ABSTRACT

SMITH, RYAN CUMMINGS. A Comparison of Middle School Students' Mathematical Arguments in Technological and Non-Technological Environments. (Under the direction of Dr. Karen Hollebrands).

Prior research on students' uses of technology has suggested it can be used to support students' development of formal justifications and proofs. The ways in which these technologies influence the construction of arguments and proofs remain uncertain. Furthermore, research has not been conducted that compares the arguments students develop while working in a technological environment to those created by students working in a non-technological environment. This study characterized and compared the arguments eighth grade mathematics students created while working in technological and non-technological environments as the students worked through a unit in which they investigated and developed definitions, properties, theorems, and classification systems related to triangles.

A teaching experiment methodology was utilized with two eighth grade mathematics classes in an urban middle school in the southeast United States. For one class, technology played an integral role using a dynamic geometry environment, Geometer's Sketchpad (Jackiw, 2001) to explore and investigate geometric concepts. For the other class, the students used non-technological mathematical tools such as snap-cubes, rulers, and protractors. The arguments created by three pairs of students in each class were documented, analyzed, and compared. Toulmin's (1958) argumentation model was used to analyze the content and structure of the arguments, including the ways in which the students used the tools (technological and non-technological). Findings from this study indicate that students in the technology class created more arguments than their counterparts in the non-technology class, which may be related to the dynamic abilities of the tools rather than merely the use of

the technology. In addition, students in both classes were less likely to make their reasoning explicit when using tools, technological or non-technological, which may be related to the task on which the students were working. When students were working on generalization or conjecturing tasks, the students were rarely actively using the tools and the warrants tended to be explicit. However, when the students were prompted to make their reasoning explicit while using the technology, the students obliged. The results from this study suggest that teachers need to design tasks and activities that utilize tools with dynamic abilities, capitalize on the dynamic affordance of the tools, and ask students to make their reasoning explicit when using these tools.

A Comparison of Middle School Students' Mathematical Arguments in Technological and Non-Technological Environments

by
Ryan Cummings Smith

A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Mathematics Education

Raleigh, North Carolina

May 11, 2010

APPROVED BY:

Dr. Karen Hollebrands Committee Chair	Dr. Hollylynne Lee
Dr. Karen Keene	Dr. Irina Kogan

DEDICATION

For Rachel

BIOGRAPHY

Ryan Cummings Smith was born in Tulsa, Oklahoma on December 31, 1975. After graduating from Mandeville High School in Mandeville, Louisiana, he enrolled at Auburn University in Auburn, Alabama in 1994 where he earned a Bachelor's of Science degree in Secondary Mathematics Education. Upon graduation from Auburn University in 1999, he enrolled at the University of Houston where he completed a Master's of Science in Applied Mathematics in 2001.

After graduating from the University of Houston, Ryan accepted a position as a mathematics teacher at Alief Hastings High School in Alief, Texas. During his five year tenure he taught Geometry, Pre-Calculus, Algebra II, Math Modeling and Applications, and Discrete Math. In 2006, Ryan returned to graduate school to pursue a doctorate in mathematics education at North Carolina State University. During his first two years, Ryan worked as a graduate teaching/research assistant teaching/co-teaching the course "Teaching Mathematics with Technology", supervising student-teachers in their culminating field experience, and working on the Integrating Dynamic Geometry Software in College Geometry Project. During his last two years, Ryan worked as a graduate researcher assistant on the Preparing to Teach Mathematics with Technology and Recognizing the Accelerated Math Potential in Underrepresented People projects.

Ryan has accepted a Visiting Assistant Professor position in the department of Mathematics and Science Education at the University of Georgia in Athens, Georgia. He will begin his job there in the fall of 2010.

ACKNOWLEDGMENTS

My deepest gratitude and appreciation to:

My advisor and dissertation chair, Dr. Karen Hollebrands, for her assistance, guidance, support, insight, and feedback on the study presented in this dissertation, for providing countless other research opportunities, for being an unending source of wisdom and encouragement both personally and professionally, and for her willingness to sacrifice so much time to further my education;

My committee, Dr. Karen Keene, Dr. Irina Kogan, and Dr. Hollylynne Stohl Lee, for their support, assistance, guidance, and wisdom on this study and my education;

Members of the PTMT and RAMP-UP research teams, Lynn Albers, Dr. Laura Bottomley, Marggie Gonzalez, James Hedges, Sarah Ives, Liz Parry, Althea Smith, and Tina Starling for their support, encouragement, and assistance in conducting this study;

Faculty and Staff at NC State, Dr. Sally Berenson, Christy Buck, Dr. Jere Confrey, Dr. Allen Maloney, Dr. Allison McCulloch, Shannon Patton, Dr. Lee Stiff, Dr. Paola Sztajn, and Eileen Williams for their encouragement and direction on this study and my education;

My fellow graduate students, Valerie Bell, Sherry Booth, Matt Campbell, Charity Cayton, Kristen Corbell, Cyndi Edgington, Richelle Dietz, Anthony Dove, Michael Downey, Peter Eley, Lisa Hervey, Krista Holstein, Rachael Kenney, Erin Krupa, Amanda Lambertus, Katie Lovin, Lodge McCammon, Gemma Mojica, Marrielle Myers, Kenny Nguyen, John Pritchett, Tyrone Washington, and Holt Wilson; for their support, encouragement, assistance, perspective, and friendship without which I would not have made it through this program;

The teacher and students with whom I conducted this study, Karen, David, Erica, Heather, Mary, Jim, Clair, Amy, Judy, Andy, Frank, Ellen, and Bob for their willingness to allow me into their classroom and share their thinking;

My family, Michael, Patricia, Sarah, Jennifer, Kyle, Allie, Colsen, Bob, Carol, Nathan, Nicole, Zachary, Jordan, Catherine, Matthew, Monette, Ed, Linda, Coleman, Jim, Cynthia, Clark, Barbara, Francis, Rhymer, Brian, Leslie, Erin, Chris, Taylor, Julie, and Babs for their continuous love and support;

My daughter, Audrey, for providing inspiration, perspective, and love; and,

My wife, Rachel, for being my biggest cheerleader, for her confidence in me even when I did not, for her unending support, for her sacrifice, and, most importantly, for her love.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

Rationale

Today's students have always lived in world where information is just a click away. This access to a large amount of information suggests that students need to be informationally literate; they need the ability to access, evaluate, organize and use information in order to learn, problem-solve, and make decisions (Bruce, 1997). The development of students' abilities to make arguments using sound reasoning is one of the fundamental goals of education across the curriculum (e.g. National Council of Teachers of Mathematics (NCTM), 2000; National Research Council (NRC), 1996). In mathematics, arguments using sound reasoning generally take the form of proofs.

Mathematical proof and justification is one of the central topics of mathematics and is important in the teaching and learning of mathematics (NCTM, 2000). In *Principles and Standards for School Mathematics*, NCTM (2000) states, "being able to reason is essential to understanding mathematics" (p. 56). Furthermore, NCTM states, "reasoning and proof should be a consistent part of students' mathematical experience in pre-kindergarten through grade 12" (p. 56). However, the difficulty students have with proof is well documented (e.g. Chazan, 1993; Edwards, 1997; Harel & Sowder, 1998; Healy & Hoyles, 2000). This difficulty is troublesome because proof is the accepted means of verification and communication in the field of mathematics (de Villiers, 1999).

de Villiers's statement that proof is the accepted means of communication in the mathematics field indicates that proof and communication are intertwined. In addition to the standard for reasoning and proof, NCTM (2000) also includes a standard titled

"Communication" which states, "communication is an essential part of mathematics and mathematics education" (p.60). NCTM (2000) indicates that students who have the opportunity to justify their solutions will gain better mathematical understanding as they attempt to convince their peers especially if others do not agree.

Definitions

Even though proof, justification, and argumentation are related, they are not synonymous. For the purpose of this study, a mathematical *proof* will be defined as a logically correct deductive argument based on given conditions, definitions, theorems, and postulates within an axiom system, which may or may not be apparent to the person building the argument. *Proving* is the process of constructing or attempting to construct a deductive argument. *Justifying* is giving reasons for a mathematical action or statement "in an attempt to communicate the legitimacy of one's mathematical activity" (Yackel & Hanna, 2003, p. 229). Justifying encompasses proving, but is not limited to it. A *justification* is the product of justifying. A mathematical *argument* may be defined as a sequence of mathematical statements that aims to convince, whereas *argumentation* may be regarded as a process in which a logically connected mathematical discourse is developed. Toulmin (1958) conceptualizes the components of arguments as data, claims, warrants, backing, qualifiers, and rebuttals that are provided in an attempt to convince an audience of the validity of a particular claim. Within argumentation, justification often occurs as arguers reveal their data, warrants, and backing to support their claims. Therefore, argumentation includes justifying and justifying includes proving which is illustrated in Figure 1.

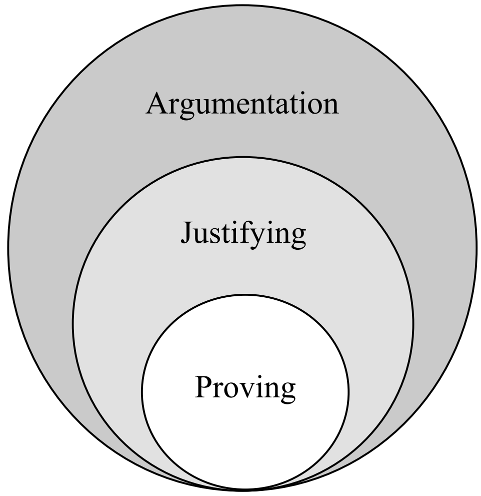


Figure 1. Relationship between proving, justifying, and argumentation.

Tool Mediation

Students' uses of tools in a mathematics classroom may influence individuals' understandings and, thus, the arguments they create. From a sociocultural perspective (e.g. Lave & Wenger, 1991; Vygotsky, 1978), the development of mathematical reasoning and understanding are culturally mediated, through language and through the use of artifacts. Because meaning can be derived from language and the use of artifacts, both are considered tools. From this perspective, there is a strong connection between students' uses of tools and their learning because tools are the instruments of access to the knowledge, activities, and practices of a classroom community (Lave & Wenger, 1991). In a community of practice, such as the mathematics classroom, understanding and knowledge are developed through students' participation, or use, of tools. Productive activity with the tools and the understanding of them are not separate. Verillon and Rabardel (1995) indicate that tools do not exist as tools until the user is able "to appropriate it for himself and has integrated it with his activity" (p. 84). The use of tools and understanding their significance interact to become a single learning process. The use of tools is an important aspect of participating in a community of practice. Similarly, the use of language also is an important aspect of participating in the community. Although it is important for students to learn from the discussions of other members of the community, it is even more important to learn how to speak in the community of practice (Lave & Wenger, 1991). The affordances of the tools dictate the understandings that one can construct within the community and, in turn, the understandings that can be developed by the classroom community. Thus, the discourse of the classroom, which includes argumentation, is based, in part, on the use of tools.

In mathematics classrooms, tools are frequently used. Traditional tools such as paper, pencil, compass, straight-edge, ruler, and scale have been used for thousands of years to explore, conjecture, and make meaning of mathematics, especially geometry. With the advent of the microprocessor, technology tools have become more prevalently used in the mathematics classroom. One type of technology tool that has become popular in the teaching and learning of geometry are dynamic geometry environments.

Dynamic Geometry Environments

Dynamic geometry environments (DGEs), such as *The Geometer's Sketchpad* (Jackiw, 2001), are technology tools in which users are able to create drawings, make measurements, and drag the drawing (or a portion of it) while the drawing maintains the dependant relationships that were formed in the construction of the drawing. Many teachers, researchers, and professional organizations have suggested the use of dynamic geometry environments to teach geometry (e.g. NCTM, 2000).

Research on DGEs include students' uses of the drag feature (e.g. Arzarello, Olivero, Paola, & Robutti, 2002), students' uses of the measurement feature (e.g. Olivero & Robutti, 2007), the strategies students employ when using this tool (e.g. Hollebrands, 2007), and the cognitive movement between the spatio-graphical field and the theoretical field (e.g. C. Laborde, 2005). Research has also been conducted on students' uses of the technological tool and the proofs, arguments, and justifications that develop from these uses (e.g. Jones, 2000; Leung & Lopez-Real, 2002; Mariotti, 2002b; Marrades & Gutiérrez, 2000). Although some researchers have shown that the use of dynamic geometry environments support students' development of arguments and proofs (e.g. Christou, Mousoulides, Pittalis, & Pitta-Pantazi, 2004; Hadas, Hershkowitz, & Schwarz, 2000; Hoyles & Healy, 1999; Jones, 2000; Mariotti, 2002b; Öner, 2008; Wares, 2007), others have warned that the uses of such tools may diminish students' views of the need and importance of proof (e.g. Allen, 1996; Hoyles & Jones, 1998).

Argumentation

Even though research on the relationship between students' uses of a DGE and proof has been conducted, little research has specifically analyzed the content and structure of the arguments students create while using this technology. The *content* of an argument includes the discourse and actions that comprise each of the constructs of Toulmin's (1958) model of argumentation which are the data, claim, warrant, backing, rebuttal, and qualifier. The *structure* of the arguments is the way students use and link the data, claim, warrant, backing, rebuttal, and qualifier. By analyzing the content and structure of an argument, one may be able to determine the relationships between the ways in which students use the tool, the arguments created while using and in response to using the tool, and whether the argument is based on sound reasoning. By comparing the content and structure of the arguments made by students using technology to those made by students using non-technological tools, one might be able to analyze whether the use of the tool influences the types of reasoning that are provided and whether these reasoning are sound.

Purpose and Research Questions

In order to study students' uses of technology and the arguments that develop from the students' uses of technology in comparison to the arguments created by students working in a non-technological environment, the participants must be of an age in which they are able to articulate their individual justifications. In addition, if students have been exposed to formal proof, then their arguments may be skewed to this particular form of mathematical argumentation. Thus, if the goal of this study is to examine the content and structure of the arguments students create, then there may be little variation in the structure of the arguments if the students have been exposed to formal proofs. Even though middle school students typically do not construct proofs, they do participate in sophisticated mathematical thinking that involves justification and argumentation (e.g. Maher, Powell, Weber, & Lee, 2006). What remains unclear is how middle school students' uses of technology influence the arguments they construct. The purpose of this study is to describe the arguments created by students enrolled in two eighth grade mathematics classes, one in which the students regularly use a dynamic geometry environment and one in which the students use non-technological tools, and compare the content a structure of the arguments created by students in each class. The following research questions will guide this study:

- What are the characteristics of the content and structure of eight grade mathematics students' arguments while working in a technological environment?
- What are the characteristics of the content and structure of eight grade mathematics students' arguments while working in a non-technological environment?
- In what ways do the arguments made by students working in a technological environment compare in content and structure to those made by students working in a non-technological environment?

Even though, a few research studies (Hollebrands, Conner, and Smith, 2010; Lavy, 2006; Maher et al., 2006) have used Toulmin's model to analyze the content and/or structure of students' arguments while working in the presence of technology, the technology used in two of the studies was not a dynamic geometry environment (Lavy, 2006; Maher et al., 2006) and the participants in the other study (Hollebrands, Conner, & Smith, 2010) were college geometry students participating in task-based interviews.

Significance of the Study

Results from this study will inform mathematics teachers and mathematics educators as well as teachers and education researchers in other disciplines. From the results of the analysis of the content and structure of the students' arguments in the presence of technology, teachers and researchers will gain a better understanding of the ways in which students are evaluating and understanding information that emerges from their use of technology. Teachers and researchers can then design activities and tasks that assist students in the evaluation and use of information in the presence of technology. In mathematics, results from the analysis of the structure of the students' arguments may also inform the design of tasks and activities whose goal is to elicit more formal and sophisticated forms of reasoning, namely mathematical proofs. Results from this study may also inform software designers in the ways students use technology and the reasoning that is elicited from these uses.

As the use of technology increases in classrooms, teachers and researchers need to understand how students' arguments change in the presence of technology. The results from the analysis of the differences between the content and structure of the arguments students construct in the presence of technology and the arguments students construct working in a non-technological environment may add to this understanding. Teachers and researchers may gain a better understanding of the relationship between the uses of a tool and the arguments students construct from these uses, which will inform the tasks and activities that teachers select.

Overview and Organization of the Study

In order to capture the development of both the students' individual understanding and the influence of the classroom, this research study is designed as a classroom teaching experiment. "A primary purpose for using teaching experiment methodology is for researchers to experience, firsthand, students' mathematical learning and reasoning" (Steffe & Thompson, 2000, p. 267). This firsthand experience will allow the researcher to document the students' uses of the geometric tools (which includes the technology and non-technological tools), and the arguments students develop while working on tasks that employ these tools. The content and structure of the arguments, including the ways in which students use tools (technological and non-technological), and how the arguments created by students working with non-technological tools compare to those created by students using technology will be reported. Through the analysis of data sources, including classroom teaching episodes and collected artifacts, the argumentation is described. Toulmin's (1958) conceptualization of argumentation to mathematics education is used as a way to analyze the content and structure of the students' arguments.

The next chapter provides a theoretical and empirical grounding for this study, expanding on ideas introduced in this chapter. A description and rationale for the classroom teaching experiment methodology chosen for this study follows in chapter 3. Chapters 4 and 5 describe the arguments students created while working in the technological and non-technological environments, respectively. The synthesis with answers to the research questions is presented in chapter 6 with implications in chapter 7.

CHAPTER 2

Literature Review

This study characterizes and compares the content and structure of the arguments eighth grade students' create while working in a dynamic geometry environment (DGE) and a non-technological environment. The diagram in Figure 2 displays the conceptual framework for this study. Researchers (e.g. Vygotsky, 1978) indicate that a tool, technological or non-technological, can act as a mediator between students' understanding and the mathematics. Students' uses and understanding of the tool will affect the mathematics that can be learned and how students think about the mathematics. Conversely, the design and affordances of the tool will affect the mathematics that students can understand and how they think about the mathematics. Because the use of a tool mediates students' thinking and understanding of mathematics, it may also influence the mathematical arguments they create.

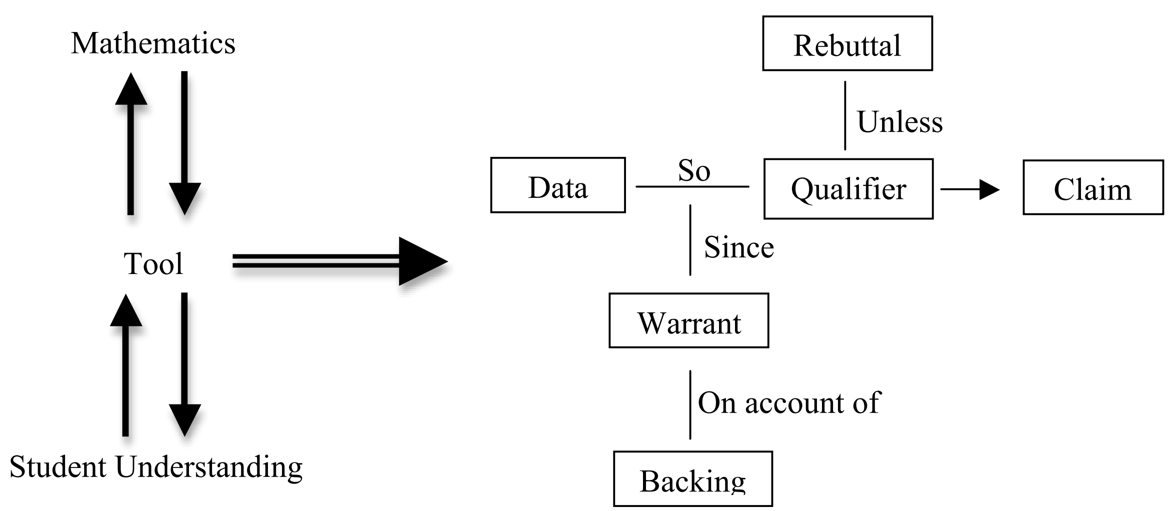


Figure 2. Tool mediation and its influence on mathematical arguments.

This chapter situates the current study using previous empirical and theoretical work in mathematics education. In this chapter, the use of mathematical manipulatives is discussed including their connection to learning. A description of the research that has been conducted on students' uses of DGEs, specifically looking at their use to support the development of arguments and proofs is provided. Then, the role of proof in mathematics and the types of students' justifications are discussed. In addition, a description of what is meant by argumentation and how Toulmin conceptualized it is given. Finally, a description of the research that has been conducted on mathematics students' argumentation is provided.

Mathematical Manipulatives

During the past century, educational researchers and theorists have been promoting the use of manipulatives to assist students in developing knowledge and understanding of mathematics. Mathematical manipulatives are objects designed to be used to concretely represent and illustrate an abstract mathematical concepts (e.g. base 10 blocks, snap-cubes). Dienes's (1969) work showed that the use of multiple and various representations of a concept, 'multiple embodiments' (which include both physical objects and symbols) were necessary to support students' understanding. Piaget (1952) suggested that many students are unable to understand abstract mathematical concepts when presented in words and symbols alone and may need experiences with physical materials and drawings for learning to occur. Bruner (1960, 1986) proposed that students exhibit their understandings in three modes of representation: enactive, iconic and symbolic. The initial mode of representation, enactive, involves encoding action-based information, including the use of physical objects, and storing it in memory. The iconic mode is where the information is stored as an image.

The symbolic mode is the last to develop and involves the storing of information as a code or symbol such as language. Vygotsky (1978) indicates that mathematical reasoning is culturally mediated through language and artifacts. The mediation is composed of two concepts, externally oriented tools (such as manipulatives) and internally oriented signs. Both Piaget and Bruner suggest that students develop knowledge and understanding in a progressive manner which initially begins through their actions with the physical environment. Thus, if students are to develop an abstract understanding of mathematics, it may be advantageous to provide students the opportunity to work with physical objects, or manipulatives.

Students who use manipulatives in their mathematics classes usually outperform those who do not (Sowell, 1989). However, manipulatives do not guarantee success (Baroody, 1989). "Although kinesthetic experience can enhance perception and thinking, understanding does not travel through the fingertips and up the arm" (Ball, 1992, p. 47). To learn mathematics from manipulatives, students need to perceive and understand relations between the manipulatives and other forms of mathematical expression (Gentner & Ratterman, 1991). Manipulatives should not be used by teachers and students as an end but as a means to that end. "Good manipulatives are those that aid students in building, strengthening, and connecting various representations of mathematical ideas," (Clements, 1999, p. 49). The use of manipulatives by students in this 'good' manner may promote the abstraction of mathematical concepts.

Manipulatives may not always be physical objects. Clements (1999) argues that even though computer-based manipulatives may not be concrete, they can provide representations that are just as meaningful to students as physical objects. And, he argues, the computer version of the representations may even be more manageable, accurate, flexible, and extensible than their physical counterparts. If this is the case, then perhaps digital representations of a geometric world may also provide students with a means to reason that can be just as meaningful as the one in which they reside.

Dynamic Geometry Environments

For centuries, geometry has been learned and taught using tools such as a compasses, rulers, and protractors. With the advent of the personal computer, technology tools were developed to assist with the teaching and learning of geometry. One type of software that has gained popularity in the mathematics education community is DGEs. There are over 40 different computer programs that are considered DGEs that provide a digital model of a variety of geometric worlds including the two-dimensional plane (e.g. *The Geometer's Sketchpad* (Jackiw, 2001), *Cabri* (Laborde & Bellemain, 1993)), the three-dimensional space (e.g. *Cabri3D* (Laborde & Bainville, 2005)), and non-Euclidean geometries such as the Poincare disk (e.g. *NonEuclid* (Castellanos, 2008)). These DGEs are characterized by their click and drag feature. "This feature allows users, after a construction is made, to move certain elements of a drawing freely and to observe how other elements respond dynamically to the altered conditions" (Goldenberg & Cuoco, 1998, p.351). The software maintains the geometric relationships used in the construction when a student "drags" one of the objects.

DGE and learning.

When students are working in a DGE, they may be cognitively moving back and forth between the abstract theoretical field, and the perceptual graphical-spatial field. Laborde

(2005) distinguishes between these two constructs by indicating that the theoretical field "denotes the theoretical referents in a geometrical theory, to theoretical objects, relations and operations on these objects as well as to judgments about them that can be expressed in various languages" (p.161); and, the graphical-spatial field "denotes the graphical entities on which it is possible to perform physical actions, and about which it is possible to express ideas, interpretations, opinions, judgments" (p.161). The nature of students' movement between these two fields may be related to the ways the tool, the DGE, mediates their learning.

From the sociocultural perspective, the development of mathematical reasoning is viewed as culturally mediated through language and artifacts. Vygotsky (1978) indicates that a mediating activity is comprised of two concepts, tools and signs.

> The tool's function is to serve as the conductor of human influence on the object of activity; it is externally oriented […]. The sign, on the other hand, changes nothing in the object of a psychological operation. It is a means of internal activity aimed at mastering oneself; the sign is internally oriented. (p. 55)

The process of internalization may transform externally oriented tools into internally oriented signs. Once the tool has been internalized into a sign, it will shape new meanings and function as a semiotic mediator (Vygotsky, 1978).

In DGEs, objects and commands may be considered externally oriented tools of what Laborde (2005) calls the geometric theoretical field, which may be internalized and become means of semiotic mediation. In other words, the objects and commands of a DGE can be internalized as signs for particular aspects of the geometric theoretical field. However, this only occurs if the instructor uses the signs in the concrete realization of classroom activity to introduce students to theoretical thinking. For example, students can use the drag feature of a DGE to assist in determining the transformation that was used to create an image from its pre-image. The ways in which the image behaves as the student drags the pre-image provides the student information about which transformation was used to create the image. Once the classroom community, including the teacher, discusses and accepts that the particular behavior of the image is connected to a certain transformation that was used to create the image, the particular behavior of the image when the pre-image is dragged becomes internalized as a sign for that transformation. Mariotti (2000) provides another example of how a DGE, a cultural artifact, mediates student learning.

> The dragging test, externally oriented at first, is aimed at testing perceptually the correctness of the drawing; as soon as it becomes part of interpersonal activities – peer interactions, dialogues with the teachers, but especially the collective discussions – it changes its function and becomes a sign referring to a meaning, the meaning of the theoretical correctness of the figure. (p. 36)

The internalization of the externally oriented tool to become a sign is, in some ways, related to what Verillon and Rabardel (1995) call *instrumental genesis*. Instrumental genesis is the process by which an artifact becomes an instrument. According to Lagrange et al. (2001, p.6) "While the artifact refers to the objective tool, the instrument refers to a mental construction of the tool by the user". Thus, an instrument consists of "the artefact and the modalities of its use, as elaborated by a particular user" (Mariotti, 2002a, p.702).

The instrumental genesis process results in the construction of "personal schemes of use" (Artigue, 2002, p.4), which function as organizers of the activity of the user and points to how the artifact has actually been used in a given situation. For example, the ways in which students construct a square in a DGE may not only depend on students' knowledge of a square, but also on the prior uses of the DGE and the internalization of these uses. Some students may construct the square using a circle, radii, and perpendicular segments. Other students may construct a square by rotating a segment ninety degrees three times.

Dragging and measuring in DGE.

Certain features of DGEs may mediate student learning and understanding. Arzarello and colleagues (2002) investigated the ways students use the drag feature in solving open-ended problems. The authors examined how students employed the drag feature, which was based on the students' purposes during the solution of a task. In particular, the authors identified seven different modalities of dragging:

- *Wandering dragging* is moving points over the screen randomly in order to discover interesting configurations.
- *Bound dragging* is moving a linked point that is bound to move only on the object it belongs to.
- *Guided dragging* is dragging points of a drawing in order to give it a particular shape.
- *Dummy locus dragging* is moving a point so that the drawing keeps a discovered property; the point moves along a path that is unknown to the student

- *Line dragging* is drawing new points along a line in order to keep the regularity of the figure
- *Linked dragging* is linking a point to an object and moving it onto that object
- *Dragging test* is the moving of points in order to see whether the drawing keeps the initial properties.

The authors found that students used these different dragging modalities to achieve different goals, which include exploring, conjecturing, validating, and justifying. In addition, the authors viewed these modalities of dragging as a hierarchical sequence. The wandering, bound, and guided dragging are used to explore and investigate. The dummy locus dragging generally indicates the creation of a conjecture. The line and linked dragging allow the students to check the conjecture. The drag test is used to validate a conjecture. The ways in which the students use the drag feature provides insight into the students' cognitive processes.

Generally, when students are working in the first three dragging modalities, they begin by considering the perceptual aspects of the drawing and then to thinking about the theoretical. For example, a student constructs a right triangle using the perpendicular tools such that point C can only be dragged on the perpendicular line through point A (see Figure 3). As the student drags point C along the perpendicular line (bounded dragging) she notices that as side AC increases, angle B also increases (see Figure 3). She may conjecture that there is a direct relationship between the length of the side of a triangle and the size of its opposite angle. In this example, the student begins by looking at the perceptual image and, as she reflects on her dragging, she begins to consider the theoretical properties of the right triangle.

Figure 3. Example of bounded dragging.

The students' use of the dummy locus dragging signals the point when the students stop thinking first in the graphical-spatial field and begin to consider the theoretical field. Using the previous example, the student drags point *A* in way such that segment *BC* remains fixed. The dragging is no longer by chance; the student is dragging in a way that is purposeful. In the final three dragging modalities, students generally consider the theoretical prior to the perceptual. In our example, the student notices that when she drags point *A* in way so that segment *BC* remains fixed, the trace of point *A* looks like a circle whose center is at the midpoint of segment *BC*. The student constructs this circle and links point *A* to this circle. She then drags point A to verify her conjecture. Her dragging is an example of linked dragging. In this episode the student considers the theoretical prior to the perceptual.

Another important aspect of DGEs is the measurement feature. Many DGEs contain measuring tools in the menu that can be used to measure distances, lengths, perimeters, areas, and angles. The measurements are linked to a diagram such that the measurements continuously change as the diagram (or some portion of it) is dragged. The changes in the measurements are not random; they update to reflect the changes in the size of the object that is being dragged and, in part, are dependent on the construction of the object. For example, if a student constructs a right triangle using the perpendicular tool and measures the interior angles of the triangle, the measure of the right angle will remain ninety degrees regardless of how (or what portion of) the right triangle is dragged. However, the other interior angles will continuously change as the triangle is dragged. Because of the continuous change and/or invariance in the measures as the diagram is dragged, students may use the measurements in different ways. In general, students use measures to either discover and make conjectures or to validate a conjecture (Olivero & Robutti, 2007).

Similar to the modalities of dragging, Olivero and Robutti (2007) classify the modalities of measuring. There are three modalities of measuring within the first category where the students are shifting from the graphical-spatial field to the theoretical field: *wandering measuring* in which students take measurements of random aspects of the construction in order to identify invariants, congruencies, and other quantitative relations; *guided measuring* in which the measurements are used to obtain a particular figure from a generic configuration; and, *perceptual measuring* in which students check the validity of a perception that may or may not transform into a conjecture. When students move from the theoretical field to the graphical-spatial field, students use two types of measuring modalities: *validation measuring* in which measurements are used to check a conjecture; and, *proof measuring* in which students use the DGE after constructing a proof in order to better understand the proof. Even though these modalities may indicate whether students are moving from the graphical-spatial field to the theoretical field (or vice-versa), they do not indicate the ways students interpret the results from their dragging and/or measuring.

Students' reasoning when using a DGE.

The ways students interpret the results from their use of the drag or measuring features of the DGE may be related to the type of strategies used by students when working on various tasks. In her qualitative study of high school geometry students working on tasks involving transformations, Hollebrands (2007) differentiates between two types of strategies used by students to solve problems using a dynamic geometry environment. A student employing a reactive strategy uses the technology to perform some type of action, and then sees how the diagram responds to this action. Reflecting on this response, the student then selects another action to perform. Students using a reactive strategy are unable to anticipate the next action without first seeing how the computer responds. In contrast, students using a proactive strategy may have certain expectations for how they want to use the tools, determine the actions needed to produce the desired result, and perform these actions. Then, the student reflects on the results from the screen. Students' use of proactive and reactive strategies may also relate to the ways students' are reasoning about a diagram in the DGE.

Students' reasoning about diagrams in DGEs may be related to whether the students are reasoning about drawings or figures. Parzysz (1998) differentiates between drawings, the material representation of a geometric object, and figures, "the geometric object which is described by the text defining it" (p. 80). In other words, a drawing is a picture and a figure is a class of related drawings with some commonality (Goldenberg & Couco, 1998). Using Parzysz work, Laborde (1998) distinguishes between drawing, figure, and the abstract geometric object. Similar to Parzysz, the drawing is the material representation and the figure is the set representations (including definitions) that refer to the abstract geometric object. Thus, the figure acts as a link between the drawing and the abstract geometrical object. In relation to the strategies students employ while working with a DGE, Hollebrands (2007) indicates that students using reactive strategies are reasoning about drawings, and those using proactive strategies are reasoning about figures.

A similar distinction can be made for the creation of objects in a DGE. Students that use the free-hand tools to create a geometric object in a DGE so that it appears to be the desired geometric object are said to be drawings. For example, if a student connects four congruent segments and places them such that two of the segments appear to be horizontal and the other two appear to be vertical, then the student has drawn a square. If a student merely draws an object, then the object will not maintain the geometric properties of the object when one or more aspects of the object are dragged. However, if the student uses the free-hand tools and menu commands to assure that the properties of the object remain invariant when any part of the object is dragged, then the student constructs the object. In order to construct an object, the student must have knowledge of the technology tool and of the underlying geometric properties of the object.

Preconstructed sketches.

One of the featrues of many DGEs (e.g. Geometer's Sketchpad, Cabri, NonEuclid) allow users to save their constructions and drawing to be used at a later time and/or date. Teachers and researchers (e.g. Accascina, Batini, Del Vecchio, Margiotta, Pietropoli, & Valenti, 2004; Sinclair, 2003, 2004) have used this feature of DGEs to create and save sketches for students to use to investigate and discover geometric properties and relationship. The saved sketches used in this manner are considered pre-constructed because the diagrams were constructed prior to the students' use of them.

Pre-constructed sketches can be beneficial for teachers and students in two ways. One, teachers can ensure all students are working with the same diagram which is constructed according to the desires of the teacher. By having all the students using the same sketch, the teacher can make certain that students are using an accurate representation of the geometric theory. If the students created their own diagrams, the students may make erroneous conjectures based on inaccuracies of their sketches. Two, the teacher may be able to limit the features of the software available to students when using a pre-constructed sketch. For example, a sketch created in Geometer's Sketchpad can be converted to html using JavaSketchpad and the html version of the sketch will maintain all relationships constructed in the original. The only feature of Geometer's Sketchpad available to students in JavaSketchpad is the ability to drag a figure or a portion of it. However, if the original sketch includes measures, animations, color, or traces, than those will be present in the html version. Thus, the teacher can decide which features will be in the html version of the sketch when he/she creates the original sketch.

In general, when using a pre-constructed sketch, the students are not creating a diagram. Rather, a diagram has been created and the students are expected to reason about what they see. Because the students were most likely not involved in the creation of the sketch, they probably have little knowledge of which geometric properties, definitions, theorems, and/or postulates were invoked in the creation of the sketch. Thus, some of the properties used in the creation of the sketch may not be apparent to the students. Accascina and colleagues (2004) found that when pre-service and in-service teachers used accurate pre-constructed sketches the students did not notice some properties and relations with in the diagrams because these properties were evident; the property is invariant when a diagram or a portion of it is dragged and one does not need to add new elements to notice the property (e.g. measures, segments). Sinclair, (2003, 2004) found that secondary students using a web-based pre-constructed sketch did not realize or ignored the fact that sketch is accurate. Those students that did recognize the accuracy of the sketch did not use the available measures or the visual evidence of the sketch to notice and interpret relationships. Sinclair (2004) concluded that this attitude is "of concern because exploring invariant properties with dynamic software requires focused attention to details that update under dragging" (p. 192). The way in which the pre-constructed sketch and accompanying materials are designed may assist in overcoming this attitude.

The design of a pre-constructed sketch not only involves decisions about which features of the software to include but also which questions to pose to students. These decisions should not be taken lightly. Sinclair (2003) states,

> In designing a pre-constructed sketch and the accompanying materials, it is not trivial to determine which details to include or which questions to pose since making too much available can remove the motivation for exploring and providing too little can make a task impossible. (p. 294)

Sinclair (2003) makes five recommendations for teachers to facilitate exploration using a pre-constructed sketch and accompanying materials:

1. If a question is to focus students' attention, the sketch must provide some visual stimulus through color, motion, or marking
2. When students are prompted to act based on a statement on the task sheet, the sketch must provide affordance for this action
3. If the explorations are open-ended, the sketches must provide alterative paths.
4. The sketch must support experimentation but be constrained such that it prevents frustration.
5. The sketch should provide students a shared image to consider and discuss.

By following these recommendations, the students may be better able to use pre-constructed sketches to investigate geometric situations, create conjectures, and discover properties and relationships. The use of these pre-constructed sketches may even motivate students to prove these discoveries.

DGE and proof.

Through the use of the measurement and dragging features of the software, some researchers are concerned that students' uses of a computer environment for geometry may diminish the need and importance of proof (e.g. Allen, 1996; Chazan, 1993; Hoyles & Jones, 1998). Allen (1996) urges caution in the use of DGEs because they tend to blur the distinction between illustration and proof and that dynamic nature of the software does not contribute to the proof needed to show that a construction is correct. Hoyles and Jones (1998) indicate that through the production of empirical evidence using a DGE, students may not appreciate the importance of logical argument and the need to produce a proof. In addition, Hoyles and Jones (1998) imply that there is danger in the use of this type of software because it may limit the mathematical work to empirical argument and pattern spotting. Chazan (1993), whose research took place in the context of a non-dynamic geometry environment, warns that extensive use of empirical evidence in geometry classes may diminish students' appreciation for the role of mathematical proof in certifying mathematical truth. The use of the drag feature may also impede students understanding on the need and function of proof. Students can be easily convinced as to the validity of a conjecture attribute by dragging a geometrical object and checking the invariance of the conjectured attribute on a whole class of objects (Hadas et. al., 2000). However, other researchers have found the use of DGEs to assist students in the process of constructing proofs, specifically the formulation of conjectures.

One strategy that has been used to assist students' construction of proof is the development of activities that require students to make conjectures and investigate the validity of these conjectures. Edwards (1997) considered these types of activities the "territory before proof". Edwards specifies five reasoning activities that can be located before proof: (1) noticing the rules or patterns that emerge; (2) describing the rule or pattern using diagrams, words, notation, or other forms of representation; (3) conjecturing the pattern or rules such that it is true in general; (4) inductive reasoning in which the pattern or rule is checked for a finite number of cases; and, (5) deductive reasoning in which justification for the reason why the generalization must hold using other mathematical results. Other researchers have also found the use of conjecturing important to assist students in developing proofs.

Hoyles and Healy (1999) indicate that exploration of geometrical concepts in DGE may motivate students to explain their empirical conjectures using proofs. The authors conducted teaching experiments in the UK at three different schools with high-achieving students whose ages ranged from fourteen to fifteen years-old. Fifteen students participated in the experiments, five from each school. The authors found that the students' uses of a DGE helped the students define and identify properties and the dependence between them. Yet, when the students worked on the proofs of these conjectures, they abandoned their constructions in the DGE. However, at least one student at each school was able to reflect on the steps they used to construct the figure in the DGE then use these steps in creating a deductive argument. Having students reflect and explain their reasoning on the steps they use to construct figures in DGE may assist students move from argumentation to proving.

Christou and colleagues (2004) conducted a study with three pre-service primary school teachers on whether the use of a DGE could help the pre-service teachers identify conjectures and whether the DGE could help the pre-service teachers search for mathematical arguments to support their conjectures and provide reasonable explanations. The authors gave the pre-service teachers an open-ended task. The authors found that pre-service teachers' use of the DGE in the phase prior to proof enabled them to make and validate conjectures, and check specific examples. During the proof phase, the pre-service teachers used the results of their investigation to justify their conjectures, specifically the measurement feature of the software afforded the teachers the ability to find explanations and gather information for justifications. In addition, the authors found that the interplay between the action in the DGE (specifically construction and measurement) and the dependent properties provided motivation for the student to explain and justify their conjectures.

Mariotti (2000, 2002b) indicates that there is a correspondence between the construction commands in Cabri and the Geometric theorems. When all the commands are available to students, the students may have difficulty determining what is given (axioms and previously proved theorems) and what must be proved (new theorems). Mariotti conducted a long term teaching experiment in which the commands in the Cabri menu were not provided to the students at the beginning of the study. Using the DGE, the students explored and made conjectures based on their interactions with the environment. Proof was used to justify why the command should be included and the students had to explain why the new command is able to perform the intended job. One of the classroom norms established by the teacher during this teaching experiment was that students were expected to defend his/her construction method in front of the class. In this manner, proof fulfills two roles, it verifies that the construction process is valid and communicates to other members of the class not only how to construct the command but also why the construction process is valid. The students' uses of the DGE contributed to their understanding of the theoretical geometry by providing a semiotic mediation that helps students explore, conjecture, and argue prior to arriving at a proof.

Jones (2000) describes an instructional unit on classification quadrilateral and made use of the precise language necessary to do so correctly. Using Cabri, students were challenged to reproduce a drawing of a quadrilateral such that it could not be "messed up" when dragged and, in some cases, satisfy other conditions. Upon completion of the construction, the students were asked to explain their reasoning as to why the construction was correctly created. This experiment provided students the change to engage in deductive reasoning. Jones indicates that students' uses of the DGE helped students formulate precise language about properties and relationships of quadrilateral. The use of deductive thought and precise languages are important precursors to proof.

In order to confront students' conviction of the validity of a conjecture based on empirical evidence, Hadas, Hershkowitz, and Schwartz (2000) designed a collection of innovative activities that were intended to cause uncertainty and surprise. The authors designed the sequence of the tasks in the activity such that the students developed expectations that turned out to be wrong when they checked them in the DGE. The disequilibrium created by the tasks compelled the students to understand the situation better rather than simply check whether they guessed incorrectly. The need for understanding and insight into why a conjecture is true fulfills one of the roles of proof.

Looking across these six studies (Christou et. al., 2004; Hadas et. al., 2000; Hoyles & Healy, 1999; Jones, 20000; Mariotti, 2000; Mariotti, 2002b) the use of the DGE seems to have shifted the focus from examining students' ability to write formal proofs to a focus on students' understanding of the need and role of proof. This view of proof as explaining empirical evidence as opposed to proof as resolving uncertainty was introduced and developed by de Villiers (1991). In his analysis of mathematicians' work, he found that in practice, mathematicians often do not begin the process of proving a statement until they have some inner conviction of the truth of the statement. Proof not only fulfills the role of validating the truth of a statement but can fulfill other roles in mathematics as well.

Proof, Argument, and Justification

Generally, in school mathematics, proof has been used to establish the truth of a statement (de Villiers, 1990). However, in the field of mathematics, proof may take on other roles. de Villiers (1999) suggests that proof has five additional mutually exclusive roles from merely establishing the validity of a mathematical claim which are explanation, discovery, systemization, communication, and intellectual challenge. One role of proof is explanation in which the proof provides insight into why the claim is true and promotes understanding of the claim. Proofs may also be a means of discovery in which within the process of proving, new results may be brought to light. The deductive nature of proofs suggests that proofs can be a means for systemization such that an axiomatic structure can be established. Proofs have also been the established tool for mathematicians to communicate new results to the field. In this communication, proofs are used to convince and persuade others as to the validity of a mathematical claim. Another role of proof is that it can challenge the intellect of an individual.

Hersch (1993) indicates that the role of proof in research is different from that in the classroom. In research, the role of proof is to convince. In the classroom, the role of proof should be to explain. However, proofs do not always convince or explain. Rodd (2000) indicates that even though students understand that proof establishes certainty of a mathematical claim, many times students do not secure this certainty through proofs. Instead, students reason upon empirical evidence or visualization to convince themselves as to validity of a mathematical claim. While this type of reasoning may not be considered proofs, they can be considered justifications.

Types of justifications.

Bell (1976) identifies two categories of students' justifications used in proof problems: *empirical* justification which is characterized by the use of examples as an element of conviction, and *deductive* justification, characterized by the use of deduction to link data with conclusion. Within these categories of justifications, Bell identifies a number of types of justification. The types of justification within the empirical category are based on the degree to which the students verify the statement in the whole set of possible examples. The types of justification within the deductive category are based on the degree of the completeness of the constructed deductive argument.

Similar to Bell, Balacheff (1988) distinguishes between two types of proofs, *pragmatic* and *conceptual*. Pragmatic proofs are based on the use of examples or on actions. Central to these types of proofs are the theorems-in-action (Vergnaud, 1981), which consists of the properties used by the student to justify their claim but are not explicitly stated. The category of pragmatic proofs includes three types: *Native empiricism*, in which a statement is justified based on a few examples; *crucial experiment*, in which a statement is justified based on the verification of a carefully selected example; and, *generic example*, in which justification is based on operations or transformations on an object that is representative of a class of objects. Conceptual proofs do not rely on action and rest on the abstract formulations of the properties and the relation between the properties of the object under question. There are two types of conceptual proofs: *thought experiment*, in which actions are internalized and disassociated from the specific examples such that the justification of the statement is based on the properties of the object; and, *calculations on statements*, in which there is no experiment per se and the proof consists of the formulation and manipulation of formalized symbolic expression based on the definitions and properties of the object.

The types of justifications used by students to warrant a claim may be influenced by the student's proof scheme. According to Harel and Sowder (1998), a proof scheme refers to what a student finds as convincing and persuading for themselves. Proof schemes can be organized within three general categories: *external conviction* proof schemes, *empirical* proof schemes, and *analytical* proof schemes. A student exhibiting an external conviction proof scheme depends on an authority such as a teacher or book, the appearance of an argument, or symbols used in a meaningless way. Empirical proof schemes rely on evidence from examples or measurements or perceptions. Analytical proof schemes include deductive reasoning and reasoning about mathematics as an axiomatic system.

The commonality between Bell (1976), Balacheff (1988), and Harel and Sowder (1998) ways of categorizing justifications is that the authors clearly distinguish between justifications based on empirical evidence (empirical, pragmatic, empirical) and those based on deductive reasoning (deductive, conceptual, analytical). However, the focus of these studies were different which is reflected in how the authors define and delineate between these two categories. In regards to the empirical category; Bell's analysis is concerned with the completeness of the sets of examples used in the justification. Balacheff focused on the type of example. And, Harel and Sowder differentiated between the use of one or more specific examples and the visual perception of the object. For the deductive categories of justifications, Bell analyzes the quality and completeness of the deductive chains. Balacheff differentiates between the means of the justification. And, Harel and Sowder describe two types of analytical proof schemes, one based on operations on the object and anticipations of the operations' result, and those based on an axiomatic system. Even though the justifications based on empirical examples would not be considered proofs, they might be considered mathematical arguments.

Argument and argumentation.

A mathematical *argument* may be defined as a sequence of mathematical statements that aims to convince, whereas *argumentation* may be regarded as a process in which a logically connected mathematical discourse is developed. Krummheuer (1995) views argument as the outcome of argumentation or a specific sub-structure within a complex argumentation. "The final sequence of statements accepted by all participants, which are more or less completely reconstructable by the participants or by an observer as well, will be called and argument" (Krummheuer, 1995, p. 247). Thus, argumentation is the process and the argument is the product. Douek (1998) shares this view of argumentation and indicates that the discursive nature of argumentation does not exclude the reference to non-discursive arguments (e.g. visual or gestural).

One framework that can be used to analyze the content and structure of arguments in the mathematics classroom is Toulmin's (1958) model of argumentation. In this work, he introduced constructs of claim, data, warrant, backing, rebuttal, and qualifier. Toulmin's model is briefly described below.

Toulmin states that a *claim* is a "conclusion whose merits we are seeking to establish" (p.90). In general, arguments are identified by the presence of a claim. An example of a claim in a mathematics classroom is a student's statement that the diagonals of a rectangle are congruent.

In order for a claim to be made, it must be based on something. Toulmin identified "the facts we appeal to as a foundation for the claim" (p. 90) as *data*. In the example of a rectangle, a student might cite the properties of a rectangle that all of the angles are ninety degrees and opposite sides are congruent. This data may be accepted by the audience or it may be questioned in which the student would need to provide an argument regarding the validity of the data. These types of arguments are called *sub-arguments*.

Often, the data of the argument are stated explicitly by the problem situation or by the one making the argument. However, the link between the data and the claim may not be (Toulmin, 1958). The "general, hypothetical statements, which can act as bridges, and authorize the sort of step to which our particular argument commits us" is called a *warrant*. In other words, the warrant is the reason the claim is made based on the data provided. In our example, a student's warrant for the claim that the diagonals of a rectangle are congruent, may be the triangle congruency postulate side-angle-side. Similar to data, the warrant can be challenged by the audience. These challenges take two basic forms; the audience can challenge the validity of the warrant or the relevance of the warrant. If the validity of the argument is challenged a sub-argument would need to be made. If the relevance of the warrant is challenged, an appeal can be made to the backing of the argument.

The *backing* of an argument is "other assurances without which the warrants themselves would possess neither authority nor currency" (p. 96). In the example, there are many unspecified backings, but one in particular may be that the student is working in the Euclidean world and thus the side-angle-side triangle congruency postulate holds.

Toulmin includes two other constructs in his argumentation model: qualifier and rebuttal. A *qualifier* “indicates the degree of strength that is conferred by the warrant,” (p. 94). Example phrases associated with the qualifier include “presumably” or “probably” (Toulmin, 1958). A *rebuttal* indicates “circumstances in which the general authority of the warrant would have to be set aside,” (p. 95). In mathematics education, these two constructs have not been traditionally used (Cobb, Stephan, McClain, & Gravemeijer, 2001; Krummheuer, 1995; Yackel, 2002). However, other researchers have found for the use of these constructs to be beneficial (Inglis, Mejia-Ramos, & Simpson, 2007).

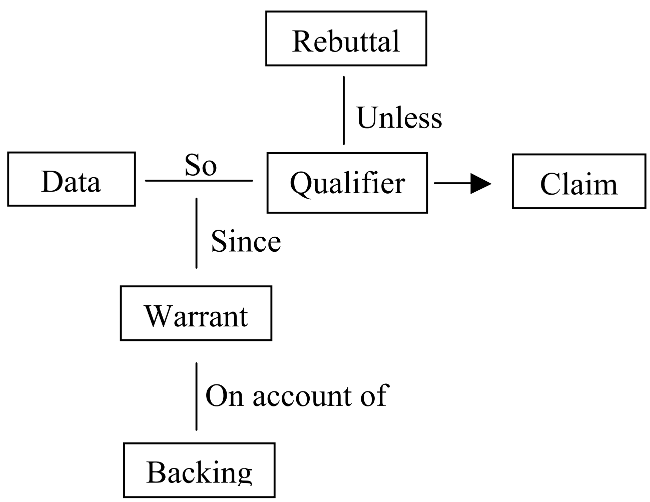


Figure 4. Toulmin’s (1958) model of argumentation.

Figure 4 shows how these six constructs (data, claim, warrant, backing, qualifier, and rebuttal) fit together. Data is provided or constructed and “so” a claim is made based on this data. This claim can be made based on this data "since" the warrant. The warrant is relevant "on account" of the backing. The claim is valid "unless" the rebuttal occurs.

Some researchers assert that through engagement in conjecturing and argumentation that students develop an understanding of mathematical proof (e.g. Boreo et al., 1996). Boreo et al. (1996) use the term "cognitive unity" to signify that continuity exists between the development of a conjecture during argumentation and the successful construction of its proof:

> During the production of conjectures, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of plausibility of his/her choice. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organizing some of the justifications ('arguments') produced during the construction of the statement according to a logical chain. (p. 113)

The reasoning that takes place during the argumentation plays a crucial role in the construction of the subsequent proof.

However, critics of the argumentation approach to proof assert that the natural language of students' argumentation conflicts with the logic associated with mathematics. Balacheff (1991) views argumentation in the mathematics classroom as a means to convince one another by whatever means the student chooses.

> The aim of argumentation is to obtain the agreement of the partner in the interaction, but not in the first place to establish the truth of some statement. As a social behavior it is an open process, in other words it allows the use of

> any kind of means; whereas, for mathematical proofs, we have to fit the requirement for the use of some knowledge taken from a common body of knowledge on which people (mathematicians) agree. (p. 188-189)

In addition, Balacheff (1999) strongly asserts that argumentation is an obstacle to the teaching of proof. In his view, arguments occur spontaneously and have the freedom to be based on anything while proofs must rely on the explicit axiomatic system.

In response to Balacheff's views of argumentation, Boreo (1999) focuses on the distinctions between proving as a process and proof as a product. Argumentation may be part of the process of proving. In addition, Boreo (1999) indicates that the nature of argumentation is dependent on the classroom culture, which includes the social norms, socio-mathematical norms, classroom practices, and the reasoning emphasized by the teacher.

Student Argumentation

As Boreo (1999) indicates, argumentation is part of the proving process. However, students, ecspecially younger students, may not create formal proofs while working on mathematical tasks. Rather, students create arguments that become more sophisticated as they work through the mathematical task. Lester (1980) reported on fifth and seventh-grade students' justifications of their problem solving strategies. He anlyzed their abilities to coordinate multiple pieces of information and reported five progressively more sophisticated strategies used by the students: trial-and-error, heterogeneous classification, local classification, partial-global classification, and global classification. Lester's results indicate that the older students had a greater proportion of students use a global classification strategy and smaller proportion use the trial-and-error and local classification strategies.

Maher and Martino (1996, 2000) also report on how a student's justifications become more sophisticated as the student ages. The authors provide a detailed case study of Stephanie as she works on combunitorics problems starting in grade 2 and finishing in grade 5. In their 1996 report, the authors anlyzed Stephanie's problem solving straetgies and justifications for those strategies. Stephanie began with what they authors considered an unsophisticated strategy and progressed such that her argument for her solution was a "proof by cases." In 2000, the authors report that Stephanie extended her earlier understanding of argument by cases to an argument by mathematical induction.

In Maher and Martino (1996, 2000) and Lester's (1980) work, the students showed more sophisticated reasoning in their approaches to solving problems and the arguments they create to justify their solution as the students grew older. This might suggest that the development of reasoning is partialy dependent on maturation. However, perhaps the tools in which the students used to create their solutions influenced the strategies they used to solve the problems and their solutions.

Argumentation and the Use of Tools

As indicated above, students' arguments increase in sophistication as the students grow older. The content and structure of the arguments may be influenced by the students' uses of the tools, specifically technological tools. Researchers (e.g. Hollebrands, Conner, and Smith, 2010; Lavy, 2006; Maher, Powell, Weber, and Lee, 2006) have employed Toulmin's model of argumentation to analyze the arguments students create while working in or based on their work with technological environments.

Lavy (2006) indicates that the content of the arguments students create while working in a technological environment are closely related to the specific setting. The study took place in an after-school elective course for seventh grade students in which the students worked on activities using an interactive computer environment. The computerized environment in this study was MicroWorlds Project Builder (MWPB), a Logo-based construction environment in which the students, at least for the study, used an electronic geoboard. The study focused on the arguments created by one pair of students. Lavy found, initially, the arguments that emerged were, in a way, setting-dependent since the justifications used by the students were screen images of geometrical shapes and relevant terms of the environment. However, later on in the exploration process, the justifications used by the students became independent of the setting, relying heavily on mathematical considerations of properties of numbers. Lavy states there were differences in the ways students used the screen images as part of the data of an argument and those that were part of the arguments' reasoning:

> When a screen image is part of the data, it might be said that it is an integral part of "external knowledge" for the student and the aim is that during the exploration process, he/she will internalize this knowledge. After the students have formulated a claim, and in the process of reasoning used concrete examples, it might be said that that these concrete examples of stars and polygons were no longer "external knowledge" for them, rather they were already internalized in the context of the geoboard setting. (p. 168)

The computerized environment enabled the construction of various arguments by serving as a mediator between the two students. “The learning environment provided both a language that enabled the students to communicate with each other, and screen images that were necessary for the development of the mathematical argumentation between them” (p. 167).

Maher and colleagues (2006) investigated the arguments students create during whole class discussions and how these arguments can facilitate the learning of mathematics. In this study, the middle school students used a computer simulation tool, *Probability Explorer*, to explore whether a hypothetical die manufacturing company made fair die, produced a poster explaining how they arrived at their conclusion, and analyzed the posters created by other groups for other hypothetical companies. After the students analyzed the posters, the students discussed which company produces fair die. Using Toulmin’s model of argumentation, the authors analyzed the content of the arguments students created during the whole class discussion. The researchers not only documented the data, claim, and warrant of each of the students’ arguments but also the challenges by other students. Analysis revealed that when the students only used the conclusions on the posters as data in their argument, the claim was challenged, not the reasoning. When the students’ claims were based, in part, on the data presented in the posters, the warrants were questioned. However, by the end of the discussion, the arguments based on the conclusions of the posters but not the data presented in the posters were challenged by other students.

Hollebrands, Conner and Smith (2010) analyzed the structure of the arguments college students created while working in a dynamic geometry environment. The authors conducted task-based interviews with students enrolled in a college geometry course. Three themes related to the structure of the arguments related to the students' uses of technology were identified. First, the students provided explicit warrants when they generally did not use technology. If students used technology and provided an explicit warrant, the students' use of technology was merely a diagram on the screen from which to reason. Second, when students used technology and did not provide an explicit warrant, the authors found differences in whether students provided qualifiers, in the ways in which technology was used, and in types of tasks on which students were working. Third, the students used technology to verify a claim when they were uncertain as to the validity of the claim.

In these three studies (Hollebrands, Conner, and Smith, 2010; Lavy, 2006; Maher et al., 2006) the authors analyzed the structure and/or content of students' arguments using Toulmin's model of argumentation. One study found the students' use of technology influences the content of the warrants. Another study found the students' use of technology influenced the structure of the argument, in particular the explicitness of the warrant. And, one study found that the content of the data, which may or may not have been collected using technology, that serves as the basis for the argument influences whether others challenge the warrant or the claim. In all of these studies, the use of technology influences the arguments students create by servings as a mediator between the students, and as a means to collect data.

Summary

From the research reported above, we see that dynamic geometry environments are technology tools that may assist students in their development of arguments and proofs. Also reported was the importance of argumentation in the mathematics classroom. The conceptual and theoretical considerations described in this chapter informed the methodology of this study. Details of the methods by which this study were conducted are found in Chapter 3.

CHAPTER 3

Methodology

This chapter describes the methodology that was used to collect and analyze data relative to the research questions for this dissertation:

1. What are the characteristics of the content and structure of eight grade mathematics students' arguments while working in a technological environment?
2. What are the characteristics of the content and structure of eight grade mathematics students' arguments while working in a non-technological environment?
3. In what ways do the arguments made by students working in a technological environment compare in content and structure to those made by students working in a non-technological environment?

To answer these questions, the researcher conducted two classroom teaching experiments with students enrolled in two eighth grade mathematics classes. Qualitative methods were used to gather and analyze data throughout the study.

Explanation of the Appropriateness of the Approach and Research Design

In order to study the content and structure of the arguments that are created while students are working in dynamic geometric environment and the how these arguments differ compared to the arguments developed from students' working in a paper-and-pencil environment, a qualitative approach is warranted. Because the study is focused on identifying and characterizing the arguments students make, a rich description of the students' behaviors, actions, and language is needed. Thus, a qualitative approach is appropriate.

Because the purpose of this study is to describe the content and structure of the arguments created by students working in a technological environment and students working in a non-technological environment and compare them to those developed by students working in a non-technological environment, the researcher utilizes a qualitative methodology. The research design for this project is based on classroom teaching experiment methodology (Cobb, 2000) and multi-case study as described by Merriam (1998). In the following sections, the classroom teaching experiment and multi-case study are described.

Classroom teaching experiment.

The classroom teaching experiment was developed as a way to study student learning in the classroom. One of the activities of the classroom while students learn is their creation of mathematical arguments. This study focuses on those arguments. The firsthand experience of studying students in the classroom allows the researcher to document how students use mathematical tools, including technology, and the arguments that students create while using mathematical tools. Because the research is situated in a cultural environment, classroom-teaching experiments can be considered a type of ethnographic methodology.

The process of conducting a classroom teaching experiment is characterized by the developmental research cycle (Cobb, 2000) (see Figure 5). In a classroom teaching experiment, both pragmatic and theoretical issues can be addressed simultaneously. Before the first teaching episode the researcher develops an initial hypothesis of the learning or mathematical development, including argumentation, that will occur in the teaching experiment. This is known as the development phase. Once the first episode of instruction is complete, the researcher will use the empirical data gathered on the students' mathematical behaviors to modify the initial hypothesis. This is known as the research phase. Within this research phase, new instructional activities are developed or modified based on the new hypothesis. These newly developed or modified activities will be grounded in the instructional theory. This recursive process of analysis and refinement continues until the end of the experiment.

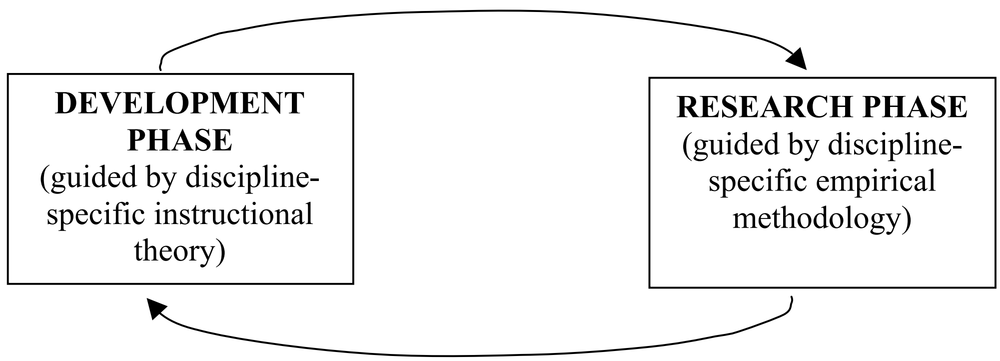


Figure 5. The Developmental Research Cycle (Cobb, 2000).

In order to compare the content and structure of the arguments created by students working in a technological environment and those working in a non-technological environment, the researcher conducted two classroom-teaching experiments. The design and implementation of the activities and tasks for one class utilized a dynamic geometry environment, *The Geometer's Sketchpad* (Jackiw, 2001), so students could use the technology to explore and investigate theorems, properties, and definitions related to triangles. For the other class, the design and implementation of the tasks and activities utilized non-technological tools such as snap-cube, protractors, rulers, and pre-cut triangles. The selection of which class to teach using technology was made randomly. For both classes, the tools provided students with a means to reason and develop new understandings about the geometry concepts.

Design of Instruction

In order to minimize the amount of variation between the two classes, the selection and sequence of tasks for each class were similar in a number of ways. First, the instructional sequence of tasks and activities for both classes were the same and based, in part, on the school district's pacing guide for the unit under study. Second, the objectives for the tasks for both classes were the same. In addition, the design and implementation of the majority of the tasks and activities for both classes were similar in nature with the major difference being the tools available to the students. In some instances, the tasks differed in order to capitalize on the affordances of the tools in each class. Another way the variation between the classes is minimized is the selection of the classes to participate in the study. The two classes participating in the study are from the same school and taught by the same teacher. Thus, it was assumed the classroom social norms, socio-mathematical norms, and classroom practices that have been established prior to the study were similar for these classrooms. In addition, for the unit under study, the researcher served as the classroom teacher and the teaching style utilized for both classes would be considered inquiry-oriented.

The researcher chose to be the classroom teacher for both of the classes during the instructional unit under study. This decision was made based on a number of factors including the familiarity with the technology, and the use of the teaching approach. The researcher was more familiar with the technological tool than the regular classroom teacher having used it as a classroom teacher himself and created curriculum based, in part, on the use of this tool. The researcher was comfortable using this tool to teach and designed tasks that capitalized on its affordances. In addition, the researcher wanted the students to be taught using an inquiry-oriented approach to provide students the opportunities to create arguments and challenge each other's arguments. The regular classroom teacher used a more traditional approach in her classroom that does not always allow the students to create arguments. For these reasons, the researcher chose to be the classroom teacher.

During the unit of study, the researcher utilized an inquiry-oriented approach to instruction for both classes, which means that students are expected to participate regularly in the mathematical activity of the classroom by making sense of mathematics, communicating their own ideas to others, and making sense of others' ideas (Cobb & Yackel, 1996). The researcher expected and encouraged the students to be actively involved in thinking about mathematics. The role of the instructor was to facilitate student thinking as they work through instructional tasks.

The research team for this study consisted of a number of individuals including the researcher/teacher (author), an experienced researcher, the regular classroom teacher, and a number of data collectors. The data collectors were not the same individuals for each class meeting. For every class meeting, there was at least one data collector who worked with the experienced researcher to operate the video cameras and take field notes.

Prior to the teaching of the unit, the researcher met with the regular classroom teacher and the experienced researcher separately to discuss and formulate the instructional sequence and the design of the tasks and activities to be used during the unit under study. During the study, the researcher and the experienced researcher met after each class meeting to discuss what occurred during that day's lesson including the argumentation the students were providing, their uses of the tools, and the teaching strategies. During these debriefing meetings, the researchers also discussed what should be taught in the next lesson. These discussions included whether and how to modify the planned tasks and activities for the next day, the teaching strategies that could and should be employed, and the anticipated students' actions and responses specifically related to promoting student argumentation. At times, data collectors would join these discussions. In addition, the researcher frequently spoke with the regular classroom teacher to get her perspective on these matters. These debriefing meetings exemplifies the developmental research cycle in Figure 1 as the classroom activity informs the research into the learning process and research informs the classroom activity.

Multi-case Study

The design of this study also draws upon the multi-case study research methodology (Merriam, 1998). Rather than solely focus on the arguments created by the classes, the researcher chose to focus the data collection and subsequent analysis on the arguments created by purposefully selected groups of students in each class. In this study, each case consists of a pair of students. Because the class was taught using an inquiry-oriented approach, the students worked in pairs such that every student had the opportunity to work with the tools and to develop their own mathematical generalizations and understandings. Multiple cases were chosen because a single case may not be representative of the class from which the pair were selected. In addition, by looking at a range of similar and contrasting cases, we can strengthen the precision, the validity and the stability of the findings (Merriam, 1998). In the next section, the selection of the site and of the pairs of students is described.

Site Selection and Sample

The researcher conducted the research study in two eighth grade mathematics classrooms at an urban public middle school in the southeast United States. The middle school is an urban middle school with students of varying ethnicities and socio-economic status. The middle school is a 1:1 laptop school in which each teacher has a class set of laptops for the students to use during class meetings. In addition, the school is magnet school partnered with a local university. The students periodically interact with university faculty and personnel. Working with researchers may not have been novel to these students.

Because this study is focused on students' mathematical arguments, eighth grade students were chosen because their abilities to reason are more developed than younger students but are generally not as formal as older students. Eighth grade mathematics students have not been introduced to formal proofs, thus their arguments are not skewed towards this form of mathematical argumentation. Yet, their abilities to reason are greater than younger students and the arguments that they develop may be more sophisticated.

The eighth grade mathematics curriculum is designed to prepare students to be successful in Algebra I in the ninth grade. Thus, the focus of the course is on functions and solving simple equations. However, the students also investigate geometric topics such as three-dimensional shapes and the Pythagorean theorem. A unit on the definitions, theorems, and properties of triangles, including the Pythagorean theorem, was chosen as the unit under study based on its wide spread application throughout the high school mathematics curriculum, and its amenability to the use of technology. The eighth grade students enrolled in this course can be considered average students. Some of their peers are taking more advanced math courses while other are taking less advanced. The majority of these students will take Algebra I in the ninth grade. The students enrolled in these eighth grade classrooms are both economically and culturally diverse.

During the study, the students in both classes were placed in pairs. By having the students work in pairs, the students were given the opportunity to have discussions with their partners while working on the tasks. These discussions were the primary focus of the study's analysis. The researcher chose to use pairs rather than larger groups because he wanted to maximize the opportunities for students to interact with the mathematical task while still having peer-to-peer discourse. Also, pairs allowed the students working in the technological environment to have clear roles. One student was responsible for moving the mouse and the other was responsible for recording the results on the task sheet. If the groups were larger than 2, then one or more of the students may not be able to interact with the task. The researcher asked the regular classroom teacher pair the students in each class such that the students would work well together. The researcher wanted pairs of students that worked well together so the students would engage in the mathematical tasks and would create few disruptions. Also, the researcher asked the classroom teacher to make the pairs of students diverse in gender, ethnicity, and mathematical ability if possible. The researcher wanted the pairs to be diverse because the students were used to working with other students from backgrounds other than their own in their regular classroom.

A sample of the pairs from each class was purposefully selected to be the focus of the data collection and subsequent analysis. According to Merriam (1998, p. 61), "purposeful sampling is based on the assumption that the investigator wants to discover, understand, and gain insight and therefore must select a sample from which the most can be learned." The selection of the purposeful sample is made by first outlining a set of criteria and then selecting participants based on those criteria (Merriam, 1998). The important criteria used for this study to select the focus groups were: whether the students elected to participate in the study and the recommendation of the regular classroom teacher. The researcher asked the teacher to identify students that would be willing and able to verbalize their thinking and work well together. The researcher also asked that these focus pairs have students representing a range in terms of their gender, race, ethnicity, and mathematical ability. However, the researcher was more interested in having students that were willing and able to discuss their thinking.

Instruction Unit for the Study

In this section, I describe the unit taught during the study. Before the first teaching episode in a classroom teaching experiment, the researcher develops a hypothesized learning trajectory (Simon, 1995). Learning trajectories include "the learning goals, the learning activities, and the thinking and learning in which students might engage" (Simon, 1995, p.133). The researcher also considered the arguments the students might create while working in this unit.

The primary learning goals for the triangles unit are for students to be able to: 1) provide a classification system of the different types of triangles; 2) state and justify theorems applicable to all triangles; 3) state and justify theorems specific to the right triangle; and, 4) apply their understandings of triangles to solve problems situated in real-world contexts.

The two classes did not meet every day for the same amount of time. Rather, the classes met four days a week. Two of the class meetings lasted approximately an hour and a half and the other class meeting lasted 50 minutes. Because the students had to walk from their regular classroom to the research classroom, the amount of time available for instruction was less. The amount of time expected to teach the unit was two weeks or 8 class meetings. Due to inclement weather, the unit was completed in 3 weeks, but still consisted of 8 class meetings. Tables 2 and 3 provide the dates, times and sequence of lessons taught for each class.

Table 1

Meeting Schedule and Topic for the Non-Technology Class

Class	Date	Time	Lesson Topic
1	January 12, 2009	12:50 PM - 2:20 PM	Triangle inequality
2	January 13, 2009	11:20 AM - 12:10 PM	Sum of the Measures of the interior angels
3	January 14, 2009	12:50 PM - 2:20 PM	Triangle Inequality
4	January 16, 2009	12:50 PM -1:40 PM	Quiz
5	January 23, 2009	12:50 PM - 2:20 PM	Side and Angle relationship
6	January 26, 2009	12:50PM - 2:20 PM	Triangle comparison, classification, and definitions
7	January 27, 2009	11:20 AM - 12:10 PM	Triangle properties
8	January 28, 2009	12:50 PM - 2:20 PM	Pythagorean Theorem

Table 2

Meeting Schedule and Topic for the Technology Class

Class	Date	Time	Lesson Topic
1	January 13, 2009	9:50 AM - 11:20 AM	Triangle inequality and Sum of the Measures of the interior angels
2	January 14, 2009	11:20 AM - 12:10 PM	Triangle inequality
3	January 15, 2009	9:50 AM - 11:20 AM	Side and Angle relationship
4	January 16, 2009	10:50 AM -11:40 AM	Quiz
5	January 22, 2009	9:50 AM - 11:20 AM	Triangle comparison and definitions
6	January 27, 2009	9:50 AM - 11:20 AM	Triangle comparison and classification
7	January 28, 2009	11:20 AM - 12:10 PM	Triangle properties
8	January29, 2009	9:50 AM - 11:20 AM	Pythagorean Theorem

Data Collection

Data collection consisted of: video and audio recordings of the two eighth grade classes both small group and whole class discussions; video-recordings of the computer screen to capture the students' uses of technology; and, artifacts which include copies of students' written work including in-class work, homework, quizzes, and exams.

Video recordings: Each class was videotaped using four to five cameras. During small group discussions, four of the video cameras recorded student participation and written work for four different identified small groups. For the purpose of continuity, the same four small groups were recorded in each class session. During whole group discussion, four cameras focused on the students in the class and capture the discourse in whole and one camera will focus on the teacher. If only four cameras were able to record, one of the cameras designated for the small group was used to focus on the teacher. In addition, a screen-capture video recording software captured the students' uses of the technology for these four groups as well as the audio recordings for each group.

Audio recordings: In addition to the video-recordings, each of the groups of students participating in the study were audio-recorded using two digital audio-recording instrument. These audio recording were used to as the audio tracks for the video recordings.

Artifacts: All written work from all participating students was collected, copied, and digitized. This work includes homework assignments, in-class work in small groups, in-class work from whole class discussions, quizzes, and the exam given at the end of the unit. Artifacts also includes researcher field notes and artifacts from the researcher planning and debriefing meetings.

Data Analysis

Of each class's eight class meetings, the researcher only analyzed the small group and whole-class discussion for three tasks. The tasks used in the analysis were the triangle inequality, the triangle sides and angle relationship, and tasks involved in the classification, comparison, and definitions of triangles (the triangle sort task for the non-technology class and the investigating triangles task for the technology class). The researcher selected these tasks to be used in the analysis for three reasons. First, all the members of the focus small groups were present these days. On other days, not all of the members were present for every small group. Second, the amount of class time for each of these tasks was at least an hour and a half. This much time allowed for the students to explore the concepts, formulate their own conjectures, test those conjectures, and discuss them in a whole class setting. In the class meetings that lasted under an hour, there was less time to complete this cycle. When selecting the tasks for analysis, the researcher reviewed the researcher field notes from all class meetings. From these research field notes, the three tasks used in the analysis appeared to have a greater number and variety of arguments than other tasks.

Even though the researcher captured video on four small groups of students for both classes, he only analyzed the work of three of the small groups. The three groups selected for each class was done based on students' attendance. For both focus groups that were not analyzed, a member of the group was absent at least one class meeting. For the small groups that the researcher analyzed, all students were present for most days. The only exception was on day 4, the quiz. One of the students in one of the focus groups in the technology class was absent. However, this data collected using the video and audio equipment was not used in the analysis.

Three pairs of students in each class were the focus of this study. In the tables below, Table 3 and 5, listed are the six groups of students whose discourse and actions were used in the analysis and the student's gender, ethnicity, and their level of academic achievement on the previous year's exam. All of the students used in the analysis either scored at achievement level II or level III on the seventh grade mathematics end-of-grade exam.

Students scoring at academic achievement level II are considered to “demonstrate inconsistent mastery of knowledge and skills in the subject area and are minimally prepared to be successful at the next grade level,” (North Carolina Department of Public Instruction (NCDPI), 2007, p. 3). Students scoring at achievement level III are considered to, “demonstrate mastery of grade level subject matter and skills and are well prepared for the next grade level,” (NCDPI, 2007, p. 3).

Table 3

Characteristics of Students in Focus Groups for Technology Class by Gender, Ethnicity, and Level of Academic Achievement (AA)

Group and Students	Gender	Ethnicity	Level of AA
Group 1			
Heather	Female	Caucasian	III
Mary	Female	Caucasian	III
Group 2			
David	Male	African-American	III
Erica	Female	Caucasian	II
Group 3			
Amy	Female	Caucasian	II
Judy	Female	African-American	II

Table 4

Characteristics of Students in Focus Groups for Non-Technology Class by Gender, Ethnicity, and Level of Academic Achievement (AA)

Group and Students	Gender	Ethnicity	Level of AA
Non-Technology Class			
Group 1			
Alex	Male	Latino	No Score
Frank	Male	Latino	III
Group 2			
Bob	Male	Caucasian	III
Ellen	Female	African-American	III
Group 3			
Clair	Female	Caucasian	III
Jim	Male	Caucasian	III

To begin the analysis, the researcher transcribed the video recordings of the whole class discussions and small group discussions. From these transcriptions, reasoning episodes were established by identifying claims. After the researcher identified the claims, he created a description of the argumentation episodes for that claim. These written descriptions included the participants' words (from the transcripts) and actions including the students' uses of the mathematical tools.

Once the researcher created the descriptions of the argumentation episodes, he identified the claims, data, warrants, backing, qualifiers, and rebuttals and created diagrams of the students' arguments. At times, all of these constructs were not specified and the researcher had to make some inferences. In these cases, the researcher noted these inferences. All non-inferred constructs were attributed to a student, the teacher, or a combination of students and/or teacher.

After the researcher created the written descriptions, each argument was diagrammed according to the model developed by Toulmin (1958) including each of the six constructs, data, claim, warrant, backing, qualifier, and rebuttal used in the argument. In the diagrams, the researcher used a box to outline each of the spoken or known constructs. If the researcher made an inference, he used a "cloud" to note the inference in the diagram. For example, Heather and Mary are working on the warm-up during the triangle inequality task using the technology trying to determine whether segments of lengths 4, 7, and 11 will form a triangle. Mary drags the endpoints of the segments towards each other to form a triangle. Failing to make the endpoints meet and form a triangle, Mary claims, "11 is not going to work." This argument is illustrated in Figure 6.

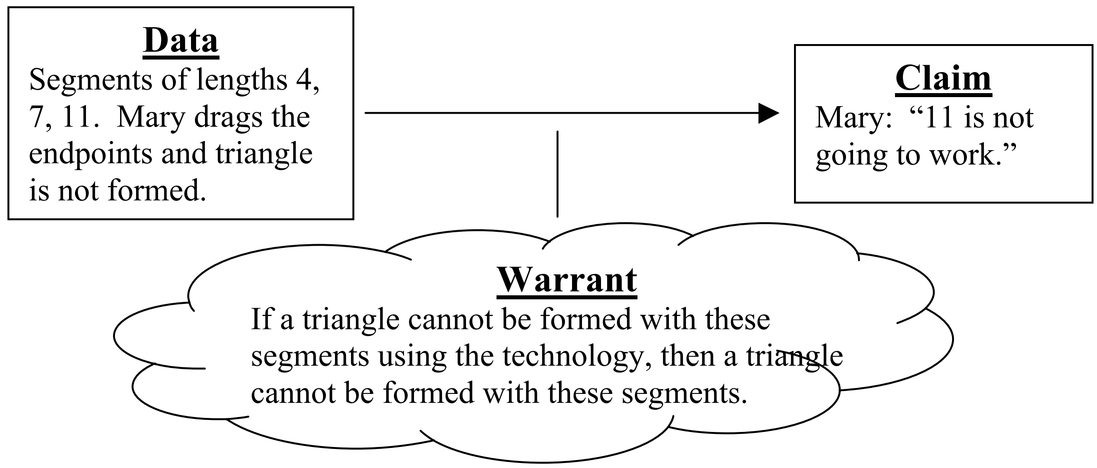


Figure 6. Mary's core argument created during task 1 with a non-explicit warrant.

The data for this argument are the segments of lengths 4, 7, and 11 and the dragging of the endpoints. Mary's claim is the three segments of lengths 4, 7, and 11 will not form a triangle. Mary did not explicitly state a warrant for this claim, but through her actions and her claim, the researcher infers the warrant to be "if a triangle cannot be formed with these segments using the technology, then a triangle can not be formed with these segments."

To identify the ways in which the students are using the mathematical tools, codes relating to these uses were created. The codes used for the students' uses of the tool(s) included the appearance of the diagram, the measures of the part(s) of the diagram, the use of the drag feature of the tools, and no use of tools. These codes were created such that they could be used to anlyze arguments created by students using both technological and non-technological tools. In addition, the codes not specific to a single task; thus, they could be used to analyze and compare students' arguments across tasks and across classes.

After the researcher diagrammed the arguments and identified the ways in which students used the tools, the researcher sorted the arguments by their basic structure (e.g. core argument, argument with additioanl data collection, linked argument, argument with a sub-argument). Within these basic structures, the researcher analyzed the arguments based on the explicitness of the warrant, the task on which the students were working, and the mathematical tool codes previously discussed. Within each structure themes based on the analysis of the arguments within the structure were developed and are discussed in the following two chapters. This process was performed for each group of students in both classes for each of the three tasks.

To demonstrate how the research questions, the data, and the method of analysis are aligned, Table 5 was created. The table provides the data and the ways in which the researcher analyzed the data for each of the research questions. The first two questions relied on the same data sources: small group and whole-class videos, classroom artifacts, and homework. The researcher analyzed these data sources using the same methods: Toulmin's model of argumentation and the students' uses of tools codes that were developed using a grounded theory approach. The data used to answer the final question includes the data to answer the first two questions and the whole-class and small group video and audio recordings for the class working in the non-technology environment. The analysis methods for the last two research questions included Toulmin's model of argumentation and the codes developed for the students' uses of tools.

Table 5

The Data Sources and Method of Analysis Used to Answer the Three Research Questions

Research Question	Data Sources	Method of Analysis
What are the characteristics of the content and structure of eight grade mathematics students' arguments while working in technological environments?	• Video and audio for technology class– small groups and whole class • In class artifacts • Homework	• Toulmin's Model of Argumentation • Students' uses of Mathematical Tools Codes
What are the characteristics of the content and structure of eight grade mathematics students' arguments while working in non-technological environments?	• Video and audio for non-technology class– small groups and whole class • In class artifacts • Homework	• Toulmin's model of Argumentation • Students' uses of Mathematical Tools Codes
In what ways do the arguments made by students working in a technological environment compare in content and structure to those made by students working in a non-technological environment?	• Video and audio for technology class– small groups and whole class • Video and audio for non-technology class– small groups and whole class	• Toulmin's model of Argumentation • Students' uses of Mathematical Tools Codes

Research Validity and Reliability

In order to ensure that that the results of this study are valid and reliable, the findings consist of a rich description of the events that transpired in the classroom. The length of the teaching experiment supports the validity of the findings. “Working with people day in and day out, for long periods of time, is what gives ethnographic research its validation and vitality” (Fetterman, 1998 as cited in Cresswell, 2007, p.208).

The data analysis draws upon multiple sources of data from multiple students including the video-recordings of the classroom, the in-class artifacts, and the homework. The variety of sources of data and variety of subjects assist in the validation of the results. “By using a combination of observations, interviewing, and document analysis, the fieldworker is able to use different data sources to validate and cross-check findings” (Patton, 1990, p. 244).

In addition to using multiple sources of data, the validity and reliability of the findings are strengthened by looking at a range of cases (Merriam, 1998). The research design of this study is partly based on the multi-case methodology, thus the validity and reliability of the findings is strengthened through the use of multiple cases.

The validity is also addressed by the researcher’s subjectivity statement. Because the reader knows the views and background of the researcher in regards to this topic, the reader can better understand how the data might have been interpreted (Merriam, 1995).

In research, reliability is concerned with the questions of the extent to ones findings will be found again. In the “hard” sciences, this issue is resolved in the replication of the measures. In qualitative research, researchers attempt to understand the work from the perspectives of those in it (Merriam, 1995). The issue is not whether the results can be replicated, but are the results consistent with the data collected. One method that can be used to ensure reliability is the use of the multiple sources of data, or triangulation. In this study, multiple sources of data will be used to triangulate the data in order for the findings to be considered valid and reliable. As previously stated, reliability of the findings will be demonstrated through the rich description of the students' discourse and behaviors in the course. In addition to the uses of multiple sources of data, the researcher also met with a member of the research team, an experienced researcher, regularly to discuss the diagramming of the arguments. This experienced researcher was very familiar with this method of analysis having employed it in the past with other data. During these meetings, the researchers discussed the analysis including whether the students actually created an argument, agreeing on the ways in which certain arguments should be diagrammed, and the students' uses of the tools. The researchers discussed the arguments created by each group of students for each of the three tasks used in the analysis.

Safeguards Against Research Bias

The possibility of a biased description by the researcher is expected. Because the researcher has a history of using technology in his work as a mathematician and as a teacher, it should be known that he is a firm supporter of using technology in the classroom. While pursuing a master's degree in applied mathematics, the researcher was a member of a research team that used an array of over fifty computers to model the production of ozone in a southern metropolitan area. As a classroom teacher, he was afforded the option of spending half of a class period in a computer lab. Using the technology to promote conceptual understanding, the researcher was able to have the students explore concepts in a way that is inaccessible using conventional methods. In addition, the students were more engaged in the class when in the computer lab using the technology. In his work as a doctoral student, the researcher taught an undergraduate teaching methods course on teaching mathematics with technology. The use of technology in his mathematical activities as a master's student and the benefits of using technology to teach cemented in his mind that technology is an integral part of today's mathematics and should be used in the mathematics classroom. Because of the reseracher's background and thoughts on why and how technology should be integrated into the mathematics classroom, this study may contain some bias.

Ethical Issues

This study proceeded with permission from the internal review board (IRB) that oversees all research at North Carolina State University. IRB approval was given and the letter of approval can be found in Appendix A. Participation in this study is voluntary and those choosing to participate signed and had one of their parents sign a consent form prior to participation in the study that assures their privacy and confidentiality of all data sources. Regardless of whether a student elected to participate in the study, all students in the class participated in the activities of the class. In the reporting of this study, all participants' names are replaced with pseudonyms and all documents were coded using these pseudonyms. Participants had the option to leave the study at any point in time, but none did. The raw digital audio- and video-recordings is stored on password protected external hard drives in a locked file cabinet at the Friday Institute. Edited video and audio clips are stored on a password-protected computer at the Friday Institute and are not available on a network server.

Because this research study is a classroom teaching experiment, the entire classroom was video and audio recorded. The research team made every attempt to capture those students that are participants in the study. During the analysis phase of this study, the non-participants' contributions to the classroom discourse were not included in the transcription of the classroom discourse.

One of the ethical issues facing this research is the instructional decision made by the researcher in the context of teaching the course. Because the researcher is acting as the teacher, the motives for the instructional decisions made during the course of instruction could be called into question. That question being: were the instructional decisions made to further the students' development and understanding or were the instructional decisions made to further the research? However, one facet of the research methodology is to engineer some type of learning. Thus, the goals for the research and instruction are aligned.

In addition, the use of focus groups could be problematic. Because the researcher was the teacher and the design of the research used focus groups during small group discussions, the researcher might have tended to focus on those groups of students and gave preferential treatment to them during whole class discussion in order to capture their thoughts and ideas on video.

Limitations of the Study

The use of only three focus groups is a limitation for this study. By only focusing on three groups' interactions with the tools and set of tasks, the data captured on how the students' uses of the tools and the small group discussions of the tasks may not provide a complete picture on how all of the students in the class used the tools.

In addition, the use of homework as an artifact may be a limitation. In general, the homework that is submitted is a finished product and may not provide much insight into how the student arrived at their solution. However, for the majority of homework problems, the teacher asked the students to provide a written description of how the student arrived at their answer.

The amount of time spent in class is a limitation to this study. Because the classes only met 8 times each, some of the learning and development took place outside of class. The use of homework and the request for the students to provide a written description and explanation of their work on the homework will assist in this limitation. However, it will not completely capture the students' development because the explanation on the homework is dependent on what the student chooses to share.

In the next two chapters, the structure and content of the arguments created by the students working in the technological and non-technological environments are described. In chapter 6, a synthesis that answers the research questions is given.

CHAPTER 4

Technology Class Arguments

This chapter details the arguments created by three groups of students working in the technological environment. For each of the three analyzed tasks, a description of the mathematical objectives, the technological tool, and the teaching of the lesson begin each task section. Thereafter, an analysis is presented of the arguments created by each group of students while working on the task. To close each task section, the researcher provides a cross-case analysis of the arguments created by the three groups of students. At the end of the chapter, the researcher provides an analysis of the arguments created by each group of students across the three tasks.

Task 1 - Triangle Inequality Task

The triangle inequality task began with an exploration and discussion about the triangle inequality theorem. The triangle inequality theorem states "if A, B, and C are three noncollinear points, then $AC < AB + BC$" (Venema, 2006, p. 103). In their previous elementary and middle school math classes, the students were expected to have studied triangles including the definition of a triangle, the differences between a triangle and other geometric figures, and the classification of triangles. However, the standard course of study for these prior mathematics courses did not indicate whether the students had explored some of the conditions needed for a triangle to exist. Rather than building on the students' notions of classification, the researcher chose to begin the unit exploring a concept that was novel to them. This was done for three reasons. First, the unit under study was on triangles in general and the researcher wanted the students to consider and understand some of the conditions needed to form a triangle. Second, the researcher wanted the students to begin the unit with an exploration rather than a recollection of facts. Third, a novel task may produce a larger number and greater variety of structures of arguments than routine tasks that ask the students to recall previously studied concepts.

The objectives for this task centered on the students' discovery and understanding of the triangle inequality. The teacher wanted the students to be able to use the technology to determine whether a given set of segments could form a triangle and use the technology to experiment with other lengths to create a conjecture about the relationship between the lengths of the sides of a triangle. The teacher also wanted the students to compare the sets of segments that were able to form a triangle and those that were not. By the end of the task, the teacher expected the students to have developed an understanding of the triangle inequality theorem.

With these objectives in mind, the researcher created a technological activity and corresponding task sheet for use in teaching this concept. The activity and task sheet were adapted from an activity in Wyatt, Lawrence, and Foletta's (1998) book "Geometry Activities for Middle School Students with The Geometer's Sketchpad." The technological activity consisted of a pre-constructed sketch using The Geometer's Sketchpad (GSP) (Jackiw, 2001).

Pre-constructed sketch.

The pre-constructed sketch was chosen for a variety of reasons. First, the students had limited exposure to the software, GSP. Even though the students had regular access to and frequently used technology in their regular class, the students' uses of this particular software were minimal at best. In fact, the students' prior uses of this software in their regular mathematics class were only to draw patterned quilts. Second, having the students construct this sketch would have taken a great deal of class time and may not have provided any insights into why the theorem was true. For these reasons, a pre-constructed sketch was used in lieu of a student-constructed sketch.

The pre-constructed sketch created for this task was one in which the students could investigate whether a set of segments could form a triangle (see Figure 7). The students would begin by dragging the right most endpoint of one of the horizontal segments (a slider) to the desired segment length, which are displayed above the sliders. When this endpoint was dragged, the linked segment in the diagram to the right would lengthen or shorten depending on whether the endpoint of the slider was dragged to the right or to the left. For example, in the image on the left in Figure 7, *a* measures 3.8 cm and segment *a* in the diagram is not touching segment *b*. In the image on the right in Figure 7, the slider has been lengthened such that *a* measures 5.5 cm and segment *a* now extends past segment *b*.

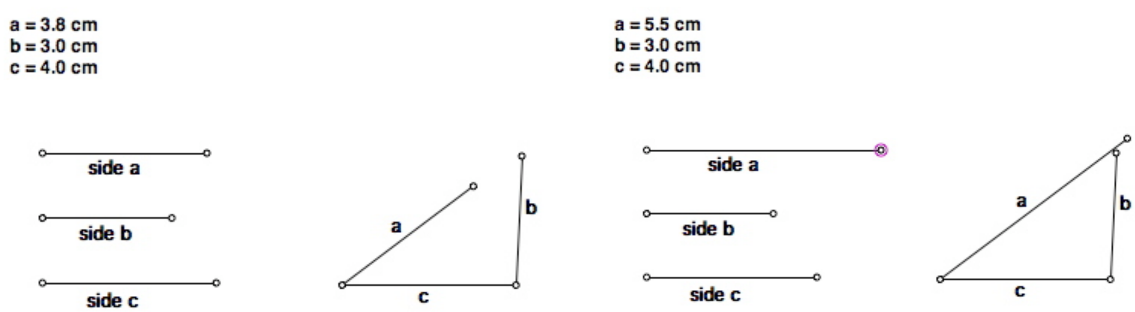


Figure 7. The pre-constructed sketch used during the triangle inequality task with side *a* adjusted from measure 3.8 to 5.5 and the corresponding change in the linked diagram

Once the students dragged the sliders to their desired lengths, the students could investigate whether a triangle could be formed with these segments. The students would accomplish this task by dragging the non-connected endpoints of the diagram on the right. If the endpoints lie exactly on top of each other, then a triangle could be formed. If not, then a triangle could not be formed. For example, compare the diagram on the right in Figure 7 to that in Figure 8. In Figure 8, the endpoint of segment *a* has been dragged down and the endpoint of segment *b* has been dragged to the right to form a triangle.

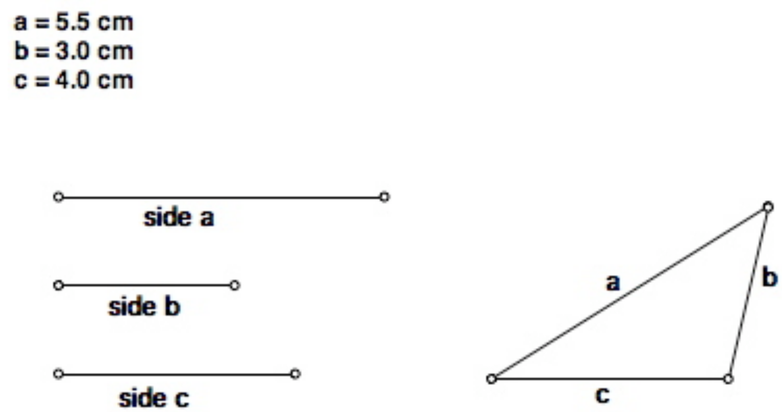


Figure 8. The endpoints of segments *a* and *b* have been dragged such that a triangle is formed.

There are a variety of sketches that could have been used to explore the concept, but this sketch was used for three important reasons. First, the sketch allows for students to explore when a triangle can be formed and when a triangle cannot be formed. Many of the sketches and student constructions used by teachers with this topic only allow for students to consider when the triangle is formed. Second, the sketch is fairly intuitive and easy to use. As previously stated, the students had limited exposure to this technology and this sketch did not require the students to use many of the features of the software. Third, the sketch allows for exploration in multiple ways. The students could have used the sketch to determine whether given sets of segments form a triangle or they could have dragged one of the sliders to find the point when a triangle could no longer be formed from the given segments.

The task sheet.

Along with the sketch, the students were given a task sheet to assist in their exploration (see Appendix B). The task sheet provided sets of segment lengths to use with the technology. The students were to adjust the sliders to the lengths on the task sheet and drag the endpoints to determine whether a triangle could be formed. The directions indicate that the students were to draw the triangle if one could be formed and to write impossible if one could not be formed. In the directions given to the class, the teacher asked the students to draw the diagram on the screen when a triangle could not be formed.

The sets of segments given to the students on the task sheet were selected for a variety of reasons. First, there are three sets of segments that are able to form a triangle and three sets that are unable to form a triangle. Second, the longest segment and the shortest segment were not always the same segment (e.g. the longest segment was not always segment *c*). Third, the value for one of the lengths of the segments was not a whole number (3.5). This was selected so the students would think the lengths of the segments must not always be whole numbers. Fourth, it was anticipated that the students were familiar with different classifications of triangles, mainly isosceles triangles. The decision to use two sets of segments that would form isosceles triangles was to draw upon this prior knowledge. Also, the isosceles triangles are a bit different; in the first set (3, 4, 4,) the non-congruent side is the shortest and in the second set (5, 5, 6) the non-congruent side is the longest. This was done in hope that the students would not develop erroneous conclusions about the relationship among the lengths of the sides of an isosceles triangle, which would have to be resolved in a later lesson. Finally, there were many example types that were omitted from this task sheet, most notably a set of segments that would form a straight line. A set of this type was purposely omitted in hope that the students may investigate this case on their own.

After the students determined whether a triangle could be formed for the sets of segments on the task sheet, the students were asked “Why was it impossible to construct a triangle with some of the given lengths?” Rather than have the students answer why it did work, the teacher wanted the students to attend to those that could not work. Thus, the students could begin comparing and finding a common attribute for those sets of segments that could not form a triangle, primarily because the length of one of the segments was much greater than the lengths of the other two segments. After the students answered this question, they were asked to “Write a conjecture about the relationship among the lengths of the three sides of a triangle.” This was given after the previous question in hope that the students would further investigate when a segment length becomes too much longer than the other two segments and a triangle was unable to be formed.

Teaching of the lesson.

The triangle inequality task was taught on the first day of the unit. The class began with a short discussion of the definition of a triangle and the students told the teacher what they knew about triangles. The teacher asked the students if any three segments could form a triangle. Some students said ‘yes’, others said ‘no’. The teacher then gave the students the example of segments with lengths 1, 1, and 1000 and asked the students whether these segments could form a triangle. The students said ‘no’. The teacher then posed the question

“When will the segments connect? Is there a rule that tells us when the segments will connect to form a triangle?” The teacher than passed out the task sheet and asked the students to open up the pre-constructed sketch on the laptops.

The teacher and students read the directions on the task sheet and worked the first set of segments (2, 3, 4) together. The teacher demonstrated how the sliders are adjusted and how the endpoints can be dragged to form a triangle. After finishing the first set of segments, the students were asked to work with their partner on the remaining sets of segments and then to answer the questions at the bottom of the task sheet. The students were given time to work on this task and to answer the questions. Once the majority of the pairs of students had completed the sheet, the students were asked to participate in a whole class discussion. The teacher asked the students whether each set was able to form a triangle and to discuss their reasons why some sets of segments were unable to form a triangle and their conjectures for the relationship among the lengths of the sides of a triangle.

During this whole class discussion, the students correctly determined whether a triangle could be formed for each of the sets of segments on the task sheet. In addition, they provided reasons why some sets of segments were unable to form a triangle. However, the conjectures they created about the relationship among the lengths of the sides of a triangle were relative to the lengths of the side lengths. For example, one group wrote, “They [side lengths] have to be close to form a triangle.” To assist the students in making their conjectures more definite, the teacher asked the students to circle the two shortest segments for each set on the task sheet. When this did not help the students state a correct relationship, the teacher asked the students to use the technology look at determine whethe the sets of segments of lengths 2, 4, 7; 2, 4, 6; and, 2, 4, 5 formed triangles. After looking at these sets of segments, one student correctly stated the conjecture. The other students agreed and verified it on their task sheet. The teacher gave the students two additional sets of segment lengths for practice and the students, using their new conjecture, were able to correctly determine whether a triangle could be formed without using the technology. Seeing this, the teacher moved on to the next task on the agenda. On that night's homework, the students were asked to determine whether a set of segments could form a triangle and state how they knew this to be true. In addition, the students were given the values for the lengths of two segments (4 and 9) and were asked to list the possible values for the length of the third segments that would be able to form a triangle, including the shortest possible side length and the longest possible side length.

On the second day of class, the students turned in their homework and were given a warm-up with four questions, two related to the triangle inequality theorem (see Appendix C). The students had difficulty answering the two questions related to the triangle inequality and the teacher decided to have the students work together as a class on this task. The teacher asked the students to recall the theorem the students had stated the previous day and to use that to assist them in answering the questions. The teacher focused the students on finding the length of the shortest side given the values of the lengths of the other two sides are 7 and 10. One student said 4 would be the smallest. Using this idea, the teacher directed the students to consider non-integer positive values less than 4 and eventually the students stated that the value had to be greater than 3. The discussion then moved to finding the largest value that can form a triangle given segments of lengths 7 and 10. The students asked if they could use the technology to investigate this task and the teacher said 'yes'. All of the pairs of students used the technology to investigate this idea choosing different segment lengths and determining whether these segments could form a triangle. The students eventually came to the conclusion that 16.9 was the largest value of the length of the third side that could form a triangle with segments of lengths 7 and 10. Class ended prior to the discussion of the fourth problem on the warm-up and was given for homework.

In the following sections, the arguments created by three pairs of students while working on these tasks are analyzed and discussed. For each pair of students, the arguments were first categorized by their basic structure. Then, the content and structure of the arguments within these basic categories were analyzed, including the students' uses of technology. The themes that emerged from these analyses are discussed for each pair of students and across the pairs of students.

Group 1's arguments while working on the triangle inequality task.

The analysis of Heather and Mary's arguments while working on the triangle inequality task can be categorized into three argument structures: core arguments, arguments in which one of the pair of students felt the need or was prompted to collect more data to verify or refute a claim, and arguments that contained an inexplicit claim. These arguments structures are discussed below.

Core arguments.

A core argument is an argument that contains data, a claim, and a warrant. At times, students may qualify their claims, but in Mary and Heather's case, this was rare. In some cases the warrants are stated explicitly and in other cases they are not. While working on the triangle inequality activities, Mary and Heather created 14 core arguments. These core arguments can be further categorized by their use of technology: their use of the drag feature, the appearance of the diagram on the screen, and the indirect use of technology.

Dragging the endpoints.

In six of the core arguments created during the triangle inequality activities, Heather and Mary dragged the endpoints of the segments to determine whether a triangle could be formed. An example of one of these arguments is Mary's core argument presented in Figure 5. In this argument, Mary collected data by dragging the endpoints of the segments to determine whether the endpoints would meet and form a triangle.

In a similar manner, Heather and Mary created a core argument regarding whether a triangle can be formed with segment lengths 7, 10, and 17. The pair adjusted the sliders accordingly and dragged the endpoints, which did not form a triangle. Both of the students made the claim, "17 doesn't [work]." This argument is illustrated in Figure 9.

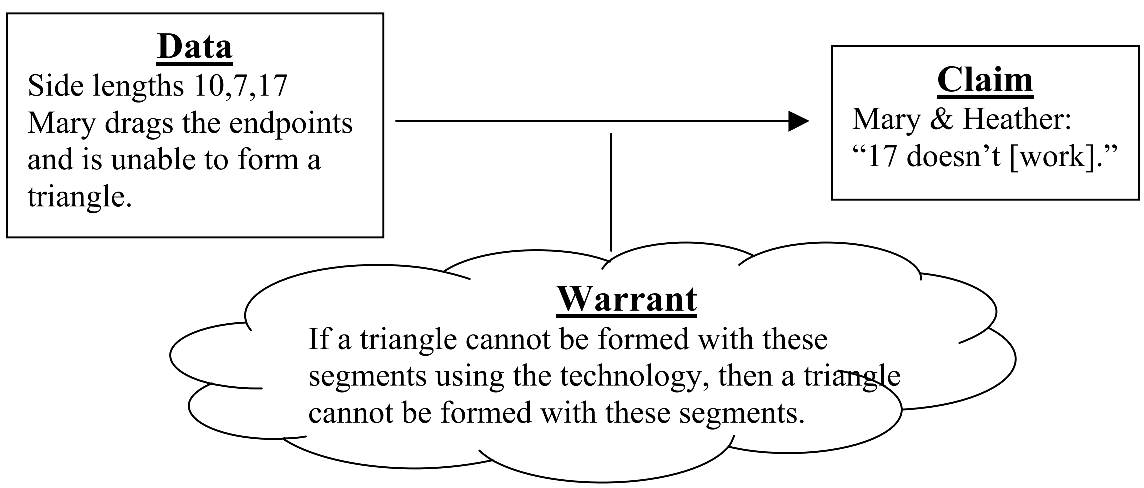


Figure 9. Heather and Mary's core argument created during task 1 with an inferred warrant and dragging as data.

The data for this argument are the segments of lengths 10, 7, and 17 and the dragging of the endpoints of the diagram. Both students claim the segments lengths 7, 10, and 17 do not form a triangle. Their warrant for this claim was not explicit and is inferred by the researcher to be the same as the previous example, "if a triangle can not be formed with these segments using the technology, then a triangle can not be formed with these segments."

There were a few similarities across these five core arguments. First, the warrants for all five of the core arguments in which the students employed the drag feature of the technology were not made explicit by the students and were inferred to be the same warrant. Second, none of the claims were qualified. And, in four of the five arguments, the students were unable to form a triangle with the given side lengths.

Appearance of the diagram on the screen.

Another use of technology in the core arguments created by Heather and Mary was the appearance of the diagram on the screen. In these five arguments, the students would adjust the sliders to the desired length, look at the corresponding diagram on the screen, and then make a claim. The students did not drag the endpoints of the diagram. For example, Heather and Mary are to determine whether a triangle can be formed with segment lengths 2, 7, and 4. They adjust the sliders accordingly and look at the triangle. Heather claims, "That's not going to work" and Mary agrees in saying, "nuh huh." This argument is illustrated in Figure 10.

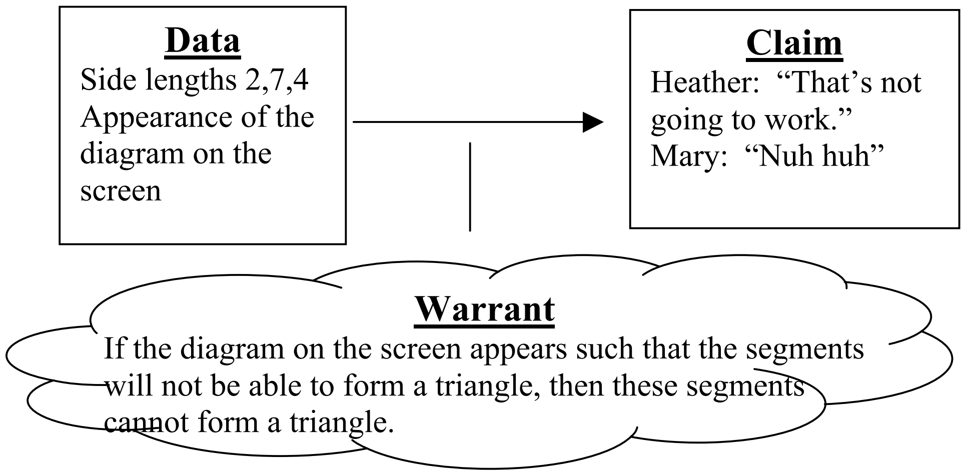


Figure 10. Heather and Mary’s core argument created during task 1 with diagram on the screen as data.

The data for this argument are the segments of lengths 2, 7, and 4 and the appearance of the corresponding diagram on the screen. Both students claim that these segments will not form a triangle. The students’ warrant was not explicit and is inferred by the researcher to be “if the diagram on the screen appears such that the segments will not be able to form a triangle, then these segments cannot form a triangle.”

Mary makes a similar argument when determining whether a triangle can be formed with segments of length 4, 7, and 12. Mary adjusts the sliders accordingly, looks at the corresponding figure, and claims, “12 won’t work.” This argument is illustrated in Figure 11.

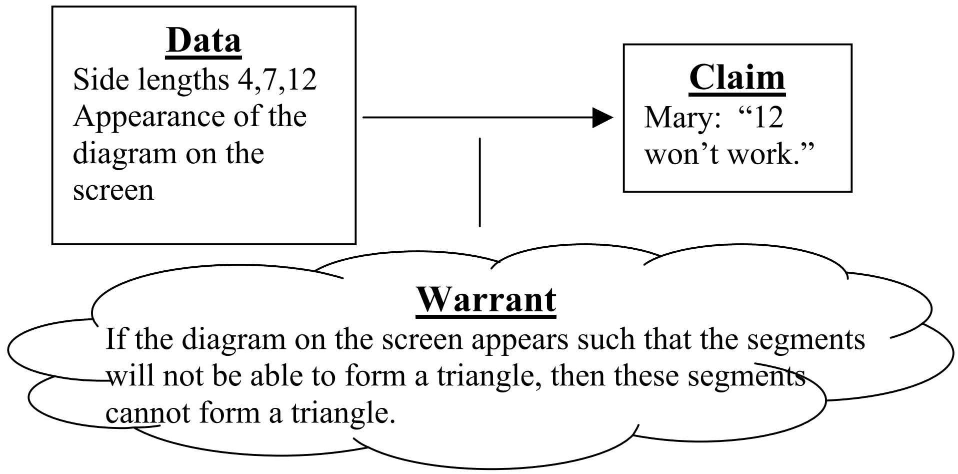


Figure 11. Mary’s core argument created during task 1 with diagram on the screen as data.

The data for this argument are the segments of lengths 4, 7, and 12 and the appearance of the corresponding diagram on the screen. Both students claim that these segments will not form a triangle. The warrant for this claim was not explicit and was inferred by the researcher to be, “if the diagram on the screen appears such that the segments will not be able to form a triangle, then these segments cannot form a triangle.”

The three other core arguments in which the appearance of the diagram on the screen is the main use of technology have similar structure and content; the data are the segment lengths and the corresponding appearance of the diagram on the screen, the claim is that a triangle cannot be formed with these segments, and the warrant is that if the appearance of the diagram looks like it will not form a triangle with the these segments, then a triangle cannot be formed. In each of these arguments, one of the segments is much longer than the other two segments (e.g. 4, 7, and 15; 4, 7, and 18). This large discrepancy in lengths was reflected in the diagrams when the students adjusted the sliders. The students may have felt it was unnecessary to drag the endpoints to determine whether a triangle could be given that one of segments was so much larger than the other two. It may have appeared to be too big, and, thus, it did not require further exploration using the technology.

Indirect use of technology.

The third category of core arguments created by Heather and Mary were those in which technology was not actively used in the argument. Even though only four of the fourteen core arguments did not involve the active use of technology, they have a unique structure that is significantly different from those in the previous two categories of core arguments in which technology was actively employed.

The first example is Heather's argument in which she claims that in order for a triangle to be formed, "they [the segments] have be zeros or fives to fit, it can't be a mixture." Heather is referring to the value of the tenths digit of the segment length. Her data for this claim is the three previous sets of segments lengths (2, 3, 4; 4, 1, 6; and, 3.5, 2, 6) and whether she was able to form a triangle (yes, no, and no, respectively). It is worth noting that the students used technology to determine whether these segment lengths could form a triangle, but her claim is not based on her active use of the technology. Rather, the claim is based on the product of her previous uses of technology. Heather qualifies her claim with "I think." She does not provide an explicit warrant for her claim and the researcher inferrs the warrant to be "a previous example in which all of the segments' length's tenths digit were the same was able to form a triangle and a different example in which the segments' length's tenths digits were not the same was unable to form a triangle." It is worth noting that Heather's claim contradicts one of the prior examples (6, 1, and 4). This core argument differs from those previously mentioned because it does not involve the active use of technology and the claim is qualified. This argument is illustrated in Figure 12.

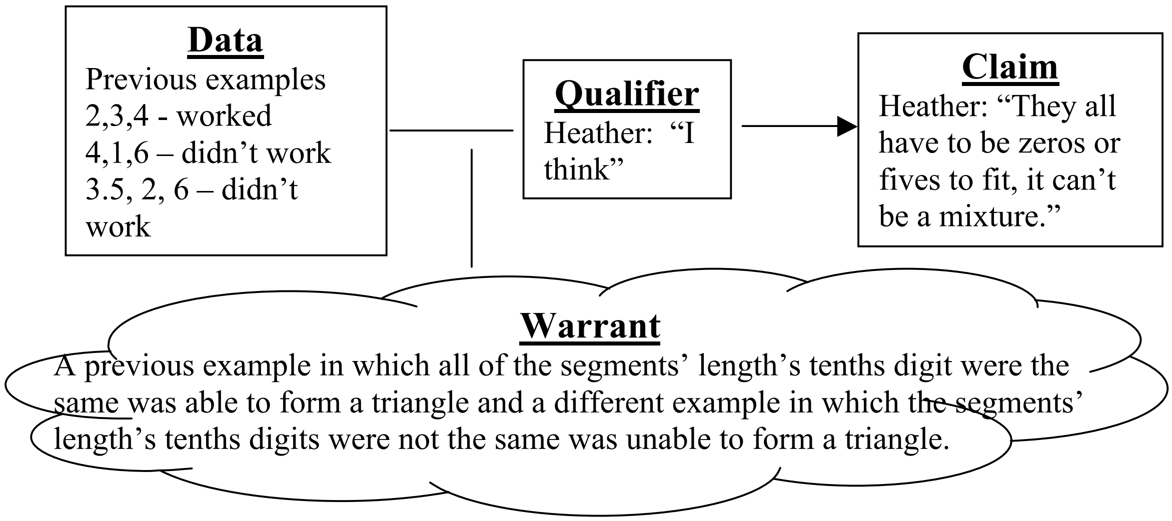


Figure 12. Heather and Mary's argument created during task 1 with a qualified claim and indirect use of technology.

Another example of a core argument in which technology is not actively employed in the creation of the argument is one provided by Heather in response to the question "Why was it impossible to construct a triangle with some of the given lengths?" Heather responds, "Because some of the sides were a lot bigger than the other two." This argument is illustrated in Figure 13.

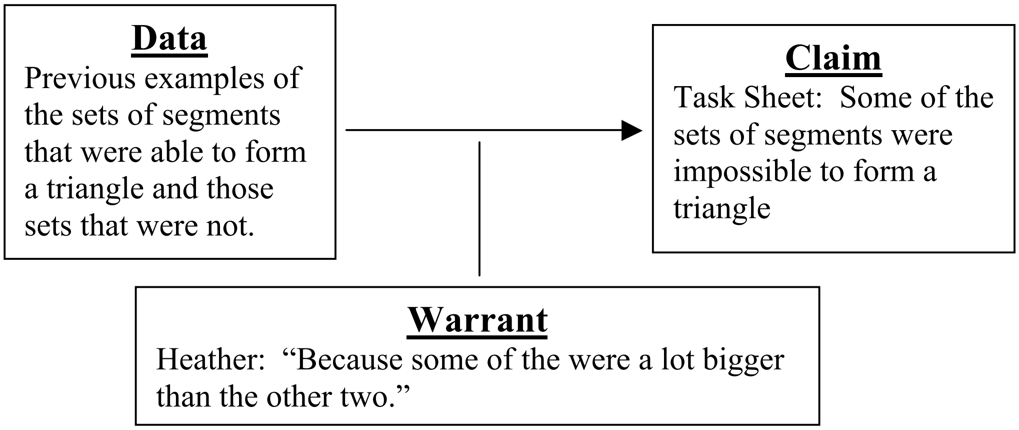


Figure 13. Heather and Mary's core argument created during task 1 with an explicit warrant in response to a claim made on the task sheet.

The claim, which is provided by the question, is that some of the side lengths were impossible. Her data for this claim are the previous examples that were able and unable to form a triangle. Even though Heather used technology to determine whether a triangle could be formed with the given segments, the data for this argument were products of these uses of technology; thus, an indirect use of the technology. Heather explicitly provides the warrant for this claim. This core argument differs from others previously discussed. Heather does not provide the claim for this argument, instead, it is provided by the task itself. The question is asking the student to provide the warrant for why some of the side lengths were impossible. In other words, the question is asking the student to make an explicit warrant. In addition, Heather's warrant is explicit. The explicitness of this warrant may be due to the nature of the task. The task is providing the claim and asks for the warrant for this claim. Given this task, it is not surprising that her warrant is explicit.

The third example of a core argument in which the students do no actively use technology is Heather and Mary's argument regarding the conditions necessary for a triangle to be formed. Heather claims, “[To make a triangle] the base has to be the longest and the others have to be relatively close to the base and the sides have to relatively close to each other.” Her claim is based on the previous examples that form triangles and those that do not form a triangle. Both Heather and Mary contribute to the warrant. Heather refers to one example and states, “Like that’s the base and those two are relatively close, they’re about one apart and it made a triangle.” She continues and states, “It’s the same here, but those two are exactly the same.” Mary says, “Yeah, like that one’s 7 and they’re like.” Heather interjects “and the other two.” Mary continues, “are completely off and like that one’s six and it’s one, four. And, that’s like six and like two.” This argument is illustrated in Figure 14.

Data
Previous examples of the sets of segments that were able to form a triangle and those sets that were not.

Claim
Heather: “[To form a triangle] the other two sides have to be relatively close to the base and the sides have to be relatively close to each other each other.”

Warrant
Heather: “Like that’s the base and those two are relatively close, they’re about one apart and it made a triangle. It’s the same here, but those two are exactly the same.”
Mary: “yeah, like that one’s 7 and they’re like”
Heather: “and the other two”
Mary: “are completely off and like that one’s 6 and its one four, and that’s like 6 and like two.”

Figure 14. Heather and Mary’s core argument created during task 1 with an explicit warrant and indirect use of technology created.

The data for this argument are the previous examples of sets of segments that were able and unable to form a triangle. Similar to the two previously discussed arguments, the students used the products of their uses of technology in their data, but not the technology itself. Heather and Mary are explicit in their warrant in two ways. First, they explicitly justify that their claim is valid by showing that the examples that were able to form a triangle are aligned with the claim. Then, they show that the examples that did not form a triangle are not aligned with the claim. In the creation of this argument, the students are explicit in their warrant and they did not actively use technology.

Conclusion.

Looking across these core arguments, two major themes emerge. First, when the students do not actively use technology, they are more likely to provide an explicit warrant. One difference between these arguments and those in which the students actively use technology is the task on which the students are working. In the majority of the arguments in which the students employ technology, the students are determining whether a triangle can be formed from the given set of segments. In the arguments in which the students do not actively use technology, the students are making a generalization regarding the conditions for the lengths of the segments to form a triangle. The students may have felt more compelled to provide an explicit warrant when making a generalization.

The second theme that emerged from the analysis of these core arguments is when Heather and Mary create a core argument based on their use of technology; they do not provide an explicit warrant. The students use technology in eleven of the fifteen core arguments; six in which the claim is based on the dragging of the endpoints and five in which the claim is based, in part, on the appearance of the diagram on the screen after the sliders have been adjusted. In addition, the students are unable to form a triangle in nine of these eleven core arguments. The lack of qualifiers and rebuttals in the structure of these arguments indicates that the students were confident in their claims. Thus, when the students are unable to form a triangle using the technology, the students are not compelled to collect additional data.

Arguments in which Heather and Mary collect additional data.

The second type of argument structure created by Heather and Mary is when the students seek additional data after an initial claim is made to verify or refute that claim. The students created ten arguments with this structure. The students' decision to seek additional data may be due to a number of factors including an explicit challenge to a claim, the uncertainty of a claim, and their uncertainty of a claim due to the lack of precision in the use of the technological tool.

Challenges.

One reason students may seek additional data to verify or refute a claim is their original claim is challenged by another person. For example, Heather and Mary overhear another student state, "10.9 works." Heather asks Mary to, "try 10.9." She is asking Mary to see if segments of lengths 4, 7, and 10.9 will form a triangle. Mary adjusts the sliders accordingly. Looking at the resulting diagram on the screen Heather claims, "10.9 works." The endpoints of the diagram are not touching and Mary, unsure whether these segments will form a triangle, states, "No it doesn't." She proceeds to collect more data by dragging the endpoints and is able to form a triangle. Mary agrees with Heather's original claim by stating, "yeah it does." This argument is illustrated in Figure 15.

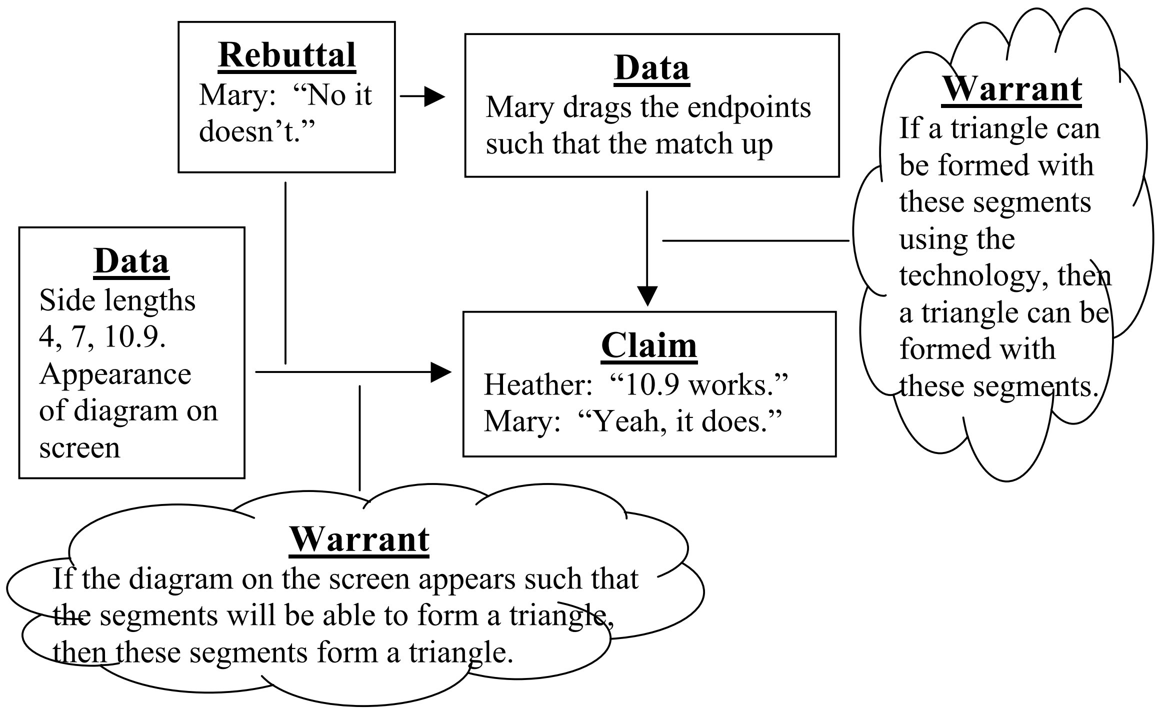


Figure 15. Heather and Mary's argument created during task 1 with additional data collection based on Mary's challenge.

In this argument, Heather's initial claim is based on the data, the segment lengths 4, 7, and 10.9 and the appearance of the corresponding diagram on the screen. Her warrant is not explicit and inferred by the researcher to be "If the diagram on the screen appears such that the segments will be able to form a triangle, then these segments form a triangle." Unlike the previously discussed core arguments, Mary challenges Heather's claim. She may not be convinced that these segment lengths will form a triangle based on the appearance of the diagram on the screen. While a challenge to the claim generally is a call for a sub-argument or backing, Mary's actions following this challenge indicate it is a rebuttal to the claim. Mary collects additional data by dragging the endpoints and is able to form a triangle, thus verifying Heather's claim. The warrant that links the additional data to the claim is not explicit and was inferred by the researchers to be “If a triangle cannot be formed with these segments using the technology, then a triangle cannot be formed with these segments.”

Uncertainty of a claim.

The decision to collect additional data may also be related to the students’ uncertainty of the claim. In one argument, Heather and Mary are asked to determine whether the segments of lengths 3.5, 2, and 6 will form a triangle. Heather adjusts the sliders accordingly and claims, “I think this works.” Heather makes the claim prior to dragging the endpoints and is based on the appearance of the corresponding diagram on the screen in which the endpoints are not touching. Heather proceeds to drag the endpoints towards each other and is unable to form a triangle. She claimed, “Nah” and Mary concurred, “It’s not going to work.” Even after this claim is made, Heather continues to drag the figure trying to get the endpoints to match up, but was unsuccessful. This argument is illustrated in Figure 16.

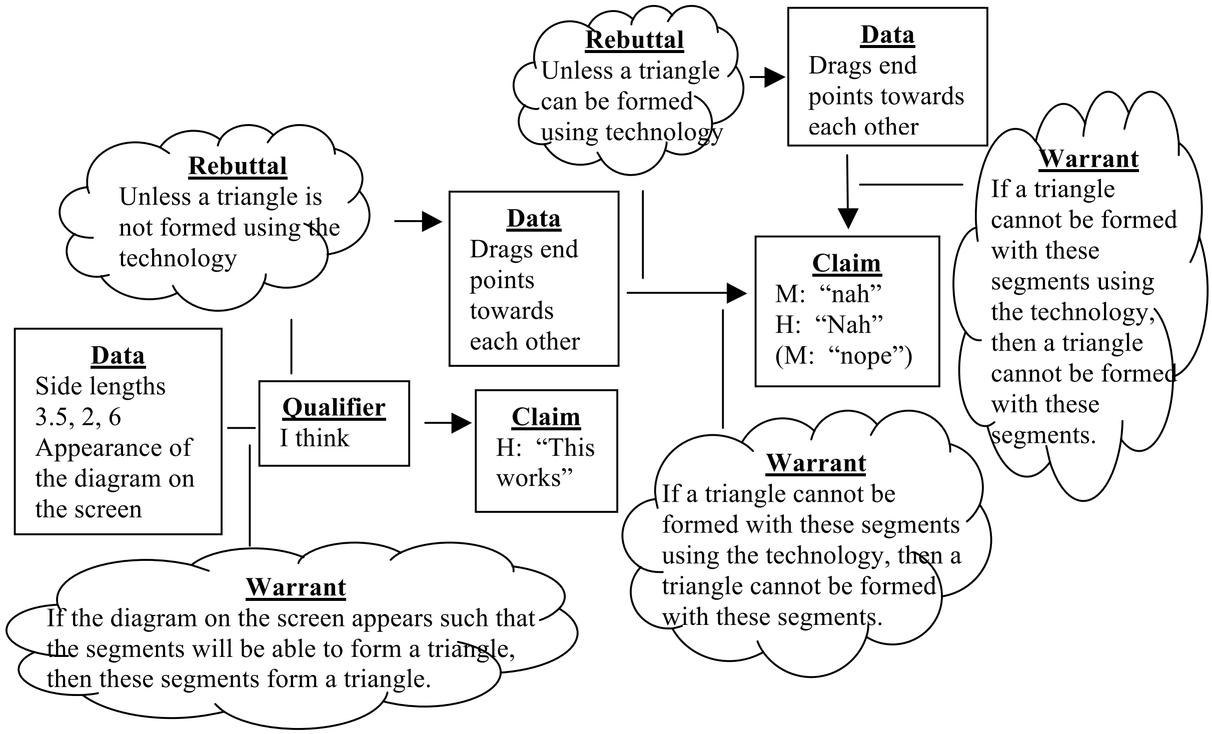


Figure 16. Heather and Mary's argument created during task 1 with additional data collection based on Heather's uncertainty about her claim.

In this argument, Heather's initial claim that the segment lengths 3.5, 2, and 6 will form a triangle, is based upon her data, the lengths of the segments and the appearance of the corresponding diagram on the screen. She does not provide an explicit warrant and is inferred by the researcher to be "if the diagram on the screen appears such that the segments will be able to form a triangle, then these segments form a triangle." She qualifies this claim with "I think" which demonstrates her uncertainty of the claim. Heather proceeds to collect additional data through dragging and based on this use of the technology, a new claim is made that refutes her original claim. Mary agrees with this new claim. Heather does not provide an explicit warrant for this claim but through her use of the technology, the researcher infers the warrant to be "If a triangle cannot be formed with these segments using the technology, then a triangle cannot be formed with these segments." Unlike the previous claim, Heather does not qualify this new claim. However, she continues to drag the diagram, which demonstrates a level of uncertainty. This dragging creates additional data that verifies the claim that a triangle cannot be formed from segments of lengths 3.5, 2, and 6. Her warrant that links this new data to the claim is not explicit and is inferred by the researcher to be the same as the previous warrant.

Precision of the tool.

The decision to collect additional data may relate to a student's uncertainty of precision needed to make a valid claim. For example, Heather and Mary are determining whether a triangle can be formed with segments lengths 7, 10, and 11. Mary adjusts the sliders accordingly and begins to drag the endpoints of the diagram. She drags the endpoints such that the endpoints are overlapping, but not coincidental. Heather claims, "it works" or, in other words, these segments form a triangle. Mary is uncertain and asks, "well no, is that close enough?" She proceeds to drag the endpoints and is able to make them coincidental. Mary verifies Heather's claim by stating, "11 works." This argument is illustrated in Figure 17.

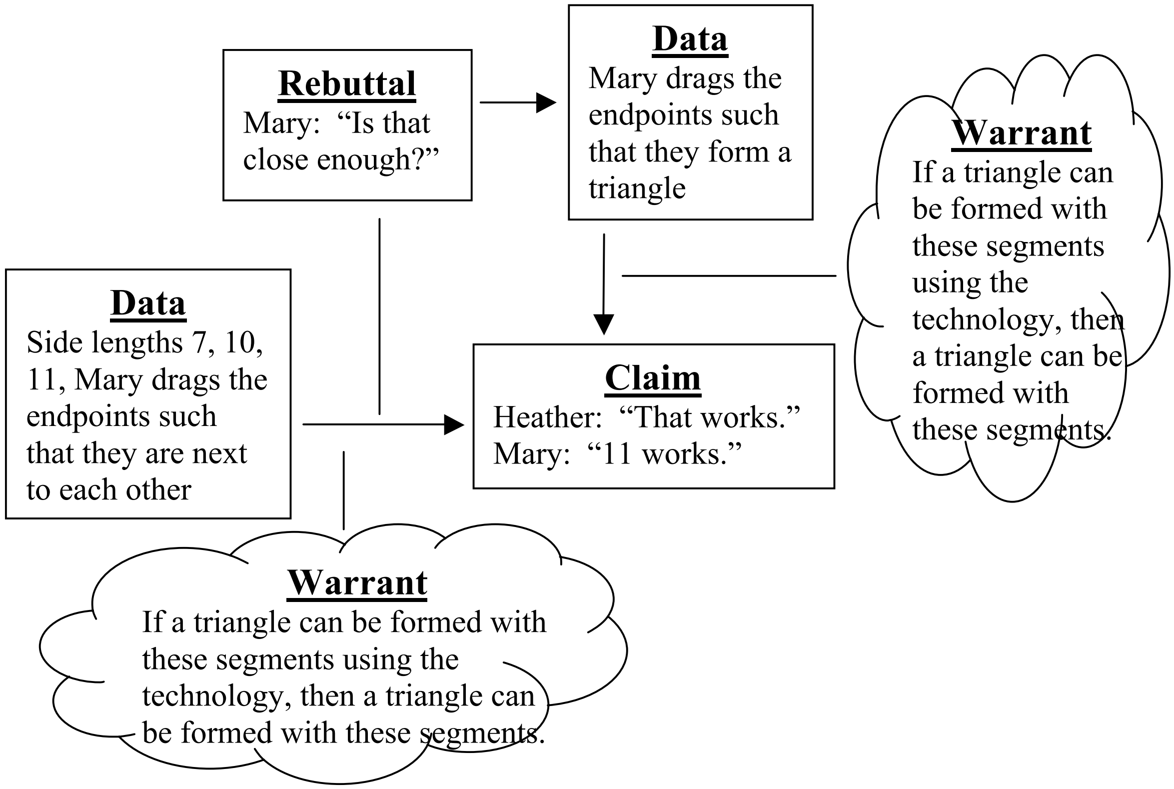


Figure 17. Heather and Mary's argument created during task 1 with additional data collection based on the uncertainty regarding the precision of the tool.

In this argument, Heather makes a claim based on the lengths of the segments and the resulting diagram after Mary has dragged the endpoints. She does not explicitly state a warrant and is inferred by the researcher to be "if a triangle can be formed with these segments using the technology, then a triangle can be formed with these segments." Mary is uncertain if this claim can be made given that the endpoints do not match up exactly. She states an explicit rebuttal by asking the question, "Is that close enough?" She proceeds to collect additional data by dragging the endpoints and the endpoints become coincidental which verifies Heather's claim. She does not provide an explicit warrant to link the data to the claim and is inferred by the researcher to be the same as the previous inferred argument.

This argument differs from the previous two in that the uncertainty does not directly rest on whether the claim is valid. Instead, Mary is uncertain whether this claim can be made from the collected data and the precision needed in order to make the claim. The technology is the source for this perturbation, and it affords the collection of additional data to resolve this issue.

Conclusion.

The common theme for this argument structure stems from the uncertainty of the student about the initial claim and his or her perceived need to collect additional data. In the first example, the one making the original claim, in this case Heather, is certain about her claim but another is not, Mary, who challenges the claim which leads to the collection of additional data. The second example illustrates that the uncertainty of a claim could stem from the one making the claim and lead to the collection of additional data. The third example differs from the previous two in that the uncertainty does not directly lie in whether the claim is valid, rather it stems from the precision of the initial data collected.

Even though the type of uncertainty and the person who held it differed in this argument structure, the technology use remained consistent. Of the ten arguments of this structure, seven of the arguments' initial claims were based on the appearance of the diagram on the screen and the additional data collected was through dragging, similar to Heather and Mary's argument illustrated in Figure 16. Due to the commonality of the content and structure of these arguments, one may infer that even though Heather and Mary make claims based on the appearance of a diagram, they also know that appearances can be deceiving and additional data is needed to support their claim.

For all of the arguments of this structure, the task on which the students are working is the same, to determine whether a set of segments would form a triangle. Of the nine arguments of this structure, the final claim for six of the arguments is that a triangle is formed and for the other three arguments, the final claim is that a triangle cannot be formed. In addition, six of the arguments result in the verification of the initial claim. In one of the arguments in which the initial claim is refuted, the second claim is verified.

Arguments with a non-explicit claim.

The third structure of arguments created by Heather and Mary are those that contain an unspoken claim. The students create two arguments of this structure. In the previous arguments, the students are attempting to determine whether a given set of segment lengths will form a triangle. However, in the arguments with a non-explicit claim, the students are extending this work and making claims beyond the scope of whether a triangle can be formed. During these arguments, the students do not explicitly state whether the given set of segment lengths will form a triangle.

One example of this structure is Heather's argument regarding the segments of lengths 4, 7, and 10.5. Mary adjusts the sliders accordingly and drags the endpoints to form a triangle. Instead of claiming that a triangle is formed, Heather states, "10.5 is the biggest number." This argument is illustrated in Figure 18.

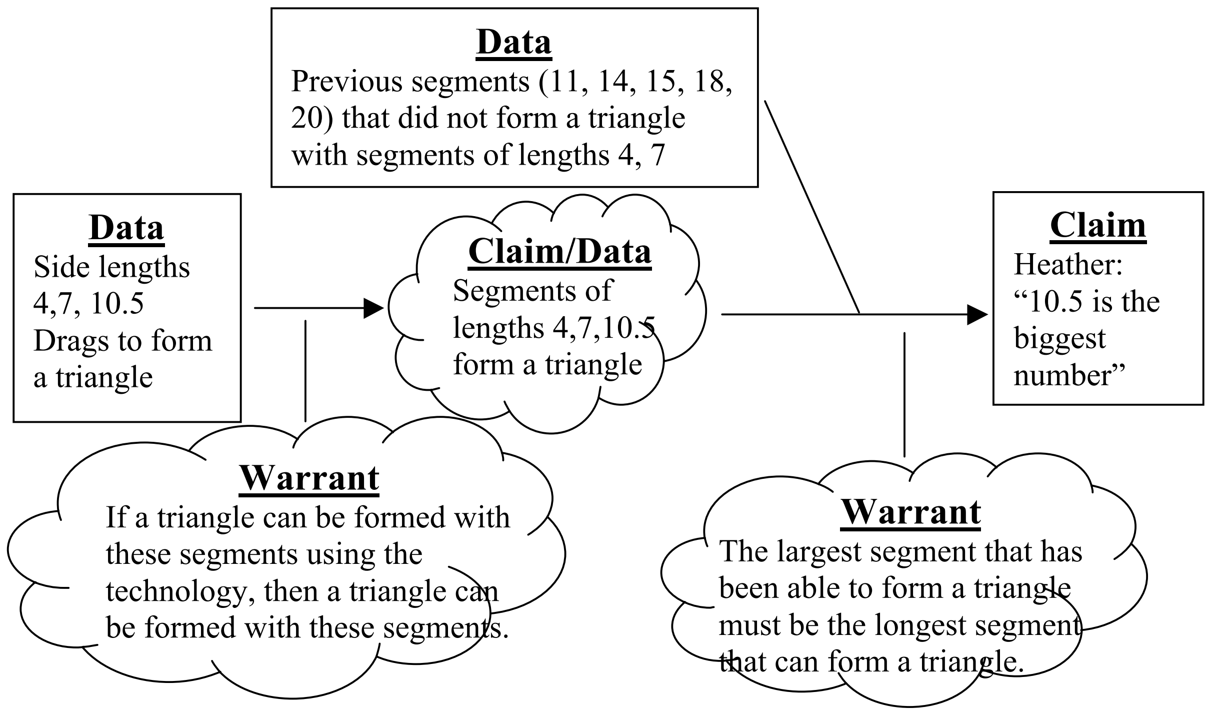


Figure 18. Heather’s argument created during task 1 with a non-explicit claim.

In this argument, the data are the segments of lengths 4, 7, and 10.5 and the use of the drag feature of the technology to form a triangle. Her explicit claim is 10.5 is the largest segment length that can be used to form a triangle with segments of length 4 and 7. In this argument, she does not explicitly claim that a triangle can be formed with segments of length 4, 7, and 10.5. However, she would be unable to make the claim that 10.5 is the largest segment length unless a triangle could be formed with the segments. Thus, the researcher infers the claim “Segments of lengths 4, 7, and 10.5 form a triangle.” The warrant that links the data to this inferred claim is not explicit and is inferred by the researcher to be “If a triangle can be formed with these segments using the technology, then a triangle can be formed with these segments.” The inferred claim becomes data for her explicit claim along with the previous examples of sets of segments that were unable to form a triangle with segments of lengths 4 and 7. Heather does not make an explicit warrant that connects this data to the explicit claim. The warrant is inferred by the researcher to be “The largest segment that has been able to form a triangle must be the longest segment that can form a triangle.”

Heather provides another example of an argument with an unspoken claim as the pair determines whether a triangle can be formed by segments of lengths 5, 5, and 6. Heather adjusts the sliders accordingly and adjusts the endpoints to form a triangle. She claims, “I think my thesis is correct.” Heather is referring to her previous argument that the tenths digits of the lengths of the segments must be the same in order for a triangle to be formed (see the argument illustrated in Figure 19).

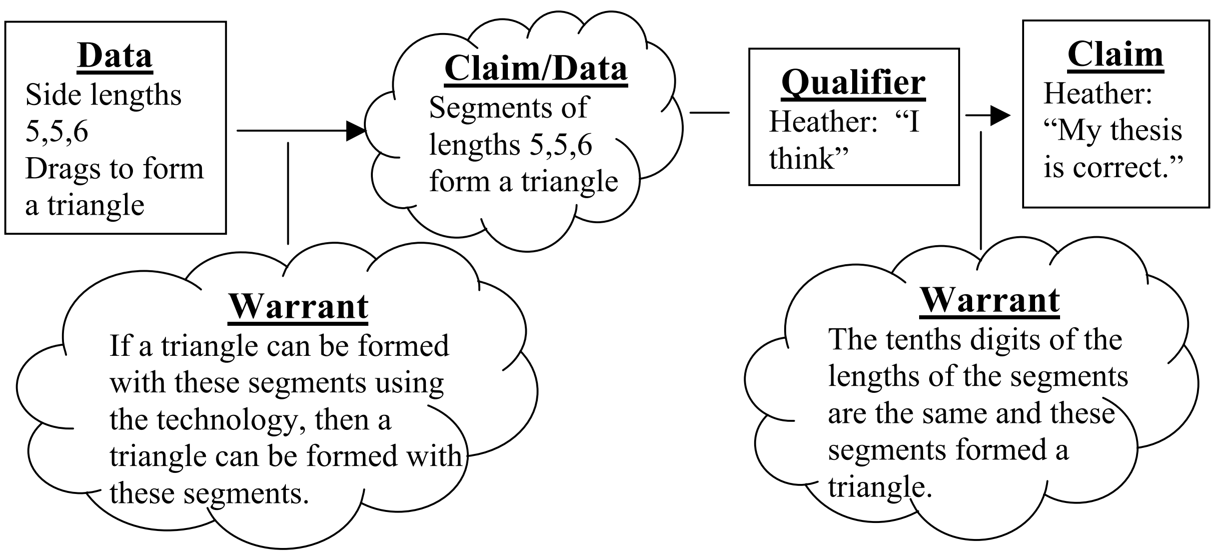


Figure 19. Heather’s argument created during task 1 with a non-explicit claim that confirms her previous generalization.

In this argument, Heather does not explicitly claim that segments of length 5, 5, and 6 will form a triangle. She only makes explicit a qualified claim regarding her conjecture. However, she is able to form a triangle using the technology. The researcher infers that in order to make the claim regarding that conjecture, she must have determined that the segments of these lengths form a triangle, which is supported by her use of technology. The non-explicit claim, "Segments of lengths 5, 5, and 6 form a triangle," is used as data for her explicit claim. The warrant linking the data to the non-explicit claim is inferred by the researcher to be, "If a triangle can be formed with these segments using the technology, then a triangle can be formed with these segments." The warrant that links the non-explicit claim to the explicit claim is not explicit and inferred by the researcher to be, "The tenths digits of the lengths of the segments are the same and these segments formed a triangle."

In both of these arguments, Heather extends the task from determining whether the segments formed a triangle to a task which used this determination to make a generalization across the examples they had previously explored. In the first example, Heather made the claim that 10.5 is the longest length that could be used to form a triangle with segments of length 4 and 7. She used the previous attempts as data to make this claim, a generalization. In the second example, Heather uses the fact that she was able to form a triangle with segments of lengths 5, 5, and 6 to support a previous conjecture, a generalization across the first four examples on the task sheet.

Though the generalizations were different in nature, the use of technology for both of these arguments was the same. Heather did not rely on the appearance of the diagrams on the screen to make her claims. Instead, the endpoints were dragged such that a triangle was formed. This use of the technology may be related to why the students felt it unnecessary to explicitly state that the triangles were formed. Because both students could see that the triangle was formed, they may not have felt compelled to explicitly make this claim. In other words, the students had a shared understanding of what constitutes a triangle being formed with the technology.

Discussion

While working on the triangle inequality tasks, Heather and Mary created arguments of various structures. Three categories of structures are noted in the analysis of these arguments: core arguments, arguments in which the students were compelled to collect additional data, and arguments with an inexplicit claim. Looking across these argument structures, three themes emerge; the lack of explicit warrants, the level of certainty in their claims, and the ways in which the students use the technology (see Table 6).

Table 6

Group 1's Arguments on the Triangle Inequality Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	10	11
	Technology Not Used	1	1
Warrants Explicit			
	Technology Used	1	2
	Technology Not Used	3	2

Of the 29 arguments created by Heather and Mary, the pair provide explicit warrants for only 7 arguments. The explicitness of the warrants may be related to the task and the students' uses of the technology. Regardless of the structure of each argument, when the students use technology they do not often provide an explicit warrant. When the students provided an explicit warrant, it was often done while the students were making a generalization and the use of technology was indirect. The students may not have felt the need to provide an explicit warrant due to the shared understanding of what they were seeing on the screen.

Heather and Mary seemed to have a high level of certainty in their claims due to the lack of qualifiers. While working on this task, the students only provide three qualifiers for their claims. The lack of qualifiers and the students' frequent uses of technology in the data seem to suggest that the students view the technology as a reliable tool. This viewpoint could also be due to the teacher's request for the students to use the tool while working on the task. In other words, the students' see the tool as reliable because the teacher would not provide the students with a tool that is faulty.

In this task, the students' uses of the technology were restricted to adjusting the sliders and dragging the endpoints of the diagram. However, the students did not always drag the endpoints of the diagram prior to making a claim. Heather and Mary often made a claim based on the appearance of the diagram on the screen after the sliders had been adjusted. In some these cases the students would collect additional data to verify or refute the claim (see the arguments illustrated in Figures 15 and 16), while in other cases the students would not (see the arguments illustrated in Figure 10). On the first day of class, the students frequently used the tool to determine whether segments of various lengths could form a triangle. By the second day, the students had become very familiar with the tool. The familiarity of the tool may have increased the frequency of claims based on the appearance of the diagram on the screen. In fact, the number of claims based on the appearance of the diagram on the screen increased from 3 on the first day to 10 on the second day. This seems to suggest that by the second day, after the students had adjusted the sliders, the students could imagine whether a triangle could be formed with the segments based on the appearance of the figure. Also, they were fairly certain about these claims due to the lack of qualification and additional data collection.

The change in the way the tool was used by these students to collect data from day one to day two suggests a change in the way they view the tool as well. At first the students mainly used the technology to gather data by dragging the endpoints of the segments to determine if a triangle could be formed. On the second day, the students were less likely to use this feature. Perhaps by the second day, these actions, the dragging of the endpoints, had become internalized and the technology becomes a tool to think with, rather than a tool to reason from.

Group 2's arguments on the triangle inequality task.

The analysis of the arguments created by David and Erica while working on the triangle inequality task can be categorized into three basic argument structures; core arguments, arguments in which the students collect additional data to verify or refute a claim, and an argument in which a claim is based, in part, on an imagined use of technology. These arguments structures are further discussed below.

Core arguments.

Though all core arguments contain the same elements, a single claim, data, and a warrant, the content of these arguments may differ. The content of David and Erica's core argument seems to depend upon whether the pair employ technology.

When David and Erica use technology in the creation of their core arguments, the students do not make their warrants explicit. The students created seven core arguments with this content. For example, David and Erica are determining whether a triangle can be formed with segments of lengths 6, 1, and 4. Erica adjusts the sliders accordingly and proceeds to drag one of the endpoints such that it makes an over-hang with the other segment (see Figure 20). Upon seeing this diagram, David claims, "It doesn't make one." This argument is illustrated in Figure 21.

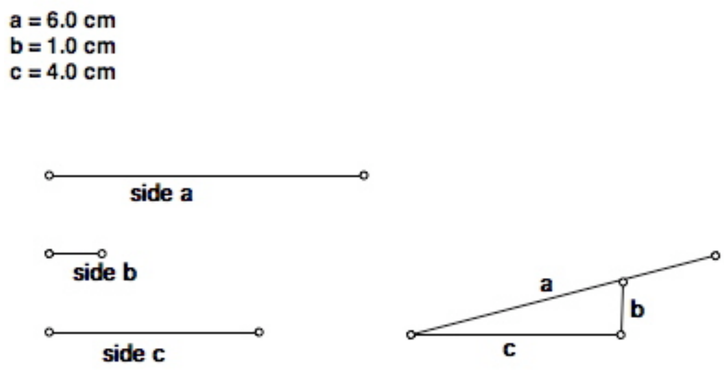


Figure 20. An example of the diagram used by David and Erica's to show the sets of segments could not form a triangle in which one of the segments extends past another segment.

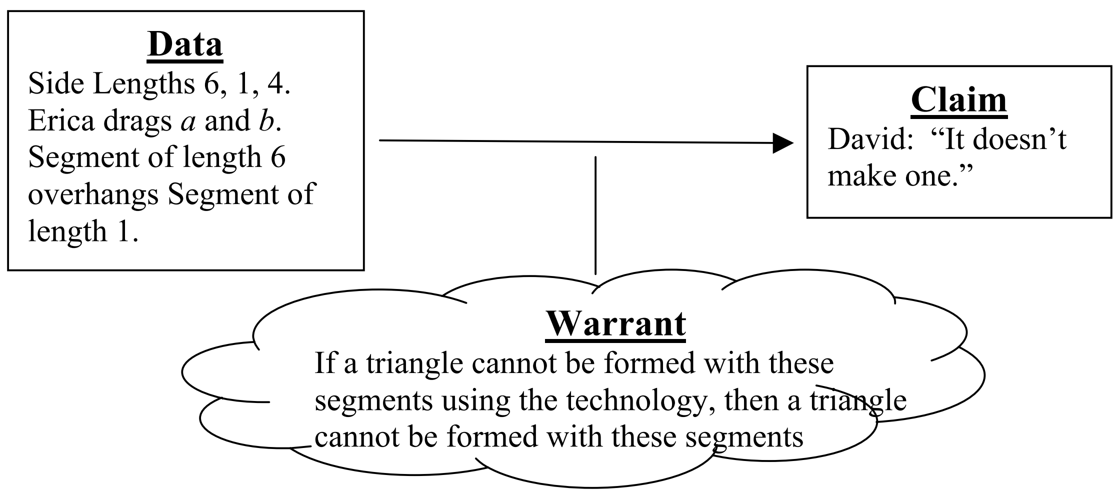


Figure 21. David's core argument created during task 1 with non-explicit warrant and use of technology.

The data for this argument are the segment lengths and the dragging of the figure. David claims, "it doesn't make one" or, in other words, these segments do not form a triangle. Neither Erica nor David provide an explicit warrant and the researcher infers it to be, "If a triangle cannot be formed with these segments using the technology, then a triangle cannot be formed with these segments."

This argument structure also occurs for those arguments in which the segments form a triangle. For example, David and Erica are given the segments of lengths 5, 5, 6 and are asked if these segments form a triangle. Erica adjusts the sliders accordingly and drags the endpoints of the diagram to form a triangle. David jokingly states, “You got it girlfriend” or, in other words, you were able to form a triangle with these segments lengths. The warrant that links the data to the claim is not explicit and inferred by the research to be the same as that in preceding example. This argument is illustrated in Figure 22.

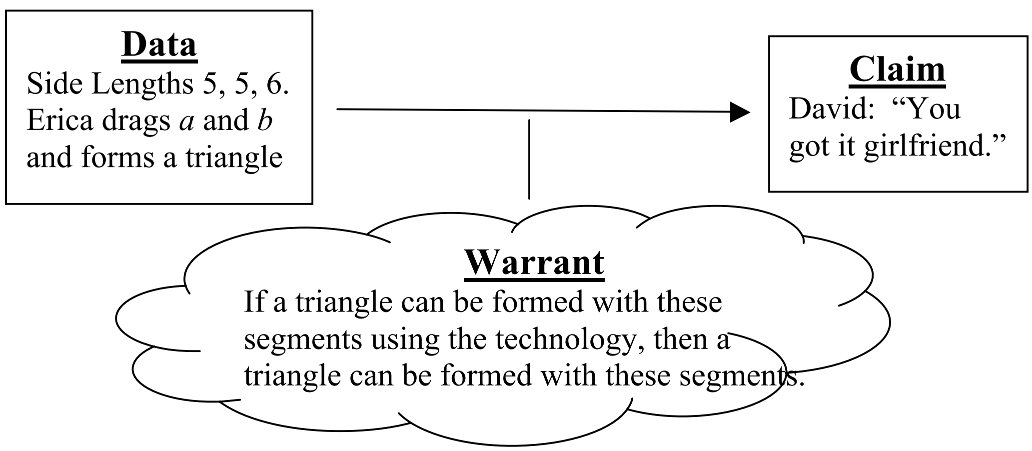


Figure 22. David’s core argument created during task 1 based on Erica’s use of technology with a non-explicit warrant.

Of the seven core arguments that employ technology, all of the warrants are not explicit. In addition, David and Erica always dragged the endpoints to determine whether a triangle could be formed. Thus, the inferred warrant is the same for all of these arguments. These students do not rely on the appearance of the diagram on the screen after the sliders are adjusted to the desired lengths. Instead, the pair drag the endpoint until they are convinced whether a triangle is or is not formed

When technology is not actively employed in the creation of a core argument, the warrants are usually explicit. The students create three arguments with this structure and content. For example, David and Erica attempt to answer the question, “Why was it impossible to construct a triangle with some of the given lengths?” Erica states, “One’s [segment] too long or too short.” This argument is illustrated in Figure 23.

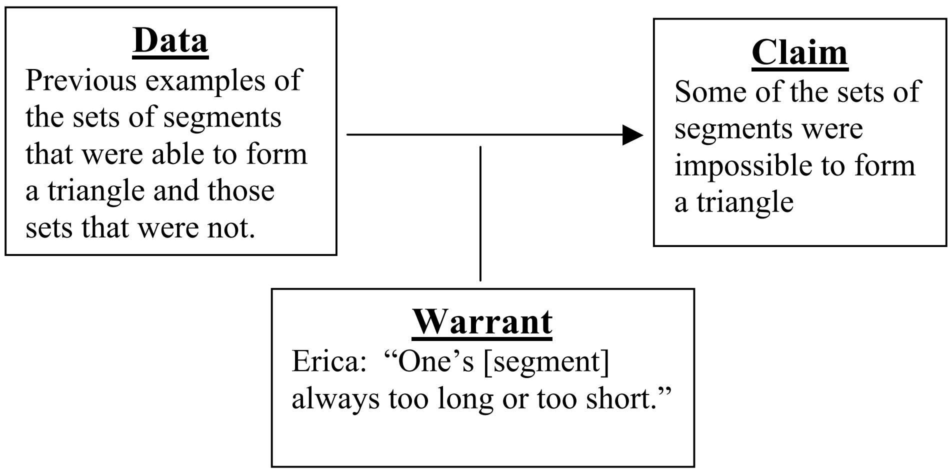


Figure 23. Erica’s core argument created during task 1 with an explicit warrant in response to a claim made on the task sheet.

The data for this argument are the previous examples that were both able and unable to form a triangle. The students do not provide the claim. Instead, the question provides the claim that some of the examples were impossible. Erica provides the warrant for this argument in her statement that one of the segments was too long or too short. For this task it is not surprising that the warrant is explicit. The question provides the claim and asks the students for the warrant. The data for this argument, which triangle could not be formed with the given segments, were the claims of prior arguments in which the students actively used technology. Hence, the data for this argument is not actively created using technology for this claim. Rather, the data for this argument is a product of the students' previous uses of technology.

Another core argument in which the students do not actively use technology and make an explicit warrant is in response to the task that asks the students to develop a conjecture about the relationship among the lengths of the sides of a triangle. David states, "Like they have to be, they have to be around each other; like the lengths got be around each other." He supports his claim by looking at specific examples and stating, "see look at this like 3.5, 2.0, 6; 3.4, 3.0, 4.0." This argument is illustrated in Figure 24.

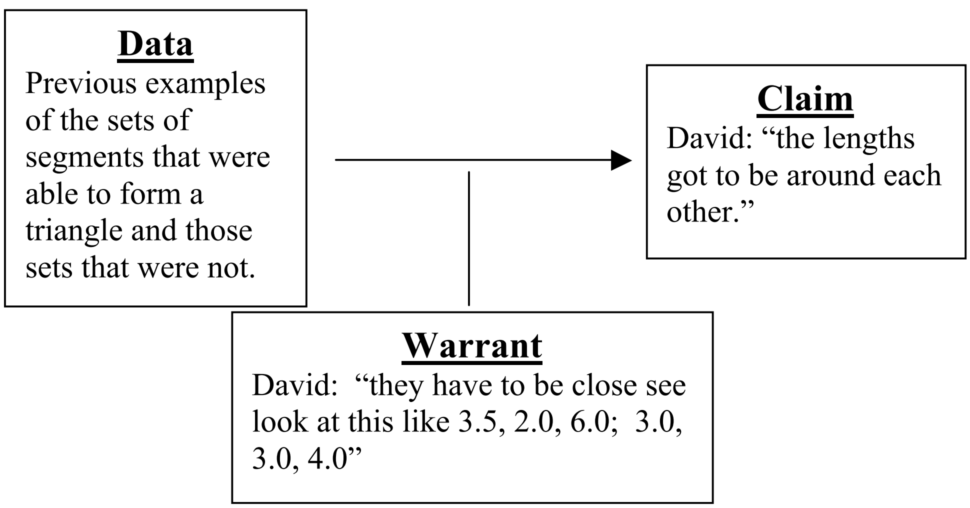


Figure 24. David's core argument created during task 1 with an explicit warrant and indirect used of technology.

In this argument, the data are the previous examples, those sets of segments that were able to form a triangle and those that were not. David claims, "the lengths got to be around each other." He warrants this claim by pointing to specific examples; specifically, a set of segments that formed a triangle, 3, 3, and 4, and a set that did not, 3.5, 2, and 6. In other words, he is warranting his claim by demonstrating that those segments whose lengths were relatively close were able to form a triangle while those that were not close to each other were unable to form a triangle. The students' use of technology is similar to the previous argument. The students do not actively use the technology to collect data. Rather, they students use the products from their previous uses.

Within the core arguments, the explicitness of students' warrants seems to be related to the students' uses of technology. When students actively use technology in the creation of their argument, the students do not explicitly state a warrant. When the students do not actively employ technology, the students provide explicit warrants. It is worth noting that in the three instances of core arguments that do not employ technology, the students were responding to questions and statements listed on the task sheet. This seems to suggest that these students will not provide explicit warrants for their claims unless prompted to do so.

Arguments in which David and Erica collect additional data.

The second argument structure created by David and Erica consists of those arguments in which a claim is made and additional data was collected in order to verify or refute a claim. Arguments of this structure can be categorized into two types; a claim is made based on the appearance of the diagram on the screen and the drag feature of the technology is employed to collect additional data, and a challenge of a claim prompts additional data collection.

At times David or Erica make a claim based on the appearance of the diagram on the screen, but the uncertainty of that claim prompts additional data collection. For example, David and Erica are determining whether a triangle can be formed from segments of lengths 2, 7, and 4. Erica adjusts the sliders accordingly and states, "That looks impossible." David requests the endpoints be dragged such that the resulting diagram is one in which the longest segment overhangs the other segments, similar to the appearance of the diagram in Figure 20. This argument is illustrated in Figure 25.

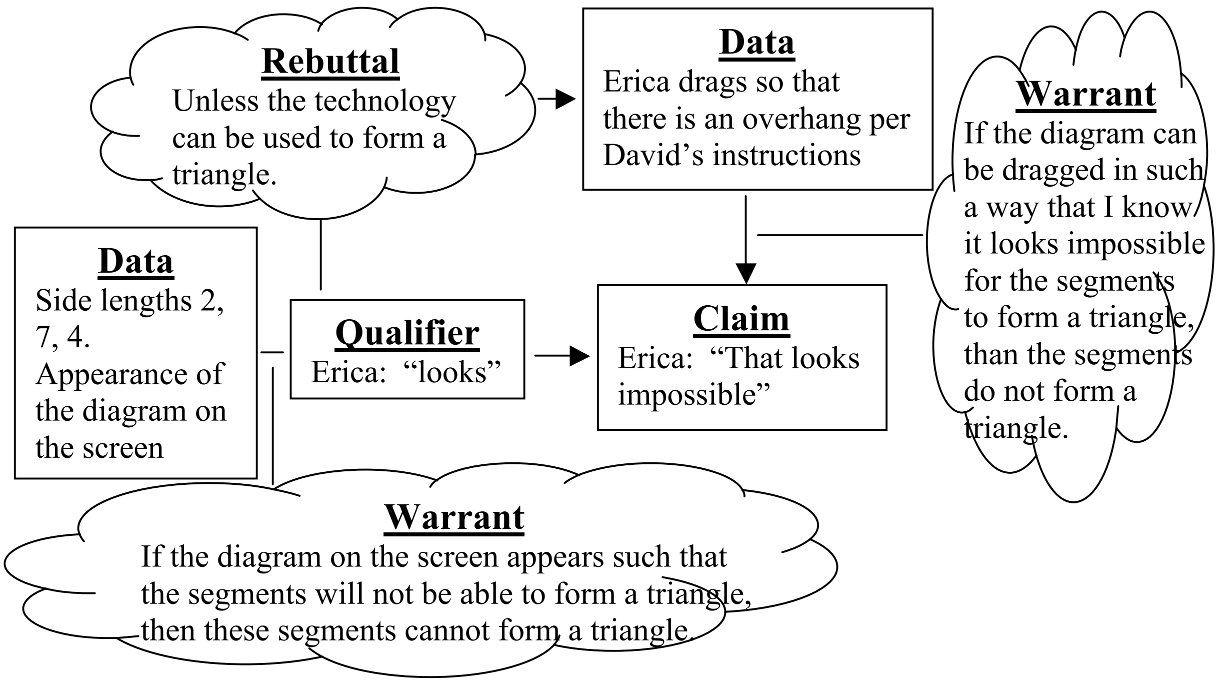


Figure 25. David and Erica's argument created during task 1 with additional data collection in which David supports Erica's claim.

In this argument, Erica makes the claim that the segments of lengths 2, 7, and 4 will not form a triangle. Her data for this claim are the segment lengths and the appearance of the corresponding diagram on the screen. She qualifies this claim with the term "looks" which indicates that she has some uncertainty whether these segments can form a triangle. Erica does not provide an explicit warrant for this claim and the researcher infers it to be, "If the diagram on the screen appears such that the segments will not be able to form a triangle, then these segments can not form a triangle." David asks Erica to drag one of the endpoints such that the longest segment extends past another segment. By making this request, David is asking for additional data and Erica obliges. Neither Erica nor David explicitly verify Erica's original claim, but they do not refute it either. It is worth noting that David asks Erica to drag one of the endpoints such that the resulting diagram's appearance is similar to that of a previous example that was unable to form a triangle. This suggests that David concurs with Erica's initial claim but wants additional data to verify that claim. Thus, when David asks Erica to drag in this manner, he asks her to drag the diagram such that the appearance of the resulting diagram is such that it verifies her initial claim. In addition, the students do not provide an explicit warrant that links this additional data to the initial claim and the researcher infers it to be, "If the diagram can be dragged in such a way that I know it looks impossible for the segments to form a triangle, than the segments do not form a triangle."

In the argument above, Erica qualifies her claim and David asks Erica to drag the diagram such that the resulting diagram verifies this claim. The qualifier provided by Erica and the request for additional data by David suggests that both students held some level of uncertainty regarding the claim. This uncertainty may due to the knowledge that the appearance of a diagram may be misleading and conclusions drawn from the appearance of a figure must be verified.

David's request for additional data in the previous argument is in support of Erica's initial claim. However, that was not always the case. At times, these students would collect additional data despite the claim made by others. For example, David and Erica seek to determine whether segments of lengths 10, 7, and 10.5 will form a triangle. Erica adjusts the sliders accordingly and drags the endpoints. Watching the endpoints being dragged and the segments not forming a triangle, David claims, "it doesn't work", or, in other words, these segments will not form a triangle. Erica states "hold on" and continues to drag the endpoints and is able to form a triangle. David changes his claim by stating, "Yes it does work. 10.5 works." This argument is illustrated in Figure 26.

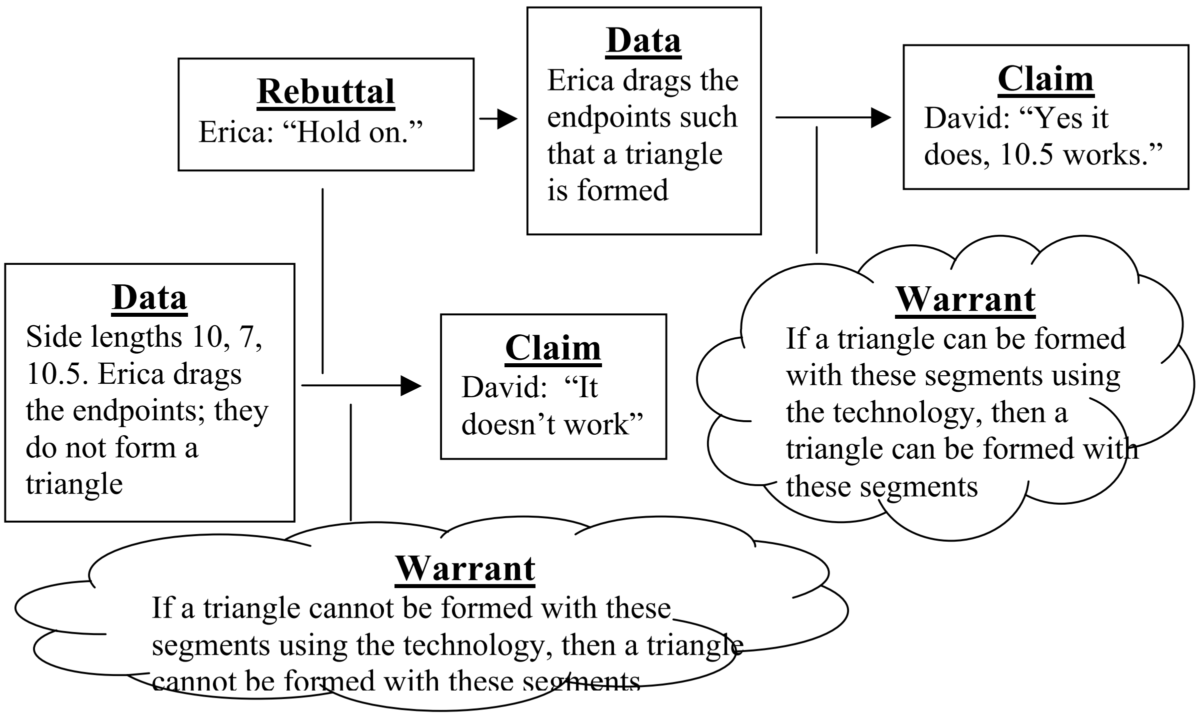


Figure 26. David and Erica's argument created during task 1 with additional data collection with a challenge by Erica.

In this argument, David's initial claim is based on the data, the segments of lengths 10, 7, and 10.5 and the appearance of the diagram as Erica drags the endpoints. David does not provide an explicit warrant for this claim and is inferred by the research to be, "If a triangle can not be formed with these segments using the technology, then a triangle can not be formed with these segments." Erica challenges this claim by saying "hold on" and continues to drag the endpoints. Through this statement and action, Erica indicates she is uncertain about David's claim and this uncertainty lies in the notion that not enough data has been collected. She collects additional data by dragging the endpoints such that a triangle is formed. David makes a new claim that these segments can form a triangle. Neither of the students provide an explicit warrant that links the additional data to the new claim and the researcher infers it to be, "If a triangle can be formed with these segments using the technology, then a triangle can be formed with these segments."

This argument differs from the previous example in many ways. First, the students' use of technology that comprises the initial data is not the same. The initial claim in the first argument is based, in part, on the appearance of the diagram on the screen and the basis for the second argument's initial claim is the dragging of the endpoints and appearance of the corresponding diagram on the screen. Second, the nature of the students' rebuttals, which leads to additional data collection, differs as well. In the first example, David does not challenge Erica's initial claim but requests additional data to support that claim. In the second argument, Erica challenges David's initial claim by indicating that not enough data been gathered to make a valid claim. These rebuttals influence the ways the data was collected. In the first example, David asks Erica to drag the endpoints in a specific way such that the resulting diagram directly supports the initial claim. In the second example, Erica continues to drag the endpoints that would not support David's initial claim. Thus, the warrants that link this new data to the claims are not the same even though the drag feature is the main use of technology. Finally, the structure of the second argument differs from the first. In the first example, the initial claim is verified. In the second example, the initial claim is refuted and a new claim is proposed. As previously indicated, this may be related to the nature of the rebuttals.

Imagined dragging.

In the previous argument structure, Erica and David collect additional data in direct support of a claim and in hope to refute a claim. In these arguments, David and Erica make claims based on their explicit use of technology. However, this is not always the case. In one argument, David makes a claim not based on the explicit use of technology, but on how the appearance of the diagram would change if the endpoints were dragged, or imagined dragging. In this argument, David and Erica are determining whether segments of lengths 15, 7, and 10 will form a triangle. Erica adjusts the sliders accordingly and the resulting diagram is such that the segment of length 15 extends past the segment of length 7. Upon seeing this diagram, David claims, "That doesn't seem to work." David takes a longer look at the figure on the screen and then exclaims, "Yes it will. Yes it will. Yes it will." Erica, unsure that these segments will form a triangle, states, "No it won't." David instructs Erica to drag the endpoint of the segment of length 7 down and then drag the segment of length 15 over. When that did not form a triangle, he claims, "No, it don't work. 15 doesn't work." David and Erica begin to work on a different task, but another student, Heather, claims segments of lengths 7, 10 and 15 form a triangle. David responds, "No it doesn't." Upon hearing David's statement, the teacher comes over to see why David is making this claim. Erica changes the sliders back to 7, 10, and 15 while David tells the teacher, "It's not even close." Erica begins to drag the endpoints attempting to make a triangle. The teacher encourages Erica to continue dragging. Erica states, "It's getting closer," and, eventually, forms a triangle. The teacher asks the students, “So, what did you find out?” David and Erica respond, “That’s a triangle.” David continues, “15 works.” This argument is illustrated in Figure 27.

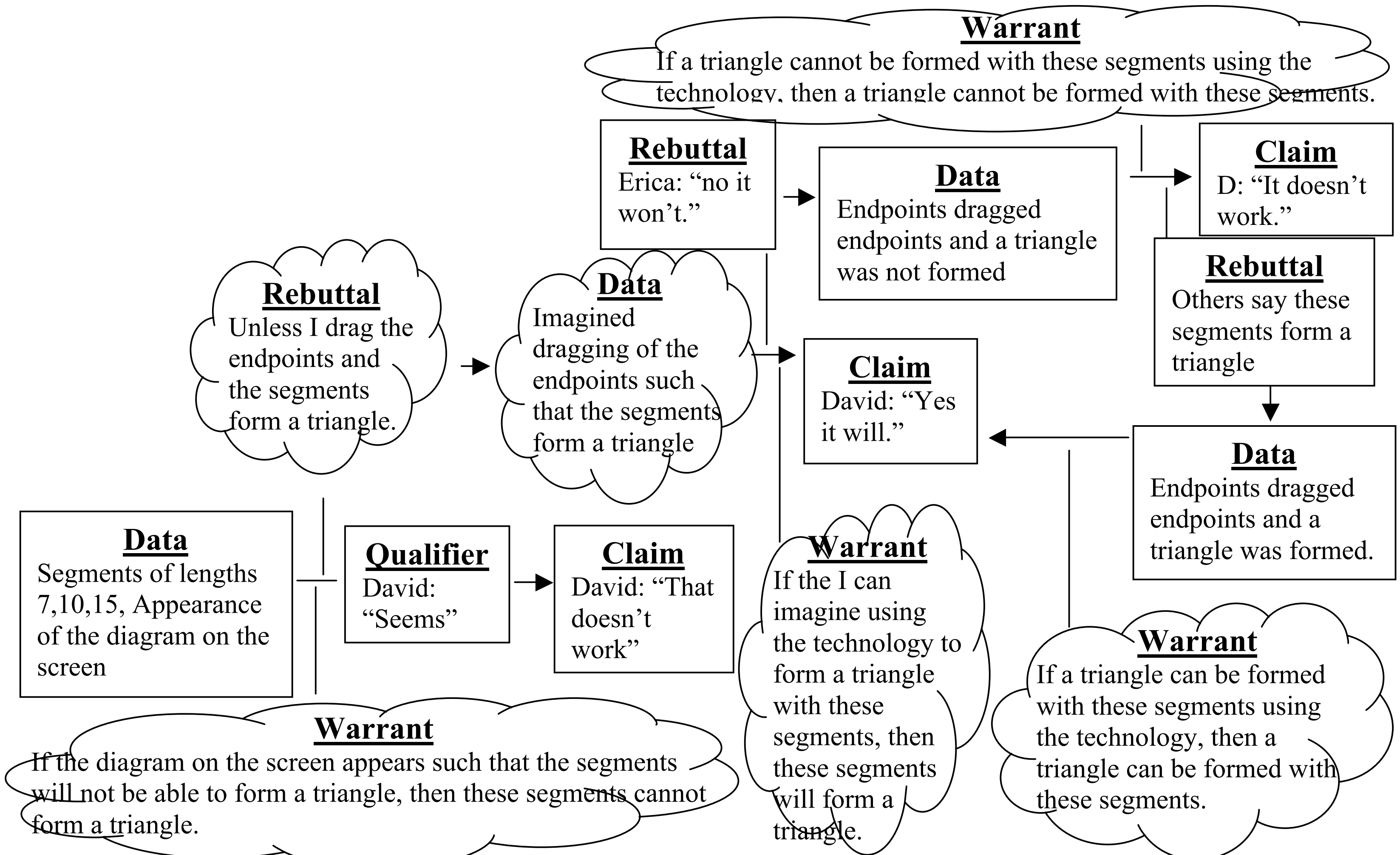


Figure 27. David and Erica's argument created during task 1 in which David imagines the endpoints being dragged.

In this argument, David makes the initial claim; a triangle can not be formed with segments of lengths 15, 7, and 10. The data for this claim are the segments and the appearance of the corresponding diagram on the screen. The appearance of the diagram is such that the segment of length 15 extends past the segment of length 7 similar to the diagram in Figure 20. In previous arguments (see the arguments illustrated in Figures 21 and 23) David and Erica drag the endpoints of the diagram such that the diagram had an overhang in order to demonstrate that a triangle cannot be formed. David does not explicitly provide a warrant that links the data to this claim and is inferred by the researcher to be "If the diagram on the screen appears such that the segments will not be able to form a triangle, then these segments cannot form a triangle." David indicates he is not entirely certain about his claim by the use of the qualifier "seems". This uncertainty leads to a rebuttal that is not explicitly provided by David and inferred by the researcher to be, "Unless I drag the endpoints and the segments form a triangle." The inferred rebuttal leads to the collection of more data, although this data is not explicitly provided. Neither David nor Erica make adjustments to the screen nor does anyone in the class make a statement that contradicts David's claim. Yet, David makes a contradictory claim and proceeds to tell Erica how to drag the endpoints of the diagram. These actions and statements suggest that David has imagined the endpoints being dragged such that the endpoints of the segments meet and form a triangle. Thus, the researcher infers the data for this new claim to be, "Imagined dragging of the endpoints such that the segments form a triangle." An alternative to the imagined dragging data could be the previous examples with segments of lengths relatively close to the 15, 7, and 10 that were able to form a triangle. The warrant that links this data to this claim is not explicit and inferred by the researcher to be "if I can imagine using the technology to form a triangle with these segments, then these segments will form a triangle." Erica challenges this claim in the form of a rebuttal and David asks her to collect additional data by dragging the figure. When they are unsuccessful in forming a triangle, David claims, "It doesn't work." Again, his warrant is not explicit and is inferred by the researcher to be, "If a triangle can not be formed with these segments using the technology, then a triangle can not be formed with these segments." Both Erica and David appear to be comfortable with this claim and begin to work another task. They only revisit this task when they overhear another student making a contradictory claim. The teacher intervenes which prompts the students to adjust their sliders back to lengths 15, 7, and 10. These statements and actions provide the inferred rebuttal "Unless these segments do form a triangle because others say they do." Erica's subsequent dragging of the endpoints to form a triangle, data, leads to a new claim. The warrant that links this data to the claim was not explicit and is inferred by the researcher to be, "If a triangle can be formed with these segments using the technology, then a triangle can be formed with these segments."

This argument contains elements of the structure of the previous type of arguments, those in which additional data is collected. In this argument, additional data is collected three times, once in form of imagined dragging and twice in the form of actual dragging. The additional data collected in the form of dragging occurs in response to challenges. The first of these challenges comes from Erica who challenged David's claim that these segments will form a triangle. The second challenge came from another student and the teacher. The data collected from imagining the diagram being dragged is not in response to an outsider's challenge. Instead, David is challenging his own initial claim. His use of a qualifier demonstrates his uncertainty and he attempts to resolve this uncertainty by imagining the diagram being dragged.

Discussion.

The structure of the arguments created by David and Erica while working on the triangle inequality task can be categorized into three types: core arguments, arguments in which additional data is collected, and arguments in which the data is inferred. Looking across these argument structures, two themes emerge, the lack of explicit warrants when using technology (see Table 7) and the ways in which the students used the technology.

Table 7

Group 2's Argument on the Triangle Inequality Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	7	6
	Technology Not Used	0	1
Warrants Explicit			
	Technology Used	0	2
	Technology Not Used	3	2

Of the 15 arguments in which the students use the technology, the students do not make their warrants explicit in 13 (87%) of the arguments. However, in the arguments the students create in which they do not employ technology, they are more likely to make their warrants explicit. This seems to suggest that the students do not feel the need to justify their claims when using technology. This could be due to the nature of the technology and the affordances it provides as a shared venue for exploration and investigation. Both students could see the diagram on the screen and how the diagram changes when the endpoints are dragged. Thus, the students may not have felt the need to justify their claims because they share in the experience of using the technology and understand the claim that is made from these uses of technology. In addition, the arguments in which the students provide explicit warrants are in response to questions and statements on the task sheet. Thus, these students may only provide explicit warrants when prompted to do so.

Two of the students' uses of technology in the creation of their arguments are worth noting. First, the students adopt the notion that when one segment extends past another segment it indicates that a triangle cannot be formed. Using the technology, the students could easily identify when a triangle is formed; the endpoints of the segments are coincident. However, David and Erica developed an idea of how to use the tool to show that the segments do not form a triangle. David and Erica develop this idea on the first day while working on the second example on the task sheet (see the argument illustrated in Figure 21). In one argument, David used this notion to support the claim that a triangle cannot be formed with segments of lengths 2, 4, and 7 by directing Erica to drag the endpoints of the diagram such that it results in an overhang (see the argument illustrated in Figure 23). In another argument, after the sliders are adjusted and the corresponding diagram on the screen appears such that there is this overhang, David's initially claims a triangle cannot be formed (see the argument illustrated in Figure 25). The students' uses of the technology in this fashion demonstrate that students may invent ways of thinking about a concept that is afforded by the tool.

The second use of technology worth noting is David's possible imagined dragging of the endpoints of the diagram to form a triangle (see the argument illustrated in Figure 27). This possible imagined dragging occurs on the second day after the students had frequently used the technology to explore the conditions needed for segments to form a triangle. These prior uses of the technology and David's possible imagined dragging suggests that the technology, at least for David, has moved from a tool for verification or refutation to one for exploration. By the second day, perhaps the use of the tool had become internalized, and the technology became a tool to think with, rather than a tool to reason from.

Group 3's arguments on the triangle inequality task.

The analysis of the arguments created by Amy and Judy while working on the triangle inequality task can be categorized into two basic argument structures: core arguments and arguments in which the students collect additional data to verify or refute a claim. These argument structures are further discussed below.

Core arguments.

Core arguments are arguments that consist of data, a claim, and a warrant that links the data to the claim. The students create twelve arguments with this structure. Even though all of Amy and Judy's core arguments have each of these elements, there are distinctions in the content of these arguments that seems to depend on the nature of the task and the use of technology.

Technology Use.

In the core arguments in which Amy and Judy employ technology, the students do not make their warrants explicit. For example, Amy and Judy are trying to determine whether segments of lengths 4, 12, and 10 will form a triangle. Judy adjusts the sliders accordingly and drags the endpoints of the diagram to form a triangle. Amy claims, "Yeah." This was interpreted to mean these segments will form a triangle. This argument is illustrated in Figure 28.

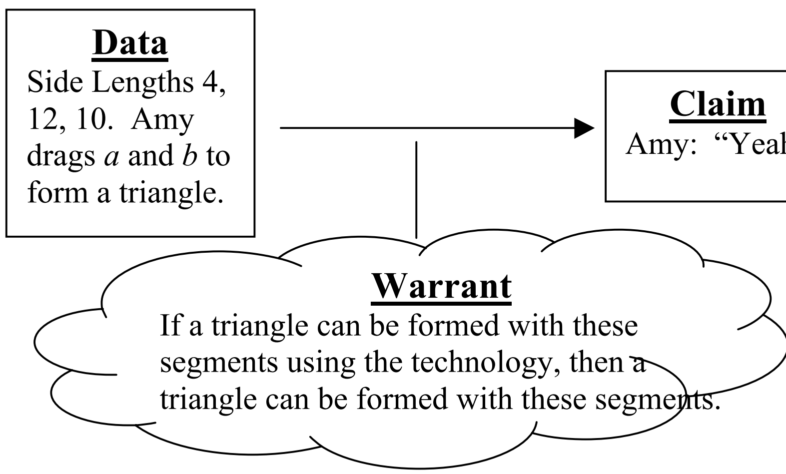


Figure 28. Amy's core argument created during task 1 with an inferred warrant and use of technology.

The data for this argument are the segments of lengths 4, 12, and 10 and the dragging of the endpoints of the diagram to form a triangle. Amy's claim is a confirmation that these segments will form a triangle. Neither Judy nor Amy provide an explicit warrant and is inferred by the researcher to be, "If a triangle can be formed with these segments using the technology, then a triangle can be formed with these segments."

Another example of an argument with this structure and content is when Judy and Amy attempt to determine whether a triangle can be formed with segments of lengths 3, 4, and 4. Amy adjusts the sliders accordingly and drags the endpoints of the diagram to form a triangle. Judy claims that a triangle can be formed by stating, “Possible” and Amy agrees by stating, “Yep.” This argument is illustrated in Figure 29

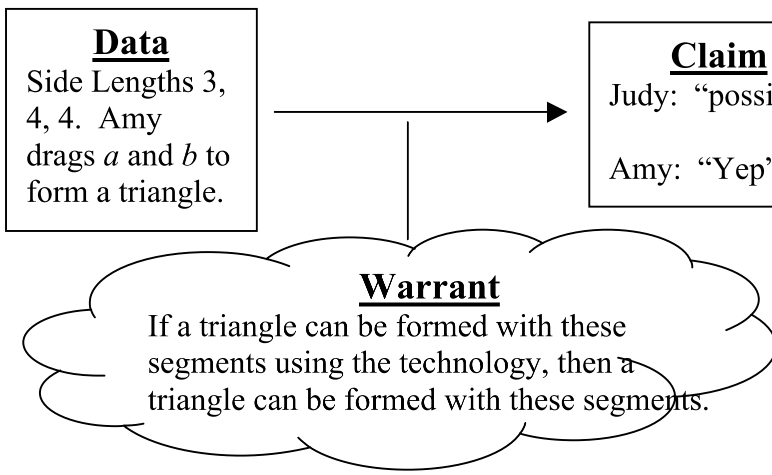


Figure 29. Amy and Judy’s core argument created during task 1 with an inferred warrant and use of technology.

In this argument, the data are the segments of lengths 3, 4, and 4 and the formation of the triangle by dragging the endpoints of the diagram. Both Amy and Judy claim a triangle can be formed with these segments. Similar to the previous argument, the pair does not provide an explicit warrant, which is inferred by the researcher to be, “If a triangle can be formed with these segments using the technology, then a triangle can be formed with these segments.”

Ten of the eleven core arguments in which the students employed technology contain similar content, including the use of technology. In these core arguments, the students provide an explicit claim, explicit data, and an inferred warrant. The only major differences between the content of the arguments are the lengths of the segments and the ways in which the students state that a triangle has been formed. The students use the drag feature of the software to adjust the sliders and drag the endpoints to form a triangle. In nine of these eleven arguments, the claims made by the students were not always the same terminology, but all had a similar message that these segments form a triangle. The students do not provide an explicit warrant and, due to the claim(s) made by the students and their use of technology, the researcher infers the warrant to be the same for all of these arguments. The lack of a qualifier in all of these arguments suggests that the students were certain about their claims. The two arguments that differ from these nine arguments have the same structure, but differed in its content. For one argument, the students' uses of technology are the same as those previously discussed but the claim differs in that the segments are unable to form a triangle. For the other argument, the students use the appearance of the diagram on the screen as data and claim that the segments can form a triangle.

Technology not used.

When the students use technology in the creation of their core arguments, the arguments had similar content and structure. However, when the students do not actively use technology in the creation of their core arguments, the structure and content differs from those in which technology is employed. In the one core argument in which the students do not actively employ technology, the students provide an explicit warrant. In this argument, the students are responding to the question on the task sheet that asks, "Why was it impossible to construct a triangle with some of the given lengths?" Amy responds, "Because some of the lengths are bigger and some double the ones." This argument is illustrated in Figure 30.

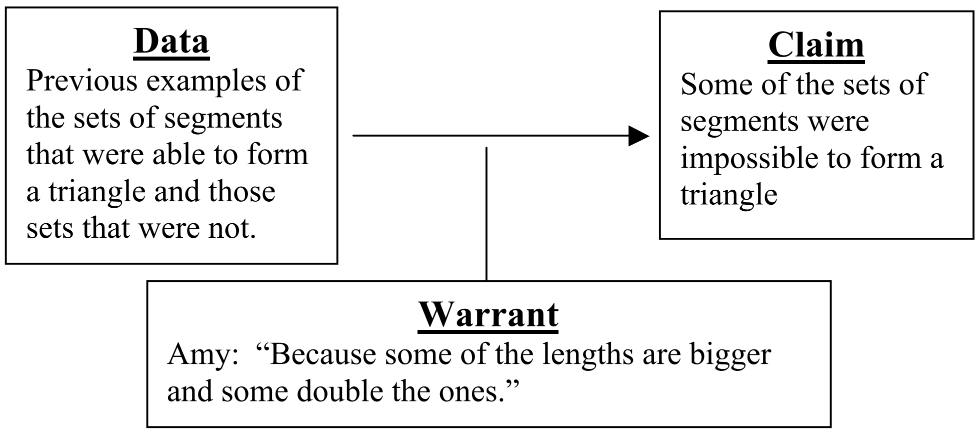


Figure 30. Amy's core argument created during task 1 with an explicit warrant in response to a claim made on the task sheet.

In this argument, the data are the previous sets of segments that have and have not formed a triangle. Even though the data was created using the technology, the students do not actively employ the technology in the creation of this argument. The claim is explicit, but is not provided by Amy or Judy. The claim that it was impossible to form a triangle with some of the sets of segments, is actually provided by the question. The question is asking the student to provide the warrant for why some of the side lengths were impossible. In other words, the question is asking the student to make an explicit warrant. Amy provides this explicit warrant that some of the lengths of the segments were bigger than the others and some even double the others in the set.

Conclusion.

Three themes are present in the core arguments created by Amy and Judy. First, the lack of qualifiers and rebuttals suggests that the students are certain about their claims. Second, the core argument in which the students actively use technology have similar structure and content. The students' uses of technology are mainly the adjustment of the sliders and the dragging of the endpoints to form a triangle. It is worth noting that for all but two of these arguments, a triangle is formed. Finally, when the students do not actively employ technology in the creation of an argument, the students provide an explicit warrant. The explicitness of the warrant may be more related to the task on which the students are working rather than solely attributing it to the students' use of technology. For those core arguments in which the students employ technology, the students are determining whether a triangle can be formed with the set of segments. In the arguments in which technology is not actively employed, the task provides the claim to the students and asks the students to provide the warrant for this claim.

Arguments in which Amy and Judy collect additional data.

The other argument structure created by Amy and Judy is one in which the students collect additional data to verify or refute that claim. Within this structure, two types of arguments emerge based on the students' uses of technology; the active use of technology to collect both the initial data and the additional data, and an indirect use of technology for the initial data and the active use of technology to collect the additional data.

Active use of technology to collect the initial data.

The first type of argument in which the students collect additional data is such that the students actively use technology to collect the initial data and the additional data. However, the use of technology is not the same for all arguments of this structure. At times, the use of the technology consists of adjusting the sliders to the desired lengths and the appearance of the corresponding diagram on the screen. For example, the students are determining whether segments of lengths 2, 7, and 4 will form a triangle. Amy drags the sliders accordingly and Judy claims, “That looks totally impossible, that is huge.” Amy immediately responds, “You don’t know that though; that’s the only thing.” Judy agrees with the sentiment and states, “I know, the weirdest looking stuff may be possible.” Amy drags the endpoints and is unable to form a triangle. She states, “It’s impossible.” This argument is illustrated in Figure 31.

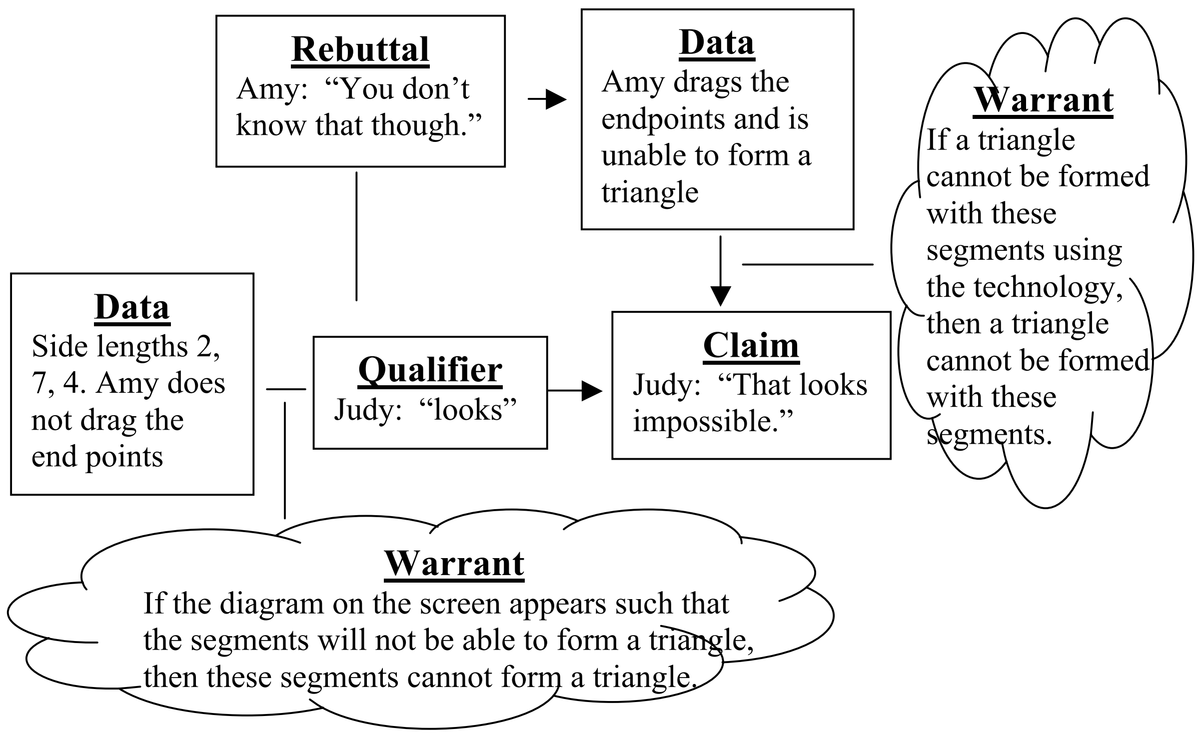


Figure 31. Judy and Amy's argument created during task 1 with additional data collection in which the initial data is based on the appearance of the diagram on the screen.

In this argument, Judy provides the initial claim that the segments will not form a triangle. Her data are the segments of lengths 2, 7, and 4 and the appearance of the diagram after the sliders are adjusted to the desired lengths. The term "looks" is a qualifier for this claim. The warrant for this claim is not explicit and the researcher infers it to be, "If the diagram on the screen appears such that the segments will not be able to form a triangle, then these segments cannot form a triangle." Amy provides the rebuttal for this argument when she indicates that the appearance of a diagram can be deceiving. Judy concurs with this statement. Amy collects additional data by dragging the endpoints and is unable to form a triangle. She verifies Judy's initial claim by stating, "It's impossible." The pair does not provide an explicit warrant, which is inferred by the researcher to be, “If a triangle can not be formed with these segments using the technology, then a triangle cannot be formed with these segments.”

In this argument, the initial claim is based, in part, on the appearance of the diagram on the screen. The verification of this claim is based on the dragging of the endpoints. It is worth noting that both Amy and Judy knew that the appearance of a mathematical diagram cannot be trusted and additional data is necessary to verify this claim. Both students demonstrate their uncertainty with the initial claim by Judy’s use of a qualifier and by both students’ contributions to the rebuttal.

Part of the initial data in the previous argument is the appearance of the diagram on the screen after the sliders have been adjusted. In other arguments of this structure, Amy and Judy make an initial claim based, in part, on the dragging of the endpoints of the diagram. For example, Amy and Judy are determining whether the segments of lengths 3.5, 2, and 6 will form a triangle. Amy adjusts the sliders accordingly and drags the endpoints trying to form a triangle. She states, “They might be [a triangle].” Amy continues to drag the endpoints and then claims, “Impossible.” However, she states, “They get closer, though” and continues to drag the endpoints. Unable to form a triangle, she stops dragging and claims, “Nope, say impossible.” This argument is illustrated in Figure 32.

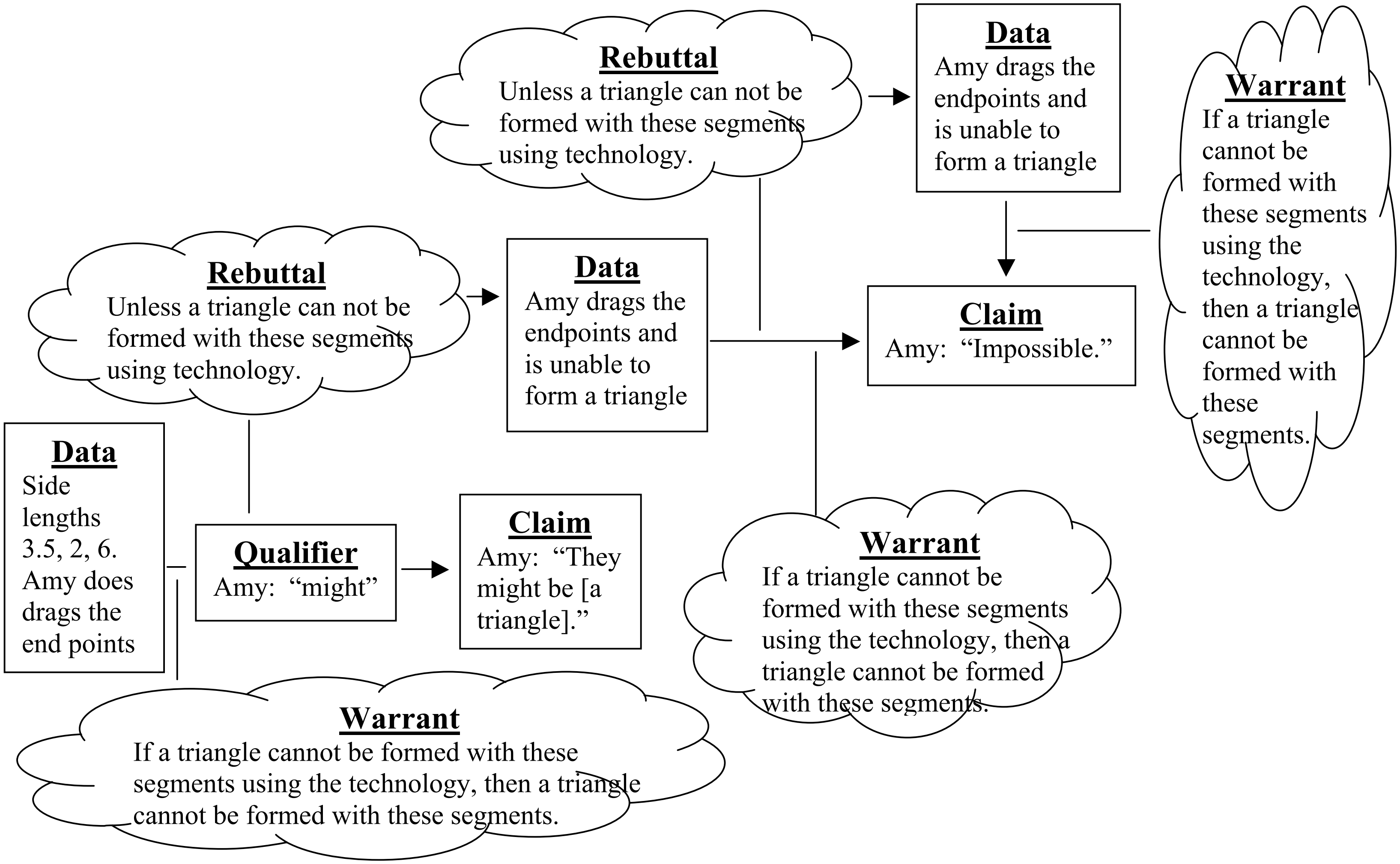


Figure 32. Amy's argument created during task 1 with additional data collection in which the initial data was gathered by dragging the endpoints.

In this argument, the data are the segments of lengths 3.5, 2, and 6 and the dragging of the endpoints of the diagram. Amy provides the initial claim that these segments are able to form a triangle. She qualifies this statement with the word "might." She does not provide an explicit warrant, which is inferred by the researcher to be, "If a triangle can be formed with these segments using the technology, then a triangle can be formed with these segments." The use of the qualifier indicates Amy's uncertainty with her claim. However, she does not provide an explicit rebuttal, which is inferred by the researcher to be, "Unless a triangle cannot be formed with these segments using technology." Amy continues to drag the endpoints, which is additional data. Unable to form a triangle using the technology, she makes a new claim that these segments will not form a triangle. Amy does not provide an explicit warrant for this claim which the researcher infers to be, "If a triangle cannot be formed with these segments using the technology, then a triangle cannot be formed with these segments." This claim is unqualified, but Amy provides an explicit rebuttal when she states, "They get closer, though" or, in other words, unless a triangle can be formed with these segments using technology. Amy continues to drag the endpoints obtaining a second set of additional data, but is unable to form a triangle. She verifies her second claim. Again, her warrant is not explicit and is inferred by the researcher to be "If a triangle cannot be formed with these segments using the technology, then a triangle cannot be formed with these segments."

Looking at the content and structure of this argument, Amy does not qualify her second claim or the verification of that second claim. By not qualifying her second claim, it seems that she is certain that these segments would not be able to form a triangle. However, she provides an explicit rebuttal for this claim, which demonstrates some uncertainty. This seems to indicate that an explicit rebuttal can exhibit some uncertainty in a claim even when a qualifier is not present. She does not qualify the verification claim, which indicates she is certain that the segments will not form a triangle.

Comparing these two arguments, the use of the technology may have influenced the structure of the argument. Even though in both arguments the students are compelled to collect additional data, the use of technology differs. In the first argument, Judy makes a claim based on the appearance of the diagram on the screen. Knowing that looks can be deceiving, Amy drags the endpoints of the diagram and does not make a claim until she is certain that a triangle cannot be formed with the segments. In the second argument, Amy makes an initial claim based on the dragging of the diagram. As previously discussed, she twice collects additional data through the use of the drag feature until she is certain about her claim.

Indirect use of technology in initial data.

The initial data for the two previous arguments is based, in part, on the active use of technology to collect data, initial and additional. However, this is not always the case. In one argument, the students make an initial claim based on the products of their previous uses of technology, collect additional data actively using the technology, which is the basis for a new claim, and use that new claim as data for another claim.

Amy and Judy are determining the longest segment that can form a triangle with segments of lengths 4 and 10. In previous attempts, they have been able to form a triangle with these segments and segments of lengths 12 and 12.9. They were unable to form a triangle with the given segments and segments of lengths 13 and 14. Judy makes the claim, "Then, 12.9 is the largest." Uncertain about this claim, Amy asks Judy to go back to 13. Amy adjusts the sliders to 4, 13, and 10 and drags the endpoints such that a triangle is formed. Amy claims, "Yeah, 13 can match up" and Judy agrees, "See, it's a match." Judy then says, "The highest you can get is 13." This argument is illustrated in Figure 33

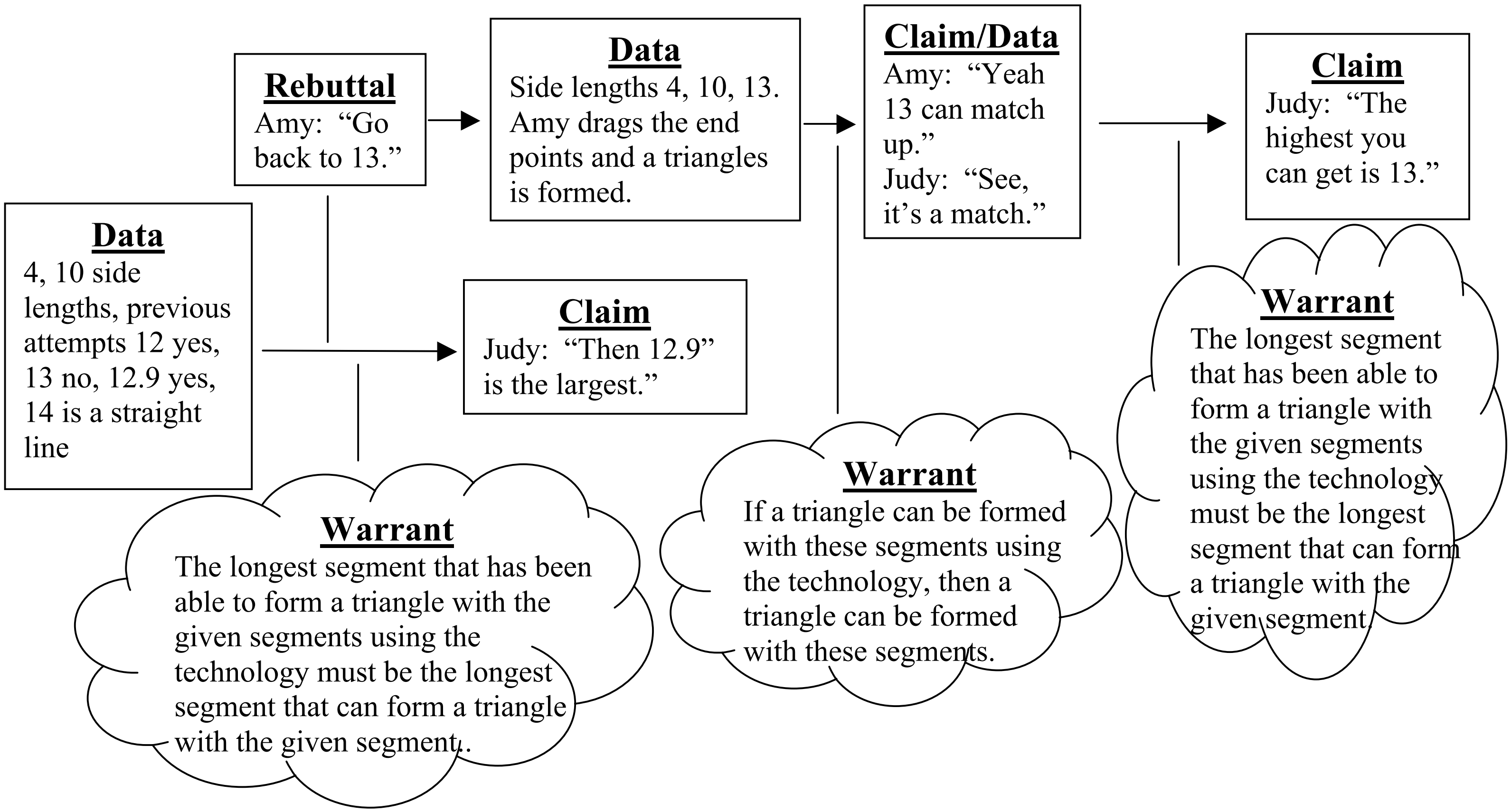


Figure 33. Judy and Amy's argument created during task 1 with additional data collection in which the initial data was based on the students' previous uses of the technology.

The initial data for this argument are the previous attempts to form a triangle with the given segments of lengths 4 and 10 and whether those sets of segments were able to form a triangle. Judy makes the initial claim that 12.9 is the length of the longest segment that can form a triangle with the given segments. Even though this data is collected using technology, the claim is not based on the students' active uses of technology. Rather, the claim is based on the product of the students' uses of technology. Thus, this data is considered an indirect use of technology. The students do not provide an explicit warrant for this claim and the researcher infers it to be, "The longest segment that has been able to form a triangle with the given segments using the technology must be the longest segment that can form a triangle with the given segment." Amy demonstrates her uncertainty with the claim in the form of a rebuttal by challenging Judy's claim. Amy gathers additional data by adjusting the sliders to lengths of 4, 13, and 10 and dragging the endpoints of the diagram to form a triangle. Amy and Judy make the claim that a triangle can be formed with these segments. Neither of the students make their warrants explicit which is inferred by the researcher to be, "If a triangle can be formed with these segments using the technology, then a triangle can be formed with these segments." This new claim becomes data and Judy makes another claim that 13 is the longest segment that can form a triangle with segments of lengths 4 and 10. Again, she does not provide an explicit warrant and the researcher infers it to be, "The longest segment that has been able to form a triangle with the given segments using the technology must be the longest segment that can form a triangle with the given segment."

Conclusion.

The commonality among the arguments that involved the collection of additional data is the ways in which the students use the technology. In all of the arguments of this structure, the students use the dragging of the endpoints of the diagram to collect additional data. Even though there may have been differences in the initial data, the additional data is collected in the same manner.

Although these arguments have this commonality, there are two distinct differences in these arguments; the use of the technology to collect the initial data and the task on which the students are working. In the first two arguments, the students base their initial claim on the use of technology, the appearance of the diagram on the screen for the first arguments and the dragging of the endpoints of the diagram for the second argument. In the third argument, the initial claim is based on the products of their previous uses. This difference may be related to the task at hand. The task for the first and second argument is to determine whether a set of segments will form a triangle. However, the task the students undertake in the third arguments is to determine the longest segment that can form a triangle with two given segments. The later task requires the students to not only determine whether the set of segments will form a triangle but also compare this finding to other sets of segments with two sets of the same lengths in order to determine whether the third length is the largest or smallest that can form a triangle.

Discussion.

While working on the triangle inequality tasks, Amy and Judy create arguments of various structures. Two categories of structures are noted in the analysis of these arguments:

core arguments, and arguments in which the students collect additional data. Looking across these argument structures, three themes emerge; the lack of explicit warrants (see Table 8), the relationship between the task and the content of an argument, and the relationship between the use of the technological tool and the type of task.

Table 8

Group 3's Argument on the Triangle Inequality Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	11	8
	Technology Not Used	0	1
Warrants Explicit			
	Technology Used	0	1
	Technology Not Used	1	1

In the 23 arguments created by Amy and Judy, explicit warrants are found in only 3. The explicitness of the warrants may be related to the task and the students' uses of the technology. Regardless of the structure of the argument, when these students use technology, they generally do not provide an explicit warrant. In fact, they only provided an explicit warrant for one argument in which the students employ technology. When the students provide an explicit warrant, it is made while the students are making a generalization and the use of technology is indirect. The students may not have felt the need to provide an explicit warrant due to the shared understanding of what they were seeing on the screen.

The task on which the students are working not only seemed to affect whether the warrants are explicit, they also seem to affect whether the students decide to collect additional data. When students are working on tasks in which they are determining whether a set of segments can form a triangle, the students create two basic argument structures: core arguments, and arguments in which additional data is collected to verify or refute an initial claim. For nine of the eleven core arguments created while working on this type of task, the claim is the segments are able to form a triangle. However, in the arguments with additional data collection created while working on this type of task, the final claim for six of the nine arguments is that a triangle cannot be formed with the segments. This difference in the content and structure of these arguments may be due to the nature of the tool. When trying to determine whether a triangle can be formed using the technology, it is fairly obvious when the segments form a triangle; the endpoints of the segments are coincidental. When these students are able to form a triangle, they may not feel compelled to collect additional data. They are certain that a triangle can be formed with the segments because they had done so using the technology. The arguments they created for these tasks are core arguments. However, to be certain a triangle cannot be formed, the students may have to drag the endpoints to different locations on the screen before reaching that conclusion. The students make an initial claim, but demonstrate their uncertainty through the use of an explicit qualifier or rebuttal. This uncertainty leads the collection of additional data.

The use of the technology allows Amy and Judy to investigate whether a set of segments can form a triangle. In one argument, the students use the technology to accomplish this task, but it is within in the context of another task. In the argument illustrated in Figure 33, Amy and Judy are determining the longest segment that can form a triangle with segments of lengths 4 and 10. Even though this task is related to their prior tasks in which they used the technology, it is a different type of task. The students chose to use the technology to assist them in accomplishing this task. This suggests that the students view this technology as more than an artifact to determine whether a triangle can be formed by given segments, but as a tool to be used to investigate other related tasks.

Cross-case analysis of arguments created while working on the triangle inequality task.

On the triangle inequality tasks, the arguments created by the three pairs of students vary in their structure and content, including the ways in which they employ the technology. Four themes emerge when looking across the arguments created by these students; the content of the argument when technology is and is not actively employed (see Table 9), the relationship between the type of task on which the students are working and the content of the argument, the collection of additional data, and the ways in which the students use the technology.

Table 9

The Combined Arguments of the Three Groups on the Triangle Inequality Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	28	25
	Technology Not Used	1	3
Warrants Explicit			
	Technology Used	1	5
	Technology Not Used	7	5

In sum, the three groups of students create 75 arguments while working on the triangle inequality task. For 57 arguments, the students do not provide explicit warrants. Additionally, the students employ technology in 59 arguments. Of the 59 arguments in which the students employ technology, they only provide an explicit warrant for 6 (10%) of these arguments. Of the 16 arguments in which technology is not actively employed, the students provide an explicit warrant for 12 (75%) of these arguments. This disparity seems to suggest that when students use technology, they are less likely to provide an explicit warrant compared to when they do not actively use technology.

The explicitness of the warrants may be related to the type of tasks on which the students were working. In general, when the students are using the technology, they are merely attempting to determine whether a triangle can be formed with a set of segments.

However, when the students are not actively using technology, the students are mainly working on generalization type tasks. For example, one of the arguments common to all the students in structure was in response to the question on the task sheet, “Why was it impossible to construct a triangle with some of the given lengths?” The question asks the students to generalize across the examples. The structure of the argument for all the groups of students is a core argument with an explicit warrant (see the arguments illustrated in Figures 13, 23, and 30). The structure of these arguments may be due to the fact that the question actually provides the claim and asks the students to provide the warrant. The data for the students are their answers to the examples sets of segments on their task sheet. To gather this data, the students use technology. However, when responding to this question, the data had been previously gathered and their reasoning is not based on their active use of technology, but on the product of their previous uses.

Many times the students collect additional data to verify or refute a previous claim. All three groups of students create arguments of this structure. The students’ decision to seek additional data may be due to a number of factors including an explicit challenge to a claim (see the arguments illustrated in Figures 15 and 26), the uncertainty of a claim (see the arguments illustrated in Figures 16 and 32), the uncertainty of a claim due to the lack of precision in the use of the technological tool (see the argument illustrated in Figure 17), and the ways in which the students use technology to collect the initial data (see the arguments illustrated in Figures 25 and 31). Even though the students collect additional data for a variety of reasons, they always use the technology to collect this additional data, usually using the drag feature of the technology. This may be related to the ease of collecting data with the technology. By using the drag feature, the students are able to easily determine whether a triangle could be formed.

In this task the uses of the technology are restricted to adjusting the sliders and dragging the endpoints of the diagram. However, the students do not always drag the endpoints of the diagram prior to making a claim. All three groups make claims based on the appearance of the diagram on the screen after the sliders have been adjusted. In some of these cases, the students collect additional data to verify or refute the claim (see the arguments illustrated in Figures 15, 25, and 31) while in other cases the students do not (see the argument illustrated in Figure 16). On the first day of class, the students frequently use the tool to determine whether segments of various lengths could form a triangle. By the second day, the students had become very familiar with the tool. The familiarity of the tool may have increased the frequency of claims based on the appearance of the diagram on the screen. In fact, the number of claims based on the appearance of the diagram on the screen increased from 5 on the first day to 14 on the second day. This seems to suggest that, by the second day, after the students had adjusted the sliders, the students could imagine whether a triangle could be formed with the segments based on the appearance of the figure. In one argument (see Figure 27), the researcher infers that David imagined the dragging of the endpoints to form a triangle.

The change in the way the tool is used by these students to collect data from day one to day two suggests a change in the way they view the tool as well. At first the students mainly use the technology to gathering data by dragging the endpoints of the segments to determine if a triangle could be formed. On the second day, the students may not feel the need to use this feature. By the second day, perhaps these actions, the dragging of the endpoints, have become internalized and the technology becomes a tool to think with, rather than a tool to reason from.

Task 2 – Triangle Side and Angle Relationship

On the third day of class, the students investigated the theorem which states "Let A, B, and C be three non-collinear points. Then $AB > BC$ if and only if $\mu(\measuredangle ACB) > \mu(\measuredangle BAC)$" (Venema, 2006, p. 102). In the two previous class meetings, the students investigated the triangle inequality theorem and the theorem that states the sum of the measures of the interior angles of a triangle is 180°. Given the students had previously explored and discussed theorems related to the angles and sides of a triangle, the teacher thought the students should learn how these elements of a triangle relate to each other.

The objectives for this task centered on the students' discovery and understanding of the relationship between the longest side and largest angle and the shortest side and smallest angle of a triangle. The teacher wanted the students to be able to use the technology to determine the longest and shortest sides and largest and smallest angles of a triangle. The teacher also wanted the students to compare the location of these sides and angles to determine the relationship between them. By the end of the task, the teacher wanted the students to have developed an understanding of this theorem.

With these objectives in mind, the researcher created a technological activity and corresponding task sheet to be used to teach this concept. The technological activity consisted of a pre-constructed sketch using *The Geometer's Sketchpad*.

Pre-constructed sketch.

The pre-constructed sketch was chosen for two primary reasons. First, even though the students had used the software the previous two class meetings, the students' knowledge and understanding of the software was still very basic. During the two previous class meetings, the students had been exposed to the pre-constructed sketch to investigate the triangle inequality, constructed a triangle, measured its angles, and used the calculate feature to sum these angle measures. Second, having the students construct these sketches would have taken a great deal of class time. While it may have been a fruitful exercise for the students in learning the definitions and properties of different triangles, it may not have provided any insights into why the theorem under study was true.

The pre-constructed sketch created for this task was one in which the students could investigate the relationship between the relative magnitude of the lengths of the sides of a triangle and the measures of the angles across from them. The sketch contained five tabs, one for each of the five triangles the students would explore (See Figure 34). The triangles were acute, obtuse, isosceles, right, and equilateral. Most of the triangles were constructed such that it maintained the desired properties when the students dragged all or a portion of the triangle. The only exception was the acute triangle. The students were given directions for this triangle to drag it such that it remained acute. It should also be noted that the way in which the obtuse triangle was constructed limited the measure of its obtuse angle to no less than 91°. Each of triangles was labeled *ABC* and the interior of the triangle was constructed and the color of each triangle was not the same. The measures of the side lengths and angles were not provided in the file.

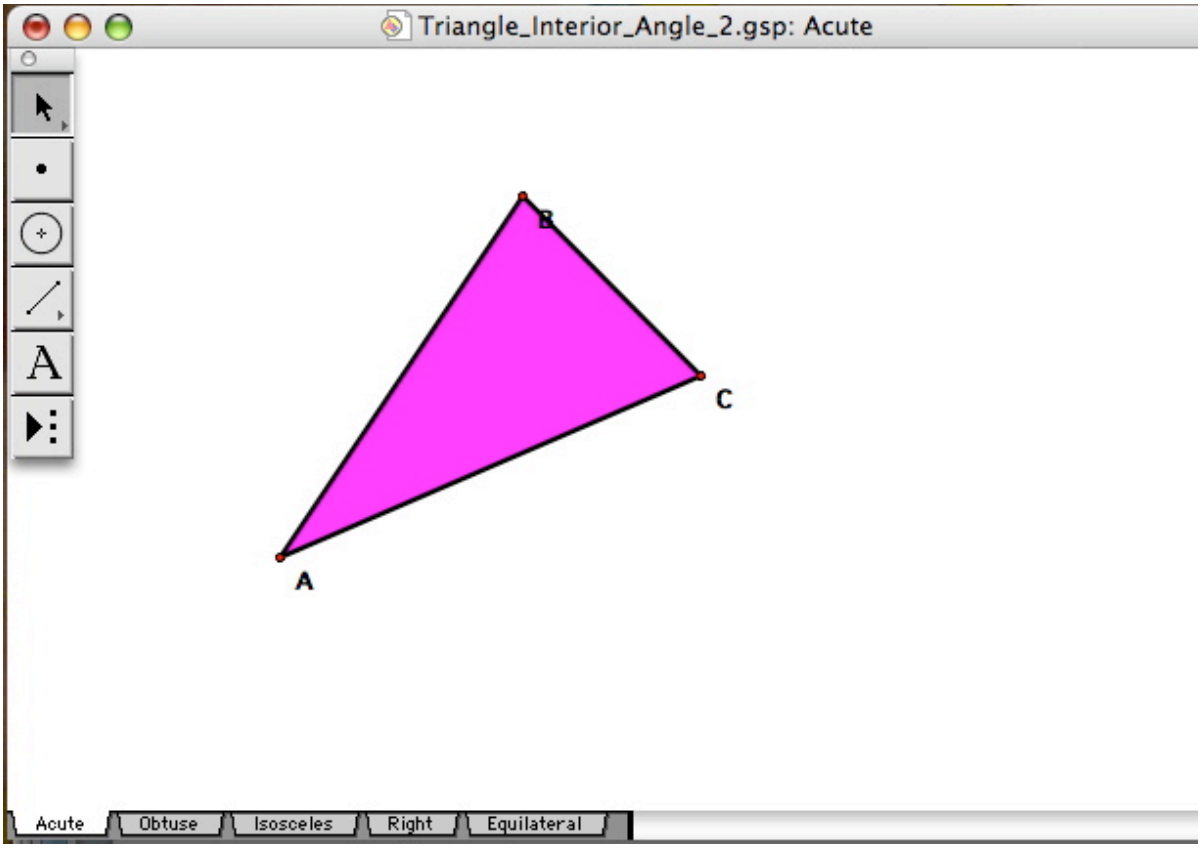


Figure 34. An example of the acute triangle in the pre-constructed sketch used during the triangle angle and side relationship task.

The decision to exclude the measures of the side lengths and the angles was two fold. First, the students had measured the angles in a previous task and measuring the lengths of the sides is not a difficult task in this program. In the opinion of the teacher, the students had gained enough knowledge of the program to be able to do these actions. Second, the teacher wanted the students to have the choice of whether to measure these components of the triangles. The students may have been able to determine the longest/shortest side and largest/smallest angle without using the measurement feature of the software (and some groups did not). If the measures had been given to the students, the students may have only relied on these measures and, thus, influenced the structure and content of the arguments they produced. The teacher wanted the students to make the decision on the type of data that was needed to correctly complete the task.

The task sheet.

Along with the sketch, the students were given a task sheet to assist in their exploration (see Appendix D). The task sheet contained directions for the activity as well as directions on how to measure the length of a side. In a previous activity, the class measured the angles of a triangle and the directions for this action were unnecessary.

On the task sheet, the students were given a table to complete. The row headings of the table were the five triangle types. The column headings were the name of the longest side, name of the largest angle, name of the shortest side, name of the smallest angle, and picture. For each of the triangles, one piece of information was given to the students. For example, for the acute triangle, the students were given $\overline{AB}$ as the longest side. Using the pre-constructed sketch, the students were to drag a vertex or side of the acute triangle such that $\overline{AB}$ was the longest side and the triangle remained acute. Once the students had positioned the triangle according to the directions, the students were to fill out the table for that triangle and draw a picture of their triangle as a reference.

After the students completed the table, the students were asked to answer two questions. The questions were, "What do you notice about the relationship between the longest side and the largest angle?" and "What do you notice about the relationship between the smallest side and the smallest angle?" These questions were intended to focus the students on these relationships while still allowing the students to arrive at their own conclusions. Using the relationships they found for these two questions, the students were asked to write a conjecture about these relationships. Then, the students were asked to explain why they think the conjecture was true.

Teaching of the lesson.

The triangle side and angle relationship task was conducted on the third day of class. At the beginning of the class period, the teacher asked the students to compare their answers on their homework assignment with their neighbors. After reviewing the solution to one of the problems, the teacher asked the students to provide a list of properties about triangles they had learned thus far. After briefly discussing the two theorems, the teacher directed the students to open the pre-constructed sketch on the computer and he distributed the task sheet. After a student read the directions, the teacher pointed out the five tabs, a feature of the program new to the students. In addition, he discussed how to measure the length of a side of a triangle, but stressed that this may not be necessary. Then, the teacher demonstrated how to drag the acute triangle such that $\overline{AB}$ was the longest side and the triangle remained acute. After this display, the teacher asked the students to complete the table. The students were given time to work on this task and to answer the questions. After some time had passed, the teacher noticed the students had finished filling out the table, but were having difficulty finding the relationship and were not discussing mathematics. The teacher asked the students to participate in a whole class discussion and began by having individuals come to the front of class and complete the teacher's table, which was displayed on the board.

As the students began filling out the teacher's table, the teacher asked the students whether they had found a pattern in the names of the angles and the sides. One student offered the following pattern; given the name and relative magnitude of a side, the corresponding angle name would be the letter not in the name of the side (e.g. for triangle *ABC*, if the name of the longest side is $\overline{AB}$, the name of the largest angle would be *C* because it is the letter missing from $\overline{AB}$). The class continued filling out the table, and the teacher asked the students to confirm this relationship. After the students had accepted this pattern, the teacher asked, “So those are good patterns and that’s right, but what does it mean, in relation to the triangle?” A student provided an explanation that the longest side opens up the largest angle. After asking if everyone agreed with this explanation, the teacher asked the students to write a conjecture. The teacher gave the students a few minutes to write their conjecture and then conducted a class discussion about their conjecture. The students came to an agreement that the largest side opens up the largest angle and the shortest side opens the smallest angle. The teacher then attempted to focus the students on the relative location of the aspects of the triangle by asking a series of questions about where these angles and side were located on the triangle. After a short discussion, one student made the claim the largest angle was across from the longest side and smallest side was across from the smallest angle. The other students immediately accepted this claim and the teacher instructed them to write it in their notes. After a brief discussion about what it means to be “across”, the teacher gave the students their homework assignment.

In the following sections, the arguments created by three pairs of students while working on this task are analyzed and discussed. For each pair of students, the arguments were first categorized by their basic structure. Then, the content and structure of the arguments within these basic categories were analyzed, including the students’ uses of technology. The themes that emerged from these analyses are discussed for each pair of students and across the pairs of students.

Group 1's arguments on the triangle side and angle relationship task. The analysis of Heather and Mary's arguments while working on the triangle side and angle relationships activity can be categorized into three argument structures: core arguments, arguments in which a claim was not explicit, and arguments where the teacher challenges a claim. These argument structures are discussed below.

Core arguments.

During the triangle sides and angle relationships task, Heather and Mary create twenty-one core arguments. For seventeen of these arguments, the students use technology to collect data and the warrants are not explicit. In one core argument, the students use technology to collect data and the warrant is explicit, but the students do not provide a claim. For the remaining three core arguments, technology is not actively used in the creation of the argument. These three argument structures are detailed below.

Non-explicit claims, technology used. For the majority of the core arguments, Heather and Mary use the technology to create data and, based on this data, make claims. However, their warrants for these claims are usually not explicit. In nine of these arguments, the data takes the form of side measures and the appearance of the triangle on the screen. For example, Heather and Mary are determining the longest side of right triangle *ABC*. Mary measures the side lengths to be $m\overline{AC} = 6.89$ cm, $m\overline{BA} = 6.58$ cm, and $m\overline{CB} = 2.04$ cm. She then states, "The longest side is $\overline{AC}$ cm." This argument is illustrated in Figure 35.

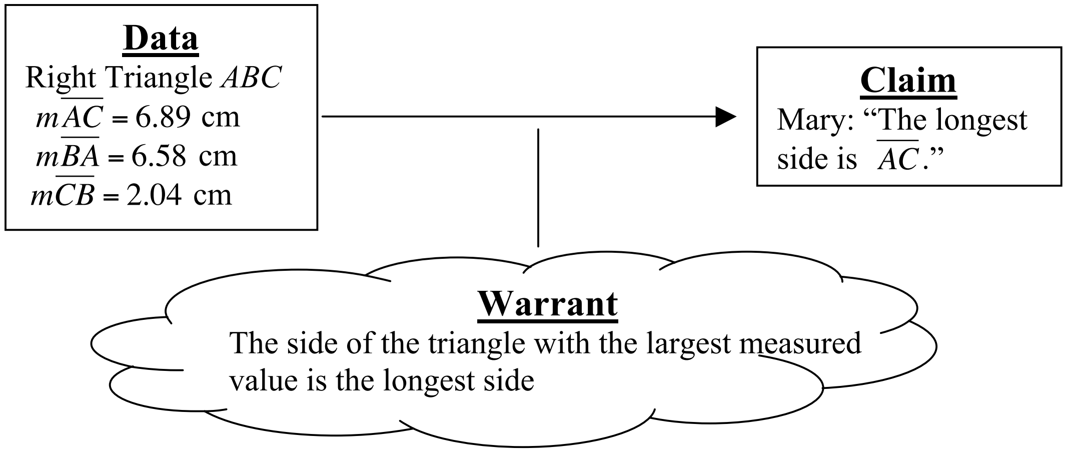


Figure 35. Heather and Mary's core argument created during task 2 with an inferred warrant and measures as data.

The data for this argument are the measures of the lengths of the sides. Mary claims side $\overline{AC}$ is the longest side of the right triangle. The warrant for this claim is not explicit and the researcher infers it to be, "The side of the triangle with the largest measure is the longest side."

In eight of the core arguments in which the students employ technology and warrants are not explicit, the students use the technology to measure the angles of a triangle to determine which angle is the largest and which is the smallest. For example, Mary and Heather are determining the largest angle of the acute triangle *ABC*. Using the measurement tool, Heather measures the angles to be $m\measuredangle BAC = 46.77°$, $m\measuredangle ABC = 49.74°$, and $m\measuredangle BCA = 83.48°$. She then states, "The largest angle is *BCA*." This argument is illustrated in Figure 36.

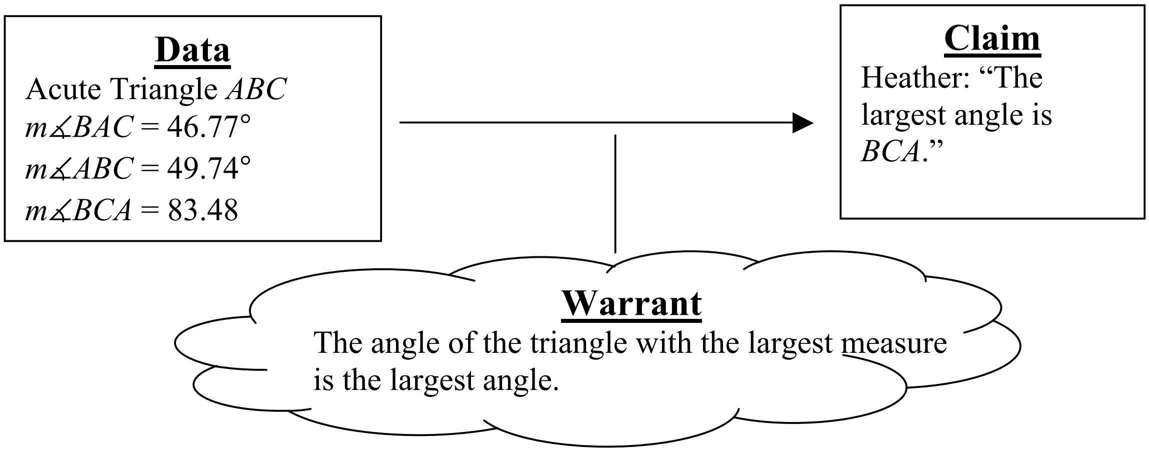


Figure 36. Heather's core argument created during task 2 with an inferred warrant and angle measures as data.

The data for this argument are the triangle's angle measure. Heather claims angle *BCA* is the largest angle of the acute triangle. The warrant for this claim is not explicit which the researcher infers to be, "The angle of the triangle with the largest measure is the largest angle."

For one core argument, the students use the drag feature of the software to create data. In this argument, Heather and Mary are working with the obtuse triangle maker. The students have not measured the angles or the lengths of the sides. Heather drags the vertices of the triangle such that the obtuse angle approaches ninety degrees and appears to be a right triangle. She states, "It's going to be a right triangle." This argument is illustrated in Figure 37.

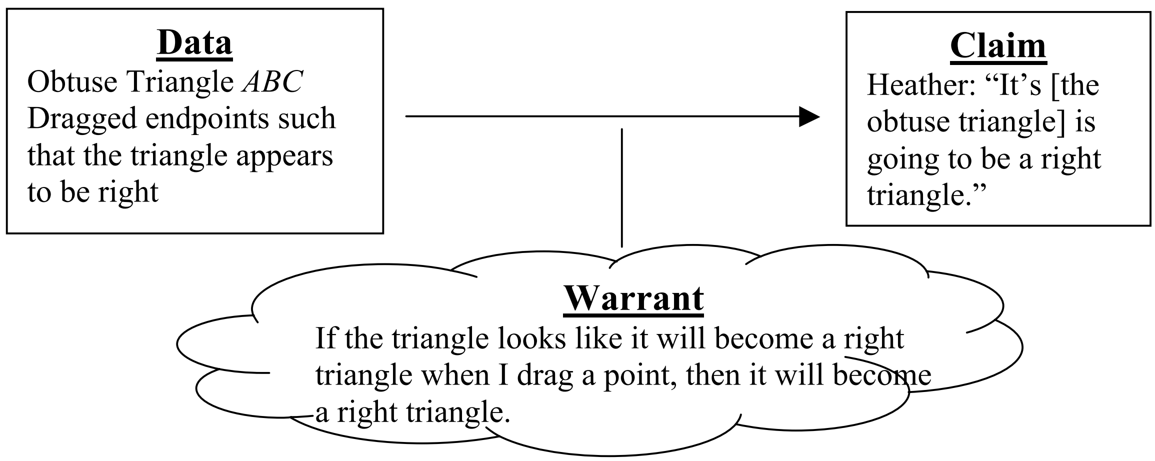


Figure 37. Heather's core argument created during task 2 with an inferred warrant and dragging as data.

The data for this argument are the dragging of the vertices of the obtuse triangle and the appearance of the obtuse triangle as it is being dragged. Heather claims the obtuse triangle will become a right triangle. The warrant is not explicit and the researcher infers it to be, "If the triangle looks like it will become a right triangle when I drag a point, then it will become a right triangle."

Across these seventeen core arguments there are many similarities. First, the structure of the arguments is the same that the claims are explicit and the warrants are inferred. Second, the data for these arguments are based, in part, on the use of technology. For seventeen of the eighteen arguments, the data consists of measures made using the technology. Third, for all but one of the arguments, the task on which the students are working is similar; to determine the longest/shortest side or largest/smallest angle of a triangle. This suggests that the students' uses of technology are directly related to the task on which they are working.

Non-explicit claim.

In one of the core arguments, the students employ technology and provide an explicit warrant, but do not provide an explicit claim. In this argument, the students are attempting to adjust the obtuse triangle such that the smallest angle of the triangle is angle C in order to determine the name of the largest angle, the shortest side, and the longest side of an obtuse triangle. In a previous argument, Mary measured the angles and claimed the angle C was not small enough. She deleted these measures and dragged the endpoints of the triangle. Then she measures again and finds that $m\measuredangle ABC = 49.74^\circ$, and $m\measuredangle BCA = 83.48^\circ$. Heather states, "Because look at how small that angle is compared to that one." This argument is illustrated in Figure 38.

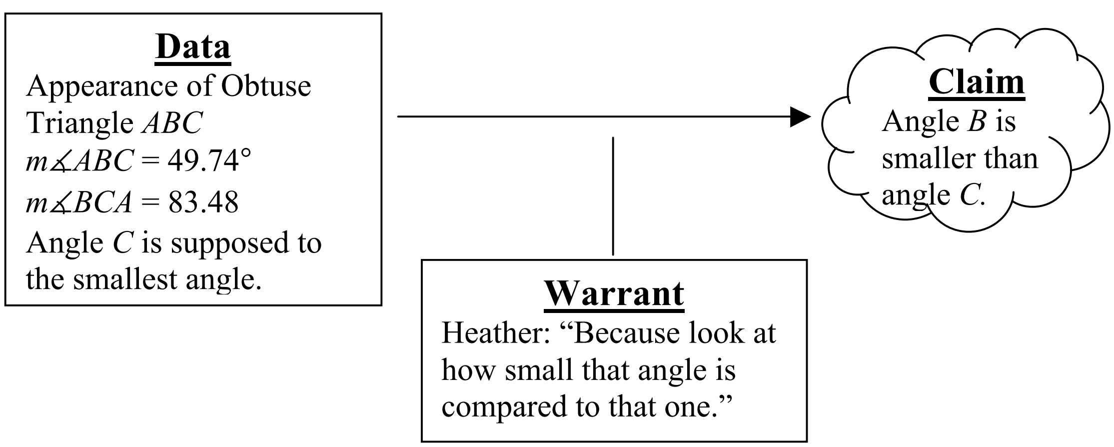


Figure 38. Heather and Mary's core argument created during task 2 with an explicit warrant and inferred claim.

The data for this argument are the measures of two of the angles of the obtuse triangle, the appearance of the obtuse triangle, and the task on which the students are working. Neither student provides an explicit claim for this argument and is inferred by the researcher to be "Angle *B* is smaller than angle *C*." This inference is made, in part, by Heather's statement, which is an explanation for why angle *B* is smaller than angle *C*. This statement is the warrant for the unstated claim.

This argument differs in structure and content than those in the previous section. In the previous section, the claims are explicit and warrants are not explicit. In this argument, the opposite occurs; the warrant is explicit, but the claim is not. This may be related to the task on which the students are working. In the previous section, the students are determining the longest/shortest side or the largest/smallest angle of a triangle. Here, the students are positioning the triangle such that it conforms to the directions on the task sheet. Even though the technology use is similar (in both types of arguments the students are using measures), Heather also relies on the appearance of the obtuse triangle on the screen in her warrant.

The unstated claim suggests that for these students, the technology provides a common venue about which to reason. Both students can see the measures of the angles as they appear on screen. Because the students are working toward the same goal, they may not feel it necessary to state what is obvious to them, that angle *B* is smaller than angle *C*. Instead, they choose to provide a reason for why this is so and to use this information to make the needed changes to the figure to accomplish this task.

Indirect use of technology.

The third category of core arguments created by Heather and Mary are those in which the students do not actively employ technology. Of the twenty-one core arguments, only three have this content. All three arguments are described below.

The first example is Heather and Mary's generalization of the relationship between the longest side and the largest angle of a triangle. The students have completed the task sheet using technology and have noticed a pattern in the names of the largest angles and the longest sides of the examples on their task sheet. Heather states, "The longest side cannot have the largest angle with it." Mary begins to rephrases Heather's claim stating, "The letters on the largest side." Heather finishes Mary's statement, "cannot be the largest angle." This argument is illustrated in Figure 39.

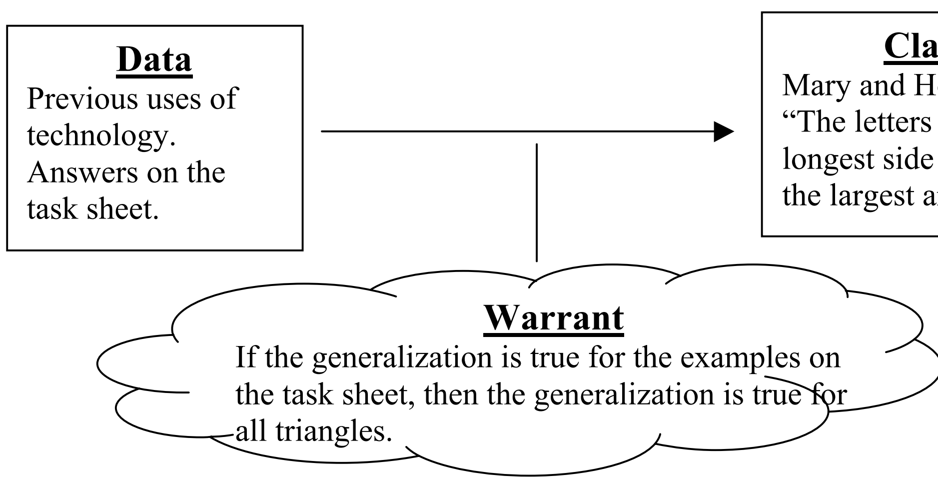


Figure 39. Heather and Mary's core argument created during task 2 with an inferred warrant and indirect use of technology.

The data for this argument are the students' responses on the task sheet, which are based, in part, on their uses of technology. Even though the students' responses are based on their uses of technology, the students are not actively using the technology in the creation of their argument. Thus, the students are using technology indirectly. Heather and Mary's claim is the relationship between the names of the largest angle and the longest side of a triangle. They do not provide an explicit warrant for this claim and the researcher infers it to be, "If the generalization is true for the examples on the task sheet, then the generalization is true for all triangles."

Another example of a core argument in which technology is not actively employed in the creation of the argument is the one provided by Heather in response to the question "If the shortest side [of a triangle] is $\overline{BC}$, what's going to be the smallest angle?" Heather responds, "*A*." When asked by the teacher why, she states, "Because it's the missing letter. This argument is illustrated in Figure 40.

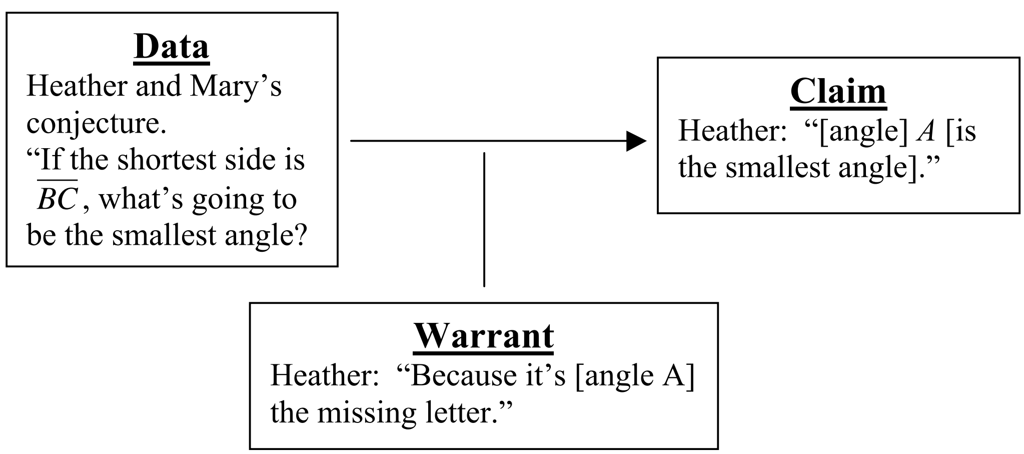


Figure 40. Heather and Mary's core argument created during task 2 with an explicit warrant with their conjecture as data.

Heather's claim is angle *A* is the smallest angle of the triangle given side $\overline{BC}$ is the shortest side. Her data for this claim are the conjecture developed by her and Mary and the teacher's question. Heather provides the warrant for this claim but only when prompted by the teacher to do so.

Heather makes a similar structured argument during the whole class discussion on the name of the longest side of a right triangle given the largest angle is angle *B*. Another student in the class states, "It's [side] $\overline{AC}$." Heather agrees stating, "Yeah, it's going to be [side] $\overline{AC}$." When the teacher asks why the student says, "Because it's long." Not liking this answer, Heather says, "No, because B is the missing letter." This argument is illustrated in Figure 41.

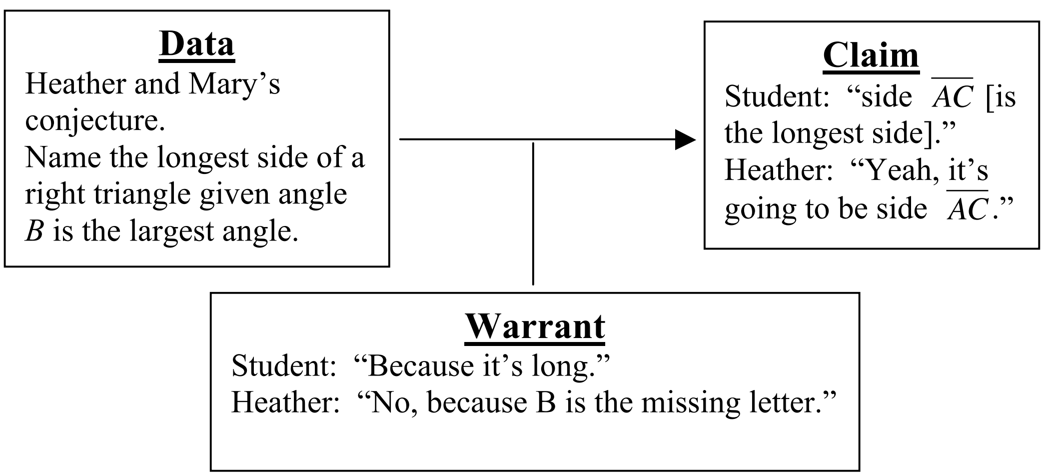


Figure 41. Heather's core argument created during task 2 with an explicit warrant and a conjecture as data.

In this argument, both Heather and another student make the claim that side $\overline{AC}$ is the longest side of a right triangle whose largest angle is angle *B*. This claim is based on the data, the question posed by the teacher (and on the task sheet) and the conjecture developed by Heather and Mary. The other student provides an explicit warrant for his claim by stating that side $\overline{AC}$ is the longest because it is long. Heather does not like this warrant and uses her conjecture to justify why side $\overline{AC}$ is the longest side.

In these final two core arguments, the students provide explicit warrants and technology is not actively used. Furthermore, for both arguments, the warrant provided by Heather is based on the conjecture developed by Heather and Mary in the argument diagrammed in Figure 39. This use of the conjecture in this manner suggests that the students are able to use a claim created while working on one task as justification in a different task.

Conclusions.

Looking across the twenty-one core arguments created by Heather and Mary while working on the triangle sides and angle relationships activity, one theme emerges. When students do not actively employ technology, they are more likely to provide an explicit warrant. Of the eighteen core arguments in which technology is used, only one warrant is explicit and that argument contained a non-explicit claim. However, two of the three arguments that do not employ technology have explicit warrants. This may be related to the task on which the students are working. In the arguments that employ technology, the students are determining the longest/shortest side or the largest/ smallest angle of a triangle. In the arguments in which technology is not actively used, the students are attempting to make a generalization or using that generalization as justification. This suggests that when students are working on task involving a generalization, they are more likely to provide explicit warrants.

Non-core arguments.

During the triangle sides and angle relationships tasks, Heather and Mary provide few arguments that have structures that are more complex than the core structure. Even though only two of the twenty-three arguments are not of the core structure, both have a unique structure and the content is significantly different from each other and those in the previous categories of core arguments. These arguments are discussed below

Linked argument with a non-explicit claim.

In the argument illustrated in Figure 38, Heather and Mary do not provide an explicit claim. This occurs in another argument, but its structure is not the core argument structure. Instead, the structure of this argument would be considered a linked argument. In this argument, Heather and Mary are positioning the obtuse triangle *ABC* such that the smallest angle is angle *C*. The students have been unsuccessful in their two previous attempts to accomplish this task (one of which is described in the argument illustrated in Figure 38). Mary drags vertex *A* (the obtuse angle), then vertex *B*. The measures of angles *C* and *B* are on the screen and adjust accordingly. When Mary stops dragging, the measures are $m\measuredangle BCA = 12.81°$, and $m\measuredangle CBA = 28.19$. Mary asks Heather, “Is that better?” and Heather agrees. Mary than states, “But that’s acute. That’s acute. That’s obtuse.” While she makes these statements, she uses the mouse to point to the three angles; angle *C*, angle *B*, and angle *A*, respectively. Heather states, “Exactly. Look what kind of triangle that is. As long as there is one angle that is obtuse.” This argument is illustrated in Figure 42.

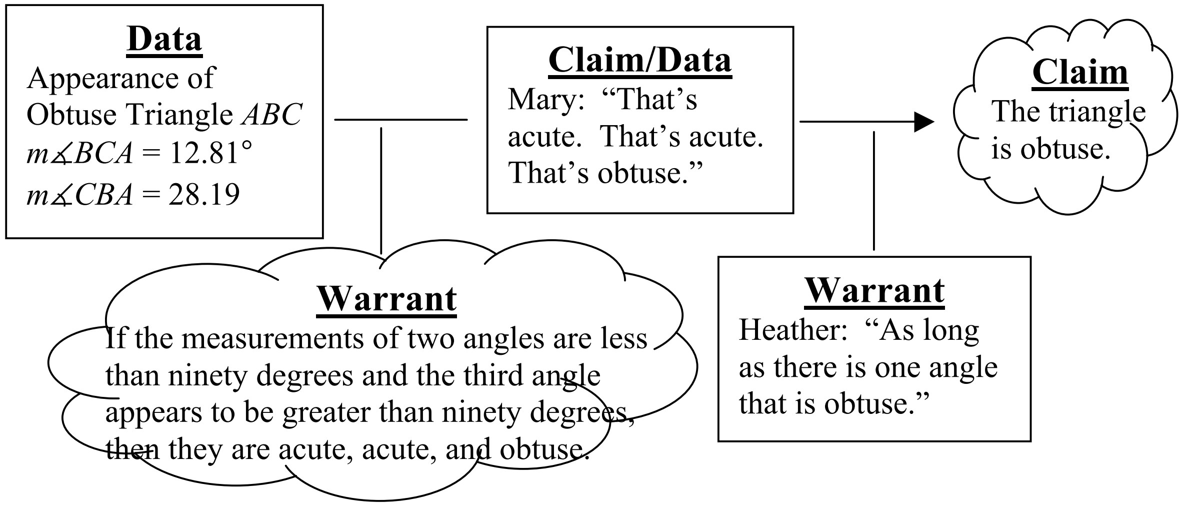


Figure 42. Heather and Mary's linked argument created during task 2 with an non-explicit claim.

In this argument, Mary makes the claim that there are two acute angles and one obtuse angle in the triangle. Her data for this claim are the measures of the two acute angles, and the appearance of the obtuse triangle on the screen. Mary does not provide an explicit warrant for this claim and is inferred by the researcher to be "If the measurements of two angles are less than ninety degrees and the third angle appears to be greater than ninety degrees, then they are acute, acute, and obtuse." Heather does not dispute Mary's claim. Instead, she uses it as data to state why the triangle is obtuse. However, she never explicitly states that the triangle is obtuse. Rather, she provides a warrant for why the triangle is obtuse. Due to the data provided by Mary and the warrant provided by Heather, the researcher infers Heather is justifying the claim "The triangle is obtuse."

Challenge by the teacher.

At the beginning of the activity, the teacher leads the students through the first example, the acute triangle. On the board is the display of the teacher's computer with the pre-constructed sketch opened to the acute tab. He asks the class, "Do we know we have an acute triangle?" Heather answers, "No, we don't have any measures." The teacher responds, "You think we need to get the measures first?" Heather answers, "Well, it's under acute so it's probably acute." This argument is illustrated in Figure 43.

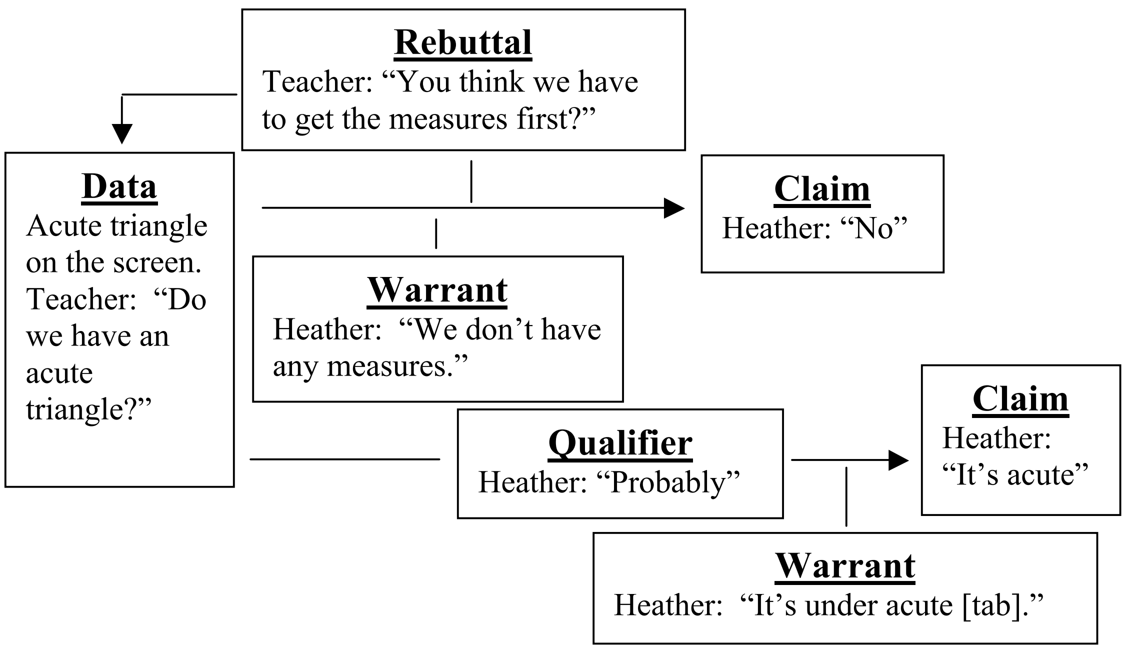


Figure 43. Heather's argument created during task 2 in which she uses the initial data to make a second claim.

In this argument, Heather claims that triangle on the screen is not an acute triangle. Her data for this claim are the triangle on the screen under the acute tab and the question posed by the teacher. She provides an explicit warrant for this claim by indicating there are not any measures on the screen to confirm that it is indeed an acute triangle. The teacher challenges this warrant by asking if measures are necessary to make this determination. This question takes the form of a rebuttal. Heather re-evaluates the initial data, and makes a new claim, “It’s acute.” She qualifies this statement with the word “probably” and provides an explicit warrant for her new claim, “It’s under the acute tab.”

Conclusions.

During the triangle sides and angle relationships activity, Heather and Mary only create two non-core arguments. The commonality of these two arguments is the task on which the students are working, to classify the triangle on the screen. In the first argument, Heather and Mary use measures and the definition of obtuse triangle to determine that the triangle on the screen is indeed obtuse. In the second argument, Heather makes an initial claim and provides a warrant that measures are needed in order to determine the type of triangle. These examples suggest that when students are working on tasks involving classification of triangles, the arguments may be more complex than the basic core structure. It is also worth noting that when the students are working on these types of tasks, they may want to use specific features of the tool. In the first argument, Heather and Mary use angle measures to determine the obtuse triangle was indeed obtuse. In the second argument, Heather wants angle measures to determine whether the triangle was acute. This suggests that these students value measures as a means to classify triangles.

Discussion.

While working on the triangle side and angle relationships activity, Heather and Mary create arguments of various structures. Three categories of structures are noted in the analysis: core arguments, arguments in which a claim is not explicitly provided, and arguments where the teacher challenges the claim. Looking across these argument structures, two themes emerge: the explicitness of the warrants (see Table 10), and the ways students use technology.

Table 10

Group 1's Argument on the Triangle Side and Angle Relationship Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	17	0
	Technology Not Used	1	0
Warrants Explicit			
	Technology Used	1	2
	Technology Not Used	2	0

Of the twenty-three arguments created by Heather and Mary, the warrant is explicit for only five arguments. The explicitness of the warrant may be related to the task and the students' uses of the technology. For the majority of the arguments, the students are engaged in the task of determining the largest/smallest angle or longest/shortest side of a triangle and use the technology to create measures as data for their claims. Only one of the warrants for the arguments created when working on this task is explicit. However, when the students create arguments while are working on the tasks in which they are developing a generalization or classifying a triangle, the majority of the warrants are explicit. This suggests that when students are working on more complex tasks, they are more likely to provide explicit warrants.

During the triangle side and angle relationships activity, Heather and Mary use technology to accomplish a variety of tasks. They use the software's measurement tool for the majority of their arguments. However, they also make claims based on the appearance of the figure, both static and when the figure is dragged. In addition, the students do not qualify or provide rebuttals for the majority of their arguments. This suggests that the students are certain about their claims. In fact, the only argument in which they qualify a claim, they use the term "definitely" which suggests they were very certain. Also, the teacher provides the only explicit rebuttal (see the argument illustrated in Figure 43). The frequent use of technology coupled with the lack of qualifiers and rebuttals might suggest that the students view the technology as a reliable tool; one that provides an accurate representation of the Euclidean plane such that the constructed figures are valid and the accompanying measures are precise. This viewpoint could also be attributed to teacher's request for the students to use the tool to complete the task. The students may view the tool as reliable because the students might believe that the teacher would not provide the students a faulty tool.

Group 2's arguments on the triangle side and angle relationship task.

The analysis of David and Erica's arguments while working on the triangle side and angle relationship task can be categorized into two argument structures: core arguments, and arguments in which the students collect additional data to verify or refute a claim. These argument structures are discussed below.

Core arguments.

During the triangle sides and angle relationships activity, David and Erica create nine core arguments. For five of these arguments, technology is used and the warrants are not explicit. In two core arguments, the students use technology and the warrants are explicit. For the remaining two core arguments, the students do not actively use technology. These three argument structures are detailed below.

Non-explicit claims, technology used.

For the majority of the core arguments, David and Erica use the technology to create data and, based on this data, make claims. However, their warrants for these claims are usually not explicit. In one of these arguments, the data takes the form of side measures and the appearance of the triangle on the screen. For example, David and Erica are determining the longest side of obtuse triangle *ABC*. David measures the side lengths to be $m\overline{CB} = 4.90$ cm, $m\overline{AC} = 2.54$ cm, and $m\overline{BA} = 3.01$ cm. He then states, "The longest side is $\overline{CB}$." This argument is illustrated in Figure 44.

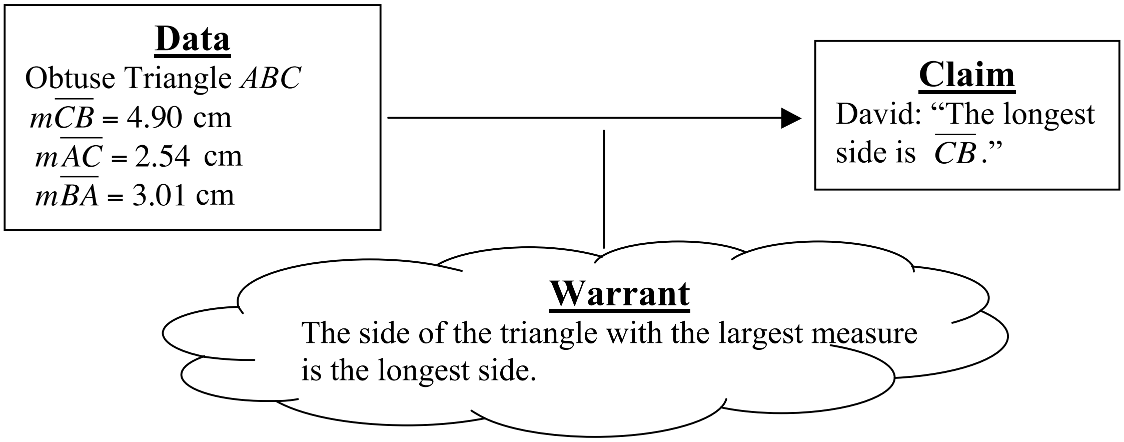


Figure 44. David and Erica's core argument created during task 2 with an inferred warrant and measures of segment lengths as data.

The data for this argument are the measures of the lengths of the sides. David claims is side $\overline{CB}$ is the longest side of the right triangle. The warrant for his claim is not explicit and inferred by the researcher to be, "The side of the triangle with the largest measure is the longest side." This is the only core argument in which the data are the measures of the lengths of the sides and the warrant is not explicit.

In four of the core arguments in which the students employ technology and warrants are not explicit, the students make claims regarding the relative size of the angle which are based on the appearance of the triangle on the screen. For example, David and Erica are determining the largest angle of the isosceles triangle *ABC*. David had previously measured the side lengths using the technology, but did not measure the angles. He looks at the triangle and states, "Name the largest angle, *BCA*." This argument is illustrated in Figure 45.

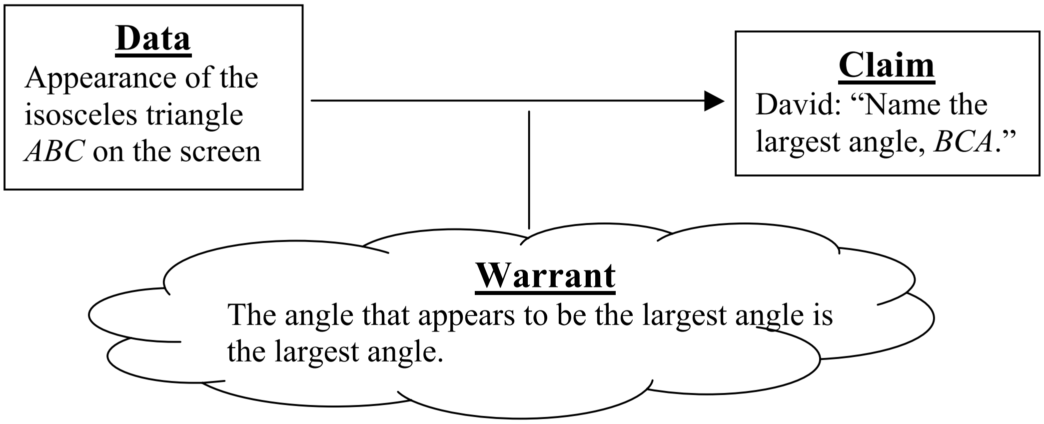


Figure 45. David’ core argument created during task 2 with an inferred warrant and the appearance of the diagram on the screen as data.

The data for this argument is the appearance of the isosceles triangle. David claims angle *BCA* is the largest angle of the isosceles triangle. The warrant for this claim is not explicit and the researcher infers it to be, “The angle that appears to be the largest angle is the largest angle.”

In these five core arguments, David and Erica do not provide explicit warrants for their claims. The pair uses technology to gather data, be it through measures or the appearance of the diagram on the screen. These arguments do not include qualifiers or rebuttals. This seems to suggest that the pair is certain about these claims.

Explicit warrants, technology used.

In two of the core arguments created by David and Erica, the students employ technology and provide an explicit warrant. For example, David and Erica are determining the longest and shortest sides of an equilateral triangle. Erica measures the lengths of sides to be $m\overline{CB} = 5.67$ cm, $m\overline{BA} = 5.67$ cm, and $m\overline{AC} = 5.67$ cm. David states surprisingly, “Oh, they're the same." Erica says, "It's because they're equal." This argument is illustrated in Figure 46.

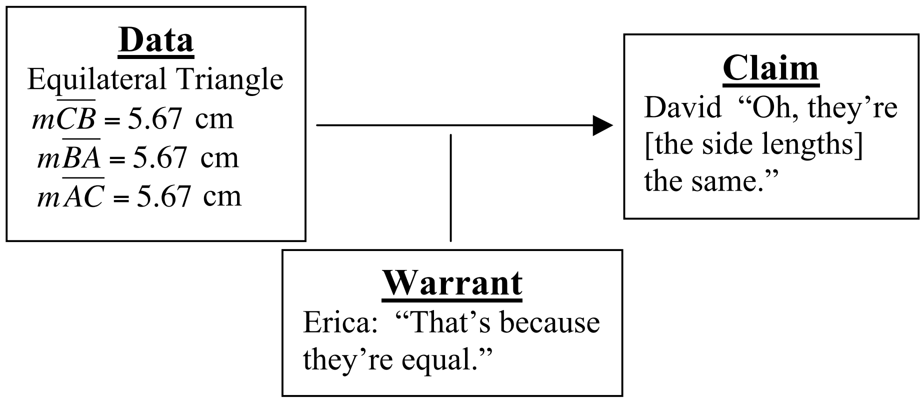


Figure 46. David and Erica's core argument created during task 2 with an explicit warrant and the measures of segment lengths as data.

The data for this argument are the measures of the side lengths, which are all 5.67 cm. David claims the side lengths are the same. Erica provides an explicit warrant indicating that the measures of the length of the sides of the equilateral triangle are equal.

David and Erica immediately follow this argument with another one of similar structure regarding the angle measures of an equilateral triangle. David and Erica are trying to determine the name of the largest angle of the equilateral triangle *ABC*. Erica states, "They [the measures of the angles] are all equal." David concurs by stating, "Yeah, everything is going to be the same." Erica follows up with, "Because it's an equilateral." This argument is illustrated in Figure 47.

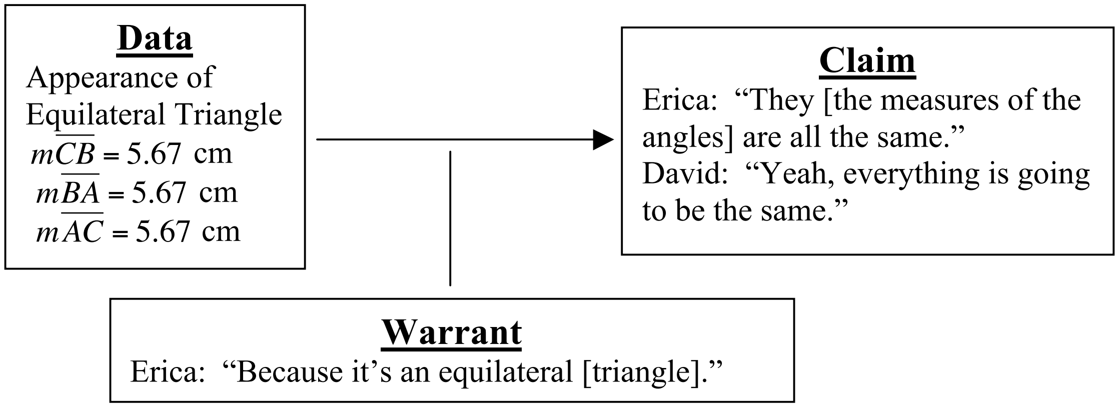


Figure 47. David and Erica’s core argument created during task 2 with an explicit warrant and the appearance of the diagram on the screen and the measures of the lengths of the segments as data.

The data for this argument are the measures of the side lengths and the appearance of the equilateral triangle on the screen. Erica claims the angle measures will be the same. It is worth noting that the pair do not measure the angles for this triangle (nor any triangle for that matter). David concurs with Erica’s claim. Erica provides an explicit warrant indicating that the measures of the angles are the same, “Because it’s an equilateral [triangle].”

For the two core arguments in which the students employ technology and provide an explicit warrant, there are two commonalities. First, the arguments do not contain qualifiers or rebuttals, which speak to the certainty of the students’ claims. Second, the content of both of these arguments is similar; the relationships among the parts of the equilateral triangle. In previous class meetings, the students were easily able to define an equilateral triangle. This suggests that when students are more familiar with the content, in this case equilateral triangles, they are more likely to provide explicit warrants, even when using technology.

Technology not used.

Of the nine core arguments created by David and Erica, two do not involve the active use of technology. For both of these arguments, the warrants are explicit. The first argument is a generalization for the relationship between the largest angle and the longest side of a triangle. David and Erica have completed the table on the task sheet. David analyzes the table and finds a pattern in the names of the longest side and the largest angle. He states, "When you go the longest side it [the largest angle] just adds a letter in the middle of it." He goes on to explain to Erica, "You know there is a pattern. First one is AB and the next one is ACB. It just adds the letter you are not using to the middle. Like, CA it adds B in the middle, CB adds A, and AB adds C in the middle." This argument is illustrated in Figure 48.

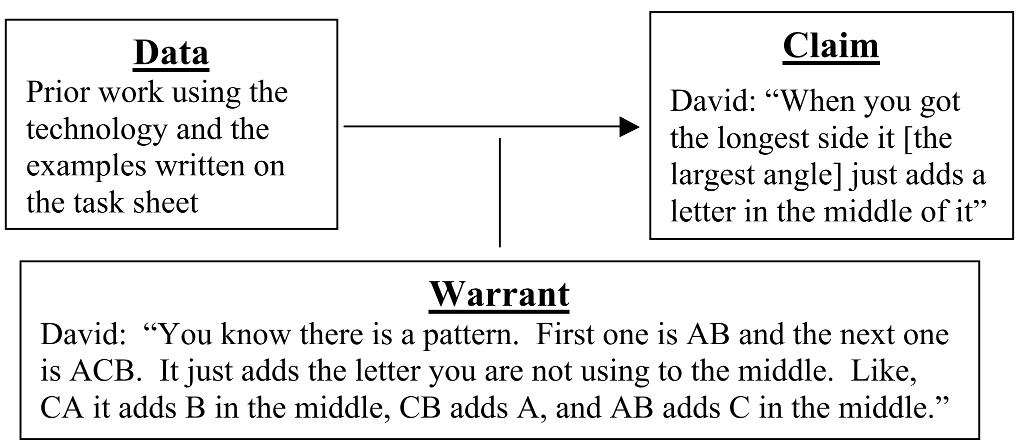


Figure 48. David and Erica's core argument created during task 2 with an explicit warrant and indirect use of technology.

The data for this argument are the examples in the table on the task sheet. David and Erica develop their responses on the task sheet based on their uses of technology. However, David is not reasoning based on his active use of the technology. Rather, he is reasoning about the product of the pair's previous uses, which is an indirect use of technology. David claims that to find the largest angle, you just add the letter missing from the name of the longest side to the middle of that name. He warrants this claim by stating the examples from the task sheet that show this pattern to be true.

Another example of a core argument in which technology is not used is in response to a question posed to the class by the teacher. The teacher asks, "If the shortest side is $\overline{BC}$, what's going to be the smallest angle?" David responds, "[angle] A [is the smallest angle]." When asked why angle A is the smallest angle, David states, "Because it's the pattern." This argument is illustrated in Figure 49.

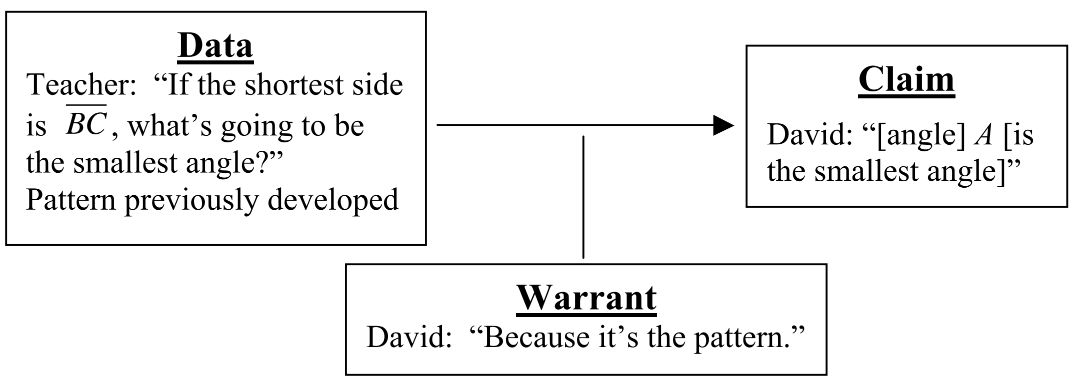


Figure 49. David's core argument created during task 2 with an explicit warrant and previously developed pattern as data.

The data for this claim are the question posed the teacher and the pattern previously developed (see prior argument). David's claim is that angle A is the smallest angle. In response to the teacher's prompt, David provides an explicit warrant, which refers to the pattern he previously developed.

In both of these core arguments, the warrants are explicit and technology is not actively used in the argument. Also, in both of these arguments, the pattern developed by David plays an important role. In the first argument, David provides the pattern and justifies it explicitly using multiple examples. In the second argument, David uses the pattern as a warrant to justify his answer in response to the teacher's inquiry. This suggests that these students are able to develop a generalization based on their uses of technology, justify the generalization, and use it to solve problems.

Conclusions.

Looking across these core arguments, two major themes emerge. First, when the students do not actively use technology, they are more likely to provide an explicit warrant. One difference between these arguments and those in which students use technology is the task on which the students are working. In the arguments in which the students employ technology, the students are determining the longest/shortest side or largest/smallest angle of a given triangle. In the arguments in which technology is not actively used, the students are making a generalization regarding the relationships between the largest angle and the longest side of a triangle or responding to a teacher's question. The students may have felt more compelled to provide an explicit warrant when making a generalization and when prompted to do so by the teacher.

The second theme that emerged from the analysis of these core arguments is when David and Erica create a core argument based on their use of technology, they are less likely to provide an explicit warrant. The students use technology in seven of the nine core arguments; two in which the claim is based on the measures of the lengths of the sides and five in which the claim is based, in part, on the appearance of the diagram on the screen. Of these seven core arguments, only two have explicit warrants. The content for both of these arguments are the relationships among the angles or sides of an equilateral triangle, a figure the students were familiar with prior to the instruction of this unit.

Arguments in which David and Erica collect additional data.

The second type of argument structure created by David and Erica is that in which the students are compelled to seek additional data after an initial claim is made to verify or refute that claim. The students' decision to seek additional data may be due to a number of factors including the ways in which the students use technology to collect the initial data and an explicit challenge to the claim.

Appearance/measures.

At times, David or Erica makes claims based on the appearance of the diagram on the screen and the uncertainty of that claim prompts additional data collection via measures. For example, David and Erica are determining the longest side of isosceles triangle *ABC*. Erica has the isosceles triangle maker on the screen and David states, "Name the longest side, $\overline{BA}$. It's got to be $\overline{BA}$." Erica measures the lengths of the sides and finds that $m\overline{CB} = 2.92$ cm, $m\overline{BA} = 4.74$ cm, and $m\overline{AC} = 2.92$ cm. David reaffirms himself stating, "*BA*, just like I said." This argument is illustrated in Figure 50.

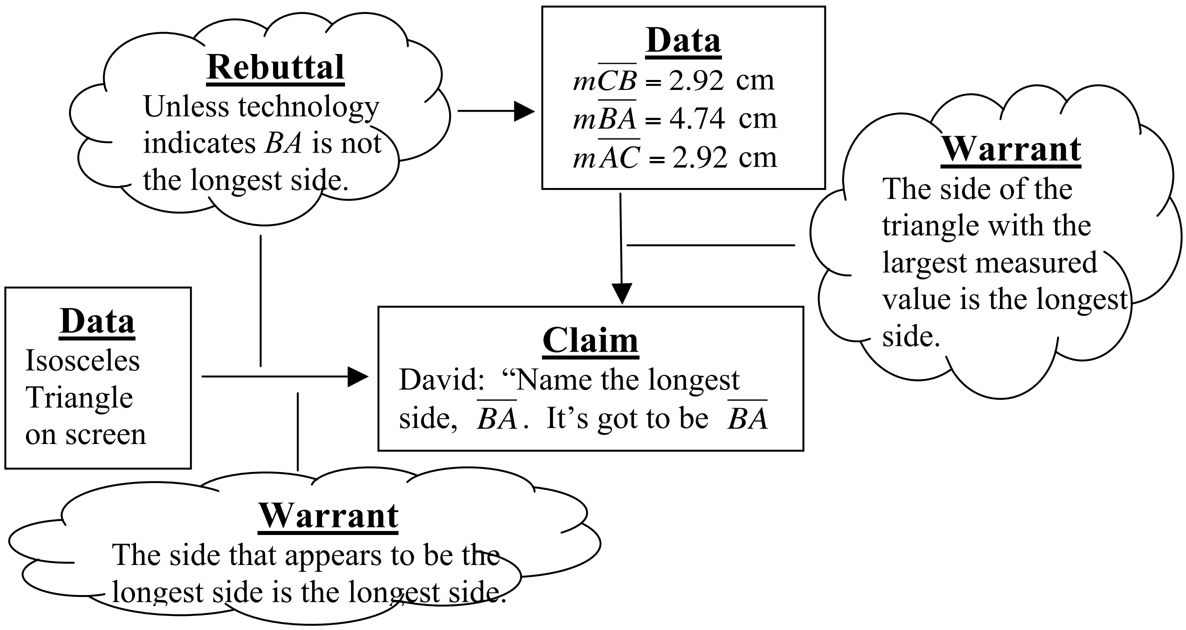


Figure 50. David and Erica's argument created during task 2 with additional data collection in which the initial data is the appearance of the diagram on the screen created and verifies the initial claim.

In this argument, the initial data is the appearance of the isosceles triangle on the screen. David claims the name of the longest side is $\overline{BA}$. He does not provide an explicit warrant for his claim and is inferred by the researcher to be "The side that appears to be the longest side is the longest side." Then, Erica measures the lengths of the sides of the isosceles triangle. By gathering additional data, Erica demonstrates her uncertainty with the claim. David verifies his initial claim. David does not provide an explicit warrant for this claim and the researcher infers it to be "The side of the triangle with the largest measured value is the longest side."

Another example of an argument in which the original claim is based upon the appearance of the diagram on the screen is one in which the students are to determine the longest side of a right triangle. Similar to the previous argument, David looks at the right triangle on the screen and claims, "$\overline{BA}$ [has] got to be the longest [side]." Erica measures the lengths of the sides and finds that $m\overline{CB} = 2.04$ cm, $m\overline{BA} = 6.58$ cm, and $m\overline{CB} = 6.89$ cm. Seeing these measures, David states, "Oh, it's $\overline{AC}$." This argument is illustrated in Figure 51.

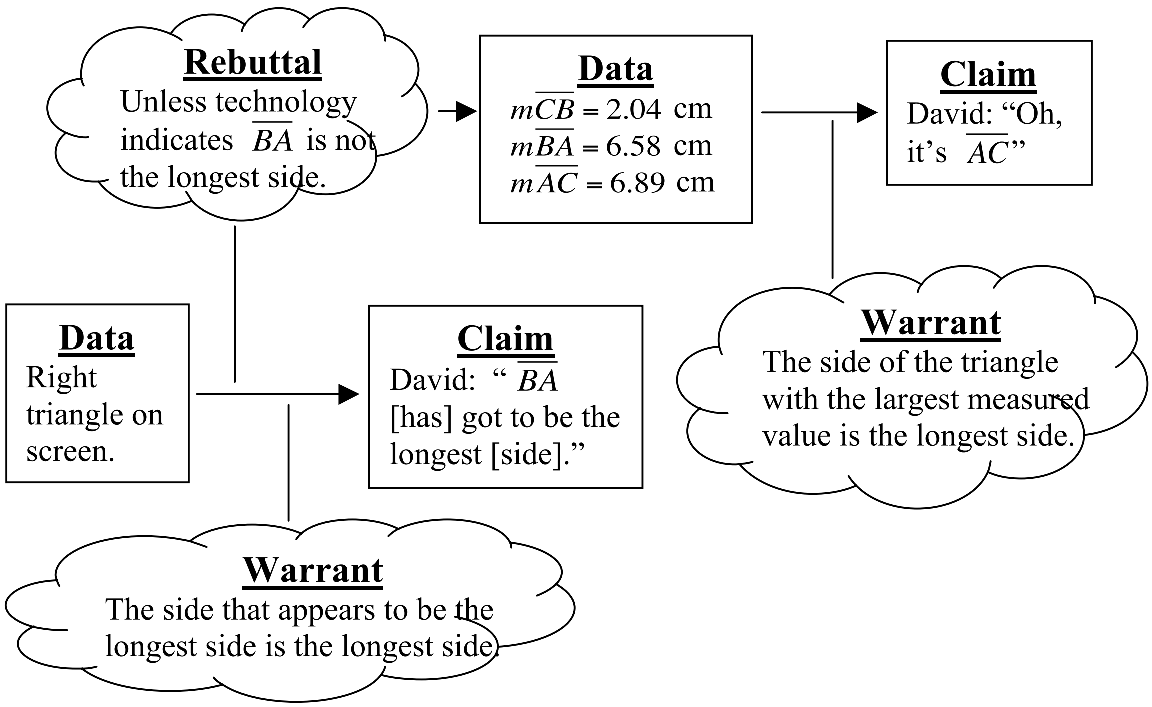


Figure 51. David's argument created during task 2 with additional data collection in which the initial data is the appearance of the diagram on the screen and a new claim is made based on the additional data collected using the technology.

The initial data for this argument is the appearance of the right triangle on the screen. Based on this data, David claims side $\overline{BA}$ is the longest side. He does not provide an explicit warrant for his claim and is inferred by the researcher to be, "The side that appears to be the longest side is the longest side." Erica measures the lengths of the sides of the right triangle. By gathering additional data, Erica demonstrates her uncertainty of the claim. David does not verify his initial claim. Instead, David provides a new claim; side $\overline{AC}$ is the longest side. He does not provide an explicit warrant for this claim and the researcher infers it to be, “The side of the triangle with the largest measured value is the longest side.”

Both of these arguments have very similar content, but slightly different structures. In both of arguments, David makes an initial claim based on the appearance of the triangles on the screen. He does not provide an explicit warrant linking his data and claim, and is inferred by the researcher to be the same for both arguments. Erica measures the length of the sides and David makes a claim based on these measures. The warrant for this claim is the same for both arguments. The difference between these arguments lies in whether he verifies his initial claim or makes a new claim based on the additional data. This suggests that while David was willing to make claims based on the appearance of the diagram on the screen, both he and Erica valued the measures as the basis for their final claims.

Challenges.

In three of the arguments, the collection of additional data is prompted by an explicit challenge from another person. For example, David and Erica are reviewing their task sheet verifying the generalization that David created. On the task sheet, David has written that the shortest side is $\overline{AB}$. Erica opens the obtuse triangle on the screen, which includes the measures of the lengths of the sides. Erica exclaims, “No it’s not. That’s wrong. That’s wrong.” She asks David, “Look at the measurements, $\overline{AC}$.” David affirms this claim stating, “That’s right.” This argument is illustrated in Figure 52.

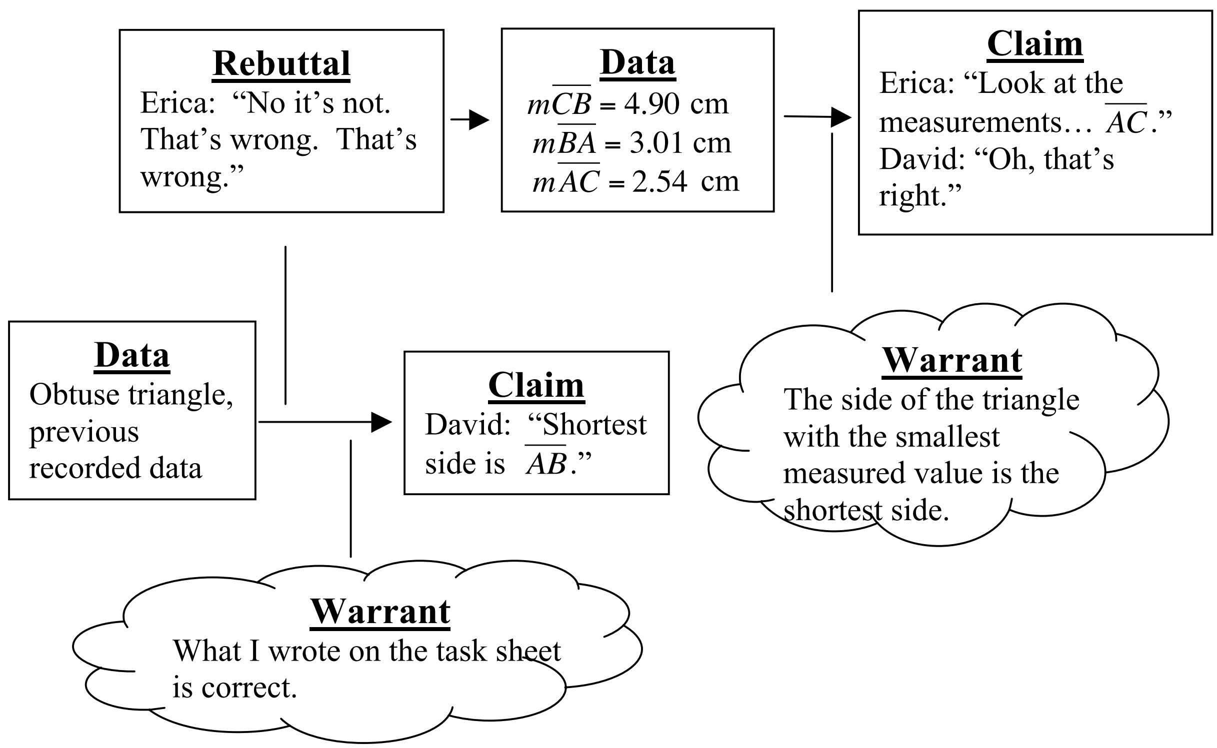


Figure 52. David and Erica's argument created during task 2 with additional data collection in which Erica challenges the initial claim.

The initial data for this argument is the recorded information on the task sheet. David claims the shortest side is $\overline{AB}$. He does not make an explicit warrant for this claim and the researcher infers it to be, "What I wrote on the task sheet is correct." Erica challenges this claim by indicating David is wrong; side $\overline{AB}$ is not the shortest side. This challenge is an explicit rebuttal. She opens the obtuse triangle tab that includes the measures of the lengths of the sides. She claims that the shortest side is $\overline{AC}$. David concurs with this claim. The students do not provide an explicit warrant and the researcher infers it to be, "The side of the triangle with the smallest measured value is the shortest side."

Another example of an argument in which Erica challenges David's initial claim is one in which David is uncertain whether the obtuse triangle on the screen is indeed obtuse. On the screen are the obtuse triangle and the measures of the lengths of the sides, $m\overline{CB} = 4.90$ cm, $m\overline{BA} = 3.01$ cm, and $m\overline{AC} = 2.04$ cm. David states, "That's not obtuse." Erica responds, "Yes it is." David says, "It's not above [90]." Erica reaffirms her statement, "Yes it is, it's obtuse." She uses the mouse to trace angle *A*, the obtuse angle. David agrees stating, "That part is but the rest isn't. This argument is illustrated in Figure 53.

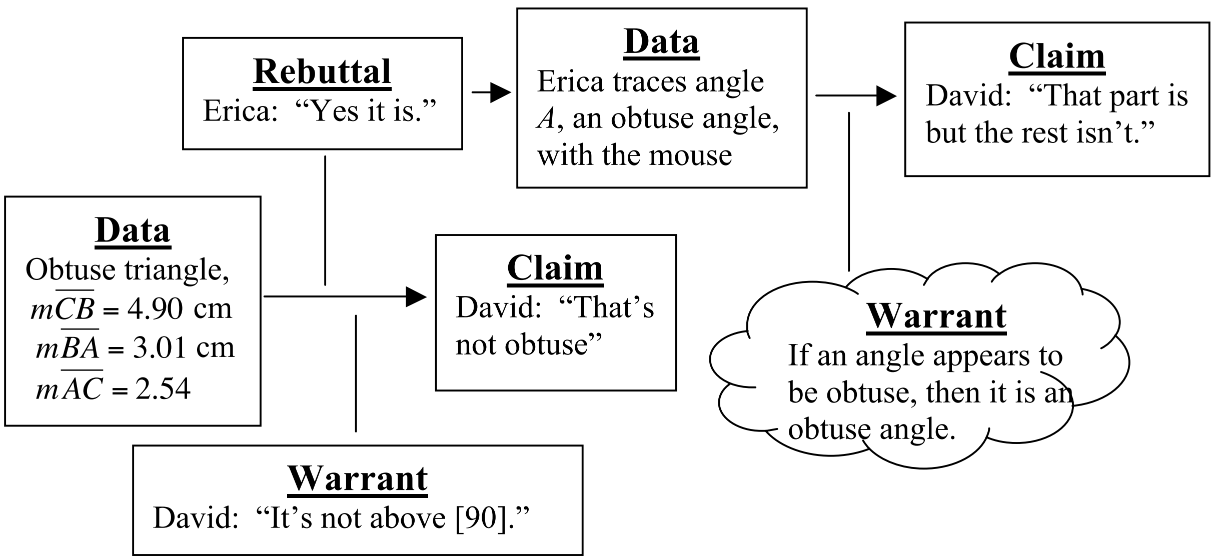


Figure 53. David and Erica's argument created during task 2 with additional data collection in which Erica challenges David's initial claim and uses the mouse as a pointer to collect additional data.

In this argument, the initial data is the appearance of the obtuse triangle on the screen and the measures of the side lengths. David claims that the triangle is not obtuse and warrants this claim by indicating that none of the measures are greater than ninety. Erica challenges his claim, which is a rebuttal. She collects additional data by using the mouse to trace obtuse angle. Seeing this, David makes a new claim the angle is obtuse. He does not provide a warrant for this claim and is inferred by the researcher to be "If an angle appears to be obtuse, then the it is an obtuse angle."

A third example of an argument in which Erica challenges David's initial claim is one in which the students are determining whether the order of the letters in the name of a side matters. Erica and David are working with the isosceles triangle. Erica measures the sides and finds $m\overline{CB} = 2.92$ cm, $m\overline{BA} = 4.74$ cm, and $m\overline{AC} = 2.92$ cm. David claims, "BC is the shortest. Erica asks, "Wouldn't it be $\overline{BC}$? Does it matter which way you put it?" Uncertain, David decides to ask the teacher. When he is unable to get the teacher's attention, he asks a fellow student, "When you're doing the shortest one does it matter which letter you put first?" The student responds, "No, as long as the letter is" and David finishes the statement "represents the line." On the task sheet, David writes $\overline{BC}$ as the shortest side. This argument is illustrated in Figure 54.

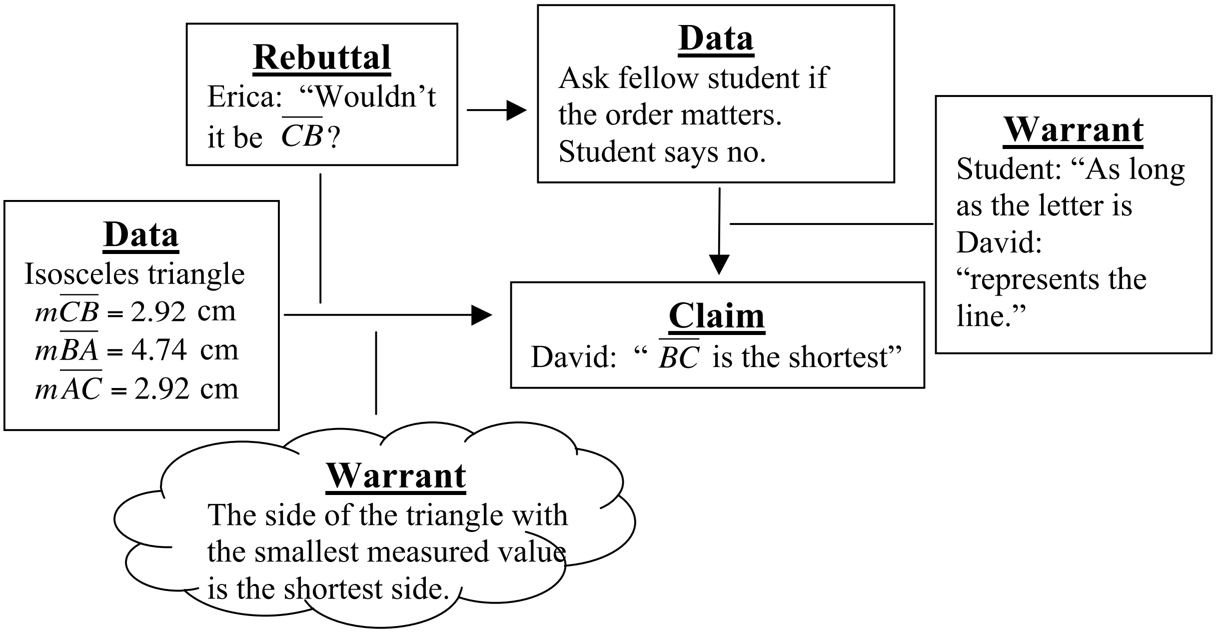


Figure 54. David and Erica's argument created during task 2 with additional data collection in which Erica challenges David's initial claim and additional data is collected by asking a peer.

The data for this argument are the appearance of the isosceles triangle on the screen and the measures of the lengths of the sides of the isosceles triangle. David claims side *BC* is the shortest side. He does not provide an explicit warrant for his claim and the researcher infers it to be, "The side of the triangle with the smallest measured value is the shortest side." Erica does not challenge the validity of the claim, rather whether the side can be written in this fashion because the technology named the side $\overline{CB}$, not $\overline{BC}$. David's uncertainty regarding Erica's challenge leads him to collect additional data by asking a fellow student whether the order of the letters of the names of side matters. The student says no which verifies David's initial claim. David and the other student provide an explicit warrant when they say, "As long as it represents the side."

Conclusions.

The arguments in which David and Erica collect additional data can be categorized into two types; those in which a claim is made based on the appearance of the figure and measures are collected as additional data to verify or refute the claim, and those in which one of the pair makes a claim and the other challenges this claim. Not only do the content and structure of these arguments differ, but so do the tasks on which the students are working. For the arguments categorized as appearance/measures, the students are attempting to find the longest or shortest sides of a triangle. For two of the three arguments categorized as challenges, the students are working on different tasks. During one argument, the students are classifying triangles. During the other argument, the students are trying to determine how to name a side of a triangle. This suggests that the nature of the task may be connected to whether it prompts a challenge.

Discussion.

While working on the triangle side and angle relationships activity, David and Erica create arguments of various structures. Two categories of structures are noted in the analysis: core arguments, and arguments in which one of the pair of students are compelled to collect additional data to verify or refute a claim. Looking across these argument structures, two themes emerge; the lack of explicit warrants (see Table 11), and the ways in which the students use the technology.

Table 11

Group 2's Argument on the Triangle Side and Angle Relationship Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	5	3
	Technology Not Used	0	0
Warrants Explicit			
	Technology Used	2	2
	Technology Not Used	2	0

Of the fourteen arguments created by David and Erica, only six contain an explicit warrant, of which four are core arguments. The explicitness of the warrant may be related to the task and the students' uses of the technology. For the majority of the arguments, the students are engaged in the task of determining the largest/smallest angle or longest/shortest side of a triangle and use the technology to create measures as data for their claims. For two of the arguments in which the students provide an explicit warrant, the students are developing a generalization and do not actively use the technology. For three of the arguments that contain an explicit warrant, the students are determining the longest/shortest side or largest/smallest side of a triangle and use technology. However, for two of the arguments, the students are working with the equilateral triangle, an object they are able to define at the beginning of the unit. For the other argument in which an explicit warrant is provided and technology is used, the students are classifying a triangle. This suggests that when students are working on more complex tasks or with familiar objects, they are more likely to provide explicit warrants.

During the triangle side and angle relationships activity, David and Erica use technology to accomplish a variety of tasks. For all but two of the arguments, they use the software's measurement tool. However, they only measure the lengths of the sides. To determine the largest or smallest angle, the students base their claims on the appearance of the diagram on the screen. In addition, the students do not qualify or provide rebuttals for the majority of their arguments. In fact, they never qualify a claim. In addition, the pair only provides explicit rebuttals when challenging the other's claim. The lack of qualifiers and explicit rebuttals to their own claims suggests that the students were fairly certain about their claims. The frequent use of technology coupled with the lack of qualifiers and rebuttals suggests that the students view the technology as a reliable tool. This viewpoint could also be attributed to teacher's request for the students to use the tool to complete the task. The students may view the tool as reliable because the teacher would not provide the students a faulty tool.

Group 3's arguments on the triangle side and angle relationship task.

The analysis of Amy and Judy's arguments while working on the triangle side and angle relationships task can be categorized into three argument structures; core arguments, arguments in which the students collect additional data to verify or refute a claim, and arguments where one of the pair of students challenges the other student's claims. These argument structures are discussed below.

Core arguments.

During the triangle side and angle relationships activity, Amy and Judy create eight core arguments. For five of these arguments, technology is used and the warrants are not explicit. In one core argument, the students use technology and the warrant is explicit. For the remaining two core arguments, technology is not actively used. These arguments are detailed below.

Non-explicit claims, technology used.

For the majority of the core arguments Amy and Judy use the technology to create data and, based on this data, make claims. However, their warrants for these claims are usually not explicit. In all of these arguments, the data takes the form of measures and the appearance of the triangle on the screen. For example, Amy and Judy are determining the shortest side of the obtuse triangle *ABC*. Judy measures the lengths of the sides. Amy states, "The shortest side would be $\overline{CA}$." This argument is illustrated in Figure 55.

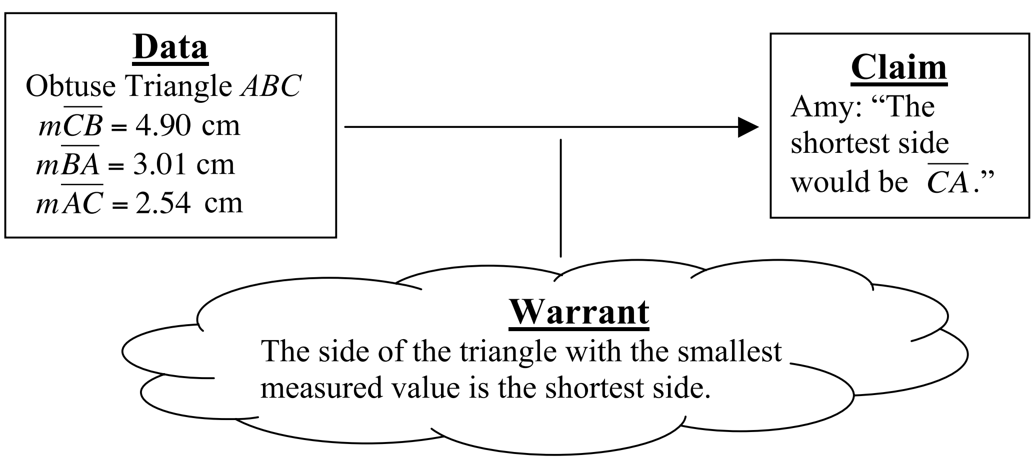


Figure 55. Amy's core argument created during task 2 with an inferred warrant and the measures of the segment lengths as data

The data for this argument are the measures of the lengths of the sides of the obtuse triangle. Amy claims side $\overline{CA}$ is the shortest side of the obtuse triangle. The warrant for her claim is not explicit and is inferred by the researcher to be, "The side of the triangle with the smallest measured value is the shortest side."

Amy creates another example of a core argument in which she employs technology in the form of measures and the warrant is not explicit. The pair is determining the largest angle of the obtuse triangle *ABC*. Amy measures the angles and states, "It's [angle] *A*." This argument is illustrated in Figure 56.

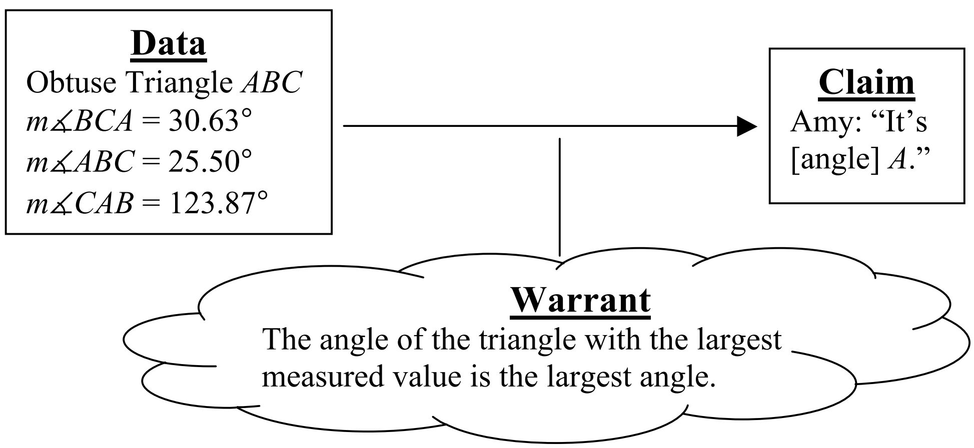


Figure 56. Amy's core argument created during task 2 with an inferred warrant and angle measures as data.

The data for this argument are the measures of the angles of the obtuse triangle. Amy claims angle *A* is the largest angle. The warrant for her claim is not explicit and is inferred by the researcher to be, "The angle of the triangle with the largest measured value is the largest angle."

Of the eight core arguments, five have this structure and similar content. In fact, for all five of these arguments, the task on which the students are working is to determine the largest/smallest angle or longest/shortest side of a triangle. In all of these arguments the students use the technology to measure parts of the triangle and use these measures as data. This seems to suggest that the structure of an argument may be related to the way in which technology is used and the task on which the students are working.

Explicit warrants, technology used.

Amy and Judy provide one notable exception to the above structure and content as they work with the equilateral triangle. In this argument, Judy has the equilateral triangle on the screen and measures the three angles whose measures are all sixty degrees. Judy claims, “It’s all the same.” Amy concurs and states, “That’s because they’re all equilateral.” This argument is illustrated in Figure 57.

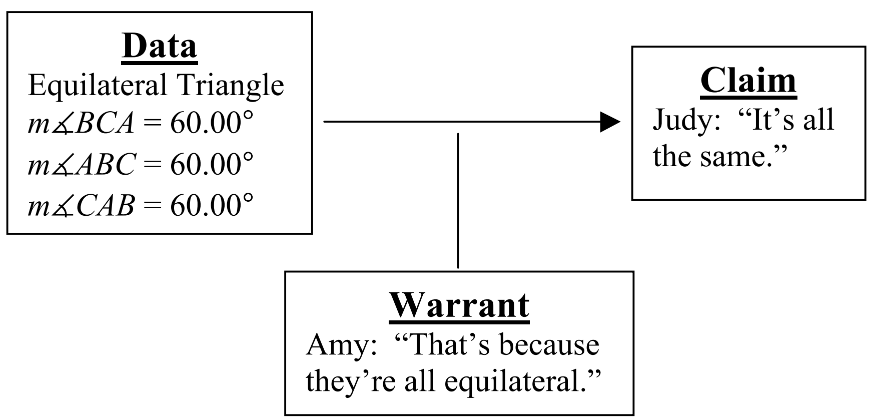


Figure 57. Amy and Judy core argument created during task 2 with an explicit warrant and angle measures as data.

The data for this argument are the equilateral triangle on the screen and the measures of the angles. Judy claims all the angles are the same, or, in other words, all the angles have equal measure. Amy provides an explicit warrant indicating that the reason the angles are all the same is because the triangle is equilateral. Even though the students' argument would not be considered mathematically correct, one can still see the students provide a justification for their claim that is not based entirely on their use of technology. Rather, the justification is based on their knowledge of equilateral triangles. This suggests that the students are able to make claims based on their uses of technology, but provide reasons for why these claims are true using a non-technology based justification.

Technology not used.

In two of the eight core arguments, the students do not actively use technology in the creation of the argument. For one of these core arguments, the students do not explicitly provide the warrant. In this argument, Judy and Amy are determining the relationship between the name of the longest side and name of the largest angles of a triangle. Amy notes, "Ok see how it's AB, AB. BC or CB, either way you know CB. AB, AB and AC, AC." What she notices is that the letters in the name of the longest side are the outside letters on the name of the largest angle. Attempting to generalize this pattern, Judy states, "The longest side is always on the side of the angle." Amy concurs by stating, "$\overline{AC}$ is the longest side and ABC is the largest angle." This argument is illustrated in Figure 58.

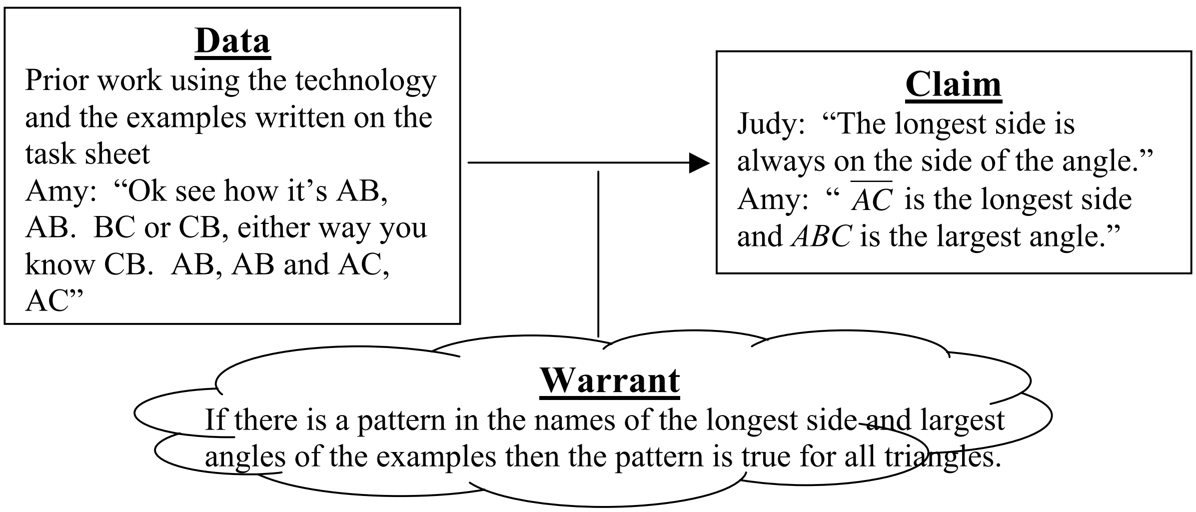


Figure 58. Amy and Judy's core argument created during task 2 with an inferred warrant and their responses on the task sheet as data.

In this argument, both Judy and Amy make similar claims related to the relationship between the longest side and the largest angle of a triangle. The data for this claim are the examples created by Amy and Judy on the task sheet and the pattern noticed by Amy. Even though the examples on the task sheet are created using the technology, specifically the measurement tool, the students do not actively employ the technology in this argument. Instead, the students use the products of their previous uses of technology which is an indirect use of technology. The students do not provide an explicit warrant for their claim and is inferred by the researcher to be "If there is a pattern in the names of the longest side and largest angles of the examples then the pattern is true for all triangles."

In the second core argument in which technology is not actively employed, the students provide an explicit warrant for their claim. In this argument, Amy and Judy are determining the reason why the longest side of a triangle is across from its largest angle. Judy states, "The longest side stretches out the angle." While she does this, Amy places her two hands such that bottom of the palms are close to each other and the fingers are away from each other forming a V. She moves her fingers away form each other widening the V and states, “Ok if its stretching out it makes the angle, here’s the angle right where my hand cups, is it getting larger?” Judy says, “Yeah, it makes it out, an angle would make it larger.” This argument is illustrated in Figure 59.

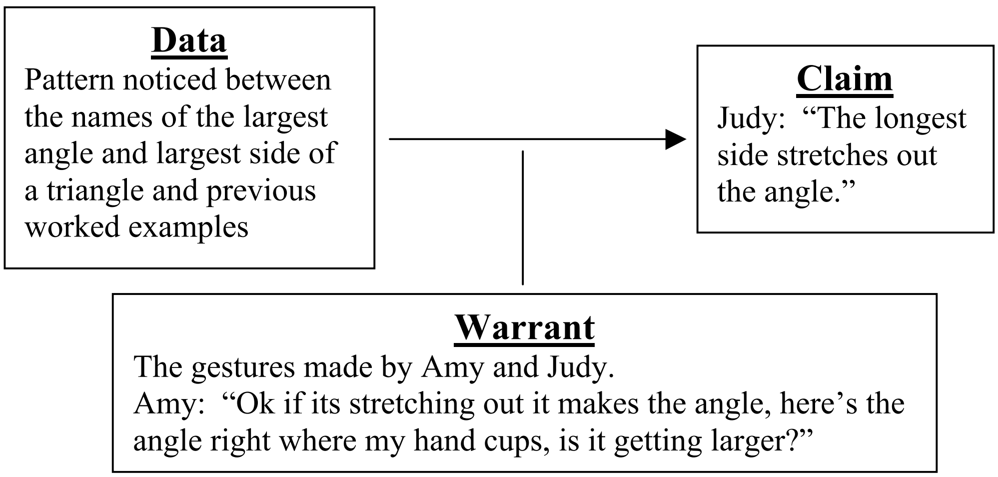


Figure 59. Amy and Judy’s core argument created during task 2 with an explicit warrant and a discovered pattern as data.

The data for this argument are the pattern noticed between the names of the largest angle and longest side and the previous examples. Judy claims that that the longest side stretches out the longest angle. Amy provides an explicit warrant for this claim using gestures. She cups her hands such that they make two sides of triangle and imagines the third side connecting her fingertips. As she moves her fingers away from each other, she indicates that the side is getting larger and so are the angles.

In both of these arguments, technology is not actively used, although it did play an indirect role. In the first argument, the students are working with the products of their uses of technology to develop a generalization. In the second argument, the students are determining why that generalization is true. These examples suggest that perhaps when students are working on a task to develop a generalization, they are not likely to use technology.

Conclusions.

Looking across the eight core arguments created by Amy and Judy while working on the triangle side and angle relationship task, one theme emerges. When students do not actively employ technology, they are more likely to provide an explicit warrant. Of the six core arguments in which technology is used, only one warrant is explicit. However, only one of the two arguments that do not actively employ technology has an explicit warrant. This may be related to the task on which the students are working. In the arguments that employ technology, the students are determining the longest/shortest side or the largest/ smallest angle of a triangle. In the arguments in which technology is not actively used, the students are attempting to make a generalization or justify that generalization. This suggests that when students are working on task involving a generalization, they are more likely to provide explicit warrants.

Arguments in which Amy and Judy collect additional data.

The second type of argument structure created by Amy and Judy is that in which the students seek additional data after an initial claim is made to verify or refute that claim. The students' decision to seek additional data may be due to the uncertainty of the claim.

At times, Amy and Judy make claims based on the appearance of the diagram on the screen and the uncertainty of that claim prompts additional data collection via measures. In one example Judy and Amy are attempting to find the longest side of obtuse triangle *ABC*. Judy clicks on the obtuse triangle tab and, without taking any measurements, Amy states, "The longest side would be CB." Judy responds, "You don't know that." She measures the lengths of the sides and finds $m\overline{CB} = 4.10$ cm, $m\overline{BA} = 3.10$ cm, and $m\overline{AC} = 2.54$ cm. Amy concludes, "It's $\overline{BC}$." This argument is illustrated in Figure 60.

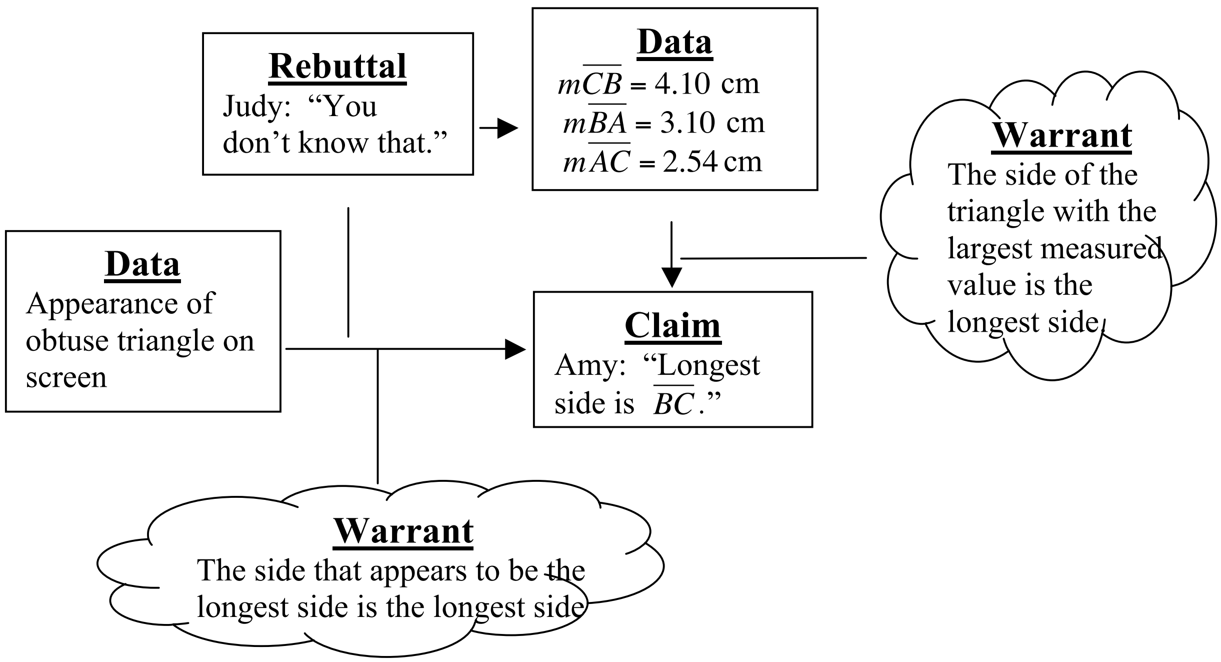


Figure 60. Amy and Judy's argument created during task 2 with additional data collection and the initial data is the appearance of the diagram on the screen.

In this argument, the initial data is the appearance of the obtuse triangle on the screen. Based on the appearance of this triangle, Amy makes the claim side $\overline{BC}$ is the longest. She does not provide an explicit warrant for this claim and is inferred by the researcher to be, "The side that appears to be the longest side is the longest side." Judy displays her uncertainty with the clam by providing an explicit rebuttal. Judy collects additional data in the form of measures. Based on these measures, Amy concludes that her initial claim was indeed correct. Amy does not provide an explicit warrant for this claim and the researcher infers it to be, "The sides of the triangle with the largest measured values is the longest side." Another example of an argument in which Amy and Judy collect additional data is one in which they are determining the longest side of an equilateral triangle. On the screen are the equilateral triangle, the measures of the angles ($m\measuredangle BCA = 60.00°$, $m\measuredangle ABC = 60.00°$, and $m\measuredangle CAB = 60.00°$) and the measure of one side length ($m\overline{CB} = 5.67$ cm). Amy claims, "They're all equal." Judy states, "They should be, they're equilateral." She proceeds to measure the two other side lengths and finds $m\overline{AC} = 5.67$ cm and $m\overline{BA} = 5.67$ cm. Judy verifies her claim by stating, "All right." This argument is illustrated in Figure 61.

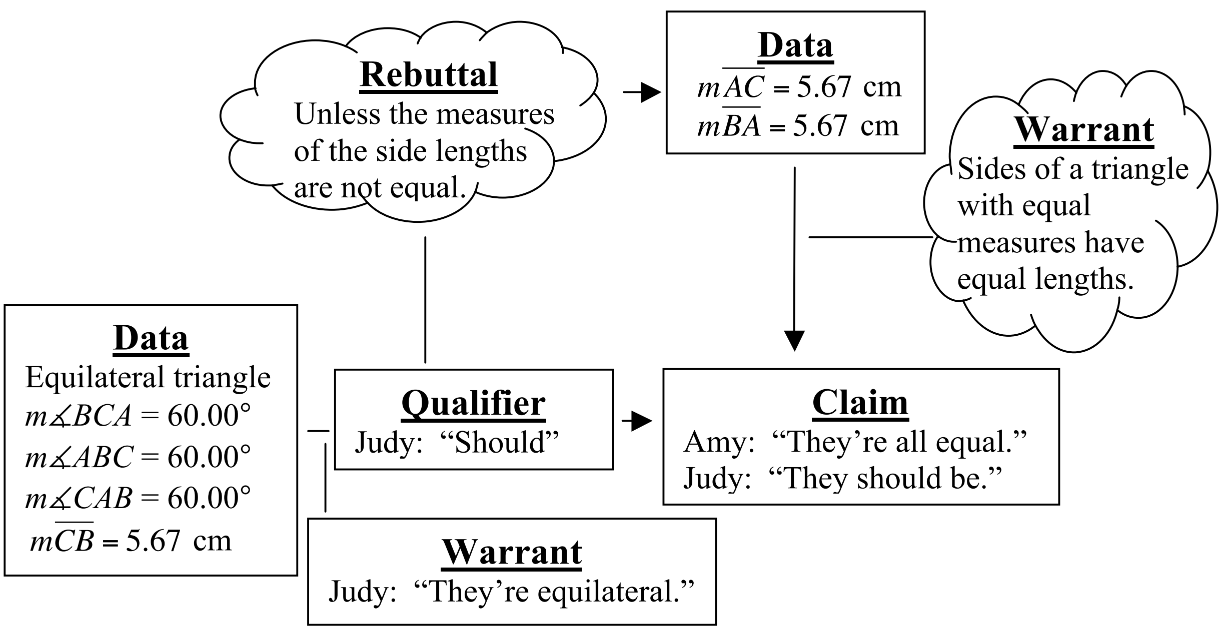


Figure 61. Amy and Judy's argument created during task 2with additional data collection with angle measures and the measure of one side length as the initial data.

The initial data for this argument are the equilateral triangle on the screen, the measures of the angles, and the measure of one of the side lengths. Both Amy and Judy claim that the measures of the side lengths are equal. Judy provides an explicit warrant for this claim by stating that the lengths of the sides are equal because it is an equilateral triangle. However, Judy demonstrates some doubt with this claim though the use of a qualifier, "should." However, she does not provide an explicit rebuttal. The researcher infers the rebuttal to be, "Unless the measures of the side lengths are not equal." Judy collects additional data in the form of the missing measures and verifies the pair's initial claim. Judy does not provide an explicit warrant for this verification and is inferred by the researcher to be, "Sides of a triangle with equal measures have equal lengths."

In both of these arguments, the decision to collect additional data is due, in part, to the uncertainty held by Judy. In the first argument, her uncertainty is with the data used to make the initial claim, the appearance of the obtuse triangle. She provides an explicit rebuttal stating that Amy could not make that claim based on that data. In the second argument, Judy uses a qualifier to indicate her uncertainty. Unlike the first argument, she does not explicitly state the claim could not be made. Instead, she agrees with the claim, but expresses some uncertainty, even though she provides an explicit warrant for why the side lengths of an equilateral triangle should be equal. This suggests, at least for Judy, that certainty for these tasks is established through the use of technology, specifically measures.

The ways in which technology is used to collect additional data in these arguments are similar; the students measure the lengths of the sides. However, the data used as the basis for the initial claim is much different which may have influenced the structure of the argument. In the first argument, the initial claim is based solely on the appearance of the obtuse triangle on the screen. This is followed by an explicit rebuttal and additional data collection. In the second argument, the students measure parts of the equilateral triangle prior to making their claim. An explicit rebuttal was not provided. Instead, the uncertainty of the initial claim is expressed in a qualifier. This suggests that the level of uncertainty with the claim, and the way it is expressed, may be related to the data; in these cases, the way in which technology is used.

Challenges.

The third category of arguments created by Amy and Judy during this task is one in which an explicit challenge to a claim is provided by one of the members of the group. Only one argument of this structure is made, but it is worth noting due to its distinct structure. In this argument, Amy and Judy are developing a generalization for the relationship between the longest side and largest angle of a triangle. The students had recorded their findings on the task sheet. Judy states, "They're always *A, B*." Amy responds, "But that's *C* [the largest angle of one of the triangles]." Judy adjusts her claim stating, "I'm saying the majority is [*AB*]. See *AB*, it's *AB.* There's an *AB*. There's an *AB*. There's an *AB*." Amy responds, "That's because it includes the whole thing; that's because you have to put every letter in there for the angle." Judy agrees with Amy but justifies her previous claim stating, ""Yeah, but still like here majority is *C* for the shortest. Here, it's *A*, *B*, or *A* whatever. I say *A*, *B* because it's the majority." This argument is illustrated in Figure 62.

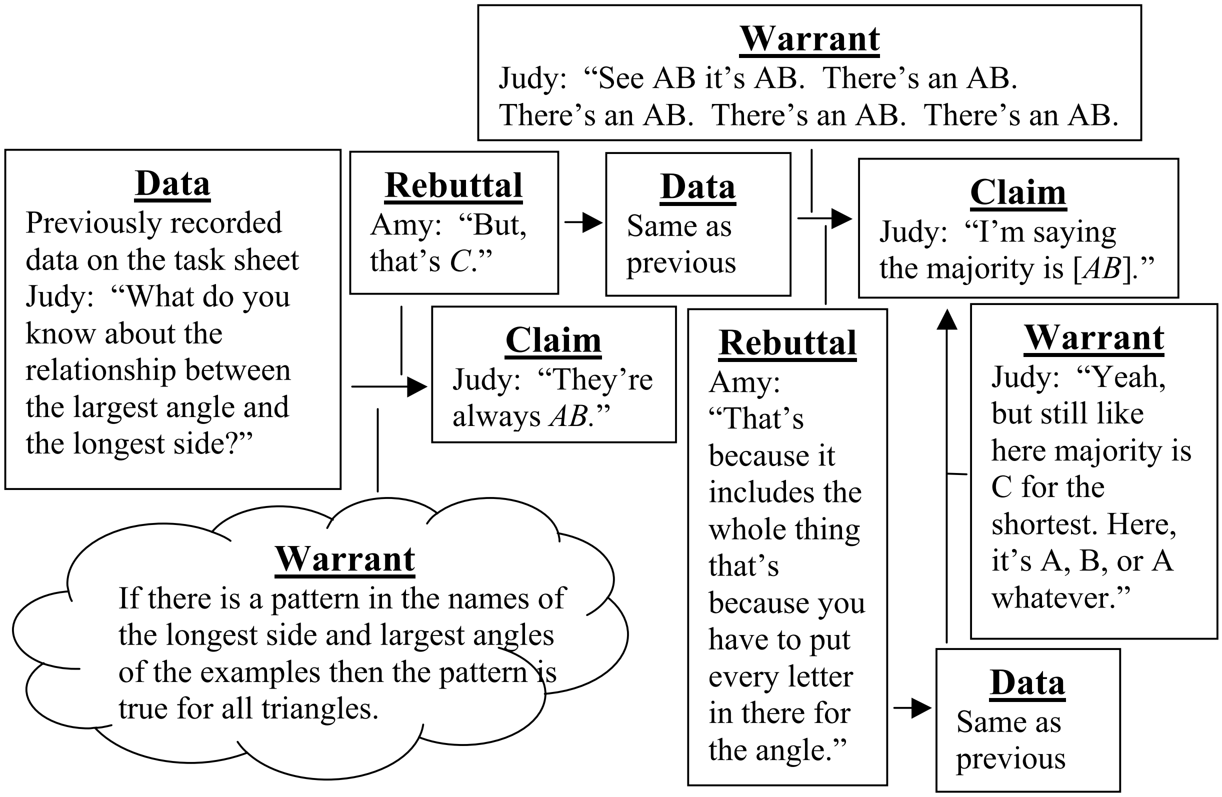


Figure 62. Amy and Judy's argument created during task 2 with additional data collection in which Amy challenges Judy's initial claim.

The initial data for this argument are the previously recorded data on the task sheet and the task on which the students are working. Judy provides an initial claim that the largest angle and the longest side always have *AB*. She does not provide an explicit warrant and is the researcher infers it to be, "If there is a pattern in the names of the longest side and largest angles of the examples then the pattern is true for all triangles." Amy provides an explicit rebuttal for this claim by pointing out a case in which neither angles *A* or *B* are the largest angle. Additional data is not collected. Instead, Judy adjusts her claim by using the term "majority." She provides an explicit warrant by using multiple examples on the task sheet in which *AB* is part of the largest angle and/or the longest side. Amy challenges this warrant in the form of a rebuttal by indicating that the angle will always have the letters A and B because you have to use all the letters when naming an angle of a triangle. Judy agrees with Amy, but verifies her original claim. She bases this claim on the same data used to make the initial claim. Judy provides an explicit warrant for this verification by indicating that C is used for the majority of the cases for the smallest angle and shortest side and that A and B are used the majority of the time in the others.

In this argument, Amy provides two challenges to Judy's claims. In the first challenge, Amy offers a counter-example to Judy's initial claim. Judy responds by adjusting her initial claim to allow for this counter-example. She also presents a justification for her new claim in the form of multiple examples. Amy offers another challenge by stating that the reason these examples appear to be valid is due to the ways in which they are named. Judy counters with more examples of why her new claim is valid. The willingness of Judy to use multiple examples as warrants for her claims suggests that she values empirical evidence.

Discussion.

While working on the triangle side and angle relationships activity, Amy and Judy create arguments of various structures. Three categories of structures emerge in the analysis: core arguments, arguments in which the students collect additional data to verify or refute a claim, and arguments where one of the pair of students challenges the other student's claims.. Looking across these argument structures, two themes emerge, the lack of explicit warrants (see Table 12), and the ways in which the students use the technology.

Table 12

Group 3's Argument on the Triangle Side and Angle Relationship Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	5	1
	Technology Not Used	1	0
Warrants Explicit			
	Technology Used	1	3
	Technology Not Used	1	1

Of the thirteen arguments created by Amy and Judy, only six contain an explicit warrant, two of which are core arguments. In addition, all of the non-core arguments have at least one non-explicit warrant. The explicitness of the warrant may be related to the task and the students' uses of the technology. For the majority of the arguments, the students are engaged in the task of determining the largest/smallest angle or longest/shortest side of a triangle and use the technology to create measures as data for their claims. For two of the arguments in which an explicit warrant is provided, the students are developing a generalization and did not actively use the technology. For the other four arguments that contain an explicit warrant, the students are determining the longest/shortest side or largest/smallest side of a triangle. However, for two of the arguments, the students are working with the equilateral triangle, an object they were able to define at the beginning of the unit. This suggests that when students are working on more complex tasks or with familiar objects, they are more likely to provide explicit warrants.

During the triangle side and angle relationships activity, Amy and Judy use technology to work on a variety of activities. They use the software's measurement tool to collect data in all of their arguments. However, they also make claims based on the static appearance of the figure. In addition, the students do not qualify or provide rebuttals for the majority of their arguments. The lack of qualifiers and rebuttals suggests that the students may have been fairly certain about their claims. The frequent use of technology coupled with the lack of qualifiers and rebuttals suggests that the students view the technology as a reliable tool. This viewpoint could also be attributed to teacher's request for the students to use the tool to complete the task. The students may view the tool as reliable because the teacher would not provide the students a faulty tool.

Cross-case analysis of arguments created while working on the triangle side and angle relationship task.

On the triangle side and angle relationship task, the arguments created by the three groups of students vary in their structure and content, including the ways in which they employ the technology. Three themes emerge when looking across the arguments created by these students; the structure of the argument when technology is and is not actively employed (see Table 13), the relationship between the type of task on which the students are working and the content of the argument, and the collection of additional data and the way in which technology is used to collect additional data.

Table 13

The Combined Arguments of the Three Groups on the Triangle Side and Angle Relationship Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	27	4
	Technology Not Used	2	0
Warrants Explicit			
	Technology Used	4	7
	Technology Not Used	5	1

While working on the triangle side and angle relationships activity, the three groups of students create 50 arguments. For 33 arguments, the students do not provide explicit warrants. Additionally, the students employ technology in 42 arguments. Of the 42 arguments in which technology is employed, the students only provide an explicit warrant in 11 of these arguments (26%). Of the 8 arguments in which technology is not actively employed, the students provide an explicit warrant for 6 of these arguments (75%). This disparity seems to suggest that when students use technology, they are less likely to provide an explicit warrant compared to when they do not actively use technology. It is worth noting that of the 11 arguments in which the students employ technology and provide an explicit warrant, the content of 4 of the arguments focuses on the equilateral triangle (see the arguments illustrated in Figures 46, 47, and 57), a figure that the students are able to define at the beginning of the unit. The students' use of technology and the explicitness of the warrant for arguments about the equilateral triangle suggests when the students are familiar with the definition of and theorems and properties related to a figure, they are able to make claims based on their uses of technology, but provide explicit warrants for these claims using the definition and/or theorems and properties related to the figure.

The explicitness of the warrants may be related to the type of tasks on which the students are working. In general, when the students are using the technology, they are merely attempting to determine the longest/shortest side or smallest/largest angle of a triangle. However, when the students are working on generalization and justification tasks, they are not actively using the technology. This suggests that when students are working on task involving a generalization, they are more likely to provide explicit warrants.

Many times, the students collect additional data to verify or refute a previous claim. However, only group 2 and group 3 create arguments of this structure. The students' decision to seek additional data may be due to a number of factors including an explicit challenge to a claim (see the argument illustrated in Figure 52), the uncertainty of a claim (see the argument illustrated in Figure 60), and the ways in which the technology is used to collect the initial data (see the argument illustrated in Figure 50). Even though the students collect additional data for a variety of reasons, they generally use the technology to collect this additional data, usually using measures. At times, the students make claims based on the appearance of the figure (see the arguments illustrated in Figures 51 and 60). Then, the students collect additional data via measures to verify or refute these initial claims. In these arguments, the students do not qualify the second claim that either verified or refuted their initial claim, which suggests they were certain about these claims. This lack of qualification coupled with the use of measures as the additional data suggests that these students are certain about claims that are based on empirical evidence. This is true whether the initial claim is verified (see the argument illustrated in Figure 60) or refuted (see the argument illustrated in Figure 51).

Task 3 – Investigating Triangles

On the sixth class meeting, the students explored whether certain triangles could share a common property. In the previous class meetings, the students completed an investigation on the six types of triangles (acute, obtuse, eight, equilateral, isosceles, and scalene) using technology that was based on the Shapemakers program (Battista, 1998). In addition, as a class, the students defined these triangles.

The objectives for this task centered on the students' discovery and understanding of common properties held by certain types of triangles. Specifically, the teacher wanted the students to understand that a right triangle and obtuse triangle could be an isosceles triangle, but not equilateral; that an isosceles triangle could be an equilateral triangle, but not vice-versa; and, a right triangle could not be an obtuse triangle and vice-versa.

With these objectives in mind, the researcher created a technological activity and corresponding task sheet to be used to teach this concept. The technological activity consisted of four pre-constructed sketches using *The Geometer's Sketchpad*.

Pre-constructed sketch.

The pre-constructed sketch was chosen for primarily for two reasons. First, even though the students had used the software the previous five class meetings, the students' knowledge and understanding of the software was still fairly basic. During the previous class meetings, the students had been taught how to measure angles and segments, how to use the calculate feature to add these measurements, and how to drag a triangle either by one if its vertices or a segment. Second, having the students construct these sketches would have taken a great deal of class time. While it may have been a fruitful exercise for the students in learning the definitions and certain properties of different triangles, the students' lack of knowledge of the software and the time needed to teach the students the different facets of the software in order to begin constructing the triangles would have taken too much time.

The pre-constructed sketch created for this task was one in which the students could explore whether a triangle type could conform to a specified property. The sketch was comprised of four files, one for each of the triangle types under investigation; equilateral, isosceles, obtuse, and right (see Figure 63). All of the triangles were constructed such that it maintained the desired properties when the students dragged all or a portion of the triangle. It should also be noted that the way in which the obtuse triangle was constructed limited the measure of its obtuse angle to no less than 91°. Each of triangles was labeled *ABC* and the interior of the triangle was constructed. The color of each triangle was not the same. The measures of the side lengths and angles were provided for each of the files. The measures of the angles were rounded to the whole degree and the measure of the sides of the triangle were rounded to the tenths digit.

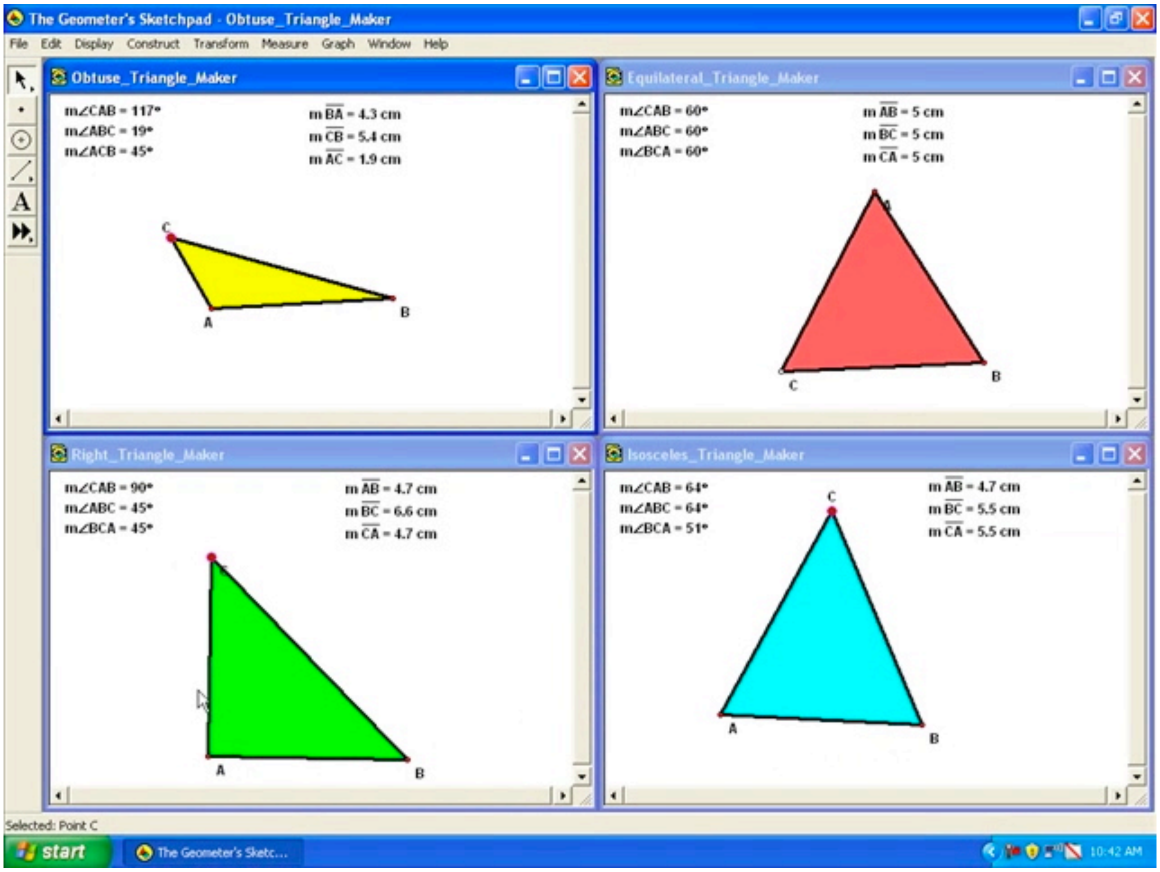


Figure 63. A screenshot of the pre-constructed sketches used during the investigating triangles task.

The decision to include the measures of the side lengths and the angles was three fold. First, even though the students had measured the angles and sides of a triangle in a previous task, the teacher was uncertain the students would be able to finish the activity if they had to make these measurements (and some were unable to finish even with the measures provided). Second, when the students had been given the option of using measures in another task, the majority of the students used those measures. The teacher anticipated the students doing the same for this activity. Third, if the students had made their own measurements, the default level of precision for the measure of the angles and the side lengths was to the hundredths digit. For one of the tasks, the teacher anticipated the students dragging the figures such that two of the angles had the same measure, including the obtuse and right triangles. If the students had this level of precision on their measures, the students may not have been able to drag these triangles such that the measures were the same. The students may have arrived at an erroneous claim. By making these measures in the pre-constructed sketches, the teacher could set the precision of the measures prior to the students using the file. For these reasons, the measures were included in the pre-constructed sketches.

The task sheet.

Along with the sketches, the students were given a task sheet to assist in their exploration (see Appendix E). The task sheet contained four activities, each centered on a given property. Those properties were: all equal sides, at least two equal angles, exactly two 45º angles, at least two equal sides. The teacher instructed the students to first predict whether each of the four triangles could make a triangle with the given property. Then, the students used the technology to determine whether their prediction was true. In addition, the students were to provide an explanation why each of the triangles was able or unable to make a triangle with the given property. Finally, the students were asked to list anything they noticed about those triangles that were able to make a triangle with the given property.

The decision to have the students predict and then check whether the triangle could have the given property was two fold. First, the teacher wanted the students to consider the definitions and their prior knowledge of these triangles to determine whether the triangle could have this property. If the students were not required to predict, then the students may not have considered the definitions of the triangles prior to using the technology. Second, the teacher hoped the predictions would elicit arguments among the pair about whether a certain triangle could have a given property.

Teaching of the lesson.

The triangle side and angle relationship task was the conducted on the sixth class meeting. In the previous class meeting, the students had created definitions for the acute, obtuse, right, and isosceles triangles. However, the class period ended prior to defining the scalene and equilateral triangle, which were assigned for homework. At the beginning of class, the class reviewed the triangles they previously defined and the teacher asked for volunteers to provide the definitions that were assigned for homework. Once the class had agreed on these definitions, they were written down in their notes. While the students copied these definitions, the teacher distributed the task sheet.

The teacher instructed the students to open the four pre-constructed sketches in GSP. Once the files were open, the teacher asked the students to use the tile feature of the software to be able to simultaneously view all four files on the screen (see Figure 63). The teacher asked the students to predict whether the four triangles could make a triangle with all equal sides. Once the students had made their predictions, the teacher told them to use the technology to verify their predictions. He cautioned the students that the measures were being rounded and instructed them to keep the triangles fairly large. The students began working in pairs on this task. While the students worked in pairs, the teacher walked around the room answering students' questions and asking the students what they found.

After all of the students had finished the first activity, and many had moved to working on the second, the teacher led a whole class discussion on the students findings and their explanations for why a given triangle was able or unable to have all the sides equal. After the discussion, the teacher instructed the students to do the second activity in the same manner they had done the first. Once all the groups finished the second activity, the teacher led another whole class discussion on the students' findings and their explanations. Then the teacher instructed the students to work on the third and fourth activities in pairs. With approximately fifteen minutes left in class, the teacher asked the students to report their findings for the third and fourth activity, even though many groups were unable to finish. After reviewing their answers and explanations, the teacher led a discussion on how a triangle can be classified by both the length of its sides and the measures of its angles. The teacher drew two triangles on the board with angle and side measures and asked the students for the best names for the triangles. With the end of class quickly approaching, the teacher passed out the homework assignment.

In the following sections, the arguments created by three pairs of students while working on this investigating triangles task are analyzed and discussed. For each pair of students, the arguments were first categorized by their basic structure. Then, the content and structure of the arguments within these basic categories were analyzed, including the students' uses of technology. The themes that emerged from these analyses are discussed for each pair of students and across the pairs of students.

Group 1's arguments on the investigating triangles task.

The analysis of Heather and Mary's arguments while working on the investigating triangles activity can be categorized into three argument structures; core arguments, arguments in which additional data is collected to verify or refute a claim, and arguments in which the students provide a second warrant. These argument structures are discussed below.

Core arguments.

During the investigating triangles activity, Heather and Mary create eleven core arguments. For three of the core arguments, technology is used and the warrants are not explicit. In four core arguments, the students use technology and the warrants are explicit. For two of the core arguments, technology is not actively employed and the warrants are explicit. For the remaining two core arguments, the students do not actively use technology and the warrants are not explicit. These four argument structures are detailed below.

Non-explicit warrants, technology used.

For three of the core arguments created by Heather and Mary while working on the investigating triangles activity, they employ technology but do not provide explicit warrants. For example, Mary and Heather are determining whether and isosceles triangle can have at least two equal angles. Mary clicks on the isosceles tab and briefly drags a vertex of the triangle such that the angle measures are $m\measuredangle BCA = 54°$, $m\measuredangle ABC = 63°$, and $m\measuredangle CAB = 63°$. She states, “Okay, then isosceles of course had to be two of the same so circle yes for that.” This argument is illustrated in Figure 64.

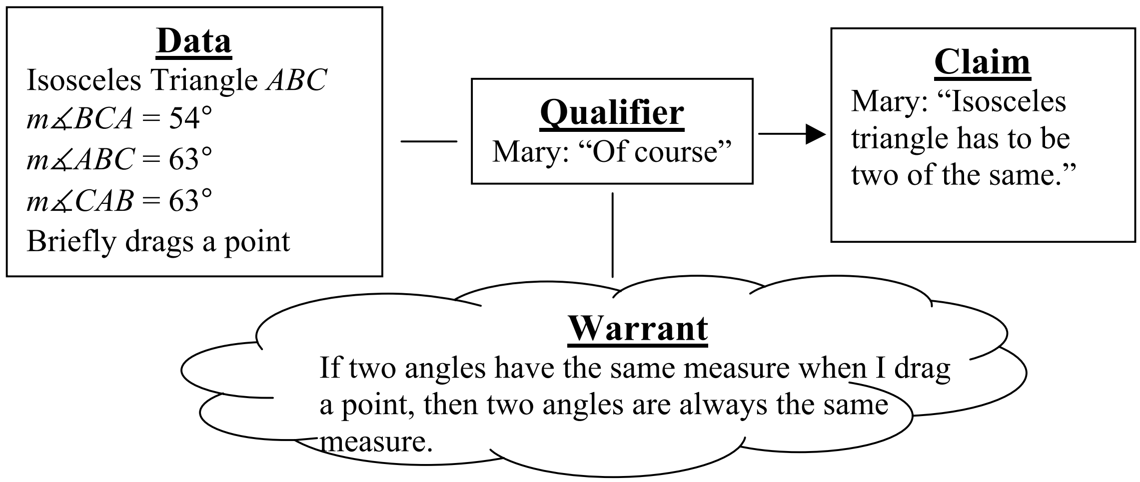


Figure 64. Mary's core argument created during task 3 with an inferred warrant, qualified claim, and angle measures and dragging as data.

The data for this argument are the isosceles triangle, the dragging of the vertex, and the measures of the angles. Mary's claim is the isosceles triangle can have two equal angles. She qualifies this claim with the phrase, "Of course." which suggests a high level of certainty with her claim. Mary does not provide an explicit warrant for her claim and is inferred by the researcher to be "If two angle have the same measure when I drag a point, then two angles are always the same measure."

The two other arguments of this structure are similar to this one. The data for all three of the arguments are the measure on the screen and the dragging of a vertex of the triangle. For one of these arguments, the claim is qualified similar to the one above.

Explicit warrants, technology used.

For four of the core arguments, technology is employed and the warrant is explicit. In one example, Heather and Mary are determining whether a right triangle can have all equal sides. Mary selects the right triangle and drags the non-right vertex. She is unable to drag the triangle such that the measures of the side lengths have the same values. She states, "It's not possible." She follows this up with, "But no matter how big you make it, it won't be the same." This argument illustrated in Figure 65.

<u>Data</u>
Right Triangle
Drag non-right vertex

<u>Claim</u>
Mary: "It's not possible."

<u>Warrant</u>
Mary: No matter how big you make it, it won't be the same."

Figure 65. Mary's core argument created during task 3 with an explicit warrant and dragging as data.

The data for this argument are the right triangle on the screen, the appearance of the right triangle when Mary drags the non-right vertex and the corresponding linked measures of the side lengths. Unable to drag such that the measures are the same, Mary claims it is not possible to make a right triangle with all equal sides. She provides an explicit warrant for this claim when she states, "No matter how big you make it, it won't be the same."

Another example of a core argument in which technology is employed and the warrant is explicit is created while Mary and Heather are determining whether an isosceles triangle can have all equal sides. Mary drags a vertex of the triangle such that the measures of the side lengths are $m\overline{AB} = 4.7$ cm, $m\overline{BC} = 4.7$ cm, and $m\overline{CA} = 4.7$ cm. She states, "Isosceles can be equilateral." She explains, "Isosceles would work because two sides have to be the same and one side is already a number but we can adjust it to be the same." This argument is illustrated in 66.

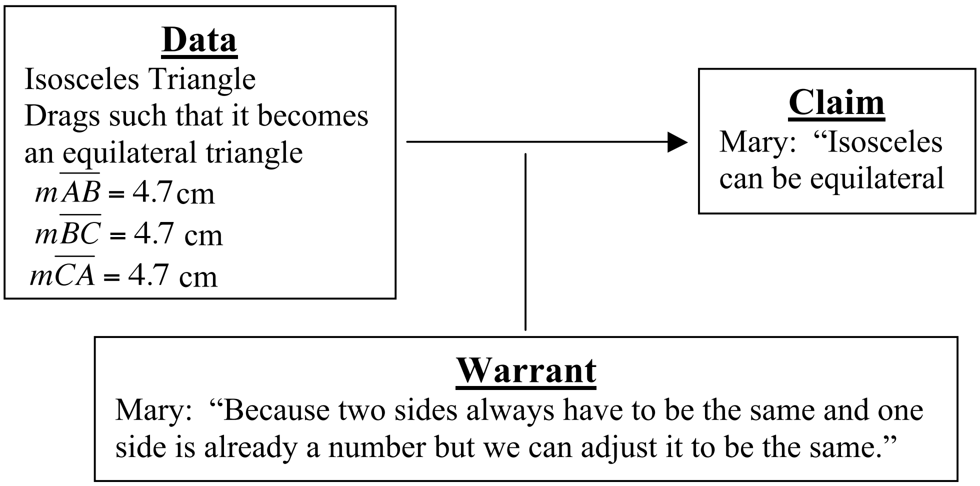


Figure 66. Mary's core argument created during task 3 with an explicit warrant and the measures of the side lengths and dragging as data.

The data for this argument are the isosceles triangle on the screen, the appearance of the isosceles triangle when Mary drags a vertex and the corresponding linked measures of the side lengths. Mary claims that the isosceles triangle can be equilateral, or, in other words, an isosceles triangle can have three equal sides. She provides an explicit warrant by stating that two of the sides are always the same and the other can be adjusted to be the same.

It is interesting to note that for these two examples, the warrants provided by Mary are in terms of her uses of technology, specifically her uses of the drag feature of the technology. In the first example, Mary states a right triangle cannot have all equal sides because she was unable to make it regardless of how large the triangle becomes. In the second example, Mary states that the isosceles triangle can have all equal sides because one is able to adjust the third side such that it has the same measure as the other two sides. For the other two core arguments in which the students use technology and provide explicit warrants, the warrants are in terms of the technology, but their content are in terms of the measures rather than the dynamic feature of the technology.

Technology not used.

During the investigating triangles activity, Heather and Mary create four core arguments in which technology is not actively employed. Of these arguments, the students only provide explicit warrants for two of them. One example of a core argument in which technology is not actively employed and the warrant is not explicit is made by Heather while developing a generalization on what the pair notices about the about the triangles that have all equal sides. Heather states, “All the angles are the same.” When asked by the teacher if this was true, Mary says, “Yes.” This argument is illustrated in Figure 67.

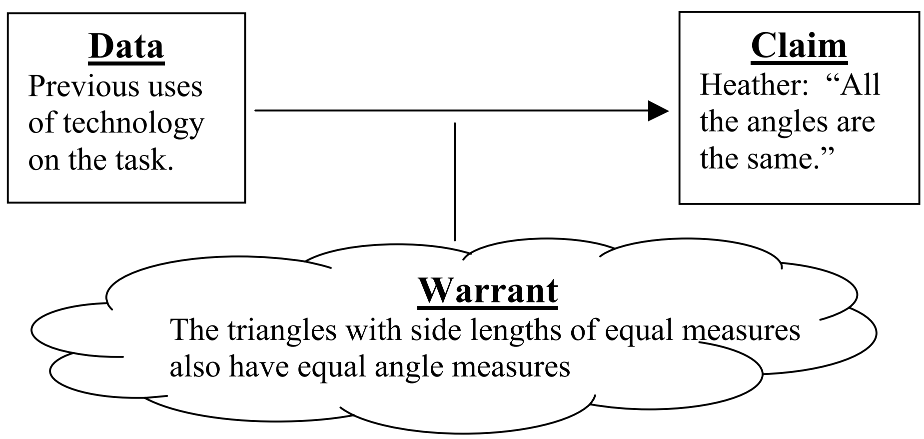


Figure 67. Heather’s core argument created during task 3 with an inferred warrant and indirect use of technology as data.

The data for this argument are the students’ previous uses of technology as they worked on the first task. Heather claims that the triangles with equal sides also have angles that are the same. She does not provide an explicit warrant for this claim and is inferred by the researcher to be, “The triangles with side lengths of equal measure also have equal angle measure.”

An example of an argument in which technology is not actively used and the warrant is explicit is one made by Heather in response to the question posed by the teacher, “Can we have two right angles in a triangle.” Heather responds, “No.” When asked why, Heather says, “90 plus 90 is 180 and that would only be two sides, two angles.” This argument is illustrated in Figure 68.

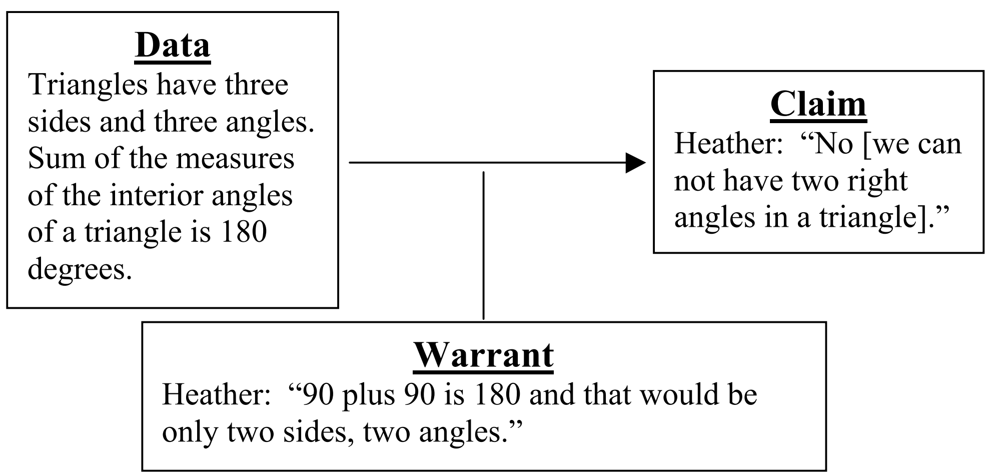


Figure 68. Heather’s core argument created during task 3 with an explicit warrant and definitions and theorems as data.

The data for this argument are the definition and properties of a triangle that they have three sides and three angles, and the theorem the sum of the measures of the interior angles of a triangle is 180 degrees. Heather claims that a triangle cannot have two right angles. Her warrant for this claim is that if a triangle did have two right angles, then the sum of the measures of these two interior angles would be 180, which would not allow for a third angle.

Heather provides the other core argument in which technology is not actively employed and the warrant is explicit as she classifies the right triangle on the board with side lengths 7, 9, and 12. Heather claims, “It’s scalene.” When asked why by the teacher, she explains, “Because they [the side lengths] are all different.” This argument is illustrated in Figure 69.

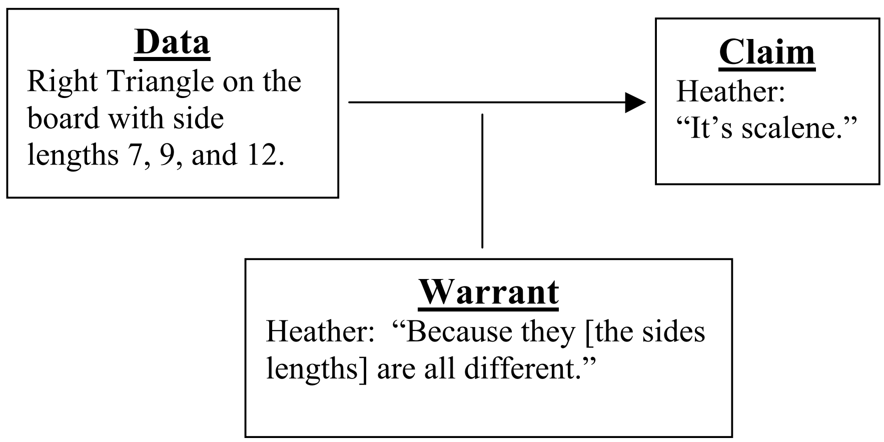


Figure 69. Heather’s core argument created during task 3 with an explicit warrant and a triangle drawn on the board with its measures of the side lengths as data.

In this argument, the data are the right triangle on the board with side lengths 7. 9, and 12. Heather claims that the triangle on the board is scalene and provides a warrant for her claim using the definition of a scalene triangle.

Of the four core arguments in which technology is not actively employed, the warrants for two of the arguments are explicit and two are not explicit. For both of the arguments in which the warrant is explicit, Heather responds to a teacher's question and only provides an explicit warrant when prompted by the teacher.

Conclusions.

Looking across the eleven core arguments created by Heather and Mary while working on the investigating triangles activity, it is worth noting that the students are just as likely to provide explicit warrants when employing technology as they are when it was not actively employed. On prior tasks discussed in this chapter, this is not the case; the students are more likely to provide explicit warrants when technology is not actively employed. This shift may have occurred due to the sociomathematical norms (Yackel & Cobb, 1996) established during the course of the unit. The two previous tasks discussed in this chapter were conducted during the first week of the triangle unit. The students work on this task during the third week of the unit. Throughout the unit, the teacher continually emphasizes that the students need to justify their work. By the third week of class, the students may have viewed justification as part of the regular mathematical activity of the classroom.

Arguments in which Heather and Mary collect additional data.

The second type of argument structure created by Heather and Mary during the investigating triangles activity is that in which the students are compelled to seek additional data after an initial claim is made to verify or refute that claim. The students' decision to seek additional data may be due to a number of factors including the uncertainty of a claim.

One reason students may seek additional data to verify or refute a claim is that they are uncertain about their initial claim. For example Heather and Mary are determining whether an isosceles triangle can have two 45° angles. Heather drags the triangle and is able to get one angle to have measure 45°. Unable to drag the triangle such that another angle's measure is 45°, she states, "I don't think isosceles can do it." She continues to drag a vertex of the triangle unable to get another angle to have measure 45° and states, "I don't think it's going to do it." The pair decides to begin working with another type of triangle. After working with other triangles, they return to working with the isosceles triangle. Heather drags a vertex of the triangle and is able to position the triangle such that the angle measures are $m\measuredangle BCA = 45°$, $m\measuredangle ABC = 45°$, and $m\measuredangle CAB = 90°$. She exclaims, "I got it, yeah!" She then states, "It's a right triangle now." Mary asks how, and Heather explains, "One angle is at 90°." and points to the measure on the screen. This argument is illustrated in Figure 70.

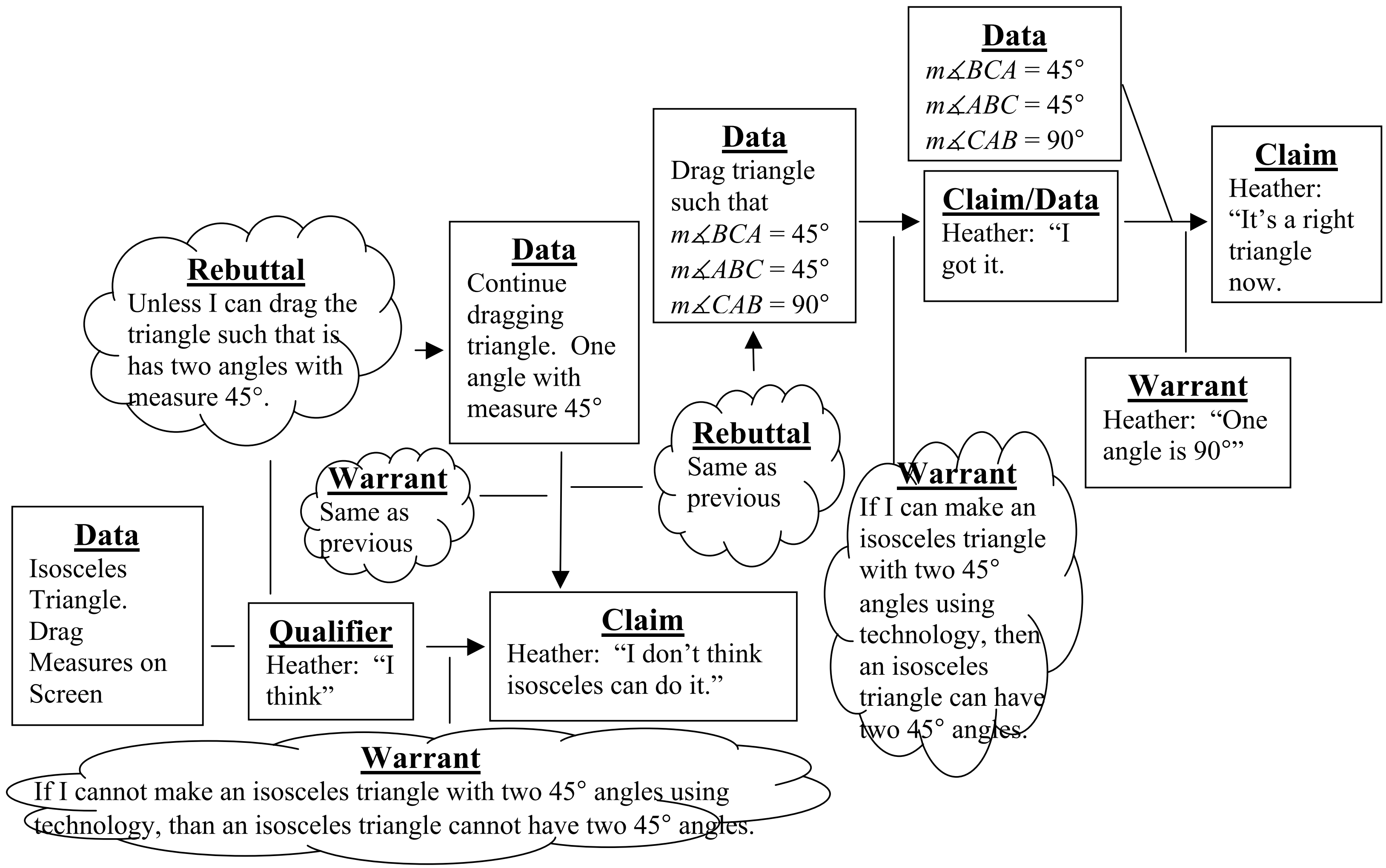


Figure 70. Heather and Mary's argument created during task 3 with additional data collection based on Heather's uncertainty with the initial claim.

211

The initial data for this argument are the isosceles triangle on the screen, the dragging of a vertex of the triangle, and the linked measures of the angles. Heather's initial claim is the isosceles triangle cannot have two 45° angles. She does not provide an explicit warrant for this claim and is inferred by the researcher to be, "If I cannot make an isosceles triangle with two 45° angles using technology, than an isosceles triangle cannot have two 45° angles." She qualifies her claim with the phrase, "I think" which demonstrates some uncertainty with her claim. She continues to drag a vertex of the triangle. This action suggests that she makes a rebuttal to her claim, which the researcher infers to be "Unless I can drag the triangle such that is has two angles with measure 45°." This dragging creates additional data. However, she is still unable to drag the triangle such that it has two angles with measures 45° and she verifies her original claim. The warrant for this is claim is not explicit and the researcher infers it to be the same as the previous warrant. Heather and Mary begin to work on the same activity with the obtuse triangle. However, they return to the isosceles triangle to see if they can make it have two angles with measures 45°. This action suggests that they make a rebuttal to the claim, which is inferred by the researcher to be the same as the previous rebuttal. Heather drags a vertex and is able to position the triangle such that the measures are $m\measuredangle BCA = 45°$, $m\measuredangle ABC = 45°$, and $m\measuredangle CAB = 90°$. She makes a new claim that the isosceles triangle can have two angles with measures 45°. She does not provide a warrant for this new claim and the researcher infers it to be, "If I can make an isosceles triangle with two 45° angles using technology, then an isosceles triangle can have two 45° angles." Using this claim as data and the angle measures on the screen, Heather makes a further claim that the isosceles triangle is now a right triangle. Mary asks Heather to justify this claim and Heather provides the warrant that one of the angles has measure 90°.

Another example of an argument in which the students are compelled to collect additional data is provided by Heather while they are determining whether an obtuse triangle can have two equal sides. Heather clicks on the obtuse triangle and sees the side lengths with the measures $m\overline{BA} = 6.2$ cm, $m\overline{CB} = 9.0$ cm, and $m\overline{AC} = 6.4$ cm. She says, "This can't have one." She drags a vertex of the obtuse triangle such that the measures become $m\overline{BA} = 4.6$ cm, $m\overline{CB} = 6.6$ cm, and $m\overline{AC} = 4.6$ cm. She claims, "Got it." This argument is illustrated in Figure 71.

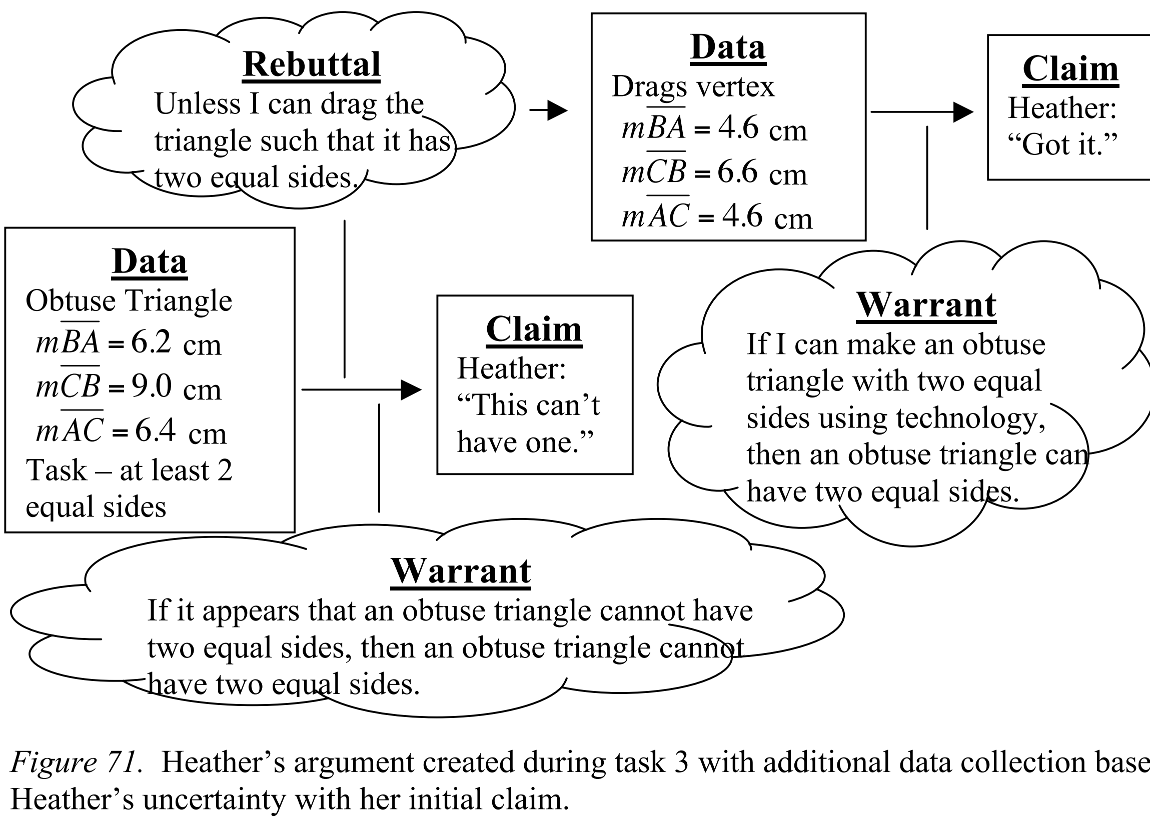


Figure 71. Heather's argument created during task 3 with additional data collection based on Heather's uncertainty with her initial claim.

In this argument, Heather's initial claim is based on the data, the obtuse triangle on the screen and the measures of the side lengths. Her warrant is not explicit and was inferred by the researcher to be, "If it appears that an obtuse triangle cannot have two equal sides, then an obtuse triangle cannot have two equal sides." Uncertain about her claim, Heather begins to drag a vertex of the triangle. This action suggests that she makes a rebuttal to her claim, which the researcher infers to be "Unless I can drag the triangle such that is has two equal sides." The dragging of the vertex creates additional data and she is able to drag the triangle such that two of the measures of the side lengths are the same. She makes a new claim that the obtuse triangle can have two equal sides. She does not provide a warrant for her claim and is inferred by the researcher to be, "If I can make an obtuse triangle with two equal sides using technology, then an obtuse triangle can have two equal sides."

The decision to collect additional data in both of these arguments is due, in part, to the uncertainty of the original claim. In the first example, Heather qualifies her initial claim with the phrase, "I think", which demonstrates some uncertainty. In the second argument, Heather's action to collect additional data shows she is uncertain her initial claim is true. The decision to collect additional data may be related to the affordances of the tool; mainly, the ability to easily collect data through the drag and measurement features.

The technology provides a venue that allows for data to be collected rather easily. Using the drag feature of the technology the students are able to look at multiple examples of specific triangles. In both of these arguments, the students' uses of technology are similar; the students collect additional data using the drag feature of the technology. This suggests that the technology provides a means to collect additional data to establish certainty.

Second warrants.

The third type of argument structure created by Heather and Mary during the investigating triangles activity is characterized by students providing a second warrant to claim. In some cases, the students make a claim based on their uses of technology and do not immediately provide a warrant for this claim. Instead, they would begin working on another activity and make new arguments. Later, the students provide an explicit warrant for their claim. The second warrant is not considered backing due to the field dependence of backing. If the students make a claim based on their use of technology and their warrant is not explicit, the researcher infers their warrant to be in terms of their uses of technology. For these arguments, the backing would indicate that the technology is a reliable and accurate representation of the Euclidean plane. However, the students did not make this type of statement. Instead, they provided a justification for why the claim is true, a warrant.

Mary provides one example of this argument structure as she determines whether an obtuse triangle can have all equal sides. Mary drags all the vertices of the obtuse triangle and the measures of the side lengths of the obtuse triangle when she stops dragging are $m\overline{BA} = 6.2$ cm, $m\overline{CB} = 9.0$ cm, and $m\overline{AC} = 6.4$ cm. She states, "Obtuse can't work." At this time, she does not provide a warrant for her claim and begins to work on the same task with the isosceles triangle. Later, Heather and Mary are filling out the section of the task sheet (see Appendix E) which asks the students to provide an explanation why the triangle maker was or was not able to form a triangle with all equal sides. Mary explains to Heather than an obtuse triangle could not have all equal angles because, "One angle has to be bigger than everything else." This argument is illustrated in Figure 72.

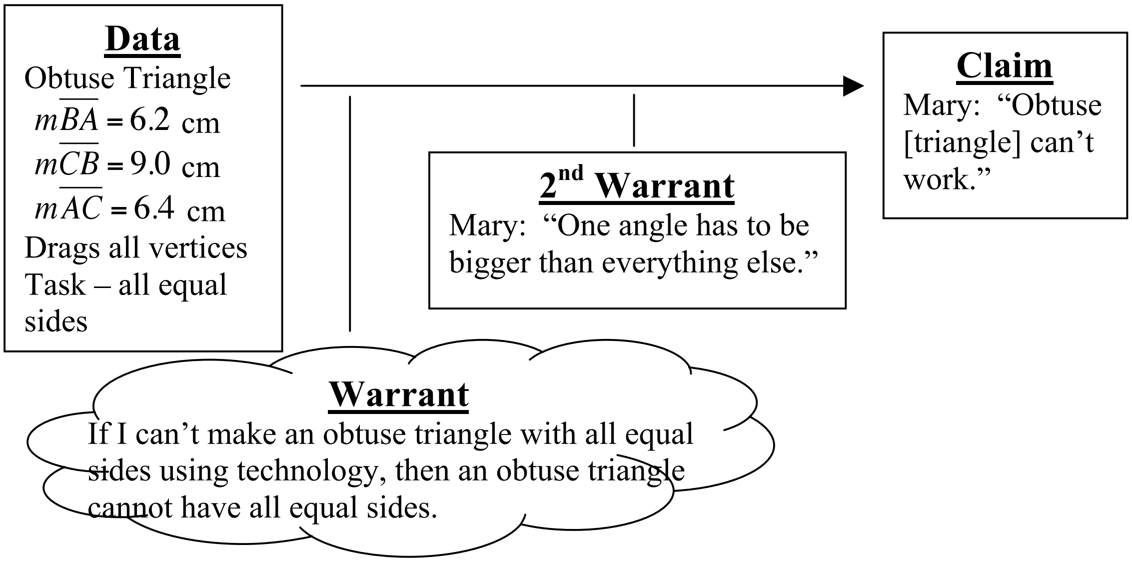


Figure 72. Heather and Mary's argument created during task 3 with an inferred first warrant and an explicit second warrant.

The data in this argument are the obtuse triangle on the screen, the dragging of a vertex of the triangle, and the linked measures of the side lengths. Unable to make the side lengths to have the same measure, Mary makes the claim that the obtuse triangle cannot have sides with all equal lengths. She does not provide a warrant for this claim and is inferred by the researcher to be, "If I can't make an obtuse triangle with all equal sides using technology, then an obtuse triangle cannot have all equal sides." Later, Mary provides an explicit second warrant when she indicates that an obtuse triangle cannot have sides with equal lengths because one of the angles has to be bigger than the other two.

Another argument in which a second warrant is offered is one in which the students collect additional data. Heather and Mary are determining whether an obtuse triangle can have two 45° angles. Heather drags the vertices of the obtuse triangle and is unable to position the triangle such that measure of two of its angles are 45°. Mary states, "Obtuse is a no." Heather says, "Wait, I almost had it." She continues to drag a vertex and is able to position the triangle such that the measures of the angles are $m\measuredangle BCA = 44°$, $m\measuredangle ABC = 45°$, and $m\measuredangle CAB = 91°$. Unable to get both angles to have measure 45°, Heather indicates that the obtuse triangle cannot have two angles with measure 45°. Mary says, "Remember what he [the teacher] said, sometimes it rounds." Heather responds, "Okay, then obtuse is a yes. Wait, how do you check if it's rounding?" Heather asks the teacher and the teacher responds, "Well look, it's 91 so it's an obtuse triangle, it's always going to be obtuse. What does it mean to be obtuse?" Heather replies, "It's going to be over 90." The teacher says, "Greater than 90, so you'll never be able to do obtuse. Heather says, "Ok." Heather and Mary proceed to work on the same task for the equilateral triangle. While filling out the section on the task sheet that ask the students to explain why the triangle maker was able or unable to make a triangle with two 45° angles, "Because obtuse is always going to be an obtuse angle so the other two angles can't be equal [to 45°]." This argument is illustrated in Figure 73.

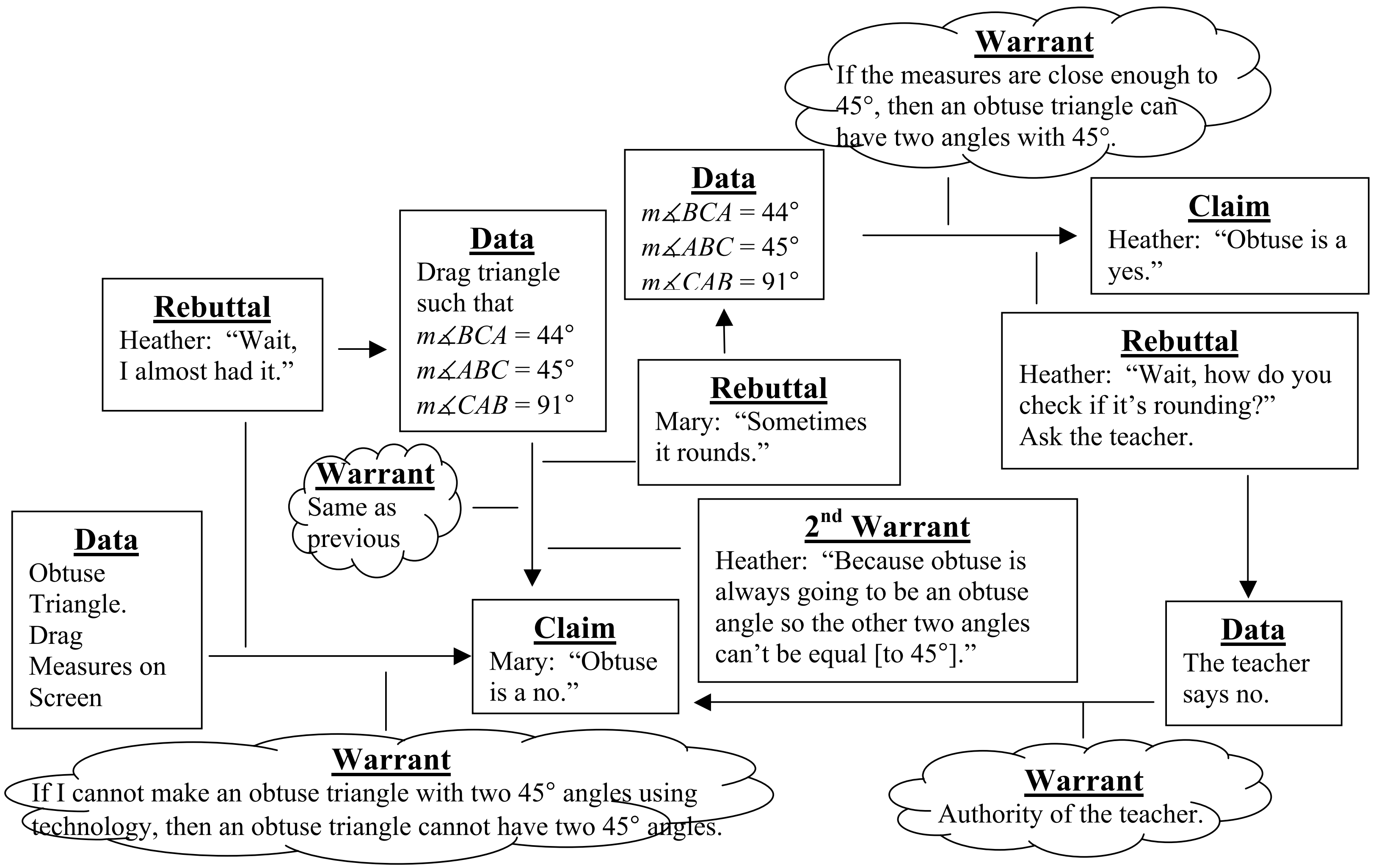


Figure 73. Heather and Mary's argument created during task 3 with additional data collection and a second warrant verifying the initial claim.

The initial data for this argument are the obtuse triangle on the screen, the dragging of a vertex of the triangle, and the linked measures of the angles. Mary's initial claim is that an obtuse triangle cannot have two angles with measures 45°. She does not provide a warrant for this claim and is inferred by the researcher to be, "If I cannot make an obtuse triangle with two 45° angles using technology, then an obtuse triangle cannot have two 45° angles." Heather provides an explicit rebuttal when she asks Mary to wait because she almost had it. Heather continues to collect additional data by dragging a vertex of the triangle and is able to position the triangle such that the measures for two of the angles very close to those desired. Heather indicates it will not work, verifying Mary's original claim. She does not provide a warrant for this verification and the researcher infers it to be the same as the previously inferred warrant. Mary provides an explicit rebuttal indicating that the measures may be rounded. Heather looks at the measures she just collected and states that an obtuse triangle could have two angles with measures 45°. She does not provide a warrant for this claim and is inferred by the researcher to be, "If the measures are close enough to 45°, then an obtuse triangle can have two angles with 45°." She immediately provides a rebuttal when she asks how to check whether the values are being rounded. She collects additional data by asking the teacher if the measures are being rounded. He responds by indicating that an obtuse triangle cannot have two angles with measures 45°. Heather acknowledges this statement, which verifies her initial claim. A warrant is not provided by Heather and is inferred by the researcher to be the authority of the teacher. Later, Heather provides an explicit second warrant when she indicates that an obtuse triangle can not have two angles with measure 45° because one of the angles will always be obtuse and the other angles can not be 45°.

During the investigating triangles activity, Heather and Mary create four arguments in which they offer an explicit second warrant for their initial claims. For three of the arguments, the original structure of the argument (prior to the second warrant) is the core argument (similar to the argument illustrated in Figure 72). However, the second warrant also occurs for arguments with a more complex structure (e.g. the argument illustrated in Figure 73). The occurrence of the second warrant is most likely related to the structure of the task sheet. On the task sheet the students are asked to first predict whether the four triangles could have the desired property. Then the students are asked to verify or refute their predictions and explain why the triangles are able or unable to conform to the desired property. During the activity, Heather and Mary would first make their predictions, then use the technology to check their predictions. After they had worked with each of the triangles, they would begin forming their explanations. By working in this manner, it is not surprising that the arguments the students created would have this structure. The researcher chose to represent the arguments in this manner to indicate that the explicit warrant was not immediately provided by the students, rather than replacing the initial warrant (whether inferred or explicit).

Discussion.

While working on the investigating triangles activity, Heather and Mary create arguments of various structures. Three categories of structures are noted in the analysis: core arguments, arguments in which the students collect additional data to verify or refute a claim, and arguments in which the students provide a second warrant. Looking across these argument structures, two themes emerge; the number of explicit warrants, (see Table 14), and the ways in which the students use the technology.

Table 14

Group 1's Argument on the Investigating Triangles Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	3	1
	Technology Not Used	2	1
Warrants Explicit			
	Technology Used	4	5
	Technology Not Used	2	1

Of the eighteen arguments created by Heather and Mary, at least one of the warrants is explicit for 12 arguments. In fact, this is the first task in which the number of core arguments with explicit warrants out numbers those with non-explicit warrants. As previously discussed, this may be attributed to the establishing of the norm that the students are to justify their claims. It also may be related to the nature of the activity. For each of the tasks within the activity, the students are expected to provide an explanation for why they are able or unable to form a triangle with the given property. This suggests, that the students are able and willing to provide justifications when prompted to do so.

During this activity, Heather and Mary frequently use the drag feature of the technology to determine whether a specific triangle could be positioned such that it conforms to the desired property. The use of the drag feature to accomplish these tasks provide students with a means to reason and provide justifications in terms of their uses of technology. Of the twelve explicit warrants provided by Heather and Mary, six would be considered in terms of the technology (e.g. the warrant provided by Mary in the core argument illustrated in Figure 66). This suggest that the students are, at times, reasoning about their uses of technology such that they translate these uses into justifications for their claims.

Group 2's arguments on the investigating triangles task.

The analysis of David and Erica's arguments while working on the investigating triangles activity can be categorized into three argument structures: core arguments, arguments in which the students collect additional data to verify or refute a claim, and arguments in which the students provide a second warrant. These argument structures are discussed below.

Core arguments.

During the investigating triangles activity, David and Erica create five core arguments. For two of the core arguments, technology is used and warrants are not explicit. In one core argument, the students use technology and the warrant is explicit. For two of the core arguments, technology is not actively employed and the warrants are explicit. These three argument structures are detailed below.

Technology used.

For two of the core arguments created by David and Erica while working on the investigating triangles activity, they employ technology but do not provide explicit warrants. For example, David is determining whether and obtuse triangle can have two equal angles. He drags a vertex of the triangle such that the angles measure $m\measuredangle BCA = 23^\circ$, $m\measuredangle ABC = 23^\circ$, and $m\measuredangle CAB = 134^\circ$. He claims, "Obtuse, yes." This argument is illustrated in Figure 74.

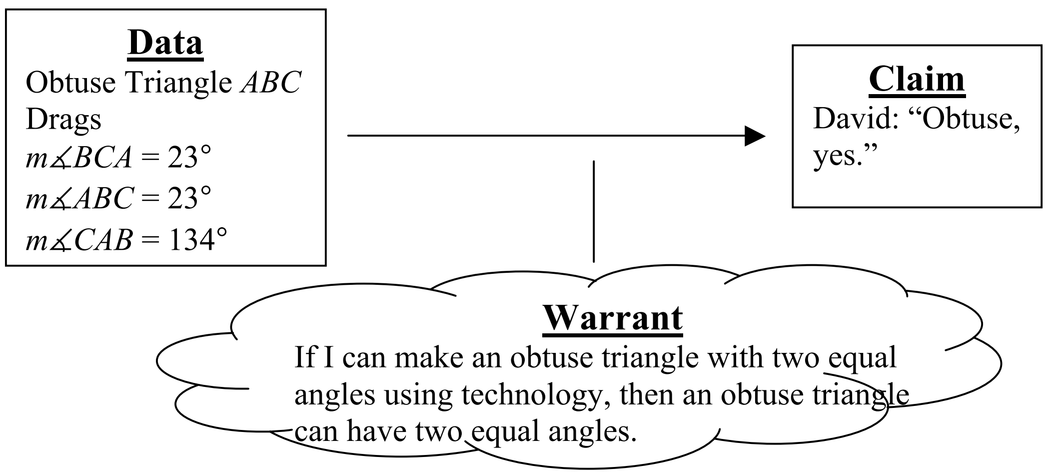


Figure 74. David and Erica's core argument created during task 3 with an inferred warrant and dragging and angle measures as data.

The data for this argument are the obtuse triangle, the dragging of the vertex, and the measures of the angles. David's claim is the obtuse triangle can have two equal angles. He does not provide a warrant for his claim and is inferred by the researcher to be, "If I can make an obtuse triangle with two equal angles using technology, then an obtuse triangle can have two equal angles."

The other argument of this structure is similar to this one. The data for both of the arguments are the measure on the screen and the dragging of a vertex of the triangle. In addition, the students do not qualify their claim or provide a rebuttal.

For one core argument, the student employ technology and the warrant is explicit. In this argument, David and Erica are determining whether an obtuse triangle can have all equal sides. David opens the obtuse triangle, looks at the figure and sates, “Obtuse can [have all equal sides].” He follows this up with, “Because right here, just look at your angle, obtuse. All you’ve got to do is longer that out and longer that out.” This argument illustrated in Figure 75.

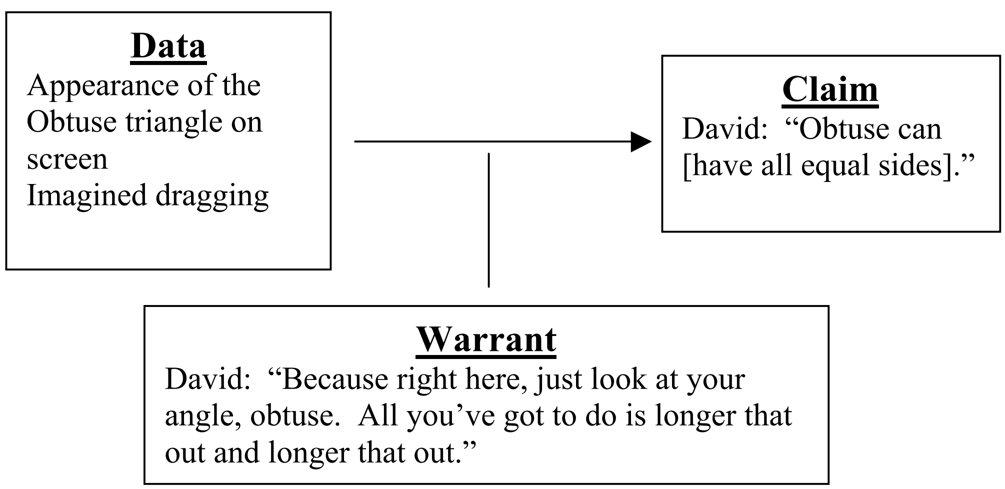


Figure 75. David’s core argument created during task 3 with an explicit warrant and the appearance of the diagram on the screen and imagined dragging as data.

David’s claim in this argument is the obtuse triangle can have all equal sides. The data are the appearance of the obtuse triangle on the screen and the imagined dragging of two of the sides such that the sides of the obtuse triangle have the same length. The imagined dragging was noted as part of the data due to his warrant. In his warrant, he makes explicit reference to changing the length of two of the sides such that the two sides are the same length as the side across from the obtuse angle.

Technology not used.

During the investigating triangles activity, David and Erica create two core arguments in which technology is not actively employed. For both of these arguments, the warrants are explicit. For example, David and Erica are determining whether the equilateral triangle can have all equal sides. David says, “An equilateral can. Equilateral is all sides are equal so why wouldn’t it do it?” This argument is illustrated in Figure 76

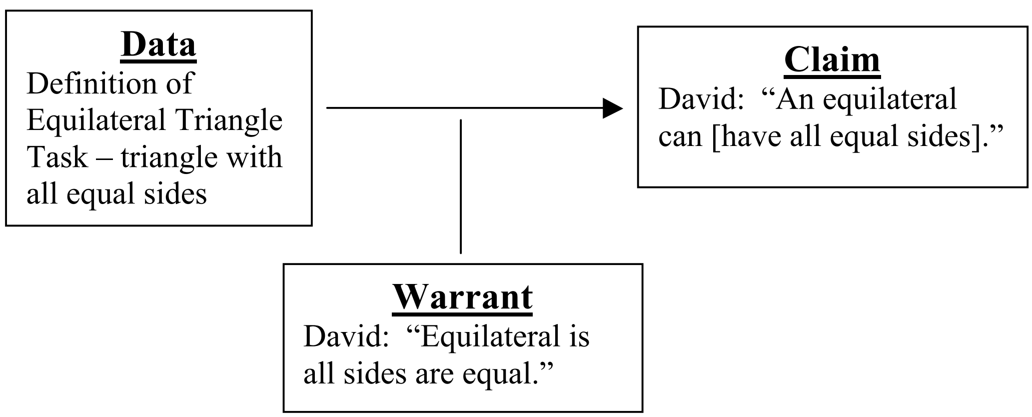


Figure 76. The first of two core arguments David created during task 3 with an explicit warrant and the definition of an equilateral triangle as part of the data.

The data for this argument are the equilateral triangle and the task itself, to determine which triangle can have all equal sides. David claims that and equilateral triangle can have all equal sides. He provides an explicit warrant by using the definition of an equilateral triangle to support his claim. During this argument David did not make use or reference to the technology.

David creates the other argument in which technology is not actively employed and the warrant is explicit as he and Erica determine whether and equilateral triangle can have at least two equal sides. David states, “Equilateral, yes. Because it has to have three equal sides.” This argument is illustrated in Figure 77.

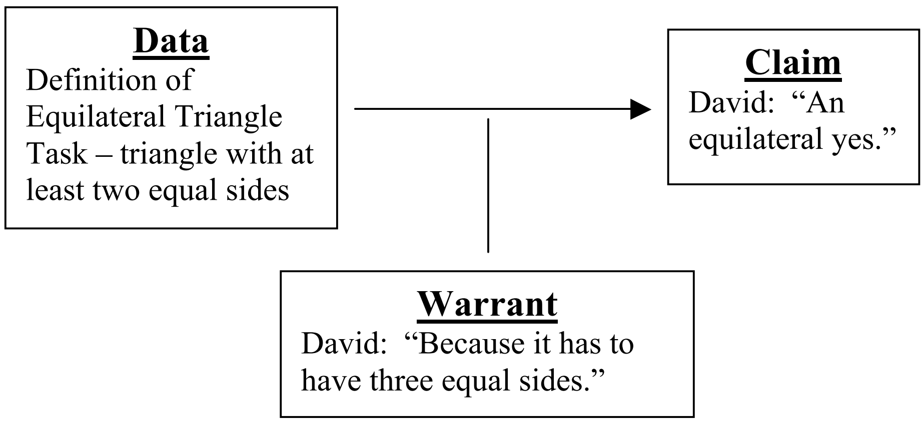


Figure 77. The second of two core arguments David created during task 3 with an explicit warrant and the definition of an equilateral triangle as part of the data.

The data for this argument are the definition of equilateral triangle and the task, to determine whether an equilateral triangle can have at least two equal sides. David claims that an equilateral triangle can have at least two equal sides. He provides an explicit warrant by stating that an equilateral triangle has three equal sides. In this argument, David does not actively use technology. However, it is worth noting that this was the final task on the activity sheet and the pair had used the technology to investigate whether other properties were true for the equilateral triangle. Of course, these prior uses would be considered an indirect use of technology. It seems David’s reasoning is based on the definition of the equilateral triangle rather than on these uses.

For both of these arguments, the students do not actively employ technology and the warrant is explicit. The other commonalities are the content of the arguments. In both arguments, the data is the definition of an equilateral triangle and the task on which the students are working, the claim is that the equilateral can have this property, and the warrant is based on the definition of the equilateral triangle. It is not surprising that these arguments are so similar because the tasks on which David and Erica were working were very similar. What is surprising is that these are the only two arguments of this structure. As previously discussed, the equilateral triangle is a figure familiar to the students. Thus, the students may not have felt the need to use the technology to explore whether the property was true because, based on the definition, they knew it to be true.

Conclusions.

Looking across the five core arguments created by David and Erica while working on the investigating triangles activity, two themes emerge. First, when the students do not actively use technology, they provide an explicit warrant. One difference between these arguments and those in which technology is actively used is the figure about which the students were reasoning. In both of the arguments that do not actively employ technology, the students are working with the equilateral triangle. As previously discussed, the explicitness of these warrants may be attributed to the equilateral triangle being a figure familiar to the students.

The second theme that emerges from the analysis of these core arguments is David and Erica' certainty of their claims. For all of the core arguments, the students do not qualify their claims nor provide rebuttals. The lack of these elements suggests that the students were fairly certain about their claims.

In addition to these two themes, it is worth noting the small number of arguments of this structure. In the two previous tasks, David and Erica create ten core arguments for the first task and nine for the second task. The decrease in frequency of these arguments suggests that the students are making more complex arguments while engaged in this task.

Arguments in which David and Erica collect additional data.

The second type of argument structure created by David and Erica during the investigating triangles task is that in which the students are compelled to seek additional data after an initial claim is made to verify or refute that claim. The students' decision to seek additional data may be due to a number of factors including the uncertainty of a claim and an explicit challenge to the claim.

One reason students may seek additional data to verify or refute a claim is that they are uncertain about their initial claim. For example, David and Erica are determining whether a right triangle can have two equal angles. David drags the vertices of the right triangle on the screen and is unable to get the measures of the angles the same; $m\measuredangle BCA = 32°$, $m\measuredangle ABC = 58°$, and $m\measuredangle CAB = 90°$. He states, "No [a right triangle cannot have two equal angles]." He continues to drag a vertex of the triangle and says, "Hold on, maybe it will." He drags the triangle such that the angle measures are $m\measuredangle BCA = 45°$, $m\measuredangle ABC = 45°$, and $m\measuredangle CAB = 90°$.. He claims, "Yes [a right triangle can have two equal angles]." This argument is illustrated in Figure 78.

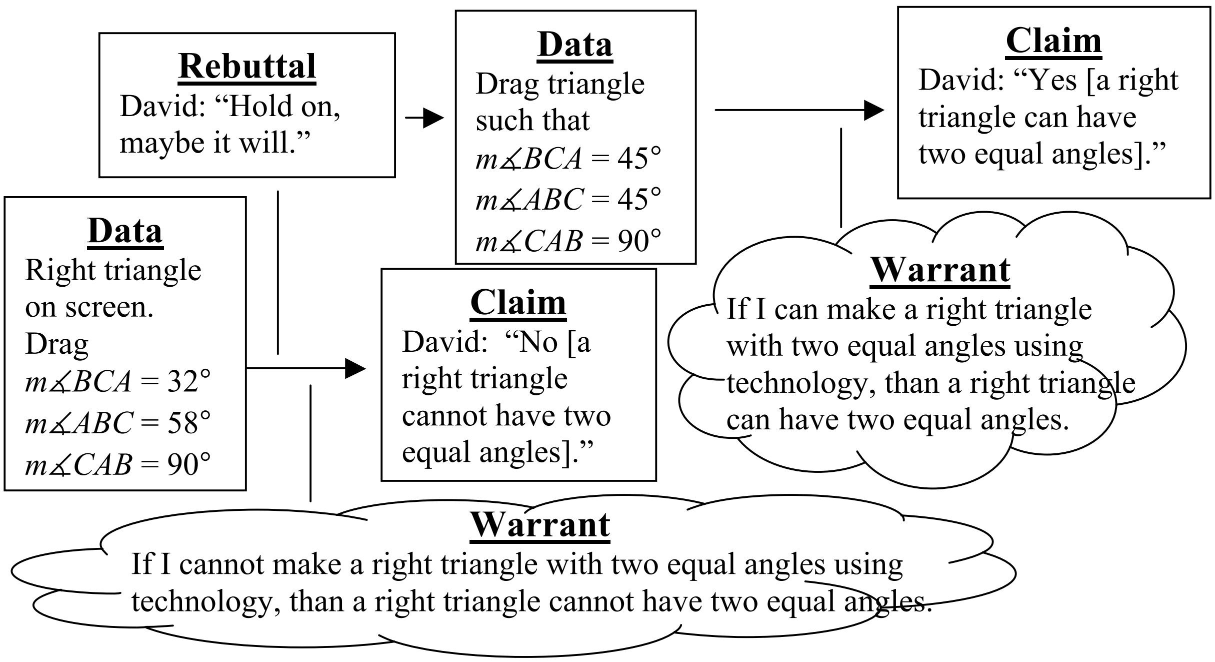


Figure 78. David's argument created during task 3 with additional data collection based on David's uncertainty with his initial claim.

The initial data for this argument are the right triangle on the screen, the dragging of a vertex of the triangle, and the linked angle measures. His initial claim is the right triangle cannot have two equal angles. He does not provide a warrant for this claim and is inferred by the researcher to be, "If I cannot make a right triangle with two equal angles using technology, than a right triangle cannot have two equal angles." Uncertain about his claim, he provides an explicit rebuttal when he says, "Hold on, maybe it will." He collects additional data by dragging a vertex of the triangle and is able to position the triangle such that two of the angles have measure 45°. He makes a new claim that the right triangle can have two equal angles. Again, he does not provide an explicit warrant and the researcher infers it to be, “If I can make a right triangle with two equal angles using technology, than a right triangle can have two equal angles.”

Students may also seek additional data because another person challenges their initial claims. For example, David and Erica are developing a generalization about the triangles that are able to have two equal sides. On their screen is the right triangle with two equal sides and two equal angles. Unsure that his reasoning is correct, David calls the teacher over and asks, “If two sides are the same length, would their angles be the same?” The teacher responds, “Why would that be true?” David says, “I don’t know that’s what I’m trying to figure out.” The teacher then says, “It’s true on this one right, on the right triangle. Was it true on the others?” David says, “I’m going to check.” He proceeds to check the other triangle using the technology and the angles are the same for the equilateral, obtuse, and isosceles triangles. David exclaims, “Mr. Smith, I’m right, I’m right.” This argument is illustrated in Figure 79.

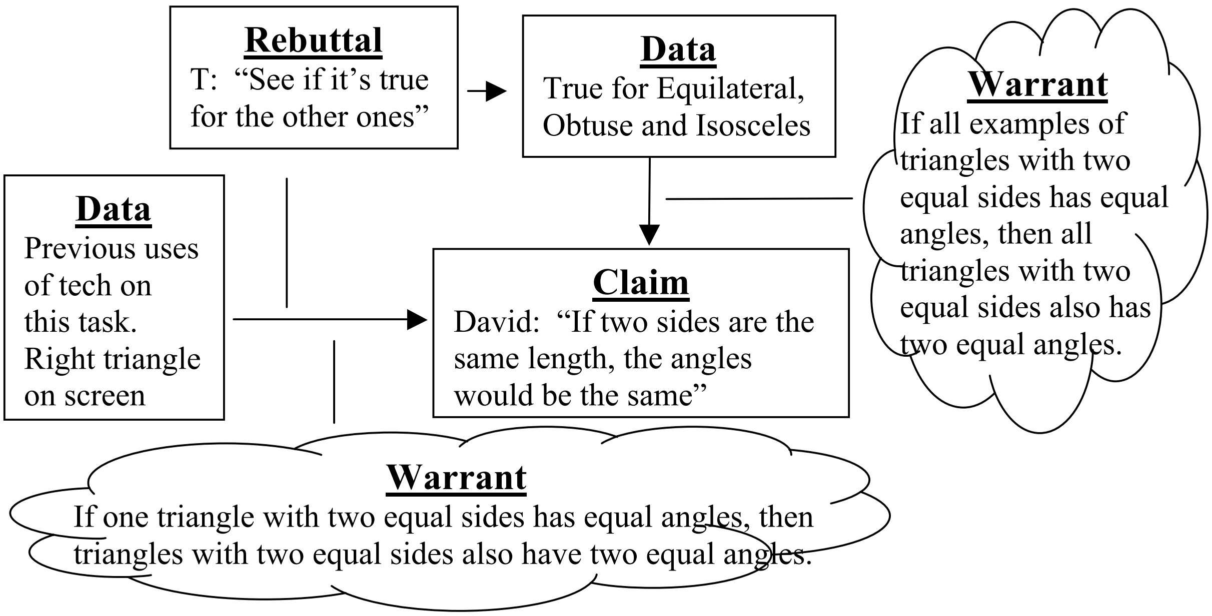


Figure 79. David's argument created during task 3 with additional data collection based on the challenge by the teacher.

The initial data for this argument are the previous uses of technology and the right triangle on the screen with two equal sides and two equal angles. David's claim is in the form of a question and asks if two sides are the same, would the angles be the same. He does not provide a warrant for this claim and is inferred by the researcher to be, "If one triangle with two equal sides has equal angles, then all triangles with two equal sides have two equal angles." The teacher provides an explicit rebuttal when he challenges David to see if his claim is true for the other types of triangles. David checks the other triangles using technology and verifies his original claim. He does not provide a warrant for his verification and is inferred by the researcher to be, "If all examples of triangles with two equal sides has equal angles, then all triangles with two equal sides also has two equal angles."

During the investigating triangle activity, David and Erica create two arguments of this structure. For both of the arguments the students use technology to collect the initial and additional data. In the first example, David uses the drag feature to collect the initial data and the additional data. In this argument, he uses the additional data as a basis for a new claim, which refutes his initial claim. In the second argument, David does not employ the drag feature. Prior to this argument, David uses the drag feature to adjust the triangles such that each of the triangles had two equal sides. Thus, he has no need to drag the triangles when exploring his claim. Instead, he merely looks at the angle measures for each of the triangles to verify his initial claim. For both of these arguments, technology is integral to the collection of additional data.

Arguments with a second warrant.

The third type of argument structure created by David and Erica during the investigating triangles activity is that in which the students provide a second warrant to claim. In some cases, the students make a claim based on their uses of technology and do not immediately provide a warrant for this claim. Instead, they work with other triangles and make new arguments. Later, the students would provide an explicit warrant for their claim. David and Erica create eleven arguments of this structure and these can be further categorized by their initial structure. Eight of the arguments with a second warrant are initially core arguments. And, for three of the arguments with a second warrant, the students collect additional data to verify or refute a claim. These two argument structures are detailed below.

Core arguments with a second warrant.

During the investigating triangles, David and Erica create eight arguments with a second warrant whose initial structure would be considered core. For example, David and Erica are determining whether an obtuse triangle can have all equal sides. Erica drags the vertices of the obtuse triangle and the measures when she finishes dragging are $m\overline{BA} = 3.5$ cm, $m\overline{CB} = 5.0$ cm, and $m\overline{AC} = 2.1$ cm. David says, “No you can’t do this[an obtuse triangle cannot have all equal sides]. This side is too long.” The pair begin to work on the same activity with the equilateral triangle. Later, they begin filling out the section of the task sheet which asks the students to explain why the triangle was able or unable to have all equal sides. David says, “When you have the two sides that make the obtuse, it spreads it out so the side you got up top is going to be longer.” This argument is illustrated in Figure 80.

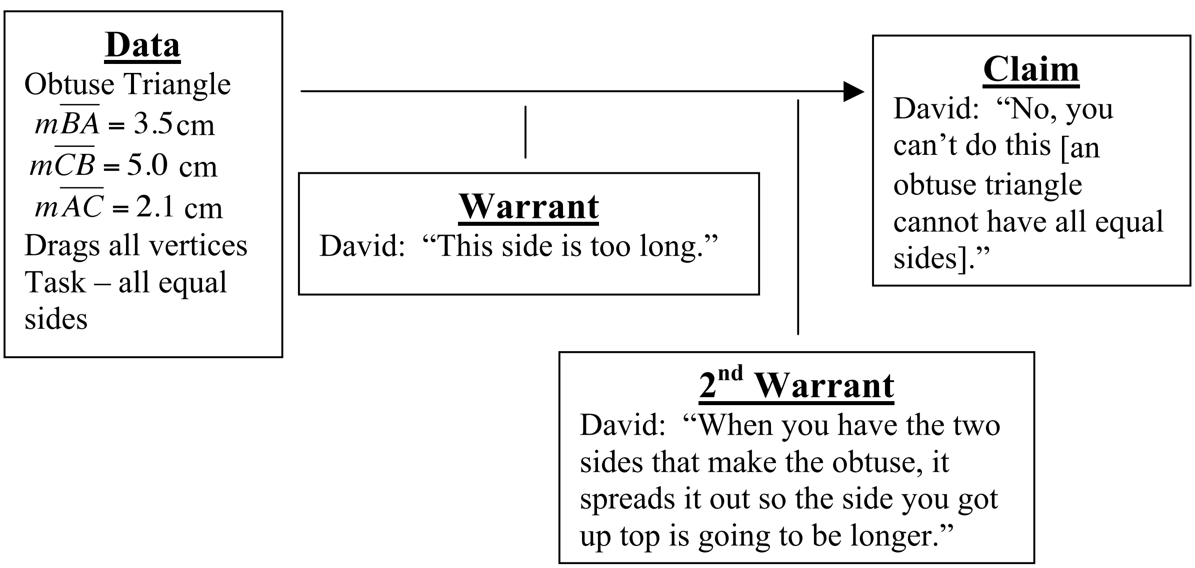


Figure 80. David’s argument created during task 3 with a second warrant whose initial structure was core and both warrants are explicit.

The data for this argument are the obtuse triangle, the dragging of the vertices of the triangle, and the linked measures of the side lengths. David claims that an obtuse triangle cannot have all equal sides. He provides an explicit warrant for this claim stating that one of the sides is too long. Later, David provides a second explicit warrant explaining why the side across from the obtuse angle will always be too long. In his reasoning, he uses an idea developed in a previous class meeting.

In the previous example, David' first warrant is explicit. This is not always the case. For example, David and Erica are determining whether an equilateral triangle can have two angles with measure 45°. David drags the vertices of the equilateral triangle and the measures of the angle remain $m\measuredangle BCA = 60°$, $m\measuredangle ABC = 60°$, and $m\measuredangle CAB = 60°$. He states, "No. Equilateral, no." David and Erica proceed to work on the same task with the right triangle. While working on the explanation section of the task sheet, David explains that the equilateral triangle is unable to have two angles with measure 45°, "Because all sides are equal. This argument is illustrated in Figure 81.

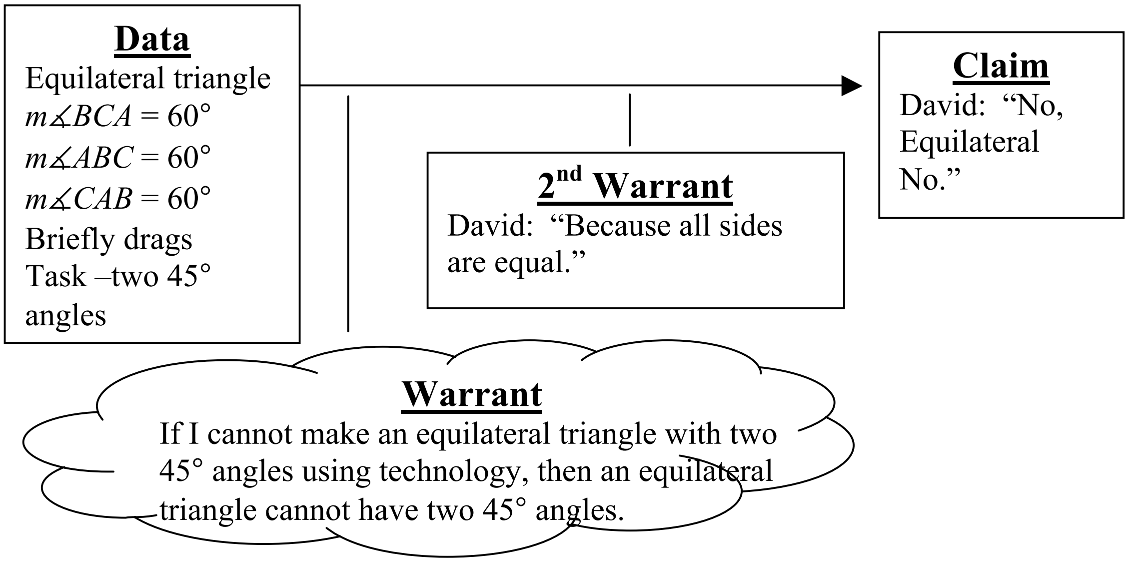


Figure 81. David's argument created during task 3 whose initial structure was core, the initial warrant was inferred and the second warrant is explicit.

The data for this argument are the equilateral triangle, the dragging of the vertices of the equilateral triangle, and the linked measures of the angles. David claims that an equilateral triangle cannot have two angles with measure 45°. He does not provide a warrant for this claim and is inferred by the researcher to be, "If I cannot make an equilateral triangle with two 45° angles using technology, then an equilateral triangle cannot have two 45° angles." Later, he provides an explicit warrant for why an equilateral triangle cannot have two angles with measure 45° when he explains that an equilateral triangle has all equal sides.

Of the eight arguments with the second warrant core structure, two were similar to the first example (an explicit first warrant) and six were similar to the second example (a non-explicit first warrant). In all of these arguments, the students use the technology to collect data; mainly the drag feature. The only instance when the students did not use the drag feature is when the triangle already possesses the desired property. In six of these arguments, the second warrants are based on known properties of the triangles. For the other two arguments of this structure, the second warrant was in terms of their uses of the drag feature of the technology.

Arguments in which the students collected additional data and created second warrants.

For three of the arguments with a second warrant, the students collect additional data. For example, David and Erica are determining whether an isosceles triangle can have two angles with measure 45°. David drags the vertices of the isosceles triangle and stops when the measures of the angles are $m\measuredangle BCA = 61°$, $m\measuredangle ABC = 60°$, and $m\measuredangle CAB = 60°$. He states, "No, isosceles [can not have two 45° angles]." He begins to drag the vertices of the isosceles triangle again and says, "Hold on, maybe not." He drags the vertices such that the angles measures are $m\measuredangle BCA = 45°$, $m\measuredangle ABC = 45°$, and $m\measuredangle CAB = 90°$. He claims, "Isosceles is yes [it can have two 45° angles]." David and Erica turn their attention to the equilateral triangle. Later, as they work on explaining why the isosceles triangle can make a triangle with two angles with measure 45°, David states, "Two sides have to equal the same and if two sides equal the same they have to have two equal degrees, which can be 45°." This argument is illustrated in Figure 82.

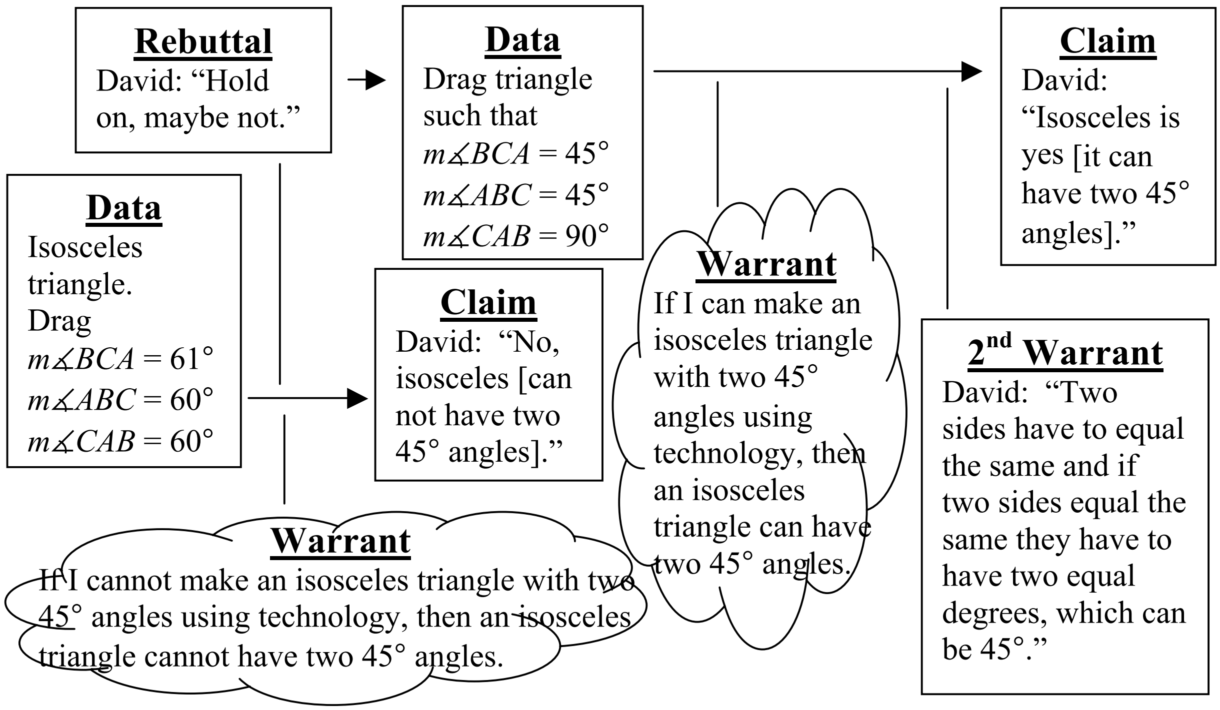


Figure 82. David's argument created during task 3 with additional data collection and a second warrant.

The initial data for this argument are the isosceles triangle, the dragging of the vertices, and the linked measures. David claims that an isosceles triangle cannot have two angles with measure of 45°. He does not provide an explicit warrant for this claim and is inferred by the researcher to be, "If I cannot make an isosceles triangle with two 45° angles using technology, then an isosceles triangle cannot have two 45° angles." Uncertain of his claim, he provides a rebuttal in which he indicates that an isosceles triangle might have this property. He collects additional data by dragging the vertices and is able to position the triangle such that it has two angles with measure 45°. He makes a new claim that an isosceles triangle can have two angles with measure 45°. An explicit warrant is not provided and is inferred by the researcher to be, "If I can make an isosceles triangle with two 45°

angles using technology, then an isosceles triangle can have two 45° angles." Later in the class session, David provides an explicit second warrant for his claim indicating the isosceles triangle has two congruent sides which implies that it also has two congruent angles and those angles can have measure 45°.

David and Erica make another example of an argument in which the pair collect additional data and provide an explicit second warrant as they determine whether and isosceles triangle can have three equal sides. David has the isosceles triangle on the screen and recalls the definition of the isosceles triangle. He says, "Two sides, two sides have to be the same. So, three sides can't be the same. Boom, that's a check baby." Erica says, "Check it. You're supposed to check it. Just move them around and figure it out." David drags the vertices of the triangle and positions the triangle such that the measures of the side lengths are $m\overline{AB} = 4.7\text{cm}$, $m\overline{BC} = 4.7\text{cm}$, and $m\overline{CA} = 4.7\text{cm}$. David claims, "It fits. Say yes." David and Erica begin working on the same activity with the obtuse triangle. Later, they work on the section of the task sheet that asks them to explain why the isosceles triangle was able to have all sides equal. David states, "Because of the definition, it said if it had at least two So, it could be more." This argument is illustrated in Figure 83.

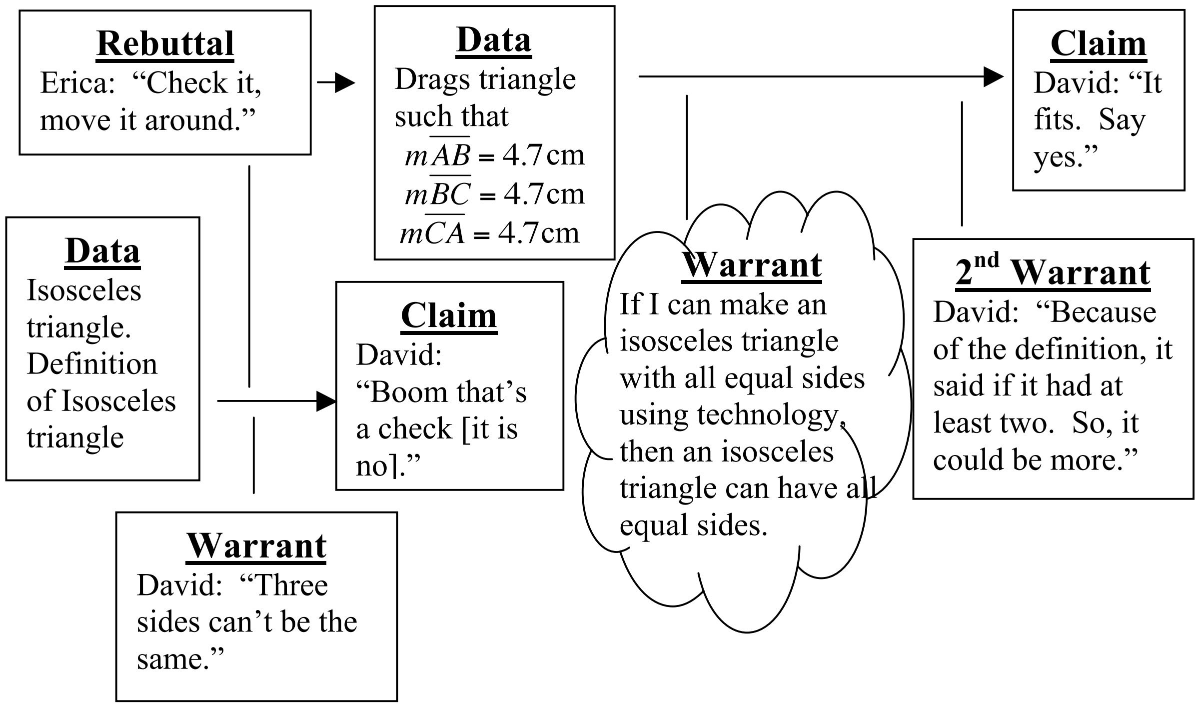


Figure 83. David and Erica's argument created during task 3 with additional data collection and a second warrant.

The initial data for this argument are the isosceles triangle and it's definition. David claims that an isosceles triangle cannot have all equal sides. He provides an explicit warrant when he says that three sides of an isosceles triangle cannot be the same because the definition of the isosceles triangle only says two sides are the same. Erica provides an explicit challenge to David's claim when she says he needs to check it using the technology. David collects additional data by dragging the vertices of the triangle such that the measures of the side lengths are the same. He makes a new claim indicating that an isosceles triangle can have all equal sides. He does not provide a warrant for this claim and is inferred by the researcher to be, "If I can make an isosceles triangle with all equal sides using technology, then an isosceles triangle can have all equal sides." Later in the class session, David provides an explicit warrant for this new claim. He states the definition of the isosceles triangle says at lest two sides, which implies that it could be more than two sides.

During the investigating triangles activity, David and Erica create eleven arguments in which they offer an explicit second warrant for their final claims. For eight of the arguments, the original structure of the argument (prior to the second warrant) is the core argument (e.g. the arguments illustrated in Figures 80 and 81). However, the second warrant also occurs for three arguments with a more complex structure (e.g. argument illustrated in Figures 82 and 83). The occurrence of the second warrant is most likely related to the structure of the task sheet. On the task sheet the students are asked to first predict whether the four triangles could have the desired property. Then the students are asked to verify or refute their predictions and explain why the triangles are able or unable to conform to the desired property. During the activity, David and Erica would first make their predictions, then use the technology to check their predictions. After they had worked with each of the triangles, they would begin forming their explanations. By working in this manner, it is not surprising that the arguments the students create would have this structure. The researcher chose to represent the arguments in this manner to indicate that the explicit warrant was not immediately provided by the students, rather than replacing the initial warrant (whether inferred or explicit).

For all of these arguments, technology plays an integral role. For all but one of the arguments, the students employ the drag feature of the triangle to attempt to position the triangle such that it conforms to the desired property. In the only argument in which the drag feature is not employed, the triangle already conforms to the desired property and does not need to be dragged. Looking at the nature of the second warrants, the majority of the warrants are based on definitions or known properties of the triangles. In fact, only one of the second warrants is in terms of the students' uses of technology (David explains that a right triangle cannot have all equal sides, "Because when you pull a corner, all the sides change."). This suggests that these students use the technology to collect data as the basis for their claims but are able to reason about their findings based on the definitions and properties of the triangle.

Discussion.

While working on the investigating triangles activity, David and Erica create arguments of various structures. Three categories of structures are noted in the analysis; core arguments, arguments in which the students collect additional data to verify or refute a claim, and arguments in which the students provide a second warrant. Looking across these argument structures, two themes emerge; the number of explicit warrants (see Table 15), and the ways in which the students use the technology.

Table 15

Group 2's Argument on the Investigating Triangles Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	2	2
	Technology Not Used	0	1
Warrants Explicit			
	Technology Used	1	11
	Technology Not Used	2	0

Of the nineteen arguments created by David and Erica, at least one of the warrants wi explicit for fourteen arguments. In fact, this is the first task in which the number of core arguments with explicit warrants out numbers those with non-explicit warrants. As previously discussed, this may be attributed to the establishing of the socio-mathematical norm that the students are to justify their claims. It also may be related to the nature of the activity. For each of the tasks within the activity, the students were expected to provide an explanation for why they are able or unable to form a triangle with the given property. This suggests, that the students are able and willing to provide justifications when prompted to do so.

During this activity, David and Erica frequently use the drag feature of the technology to determine whether a specific triangle could be positioned such that it conforms to the desired property. The use of the drag feature to accomplish these tasks provides students with a means to discover whether the triangle could conform to the desired property. However, David and Erica do not frequently justify their claims in terms of their uses of the technology. Instead, they base their justifications on the definitions and known properties of the triangles. Of the seventeen explicit warrants provided by David and Erica, only three would be considered in terms of their uses of technology (e.g. the warrant provided by David in the core argument illustrated in Figure 75). This suggest that these students are able to use the technology in meaningful ways to discover relationships among different types of triangles and justify these relationships using the definitions and properties of the triangles rather than only in terms of their uses of technology.

Group 3's arguments on the investigating triangles task.

The analysis of Amy and Judy's arguments while working on the investigating triangles activity can be categorized into three argument structures; core arguments, arguments in which the students collect additional data to verify or refute a claim, and arguments in which the students provide a second warrant. These argument structures are discussed below.

Core arguments.

During the investigating triangles activity, Amy and Judy create seven core arguments. For one core argument, technology is used and the warrant is not explicit. In five core arguments, the students use technology and the warrants are explicit. For one core argument, technology is not actively employed and the warrant is explicit. These three argument structures are detailed below.

Technology used.

For one core argument created by Amy and Judy while working on the investigating triangles activity, they employ technology but do not provide an explicit warrant. In this argument, the students are determining whether an isosceles triangle can have all equal sides. Amy drags the vertices of the triangle such that measures of the side lengths are $m\overline{AB} = 5.0$ cm, $m\overline{BC} = 5.0$ cm, and $m\overline{CA} = 5.0$ cm. Judy states, “So, yes for isosceles.” This argument is illustrated in Figure 84.

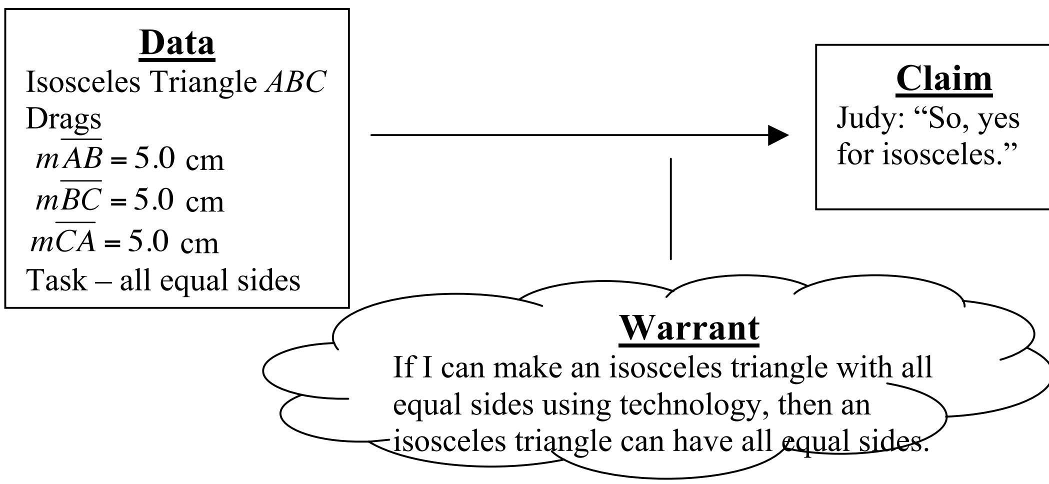


Figure 84. Judy’s core argument created during task 3 with an inferred warrant and dragging and the measures of side lengths as data.

The data for this argument are the isosceles triangle, the dragging of the vertices, and the linked measures of the side lengths. Judy claims the isosceles can have all equal sides. She does not provide an explicit warrant for this claim and is inferred by the researcher to be, “If I can make an isosceles triangle with all equal sides using technology, then an isosceles triangle can have all equal sides.”

For five of the core arguments, technology is employed and the warrants are explicit. In one example, Amy and Judy are determining whether an equilateral triangle can have two angles whose measures are 45°. Amy briefly drags the equilateral triangle and the angle measures remain 60°. Amy states, "You can't change it because if you want 45 this one equilateral, you can't change the sides if you want 45 as one angle the other one has to equal it and it all equals up to 135 which doesn't equal 180 so it won't make a triangle." This argument is illustrated in Figure 85.

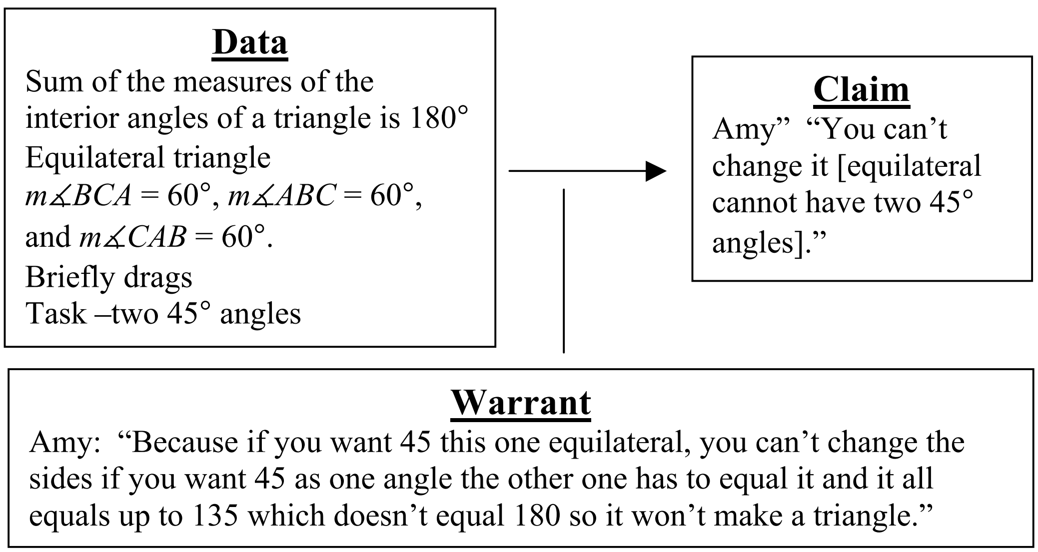


Figure 85. The first of two core argument created by Judy during task 3 with an explicit warrant and a theorem, angle measures, and dragging as data.

The data for this argument are: the equilateral triangle, the dragging of the vertices, the linked measure, and the property the sum of the measures of the interior angles of a triangle is 180°. Amy claims the angle measures of the equilateral cannot change or, in other words, and equilateral triangle cannot have two 45° angles. She provides an explicit warrant for this claim by indicating that if the three angles of an equilateral triangle must be the same and if they were 45° then their sum would be 135° which would not form a triangle.

Another example of a core argument in which technology is employed and the warrant is explicit is one where Amy and Judy are determining whether an obtuse triangle can have two equal angles. Judy drags the vertices of the triangle such that the angle measures are $m\measuredangle BCA = 39°$, $m\measuredangle ABC = 39°$, and $m\measuredangle CAB = 102°$. She uses the calculate feature of the software to add the three angle measures whose sum is 180°. She states, "Oh it works." When Amy asks why it works, Judy says, "Because 39° plus 39° plus 102 equals 180 which makes a triangle." This argument is illustrated in Figure 86.

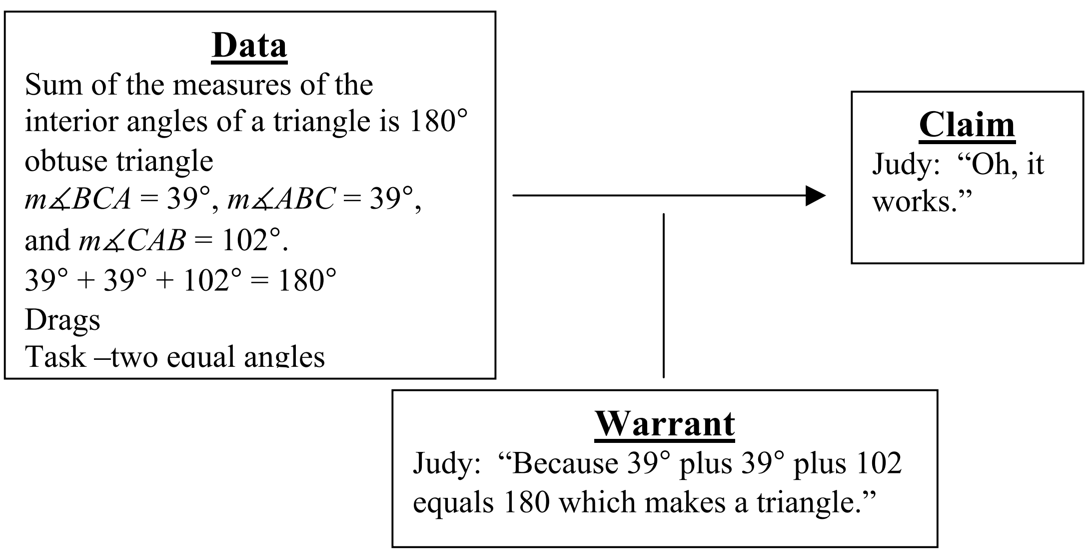


Figure 86. The second of two core argument created by Judy during task 3 with an explicit warrant and a theorem, angle measures, and dragging as data.

The data for this argument are: the obtuse triangle, the dragging of the vertices, the linked measure, the equation 39° + 39° + 102° = 180°, and the property the sum of the measures of the interior angles of a triangle is 180°. Judy claims an obtuse triangle can have two angles with equal measure. Her warrant for this claim is the sum of the measures of the interior angle for this obtuse triangle is 180°, which makes a triangle.

These two examples typify the other three arguments of this structure. Even though the students work on different activities with different triangles, all of the warrants, for these arguments involved the use of the theorem the sum of the measures of the interior angles of a triangle is 180°. This suggests, at least for these students, that this theorem is the determining factor whether a triangle could or could not make the desired property on the activity sheet.

Technology not used.

During the investigating triangles activity, Amy and Judy create one core argument in which the students do not actively employ the technology. In this argument, the warrant is explicit. In this argument, the teacher asks the class, "Can we have two right angles in a triangle?" Judy says, "No." When asked why by the teacher, Judy replies, "Because it would look like a square." This argument is illustrated in Figure 87.

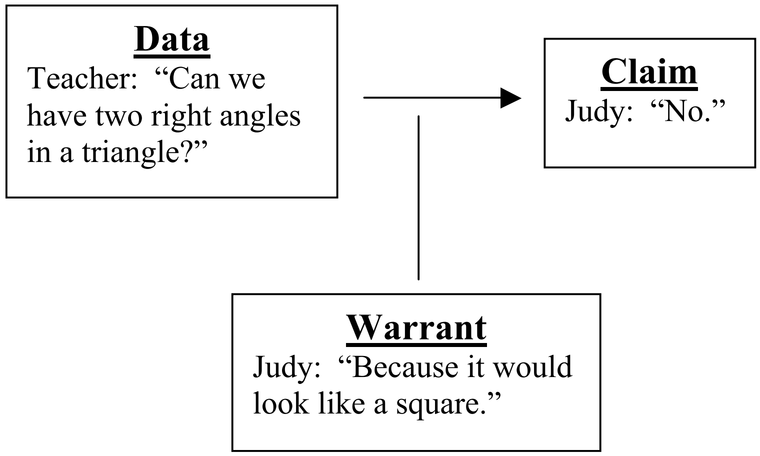


Figure 87. Judy's core argument created during task 3 with an explicit warrant and the data is a question posed by the teacher.

The data for this argument is the question posed by the teacher. Judy's claim is her response to this question; a triangle cannot have two right angles. Her warrant for this claim is the appearance of a triangle with two right angles would look like a square.

Conclusions.

Looking across the seven core arguments created by Amy and Judy while working on the investigating triangles activity, three themes emerge. First, the students frequently provide explicit warrants in their core arguments. In fact, in six of the seven core arguments, the warrants are explicit. Second, the students frequently use technology, mainly the drag feature and the linked measures. Only one of the core arguments do not involve the use of technology and that argument is in response to a teacher's inquiry. This frequent use of technology may be related to the nature of the activity, which asks the students to explore whether a given property would hold for the four triangle types.

The third theme that emerges from the analysis of these core arguments is Amy and Judy's certainty of their claims. For all of the core arguments, the students do not qualify their claims or provide rebuttals. The lack of these elements suggests that the students are fairly certain about their claims.

Arguments in which Amy and Judy collect additional data.

The second type of argument structure created by Amy and Judy during the investigating triangles activity is that in which the students are compelled to seek additional data after an initial claim is made to verify or refute that claim. The students' decision to seek additional data may be due to a number of factors including the uncertainty of a claim and an explicit challenge to the claim.

One reason students may seek additional data to verify or refute a claim is that they are uncertain about their initial claim. For example, Amy and Judy are determining whether an equilateral triangle can have all equal sides. Amy clicks on the equilateral triangle window, which has the measures of the side lengths as $m\overline{AB} = 4$ cm, $m\overline{BC} = 4$ cm, and $m\overline{CA} = 4$ cm. Amy says, "Well, they're already equal." She begins to drag one of the vertices of the triangle and observes that the measures of the side lengths remain equal. She concludes, "Yep, they're all equal." This argument is illustrated in Figure 88.

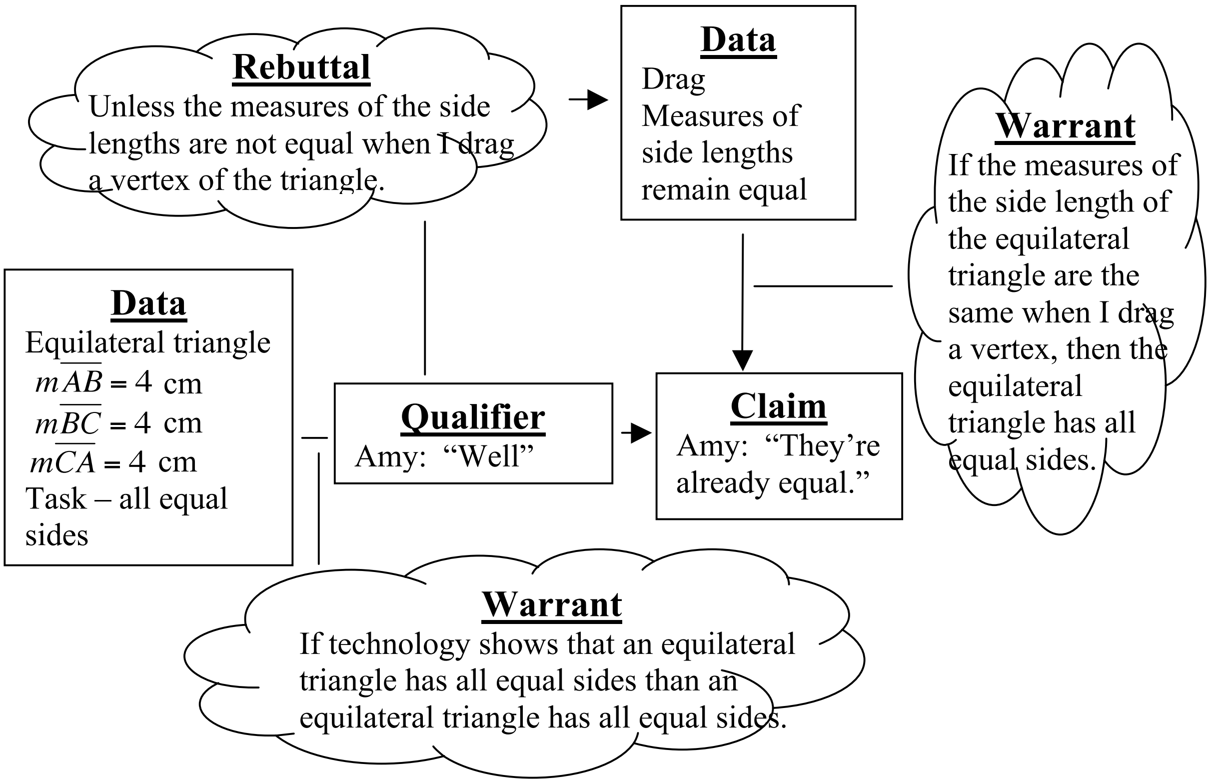


Figure 88. Amy's argument- created during task 3 with additional data collection based on Amy's uncertainty.

The initial data for this argument are the equilateral triangle and the measures of the side lengths when the window is opened. Amy claims the lengths of the sides of the equilateral triangle are equal. She does not provide an explicit warrant for her claim and is inferred by the researcher to be, "If technology shows that an equilateral triangle has all equal sides than an equilateral triangle has all equal sides." Amy demonstrates some uncertainty wit this claim with the term, "Well." She then begins to collect additional data by dragging a vertex of the triangle. This action coupled with the use of a qualifier suggests that Amy makes a rebuttal, which is the researcher infers to be, "Unless the measures of the side lengths are not equal when I drag a vertex of the triangle." When Amy collects additional data by dragging a vertex of the triangle she notices the measures of the side lengths remain equal. She verifies her original claim. She does not provide a warrant for this verification and is inferred by the researcher to be, “If the measures of the side length of the equilateral triangle are the same when I drag a vertex, then the equilateral triangle has all equal sides.”

Amy and Judy make another example of an argument in which the uncertainty of an initial claim leads to the collection of additional data as they determine whether a right triangle can have two angles with measure 45°. Judy clicks on the right triangle window and begins to drag the vertices of the right triangle. She stops dragging when the angle measures are $m\measuredangle BCA = 50°$, $m\measuredangle ABC = 40°$, and $m\measuredangle CAB = 90°$. She turns to Amy and asks, “What’s 45 plus 45?” Amy adds the values on a piece of paper and responds, “90.” Judy then asks, “What’s 90 plus 90?” Amy says, “180.” Judy exclaims, “Thank you. A right triangle can do it but it just… Wait, no you can’t.” Amy agrees, “No, you can’t.” Judy then says and points to the angles on the screen, “Wait, what if this angle equals 45 degrees and this angle equals 45 degrees and this angle equals 90? You still can do it, exactly.” She calls over the regular classroom teacher and presents this scenario to her. The regular classroom teacher does not say anything and gestures for the teacher to assist Amy and Judy. The teacher is unable to attend to these students immediately. Meanwhile, Judy drags the vertices of the right triangle and states, “45 plus 45 equals 90 plus another 90 equals 180. I cannot change it 45. That’s what’s irritating. I don’t know how to…” Amy suggests that she use the calculate feature of the program. The teacher arrives and Judy drags a vertex of the triangle such that the angle measures are $m\measuredangle BCA = 45°$, $m\measuredangle ABC = 45°$, and $m\measuredangle CAB = 90°$. The teacher says, “There you go." Judy exclaims, "I told you, what did I say? I told you so it's yes." This argument is illustrated in Figure 89.

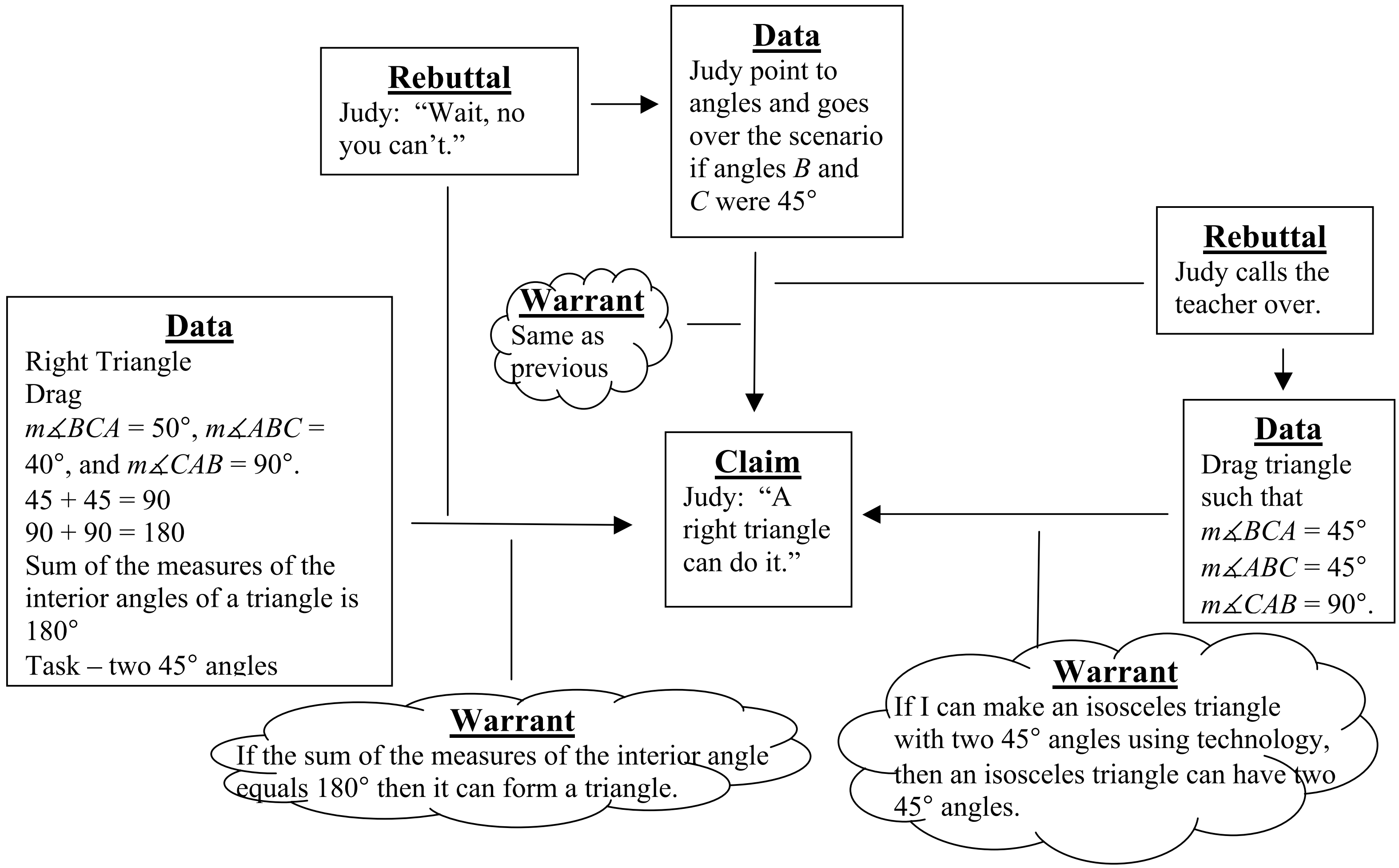


Figure 89. Amy and Judy's argument created during task 3 with additional data collection based on Judy's uncertainty.

The initial data for this argument are the right triangle, the dragging of the vertices, the linked measures, the equations 45 + 45 = 90 and 90 + 90 = 180, and the property the sum of the measures of the interior angles of a triangle is 180°. Judy claims the right triangle can have two angles with measures 45°. She does not provide a warrant for this claim and is inferred by the researcher to be, "If the sum of the measures of the interior angle equals 180° then it can form a triangle." Judy demonstrates some uncertainty about her claim when she says, "No, it can't." This statement is a rebuttal. She then begins to collect additional data by pointing to the screen and presents the scenario that if the measures of angles *B* and *C* are 45° and angle *A* is 90 then it would form a right triangle. Judy claims that this scenario would work, which is a verification of her original claim. She does not provide a warrant for this verification and is inferred by the researcher to be the same as the previous warrant. Judy is still uncertain about her claim and would like to discuss it with the teacher. This action is a rebuttal to her claim. While attempting to gain the teacher's attention, Judy collects additional data by dragging a vertex of the triangle and when the teacher arrives, she is able to position the triangle such that two of the measures of the angles are 45°. Judy provides a verification of her initial claim. However, she does not provide a warrant and is inferred by the researcher to be, "If I can make an isosceles triangle with two 45° angles using technology, then an isosceles triangle can have two 45° angles."

Another reason students may seek additional data is due to another person's challenge of a claim. In one such argument, Amy and Judy are making their predictions whether an isosceles triangle can have all equal sides. Judy selects the isosceles window and the side length measures are $m\overline{AB} = 4.7$ cm, $m\overline{BC} = 2.9$ cm, and $m\overline{CA} = 2.9$ cm. She states, "Well, Isosceles, no because two sides are different." Amy says, "I think it can." Judy says, "Well, possible I think it can too." Amy, pointing to the sides of the equilateral triangle, explains, "Because it's like these two stay the same and it connects these two." This argument is illustrated in Figure 90.

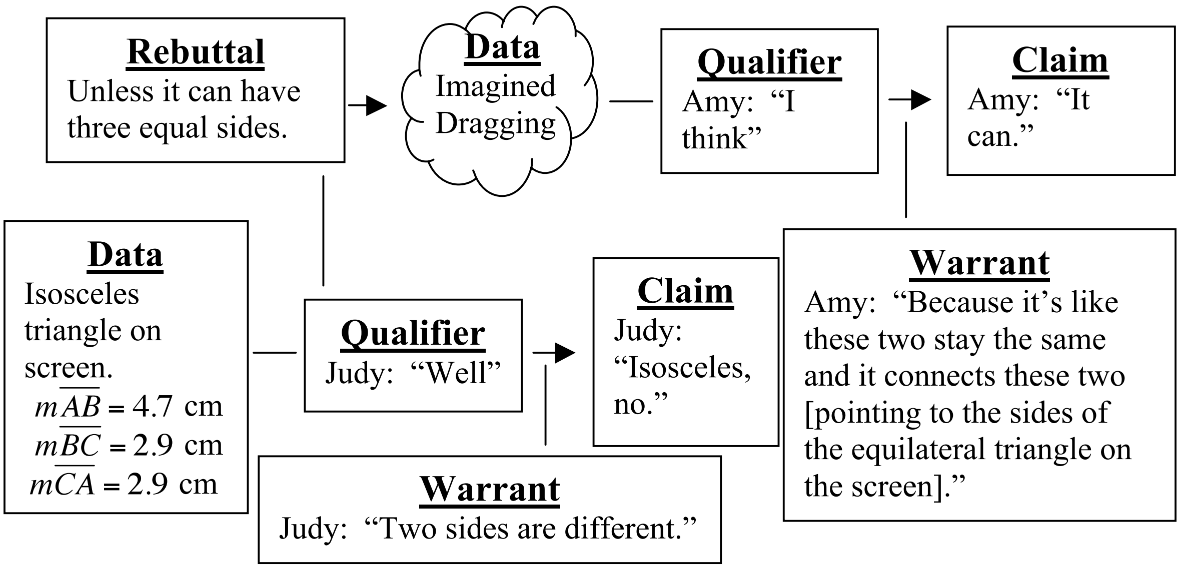


Figure 90. Amy and Judy's argument created during task 3 with imagined dragging as the additional data collected.

The data for this argument are the isosceles triangle and the measures of the lengths of the sides on the screen. Judy claims that an isosceles triangle cannot have three equal sides. She demonstrates some uncertainty with this claim with the term "Well." Her warrant is that two of the sides are different or, in other words, two of the sides have different measures. Amy challenges this claim when she says she thinks it can have equal sides. The challenge takes the form of the rebuttal. She collects additional data by imagining the isosceles triangle dragging such that the sides have equal measure. Amy's claim is that the isosceles triangle can have all equal sides. She qualifies this claim with the phrase "I think."

She provides an explicit warrant when she indicates that she can imagine the congruent sides of the isosceles triangle positioned such that the third side is the same length. The researcher infers the additional data in this argument because the students do not actually drag the figure or make any changes to the screen. Instead, Amy is viewing and using the figure on the screen in a new manner, which makes it additional data. However, the lack of change in the screen makes it inferred.

In these three arguments the students collect additional data to verify or refute a claim. Looking across these arguments, two themes emerge. When the students are uncertain about an initial claim, they use the drag feature of the software to collect additional data. However, they establish certainty in different ways. In the first argument, dragging the triangle and seeing the measures of the side lengths remain equal establishes certainty. In the second argument, certainty is not established until the students position the triangle such that it has two 45° angles even though she provides an appropriate mathematical justification why the right triangle could have two 45° angles. This suggests that the students not only use the tool for exploration but also use it to establish certainty.

Second, when the students do not make use of the drag feature of the technology, they imagined its use to make a claim. In the third example argument, Amy is able to see the isosceles as a dynamic object and can imagine the sides changing such that all three sides are the same length. This suggests that, at least for Amy, the technology has become more than something she can use to accomplish a task (an artifact) to one in which she reason about it's use (a tool).

Second warrants.

The third type of argument structure created by Amy and Judy during the investigating triangles activity is that in which the students provide a second warrant to claim. In some cases, the students make a claim based on their uses of technology and do not immediately provide a warrant. Instead, they begin working with other triangle types and make new arguments. Later, the students provide an explicit warrant. Amy and Judy create three arguments of this structure, all of which would initially be considered core arguments. In the first argument the pair are determining whether a right triangle can have all equal sides. Amy briefly drags the vertices of the right triangle and the measures of the sides when she stops dragging are $m\overline{AB} = 6.6$ cm, $m\overline{BC} = 7.1$ cm, and $m\overline{CA} = 2.8$ cm. She states, "It's going to be no for right triangle." Judy and Amy begin working on the same activity with the obtuse triangle. Later, the students are filling out the section that asks them to explain why they are unable to make a right triangle with all equal sides. Judy states, "The side of a right triangle is 90°, it equals up to 90°. So if you try to change the other side it has to equal that side. So they would both be 90° and it would look like a square." This argument is illustrated in Figure 91.

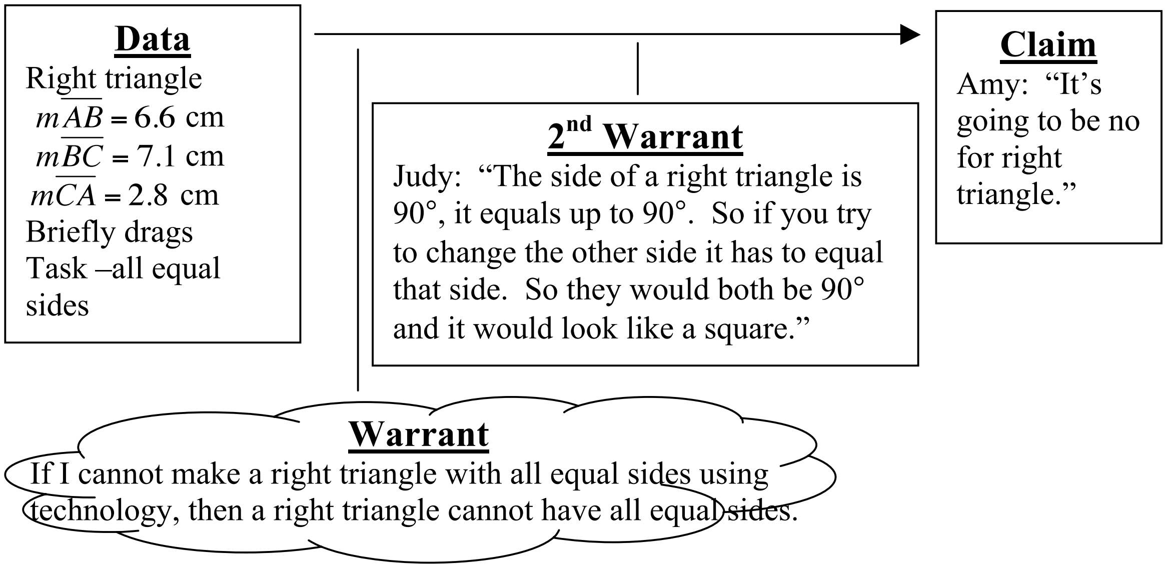


Figure 91. Amy and Judy argument created during task 3 with an inferred initial warrant and explicit second warrant.

The data for this argument are the right triangle, the dragging of the vertices of the right triangle, and the linked measures of the sides. Amy claims the right triangle is unable to have all equal sides. She does not provide a warrant for her claim and is inferred by the researcher to be, "If I cannot make a right triangle with all equal sides using technology, then a right triangle cannot have all equal sides." Later Judy provides a second warrant for this claim when she indicates that if all the sides were the same, then two of the angles would have measure 90° and that it would look like a square.

Amy and Judy create another argument with a second warrant while determining whether an obtuse triangle can have all equal sides. Amy opens the obtuse triangle window and briefly drags the vertices of the triangles. When she stops dragging the measures of the side lengths are $m\overline{AB} = 6.0$ cm, $m\overline{BC} = 9.0$ cm, and $m\overline{CA} = 4.1$ cm. Amy states, "Obtuse is a no." She and Judy begin working on the same task with the equilateral triangle. Later, when completing the section that asks the students to explain why the obtuse triangle is unable to have all equal sides, Amy states, “If you try to change like one of the sides, the other will get smaller or bigger.” This argument is illustrated in Figure 92.

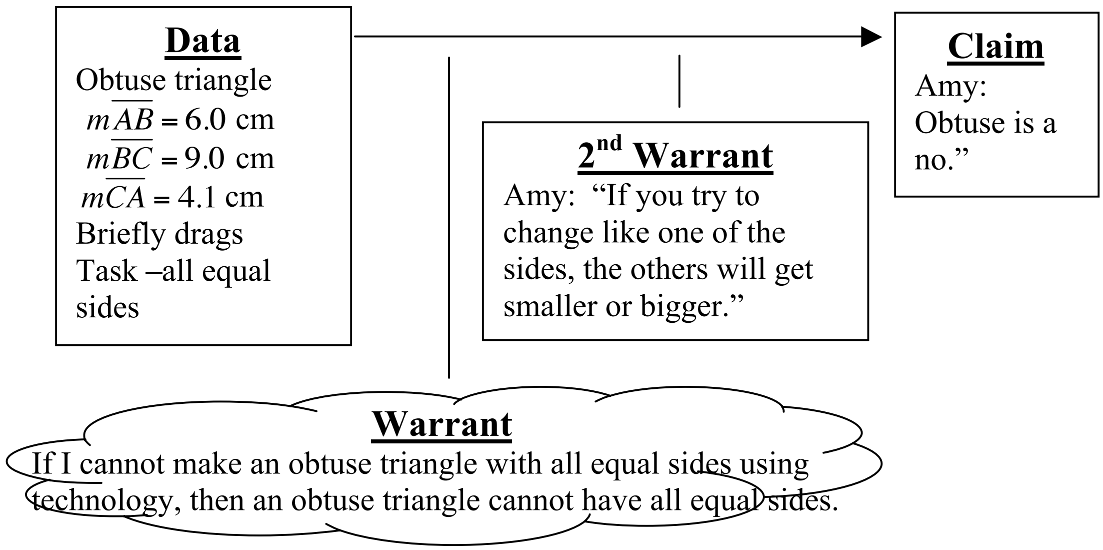


Figure 92. Amy’s argument created during task 3 with an inferred initial warrant and explicit second warrant.

The data for this argument are the obtuse triangle, the dragging of the vertices of the obtuse triangle, and the linked measures of the sides. Amy claims the obtuse triangle is unable to have all equal sides. She does not provide a warrant for her claim and is inferred by the researcher to be, “If I cannot make an obtuse triangle with all equal sides using technology, then an obtuse triangle cannot have all equal sides.” Later Amy provides a second warrant for this claim when she indicates that when you drag one of the vertices of the obtuse triangle two sides will change in length.

The third argument with a second warrant slightly differs from the previous two. In this argument, the initial warrant is explicit. In this argument Amy and Judy are determining whether a right triangle can have two angles with each measuring 45°. When Amy opens the right triangle window, the measures of the angles are $m\measuredangle BCA = 45°$, $m\measuredangle ABC = 45°$, and $m\measuredangle CAB = 90°$. Amy says to Judy, "You should have said yes on this one, put yes." Judy says, "Yeah, because I did that. It already equals 45°. Then Judy asks Amy what she should write for the explanation. Amy begins, "Because it already equals 90°. Because you can have the other two angles for 45°." Judy finishes this thought saying, "is 90 plus 90 is 180." This argument is illustrated in Figure 93.

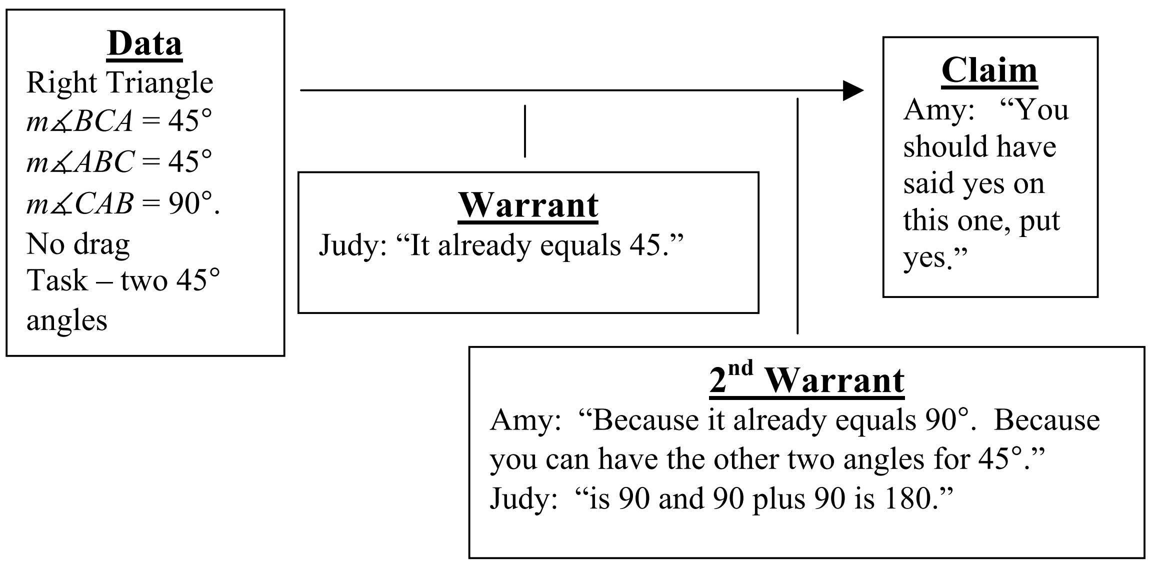


Figure 93. Amy and Judy's argument created during task 3 with explicit initial and second warrants.

The data for this argument are the right triangle and the measures of the angles on the screen. The students did not drag the triangle because they triangle already posses the desired property. Amy claims that a right triangle can have two angles with measure 45°. Judy provides the warrant for this claim indicating that the measures are on the screen were 45°. However, this was not the explanation they wrote on the activity sheet. Instead, they indicate that a right triangle has a 90° angle and the other two angles can equal 45 and the sum of these angles is 180°, which forms a triangle.

During the investigating triangles activity, Amy and Judy create three arguments in which they offer an explicit second warrant for their initial claims. For all of the arguments, the original structure of the argument (prior to the second warrant) is the core argument (similar to the argument illustrated in Figure 91). The occurrence of the second warrant is most likely related to the structure of the activity sheet. On the task sheet the students are asked to first predict whether the four triangles could have the desired property. Then the students are asked to verify or refute their predictions by using the drag feature of the technology to adjust the triangle such that the measures meet the desired property. Finally, the teacher asks the students to provide an explanation on why the triangle is or is not able to have the desired property. For the first activity (all equal sides), Amy and Judy would first make their predictions, and then use the technology to check their predictions. After they worked with each of the triangles, they begin forming their explanations. By working in this manner, it is not surprising that the arguments the students created would have this structure. The researcher chose to represent the arguments in this manner to indicate that the explicit warrant is not immediately provided by the students, rather than replacing the initial warrant (whether inferred or explicit). However, this activity sequence is not the case for the third argument. In this episode, the pair provide two different explicit warrants; one based on their uses of the technology and one based on properties and definitions. The researchers chose to represent this episode in this fashion due to the distinct nature of the warrants. For the other activities on the task sheet, Amy and Judy would first make their predictions, and then check their predictions using the technology which was immediately followed by an explanation for why the triangle is able or unable to conform to the desired property. Thus, Judy and Amy create fewer arguments with second warrants than the other groups of students.

Discussion.

While working on the investigating triangles activity, Amy and Judy create arguments of various structures. Three categories of structures emerged in the analysis; core arguments, arguments in which the students collect additional data to verify or refute a claim, and arguments in which the students provide a second warrant. Looking across these argument structures, the two themes that emerge are the number of explicit warrants (see Table 16), and the ways in which the students use the technology.

Table 16

Group 3's Argument on the Investigating Triangles Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	1	2
	Technology Not Used	0	0
Warrants Explicit			
	Technology Used	5	6
	Technology Not Used	1	0

Of the fifteen arguments created by Amy and Judy, at least one of the warrants is explicit for 12 arguments. In fact, this is the first task for this group of students in which the number of core arguments with explicit warrants out numbers those with non-explicit warrants. As previously discussed, this may be attributed to the establishing of the socio-mathematical norm that the students are to justify their claims. It also may be related to the nature of the activity. For each of the tasks within the activity, the students are expected to provide an explanation for why they are able or unable to form a triangle with the given property. This suggests, that the students are able and willing to provide justifications when prompted to do so.

The students frequently use technology during this activity. Only one of the arguments created by Amy and Judy do not involve the use of technology, which is in response to the teacher's inquiry (see Figure 87). Analyzing the fourteen explicit warrants Amy and Judy provide during this activity, eight are based on properties and definitions and six are in terms of their uses of the technology. This seems to suggest that these students are able to use the technology to investigate and discover relationship but also justify their findings using definitions and properties.

Cross-case analysis of arguments created while working on the investigating triangles task.

During the investigating triangles activity, the arguments created by the three groups of students vary in their structure and content, including the ways in which they employ the technology. Three themes emerge when looking across the arguments created by these students on this task; the structure of the argument when technology is and is not actively employed (see Table 17), the relationship between the structure of the task and the structure and content of the arguments, and the certainty of claims based on the use of technology.

Table 17

The Combined Arguments of the Three Groups on the Investigating Triangles Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	6	5
	Technology Not Used	2	2
Warrants Explicit			
	Technology Used	10	22
	Technology Not Used	5	1

While working on the investigating triangles activity, the three groups of students create 53 arguments. For 15 arguments, the students do not provide explicit warrants. Additionally, the students employ technology in 43 arguments. Of the 43 arguments in which technology is employed, the students provide an explicit warrant in 32 of these arguments (approximately 74%). Of the 10 arguments in which technology is not actively employed, the students provide an explicit warrant for 6 of these arguments (60%). Unlike the previous two tasks, the students are more likely to provide an explicit warrant when using technology than when not using the technology.

The increase in frequency of explicit warrants may be related to the norms established by the teacher throughout the instructional unit. The tasks previously analyzed, the triangle inequality activity and the triangle side and angle relationships activity, occur at the beginning of the unit. During these tasks, the teacher is attempting to establish the norm that students are to justify their claims. The investigating triangles task occurs near the end of the unit. By this time, the students knew and were expected to provide justifications for their claims.

The explicitness of the warrants when using technology may be related to the structure of the task. In this task, the teacher instructs the students to first predict whether each of the four triangles could make a triangle with the given property. Then, the students use the technology to determine whether their prediction was true. In addition, the students are to provide an explanation why each of the triangles is able or unable to make a triangle with the given property. By having students write an explanation, the teacher asks for the students to provide an explicit warrant for their claims. This suggests that these students are able to provide explicit justifications when prompted to do so.

What is note worthy is the content of the explicit warrants. Even when the students use the technology, their explanations were not always in terms of their uses of technology. This is evident in the arguments whose structure includes a second warrant. Arguments of this structure are created by all three groups of students, but the nature of the content of the second warrant differs between the groups. The majority of the second warrants offered by group 1, Heather and Mary, are generally in terms of the technology (see the argument illustrated in Figure 72). However, the second warrants for group 2, David and Erica, are generally based on the definitions and properties of the triangles (see the argument illustrated in Figure 82). The warrants for group 3, Amy and Judy, are mixed between the two (see the arguments illustrated in Figures 91 and 93). Even when the students use technology to determine whether a property was valid for a given type of triangle, many times they are able to make justifications based on the known definitions, theorems, and properties.

The third theme across the three groups is the lack of qualifiers in the arguments. The students employ technology in 43 of the 53 arguments. Of those 43, only 5 have qualified claims. The high frequency of technology use and the low number of qualifiers suggests that these students were certain about their claims whose data are based, in part, on their uses of technology.

Discussion

During the three tasks under analysis (the triangle inequality task, the triangles sides and angles relationship task, and the investigating triangles task), the arguments created by the three groups of students vary in their structure and content, including the ways in which they employ the technology. In this section, the structure and content of the arguments created by each of the three groups across the three tasks are discussed.

Summary for group 1.

While working on the three tasks, Heather and Mary create 73 arguments of various structures and content (see Table 18). Four themes emerge when looking across the arguments created by these students on these tasks; the discrepancy in the number of arguments in which the students actively employ technology to those in which they do not, the rather simplistic structure of the majority of their arguments, the relationship between the structure of the argument and the explicitness of the warrant when the students employ technology, and the explicitness of the warrant when the students do not actively use technology.

Table 18

Group 1's Arguments Across the Three Tasks by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	30	12
	Technology Not Used	4	2
Warrants Explicit			
	Technology Used	6	9
	Technology Not Used	7	3

Of the 73 arguments Heather and Mary create while working on the three tasks, the students employ technology for 57 (78%) of them. Of these 57 arguments in which the students employ technology, the structure of 36 (63%) of the arguments are core and 21 (37%) are non-core. Of the 16 arguments in which the students do not actively employ technology, 11 (69%) have the core argument structure and 5 (31%) have the non-core structure. Regardless of whether the students use technology, the students are more likely to create arguments with the core structure.

When Heather and Mary employ the technology, they create 36 core arguments, 6 (17%) of which have warrants are explicit. In addition, they create 21 arguments while employing technology whose structure is non-core. Of these 21 non-core arguments, 9 (43%) of the arguments have at least one warrant that is non-explicit. This suggests when these students use technology and create a more complex structure, the students are more likely to provide an explicit warrant than when the students use technology and create an argument with the core structure. This may be due to the number of opportunities to provide an explicit warrant. Heather and Mary create many non-core arguments including those in which they collect additional data (e.g. the argument illustrated in Figure 70), use a claim as data (e.g. the argument illustrated in Figure 42), and provide a second warrant (e.g. the argument illustrated in Figure 72). In all of these structures, Heather and Mary have the opportunity to provide more than one warrant. In the core arguments, this is not the case. Another possible reason why the non-core arguments have a higher percentage of arguments with an explicit warrant may be due to the challenge of a claim. At times, Heather, Mary, or the teacher challenges a claim. This challenge may prompt the students to provide an explicit warrant. If a core argument remains unchallenged, then the students may not be compelled to provide an explicit warrant.

When the students do not actively use technology in the creation of their arguments, the students are more likely to provide an explicit warrant. In fact, of the 16 arguments in which technology is not actively employed, Heather and Mary provide an explicit warrant for 10 of the arguments (63%). As previously discussed, many of the tasks on which the students are working while creating these arguments would be considered generalization or justification type tasks. Thus, the students may be more likely to provide, or are asked to provide, a justification for their claims when working on these types of tasks.

In summary, Heather and Mary are more likely to create arguments in which the structure is core, the warrant is not explicit, and the students use the technology.

Summary for group 2.

While working on the three tasks, David and Erica create 55 arguments of various structures and content (see Table 19). Four themes emerge when looking across the arguments created by these students on this task; the discrepancy in the number or arguments in which the students actively employ technology to those in which they do not, the complex structure of their arguments, the relationship between the structure of the argument and the explicitness of the warrant when the students employ technology, and the explicitness of the warrant when the students do not actively use technology.

Table 19

Group 2's Arguments Across the Three Tasks by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	14	11
	Technology Not Used	0	2
Warrants Explicit			
	Technology Used	3	16
	Technology Not Used	7	2

Of the 55 arguments David and Erica create while working on the three tasks, the students employ technology for 44 (80%) of them. Of these 44 arguments in which the students employ technology, the structure of 17 (39%) of the arguments are core and 27

(61%) are non-core. Of the 11 arguments in which the students do not actively employ technology, 7 (64%) have the core argument structure and 4 (36%) have the non-core structure. This suggests when David and Erica are using technology, the arguments they create have a more complex structure compared to those they create when not actively using the technology.

When David and Erica employ the technology, they create 17 core arguments, 3 (18%) of which the warrant is explicit. In addition, they create 27 arguments while employing technology whose structure is non-core. Of these 27 non-core arguments, 16 (59%) of the arguments have at least one warrant that is explicit. This suggests when the students use technology and create an argument with a more complex structure, the students are more likely to provide an explicit warrant. This may be due to the number of opportunities to provide an explicit warrant. David and Erica create many non-core arguments including those in which they collect additional data (e.g. the argument illustrated in Figure 25), imagine using the technology (e.g. the argument illustrated in Figure 27) and provide a second warrant (e.g. the argument illustrated in Figure 82). In all of these structures, David and Erica have the opportunity to provide more than one warrant. In the core arguments, this is not the case. Another possible reason why the non-core arguments have a higher percentage of arguments with an explicit warrant may be due to the challenge of a claim. At times, David, Erica, or the teacher challenges a claim. This challenge may prompt the students to provide an explicit warrant. If a core argument remains unchallenged, then the students may not be compelled to provide an explicit warrant.

When the students do not actively use technology in the creation of their arguments, the students are more likely to provide an explicit warrant. In fact, of the 11 arguments in which technology is not actively employed, David and Erica provide an explicit warrant for 9 (82%) of the arguments. As previously discussed, many of the tasks on which the students are working while creating these arguments would be considered generalization or justification type tasks. Thus, the students may be more likely to provide, or were asked to provide, a justification for their claims when working on these types of tasks.

In summary, David and Erica create more arguments with at least one explicit warrant (28) than arguments in which the warrants are inferred (27). In addition, the students are more likely to create an argument with a more complex structure than the core structure and use the technology.

Summary for group 3.

While working on the three tasks, Amy and Judy create 51 arguments of various structures and content (see Table 20). Three themes emerge when looking across the arguments created by these students on this task; the discrepancy in the number or arguments in which the students actively employ technology to those in which they do not, the relationship between the structure of the argument and the explicitness of the warrant when the students employ technology, and the explicitness of the warrant when the students do not actively use technology.

Table 20

Group 3's Arguments Across the Three Tasks by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	17	11
	Technology Not Used	1	1
Warrants Explicit			
	Technology Used	6	10
	Technology Not Used	3	2

Of the 51 arguments Amy and Judy create while working on the three tasks, the students employ technology for 44 (86%) of them. Of these 44 arguments in which the students employ technology, the structure of 28 (64%) of the arguments are core and 16 (36%) are non-core. Of the 7 arguments in which the students do not actively employ technology, 4 (57%) have the core arguments and 3 (43%) have the non-core structure. Regardless of whether the students use technology, the students are more likely to create arguments with the core structure.

When Amy and Judy employ the technology, they create 23 core arguments, 6 (26%) of which the warrant is explicit. In addition, they create 21 arguments while employing technology whose structure is non-core. Of these 21 non-core arguments, 10 (48%) of the warrants are explicit. This suggests when the students use technology and create a more complex structure, the students are more likely to provide an explicit warrant. This may be due to the number of opportunities to provide an explicit warrant. Amy and Judy create many non-core arguments including those in which they collect additional data (e.g. the argument illustrated in Figure 33), and provide a second warrant (e.g. the argument illustrated in Figure 91). In all of these structures, Amy and Judy have the opportunity to provide more than one warrant. In the core arguments, this was not the case. Another possible reason why the non-core arguments have a higher percentage of arguments with an explicit warrant may be due to the challenge of a claim. At times, Amy, Judy, or the teacher challenges a claim. This challenge may prompt the students to provide an explicit warrant. If a core argument remains unchallenged, then the students may not be compelled to provide an explicit warrant.

When the students do not actively use technology in the creation of their arguments, the students are more likely to provide an explicit warrant. In fact, of the 7 arguments in which technology is not actively employed, Amy and Judy provide an explicit warrant for 5 (71%) of the arguments. As previously discussed, many of the tasks on which the students are working while creating these arguments would be considered generalization or justification type tasks. Thus, the students may be more likely to provide, or were asked to provide, a justification for their claims when working on these types of tasks.

In summary, Amy and Judy are more likely to create arguments in which structure is core, the warrant is not explicit, and the students use the technology.

CHAPTER 5

The Arguments Created by Students in the Non-Technology Class

This chapter details the arguments created by the students working in the non-technological environment. For each of the three analyzed tasks, a description of the mathematical objectives, the tools and task sheet, and the teaching of the lesson begin each task section. Thereafter, an analysis is presented of the arguments created by each of the three groups of students while working on the task. To close each task section, the researcher provides a cross-case analysis of the arguments created by the three groups of students. At the end of the chapter, the researcher provides an analysis of the arguments created by each group of students across the three tasks.

Task 1 - Triangle Inequality Task

The task began with an exploration and discussion about the triangle inequality theorem. The triangle inequality theorem states "if A, B, and C are three noncollinear points, then $AC < AB + BC$" (Venema, 2006, p. 103). In their previous elementary and middle school math classes, the students were expected to have studied triangles. This includes the definition of a triangle, the differences between a triangle and other geometric figures, and the classification of triangles. However, the standard course of study for these prior mathematics courses did not indicate whether the students had explored some of the conditions needed for a triangle to exist. Rather than building on the students' notions of classification, the researcher chose to begin the unit exploring a concept that was novel to them. This was done for three reasons. First, the unit under study was on triangles in general and the researcher wanted the students to consider and understand some of the conditions needed to form a triangle. Second, the researcher wanted the students to begin the unit with an exploration rather than a recollection of facts. Third, a novel task may produce a larger number and greater variety of structures of arguments than tasks that ask the students to recall previously studied concepts.

The objectives for this task centered on the students' discovery and understanding of the triangle inequality. The researcher wanted the students to be able to use the non-technological tools to determine whether a given set of segments could form a triangle and use the tools to experiment with other lengths to create a conjecture about the relationship among the lengths of the sides of a triangle. The teacher also wanted the students to compare the sets of segments that were able to form a triangle and those that were not. By the end of the task, the researcher wanted the students to state the triangle inequality theorem, provide a justification for this theorem, and use the theorem to solve problems.

With these objectives in mind, the researcher created an activity and corresponding task sheet to be used to teach this concept. The activity and task sheet were similar to that given to the class working with technology. Instead of using technology to investigate, the students in this class used snap-cubes.

The tools and task sheet.

The students were given a set of snap-cubes (approximately 20) and a task sheet to assist in their exploration (see Appendix F). The task sheet provided sets of three segment lengths to use with the snap-cubes. Using the snap-cubes, the students were to create segments according to the lengths on the task sheet and arrange the segments to determine whether a triangle could be formed. The directions indicate that the students were to draw the triangle if one could be formed and to write impossible if one could not be formed. In the directions given to the class, the teacher asked the students to draw the appearance of the figure they created when a triangle could not be formed.

The sets of segments given to the students on the task sheet were selected for a variety of reasons. First, there are three sets of segments that are able to form a triangle and three sets that are unable to form a triangle. Second, the longest segment and the shortest segment were not always the same segment (e.g. the longest segment was not always segment *c*). Third, it was anticipated that the students were familiar with different classifications of triangles, mainly isosceles triangles. The decision to use two sets of segments that would form isosceles triangles was to draw upon this prior knowledge. Also, the isosceles triangles are a bit different; on the first set (3, 4, 4,) the non-congruent side is the shortest and the second set (5, 5, 6) the non-congruent side is the longest. This was done in hope that the students would not develop erroneous conclusions about the relationship among the lengths of the sides of an isosceles triangle, which would have to be resolved in a later lesson. Finally, there were many example types that were omitted from this task sheet, most notably a set of segments that would form a line. A set of this type was purposely omitted in hopes that the students may investigate this case on their own.

After the students determined whether a triangle could be formed for the sets of segments on the task sheet, the students were asked "Why was it impossible to construct a triangle with some of the given lengths?" Rather than have the students answer why it did work, the teacher wanted the students to attend to those that didn't work. This was done so the students could begin comparing and finding a common attribute for those sets of segments that could not form a triangle, mainly that the length of one of the segments was much greater than the lengths of the other two segments. After the students answered this question, they were asked to "Write a conjecture about the relationship among the lengths of the three sides of a triangle." This was given after the previous question in hope that the students would further investigate when a segment length becomes longer than the other two segments and a triangle was unable to be formed.

Teaching of the lesson.

The triangle inequality task was taught during the first class meeting of the instructional unit. The class began with a short discussion of the definition of a triangle and the students told the teacher what they knew about triangles. The teacher told the students they would be investigating a relationship among the sides of a triangle. He than distributed the task sheet and asked the students to open up the pre-constructed sketch on the laptops.

The teacher and students read the directions on the task sheet and worked the first set of segments (2, 3, 4) together. The teacher demonstrated how to create segments using snap-cubes, and how the ends of the segments formed with snap-cubes should be placed to determine whether a triangle could be formed. After finishing the first set of segments, the teacher asked the students to work with their partner to complete the task sheet. The students were given time to work on this task and to answer the questions. Once the majority of the pairs of students had completed the sheet, the teacher asked the students to participate in a whole class discussion. The teacher asked the students whether each set was able to form a triangle, discuss their reasons for why some sets of segments were unable to form a triangle, and state their conjectures for the relationship among the lengths of the sides of a triangle.

During this whole class discussion, the students correctly determined whether a triangle could be formed for each of the sets of segments on the task sheet. In addition, they provided reasons for why some sets of segments were unable to form a triangle. However, the conjectures they created about the relationship among the lengths of the sides of a triangle were relative to the lengths of the side lengths. To assist the students in making their conjectures more definite, the teacher asked the students to create segments of lengths 2, 4, 7 using the snap-cubes and determine whether these snap-cube segments form a triangle. The class agreed that a triangle could not be formed. The teacher asked the students to remove a snap-cube from the segment of length 7 and determine whether a triangle could be formed with this new segment. The teacher requested the students to continue to perform this action until they were able to form a triangle. After the students had found a segment that could form a triangle with segments of lengths 4 and 2, the teacher led a discussion about what they found. The class discussed whether a triangle could be formed from segments of lengths 2, 4, and 6. After some deliberation, the students realized that these segments would form a line. The teacher then asked the students to consider the values of these lengths. The students said 2 plus 4 is 6. Another student said the 4 needs to be a five so that it "takes up space by going up." The class then discussed which values would make a flat figure and then those that would take up space. The teacher focused the students' attention on the values of the lengths of the sets of segments on their task sheet. While looking at these values, one student stated the triangle inequality theorem. The other students agreed and verified it on their task sheet. Next, the teacher led a discussion on how to determine whether a set of segments could form a triangle. Then, the teacher moved on to the next task on the agenda.

On that night's homework (see Appendix G), the students were asked to determine whether a set of segments could form a triangle and state how they knew this to be true. In addition, the students were given the values for the lengths of two segments (4 and 9) and were asked to list the possible values for the length of the third segment that would be able to form a triangle, including the shortest possible side length and the longest possible side length.

On the second day of class, the teacher asked the students to compare their answers on the homework. The teacher walked around the room to check whether the students completed the assignment. After seeing the students had some difficulty, the teacher asked students to show their solution to the first two homework problems. After this discussion, the class worked on the task they did not finish the previous day.

On the third day of class the students turned in their homework and were given a warm-up with four questions, two related to the triangle inequality theorem (see Appendix C). The students had difficulty answering the two questions related to the triangle inequality. The teacher decided to have the students work together as a class on these problems and began with number 3. The problem asked the students to list the possible values of the length of the segment that could form a triangle with segments of lengths 10 and 7. The teacher asked the student to look at specific lengths of sides, specifically 8 and 20. The students were able to correctly state whether these lengths would form a triangle with the given segments. Then, the teacher focused the students on listing the lengths of the segments that would form a triangle with the given segments. The students began calling out values and one student indicated the length of the third segment just had to be bigger than 4. The teacher asked the class if segments of lengths 10, 7, and 1000 would form a triangle. The students said no and began discussing whether there was limit to the longest segment that could form a triangle with the given segments. The teacher asked the class to discuss these ideas with the partners. During this time, the teacher walked around the classroom and questioned the students as to whether the values of the lengths of the sides must be while numbers and what is the smallest and largest possible value. After some time had passed, the teacher led a class discussion. The students discussed the different possible values and begin listing them on the board. The teacher asked the class if 3 would form a triangle with the given segments. The students said no and attempted to find the lowest possible value that would form a triangle. The students stated that it only has to be more than 3. The teacher then turned the students attention toward the largest value. Eventually the students realized if you add the two values of the given segment lengths, the sum is the limit for the highest possible length that can form a triangle. Also, if the values of the given segments' lengths are subtracted, then the difference is the limit for the smallest possible value of the segment length that can form a triangle with the given segments. Then the teacher asked them to work on the fourth problem on the warm-up. The students worked on this problem and then the teacher led a discussion on their answers. The majority of the students in the class were able to solve the problem correctly.

In the following sections, the arguments created by three pairs of students while working on these activities are analyzed and discussed. For each pair of students, the arguments were first categorized by their basic structure. Then, the content and structure of the arguments within these basic categories were analyzed, including the students' uses of tools. The themes that emerged from these analyses are discussed for each pair of students and across the pairs of students.

Group 1's arguments while working on the triangle inequality task.

The analysis of Andy and Frank's arguments while working on the triangle inequality task can be categorized into three argument structures; core arguments, arguments in which the one of the pair of students collect additional data to verify or refute a claim, and arguments in which a claim is made and then used as data to make another claim. These arguments structures are discussed below.

Core arguments.

A core argument is an argument that contains data, a claim, and a warrant. At times, students may qualify their claims, but in Andy and Frank's case, this did not occur. The structure of their core arguments seems to depend on whether the pair uses the tools, in this case the snap-cubes. For six core arguments, the students use the tools and the warrants are not explicit. For four core arguments, the students do not actively use the tools and the warrants are explicit. These two argument structures are discussed below.

Non-explicit warrants, tools used.

For many of the core arguments, Andy and Frank use the tools. However, their warrants for these claims are not explicit. In all of these arguments, the data are the lengths of the three segments, the arranging of the snap-cube segments with these lengths, and the appearance of the figure the students form using the snap-cube segments. For example, Andy and Frank determine whether a triangle can be formed with segments of length 3, 2, and 6. Andy creates the segments using the snap-cubes and arranges them such that the one end of each of the segments of lengths 2 and 6 are touching the ends of the segment of length 3. Andy moves the segments of lengths 2 and 6 while keeping one endpoint of each segment touching the segment of length 3. He is unable to get the free endpoints of segments of lengths 2 and 6 to meet and form a triangle. Andy states, "That's impossible." This argument is illustrated in Figure 94.

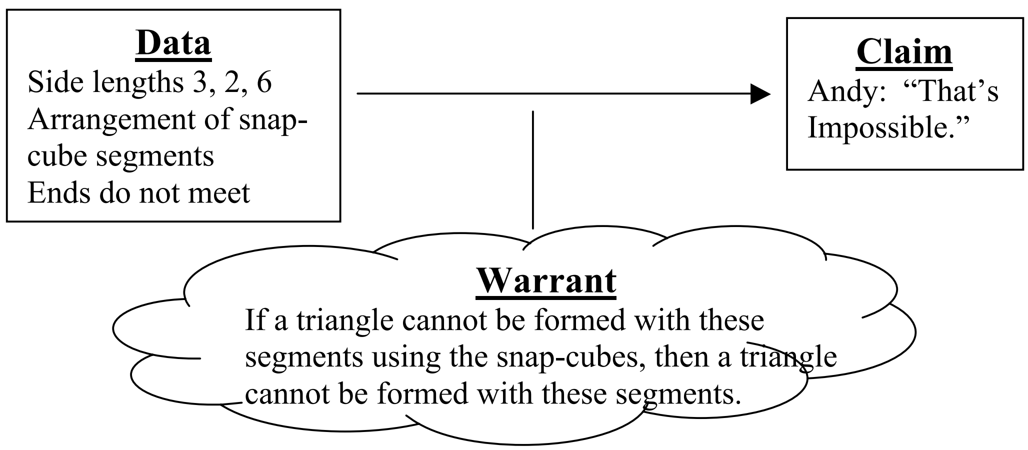


Figure 94. Frank and Andy's core argument created during task 1 with an inferred warrant and tool use in the data.

The data for this argument are the lengths of the segments, and the arrangement of the snap-cube segments such that a triangle is not formed. Andy claims that it is impossible to make a triangle with segments of lengths 3, 2, and 6. The warrant for this claim is not explicit and is inferred by the researcher to be, "If a triangle cannot be formed with these segments using the snap-cubes, then a triangle cannot be formed with these segments."

This argument structure also occurs for those arguments in which the segments form a triangle. For example, Andy and Frank are given segment lengths 4 and 2 and are asked to find the length of a segment that would form a triangle with the given segments. Using the snap-cubes, Frank creates and arranges segments of lengths 4, 2, and 4 to form a triangle. Andy claims, “That works.” This argument is illustrated in Figure 95.

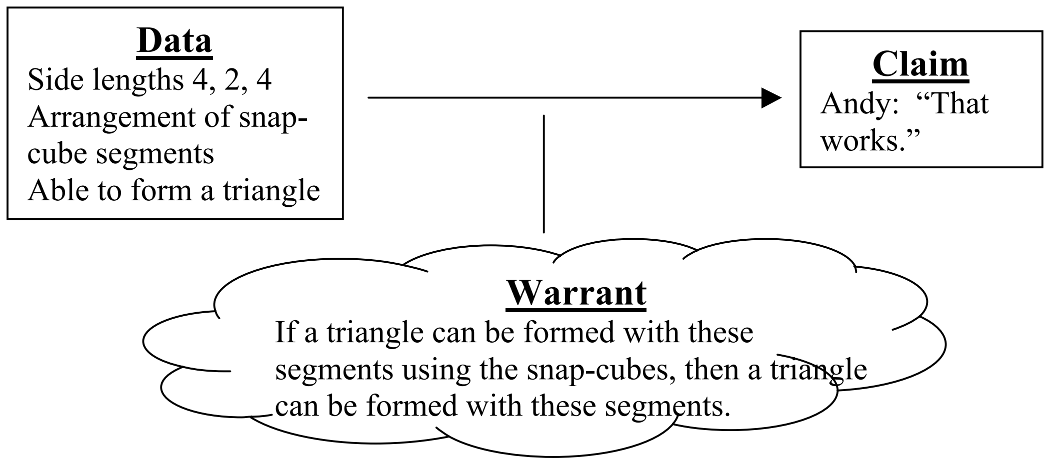


Figure 95. Andy’s core argument created during task 1 with an inferred warrant and tool use as data.

The data for this argument are the lengths of the segments, and the arrangement of the snap-cube segments such that a triangle is formed. Andy claims that a triangle with segments of lengths 4, 2, and 4 is possible. The warrant for this claim is not explicit and the researcher infers it to be, “If a triangle can be formed with these segments using the snap-cubes, then a triangle can be formed with these segments.”

Across these six core arguments there are many similarities. First, the structure of the arguments is the same; the claims are explicit and the warrants are not explicit. Second, the data for these arguments are based, in part, on the use of tools, the snap-cubes. Third, the task on which the students are working is the same; to determine whether the given segments form a triangle.

Absence of tools.

In four of the core arguments, Andy and Frank do not use the tools and the warrants are explicit. For example, the pair attempt to answer the question, "Why was it impossible to construct a triangle with some of the given lengths?" Frank states, "They weren't long enough to make it." Andy concurs, "It was impossible because the lengths weren't long enough." This argument is illustrated in Figure 96.

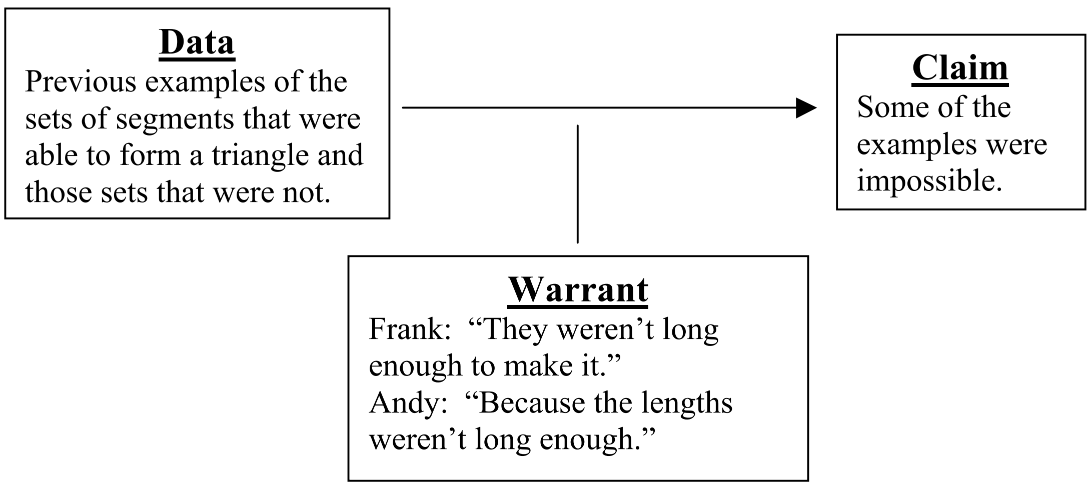


Figure 96. Frank and Andy's core argument created during task 1 with an explicit warrant in response to a question on the task sheet.

The data for this argument are the previous examples of sets of segment lengths that are both able and unable to form a triangle. The students do not explicitly provide the claim. Instead, the question provides the claim that some of the examples were impossible. Both Andy and Frank provide a warrant for this argument in their statements that the segments were not long enough. For this argument it is expected that the warrant is explicit. The question provides the claim and asks the students for the warrant. The data for this argument, which triangles could and could not be formed with the given segments, are the claims of prior arguments in which the students used the tools. Hence, the students do not actively use the tools in the creation of this argument. Rather, the data for this argument are the products of the students' previous uses of tools, an indirect use of the tools.

Another core argument in which the students do not actively use the tools and provide an explicit warrant is in response to the homework problem that asks the students to determine whether the segments of lengths 7, 7, and 2 could form a triangle. The pair of students compare their answers on the homework. For this problem, Andy and Frank do not have the same answer. Andy indicates that the segments of these lengths can make a triangle while Frank disagrees. Andy explains, "See you [make] the same and then you just got the smallest and you can narrow it." As he makes this explanation, Andy places his fingers together and draws his palms inward. Frank agrees, "Yeah." This argument is illustrated in Figure 97.

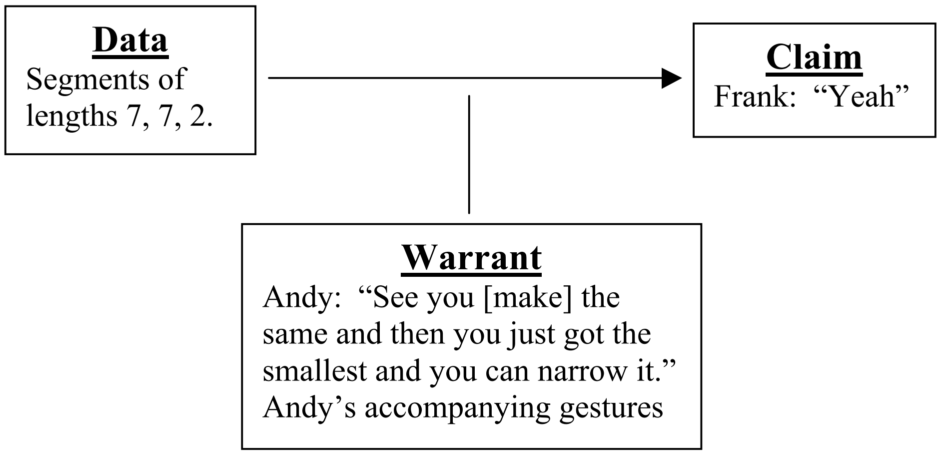


Figure 97. Frank and Andy's core argument created during task 1with an explicit warrant and tools not used.

The data for this argument are the segments of lengths 7, 7, and 2. Both Andy and Frank claim that a triangle can be formed with these segments. Andy provides an explicit warrant when he indicates, through his words and hand gestures, that a triangle can be formed with these segments by taking the two equal length segments and moving them towards each other such that the bottom segment is relatively short.

Conclusions.

Looking across these core arguments, two major themes emerge. First, when the students do not actively use the tools, they provided explicit warrants. One difference between these arguments and those in which the tools are actively employed is the request for a warrant. Of the six core arguments that do not involve the active use of tools, two are made in response to a problem on the homework in which the students are asked to explain their solution (see argument illustrated in Figure 97), three are made in response to a teacher's question, and one is in response to a question on the task sheet that asks the students for an explicit warrant (see the argument illustrated in Figure 96). The students may feel compelled to provide an explicit warrant when making a generalization and when prompted to do so by the teacher either in person or on the homework.

The second theme that emerges from the analysis of these core arguments is when Andy and Frank create a core argument based on their uses of tools they do not provide an explicit warrant. However, in all of these arguments, the students are working on items on the task sheet in which they are determining whether a triangle could be formed with the given set of segment lengths. These items do not ask the students to justify their responses, so they may not have felt compelled to do so.

Additional data collected by Andy and Frank.

The second type of argument structure created by Andy and Frank is that in which the students are compelled to seek additional data after an initial claim is made to verify or refute that claim. In this structure, three types of arguments are identified; those in which the students collect additional data using the tools, those in which they look at alternative features of their data, and those in which the decision to collect additional data is based on an explicit challenge. These three argument structures are detailed below.

Collection using tools.

In one argument, Andy and Frank use the tools to gather additional data after an initial claim is made. In this argument, Andy and Frank are determining whether a triangle can be formed with segments of lengths 3, 4, and 4. Frank creates the segments using the snap-cubes and builds a triangle such that the segment of length 3 is on the bottom and between the segments of lengths 4. Frank says, “Yeah.” Andy states, “I think he wants you to put it in order." Frank rearranges the segments such that segment of length 3 is no longer on the bottom and he is able to form a triangle. Andy concludes, "Yeah. So you write yes." This argument is illustrated in Figure 98.

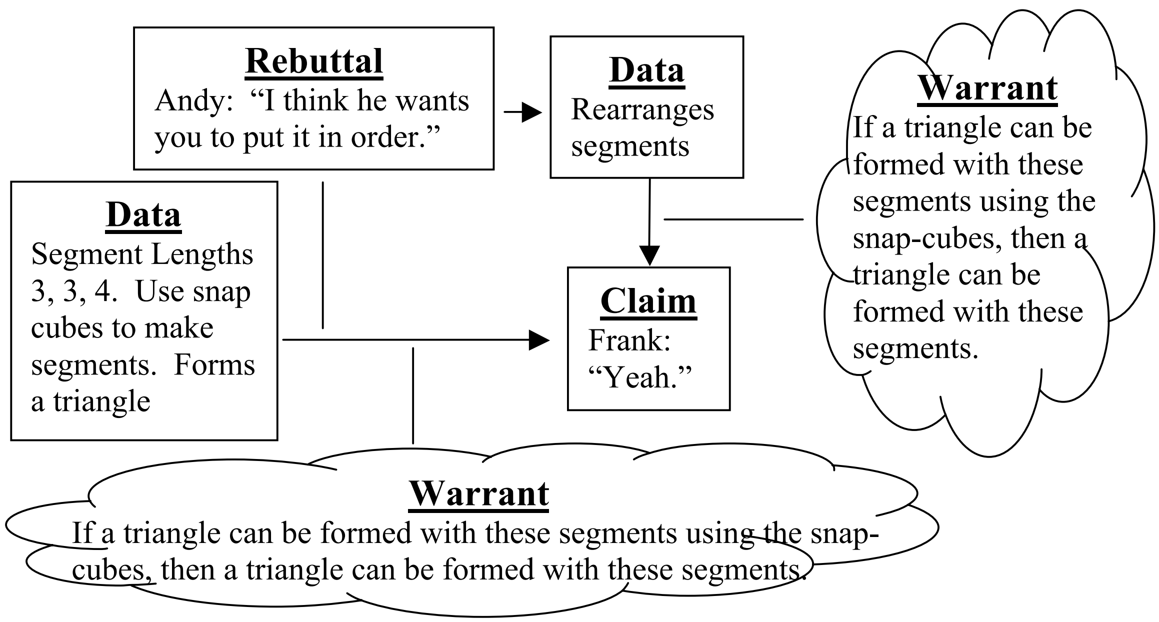


Figure 98. Frank and Andy's argument created during task 1 with additional data collection in which the tools was used to collect the initial and additional data.

The initial data for this argument are the segments of lengths 3, 3, and 4, the uses of the tools to form these segments, and the arrangement of the snap-cube segments to form a triangle. Frank claims that a triangle can be formed with these segments. He does not provide an explicit warrant for his claim and is inferred by the researcher to be, "If a triangle can be formed with these segments using the snap-cubes, then a triangle can be formed with these segments." Andy provides and explicit rebuttal by indicating that the segments must be in the desired order. Frank collects additional data by rearranging the segments and forming a triangle. He verifies his initial claim. Frank does not provide an explicit warrant for his verification and the researcher infers it to be the same as the previous warrant.

Look at other features of the initial data.

In two of the arguments in which additional data is collected, Andy and Frank do not actively collect additional data. Instead, they look at other features of the initial data and make claims based on this data to verify or refute a claim. For example, Andy and Frank are determining whether a triangle can be formed with segments of lengths 6, 1, and 4. Frank uses the snap-cubes to create these segments and tries to arrange the segments such that a triangle is formed. He is unsuccessful and states, “Write impossible. You can’t make a triangle with that.” The students have the segments arranged such that the segments of lengths 1 and 4 form a right angle and the segment of length 6 extends past the segment of length 1. Andy says, “Yeah you can, it still got the right angle, it’s still the three points and stuff. Ask him.” The teacher stops at their table and asks how they are doing. Frank says, “We think it can make a triangle.” The teacher responds, “Is this point lining up with this?” pointing to the two endpoints of the segments that are not touching. Both Frank and Andy say, “No.” This argument is illustrated in Figure 99.

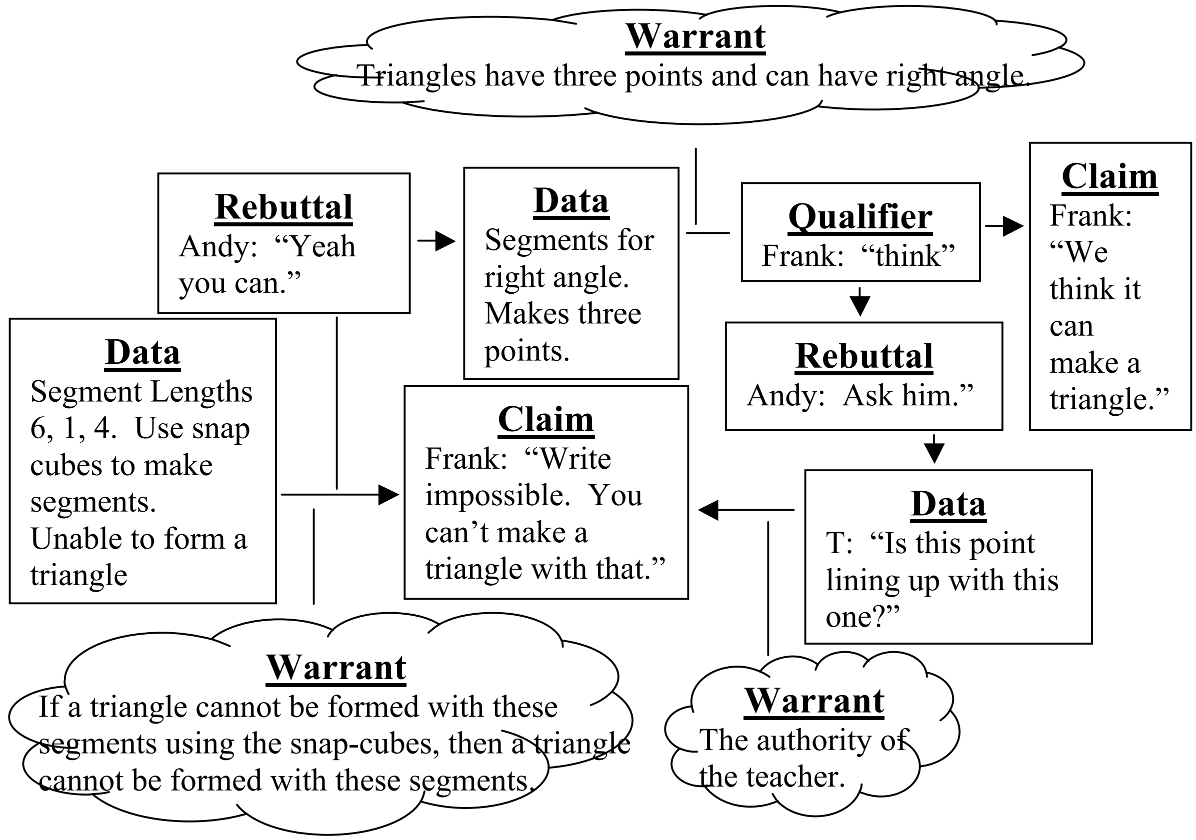


Figure 99. Frank and Andy's argument created during task 1 with additional data collection which is collected using the tools and by asking the teacher.

The initial data for this argument are the segments of lengths 6, 1, and 4, the uses of the tools to form these segments, and the arrangement of the snap-cube segments that do not form a triangle. Frank claims that a triangle cannot be formed with these segments. He does not provide an explicit warrant for his claim and is inferred by the researcher to be, "If a triangle cannot be formed with these segments using the snap-cubes, then a triangle cannot be formed with these segments." Andy provides an explicit rebuttal by stating the segments can form a triangle. He points out the right angle formed by the segments of lengths 1 and 4, and the three points formed by these segments. These are additional data in this argument.

Frank adjusts his claim and states that a triangle can be formed with these segments. He qualifies his claim with the term "think" which suggests some uncertainty. The students do not provide an explicit warrant and is inferred by the researcher to be "Triangles have three points and can have right angles." Andy provides an explicit rebuttal by saying, "Ask him [the teacher]". The teacher provides additional data by pointing to the segments that are not touching and asking if they are touching. Frank and Andy claim "no" which verifies Frank's initial claim. Neither Frank nor Andy provides an explicit warrant for this verification and the researcher infers it to be "The authority of the teacher."

In the two previous arguments, the students collect the initial data in the same manner; the students create the segment lengths using the snap-cubes and arrange the segments trying to form a triangle. However, the ways in which the students collect additional data in these two arguments differ. In the first argument, the students gather additional data by rearranging the segments. In the second argument, the students collect additional data by noticing and pointing to features of the arrangement of the segments. In both arguments, the claims the students make based on their additional data collection verify their initial claim.

Challenges.

At times, the students collect additional data due to the challenge of another person. In seven arguments, the collection of additional data is prompted by an explicit challenge from another person. In three of these cases, the challenge comes from the teacher. For example, Andy and Frank are determining the possible lengths of the third side of a triangle given sides of lengths 4 and 7. Andy states, "1 is possible. 1 is still a measure." The teacher asks, “1 is possible? 1 is still a measure? So show me.” Andy makes a sketch on his paper and says, “Yeah, yeah you’re right. 1 can’t because it would be like right there.” This argument is illustrated in Figure 100.

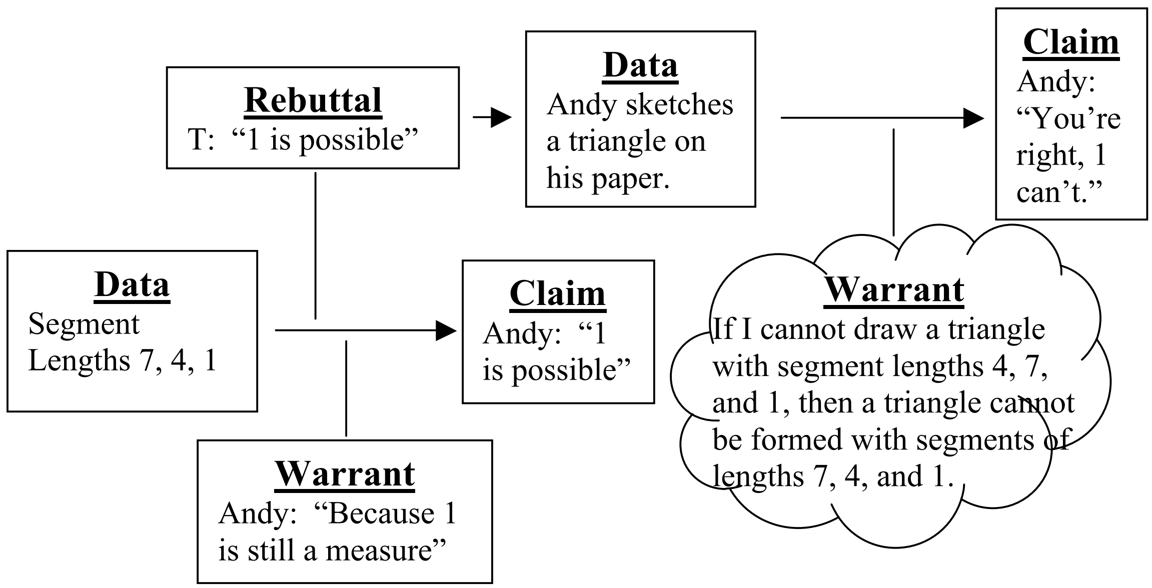


Figure 100. Andy’s argument created during task 1 with additional data collection in response to a challenge from the teacher.

The initial data for this argument are the segments of lengths 7, 4, and 1. Andy claims that a triangle can be formed with these segments. He provides an explicit warrant by stating that 1 is still a measure. The teacher challenges this claim by asking if 1 is possible, which is a rebuttal. Andy collects additional by drawing a sketch of the segments on his paper. Based on his sketch, Andy makes a new claim that segments of lengths 7, 4, and 1 cannot form a triangle. He does not provide an explicit warrant for his claim and is inferred by the researcher to be, “If I cannot draw a triangle with segment lengths 4, 7, and 1, then a triangle cannot be formed with segments of lengths 7, 4, and 1.”

In the previous argument, the challenge comes from the teacher. However, this is not always the case. As the argument illustrated Figure 99 demonstrates, another student can make the challenge. In this argument, Andy challenges Frank's claim that segments of lengths 6, 1, and 4 cannot form a triangle.

Conclusions.

The arguments in which Andy and Frank collect additional data can be categorized into three types; those in which the students collect additional data using the tools, those in which they look at alternative features of their original data, and those in which the decision to collect additional data is based on an explicit challenge. Of the eight arguments in which the students gather additional data, seven of the initial claims are challenged by another person; three by the teacher and four by a student. All of the initial claims for these seven arguments are not qualified which demonstrates the students are certain about their claims.

Linked argument.

The third type of argument structure created by Andy and Frank is that in which the students make an initial claim and use that claim as data to make a further claim, or a linked argument. For example, Andy first explains to the class that when three segments form a triangle, the two non-bottom segments meet above the bottom segment, which takes up space. He uses segments of lengths 5, 4, and 6 as an example to assist him in his explanation. The teacher asks Andy, "So when would it not take up space?" Andy replies, "When it's down." The teacher asks a second question, "What numbers would make it flat?" Andy states, "When it would be nine. Because 5 and 4 equal 9." This argument is illustrated in Figure 101.

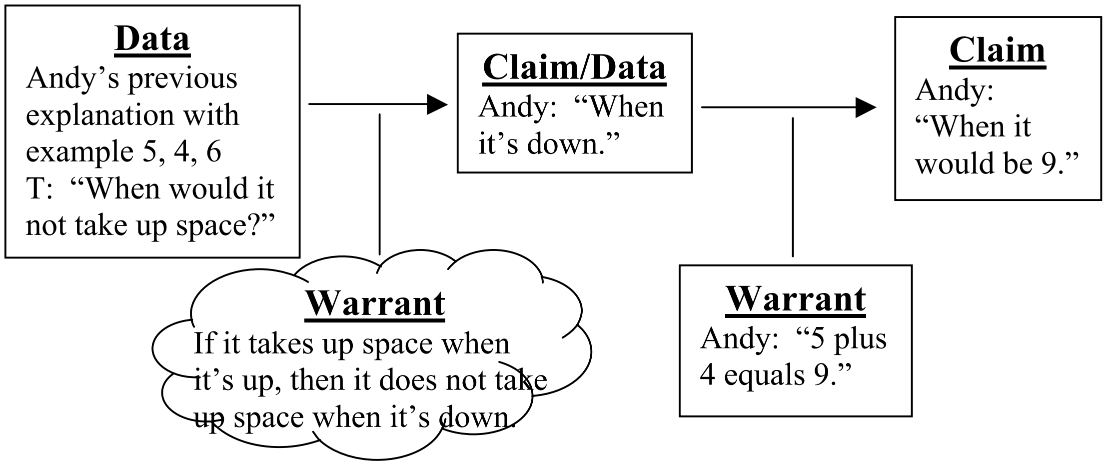


Figure 101. Frank and Andy's linked argument created during task 1.

The initial data for this argument are Andy's previous explanation with the example using segments of lengths 5, 4, and 6 and the teacher's question. Andy claims the segments would not take up space when the segments are down, or close to the bottom segment. He does not provide a warrant for his claim and is inferred by the researcher to be, "If it takes up space when it's up, then it does not take up space when it's down." This claim becomes data for Andy's second claim, that the two segments are down, close to the bottom, when the bottom segment is 9 given the other two segments are 5 and 4. Andy provides an explicit warrant for this claim when he says 5 plus 4 equals 9.

Even though only two arguments are categorized as having this link structure, this structure is noteworthy. These students are able to use a claim as data to make another claim. Perhaps, when students create arguments of this structure, they demonstrate they are able to think and reason in a deductive manner.

Discussion.

While working on the triangle inequality task, Andy and Frank create arguments of various structures. Three categories of structures emerge in the analysis: core arguments, arguments in which the students gather additional data to verify or refute a claim, and linked arguments. Looking across these argument structures, two themes emerge: the relationship between the number of explicit warrants and the use of tools (see Table 21), and the prompts for explicit warrants.

Table 21.

Group 1's Arguments on the Triangle Inequality Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Tools	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	4	2
	Tools Not Used	0	1
Warrants Explicit			
	Tools Used	0	0
	Tools Not Used	6	6

Of the 19 arguments created by Andy and Frank, the students employ the tools in 6 (32%) of the arguments and none of the warrants in these arguments are explicit. Of the 13 arguments in which the students do not actively employ the tools, the warrants for 12 (92%) of the arguments are explicit. This seems to suggest that when the students do not actively use tools in the creation of their arguments, they are more likely to provide an explicit warrant.

However, the explicitness of the warrant may be due to the nature of the task and whether the students are prompted to provide a justification. Of the 6 arguments in which the tools are used, the task on which the students are working is the same, to determine whether a given set of segments would form a triangle. Five of these six arguments are in response to items on the task sheet, which did not require the students to justify their claims. For the 13 arguments in which the students do not actively employ the tools, the tasks on which the students are working request a justification. The students create one of the arguments in response to a question on the task sheet in which the students are given the claim and asked to provide a warrant (see the argument illustrated in Figure 96). Five of the arguments are made in response to items on the homework, all of which ask the students to explain their solutions. Seven of the arguments are made in response to the teacher's questions or statements and, in most cases, the teacher asks the students to justify their claim. Thus, whether the pair provides an explicit warrant in an argument might be better related to the nature of the task than the use of the tool.

Group 2's arguments while working on the triangle inequality task.

The analysis of Bob and Ellen's arguments while working on the triangle inequality task can be categorized into three argument structures; core arguments, arguments in which the students collect additional data to verify or refute a claim, and arguments in which the students make a claim and then use that claim as data to make another claim. These arguments structures are discussed below.

Core arguments.

Bob and Ellen create eleven core arguments while working on the triangle inequality task. The structure of their core arguments seems to depend on whether the pair use the tools, in this case the snap cubes. For two core arguments, the students use the tools and the warrants are not explicit. For four core arguments, the students do not actively use the tools and the warrants are not explicit. For five core arguments, the students do not actively employ the tools and the warrants are explicit. These three argument structures are discussed below.

Non-explicit warrants, tools used.

For the two of the core arguments, Bob and Ellen use the tools to create data and, based on this data, make claims. However, the students do not provide explicit warrants. In all of these arguments, the data are the lengths of the three segments, the arranging of the snap-cube segments with these lengths, and the appearance of the figure the students form using the snap-cube segments. For example, Bob and Ellen are determining whether a triangle can be formed with segments of length 3, 2, and 6. Bob makes the segments using the snap-cubes, arranges them on the table, and moves the segments of lengths 2 and 6 while keeping one endpoint of each segment touching the segment of length 3. He is unable to get the free endpoints of segments of lengths 2 and 6 to meet and form a triangle. Bob states, “That’s not going to work.” This argument is illustrated in Figure 102.

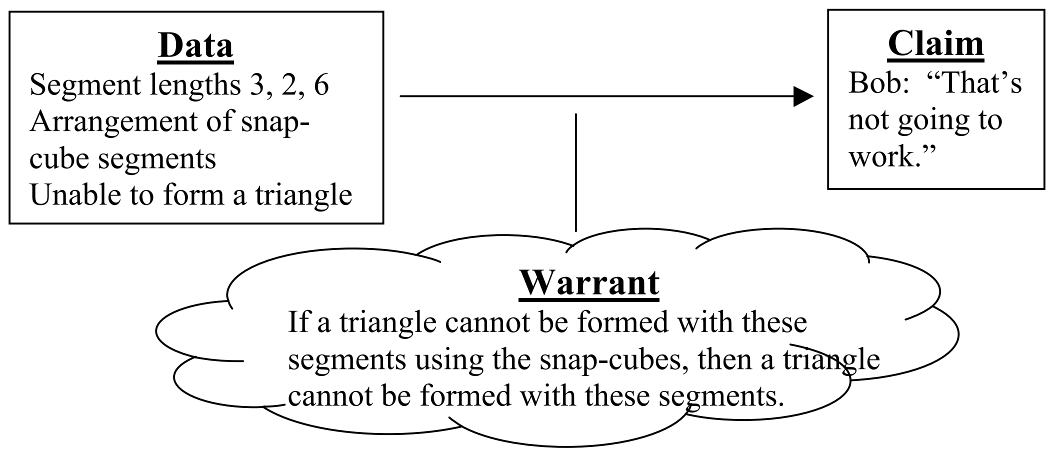


Figure 102. Bob and Ellen's core argument created during task 1 with an inferred warrant and tool use.

The data for this argument are the lengths of the segments, and the arrangement of the snap-cube segments such that a triangle is not formed. Bob claims it is impossible to make a triangle with segments of lengths 3, 2, and 6. The warrant for this claim is not explicit and the researcher infers it to be, "If a triangle cannot be formed with these segments using the snap-cubes, then a triangle cannot be formed with these segments."

Absence of tools – non-explicit warrants.

For four core arguments, Bob and Ellen do not actively employ the tools and their warrants are not explicit. Ellen provides one example of this structure as she determines the longest segment that could be used to form a triangle with segments of lengths 17 and 31. Ellen simply states, "It can't be more than 17 plus 31." This argument is illustrated in Figure 103.

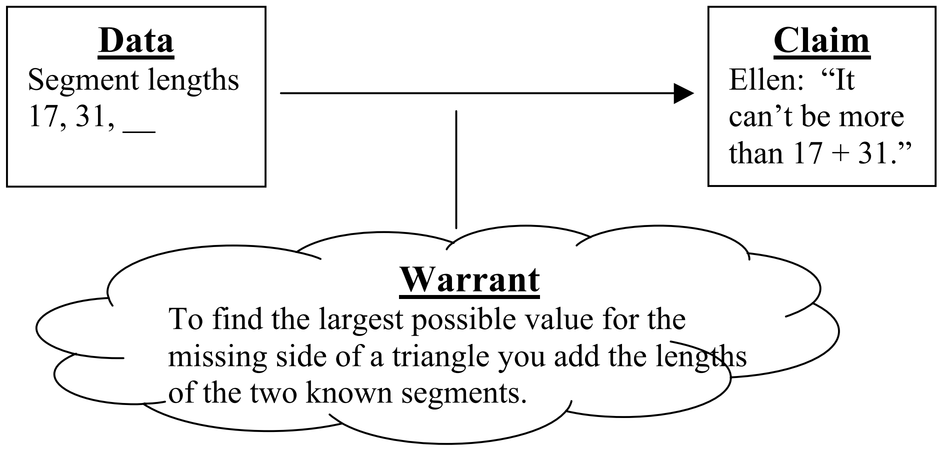


Figure 103. Ellen's core argument created during task 1 with an inferred warrant and no tool use.

The data for this argument are the segment lengths 17 and 31. Ellen claims that the missing segment cannot be greater than the sum of the two known segments, 17 and 31. She does not provide an explicit warrant for her claim and is inferred by the researcher to be, "To find the largest possible value for the missing side of a triangle you add the lengths of the two known segments."

Absence of tools – explicit warrants.

In five of the core arguments, Bob and Ellen do not actively use the tools and their warrants are explicit. For example, the pair attempt to answer the question, "Why was it impossible to construct a triangle with some of the given lengths?" Ellen states, "Because the other sides would not be as long as the base." Bob says, "They were too short to meet." This argument is illustrated in Figure 104.

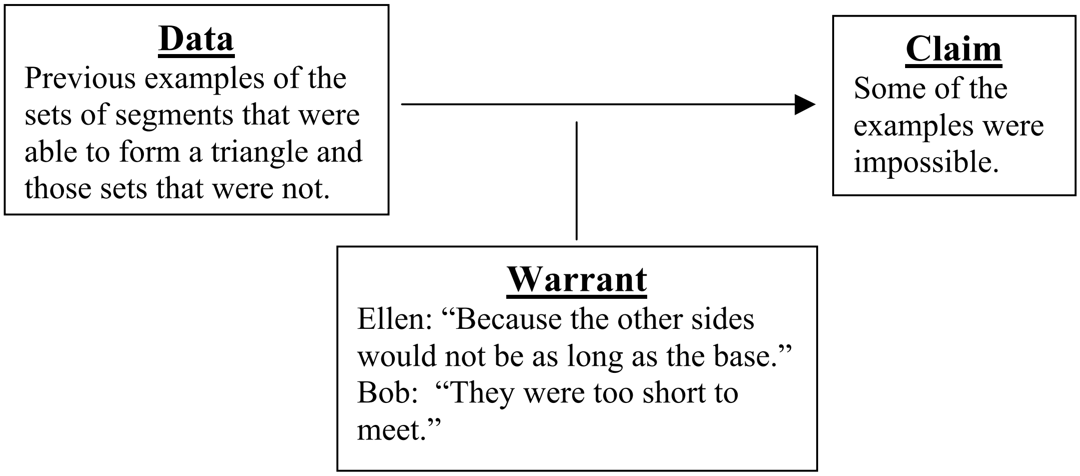


Figure 104. Bob and Ellen’s core argument created during task 1 with an explicit warrant in response to a question on the task sheet.

The data for this argument are the previous examples of sets of segment lengths that were able and unable to form a triangle. The students do not explicitly provide the claim. Instead, the question provides the claim that some of the examples were impossible. Both Bob and Ellen provide a warrant for this argument in their statements that the segments are not long enough. For this argument it was expected that the warrant is explicit. The question provides the claim and asks the students for the warrant. The data for this argument, which triangle could and could not be formed with the given segments, are the claims of prior arguments in which the students use the tools. The students do not actively use the tools in this argument. Rather, the data for this argument are the products of the students’ previous uses of tools. This is considered an indirect use of the tools.

Bob and Ellen create another example of a core argument in which tools are not actively used and the warrant is explicit as they answer the question posed by the teacher. The teacher asks the class whether a triangle can be formed with segments of lengths 10, 7, and 8. Ellen and Bob say, "Yes." The teacher then asks, "Who can explain to me why this will make a triangle? Bob states, "Because the two lowest numbers 7 and 8 add up to be greater than 10." This argument is illustrated in Figure 105.

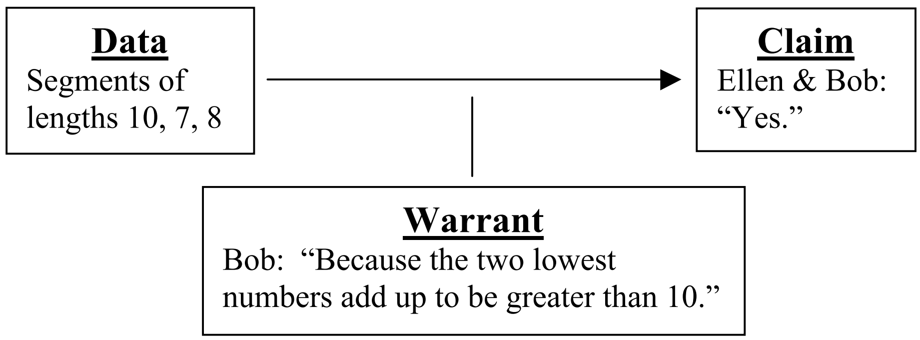


Figure 105. Bob and Ellen's core argument created during task 1 with an explicit warrant and no tool use in response to a task posed by the teacher.

The data for this argument are the segments of lengths 10, 7, and 8. Bob and Ellen indicate that these segments can form a triangle. Bob provides an explicit warrant indicating that when the sum of the two shortest segments, in this case 7 and 8, is greater than the longest segment, a triangle can be formed. The students do not use the tools during this argument.

Conclusions.

Looking across these core arguments, three themes emerge. First, the students do not qualify their claims or provide explicit rebuttals. This seems to suggest that the students are fairly certain about their claims. Second, when the students use the tools, they do not provide explicit warrants. This may be related to the task on which the students are working. Both of these arguments are made in response to questions on the task sheet in which the students are asked whether a triangle can be formed with the given segments lengths using the tools.

Perhaps, because a justification is not required the students do not feel the need to provide one. The third theme that emerges from the analysis of these core arguments is when Bob and Ellen do not actively use the tools in the creation of their arguments, they are more likely to provide explicit warrants. For five of the nine core arguments in which the tools are not actively employed, Bob and Ellen provide an explicit warrant. This may be related to the tasks on which the students are working. For the five arguments in which the students do not actively employ the tools and the warrants are explicit, the tasks on which the students are working requested a justification. The students create one argument in response to a question on the task sheet in which the students are given the claim and asked to provide a warrant (see the argument illustrated in Figure 104). For the other four arguments, the students create arguments in response to the teacher's questions or statements. In general, the teacher asks the students for a justification (see the argument illustrated in Figure 105). Of those arguments where the tools are not actively used and the warrants are not explicit, two are in response to questions and statements posed by the teacher and two are made while working on the warm-up.

Additional data collected by Ellen and Bob.

The second type of argument structure created by Bob and Ellen is that in which the students seek additional data to verify or refute a claim. In this structure, two types of arguments are identified: those in which the students use the tools to collect additional data in response to a challenge to the claim, and those in which the students' uncertainty of his/her own claim motivate the students to collect additional data. These two argument structures are detailed below.

Tools and challenge.

In one argument, Ellen uses a tool to gather additional data after an initial claim is made. The decision to collect additional data is prompted by a challenge made by the teacher. In this argument, Ellen has written on her paper that segments of lengths 10, 7, and 1 will form a triangle. She states, "It's going to work." The teacher challenges this claim asking, "One's going to work? So 10, 7, 1 works?" Ellen replies, "All day long. It's going to work today, work with me." The teacher asks, "It's going to work today? You are going to change the laws of geometry and physics?" Ellen replies, "Fine, I'll just draw it myself. This will do." She uses the ruler to draw a triangle on the back of her paper with side lengths 10 in., 7 in., and 1 in. Unable to make a triangle with these segments, she states, "Ugh, it won't work." This argument is illustrated in Figure 106.

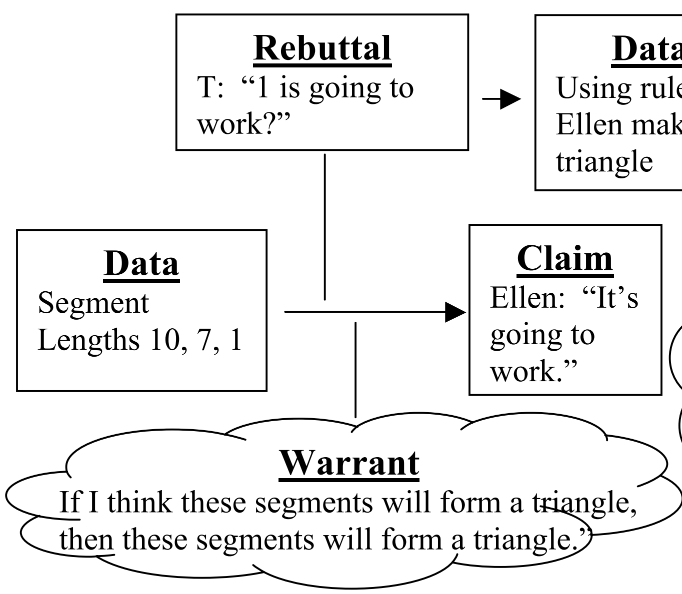


Figure 106. Ellen & Bob's argument created during task 1 with additional data collection in response to a challenge from the teacher.

The initial data for this argument are the segments of lengths 10, 7, and 1. Ellen claims that a triangle can be formed with these segments. She does not provide an explicit warrant for her claim and is inferred by the researcher to be, “If I think these segments will form a triangle, then these segments will form a triangle.” The teacher provides an explicit rebuttal when he challenges this claim by asking if these segments will form a triangle. Ellen collects additional data by using a tool, in this case a ruler and a pencil, to measure the segments and draw a triangle. Unable to draw a triangle with these segments, she makes a new claim that the segments will not form a triangle. She does not provide an explicit warrant for this claim and is inferred by the researcher to be, “If I cannot draw a triangle with segment lengths 10, 7, and 1, then a triangle cannot be formed with segments of lengths 10, 7, and 1.”

In the previous argument, the teacher makes the challenge. However, that is not always the case. For example, Bob and Ellen are using the snap-cubes to determine whether segments of lengths 2, 7, and 4 will form a triangle. Bob creates the segments using the snap cubes, arranges them according to the directions provided by the teacher, and is unable to form a triangle. He states, “Impossible.” Ellen, uncertain about Bob’s claim, states, “I got this. You don’t know how to make things.” She rearranges the segments and is unable to form a triangle. She says, “Oh shoot.” Bob smugly states, “Told you.” Ellen reluctantly agrees, “Ok anyway, impossible.” This argument is illustrated in Figure 107.

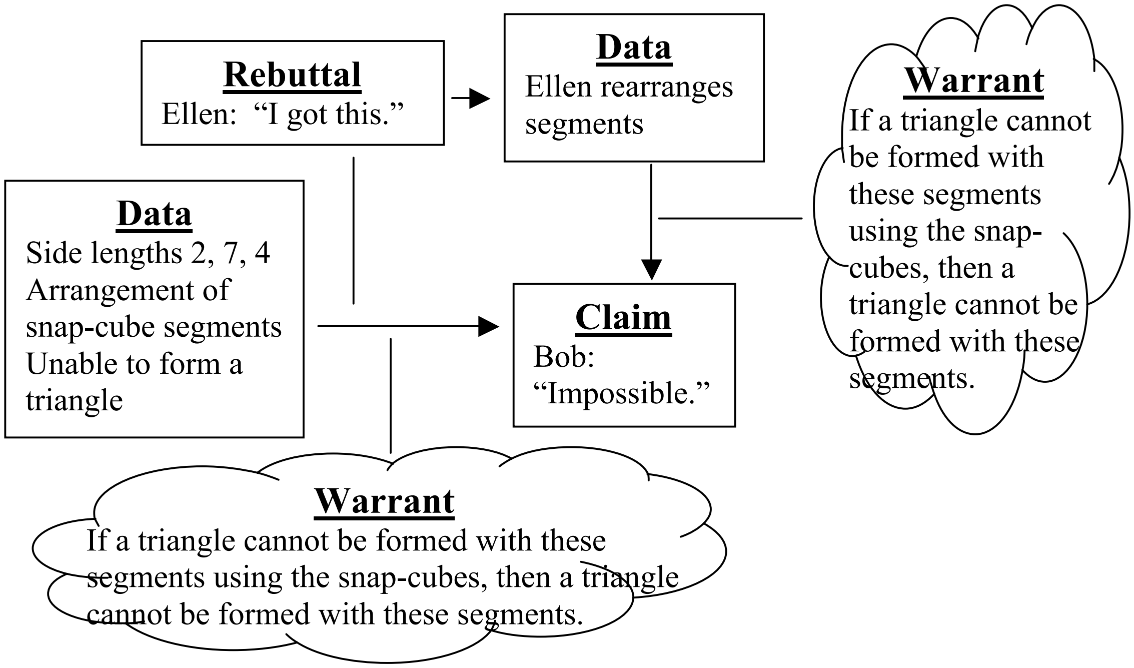


Figure 107. Ellen & Bob's argument created during task 1 with additional data collection in response to a challenge from Ellen.

The initial data for this argument are the segments of lengths 2, 7, and 4, the uses of the tools to form these segments, and the arrangement of the snap-cube segments to form a triangle. Bob claims that it is impossible to form a triangle with these segments. He does not provide an explicit warrant for his claim and is inferred by the researcher to be, "If a triangle cannot be formed with these segments using the snap-cubes, then a triangle cannot be formed with these segments." Ellen provides an explicit rebuttal when she makes the challenge that she can form a triangle with segments. She collects additional data by rearranging the segments. Unable to form a triangle, she verifies Bob's initial claim. She does not provide an explicit warrant for his claim and the researcher infers it to be the same as the previous inferred warrant.

Uncertainty.

In the previous argument, Ellen's uncertainty with Bob's ability to use the tools and his claim leads to an explicit challenge. In two arguments, Ellen's uncertainty about her initial claim compels her to collect additional data which ultimately leads her to refute her initial and make a new claim. For example, Ellen writes on her paper that a triangle can be formed with segments of lengths 10, 7, 24. The teacher asks, "So, you think 24?" She responds, "As long as it's more than 17, it don't [sic] matter." The teacher simply states, "So 10, 7, 24…okay, okay." Ellen asks, "Hold on is it more or less?" The teacher points to the board and asks, "What does it say up there?" Written on the board is the triangle inequality theorem. Ellen says, "Oh crap." and begins erasing her answer. This argument is illustrated in Figure 108.

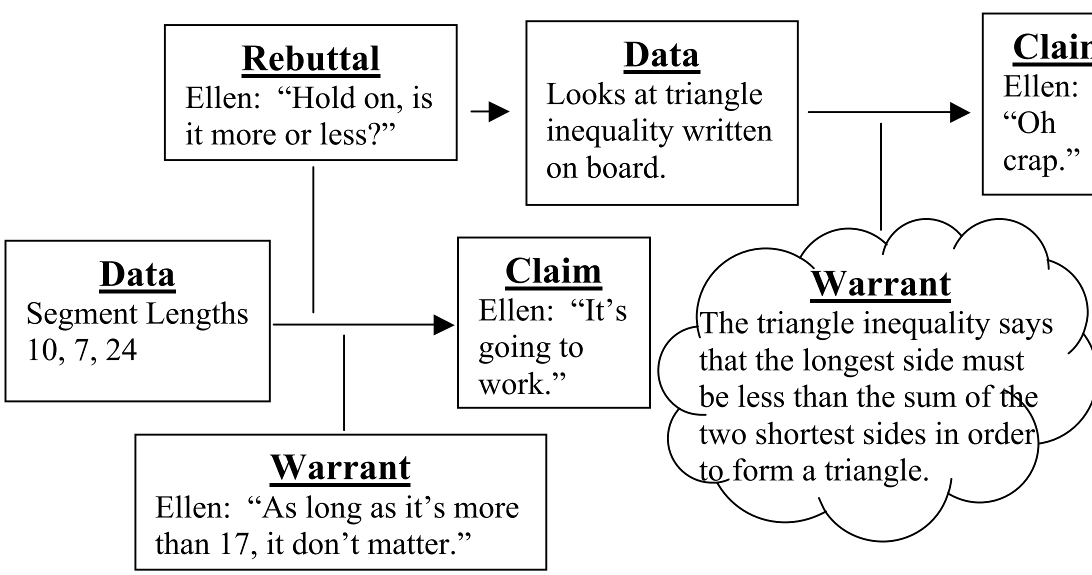


Figure 108. Ellen's argument created during task 1 with additional data collection based on Ellen's uncertainty.

The initial data for this argument are the segments of lengths 10, 7, and 24. Ellen claims these segments will form a triangle. She provides an explicit warrant indicating that as long as the longest segments is greater than the sum of the two shortest segments a triangle will be formed. Even though the teacher makes a statement about these segments, he does not provide a challenge in the form of a question nor does he state that she is incorrect. Instead, she provides the rebuttal to her own claim when she asks whether the longest segment has to be greater than or less than the sum of the two shortest segments in order to form a triangle. The teacher draws her attention to the triangle inequality theorem written on the board. The researcher determines Ellen collects additional data by reading this theorem. Realizing that her initial claim is incorrect she makes a new claim by indicating she made a mistake. She does not provide an explicit warrant for her new claim and is inferred by the researcher to be, "The triangle inequality says that the longest side must be less than the sum of the two shortest sides in order to form a triangle."

Conclusions.

The arguments in which Bob and Ellen collect additional data can be categorized into two types: those in which the students collect additional data using tools due to a challenge, and those in which the additional data collected was motivated by the student's uncertainty of his/her own claim. It is worth noting that in the two arguments of the second type, the additional data collected is a theorem (see the argument illustrated in Figure 108) and a previous example. The students do not make use of the tools in these arguments. Perhaps, when these students' initial claims are challenged, the students feel the need to collect additional data using tools to demonstrate that they are correct. However, when these students are uncertain about their own claims, they question the basis for their reasoning and seek additional data to determine whether their reasoning is correct and, hence, whether their claims are valid. In other words, when a claim is challenged, the students use tools to collect additional data in order to resolve the challenge. When the students are uncertain about their claim, they collect additional data to determine whether they are using sound reasoning.

Linked Argument.

The third type of argument structure created by Bob and Ellen is that in which the students made an initial claim and used that claim as data to make an additional claim, or a linked argument. For example, Ellen and Bob are determining the longest possible segment that can form a triangle with segment of lengths 10 and 7. Bob is having trouble answering this question. Ellen says, “You don’t get this?” She explains to Bob, “Okay, 10 plus 7. Because those are the two sides, you have to find the base and the base has to be shorter or they’re not going to touch, remember? So it has to be less than 17 because 10 plus 7 is…” Understandingly, Bob says, “Oh.” Ellen continues, “16, 15, like that.” This argument is illustrated in Figure 109.

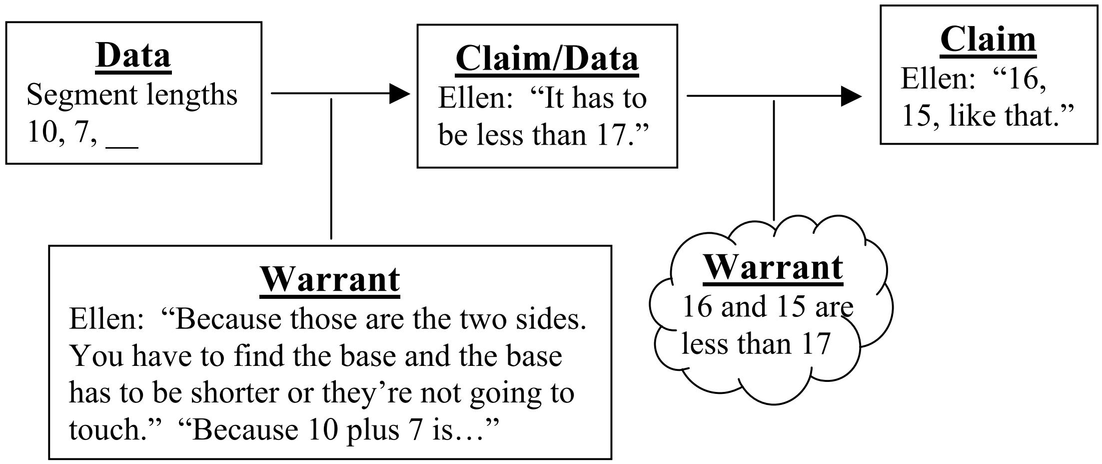


Figure 109. Bob and Ellen's linked argument created during task 1.

The initial data for this argument is the task to determine the length of the longest segment that can form a triangle with segments of lengths 10 and 7. Ellen claims that the longest side has to be less than 17. She provides an explicit warrant indicating the longest segment, she calls it the base, has to be shorter than the sum of the two shortest segments or the two shortest segments will not touch and form a triangle. This claim becomes data and she makes an additional claim that 16 and 15 would form a triangle. She does not provide an explicit warrant for this claim and is inferred by the researcher to be, "16 and 15 are less than 17."

Even though only one of Bob and Ellen's arguments can be categorized as having this link structure, this structure is noteworthy. These students are able to use a claim as data to make a claim. Perhaps, when students create arguments of this structure, they demonstrate they are able to think and reason in a deductive manner.

Discussion.

While working on the triangle inequality activity, Bob and Ellen create arguments of various structures. Three categories of structures are noted in the analysis: core arguments, arguments in which the students are compelled to gather additional data to verify or refute a claim, and linked arguments. Looking across these argument structures, two themes emerge; the relation between the use of tools and the explicitness of the warrants (see Table 22), and the prompting for explicit warrants.

Table 22.

Group 2's Arguments on the Triangle Inequality Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Tools	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	2	2
	Tools Not Used	4	1
Warrants Explicit			
	Tools Used	0	0
	Tools Not Used	5	2

Of the 16 arguments created by Bob and Ellen, the students employ the tools in 4 (25%) of the arguments and none of the warrants in these arguments are explicit. This suggests that when the students use tools, they do not provide an explicit warrant. However, the lack of explicit warrants may also be due to the nature of the task on which the students are working. For three of these four arguments, the students are working on the items on the task sheet which asks the students to determine whether they can form a triangle with the given segments using the snap-cubes. These items do not ask the students to justify their answers and, thus, the students may not have felt compelled to do so.

Conversely, of the seven arguments in which the students provide at least one explicit warrant, the students are prompted to do so either by the teacher or the task sheet for six of the arguments. For five of the seven arguments, the teacher prompts the students to provide an explicit warrant by asking the students why their claim is true (e.g. the arguments illustrated in Figures 105 and 108). The students also create one argument in response to a question on the task sheet in which the students are given the claim and asked to provide a warrant (see the argument illustrated in Figure 104). The high proportion of arguments with explicit warrants that are prompted combined with the nature of the tasks on which the students are working when they create arguments using a tool suggests the explicitness of warrants might be better attributed to whether the students are prompted to justify their claims, regardless of whether the students use the tools.

Group 3's arguments on the triangle inequality task.

The analysis of Clair and Jim's arguments while working on the triangle inequality task can be categorized into three argument structures; core arguments, arguments in which the students collect additional data to verify or refute a claim, and arguments in which the students make a claim and then use that claim as data to make another claim. These argument structures are discussed below.

Core arguments.

Clair and Jim create sixteen core arguments while working on the triangle inequality task. The structure of their core arguments seems to depend on whether the pair use the tools, in this case the snap-cubes. For five core arguments, the students use the tools and the warrants are not explicit. In one core argument, the students use the tool and the warrant is explicit. For nine of the core arguments, the students do not actively use the tools and the warrants are explicit. For one of the core arguments, the students did not actively employ the tools and the warrant is not explicit. These argument structures are discussed below.

Non-explicit warrants, tools used.

For the five of the core arguments, Clair and Jim use the tools to create data and, based on this data, make claims. However, their warrants for these claims are not explicit. . In all of these arguments, the data are the lengths of the three segments, the arranging of the snap-cube segments with these lengths, and the appearance of the figures the students form using the snap-cube segments. For example, Jim and Clair are determining whether a triangle can be formed with segments of length 3, 4, and 4. Jim and Clair work together to form the three segments using the snap-cubes. Clair arranges the snap-cube segments and is able to form a triangle. Jim says, “Possible.” Clair agrees saying, “Yep.” This argument is illustrated in Figure 110.

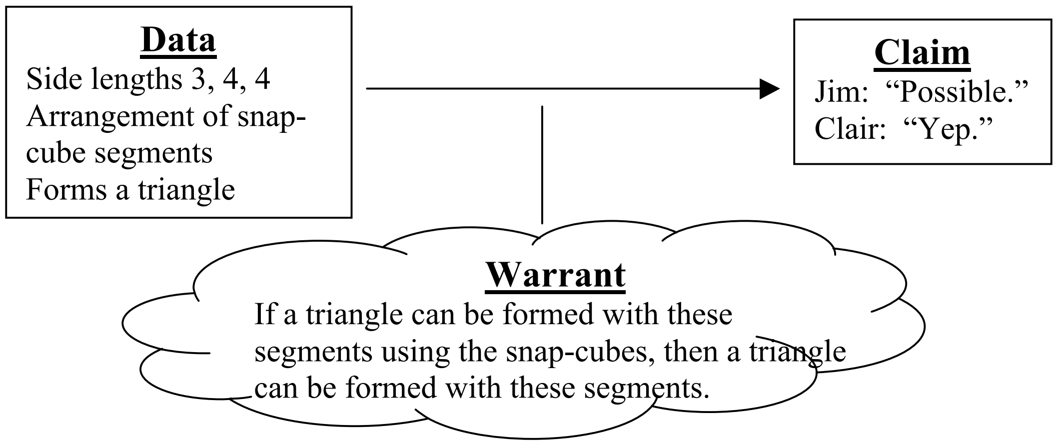


Figure 110. Jim and Clair's core argument created during task 1 with an inferred warrant, tool use, and the claim is a triangle can be formed.

The data for this argument are the lengths of the segments, and the arrangement of the snap-cube segments such that a triangle is formed. Jim claims that it is impossible to make a triangle with segments of lengths 3, 4, and 4 and Clair confirms this claim. The warrant for this claim is not explicit and is inferred by the researcher to be, "If a triangle can be formed with these segments using the snap-cubes, then a triangle can be formed with these segments."

This argument structure also occurs for those arguments in which the segments do not form a triangle. For example, Jim and Clair are given segment lengths 2, 7, and 4. Using the snap-cubes, Clair creates segments of lengths 7, 2, and 4. She arranges the snap-cube segments such that the ends of the two shorter segments touch the ends of the segment of length 7. She swivels the segments and is unable to make them meet. Clair claims, "It's not possible." Jim agrees, "Not possible." This argument is illustrated in Figure 111.

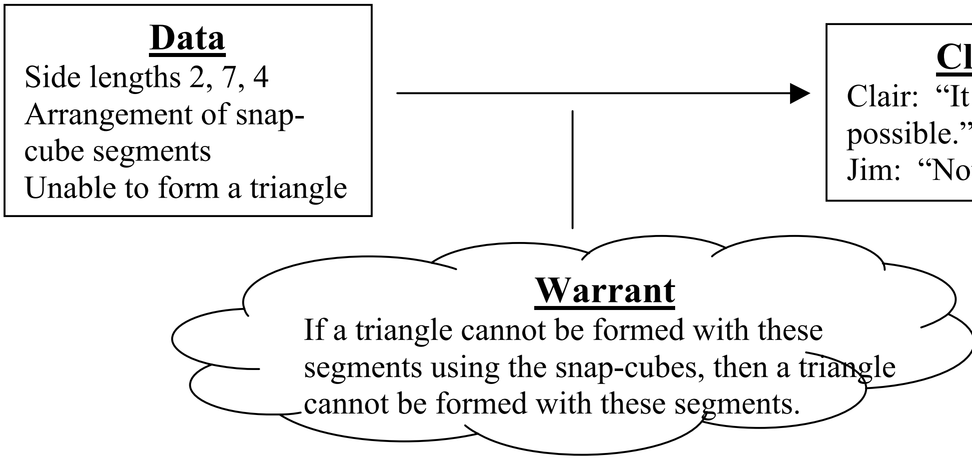


Figure 111. Clair and Jim's core argument created during task 1 with an inferred warrant, tool use, and the claim is a triangle cannot be formed.

The data for this argument are the lengths of the segments, and the arrangement of the snap-cube segments such that a triangle is not formed. Clair claims that a triangle with segments of lengths 2, 4, and 7 is not possible and Jim agrees. The warrant for this claim is not explicit and is inferred by the researcher to be, "If a triangle cannot be formed with these segments using the snap-cubes, then a triangle cannot be formed with these segments."

Across these five core arguments there are many similarities. First, the structure of the arguments is the same; the claims are explicit and the warrants are not explicit. Second, the data for these arguments are based, in part, on the use of tools, the snap-cubes. Third, the task on which the students are working is the same; to determine whether the given segments form a triangle.

Use of tools – explicit warrants.

In one core argument, Jim and Clair employ the tools, the snap-cubes, and provide an explicit warrant. In this argument, Jim and Clair are attempting to find the longest segment length that will form a triangle given segments of lengths 2 and 4. The teacher instructs the students to begin by making segments of lengths 2, 4, and 7 using the snap-cubes. Clair makes the segments using the snap-cubes, positions them such that the segment of length 7 extends beyond the segment of length 2. She states, “It doesn’t match up.” Jim says, “It pretty much is way too big.” This argument is illustrated in Figure 112.

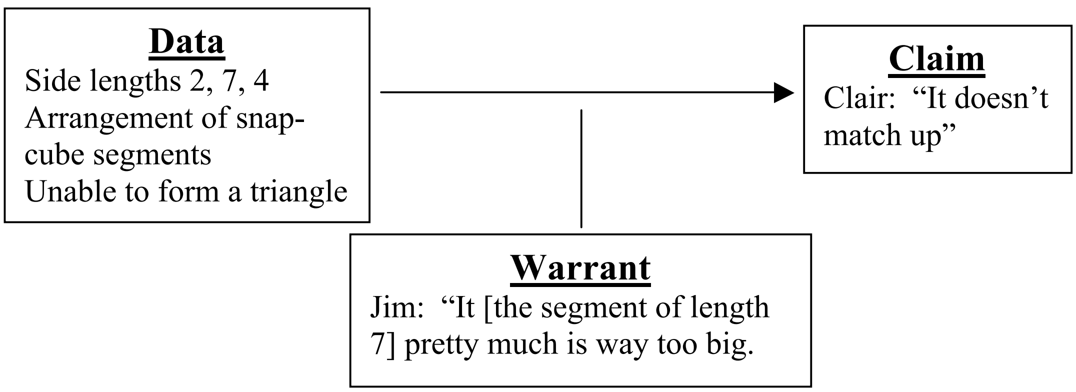


Figure 112. Clair and Jim’s core argument created during task 1 with an explicit warrant and tool use.

The data for this argument are the lengths of the sides, and the arrangement of the snap-cube segments such that a triangle is not formed. Clair claims that a triangle with segments of lengths 2, 4, and 7 is not possible. Jim agrees and provides an explicit warrant for this claim explaining that the segment of length seven is too long.

In this argument, the students use the tools and provide an explicit warrant. This is the only argument provided by Clair and Jim of this structure and content. It may be due to the fact that the pair had previously determined this set of segment lengths is unable to form a triangle. In the argument illustrated in Figure 111, Jim and Clair use the tools to investigate whether the set of segments of lengths 2, 4, and 7 form a triangle. In this argument, they do not provide an explicit warrant. When they investigate this set of segments a second time, Jim provides an explicit warrant to justify why these segments are unable to form a triangle. Perhaps, when students revisit the same problems using tools, they are more likely to provide an explicit warrant for their claims.

Absence of tools – explicit warrants.

In nine of the core arguments, Clair and Jim do not actively use the tools and their warrants are explicit. For example, the pair answer the question, "Why was it impossible to construct a triangle with some of the given lengths?" Clair states, "Well, some lengths wouldn't reach the others." This argument is illustrated in Figure 113.

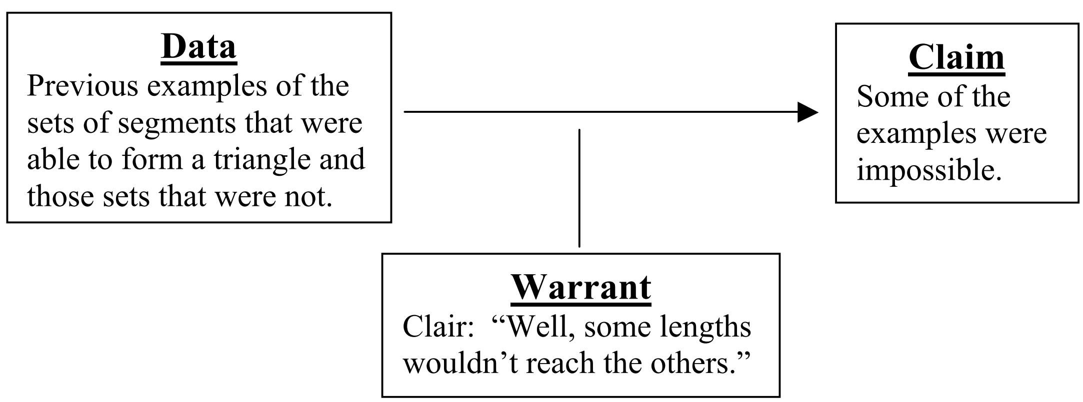


Figure 113. Clair's core argument created during task 1 with an explicit warrant and indirect use of tools.

The data for this argument are the previous examples that were both able and unable to form a triangle. The students do not explicitly provide the claim. Instead, the question provides the claim that some of the examples were impossible. Clair provides a warrant for this argument in her statement that the segments were not long enough. For this argument it is expected that the warrant is explicit because this question provides the claim and asks the students for the warrant. The data for this argument, the sets of segment lengths that were able and unable to form a triangle, are the claims of prior arguments in which the students used the tools. The students are not actively using the tools in this argument. Rather, the data for this argument are the products of the students' previous uses of tools, an indirect use.

Clair and Jim create another example of a core argument in which they do not actively use the tools and the warrant is explicit as they discuss their solutions to a homework problem. The problem asks the students to determine whether a triangle can be formed with segments of lengths 6, 4, and 10. Clair says, "The first one you said no and I agreed. I said no because there would be,like it wouldn't match up because 6 + 4 = 10." Jim adds, "And that would be a straight line." This argument is illustrated in Figure 114.

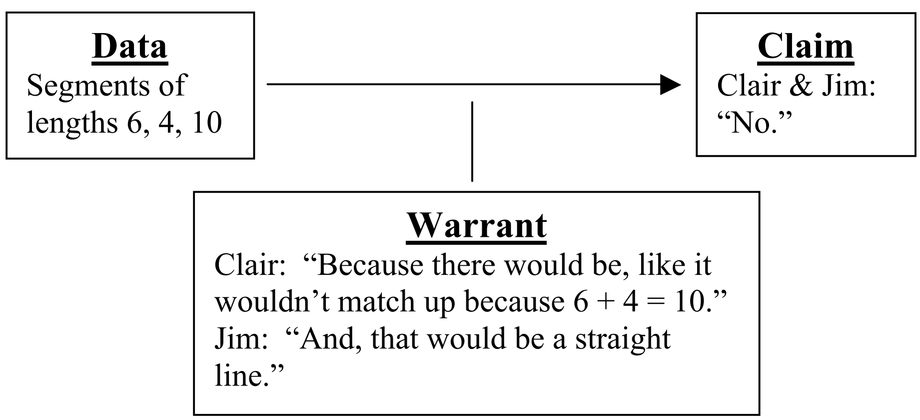


Figure 114. Clair and Jim's core argument created during task 1 with an explicit warrant and no tool use.

The data for this argument are the segments of lengths 6, 4, and 10. Clair and Jim indicate that these segments cannot form a triangle. Clair provides an explicit warrant indicating that when you add the two shortest segments, in this case 6 and 4, and the sum is equal to the longest segment, a triangle cannot be formed. Jim adds, that when they are equal, the segments form a line.

Conclusions.

Looking across these core arguments, three themes emerge. First, the students do not qualify their claims or provide explicit rebuttals. This seems to suggest that the students are fairly certain about their claims. Second, when the students use the tools, they generally do not provide an explicit warrant. This may be related to the task on which the students are working. Of the six core arguments in which the pair use the tools, four are in response to questions on the task sheet in which they are asked whether they can form a triangle with the given segments lengths using the tools and their warrants for these arguments are not explicit. Perhaps, because the task sheet does not require the student justify or explain their findings, the students do not feel the need to provide them.

The third theme that emerges from the analysis of these core arguments is when Clair and Jim do not actively use the tools in the creation of their arguments, they are more likely to provide an explicit warrant. For nine of the ten core arguments in which the students do not actively employ the tools, Clair and Jim provide an explicit warrant. This may be related to the tasks on which the students are working. For seven of the arguments in which the students do not actively use the tools and the warrants are explicit, the tasks on which the students are working request a justification or explanation. The students also create two of the arguments in response to the teacher's questions or statements. In general, the teacher asks the students to provide a justification for their claims.

Additional data collected by Clair and Jim.

The second type of argument structure created by Clair and Jim is that in which the students seek additional data to verify or refute that claim. In this structure, two types of arguments are identified; those in which the students use the tools to collect additional data in response to a challenge to a claim, and those in which the students' uncertainty of his/her own claim motivated the students to collect additional data. These two argument structures are detailed below.

Tools and challenge.

In one argument, Jim uses a tool to gather additional data after an initial claim is made. The decision to collect additional data is prompted by a challenge. In this argument, Clair and Jim are to determine whether segments of lengths 3, 2, and 6 will form a triangle. Clair tells Jim to make segments with, "3 cubes, 1 cube and 6." Jim confirms, "3, 1, 6." Clair creates the segments using the snap-cubes and is unable to arrange them such that they form a triangle. Jim says, "No." Clair agrees, "Nope." Jim notices that the set of segments on the task sheet is 3, 2, 6 and brings this to Clair's attention saying, "Wait, it's 3, 2, and 6." Clair says, "We already did that one." Jim replies, "No. Let me just try this out." Jim adds a cube to the segment of length 1 and tries to arrange the segments to form a triangle. Unable to do so, he states, "Nah, it's not possible." This argument is illustrated in Figure 115.

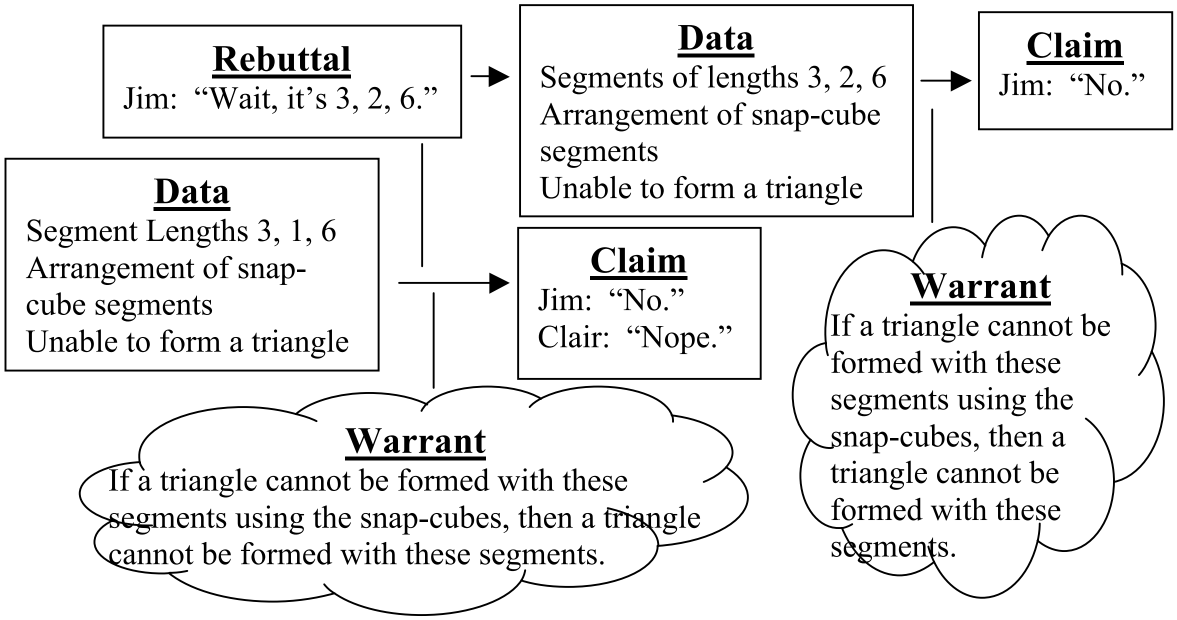


Figure 115. Jim and Clair's argument with additional data collection in response to a challenge by Jim.

The initial data for this argument are the segments of lengths 3, 1, and 6 and the arrangement of the snap-cube segments such that a triangle is not formed. Both Clair and Bob claim that a triangle cannot be formed with these segments. Neither student provides an explicit warrant for their claim and is inferred by the researcher to be, "If a triangle cannot be formed with these segments using the snap-cubes, then a triangle cannot be formed with these segments." Jim provides an explicit rebuttal when he indicates that the set of segments should be 3, 2, 6 not 3, 1, 6. Jim collects additional data by using the snap-cubes to create the segment of length 2 and arranging them to determine whether a triangle can be formed. Unable to form a triangle, he makes a new claim that these segments will not form a triangle. He does not provide an explicit warrant for this claim and is inferred by the researcher to be the same as the previous warrant.

In the previous argument, Jim makes the challenge. However, that is not always the case. In one argument, the teacher makes the challenge. In this argument, the teacher asks the students to make segments of lengths 2, 4, and 7 using the snap-cubes and asks whether these segments form a triangle. After the class agrees this is impossible, the teacher asks the students to begin removing blocks from the segment of length seven until they are able to form a triangle. Clair takes away three blocks and is able to form a triangle with the snap-cube segments of lengths 2, 4, and 4. The teacher stops by the table and asks Clair, " You got it?" Clair responds, "Yes." The teacher asks, "How many did you take away? Clair replies, "3." The teacher says, "Three of them. Could you take, like, only one of them away?" Clair replaces one of the blocks, arranges the snap-cube segments of lengths 2, 4, and 5 such that a triangle is formed. She claims, "Two of them work to. Let's try one." She then adds another block to segment of length 5 and tries to form a triangle with the snap-cube segments of lengths 2, 4, and 6. Unable to form a triangle, she says, "No you cannot take one away. This argument is illustrated in Figure 116.

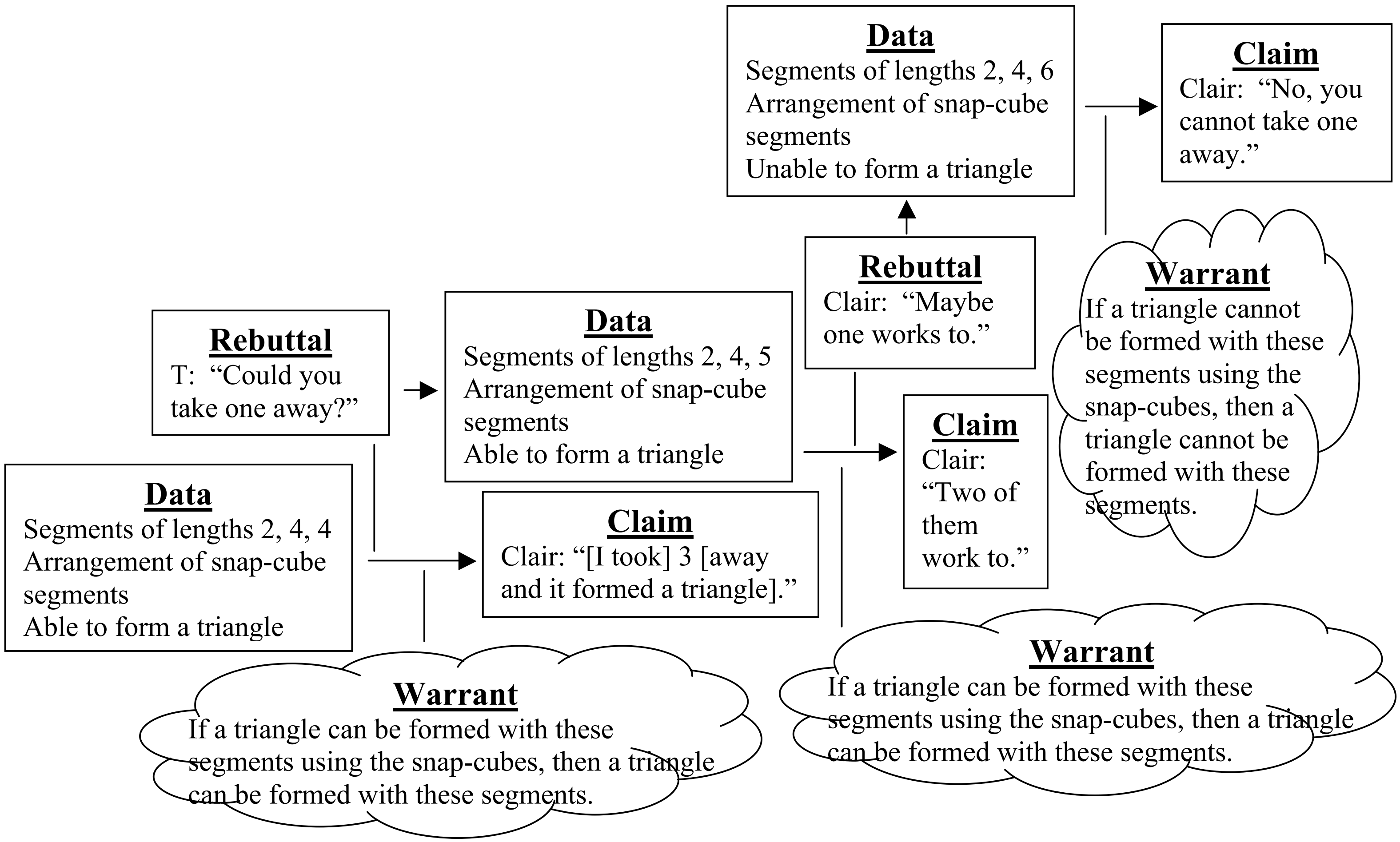


Figure 116. Clair and Jim's argument created during task 1 with additional data collection in response to a challenge made by the teacher.

The initial data for this argument are the segments of lengths 2, 4, and 4, the uses of the tools to create these segments, and the arrangement of the snap-cube segments to form a triangle. Clair claims that it is possible to form a triangle with these segments. She does not provide an explicit warrant for her claim and the researcher infers it to be, "If a triangle can be formed with these segments using the snap-cubes, then a triangle can be formed with these segments." The teacher provides an explicit rebuttal when he makes the challenge that a triangle may be formed with fewer removed blocks. Clair collects additional data by adding a block back to one of the snap-cube segments such that the lengths of the segments are 2, 4, and 5, and arranges the snap-cube segments such that a triangle is formed. She claims a triangle is formed when two of the blocks are removed. Clair does not provide an explicit warrant and is inferred by the researcher to be the same as the previous warrant. Clair provides an explicit rebuttal when she questions whether a removal of only one block will form a triangle. She collects additional data by adding a block to the snap-cube segment of length 5 such that the lengths of the snap-cube segments are 2, 4, and 6. She attempts to form a triangle with these snap-cube segments but is unsuccessful. She makes a new claim that these segments will not form a triangle. Clair does not provide an explicit warrant for his claim and is inferred by the researcher to be, "If a triangle cannot be formed with these segments using the snap-cubes, then a triangle cannot be formed with these segments."

In both of these arguments, the decision to collect additional data is prompted by a challenge. In the first argument, the challenge comes from a peer. In the second argument, the challenge comes from the teacher. In both of these cases, neither the warrant nor the claim is challenged. Instead, the challenge is to the relationship between the data and the task on which the students are working. In the first argument, Jim does not challenge the claim that segments of lengths 3, 1, and 6 will not form a triangle. Rather, he challenges whether the data matches that on the task sheet. In the second argument, the teacher does not dispute Clair's claim that a triangle will be formed with segments of lengths 2, 4, and 4. Instead he asks her if she had only removed one of the snap cubes, could she form a triangle.

Uncertainty.

In the previous arguments, the decision to collect additional data is based on a challenge. This is not always the case. In one argument, the student's uncertainty leads to additional data collection to verify a claim. In this argument, Clair and Jim are to determine whether segments of lengths 6, 1, and 4 will form a triangle. Clair and Jim use the snap-cubes to create the snap-cube segments and arrange them attempting to form a triangle. Unsuccessful, Clair claims, "I don't think we can do it. Do you?" Jim replies, "No." Clair continues to rearrange the blocks attempting to form a triangle. She then states, "I don't think it's possible." Jim agrees, "Impossibility." Clair says, "So let's write no possibility." This argument is illustrated in Figure 117.

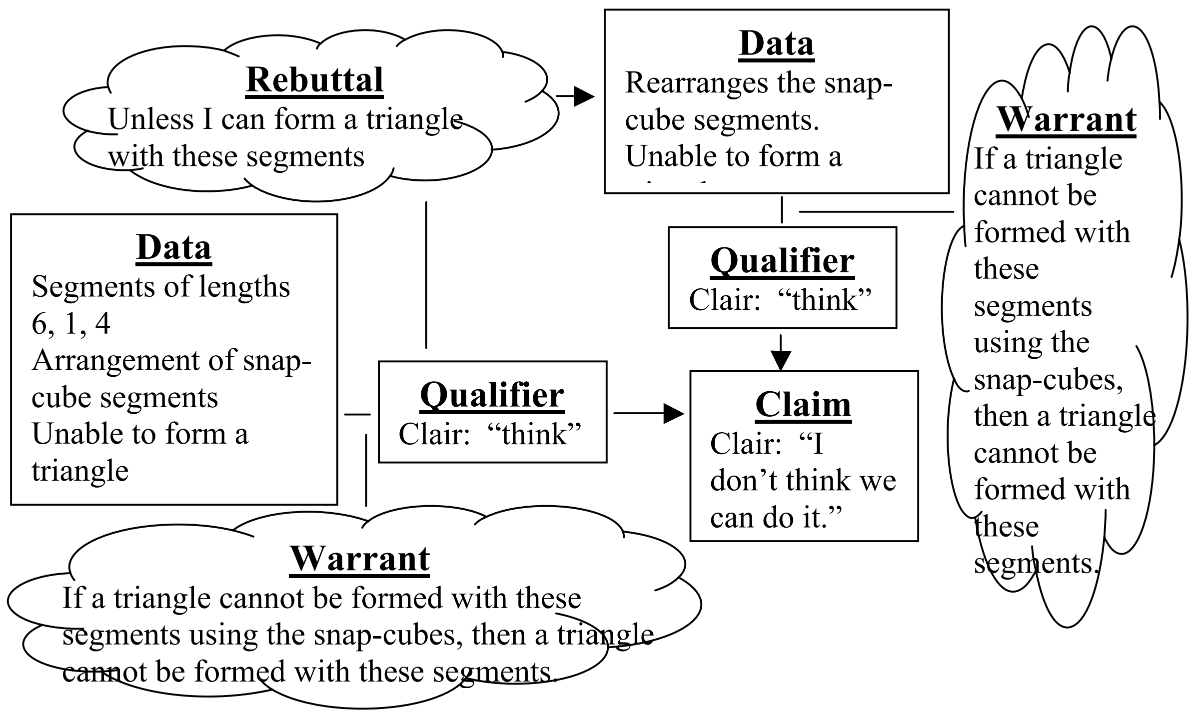


Figure 117. Clair's argument created during task 1 with additional data collection based on Clair's uncertainty.

The initial data for this argument are the segments of lengths 6, 1, and 4, the uses of the tools to form these segments, and the arrangement of the snap-cube segments to form a triangle. Clair claims that it is not possible to form a triangle with these segments. She does not provide an explicit warrant for her claim and is inferred by the researcher to be, "If a triangle cannot be formed with these segments using the snap-cubes, then a triangle cannot be formed with these segments." She qualifies her claim with the term "think" which suggests she is uncertain her claim is correct. However she does not provide an explicit rebuttal. Instead she continues to arrange the snap-cube segments. Based on these actions, the researcher infers her rebuttal to be "Unless I can form a triangle with these segments." Clair's rearrangement of the segments and her inability to form a triangle is additional data.

She verifies her initial claim but still qualifies it that suggests she is still uncertain. She does not provide an explicit warrant and the researcher infers it to be the same as the previous.

Conclusions.

The arguments in which Clair and Jim collect additional data can be categorized into two types; those in which the students collect additional data using tools in response to a challenge, and those in which the additional data collected using tools is motivated by the student's uncertainty of his/her own claim. It is worth noting that in all of the arguments in which the students collect additional data, the students use the tools, the snap-cubes. This suggests that these students may view the tools as a means to collect additional data to verify or refute initial claims.

Linked argument.

The third type of argument structure Clair and Jim create is that in which the students make a claim and use that claim as data to make a further claim, or a linked argument. For example, Clair and Jim are determining the shortest possible segment that can form a triangle with segment of lengths 10 and 7. Jim previously stated that 4 is the smallest possible length of a segment that will form a triangle with segments of lengths 10 and 7. The teacher writes 3.5 on his paper and asks, "Does it have to be a whole number?" Jim replies, "It doesn't have to be a whole number." He asks himself, "Does 3.5 work? So 10, 7, and 3.5. Ok 3.5 and 7 that is…yes it does work." Jim then states, "3.1 would work because if 3.1 and 7 would be 10.1 which is bigger than 10. So the smallest is 3.1. This argument is illustrated in Figure 118.

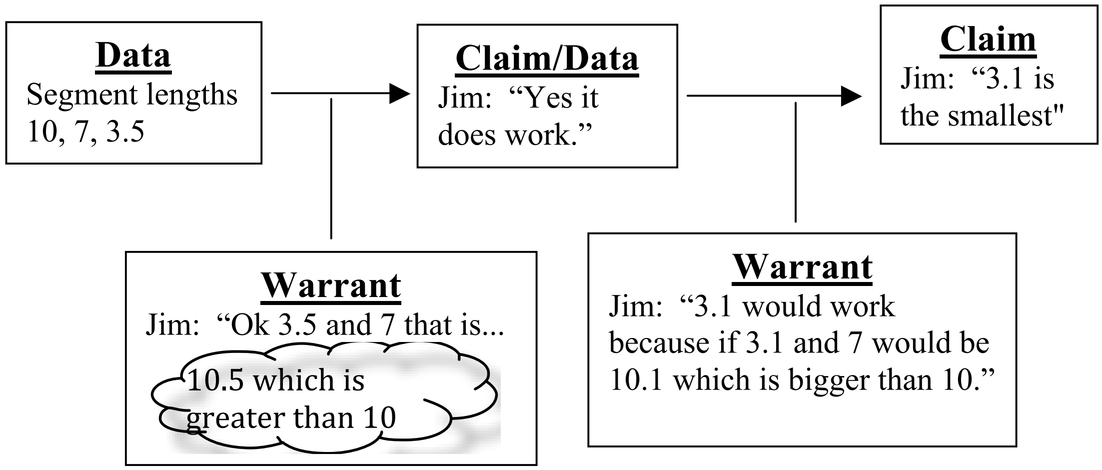


Figure 118. Clair and Jim's linked argument.

The initial data for this argument are the segments of lengths 10, 7, and 3.5 and the task, which is to determine the length of the shortest segment that can form a triangle with segments of lengths 10 and 7. Jim claims that these segments will form a triangle. He provides an explicit warrant by indicating that the sum of the two shortest segments is greater than the longest segment. This claim becomes data and Jim considers other segments whose lengths are not whole numbers. He claims that 3.1 would be the shortest segment length that could form a triangle with segments of lengths 10 and 7. He provides an explicit warrant for his claim by stating the sum of 3.1 and 7 is greater than the value of the length of the longest side.

Even though only two arguments are categorized as having the linked argument structure, this structure is noteworthy. These students are able to use a claim as data to make a claim. Perhaps, when students create arguments of this structure, they demonstrate they are able to think and reason in a deductive manner.

Discussion.

While working on the triangle inequality activity, Clair and Jim create arguments of various structures. Three categories of structures are noted in the analysis: core arguments, arguments in the students are compelled to gather additional data to verify or refute a claim, and linked arguments. Looking across these argument structures, two themes emerge; the relation between the use of tools and the explicitness of the warrants (see Table 23), and, the prompting for explicit warrants.

Table 23.

Group 3's Arguments on the Triangle Inequality Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Tools	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	5	3
	Tools Not Used	1	0
Warrants Explicit			
	Tools Used	1	0
	Tools Not Used	9	2

Of the 21 arguments created by Clair and Jim, the students employ the tools in 9 (43%) of the arguments. In 8 (89%) of these 9 arguments, the students do not provide an explicit warrant. In fact, the only argument in which the students use the tools and offer an explicit warrant is the argument in which they revisit a previous example (see the argument illustrated in Figure 112). This seems to suggest that when the students use tools, they are unlikely to provide an explicit warrant. However, this lack of explicit warrants may also be due to the nature of the task on which the students are working. For six of the nine arguments in which the students are using the tools, the students are working on items on the task sheet. These items ask the students to determine whether they can form a triangle with the given segment lengths using the snap-cubes. These items do not ask the students to justify their answers. Thus, the students may not have felt compelled to do so.

Conversely, of the twelve arguments in which the students provide at least one explicit warrant, the students are prompted to do so either by the teacher, the task sheet, or their homework for eleven of the arguments. For five of these eleven arguments, the teacher prompts the students to provide an explicit warrant by asking the students to explain why their claim is true. For another five arguments, the students are comparing their answers on the homework, which asks the students to provide an explanation stating how they arrive at their answer. And, one of these eleven arguments is created in response to a question on the task sheet in which the students are given the claim and asked to provide a warrant (see the argument illustrated in Figure 113). The high proportion of arguments with explicit warrants that are prompted combined with the nature of the tasks on which the students are working when they create arguments using a tool suggests the explicitness of warrants might be better attributed to whether the students are prompted to justify their claims, regardless of whether the students use the tools.

Cross-case analysis of arguments created while working on the triangle inequality task.

On the triangle inequality task, the arguments created by the three pairs of students vary in structure and content. Three themes emerge when looking across the arguments created by these students; the structure of the argument when the tools are and are not actively employed (see Table 24), the relationship between the type of task on which the students are working and the content of the argument, and the collection of additional data.

Table 24.

The Combined Arguments of the Three Groups on the Triangle Inequality Task by Structure, Use of Technology, and Explicitness of the Warrant

		Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	11	7
	Tools Not Used	5	2
Warrants Explicit			
	Tools Used	1	0
	Tools Not Used	20	10

In sum, the three groups of students create 56 arguments while working on the triangle inequality task. For 25 (45%) arguments, the students do not provide explicit warrants. Additionally, the students employ the tools in 19 (34%) arguments. Of the 19 arguments in which the students use tools, they only provided an explicit warrant for 1 (5%)

of these arguments. Of the arguments in which tools are not actively employed, the students provide an explicit warrant for 30 (81%) of these arguments . This disparity seems to suggest that when students use the tools, they are less likely to provide an explicit warrant compared to when they do not actively use the tools.

The explicitness of the warrants may be related to the type of tasks on which the students are working. In general, when the students are using the tools, they are attempting to determine whether a triangle could be formed with the given segments. However, when the students are not actively using the tools, the students are mainly working on generalization type tasks. For example, one of the arguments common to all the students in structure is in response to the question on the task sheet, "Why was it impossible to construct a triangle with some of the given lengths?" The question is asking for the students to generalize across the examples. The structure of the argument for all the groups of students is a core argument with an explicit warrant (see the arguments illustrated in Figures 96, 104, and 113). The structure of these arguments may be due to the fact that the question actually provides the claim and asks the students to provide the warrant. The data for the students are their responses on their task sheet. To gather this data, the students use the tools, the snap-cubes. However, when responding to this question, the data had been previously gathered and their reasoning is not based on their active use of tools, but on the product of their previous uses.

Many times, the students collect additional data to verify or refute a previous claim. All three groups of students create arguments of this structure. The students' decision to seek additional data may be due to a number of factors including an explicit challenge to a claim (e.g. the arguments illustrated in Figures 100 and 106), and the uncertainty of a claim (e.g. the arguments illustrated in Figures 108 and 117). Generally, the students use the tools to collect additional data by rearranging the snap-cube segments to determine whether a triangle can be formed (e.g. the argument illustrated in Figure 115), using a ruler to make a sketch to determine if a triangle can be formed (e.g. the argument illustrated in Figure 106), or focusing on different features of the tool (e.g. the argument illustrated in Figure 99). Even though the students mainly use the tools to collect additional data, this was not always the case. At times, the students use known facts such as definitions and theorem as additional data to verify or refute a claim (e.g. the argument illustrated in Figure 108). For one group, group 2, there seems to be a reltionship between the content of the additional data and the reasons for collecting it. When these students' initial claims are challenged, the students feel the need to collect additional data using tools to demonstrate that they are correct. However, when these students are uncertain about their own claims, they question the basis for their reasoning and seek additional data to determine whether their reasoning is correct and, hence, whether their claims are valid. In other words, when a claim is challenged, the students use tools to collect additional data in order to resolve the challenge. When the students are uncertain about their claim, they collect additional data to determine whether they are using sound reasoning

Task 2 - Triangle Side And Angle Relationship Task

On the fifth day of class, the students in the non-technology class investigated the theorem which states "Let A, B, and C be three non-collinear points. Then $AB > BC$ if and only if $\mu(\measuredangle ACB) > \mu(\measuredangle BAC)$" (Venema, 2006, p. 102). In the three of the previous class meetings, the students investigated the triangle inequality theorem and the theorem that states the sum of the measures of the interior angles of a triangle is 180°. Given the students had previously explored and discussed theorems related to the angles and sides of a triangle, the teacher thought the students should learn how these elements of a triangle relate to each other.

The objectives for this task centered on the students' discovery and understanding of the relationship between the longest side and largest angle and the shortest side and smallest angle. The teacher wanted the students to be able to use the tools, ruler and protractor, to determine the longest and shortest sides and largest and smallest angles of a triangle. The teacher also wanted the students to compare the location of these sides and angles to determine the relationship between them. By the end of the task, the teacher wanted the students to have developed an understanding of this theorem. With these objectives in mind, the researcher created an activity and corresponding task sheet to be used to teach this concept.

The tools.

The tools the students used while working on this task were five pre-cut example triangles (See Tables 26 and 27). The teacher selected these triangles from a larger set of fifteen precut triangles that would be used in a later activity. The teacher chose these five triangles for a number of reasons. First, the set captured each of the six classifications of triangles. Triangle 1 is obtuse and isosceles, triangle 3 is acute and isosceles, triangle 9 is acute and scalene, triangle 5 is acute and equilateral, and triangle 15 is right and scalene. Even though the task was not focused on the classification of these triangles, the teacher wanted the students to work with a variety of triangle types in hope that the students would not form the notion that the relationships are only true for a given type of triangle. Second, the teacher expected the students to have some difficulty measuring the triangles, especially the angles. In the previous class meeting, the teacher asked the students to measure the interior angles and length of the sides of the five example triangles. The teacher was uncertain the students had used protractors to measure angles in their previous mathematics courses, and if they had, he was unsure whether they recalled how to use the tool. Using angle measures that either ended in 5 or 0, the teacher hoped the students would have less difficulty correctly measuring the angles. The teacher did not expect the students to have much difficulty measuring the side lengths because he expected the students to have frequently used rulers to measure lengths in their previous mathematics classes. To assist the students with their measuring, the teacher provided a short tutorial on how to measure angles and sides using a protractor and a ruler, respectively.

Table 25.

Measures of the angles of the example triangle used in the triangle sides and angle relationships tasks

Triangle #	Angle	Angle	Angle
1	$m\measuredangle A = 30°$	$m\measuredangle B = 120°$	$m\measuredangle C = 30°$
3	$m\measuredangle G = 40°$	$m\measuredangle H = 70°$	$m\measuredangle I = 70°$
5	$m\measuredangle M = 60°$	$m\measuredangle N = 60°$	$m\measuredangle O = 60°$
9	$m\measuredangle Y = 45°$	$m\measuredangle Z = 50°$	$m\measuredangle A = 85°$
13	$m\measuredangle E = 90°$	$m\measuredangle L = 75°$	$m\measuredangle P = 15°$

Table 26

Measures of the lengths of the sides of the example triangles using in the triangle sides and angles relationship task

Triangle #	Side	Side	Side
1	AC = 16.0 cm	AB = 9.3 cm	BC = 9.3 cm
3	IH = 6.5 cm	IG = 9.3cm	GH = 9.3cm
5	MN = 7.7 cm	NO = 7.7 cm	MO = 7.7 cm
9	AZ = 8.2 cm	YZ = 11.5 cm	AY = 8.8 cm
13	EP = 15.9 cm	EL = 4.1 cm	LP = 16.4 cm

The task sheet.

Along with the example triangles, the students were given a task sheet to assist in their exploration (see Appendix H). The task sheet contained directions for the activity. On the task sheet, the students were given a table to complete. The column headings of the table were the numbers for the five example triangles. The row headings were the name of the longest side, name of the largest angle, name of the shortest side, name of the smallest angle, and picture. The students were to look at the example triangles, fill out the table for that triangle and draw a picture of their triangle as a reference.

After the students completed the table, the students were asked to answer two questions. The questions were, "What do you notice about the relationship between the longest side and the largest angle?" and "What do you notice about the relationship between the smallest side and the smallest angle?" These questions were intended to focus the students on these relationships while still allowing the students to arrive at their own conclusions. Using the relationships they found for these two questions, the students were asked to write a conjecture about these relationships. Then, the students were asked to explain why they thought the conjecture was true.

Teaching of the lesson.

The triangle side and angle relationship task was conducted on the fifth day of class. At the beginning of the class period, the teacher asked the students to compare their answers on their homework assignment with their neighbors. After reviewing the solution to one of the problems, the teacher asked the students to provide a list of properties about triangles they had learned thus far. After briefly discussing the two theorems, the teacher directed the students to take out triangles 1, 3 ,5, 9, and 13. In a previous class meeting, the students measured the example triangles' angles and sides. So, the students should have had the measures already written on these triangles. Unsure the students would have the correct measures, the teacher displayed a table of the measures on the board. He directed the students to check their measures with those on the board and, for those that had not finished measuring, to copy the measurements on their triangles. Once the students finished this activity, the teacher asked a student to read the directions at the top of the task sheet. The teacher led the class as they filled out the table for example triangle number 1. Then, the teacher asked the students to complete the rest of the table and answer the questions at the bottom of the task sheet. The students began working on the task in pairs.

While the students worked on this task, the teacher moved from one group to another answering and posing questions to the students. After approximately 20 minutes, the teacher led a whole class discussion. At the beginning of the discussion, the teacher asked for volunteers to present their findings for their responses in the table. These volunteers would write their answers on the teacher's task sheet that was projected on the board. While the volunteer student was filling out the table, the teacher would ask questions to the volunteer students and to the class about the responses.

Once the last row in the table had been filled out by the students, the teacher asked the students what they found. One student said that he and his partner found, "The largest side was across from the largest angle." The teacher asked if other groups noticed the same relationship. The majority of the class responded affirmatively. The teacher then asked if they found any other relationships. Another student said, "The smallest side is across from the smallest angle." The teacher asked if this was true for all the triangles. One student said it was not true for example triangle 5, the equilateral triangle. The teacher asked the class why this student thinks it is not true for this triangle. The students conjectured that the relationship did not hold for this triangle because all the measures were the same. With time winding down in class, the teacher asked the students why these relationships were true. One student came to the board and drew an obtuse angle whose rays would be two of the sides of the triangles. He said, "So this side and this are the smallest so they have to make a bigger line for it to close up so that's and so the two shortest ones have to make a bigger line for them to close up and make a triangle." The teacher led a whole class discussion on this student's idea and how it applies to the relationship between the shortest side and smallest angle of a triangle. Once the teacher felt the students had an understanding of the reasons why these relationships were true, he wrote the theorems on the board and asked the students to copy them in their notes. Next, the teacher distributed the homework worksheet.

In the following sections, the arguments created by three pairs of students while working on these activities are analyzed and discussed. For each pair of students, the arguments were first categorized by their basic structure. Then, the content and structure of the arguments within these basic categories were analyzed, including the students' uses of tools. The themes that emerged from these analyses are discussed for each pair of students and across the pairs of students.

Group 1's arguments on the triangle side and angle relationship task.

The analysis of Andy and Frank's arguments while working on the triangle side and angle relationships task can be categorized into three argument structures; core arguments,

arguments in which the students are compelled to collect additional data to verify or refute a claim, and arguments with a sub-argument. These arguments structures are discussed below.

Core argument.

While working on the triangle side and angle relationship task, Andy and Frank create four core arguments, two in which the warrant is explicit and two in which the warrant is not explicit. These two argument structures are discussed below.

Non-explicit warrants.

For two of the core arguments, Andy and Frank use measures of the triangles as data and, based on this data, make claims. However, the pair do not provide an explicit warrant. In one argument, Andy and Frank are looking for patterns among the lengths of the sides and the measures of the angles using the table on the task sheet they just completed. Andy states, "It can't go lower than 6.5." Andy is referring to the measurements of the shortest side length of the triangles. This argument is illustrated in Figure 119.

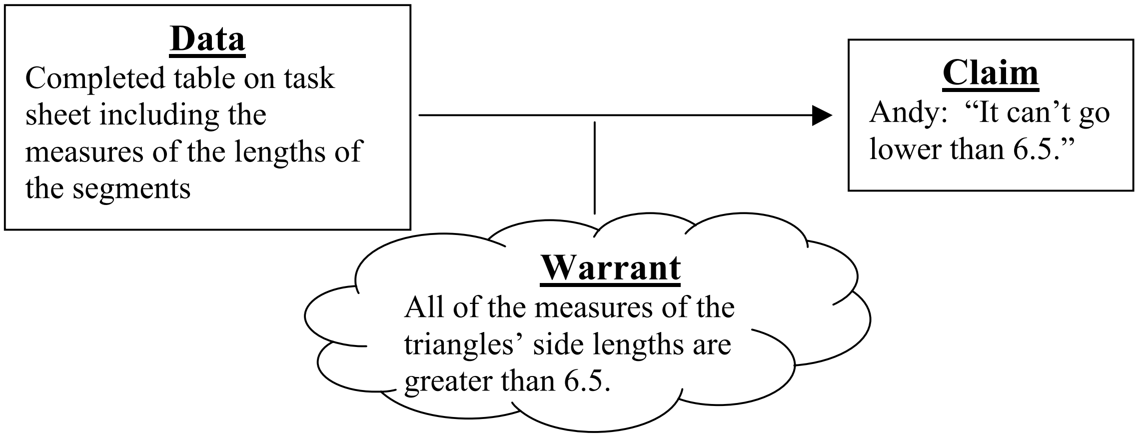


Figure 119. Frank and Andy's core argument created during task 2 with an inferred warrant and tool use.

The data for this argument are the measures of the lengths of the segments and the completed table on the task sheet. Andy claims that the measure of the lengths of the sides of the triangles must be less than 6.5. The warrant for this claim is not explicit and is inferred by the researcher to be, "All of the measures of the triangles' side lengths are greater than 6.5."

The second argument that contains this structure is one in which Andy and Frank are being assisted by the teacher as they attempt to formulate a conjecture. The teacher asks the pair, "Andy, where is the largest angle?" Frank points to the largest measured angle and replies, "Right there." The teacher responds, "It's over here. So look at your numbers and see if you can figure it out. Ok? Here's the smallest and the smallest angle's over here." Frank asks, "Yes, so it's the opposite of?" The teacher directs him, "There you go. Write it." Frank asks, "How would you put it though?" The teacher begins, "The largest side is…" Frank interjects, "opposite of" and the teacher gives his approval, "There you go. Write it up." The teacher leaves the pair to write the conjecture. Frank says, "The largest side is opposite." Andy interjects, "Right here? The largest?" Losing his concentration, Frank asks, "What did he say? Wait, the largest side is opposite the angle. Wait, yeah. Yeah. Yeah. It's going to be, it's always going to be the opposite side." This argument is illustrated in Figure 120.

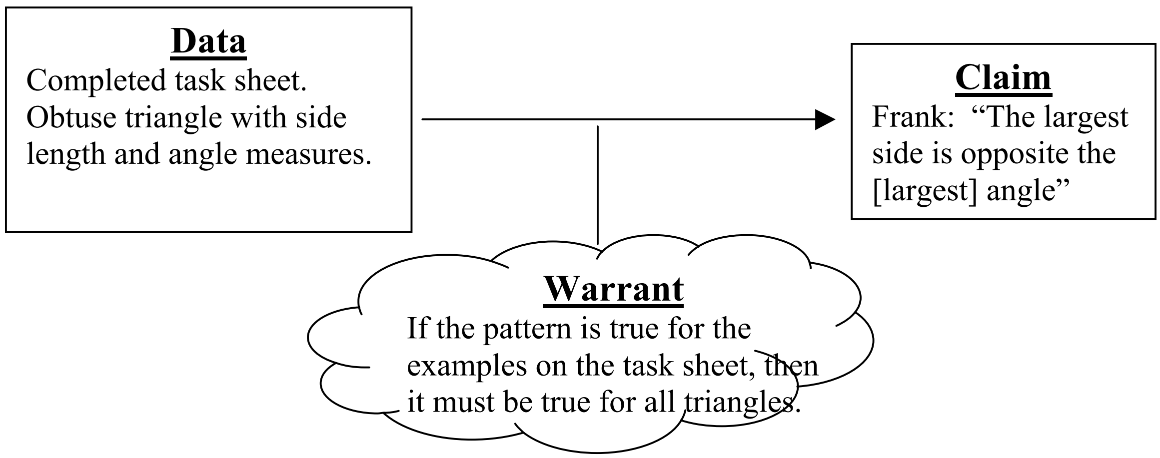


Figure 120. Frank's core argument created during task 2 with an inferred warrant and tool use.

The data for this argument are the measures of the lengths of the segments and the completed table on the task sheet. Frank claims the longest side is across from the largest angle. The warrant for this claim is not explicit and is inferred by the researcher to be, "If the pattern is true for the examples on the task sheet, then it must be true for all triangles."

Across these two core arguments there are many similarities. First, the structure of the arguments is the same; the claims are explicit and the warrants are not explicit. Second, the data for these arguments are based, in part, on the use of the measures of the lengths of the sides, the measures of the angles, and the completed task sheet. Third, the task on which the students are working is the same, to determine a generalization from the examples on the task sheet.

Explicit warrants.

In two of the core arguments, Andy and Frank provide an explicit warrant. In one argument, the pair attempts to determine the largest angle for the equilateral triangle with angle measures 60 degrees. Andy states, “Yeah, for the largest angle, it doesn’t matter which one cause they’re all the same.” This argument is illustrated in Figure 121.

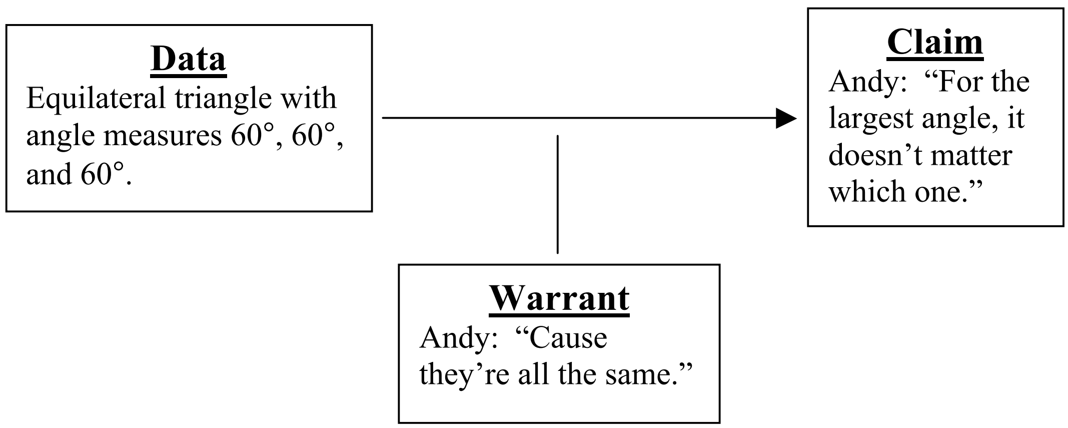


Figure 121. Frank and Andy’s core argument created during task 2 with an explicit warrant and tool use.

The data for this argument are the equilateral triangle with all three angles having measure 60°. Andy claims that it does not matter which angle is selected as the largest angle. He provides an explicit warrant stating that the angles are all the same measure.

In the second core argument with an explicit warrant, Frank explains why the longest side must be across from the largest angle. He draws a picture on the board (see Figure 122) and states, “So this side and this are the smallest so they have to make a bigger line for it to close up.” This argument is illustrated in Figure 123.

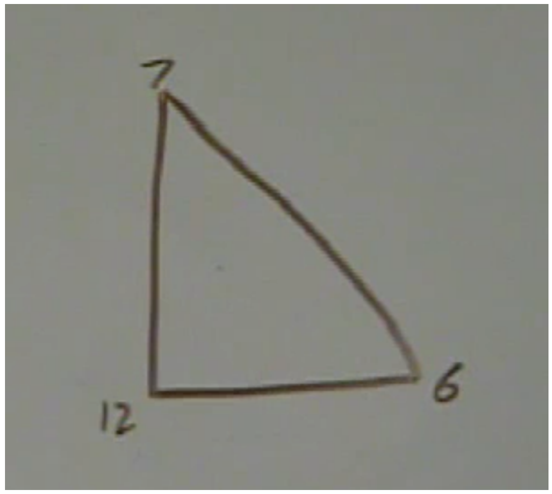


Figure 122. The figure Frank draws on the board during task 2.

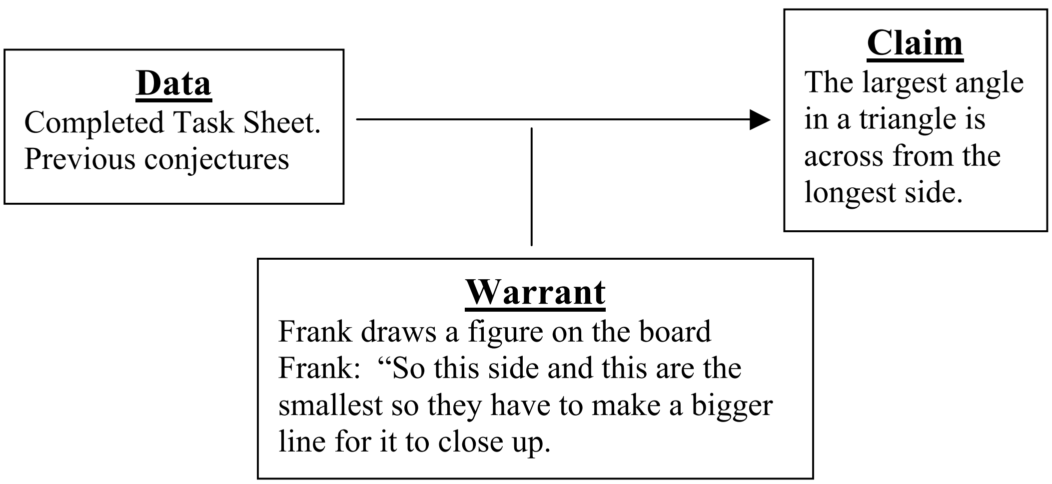


Figure 123. Frank's core argument created during task 2 with an explicit warrant and no tool use.

The data for this argument are the completed task sheet and the previous conjectures offered during the whole class discussion. The claim is the largest angle in a triangle is across from the longest side. However, Frank does not explicitly provide the claim. Rather, the class develops the claim during the whole class discussion and the teacher asks Frank to provide the warrant. Frank draws a figure on the board (see Figure 122) and indicates that a line segment longer than the two given segments would be needed in order to connect the two shorter segments to form a triangle.

Conclusions.

Looking across these core arguments, two major themes emerge. First, in both categories of core arguments, Andy and Frank use the measures of the angles and/or sides as data for their claims (e.g. the arguments illustrated in Figures 119, 120, and 121). The example triangles are considered the tools for this task and the pair use these tools as data for their arguments. Second, the claims for three of the four core arguments are conjectures and/or generalizations (e.g. the arguments illustrated in Figures 119, 120, and 123). Although the claims of the core arguments with an explicit warrant are generalizations and the data of the core arguments with a non-explicit warrant contain measures, the small number of arguments of this structure do not allow for a blanket generalization.

Andy and Frank's non-core arguments.

While working on the triangle side and angle relationship activity, Andy and Frank create three arguments that are not of the core structure. These non-core structures include arguments in which the students collect additional data in response to a challenge to the initial claim, arguments in which the students collect additional data in response to their uncertainty with the initial claim, and arguments in which the warrant is challenged and a sub-argument is created. These three argument structures are discussed below.

Arguments with additional collection, challenge

In one argument, Andy and Frank attempt to determine a pattern in the names of the triangles. Andy states, "Maybe they're going by 3 because 3 letters GEF, ABCDEFGHIJKL and then you go to the next one. It keeps skipping three letters." Frank asks, "Where is the L at?" Andy responds, "Like there is no L because it skips it. Like ABC you skip 3, DEF, then you got GHI. There is none." Frank agrees, "All right, yeah." Andy continues, "Then it skips 3, then it goes to the next three letters, skip 3, go to the next three letters, skip 3 go to…" Frank interjects, "Yeah, but what about A?" Andy replies, "Well, it starts over, A, BCD, cause it starts over. Frank says, "All right, YZA?" Andy replies, "Yes." Looking at the name of the final triangle, Frank asks, "LBD? That skips a lot of letters, so you can't do that. Andy agrees, "Yeah." This argument is illustrated in Figure 124.

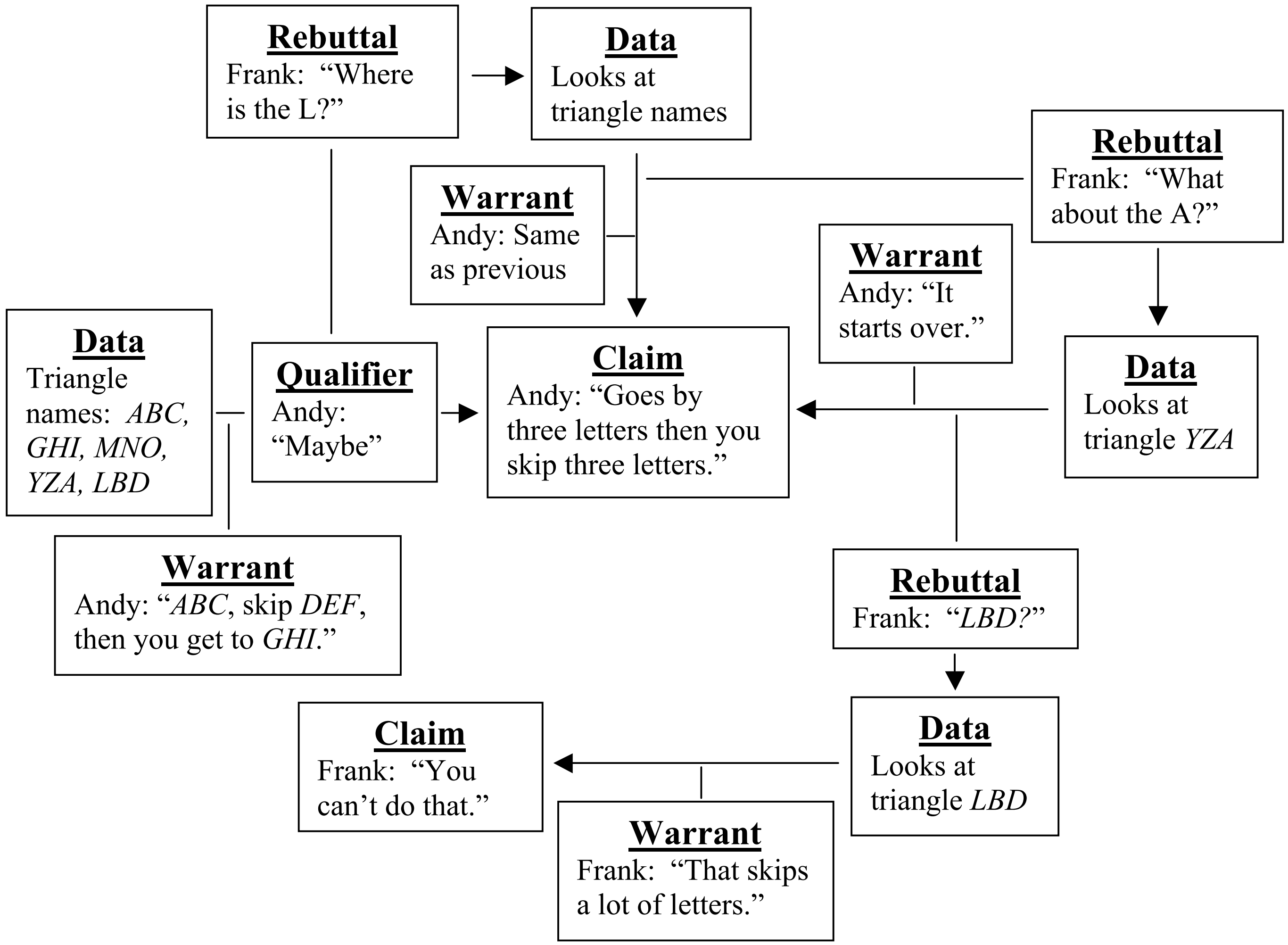


Figure 124. Andy and Frank's argument created during task 2 with additional data collection in response to challenges by Frank.

The initial data for this argument are the names of the example triangles; Δ*ABC*, Δ*GHI*, Δ*MNO*, Δ*YZA*, and Δ*LBD*. Andy claims that the pattern in the naming of the example triangle is the use of three consecutive letters, beginning with *A*, to name the fist triangle, then three letters are skipped, and then the next three consecutive letters are used to name the next triangle. Andy qualifies this initial claim with the term "maybe." He provides an explicit warrant by pointing toward the examples triangle Δ*ABC* and Δ*GHI*. Frank challenges this claim by asking Andy about the use of the letter *L* in the names of the triangles, which is an explicit rebuttal. Andy collects additional data by reviewing the list of triangle names and notices that *L* is not used. He claims that the letter *L* is one of the letters that is skipped which verifies is initial claim. Andy's warrant for this verification is the same as his previous warrant. Again, Frank challenges Andy's claim through a rebuttal by asking about the use of the letter *A* in the triangle Δ*YZA*. Andy collects additional data by reviewing the name of the triangle. Once again, he verifies his initial claim by indicating that the pattern still holds. He provides an explicit warrant indicating that the letters start over at *A* when *Z* is used. Frank challenges the claim in the form of a rebuttal by asking Andy about the name of triangle *LBD*. Frank collects additional data by studying the name of this triangle and Andy's pattern. He claims Andy's pattern does not hold. He provides an explicit warrant stating many letters are skipped in the naming of triangle *LBD*. Andy concurs.

In this argument, Andy develops a potential pattern among the names of the five example triangles, which is the initial claim. Four times, Frank challenges this claim. And, Andy is able to justify that his pattern holds for three of these challenges. In the end, Frank is able to demonstrate that the pattern is not true for these triangles. In this episode, Frank and Andy demonstrate their recognition that in order for a pattern to be valid, it must hold for all examples.

Argument with additional data collection, uncertainty.

Another argument in which Andy and Frank collect additional data is in response to the students' uncertainty to validity of their initial claim. In this episode, Frank and Andy discover a possible relationship between the longest/shortest side and largest/smallest angle of a triangle. Frank explains, "Because look this, this is a pattern. Side like *A* and *B*, I mean *A* and *C* and then that's the one *B*. So, you see this is *A* and *C* and then the opposite is *A* and *B* and the *B* is right here and the *C* is right here. So, then *GH*, *I*; *HI*, *G*." Later in the class, the teacher asks the pair, "Did y'all figure it out?" "Uh, we kind of have a pattern," replies Frank. The teacher asks Frank to explain his pattern and Franks states, "Like from here like the angle not used in this [the longest or shortest side] is this [the largest or smallest angle]." The teacher says, "Ok." Frank continues, "But we messed up," and points to the equilateral triangle. "Well, what's special about this one?" asks the teacher. Frank replies, "Well, we like didn't um, um like on here instead of being like *N* it had to be *O*." "Yeah, but what's special about this triangle?" asks the teacher. "Everything's the same," answers Frank. This argument is illustrated in Figure 125.

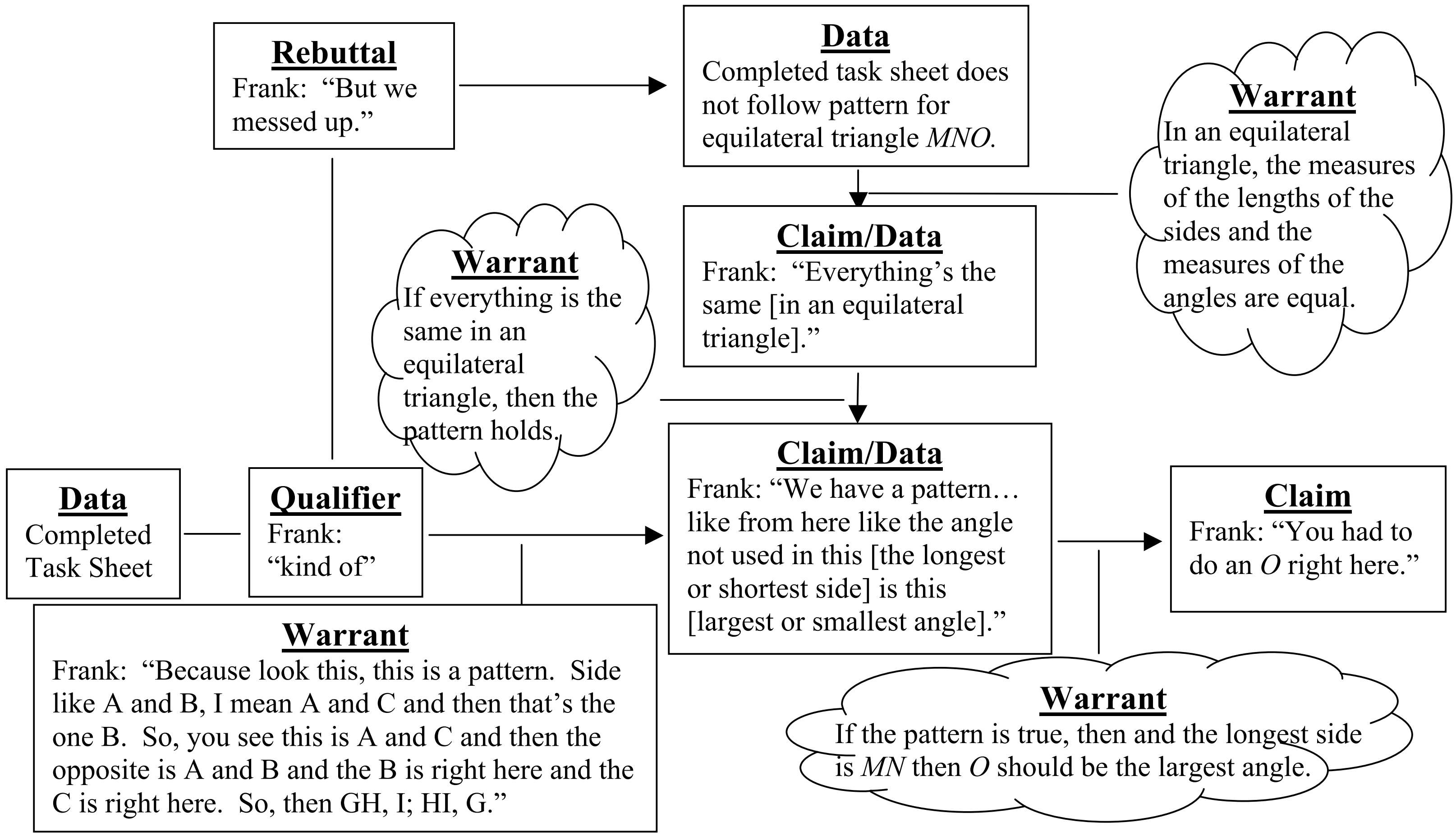


Figure 125. Andy and Frank's argument created during task 2 with additional data collection in response to Frank's uncertainty.

The initial data of this argument is the completed task sheet. Frank makes the initial claim that the pair have a pattern which is the letter not used in naming the longest/shortest side is the name of the largest/smallest angle. He qualifies this claim with the phrase, "kind of." Frank provides an explicit warrant for this claim by referring to the first two examples on the task sheet. However, he recognizes his third example does not follow this pattern. He states that he and Andy may have messed up which is a rebuttal. He collects additional data by directing his and the teacher's attention to his responses on the task sheet for triangle number 5, the equilateral triangle. When asked what is special about this triangle, Frank makes the claim everything is the same. He does not provide a warrant for this claim and is inferred by the researcher to be, "In an equilateral triangle, the measures of the lengths of the sides and the measures of the angles are equal." Frank's new claim becomes data for the verification of Frank's initial claim. He does not provide an explicit warrant for this verification and is inferred by the research to be, "If everything is the same in an equilateral triangle, then the pattern holds." The initial claim then become data for Frank as he makes the claim the largest angle for the equilateral triangle should be *O* given the longest side is *MN*. He does not provide an explicit warrant for this claim and is inferred by the researcher to be, "If the pattern is true, then and the longest side is segment *MN* then *O* should be the largest angle."

The final claim in this argument is the name of the longest angle for one of the examples on the students' task sheet should be changed. Similar to the argument illustrated in Figure 124, Frank recognizes that a pattern should hold for all examples, not just a select few. This notion coupled with the responses on the task sheet for the equilateral triangle that do not follow the pattern developed by he and Andy provides a cognitive conflict. Thus, Frank has some uncertainty with his claim. When speaking with the teacher, Frank realizes that his responses on the task sheet can be modified to fit the pattern without making the responses incorrect. By making this change, he is able to show that this pattern is true for all examples.

Argument with a sub-argument.

During the whole class discussion, the class formulated the relationship that the shortest side of a triangle is always across from the smallest angle. The teacher asked if this is true from all of the examples. Frank claimed, “No, except for [triangle] number 5.” The teacher asked, “Now why do you say number 5 doesn’t work?” Frank replied, “Because they’re all the same measurements so there’s not an angle or a side bigger than another.” This argument is illustrated in Figure 126.

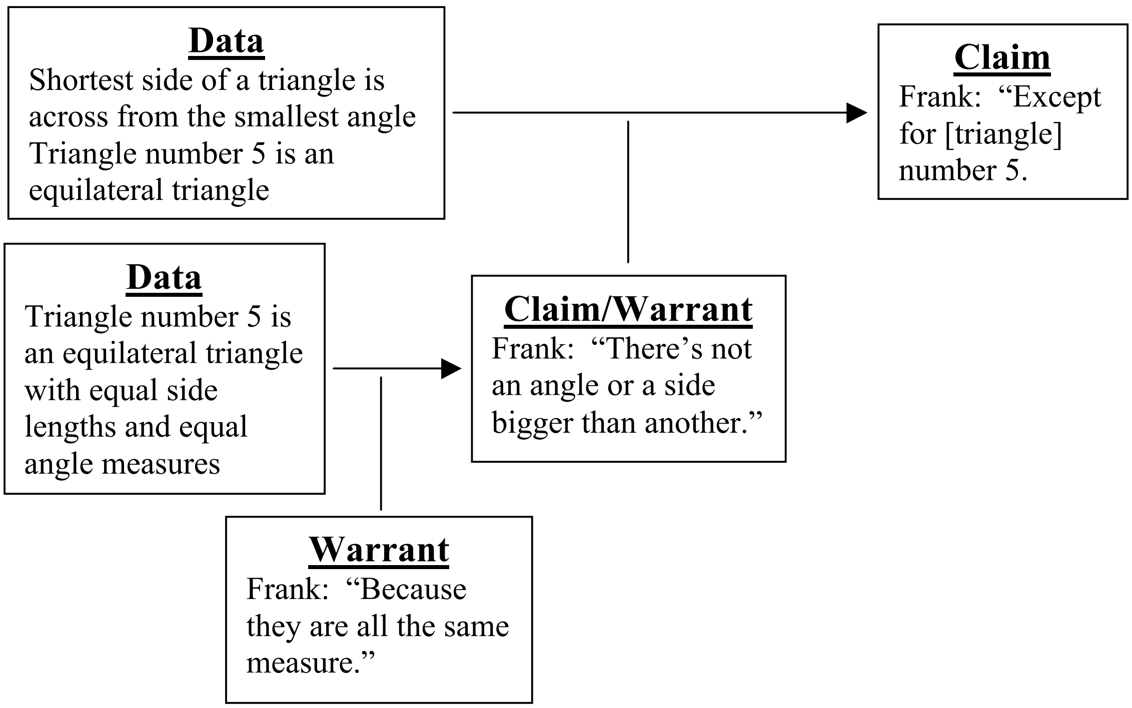


Figure 126. Frank's argument created during task 2 with a teacher prompted sub-argument.

The initial data for this argument is the relationship the shortest side of a triangle is across from the smallest angle and the example triangle number 5 is an equilateral triangle. Frank claims that the relationship is true for all examples except for triangle number 5. Initially, Frank did not provide an explicit warrant. Only when prompted by the teacher does he provide support for his claim in the form of a sub-argument. The data for this sub-argument is triangle number 5 is an equilateral triangle and the measures of the side lengths and angles. The claim for his sub-argument, which is the warrant for his main argument, is that an angle or a side is not bigger than another one. He provides an explicit warrant for this claim stating the measures are the same.

Group 1's summary for triangle side and angle relationship task.

While working on the triangle side and angle relationship activity, Andy and Frank create arguments of various structures. Three categories of structures are noted in the analysis: core arguments, arguments in which the students collect additional data to verify or refute a claim, and arguments with a sub-argument. Looking across these argument structures, two themes emerge; the number of arguments (see Table 27), and the explicitness of the warrants in the non-core arguments.

Table 27.

Group 1's Argument on the Triangle Side and Angle Relationship Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Tools	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	2	0
	Tools Not Used	0	0
Warrants Explicit			
	Tools Used	2	3
	Tools Not Used	0	0

While working on the triangle sides and angle relationship activity Andy and Frank only create seven arguments. This may be due to the nature of the task sheet. When completing the task sheet, the students do not discuss their findings. They simply look at the triangles and record the names for the longest/shortest side and the largest/smallest angle.

The students do not make their responses audible nor did the students question each other about their responses. Thus, the claims the students make on their task sheet could not be documented as arguments.

However, when the students do create arguments, they are more likely to be explicit about their warrants. Of the seven arguments created by Andy and Frank, five of the arguments have at least one explicit warrant, and all of the non-core arguments have at least one explicit warrant. The explicitness of the warrants for these non-core arguments may be related to the nature of the argument. In the argument illustrated in Figure 124, Frank challenges Andy's claim four times. In response to these challenges, Andy defends his claim by pointing out why his pattern is valid and accommodates Frank's challenges. By defending his claim in showing how the pattern accommodates Frank's challenges, Andy provides an explicit warrant. Due to the challenges of Frank and the defense by Andy, it is not surprising that there are explicit warrants in this argument. In the argument illustrated in Figure 126, Frank makes a claim and provides an explicit warrant for his claim. The teacher asks Frank to justify his claim and Frank provides support for his claim in the form of a sub-argument. The explicitness of the warrant in this argument is not surprising because the teacher is requesting Frank provide a justification for his claim. In both of these arguments, another person is either challenging the claim or requesting a warrant. Perhaps, when students' claims are challenged or when prompted to justify their claims, the students are more likely to provide an explicit warrant.

Group 2's arguments on the triangle side and angle relationship task.

The analysis of Bob and Ellen's arguments while working on the triangle side and angle relationship task can be categorized into one argument structure: a core argument with an explicit warrant. This argument structure is discussed below.

Core argument.

While working on the triangle side and angle relationship task, the pair of students created three core arguments, all with explicit warrants. In one argument, the teacher asks the class to name largest angle in obtuse triangle *ABC*. Bob responds, "*B*." "How do you know it's angle *B*?" asks the teacher. Bob responds, "Because it's 120." This argument is illustrated in Figure 127.

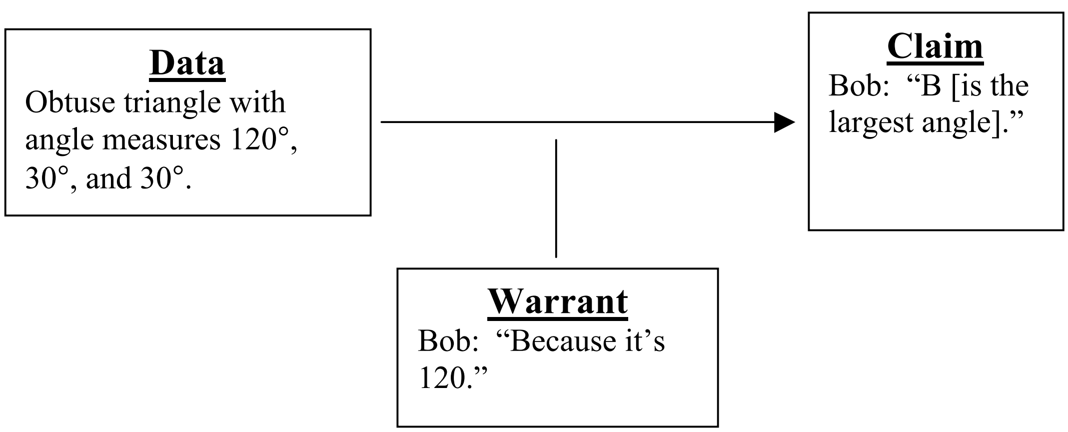


Figure 127. Bob's core argument created during task 2 with an explicit warrant and tool use.

The data for this argument are the obtuse triangle with angle measures 120°, 30°, and 30°. Bob claims that angle *B* is the largest angle. The teacher prompts him to provide an explicit warrant and he indicates that the measure of angle *B* is 120, which is the largest angle.

Bob provides another example of a core argument with an explicit warrant in response to the teacher's question, "Why is the largest angle across from the longest side?" Bob replies, "Cause there's three sides and two were taken up." This argument is illustrated in Figure 128.

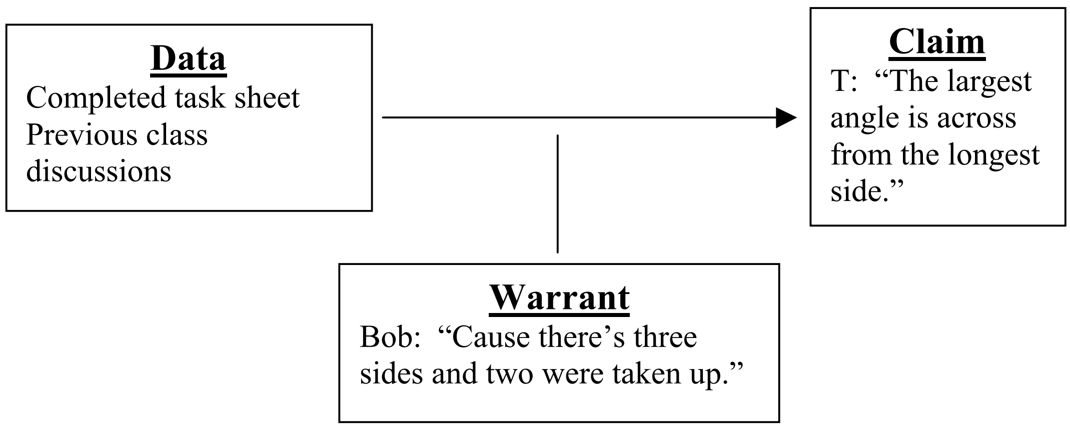


Figure 128. Bob's core argument created during task 2 with an explicit warrant and a teacher provided claim .

The data for this argument are the completed task sheet and the previous class discussions. Bob does not make the claim in this argument. Rather, the teacher makes the claim and asks Bob to provide the warrant for the claim. Bob indicates that the claim is true because there are three sides in a triangle and two of the sides had been used to form the largest angle.

Group 2's summary for triangle side and angle relationship activity.

While working on the triangle side and angle relationship activity, Bob and Ellen create three arguments, all of the same structure. Looking across these arguments, two themes emerge; the number of arguments (see Table 28), and the relationship between the structure of the argument and the role of the teacher.

Table 28.

Group 2's Argument on the Triangle Side and Angle Relationship Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Tools	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	0	0
	Tools Not Used	0	0
Warrants Explicit			
	Tools Used	1	0
	Tools Not Used	2	0

While working on the triangle sides and angle relationship activity, Ellen and Bob only create three arguments. This may due to the lack of communication between the pair as they complete the task sheet. Bob and Ellen take turns filling out the task sheet rather than working together. They do not discuss how they arrive at their conclusions. Thus, they are not making audible arguments. In addition, the low number of arguments may be attributed to the students' reluctance and inability to develop a conjecture. The students have a great deal of difficulty finding a relationship between the names of the sides and angles of the triangles. In fact, Bob's claim in the argument not discussed in this section is "There really is nothing."

Even though Bob and Ellen only create three arguments, all have the same structure, a core argument with an explicit warrant. The explicitness of the warrant may be related to the role of the teacher. In the arguments discussed in this section, the teacher asks the students to provide an explicit warrant. In the argument illustrated in Figure 127, the teacher asks the class to make a claim and then asks Bob how he arrives at his conclusion. In the argument illustrated in Figure 128, the teacher provides the claim and asks the students to provide an explicit warrant. If the teacher asks the students to justify their claims or furnishes the claims and asks for a justification, then it is not surprising that the students provide explicit warrants. Thus, the students are willing to make their warrants explicit, but may need to be prompted by the teacher to do so.

Group 3's arguments on the triangle side and angle relationship task.

The analysis of Clair and Jim's arguments while working on the triangle side and angle relationships task can be categorized into two argument structures: core arguments, and arguments in which the students collect additional data to verify or refute a claim. These arguments structures are discussed below.

Core argument.

While working on the triangle side and angle relationship task, Clair and Jim create three core arguments, two in which the warrant is explicit and one in which the warrant is not explicit. These two argument structures are discussed below.

Non-explicit warrants.

For one of the core arguments, Clair and Jim use the measures of the side lengths of the triangles as data and based on this data, make a claim. However, the pair does not provide an explicit warrant. In this argument, Jim and Clair are to determine the longest side of triangle *GHI*. Jim and Clair measured the side lengths at the end of class the previous day and these measures are written on the triangle. Clair states, "*GI*, no it [the longest side] could be either [*GI* or *GH*], it could be either." This argument is illustrated in Figure 129.

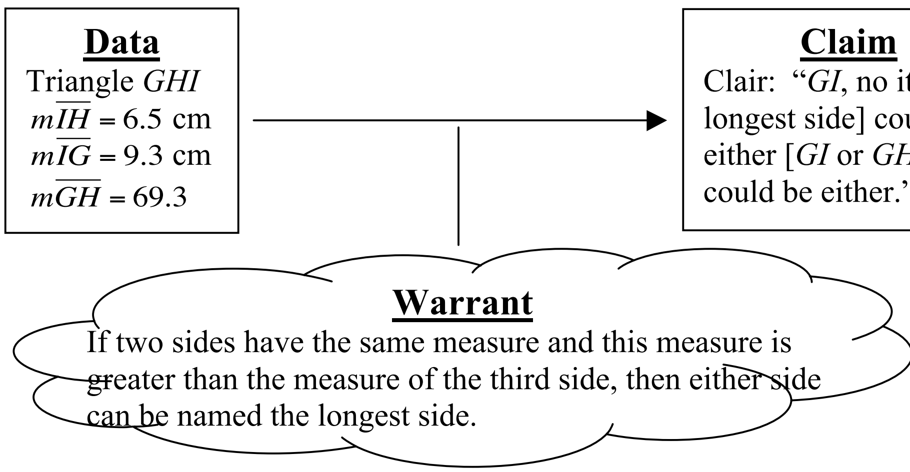


Figure 129. Clair's core argument created during task 2 with an inferred warrant and measures of the side lengths as data.

The data for this argument are the triangle *GIH* and the measures of its side lengths. Clair claims either $\overline{GI}$ or $\overline{GH}$ could be used as the largest side. She does not provide an explicit warrant and is inferred by the researcher to be, "If two sides have the same measure and this measure is greater than the measure of the third side, then either side can be named the longest side."

Explicit warrants.

In two of the core arguments, Clair and Jim provide an explicit warrant. In one argument, the pair attempts to determine the largest angle for the equilateral triangle with angle measures 60 degrees. Clair states, “It [the largest angle] could be either one cause all the angles are the exact same.” This argument is illustrated in Figure 130.

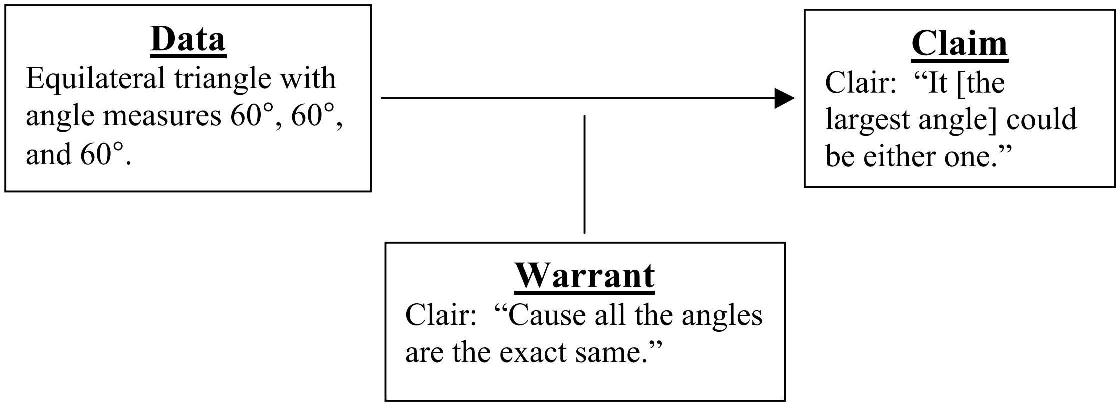


Figure 130. Clair’s core argument created during task 2 with an explicit warrant and angle measures as data.

The data are the equilateral triangle with all three angles having measure 60°. Clair claims that it does not matter which angle is selected as the largest angle. She provides an explicit warrant stating the angles have the same measure.

In the second core argument with an explicit warrant, Clair explains why the longest side must be across from the largest angle. Previously, Clair and Jim developed the conjecture the largest/smallest angle is opposite of the longest/shortest side, respectively. She says to Jim, “You see how they’re opposite [point to triangle 1]?” Jim replies, “Yeah.” Clair states, “Well, maybe if they were beside each other, it wouldn’t be that big.” “Yeah, maybe, that’s true,” replies Jim. Clair further explains, “Maybe like [pointing to triangle 1], maybe if the angle was down here, it might not have been able to connect.” Jim agrees saying, “Yeah.” This argument is illustrated in Figure 131.

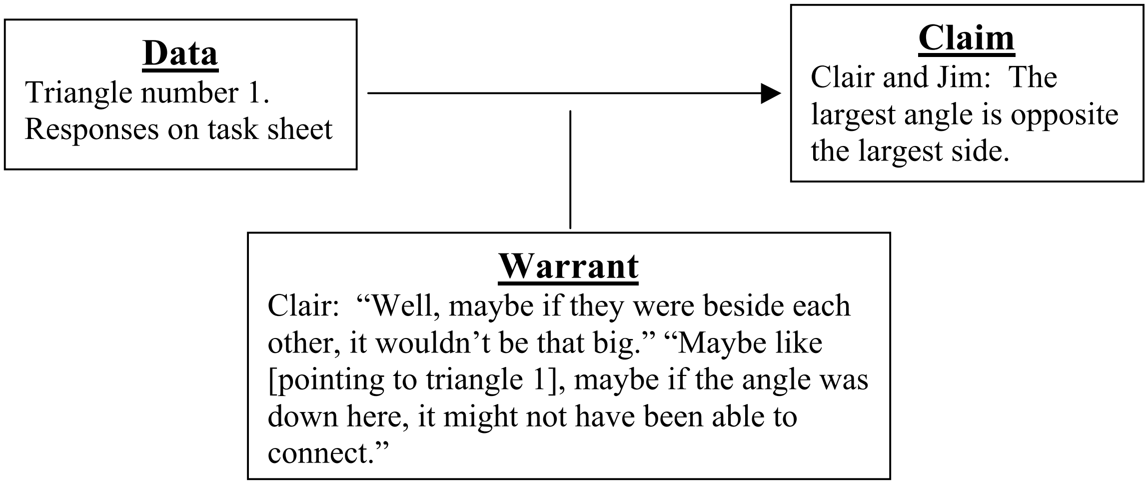


Figure 131. Clair and Jim’s core argument created during task 2 with an explicit warrant and use of tools.

The data for this argument are the students’ responses on the task sheet and the first example triangle, an obtuse isosceles triangle. The students claim the largest angle is opposite the largest side. Clair provides an explicit warrant when she poses a hypothetical situation in which the largest angle was moved such that it was adjacent to the longest side and indicates, in this situation, the largest angle would cease being the largest angle or that a triangle would not be formed in this configuration.

Clair and Jim’s non-core arguments.

While working on the triangle side and angle relationship task, Clair and Jim create two arguments that are not the core structure. In both of these arguments, the students collect additional data in response to their uncertainty with the initial claim. These two arguments are discussed below.

In one argument, Jim and Clair attempt to develop a relationship between the longest side and largest angle in a triangle. Jim looks at the example triangles and states, “They’re pretty much both on the same side." Jim and Clair look at the example triangles and Jim points to the longest side on each triangle. Jim then states, "Well, pretty much it's always opposite the largest angle because the angle's right there and the side's right there [triangle 1 – pointing]; angle, side [triangle 3]; angle, side [triangle 5]; angle, side [triangle 9]; angle, side [triangle 13]." This argument is illustrated in Figure 132.

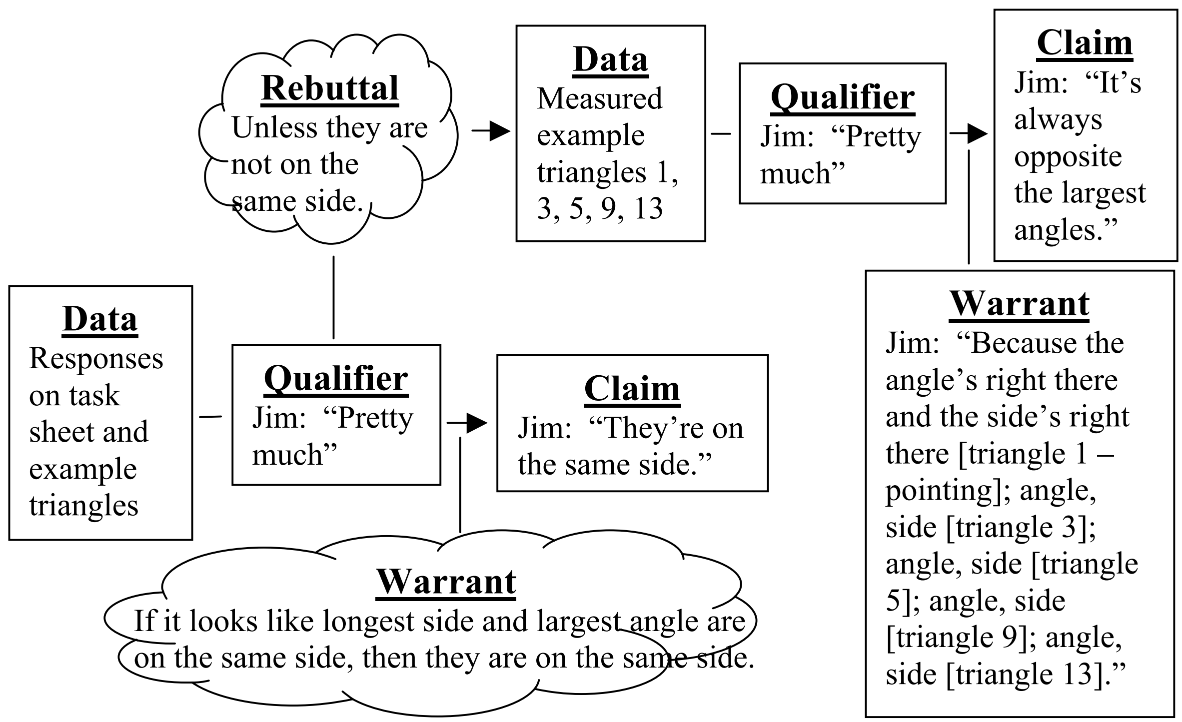


Figure 132. Jim's argument created during task 2 with additional data collection in response to Jim's uncertainty.

The initial data for this argument are the students' responses on the task sheet and the measured example triangles. Jim makes the initial claim that the largest angle and longest side are on the same side, or adjacent to each other. He provides a qualifier to this claim with the term "pretty much". Jim does not provide an explicit warrant for his claim and is inferred by the researcher to be "If it looks like longest side and largest angle are on the same side, then they are on the same side." Jim and Clair continue to look at the example triangles with Jim pointing out the largest side on each example triangles. This action coupled with the use of qualifier demonstrates that the pair has some uncertainty regarding Jim's initial claim. Thus, the researcher infers the rebuttal, "Unless they are not on the same side." Based on the additional data collected by Jim and Clair, Jim refutes his initial claim and makes a new claim, that the longest side is always opposite the largest angle. He qualifies this claim with the same phrase he used to qualify his initial claim, "pretty much." Jim provides an explicit warrant for this claim by pointing to the location of the longest side and largest angle for each of the five example triangles and demonstrates they are opposite each other for all of the example triangles.

In the other argument in which the students collect additional data, Jim and Clair are to determine the relationship between the shortest side and smallest angle of a triangle. Clair reads the question on the task sheet, "What do you notice about the relationship between the smallest side and the smallest angle." She remarks, "Well, they're probably opposite to." Jim says, "Well, yeah, because the smallest angle's right here like that, that, that, that, that, that, that, that, that [points to the smallest angle and side for each of the example triangle]. So yeah it's the opposite." This argument is illustrated in Figure 133.

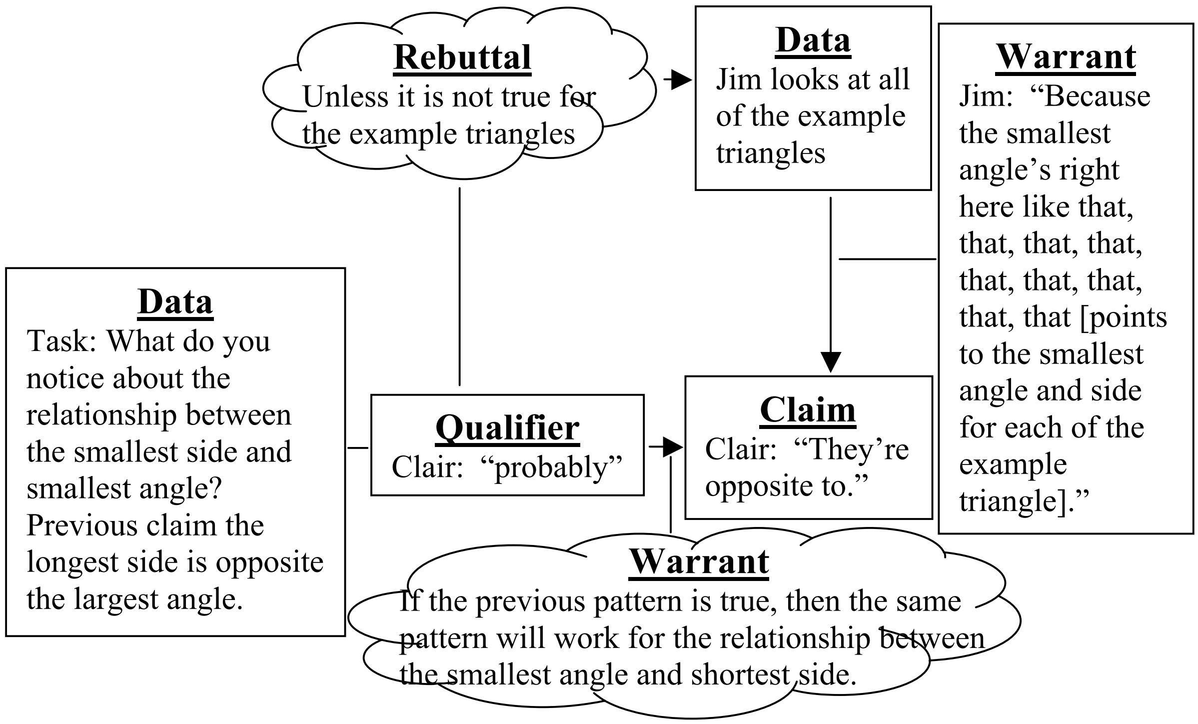


Figure 133. Clair and Jim's argument created during task 2 with additional data collection in response to Clair's uncertainty.

The initial data for this argument are the question, "What do you notice about the relationship between the smallest side and the smallest angle?" and the students' previous claim the longest side of a triangle is opposite the largest angle. Clair makes the claim the shortest side and the smallest angle are opposite each other as well. She qualifies this claim with the term "probably." Clair does not provide an explicit warrant for this claim and is inferred by the researcher to be, "If the previous pattern is true, then the same pattern will work for the relationship between the smallest angle and shortest side." Jim collects additional data by looking at the example triangles and pointing to the smallest angle and shortest side. This action coupled with the use of qualifier demonstrates that the pair had some uncertainty regarding Clair's initial claim. Thus, the researcher infers the rebuttal, "Unless it is not true for the example triangles." Jim verifies Clair's claim and provides an explicit warrant demonstrating that the shortest side is opposite the smallest angle for each of the example triangles.

Group 3's summary for triangle side and angle relationship task.

While working on the triangle side and angle relationship task, Clair and Jim create five arguments. Two categories of structures are noted in the analysis: core arguments, and arguments in which the students are compelled to gather additional data to verify or refute a claim. Looking across these arguments, two themes emerge; the number of arguments (see Table 29), and the frequency of the use of the tools to create data.

Table 29.

Group 3's Argument on the Triangle Side and Angle Relationship Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Tools	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	1	1
	Tools Not Used	0	0
Warrants Explicit			
	Tools Used	2	1
	Tools Not Used	0	0

While working on the triangle side and angle relationship activity, Clair and Jim only create five arguments. This may be due to the nature of the task sheet. When completing the task sheet, the students do not discuss their findings. They simply look at the triangles and write down the names for the longest/shortest side and the largest/smallest angle. The students do not make their responses audible nor do the students question each other about their responses. Thus, the claims made on their task sheet cannot be documented as arguments.

Even though the students only create five arguments, the students actively use the tools to collect data in all of the arguments. The tools in this argument are the measured example triangles. For all of the core arguments the students use the measures of the example triangle as data. In the two arguments in which the pair collected additional data, the students use the tools, the measured example triangles, to collect additional data.

Cross-case analysis of the arguments created by the groups while working on the triangle side and angle relationship task.

On the triangle side and angle relationships activity, the arguments created by the three groups of students vary in their structure and content. Three themes emerge when looking across the arguments created by these students; the structure of the argument when the tools are and are not actively employed (see Table 13), the number of arguments, and the relationship between the type of task on which the students are working and the content of the argument.

Table 30.

The Combined Arguments of the Three Groups on the Triangle Side and Angle Relationship Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Tools	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	3	1
	Tools Not Used	0	0
Warrants Explicit			
	Tools Used	5	4
	Tools Not Used	2	0

While working on the triangle side and angle relationship task, the three groups of students create 15 arguments. For 4 (27%) arguments, the students do not provide explicit warrants. Additionally, the students employ the tools in 13 (87%) arguments. Of the 13 arguments in which the students employ the tools, they provide an explicit warrant in 9 (69%) of these arguments. For both of the arguments in which the tools are not actively employed, the students provide an explicit warrant (100%). The disparity between the percentages may suggest when students use the tools, they are less likely to provide an explicit warrant compared to when they do not actively use the tools.

The explicitness of a warrant may be better attributed to the activity on which the students are working rather than the active use of tools. Of the 15 arguments created while working on the triangle sides and angle relationship task, 10 are related to a pattern or conjecture developed by the students, 8 (80%) of which have an explicit warrant. Of the 5 arguments that are not about a pattern or conjecture, only 3 (60%) arguments have explicit warrants. When the students are creating arguments about patterns and conjectures, the students were forming generalizations and were likely to make their reasoning explicit.

The lack of arguments may be due to the nature of the task sheet. When completing the task sheet, the students do not discuss their findings. They simply look at the example triangles and write down the names for the longest/shortest side and the largest/smallest angle. The students do not make their responses audible nor do the students question each other about their responses. Thus, the claims made on their task sheet can not be documented as arguments.

Task 3 - Triangle Sort Task

On the sixth day of class, the students in the non-technology class investigated the classification of triangles. In the previous class meetings, the students investigated the triangle inequality theorem, the theorem that states the sum of the measures of the interior angles of a triangle is 180°, and the relationship between the relative magnitudes of the length of a side of a triangle and the measure of the triangle's interior angle across from it. Given the students had previously explored and discussed theorems that applied to all triangles, the teacher thought the students should learn how triangles are classified and the definitions of those classifications.

The objectives for this task centered on the students' understanding and application of the definitions of the six triangle classifications (right, acute, obtuse, isosceles, equilateral, and scalene). The teacher wanted the students to be able to use the tools, ruler and protractor, to determine the length of the sides and measure of the angles of a triangle. The teacher also wanted the students to study the appearance and measures of a triangle and decide the best name for the triangle. By the end of the task, the teacher wanted the students to have developed an understanding of the definitions of these six triangle types. With these objectives in mind, the researcher created an activity and corresponding task sheet to be used to teach this concept.

The tools.

The activity consisted of students sorting measured precut triangles (See the tables in Appendix I), the tools of the task. The teacher selected these fifteen triangles for a number of reasons. First, the set captured every possible combination of the six classifications of triangles. For example, Triangle 1 is obtuse and isosceles, triangles 3 is acute and isosceles, triangle 9 is acute and scalene, triangle 14 is obtuse and scalene, triangle 5 is acute and equilateral, triangle 2 is right and isosceles, and triangle 15 is right and scalene. In addition, there was more than one example for all but one of the classifications listed above. The lone exception was the right isosceles triangle. For the triangles within the same classification, the lengths of the sides and/or the measures of the angles differed between triangles (e.g. triangles 4 and 13, both right scalene triangles). Second, the angle measures for all of the angles ended in a 0 or 5. When the teacher created the examples triangles, he was uncertain the students had used protractors to measure angles in their previous mathematics courses, and if they had, he was unsure whether they recalled how to use the tool. Using angle measures that either ended in 5 or 0, the teacher hoped the students would have less difficulty with their measures. The teacher was more certain about the students' abilities to use a ruler to measure the side lengths, which is why the lengths of the sides did not have the same restrictions.

In the planning of the lesson, the teacher hoped the students would measure every triangle. However, this did not occur. In previous class meetings, the teacher had instructed the students to measure the angles and side lengths for these triangles when they finished other tasks. However, all of the groups did not finish their measures prior to the triangle sort task. Rather then have the students spend class time measuring, the teacher wanted the students to have time to sort the triangles three times. If the students spent time measuring, then they might not have time to complete the task. The teacher decided to write the measures on the triangles for the students. This way, the students would have the measures on the triangles when they began the task and could sort the triangles immediately.

The task sheet.

Along with the example triangles, the students were given a task sheet to assist in their exploration (see Appendix J). Three sorting activities comprised the task sheet. The directions for the first sorting activity asked the students to sort the fifteen example triangles into two or more groups. The students were free to sort the triangles based on criteria of their choosing. Once the students sorted their triangles, the students were to complete the table on the task sheet. The table's column headings are the sorting groups. The table only allowed the students to sort the triangles into, at most, six groups. The row headings are description and triangle number. The students were directed to write the characteristics of the group in the description cell for each of their groups and then record the triangles that belonged to the group in the triangle number cell. After completing the table, the students were asked to answer the question, “What characteristics of the triangles did you focus on when you sorted the triangles?” This question was posed to focus the students’ attention to the properties of certain triangles.

The next sorting activity asked the students to sort the triangle based on the measures of the triangle’s interior angles. The same table and directions were given for the students to record the way the students sorted the triangles. After the students completed the sort and recorded their results in the table, they were asked to answer the questions “How did you select the number of groups?” and “Which triangle(s) were the most difficult to sort?” The first question was posed to have the students consider the relationship between the types of angles and the classification of the triangles by their angles. The latter question was posed in the hope of prompting a discussion between the group members.

The final sorting activity asked the students to sort the triangle based on the length of the sides. The same table and directions on how to complete the table were given to the students. Once the students finished sorting the triangles and recording their findings, the students were posed the same two questions that were given in the previous sorting activity.

At the bottom of the task sheet, the teacher provided a table to write the classifications for each of the example triangles. The teacher hoped the students would draw upon their actions while working on the sorting activities to provide the classification of each of the example triangles.

Teaching of the lesson.

The triangle sort task was conducted on the sixth class meeting. At the beginning of the class period, the teacher gave the students a warm-up. After giving the students time to work on the problems and reviewing their solutions, the teacher asked the students to state the classifications of different triangles. The students named acute, obtuse, isosceles, right, and equilateral. Then, the teacher asked the students to take out their example triangles. He told the students he had written the measures on the triangles and that they did not need to measure them as he distributed the task sheet. The teacher provided directions to the students for the first sorting activity. He provided an example of how to sort the triangles using an odd triangle number as one group and an even triangle number as the other group. He told the students that they had to use at least two groups, but the number of groups and characteristics of their groups were their choice. He did tell them not to use even and odd as the groups. While the students worked the teacher walked around the room answering questions from the students and posing questions about how the students determined their groups and why a triangle belongs to a certain group and not another. When the students finished sorting and answering the questions below the table, the teacher asked the students to go to the board and transfer their findings on the table to the board so all members of the class would be able to view their work. Once all the groups had recorded their findings, the teacher led a whole class discussion on the different ways the students sorted the triangles.

After this discussion, the teacher directed the students to begin the next sorting activity, sorting the example triangles by the measures of the angles. During the previous whole-class discussion, the teacher made a point to stress the way one group had used the measures of the angles to sort the triangles. Many other groups also used the same classification system. The students worked in pairs and completed the activity. The teacher led a whole class discussion and asked the students to state their findings while he wrote them on the board. If one pair of students had a different response, the teacher and classmates discussed why the particular triangle was part of a specific group.

Next, the teacher directed the students to sort the example triangles according to the lengths of the sides. Anticipating the students may sort the triangles by the specific length (e.g. the length of all the sides are greater than 4) the teacher told the students to think about the relationship among the sides rather than the specific values. The students worked in pairs to sort the triangles and fill out the third table on the task sheet. When finished with this activity, the teacher led a discussion of their findings.

With time running out, the teacher decided not to have the students complete the table at the bottom of the task sheet that asked the students to classify the example triangles. Instead, the teacher led a classroom discussion on the definitions of the six types of triangles. The teacher would provide the name of a triangle type and ask the students to provide a definition. The teacher would question the students and rephrase their statements until the definition was correct. Once the teacher and students agreed on the definition, the teacher would write it on the board. The teacher asked the students to write the definitions in their notes. After the students and teacher defined each of the six triangle types, the teacher distributed the homework.

In the following sections, the arguments created by three pairs of students while working on the triangle sort task are analyzed and discussed. For each pair of students, the arguments were first categorized by their basic structure. Then, the content and structure of the arguments within these basic categories were analyzed, including the students' uses of tools. The themes that emerged from these analyses are discussed for each pair of students and across the pairs of students.

Group 1's arguments while working on the triangle sort task.

The analysis of Andy and Frank's arguments while working on the triangle sort task can be categorized into three argument structures: core arguments, arguments in which the students collect additional data to verify or refute a claim, and arguments in which the students make a claim and then use that claim as data to make another claim. These arguments structures are discussed below.

Core arguments.

While working on the triangle sort task, the students create two categories of core arguments: core arguments with an explicit warrant and core arguments with a non-explicit claim. These two categories of core arguments are detailed below.

Core arguments with an explicit claim and warrant.

While working on the triangle sort task, Andy and Frank create five core arguments with an explicit warrant and claim. For example, Andy and Frank are sorting the fifteen example triangles into groups they created, one of which is right triangle. Example triangle number 2 has angle measures $m\measuredangle D = 45°$, $m\measuredangle E = 45°$, and $m\measuredangle F = 90°$. Frank states, "All right, let's see. Where's number 2?" Andy says, "Two has a right angle." Frank concurs, "90, right." This argument is illustrated in Figure 134.

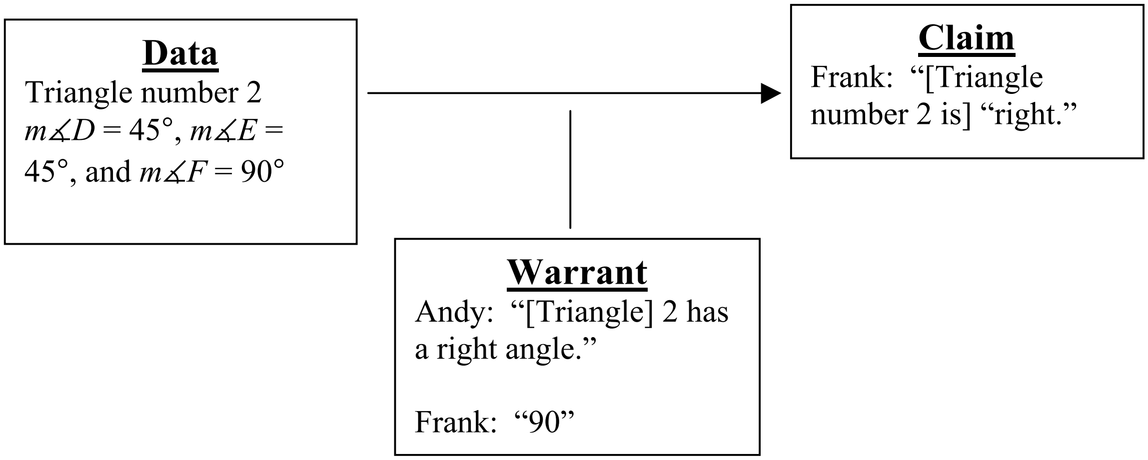


Figure 134. Andy and Frank's core argument created during task 3 with an explicit warrant and angle measures as data.

The data for this argument are the example triangle number 2 and its angle measures. Frank makes the claim triangle number 2 is a right triangle. Andy provides the warrant stating triangle number 2 has a right angle.

Andy and Frank create another example of a core argument with an explicit claim and warrant while sorting the triangles based on the lengths of their sides. For this sorting activity, the pair create three groups; equilateral, isosceles, and scalene. Andy attempts to sort example triangle number 12. The measures of the side lengths for this triangle are $m\overline{KD} = 5.7$ cm, $m\overline{KG} = 11.0$cm, and $m\overline{DG} = 16.2$ cm. Andy asks Frank, "Where do [sic] 12 go?" Frank replies, "Scalene. Because look 16.2, 5 point something." Andy interrupts Frank saying, "Oh yeah." and places the triangle in the scalene group. This argument is illustrated in Figure 135.

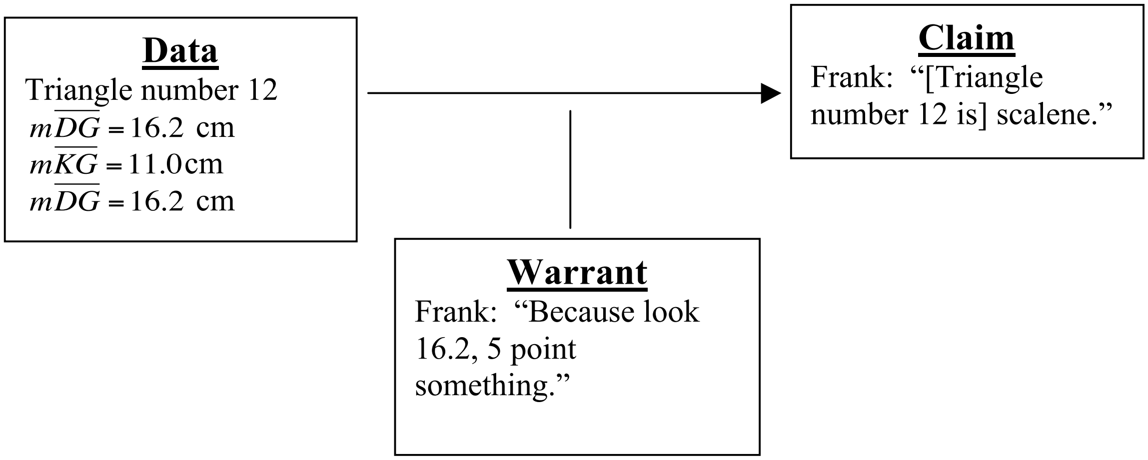


Figure 135. Andy and Frank's core argument created during task 3 with an explicit warrant and the measures of segment lengths as data.

The data for this argument are the example triangle number 12 and the measures of its side lengths. Frank makes the claim triangle number 12 is a scalene triangle. He provides an explicit warrant stating the measures of the side lengths of the example triangle.

Looking across the five core arguments with an explicit claim and warrant, three themes emerge. First, the data for all of these arguments are a single example triangle and its measures. In all but one of the arguments, (the argument discussed in the preceding paragraph) the students use the angle measures as data. Second, the claims for four of these core argument concern whether the example triangle is a right triangle. Third, in all of these arguments, the students use the measures of the triangle's angles or side lengths as a justification for classifying the triangle.

Core argument with an inferred claim.

In one core argument, Andy and Frank do not make an explicit claim but provide an explicit warrant. In this argument, Andy and Frank are sorting the example triangles into groups, one of which is right triangle. They are attempting to sort example triangle number 4 which has angle measures $m\measuredangle J = 90°$, $m\measuredangle K = 60°$, and $m\measuredangle L = 30°$. Franks asks, "4?" Andy replies, "4, right here's 90." Frank states, "4." This argument is illustrated in Figure 136.

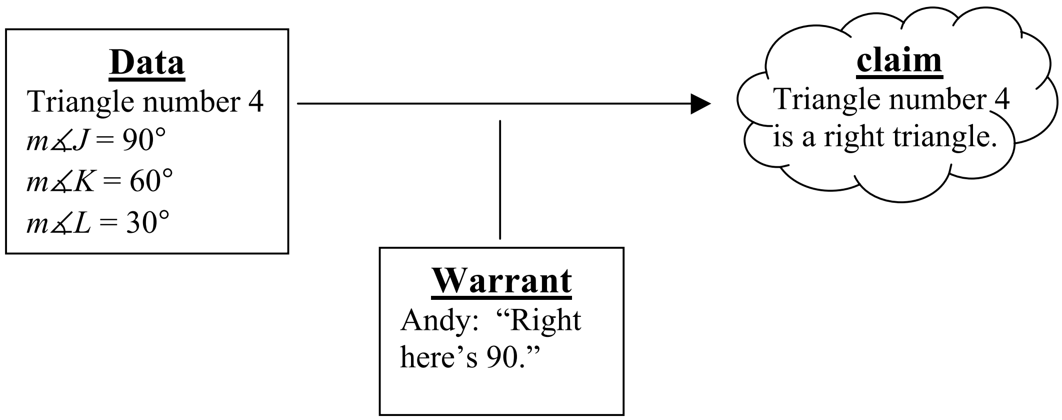


Figure 136. Andy and Frank's core argument created during task 3 with an inferred claim.

The data for this argument are the example triangle number 4 and its angle measures. Neither Andy nor Frank provides an explicit claim. The researcher infers the claim to be, "Triangle number 4 is a right triangle." This researcher makes this inference for two reasons. On the students' task sheet, example triangle 4 is placed under the right triangle heading for this sorting activity, and the explicit warrant provided by Andy indicates that he identified the right angle in the triangle.

Core arguments summary.

While working on the triangle sort task, Andy and Frank create 6 core arguments, one of which has a non-explicit claim. The data for these arguments are similar, a single example triangle and its measures. In addition, the claims and warrants are similar as well. The claims are whether a triangle can be classified as a certain type of triangle. The warrants are based on the measures of the example triangles. However, unstated and underlying the arguments' warrants are the definitions of the different triangle types. For example, Frank uses the measures of the side lengths for example triangle 12 to justify why this triangle should be classified as a scalene triangle in the argument illustrated in Figure 135. Even though Frank never states the definition of scalene triangle, he uses the measures to show that the triangle is scalene. By using the measures in this way, he demonstrates that he has some understanding of the definition of scalene. Perhaps, when the students use the measures in this fashion, they not only demonstrate their understanding of the definition but also their ability to apply the definition.

Argument in which Andy and Frank collect additional data.

Andy and Frank create one argument in which the students collect additional data to verify or refute an initial claim. In this argument, Frank had just arrived and Andy had been working with Bob. While working with Bob, Andy writes on his task sheet example triangle 2, 4, 9, and 13 are right triangles. The teacher approaches the pair and asks, "2, 4, 9, and 13?" Frank nods his head and says "Uh huh." The teacher asks, "9 is [right]?" Frank replies, "Yeah." The teacher asks the students to find the example triangle and asks, "What's the measure of the angles? It's on the triangle." Frank indicates he did not measure these triangles. The teacher asks, "85, is that 90? No it's 85." Frank concludes, "All right then, 9's not one of them." This argument is illustrated in Figure 137.

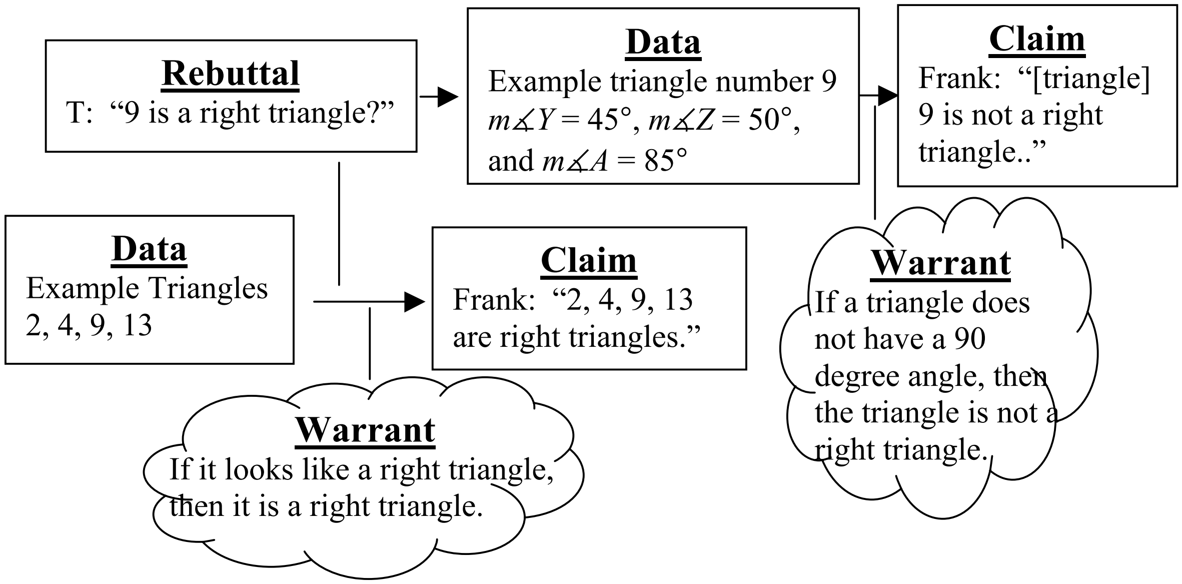


Figure 137. Frank and Andy's argument created during task 3 with additional data collection in response to a challenge made by the teacher.

The initial data for this argument are the example triangles 2, 4, 9, and 13. Frank claims that these example triangles are right triangles. He does not provide an explicit warrant and is inferred by the researcher to be, "If it looks like a right triangle, then it is a right triangle." The teacher challenges part of the claim by asking whether example triangle number 9 is a right triangle. This challenge is a rebuttal to the claim. Frank, Andy and the teacher collect additional data by looking at the measure of the angles. The teacher states the measure of the angle is 85 degrees, not 90 degrees. Frank claims example triangle number 9 is not a right triangle. He does not provide an explicit warrant and the researcher infers it to be, "If a triangle does not have a 90 degree angle, then the triangle is not a right triangle."

Linked argument.

While working on the triangle sort task, Andy and Frank create an argument in which a student's claim is used as data to make an additional claim, a linked argument. The students are sorting triangles according to the groups of their choosing. Andy attempts to classify example triangle number 6, an equilateral triangle. Andy says, “6 is um…” Frank states, “It’s acute to.” “Yeah…is, isn’t this equilateral?” asks Andy. Frank responds, “No, look there’s 60.” Andy concludes, “So all 60 are acute.” Frank isn’t sure about Andy’s conclusion. So, the pair calls the teacher over and Franks asks him, “Is an equilateral the one that equals all the angles are the same? So…couldn’t it be on those [in the equilateral group]?” The teacher answers, “Yeah like that. Look all the sides are the same.” Andy agrees saying, “Yeah.” Frank asks, “Wouldn’t it be an acute to?” “Right, an equilateral is also acute,” answers the teacher. Frank responds, “But, you said it could only be on one.” The teacher replies, “Put it in one. So what’s the best name for it?” Frank claims, “Equilateral.” This argument is illustrated in Figure 138.

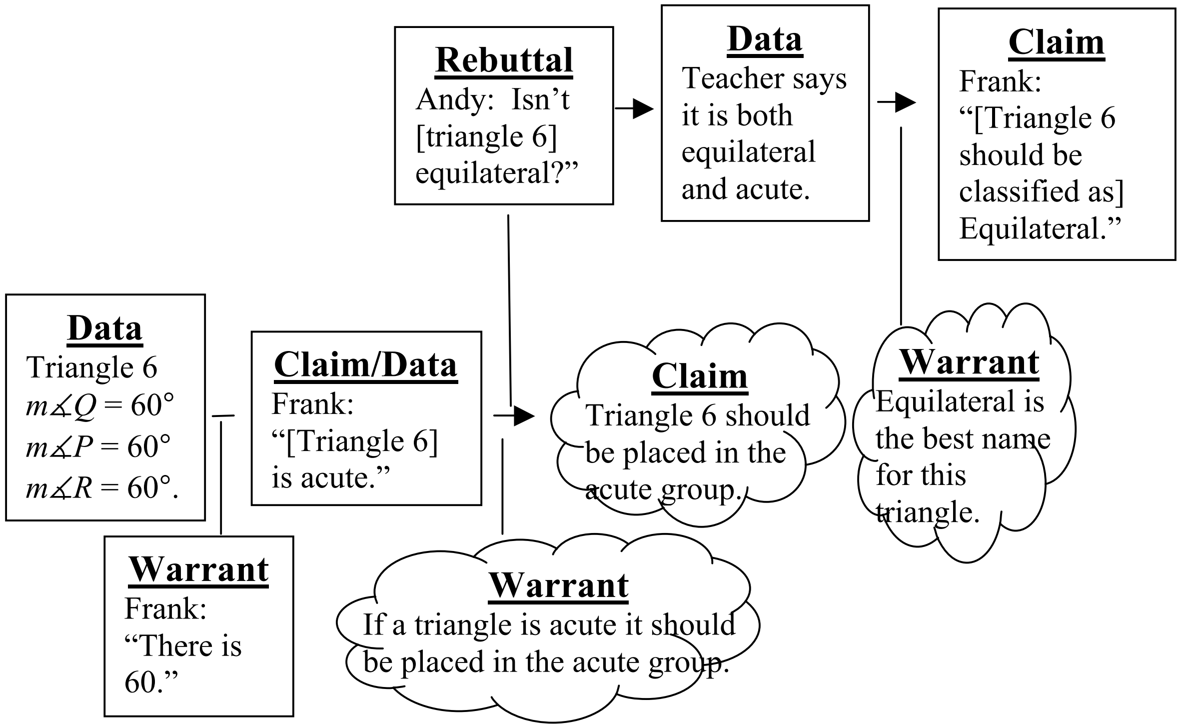


Figure 138. Frank and Andy's linked argument created during task 3.

The initial data for this argument are the example triangle number 6 and its angle measures. Frank claims the triangle is acute. He provides an explicit warrant indicating the angle measures are 60 degrees. The claim, the example triangle is acute, is used as data for a new claim that the triangle should be classified as acute. However, neither student explicitly states this claim. The researcher infers this claim for two reasons. First, the activity on which the students are working is to sort the triangles into categories of their choosing, which includes classifying the triangle and then placing it in the appropriate group. The students create an argument for the classification as acute, but do not explicitly say the triangle should be placed in the acute group. The second reason for inferring the warrant is due to the remainder of the argument that discusses whether the triangle should be placed in the acute group or the equilateral group. Because the students are determining the group in which the triangle should be placed and not the classification of the triangle, whether it is acute or equilateral, the inferred claim is needed in order for this to be an accurate representation of the students' argument. The warrant which links the initial claim to the inferred claim is not explicit and is inferred by the researcher to be, "If a triangle is acute it should be placed in the acute group." Andy makes an explicit rebuttal by challenging the claim that it should be placed in the equilateral group by indicating the triangle is also equilateral. The pair recognizes that this triangle can be placed in both groups and discuss their problem with the teacher. The discussion with teacher is additional data and he tells the students the triangle is both equilateral and acute. Frank makes a new claim the triangle should be placed in the equilateral group. However, he does not provide an explicit warrant and the researcher infers it to be, "Equilateral is the best name for this triangle."

Group 1's summary for the triangle sort task.

While working on the triangle sort task, Andy and Frank create arguments of various structures. Three categories of structures are noted in the analysis: core arguments, arguments in which the students collect additional data to verify or refute a claim, and arguments in which the students make a claim and then the claim is used as data to make an additional claim. Looking across these argument structures, three themes emerge; the number of arguments (see Table 31), the use of tools in the arguments, and the explicitness of the warrants in the core arguments.

Table 31.

Group 1's Argument on the Triangle Sort Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Tools	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	0	1
	Tools Not Used	0	0
Warrants Explicit			
	Tools Used	6	1
	Tools Not Used	0	0

While working on the triangle sort task, Andy and Frank only create eight arguments. This may be due to the nature of the task. When sorting the triangles, the students rarely discuss their reasons for placing an example triangle in a group. While sorting the triangles, Andy and Frank select an example triangle and place it in a group while saying the name of the group. Even tough the researcher documents these as claims, he could not determine the specific data the students use to make these claims. The data may have been the appearance of the example triangle or its measures. Due to the researcher's uncertainty about the data of these claims, he is unable to consider them arguments. Generally, when the researcher is able to document an argument, it is due to a student's uncertainty about the classification of a triangle, when one of the students do not agree with the classification of an example triangle, or when a claim is challenged.

However, when the students do create arguments, they always use the tools. The tools for this activity are the fifteen measured example triangles. In particular, the students use the measures of the triangles as data. In some arguments, the students also use the measures as a warrant to classify a given triangles (e.g. the arguments illustrated in Figures 134, 135, 136, and 138).

For the core arguments, the warrants are all explicit and the students use measures in their warrants for their claims. This may be related to the task on which the students are working. In these episodes, the students are classifying the triangles. The claims of the arguments are a triangle's classification. The data are the example triangle and its measures. The warrants are also measures. However, the students focus on particular measures in the warrant. For example, the warrant for the argument illustrated in Figure 134 is Andy's statement that the example triangle has a right angle. Even though the example triangle has other measures written on it, Frank uses the fact that it has a right angle to justify its inclusion in the right triangle group. Perhaps, when the students use the measures in this fashion, they not only demonstrate their understanding of the definition but also their ability to apply the definition.

Group 2's arguments on the triangle sort task.

The analysis of Bob and Ellen's arguments while working on the triangle sort task can be categorized into two argument structures: core arguments, and arguments in which the students collect additional data. These argument structures are discussed below.

Core argument.

While working on the triangle side and angle relationship activity, the students create two core arguments, one with an explicit warrant and one with an inferred warrant. These two arguments are discussed below.

Core argument with an explicit warrant.

In one episode, Bob creates a core argument with an explicit warrant and a qualified claim. Bob is sorting the example triangles based on the lengths of the sides of the triangles. He has grouped the example triangles with all equal sides and is attempting to find another group. He places triangles 4 and 13 next to each other. Triangle 4 has angle measures $m\measuredangle J = 90°$, $m\measuredangle K = 60°$, and $m\measuredangle L = 30°$. and side lengths $KL = 14.0$ cm, $KJ = 7.0$ cm, and $JL = 12.3$ cm. Triangle 13 has angle measures $m\measuredangle E = 90°$, $m\measuredangle L = 75°$, and $m\measuredangle P = 15°$. and side lengths $EP = 15.9$ cm, $EL = 4.1$ cm, and $LP = 16.4$ cm. The teacher asks Bob, “So which ones are you putting together? You have these two with all sides the same. Right so which ones could there also be?” Bob replies, “I know these because they kind of look the same.” This argument is illustrated in Figure 139.

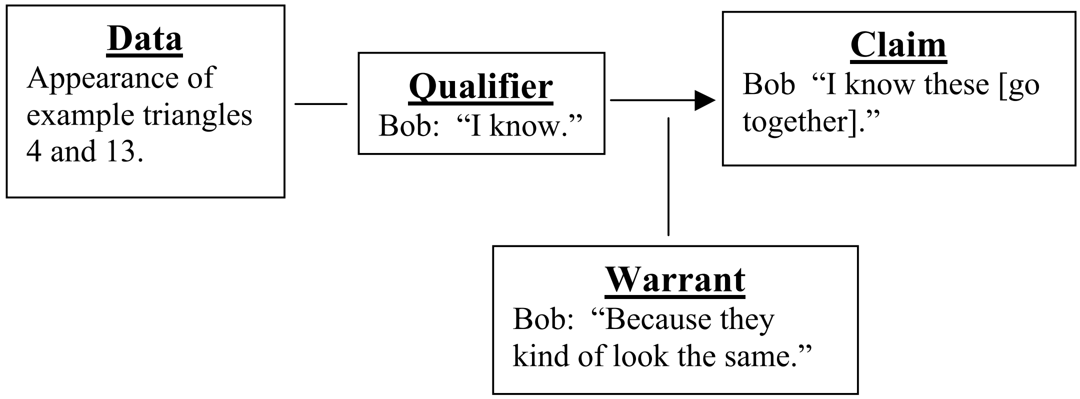


Figure 139. Bob's core argument created during task 3 with explicit warrant, qualified claim, and appearance of the triangles as data.

The data for this argument are the example triangles 4 and 13 and the appearance of these example triangles. Bob claims that these triangles will be grouped together. He qualifies this claim using the phrase "I know" which indicates a high level of certainty with his claim. Bob provides an explicit warrant stating that the two triangles will be grouped together because the triangles appear to be similar.

Core argument with an inferred warrant.

In addition to creating a core argument with an explicit claim and an explicit warrant, Bob creates a second core argument, but does not provide an explicit warrant. Bob is sorting the triangles into categories he created. He creates a group with triangle 2, 4, 6, and 9. Andy joins Bob because Ellen has not arrived and asks him, "What group is this?" Bob answers, "Right." Andy replies, "Duh, 2, 4, 6, and 9." This argument is illustrated in Figure 140.

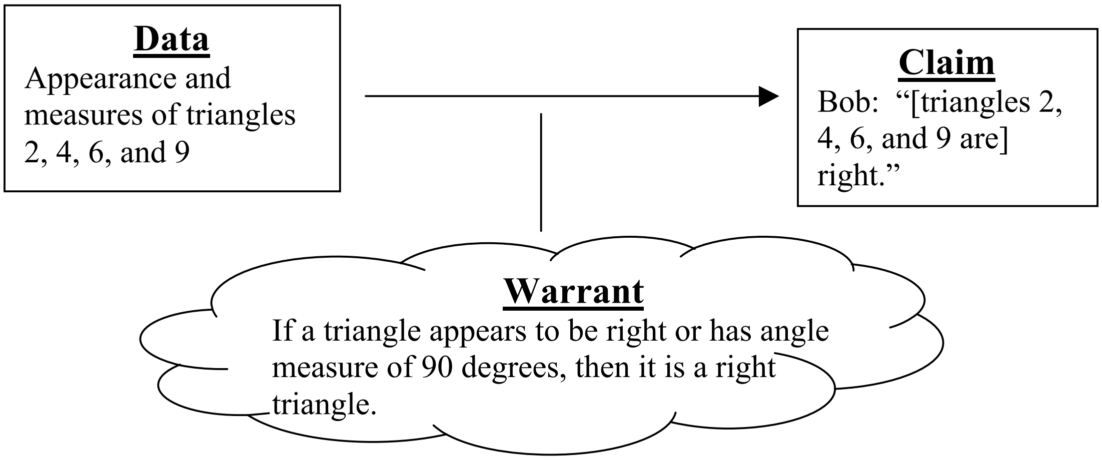


Figure 140. Bob's core argument created during task 3 with an inferred warrant and the appearance of the triangles as data.

The data for this argument are the appearance and measures of triangles 2, 4, 6, and 9. Bob claims that these triangles are right triangles. He does not provide an explicit warrant for this claim and is inferred by the researcher to be, "If a triangle appears to be right or has angle measure of 90 degrees, then it is a right triangle."

Argument with additional data collection.

Bob creates one argument in which the he collects additional data to verify or refute a claim. During the whole class discussion, Bob realizes he has classified triangle number 9 as a right triangle which differs from the rest of the class. The angle measures written on his triangle are $m\measuredangle Y = 45°$, $m\measuredangle Z = 50°$, and $m\measuredangle A = 90°$. Bob explains, "I put 9 in there because one side." The teacher interrupts, "What's the measurements for 9?" Bob replies, "It's got a right angle though?" The teacher repeats, "What are the measurements for 9?" "90 degrees 45 degrees and 50 degrees," responds Bob. The teacher questions, "For yours? How about other people's 9?" Frank states, "Mr. Smith I got 85, 50, 45." The teacher says, "85, 50, 45." Bob begins to erase his paper. This argument is illustrated in Figure 141.

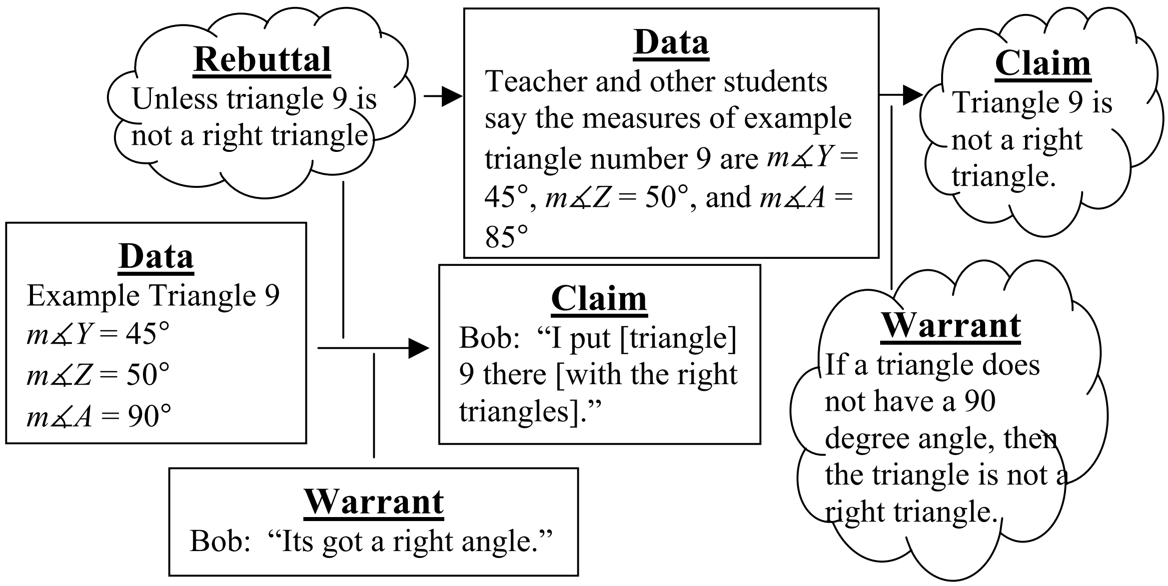


Figure 141. Bob's argument created during task 3 with additional data collection.

The initial data for this argument are the example triangle number 9 and the angle measures written on the triangle. Bob claims that this triangle is a right triangle. He provides an explicit warrant indicating the triangle has a right angle. Bob does not provide an explicit rebuttal and the researcher inferred it to be, "Unless triangle 9 is not a right triangle." The researcher makes this inference for two reasons. First, Bob expresses some uncertainty with his initial claim because it differs from the claims of other students and is seeking clarification from the teacher. Second, the teacher asks other students for their angle measures, which is additional data collection. This additional data collection leads to a new claim, but one that is not explicitly stated by Bob. The researcher infers this new claim to be, "Triangle 9 is not a right triangle." This researcher makes this inference due to the actions performed by Bob when the teacher confirms the other students' measures are correct. An explicit warrant is not provided by Bob and is inferred by the researcher to be, “If a triangle does not have a 90 degree angle, then the triangle is not a right triangle.”

Group 2’s summary for the triangle sort task.

While working on the triangle sort task, Bob and Ellen create arguments of various structures. Two categories of structures are noted in the analysis: core arguments and arguments in which the students were compelled to collect additional data. Looking across these argument structures, two themes emerge; the number of arguments (see Table 32), and the use of tools in the arguments.

Table 32.

Group 2’s Argument on the Triangle Sort Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Tools	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	1	0
	Tools Not Used	0	0
Warrants Explicit			
	Tools Used	1	1
	Tools Not Used	0	0

While working on the triangle sort task Bob and Ellen only create three arguments. This may be due Bob working alone for part of the class period and Ellen’s reluctance to participate in the task when she arrives. The lack of arguments may also be due to the nature of the task. When sorting the triangles, the students rarely discuss their reasons for putting a triangle in a group. Bob selects a triangle and places it in a group while saying the name of the group. Even though the researcher documents these as claims, he could not determine the specific data the students use to make these claims. The data may have been the appearance of the example triangle or its measures. Due to the researcher's uncertainty about the data of these claims, he is unable to consider them arguments.

However, when Bob and Ellen do create arguments, they always actively use the tools. The tools for this activity are the fifteen measured example triangles. The students use the measures and the appearance of the triangles as data. In some arguments, the students also use the measures as a warrant to classify a given triangles (e.g. the argument illustrated in Figure 141).

Group 3's arguments on the triangle sort task.

The analysis of Jim and Clair's arguments while working on the triangle sort task can be categorized into three argument structures: core arguments, arguments in which the students collect additional data, and arguments in which the pair reevaluate the initial data. These argument structures are discussed below.

Core argument.

While working on the triangle sort task, the pair of students create two core arguments, both with an explicit warrant. For example, Jim and Clair are sorting the example triangles based on the angles. They create three group; acute, right, and obtuse. Jim selects example triangle number 10 whose angle measures are $m\measuredangle B = 100°$, $m\measuredangle E = 30°$, and $m\measuredangle H$

= 15°. Jim says, “This one is an obtuse because it has a 100 degree angle.” This argument is illustrated in Figure 142.

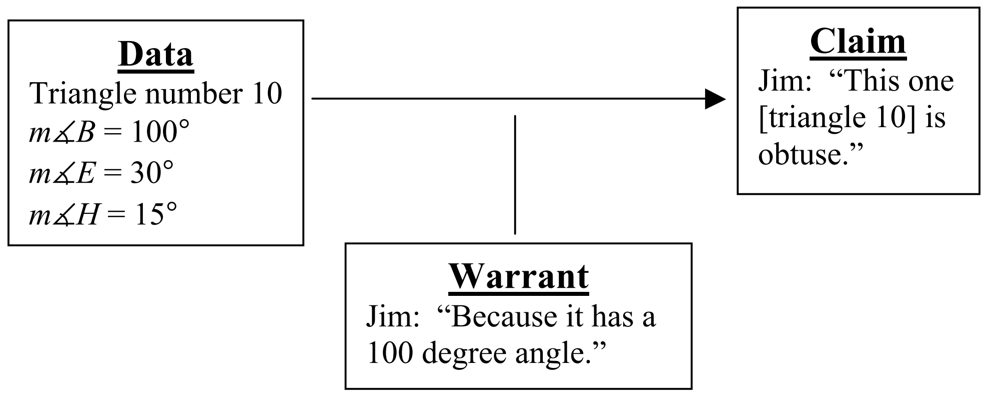


Figure 142. Jim’s core argument created during task 3 with an explicit warrant and angle measures as data.

The data for this argument are the example triangle number 10 and its angle measures. Jim claims that the triangle is obtuse. He provides an explicit warrant stating one of the angle measures is 100 degrees.

The other core argument with an explicit warrant created by Clair and Jim is similar to the one above. The data is an example triangle and it’s measures. The claim is the classification of the example triangle, and the warrant is the triangle’s measures. What is interesting about these arguments is the role the definitions. In the preceding argument, neither Jim nor Clair states the definition of an obtuse triangle. Yet, Jim is applying that definition to reason that the example triangle is obtuse. Perhaps, when students use measures in this fashion, it demonstrates some understanding of the definition through their application of it.

Arguments with additional data collection.

In one argument created by Jim and Clair, additional data is collected. The students sort the example triangle according to the groups of their choosing which are equilateral, isosceles, right, obtuse, and acute. The teacher asks the pair, “So, number 1. Here’s the thing. Why did you put it with isosceles?” Jim replies, “Because it had two angles, two angles that were the same.” The teacher responds, “So, this one has an obtuse [angle]. Which one does it belong to?” Jim says, “Both.” The teacher says, “Ok, so why did you pick isosceles? Could it have gone to obtuse?” Jim answers, “It could of.” This argument is illustrated in Figure 143.

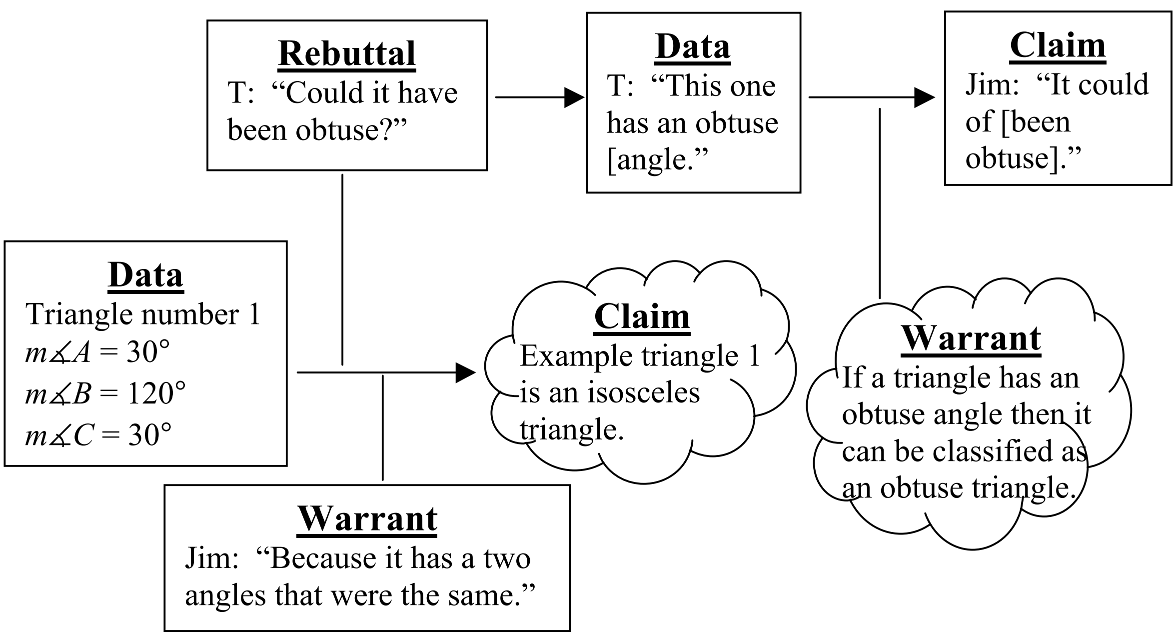


Figure 143. Jim’s argument created during task 3 with additional data collection an inferred initial claim.

The initial data for this argument are the example triangle number 1 and its angle measures. Jim does not explicitly state the triangle is isosceles, but places the example triangle in the isosceles group. Due to this action, the researcher infers the claim, "Example triangle 1 is an isosceles triangle." Jim provides an explicit warrant indicating the triangle is isosceles because two of the angles have the same measure. The teacher challenges Jim's unstated claim by asking him whether the triangle can be classified as obtuse. This challenge is an explicit rebuttal. The teacher provides additional data by telling the pair the example triangle has an obtuse angle. Jim claims the example triangle could have been classified as obtuse. He does not provide an explicit warrant for his claim and is inferred by the researcher to be, "If a triangle has an obtuse angle then it can be classified as an obtuse triangle."

Arguments with a reevaluation of data.

In some episodes, one of the pair would challenge a claim made by the other. However, the challenge would not lead to additional data collection. Instead, the pair would revaluate the initial data and either verify the initial claim or make a new claim. For example, Jim and Clair are sorting the triangle based on the angles. Jim states, "9 is an acute triangle." Clair says, "So is 6." Jim responds, "6 is equilateral." Clair replies, "Yeah, but it's an acute one." This argument is illustrated in Figure 144.

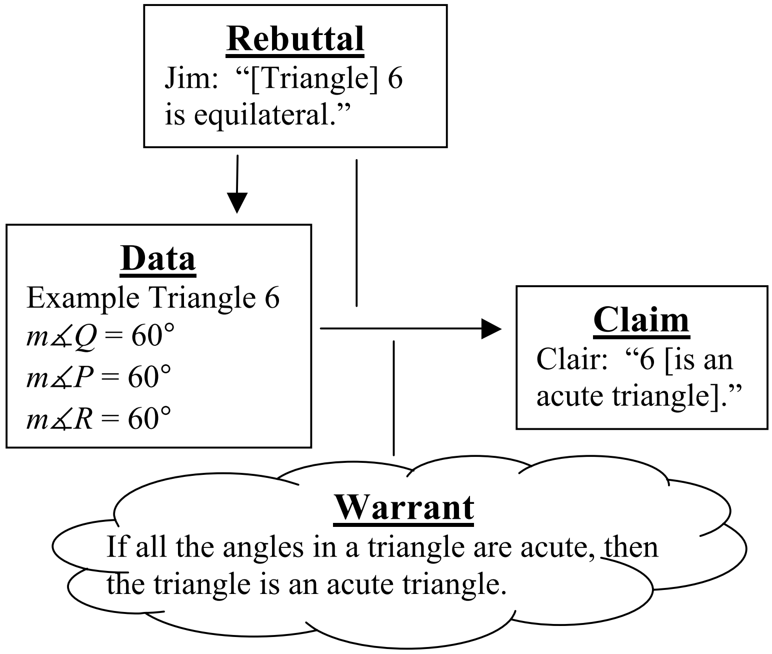


Figure 144. Jim and Clair's argument created during task 3 with the reevaluation of data and challenge from Jim.

The data for this argument are the example triangle number 6 and its angle measures. Clair claims the example triangle is an acute triangle. She does not provide an explicit warrant and is inferred by the researcher to be, “If all the angles in a triangle are acute, then the triangle is an acute triangle.” Jim challenges this claims by stating the example triangle is equilateral. This challenge is an explicit rebuttal. Clair revaluates the measures and restates her initial claim. She does not provide an explicit warrant for this verification and is inferred by the researcher to be the same as the previous inferred warrant.

The reevaluation of data does not always lead to a verification of the initial claim. At times, the revaluation of the data leads to a new claim. For example, Jim and Clair are sorting the triangles based the groups of their choosing. Jim selects example triangle 2 and says, “[Example triangle 2 is] Isosceles.” Clair states, “That could be a right angle.” Jim agrees and says, “Yeah make that right.” This argument is illustrated in Figure 145.

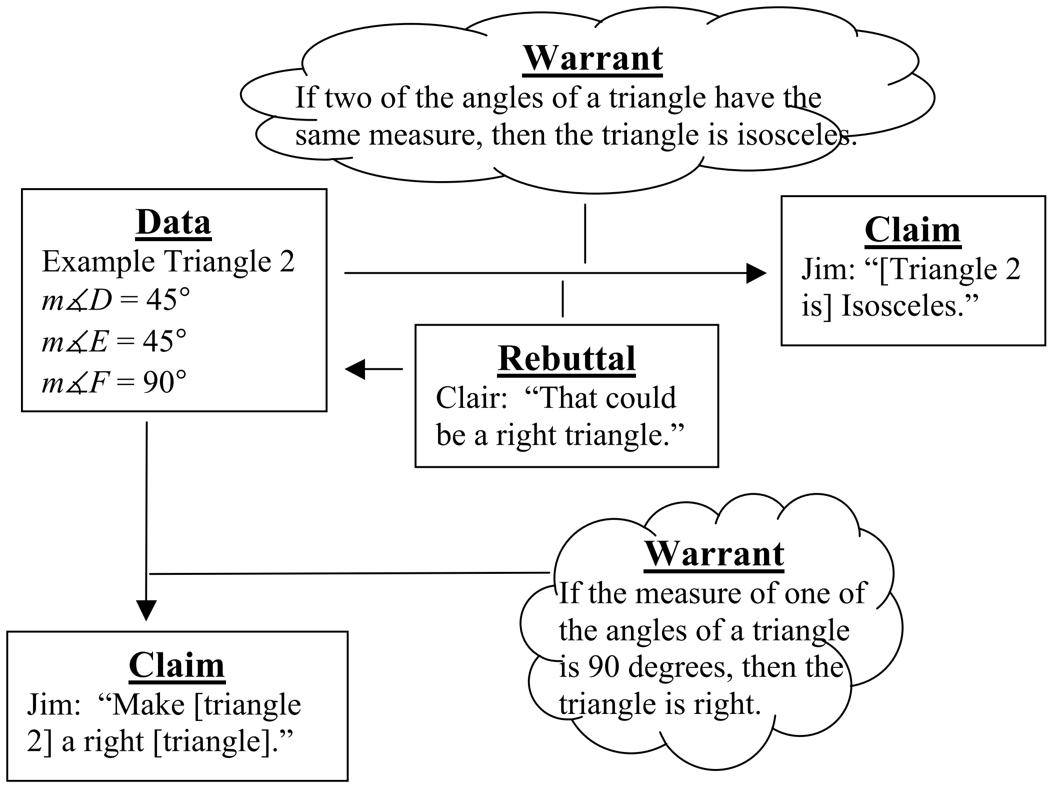


Figure 145. Jim and Clair’s argument created during task 3 with the reevaluation of data and a challenge from Clair leads to a new claim.

The data for this argument are the example triangle number 2 and its angle measures. Jim claims that example triangle is isosceles. He does not provide an explicit warrant for his claim and is inferred by the researcher to be, “If two of the angles of a triangle have the same measure, then the triangle is isosceles.” Clair challenges this claim stating the example triangle could be considered a right triangle. This challenge is a rebuttal. Jim reevaluates the data and claims the example triangle is right. He does not provide an explicit warrant and is inferred by the researcher to be, "If the measure of one of the angles of a triangle is 90 degrees, then the triangle is right."

Group 3's summary for the triangle sort task.

While working on the triangle sort task, Jim and Clair create arguments of various structures. Three categories of structures are noted in the analysis: core arguments, arguments with additional data collection, and arguments in which the pair reevaluate the same data. Looking across these argument structures, three themes emerge; the number of arguments (see Table 33), the use of tools in the arguments, and the explicitness of the warrants in the core arguments.

Table 33.

Group 3's Argument on the Triangle Sort Task by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Tools	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	0	2
	Tools Not Used	0	0
Warrants Explicit			
	Tools Used	3	1
	Tools Not Used	0	0

While working on the triangle sort task, Clair and Jim only create six arguments. This may be due to the nature of the task. When sorting the triangles, the students rarely discuss their reasons for putting an example triangle in a group. Clair and Jim select an example triangle and place it in a group while saying the name of the group. Even tough the researcher documents these as claims, he could not determine the specific data the students use to make these claims. The data may have been the appearance of the example triangle or its measures. Due to the researcher's uncertainty about the data of these claims, he is unable to consider them arguments. Generally, when the researcher can document an argument, it is due to a student's uncertainty about the classification of a triangle, when one of the students do not agree with the classification of an example triangle, or when a claim is challenged.

However, when the students do create arguments, they always actively use the tools. The tools for this activity are the fifteen measured example triangles. In particular, the students use the measures of the triangles as data. In some arguments, the students also use the measures as a warrant to classify a given triangles (e.g. the argument illustrated in Figure 142).

For the core arguments, the warrants are all explicit and the students use measures as a warrant for their claims. This may be related to the task on which the students are working. In these episodes, the students are classifying the example triangles. The claims of the arguments are a triangle's classification. The data are the example triangle and its measures. The warrants are also measures. However, the students focus on particular measures in the warrant. For example, the warrant for the argument illustrated in Figure 142 is Jim's statement that one of the angle measures is 100°. Even though the example triangle has other measures written on it, Frank uses the fact that it has an obtuse angle to justify its inclusion in the obtuse triangle group. Perhaps, when the students use the measures in this fashion, they not only demonstrate their understanding of the definition but also their ability to apply the definition.

Cross-case analysis of the arguments created while working on the triangle sort task.

On the triangle sort task, the arguments created by the three groups of students vary in structure and content. Two themes emerge when looking across the arguments created by these students; the explicitness of the warrants (see Table 34), and the use of the tools as data and warrants.

Table 34.

The Combined Arguments of the Three Groups on the Triangle Sort Task by Structure, Use of Technology, and Explicitness of the Warrant

		Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	1	3
	Tools Not Used	0	0
Warrants Explicit			
	Tools Used	10	3
	Tools Not Used	0	0

While working on the triangle sort task, the three groups of students only create 17 arguments. Of these 17 arguments, the students provided at least one explicit warrant in 13 (76%) of the arguments. The explicitness of the warrant may be related to the researcher's ability to document a claim as an argument. While working on the triangle sort task, all three groups make claims about the classification of the example triangles. However, the students do not often state anything other than the classification of the triangle. The researcher is unable to document these as arguments because he is unable to determine the data on which the claim is based. The data could be the appearance of the example triangle or the measures written on the example triangle. When the researcher is able to document a claim as an argument, it generally was due to the students providing an explicit warrant. Thus, it is not surprising the students have a high percentage of arguments with an explicit warrant.

Even though the students only created 17 arguments, the students employ the tools, the measured example triangles, in all 17 arguments. In general, the students use the measures as data. At times, the students also use the measures as a warrant (e.g. the arguments illustrated in Figures 134, 135, 142). The use of measures as justification is generally frowned upon in mathematics. However, the use of measures for these claims is based on sound reasoning. In these arguments, the students are not making claims that would be considered generalizations across a family of triangles. Rather, the students are making a claim about a specific example triangle. The warrants are resting upon unstated facts, which are the definitions of the different triangle types. The warrants are not merely measures, but could be considered applications of the definitions of the triangle types.

Discussion

During the three tasks under analysis (the triangle inequality task, the triangle side and angle relationship task, and the triangle sort task), the arguments created by the three groups of students vary in their structure and content. In this section, the structure and content of the arguments created by each of the three groups across the three tasks are discussed.

Summary for group 1 across the three tasks.

While working on the three tasks, Andy and Frank create 34 arguments of various structures and content (see Table 35). Two themes emerge when looking across the arguments created by these students on these tasks; the relationship between the structure of the argument and the explicitness of the warrant when the students use tools, and the explicitness of the warrant when the students do not actively use tools.

Table 35.

Group 1's Arguments Across the Three Tasks by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Tools	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	5	3
	Tools Not Used	1	1
Warrants Explicit	Tools Used	7	3
	Tools Not Used	7	7

Of the 34 arguments Andy and Frank create while working on the three tasks, the students use the tools in 18 (53%) of them. Of these 18 arguments in which the students use the tools, the structure of 12 (67%) of the arguments are core and 6 (33%) are non-core. Of the 16 arguments in which the students do not actively use the tools, 8 (50%) have the core argument structure and 8 (50%) have the non-core structure. This suggests that when the students use tools, they are more likely to create more simplistic arguments than when they do not use the tools.

When Andy and Frank use the tools, they create 12 core arguments, 7 (58%) of which have explicit warrants. In addition, they create 6 non-core arguments while using the tools. Of these 6 non-core arguments, 3 (50%) of the arguments have at least one warrant that is explicit. This suggests when these students use the tools and create a more complex structure, the students are less likely to provide an explicit warrant than when the students use tools and create an argument with the core structure.

When the students do not actively use the tools in the creation of their arguments, the students are more likely to provide an explicit warrant. In fact, of the 16 arguments in which the tools are not actively employed, Andy and Frank provide an explicit warrant for 14 (88%). of the arguments. As previously discussed, many of the tasks on which the students are working while creating these arguments would be considered generalization or justification type tasks. Thus, the students may be more likely to provide, or are asked to provide, a justification for why their claims are true.

In summary, Andy and Frank are more likely to create arguments in which the structure is core, the warrant is explicit, and the students use the tools.

Summary for group 2.

While working on the three tasks, Bob and Ellen create 20 arguments of various structures and content (see Table 36). Two themes emerge when looking across the arguments created by these students on these tasks; the relationship between the structure of the argument and the explicitness of the warrant when the students use the tools, and the explicitness of the warrant when the students do not actively use tools.

Table 36.

Group 2's Arguments Across the Three Tasks by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Tools	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	1	2
	Tools Not Used	4	1
Warrants Explicit			
	Tools Used	2	1
	Tools Not Used	7	2

Of the 20 arguments Bob and Ellen create while working on the three tasks, the students use the tools in 6 (30%) of them. Of these 6 arguments in which the students use the tools, the structure of 3 (50%) of the arguments are core and 3 (50%) are non-core. Of the 14 arguments in which the students do not actively use the tools, 11 (79%) have the core argument structure and 3 (21%) have the non-core structure. This suggests that when the students use tools, they are more likely to create more complex arguments than when they do not use the tools.

When Bob and Ellen use the tools, they create 3 core arguments, 2 (67%) of which have warrants are explicit. In addition, they create 3 arguments while using the tools whose structure is non-core. Of these 3 non-core arguments, 1 (33%) of the arguments have at least one warrant that is explicit. This suggests when these students use the tools and create a more complex structure, the students are less likely to make their reasoning explicit than when the students use tools and create an argument with the core structure.

When the students do not actively use the tools in the creation of their arguments, the students are more likely to provide an explicit warrant. In fact, of the 14 arguments in which the tools are not actively employed, Bob and Ellen provide an explicit warrant for 9 (64%). of the arguments. As previously discussed, many of the tasks on which the students are working while creating these arguments would be considered generalization or justification type tasks. Thus, the students may be more likely to provide, or are asked to provide, a justification for why their claims are true.

In summary, Bob and Ellen are more likely to create arguments in which the structure is core, the warrant is explicit, and the students do not use the tools.

Summary for group 3.

While working on the three tasks, Clair and Jim create 32 arguments of various structures and content (see Table 37). Two themes emerge when looking across the arguments created by these students on these tasks; the relationship between the structure of the argument and the explicitness of the warrant when the students use the tools, and the explicitness of the warrant when the students do not actively use tools.

Table 37.

Group 3's Arguments Across the Three Tasks by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Tools	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	6	6
	Tools Not Used	1	0
Warrants Explicit			
	Tools Used	6	2
	Tools Not Used	9	2

Of the 32 arguments Clair and Jim create while working on the three tasks, the students use the tools in 20 (63%) of them. Of these 20 arguments in which the students use the tools, the structure of 12 (60%) of the arguments are core and 8 (40%) are non-core. Of the 12 arguments in which the students do not actively use the tools, 10 (83%) have the core argument structure and 2 (17%) have the non-core structure. This suggests that when the students use tools, they are more likely to create more complex arguments than when they do not use the tools.

When Clair and Jim use the tools, they create 12 core arguments, 6 (50%) of which have warrants are explicit. In addition, they create 8 arguments while using the tools whose structure is non-core. Of these 8 non-core arguments, 2 (25%) of the arguments have at least one warrant that is explicit. This suggests when these students use the tools and create a more complex structure, the students are less likely to provide an explicit warrant than when the students use tools and create an argument with the core structure.

When the students do not actively use the tools in the creation of their arguments, the students are more likely to provide an explicit warrant. In fact, of the 12 arguments in which the tools are not actively employed, Clair and Jim provide an explicit warrant for 11 (92%). of the arguments. As previously discussed, many of the tasks on which the students are working while creating these arguments would be considered generalization or justification type tasks. Thus, the students may be more likely to provide, or are asked to provide, a justification for why their claims are true.

In summary, Clair and Jim are more likely to create arguments in which the structure is core, the warrant is explicit, and the students use the tools.

CHAPTER 6

Cross-Case Synthesis and Conclusions

This chapter involves looking across the cases to answer the research questions presented in Chapter 1 (and repeated here).

1. What are the characteristics of the content and structure of eight grade mathematics students' arguments while working in a technological environment?
2. What are the characteristics of the content and structure of eight grade mathematics students' arguments while working in a non-technological environment?
3. In what ways do the arguments made by students working in a technological environment compare in content and structure to those made by students working in a non-technological environment?

First, I characterize the content and structure of the arguments the three groups of students created while working in the technological environment across the three tasks (the triangle inequality task, the triangle side and angle relationship task, and the investigating triangles task). Then, I characterize the content and structure of the arguments the three groups of students created while working in the non-technological environment across the three tasks (the triangle inequality task, the triangle side and angle relationship task, and the triangle sort task). Finally, I compare the content and structure of the arguments the three groups of students created while working in the technological environment to the arguments the three groups of students created while working in the non-technological environment for each of the three tasks and across the three tasks.

The Content and Structure of the Arguments Students Create While Working in the Technological Environment

During the instructional unit, the three groups of students in the technology class create arguments of various structures and content including the ways in which the students use the technology. The basic structures of the arguments the groups of students are core arguments (with and without explicit warrants), arguments in which the student collect additional data, arguments with non-explicit claims, arguments with non-explicit data, linked arguments, and arguments with a second warrant. Looking across the arguments the three groups of students create while working on the three tasks, four themes emerge: the explicitness of the warrant and the use of technology (see Table 38); the complexity of the structure of the arguments and the use of technology; the use of technology to collect additional data, and the change in the explicitness of the warrants during the unit.

Table 38

The Combined Arguments of the Three Groups of Students Working in the Technology Class Across the Three Tasks by Structure, Use of Technology, and Explicitness of the Warrant

Warrants	Use of Technology	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Technology Used	61	34
	Technology Not Used	5	5
Warrants Explicit			
	Technology Used	15	35
	Technology Not Used	17	7

Explicitness of the warrant and the use of technology.

Of the 179 arguments the three groups of students create while working on the three tasks, the students employ technology for 145 (81%) of them. Of these 145 arguments in which the students use technology, the warrants are explicit for 50 (34%) arguments. Of the 34 arguments in which the students do not actively use technology, 24 (71%) of the warrants are explicit. This difference suggests that when students do not actively employ technology, they are more likely to provide an explicit warrant. The explicitness of the warrant may be related to the task on which the students are working.

Across the three tasks, when the students do not actively employ technology, the students are generally working on generalization and explanation type tasks. When the students use the technology, they were working on tasks which involved completing the task sheet. For example, the task sheet used in the triangle inequality task asks the students to develop a conjecture after the students use the technology to determine whether given sets of segment lengths are able to form a triangle. All of the groups of students actively use the technology to determine whether each set of segments form a triangle, however they do not actively use the technology as they developed their conjecture. Another example occurs when the students are working on the triangle side and angle relationship task. The students use the technology to measure part(s) of the triangle and position the triangle such that it conforms to the instructions on the task sheet. Once the students use the technology they record their findings. When developing their generalizations, the students use their responses on the task sheet and do not actively use the technology. On the investigating triangles task, the task sheet asks the students to state what they noticed while working with the four triangles for each of the desired properties. Many times, the students make a generalization based on their findings for each of the four triangles. When making these generalizations, the students do not always use the technology.

The students aslo create arguments during whole class discussions for each of the three tasks. These discussions generally focus on the students' generalizations, whether these generalizations are valid, and why they are true. During these discussions, the students may rely upon their prior uses of technology, but do not actively use it. The teacher usually asks the students to justify their claims during these discussions. Perhaps, when the students are creating a generalization or conjecture, they are compelled to provide an explicit warrant either because they are prompted to do so or to make their reasoning clear to others.

In contrast, when the students are using technology to complete the task sheet, the students rarely provide an explicit warrant. The lack of explicit warrants when students use the technology may be due to the lack of prompting on the task sheet. It may also be attributed to the shared nature of the technology. Because both students can view the screen and the ways in which the technology is used, the reasoning for the students' claims may be apparent to both individuals and does not need to be stated.

The use of technology in non-core arguments.

Of the 145 arguments in which the three groups of students employ technology, the structure of 76 (52%) of the arguments is core and 69 (48%) are non-core. Of the 34 arguments in which the students do not actively employ technology, 22 (65%) have the core structure and 12 (35%) have the non-core structure. This suggests when the students use technology in their arguments, the students are more likely to create a more complex structure than when the students do not use technology.

When the three groups of students employ the technology, they create 76 core arguments, 15 (20%) have explicit warrants. In addition, they create 69 arguments while employing technology whose structure is non-core. Of these 69 non-core arguments, 35 (51%) have explicit warrants. This suggests when the students use technology and create an argument with a more complex structure, the students are more likely to provide an explicit warrant. This may be due to the number of opportunities to provide an explicit warrant. The three groups of students create many non-core arguments including those in which they collect additional data (e.g. the arguments illustrated in Figures 15, 26, 62), provide a second warrant (e.g. the argument illustrated in Figures 72, 80, 91), or use a claim as data to make additional claims, (e.g. the argument illustrated in Figures 18, 42). When the students create arguments of these structures, the students have the opportunity to provide more than one warrant. In the core arguments, this is not the case.

Another possible reason why the non-core arguments have a higher percentage of arguments with an explicit warrant may be due to the challenge of a claim. At times, another student or the teacher challenges a student's claim. This challenge may have prompted the students to provide an explicit warrant. If a core argument remains unchallenged, then the students may not be compelled to provide an explicit warrant.

The use of technology to collect additional data.

One of the structures of arguments the students create while working on these tasks is that in which the students collect additional data. The students' decision to collect additional data may have been due to a number of factors including the uncertainty of a claim (e.g. the arguments illustrated in Figures 16, 78, 88), a challenge to the claim (e.g. the arguments illustrated in Figures 26, 43, 90), the uncertainty with the precision of the tools (e.g. the argument illustrated in Figure 17) and the ways in which the students use the technology to collect the initial data (e.g. the arguments illustrated in Figures 25, 31, 50, 60).

In most cases, the students collect additional data using the technology. The use of technology to collect additional data may be related to the ease of collecting data with the technology. While working on the triangle inequality task, the students employ the technology's drag feature to collect additional data for all arguments of this structure (e.g. the arguments illustrated in Figures 16, 26, 32). While working on the triangle side and angle relationship task, the students collect additional data through the use of measures for the majority of their arguments of this structure (e.g. see the arguments illustrated in Figures 51, 60). And, for the investigating triangles task, the students collect additional data by dragging a vertex of the triangle and viewing the linked measures (e.g. see the arguments illustrated in Figures 70, 78, 83). In some episodes, the students make a claim based on the appearance of the diagram on the screen and collect additional data using the drag feature (e.g. see the arguments illustrated in Figures 25, 31), the measurement tool (e.g. see the arguments illustrated in Figures 50, 60) or using both features of the technological tool (e.g. see the argument illustrated in Figure 71). In all of these arguments, the students use the drag and/or measurement features of the technological tool to validate or refute their initial claims.

The change in the explicitness of the warrant during the instructional unit

Although the number of arguments with non-explicit warrants is greater than those with at least one explicit warrant, the proportion of arguments provided by the students that include an explicit warrant increases during the instruction of the unit. While working on the triangle inequality task, the three groups of students create 73 arguments and 25 (34%) of the arguments have at least one explicit warrant. The three groups of students create 50 arguments while working on the triangle side and angle relationship task and 17 (34%) of the arguments have at least one explicit warrant. While working on the investigating triangles task, the three groups of students create 53 arguments and 38 (72%) of the arguments have at least one explicit warrant. The triangle inequality and triangle side and angle relationship tasks were taught during the first three days of instruction and the investigating triangles task was taught during the sixth day of instruction. Perhaps, by the six day, the classroom norm has been established in which the students are expected to provide justifications and explanations for their claims.

The increase in explicit warrants may also be attributed to the activities of the task sheet. On the task sheets for the triangle inequality and triangle side and angle relationship tasks, the students are asked to work through a given number of examples and then asked to make one or more generalizations based on their findings. The task sheet for the investigating triangles task asks the students to provide an explanation on whether the triangles could conform to the desired properties. The increase in the proportion of arguments with an explicit warrant coupled with the activities of the task sheet for the investigating triangles task suggests students are able and willing to provide justifications for their claims when prompted to do so.

The Content and Structure of the Arguments Students Create While Working in the Non-Technological Environment

During the instructional unit, the three groups of students in the non-technology class create arguments of various structures and content. The basic structures of the arguments the students created are core arguments (with and without explicit warrants), arguments in which the student collect additional data, arguments with non-explicit claims, linked arguments, and arguments with a sub-argument. Looking across the arguments the three groups of students create while working on the three tasks, three themes emerge: the number of arguments (see Table 39), the explicitness of the warrant and the use of tools, and the ways in which the students collect additional data.

Table 39

The Combined Arguments of the Three Groups of Students Working in the Non-Technology Class Across the Three Tasks by Structure, Use of Technology, and Explicitness of the Warrant

Warrant	Use of Tools	Core Arguments	Non-Core Arguments
Warrants Not Explicit			
	Tools Used	15	11
	Tools Not Used	5	2
Warrants Explicit			
	Tools Used	16	7
	Tools Not Used	22	10

The number of arguments.

While working on the three tasks, the three groups of students create 88 arguments. However, the students create 56 (64%) of the arguments while working on the triangle inequality task. As discussed in the previous chapter, the lack of arguments while working on the triangle side and angle relationship and triangle sort tasks may be attributed to the nature of the task and the incapability of the researcher to document some claims as arguments or to the students unwillingness to verbally express their claims. Though, the difference in the number of arguments between the tasks may also be related to the nature of the tools used in the tasks. The tools the students in the non-technology class use while working on the triangle inequality task are the snap-cubes. The students use the snap-cubes to create segments of specific lengths and arrange these segments to determine whether a triangle can be formed. The students are able to adjust the lengths of the segments by adding or removing a snap-cube to or from the segments and change the appearance of the diagram by moving the segments. In some respects, the students' uses of the tools are dynamic due to their abilities to adjust the lengths of the segments and rearrange the segments. The tools the non-technology students use for the triangle side and angle relationship and the triangle sort task are the measured triangles. These tools are static; the students could not adjust the appearance of the triangle or its measures. Perhaps, students create more arguments when working with tools that can be used dynamically.

Explicitness of the warrant and the use of tools.

Of the 88 arguments the three groups of students create while working on the three tasks, the students employ the tools for 44 (50%) of them. Of these 44 arguments in which the students use the tools, the warrants are explicit for 21 (47%) arguments. Of the 44 arguments in which the students do not actively use the tools, 34 (77%) of the warrants are explicit. The explicitness of the warrant may be related to the task on which the students are working.

Across the three tasks, when the three groups of students do not actively employ the tools, the students are generally working on generalization and explanation tasks. The task sheet used in the triangle inequality task asks the students to develop a conjecture after the students use the tools, the snap-cubes, to determine whether given sets of segment lengths are able to form a triangle. None of the three groups of students actively use the tools as they develop their conjecture. While working on the triangle side and angle relationship task, the students use the tools, the measured example triangles, to determine the names of the largest and smallest angles and the names of the longest and shortest sides of the example triangles. The students record their findings on the task sheet and then use those findings to develop generalizations. The students do not actively use the tools while conjecturing. When the students are making these conjectures, the students are not actively using tools and providing explicit warrants.

The arguments the students make during the whole class discussions generally do not involve the active use of tools and these discussions focus on the students' generalizations, whether these generalizations are valid, and why they are true. In addition, the teacher usually asks students to justify their claims during whole class discussions. In general, the arguments the students created during the whole class discussion are based on the students' generalizations and conjectures and the students are asked by the teacher to justify their claims.

The ways in which the students collect additional data.

While working on the three tasks, the students create arguments in which they collected additional data to verify or refute a previous claim. All three groups of students create arguments of this structure and at least one group of students create an argument of this structure while engaged in each of the three tasks. The students' decision to seek additional data may be due to a number of factors including an explicit challenge to a claim (e.g. the arguments illustrated in Figures 106, 124, and 143), and the uncertainty of a claim (e.g. the arguments illustrated in Figures 117, 132, and 141. Even though the students collect additional data for a variety of reasons, they generally use the tools to collect additional data (e.g. the arguments illustrated in Figures 107, 132, and 137), but this was not always the case.

At times, the students use known facts such as definitions and theorem as additional data to verify or refute a claim (e.g. the arguments illustrated in Figure 108). The use of definitions and theorems as additional data indicates that the students understand and are able to use these mathematical ideas to make claims and solve problems.

Comparison

In this section, the structures and content of the arguments created by the students working in the technological environment are compared to those created by the students working in the non-technological environment. A comparison is made for each of the three tasks and across the three tasks.

Comparison for task 1.

While working on the triangle inequality tasks, both groups of students create arguments of various structures and contents. Four themes emerge in the comparison of the arguments: the difference in the number of arguments between the two classes, the frequency in the use of the technology/tools, the explicitness of the warrants and the use of technology/tools, and the content of the additional data when students collected additional data.

Difference in number of arguments.

The students working in the technological environment create 75 arguments while engaged in the activities of the triangle inequality task (see Table 9). While working on a similar task, the students working in the non-technological environment create 56 arguments (see Table 24). This difference is even greater when considering the students in the non-technology class had twenty additional minutes to create arguments. In both classes, the students spend part of the first class meeting working on this task as well as another task. The students in both classes also spend fifty minutes working on a warm-up which included problems asking the students to apply the triangle inequality theorem. However, the students in the non-technological class spend part of one class meeting comparing their answers to problems on the homework with their partners, which includes problems which asks the students to determine whether a set of three segment lengths will form a triangle. Removing the arguments the students create while comparing their answers on the homework, the students in the non-technological class only create 44 arguments while working on the triangle inequality task.

The frequency of the use of the technology/tools.

In addition to the discrepancy in the number of arguments, there is also a difference in the proportion of arguments in which the students use the technology/tools. While working on the triangle inequality task, the students in the technology class employ the technology in 59 (79%) of their arguments. The students in the non-technology class create 19 (34%) arguments in which they use the tools. The higher proportion of arguments in which the technology is employed for the technology class is most likely due to its use during the warm-up activity. While working on the warm-up, the teacher does not instruct the students to use the technology to solve the problems. However, the students request to use the technology to assist them in solving the problems. The students in the non-technology class did not request to use the snap-cubes to solve these warm-up problems. At times, the students in the non-technology class would use a ruler to make a sketch, but this was infrequent. Perhaps, the students in the technology class view the technology as a tool that can assist them in solving problems. The students in the non-technology class may not view the snap-cubes in this manner. Rather, the students may see the snap-cubes as a means to create data to make a generalization, but not to solve problems.

Explicitness of the warrant and the use of technology/tool.

While working on the triangle inequality tasks, the students in both classes are more likely to provide an explicit warrant for their arguments when not actively using the tools (technological and non-technological) than when the students create an argument using the tools. Of the 59 arguments in which the students in the technology class employ technology, they only provide an explicit warrant for 6 (10%) of these arguments. Of the 16 arguments in which technology is not actively employed, the students in the technology class provide an explicit warrant for 12 (75%) of these arguments. Of the 19 arguments in which the students use the tools, they only provide an explicit warrant for 1 (5%) of these arguments. Of the arguments in which tools are not actively employed, the students provide an explicit warrant for 30 (81%) of these arguments.

The explicitness of the warrants may be related to the type of tasks on which the students are working. In general, when the students are using the tools (technological and non-technological), they are attempting to determine whether a triangle could be formed with the given segments. However, when the students are not actively using the tools, the students are mainly working on generalization type tasks. For example, one of the arguments common to all students in structure was in response to the question on the task sheets, "Why was it impossible to construct a triangle with some of the given lengths?" The question is asking for the students to generalize across the examples. The structure of the argument for all the groups of students is a core argument with an explicit warrant (see the arguments illustrated in Figures 13, 23, 30, 96, 104, and 113). The structure of these arguments may be due to the fact that the question actually provides the claim and asks the students to provide the warrant. The data for the students are their answers to the examples sets of segments on their task sheet. To gather this data, the students use the tools (technological and non-technological). However, when responding to this question, the data had been previously gathered and their reasoning is not based on their active use of the tools, but on the product of their previous uses.

Content of the additional data.

Many times the students in both classes collected additional data to verify or refute a previous claim. All three groups of students in both classes create arguments of this structure. The students' decision to seek additional data may be due to a number of factors including an explicit challenge to a claim (see the arguments illustrated in Figures 15, 26, 100 and 106), the uncertainty of a claim (see the arguments illustrated in Figures 15, 31, 108, and 117), the uncertainty of a claim due to the lack of precision in the use of the technological tool (see the argument illustrated in Figure 17), and the ways in which the technology was used to collect the initial data (see the arguments illustrated in Figures 25 and 31). Even though the students collect additional data for a variety of reasons, the students in the technology class always use technology to collect this additional data, usually using the drag feature of the technology. Generally, the students in the non-technology class use the tools to collect additional data, but this is not always the case. At times, the students use known facts such as definitions and theorems as additional data to verify or refute a claim (e.g. the argument illustrated in Figure 108).

Comparison for task 2.

When working on the triangle side and angle relationship tasks, both groups of students create arguments of various structures and contents. Three themes emerge in the comparison of the arguments: the difference in the number of arguments between the two classes, the frequency in the use of the tools (technological and non-technological), the explicitness of the warrants and the use of tools (technological and non-technological), and the creation of arguments with a complex structure.

Difference in number of arguments.

The students working in the technological environment create 50 arguments while engaged in the activities of the triangle side and angle relationship task (see Table 13). While working on a similar task, the students working in the non-technological environment create 15 arguments (see Table 30). The lack of arguments created by the students in the non-technology class may be due to the nature of the task sheet. When completing the task sheet, the students do not discuss their findings. They simply look at the example triangles and write down the names for the longest/shortest side and the largest/smallest angle. The students do not make their responses audible nor do the students question each other about their responses. Thus, the claims the students make on their task sheet cannot be documented as arguments.

The difference in the number of arguments between the groups of students in the two classes may also be attributed to the nature of the tools. The tools used by the students in the non-technology class were the five measured example triangles. Written on the triangles are the measures of the angles and the side lengths. In previous class meetings, the students make the measures using protractors and rulers. Per the instructions on the task sheet, the students determine the name of longest/shortest side and largest/smallest angle for each of the example triangles and write it on the task sheet. The triangles are static; the students are unable to change the appearance or the measures of the triangles.

In contrast, the instructions for the task sheet used with the students in the technology class asks the students to drag part(s) of the five triangles such that each meet the desired criteria on the task sheet (e.g. side *BC* is the longest side of the obtuse triangle *ABC*). The students also have the option of using the measurement features of the software to determine the longest/shortest side and/or largest/smallest angle. Because the students are able to make use of these dynamic features, perhaps the students are more actively engaged in the task, thus creating more arguments.

Explicitness of the warrant and the use of technology/tool.

While working on the triangle side and angle relationship tasks, the students in both classes are more likely to provide an explicit warrant when not actively using the tools (technological and non-technological) than when the students create an argument using the tools (technological and non-technological). Of the 42 arguments in which the students in the technology class employ technology, they only provide an explicit warrant for 11 (26%) of these arguments. Of the 8 arguments in which technology is not actively employed, the students in the technology class provide an explicit warrant for 6 (75%) of these arguments. Of the 13 arguments in which the students in the non-technology class use the tools, they only provide an explicit warrant for 9 (69%) of these arguments. Of the 2 arguments in which tools are not actively employed, the students provide an explicit warrant for both (100%) of these arguments.

The explicitness of the warrants may be related to the type of tasks on which the students are working. In general, when students in the technology class are using the technology, they are determining the longest/shortest side or smallest/largest angle of triangle. When these students are working on generalization and justification tasks, they generally are not actively using technology. The students in the non-technology class make few arguments concerning the longest/shortest side or largest/smallest angle of a triangle. However, students in this class frequently use the tools as they create conjectures and generalizations and their warrants for most of these arguments are explicit. This suggests that when students are working on tasks involving a generalization, they were more likely to provide explicit warrants.

Arguments with a complex structure.

The students in both classes create arguments with structures that are more complex than the core structures. In the technology class, the students create arguments in which the students collect additional data, (e.g. the arguments illustrated in Figures 53 and 60), linked arguments (e.g. the argument illustrated in Figure 42), and arguments in which the students revisit the data and make an alternate claim (e.g. the argument illustrated in Figure 43). In the non-technology class, the students create arguments in which the students collect additional data (e.g. the arguments illustrated in Figure 124 and 133), and arguments with a teacher prompted sub-argument (e.g. the argument illustrated in Figure 126). The students in the technology class create 12 (24%) arguments with a non-core structure. Of these 12 arguments, the students employed the technology in 11 (92%) of the arguments. The students in the non-technology class create 5 (33%) arguments with a non-core structure. Of these 5 arguments, the students employ the non-technology tools in all (100%) of the arguments. Even though the students in both classes create arguments with different non-core structures, there does not seem to be a great difference in the proportion of arguments with a complex structure or the proportion of these non-core arguments in which the students employed the technology/tools.

Comparison for task 3.

While working on the investigating triangle task and the triangle sort task, the students in the technology class and the students in the non-technology class, respectively, create arguments of various structures and contents. Three themes emerge in the comparison of the arguments: the difference in the number of arguments created by the students in the two classes, the explicitness of the warrants and the use of tools (technological and non-technological), and the content of the explicit warrants.

Difference in number of arguments.

The students working in the technological environment create 53 arguments while engaged in the activities of the triangle side and angle relationship task (see Table 17). While working on a similar task, the students working in the non-technological environment create 17 arguments (see Table 34). The lack of arguments created by the students in the non-technology class may be due to the nature of the task. While working on the triangle sort task, all three groups in the non-technology class make claims about the classification of the example triangles. However, the students do not often state anything other than the classification of the triangle. The researcher is unable to document these as arguments because he is unable to determine the data on which the claim is based. The data could be the appearance of the example triangle or the measures written on the example triangle. Thus, these claims could not be documented as arguments.

The difference in the number of arguments between the groups of students in the two classes may also be attributed to the nature of the tools. The tools used by the students in the non-technology class are the measured example triangles. Written on the triangles are the measures of the angles and the side lengths. In previous class meetings, the students make the measures using protractors and rulers for most of the triangles. For the triangles the students did not measure, the teacher wrote the measures on the triangles. Per the instructions on the task sheet, the students are to sort the triangles three times, based on differing criteria. The triangles are static, as the students are unable to change the appearance or the measures of the triangles.

In contrast, the instructions for the task sheet used with the students in the technology class asks the students to determine whether the constructed triangles on the screen (equilateral, isosceles, obtuse and right) can conform to the desired property (e.g. at least two equal angles). For each of the constructed triangles, the measures of the angles and side lengths are provided to the students. The task sheet asks the students to first make predictions whether each of the constructed triangles can conform to the property, use the technology to verify their predictions, and explain why each of the triangles was able or unable to conform to the property. The students use the drag feature, along with the linked measures, of the technology to determine whether the triangles can conform to the properties. Because the students are able to make use of these dynamic features, perhaps the students are more actively engaged in the task, thus creating more arguments.

Explicitness of the warrant and the use of technology/tool.

While working on the investigating triangles task and the triangle sort task, the students in the technology class and those in the non-technology class, respectively, are more likely to provide an explicit warrant regardless of whether they used the tools (technological and non-technological). Of the 43 arguments in which the students in the technology class employ technology, they provide an explicit warrant for 32 (74%) of these arguments. Of the 11 arguments in which technology is not actively employed, the students in the technology class provide an explicit warrant for 6 (55%) of these arguments. Of the 17 arguments in which the students in the non-technology class use the tools (which is all of the arguments), they provide an explicit warrant for 13 (76%) of these arguments.

The high proportion of arguments with an explicit warrant may be related to the nature of the tasks. For the technology class, the task sheet asks the students to provide an explanation why each of the triangles is or is unable to conform to the property. By having students write an explanation, the teacher asks for the students to provide an explicit warrant for their claims. For the non-technology class, the researcher is unable to document many of their claims as arguments because he is unable to determine the data the students used as a basis for their claims. When the researcher is able to document a claim as an argument, it generally is due to the students providing an explicit warrant. Thus, it is not surprising the students in both classes have a high proportion of arguments with explicit warrants.

The content of the students' explicit warrants.

Though both classes have a high proportion of arguments with explicit warrants, the content of the explicit warrants differ. The explicit warrants in the technology class consist of theorems (e.g. the arguments illustrated in Figures 80, 85, 93), definitions (e.g. the arguments illustrated in Figures 70, 83), and were in terms of the students' uses of technology (e.g. the arguments illustrated in Figures 66, 75, 90).

The explicit warrants in the non-technology class consists of measures (e.g. the arguments illustrated in Figures 135, 142) and the appearances of the triangles (e.g. the argument illustrated in Figure 139). Even though the students use measures in their explicit warrants, it is based on sound reasoning. In these arguments, the students are not making claims that would be considered generalizations across a family of triangles. Rather, the students make claims about specific example triangles. Their warrants are relying upon unstated facts, which are the definitions of the different triangle types. The warrants were not merely measures, but could be considered applications of the definitions of the triangle types. For example, in argument illustrated in Figure 142, Jim's states that example triangle 10 is an obtuse triangle "because it has a 100 degree angle."

Comparison across tasks.

Across the three tasks, the students in both classes create arguments of varying structures and content. Looking across the three tasks for the two classes, three themes emerge: the difference in the number of arguments, the explicitness of the warrants when using the technology/tools, and the proportion of arguments with a complex structure. These themes are discussed below

The difference in the number of arguments.

The students working in the technological environment create 179 arguments while engaged in the activities of the three tasks (see Table 38). While working on similar tasks, the students working in the non-technological environment create 88 arguments (see Table 39). For each of the three comparison tasks, the students in the technology class create more arguments than the students in the non-technology class.

As previously discussed, the low number of arguments created by the groups of students in the non-technology class may be due to the nature of the task for the triangle side and angle relationship and triangle sort tasks. It could also possibly be due to the nature of the tools. The tools the students in the non-technology class use while working on the triangle inequality task are the snap-cubes. The students use the snap-cubes to create segments of specific lengths and rearrange the segments to determine whether a triangle could be formed. The students are able to adjust the lengths of the segments by adding or removing a snap-cube to or from the segments. In some respects, the students' uses of the tools are dynamic due to their abilities to adjust the lengths of the segments and arrange the segments. The tools the non-technology students use for the triangle side and angle relationship and the triangle sort task are the measured triangles. These tools are static, as the students cannot adjust the appearance of the triangle or its measures.

The pre-constructed sketches the students in the technology class utilize some of the dynamic features of the software for each of the three tasks. For the triangle inequality task, the students can adjust the sliders to change the lengths of the segments and drag the endpoints of the segments to determine whether a triangle can be formed. For the triangle side and angle relationship and investigating triangles tasks, the students can drag all or a portion of the triangles and view the change or invariance of the linked measures.

During the triangle inequality task, the students in the non-technology class create 19 (25%) fewer arguments than the students in the technology class. While working on the triangle side and angle relationship task, the students in the non-technology class create 35 (70%) fewer arguments than the students in the technology class. On the triangle sort task, the students in the non-technology class create 36 (68%) fewer arguments than the students in the technology class. The increase in the percentage of fewer arguments from the triangle inequality task to the triangle side and angle relationship and triangle sort tasks coupled with the dynamic/static nature of the tools being used in these tasks suggests that the frequency of the arguments can also be attributed to the nature of the tool. Perhaps, when students use technological or non-technological tools in a dynamic manner, they are more actively engaged in the task, thus creating more arguments.

Explicitness of the warrant and the use of technology/tool.

While working on the three tasks, the students in both classes are more likely to provide an explicit warrant when not actively using the tools (technological and non-technological) than when the students created an argument using the tools (technological and non-technological). Of the 145 arguments in which the students in the technology class employ technology, they only provide an explicit warrant for 51 (35%) of these arguments. Of the 34 arguments in which technology is not actively employed, the students in the technology class provide an explicit warrant for 24 (71%) of these arguments. Of the 49 arguments in which the students in the non-technology class use the tools, they provide an explicit warrant for 23 (47%) of these arguments. Of the 39 arguments in which the students in the non-technology class do not actively employ the tools, the students provide an explicit warrant for 32 (82%) of these arguments. The warrants for the arguments created while employing the technology/tools are more likely to be non-explicit regardless of whether the students used technology or non-technological tools.

The explicitness of the warrants may be related to the type of tasks on which the students are working. In general, when the students are using the tools (technological and non-technological), they are completing the tables on task sheets or working on the problems on the warm-up. However, when the students are not actively using the tools (technological and non-technological), the students are mainly working on generalization type tasks. Thus, the students may have been more compelled to provide an explicit warrant while working on these types of tasks.

The proportion of arguments with a complex structure.

On all three tasks, the groups of students in each class create arguments whose structures are more complex than the core structure. These structures include arguments in which the students are compelled to collect additional data, linked arguments, arguments with a sub-argument, arguments in which the students reevaluate the same data, and arguments with a second warrant. The students in the technology class created 81 (45%) non-core arguments while working on the three tasks. The students in the non-technology class create 30 (35%) non-core arguments. Although this difference may not be great, it suggests that when students use technology they tend to create more arguments with a complex structure than students who use non-technological tools.

The difference may be attributed to the use of the tools (technological and non-technological). Of the 81 non-core arguments created by the students in the technology class, the students employ the technology for 69 (85%) of the arguments. The students in the non-technology class create 30 non-core arguments and use the tools in 18 (60%) of the arguments. Perhaps, when students use technology they are not only more likely to create arguments with complex structures but also more likely to use the technology in their arguments. This may be due to the ways in which students can collect additional data using the technology. When the students in the technology class are uncertain about their claim or when a claim is challenged, they are able to collect additional data by measuring and/or by dragging the object or a portion of it to create additional examples. The affordances of the technology to easily collect these types of data may contribute the students' ability to establish certainty or to reconcile a challenge.

In the next chapter, the findings in this chapter will be discussed including their connection to the research literature and the implications of these finding for teacher, teacher educators, and researchers. In addition, the next chapter discusses future directions for research related to students use of technology and the arguments that students create based on upon these uses.

CHAPTER 7

Discussion

In this chapter, the major findings mentioned in the previous chapter are discussed and implications of these findings for teachers, teacher educators, and researchers are articulated. At the end of this chapter, I discuss the questions that emerged during this study and possible future directions for research on argumentation and the use of technology.

Explicitness of Warrants and Use of Technology/Tools

One major finding of this study is when students actively used the technological or non-technological tools, they were more likely to create arguments with non-explicit warrants. In contrast, when the students did not actively use the tools, they were more likely to create arguments with an explicit warrant. Hollebrands, Conner, and Smith (2010) had similar findings in their study of the arguments college geometry students create when working with technology. When the college geometry students provided explicit warrants for their claims, the students were generally not using technology. The authors attributed this finding to the students' prior experiences in learning mathematics at the collegiate level where the students were expected to provide formal proofs, which require explicit warrants. Because of the unlikeliness that the middle school students in this study had been exposed to formal proofs, this same attribution cannot be made. Rather, the lack of explicit warrants when the middle school students were using the technological and non-technological tools may be attributed to the task on which the students were working, the establishment of classroom social and sociomathematical norms, and the visual nature of the technology.

Task on which the students were working and the use of technology.

In general, when the students are using the technological or non-technological tools, they were completing the tables on task sheets. However, when the students were not actively using the tools, the students were mainly working on generalization type tasks. Perhaps, the students may have been more compelled to provide an explicit warrant while working on these generalization type tasks. At times, the students were prompted to provide an explicit warrant when working on generalization tasks either by the teacher or the task sheet. For example, the triangle inequality task sheet for both classes asks the students, "Why was it impossible to construct a triangle with some of the given lengths?" The question asks the students to generalize across the examples. The structure of the argument for all the groups of students was a core argument with an explicit warrant (see the arguments illustrated in Figures 13, 23, 30, 96, 104, and 113). The structure of these arguments may be due to the fact that the question actually provided the claim and asked the students to provide the warrant. The data for the students are their answers to the examples sets of segments on their task sheet. To gather this data, the students used the technological or non-technological tools. However, when responding to this question, the data had been previously gathered and their reasoning is not based on their active use of the technology/tools, but on the product of their previous uses.

Even though the students in the current study did not actively use the technology when working on generalization type tasks, other researchers (e.g. Healy & Hoyles, 2000) found that students will create generalizations while using technology. Some pairs of students in Healy and Hoyle's (2000) study used a DGE to investigate relationships among the angle bisectors of a quadrilateral. The pairs of students did not create the same constructions and did not arrive at the same conclusions. However, those students that were successful were able to construct and measure aspects of their diagrams and developed generalizations while using the DGE. The students in the current study did not have the option of creating their own diagram. Instead, the students in the current study used teacher-generated pre-constructed sketches that limited the students in how they could modify and/or measure aspects of the diagram. Perhaps, the students in the current study would have been more likely to use the technology while working on generalization type tasks if they had been given the opportunity to create their own diagrams in the DGE.

Across the three tasks, the students in the technology class were less likely to provide an explicit warrant when using technology. However, for one of the three tasks, the students in the technology class were more likely to provide an explicit warrant when using technology. In fact, on the third task, the students were more likely to provide an explicit warrant when using the technology then when the students did not use the technology. This may be related to the design of the activity and task sheet. On the investigating triangles task sheet (see Appendix E), the students were asked to explain why each of the four triangle types was able or unable to conform to the desired property. On the two previous tasks, the students were expected to fill out a table using the technology and then create a generalization across the examples in the table. During the first two tasks, the students were asked to justify their conjectures and generalizations, but not their reasoning while they were using the technology. The increase in the proportion of arguments in which the students used technology and provided an explicit warrant on the investigating triangles task compared to the same proportions in the previous tasks coupled with the nature of the task sheets suggests that students will provide explicit warrants while using technology when prompted to do so.

The difference in the proportion of explicit warrants in the third task compared to the previous two coupled with the change in the design and nature of the third task suggests that the design of the task may have influenced the content and structure of the arguments created by the students. Mathematical tasks not only provide the context in which students learn about the content matter but also communicate what mathematics is and how it is done (NCTM, 1991). "Thus, the nature of tasks can potentially influence and structure the way students think and can serve to limit or broaden their views of the subject matter with which they are engaged" (Henningsen & Stein, 1997, p. 205). In the current study, it appears that the design of the tasks influenced the way students think and, thus, the arguments they create.

Establishment of norms.

Even though the task sheet for the investigating triangles task prompts the students to make their reasoning explicit, the increase in the proportion of the arguments with an explicit warrant and active use of technology for the third task may relate to when the task was conducted and the classroom social and sociomathematical norms that had been established prior to the task. The classroom social norms are the regularities in the interaction patterns that regulate social interaction in the classroom and influence individuals' beliefs about their role in the classroom and the nature of mathematics (Yackel & Rasmussen, 2002) and the sociomathematical norms are the regularities in the classroom interaction patterns that are specific to mathematics (Yackel, 2002). Prior to the teaching of the triangles unit, the classroom community, including the students and regular classroom teacher, had established classroom social and sociomathematical norms during the course of the school year. During the triangles unit, the researcher/teacher emphasized students to justify their reasoning and challenge each other's claims. The researcher/teacher attempted to change, or renegotiate, the classroom social and sociomatheamtical norms for the classroom community. The classroom social and sociomathematical norms do not change instantaneously. Rather, they develop and evolve as the classroom community negotiates these norms (Cobb & Yackel, 1996). At the beginning of the triangles unit, the two previously mentioned classroom social norms were beginning to emerge, but had not been established. Thus, the proportion of arguments in which technology was used and the warrants were explicit for the first two tasks was low. By the sixth class meeting, perhaps these two classroom social norms may have been bettered established and contributed to the increase in the proportion of arguments with an explicit warrant and use of technology.

Affordances of the technology.

The students' lack of explicit warrants in their arguments created while actively using the technological and non-technological tools may also be related to the visual nature of the tools. Each student did not have his or her own personal set of tools. Rather the pairs of students shared the tools and would alternate between using them. Thus, the students shared the same visual display presented by the students' uses of the tools (the screen for the technology class and the arrangement of the snap-cube segments and example measured triangles for the non-technology class). Because the students had the same visual referent, the students may not have been compelled to provide an explicit warrant. Lavy (2006) investigated the types of arguments middle school students created while using an interactive computerized environment. The author found that the students not only used images on the screen and commands as data, but also in their reasoning. Lavy concluded, “In a visualized environment, it is obvious that visual evidences can serve as reasoning in an argument, since all the work in this setting has the same character” (p. 168).

In the current study, each pair of students had a common visual environment and was able to view how the tools were used. When a student makes a claim and actively uses the tools, the students may not provide an explicit warrant because their uses of the tools is the warrant. At times, the students would provide explicit warrants based on their uses of the tools (e.g. see the arguments illustrated in Figures 66, 75, and 90), but mostly occurred during the third task when the students were asked to provide an explanation for their findings.

Conclusions and implications.

NCTM (2000) indicates that reasoning, communication, and the use of technology are essential to the learning of mathematics. If students are expected to employ technology or manipulatives, use sound reasoning, and effectively communicate as they learn mathematics, then teachers need to be aware that students are less likely to communicate their reasoning explicitly when using technology. The lack of communication may be due to the content of the students’ reasoning which include the uses of the tools. Teachers need to design activities and tasks that prompts students to make their reasoning explicit, including the ways in which they used the tools and the reasoning that connects the tool usage to their claims. Doerr and English (2006) suggest tasks be designed to allow and encourage students to use representations and justifications as a means for others to view their mathematical reasoning.

Teachers also need to establish classroom social and sociomathematical norms that encourage students to make their reasoning explicit and to challenge each other's claims. And, a teacher may need to allow students the opportunity to create their own sketches while using a DGE, although other researchers (Sinclair, 2003) have found pre-constructed sketches to be beneficial while working on generalization type tasks.

The Number of Arguments and the Dynamic Nature of the Tools

Another major finding of this study is that the number of arguments students create appears to be related to whether the students use technology. The students in the technology class created more arguments than students in the non-technology class. This may speak to the level of the students' engagement with the mathematics. Student engagement can be defined as "a psychological process, specifically, the attention, interest, investment, and effort students expend in the work of learning," (Marks, 2000, p. 155). Perhaps, students that are more engaged with the activities of the classroom create more arguments. In their summary of the research literature on the use of technology to teach mathematics to middle school students, Guerrero, Walker, and Dugdale (2004) found that when technology is used appropriately with middle school mathematics students it can positively effect the students' confidence in doing and attitudes towards mathematics, engagement with the mathematics, conceptual understanding, and achievement.

The discrepancy in the number of arguments between the two classes corroborates the findings of Guerrero and colleagues (2004). However, these researchers indicate that the ways in which a teacher integrates the technology into the classroom must be based on sound pedagogical principles in order for it to positively effect students' engagement with the activities of the classroom. In the current study, the researcher/teacher had experience and knowledge of how to effectively integrate technology into the mathematics curriculum having used technology to teach high school students and to teach a mathematics teaching methods courses on how to effectively integrate technology mathematics classroom. The difference in the number of arguments between the two classes may not have occurred if the teacher of the classes was an inexperienced user of technology and/or did not have the knowledge of how to effectively integrate technology into the classroom. Teacher educators need to devote resources to developing teachers' abilities to effectively integrate technology into the mathematics classroom.

The level of students' engagement may not be only related to whether the students used technology, but to the affordances of the tools. As discussed in the previous chapter, the frequency of the arguments may also be attributed to the dynamic nature of the tools, technological and non-technological. During the three tasks, the students had the opportunity to use the technology in a dynamic manner. When the non-technological tools could be used in a dynamic manner, the students in the non-technology class created more arguments than when the tools could only be used as static objects. This finding may be related to the findings of a study conducted by Martin, Lukong, and Reaves (2007) who compared the ways in which students solved geometry problems with images and with manipulatives. The authors found that the students in the two groups correctly classified the shapes at similar rates and provided similar justifications. But, the students working with the manipulatives used more sophisticated means to solve problems. While the authors of the study do not focus on the arguments the students created, it does indicate that the physical action of using a manipulative may influence the ways students solve problems and that the physical action of using a tool may promote more sophisticated thinking and reasoning.

In the previous section, I discussed the need for teacher to design tasks that prompts students to make their reasoning explicit. When designing tasks, teachers also need to consider the tools the students will use during the task. The findings of this study suggest that the teacher need to select tools that afford the students the ability to use the tools in a dynamic manner so the students are more likely to be engaged with the mathematics.

Although it appears from the results that teachers merely need to select tools that have dynamic abilities, teacher still need to integrate the tools in an appropriate manner in order to have a positive effect on students (Guerrero, Walker, and Dugdale, 2004). Thus, teacher educators need to stress the design of tasks such that they capitalize on the dynamic affordances of the technology/tools and implement the tasks in an appropriate manner.

Toulmin's Model of Argumentation

In the current study, Toulmin's (1958) model of argumentation was used to analyze the arguments created by students. However, there were limitations to its use, mainly the type of discourse that could be considered arguments. Communications that do not fall into the data-claim-warrant framework are not documented, but they may contribute to students' arguments. Thus, the complete picture of how a claim is argued is not presented and cannot be analyzed. In turn, another limitation is that the framework is fairly dependent on discourse. Many times, a student makes a claim, but the researcher is unable to determine the data the student uses as the basis of the claim nor the reasoning to link the data in the claim. This occurred fairly frequently while the students in the non-technology class worked on the second and third tasks. In these instances, the lack of discourse prevents the researcher for documenting the claim as an argument.

However, other researchers have used Toulmin's model to analyze whole class arguments (e.g. Maher et al., 2006; Stephan & Rasmussen, 2002, Yackel, 2002), the relationship between the ways teachers' fostered whole arguments and their conceptions of proof (Conner, 2007), and the nature of the arguments when students use a technological tool (Hollebrands, Conner, & Smith, 2010). In the current study, Toulmin's model of argumentation was used to analyze the structure and content of students' arguments as they used technological and non-technological tools. One of the findings previously discussed in this chapter connects the structure and content of students' arguments to the tasks on which the students were working. Researchers may be able to use Toulmin's model to analyze tasks by looking at the structure and content of the arguments the students create in response to certain tasks. For example, researchers may be able to determine the cognitive challenge of a task by looking at the complexity of the argument students create in response to the task.

Future Research

During this study, the students in the technology class used a dynamic geometry environment, *Geometer's Sketchpad* (Jackiw, 2001), to investigate properties, definitions, and theorems related to triangles. For reasons previously discussed, the researcher purposefully chose to have the students use pre-constructed sketches rather than creating their own sketches. The use of the pre-constructed sketches limited the students on how they could use the DGE. Perhaps, the structure and content of the arguments that students developed from the uses of pre-constructed sketches were limited to those documented. If the students were allowed to construct their own figures using the DGE, would the content and structure of the arguments be similar to those documented in this study? What argument structures would the students create when they are given the opportunity to create their own diagrams?

The participants in this study were eighth grade mathematics students who had not been exposed to formal proofs. Would students of a different age and/or mathematical ability create arguments with similar structure and content? Specifically, do students enrolled in a secondary geometry course where formal proof is expected to be part of the curriculum create arguments of similar structure to those students in this study when working in similar environments? What is there a relationship between the mathematical ability of the students and the content and structure of the arguments they create?

In this study, the dynamic nature of the technology/tools seemed to influence the number of arguments students created. In what ways do the structure and content of the arguments students create when using a different technology tool differ to those created by the students in this study who used a dynamic geometry environment? How do the content and structure of arguments created by students using a different technology compare to the arguments students create that do not use technology?

For many of the arguments illustrated in chapters 4 and 5, the teacher played a role in the arguments either by challenging claims (see the arguments illustrated in Figures 43, 79, 116, and 137), requesting explicit warrants (see the argument illustrated in Figure 126), or providing data (see the arguments illustrated in Figures 73 and 99). What teaching behaviors are needed in order to facilitate argumentation? Are these behaviors different when students are using technology? How can teacher educators develop teachers' abilities to foster argumentation?

Conclusion

"In this changing world, those who understand and can do mathematics will have significantly enhanced opportunities and options for shaping their future" (NCTM, 2000, p. 5). Thus, students need to be provided the opportunity to be engaged in tasks that promote understanding and doing mathematics. One part of doing mathematics is communicating the reasoning and strategies used while engaged in a task (NCTM, 2000). These communications can vary in its sophistication and means of expression, but provide insight into students' thinking and reasoning.

Mathematical arguments are one form of communication provided by students as they engage in mathematical tasks. These mathematical arguments not only provide insight into how students are thinking and reasoning, but also how they may view doing mathematics and how they are interpreting the feedback generated by their use of a tool. The current study analyzed the content and structure of the students' arguments and the findings indicate that the dynamic nature of the tool, the design of the task, and the emergence of sociomathematical and classroom social norms may have influenced the students' arguments as they worked in technological and non-technological environments. As previously discussed, other researchers have found dynamic nature of tools (e.g. Guerrero et al., 2004), task design (e.g. Henningsen & Stein, 1997), and the establishment of norms (e.g. Yackel, 2002) to influence students understanding of and abilities to do mathematics. Thus, the analysis of arguments may provide insight into how students are thinking, reasoning, understanding, and doing mathematics.

REFERENCES

Accascina, G., Batini, M., Del Vecchio, F., Margiotta, G., Pietropoli, E., Valenti D. (2004). Problem posing e problem solving con Cabri. *Progetto Alice, 14*, 217-242.

Allen, F. B. (1996, November 8, 2008). A program for raising the level of student achievement in secondary level mathematics.

Artigue, M. (2002) Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*, 245-274.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *ZDM, 34*(3), 66-72.

Balacheff, N. (1988). Aspects of proof in pupil's practice of school mathematics. In D. Primm (Ed.), *Mathematics, Teachers, and Children* (pp. 216-238). London: Hodder and Stoughton.

Balacheff, N. (1991). The benefits and limits of social interaction: The case of mathematical proof. In A. Bishop, S. Mellin-Olsen & J. van Dormolen (Eds.), *Mathematical Knowledge: Its Growth Through Teaching* (pp. 175-194). Dordrecht, The Netherlands: Kluwer Academic.

Balacheff, N. (1999). Is argumentation an obstacle? Invitation to a debate... *International Newsletter on the Teaching and Learning of Mathematical Proof*(May/June).

Ball, D.L. (1992). Magical hopes: Manipulatives and the reform of math education. *American Educator 16*(2), 14–18, 46–47.

Baroody, A.J. (1989). Manipulatives don't come with guarantees. *Arithmetic Teacher 37*(2), 4–5.

Battista, M. (1998). *Shape makers: Developing geometric reasoning in middle school students with the geometer's sketchpad*. Emeryville, CA: Key Curriculum Press.

Bell, A. W. (1976). A study of pupils' proof--explanations in mathematical situations. *Educational Studies in Mathematics, 7*, 23-40.

Boero, P. (1999). Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on the Teaching and Learning of Mathematical Proof*(July/August).

Boero, P., Garuti, R., & Mariotti, M. (1996). Some dynamic mental processes underlying producing and proving conjectures. Paper presented at the 2*0th conference of the International Group for the Psychology of Mathematics Education*, Valencia, Spain.

Bruce, C. S. (1997). *Seven Faces of Information Literacy*. Adelaide: AULSIB Press.

Bruner, J.S. (1960). *The Process of Education.* Cambridge, MA: Harvard University Press.

Castellanos, J. (2008). NonEuclid (Version 2007.04) [Computer Software]. Retrieved April 17, 2006 from http://www.cs.unm.edu/~joel/NonEuclid/NonEuclid.html

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24*(4), 359-387.

Christou, C., Mousoulides, N., Pittalis, M., & Pitta-Pantazi, D. (2004). Proofs through exploration in dynamic geometry environments. *International Journal of Science and Mathematics Education, 2*(3), 339-352.

Clements, D.H. (1999). 'Concrete' manipulatives, concrete ideas. *Contemporary Issues in Early Childhood 1*(1), 45-60.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. Kelly & R. Lesh (Eds.), *Research Design in Mathematics and Science Education*. Mahwah, NJ: Lawrence Erlbaum Associates.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *Journal of the Learning Sciences, 10*(1), 113 - 163.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3/4), 175-190.

Conner, A. (2007). *Student teachers' conceptions of proof and facilitation of argumentation in secondary mathematics classrooms*. (Doctoral dissertation). Available from ProQuest Dissertations and theses database.(AAT 3266090).

Creswell, J. W. (2007). *Qualitative inquiry and research design*. Thousand Oaks, CA: Sage.

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17-24.

de Villiers, M. (1999). *Rethinking proof with the geometer's sketchpad*. Emeryville, CA: Key Curriculum Press.

Dienes, Z.P. (1969). *Building Up Mathematics*. London: Hutchison Education.

Douek, N. (1999). Argumentation and conceptualization in context: a case study on sunshadows in primary school. *Educational Studies in Mathematics, 39*(1), 89-110.

Edwards, L. (1997). Exploring the territory before proof: Student's generalizations in a computer microworld for transformation geometry. *International Journal of Computers for Mathematical Learning, 2*(3), 187-215.

Gentner, D. & Rattennan, M.J. (1991). Language and the career of similarity. In S.A. Gelman & J.P. Byrnes (Eds.), *Perspectives on thought and language: Interrelations in development* (pp. 225-277). London: Cambridge University Press.

Goldenberg, E., & Cuoco, A. (1998). What is dynamic geometry? In R. Lehrer & D. Chazan (Eds.), *Designing Learning Environments for Developing Understanding of Geomery and Space* (pp. 351-367). Hillsdale, NJ: Lawrence Erlbaum Associates.

Guerrero, G., Walker, N., & Dugdale, S. (2004). Technology in Support of Middle Grade Mathematics: What Have We Learned? *Journal of Computers in Mathematics and Science Teaching*, *23*(1), 5-20.

Hadas, N., Hershkowitz, R., & Schwarz, B. B. (2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments. *Educational Studies in Mathematics, 44*(1/2), 127-150.

Harel, G., & Sowder, L. (1998). Students' proof schemes: results from exploratory studies. In A. Schoenfeld, J. Kaput & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 234-283). Washington DC: Mathematical Association of America.

Healy, L., & Hoyles, C. (2000). A study of proof conception in algebra. *Journal for Research in Mathematics Education, 31*(4), 396-428.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support or inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28*, 524-549.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics, 24*(4), 389-399.

Hollebrands, K. F. (2007). The role of a dynamic software program for geometry in the strategies high school mathematics students employ. *Journal for Research in Mathematics Education, 38*(2), 164-192.

Hollebrands, K. F., Conner, A., & Smith, R.C. (2010). The Nature of Arguments Provided by College Geometry Students With Access to Technology While Solving Problems *Journal for Research in Mathematics Education, 41*(3), 324-350.

Hoyles, C., & Healy, L. (1999). Linking informal argumentation with formal proof through computer-integrated teaching experiments. Paper presented at the *23rd Conference of the International Group for the Psychology of Mathematics Education*, Haifa, Israel.

Hoyles, C., & Jones, K. (1998). Proof in dynamic geometry contexts. In C. Mammana & V. Villani (Eds.), *Perspectives on the Teaching of Geometry for the 21st century* (pp. 121-128). Dordrecht: Kluwer.

Inglis, M., Mejia-Ramos, J., & Simpson, A. (2007). Modeling mathematical argumentation: the importance of qualification. *Educational Studies in Mathematics, 66*(1), 3-21.

Jackiw, N. (1991). The Geometer's Sketchpad (Version 4) [Computer Software]. Emeryville, CA: Key Curriculum Press.

Jones, K. (2000). Providing a foundation for deductive reasoning: students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics, 44*(1), 55-85.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229-269). Hillsdale, NJ: Lawrence Erlbaum Associates.

Laborde, C. (1998). Visual phenomena in the teaching/learning of geometry on a computer-based environment. In *Perspectives on the Teaching of Geometry for the 21st Century* (pp. 113-120). Boston: Kluwer Academic Publishers.

Laborde, C. (2005). The hidden role of diagrams in students' construction of meaning in geometry. In *Meaning in Mathematics Education* (pp. 159-179).

Laborde, J. M., & Bainville, E. (2005). *Cabri 3D.* (Version 2.0) [Computer Software]. Grenoble: Cabrilog.

Laborde, J. M., & Bellemain, F. (1993). *Cabri II.* [Computer Software]. Dallas, TX: Texas Instruments.

Lagrange, J.B. et al. (2001). A meta study on IC technologies in education. in, M. van den Heuvel-Panhuizen. *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education,* vol. 1 (pp. 111–122). Utrecht: Freudenthal Institute.

Lave, J., & Wenger, E. (1991). *Situated learning, legitimate peripheral participation*. New York: Cambridge University Press.

Lavy, I. (2006). A case study of different types of arguments emerging from explorations in an interactive computerized environment. *Journal of Mathematical Behavior*, 25, 153-169.

Leung, A., & Lopez-Real, F. (2002). Theorem justification and acquisition in dynamic geometry: A case of proof by contradiction. *International Journal of Computers for Mathematical Learning, 7*(2), 145-165.

Lester, F. K. (1980). A procedure for studying the cognitive processes used during problem solving. *Journal for Experimental Education, 48*, 323-327.

Maher, C. A., & Martino, A. M. (1996). The development of the idea of a mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27*, 29–63.

Maher, C. A., & Martino, A. M. (2000). From patterns to theories: Conditions for conceptual change. *The Journal of Mathematical Behavior, 19*, 247–271.

Maher, C. A., Powell, A. B., Weber, K., Lee, H. S. (2006). Tracing middle school students' arguments. In S. Alatorre, J. L. Cortina, M. Sáiz, and A. Méndez (Eds.), *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 403-410). Mérida, México: Universidad Pedagógica Nacional.

Marks, H. (2000). Student Engagement in Instructional Activity: Patterns in the Elementary, Middle, and High School Years. *American Educational Research Journal*, *37*(1) 153-184.

Martin, T., Lukong, A., & Reaves, R. (2007). The Role of Manipulatives in Arithmetic and Geometry Tasks. *Journal of Education and Human Development, 1*(1)

Mariotti, M. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics, 44*(1), 25-53.

Mariotti, M. A. (2002a). Influence of technologies advances on students' maths learning. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of* *international research in mathematics education* (pp. 695–721). Mahwah, NJ: Lawrence Erlbaum Associates.

Mariotti, M. (2002b). Justifying and proving in the cabri environment. *International Journal of Computers for Mathematical Learning, 6*(3), 257-281.

Marrades, R., & Gutiérrez, Á. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics, 44*(1), 87-125.

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. San Francisco: Jossey-Bass.

Merriam, S. B. (1995). What can you tell from an n of 1?: Issues of validity and reliability in qualitative research. *PAACE Journal of Lifelong Learning, 4*, 51-60.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). (1991). *Professional Standards for Teaching Mathematics*. Reston, VA: Author.

National Research Council (NRC). (1996). *National Science Education Standards*. Washington D.C.: National Academy Press.

North Carolina Department of Public Instruction (NCDPI) (2007). *Mathematics EOG achievement level descriptors—Grade 8.* Rerieved from http://www.ncpublicschools.org/docs/accountability/testing/eog/eogmathgr6to8ald.pdf

Olivero, F., & Robutti, O. (2007). Measuring in dynamic geometry environments as a tool for conjecturing and proving. *International Journal of Computers for Mathematical Learning, 12*(2), 135-156.

Öner, D. (2008). Supporting students' participation in authentic proof activities in computer supported collaborative learning (CSCL) environments. *International Journal of Computer-Supported Collaborative Learning, 3*(3), 343-359.

Parzysz, B. (1988). "Knowing" vs "seeing". problems of the plane representation of space geometry figures. *Educational Studies in Mathematics, 19*(1), 79-92.

Patton, M. (1990). *Qualitative evaluation methods (2nd ed.)*. Thousand Oaks, CA: Sage.

Piaget, J. (1952). *The Child's Conception of Number*. New York: Humanities Press.

Rodd, M. M. (2000). On mathematical warrants: Proof does not always warrant, and a warrant may be other than a proof. *Mathematical Thinking and Learning, 2*(3), 221 - 244.

Simon, M. A. (1995). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Sinclair, M. P. (2004). Working with accurate representations: The case of preconstructed dynamic geometry sketches. *Journal of Computers in Mathematics and Science Teaching, 23*(2), 191-208.

Sinclair, M. P. (2003). Some implications of the results of a case study for the design of pre-constructed, dynamic geometry sketches and accompanying materials. *Educational Studies in Mathematics 52*(3), 289–317.

Sowell, E.J. (1989). Effects of manipulative materials in mathematics instruction. *Journal* *for Research in Mathematics Education 20*(5), 498–505.

Steffe, L., & Thompson (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 267-306). Mahwah, NJ: Lawrence Erlbaum Associates.

Stephan, M. & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *Journal of Mathematical Behavior, 21*, 459-490.

Toulmin, S. (1958). *The uses of argument*. New York: Cambridge University Press.

Venema, G. (2002). *Foundations of Geometry.* Upper Saddle River, NJ: Pearson Prentice Hall.

Vergnaud, G. (1981). A classification of cognitive tasks and operations of thought involved in addition and subtraction problemsrs. In T. P.Carpenter, J. M.Moser and T.Romberg (Eds.). *Addition and Subtraction: Developmental Perspective* (pp.). Hillsdale, NJ: Lawrence Erlbaum Associates.

Vérillon, P., & Rabardel, P. (1995). Cognition and artefacts: A contribution to the study of though [sic] in relation to instrumented activity. *European Journal of Psychology of Education, 10*, 77–103.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

Wares, A. (2007). Using dynamic geometry to stimulate students to provide proofs. *International Journal of Mathematical Education in Science and Technology, 38*(5), 599-608.

Wyatt, Karen W., Lawrence, Ann and Foletta, Gina M. (1998). *Geometric activities for middle school students: with the Geometer's Sketchpad*. Emeryville, California: Key Curriculum Press.

Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *The Journal of Mathematical Behavior, 21*(4), 423-440.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 258-277.

Yackel, E., & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics* (pp. 227-236). Reston, VA: NCTM.

Yackel, E. & Rasmussen, C. (2002). Beliefs and norms in the mathematics classroom. In G. Toerner, E. Pehkonen, & G. Leder (Eds.). *Mathematical beliefs and implications for teaching and learning* (pp. 313–330). Dordrecht, The Netherlands: Kluwer.

APPENDICES

APPENDIX A

North Carolina State University is a land-grant university and a constituent institution of The University of North Carolina

Office of Research and Graduate Studies

NC STATE UNIVERSITY

Sponsored Programs and
Regulatory Compliance
Campus Box 7514
2701 Sullivan Drive
Raleigh, NC 27695-7514

919.515.2444
919.515.7721 (fax)

From: Joseph Rabiega, IRB Coordinator
North Carolina State University
Institutional Review Board

Date: November 11, 2008

Project Title: A comparison of middle school students' mathematical arguments in technological and traditional environments

IRB#: 630-08-11

Dear Ryan:

The project listed above has been reviewed by the NC State Institutional Review Board for the Use of Human Subjects in Research, and is approved for one year. **This protocol expires on November 11, 2009, and will need continuing review before that date.**

NOTE:

1. You must use the attached consent form which has the approval and expiration dates of your study.

2. This board complies with requirements found in Title 45 part 46 of The Code of Federal Regulations. For NCSU the Assurance Number is: FWA00003429.

3. Any changes to the attached protocol and supporting documents must be submitted and approved by the IRB prior to implementation.

4. If any unanticipated problems occur, they must be reported to the IRB office within 5 business days.

5. Your approval for this study lasts for one year from the review date. If your study extends beyond that time, including data analysis, you must obtain continuing review from the IRB.

Please provide a copy of this letter to your faculty sponsor.

Sincerely,

Joseph Rabiega

Joseph Rabiega
NCSU IRB

APPENDIX B

Triangle Inequality

(Adapted from Geometry Activities for Middle School Students with the Geometer's Sketchpad, 1998)

Open the sketch Triangle_Inequality.gsp

Try to make a triangle using the lengths of sides *a, b,* and *c* in the table below. To adjust the length of *a, b,* or *c,* drag the right most endpoint of the parallel segments labeled "side a", "side b", or "side c". Then, swing the endpoints of the figure to see whether you can make a triangle. The endpoints must meet to form the vertices of the triangle. If a triangle is formed draw a picture of it in the space provided. If a triangle cannot be formed, write *impossible*.

#	**Length of Side *a***	**Length of Side *b***	**Length of Side *c***	**Triangle?**
1	2.0 cm	3.0 cm	4.0 cm	
2	6.0 cm	1.0 cm	4.0 cm	
3	3.5 cm	2.0 cm	6.0 cm	
4	3.0 cm	4.0 cm	4.0 cm	
5	5.0 cm	5.0 cm	6.0 cm	
6	2.0 cm	7.0 cm	4.0 cm	

Why was it impossible to construct a triangle with some of the given lengths?

Write a conjecture about the relationship among the lengths of the three sides of a triangle.

APPENDIX C

Warm-up – Triangle Inequality and Triangle Interior Angle Relationships

1. If the measures of two of the interior angles of a triangle are 40º and 20º, what is the measure of the third interior angle?

2. Given the measure of an angle of a triangle is 50º, what are some possible measures for the other two angles? How do you know?

3. If two of the side lengths of a triangle are 10 and 7, what are some of the possible lengths for the third side? What is the shortest length? What is the longest length?

4. If two of the side lengths of a triangle are 6 and 6, what are some possible lengths for the third side? What is the shortest length? What is the longest length?

APPENDIX D

Triangle Interior Angle Relationships 2

Open the file Triangle_Interior_Angle_2.gsp Notice that the file has five tabs, one for each type of triangle in the table below. Today, we will also investigate the relationship between the lengths of the sides of a triangle and the measure of the angle of the triangle. Yesterday, you learned how to measure angles. To measure a side length, select one of the segments of the triangle. Then, in the Measure Menu, choose Length. Drag a vertex of the triangle to make an acute triangle with the longest side . Record which angle is the largest, which side is the shortest, and which angle is the smallest. For the next one, the obtuse triangle, click on the obtuse tab at the bottom of your file and drag the triangle such that the smallest angle is . Continue and fill in the rest of table. Answer the questions that follow.

Triangle Type	Name of Longest Side	Name of Largest Angle	Name of Shortest Side	Name of Smallest Angle	Picture
Acute	$\overline{AB}$				
Obtuse				$\angle ACB$	
Isosceles			$\overline{BC}$		
Right		$\angle ABC$			
Equilateral					

What do you notice about the relationship between the longest side and the largest angle?

____

____

What do you notice about the relationship between the smallest side and the smallest angle?

____

Write a conjecture about these relationships.

Explain why you think this relationship is true.

APPENDIX E

Investigating Triangles

(Adapted from ShapeMakers, 1998)

1. Predict which measured Triangle Makers can make a triangle with *all of its sides equal*. Check answers with the measure Triangle Makers.

Measured Triangle Maker	Predict	Check	Explain why you think the Triangle Maker was or was not able to make a triangle with all equal sides.
Isosceles	Y N	Y N	
Equilateral	Y N	Y N	
Right	Y N	Y N	
Obtuse	Y N	Y N	

What do you notice about triangles that have all sides equal?

____

2. Predict which measured Triangle Makers can make a triangle with *at least two equal angles*. Check answers with the measure Triangle Makers.

Measured Triangle Maker	Predict	Check	Explain why you think the Triangle Maker was or was not able to make a triangle with at least two equal angles.
Isosceles	Y N	Y N	
Equilateral	Y N	Y N	
Right	Y N	Y N	
Obtuse	Y N	Y N	

What do you notice about triangles that have at least two equal angles?

____

3. Predict which measured Triangle Makers can make a triangle with *exactly two 45º angles.* Check answers with the measure Triangle Makers.

Measured Triangle Maker	Predict	Check	Explain why you think the Triangle Maker was or was not able to make a triangle with exactly two 45º angles.
Isosceles	Y N	Y N	
Equilateral	Y N	Y N	
Right	Y N	Y N	
Obtuse	Y N	Y N	

What do you notice about triangles that have exactly two 45º angles?

____

4. Predict which measured Triangle Makers can make a triangle with *at least two sides equal.* Check answers with the measure Triangle Makers.

Measured Triangle Maker	Predict	Check	Explain why you think the Triangle Maker was or was not able to make a triangle with at least two sides equal.
Isosceles	Y N	Y N	
Equilateral	Y N	Y N	
Right	Y N	Y N	
Obtuse	Y N	Y N	

What do you notice about triangles with at least two sides equal?

____

APPENDIX F

Triangle Inequality

Using the snap cubes, try to make a triangle using the lengths of sides *a, b,* and *c* in the table below. The endpoints must meet to form the vertices of the triangle. If a triangle is formed draw a picture of it in the space provided. If a triangle cannot be formed, write *impossible*.

#	**Length of Side *a***	**Length of Side *b***	**Length of Side *c***	**Triangle?**
1	2 cubes	3 cubes	4 cubes	
2	6 cubes	1 cubes	4 cubes	
3	3 cubes	2 cubes	6 cubes	
4	3 cubes	4 cubes	4 cubes	
5	5 cubes	5 cubes	6 cubes	
6	2 cubes	7 cubes	4 cubes	

Why was it impossible to construct a triangle with some of the given lengths?

____

____

Write a conjecture about the relationship among the lengths of the three sides of a triangle.

____

____

APPENDIX G

Triangle Inequality and Triangle Interior Angle Relationships HW

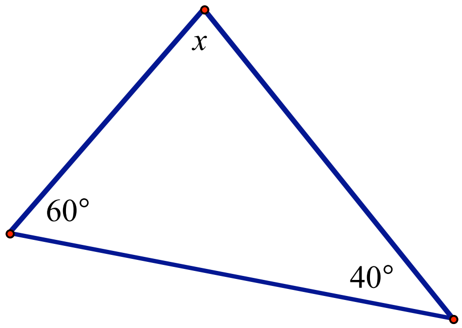
For each set of lengths, determine whether it is possible to draw a triangle with sides of the given measure. If possible, circle YES and explain how you arrived at this conclusion. If not possible, circle NO, explain why it is not possible.

1. 6 m, 10 m, 4 m YES / NO Explanation:	2. 6 ft, 2 ft, 5 ft YES / NO Explanation:
3. 0.5 cm, 1.2 cm, 0.6 cm YES / NO Explanation:	4. 3.5", 4.5", 7" YES / NO Explanation:
5. 7 in, 7, in, 2 in YES / NO Explanation:	6. 132 mm, 196 mm, 322 mm YES / NO Explanation:

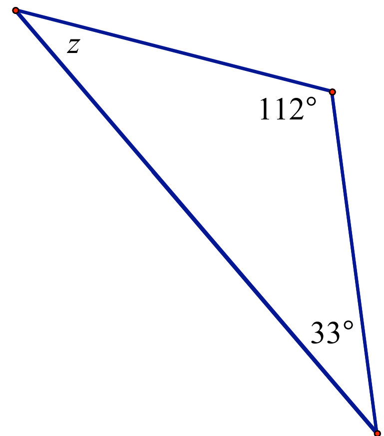
7. If you wanted to make a triangle and two of the sides had to be 4 in and 9 in, what are some of the possible measurements for the third side? What could be the smallest value? What could be largest value? Explain how you arrived at these answers.

Find the missing angle in the following triangles. Show your work.

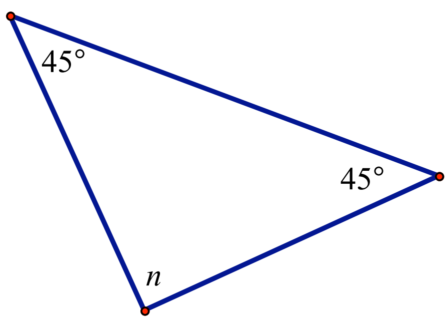
8. x = ______



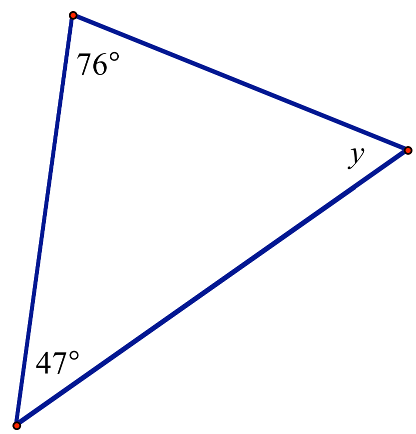
9. z = ______



10. n = ______



11. y = ______



12. Given what you know about the sum of the measures of the interior angles of triangle, what do you think is the sum of the measures of the interior angles of a quadrilateral (a four sided figure). Explain why you think this is the case?

APPENDIX H

Triangle Interior Angle Relationships 2

Using your ruler, measure the length (cm) of the sides for triangles 1, 3, 5, 9, and 13. Record the lengths of the sides on the triangles and fill in the table below

Triangle	Name of Longest Side	Name of Largest Angle	Name of Shortest Side	Name of Smallest Angle	Picture
1					
3					
5					
9					
13					

What do you notice about the relationship between the longest side and the largest angle?

____

____

What do you notice about the relationship between the smallest side and the smallest angle?

____

____

Write a conjecture about these relationships.

____

____

Explain why you think this relationship is true.

____

____

APPENDIX I

Measures of the angles of the example triangle used in the triangle sort task

Triangle #	Angle	Angle	Angle
1	$m\measuredangle A = 30°$	$m\measuredangle B = 120°$	$m\measuredangle C = 30°$
2	$m\measuredangle D = 45°$	$m\measuredangle E = 45°$	$m\measuredangle F = 90°$
3	$m\measuredangle G = 40°$	$m\measuredangle H = 70°$	$m\measuredangle I = 70°$
4	$m\measuredangle J = 90°$	$m\measuredangle K = 60°$	$m\measuredangle L = 30°$
5	$m\measuredangle M = 60°$	$m\measuredangle N = 60°$	$m\measuredangle O = 60°$
6	$m\measuredangle Q = 60°$	$m\measuredangle P = 60°$	$m\measuredangle R = 60°$
7	$m\measuredangle S = 80°$	$m\measuredangle T = 50°$	$m\measuredangle U = 50°$
8	$m\measuredangle V = 50°$	$m\measuredangle W = 50°$	$m\measuredangle X = 80°$
9	$m\measuredangle Y = 45°$	$m\measuredangle Z = 50°$	$m\measuredangle A = 85°$
10	$m\measuredangle B = 100°$	$m\measuredangle E = 30°$	$m\measuredangle H = 50°$
11	$m\measuredangle C = 40°$	$m\measuredangle F = 40°$	$m\measuredangle J = 100°$
12	$m\measuredangle D = 20°$	$m\measuredangle G = 10°$	$m\measuredangle K = 150°$
13	$m\measuredangle E = 90°$	$m\measuredangle L = 75°$	$m\measuredangle P = 15°$
14	$m\measuredangle M = 25°$	$m\measuredangle R = 135°$	$m\measuredangle V = 20°$
15	$m\measuredangle N = 65°$	$m\measuredangle Q = 55°$	$m\measuredangle W = 60°$

Measures of the lengths of the sides of the example triangles used in the triangle sort task

Triangle #	Side	Side	Side
1	$AC = 16.0$ cm	$AB = 9.3$ cm	$BC = 9.3$ cm
2	$DE = 10.8$ cm	$EF = 7.7$ cm	$DF = 7.7$ cm
3	$IH = 6.5$ cm	$IG = 19.3$ cm	$GH = 9.3$ cm
4	$KL = 14.0$ cm	$KJ = 7.0$ cm	$JL = 12.3$ cm
5	$MN = 7.7$ cm	$NO = 7.7$ cm	$MO = 7.7$ cm
6	$QP = 12.9$ cm	$QR = 12.9$ cm	$PR = 12.9$ cm
7	$ST = 8.7$ cm	$SU = 8.7$ cm	$UT = 11.3$ cm
8	$WV = 14.0$ cm	$WX = 10.8$ cm	$XV = 10.8$ cm
9	$AZ = 8.2$ cm	$YZ = 11.5$ cm	$AY = 8.8$ cm
10	$BH = 6.5$ cm	$BE = 9.8$ cm	$EH = 12.7$ cm
11	$CF = 13.1$ cm	$JC = 8.8$ cm	$FJ = 8.8$ cm
12	$KD = 5.7$ cm	$KG = 11.0$ cm	$DG = 16.2$ cm
13	$EP = 15.9$ cm	$EL = 4.1$ cm	$LP = 16.4$ cm
14	$MR = 8.0$ cm	$RV = 10.0$ cm	$VM = 16.7$ cm
15	$NQ = 9.7$ cm	$WQ = 10.1$ cm	$NW = 9.1$ cm

APPENDIX J

Triangle Sort

Sort the 15 triangles into 2 or more groups. In the table below, write the characteristics of the group and the triangles that belong to the group. Triangles can only belong to one group.

	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
Description						
Triangle #						

What characteristics of the triangles did you focus on when you sorted the triangles?

____

____

Combine the triangles again and this time, sort them based on their measure of the interior angles. If you previously sorted them this way, try making a different number of groups.

	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
Description						
Triangle #						

How did you select the number of groups? Which triangle(s) were the most difficult to sort?

____

____

Combine the triangles again and this time, sort them based on the length of the sides. If you previously sorted them this way, try making a different number of groups.

	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
Description						
Triangle #						

How did you select the number of groups? Which triangle(s) were the most difficult to sort?

____

____

Write the name the triangles in the table and on the back of the triangle.

Triangle	Name	Triangle	Name	Triangle	Name
1		6		11	
2		7		12	
3		8		13	
4		9		14	
5		10		15	