms that are doubled.

Now, if we use the common subscripts i and j, we have that $\delta = j(i-1)$ for $f_i \in Hom(V^{\otimes i}, V)_{i-1}$ and $l_j \in Hom(V^{\otimes j}, V)_{j-2}$. ■

Chapter 3

Developing a New L_∞ Algebra Structure

Suppose we are given two L_∞ algebra structures over the graded vector space V, say (V, l_n) and (V, l'_n). Then l_n and l'_n are in $B_{-1}(V)$. Now consider an L_∞ morphism $\{f_i\}$, which are a collection of skew symmetric maps $f_k : (V, l_n) \to (V, l'_n)$ of degree $k-1$ in $B_0(V)$. Then $f\langle l \rangle$ and $l' \langle f \rangle$ would both be in $B_{-1}(V)$, the same space that our original L_∞ algebra structures are in. This fact involving symmetric braces is useful. Suppose we don't know the maps of the second structure, $\{l'_n\}$?

What if we want to define l'_n in terms of f and l? It turns out that this is possible! We don't even have to begin with the assumption that the $\{f_i\}$ collection is a morphism, although that will follow.

Already, it has been shown that symmetric braces can help tremendously in L_∞ algebra proofs. Somehow, I need to convert the previous L_∞ morphism definition to one using symmetric braces. To do so, the intricacies and problems of unshuffling must be examined further.

3.1 Unshuffle Argument

3.1.1 Lemma

For $i_1 = \cdots = i_{t_1}$ and $k = i_1 + ... + i_{t_1}$, with the following summation over all $i_1, i_2, ..., i_{t_1}$-unshuffles ,

$$\sum \frac{1}{(t_1)!}(f_{i_1} \otimes \cdots \otimes f_{i_{t_1}})(v_{\sigma(1)}, ..., v_{\sigma(k)})$$

is equivalent to

$$\sum_{\sigma \in S^{<}_{i_1,...,i_{t_1}}} (f_{i_1} \otimes \cdots \otimes f_{i_{t_1}})(v_{\sigma(1)}, ..., v_{\sigma(k)})$$

where $S^{<}_{i_1,...,i_{t_1}}$ denotes the $i_1, ..., i_{t_1}$-unshuffles satisfying $\sigma(i_1+\cdots+i_{k-1}+1) < \sigma(i_1+\cdots+i_k+1)$ if $i_k = i_{k+1}$.

Proof: This proof and lemma do not include any sign changes for the unshuffles. We are assuming that $f_{i_1} = f_{i_2} = \cdots = f_{i_{t_1}}$. Start with $\sum(f_{i_1} \otimes \cdots \otimes f_{i_{t_1}})(v_{\sigma(1)}, ..., v_{\sigma(k)})$ over unrestricted unshuffles. Without loss of generality, look at the identity unshuffle

$$v_1 < \cdots < v_{i_1}, v_{i_1+1} < \cdots < v_{i_1+i_2}, ..., v_{k-i_{t_1}+1} < \cdots < v_k$$

Evaluate the unshuffle in this order to get

$$f_{i_1}(v_1, ..., v_{i_1}) \otimes f_{i_2}(v_{i_1+1}, ..., v_{i_1+i_2}) \otimes \cdots \otimes f_{i_{t_1}}(v_{k-i_{t_1}+1}, ..., v_k)$$

To complete all possible permutations of this identity unshuffle grouping, the set $v_1, ..., v_{i_1}$ moves to each f_{i_j}. Since $i_1 = \cdots = i_{t_1}, f_{i_1}(v_1, ..., v_{i_1}) = f_{i_2}(v_1, ..., v_{i_1}) = \cdots = f_{i_{t_1}}(v_1, ..., v_{i_1})$ and we end up with the equivalent of

$$t_1 f_{i_1}(v_1, ..., v_{i_1}) \otimes f_{i_2}(v_{i_1+1}, ..., v_{i_1+i_2}) \otimes \cdots \otimes f_{i_{t_1}}(v_{k-i_{t_1}+1}, ..., v_k)$$

We do not want to have multiples. There are two ways to deal with the multiple terms: divide by t_1 or hold the unshuffle group $v_1, ..., v_{i_1}$ fixed with f_{i_1} so that no other f map evaluates this group, which is what $S^<$ does.

The unshuffles $v_{i_1+1} < ... < v_{i_1+i_2}, \cdots, v_{k-i_{t_1}+1} < \cdots < v_k$ are still causing repeats. We can either fix the unshuffle group $v_{i_1+1}, ..., v_{i_1+i_2}$ to f_{i_2} or divide by $t_1 - 1$.

Continue this pattern until the unshuffle group $v_{k-i_{t_1}+1}, ..., v_k$ is fixed with $f_{i_{t_1}}$. Overall, we either did not restrict unshuffles but instead divided by $(t_1)!$. Or we dealt with the problem by the other method. Each unshuffle "group" already must be in increasing order. We can fix/arrange the overall order by making the first element of each unshuffle group less than the leading element of the following unshuffle group, with ranking based on the original order. This is the same as using the restricted symmetric group $S^<$ for σ above. ■

3.1.2 Application Example

Ignoring signs,

$$\begin{aligned}
\langle f_2, f_2 \rangle (x_1, x_2, x_3, x_4) &= (f_2 \otimes f_2)(x_1, x_2, x_3, x_4) \\
&= f_2(x_1, x_2) \otimes f_2(x_3, x_4) + f_2(x_1, x_3) \otimes f_2(x_2, x_4) \\
&\quad + f_2(x_1, x_4) \otimes f_2(x_2, x_3) + f_2(x_2, x_3) \otimes f_2(x_1, x_4) \\
&\quad + f_2(x_2, x_4) \otimes f_2(x_1, x_3) + f_2(x_3, x_4) \otimes f_2(x_1, x_2) \\
&= 2[f_2(x_1, x_2) \otimes f_2(x_3, x_4) + f_2(x_1, x_3) \otimes f_2(x_2, x_4) \\
&\quad + f_2(x_1, x_4) \otimes f_2(x_2, x_3)]
\end{aligned}$$

To avoid multiples, multiply by 1/2 (which is $\frac{1}{(t_1)!}$ in this case). In general, we have to multiply by $\frac{1}{n!}$ for n number of repeating f_i's. Or the $S^<$ method implies that

$$\begin{aligned} \langle f_2, f_2 \rangle (x_1, x_2, x_3) &= f_2(x_1, x_2) \otimes f_2(x_3, x_4) \\ &\quad + f_2(x_1, x_3) \otimes f_2(x_2, x_4) \\ &\quad + f_2(x_1, x_4) \otimes f_2(x_2, x_3) \end{aligned}$$

3.2 Symmetric Brace Version of Morphism

I will use the definition of an L_∞ morphism, but in terms of symmetric braces. So, I'm essentially setting up the relation $f \langle l \rangle = l' \langle f \rangle$. I actually want to define these new l' maps in terms of the known l's and f's, but that is a later goal.

3.2.1 Theorem

For $f_i \in Hom(V^{\otimes i}, V)^{as}_{i-1}$, $l_i \in Hom(V^{\otimes i}, V)^{as}_{i-2}$, $l'_i \in Hom(V^{\otimes i}, V)^{as}_{i-2}$ and $(v_1, ..., v_n) \in V^{\otimes n}$, $\{f_i\}_{i \geq 1}$ is an L_∞ morphism if

$$\sum_{i=1}^{n} f_i \langle l_{n-i+1} \rangle = \sum_{j=1}^{n} \sum_{\substack{i^1_1 = ... = i^1_{t_1} < i^2_1 = ... = i^2_{t_2} < i^m_1 = ... = i^m_{t_m} \\ i^1_1 + i^1_2 + ... + i^m_{t_m} = n \\ t_1 + t_2 + ... + t_m = j}} C l'_j \langle f_{i^1_1}, ..., f_{i^1_{t_1}}, f_{i^2_1}, ..., f_{i^2_{t_2}}, ..., f_{i^m_1}, .., f_{i^m_{t_m}} \rangle$$

where $C = \prod \frac{1}{(t_a)!}, a = 1, ..., m$

and the sum is over all $(i^1_1, i^1_2, ..., i^m_{t_m})$ unshuffles when evaluated on elements.

We need the coefficient C to keep from repeating terms during our unshuffling, which happens when there is more than one of the same f_i map, as shown in the last lemma.

Proof: Show that this definition of an L_∞- morphism is equivalent to Definition 2.3.1. First, show that the left sides are equivalent. Start with the left side the theorem stated above.

$$\sum_{i=1}^{n} f_i\langle l_{n-i+1}\rangle(v_1, ..., v_n) = \sum_{i=1}^{n}\left[(-1)^{\delta}\sum \chi f_i(l_{n-i+1})(v_{i_1}, ..., v_{i_n})\right]$$

where the second summation is over unshuffles $i_1 < \cdots < i_{n-i+1}, i_{n-i+2} < \cdots i_n$ (this could be described as $n-i+1, i-1$ unshuffles as well). This is true from Definition 2.1.1, where $r = n-i+1+i-1 = n$. Continuing:

$$= \sum_{i+j=n+1}(-1)^{j(i-1)}\sum_{\sigma\in S_{j,i-1}}\chi(\sigma)f_i\left(l_j(v_{\sigma(1)}, ..., v_{\sigma(j)}), v_{\sigma(j+1)}, ..., v_{\sigma(n)}\right)$$

where $S_{j,i-1}$ denotes the set of j,i-1 unshuffles and $\delta = j(i-1)$ as shown in Lemma 2.4.1. Now this is the same as the left side in Definition 2.3.1.

Now, start with the right side in this theorem, and show it is equivalent to the right side in Definition 2.3.1.

$$\sum_{j=1}^{n}\sum_{\substack{i_1^1=...=i_{t_1}^1<i_1^2=...=i_{t_2}^2<i_1^m=...=i_{t_m}^m\\ i_1^1+i_2^1+...+i_{t_m}^m=n\\ t_1+t_2+...+t_m=j}} Cl'_j\langle f_{i_1^1}, ..., f_{i_{t_1}^1}, f_{i_1^2}, ..., f_{i_{t_2}^2}, ..., f_{i_1^m}, .., f_{i_{t_m}^m}\rangle(v_1, ..., v_n)$$

$$= C\sum_{j=1}^{n}\sum_{\substack{i_1^1=...=i_{t_1}^1<i_1^2=...=i_{t_2}^2<i_1^m=...=i_{t_m}^m\\ i_1^1+i_2^1+...+i_{t_m}^m=n\\ t_1+t_2+...+t_m=j}}$$

$$\sum_{\sigma\in S_{i_1^1,i_2^1,...,i_{t_m}^m}}(-1)^{\delta}\chi l'_j\left(f_{i_1^1}\otimes\cdots\otimes f_{i_{t_1}^1}\otimes f_{i_1^2}\otimes\cdots\otimes f_{i_{t_2}^2}\otimes\cdots\otimes f_{i_1^m}\otimes\cdots\otimes f_{i_{t_m}^m}\right)(v_{\sigma(1)}, ..., v_{\sigma(n)})$$

where the last summation is over all $i_1^1, i_2^1, ..., i_{t_m}^m$ unshuffles. Now use Lemma 3.1.1 to show that the above equals:

$$= \sum_{j=1}^{n} \sum_{\substack{i_1^1 \le i_2^1 \le \cdots \le i_{t_m}^m \\ i_1^1 + i_2^1 + \ldots + i_{t_m}^m = n \\ t_1 + t_2 + \ldots + t_m = j}} (-1)^{\delta}$$

$$\sum_{\sigma \in S^<_{i_1^1, \ldots, i_{t_m}^m}} \chi(\sigma)\beta(\sigma) l'_j \left(f_{i_1^1}(v_{\sigma(1)}, \ldots, v_{\sigma(i_1^1)}) \otimes \cdots \otimes f_{i_{t_m}^m}(v_{\sigma(n - i_{t_m}^m + 1)}, \ldots, v_{\sigma(n)}) \right)$$

There is no longer an additional coefficient when f maps are equal so C is gone, but the necessary restriction on the unshuffles is there instead. The beta sign (same as in Definition 2.3.1) appeared from permuting some v elements over the f maps, and accounting for any sign changes that ensued. By Lemma 2.4.1, delta is the same as the sign in Definition 2.3.1, so now just rename the subscripts of the f's to get the rest:

$$= \sum_{j=1}^{n} \sum_{\substack{i_1 + \cdots + i_j = n \\ i_1 \le \cdots \le i_j}} (-1)^{\frac{j(j-1)}{2} + \sum_{l=1}^{j-1} i_l (j-l)}$$

$$\sum_{\sigma \in S^<_{i_1, \ldots, i_j}} \chi(\sigma)\beta(\sigma) l_j (f_{i_1}(v_{\sigma(1)}, \ldots, v_{\sigma(i_1)}), \cdots, f_{i_j}(v_{\sigma(n-i_j+1)}, \ldots, v_{\sigma(n)}))$$

where $S^<_{i_1, \ldots, i_j}$ denotes the set of $i_1, \ldots, i_j$-unshuffles satisfying $\sigma(i_1 + \cdots + i_{k-1} + 1) < \sigma(i_1 + \cdots + i_k + 1)$ if $i_k = i_{k+1}$. ■

Example: For $n = 4$, $f_1 \langle l_4 \rangle + f_2 \langle l_3 \rangle + f_3 \langle l_2 \rangle + f_4 \langle l_1 \rangle = l'_1 \langle f_4 \rangle + l'_2 \langle f_1, f_3 \rangle + \frac{1}{2!} l'_2 \langle f_2, f_2 \rangle + \frac{1}{2!} l'_3 \langle f_1, f_1, f_2 \rangle + \frac{1}{4!} l'_4 \langle f_1, f_1, f_1, f_1 \rangle$

3.3 More on Symmetric Braces

Special Case: Suppose we let f_1=identity. Then when we are looking at a term like $\frac{1}{n!}l'_n \langle f_1, f_1, ..., f_1 \rangle$ with n number of f_1 maps and $C = \frac{1}{n!}$. It would be much simpler if we could leave off the identity maps. But in doing so, we lose n! unshuffles, so we must multiply by that number in order to simplify to just l'_n. Thus, the coefficients will cancel. Any time we drop f_1 maps from a term, the associated $\frac{1}{t_1!}$ coefficient will be canceled.

Another interesting note about symmetric braces in the example $B_*(V)$:
$l_1 \langle l_2 \langle f_2 \rangle \rangle$ is defined when acting on (x_1, x_2, x_3), but $l_1 \langle l_2 \langle f_2 \rangle \rangle = l_1 \langle l_2 \rangle \langle f_2 \rangle - l_1 \langle l_2, f_2 \rangle$ and $l_1 \langle l_2, f_2 \rangle$ is not defined on those elements. For now, we must discount that last term, and say that $l_1 \langle l_2 \langle f_2 \rangle \rangle = l_1 \langle l_2 \rangle \langle f_2 \rangle$. More generally, any $x_j \langle y_{i_1}, y_{i_2}, ..., y_{i_m} \rangle$ is undefined when $m > j$.

3.4 Defining the l' Maps

The l'_n maps in Theorem 3.2.1 generate a new L_∞ algebra structure. I want to know how these new l' maps are defined, so simply solve for l'_n. The following show the maps up to n=4, also letting f_1=identity.

- $f_1 \langle l_1 \rangle = l'_1 \langle f_1 \rangle$

$\Rightarrow \mathbf{l'_1 = l_1}$

- $f_1 \langle l_2 \rangle + f_2 \langle l_1 \rangle = l'_1 \langle f_2 \rangle + \frac{1}{2!} l'_2 \langle f_1, f_1 \rangle$

$\Rightarrow l_2 + f_2 \langle l_1 \rangle = l_1 \langle f_2 \rangle + (2!)\frac{1}{2!} l'_2$

$\Rightarrow \mathbf{l'_2 = l_2 + f_2 \langle l_1 \rangle - l_1 \langle f_2 \rangle}$

The fraction $\frac{1}{2!}$ is the application of C from Theorem 3.2.1. I multiply by 2! because of the loss of unshuffles from leaving off $f_1's$.

- $f_1 \langle l_3 \rangle + f_2 \langle l_2 \rangle + f_3 \langle l_1 \rangle = l'_1 \langle f_3 \rangle + l'_2 \langle f_1, f_2 \rangle + \frac{1}{3!} l'_3 \langle f_1, f_1, f_1 \rangle$

$\Rightarrow l_3 + f_2 \langle l_2 \rangle + f_3 \langle l_1 \rangle = l_1 \langle f_3 \rangle + l'_2 \langle f_2 \rangle + (3!) \frac{1}{3!} l'_3$

$\Rightarrow \mathbf{l'_3 = l_3 + f_2 \langle l_2 \rangle + f_3 \langle l_1 \rangle - l_1 \langle f_3 \rangle - l'_2 \langle f_2 \rangle}$

- $f_1 \langle l_4 \rangle + f_2 \langle l_3 \rangle + f_3 \langle l_2 \rangle + f_4 \langle l_1 \rangle = l'_1 \langle f_4 \rangle + l'_2 \langle f_1, f_3 \rangle + \frac{1}{2!} l'_2 \langle f_2, f_2 \rangle + \frac{1}{2!} l'_3 \langle f_1, f_1, f_2 \rangle$

$+ \frac{1}{4!} l'_4 \langle f_1, f_1, f_1, f_1 \rangle$

$\Rightarrow l_4 + f_2 \langle l_3 \rangle + f_3 \langle l_2 \rangle + f_4 \langle l_1 \rangle = l_1 \langle f_4 \rangle + l'_2 \langle f_3 \rangle + \frac{1}{2!} l'_2 \langle f_2, f_2 \rangle + l'_3 \langle f_2 \rangle + l'_4$

$\Rightarrow \mathbf{l'_4 = l_4 + f_2 \langle l_3 \rangle + f_3 \langle l_2 \rangle + f_4 \langle l_1 \rangle - l_1 \langle f_4 \rangle - l'_2 \langle f_3 \rangle - \frac{1}{2!} l'_2 \langle f_2, f_2 \rangle - l'_3 \langle f_2 \rangle}$

Chapter 4

Main Theorem

4.1 Theorem

Assume that the collection $\{l_i\} \in B_{-1}(V)$ is an L_∞ algebra structure and therefore $l\langle l\rangle = 0$. Suppose there exists a collection $\{f_i\} \in B_0(V)$ such that

$$\sum_{i=1}^{n} f_i \langle l_{n-i+1}\rangle = \sum_{j=1}^{n} \sum_{\substack{i_1^1=...=i_{t_1}^1<i_1^2=...=i_{t_2}^2<i_1^m=...=i_{t_m}^m \\ i_1^1+i_2^1+...+i_{t_m}^m=n \\ t_1+t_2+...+t_m=j}} Cl'_j\langle f_{i_1^1}, ..., f_{i_{t_1}^1}, f_{i_1^2}, ..., f_{i_{t_2}^2}, ..., f_{i_1^m}, .., f_{i_{t_m}^m}\rangle$$

for some $l'_i \in B_{-1}(V)$, where $C = \prod \frac{1}{(t_a)!}, a = 1, ..., m$. Also, let f_1 =identity.

Then the collection $\{l'_i\}$ is an L_∞ algebra structure.

Additionally, as a consequence of the above equation, the collection $\{f_i\}$ is an L_∞ morphism.

4.2 Proof

Show that $l'\langle l'\rangle = 0$ for $l' = l'_1 + l'_2 + ... + l'_n$. To simplify,

$$l'\left\langle l'\right\rangle = 0 \Rightarrow l'_1\left\langle l'_n\right\rangle + l'_2\left\langle l'_{n-1}\right\rangle + l'_3\left\langle l'_{n-2}\right\rangle + \cdots + l'_{n-1}\left\langle l'_2\right\rangle + l'_n\left\langle l'_1\right\rangle = 0$$

Solve for l'_n to get

$$l'_n = \sum_{i=1}^{n} f_i\langle l_{n-i+1}\rangle - \sum_{j=1}^{n-1} \sum_{\substack{i_1^1=...=i_{t_1}^1<i_1^2=...=i_{t_2}^2<i_1^m=...=i_{t_m}^m \\ i_1^1+i_2^1+...+i_{t_m}^m=n \\ t_1+t_2+...+t_m=j}} Cl'_j\langle f_{i_1^1},...,f_{i_{t_1}^1}, f_{i_1^2},...,f_{i_{t_2}^2},...,f_{i_1^m},..,f_{i_{t_m}^m}\rangle$$

This proof will be completed in four parts.

4.2.1 Part I

Write out the pairs $l'_1\langle l'_n\rangle + l'_2\left\langle l'_{n-1}\right\rangle + l'_3\left\langle l'_{n-2}\right\rangle + \cdots + l'_{n-1}\langle l'_2\rangle + l'_n\langle l'_1\rangle$ and substitute the correct formula for the second map. The positive terms are from the first summation, $\sum_{i=1}^{n} f_i\langle l_{n-i+1}\rangle$. The negative terms follow from the double summation.

$$1.)\ l'_1\langle l'_n\rangle = l_1\overset{a}{\langle l_n\rangle} + l_1\langle f_2\overset{b}{\langle l_{n-1}\rangle}\rangle + l_1\langle f_3\overset{c}{\langle l_{n-2}\rangle}\rangle + ... + l_1\langle \overset{d}{f_n}\langle l_1\rangle\rangle - \cancel{l_1\langle l_1\langle f_n\rangle\rangle}$$

$$-l_1\left\langle C\overset{e}{\sum_{\substack{i\le j \\ i+j=n}}} l'_2\langle f_i, f_j\rangle\right\rangle - l_1\left\langle C\sum_{\substack{i_1\le i_2\le i_3 \\ i_1+i_2+i_3=n}} \overset{f}{l'_3}\langle f_{i_1}, f_{i_2}, f_{i_3}\rangle\right\rangle$$

$$-... - \frac{1}{(n-2)!} l_1\langle \overset{g}{l'_{n-1}}\langle f_1, ..., f_1, f_2\rangle\rangle$$

2.) $$l_2'\left\langle l_{n-1}'\right\rangle = l_2'\overset{a}{\langle l_{n-1}\rangle} + l_2'\left\langle f_2\overset{b}{\langle l_{n-2}\rangle}\right\rangle + l_2'\left\langle f_3\overset{c}{\langle l_{n-3}\rangle}\right\rangle + ... + l_2'\overset{d}{\left\langle f_{n-1}\langle l_1\rangle\right\rangle} - l_2'\left\langle l_1\overset{e}{\langle f_{n-1}\rangle}\right\rangle$$

$$-l_2'\left\langle C\overset{f}{\sum_{\substack{i\leq j\\ i+j=n-1}}} l_2'\langle f_i, f_j\rangle\right\rangle - l_2'\left\langle C\overset{g}{\sum_{\substack{i_1\leq i_2\leq i_3\\ i_1+i_2+i_3=n-1}}} l_3'\langle f_{i_1}, f_{i_2}, f_{i_3}\rangle\right\rangle$$

$$-... - \frac{1}{(n-3)!}l_2'\overset{h}{\left\langle l_{n-2}'\langle f_1, f_1, ..., f_1, f_2\rangle\right\rangle}$$

3.) $$l_3'\left\langle l_{n-2}'\right\rangle = l_3'\overset{a}{\langle l_{n-2}\rangle} + l_3'\left\langle f_2\overset{b}{\langle l_{n-3}\rangle}\right\rangle + l_3'\left\langle f_3\overset{c}{\langle l_{n-4}\rangle}\right\rangle + ... + l_3'\overset{d}{\left\langle f_{n-2}\langle l_1\rangle\right\rangle} - l_3'\left\langle l_1\overset{e}{\langle f_{n-2}\rangle}\right\rangle$$

$$-l_3'\left\langle C\overset{f}{\sum_{\substack{i\leq j\\ i+j=n-2}}} l_3'\langle f_i, f_j\rangle\right\rangle - l_3'\left\langle C\overset{g}{\sum_{\substack{i_1\leq i_2\leq i_3\\ i_1+i_2+i_3=n-2}}} l_3'\langle f_{i_1}, f_{i_2}, f_{i_3}\rangle\right\rangle$$

$$-... - \frac{1}{(n-4)!}l_3'\overset{h}{\left\langle l_{n-3}'\langle f_1, f_1, ..., f_1, f_2\rangle\right\rangle}$$

$$\vdots$$

4.) $$l_k'\left\langle l_{n-k+1}'\right\rangle = l_k'\overset{a}{\langle l_{n-k+1}\rangle} + l_k'\left\langle f_2\overset{b}{\langle l_{n-k}\rangle}\right\rangle + l_k'\left\langle f_3\overset{c}{\langle l_{n-k-1}\rangle}\right\rangle + ... + l_k'\overset{d}{\left\langle f_{n-k+1}\langle l_1\rangle\right\rangle}$$

$$-l_k'\left\langle l_1\overset{e}{\langle f_{n-k+1}\rangle}\right\rangle - l_k'\left\langle C\overset{f}{\sum_{\substack{i\leq j\\ i+j=n-k+1}}} l_2'\langle f_i, f_j\rangle\right\rangle - l_k'\left\langle C\overset{g}{\sum_{\substack{i_1\leq i_2\leq i_3\\ i_1+i_2+i_3=n-k+1}}} l_3'\langle f_{i_1}, f_{i_2}, f_{i_3}\rangle\right\rangle$$

$$-... - \frac{1}{(n-k-1)!}l_k'\overset{h}{\left\langle l_{n-k}'\langle f_1, f_1, ..., f_1, f_2\rangle\right\rangle}$$

$$\vdots$$

5.) $l'_{n-1}\langle l'_2\rangle = l'_{n-1}\langle l_2\rangle + l'_{n-1}\langle f_2\langle l_1\rangle\rangle - l'_{n-1}\langle l_1\langle f_2\rangle\rangle$

6.) $l'_n\langle l'_1\rangle = l'_n\langle l_1\rangle$

I will use the roman numeral part number (I), the numbers 1-5, and the letters within each number for reference purposes. For instance, the notation I.2.a means $l'_2\langle l_{n-2}\rangle$. Since l_1 is a differential, $l_1^2 = 0$ and we can ignore anything with consecutive l_1 maps. Also, I have set $f_1 = id$ and I do not want to have to keep track of identity maps. Therefore, drop the f'_1s, and in doing so the coefficients $\frac{1}{(t_1)!}$ will cancel with the new coefficient $(t_1)!$ that I must multiply by in order to account for the loss of unshuffles. For example, from I.1.g there must be $n-2$ "f_1"s in order for the total number of "f"s to be $n-1$ and $C = \frac{1}{(n-2)!}$. But drop the "f_1"s and multiply by $(n-2)!$ to get $l'_{n-1}\langle f_2\rangle$.

4.2.2 Part II

Step A

Expand $l'_k\langle l_{n-k+1}\rangle$ for $k = 2, ..., n$ by replacing l'_k with its definition. These are the Part I.a terms. The motivation behind this is that they are similar to $l_1\langle l_n\rangle + l_2\langle l_{n-1}\rangle + \cdots + l_{n-1}\langle l_2\rangle + l_n\langle l_1\rangle$. These are exactly the terms from $l\langle l\rangle$, which is zero since the "l"s make up an L_∞ algebra structure.

1.) (I.1.a): $l_1\langle l_n\rangle = l_1\overset{a}{\langle l_n\rangle}$

2.) (I.2.a): $l'_2\langle l_{n-1}\rangle = l_2\overset{a}{\langle l_{n-1}\rangle} + f_2\langle l_1\overset{b}{\rangle}\langle l_{n-1}\rangle - l_1\langle f_2\overset{c}{\rangle}\langle l_{n-1}\rangle$

3.) (I.3.a): $l'_3\langle l_{n-2}\rangle = l_3\overset{a}{\langle l_{n-2}\rangle} + f_2\langle l_2\overset{b}{\rangle}\langle l_{n-2}\rangle + f_3\langle l_1\overset{c}{\rangle}\langle l_{n-2}\rangle - l_1\langle f_3\overset{d}{\rangle}\langle l_{n-2}\rangle - l'_2\langle f_2\overset{e}{\rangle}\langle l_{n-2}\rangle$

$$\vdots$$

4.) (I.4.a): $l'_k \langle l_{n-k+1} \rangle = l_k \overset{a}{\langle l_{n-k+1} \rangle} + f_2 \langle l_{k-1} \overset{b}{\rangle} \langle l_{n-k+1} \rangle + f_3 \langle l_{k-2} \overset{c}{\rangle} \langle l_{n-k+1} \rangle + \ldots + f_k \langle l_1 \rangle \overset{d}{\langle l_{n-k+1} \rangle}$

$$-l_1 \langle f_k \rangle \overset{e}{\langle l_{n-k+1} \rangle} - l'_2 \langle f_{k-1} \overset{f}{\rangle} \langle l_{n-k+1} \rangle - C \sum_{\substack{i_1+i_2=k \\ i_1 \neq 1}} \overset{g}{l'_2} \langle f_{i_1}, f_{i_2} \rangle \langle l_{n-k+1} \rangle$$

$$\overset{h}{-\sum_{j=3}^{k-1} \sum_{\substack{i^1_1=\ldots=i^1_{t_1}<i^2_1=\ldots=i^2_{t_2}<i^m_1=\ldots=i^m_{t_m} \\ i^1_1+i^1_2+\cdots+i^m_{t_m}=k \\ t_1+t_2+\cdots+t_m=j}} Cl'_j \left\langle f_{i^1_1}, \ldots, f_{i^1_{t_1}}, f_{i^2_1}, \ldots, f_{i^2_{t_2}}, \ldots, f_{i^m_1}, \ldots, f_{i^m_{t_m}} \right\rangle \langle l_{n-k+1} \rangle}$$

I purposely separated terms f and g, although they derive from the same summation, because the terms that have only a single f map will usually cancel in a different way than the rest.

$$\vdots$$

5.) (I.5.a): $l'_{n-1} \langle l_2 \rangle = \overset{a}{l_{n-1}} \langle l_2 \rangle + f_2 \overset{b}{\langle l_{n-2} \rangle} \langle l_2 \rangle + f_3 \overset{c}{\langle l_{n-3} \rangle} \langle l_2 \rangle + \ldots. + f_{n-1} \overset{d}{\langle l_1 \rangle} \langle l_2 \rangle - l_1 \overset{e}{\langle f_{n-1} \rangle} \langle l_2 \rangle$

$$-C \sum_{\substack{i \leq j \\ i+j=n-1}} \overset{f}{l'_2} \langle f_i, f_j \rangle \langle l_2 \rangle$$

$$\overset{g}{-\sum_{j=3}^{n-2} \sum_{\substack{i^1_1=\ldots=i^1_{t_1}<i^2_1=\ldots=i^2_{t_2}<i^m_1=\ldots=i^m_{t_m} \\ i^1_1+i^1_2+\ldots+i^m_{t_m}=n-1 \\ t_1+t_2+\ldots+t_m=j}} Cl'_j \langle f_{i^1_1}, \ldots, f_{i^1_{t_1}}, f_{i^2_1}, \ldots, f_{i^2_{t_2}}, \ldots, f_{i^m_1}, .., f_{i^m_{t_m}} \rangle \langle l_2 \rangle}$$

6.) (I.6.a): $l'_n \langle l_1 \rangle = l_n \overset{a}{\langle l_1 \rangle} + f_2 \overset{b}{\langle l_{n-1} \rangle} \langle l_1 \rangle + f_3 \overset{c}{\langle l_{n-2} \rangle} \langle l_2 \rangle + \ldots + \cancel{f_n \langle l_1 \rangle \langle l_1 \rangle} - l_1 \overset{d}{\langle f_n \rangle} \langle l_1 \rangle$

$$-C \sum_{i_1+i_2=n} \overset{e}{l'_2} \langle f_{i_1}, f_{i_2} \rangle \langle l_1 \rangle$$

$$f$$

$$-\sum_{j=3}^{n-1} \sum_{\substack{i^1_1=...=i^1_{t_1}<i^2_1=...=i^2_{t_2}<i^m_1=...=i^m_{t_m}\\ i^1_1+i^1_2+...+i^m_{t_m}=n \\ t_1+t_2+...+t_m=j}} Cl'_j\langle f_{i^1_1},...,f_{i^1_{t_1}}, f_{i^2_1},...,f_{i^2_{t_2}},...,f_{i^m_1},..,f_{i^m_{t_m}}\rangle\langle l_1\rangle$$

Step B

All of the "a" terms in Part II, Step A are the relations that cancel since I am given an L_∞ algebra structure. Now rewrite all II.A.$b, c, d, ...$ terms using symmetric brace relations. An example of the motivation for this comes from the b terms, within which we have $\langle l_1\rangle\langle l_{n-1}\rangle + \langle l_2\rangle\langle l_{n-2}\rangle + \cdots + \langle l_{n-1}\rangle\langle l_2\rangle$, which again is similar to the L_∞ algebra relation (using induction) that proves it to be zero.

II.2.b: ~~$f_2\langle l_1\rangle\langle l_{n-1}\rangle$~~$= f_2\langle l_1\langle l_{n-1}\rangle\rangle + f_2\langle l_1, l_{n-1}\rangle$

II.2.c: ~~$-l_1\langle f_2\rangle\langle l_{n-1}\rangle$~~$= -l_1\langle f_2\langle l_{n-1}\rangle\rangle$

Recall from the note in 3.3 that $l_1\langle f_2, l_{n-1}\rangle$ is not defined, so I do not include that or similar terms.

II.3.b: ~~$f_2\langle l_2\rangle\langle l_{n-2}\rangle$~~$= f_2\langle l_2\langle l_{n-2}\rangle\rangle + f_2\langle l_2, l_{n-2}\rangle$

II.3.c: ~~$f_3\langle l_1\rangle\langle l_{n-2}\rangle$~~$= f_3\langle l_1\langle l_{n-2}\rangle\rangle + f_3\langle l_1, l_{n-2}\rangle$

II.3.d: ~~$-l_1\langle f_3\rangle\langle l_{n-2}\rangle$~~$= -l_1\langle f_3\langle l_{n-2}\rangle\rangle$

II.3.e: ~~$-l'_2\langle f_2\rangle\langle l_{n-2}\rangle$~~$= -l'_2\langle f_2\langle l_{n-2}\rangle\rangle - l'_2\langle f_2, l_{n-2}\rangle$

II.4.b: ~~$f_2\langle l_{k-1}\rangle\langle l_{n-k+1}\rangle$~~ $= f_2\langle l_{k-1}\langle l_{n-k+1}\rangle\rangle + f_2\langle l_{k-1}, l_{n-k+1}\rangle$

II.4.c: ~~$f_3\langle l_{k-2}\rangle\langle l_{n-k+1}\rangle$~~ $= f_3\langle l_{k-2}\langle l_{n-k+1}\rangle\rangle + f_3\langle l_{k-2}, l_{n-k+1}\rangle$

II.4.d: ~~$f_k\langle l_1\rangle\langle l_{n-k+1}\rangle$~~ $= f_k\langle l_1\langle l_{n-k+1}\rangle\rangle + f_k\langle l_1, l_{n-k+1}\rangle$

II.4.e: ~~$-l_1\langle f_k\rangle\langle l_{n-k+1}\rangle$~~ $= -l_1\langle f_k\langle l_{n-k+1}\rangle\rangle$

II.4.f: ~~$-l'_2\langle f_{k-1}\rangle\langle l_{n-k+1}\rangle$~~ $= -l'_2\langle f_{k-1}\langle l_{n-k+1}\rangle\rangle - l'_2\langle f_{k-1}, l_{n-k+1}\rangle$

II.4.g: $-C\sum_{\substack{i_1+i_2=k\\ i_1\neq 1}}$ ~~$l'_2\left\langle f_{i_1},f_{i_2}\right\rangle\left\langle l_{n-k+1}\right\rangle$~~

$$=-C\sum_{\substack{i_1+i_2=k\\ i_1\neq 1}}\left[l'_2\left\langle f_{i_1}\left\langle l_{n-k+1}\right\rangle,f_{i_2}\right\rangle+l'_2\left\langle f_{i_1},f_{i_2}\left\langle l_{n-k+1}\right\rangle\right\rangle\right]$$

II.4.h:

$-C\sum_{j=3}^{k-1}\sum_{\substack{i^1_1=\ldots=i^1_{t_1}<i^2_1=\ldots=i^2_{t_2}<i^m_1=\ldots=i^m_{t_m}\\ i^1_1+i^1_2+\cdots+i^m_{t_m}=k\\ t_1+t_2+\cdots+t_m=j}}$ ~~$l'_j\left\langle f_{i^1_1},...,f_{i^1_{t_1}},f_{i^2_1},...,f_{i^2_{t_2}},...,f_{i^m_1},...,f_{i^m_{t_m}}\right\rangle\left\langle l_{n-k+1}\right\rangle$~~

$$=-C\sum_{j=3}^{k-1}\sum_{\substack{i^1_1=\ldots=i^1_{t_1}<i^2_1=\ldots=i^2_{t_2}<i^m_1=\ldots=i^m_{t_m}\\ i^1_1+i^1_2+\cdots+i^m_{t_m}=k\\ t_1+t_2+\cdots+t_m=j}}[l'_j\left\langle f_{i^1_1}\left\langle l_{n-k+1}\right\rangle,...,f_{i^1_{t_1}},f_{i^2_1},...,f_{i^2_{t_2}},...,f_{i^m_1},...,f_{i^m_{t_m}}\right\rangle$$

$$+l'_j\left\langle f_{i^1_1},f_{i^1_2}\left\langle l_{n-k+1}\right\rangle,...,f_{i^1_{t_1}},f_{i^2_1},...,f_{i^2_{t_2}},...,f_{i^m_1},...,f_{i^m_{t_m}}\right\rangle$$

$$+\cdots+l'_j\left\langle f_{i^1_1},...,f_{i^1_{t_1}},f_{i^2_1},...,f_{i^2_{t_2}},...,f_{i^m_1},...,f_{i^m_{t_m}}\left\langle l_{n-k+1}\right\rangle\right\rangle]$$

II.5.b: ~~$f_2\left\langle l_{n-2}\right\rangle\left\langle l_2\right\rangle$~~$=f_2\left\langle l_{n-2}\left\langle l_2\right\rangle\right\rangle+f_2\left\langle l_{n-2},l_2\right\rangle$

II.5.c: ~~$f_3\left\langle l_{n-3}\right\rangle\left\langle l_2\right\rangle$~~$=f_3\left\langle l_{n-3}\left\langle l_2\right\rangle\right\rangle+f_3\left\langle l_{n-3},l_2\right\rangle$

II.5.d: ~~$f_{n-1}\left\langle l_1\right\rangle\left\langle l_2\right\rangle$~~$=f_{n-1}\left\langle l_1\left\langle l_2\right\rangle\right\rangle+f_{n-1}\left\langle l_1,l_2\right\rangle$

II.5.e: ~~$-l_1\left\langle f_{n-1}\right\rangle\left\langle l_2\right\rangle$~~$=-l_1\left\langle f_{n-1}\left\langle l_2\right\rangle\right\rangle$

II.5.f: ~~$-C\sum_{i_1+i_2=n-1}l'_2\left\langle f_{i_1},f_{i_2}\right\rangle\left\langle l_2\right\rangle$~~$=$

$$-C\sum_{i_1+i_2=n-1}l'_2\left\langle f_{i_1}\left\langle l_2\right\rangle,f_{i_2}\right\rangle-l'_2\left\langle f_{i_1},f_{i_2}\left\langle l_2\right\rangle\right\rangle$$

II.5.g:

$-C\sum_{j=3}^{n-2}\sum_{\substack{i_1^1=...=i_{t_1}^1<i_1^2=...=i_{t_2}^2<i_1^m=...=i_{t_m}^m \\ i_1^1+i_2^1+\cdots+i_{t_m}^m=n-1 \\ t_1+t_2+\cdots+t_m=j}} \cancel{l_j'\left\langle f_{i_1^1},...,f_{i_{t_1}^1},f_{i_1^2},...,f_{i_{t_2}^2},...,f_{i_1^m},...,f_{i_{t_m}^m}\right\rangle\langle l_2\rangle}$

$$= -C\sum_{j=3}^{n-2}\sum_{\substack{i_1^1=...=i_{t_1}^1<i_1^2=...=i_{t_2}^2<i_1^m=...=i_{t_m}^m \\ i_1^1+i_2^1+\cdots+i_{t_m}^m=n-1 \\ t_1+t_2+\cdots+t_m=j}} [l_j'\left\langle f_{i_1^1}\langle l_2\rangle,...,f_{i_{t_1}^1},f_{i_1^2},...,f_{i_{t_2}^2},...,f_{i_1^m},...,f_{i_{t_m}^m}\right\rangle$$

$$+l_j'\left\langle f_{i_1^1},f_{i_2^1}\langle l_2\rangle,...,f_{i_{t_1}^1},f_{i_1^2},...,f_{i_{t_2}^2},...,f_{i_1^m},...,f_{i_{t_m}^m}\right\rangle$$

$$+\cdots+l_j'\left\langle f_{i_1^1},...,f_{i_{t_1}^1},f_{i_1^2},...,f_{i_{t_2}^2},...,f_{i_1^m},...,f_{i_{t_m}^m}\langle l_2\rangle\right\rangle]$$

II.6.b: $\cancel{f_2\langle l_{n-1}\rangle\langle l_1\rangle}=f_2\langle l_{n-1}\langle l_1\rangle\rangle+f_2\langle l_{n-1},l_1\rangle$

II.6.c: $\cancel{f_3\langle l_{n-2}\rangle\langle l_1\rangle}=f_3\langle l_{n-2}\langle l_1\rangle\rangle+f_3\langle l_{n-2},l_1\rangle$

II.6.d: $\cancel{-l_1\langle f_n\rangle\langle l_1\rangle}=-l_1\langle f_n\langle l_1\rangle\rangle$

II.6.e: $\cancel{-C\sum_{i_1+i_2=n} l_2'\langle f_{i_1},f_{i_2}\rangle\langle l_1\rangle}=$

$$-C\sum_{i_1+i_2=n} l_2'\langle f_{i_1}\langle l_1\rangle,f_{i_2}\rangle-l_2'\langle f_{i_1},f_{i_2}\langle l_1\rangle\rangle$$

II.6.f: $-C\sum_{j=3}^{n-1}\sum_{\substack{i_1^1=...=i_{t_1}^1<i_1^2=...=i_{t_2}^2<i_1^m=...=i_{t_m}^m \\ i_1^1+i_2^1+\cdots+i_{t_m}^m=n \\ t_1+t_2+\cdots+t_m=j}} \cancel{l_j'\left\langle f_{i_1^1},...,f_{i_{t_1}^1},f_{i_1^2},...,f_{i_{t_2}^2},...,f_{i_1^m},...,f_{i_{t_m}^m}\right\rangle\langle l_1\rangle}$

$$= -C\sum_{j=3}^{n-1}\sum_{\substack{i_1^1=...=i_{t_1}^1<i_1^2=...=i_{t_2}^2<i_1^m=...=i_{t_m}^m \\ i_1^1+i_2^1+\cdots+i_{t_m}^m=n \\ t_1+t_2+\cdots+t_m=j}} [l_j'\left\langle f_{i_1^1}\langle l_1\rangle,...,f_{i_{t_1}^1},f_{i_1^2},...,f_{i_{t_2}^2},...,f_{i_1^m},...,f_{i_{t_m}^m}\right\rangle$$

$$+l_j'\left\langle f_{i_1^1},f_{i_2^1}\langle l_1\rangle,...,f_{i_{t_1}^1},f_{i_1^2},...,f_{i_{t_2}^2},...,f_{i_1^m},...,f_{i_{t_m}^m}\right\rangle$$

$$+\cdots+l'_j\left\langle f_{i^1_1},...,f_{i^1_{t_1}},f_{i^2_1},...,f_{i^2_{t_2}},...,f_{i^m_1},...,f_{i^m_{t_m}}\langle l_1\rangle\right\rangle]$$

Step C

Some patterns should be evident.

i) From the "a"s in Part II, Step A we get $\sum_{k=1}^{n} l_k\langle l_{n-k+1}\rangle$ which $=0$ since the "l"s make up an L_∞-algebra structure.

ii) From the "b"s, we get $\sum_{k=1}^{n-1} f_2\langle l_k\langle l_{n-k}\rangle\rangle = f_2\left\langle \sum_{k=1}^{n-1} l_k\langle l_{n-k}\rangle\right\rangle = 0$ since "l"s are an L_∞ algebra structure.

iii) Similarly, from the "c"s, we get $\sum_{k=1}^{n-2} f_3\langle l_k\langle l_{n-k+1}\rangle\rangle = 0$. This pattern continues for all

$$\sum_{i=2}^{n-1}\sum_{k=1}^{n-i+1} f_i\langle l_k\langle l_{n-k-i+2}\rangle\rangle = 0$$

and results in the cancellation of all of the first terms in the expansions of the $f_i\langle l_k\rangle\langle l_{n-k-i+2}\rangle$ type symmetric braces.

iv) Those were not the only terms that appeared from the expansions of b, c, and d. The others are:

$$\sum_{i=2}^{n-1}\sum_{k=1}^{n-i+1} f_i\langle l_k, l_{n-k-i+2}\rangle$$

These are automatically 0 if $k = n-k-i+2$ because of the symmetry of symmetric braces combined with the Koszul sign. For example, $\langle l_2, l_2\rangle = (-1)^{|l_2||l_2|}\langle l_2, l_2\rangle = -\langle l_2, l_2\rangle = 0$. Here I am using the symmetric brace degree for the maps, which is -1 since $l_i \in B_{-1}(V)$. Notice that k and $n-k-i+2$ run over the same numbers, so these will cancel with each other!

Example: From II.2.b we got $f_2\langle l_1, l_{n-1}\rangle$ and we will also see $f_2\langle l_{n-1}, l_1\rangle$ from II.6.b. Then $f_2\langle l_{n-1}, l_1\rangle + f_2\langle l_1, l_{n-1}\rangle = -f_2\langle l_1, l_{n-1}\rangle + f_2\langle l_1, l_{n-1}\rangle = 0$

v) From II.2.c, II.3.d, II.4.e, II.5.e, II.6.d, $\sum_{k=2}^{n} -l_1\langle f_k\langle l_{n-k+1}\rangle\rangle$ will cancel with the same terms of the opposite sign in I.2.b, c, d.

vi) From II.3.e, II.4.f, II.5.f, II.6.e, $\sum_{k=2}^{n-1} -l_2'\langle f_k\langle l_{n-k}\rangle\rangle$ will cancel with the same terms of the opposite sign in I.3.b, c.

vii) Continue this pattern with $\sum_{k=2}^{n-2} -l_3'\langle f_k\langle l_{n-k+1}\rangle\rangle$ through $-l_{n-1}'\langle f_2\langle l_1\rangle\rangle$ (this term appears because the other $n-2$ f maps must be f_1), which cancels with $l_{n-1}'\langle f_2\langle l_1\rangle\rangle$ in $l_{n-1}'\langle l_2'\rangle$ from Part I.

viii) We still have $-\sum_{k=2}^{n-1} l_2'\langle f_k, l_{n-k}\rangle - \sum_{k=3}^{n-2} l_3'\langle f_k, l_{n-k+1}\rangle - ... - l_{n-1}'\langle f_2, l_1\rangle$. (i.e. the big summations from Step B with all but one f_i being identity maps.)

ix) We also still have all of the cases where there was more than one f map (once we leave off the f_1's).

$$C\sum_{k=2}^{n-2} l_k'\left\langle f_{i_1^1}, ..., f_{i_{t_m}^m}\right\rangle\langle l_j\rangle$$

This was rewritten using symmetric brace relations, such as with II.4.g, and some of II.4.h, II.5.h, and II.6.e, f.

Therefore, from Part II, I have viii and ix to deal with and cancel at some point.

4.2.3 Part III

There are still a lot of terms from Part I that have not canceled yet. Return to those now.

Step A: Back to $l'_1 \langle l'_n \rangle$ (i.e. Part I.1)

Everything up until I.1.e should be gone. We can't cancel the rest of the terms $e - h$ as they are, so try rewriting them using symmetric brace relations.

1.) (I.1.e):

$$-l_1 \left\langle C \sum_{\substack{i \le j \\ i+j=n}} l'_2 \langle f_i, f_j \rangle \right\rangle = -C \sum_{\substack{i \le j \\ i+j=n}} \overset{a}{l_1} \langle l'_2 \rangle \langle f_i, f_j \rangle$$

I want to find

$$-C \sum_{\substack{i \le j \\ i+j=n}} l'_2 \langle l_1 \rangle \langle f_i, f_j \rangle$$

because the two added together equal 0 by induction on the l' maps composing an L_∞ structure.

2.) (I.1.f):

$$-l_1 \left\langle C \sum_{\substack{i_1 \le i_2 \le i_3 \\ i_1+i_2+i_3=n}} l'_3 \langle f_{i_1}, f_{i_2}, f_{i_3} \rangle \right\rangle = -C \sum_{\substack{i_1 \le i_2 \le i_3 \\ i_1+i_2+i_3=n}} \overset{a}{l_1} \langle l'_3 \rangle \langle f_{i_1}, f_{i_2}, f_{i_3} \rangle$$

so I need to find

$$-C \sum_{\substack{i_1 \le i_2 \le i_3 \\ i_1+i_2+i_3=n}} \left(l'_2 \langle l'_2 \rangle + l'_3 \langle l_1 \rangle \right) \langle f_{i_1}, f_{i_2}, f_{i_3} \rangle$$

to cancel.

$\vdots$

3.) Generally, for j=2,...,n-1:

$$-C\sum_{\substack{i_1^1+...+i_{t_m}^m=n\\ t_i+...+t_m=j\\ i_1^1\le...\le i_{t_m}^m}}\sum_{j=2}^{n-1}l_1\left\langle l_j'\left\langle f_{i_1^1},...,f_{i_{t_m}^m}\right\rangle\right\rangle=-C\sum_{\substack{i_1^1+...+i_{t_m}^m=n\\ t_i+...+t_m=j\\ i_1^1\le...\le i_{t_m}^m}}\sum_{j=2}^{n-1}l_1\left\langle l_j'\right\rangle\left\langle f_{i_1^1},...,f_{i_{t_m}^m}\right\rangle$$

I will need $-C[l_2'\left\langle l_{j-1}'\right\rangle+l_3'\left\langle l_{j-2}'\right\rangle+...+l_j'\left\langle l_1\right\rangle]$ with the appropriate associated f's. Note that the C coefficient should match up when these terms appear, since the f's will be the same. There are no remaining terms in $l_1'\left\langle l_n'\right\rangle$ that have not been considered.

Step B: Back to $l_2'\left\langle l_{n-1}'\right\rangle$ (i.e. Part I.2)

1.) (I.2.e): $-l_2'\left\langle l_1\left\langle f_{n-1}\right\rangle\right\rangle=-l_2'\overset{a}{\left\langle l_1\right\rangle}\left\langle f_{n-1}\right\rangle+l_2'\left\langle l_1,\overset{b}{f_{n-1}}\right\rangle$ Term a is one of the induction terms needed. Also, the second term, b, will cancel with what was found in part viii of Step C when $k=n-1$ in the first summation.

2.) (I.2.f):

$$-l_2'\left\langle C\sum_{\substack{i\le j\\ i+j=n-1}}l_2'\left\langle f_i,f_j\right\rangle\right\rangle=-C\sum_{\substack{i\le j\\ i+j=n-1}}\left(l_2'\overset{a}{\left\langle l_2'\right\rangle}\left\langle f_i,f_j\right\rangle+l_2'\left\langle l_2'\overset{b}{\left\langle f_i\right\rangle},f_j\right\rangle+l_2'\left\langle l_2'\overset{c}{\left\langle f_j\right\rangle},f_i\right\rangle\right)$$

Again, a is an induction term. The other two (b and c) will need to be manipulated later (the beginning of Part IV) to cancel.

⋮

3.) Similar for

$$-C\sum_{\substack{i_1\le\cdots\le i_j\\ i_1+\cdots+i_j=n-1}}\sum_{j=3}^{n-2}l_2'\left\langle l_j'\left\langle f_{i_1},...,f_{i_j}\right\rangle\right\rangle$$

$$= C \sum_{\substack{i_1 \leq \cdots \leq i_j \\ i_1 + \cdots + i_j = n-1}} \sum_{j=3}^{n-2} \left[-l'_2 \left\langle l'_j \right\rangle \overset{a}{\left\langle f_{i_1}, ..., f_{i_j} \right\rangle} + \sum_{\sigma} l'_2 \left\langle l'_j \left\langle f_{\sigma(i_1)}, \overset{b}{...}, f_{\sigma(i_{j-1})} \right\rangle, f_{\sigma(i_j)} \right\rangle \right]$$

where σ runs over all unshuffles such that $\sigma(i_1) < \cdots < \sigma(i_{j-1}), \sigma(i_j)$.

I will not write out this part of the proof for the Part I.3 section, because it is so similar to the other steps here.

Step C: Back to $l'_k \left\langle l'_{n-k+1} \right\rangle$ (i.e. Part I.4)

1.) (I.4.e):

$$-l'_k \left\langle l_1 \left\langle f_{n-k+1} \right\rangle \right\rangle = -l'_k \left\langle l_1 \right\rangle \overset{a}{\left\langle f_{n-k+1} \right\rangle}$$

This is another induction term.

2.) (I.4.f):

$$-l'_k \left\langle C \sum_{\substack{i \leq j \\ i+j=n-k+1}} l'_2 \left\langle f_i, f_j \right\rangle \right\rangle$$

$$= -C \sum_{\substack{i \leq j \\ i+j=n-k+1}} \left(l'_k \left\langle l'_2 \right\rangle \overset{a}{\left\langle f_i, f_j \right\rangle} + l'_k \left\langle l'_2 \overset{b}{\left\langle f_i \right\rangle}, f_j \right\rangle + l'_k \left\langle l'_2 \overset{c}{\left\langle f_j \right\rangle}, f_i \right\rangle \right)$$

3.) (I.4.g):

$$-C \sum_{\substack{i_1 \leq i_2 \leq i_3 \\ i_1+i_2+i_3=n-k+1}} l'_k \left\langle l'_3 \left\langle f_{i_1}, f_{i_2}, f_{i_3} \right\rangle \right\rangle$$

$$=C \sum_{\substack{i_1 \le i_2 \le i_3 \\ i_1+i_2+i_3=n-k+1}} [-l'_k \langle l'_3 \rangle \langle f_{i_1}, f_{i_2}, f_{i_3} \rangle$$

$$+ \sum_{\sigma} l'_k \left\langle l'_3 \left\langle f_{\sigma(i_1)}, f_{\sigma(i_2)} \right\rangle, f_{\sigma(i_3)} \right\rangle + \sum_{\sigma} l'_k \left\langle l'_3 \left\langle f_{\sigma(i_1)} \right\rangle, f_{\sigma(i_2)}, f_{\sigma(i_3)} \right\rangle]$$

$\vdots$

4.) (I.4.h): $-l'_k \left\langle l'_{n-k} \langle f_2 \rangle \right\rangle = -l'_k \left\langle l'_{n-k} \right\rangle \langle f_2 \rangle$

Once this is completed, we can cancel:

∘ From III.A.1.a +III.B.1.a: $(-l_1 \langle l'_2 \rangle - l'_2 \langle l_1 \rangle) \langle f_{n-1} \rangle = 0$

∘ From III.A.2.a and others not explicitly written: $(-l_1 \langle l'_3 \rangle - l'_2 \langle l'_2 \rangle - l'_3 \langle l_1 \rangle) \langle f_{n-2} \rangle = 0$

$\vdots$ Continue for $f_{n-3} ... f_3$ and finally

∘ $\left[-\sum_{k=1}^{n-1} l'_k \left\langle l'_{n-k} \right\rangle \right] \langle f_2 \rangle = 0$

Some more work is required to use this trick when there is more than one f. Notice that for 1.a above, we only have the induction term for f_{n-k+1} since l_1 originally couldn't act on more than one f. Similarly, we are missing $l'_k \langle l'_2 \rangle$ on more than two "f"s, and $l'_k \langle l'_3 \rangle$ on more than three "f"s, and so on.

If we look at the individual "k"s from the a pieces of the rewritten symmetric braces in Part III, we have the following pieces that can be used for the induction argument on the l's being L_∞ maps. (leaving off the summations):
$l'_j \langle l'_1 \rangle \langle f \rangle$ for $j = 1, ..., n-1$

$l'_j \langle l'_2 \rangle \langle f, f \rangle$ for $j = 1, ..., n-2$

$l'_j \langle l'_3 \rangle \langle f, f, f \rangle$ for $j = 1, ..., n-3$

$\vdots$

$l'_j \left\langle l'_{n-2} \right\rangle \langle f, ..., f \rangle$ for $j = 1, 2$, where there are $n-2$ "f"s.

$l'_j \left\langle l'_{n-1} \right\rangle \langle f, ..., f \rangle$ for $j = 1$, where there are $n-1$ "f"s.

Much progress has been made, but I still need the other induction terms where there are more "f" maps, and the $b, c, ...$ terms from Part III where the l'_i map within the symmetric braces was not l_1. If it was l_1, then it canceled with II.C.viii. I still have left other cases from II.viii and ix.

4.2.4 Part IV

Before finding the rest of the induction terms, note that $l'_2 \langle l'_2, f_{n-2} \rangle$ appears when $f_{i_1} = f_1$ in III.B.2.b, which is similar to $-l'_2 \langle f_{n-2}, l_2 \rangle$ from II.C.viii.. How to use this:
Expand the second l'_2, by its definition, in the first term mentioned above.

1.) ~~$l'_2 \langle l'_2, f_{n-2} \rangle$~~$= l'_2 \langle l_2, \overset{a}{f_{n-2}} \rangle + l'_2 \langle f_2 \langle \overset{b}{l_1} \rangle, f_{n-2} \rangle - l'_2 \langle l_1 \langle \overset{c}{f_2} \rangle, f_{n-2} \rangle$

- a cancels with $-l'_2 \langle f_{n-2}, l_2 \rangle$ as described above.
- b cancels with the II.6.e expansion in Part II.B where $i_1 = 2, i_2 = n-2, j = 1$.
- c is going to give us an induction term:

~~$l'_2 \langle l_1 \langle f_2 \rangle, f_{n-2} \rangle$~~$= -l'_2 \langle l_1 \rangle \langle f_2, f_{n-2} \rangle + l'_2 \langle l_1 \langle f_{n-2} \rangle, f_2 \rangle$
and then the second term, $l'_2 \langle l_1 \langle f_{n-2} \rangle, f_2 \rangle$, will cancel with a similar step when we expand $l'_2 \left\langle l'_{n-2}, f_2 \right\rangle$ and get $-l'_2 \langle l_1 \langle f_{n-2} \rangle f_2 \rangle$. Notice that we won't need to rewrite that part of the expansion (as was necessary in this step for c) to get the induction term. So it will actually just cancel with the leftover term from c without any rewriting.

2.) Expand III.B.3.b when $\sigma(i_1) = \sigma(i_2) = \cdots = \sigma(i_{j-1}) = 1$.

$$\sum_{j=3}^{n-2} l'_2 \left\langle l'_j, f_{n-j} \right\rangle = \sum_{j=3}^{n-2} l'_2 \overset{a}{\left\langle l_j, f_{n-j} \right\rangle} + l'_2 \overset{b}{\left\langle f_2 \left\langle l_{j-1} \right\rangle, f_{n-j} \right\rangle} + \cdots + l'_2 \overset{c}{\left\langle f_j \left\langle l_1 \right\rangle, f_{n-j} \right\rangle} - l'_2 \overset{d}{\left\langle l_1 \left\langle f_j \right\rangle, f_{n-j} \right\rangle}$$

$$-l'_2 \overset{e}{\left\langle \sum_{\substack{i_1+i_2 \\ i_1 \le i_2}} C l'_2 \left\langle f_{i_1}, f_{i_2} \right\rangle, f_{n-j} \right\rangle} - \cdots - l'_2 \overset{f}{\left\langle l'_{j-1} \left\langle f_2 \right\rangle, f_{n-j} \right\rangle}$$

• a cancels with II.C.viii.

• b cancels with II.4.g expansion in Part II.B

• c cancels with II.6.e in Part II.B

• d: For $j = 3, ..., \frac{n-2}{2}$ (if n even) or $\frac{n-3}{2}$ (if n odd), rewrite $-l'_2 \left\langle l_1 \left\langle f_j \right\rangle, f_{n-j} \right\rangle = -l'_2 \left\langle l_1 \right\rangle \left\langle f_j f_{n-j} \right\rangle + \overset{x}{l'_2 \left\langle l_1 \left\langle f_{n-j} \right\rangle, f_j \right\rangle}$. Once j is large enough (the j and $n-j$ subscripts will have "switched places") then we will already have all of the induction terms needed. So instead of expanding these d terms, they will cancel with the leftovers from the expansions for the smaller "j"s (before they switched places), as was described in 1.c above. Note the opposite sign for x compared to d.

• e: If none of the three "f"s are the identity, then

$$l'_2 \left\langle \sum_{\substack{i_1+i_2 \\ i_1 \le i_2}} C l'_2 \left\langle f_{i_1}, f_{i_2} \right\rangle, f_{n-j} \right\rangle$$

$$= -C \sum_{\substack{i_1+i_2=j \\ i_1 \le i_2}} [\overset{x}{l'_2 \left\langle l'_2 \right\rangle \left\langle f_{i_1}, f_{i_2}, f_{n-j} \right\rangle} + \overset{y}{l'_2 \left\langle l'_2 \left\langle f_{i_1}, f_{n-j} \right\rangle, f_{i_2} \right\rangle} + \overset{z}{l'_2 \left\langle l'_2 \left\langle f_{i_2}, f_{n-j} \right\rangle, f_{i_1} \right\rangle}$$

If one $f = f_1$, then do not rewrite and go ahead and cancel with Part III

◦ x is needed for induction.

◦ y and z will cancel with later terms from this step once all induction pieces have been found and expansion no longer needed for e.

• f: No need to rewrite this. It will cancel with III.B.3.

3.) Expand similarly for all terms of the type $l'_k \left\langle l'_j, f, f, f... \right\rangle$, $k = 3, ..., n$, which come from III.C.3 when all f's in the piece $l'_j \left\langle f_{\sigma(i_1)}, ..., f_{\sigma(i_l)} \right\rangle$ are f_1 =identity.

$$-C \sum_{\sigma(i_{l+1}) \leq \cdots \leq \sigma(i_j)} l'_k \left\langle l'_j, f_{\sigma(i_{l+1})}, .., f_{\sigma(i_j)} \right\rangle$$

$$= -C \sum_{j=1}^{k-1} \sum_{\sigma(i_{l+1}) \leq \cdots \leq \sigma(i_j)} [\overset{a}{l'_k \left\langle l_j, f_{\sigma(i_{l+1})}, ..., f_{\sigma(i_j)} \right\rangle} + \overset{b}{l'_k \left\langle f_2 \left\langle l_{j-l+1} \right\rangle, f_{\sigma(i_{l+1})}, ..., f_{\sigma(i_j)} \right\rangle}$$

$$+ \cdots + \overset{c}{l'_k \left\langle f_j \left\langle l_1 \right\rangle, f_{\sigma(i_{l+1})}, ..., f_{\sigma(i_j)} \right\rangle}$$

$$\overset{d}{-l'_k \left\langle l_1 \left\langle f_j \right\rangle, f_{\sigma(i_{l+1})}, ..., f_{\sigma(i_j)} \right\rangle} \overset{e}{- l'_k \left\langle C \sum_{\substack{r_1 \leq r_2 \\ r_1 + r_2 = j}} l'_2 \left\langle f_{r_1}, f_{r_2} \right\rangle, f_{i_1}, ..., f_{i_m} \right\rangle}$$

$$\overset{f}{-l'_k \left\langle C \sum_{\substack{r_1 \leq r_2 \leq r_3 \\ r_1 + r_2 + r_3 = j}} l'_3 \left\langle f_{r_1}, f_{r_2}, f_{r_3} \right\rangle, f_{\sigma(i_{l+1})}, ..., f_{\sigma(i_j)} \right\rangle} - ... - \overset{g}{l'_k \left\langle l'_{j-1} \left\langle f_2 \right\rangle, f_{i_1}, ..., f_{i_m} \right\rangle}]$$

• a cancels with II.6.f in Part II.B, when some "f"s $= f_1$.

• b and c cancel with II.C.viii if only one $f \neq f_1$ and III.C.ix otherwise.

• d must be rewritten as:

$$-l'_k \langle l_1 \rangle \left\langle f_j, f_{\sigma(i_{l+1})}, ..., f_{\sigma(i_j)} \right\rangle + l'_k \left\langle l_1 \left\langle f_{\sigma(i_{l+1})} \right\rangle, f_j, ..., f_{\sigma(i_j)} \right\rangle$$

$$+ \cdots + l'_k \left\langle l_1 \left\langle f_{\sigma(i_j)} \right\rangle, f_j, f_{\sigma(i_{l+1})}, ..., f_{\sigma(i_j - 1)} \right\rangle$$

All terms, excluding the first one which is used for induction, will cancel in this same process for a different j. We won't have to rewrite to get the induction term for these f maps again.

Continue this process for e, f, and g, using the first term for induction and canceling the rest as different j values are used. Eventually, this will take care of anything left in II.viii and ix as well as any other remaining terms from Part III. ■

Chapter 5

General Example Proof for $n = 4$

The proof of the theorem will be greatly illuminated if it is demonstrated for a reasonably small n. The smallest value that still allows insight into the complexities of the proof is $n = 4$. I will follow the outline given in chapter 4 for this example proof. The theorem gives:

$$\sum_{i=1}^{4} f_i \langle l_{5-i} \rangle = \sum_{j=1}^{4} \sum_{\substack{i_1^1=...=i_{t_1}^1<i_1^2=...=i_{t_2}^2<i_1^m=...=i_{t_m}^m \\ i_1^1+i_2^1+...+i_{t_m}^m=4 \\ t_1+t_2+...+t_m=j}} Cl'_j \left\langle f_{i_1^1}, ..., f_{i_{t_1}^1}, f_{i_1^2}, ..., f_{i_{t_2}^2}, ..., f_{i_1^m}, .., f_{i_{t_m}^m} \right\rangle$$

for $C = \prod \frac{1}{(t_a)!}, a = 1, ..., m$ with f_1 =identity.

The formulas for l'_1, l'_2, l'_3, and l'_4 were found in section 3.4. I relist them here:

- $l'_1 = l_1$

- $l'_2 = l_2 + f_2 \langle l_1 \rangle - l_1 \langle f_2 \rangle$

- $l'_3 = l_3 + f_2 \langle l_2 \rangle + f_3 \langle l_1 \rangle - l_1 \langle f_3 \rangle - l'_2 \langle f_2 \rangle$

- $l'_4 = l_4 + f_2 \langle l_3 \rangle + f_3 \langle l_2 \rangle + f_4 \langle l_1 \rangle - l_1 \langle f_4 \rangle - l'_2 \langle f_3 \rangle - \frac{1}{2!} l'_2 \langle f_2, f_2 \rangle - l'_3 \langle f_2 \rangle$

5.1 Proof

To prove that these maps compose an L_∞ algebra structure, I have to show $l'_1\langle l'_4\rangle + l'_2\langle l'_3\rangle + l'_3\langle l'_2\rangle + l'_4\langle l'_1\rangle = 0$. I want to start by expanding the second map using its definition as given above.

5.1.1 Part I

1.) $\mathbf{l'_1}\langle\mathbf{l'_4}\rangle = l_1\left\langle l_4 + f_2\langle l_3\rangle + f_3\langle l_2\rangle + f_4\langle l_1\rangle - l_1\langle f_4\rangle - \frac{1}{2!}l'_2\langle f_2, f_2\rangle - l'_2\langle f_3\rangle\right\rangle$

$= l_1\overset{a}{\langle} l_4\rangle + l_1\langle f_2\overset{b}{\langle} l_3\rangle\rangle + l_1\langle f_3\overset{c}{\langle} l_2\rangle\rangle + l_1\langle f_4\overset{d}{\langle} l_1\rangle\rangle -$ ~~$l_1\langle l_1\langle f_4\rangle\rangle$~~ $-\frac{1}{2!}l_1\langle l'_2\overset{e}{\langle} f_2, f_2\rangle\rangle - l_1\langle l'_2\overset{f}{\langle} f_3\rangle\rangle$

$-l_1\langle l'_3\overset{g}{\langle} f_2\rangle\rangle$

2.) $\mathbf{l'_2}\langle\mathbf{l'_3}\rangle = l'_2\langle l_3 + f_2\langle l_2\rangle + f_3\langle l_1\rangle - l_1\langle f_3\rangle - l'_2\langle f_2\rangle\rangle$

$= l'_2\overset{a}{\langle} l_3\rangle + l'_2\langle f_2\overset{b}{\langle} l_2\rangle\rangle + l'_2\langle f_3\overset{c}{\langle} l_1\rangle\rangle - l'_2\langle l_1\overset{d}{\langle} f_3\rangle\rangle - l'_2\langle l'_2\overset{e}{\langle} f_2\rangle\rangle$

3.) $\mathbf{l'_3}\langle\mathbf{l'_2}\rangle = l'_3\langle l_2 + f_2\langle l_1\rangle - l_1\langle f_2\rangle\rangle$

$= l'_3\overset{a}{\langle} l_2\rangle + l'_3\langle f_2\overset{b}{\langle} l_1\rangle\rangle - l'_3\langle l_1\overset{c}{\langle} f_2\rangle\rangle$

4.) $\mathbf{l'_4}\langle\mathbf{l'_1}\rangle = l'_4\overset{a}{\langle} l_1\rangle$

Note that $l_1 \circ l_1$ should always be zero because l_1 is a differential.

5.1.2 Part II

Step A

To use the fact that I have an L_∞ algebra structure, I need to find the terms that cancel due to $l\langle l\rangle = 0$ for $l = l_1 + l_2 + l_3 + l_4$. Try expanding $l_2'\langle l_3\rangle, l_3'\langle l_2\rangle$, and $l_4'\langle l_1\rangle$ (the a terms), using the definitions for the first maps.

1.) $\mathbf{l_2'}\langle \mathbf{l_3}\rangle = (l_2 + f_2\langle l_1\rangle - l_1\langle f_2\rangle)\langle l_3\rangle$

$$= l_2 \overset{a}{\langle l_3\rangle} + f_2 \overset{b}{\langle l_1\rangle}\langle l_3\rangle - l_1 \overset{c}{\langle f_2\rangle}\langle l_3\rangle$$

2.) $\mathbf{l_3'}\langle \mathbf{l_2}\rangle = (l_3 + f_2\langle l_2\rangle + f_3\langle l_1\rangle - l_1\langle f_3\rangle - l_2'\langle f_2\rangle)\langle l_2\rangle$

$$= l_3 \overset{a}{\langle l_2\rangle} + f_2 \overset{b}{\langle l_2\rangle}\langle l_2\rangle + f_3 \overset{c}{\langle l_1\rangle}\langle l_2\rangle - l_1 \overset{d}{\langle f_3\rangle}\langle l_2\rangle - l_2' \overset{e}{\langle f_2\rangle}\langle l_2\rangle$$

3.) $\mathbf{l_4'}\langle \mathbf{l_1}\rangle = (l_4 + f_2\langle l_3\rangle + f_3\langle l_2\rangle + f_4\langle l_1\rangle - l_1\langle f_4\rangle - \frac{1}{2!}l_2'\langle f_2, f_2\rangle - l_2'\langle f_3\rangle)\langle l_1\rangle$

$$= l_4 \overset{a}{\langle l_1\rangle} + f_2 \overset{b}{\langle l_3\rangle}\langle l_1\rangle + f_3 \overset{c}{\langle l_2\rangle}\langle l_1\rangle + \cancel{f_4\langle l_1\rangle\langle l_1\rangle} - l_1 \overset{d}{\langle f_4\rangle}\langle l_1\rangle - l_2' \overset{e}{\langle f_3\rangle}\langle l_1\rangle - \frac{1}{2!}l_2' \overset{f}{\langle f_2, f_2\rangle}\langle l_1\rangle$$

$$-l_3' \overset{g}{\langle f_2\rangle}\langle l_1\rangle$$

Step B

To find more terms to cancel, rewrite all Part II.A.$b, c, ..., g$ terms using the definition of symmetric braces.

II.1.b: $\cancel{f_2 \langle l_1 \rangle \langle l_3 \rangle} = f_2 \langle l_1 \langle l_3 \rangle \rangle + f_2 \langle l_1, l_3 \rangle$

II.1.c: $\cancel{-l_1 \langle f_2 \rangle \langle l_3 \rangle} = -l_1 \langle f_2 \langle l_3 \rangle \rangle$

II.2.b: $\cancel{f_2 \langle l_2 \rangle \langle l_2 \rangle} = f_2 \langle l_2 \langle l_2 \rangle \rangle + f_2 \langle l_2, l_2 \rangle$

II.2.c: $\cancel{f_3 \langle l_1 \rangle \langle l_2 \rangle} = f_3 \langle l_1 \langle l_2 \rangle \rangle + f_3 \langle l_1, l_2 \rangle$

II.2.d: $\cancel{-l_1 \langle f_3 \rangle \langle l_2 \rangle} = -l_1 \langle f_3 \langle l_2 \rangle \rangle$

II.2.e: $\cancel{-l_2' \langle f_2 \rangle \langle l_2 \rangle} = -l_2' \langle f_2 \langle l_2 \rangle \rangle - l_2' \langle f_2, l_2 \rangle$

II.3.b: $\cancel{f_2 \langle l_3 \rangle \langle l_1 \rangle} = f_2 \langle l_3 \langle l_1 \rangle \rangle + f_2 \langle l_3, l_1 \rangle$

II.3.c: $\cancel{f_3 \langle l_2 \rangle \langle l_1 \rangle} = f_3 \langle l_2 \langle l_1 \rangle \rangle + f_3 \langle l_2, l_1 \rangle$

II.3.d: $\cancel{-l_1 \langle f_4 \rangle \langle l_1 \rangle} = -l_1 \langle f_4 \langle l_1 \rangle \rangle$

II.3.e: $\cancel{-l_2' \langle f_3 \rangle \langle l_1 \rangle} = -l_2' \langle f_3 \langle l_1 \rangle \rangle - l_2' \langle f_3, l_1 \rangle$

II.3.f: $\cancel{-\frac{1}{2!} l_2' \langle f_2, f_2 \rangle \langle l_1 \rangle} = -\frac{1}{2} l_2' \langle f_2 \langle l_1 \rangle, f_2 \rangle - \frac{1}{2} l_2' \langle f_2, f_2 \langle l_1 \rangle \rangle = -l_2' \langle f_2 \langle l_1 \rangle, f_2 \rangle$

II.3.g: $\cancel{-l_3' \langle f_2 \rangle \langle l_1 \rangle} = -l_3' \langle f_2 \langle l_1 \rangle \rangle - l_3' \langle f_2, l_1 \rangle$

Step C

Now there is enough information for some patterns to become noticeable:

i) From the a's in Part II we get $l_1 \langle l_4 \rangle + l_2 \langle l_3 \rangle + l_3 \langle l_2 \rangle + l_4 \langle l_1 \rangle$ which $= 0$ since these maps make up an L_∞ algebra.

ii) From the b's in Part II we get $f_2 \langle l_1 \langle l_3 \rangle \rangle + f_2 \langle l_2 \langle l_2 \rangle \rangle + f_2 \langle l_3 \langle l_1 \rangle \rangle = f_2 \langle l_1 \langle l_3 \rangle + l_2 \langle l_2 \rangle + l_3 \langle l_1 \rangle \rangle = 0$

iii) Similarly, from II.2.c+II.3.c we get $f_3 \langle l_1 \langle l_2 \rangle\rangle + f_3 \langle l_2 \langle l_1 \rangle\rangle = f_3 \langle l_1 \langle l_2 \rangle + l_2 \langle l_1 \rangle\rangle = 0$

iv) From II.2.b, $f_2 \langle l_2, l_2 \rangle = 0$ by symmetry of the braces (recall that the brace degree for l_2 is -1 since $l_2 \in B_{-1}(V)$). Also, $f_2 \langle l_1, l_3 \rangle$ and $f_2 \langle l_3, l_1 \rangle$ cancel and $f_3 \langle l_1, l_2 \rangle$ will cancel with $f_3 \langle l_2, l_1 \rangle$, which takes care of all of the expansion terms from II.1.b, II.2.c, and II.3.b, c.

v) Notice that we can cancel $-l_1 \langle f_2 \langle l_3 \rangle\rangle$ from II.1.c with I.1.b, $-l_1 \langle f_3 \langle l_2 \rangle\rangle$ from II.2.d with I.1.c., and finally $-l_1 \langle f_4 \langle l_1 \rangle\rangle$ from II.3.d with I.1.d.

vi) Similarly, $-l_2' \langle f_2 \langle l_2 \rangle\rangle$ (II.2.e) and $-l_2' \langle f_3 \langle l_1 \rangle\rangle$ (II.3.e) cancel with the terms of the opposite sign in $l_2' \langle l_3' \rangle$ (I.2.b, c).

vii) $-l_3' \langle f_2 \langle l_1 \rangle\rangle$ (II.3.g) cancels with $l_3' \langle f_2 \langle l_1 \rangle\rangle$ (I.3.b) from $l_3' \langle l_2' \rangle$.

viii) We still have $-l_2' \langle f_2, l_2 \rangle, -l_3' \langle f_2, l_1 \rangle$ and $-l_2' \langle f_3, l_1 \rangle$. I will return to these terms later.

ix) There is still the expansion of the term that had more than one f map, leaving us with $-l_2' \langle f_2 \langle l_1 \rangle, f_2 \rangle$ (II.3.f).

5.1.3 Part III

Step A: Back to $l_1' \langle l_4' \rangle$ (i.e. Part I.1)

1.) I.1.e: ~~$-\frac{1}{2!} l_1 \langle l_2' \langle f_2, f_2 \rangle\rangle$~~ $= -\frac{1}{2} l_1 \langle l_2' \rangle \langle f_2, f_2 \rangle$

I need to find $-\frac{1}{2} l_2' \langle l_1 \rangle \langle f_2, f_2 \rangle$ to cancel by induction on the l' maps composing an L_∞

algebra structure. Really, this only uses that the maps make up an L_2 algebra structure.

2.) I.1.f: ~~$-l_1 \langle l'_2 \langle f_3 \rangle\rangle$~~$= -l_1 \langle l'_2 \rangle \langle f_3 \rangle$

I want to find $-l'_2 \langle l_1 \rangle \langle f_3 \rangle$ for the induction argument.

3.) I.1.g: ~~$-l_1 \langle l'_3 \langle f_2 \rangle\rangle$~~ $= -l_1 \langle l'_3 \rangle \langle f_2 \rangle$

Therefore, I need $-l'_2 \langle l'_2 \rangle \langle f_2 \rangle - l'_3 \langle l_1 \rangle \langle f_2 \rangle$.

Step B: Back to $l'_2 \langle l'_3 \rangle$ (i.e. Part I.2)

1.) I.2.d: ~~$-l'_2 \langle l_1 \langle f_3 \rangle\rangle$~~ $= -l'_2 \langle l_1 \rangle \langle f_3 \rangle + l'_2 \langle l_1, f_3 \rangle$.

The first term is what I needed for induction with III.A.1! The second term cancels with one of the parts in II.C.viii

2.) I.2.e: ~~$-l'_2 \langle l'_2 \langle f_2 \rangle\rangle$~~ $= -l'_2 \langle l'_2 \rangle \langle f_2 \rangle + l'_2 \langle l'_2, f_2 \rangle$

Again, the first term is part of what was needed for III.A.3. I will return to $l'_2 \langle l'_2, f_2 \rangle$ later.

Step C: Back to $l'_3 \langle l'_2 \rangle$ (i.e. Part I.3)

1.) I.3.c: ~~$-l'_3 \langle l_1 \langle f_2 \rangle\rangle$~~ $= -l'_3 \langle l_1 \rangle \langle f_2 \rangle + l'_3 \langle l_1, f_2 \rangle$

Now all of the parts for III.A.3 are together. The second term cancels with $-l'_3 \langle f_2, l_1 \rangle$ from

II.C.viii.

To summarize the cancellation by induction:

- $-l_1 \langle l_2' \rangle \langle f_3 \rangle - l_2' \langle l_1 \rangle \langle f_3 \rangle = (-l_1 \langle l_2' \rangle - l_2' \langle l_1 \rangle) \langle f_3 \rangle = 0.$

- $-l_1 \langle l_3' \rangle \langle f_2 \rangle - l_2' \langle l_2' \rangle \langle f_2 \rangle - l_3' \langle l_1 \rangle \langle f_2 \rangle = -(l_1 \langle l_3' \rangle + l_2' \langle l_2' \rangle + l_3' \langle l_2 \rangle) \langle f_2 \rangle = 0$

What terms are left?

- $-l_2' \langle f_2, l_2 \rangle$ from II.C.viii
- $-l_2' \langle f_2 \langle l_1 \rangle, f_2 \rangle$ from ix
- $-\frac{1}{2} l_1 \langle l_2' \rangle \langle f_2, f_2 \rangle$ from III.A.1
- $l_2' \langle l_2', f_2 \rangle$ from III.B.2.

5.1.4 Part IV

Expand the second l_2' in $l_2' \langle l_2', f_2 \rangle$ (from III.B.2) by the definition of the map.

$$\cancel{l_2' \langle l_2', f_2 \rangle} = l_2' \langle (l_2 + f_2 \langle l_1 \rangle - l_1 \langle f_2 \rangle), f_2 \rangle = l_2' \langle l_2, f_2 \rangle + l_2' \langle f_2 \langle l_1 \rangle, f_2 \rangle - l_2' \langle l_1 \langle f_2 \rangle, f_2 \rangle$$

The first two terms cancel immediately with II.C.viii and ix.

Finally, rewrite $-l_2' \langle l_1 \langle f_2 \rangle, f_2 \rangle$ as $-\frac{1}{2!} l_2' \langle l_1 \rangle \langle f_2, f_2 \rangle$, and it will cancel with the only other remaining term (III.A.1) because of induction on l''s being L_∞. This works since

$$l_2' \langle l_1 \rangle \langle f_2, f_2 \rangle = l_2' \langle l_1 \langle f_2 \rangle, f_2 \rangle + l_2' \langle l_1 \langle f_2 \rangle, f_2 \rangle = 2l_2' \langle l_1 \langle f_2 \rangle, f_2 \rangle \blacksquare$$

Chapter 6

Concrete Example for $n = 4$

6.1 Setting Up the Example

Using a known L_∞ algebra structure and a collection of $\{f_i\}$ maps that I concocted, I will demonstrate the theorem for $n = 4$.

Original L_∞ Algebra Structure: Use the L_∞ algebra (up to $n = 4$) given by Daily in [1].

Let $V = V_0 \oplus V_{-1}$, $V_0 =< v_1, v_2 >$ and $V_{-1} =< w >$ with the following relations:

- $l_1(v_1) = l_1(v_2) = w$
- $l_2(v_1, v_2) = v_1$
- $l_2(v_1, w) = w$
- $l_3(w, w, v_2) = w$
- $l_4(w, w, w, v_2) = w$
- l_i, $i = 1, 2, 3, 4$ defined to be 0 on any element not listed above.

Every l map is skew symmetric, implying that $l_2(v_2, v_1) = -l_2(v_1, v_2) = -v_1$.

Collection $\{f_i\}$: Use L_∞ morphism $f \in Hom(V^{\otimes k}, V)^{as}$ defined by:

- $f_1 = identity$

- $f_i(w^{\otimes i}) = w$ for $i = 2, 3, 4$
- $f_i(w^{\otimes i-1} \otimes v_1) = v_2$, $i = 2, 3, 4$
- $f_i(w^{\otimes i-1} \otimes v_2) = v_1$, $i = 2, 3, 4$
- f_i, i=2,3,4 defined to be 0 on any element not listed above.

Every f map is skew symmetric.

New L_∞ Algebra Structure: I want to prove that the following maps make up an L_4 algebra by showing $l' \langle l' \rangle = 0$ for $l' = l'_1 + l'_2 + l'_3 + l'_4$.

- $l'_1 = \quad l_1$
- $l'_2 = \quad f_1 \langle l_2 \rangle + f_2 \langle l_1 \rangle - l_1 \langle f_2 \rangle$
  $= \quad l_2 + f_2 \langle l_1 \rangle - l_1 \langle f_2 \rangle$
- $l'_3 = \quad f_1 \langle l_3 \rangle + f_2 \langle l_2 \rangle + f_3 \langle l_1 \rangle - l_1 \langle f_3 \rangle - l'_2 \langle f_2, f_1 \rangle$
  $= \quad l_3 + f_2 \langle l_2 \rangle + f_3 \langle l_1 \rangle - l_1 \langle f_3 \rangle - l'_2 \langle f_2 \rangle$
- $l'_4 = \quad f_1 \langle l_4 \rangle + f_2 \langle l_3 \rangle + f_3 \langle l_2 \rangle + f_4 \langle l_1 \rangle - l_1 \langle f_4 \rangle - l'_2 \langle f_3, f_1 \rangle - \frac{1}{2} l'_2 \langle f_2, f_2 \rangle - l'_3 \langle f_2, f_1, f_1 \rangle$
  $= \quad l_4 + f_2 \langle l_3 \rangle + f_3 \langle l_2 \rangle + f_4 \langle l_1 \rangle - l_1 \langle f_4 \rangle - l'_2 \langle f_3 \rangle - \frac{1}{2} l'_2 \langle f_2, f_2 \rangle - l'_3 \langle f_2 \rangle$

Note that the degrees of the elements are $|w| = -1$ and $|v_1| = |v_2| = 0$. Also, the degree of f_i is $|f_i| = i - 1$ and the degree of l'_i is the same as the degree of l_i, which is $i - 2$. This will be important for calculating signs.

6.1.1 Signs to Consider

There are several factors to consider before starting, mostly involving sign changes.

Evaluating Sign

Remember that permuting elements and maps (i.e. moving elements over a map) causes a sign change.

Example: $(f_2 \otimes f_2)(w, w, v_1, v_2) = (-1)^{|f_2|(|w|+|w|)} f_2(w, w) \otimes f_2(v_1, v_2)$

For this reason, it is helpful to rearrange the subscripts of the "f"s in decreasing order (as I did above), so that the f_1 identity maps are last and won't change the sign if elements are moved over them. Then we can essentially ignore f_1.

Koszul Sign

We must also use the antisymmetric Koszul sign whenever we permute the elements.

Examples:

$(w, v_1) = -(-1)^{|w||v_1|}(v_1, w) = -(v_1, w)$

$(w, v_2) = -(-1)^{|w||v_2|}(v_2, w) = -(v_2, w)$

$(v_1, v_2) = -(-1)^{|v_1||v_2|}(v_2, v_1) = -(v_2, v_1)$

$(w, w) = -(-1)^{2|w|}(w, w) = (w, w)$

Delta

When evaluating symmetric braces, a δ sign factors in. This can happen in three cases:

1) For $f_i \langle l_j \rangle$ use $j(i-1)$.

2) For $l'_j \langle f_{r_1}, ..., f_{r_j} \rangle$ use $\frac{j(j-1)}{2} + \sum_{i=1}^{j-1} r_i(j-i)$.

3) For $l'_i \left\langle l'_j \right\rangle$ use the actual δ formula found in Definition 2.1.1. As shown in Lemma 2.4.1, this matches the sign in the L_∞-algebra definition which is good since we are proving that the l''s make up such an algebra.

6.1.2 Evaluating l'

This example will demonstrate the case where l' is evaluated on (w, w, v_1, v_2) because it is the most interesting case. It will be easier to prove if we already know what the l' maps are once evaluated on certain elements. I will work out the first few terms in detail, but then will only include ones (after permuting with Koszul sign) that are nonzero when evaluated, instead of listing all of the unshuffles.

Cases we need to know:

- $l_1'(v_1) = \quad l_1'(v_2) = l_1(v_1) = l_1(v_2) = w$
- $l_1'(w) = \quad l_1(w) = 0$
- $$\begin{aligned} l_2'(w, v_1) &= (l_2 + f_2 \langle l_1 \rangle - l_1 \langle f_2 \rangle)(w, v_1) \\ &= -l_2(v_1, w) + f_2 \langle l_1 \rangle (w, v_1) - l_1 \langle f_2 \rangle (w, v_1) \\ &= -w + (-1)^{1(2-1)} f_2 l_1(w, v_1) - (-1)^{1(1-1)/2} l_1 f_2(w, v_1) \\ &= -w - f_2((l_1(w), v_1) - (l_1(v_1), w)) - l_1 f_2(w, v_1) \\ &= -w - f_2(0 - (w, w)) - l_1(v_2) \\ &= -w + w - w \\ &= -w \end{aligned}$$
- $$\begin{aligned} l_2'(w, v_2) &= (l_2 + f_2 \langle l_1 \rangle - l_1 \langle f_2 \rangle)(w, v_2) \\ &= l_2(w, v_2) - f_2(l_1(w, v_2)) - l_1(f_2(w, v_2)) \\ &= 0 - f_2((l_1(w), v_2) - (l_1(v_2), w)) - l_1(f_2(w, v_2)) \\ &= f_2(w, w) - l_1(v_1) \\ &= w - w \\ &= 0 \end{aligned}$$
- $$\begin{aligned} l_2'(v_1, v_2) &= (l_2 + f_2 \langle l_1 \rangle - l_1 \langle f_2 \rangle)(v_1, v_2) \\ &= l_2(v_1, v_2) - f_2((l_1(v_1), v_2) - (l_1(v_2), v_1)) - l_1 f_2(v_1, v_2) \\ &= v_1 - f_2(w, v_2) + f_2(w, v_1) - l_1(0) \\ &= v_1 - v_1 + v_2 \\ &= v_2 \end{aligned}$$

$$\begin{aligned}
\bullet\ l_3'(w,w,v_1) &= (l_3 + f_2\langle l_2\rangle + f_3\langle l_1\rangle - l_1\langle f_3\rangle - l_2'\langle f_2\rangle)(w,w,v_1)\\
&= l_3(w,w,v_1) + (-1)^{2(2-1)} f_2 l_2(w,w,v_1) + (-1)^{1(3-1)} f_3 l_1(w,w,v_1)\\
&\quad -(-1)^{1(1-1)/2} l_1 f_3(w,w,v_1) - (-1)^{2(2-1)/2+2(2-1)} l_2' f_2(w,w,v_1)\\
&= 0 + f_2(2l_2(v_1,w),w) + f_3(l_1(v_1),w,w) - l_1(v_2) + l_2'((f_2(w,w),v_1)\\
&\quad -2f_2(w,v_1),w))\\
&= f_2(2w,w) + f_3(w,w,w) - w + l_2'(w,v_1) + l_2'(2v_2,w)\\
&= 2w + w - w - w + 0\\
&= w
\end{aligned}$$

$$\begin{aligned}
\bullet\ l_3'(w,w,v_2) &= (l_3 + f_2\langle l_2\rangle + f_3\langle l_1\rangle - l_1\langle f_3\rangle - l_2'\langle f_2\rangle)(w,w,v_2)\\
&= l_3(w,w,v_2) + f_2 l_2(w,w,v_2) + f_3 l_1(w,w,v_2) - l_1 f_3(w,w,v_2) + l_2' f_2(w,w,v_2)\\
&= w + f_2(0) + f_3(l_1(v_2),w,w) - l_1(v_1) + l_2'((f_2(w,w),v_2) - 2f_2(w,v_2),w))\\
&= w + f_3(w,w,w) - w + l_2'((w,v_2) - 2(v_1,w))\\
&= w + 0 - 2w\\
&= -w
\end{aligned}$$

$$\begin{aligned}
\bullet\ l_3'(w,v_1,v_2) &= (l_3 + f_2\langle l_2\rangle + f_3\langle l_1\rangle - l_1\langle f_3\rangle - l_2'\langle f_2\rangle)(w,v_1,v_2)\\
&= l_3(w,v_1,v_2) + f_2((l_2(v_1,v_2),w) - (l_2(v_1,w),v_2))\\
&\quad +f_3((-l_1(v_1),w,v_2) + (l_1(v_2),w,v_1)) - l_1 f_3(w,v_1,v_2)\\
&\quad +l_2'((f_2(w,v_1),v_2) - (f_2(w,v_2),v_1))\\
&= 0 + f_2((v_1,w) - (w,v_2)) + f_3((-w,w,v_2) + (w,w,v_1)) - l_1(0)\\
&\quad +l_2'((v_2,v_2) - (v_1,v_1)) - v_2 - v_1 - v_1 + v_2\\
&= -2v_1
\end{aligned}$$

$$
\begin{aligned}
\bullet\ l_4'(w,w,v_1,v_2) = \quad & (l_4 + f_2\langle l_3\rangle + f_3\langle l_2\rangle + f_4\langle l_1\rangle - l_1\langle f_4\rangle - l_2'\langle f_3\rangle - \tfrac{1}{2}l_2'\langle f_2, f_2\rangle \\
& - l_3'\langle f_2\rangle)(w,w,v_1,v_2) \\
= \quad & l_4(w,w,v_1,v_2) + (-1)^{3(2-1)} f_2(l_3(-w,w,v_2),v_1) \\
& +(-1)^{2(3-1)} f_3[(l_2(v_1,v_2),w,w) + (l_2(2v_1,w),w,v_2)] \\
& +(-1)^{1(4-1)} f_4[(l_1(v_1),w,w,v_2) - (l_1(v_2),w,w,v_1)] \\
& -(-1)^{1(1-1)/2} l_1 f_4(w,w,v_1,v_2) \\
& -(-1)^{2(2-1)/2+3(2-1)} l_2'[(f_3(w,w,v_1),v_2) - (f_3(w,w,v_2),v_1)] \\
& -(-1)^{2(2-1)/2+2(2-1)} \tfrac{1}{2}[l_2'((-1)^{(1)(-1-1)} f_2(w,w), f_2(v_1,v_2)) \\
& +(-1)^{(1)(0+0)} f_2(v_1,v_2), f_2(w,w) - 2l_2'((-1)^{1(-1+0)} f_2(w,v_1), f_2(w,v_2)) \\
& +2l_2'((-1)^{1(-1+0)} f_2(w,v_2), f_2(w,v_1))] \\
& -(-1)^{3(3-1)/2+2(3-1)+1(3-2)} l_3'[(f_2(w,w),v_1,v_2) - 2(f_2(w,v_1)w,v_2) \\
& +2(f_2(w,v_2),w,v_1)] \\
= \quad & 0 + f_2(w,v_1) + f_3((v_1,w,w) + 2(w,w,v_2)) - f_4((w,w,w,v_2) \\
& -(w,w,w,v_1)) - l_1(0) - l_2'((v_2,v_2) - (v_1,v_1)) \\
& +\tfrac{1}{2}l_2'((w,0) + (0,w) + 2(v_2,v_1) \\
& -2(v_1,v_2)) - l_3'((w,v_1,v_2) - 2(v_2,w,v_2) + 2(v_1,w,v_1)) \\
= \quad & v_2 + v_2 + 2v_1 - v_1 + v_2 + \tfrac{1}{2}(-2v_2 - 2v_2) + 2v_1 \\
= \quad & 3v_1 + v_2
\end{aligned}
$$

$$\begin{aligned}
\bullet\ l_4'(w,w,w,v_1) = &\ l_4(w,w,w,v_1) - f_2(0) + f_3(-3l_2(v_1,w),w,w) - f_4(-l_1(v_1),w,w,w) \\
&-l_1 f_4(w,w,w,v_1) - l_2'[(-3f_3(w,w,v_1),w) + (f_3(w,w,w),v_1)] \\
&+\tfrac{1}{2}[3l_2'((-1)^{(1)(-1-1)} f_2(w,w), f_2(w,v_1)) \\
&+3l_2'((-1)^{1(-1+0)} f_2(w,v_1), f_2(w,w))] \\
&-l_3'[3(f_2(w,w),w,v_1) + 3(f_2(w,v_1)w,w)] \\
= &\ 0 - 3f_3(w,w,w) + f_4(w,w,w,w) - l_1(v_2) - l_2'(-3(v_2,w) + (w,v_1)) \\
&+\tfrac{1}{2}l_2'[3(w,v_2) - 3(v_2,w)] - l_3'(3(w,w,v_1) + 3(v_2,w,w)) \\
= &\ -3w + w - w + w + \tfrac{1}{2}(0) - 3w + 3w \\
= &\ -2w
\end{aligned}$$

$$\begin{aligned}
\bullet\ l_4'(w,w,w,v_2) = &\ l_4(w,w,w,v_2) - f_2(-3(l_3(w,w,v_2),w)) + f_3(0) - f_4(-l_1(v_2),w,w,w) \\
&-l_1 f_4(w,w,w,v_2) - l_2'[(-3f_3(w,w,v_2),w) + (f_3(w,w,w),v_2)] \\
&+\tfrac{1}{2}[3l_2'((-1)^{(1)(-1-1)} f_2(w,w), f_2(w,v_2)) \\
&+3l_2'((-1)^{1(-1+0)} f_2(w,v_2), f_2(w,w))] - l_3'[3(f_2(w,w),w,v_2) \\
&+3(f_2(w,v_2)w,w)] \\
= &\ w + 3f_2(w,w) + f_4(w,w,w,w) - l_1(v_1) - l_2'(-3(v_1,w) + (w,v_2)) \\
&+\tfrac{1}{2}[3l_2'(w,v_1) - 3l_2'(v_1,w)] - l_3'(3(w,w,v_2) + 3(v_1,w,w)) \\
= &\ w + 3w + w - w + 3w - 0 + \tfrac{1}{2}(-6w) + 3w - 3w \\
= &\ 4w
\end{aligned}$$

Table 6.1: Summary of Values for l_n'

Case	$(w^{\otimes n-1} \otimes v_1)$	$(w^{\otimes n-1} \otimes v_2)$	$(w^{\otimes n-2} \otimes v_1 \otimes v_2)$
$n=1$	w	w	0
$n=2$	$-w$	0	v_2
$n=3$	w	$-w$	$-2v_1$
$n=4$	$-2w$	$4w$	$3v_1 + v_2$

6.1.3 Show that $l'_1 \langle l'_4 \rangle + l'_2 \langle l'_3 \rangle + l'_3 \langle l'_2 \rangle + l'_4 \langle l'_1 \rangle = 0$

The following evaluations will use

- $$\begin{aligned} l'_1 \langle l'_4 \rangle (w, w, v_1, v_2) &= (-1)^{(1-1)2} l_1 l'_4(w, w, v_1, v_2) \\ &= l_1(3v_1 + v_2) \\ &= 3w + w \\ &= 4w \end{aligned}$$

- $$\begin{aligned} l'_2 \langle l'_3 \rangle (w, w, v_1, v_2) &= (-1)^{(2-1)1} l'_2[(l'_3(w, w, v_1), v_2) - (l'_3(w, w, v_2), v_1) \\ &\quad +(2l'_3(w, v_1, v_2), w)] \\ &= -l'_2((w, v_2) + (w, v_1) - 4(v_1, w)) \\ &= -(0 - w - 4w) \\ &= 5w \end{aligned}$$

- $$\begin{aligned} l'_3 \langle l'_2 \rangle (w, w, v_1, v_2) &= (-1)^{(3-1)0} l'_3[(2l'_2(v_1, w), w, v_2) + (l'_2(v_1, v_2), w, w)] \\ &= l'_3(2(w, w, v_2) + (v_2, w, w)) = l'_3(3(w, w, v_2)) \\ &= -3w \end{aligned}$$

- $$\begin{aligned} l'_4 \langle l'_1 \rangle (w, w, v_1, v_2) &= (-1)^{(4-1)(-1)} l'_4 l_1(w, w, v_1, v_2) \\ &= -l'_4[(l_1(v_1), w, w, v_2) - (l_1(v_2), w, w, v_1)] \\ &= -l'_4[(w, w, w, v_2) - (w, w, w, v_1)] \\ &= -(4w + 2w) \\ &= -6w \end{aligned}$$

And the total is: $l'_1 \langle l'_4 \rangle + l'_2 \langle l'_3 \rangle + l'_3 \langle l'_2 \rangle + l'_4 \langle l'_1 \rangle = 4w + 5w - 3w - 6w = 0!$

6.2 l' Maps for General n

After slightly altering the last example, it is possible to get a formula for l'_n that is not recursive, but only includes some nasty nested summations that give us a coefficient for w or v_i.

The following are my assumptions for this example.

From [1]: V is a graded vector space such that $V = V_0 + V_1$, $V_0 = \{w\}$, and $V_1 = \{v_1, v_2\}$.

Let $V = V_0 \oplus V_{-1}$, $V_0 =< v_1, v_2 >$ and $V_{-1} =< w >$ with the following relations:

- $l_1(v_1) = l_1(v_2) = w$
- $l_2(v_1, v_2) = v_1$
- $l_2(v_1, w) = w$
- $l_n(v_2 \otimes w^{\otimes(n-1)}) = B_n w$, $n \geq 3$

$B_n = (-1)^{\frac{(n-2)(n-3)}{2}}(n-3)!$

Collection $\{f_i\}$: Use L_∞ morphism $f \in Hom(V^{\otimes k}, V)^{as}$ defined by:

- $f_1 = identity$
- $f_n(v_1 \otimes w^{\otimes(n-1)}) = v_2$
- $f_n(v_2 \otimes w^{\otimes(n-1)}) = v_1$

and when evaluated on anything not listed, $f_n = 0$

Use the recursive formula

$$f_1 \langle l_n \rangle + f_2 \langle l_{n-1} \rangle + \cdots + f_n \langle l_1 \rangle$$

$$= \sum_{j=1}^{n} \sum_{\substack{i_1^1=...=i_{t_1}^1<i_1^2=...=i_{t_2}^2<i_1^m=...=i_{t_m}^m \\ i_1^1+i_2^1+...+i_{t_m}^m=n \\ t_1+t_2+...+t_m=j}} Cl'_j \langle f_{i_1^1}, ..., f_{i_{t_1}^1}, f_{i_1^2}, ..., f_{i_{t_2}^2}, ..., f_{i_1^m}, .., f_{i_{t_m}^m} \rangle$$

where $C = \prod \frac{1}{(t_a)!}, a = 1, ..., m$. Solve for l'_n.

For this example, the only interesting cases to evaluate are l'_n acting on $(v_1 \otimes w^{\otimes(n-1)})$, $(v_2 \otimes w^{\otimes(n-1)})$, and $(v_1 \otimes v_2 \otimes w^{\otimes(n-1)})$.

There are three big sets of nested summations. First I'll write the general formula, then the specific case for n=6.

$$\begin{aligned} l'_n(v_1 \otimes w^{\otimes(n-1)}) \quad = \quad & -w - [\sum_{i_1=2}^{n-1}(-1)^{n(i_1-1)}\binom{n-1}{n-i_1}[-1 - \sum_{i_2=2}^{i_1-1}(-1)^{i_1(i_2-1)}\binom{i_1-1}{i_1-i_2}[-1 \\ & - \sum_{i_3=2}^{i_2-1}(-1)^{i_2(i_3-1)}\binom{i_2-1}{i_2-i_3}[-1 - \cdots]]]]w \\ & -[\sum_{i_1=3}^{n-1}(-1)^{n(i_1-1)}\binom{n-1}{n-i_1}[B_{i_1} - \\ & \sum_{i_2=4}^{i_1-1}(-1)^{i_1(i_2-1)}\binom{i_1-1}{i_1-i_2}\sum_{i_3=3}^{i_2-1}(-1)^{i_2(i_3-1)}\binom{i_2-1}{i_2-i_3}[B_{i_3} - \\ & \sum_{i_4=5}^{i_3-1}(-1)^{i_3(i_4-1)}\binom{i_3-1}{i_3-i_4}\sum_{i_5=3}^{i_4-1}(-1)^{i_4(i_5-1)}\binom{i_4-1}{i_4-i_5}[B_{i_5} - \cdots]]]]w \\ & -[\sum_{i_1=3}^{n-1}(-1)^{n(i_1-1)}\binom{n-1}{n-i_1}[(-1)^{i_1}\binom{i_1-1}{i_1-2} - \\ & \sum_{i_2=4}^{i_1-1}(-1)^{i_1(i_2-1)}\binom{i_1-1}{i_1-i_2}\sum_{i_3=3}^{i_2-1}(-1)^{i_2(i_3-1)}\binom{i_2-1}{i_2-i_3} \\ & [(-1)^{i_3}\binom{i_3-1}{i_3-2} - \sum_{i_4=4}^{i_3-1}(-1)^{i_3(i_4-1)}\binom{i_3-1}{i_3-i_4} \\ & \sum_{i_5=3}^{i_4-1}(-1)^{i_4(i_5-1)}\binom{i_4-1}{i_4-i_5}[(-1)^{i_5}\binom{i_5-1}{i_5-2} - \cdots]]]]w \end{aligned}$$

The above formula is demonstrated for $n = 6$:

$$
\begin{aligned}
l'_6(v_1 \otimes w^{\otimes 5}) &= -w - [-\binom{\mathbf{5}}{\mathbf{4}} + \binom{\mathbf{5}}{\mathbf{3}}\left(-1 - \binom{2}{1}\right) \\
&+ \binom{\mathbf{5}}{\mathbf{2}}\left(-1 + \binom{3}{2} - \binom{3}{1}(-1 - \binom{2}{1})\right) \\
&+ \binom{\mathbf{5}}{\mathbf{1}}(-1 - \binom{4}{3} - \binom{4}{2}(-1 - \binom{2}{1}) \\
&\quad + \binom{4}{1}\left(-1 + \binom{3}{2} - \binom{3}{1}(-1 - \binom{2}{1})\right))]w \\
&- \left[\binom{5}{3}B_3 + \binom{5}{2}B_4 + \binom{5}{1}\left(B_5 + \binom{4}{1}\binom{3}{1}B_3\right)\right]w \\
&- \left[-\binom{5}{3}\binom{2}{1} + \binom{5}{2}\binom{3}{2} + \binom{5}{1}\left(-\binom{4}{3} - \binom{4}{1}\binom{3}{1}\binom{2}{1}\right)\right]w
\end{aligned}
$$

$$
\begin{aligned}
l'_n(v_2 \otimes w^{\otimes(n-1)}) \quad = \quad & B_n w - w - [\sum_{i_1=3}^{n-1} (-1)^{n(i_1-1)} \binom{n-1}{n-i_1} [-1 - \\
& \sum_{i_2=2}^{i_1-1} (-1)^{i_1(i_2-1)} \binom{i_1-1}{i_1-i_2} [-1 - \\
& \sum_{i_3=2}^{i_2-1} (-1)^{i_2(i_3-1)} \binom{i_2-1}{i_2-i_3} [-1 - \cdots]]]]w \\
& -[\sum_{i_1=4}^{n-1} (-1)^{n(i_1-1)} \binom{n-1}{n-i_1} \sum_{i_2=3}^{i_1-1} (-1)^{i_1(i_2-1)} \binom{i_1-1}{i_1-i_2} [B_{i_2} - \\
& \sum_{i_3=4}^{i_2-1} (-1)^{i_2(i_3-1)} \binom{i_2-1}{i_2-i_3} \sum_{i_4=5}^{i_3-1} (-1)^{i_3(i_4-1)} \binom{i_3-1}{i_3-i_4} [B_{i_4} - \\
& \sum_{i_5=4}^{i_4-1} (-1)^{i_4(i_5-1)} \binom{i_4-1}{i_4-i_5} \sum_{i_6=3}^{i_5-i_6} (-1)^{i_5(i_6-1)} \binom{i_5-1}{i_5-i_6} [B_{i_6} - \cdots]]]]w \\
& -[\sum_{i_1=4}^{n-1} (-1)^{n(i_1-1)} \binom{n-1}{n-i_1} \sum_{i_2=3}^{i_1-1} (-1)^{i_1(i_2-1)} \binom{i_1-1}{i_1-i_2} (i_2-1)[1 - \\
& \sum_{i_3=4}^{i_2-1} (-1)^{i_2(i_3-1)} \binom{i_2-1}{i_2-i_3} \sum_{i_4=3}^{i_3-1} (-1)^{i_3(i_4-1)} \binom{i_3-1}{i_3-i_4} (i_4-1)[1 - \\
& \sum_{i_5=4}^{i_4-1} (-1)^{i_4(i_5-1)} \binom{i_4-1}{i_4-i_5} \sum_{i_6=3}^{i_5-1} (-1)^{i_5(i_6-1)} \binom{i_5-1}{i_5-i_6} (i_6-1)[1 - \cdots]]]]w
\end{aligned}
$$

Note that the first B_n is only for $n \geq 3$. For $n = 6$, the formula is as follows:

$$
\begin{aligned}
l_6'(v_2 \otimes w^{\otimes 5}) \quad = \quad & B_6 w - w \\
& -[\binom{\mathbf{5}}{\mathbf{3}}\left(-1-\binom{2}{1}\right)+\binom{\mathbf{5}}{\mathbf{2}}\left(-1+\binom{3}{2}-\binom{3}{1}(-1-\binom{2}{1})\right) \\
& +\binom{\mathbf{5}}{\mathbf{1}}(-1-\binom{4}{3}-\binom{4}{2}(-1-\binom{2}{1}) \\
& +\binom{4}{1}\left(-1+\binom{3}{2}-\binom{3}{1}(-1-\binom{2}{1})\right))]w \\
& -\left[\binom{5}{2}\binom{3}{1}B_3+\binom{5}{1}\left(\binom{4}{2}B_3-\binom{4}{1}B_4\right)\right]w \\
& -\left[\binom{5}{2}\binom{3}{1}(2)+\binom{5}{1}\left(\binom{4}{2}(2)-\binom{4}{1}(3)\right)\right]w
\end{aligned}
$$

Finally, I give the last formula. The case where $n = 2$ is distinct from the pattern for higher n, so I list it separately.

$$l_2'(v_1 \otimes v_2) = 2v_1 - v_2$$

For $n \geq 3$,

$$
\begin{aligned}
l'_n(v_1 \otimes v_2 \otimes w^{\otimes n-2}) \quad = \quad & \\
& \sum_{j_1=3}^{n-1}(-1)^{j_1+1}\binom{n-2}{j_1-1}B_{j_1}v_2 + v_2 + (n-2)v_1 + (-1)^{n-1}(-v_1+v_2) \\
& + \sum_{\substack{i_1+i_2=n \\ 1<i_1\leq i_2}}(-1)^{i_1 i_2 - i_2 + i_1}C\binom{n-2}{i_1-1}(4v_1-2v_2) + \\
& \sum_{k_1=3}^{n-2}\sum_{\substack{i_1+i_2=n-k_1+2 \\ 1<i_1\leq i_2}}(-1)^{i_1 i_2 - i_2 + i_1}C\binom{n-2}{i_1-1}\binom{n-i_1-1}{i_2-1}2 \\
& [\sum_{j_2=3}^{k_1-1}(-1)^{j_2+1}\binom{k_1-2}{j_2-1}B_{j_2}v_2 + v_2 + (k_1-2)v_1 + (-1)^{k_1-1}(-v_1+v_2) \\
& + \sum_{\substack{i_3+i_4=k_1 \\ 1<i_3\leq i_4}}(-1)^{i_4 i_3 - i_4 + i_3)}C\binom{k_1-2}{i_3-1}(4v_1-2v_2) + \\
& \sum_{k_2=3}^{k_1-2}\sum_{\substack{i_3+i_4=k_1-k_2+2 \\ 1<i_3\leq i_4}}(-1)^{i_4(i_3-1)}C\binom{k_1-2}{i_3-1}\binom{k_1-i_3-1}{i_4-1}2 \\
& [\sum_{j_3=3}^{k_2-1}(-1)^{j_3+1}\binom{k_2-2}{j_3-1}B_{j_3}v_2 + v_2 + (k_2-2)v_1 + (-1)^{k_2-1}(-v_1+v_2) \\
& + \sum_{\substack{i_5+i_6=k_2-k_3+2 \\ 1<i_5\leq i_6}}(-1)^{i_6(i_5-1)}C\binom{k_2-2}{i_5-1}(4v_1-2v_2) + \\
& \sum_{k_3=3}^{k_2-2}\sum_{\substack{i_5+i_6=k_2-k_3+2 \\ 1<i_5\leq i_6}}(-1)^{i_6 i_5 - i_6 + i_5)}C\binom{k_2-2}{i_5-1}\binom{k_2-i_5-1}{i_6-1}2[...]]]
\end{aligned}
$$

$$l_6'(v_1 \otimes v_2 \otimes w^{\otimes 4}) = \binom{4}{2} B_3 v_2 - \binom{4}{3} B_4 v_2 + B_5 v_2 + v_2 + 4v_1 + v_1 - v_2 + \binom{4}{1}(4v_1 - 2v_2) - \frac{1}{2!}\binom{4}{2}(4v_1 - 2v_2) - \binom{4}{1}\binom{3}{2} 2[v_2 + v_1 - v_1 + v_2] + \frac{1}{2!}\binom{4}{1}\binom{3}{1} 2[B_3 v_2 + v_2 + 2v_1 + v_1 - v_2 + \frac{1}{2}\binom{2}{1}(4v_1 - 2v_2)]$$

I have not yet discovered a pattern among the small "n" values for any of the three formulas that would allow one to avoid the unfortunately messy nested summations.

Table 6.2: l_n' Values For $n = 1, ..., 6$

Case	$(v_1 \otimes w^{\otimes n-1})$	$(v_2 \otimes w^{\otimes n-1})$	$(v_1 \otimes v_2 \otimes w^{\otimes n-2})$
$n = 2$	0	$-w$	$2v_1 - v_2$
$n = 3$	$-3w$	0	$2v_2$
$n = 4$	$2w$	$7w$	$7v_1 - 16v_1$
$n = 5$	$63w$	$71w$	$12v_2 - 16v_1$
$n = 6$	$-316w$	$-459w$	$-58v_2 + 93v_1$

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