ABSTRACT

THERKELSEN, RYAN KENNETH. The Conjugacy Poset of a Reductive Monoid. (Under the direction of Mohan S. Putcha).

The Bruhat-Renner decomposition of a reductive monoid tells us that a reductive monoid may be partitioned into a union of double cosets with respect to a Borel subgroup, indexed by elements from the Renner monoid. A more recent object of study is the conjugacy decomposition of a reductive monoid. Here the disjoint subsets in the decomposition are unions of conjugates of the double cosets by elements of the unit group. The conjugacy poset of a reductive monoid is the indexing set for this decomposition, with a partial order we call the conjugacy order.

The purpose of the research presented in this paper is to provide a better understanding of this poset. In our attempt to do so, we follow an approach analogous to that for the study of the Renner monoid under the Bruhat-Chevalley order. We begin by studying the order within classes indexed by certain idempotent elements in the monoid. New results for the general case are presented before providing a more thorough analysis for three well-studied types of reductive monoids: the set of $n \times n$ matrices and the so-called canonical and dual canonical monoids. We then examine the order between classes for these three types, finding order-preserving maps that are generalizations of the maps between classes in the Renner monoid that have proven quite useful in the study of the Bruhat-Chevalley order.

The Conjugacy Poset of a Reductive Monoid

by
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DEDICATION

For K. Larry Nelson,

March 22, 1950 – December 13, 2007

"The clergy know that I know that they know that they do not know."

-Robert G. Ingersoll

BIOGRAPHY

Ryan Therkelsen was born in Iowa, in the late 1970s. He remained in that state for approximately two decades and still considers it home, though it is unlikely that he will ever live there again. Ryan has attended many schools, receiving degrees from the University of Iowa in 2001 (B.A., Mathematics) and San Diego State University in 2004 (M.A., Mathematics). In whatever spare time he finds, he enjoys watching and playing soccer, traveling, attending music concerts, and trying to figure it all out.

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Chapter 1

Introduction

The study of linear algebraic monoids began about 30 years ago, independently, by Mohan Putcha and Lex Renner. The theory is a blend of topics from semigroups, algebraic geometry, and algebraic groups. The linear algebraic monoids we consider in this dissertation are reductive monoids. Reductive monoids are to the theory of linear algebraic monoids what reductive groups are to linear algebraic groups. Namely, they are the most important and well-studied objects, with the nicest structure. Like reductive groups, reductive monoids are important in several areas, such as representation theory and embedding theory, though they also have a beautiful structure that makes them worthwhile objects of study on their own.

The first major result on reductive monoids concerned their structure: reductive monoids are regular. A property of semigroups, regular here means that the monoid is determined by its unit group and set of idempotents. With this result, the theory really began to develop. We will summarize several of the bigger results relating to the structure theory in greater detail in an upcoming chapter. For now, we provide just enough background information to motivate the dissertation topic.

Motivation

It is well known that for a reductive group, the Weyl group plays a decisive role in the structure theory. For reductive monoids, the analogous object is the Renner

monoid. This monoid decomposes nicely into classes defined in terms of its unit group and a certain subset of its idempotents. In semigroup jargon, these are called the $\mathcal{J}$-classes of the Renner monoid (from Green's relations, in semigroup theory). In more algebraic terms, these are the double cosets of the idempotent set by the unit group of the Renner monoid. In [23], the structure of these classes is studied in detail. In [24], the order between these classes is examined. In particular, an order-preserving map is defined giving a precise description of the order between elements of two classes.

While the ideas above were being developed, the study of conjugacy classes in reductive monoids was proceeding, having been inititated by Putcha in [17]. Theorem 4.1 of [17] shows there is an analogue of the Fitting decomposition for reductive monoids. After the proof of the theorem, we find the following remark:

> ...thus the conjugacy problem in M reduces to the following three problems:
>
> (A) Conjugacy problem for idempotents.
> (B) Conjugacy problem within a reductive group.
> (C) Conjugacy problem for nilpotent elements.
>
> Much is known about (A) and (B). So we are left with problem (C).

The "attack" on problem (C) began with [18], followed by [21]. These contributions laid the groundwork for a decomposition of a reductive monoid in terms of unions of certain conjugacy classes, called the conjugacy decomposition. In [22], this decomposition was formally defined and with it an associated partial order, the conjugacy order. This decomposition proved useful in the study of the variety consisting of nilpotent elements of a reductive monoid (that is, the so-called nilpotent variety). For example, it is used to obtain the irreducible components of the nilpotent variety for certain classes of reductive monoids in [22], [26].

Chapter Outline

Chapter 2 is a brief summary of the background material necessary for the study of reductive monoids. In addition to the basics from semigroups, algebraic geometry,

and algebraic group theory, we cover the necessities regarding posets and Coxeter groups, as well as an introduction to linear algebraic monoids. Chapter 3 continues the background, with a more detailed description of the theory of reductive monoids. In the final section of the chapter, we introduce a new description of the maps from [24] for matrices.

The focus of this dissertation is on the poset associated with the conjugacy decomposition of a reductive monoid. Chapter 4 provides an introduction to this topic. We summarize the theory up to this point and provide a few new general results. The two chapters that follow contain the bulk of the new results. The classes in the decomposition from Chapter 4 are indexed by certain idempotent elements of the reductive monoid. In Chapter 5, we examine the order within these classes, beginning with a result for general reductive monoids and following with a more complete description for three types of reductive monoids. Chapter 6 follows a similar approach to Chapter 5, though we now examine the order between the classes. Our primary results occur here, where for three types of reductive monoids we provide a definitive answer to a conjecture from [25], regarding the generalization of the maps from [24]. The final chapter is Chapter 7, in which we summarize the results obtained and address the next steps to undertake in the theory.

Chapter 2

Preliminaries

This chapter outlines the basic background material necessary in the study of reductive monoids. We begin with a review of posets. The material presented here can be found in [33]. The general theory of linear algebraic monoids is a blend of ideas from semigroup theory, algebraic geometry, and algebraic group theory. These topics are the next to be reviewed. For semigroups, we rely heavily on [8] and [12]. For algebraic geometry, we use a number of sources, including [9], [11], and [29]. For algebraic groups, we use [3], [6], [9], and [32]. Coxeter groups play an important role in the theory of reductive monoids, and so we briefly review ideas central to that theory, drawing largely from [1] and [10]. Finally, we close with a brief introduction to linear algebraic monoids, using [19] and [32], in preparation for a more thorough description of reductive monoids in the following chapter.

2.1 Posets

A *partially ordered set* (or *poset*) is a pair $(P, \leq)$ where P is a set and $\leq$ is a *partial order*. That is, $\leq$ satisfies the following conditions:

1. For all $x \in P$, $x \leq x$. (Reflexive)

2. For all $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$. (Antisymmetric)

3. For all $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$. (Transitive)

Usually, we will refer to $(P, \leq)$ simply as P, as in most cases the partial order $\leq$ is understood. If ambiguities arise, we will denote the partial order on P by $\leq_P$. Additionally, by $x \geq y$ we mean $y \leq x$, by $x < y$ we mean $x \leq y$ and $x \neq y$, and so on. The elements x and y in P are said to be *comparable* if either $x \leq y$ or $y \leq x$, otherwise they are called *incomparable*. If every pair of elements in P are comparable, we say $\leq$ is a *total order* and $(P, \leq)$ is a *totally ordered set*, or *chain*. Throughout this dissertation, by "order" we will mean partial order (not total order).

If $x \in P$ is such that $y \in P$ with $y \leq x$ implies $y = x$, then x is called a *minimal element* of P. Reversing the inequality gives the definition of *maximal element*. An element $x \in P$ is a *minimum* if $x \leq y$ for all $y \in P$. Likewise, x is a *maximum* if $y \leq x$ for all $y \in P$. We denote the minimum and maximum elements of a poset by $\hat{0}$ and $\hat{1}$, respectively, or just 0 and 1 if the context is clear.

Example 2.1.1. *The natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ form a partially ordered set, under the usual order for integers. $\mathbb{N}$ is a chain with minimum element 1 and no maximal elements.*

A *partition* of $n \in \mathbb{N}$ is a sequence $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_{\geq 0}^k$ such that $\sum \alpha_i = n$ and $\alpha_1 \geq \ldots \geq \alpha_k$. We consider two partitions to be the same if they differ only in the number of terminal 0's. If α is a partition of n we write $\alpha \vdash n$ and we denote the set of all partitions of n by $\mathrm{Par}(n)$.

Example 2.1.2. *Given $\alpha, \beta \in Par(n)$, define $\alpha \leq \beta$ if either $\alpha = \beta$ or for the smallest i such that $\alpha_i \neq \beta_i$, $\alpha_i < \beta_i$. This order is called the lexicographic order. $Par(n)$ with this partial order is a chain for all n.*

Example 2.1.3. *Given $\alpha, \beta \in Par(n)$, define $\alpha \trianglelefteq \beta$ if $\alpha_1 + \cdots + \alpha_i \leq \beta_1 + \cdots + \beta_i$ for $1 \leq i \leq n$. This order is called the dominance order. $Par(n)$ with this partial order is a chain if and only if $n \leq 5$. A generalization of this order will appear later in the examination of the conjugacy poset of $M_n(k)$.*

A map $\varphi : P \to Q$ of posets is *order-preserving* if $x \leq_P y$ implies $\varphi(x) \leq_Q \varphi(y)$ and *order-reversing* if $x \leq_P y$ implies $\varphi(y) \leq_Q \varphi(x)$. An *isomorphism* of posets is an order-preserving bijection whose inverse is also order-preserving. A bijection $\varphi : P \to P$ is an *automorphism* if φ and φ^{-1} are order-preserving and an *antiautomorphism* if φ and φ^{-1} are order-reversing. Given a poset P, let P^* be the poset such that $P^* = P$ as sets and $x \leq y$ in P^* if and only if $y \leq x$ in P. We then call P^* the *dual* of P. If P and P^* are isomorphic, P is called *self-dual*.

An *induced subposet* of a poset P is a subset Q of P with the ordering on Q such that $x \leq y$ in Q if and only if $x \leq y$ in P. The order on Q is called the *induced order*. There is another notion of subposet, called a *weak subposet*. In this dissertation, by "subposet" we will mean induced subposet. For $x, y \in P$ with $x \leq y$, we have an important type of subposet, the *closed interval* $[x, y] = \{z \in P \mid x \leq z \leq y\}$. For $x, y \in P$, we say that y *covers* x if $x < y$ and $[x, y] = \{x, y\}$, and write $x \lessdot y$.

The *length* of a chain C is defined to be $\ell(C) = |C| - 1$. For a (finite) poset P, we define the length to be $\ell(P) = \max\{\ell(C) \mid C \text{ is a chain}\}$. The length of a finite interval is denoted $\ell(x, y)$. If every maximal chain in P has the same length, then we say P is *graded*. If P is graded, then there exists a unique rank function $\rho : P \to \mathbb{Z}$ such that $\rho(x) = 0$ if x is a minimal element of P and $\rho(y) = \rho(x) + 1$ if y covers x. In a graded poset, $\ell(x, y) = \rho(y) - \rho(x)$.

Example 2.1.4. *Let X be a finite set, say $|X| = n$, and consider its power set $2^X = \{Y \mid Y \subseteq X\}$. 2^X is a poset, with order given by set inclusion. This poset has a minimum element, $\emptyset$, as well as a maximum element, X itself. Additionally, 2^X is graded, as the length of every maximal chain is n.*

It is often useful to represent a (finite) poset via a *Hasse diagram*. This is a graph whose vertices are the elements of the poset and whose edges are the cover relations, following the rule that if y covers x then y is drawn "above" x.

Example 2.1.5. *Let $P = \{1, A, B, AB, BA, ABA\}$ with the partial order $\leq$ given by the rule that $x \leq y$ if we may remove letters in y to obtain x, with the understanding*

that "no letters" is represented by 1*. The Hasse diagram for* $(P, \leq)$ *is given in Figure 2.1.*
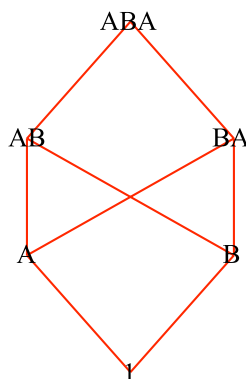
Figure 2.1: The Hasse diagram for $(P, \leq)$.

Let P be a poset and $x, y \in P$. If $z \in P$ is such that $x \leq z$ and $y \leq z$, then z is called an *upper bound* of x and y. If there is an upper bound z of x and y such that every other upper bound w of x and y satisfies $z \leq w$, z is called a *least upper bound*, or *join*, of x and y, denoted $x \vee y$. If the join of x and y exists, it is unique. Reversing the inequalities defines a *lower bound* and *greatest lower bound*, or *meet*, denoted $x \wedge y$. If the meet of x and y exists, it is unique. A *lattice* is a poset such that every pair of elements has a join and meet. All lattices we will consider will be finite lattices (that is, $|P| < \infty$). Note that all finite lattices have a $\hat{0}$ and $\hat{1}$.

Example 2.1.6. $(Par(n), \trianglelefteq)$, *from Example 2.1.3, is a self-dual lattice. It is graded if and only if $n \leq 6$. See [34] for details.*

Example 2.1.7. *The poset from Example 2.1.4 is a lattice. For $U, V \in 2^X$, $U \vee V = U \cup V$ and $U \wedge V = U \cap V$. Since AB and BA both cover A and B and are incomparable, the poset from Example 2.1.5 is not a lattice.*

2.2 Semigroups

A *semigroup* is a pair $(S, \cdot)$, consisting of a non-empty set S and an associative operation, $\cdot$. Unless otherwise noted, we will write $a \cdot b$ as ab and $(S, \cdot)$ as S, since it is usually clear how the operation is defined. If there exists $1 \in S$ such that $1x = x1 = x$ for all $x \in S$, then 1 is called an *identity element* of S and S is called a *monoid*. If S has no identity element, then we may adjoin an element 1 to S, defining $1s = s1 = s$ for all $s \in S$ and $11 = 1$. Now $S \cup \{1\}$, usually denoted S^1, is a monoid. If a semigroup S with $|S| > 1$ has an element 0 such that $0x = x0 = 0$ for all $x \in S$, then 0 is called a *zero element* of S and S is called a *semigroup with* 0. As with identity elements, if S has no zero element, we can adjoin a zero to S to make it a semigroup with 0, with multiplication defined as $0s = s0 = 0$ for all $s \in S$ and $00 = 0$. Identity elements and zero elements are unique and, since $|S| > 1$ for S to have a zero element, a zero element is not an identity element.

Example 2.2.1. $S = \{\ldots, -2, 0, 2, 4, \ldots\}$, *under multiplication, is a semigroup. It has a zero element but no identity element.* $S \cup \{1\}$, *where* $1 \in \mathbb{Z}$, *is a monoid.*

If A and B are non-empty subsets of a semigroup S, then by AB we mean $\{ab \mid a \in A, b \in B\}$. If $A = \{a\}$, we'll write aB instead of $\{a\}B$ (and likewise for $B = \{b\}$). If T is a non-empty subset of a semigroup S such that for all $x, y \in T$ we have $xy \in T$, then T is called a *subsemigroup*. Equivalently, T is a subsemigroup if $T^2 \subseteq T$. Note that a subsemigroup T is itself a semigroup. If $e \in S$ is such that $e^2 = e$, then e is called an *idempotent*. Let $E(S)$ denote the set of all idempotent elements of S and note that $E(S)$ is a subsemigroup of S. There is a natural partial order on $E(S)$, defined for $e, f \in E(S)$ as:

$$e \leq f \iff ef = e = fe \tag{2.1}$$

If A is a non-empty subset of S, we call A a *left ideal* if $SA \subseteq A$, a *right ideal* if $AS \subseteq A$, and a *two-sided ideal* (or just *ideal*) if A is both a left and right ideal. If A is an ideal such that $\{0\} \subset A \subset S$, it is called a *proper* ideal.

Let S be a semigroup with $x, y \in S$. The study of ideals in S leads naturally to the following equivalence relations, called *Green's relations*:

1. $x\mathcal{L}y$ if $S^1 x = S^1 y$.

2. $x\mathcal{R}y$ if $xS^1 = yS^1$.

3. $x\mathcal{J}y$ if $S^1 x S^1 = S^1 y S^1$.

4. $x\mathcal{H}y$ if $x\mathcal{R}y$ and $x\mathcal{L}y$.

5. $x\mathcal{D}y$ if $x\mathcal{R}z$ and $z\mathcal{L}y$, for some $z \in S^1$.

We denote the $\mathcal{L}$-class of x by L_x, and likewise for $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$, and $\mathcal{D}$. Additionally, we note that there is a partial order on the set of $\mathcal{J}$-classes of S^1, defined by

$$\mathcal{J}_a \le \mathcal{J}_b \iff a \in S^1 b S^1. \tag{2.2}$$

Example 2.2.2. *Let $M = M_n(k)$. Then $x\mathcal{L}y$ if and only if x and y are row equivalent, $x\mathcal{R}y$ if and only if x and y are column equivalent, and $x\mathcal{J}y$ if and only if $\mathrm{rk}(x) = \mathrm{rk}(y)$.*

Green's relations were first studied by J. A. Green in [7] and are of great importance in the theory of semigroups. In the structure theory of reductive monoids, presented in the next chaper, the $\mathcal{J}$-classes will play a crucial role.

Let S be a semigroup. An element $s \in S$ is said to have an *inverse* if there exists an $x \in S$ such that $s = sxs$ and $x = xsx$. In contrast to the group setting, inverses need not be unique. However, if every element $s \in S$ has a unique inverse, denoted s^{-1}, then S is called an *inverse semigroup*.

Example 2.2.3. *Let $N \subseteq M_n(k)$ be the set of $n \times n$ matrices over k having at most one non-zero entry in each row or column. N is an inverse semigroup, under multiplication.*

2.3   Algebraic Geometry

For this paper, we will always assume that the fields we consider are algebraically closed and that all rings are commutative with identity.

A ring R satisfies the *ascending chain condition* (ACC) on ideals if for every chain $I_1 \subseteq I_2 \subseteq \cdots$ of ideals there is an n such that $I_j = I_n$ for all $j \geq n$. Such a ring is called *noetherian*. A proof for the following can be found in [11].

Proposition 2.3.1. *The following are equivalent.*

1. *R is noetherian.*

2. *For every ideal I of R, I and R/I are noetherian.*

3. *Every ideal of R is finitely generated.*

4. *Every non-empty collection of ideals has a maximal element (with respect to set inclusion).*

The following important result is also not proved here, though a proof can be found in any book on algebraic geometry.

Theorem 2.3.2 (Hilbert Basis Theorem)**.** *If R is a commutative noetherian ring with identity, then so is $R[x_1, \ldots, x_n]$.*

Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Since k is noetherian, $k[x_1, \ldots, x_n]$ is noetherian and hence I is finitely generated, say $I = (f_1, \ldots, f_m)$. Define the *zero set* of I, denoted $\mathcal{V}(I)$, as:

$$\mathcal{V}(I) = \{a = (a_1, \ldots, a_n) \in k^n \mid f(a) = 0 \ , \ \forall f \in I\}.$$

Since I is finitely generated, $\mathcal{V}(I)$ is therefore the set of points $a \in k^n$ such that $f_1(a) = f_2(a) = \cdots = f_m(a) = 0$.

An *affine variety* is a set $V \subseteq k^n$ that is the zero set of an ideal of $k[x_1, \ldots, x_n]$. Let X and Y be affine varieties and $k[X] = k[x_1, \ldots, x_n]/I$, where $I = \{f \in k[x_1, \ldots, x_n] \mid f(X) = 0\}$. A *morphism* is a mapping $\varphi : X \to Y$, with $\varphi(x) = \varphi(x_1, \ldots, x_n) = (\psi_1(x), \ldots, \psi_m(x))$, where each $\psi_i \in k[X]$.

Example 2.3.3. *Let $f \in k[x]$. Then either $f = 0$ or f has finitely many zeros. That is, either $f(x) = 0$ for all x or $f(x) = \prod_{i=1}^{n}(x - c_i)$ with $c_i \in k$. Thus k and all finite sets of k are affine varieties.*

Example 2.3.4. *Let $X = \{(x, y) \mid xy = 0\} \subseteq k^2$. X is the zero set of $f(x, y) = xy$ and so it is an affine variety.*

Example 2.3.5. *The special linear group $SL_n(k) = \{(a_{ij}) \in k^{n^2} \mid \det(a_{ij}) = 1\}$ is an affine variety, as we see by this definition that it is the set of zeros of a single polynomial equation.*

Example 2.3.6. *The general linear group $GL_n(k) = \{(a_{ij}) \in k^{n^2} \mid \det(a_{ij}) \neq 0\}$ is an affine variety. To see this, we observe that $GL_n(k) \cong \{(a_{ij}, b) \in k^{n^2+1} \mid b \det(a_{ij}) = 1\}$ and so it is (isomorphic to) the set of zeros of a single polynomial equation.*

Suppose that $X \in k^n$. Let $\mathcal{I}(X)$ be the set of all polynomials in $k[x_1, \ldots, x_n]$ having X as its zero set. Both $\mathcal{V}$ and $\mathcal{I}$ are reverse inclusion maps. That is, if $I \subseteq I'$ then $\mathcal{V}(I) \supseteq \mathcal{V}(I')$ and if $X \subseteq X'$ then $\mathcal{I}(X) \supseteq \mathcal{I}(X')$. Additionally, $\mathcal{I}(X)$ is an ideal of $k[x_1, \ldots, x_n]$ and so we have the following set inclusions:

$$
\begin{aligned}
X &\subseteq \mathcal{V}(\mathcal{I}(X)), \\
I &\subseteq \mathcal{I}(\mathcal{V}(I)).
\end{aligned}
$$

We have equality in the first case if and only if X is a variety. The conditions for equality in the second case are given by Hilbert's Nullstellensatz. Before stating this result, we first note that the *radical* of an ideal I in R, denoted $\sqrt{I}$, is the set

$$
\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N}\}.
$$

This brings us to the aforementioned result:

Theorem 2.3.7 (Hilbert's Nullstellensatz)**.** *If I is any ideal in $k[x_1, \ldots, x_n]$, then*

$$
\sqrt{I} = \mathcal{I}(\mathcal{V}(I)).
$$

I is called a *radical ideal* if $\sqrt{I} = I$. So, there is a one-to-one correspondence between the set of radical ideals of $k(x_1, \ldots, x_n)$ and the affine varieties in k^n.

The next step is to define a suitable topology on k^n. We do so by defining the *closed sets* to be exactly the affine varieties of k^n. It is a straight-forward matter to check that the axioms for a topology are satisfied, [9]. This is called the *Zariski topology*. In this topology, points are closed and every open cover contains a finite subcover. However, all open sets of k^n are dense, so any two non-empty open sets intersect and thus this space is not Hausdorff. A topological space is *noetherian* if it satisfies the descending chain condition on closed sets. While k^n is not Hausdorff, as noted, we do have the following:

Proposition 2.3.8. k^n, *with the Zariski topology, is a noetherian topological space.*

A topological space is *irreducible* if it is not the union of two proper, non-empty, closed sets. Hence a space X is irreducible if and only if any two non-empty open sets in X have non-empty intersection, or equivalently, if any non-empty open set is dense. A variety is therefore *irreducible* if it is non-empty and not the union of two proper subvarieties.

Example 2.3.9. $X = \{(x, y) \mid xy = 0\} \subseteq k^2$, *from Example 2.3.4 above, is not irreducible. Let* $X_1 = \{(x, y) \mid x = 0\} \subseteq k^2$ *and* $X_2 = \{(x, y) \mid y = 0\} \subseteq k^2$. *Both* X_1 *and* X_2 *are proper subvarieties of* X, *and* $X = X_1 \cup X_2$.

The following can be found in [9]:

Proposition 2.3.10. *A noetherian topological space* X *has only finitely many maximal irreducible subspaces. These subspaces are closed and their union is* X.

These maximal irreducible subspaces are called the *irreducible components* of X.

An ideal I of a ring R is called *prime* if $fg \in I$ implies $f \in I$ or $g \in I$. The following is shown in [9]:

Proposition 2.3.11. *A closed set* X *in* k^n *is irreducible if and only if its ideal* $\mathcal{I}(X)$ *is prime. In particular, this means* k^n *itself is irreducible.*

Example 2.3.12. *Let $I = (x^2) \subset k[x]$. $X = \mathcal{V}(I) = \{0\}$ and so X is clearly a closed set and since it consists of a single point it must be irreducible. Now, I is clearly not a prime ideal (as $x^2 = xx \in I$ though $x \notin I$), however $\mathcal{I}(X) = (x) \neq (x^2) = I$ and (x) is prime, so $\{0\}$ is indeed irreducible.*

Finally, we note that since k^n is irreducible any open set X in k^n is dense. That is, $\overline{X} = k^n$. This fact will be useful to keep in mind in the discussion of linear algebraic monoids a few sections ahead.

2.4   Algebraic Groups

Let k be an algebraically closed field. An *affine algebraic group* over k is a set G which is both an affine variety and a group such that the maps $\mu : G \times G \to G$, defined as $\mu(x, y) = xy$, and $\iota : G \to G$, defined as $\iota(x) = x^{-1}$, are morphisms of varieties. A map $\varphi : G_1 \to G_2$ is a *homomorphism* of affine algebraic groups if it is a morphism of varieties and a homomorphism of groups. φ is an *isomorphism* of affine algebraic groups if it is a bijection such that φ and φ^{-1} are homomorphisms of affine algebraic groups.

Example 2.4.1. *In the previous section, we observed that $SL_n(k)$ and $GL_n(k)$ are both affine varieties. The usual multiplication and inverse maps applied to elements of these sets are morphisms and so both are affine algebraic groups.*

A *linear algebraic group* is a closed subgroup of $GL_n(k)$, for some n. Every linear algebraic group is therefore an affine algebraic group. On the other hand, it turns out that every affine algebraic group is isomorphic to a linear algebraic group (for a particularly nice proof of this, see Section 2 in MacDonald's part of [4]). The terms *linear* and *affine* are therefore interchangeable.

A linear algebraic group G, considered as an affine variety, is a union of irreducible components. This union forms a nice structure, as we see in the following, from [4]:

Theorem 2.4.2. *Let G be a linear algebraic group. Then G has a unique irreducible component G° containing the identity element 1, and G° is a closed normal subgroup of*

finite index in G. The irreducible components of G are also the connected components of G, and are the cosets of G° in G.

Thus for linear algebraic groups, irreducible and connected components are identical (this is not the case for linear algebraic monoids, as we will see). If $G = G^\circ$, we say G is *connected* (instead of irreducible). Unless otherwise noted, we will always assume a linear algebraic group G is connected.

The subgroups of a linear algebraic group that we are interested in are closed subgroups. By this we mean that the subgroup is a closed set, with respect to the Zariski topology. Any closed subgroup of a linear algebraic group is itself a linear algebraic group.

Example 2.4.3. *The following examples are important subgroups of $GL_n(k)$. All are connected.*

1. *The group of diagonal matrices:*

$$D_n(k) = \{(x_{ij}) \in GL_n(k) \mid x_{ij} = 0 \text{ if } i \neq j\}.$$

2. *The group of upper triangular matrices:*

$$B_n(k) = \{(x_{ij}) \in GL_n(k) \mid x_{ij} = 0 \text{ if } i > j\}.$$

3. *The group of upper unipotent matrices:*

$$U_n(k) = \{(x_{ij}) \in GL_n(k) \mid x_{ij} = 0 \text{ if } i > j; x_{ii} = 1\}.$$

Example 2.4.4. *The classical matrix groups are linear algebraic groups. In addition to $GL_n(k)$ and $SL_n(k)$, these include:*

1. *The* symplectic group*:*

$$Sp_{2n}(k) = \{x \in GL_{2n}(k) \mid x^T J x = J\},$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ with I_n the $n \times n$ identity matrix.

2. *The* special orthogonal group, *for char $k \neq 2$:*

$$SO_{2n+1}(k) = \{x \in SL_{2n+1}(k) \mid x^T s x = s\},$$

where $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & J \\ 0 & J & 0 \end{pmatrix}$.

3. *The* special orthogonal group, *for char $k \neq 2$:*

$$SO_{2n}(k) = \{x \in SL_{2n}(k) \mid x^T s x = s\},$$

where now $s = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$.

All examples here are connected, though this fact is not immediate (see [9]).

Let G be a connected linear algebraic group. G contains a unique maximal connected solvable normal subgroup $R(G)$, called the *radical* of G. An element x of G is called *unipotent* if the only eigenvalue of x is 1. The set of all unipotent elements of $R(G)$ is a subgroup of G. This subgroup, denoted $R_u(G)$, is the unique maximal connected normal unipotent subgroup of G, and is called the *unipotent radical* of G. If $G \neq \{1\}$, then G is called *semisimple* if $R(G) = \{1\}$ and *reductive* if $R_u(G) = \{1\}$. Hence any semisimple group is reductive, though not conversely.

Example 2.4.5. *Both $GL_n(k)$ and $SL_n(k)$ are reductive groups.*

Example 2.4.6. *The set of invertible $n \times n$ upper triangular matrices, $B_n(k)$ from Example 2.4.3, is not a reductive group for $n \geq 2$. The unipotent radical is the set of $n \times n$ upper unipotent matrices, $U_n(k)$ from Example 2.4.3, which contains more than one element for $n \geq 2$.*

An algebraic group isomorphic to $k^* \times \cdots \times k^*$ is called a *torus*. As a linear group, this means it is isomorphic to a subgroup of $D_n(k)$, for some n. A *maximal torus* is a torus not properly contained in a larger torus. A *Borel subgroup* is a maximal connected solvable subgroup of G (such a subgroup is closed). As a linear group, this

means it is isomorphic to a subgroup of $B_n(k)$, for some n. The following results are proven in [9]:

Theorem 2.4.7. *Let B be any Borel subgroup of G. Then all other Borel subgroups are conjugate to B.*

Corollary 2.4.8. *The maximal tori of G are those of the Borel subgroups of G, and are all conjugate.*

If B and B^- are Borel subgroups of G such that $B \cap B^- = T$ is a torus, then B^- is called the *opposite Borel subgroup of B relative to T*. The following is from [9]:

Proposition 2.4.9. *Let G be a reductive group and T a torus in G. Then every Borel subgroup containing T has a unique opposite Borel subgroup, relative to T.*

Example 2.4.10. *Let $G = GL_n(k)$. As noted in Example 2.4.5, G is reductive. $B = B_n(k)$ is a Borel subgroup of G containing maximal torus $T = D_n(k)$. For this choice of B and T, the opposite Borel subgroup B^- is the set of lower triangular matrices: $B_n^-(k) = \{(x_{ij}) \in GL_n(k) \mid x_{ij} = 0 \text{ if } i < j\}$. Clearly, the intersection of $B_n(k)$ and $B_n^-(k)$ is $D_n(k)$.*

Given a group G with $X \subseteq G$, we next recall two important subgroups of G:

$$
\begin{aligned}
N_G(X) &= \{g \in G \mid g^{-1}Xg = X\}, \\
C_G(X) &= \{g \in G \mid gx = xg \text{ for all } x \in X\}.
\end{aligned}
$$

$N_G(X)$ is called the *normalizer* of X in G and $C_G(X)$ is called the *centralizer* of X in G. The *center* of G is $C_G(G)$, denoted $C(G)$. If $x, y \in G$, then x is *conjugate* to y, denoted $x \sim y$, if $y = x^g = g^{-1}xg$ for some $g \in G$. If $X, Y \subseteq G$, then X is *conjugate* to Y, denoted $X \sim Y$, if every element of X is conjugate to an element of Y, and vice versa.

To close this section, we describe the notion of a group with a BN-pair and state important properties of such groups.

Let G be a group. G is called a *group with a BN-pair* if there are subgroups $B, N \subseteq G$ such that the following conditions hold:

(BN1) G is generated by B and N.

(BN2) $T = B \cap N$ is normal in N, and the quotient group $W = N/T$ is a finite group generated by a set S of elements of order 2.

(BN3) $n_s B n_s \neq B$ if $s \in S$ and n_s is a representative of s in N.

(BN4) $n_s B n \subseteq B n_s n B \cup B n B$ for any $s \in S$ and $n \in N$.

(BN5) $\bigcap\limits_{n \in N} n B n^{-1} = T$.

The description given here is that of [6]. The condition (BN5) is sometimes omitted from the definition of a BN-pair (for example, it is not included in [9]).

W is called the *Weyl group* of G. We will have more to say regarding these groups in the subsequent section. The important fact for us to note is that every reductive group G has a BN-pair, where B is a Borel subgroup of G with $N = N_G(T)$ for T a maximal torus contained in B.

Example 2.4.11. *Let $G = GL_n(k)$, with $B = B_n(k)$ and $T = D_n(k)$. The normalizer of T in G is the set of monomial matrices in G (that is, the set of invertible matrices having exactly one nonzero entry in each row and column). The Weyl group $W = N_G(T)/T$ is isomorphic to the symmetric group on n elements, S_n.*

Any reductive group G, as a group with a BN-pair, can be written as $G = BNB$. One of the most important properties of a reductive group is that it can be decomposed into a disjoint union of double cosets, indexed by elements from the Weyl group $W = N/T$. This is called the *Bruhat decomposition*. Formally, we have:

Theorem 2.4.12. *Let G be a reductive group, with Borel subgroup B containing a maximal torus T and Weyl group $W = N_G(T)/T$. Then*

$$G = \bigsqcup_{w \in W} BwB \tag{2.3}$$

with $BwB = Bw'B$ if and only if $w = w'$ in W.

There is a partial order on W, the *Bruhat-Chevalley order*, described in terms of these $B \times B$-orbits as follows:

$$x \leq y \iff BxB \subseteq \overline{ByB} \qquad (2.4)$$

where $x, y \in W$, and closure is with respect to the Zariski topology. We present an alternate description, of a more combinatorial flavor, in the subsequent section.

Let G be a group with BN-pair with S the generating set of W. Let W_I denote the subgroup of W generated by $I \subseteq S$ and let $P_I = BW_I B$ (so $P_\emptyset = B$ and $P_S = G$). A subgroup of G is called *parabolic* if it contains a Borel subgroup. The following is from [9]:

Theorem 2.4.13. *1. The only subgroups of G containing B are those of the form P_I, with $I \subseteq S$.*

2. If P_I is conjugate to P_J, then $P_I = P_J$.

3. $N_G(P_I) = P_I$.

4. If $P_I \subseteq P_J$, then $I \subseteq J$.

According to the theorem, every parabolic subgroup of G is conjugate to P_I for some $I \subseteq S$. Additionally, given a Borel subgroup B of G, the subgroups of G containing B form a lattice, isomorphic to 2^S (ordered by set inclusion).

2.5 Coxeter Groups

Let S be a set and W the group generated by S, subject only to relations of the form $(ss')^{m(s,s')} = 1$, for all $s, s' \in S$, where $m(s,s) = 1$ and $m(s,s') = m(s',s) \geq 2$ for $s \neq s'$. If no relation occurs for s, s', we write $m(s,s') = \infty$. The pair (W, S) is called a *Coxeter system*. $|S|$ is called the *rank* of (W, S) and we call W a *Coxeter group* when the presentation given by S and m is understood. Unless otherwise noted, the Coxeter systems we will consider will be such that $|S| < \infty$ and $m(s,s') < \infty$ for all $s, s' \in S$. By [10], we have the following:

Proposition 2.5.1. *Let (W, S) be a Coxeter system.*

1. *If $s \in S$, then s has order 2 in W.*

2. *If $s, s' \in S$, then ss' has order $m(s, s')$ in W.*

Example 2.5.2. *The example to keep in mind throughout any discussion of Coxeter groups is the symmetric group, S_n. This group forms a Coxeter system with S the set of adjacent transpositions $S = \{s_i = (i, i + 1) \mid 1 \leq i \leq n - 1\}$. For $n = 4$, we have $W = S_4$ with $S = \{(12), (23), (34)\}$.*

Associated with a Coxeter system is the *Coxeter graph*. This is an undirected graph having vertex set S with an edge connecting s and s' if $m(s, s') \geq 3$. The edges are labeled $m(s, s')$ with the convention that the label is omitted for $m(s, s') = 3$. A Coxeter system is called *irreducible* if its Coxeter graph is connected. As noted in the example above, the Coxeter group we want to generally keep in mind is the symmetric group, S_n. The Coxeter graph for S_n is a chain of $n - 1$ vertices, corresponding to the $n - 1$ adjacent transpositions that form S. In the classification of irreducible Coxeter systems, this corresponds to Type A_{n-1}.

The finite Coxeter groups for which $m(s, s') \in \{2, 3, 4, 6\}$ for all distinct s, s' in S are called *Weyl groups*. If $m(s, s') \in \{2, 3\}$ for distinct s, s' the Coxeter group is said to be *simply-laced*. All Coxeter groups considered in this paper will be Weyl groups and all examples will, in addition, be simply-laced.

Let (W, S) be a Coxeter system. Any element $w \in W$ can be written as a product of generators, $w = s_1 s_2 \cdots s_k$ for some $s_i \in S$. If k is as small as possible we say the *length* of w is k, written $\ell(w) = k$, and say $s_1 s_2 \cdots s_k$ is a *reduced expression* for w. Reduced expressions are, generally, not unique. The following properties of the length function can be found in both [1] and [10]:

1. $\ell(w) = \ell(w^{-1})$.

2. $\ell(sw) = \ell(w) \pm 1$.

3. $\ell(ww') \equiv \ell(w) + \ell(w') \pmod 2$.

4. $\ell(ww') \leq \ell(w) + \ell(w')$.

We emphasize that equality does not hold for the last property above, in general.

Assume now that W is an arbitrary group and that $S \subseteq W$ is a generating subset of W such that all elements of S have order 2. For $w \in W$, define length and reduced expression as above and let $\hat{s}_i$ denote the omission of s_i from an expression. We now describe two important properties that (W, S) may have.

Exchange Property: Let $w = s_1 s_2 \cdots s_k$ be a reduced expression for w and $s \in S$. If $\ell(sw) < \ell(w)$ implies $sw = s_1 \cdots \hat{s}_i \cdots s_k$ for some $1 \leq i \leq k$, we say that (W, S) has the *Exchange Property*.

Deletion Property: Let $w = s_1 s_2 \cdots s_k$ be an expression for w. If $\ell(w) < k$ implies $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$ for some $1 \leq i < j \leq k$, we say that (W, S) has the *Deletion Property*.

These properties are fundamental in the theory of Coxeter groups, in the sense that they (each) characterize Coxeter groups. We make this explicit in the following theorem. For additional details, see [1] or [10].

Theorem 2.5.3. *Let W be a group and S a set of generators of order 2. Then the following are equivalent:*

1. *(W, S) is a Coxeter system.*

2. *(W, S) has the Exchange Property.*

3. *(W, S) has the Deletion Property.*

Let (W, S) be a Coxeter system and T the set of reflections of W. That is, $T = \{wsw^{-1} \mid w \in W, s \in S\}$. For $u, w \in W$, we write $u \to w$ if $\ell(u) < \ell(w)$ and $w = tu$ for some $t \in T$. Define $u < w$ if there exist $w_i \in W$ such that $u = w_0 \to w_1 \to \cdots \to w_{k-1} \to w_k = w$. This defines a partial order on W, called the *Bruhat-Chevalley order*.

A useful way to describe Bruhat order is in terms of subwords. A *subword* of a reduced expression $s_1 s_2 \cdots s_k$ is an expression $s_{i_1} s_{i_2} \cdots s_{i_j}$ such that $1 \leq i_1 < \cdots < i_j \leq k$. The following important result is from [1]:

Theorem 2.5.4 (Subword Property). *Let $w = s_1 s_2 \cdots s_k$ be a reduced expression. Then $u \leq w$ if and only if there exists a reduced expression $u = s_{i_1} s_{i_2} \cdots s_{i_j}$, with $1 \leq i_1 < \cdots < i_j \leq k$.*

Example 2.5.5. *The set P from Example 2.1.5 is a Coxeter group, with Coxeter graph $A - B$.*

The identity element of a Coxeter group is a minimum element, with respect to Bruhat order. In the case that the group is finite, Theorem 2.5.4 tells us that there is also a maximum element. The usual notation for this element is w_0. The following properties of w_0 are proved in [1]:

1. $w_0^2 = 1$.

2. $\ell(w w_0) = \ell(w_0) - \ell(w)$.

3. $\ell(w_0 w) = \ell(w_0) - \ell(w)$.

4. $\ell(w_0 w w_0) = \ell(w)$.

5. $\ell(w_0) = |T|$.

6. $w \mapsto w w_0$ and $w \mapsto w_0 w$ are antiautomorphisms.

7. $w \mapsto w_0 w w_0$ is an automorphism.

Example 2.5.6. *If $W = S_n$, then w_0 is the permutation mapping i to $n + 1 - i$. For $n = 4$, this is $w_0 = (14)(23)$.*

Let (W, S) be a Coxeter system. Recall from the previous section that W_I is the subgroup of W generated by $I \subseteq S$. Such a group is called a *parabolic subgroup* of W. Let ℓ_I be the length function of W_I, with respect to I. The following is from [1]:

Proposition 2.5.7. *1. (W_I, I) is a Coxeter system.*

2. *$\ell_I(w) = \ell(w)$, for all $w \in W_I$.*

3. *$W_I \cap W_J = W_{I \cap J}$.*

4. *$\langle W_I \cup W_J \rangle = W_{I \cup J}$.*

5. *$W_I = W_J \Rightarrow I = J$.*

Finally, we note that cosets of parabolic subgroups have a unique member of shortest length. The sets of shortest length coset representatives are called *quotients* and are also very important in the structure theory of Coxeter groups. Given W_I, the parabolic subgroup of W generated by $I \subseteq S$, we will denote these sets by D_I and D_I^{-1}, where

$$\begin{aligned}
D_I &= \{x \in W \mid \ell(sw) = \ell(s) + \ell(w) \text{ for all } w \in W_I\}, \\
D_I^{-1} &= \{x \in W \mid \ell(ws) = \ell(w) + \ell(s) \text{ for all } w \in W_I\}.
\end{aligned}$$

These sets are denoted W^I and $^I W$, respectively, in [1].

The following result illustrates why parabolic subgroups and their associated quotients are so important.

Proposition 2.5.8. *Let $I \subseteq S$. Every $w \in W$ has a unique factorization $w = xy$ such that $x \in W_I$ and $y \in D_I^{-1}$, and $\ell(w) = \ell(x) + \ell(y)$.*

2.6 Linear Algebraic Monoids

Let k be an algebraically closed field. A *linear algebraic monoid* M is an affine variety together with an associative morphism $\mu : M \times M \to M$ and an identity element $1 \in M$ for μ. Recall that for linear algebraic groups, our standard example is $GL_n(k)$. For linear algebraic monoids, our example to remember is $M_n(k)$. We have seen that any affine algebraic group is isomorphic to a closed subgroup of $GL_n(k)$, for some n. The analogous result holds for monoids, as shown in [19].

Theorem 2.6.1. *Let M be a linear algebraic monoid. Then M is isomorphic to a closed submonoid of $M_n(k)$ for some n.*

A closed submonoid of $M_n(k)$ is a linear algebraic monoid, so the converse to this theorem holds as well.

If G is a linear algebraic group, then it is a closed subgroup of $GL_n(k)$, for some n, and thus is contained in $M_n(k)$. The closure of G in $M_n(k)$, with respect to the Zariski topology, is therefore a linear algebraic monoid. What's more, the unit group of M is exactly this group G.

Example 2.6.2. *The following are linear algebraic monoids.*

1. *The set of diagonal matrices:*

$$\overline{D_n(k)} = \{(x_{ij}) \in M_n(k) \mid x_{ij} = 0 \text{ if } i \neq j\}.$$

2. *The set of upper triangular matrices:*

$$\overline{B_n(k)} = \{(x_{ij}) \in M_n(k) \mid x_{ij} = 0 \text{ if } i > j\}.$$

A linear algebraic monoid is *irreducible* if it cannot be expressed as the union of two proper, closed, non-empty subsets. The *irreducible components* of M are the maximal, irreducible subsets of M. In our review of linear algebraic groups, we noted that the irreducible and connected components are identical. This is not the case for linear algebraic monoids. In general, irreducible implies connected, though not conversely. The following example, from [19], illustrates this point.

Example 2.6.3. *Let $M = \{(a, b) \in k^2 \mid a^2 = b^2\}$, with multiplication $(a, b) \cdot (c, d) = (ac, bd)$. M is a monoid and it is the zero set for the ideal I generated by $f(x, y) = (x - y)(x + y) \in k[x, y]$, hence it is closed. M is therefore a linear algebraic monoid. It is clearly connected though not irreducible (since I is not a prime ideal).*

We can now reformulate Green's relations for linear algebraic monoids. The following is from [19], as presented in [32].

Proposition 2.6.4. *Let M be an irreducible linear algebraic monoid with $a, b \in M$ and G the unit group of M. Then*

1. *$a\mathcal{R}b$ if and only if $aG = bG$.*

2. *$a\mathcal{L}b$ if and only if $Ga = Gb$.*

3. *$a\mathcal{J}b$ if and only if $GaG = GbG$.*

The third case is of most interest to us. This states that a and b are in the same $\mathcal{J}$-class if and only if they are in the same $G \times G$ orbit. Using our initial description of Green's relations in Section 2.2 we observe that

$$a\mathcal{J}b \iff GaG = GbG \iff MaM = MbM. \tag{2.5}$$

Additionally, the partial order (2.2) can now be described as:

$$\mathcal{J}_a \leq \mathcal{J}_b \iff GaG = \overline{GbG} \iff a \in MbM. \tag{2.6}$$

where closure is with respect to the Zariski topology.

M is called a *reductive monoid* if M is an irreducible linear algebraic monoid whose unit group G is a reductive group. M is *regular* if for each $a \in M$ there exists $x \in M$ such that $axa = a$ and *unit regular* if $M = GE(M)$, where $E(M)$ is the set of idempotents of M. By [19], any regular irreducible linear algebraic monoid is unit regular. The following result, proved for the characteristic 0 case in [16] and the characteristic p case in [30], is the first important result on the structure of a reductive monoid.

Theorem 2.6.5. *Let M be an irreducible monoid with unit group G and zero element $0 \in M$. The following are equivalent:*

1. *G is reductive.*

2. *M is regular.*

3. *M has no non-zero nilpotent ideals.*

Our interest is in reductive monoids. This theorem tells us that if M is a reductive monoid, then $M = GE(M)$. This is the only the starting point. It turns out that a certain subset of $E(M)$ is enough to give a nice description of M in terms of this set of idempotents and the unit group of M. Additionally, there is a decomposition of M and resulting partial order that generalize (2.3) and (2.4) above. We will pursue these points in the next chapter.

Chapter 3

Reductive Monoids

In the previous chapter, we observed that reductive groups have a very nice structure. Reductive monoids do as well, with the description given in terms of the (reductive) unit group and idempotent set. In this chapter, we provide a description of the general theory of reductive monoids. An attempt has been made to find a balance between brevity and thoroughness. As a result, the basics required for this disseration have been covered, though little more. For more details, one should consult [19], [32], or [35].

Unless otherwise noted, for the remainder of this paper M will be a reductive monoid with G its unit group. Additionally, $B \subseteq G$ will be a Borel subgroup of G with $T \subseteq B$ a maximal torus contained in B. Finally, all closure is with respect to the Zariski topology, in M.

3.1 Background

In this section we provide a brief introduction to the general theory of reductive monoids. We begin with the description of a set of idempotents (the cross-section lattice) and semigroup (the Renner monoid) that form the foundation upon which the structure theory of reductive monoids is built. We then describe the generalization of the Bruhat decomposition, (2.3), to reductive monoids. Next, we define a certain

subset of a reductive monoid that will be important in Chapter 4, where we examine the partial order associated with the conjugacy decomposition of a reductive monoid. Finally, we describe two classes of monoids that, along with $M_n(k)$, will be the primary objects of our more detailed examination of the conjugacy poset in the remaining chapters.

3.1.1 Cross-Section Lattices

Let M be a reductive monoid and define $\mathcal{U}(M)$ to be the set of $\mathcal{J}$-classes of M. Recall that these are the $G \times G$-orbits of M, (2.5). By [15], $\mathcal{U}(M)$ is a finite lattice, with order (2.6), and there is a diagonal idempotent cross-section $\Lambda \subseteq E(\overline{T})$ of $\mathcal{U}(M)$ that preserves this order. That is, there is a set Λ such that for all $J_i \in \mathcal{U}(M)$, $|\Lambda \cap J_i| = 1$ and

$$e_1 \leq e_2 \iff J_1 \leq J_2, \tag{3.1}$$

where $e_i \in J_i \cap \Lambda$ and $e_1 \leq e_2$ means $e_1 e_2 = e_1 = e_2 e_1$, as in (2.1).

Λ is a finite lattice, called a *cross-section lattice* of M. The following result on cross-section lattices first appears in [15].

Theorem 3.1.1. *Let M be a reductive monoid with W the Weyl group of the unit group G and T a maximal torus in G. Then*

1. *Cross-section lattices exist.*

2. *Any two cross-section lattices are conjugate by an element of W.*

3. *There is a one-to-one correspondence between cross-section lattices and Borel subgroups of G containing T.*

Given a Borel subgroup B with $T \subseteq B$, the correspondence from the theorem above yields the following cross-section lattice:

$$\Lambda = \{e \in E(\overline{T}) \mid Be = eBe\}. \tag{3.2}$$

Since we usually have a specific B and T in mind when considering a reductive monoid M, in any subsequent discussions by *the* cross-section lattice of M we mean the one corresponding to B and T as in (3.2).

We may consider M as a union of its $\mathcal{J}$-classes. A particularly nice way to do this is by choosing the $\mathcal{J}$-classes of the elements of the cross-section lattice Λ. This gives us a disjoint union which, by (2.5), we may write as:

$$M = \bigsqcup_{e \in \Lambda} GeG. \tag{3.3}$$

Example 3.1.2. *Let $M = M_n(k)$. Then $G = GL_n(k)$. If $B = B_n(k)$ with $T = D_n(k)$, then $\Lambda = \{I_r \oplus 0_{n-r} \mid 0 \leq r \leq n\}$, where I_r is the $r \times r$ identity matrix and 0_k is the $k \times k$ zero matrix. We note that Λ is a chain and that each $G \times G$ orbit consists of matrices of a given rank.*

3.1.2 The Renner Monoid and Bruhat-Renner Decomposition

Let M be a reductive monoid with unit group G. Define $R = \overline{N_G(T)}/T$, where $\overline{N_G(T)}$ is the Zariski closure of $N_G(T)$ in M. R is a monoid, called the *Renner monoid* of M. This monoid was introduced in [31], where it is shown that R is a finite inverse semigroup with unit group W and idempotent set $E(\overline{T})$. The Renner monoid is an important component of the theory. It plays the role in a reductive monoid that the Weyl group plays in a reductive group.

Example 3.1.3. *Let $M = M_n(k)$. Then the Renner monoid is the set of $n \times n$ partial permutation matrices (or rook monoid). By this we mean the set of $\{0,1\}$-matrices having at most one 1 in each column and each row. The unit group is the set of permutation matrices, S_n. For the case of $M_n(k)$, we denote the Renner monoid by R_n.*

Given $\sigma \in R_n$, we associate a sequence $(s_1 \ s_2 \ \ldots \ s_n)$ with σ such that for $1 \leq i \leq n$, $s_i = 0$ if σ has all zeros in the ith column and $s_i = j$ if $\sigma_{ji} = 1$. We call this

the *one-line notation* for elements of R_n. As many of our results and examples come from $M_n(k)$, we will use this notation often throughout the remainder of the paper.

Example 3.1.4. *Using the one-line notation, we identify*

$$(3\ 0\ 4\ 1) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The Bruhat decomposition for reductive groups, (2.3), extends in a natural way to reductive monoids. In this new setting, the $B \times B$-orbits are indexed by elements of the Renner monoid. By [31], we may decompose M as:

$$M = \bigsqcup_{\sigma \in R} B\sigma B. \tag{3.4}$$

This is called the *Bruhat-Renner decomposition* of M. Furthermore, we may decompose R into $W \times W$-orbits, as follows:

$$R = \bigsqcup_{e \in \Lambda} WeW. \tag{3.5}$$

As usual, W is the Weyl group of G and Λ is the cross-section lattice of M. We will often denote WeW by $R(e)$.

Upon introduction of the Bruhat decomposition in the previous chapter, we defined a partial order on the Weyl group in terms of $B \times B$-orbits. This order, the *Bruhat-Chevalley order*, (2.4), also extends to R in a natural way. For $\sigma, \theta \in R$, we have:

$$\sigma \leq \theta \iff B\sigma B \subseteq \overline{B\theta B}. \tag{3.6}$$

Figure 3.1 is the Hasse diagram for R_3, the Renner monoid of $M_3(k)$, under Bruhat-Chevalley order. The elements of R_3 are labeled using the one-line notation. In the next section, we will provide a description of this order in terms of the respective orders on W and Λ.
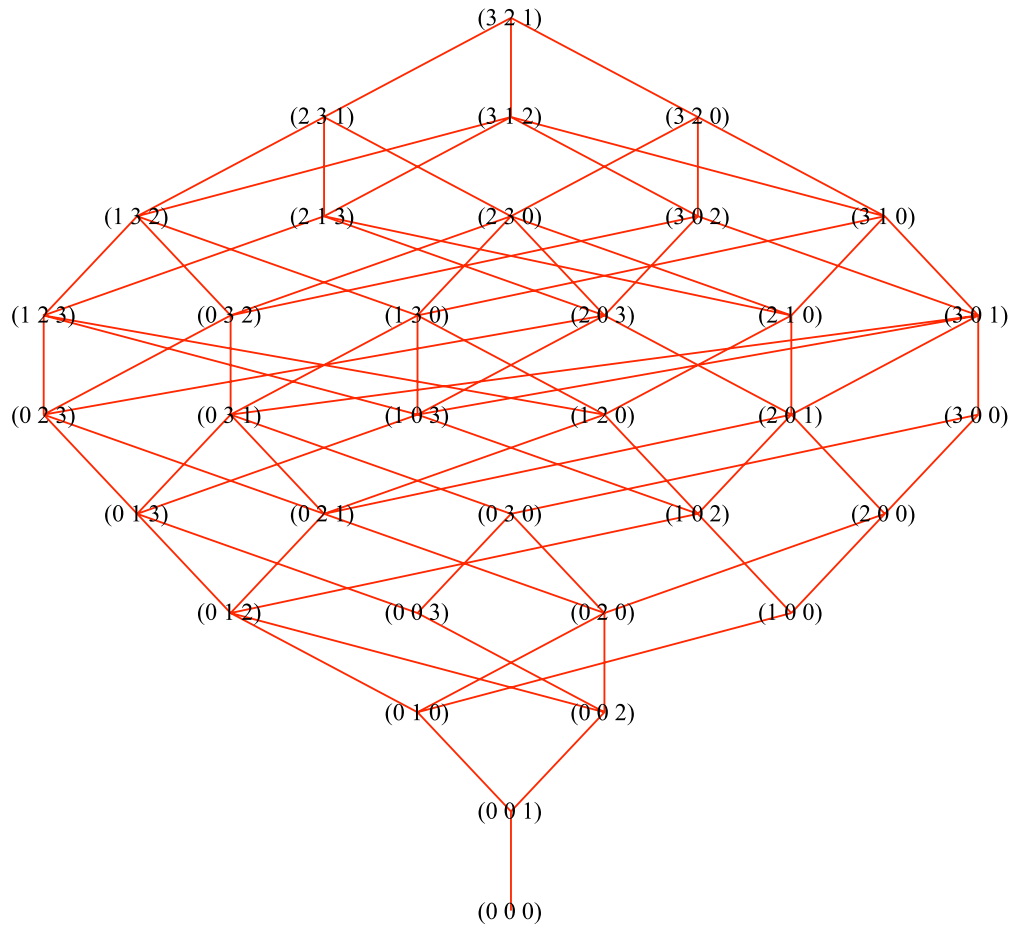
Figure 3.1: R_3 under Bruhat-Chevalley Order.

3.1.3 The Type Map and Parabolic Subgroups of W

Let S be the set of simple reflections of the Weyl group W of G. The *type map* of M is the map $\lambda : \Lambda \to 2^S$ such that $\lambda(e) = \{s \in S \mid es = se\}$. Closely related to the type map are two maps, denoted λ^* and λ_*, defined as follows:

$$
\begin{aligned}
\lambda^*(e) &= \bigcap_{f \geq e} \lambda(f), \\
\lambda_*(e) &= \bigcap_{f \leq e} \lambda(f).
\end{aligned}
$$

In Section 2.5, we denoted the subgroup of W generated by $I \subseteq S$ by W_I. For reductive monoids, we will describe subsets of S in terms of the type map and therefore use the following notation for parabolic subgroups of W:

$$
\begin{aligned}
W(e) &= W_{\lambda(e)} = \{w \in W \mid ew = we\}, \\
W^*(e) &= W_{\lambda^*(e)}, \\
W_*(e) &= W_{\lambda_*(e)} = \{x \in W \mid ex = e = xe\},
\end{aligned}
$$

and note that

$$
W(e) = W^*(e) \times W_*(e), \tag{3.7}
$$

as shown in Chapter 10 of [19].

Elements of $W(e)$ are the Weyl group elements that commute with the idempotent $e \in \Lambda$. $W_*(e)$ is the set of commuting elements which are absorbed by e and $W^*(e)$ consists of those that commute where nothing is absorbed. Additionally, we note that for $e, f \in \Lambda$ with $e \leq f$,

$$
W^*(e) \subseteq W^*(f) \text{ and } W_*(f) \subseteq W_*(e). \tag{3.8}
$$

As noted in Example 3.1.2, if $M = M_n(k)$ with $B = B_n(k)$ and $T = D_n(k)$, $e \in \Lambda$ is of the form $e = \begin{pmatrix} I_j & 0 \\ 0 & 0 \end{pmatrix}$. For $M = M_n(k)$, such an idempotent will be denoted by

e_j. Then

$$
\begin{aligned}
W(e_j) &= \left\{ \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \right\}, \\
W^*(e_j) &= \left\{ \begin{pmatrix} P & 0 \\ 0 & I_{n-j} \end{pmatrix} \right\}, \\
W_*(e_j) &= \left\{ \begin{pmatrix} I_j & 0 \\ 0 & Q \end{pmatrix} \right\},
\end{aligned}
$$

where P and Q are permutation matrices, of sizes $j \times j$ and $(n-j) \times (n-j)$, respectively.

Example 3.1.5. *Let* $W = S_5$ *and* $e_3 \in \Lambda$. *Then* $W(e_3) = \langle (12), (23), (45) \rangle$ *with* $W^*(e_3) = \langle (12), (23) \rangle$ *and* $W_*(e_3) = \langle (45) \rangle$. $(13)(45)$ *is an element of* $W(e_3)$ *with* $(13) \in W^*(e_3)$ *and* $(45) \in W_*(e_3)$. *That is,* $e_3(13) = (13)e_3$ *and* $e_3(45) = e_3 = (45)e_3$.

We illustrate as follows:

$$e_3(13)(45) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$e_3(45)(13) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e_3(13) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e_3(13) \;=\; \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We use a corresponding notation for quotients of parabolic subgroups:

$$D(e) \;=\; D_{\lambda(e)} = \{x \in W \mid \ell(xw) = \ell(x) + \ell(w) \;\; \forall w \in W(e)\}\,,$$
$$D_*(e) \;=\; D_{\lambda_*(e)} = \{x \in W \mid \ell(xw) = \ell(x) + \ell(w) \;\; \forall w \in W_*(e)\}\,.$$

Observe that $x \in D(e)$ is equivalent to x having minimal length in the coset $xW(e)$. Likewise, $x \in D_*(e)$ is equivalent to x having minimal length in the coset $xW_*(e)$.

Additionally, we define the set $D(e)^{-1}$ as:

$$
\begin{aligned}
D(e)^{-1} &= \{x \in W \mid \ell(wx) = \ell(w) + \ell(x) \ \ \forall w \in W(e)\} \\
&= \{x \in W \mid x^{-1} \in D(e)\}.
\end{aligned}
$$

Example 3.1.6. *In the previous example, $D(e)^{-1}$ is:*

$$D(e_3)^{-1} = \{1, (34), (243), (345), (1432), (2453), (14532), (24)(35), (142)(35), (14253)\}.$$

Now, $(1325) \in S_5$. We may write this as $(13)(45)(2453)$, where $(13)(45) \in W(e_3)$ and $(2453) \in D(e)^{-1}$. As noted in Proposition 2.5.8, this factorization is unique.

3.1.4 Gauss-Jordan Elements

Given an $n \times n$ matrix, the Gauss-Jordan algorithm produces a matrix in reduced row echelon form. Such matrices classify the orbits corresponding to the action of $G = GL_n(k)$ on $M_n(k)$ by left multiplication. That is, if we call the set of reduced row echelon matrices GJ then for any $x \in M_n(k)$ we have $|Gx \cap GJ| = 1$. In [31], the analogous subset is defined for a general reductive monoid. Our concern will be with such elements that are also in the Renner monoid.

Let $\mathcal{GJ} = \{x \in R \mid Bx \subseteq xB\}$. This is called the set of *Gauss-Jordan elements* of R. By [31], $W \cdot \mathcal{GJ} = R$ and for each $x \in R$, $|Wx \cap \mathcal{GJ}| = 1$. Gauss-Jordan elements are a key part of the conjugacy decomposition of M, as we will show in the next chapter. For now, we give an alternate description of $\mathcal{GJ}$, in terms of quotients of parabolic subgroups of W:

$$\mathcal{GJ} = \{ey \in R \mid e \in \Lambda \ , \ y \in D(e)^{-1}\}. \tag{3.9}$$

We denote the Gauss-Jordan elements in WeW by $\mathcal{GJ}(e)$. Thus $\mathcal{GJ} = \bigsqcup_{e \in \Lambda} \mathcal{GJ}(e)$.

If $M = M_n(k)$, then the set of Gauss-Jordan elements of R_n, denoted $\mathcal{GJ}_n$, is the set of $n \times n$ partial permutation matrices in row echelon form. The following example illustrates.

Example 3.1.7. *Let $M = M_3(k)$. Then*

$$
\begin{aligned}
D(e_0)^{-1} &= \{1\}, \\
D(e_1)^{-1} &= \{1, (12), (123)\}, \\
D(e_2)^{-1} &= \{1, (23), (132)\}, \\
D(e_3)^{-1} &= \{1\}.
\end{aligned}
$$

The Gauss-Jordan elements are therefore:

$$
e_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad
e_1(12) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad
e_1(123) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad
e_2(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad
e_2(132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
e_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

3.1.5   Canonical and Dual Canonical Monoids

The example to keep in mind when considering reductive monoids is $M_n(k)$. There are, however, two other well-studied classes of reductive monoids that we will consider.

Let Λ be a cross-section lattice for a reductive monoid M, with type map $\lambda : \Lambda \to 2^S$. As Λ is a finite lattice, it contains a minimum and maximum element, which we denote 0 and 1, respectively. Let Λ_{min} denote the minimal elements of $\Lambda \setminus \{0\}$ and Λ_{max} the maximal elements of $\Lambda \setminus \{1\}$.

Suppose $|\Lambda_{min}| = 1$, say $\Lambda_{min} = \{e_0\}$, with $\lambda(e_0) = I$. M is then called a $\mathcal{J}$-*irreducible monoid of type I*. Since its cross-section lattice is a chain, $M_n(k)$ is an example of such a monoid, in this case of type $S \setminus \{(12)\}$. If M is a $\mathcal{J}$-irreducible monoid of type $\emptyset$, M is called a *canonical monoid*. Canonical monoids are one of the three classes of reductive monoids we consider in detail.

The following result appears in [27] for general $\mathcal{J}$-irreducible monoids. We have considered only the canonical monoid case here (that is, $I = \emptyset$), and have modified the conclusions accordingly.

Theorem 3.1.8. *Let M be a canonical monoid. Then*

1. *If $e, f \in \Lambda \setminus \{0\}$, then $e \leq f$ if and only if $\lambda^*(e) \subseteq \lambda^*(f)$.*

2. *If $e \in \Lambda \setminus \{0\}$, then $\lambda_*(e) = \emptyset$.*

3. *If $K \subseteq S$, then $K = \lambda^*(e)$ for some $e \in \Lambda \setminus \{0\}$.*

Thus for a canonical monoid, $\Lambda \setminus \{0\}$ is isomorphic to $2^{|S|}$, ordered by inclusion. Canonical monoids were first studied in [28]. Their construction was modeled by the canonical compactification of a reductive group, as in [5].

Example 3.1.9. *Let $G_0 = \{A \otimes (A^{-1})^t \mid A \in SL_3(k)\}$ and let $M = \overline{kG_0} \subseteq M_9(k)$. Then M is a canonical monoid with $W = S_3$. This example appears in [19], [24], and [26].*

Suppose now that $|\Lambda_{max}| = 1$, say $\Lambda_{max} = \{e_0\}$, with $\lambda(e_0) = I$. M is then called a $\mathcal{J}$-*coirreducible monoid of type I*. $M_n(k)$ is also an example of such a monoid, of type $S \setminus \{(n-1, n)\}$. If M is a $\mathcal{J}$-coirreducible monoid of type $\emptyset$, we call M a *dual canonical monoid* (or *cocanonical monoid*). Dual canonical monoids are the third, and final, class of reductive monoids we will study.

The result from [27] referred to above dualizes for $\mathcal{J}$-coirreducible monoids. We consider only the dual canonical monoid case here (that is, $I = \emptyset$), and have modified the conclusions accordingly, similar to above.

Theorem 3.1.10. *Let M be a dual canonical monoid. Then*

1. *If $e, f \in \Lambda \setminus \{1\}$, then $e \leq f$ if and only if $\lambda_*(f) \subseteq \lambda_*(e)$.*

2. *If $e \in \Lambda \setminus \{1\}$, then $\lambda^*(e) = \emptyset$.*

3. *If $K \subseteq S$, then $K = \lambda_*(e)$ for some $e \in \Lambda \setminus \{1\}$.*

Hence for a dual canonical monoid, $\Lambda \setminus \{1\}$ is isomorphic to the dual of $2^{|S|}$, ordered by inclusion. Dual canonical monoids also arise naturally, as the following example shows.

Example 3.1.11. *Let $G_0 = \{A \oplus (A^{-1})^t \mid A \in SL_3(k)\}$ and let $M = \overline{kG_0} \subseteq M_6(k)$. Then M is a dual canonical monoid with $W = S_3$. This example appears in [19], [24], and [26].*

Remark 3.1.12. *From Theorems 3.1.8 and 3.1.10, it follows that:*

1. *The dual of a cross-section lattice for a canonical monoid is a cross-section lattice for a dual canonical monoid.*

2. *If M is a canonical monoid, then $W(e) = W^*(e)$ and $W_*(e) = \{1\}$ for all $e \in \Lambda \setminus \{0\}$.*

3. *If M is a dual canonical monoid, then $W(e) = W_*(e)$ and $W^*(e) = \{1\}$ for all $e \in \Lambda \setminus \{1\}$.*

Examples 3.1.9 and 3.1.11 show that there exist canonical and dual canonical monoids having $W = S_3$ as a Weyl group. Let $A - B$ be the Coxeter graph of W with $\lambda(\emptyset) = e0$, $\lambda(\{A\}) = eA$, $\lambda(\{B\}) = eB$, and $\lambda(\{A, B\}) = eAB$. The cross-section lattice of the canonical monoid with $W = S_3$ is shown in Figure 3.2. Figure 3.3 is the cross-section lattice of the dual canonical monoid with $W = S_3$.
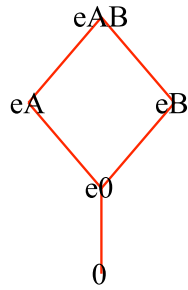
Figure 3.2: The cross-section lattice for the canonical monoid with $W = S_3$.
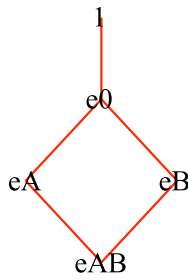
Figure 3.3: The cross-section lattice for the dual canonical monoid with $W = S_3$.

3.2 The Bruhat-Chevalley Order in a Reductive Monoid

Let R be the Renner monoid of M with $\sigma \in R$. By (3.5), $\sigma \in WeW$ for a unique idempotent $e \in \Lambda$. Thus $\sigma = w_1 e w_2$ for some $w_1, w_2 \in W$. We may write w_2 as $w_2 = uy$ for $u \in W(e)$ and $y \in D(e)^{-1}$, by Proposition 2.5.8, and so $\sigma = w_1 uey$. Now we may write $w_1 u$ as $w_1 u = xv$ for $x \in D_*(e)$ and $v \in W_*(e)$, again by Proposition 2.5.8, and so $\sigma = xey$. An element of the Renner monoid is said to be in *standard form* if it is written this way. That is, the standard form of σ is:

$$\sigma = xey \tag{3.10}$$

for a unique $x \in D_*(e)$ and $y \in D(e)^{-1}$. If $M = M_n(k)$, y has the effect of shifting certain non-zero columns of e to the right while x switches various rows with each other.

We may now give a more combinatorial description of the Bruhat-Chevalley order on R, in terms of the Weyl group and cross-section lattice. This was first identified in [14].

Theorem 3.2.1. *Let $\sigma = xey$ and $\theta = ufv$ be in standard form. Then $\sigma \leq \theta$ if and only if $e \leq f$ and there exists $w \in W(f)W_*(e)$ such that $x \leq uw$ and $w^{-1}v \leq y$.*

Example 3.2.2. *Let $M = M_4(k)$ with $B = B_4(k)$ and $T = D_4(k)$. Then $W = S_4 = \langle (12), (23), (34) \rangle$. For $e_2 = \Lambda$, we have:*

$$
\begin{aligned}
W(e_2) &= \{(1), (12), (34), (12)(34)\}, \\
W_*(e_2) &= \{(1), (34)\}, \\
D(e_2)^{-1} &= \{(1), (23), (234), (132), (1342), (13)(24)\}, \\
D_*(e_2) &= \{(1), (12), (23), (123), (132), (243), \\
&\qquad (13), (1243), (1432), (143), (13)(24), (1423)\}.
\end{aligned}
$$

For $e_3 = \Lambda$, we have:

$$
\begin{aligned}
W(e_3) &= \{(1), (12), (23), (123), (132), (13)\}, \\
W_*(e_3) &= \{(1)\}, \\
D(e_3)^{-1} &= \{(1), (34), (243), (1432)\}, \\
D_*(e_3) &= W.
\end{aligned}
$$

Additionally, we note that

$$
\begin{aligned}
W(e_3)W_*(e_2) &= \{(1), (12), (23), (123), (132), (13), (34), \\
&\quad\ (12)(34), (234), (1234), (1342), (134)\}.
\end{aligned}
$$

Let $\sigma \in R_4$, with factorization in standard form as follows:

$$
\begin{aligned}
\sigma &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
&= (243)e_2(13)(24) \\
&= xe_2y
\end{aligned}
$$

Let $\theta \in R_4$, with factorization in standard form as follows:

$$\theta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$= (143)e_3(1432)$$

$$= ue_3v$$

Obviously $e_2 \leq e_3$. Choose $w = (34) \in W(e_3)W_*(e_2)$. Then

$$x = (243) = (34)(23) \leq (14) = (12)(34)(23)(12)(34) = (143)(34) = uw$$

and

$$\begin{aligned} w^{-1}v &= (34)^{-1}(1432) \\ &= (34)(34)(23)(12) \\ &= (23)(12) \\ &\leq (23)(34)(12)(23) \\ &= (13)(24) \\ &= y. \end{aligned}$$

So $\sigma \leq \theta$.

Suppose instead that we compare σ with $\psi \in R_4$, where $\psi = (1)e_3(1)$ in standard form. Since $e_3 \not\leq e_2$, $\psi \not\leq \sigma$. If $\sigma \leq \psi$, then there is a $w \in W(e_3)W_*(e_2)$ such that $(243) \leq w$ and $w^{-1} \leq (13)(24)$. We rewrite the elements $W(e_3)W_*(e_2)$ as follows:

$$\begin{aligned} W(e_3)W_*(e_2) = \; & \{(1), (12), (23), (12)(23), (23)(12), (23)(12)(23), (34), \\ & (12)(34), (23)(34), (12)(23)(34), (23)(12)(34), (23)(12)(23)(34)\}. \end{aligned}$$

Upon inspection, we observe that there is no $w \in W(e_3)W_*(e_2)$ such that $(243) = (34)(23) \leq w$. So $\sigma \not\leq \psi$ and thus σ and ψ are incomparable.

3.2.1 The Structure of $W \times W$-orbits

If $\sigma, \theta \in WeW$ with $\sigma = xey$ and $\theta = uev$ in standard form, then Theorem 3.2.1 simplifies to:

$$\sigma \leq \theta \iff x \leq uw, w^{-1}v \leq y \text{ for some } w \in W(e) \tag{3.11}$$

Describing the structure of WeW using this order is manageable, however a more useful description is given in [23]. For this description, we let $I = \lambda(e)$ and $K = \lambda_*(e)$ and define

$$\mathcal{W}^*_{I,K} = D_I \times W_{I \backslash K} \times D_I^{-1}.$$

Then, for $\sigma = (x, w, y), \theta = (u, v, z) \in \mathcal{W}^*_{I,K}$, define

$$\sigma \leq \theta \text{ if } w = w_1 * w_2 * w_3 \text{ with } xw_1 \leq u, w_2 \leq v, w_3y \leq z. \tag{3.12}$$

To be clear, $\leq$ here is not the Bruhat-Chevalley order on R. However, $\mathcal{W}^*_{I,K}$ with this order is isomorphic to the dual of WeW, by [23].

3.2.2 Order Between $W \times W$-orbits

Suppose now that we're considering elements from different $W \times W$-orbits, say $\sigma \in WeW$ and $\theta \in WfW$. Theorem 3.2.1 tells us how to determine if σ and θ are comparable. To begin, e and f must be comparable. This is usually easy to check. Verifying the remaining conditions from the theorem, however, is often much more difficult. In [24], a description is obtained via maps between $W \times W$-orbits.

We describe these maps and summarize the main result of [24], illustrating why the maps are useful. Before doing so, we need a few definitions.

To begin, if $x_1, \ldots, x_n \in W$, then let

$$x_1 * \cdots * x_n = \begin{cases} x_1 \ldots x_n & \text{if } \ell(x_1 \ldots x_n) = \ell(x_1) + \cdots + \ell(x_n) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Note that, by Proposition 2.5.8, if $x \in W(e)$ and $y \in D(e)^{-1}$ then $x * y = xy$.

Next, for $x, y \in W$, define $x \circ y, x \triangle y \in W$ as:

$$\begin{aligned} x \circ y &= \max\{xy' \mid y' \leq y\}, \\ x \triangle y &= \min\{xy' \mid y' \leq y\}. \end{aligned}$$

The following properties of these new operations on W are proved in [24]:

Lemma 3.2.3. *Let* $x, y \in W$. *Then*

1. $x \circ y = x_1 * y = x * y_1$ *for some* $x_1 \leq x$, $y_1 \leq y$.

2. $x \circ y = \max\{xy' \mid y' \leq y\} = \max\{x'y \mid x' \leq x\} = \max\{x'y' \mid x' \leq x, y' \leq y\}$.

3. $x \triangle y = \min\{xy' \mid y' \leq y\} = \min\{x'y \mid x' \geq x\} = \min\{x'y' \mid x' \geq x, y' \leq y\}$.

Now we are equipped to define the aforementioned maps. Let $e, f \in \Lambda$ with $e \leq f$, and let $\sigma = xey$ in standard form. Let z_e denote the longest element in $W_*(e)$ and let $z_e y = u y_1$, with $u \in W(f)$ and $y_1 \in D(f)^{-1}$. The *projection* $p_{e,f} : WeW \to WfW$ is defined as:

$$p_{e,f}(\sigma) = (x \triangle u) f y_1 \tag{3.13}$$

As noted in [24], $(x \triangle u) f y_1 \in WfW$ is in standard form.

Using these projection maps, a new description of the order between $W \times W$-orbits of R is given by the following theorem, from [24]:

Theorem 3.2.4. *Let* $e, f \in \Lambda$, $e \leq f$. *Then*

1. $p_{e,f} : WeW \to WfW$ *is order-preserving and* $\sigma \leq p_{e,f}(\sigma)$ *for all* $\sigma \in WeW$.

2. *If* $\sigma \in WeW$, $\theta \in WfW$, *then* $\sigma \leq \theta$ *if and only if* $p_{e,f}(\sigma) \leq \theta$.

3. *If* $h \in \Lambda$ *with* $e \leq h \leq f$, *then* $p_{e,f} = p_{h,f} \circ p_{e,h}$.

4. $p_{e,f}$ *is onto if and only if* $\lambda_*(e) \subseteq \lambda_*(f)$.

5. $p_{e,f}$ *is one-to-one if and only if* $\lambda(f) \subseteq \lambda(e)$.

Remark 3.2.5. *The above theorem, combined with Theorems 3.1.8 and 3.1.10, tells us that:*

1. *If M is a canonical monoid, then $p_{e,f}$ is onto.*

2. *If M is a dual canonical monoid, then $p_{e,f}$ is one-to-one.*

3.2.3 Pennell's Description for $M_n(k)$

We begin by noting the following from [13], as presented in [32]. These results will be used at several points in the next section, where we provide an alternate description of the projection maps for $M = M_n(k)$.

Theorem 3.2.6 (Pennell's Theorem). *Let $\sigma, \theta \in R$. Then $\sigma \leq \theta$ if and only if there exist $\theta_0, \theta_1, \ldots, \theta_m \in R$ such that*

$$\sigma = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_m = \theta$$

and, for each k, either $\theta_k \in \overline{B}\theta_{k+1}\overline{B}$ or else θ_{k+1} is obtained from k by a "Bruhat interchange".

If $M = M_n(k)$, then $A < B$ via a Bruhat interchange if and only if B is obtained from A by interchanging two non-zero rows of A and, in the process of doing so, a 2×2 submatrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ of A ends up as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We are concerned here with the case of $M = M_n(k)$ and so we will refrain from defining a Bruhat interchange for the general setting (see [13] or [32] for details).

Recall that for $M = M_n(k)$ we denote the Renner monoid of M by R_n. Using the theorem, we therefore have the following description of the Bruhat-Chevalley order on R_n.

Theorem 3.2.7. *Let $\sigma, \theta \in R_n$ with $\ell(\sigma) = \ell(\theta) - 1$. Then $\sigma \leq \theta$ if and only if one of the following holds:*

1. *θ is obtained from σ by setting some zero entry of σ to a non-zero.*

2. θ is obtained from σ by moving a non-zero entry either downward or to the left.

3. θ is obtained from σ via a Bruhat interchange.

Using the *one-line notation*, we have the following combinatorial description of the Bruhat-Chevalley order on R_n.

Theorem 3.2.8. *Let* $\sigma = (\delta_1 \ \ldots \ \delta_n), \theta = (\epsilon_1 \ \ldots \ \epsilon_n) \in R_n$. *Then* $\leq$ *is the smallest partial order on* R_n *generated by declaring* $\sigma < \theta$ *if either*

1. $\delta_j = \epsilon_j$ *for* $j \neq i$ *and* $\delta_i < \epsilon_i$, *or*

2. *(a)* $\delta_k = \epsilon_k$ *if* $k \neq \{i, j\}$,

   (b) $i < j$, *and*

   (c) $\delta_i = \epsilon_j$, $\delta_j = \epsilon_i$ *and* $\epsilon_i > \epsilon_j$.

The next example follows these results, as they are presented in [32].

Example 3.2.9. *Let* $\sigma = (2\ 1\ 4\ 0\ 3)$ *and* $\theta = (3\ 5\ 2\ 0\ 1)$ *in* R_5. *Then* $\sigma < \theta$ *since*

$$(2\ 1\ 4\ 0\ 3) < (3\ 1\ 4\ 0\ 2) < (3\ 4\ 1\ 0\ 2) < (3\ 5\ 1\ 0\ 2) < (3\ 5\ 2\ 0\ 1).$$

The first, second, and fourth inequalities follow from condition 2 in the theorem. The third is by condition 1.

Finally, we revisit Example 3.2.2, using this new approach.

Example 3.2.10. *Let* $\sigma, \theta \in R_4$ *be*

$$\sigma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \theta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Using the one-line notation, $\sigma = (0\ 0\ 1\ 4)$ *and* $\theta = (0\ 4\ 2\ 1)$. *Then* $\sigma < \theta$ *since*

$$(0\ 0\ 1\ 4) < (0\ 1\ 0\ 4) < (0\ 1\ 4\ 0) < (0\ 4\ 1\ 0) < (0\ 4\ 1\ 2) < (0\ 4\ 2\ 1).$$

We note that for $\sigma < \theta$ in R_n, the sequence of inequalities from σ to θ is in general not unique. Figure 3.4 illustrates this for the previous example (the solid lines follow the path we chose). In addition, note that it is not necessary for all the relations to be covers. For example,

$$(0\ 0\ 1\ 4) < (0\ 4\ 1\ 0) < (0\ 4\ 1\ 2) < (0\ 4\ 2\ 1)$$

satisfies the conditions of Theorem 3.2.6.
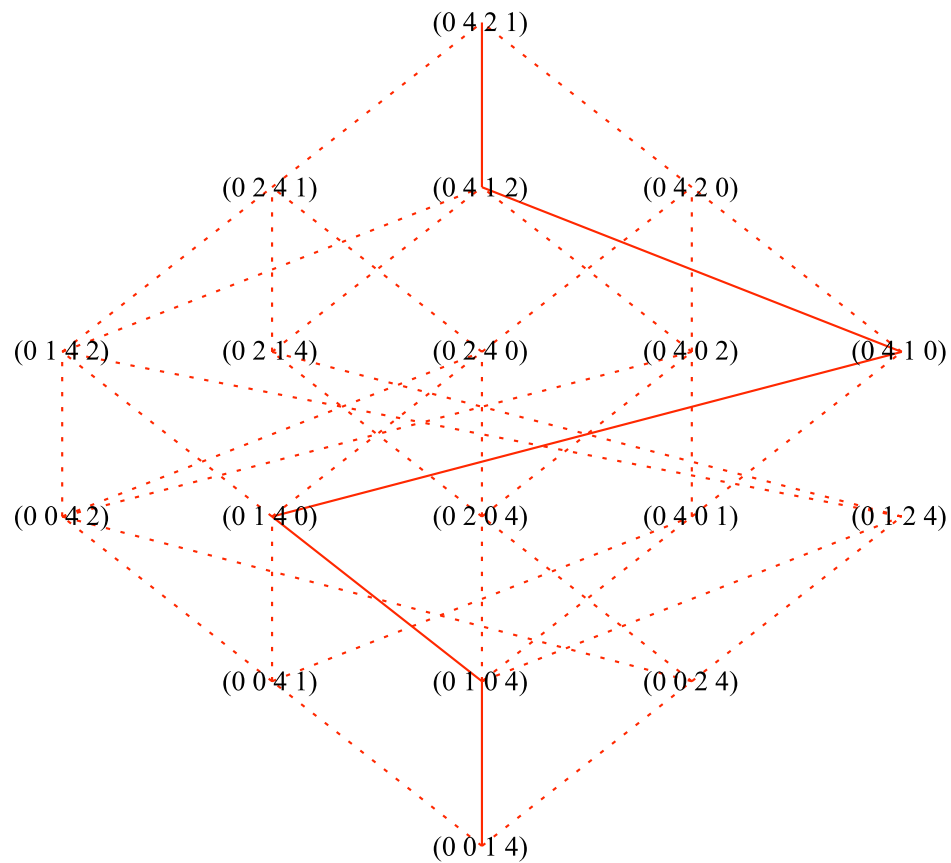
Figure 3.4: The interval $[(0\ 0\ 1\ 4), (0\ 4\ 2\ 1)]$ in R_4.

3.3 An Alternate Description of $W \times W$ Projections for $M_n(k)$

In this section, we describe (3.13) in terms of the combinatorial description of R_n from the end of the last section. The procedure we develop is for $We_iW \to We_{i+1}W$ whereas for (3.13) $e \leq f$ though e need not be covered by f. However, the procedure can be repeated, as necessary, to find the projection $WeW \to WfW$. We begin by describing the procedure, with examples, before going on to verify that it is satisfies the first two parts of Theorem 3.2.4.

3.3.1 The k-insertion Algorithm

Let $\sigma \in R_n$. Then $\sigma = (\alpha_1 \ \alpha_2 \ \ldots \ \alpha_n)$ where $\alpha_j \in \{0, 1, 2, \ldots, n\}$ and for $\alpha_i, \alpha_j \neq 0$, $\alpha_i \neq \alpha_j$ if $i \neq j$. Set $k = \min\{c \in \mathbb{Z}_{\geq 0} \mid c \neq \alpha_j, 1 \leq j \leq n\}$ and let $\{b_i\}$ be the longest decreasing subsequence, from right to left, in $\sigma = (\alpha_1 \ \ldots \ \alpha_n)$ such that $b_1 < k$. If $k = 0$, we do nothing. If $k \neq 0$, then we shift each term in the subsequence one spot to the left in the subsequence. For both cases the last term is 0, so the final step is to shift a 0 "out" of the sequence initially describing σ.

We call this the *k-insertion algorithm*. Given σ we denote the result of this process by $\beta(\sigma)$. By construction, if $\sigma \in We_iW$ then $\beta(\sigma) \in We_{i+1}W$, unless $i = n$ in which case $\beta(\sigma) = \sigma$. Note that $i = n$ if and only if $k = 0$.

In several of the proofs that follow, it will be convenient to keep track of what is being inserted into the sequence of terms. We use a bar over the entry to show that it is the entry being considered and denote what's being inserted inside square brackets. Note that from one step to the next, the corresponding sequence of $[i]$'s is decreasing (they are the terms from $\{b_i\}$). The following example illustrates this process.

Example 3.3.1. *Suppose $\sigma = (0\ 1\ 0\ 0\ 2\ 5\ 3\ 6)$. Then $k = 4$ and $b_1 = 3 > b_2 = 2 >$*

$b_3 = 0$. *We therefore obtain $\beta(\sigma)$ as follows:*

$$
\begin{aligned}
\sigma &= & (0\ 1\ 0\ 0\ 2\ 5\ 3\ 6) \\
&= & (0\ 1\ 0\ 0\ 2\ 5\ \bar{3}\ 6)[4] \\
&\rightarrow & (0\ 1\ 0\ 0\ \bar{2}\ 5\ 4\ 6)[3] \\
&\rightarrow & (0\ 1\ 0\ \bar{0}\ 3\ 5\ 4\ 6)[2] \\
&\rightarrow & (0\ 1\ 0\ 2\ 3\ 5\ 4\ 6)[0] \\
&= & (0\ 1\ 0\ 2\ 3\ 5\ 4\ 6) \\
&= & \beta(\sigma)
\end{aligned}
$$

We emphasize that the bars and brackets are strictly for bookkeeping purposes. For example, for $(0\ 1\ 0\ 0\ \bar{2}\ 5\ 4\ 6)[3]$ the element we are considering is $(0\ 1\ 0\ 0\ 2\ 5\ 4\ 6)$. This will be important to keep in mind for future work involving inequalities.

3.3.2 Verifying the Alternate Description

If $\sigma \in R_n$ is a Gauss-Jordan element, it is not difficult to see that this new description coincides with the projection maps from [24]. To do so, we observe that $\sigma = e_i y$ for some $e_i \in \Lambda$ and $y \in D(e_i)^{-1}$, (3.9). Since $W = S_n$, y is a permutation matrix of the form $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, where y_1 consists of i rows in row echelon form and the non-zero columns of y_2 are in row echelon form. If z_{e_i} is the longest element in $W_*(e_i)$, then $z_{e_i} = I_i \oplus J_{n-i}$, where J_{n-i} is the $(n-i) \times (n-i)$ permutation matrix with 1's on the top-right to bottom-left diagonal.

Row $(i+1)$ of $z_{e_i} y$ therefore has a 1 in the rightmost column containing all 0's in rows 1 through i. In the one-line notation, this means the rightmost 0 in σ now has an entry of $i+1$. Factoring to $u y'$, with $u \in W(e_{i+1})$ and $y' \in D(e_{i+1})^{-1}$, has the effect of putting the first $i+1$ rows of y' in row echelon form (the permutation matrix that does this, by switching appropriate rows, is u). This amounts to putting the non-zero entries in increasing order, in the one-line notation. Hence $p_{e_i, e_{i+1}}$ and the k-insertion algorithm yield the same output on Gauss-Jordan elements of R_n.

Example 3.3.2. *Let* $M = M_4(k)$. *If* $\sigma = (1\ 0\ 0\ 2) \in \mathcal{GJ}$, *then* $\beta(\sigma) = (1\ 0\ 2\ 3)$. *In terms of idempotents and quotient elements, we have* $\sigma = e_2(234)$, *or*

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proceeding with the projection map procedure, we factor $z_{e_2}y$ *as follows:*

$$
\begin{aligned}
z_{e_2}y &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\[2mm]
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\[2mm]
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\[2mm]
&= uy'
\end{aligned}
$$

That is, $uy' = (23)(243)$ *and so* $p_{e_2,e_3}(\sigma) = e_3(243) = \beta(\sigma)$.

We now show that the k-insertion algorithm and the projection maps from [24] coincide for all of R_n. To begin, we have the following result.

Proposition 3.3.3. *For* $\sigma \in R_n$, $\sigma \le \beta(\sigma)$.

Proof. The k-insertion algorithm is a repeated application of the first condition of Theorem 3.2.8. Hence $\sigma \le \beta(\sigma)$. $\qquad\square$

We next want to show that β is order-preserving on $W \times W$-orbits. That is, we want to show that for $\sigma, \theta \in We_iW$, if $\sigma \leq \theta$ then $\beta(\sigma) \leq \beta(\theta)$. Since $\beta(\sigma) = \sigma$ if $i = n$, we will assume $i < n$.

If $\sigma \leq \theta$ then $\sigma = \theta_0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_t = \theta$, where θ_{m+1} is obtained from θ_m via a change corresponding to one of the two conditions given in Theorem 3.2.6. The conditions given in Theorem 3.2.8 for this scenario can be stated equivalently as follows:

1. $\theta_m = (\alpha_1 \ \ldots \ \alpha_i \ \ldots \ \alpha_n)$ and $\theta_{m+1} = (\alpha_1 \ \ldots \ (\alpha_i + c) \ \ldots \ \alpha_n)$ for a permitted $c \in \mathbb{N}$.

2. $\theta_m = (\alpha_1 \ \ldots \ \alpha_i \ \ldots \ \alpha_j \ \ldots \ \alpha_n)$ and $\theta_{m+1} = (\alpha_1 \ \ldots \ \alpha_j \ \ldots \ \alpha_i \ \ldots \ \alpha_n)$ with $\alpha_i > \alpha_j$.

To show that order is preserved, we will show that $\beta(\theta_m) \leq \beta(\theta_{m+1})$ in both cases, as it then follows that $\beta(\sigma) \leq \beta(\theta)$. In doing so, we will need the following lemma.

Lemma 3.3.4. *Let* $\sigma = (\alpha_1 \ \alpha_2 \ \ldots \ \alpha_n) \in R_n$ *and* $c, d \in \{0, 1, 2, \ldots, n\} \backslash \{\alpha_i \mid \alpha_i \neq 0\}$, *such that* $0 \leq c < d \leq n$. *Let* σ_c *denote* σ *after* c *is inserted via the process described above (even though* c *might not be* k*), and* σ_d *likewise. Then* $\sigma_c < \sigma_d$.

Proof. If $c > \alpha_n$ then

$$(\alpha_1 \ \alpha_2 \ \ldots \ \alpha_{n-1} \ \overline{\alpha_n})[c] \ = \ (\alpha_1 \ \alpha_2 \ \ldots \ \alpha_{n-1} \ \overline{\alpha_n})[d]$$
$$(\alpha_1 \ \alpha_2 \ \ldots \ \overline{\alpha_{n-1}} \ c)[\alpha_n] \ < \ (\alpha_1 \ \alpha_2 \ \ldots \ \overline{\alpha_{n-1}} \ d)[\alpha_n]$$

where the inequality follows by Theorem 3.2.8 since $c < d$. For the rest of the process, we end up with the same entries and so the claim holds for this case.

If $d > \alpha_n > c$ then

$$(\alpha_1 \ \alpha_2 \ \ldots \ \alpha_{n-1} \ \overline{\alpha_n})[c] \ = \ (\alpha_1 \ \alpha_2 \ \ldots \ \alpha_{n-1} \ \overline{\alpha_n})[d]$$
$$(\alpha_1 \ \alpha_2 \ \ldots \ \overline{\alpha_{n-1}} \ \alpha_n)[c] \ < \ (\alpha_1 \ \alpha_2 \ \ldots \ \overline{\alpha_{n-1}} \ d)[\alpha_n]$$

where the inequality follows by Theorem 3.2.8 since $\alpha_n < d$, and we start the process over, now at α_{n-1} (inserting c and a_n, respectively, as indicated). At some point,

$\alpha_i < c$ and so we have the case described above. For this next step, the relation will not be equality, however the desired inequality is maintained.

If $\alpha_n > d$ then nothing changes in the first step, so we start the process over, now at α_{n-1} (still inserting c and d). At some point, $\alpha_i < d$ and so one of the two cases above holds. Again, the relation will not be equality at the next step, though we do have the inequality we want.

This procedure eventually terminates and we have the inequality, as claimed. $\square$

Proposition 3.3.5. *Let* $\theta_m = (\alpha_1 \ \alpha_2 \ \ldots \ \alpha_i \ \ldots \ \alpha_n)$ *and* $\theta_{m+1} = (\alpha_1 \ \alpha_2 \ \ldots \ (\alpha_i + c) \ \ldots \ \alpha_n)$, *for* $c \in \mathbb{N}$ *with* $0 < c \leq n - \alpha_i$ *such that* $\alpha_i + c \neq \alpha_j$ *for* $1 \leq j \leq n$. *Then* $\beta(\theta_m) \leq \beta(\theta_{m+1})$.

Proof. Let $k = \min\{d \in \mathbb{Z}_{\geq 0} \mid d \neq \alpha_j, 1 \leq j \leq n\}$ and let k' be likewise, for θ_{m+1}.
Case 1: $k = k'$

Everything proceeds the same for both until the ith entry, for some i. Therefore, without loss of generality, we will assume $i = n$.

If $k > \alpha_n + c$, then we have:

$$(\alpha_1 \ \alpha_2 \ \ldots \ \alpha_{n-1} \ \overline{\alpha_n})[k] \quad \leq \quad (\alpha_1 \ \alpha_2 \ \ldots \ \alpha_{n-1} \ \overline{(\alpha_n + c)})[k]$$
$$(\alpha_1 \ \alpha_2 \ \ldots \ \overline{\alpha_{n-1}} \ k)[\alpha_n] \quad \leq \quad (\alpha_1 \ \alpha_2 \ \ldots \ \overline{\alpha_{n-1}} \ k)[(\alpha_n + c)]$$

Note that at this stage, we actually have equality. However, by Lemma 3.3.4, at the end of the procedure we will have a (strict) inequality.

Suppose that $\alpha_n + c > k > \alpha_n$. Then we have:

$$(\alpha_1 \ \alpha_2 \ \ldots \ \alpha_{n-1} \ \overline{\alpha_n})[k] \quad \leq \quad (\alpha_1 \ \alpha_2 \ \ldots \ \alpha_{n-1} \ \overline{(\alpha_n + c)})[k]$$
$$(\alpha_1 \ \alpha_2 \ \ldots \ \overline{\alpha_{n-1}} \ k)[\alpha_n] \quad \leq \quad (\alpha_1 \ \alpha_2 \ \ldots \ \overline{\alpha_{n-1}} \ (\alpha_n + c))[k]$$

where again the inequality is preserved by Lemma 3.3.4, since what remains is equivalent to

$$(\alpha_1 \ \alpha_2 \ \ldots \ \overline{\alpha_{n-1}})[\alpha_n] \leq (\alpha_1 \ \alpha_2 \ \ldots \ \overline{\alpha_{n-1}})[k].$$

Finally suppose that $k < \alpha_n$. We then have:

$$(\alpha_1 \; \alpha_2 \; \ldots \; \alpha_{n-1} \; \overline{\alpha_n})[k] \;\; \leq \;\; (\alpha_1 \; \alpha_2 \; \ldots \; \alpha_{n-1} \; \overline{(\alpha_n + c)})[k]$$

$$(\alpha_1 \; \alpha_2 \; \ldots \; \overline{\alpha_{n-1}} \; \alpha_n)[k] \;\; \leq \;\; (\alpha_1 \; \alpha_2 \; \ldots \; \overline{\alpha_{n-1}} \; (\alpha_n + c))[k]$$

and for the remaining steps everything proceeds the same for both sides of the inequality, and so the inequality is preserved. We've considered all possibilities and have shown the inequality holds, moving one step left. We repeat as necessary, eventually obtaining $\beta(\theta_m) \leq \beta(\theta_{m+1})$ for this case.

Case 2: $k \neq k'$

Suppose $\alpha_i \neq 0$. Then $k' = \alpha_i$ and $k' < k$. We will assume that c is the smallest value in $\{1, 2, \ldots, n - \alpha_i\}$ such that $\alpha_i + c \neq \alpha_j$ for $1 \leq j \leq n$. For example, we would consider

$$(0 \; 1 \; 2 \; 0 \; 3) \rightarrow (0 \; 5 \; 2 \; 0 \; 3)$$

as first

$$(0 \; 1 \; 2 \; 0 \; 3) \rightarrow (0 \; 4 \; 2 \; 0 \; 3)$$

and then

$$(0 \; 4 \; 2 \; 0 \; 3) \rightarrow (0 \; 5 \; 2 \; 0 \; 3).$$

This assumption forces $k = \alpha_i + c$. Hence we want to show that the inequality is preserved in

$$(\alpha_1 \; \alpha_2 \; \ldots \; \alpha_i \; \ldots \overline{\alpha_n})[\alpha_i + c] \leq (\alpha_1 \; \alpha_2 \; \ldots \; (\alpha_i + c) \; \ldots \overline{\alpha_n})[\alpha_i].$$

Let $\{b_j\}$ be the largest decreasing subset of values in $(\alpha_1 \; \ldots \; \alpha_i \; \ldots \; \alpha_n)$ from right to left, starting with $\alpha_i + c$. That is, $\alpha_i + c > b_1 > \ldots > 0$. Let $\{d_j\}$ be likewise for $(\alpha_1 \; \ldots \; (\alpha_i + c) \; \ldots \; \alpha_n)$, starting with α_i. Note that $\{d_j\}$ is a subsequence of $\{b_j\}$. In fact, the last $|\{d_j\}|$ terms of $\{b_j\}$ are exactly $\{d_j\}$.

Suppose $\{d_j\}$ is a proper subsequence of $\{b_j\}$ and let b_ℓ be the smallest element in $\{b_j\}$ larger than d_1. For example, if we were considering $(0 \; 1 \; 2 \; 0 \; 3)$ and $(0 \; 4 \; 2 \; 0 \; 3)$, then $\alpha_i = k' = 1$ and $\alpha_i + c = k = 4$. The desired subsequences are $4 > b_1 = 3 > b_2 = 0$ and $1 > d_1 = 0$ and $b_\ell = b_1 = 3$.

Suppose $\alpha_i \notin \{b_j\}$. If d_1 is to the right of α_i then:

$$(\alpha_1 \ \ldots \ \alpha_i \ \ldots \ d_1 \ \ldots \ b_\ell \ \ldots \ b_2 \ \ldots \ b_1 \ \ldots \ \overline{\alpha_n})[\alpha_i + c]$$
$$\vdots$$
$$\leq \ (\alpha_1 \ \ldots \ \alpha_i \ \ldots \ b_\ell \ \ldots \ b_{\ell-1} \ \ldots \ b_1 \ \ldots \ (\alpha_i + c) \ \ldots \ \alpha_n)[d_1]$$
$$\vdots$$
$$\leq \ (\alpha_1 \ \ldots \ \alpha_i \ \ldots \ b_\ell \ \ldots \ b_{\ell-1} \ \ldots \ b_1 \ \ldots \ (\alpha_i + c) \ \ldots \ \alpha_n)$$
$$= \ \beta(\theta_m).$$

However, we may "undo" the changes in these extra b_j's, obtaining the following chain of inequalities:

$$\beta(\theta_m) \ = \ (\alpha_1 \ \ldots \ \alpha_i \ \ldots \ b_\ell \ \ldots \ b_{\ell-1} \ \ldots \ b_1 \ \ldots \ (\alpha_i + c) \ \ldots \ \alpha_n)$$
$$\leq \ (\alpha_1 \ \ldots \ \alpha_i \ \ldots \ b_\ell \ \ldots \ b_{\ell-1} \ \ldots \ (\alpha_i + c) \ \ldots \ b_1 \ \ldots \ \alpha_n)$$
$$\vdots$$
$$\leq \ (\alpha_1 \ \ldots \ \alpha_i \ \ldots \ b_\ell \ \ldots \ (\alpha_i + c) \ \ldots \ b_2 \ \ldots \ b_1 \ \ldots \ \alpha_n)$$
$$\leq \ (\alpha_1 \ \ldots \ \alpha_i \ \ldots \ (\alpha_i + c) \ \ldots \ b_\ell \ \ldots \ b_2 \ \ldots \ b_1 \ \ldots \ \alpha_n)$$
$$\leq \ (\alpha_1 \ \ldots \ (\alpha_i + c) \ \ldots \ \alpha_i \ \ldots \ b_\ell \ \ldots \ b_2 \ \ldots \ b_1 \ \ldots \ \alpha_n)$$
$$= \ \beta(\theta_{m+1}).$$

If d_1 is to the left of α_i then we would actually have $\alpha_i \in \{b_j\}$, which we initially assumed was not the case.

Suppose then that $\alpha_i \in \{b_j\}$. Then $b_\ell = \alpha_i$ and

$$(\alpha_1 \ \ldots \ d_1 \ \ldots \ \alpha_i \ \ldots \ b_{\ell-1} \ \ldots \ b_2 \ \ldots \ b_1 \ \ldots \ \overline{\alpha_n})[\alpha_i + c]$$
$$\vdots$$
$$\leq \ (\alpha_1 \ \ldots \ d_1 \ \ldots \ \overline{\alpha_i} \ \ldots \ b_{\ell-2} \ \ldots \ b_1 \ \ldots \ (\alpha_i + c) \ \ldots \ \alpha_n)[b_{\ell-1}]$$
$$\leq \ (\alpha_1 \ \ldots \ \overline{d_1} \ \ldots \ b_{\ell-1} \ \ldots \ b_{\ell-2} \ \ldots \ b_1 \ \ldots \ (\alpha_i + c) \ \ldots \ \alpha_n)[\alpha_i].$$

We next undo the changes due to $b_1 > \ldots > b_{\ell-1}$

$$(\alpha_1 \ldots \overline{d_1} \ldots b_{\ell-1} \ldots b_{\ell-2} \ldots b_1 \ldots (\alpha_i + c) \ldots \alpha_n)[\alpha_i]$$
$$\leq (\alpha_1 \ldots \overline{d_1} \ldots b_{\ell-1} \ldots b_{\ell-2} \ldots (\alpha_i + c) \ldots b_1 \ldots \alpha_n)[\alpha_i]$$
$$\vdots$$
$$\leq (\alpha_1 \ldots \overline{d_1} \ldots (\alpha_i + c) \ldots b_{\ell-1} \ldots b_2 \ldots b_1 \ldots \alpha_n)[\alpha_i]$$
$$= (\alpha_1 \ldots (\alpha_i + c) \ldots \overline{\alpha_n})[\alpha_i].$$

Note that if $\{d_j\} = \{b_j\}$ then there is no "undoing" of the extra b_i's and we arrive at

$$(\alpha_1 \ldots (\alpha_i + c) \ldots b_2 \ldots \alpha_i \ldots \alpha_n)[b_1]$$
$$\leq (\alpha_1 \ldots \alpha_i \ldots b_2 \ldots (\alpha_i + c) \ldots \alpha_n)[b_1].$$

Hence, if $\alpha_i \neq 0$ we've shown $\beta(\theta_m) \leq \beta(\theta_{m+1})$.

Suppose that $\alpha_i = 0$. Let k and k' be as usual. If there exists $\alpha_j = 0$ for $i \neq j$ then $k' > k = c$. That is

$$(\alpha_1 \ldots \alpha_i \ldots \overline{\alpha_n})[k] = (\alpha_1 \ldots 0 \ldots \overline{\alpha_n})[c]$$

and

$$(\alpha_1 \ldots (\alpha_i + c) \ldots \overline{\alpha_n})[k'] = (\alpha_1 \ldots c \ldots \overline{\alpha_n})[k'].$$

But now we have

$$(\alpha_1 \ldots 0 \ldots \overline{\alpha_n})[c] \leq (\alpha_1 \ldots 0 \ldots \overline{\alpha_n})[k'] \leq (\alpha_1 \ldots c \ldots \overline{\alpha_n})[k']$$

where the first inequality holds by Lemma 3.3.4 and the second by **Case 1** above.

If there is no $\alpha_j = 0$ for $i \neq j$, then $k = c$ and $k' = 0$. Let $\{b_i\}$ be as usual, where

$c > b_1 > \ldots > b_\ell > 0 = \alpha_i$. Then

$$(\alpha_1 \ \ldots \ 0 \ \ldots \ b_2 \ \ldots \ b_1 \ \ldots \ \overline{\alpha_n})[c]$$
$$\vdots$$
$$\leq \ (\alpha_1 \ \ldots \ 0 \ \ldots \ b_2 \ \ldots \ \overline{b_1} \ \ldots \ \alpha_n)[c]$$
$$\leq \ (\alpha_1 \ \ldots \ 0 \ \ldots \ \overline{b_2} \ \ldots \ c \ \ldots \ \alpha_n)[b_1]$$
$$\vdots$$
$$\leq \ (\alpha_1 \ \ldots \ b_\ell \ \ldots \ b_1 \ \ldots \ c \ \ldots \ \alpha_n)$$
$$= \ \beta(\theta_m)$$

and

$$\beta(\theta_m) \ = \ (\alpha_1 \ \ldots \ b_\ell \ \ldots \ b_1 \ \ldots \ c \ \ldots \ \alpha_n)$$
$$\leq \ (\alpha_1 \ \ldots \ b_\ell \ \ldots \ c \ \ldots \ b_1 \ \ldots \ \alpha_n)$$
$$\vdots$$
$$\leq \ (\alpha_1 \ \ldots \ c \ \ldots \ b_2 \ \ldots \ b_1 \ \ldots \ \alpha_n)$$
$$= \ (\alpha_1 \ \ldots \ c \ \ldots \ \overline{\alpha_n})[0]$$
$$= \ \beta(\theta_{m+1}).$$

So, if $\alpha_i = 0$ we've shown $\beta(\theta_m) \leq \beta(\theta_{m+1})$.

Now, finally, all cases have been covered and so the proof is complete. $\qquad\square$

Example 3.3.6. *Suppose $\theta_m = (0\ 2\ 0\ 3\ 1\ 4\ 5)$ and $\theta_{m+1} = (0\ 2\ 0\ 6\ 1\ 4\ 5)$. Then $k = 6$ and $k' = 3$ and we have the sequences $b_1 = 5 > b_2 = 4 > b_3 = 1 > b_4 = 0$ and $d_1 = 1 > d_2 = 0$. We note that $d_1 = 1$ is to the right of $\alpha_i = 3$ (and so $\alpha_i \notin \{b_i\}$). Thus*

$$\theta_m = (0\ 2\ 0\ 3\ 1\ 4\ 5) \to (0\ 2\ 1\ 3\ 4\ 5\ 6) = \beta(\theta_m)$$

and

$$\theta_{m+1} = (0\ 2\ 0\ 6\ 1\ 4\ 5) \to (0\ 2\ 1\ 6\ 3\ 4\ 5) = \beta(\theta_{m+1}).$$

Since $\{b_i\} \setminus \{d_i\} = \{5, 4\}$, we "undo" as follows (bold added for emphasis)

$$\beta(\theta_m) = (0\ 2\ 1\ 3\ \mathbf{4}\ \mathbf{5}\ \mathbf{6}) \leq (0\ 2\ 1\ 3\ \mathbf{4}\ \mathbf{6}\ 5) \leq (0\ 2\ 1\ 3\ \mathbf{6}\ \mathbf{4}\ 5)$$

and

$$(0\ 2\ 1\ 3\ 6\ 4\ 5) \leq (0\ 2\ 1\ 6\ 3\ 4\ 5) = \beta(\theta_{m+1}).$$

Example 3.3.7. *Suppose* $\theta_m = (0\ 0\ 1\ 0\ 2\ 3\ 6)$ *and* $\theta_{m+1} = (0\ 0\ 1\ 0\ 4\ 3\ 6)$. *Then* $k = 4$ *and* $k' = 2$ *and we have the sequences* $b_1 = 3 > b_2 = 2 > b_3 = 0$ *and* $d_1 = 0$. *We note that* $\alpha_i = 2 \in \{b_i\}$. *Thus*

$$\theta_m = (0\ 0\ 1\ 0\ 2\ 3\ 6) \rightarrow (0\ 0\ 1\ 2\ 3\ 4\ 6) = \beta(\theta_m)$$

and

$$\theta_{m+1} = (0\ 0\ 1\ 0\ 4\ 3\ 6) \rightarrow (0\ 0\ 1\ 2\ 4\ 3\ 6) = \beta(\theta_{m+1}).$$

Since $\{b_i\} \setminus \{d_i\} = \{3, 2\}$ *and* $\alpha_i = 2$, *we "undo" as follows (again, bold added for emphasis)*

$$\beta(\theta_m) = (0\ 0\ 1\ 2\ \mathbf{3}\ \mathbf{4}\ 6) \leq (0\ 0\ 1\ 2\ \mathbf{4}\ \mathbf{3}\ 6) = \beta(\theta_{m+1}).$$

Proposition 3.3.5 tells us that $\beta(\theta_m) \leq \beta(\theta_{m+1})$ for the first case described above. The following proposition addresses the second case.

Proposition 3.3.8. *Suppose*

$$\theta_m = (\alpha_1\ \alpha_2\ \ldots\ \alpha_i\ \ldots\ \alpha_j\ \ldots\ \alpha_n) < (\alpha_1\ \alpha_2\ \ldots\ \alpha_j\ \ldots\ \alpha_i\ \ldots\ \alpha_n) = \theta_{m+1}$$

where $\alpha_i < \alpha_j$. *Then* $\beta(\theta_m) \leq \beta(\theta_{m+1})$.

Proof. Let k, respectively k', be defined as in Proposition 3.3.5 for θ_m, respectively θ_{m+1}. Note that $k = k'$. We start the process, noting that all is the same until we get to the jth entry. So, without loss of generality, assume that $j = n$.
Case 1: $k > \alpha_n$

In this case we have:

$$(\alpha_1\ \alpha_2\ \ldots\ \alpha_i\ \ldots\ \alpha_{n-1}\ \overline{\alpha_n})[k] \ \leq \ (\alpha_1\ \alpha_2\ \ldots\ \alpha_n\ \ldots\ \alpha_{n-1}\ \overline{\alpha_i})[k]$$
$$(\alpha_1\ \alpha_2\ \ldots\ \alpha_i\ \ldots\ \overline{\alpha_{n-1}}\ k)[\alpha_n] \ \leq \ (\alpha_1\ \alpha_2\ \ldots\ \alpha_n\ \ldots\ \overline{\alpha_{n-1}}\ k)[\alpha_i]$$

where the inequality holds since $\alpha_i < \alpha_n$. But this is the same as

$$(\alpha_1\ \alpha_2\ \ldots\ \alpha_i\ \ldots\ \alpha_{n-1}\ \overline{k})[\alpha_n] \leq (\alpha_1\ \alpha_2\ \ldots\ \alpha_n\ \ldots\ \alpha_{n-1}\ \overline{k})[\alpha_i].$$

Considering α_n as $\alpha_i + c$, we know by Proposition 3.3.5 that the inequality is preserved throughout the process. That is, $\beta(\theta_m) \leq \beta(\theta_{m+1})$.

Case 2: $k < \alpha_i$

In this case, α_i and α_n remain fixed during the process. That is, if we let $\{b_i\}$ be as in Proposition 3.3.5 then $\alpha_n > \alpha_i > k > b_1 > b_2 > \ldots > 0$. Hence,

$$(\alpha_1 \ \ldots \ b_\ell \ \ldots \ \alpha_i \ \ldots \ b_2 \ \ldots \ b_1 \ \ldots \ \overline{\alpha_n})[k] \ \leq \ (\alpha_1 \ \ldots \ b_\ell \ \ldots \ \alpha_n \ \ldots \ b_2 \ \ldots \ b_1 \ \ldots \ \overline{\alpha_i})[k]$$
$$(\alpha_1 \ \ldots \ b_\ell \ \ldots \ \alpha_i \ \ldots \ \overline{b_2} \ \ldots \ k \ \ldots \ \alpha_n)[b_1] \ \leq \ (\alpha_1 \ \ldots \ b_\ell \ \ldots \ \alpha_n \ \ldots \ \overline{b_2} \ \ldots \ k \ \ldots \ \alpha_i)[b_1]$$
$$\vdots$$
$$(\alpha_1 \ \ldots \ b_{\ell-1} \ \ldots \ \alpha_i \ \ldots \ b_1 \ \ldots \ k \ \ldots \ \alpha_n) \ \leq \ (\alpha_1 \ \ldots \ b_{\ell-1} \ \ldots \ \alpha_n \ \ldots \ b_1 \ \ldots \ k \ \ldots \ \alpha_i)$$

And so for this situation, $\beta(\theta_m) \leq \beta(\theta_{m+1})$.

Case 3: $\alpha_i < k < \alpha_n$

In this case we have:

$$(\alpha_1 \ \alpha_2 \ \ldots \ \alpha_i \ \ldots \ \alpha_{n-1} \ \overline{\alpha_n})[k] \ \leq \ (\alpha_1 \ \alpha_2 \ \ldots \ \alpha_n \ \ldots \ \alpha_{n-1} \ \overline{\alpha_i})[k]$$
$$(\alpha_1 \ \alpha_2 \ \ldots \ \alpha_i \ \ldots \ \overline{\alpha_{n-1}} \ \alpha_n)[k] \ \leq \ (\alpha_1 \ \alpha_2 \ \ldots \ \alpha_n \ \ldots \ \overline{\alpha_{n-1}} \ k)[\alpha_i].$$

However,

$$(\alpha_1 \ \alpha_2 \ \ldots \ \alpha_i \ \ldots \ \alpha_{n-1} \ \overline{\alpha_n})[k] \ \leq \ (\alpha_1 \ \alpha_2 \ \ldots \ k \ \ldots \ \alpha_{n-1} \ \overline{\alpha_n})[\alpha_i]$$

where the inequality is maintained throughout the process, by Proposition 3.3.5 (considering k as $\alpha_i + c$). Additionally,

$$(\alpha_1 \ \alpha_2 \ \ldots \ k \ \ldots \ \alpha_{n-1} \ \overline{\alpha_n})[\alpha_i] \ \leq \ (\alpha_1 \ \alpha_2 \ \ldots \ \alpha_n \ \ldots \ \alpha_{n-1} \ \overline{k})[\alpha_i]$$

where the inequality is maintained throughout the process, by **Case 2** (since α_i is smaller than k and α_n). Thus we have:

$$\beta(\theta_m) \leq \beta(\alpha_1 \ \alpha_2 \ \ldots \ k \ \ldots \ \alpha_{n-1} \ \alpha_n) \leq \beta(\theta_{m+1}).$$

All the cases have been covered, hence $\theta_m \leq \theta_{m+1}$. □

As noted above, Propositions 3.3.5 and 3.3.8 imply the following.

Theorem 3.3.9. *For $\sigma, \theta \in We_iW$, $\sigma \leq \theta \Rightarrow \beta(\sigma) \leq \beta(\theta)$.*

Example 3.3.10. *Suppose $\theta_m = (0\ 1\ 0\ 2\ 3\ 5)$ and $\theta_{m+1} = (0\ 5\ 0\ 2\ 3\ 1)$. Then $\alpha_2 = 1 \leq k = 4 \leq \alpha_6 = 5$ and we have*

$$
\begin{aligned}
(0\ 1\ 0\ 2\ 3\ \bar{5})[4] &\leq (0\ 5\ 0\ 2\ 3\ \bar{1})[4] \\
(0\ 1\ 0\ 2\ \bar{3}\ 5)[4] &\leq (0\ 5\ 0\ 2\ \bar{3}\ 4)[1]
\end{aligned}
$$

but

$$
\begin{aligned}
(0\ 1\ 0\ 2\ 3\ \bar{5})[4] &\leq (0\ 4\ 0\ 2\ 3\ \bar{5})[1] \\
&\ \ \vdots \\
(0\ 1\ 2\ 3\ 4\ 5) &\leq (0\ 4\ 1\ 2\ 3\ 5)
\end{aligned}
$$

and

$$
\begin{aligned}
(0\ 4\ 0\ 2\ 3\ \bar{5})[1] &\leq (0\ 5\ 0\ 2\ 3\ \bar{4})[1] \\
&\ \ \vdots \\
(0\ 4\ 1\ 2\ 3\ 5) &\leq (0\ 5\ 1\ 2\ 3\ 4)
\end{aligned}
$$

and so

$$
\beta(\theta_m) = (0\ 1\ 2\ 3\ 4\ 5) \leq (0\ 5\ 1\ 2\ 3\ 4) = \beta(\theta_{m+1})
$$

This brings us to our final result.

Theorem 3.3.11. *For $\sigma \in We_iW$, $\theta \in We_{i+1}W$, $\sigma \leq \theta \iff \beta(\sigma) \leq \theta$.*

Proof. Suppose $\beta(\sigma) \leq \theta$. Then, by Proposition 3.3.3, $\sigma \leq \beta(\sigma)$ and so $\sigma \leq \theta$.

Suppose then that $\sigma \leq \theta$. By Theorem 3.2.6, there exist $\theta_0, \theta_1, \ldots, \theta_m \in R$ such that

$$
\sigma = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_m = \theta, \tag{3.14}
$$

where each term in the string of inequalities is obtained from the previous one via one of the two described methods.

Considering (3.14), we observe that there must exist a j, $0 \leq j \leq m - 1$, such that $\theta_j \in W e_i W$ and $\theta_{j+1} \in W e_{i+1} W$. θ_{j+1} is therefore obtained from θ_j by changing a 0 in θ_j to a suitable non-zero. That is,

$$\theta_j = (\alpha_1 \; \alpha_2 \; \ldots \; 0 \; \ldots \; \alpha_n)$$
$$\theta_{j+1} = (\alpha_1 \; \alpha_2 \; \ldots \; c \; \ldots \; \alpha_n).$$

Now $c \geq k$, where $k = \min\{a \in \mathbb{Z}_{\geq 0} \mid a \neq \alpha_j, 1 \leq j \leq n\}$, as previously defined. If $c > k$, then $\theta_j \leq \theta_j^* \leq \theta_{j+1}$, where

$$\theta_j^* = (\alpha_1 \; \alpha_2 \; \ldots \; k \; \ldots \; \alpha_n).$$

Thus, without loss of generality, we may assume that $c = k$ and so

$$\theta_{j+1} = (\alpha_1 \; \alpha_2 \; \ldots \; k \; \ldots \; \alpha_n).$$

Suppose there is a 0 to the right of the 0 identified in θ_j. Then $\theta_j \leq \theta_j^* \leq \theta_{j+1}$, where

$$\theta_j = (\alpha_1 \; \alpha_2 \; \ldots \; 0 \; \ldots \; 0 \; \ldots \; \alpha_n)$$
$$\theta_j^* = (\alpha_1 \; \alpha_2 \; \ldots \; 0 \; \ldots \; k \; \ldots \; \alpha_n)$$
$$\theta_{j+1} = (\alpha_1 \; \alpha_2 \; \ldots \; k \; \ldots \; 0 \; \ldots \; \alpha_n).$$

Thus, without loss of generality, we may assume that there is no 0 to the right of the 0 identified in θ_j.

Let $\{b_i\}$ be the longest decreasing subsequence from right to left in θ_j, with $b_1 < k$. The final term at the end of this sequence, $b_\ell = 0$, corresponds to the 0 under discussion in θ_j. Now

$$\theta_j = (\alpha_1 \; \alpha_2 \; \ldots \; 0 \; \ldots \; b_2 \; \ldots \; b_1 \; \ldots \; \alpha_n)$$

and

$$\begin{aligned} \beta(\theta_j) &= (\alpha_1 \ \alpha_2 \ \ldots \ b_{\ell-1} \ \ldots \ b_1 \ \ldots \ k \ \ldots \ \alpha_n) \\ &< (\alpha_1 \ \alpha_2 \ \ldots \ b_{\ell-1} \ \ldots \ k \ \ldots \ b_1 \ \ldots \ \alpha_n) \\ &\vdots \\ &< (\alpha_1 \ \alpha_2 \ \ldots \ k \ \ldots \ b_2 \ \ldots \ b_1 \ \ldots \ \alpha_n) \\ &= \theta_{j+1}. \end{aligned}$$

So $\beta(\theta_j) \leq \theta_{j+1}$, and thus

$$\sigma \leq \theta_1 \leq \cdots \leq \theta_j \leq \beta(\theta_j) \leq \theta_{j+1} \leq \cdots \leq \theta_m = \theta.$$

Now $\sigma \leq \theta_j$ with $\sigma, \theta_j \in We_iW$ and so, by Theorem 3.3.9, $\beta(\sigma) \leq \beta(\theta_j)$ and thus $\beta(\sigma) \leq \theta$. This completes the proof. $\qquad \square$

Table 3.1 shows the projections from rank 1 elements to rank 2 elements in R_3. The first two columns are found using the method we've described. Observe that they are indentical to the results of the last two columns, as given in [24]. Figure 3.5 is the Hasse diagram of R_3 from Figure 3.1, but now with the $W \times W$ orbits in different colors with the projections between the orbits in dotted lines.

Table 3.1: Projections from rank 1 to rank 2, in R_3.

| σ	$\beta(\sigma)$	σ	$p_{e_1,e_2}(\sigma)$
$(1\ 0\ 0)$	$(1\ 0\ 2)$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$(0\ 1\ 0)$	$(0\ 1\ 2)$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$(0\ 0\ 1)$	$(0\ 1\ 2)$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$(2\ 0\ 0)$	$(2\ 0\ 1)$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$(0\ 2\ 0)$	$(0\ 2\ 1)$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$(0\ 0\ 2)$	$(0\ 1\ 2)$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$(3\ 0\ 0)$	$(3\ 0\ 1)$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$(0\ 3\ 0)$	$(0\ 3\ 1)$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$(0\ 0\ 3)$	$(0\ 1\ 3)$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
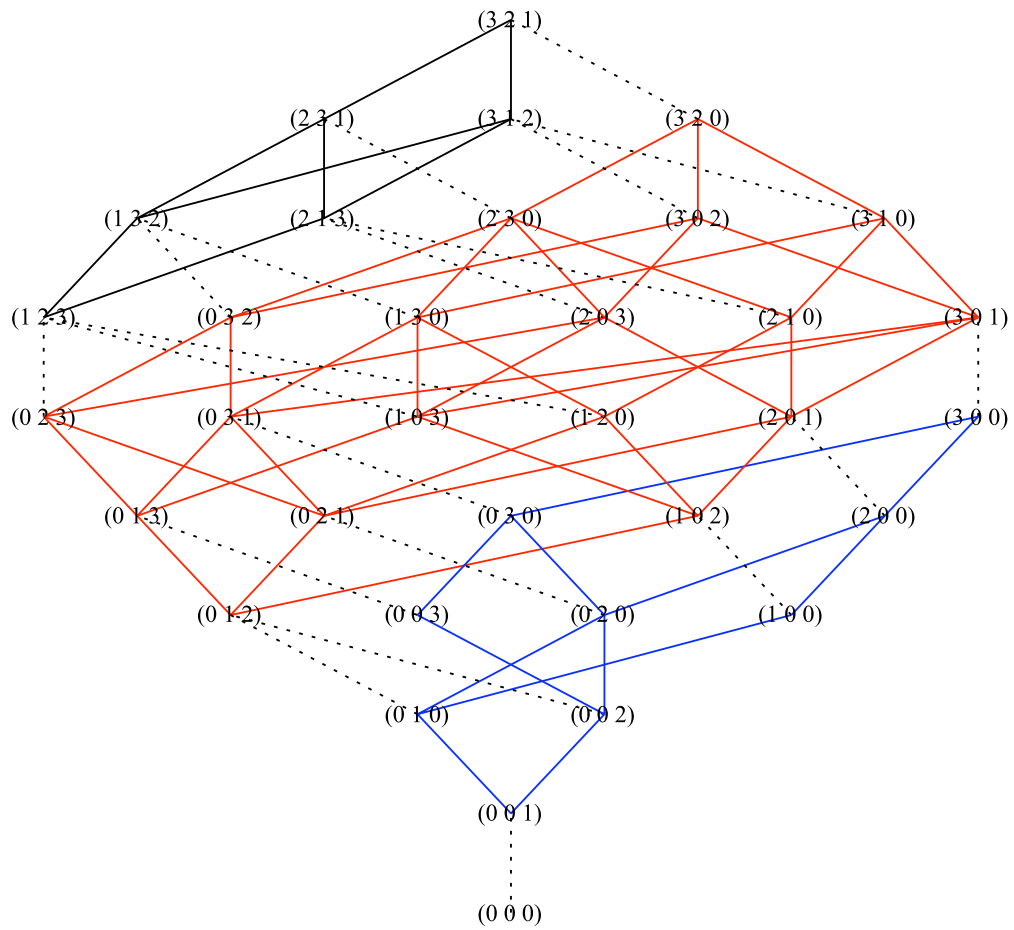
Figure 3.5: R_3 under Bruhat-Chevalley Order.

Chapter 4

The Conjugacy Poset

The study of conjugacy classes of a reductive monoid was initiated by Putcha in [18], with subsequent contributions in [21], [22], and [25]. In [18], for each $(e, w) \in \Lambda \times W$ a subset $M_{e,w} \subseteq M$ is defined such that every $x \in M$ is conjugate to some $y \in M_{e,w}$, for some $(e, w) \in \Lambda \times W$. Furthermore, for each $M_{e,w}$ there exists a reductive group $G_{e,w}$ with $\xi : M_{e,w} \to G_{e,w}$ such that if x and y are in $M_{e,w}$, then $x \sim y$ in M if and only if $\xi(x)$ and $\xi(y)$ are σ-conjugate in $G_{e,w}$ (for an automorphism σ of $G_{e,w}$). It turns out that

$$M = \bigcup_{(e,w)\in\Lambda\times W,\ g\in G} gM_{e,w}g^{-1}.$$

In [22], a subset of $\Lambda \times W$ is found such that the union is disjoint. Additionally, a partial order is defined on this subset. The resulting decomposition is therefore called the *conjugacy decomposition* of M and we call the partial order the *conjugacy order*.

As motivation for the general theory, in Section 4.1 we describe this decomposition and the associated partial order for the case that $M = M_n(k)$. After this brief introduction, we go on to describe the general theory, with some initial results, in Section 4.2. The final three sections of the chapter consist of an initial analysis for the three classes of monoids we will consider in this paper: $M_n(k)$, dual canonical monoids, and canonical monoids.

4.1   The Motivating Example

Let $M = M_n(k)$ with $a, b \in M_n(k)$. Define $a \equiv b$ if the rank of a^i equals the rank of b^i for $1 \le i \le n$ or, equivalently, if a and b have the same nilpotent blocks in the Jordan form. This is an equivalence relation and we index the equivalence classes by the partitions $\mathcal{P} = \{\alpha \vdash m \mid 0 \le m \le n\}$. Specifically, if we let $X(\alpha)$ denote the set of matrices whose nilpotent blocks in the Jordan form correspond to α, we obtain the following decomposition:

$$M_n(k) = \bigsqcup_{\alpha \in \mathcal{P}} X(\alpha) \tag{4.1}$$

This is the *conjugacy decomposition* of $M_n(k)$.

Given $\alpha, \beta \in \mathcal{P}$ with $a \in X(\alpha)$ and $b \in X(\beta)$, we define a partial order on $\mathcal{P}$ by declaring $\alpha \le \beta$ if $\mathrm{rk}(a^i) \le \mathrm{rk}(b^i)$ for $1 \le i \le n$. In terms of partitions, $\alpha \le \beta$ if for $\alpha \vdash m$ with $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta \vdash \ell$ with $\beta = (\beta_1, \beta_2, \ldots)$, we have:

$$
\begin{aligned}
n - m &\le n - \ell \\
n - m + \alpha_1 &\le n - \ell + \beta_1 \\
n - m + \alpha_1 + \alpha_2 &\le n - \ell + \beta_1 + \beta_2 \\
&\vdots
\end{aligned}
$$

The poset $(\mathcal{P}, \le)$ is the *conjugacy poset* of $M_n(k)$. We will revisit this case in Section 4.3 before going on to a more thorough analysis in the following chapters.

4.2   Background

The first in depth analysis of the conjugacy decomposition of a reductive monoid M appears in [22]. We therefore begin with a summary of these main results. We follow with the description of the conjugacy order, as presented in [22]. Before going on to describe new findings, we summarize a few results from [25] that describe the order. In doing so, it will be necessary to restate some results presented above that appear in [25] in a slightly different manner.

4.2.1 The Conjugacy Decomposition of M

Let M be a reductive monoid, with unit group G and W the Weyl group of G. Let $S \subseteq W$ be the set of simple reflections of W. By $K \triangleleft I$, for $I \subseteq S$, we mean that K is a union of connected components of I. Let $I, J \subseteq S$ and $y \in D_I^{-1}$. Then, by [2], $W_I \cap yW_Jy^{-1}$ is a standard parabolic subgroup of W. In particular, we have:

$$
\begin{aligned}
W_I \cap yW_Iy^{-1} &= W_{I_1}, & I_1 \subseteq I \\
W_I \cap yW_{I_1}y^{-1} &= W_{I_2}, & I_2 \subseteq I_1 \\
W_I \cap yW_{I_2}y^{-1} &= W_{I_3}, & I_3 \subseteq I_2 \\
&\vdots
\end{aligned}
$$

If $K \triangleleft I$ we therefore have:

$$
\begin{aligned}
W_I \cap yW_Ky^{-1} &= W_{K_1}, & K_1 \triangleleft I_1 \\
W_I \cap yW_{K_1}y^{-1} &= W_{K_2}, & K_2 \triangleleft I_2 \\
W_I \cap yW_{K_2}y^{-1} &= W_{K_3}, & K_3 \triangleleft I_3 \\
&\vdots
\end{aligned}
$$

Using this, we now define a set $D_I^*(K) \subseteq D_I^{-1}$ that will be crucial to the theory:

$$
D_I^*(K) = \{y \in D_I^{-1} \mid y \in D_{K_j} \text{ for all } j \geq 0\}.
$$

Remark 4.2.1. *Observe that $D_I^*(\emptyset) = D_I^{-1}$ and $D_I^*(I) = D_I \cap D_I^{-1}$.*

To emphasize the idempotents involved, we use a previously described notation for parabolic subgroups of W. That is, we let $W(e) = W_I$, where $I = \lambda(e)$ for some $e \in \Lambda$ and $W_*(e) = W_K$ for $K = \lambda_*(e) \triangleleft I$. Additionally, let

$$
\begin{aligned}
D(e) &= D_I, & (4.2) \\
D^*(e) &= D_I^*(K), & (4.3) \\
R^* &= \{ey \mid e \in \Lambda, y \in D^*(e)\}. & (4.4)
\end{aligned}
$$

Then, for $y \in D(e)^{-1}$ we define

$$
\begin{aligned}
H &= C_G(zez^{-1} \mid z \in \langle y \rangle), \\
M(ey) &= eHy, \\
X(ey) &= \bigcup_{g \in G} gM(ey)g^{-1}.
\end{aligned}
$$

This brings us to the following theorem, from [22]. As usual, G is the unit group of M with B a Borel subgroup of G.

Theorem 4.2.2. *Let $e \in \Lambda$. Then*

1. *If $y \in D(e)^{-1}$, then $X(ey) = \bigcup_{g \in G} gBeyBg^{-1}$.*

2. *$GeG = \bigsqcup_{y \in D^*(e)} X(ey)$.*

3. *$M = \bigsqcup_{\sigma \in R^*} X(\sigma)$.*

The third part of the theorem describes the conjugacy decomposition. It follows immediately from the second part and (3.3). Note the similarity of this decomposition with the Bruhat-Renner decomposition of R, (3.4). Specifically, for the former decomposition we have double cosets $B\sigma B$ while for the latter we have unions of conjugates of $B\sigma B$.

4.2.2 The Conjugacy Order of M

As previously noted, there is a partial order defined on the indexing set for the conjugacy decomposition. Before describing this order, we first define the transitive relation $\preceq$ on R generated by the following:

1. If $\sigma \leq \theta$, then $\sigma \preceq \theta$.

2. If $y \in D(e)^{-1}$, $x \in W$, then $eyx \preceq xey$.

This relation is very important to us. Namely, it describes the conjugacy order, once we pick the correct subset of R. The following theorem, from [22], makes this clear.

Theorem 4.2.3.   *1. $\preceq$ is a partial order on R^*.*

2. If $\sigma, \theta \in R^*$, then $\sigma \preceq \theta$ if and only if $X(\sigma) \subseteq \overline{X(\theta)}$.

3. If $\sigma \in R^*$, then $\overline{X(\sigma)} = \bigsqcup_{\sigma' \preceq \sigma} X(\sigma')$.

Thus R^* serves as our indexing set for the classes in the conjugacy decomposition of M. Our first description of the Bruhat-Chevalley order on R, (3.6), said that if $\sigma, \theta \in R$, then $\sigma \leq \theta$ if $B\sigma B \subseteq \overline{B\theta B}$. The conjugacy order is therefore the natural generalization: for $\sigma, \theta \in R^*$, $\sigma \preceq \theta$ if $\bigcup_{g \in G} gB\sigma Bg^{-1} \subseteq \overline{\bigcup_{g \in G} gB\theta Bg^{-1}}$. We emphasize that the relation $\preceq$ is a partial order on R^*, not all of R.

Example 4.2.4. *Let $M = M_3(k)$ with $\sigma = e_1(123) = e_1(12)(23)$ and $\theta = e_1(12)$. Since $\sigma \leq \theta$, $\sigma \preceq \theta$. On the other hand, since $(23) \in W_*(e_1)$, we have*

$$\theta = \sigma(23) \preceq (23)\sigma = \sigma.$$

As $\sigma \neq \theta$, we see that anti-symmetry does not hold and thus $\preceq$ is not a partial order on R.

The structure theory of the conjugacy decomposition, as presented thus far, is therefore the study of R^* under $\preceq$. In [25], the order is described in terms of Gauss-Jordan elements (of which R^* is a subset). We explore this approach now.

4.2.3   A Second Description of the Conjugacy Order

Let M be a reductive monoid, with unit group G, Borel subgroup B, and Renner monoid R. For $\sigma \in R$, we let $X(\sigma)$ be as before. That is,

$$X(\sigma) = \bigcup_{x \in G} x^{-1} B\sigma Bx$$

and so

$$\overline{X(\sigma)} = \bigcup_{\theta \leq \sigma} X(\theta).$$

As noted in [22], $\overline{X(\sigma)}$ is a closed irreducible subset of M. If $\sigma, \theta \in R$, we define $\sigma \approx \theta$ if $\overline{X(\sigma)} = \overline{X(\theta)}$ and $\sigma \preceq \theta$ if $X(\sigma) \subseteq \overline{X(\theta)}$. Using this, we observe the following, from [25]:

Theorem 4.2.5. *1. If $\sigma \in R$, then $\sigma \approx \theta$ for some $\theta \in \mathcal{GJ}$.*

2. If $\sigma, \theta \in \mathcal{GJ}$, then

$$\sigma \approx \theta \iff \sigma \sim \theta \text{ in } R \iff X(\sigma) = X(\theta) \tag{4.5}$$

In the second part of the theorem, by $\sigma \sim \theta$ in R we mean that $w\sigma w^{-1} = \sigma'$ for some $w \in W$. Using this relation, define $\tilde{R} = \mathcal{GJ}/\sim$ and denote the class containing $\sigma \in \mathcal{GJ}$ by $[\sigma]$. That is, let

$$[\sigma] = \{\sigma' \in \mathcal{GJ} \mid \sigma \sim \sigma'\}. \tag{4.6}$$

By (3.9), if $\sigma, \sigma' \in \mathcal{GJ}$, then $\sigma = ey$ and $\sigma' = fz$ for some $e, f \in \Lambda$, $y \in D(e)^{-1}$, and $z \in D(f)^{-1}$. If $\sigma \sim \sigma'$, then $w\sigma w^{-1} = \sigma'$ and so we must have $e = f$ and $w \in W(e)$. We may therefore write $[\sigma]$ as:

$$[ey] = \{ey' \mid y' \in D(e)^{-1} , \ ey \sim ey'\}. \tag{4.7}$$

Now, for $\sigma \in R$ we let $\tilde{p}(\sigma) = [\sigma_1]$, where $\sigma_1 \in \mathcal{GJ}$ with $\sigma \approx \sigma_1$. This is well defined, since if $\sigma \approx \sigma_2 \in \mathcal{GJ}$ as well, then $\sigma_1 \approx \sigma_2$ and $\sigma_1 \sim \sigma_2$ in R, by (4.5), and hence $[\sigma_1] = [\sigma_2]$. By Theorem 4.2.5, $\preceq$ induces a partial order $\leq$ on $\tilde{R}$:

$$[\sigma_1] \leq [\sigma_2] \text{ if } \sigma_1 \preceq \sigma_2. \tag{4.8}$$

This brings us to the following result, from [25].

Theorem 4.2.6. *1. $\tilde{p} : R \to \tilde{R}$ is an order-preserving map.*

2. $M = \bigsqcup_{[\sigma] \in \tilde{R}} X(\sigma).$

3. If $[\sigma] \in \tilde{R}$, then $\overline{X(\sigma)} = \bigsqcup_{[\theta] \leq [\sigma]} X(\theta).$

4. *The decomposition in each part above is independent of the choice of the Borel subgroup B.*

As noted in [25], Theorem 4.2.6 yields an equivalence relation on M:

$$a \equiv b \ \text{ if } \ a, b \in X(\sigma) \ \text{ for some } \ [\sigma] \in \tilde{R}. \tag{4.9}$$

The theorem also tells us that the elements of $\tilde{R}$ serve as an indexing set for the conjugacy decomposition of M. This is convenient because of the next theorem, from [25], which tells how we may describe the order on $\tilde{R}$ in terms of the Bruhat-Chevalley order on R.

Theorem 4.2.7. *Let $[\sigma], [\theta] \in \tilde{R}$. Then $[\sigma] \leq [\theta]$ if and only if $a\sigma a^{-1} \leq \theta$ for some $a \in W$.*

We can refine this, using the result of [14] described in Theorem 3.2.1:

Corollary 4.2.8. *Let $[\sigma] \in \tilde{R}(e)$, $[\theta] \in \tilde{R}(f)$. Then $[\sigma] \leq [\theta]$ if and only if $a\sigma a^{-1} \leq \theta$ for some $a \in W(f)W_*(e)$.*

Proof. Suppose $[\sigma] \leq [\theta]$ with $[\sigma] \in \tilde{R}(e)$ and $[\theta] \in \tilde{R}(f)$. Then $[\sigma] = [ey]$ and $[\theta] = [fy']$ for some $y \in D(e)^{-1}$ and $y' \in D(f)^{-1}$. By Theorem 4.2.7, $aeya^{-1} \leq fy'$ for some $a \in W$. Then $aeya^{-1} = aa_1ey_1$, where $a_1 \in W(e)$, and $y_1 \in D(e)^{-1}$. By Theorem 3.2.1, there exists $w \in W(f)W_*(e)$ such that $aa_1 \leq w$, and hence $a \in W(f)W_*(e)$.

The other direction follows from Theorem 4.2.7 above, and is proved in [25]. □

We are now equipped for our analysis of the conjugacy poset. Because of Corollary 4.2.8, when discussing the conjugacy poset of M, we usually have $(\tilde{R}, \leq)$ in mind rather than $(R^*, \preceq)$, though R^* will be important in upcoming results.

In studying the Bruhat-Chevalley order on the Renner monoid, a successful approach has been to first examine WeW for $e \in \Lambda$, [24], and then find maps connecting WeW to WfW, for $e, f \in \Lambda$, [25]. We follow a similar approach here. That is, we analyze the structure within $\tilde{R}(e)$ in Chapter 5 before attempting to make connections between $\tilde{R}(e)$ and $\tilde{R}(f)$ in Chapter 6. However, as promised, to finish this chapter

we provide further descriptions of $\tilde{R}$ for the three classes of reductive monoids we will pursue in the following chapters: $M_n(k)$, dual canonical monoids, and canonical monoids.

4.3 Matrices

The first class of reductive monoids we consider is the most familiar one, $M_n(k)$. Let $a, b \in M_n(k)$ and $[a]$ denote the $\equiv$-class of a, where $\equiv$ is as in (4.9). For $m \geq 1$, let $J_m(0)$ denote the $m \times m$ nilpotent Jordan block. Then

$$
\begin{aligned}
a &\equiv I_\ell \oplus J_{\alpha_1}(0) \oplus \cdots \oplus J_{\alpha_r}(0) \ , \ \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \\
b &\equiv I_{\ell'} \oplus J_{\beta_1}(0) \oplus \cdots \oplus J_{\beta_r}(0) \ , \ \beta_1 \geq \beta_2 \geq \cdots \geq \beta_s.
\end{aligned}
$$

The following result is the combination of two theorems from Section 3 of [25].

Theorem 4.3.1. *The following conditions are equivalent:*

1. $a \equiv b$.

2. $\mathrm{rk}(a^i) \leq \mathrm{rk}(b^i)$ for $i \geq 1$.

3. $\ell \leq \ell'$, $\ell + \alpha_1 \leq \ell' + \beta_1$, $\ell + \alpha_1 + \alpha_2 \leq \ell' + \beta_1 + \beta_2$, ...

4. $[a] \subseteq \overline{[b]}$.

This theorem tells us that the theory developed in Section 4.2 does in fact coincide with the description of the conjugacy poset of $M_n(k)$ in Section 4.1 that motivated the theory. In the remainder of this section, we develop new ideas that when used with this description will provide insight into the structure of $\tilde{R}$.

Conjugacy Classes of Gauss-Jordan elements in R_n

We begin with a description of the conjugacy classes of the Gauss-Jordan elements in the Renner monoid R_n of $M_n(k)$. Recall that these elements are the partial permutation matrices of $M_n(k)$, in row echelon form. Every element of $M_n(k)$ is similar

to a unique matrix in the Jordan form. For $\sigma \in \mathcal{GJ}_n$, we have

$$\sigma \sim J_1(1) \oplus \cdots \oplus J_1(1) \oplus J_{\alpha_1}(0) \oplus \cdots \oplus J_{\alpha_k}(0) \tag{4.10}$$

where $J_1(1) \oplus \cdots \oplus J_1(1) = I_d$ for some d and $\alpha = (\alpha_1, \ldots \alpha_k)$ is a partition of $n - d$.

When considering $\sigma \in R_n$, it will be useful to know the size of the diagonalizable part (d above) in addition to the partition of the nilpotent part (α above). We record this data by assigning to σ the pair $(d; \alpha)$, which we will usually write as $(d; \alpha_1, \alpha_2, \ldots, \alpha_k)$. Although this $(k + 1)$-tuple is usually not a partition, as $d < \alpha_1$ is possible, we nevertheless call this the *Jordan partition* of σ. Two Gauss-Jordan elements are conjugate if and only if they have the same Jordan partition, and so when referring to the Jordan partition of $[\sigma]$, we mean the Jordan partition of σ (or any $\theta \in \mathcal{GJ}$ with $\theta \sim \sigma$).

Let $\sigma = (a_1 \ a_2 \ \ldots \ a_n) \in R_n$. We say the entries in σ are *z-increasing* if for $a_i, a_j \neq 0$ with $i < j$ we have $a_i < a_j$. That is, the non-zero entries are increasing, from left to right. Gauss-Jordan elements in R_n are elements with z-increasing entries.

Suppose $\sigma \in R_n$ with $\mathrm{rk}(\sigma)$ the rank of σ. Define the *rank signature* of σ, denoted $\mathrm{sig}(\sigma)$, to be the n-tuple with i^{th} entry equal to the rank of the ith power of σ. We write this as:

$$\mathrm{sig}(\sigma) = [\mathrm{rk}(\sigma), \mathrm{rk}(\sigma^2), \ldots, \mathrm{rk}(\sigma^n)]. \tag{4.11}$$

Using this new notation, we observe that for $\sigma, \theta \in \mathcal{GJ}_n$, $\sigma \sim \theta$ if and only if $\mathrm{sig}(\sigma) = \mathrm{sig}(\theta)$. As with Jordan partitions, since the rank signature is the same for any class representative, by $\mathrm{sig}([\sigma])$ we mean $\mathrm{sig}(\sigma)$. We will usually describe elements of $\tilde{R}_n$ in terms of either the associated Jordan partition or rank signature (it turns out both will be useful).

Example 4.3.2. *Let $\sigma = (0 \ 1 \ 0 \ 2 \ 0 \ 0)$. Then $\sigma^2 = (0 \ 0 \ 0 \ 1 \ 0 \ 0)$ and $\sigma^m = (0 \ 0 \ 0 \ 0 \ 0 \ 0)$ for $m \geq 3$, so $\mathrm{rk}(\sigma) = 2$, $\mathrm{rk}(\sigma^2) = 1$, and $\mathrm{rk}(\sigma^k) = 0$ for $k \geq 3$. Thus $\mathrm{sig}(\sigma) = [2, 1, 0, 0, 0, 0]$. The Jordan partition of σ is $(0; 3, 1, 1, 1)$.*

An element of $\tilde{R}_n$ is a set of conjugate Gauss-Jordan elements of R_n, and so when considering elements of $\tilde{R}_n$ we will often need a representative of the class. Such

representatives are elements of R_n, so we use the one-line notation to describe them. In our analysis of the conjugacy poset, we will often need to move between rank signatures, Jordan partitions, and class representatives in the one-line notation. We now describe how to get from one to the other, first considering rank signatures and the one-line notation.

Let $\sigma \in \mathcal{GJ}_n$. If i is in the jth position of σ, in the one-line notation, then i occupies the position in σ^{m+1} that j occupied in σ^m. Since the entries of σ are z-increasing, if i is not in the ith position then it moves to the right with successive powers of σ. Using this observation, we may obtain the rank signature of σ by quick inspection. To begin, the rightmost non-zero value in the one-line notation is r_1. If we multiply σ by itself, every non-zero value in the last $n - r_1$ positions is lost. So the rank of σ^2 (that is, r_2 in the rank signature) is the rightmost non-zero value in the first r_1 positions. In general, r_j is the rightmost non-zero value in the first r_{j-1} positions. We illustrate this process in the following example.

Example 4.3.3. *Suppose $\sigma = (1\ 2\ 0\ 3\ 4\ 0\ 5\ 0\ 6\ 0) \in \mathcal{GJ}_{10}$. Then $r_1 = 6$ and so we consider*

$$(1\ 2\ 0\ 3\ 4\ 0\ *\ *\ *\ *)$$

Now the rightmost non-zero value is 4, so $r_2 = 4$ and we consider

$$(1\ 2\ 0\ 3\ *\ *\ *\ *\ *\ *)$$

The rightmost non-zero value is 3, so $r_3 = 3$ and we consider

$$(1\ 2\ 0\ *\ *\ *\ *\ *\ *\ *)$$

The rightmost non-zero value is now 2, so $r_4 = 2$. Since the rightmost non-zero value in the first two positions is again 2, we know that $r_j = 2$ for all remaining j.

Thus $\text{sig}(\sigma) = [6, 4, 3, 2, 2, 2, 2, 2, 2, 2]$.

On the other hand, if we know the rank signature of σ, we know roughly how σ "looks". Specifically, if $\text{sig}(\sigma) = [r_1, \ldots, r_n]$, then $r_2 + 1, r_2 + 2, \ldots, r_1$ must occur in the rightmost $n - r_1$ positions (again, in the one-line notation); $r_3 + 1, r_3 + 2, \ldots, r_2$

must occur in the next $r_1 - r_2$ positions; and so on. The rank of σ^n is r_n, and so the first r_n positions are filled with $1, 2, \ldots, r_n$. After all this, we assign a 0 to all remaining open positions. Any element of $\mathcal{GJ}$ with this rank signature must satisfy these conditions, hence we have a way of describing all elements of $[\sigma]$, given $\mathrm{sig}(\sigma)$.

The following example illustrates this idea. In this example, we separate the $r_i - r_{i+1}$ positions with a vertical line for emphasis.

Example 4.3.4. *Suppose $n = 8$ with $\mathrm{sig}(\sigma) = [5, 3, 2, 1, 1, 1, 1, 1]$. Then 4 and 5 must appear in the three rightmost positions, 3 must occur in one of the next two rightmost positions, 2 must occur in the next position to the right, and 1 is in the first position. Any element of $[\sigma]$ must therefore have the form*

$$(1 * \mid 2 \mid * * \mid * * *)$$

with 4 and 5 in the rightmost part and 3 in the next part.

The elements of $[\sigma]$ are therefore:

$$(1\ 0\ 2\ 3\ 0\ 4\ 5\ 0)$$
$$(1\ 0\ 2\ 3\ 0\ 4\ 0\ 5)$$
$$(1\ 0\ 2\ 3\ 0\ 0\ 4\ 5)$$
$$(1\ 0\ 2\ 0\ 3\ 4\ 5\ 0)$$
$$(1\ 0\ 2\ 0\ 3\ 4\ 0\ 5)$$
$$(1\ 0\ 2\ 0\ 3\ 0\ 4\ 5).$$

Suppose, in the process described above, that when given the choice for placing non-zero values we always choose the rightmost allowable positions. Denote the element of $[\sigma]$ obtained this way by σ_m. By Theorem 3.2.8, if $\sigma, \theta \in \mathcal{GJ}$, then $\sigma < \theta$ if θ may be obtained from σ by moving non-zero entries of σ to the left, while maintaining the z-increasing condition on the entries. Every element of $[\sigma]$ may be obtained from σ_m in this manner. Hence we have the following:

Proposition 4.3.5. *For $M = M_n(k)$, every $[\sigma] \in \tilde{R}_n$ has a minimum element.*

Recall that $\sigma \in \mathcal{GJ}$ can be written as ey for some $e \in \Lambda$ and $y \in D(e)^{-1}$. We note that the set $\{z \in D(e)^{-1} \mid [ez] = [\sigma]\}$ has a minimum element, say y^*, and this element is in $D^*(e)$, by [22]. From this observation, and Theorem 3.2.1, it follows that $ey^* \geq ez$ for any $ez \sim ey^*$. That is, every $[\sigma]$ has a maximum element. We will denote this element by σ_M and note that it may be obtained via the process described above by choosing the leftmost allowable position at each step.

Example 4.3.6. *In the previous example, $\sigma_m = (1\ 0\ 2\ 0\ 3\ 0\ 4\ 5)$ and $\sigma_M = (1\ 0\ 2\ 3\ 0\ 4\ 5\ 0)$. Figure 4.1 is the Hasse diagram for the elements of $[\sigma]$, as listed in that example, under Bruhat-Chevalley order.*
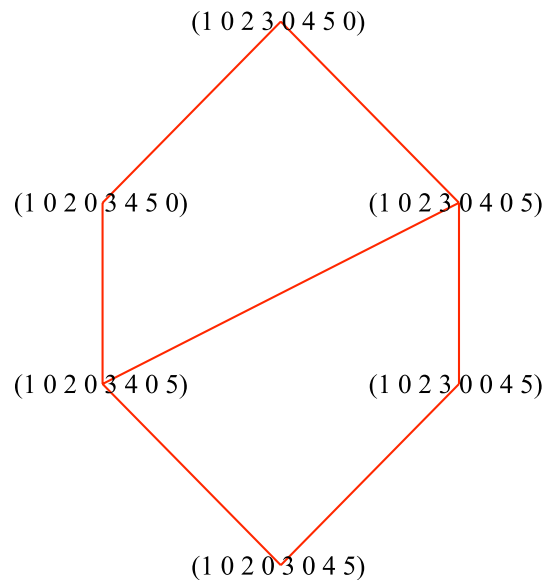
Figure 4.1: The interval $[(1\ 0\ 2\ 0\ 3\ 0\ 4\ 5), (1\ 0\ 2\ 3\ 0\ 4\ 5\ 0)]$ in $\mathcal{GJ}_8$.

We have shown that we may obtain the rank signature of a Gauss-Jordan element from its one-line notation, and vice versa (up to conjugacy), with fairly minimal effort. We now show how the Jordan partition description fits in. It turns out that the rank signature and the Jordan partition are connected in a nice way.

Let σ have Jordan partition $(d; \alpha)$. Form a Young diagram corresponding to the partition α (with the rows of the diagram corresponding to components of the

partition). For a nilpotent Jordan block, successive powers decrease the rank by 1, until we reach 0. Thus we may picture successive ranks of σ, in terms of the Young diagram, as the number of boxes remaining in the diagram after removing the leftmost column. The difference in rank between powers is the number of blocks we remove, which corresponds to the size of the row for the Young diagram of the conjugate partition of α. That is, the ith row of the conjugate partition of α is $r_{i-1} - r_i$ where $[r_1, r_2, \ldots, r_n]$ is the rank signature of σ, with $r_0 = n$. Thus we may recover the rank signature from the Jordan partition (and vice versa, working backwards).

The following example should clarify this process.

Example 4.3.7. *Let $\sigma \in \mathcal{GJ}_{15}$ with Jordan partition $(d; \alpha) = (1; 5, 3, 2, 2, 1, 1)$. The ranks of successive powers of σ are therefore:*

$$
\begin{aligned}
\mathrm{rk}(\sigma) &= 1 + 4 + 2 + 1 + 1 = 9 \\
\mathrm{rk}(\sigma^2) &= 1 + 3 + 1 = 5 \\
\mathrm{rk}(\sigma^3) &= 1 + 2 = 3 \\
\mathrm{rk}(\sigma^4) &= 1 + 1 = 2 \\
\mathrm{rk}(\sigma^k) &= 1
\end{aligned}
$$

for $k \geq 5$. Hence $\mathrm{sig}(\sigma) = [9, 5, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$.

On the other hand, the conjugate partition of α is $(6, 4, 2, 1, 1)$ and so, by the discussion above, we have

$$
\begin{aligned}
n - r_1 &= 6 \\
r_1 - r_2 &= 4 \\
r_2 - r_3 &= 2 \\
r_3 - r_4 &= 1 \\
r_4 - r_5 &= 1 \\
r_k - r_{k+1} &= 0
\end{aligned}
$$

for $k \geq 5$. Since $n = d + \sum \alpha_i$, we see that $n = 1 + 14 = 15$. We may now find the

r_i's recursively:

$$
\begin{aligned}
r_1 &= 15 - 6 = 9 \\
r_2 &= 9 - 4 = 5 \\
r_3 &= 5 - 2 = 3 \\
r_4 &= 3 - 1 = 2 \\
r_5 &= 2 - 1 = 1 \\
r_6 &= 1 - 0 = 1 \\
&\vdots \\
r_{15} &= 1
\end{aligned}
$$

Hence $\operatorname{sig}(\sigma) = [9, 5, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$, *as observed above.*

4.4 Dual Canonical Monoids

We introduced dual canonical monoids in Section 3.1. In the literature these are called $\mathcal{J}$-coirreducible monoids of type $\emptyset$. This means that the cross-section lattice Λ of M is such that $\Lambda \setminus \{1\}$ has a unique maximal element, e_0, with $\lambda(e_0) = \emptyset$. Theorem 4.2.2 tells us that the conjugacy decomposition of M is indexed by elements of $R^* = \{ey \mid e \in \Lambda, y \in D^*(e)\}$. If M is a dual canonical monoid, then Theorem 3.1.10 states that $\lambda(e) = \lambda_*(e)$ for all $e \in \Lambda \setminus \{1\}$. Hence if $\lambda(e) = I$ then $D^*(e) = D_I^*(I) = D_I \cap D_I^{-1}$, as noted in Remark 4.2.1. For a dual canonical monoid then, there is a one-to-one correspondence between $\tilde{R}$ and the set of elements that are both left and right quotient elements for a parabolic subgroup of W.

Example 4.4.1. *Let M be the dual canonical monoid from Example 3.1.11. That is, the Weyl group W of M is isomorphic to S_3. Let $A - B$ be the Coxeter graph of W. Table 4.1 shows the elements of $D_I^* = D_I \cap D_I^{-1}$ for all $I \subseteq S$.*

If M is a dual canonical monoid with $e \leq f$ for $e, f \in \Lambda \setminus \{1\}$, then $\lambda(f) = \lambda_*(f) \subseteq \lambda_*(e) = \lambda(e)$, by Theorem 3.1.10. Thus $W(f) \subseteq W(e)$. If $[\sigma] \in \tilde{R}(e)$, $[\theta] \in \tilde{R}(f)$, then

Table 4.1: Elements of R^* for the dual canonical monoid M with $W = S_3$.

$I \subseteq S = \{A, B\}$	$D_I^* = D_I \cap D_I^{-1}$
$\emptyset$	$\{1, A, B, AB, BA, ABA\}$
A	$\{1, B\}$
B	$\{1, A\}$
$\{A, B\}$	$\{1\}$

Corollary 4.2.8 now says that $[\sigma] \leq [\theta]$ if and only if $a^{-1}\sigma a \leq \theta$ for some $a \in W(e)$. We may choose any representative of $[\sigma]$ when checking this inequality, and so we choose $ey \sim \sigma$ with $y \in D^*(e) = D(e) \cap D(e)^{-1}$. With this choice of y, $a^{-1}eya = eya$ is in standard form for any $a \in W(e)$. Since $\theta = fz$ for some $z \in D(f)^{-1}$, we now have $[\sigma] \leq [\theta]$ if and only if $eya \leq fz$ for some $a \in W(e)$. For this situation, Theorem 3.2.1 tells us that $eya \leq fz$ if and only if there exists $w \in W(e)$ such that $wz \leq ya$. Thus

$$[\sigma] \leq [\theta] \iff wz \leq ya \tag{4.12}$$

where $w, a \in W(e)$, $\theta = fz$, and ey is the unique element such that $ey \sim \sigma$ with $y \in D^*(e)$. We will have more to say about this in the following chapters. In particular, in Section 5.2 we consider the case that $e = f$ before going on to consider the case that $e < f$ in Section 6.2.

Using Corollary 4.2.8, it is not difficult to construct $\tilde{R}$ for small cases. For a dual canonical monoid, we have just shown that there are even fewer calculations to perform. We therefore conclude this discussion with such a construction. Figure 4.2 is the conjugacy poset for M from Example 4.4.1. By e0 we mean the idempotent corresponding to $\emptyset$, via the type map, by eA we mean the idempotent corresponding to $\{A\}$, by e0_AB we mean the element $e_\emptyset AB$, and so on. Additionally, the representative for each vertex in the diagram is the (unique) element in the class that is also in R^*.

4.5 Canonical Monoids

The cross-section lattice of a dual canonical monoid is the dual of the cross-section lattice of a canonical monoid, as noted in Remark 3.1.12. Thus for a canonical
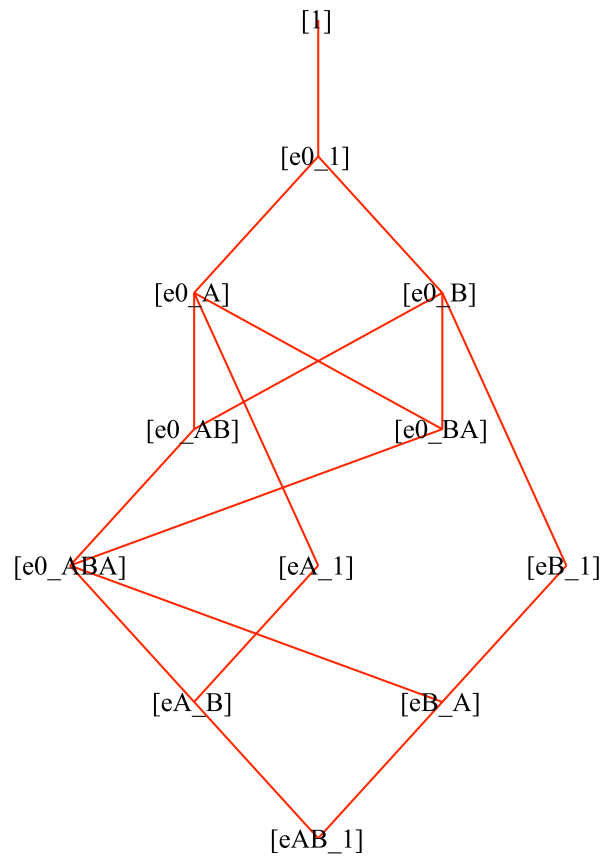
Figure 4.2: $\tilde{R}$ for the dual canonical monoid with W of Type A2.

monoid $\Lambda \setminus \{0\}$ has a unique minimal element, e_0, with $\lambda(e_0) = \emptyset$. More generally, we call a canonical monoid a $\mathcal{J}$-irreducible monoid of type $\emptyset$. For a canonical monoid, $\lambda(e) = \lambda^*(e)$ for all $e \in \Lambda \setminus \{0\}$, by Theorem 3.1.8, hence if $\lambda(e) = I$ then $D^*(e) = D_I^*(I) = D_I^{-1}$, as noted in Remark 4.2.1. Thus for a canonical monoid there is a one-to-one correspondence between $\tilde{R}$ and the set of Gauss-Jordan elements $\mathcal{GJ}$. That is, for $[\sigma] \in \tilde{R}$,

$$[\sigma] = \{\sigma\}.$$

Example 4.5.1. *Let M be the canonical monoid from Example 3.1.9. That is, the Weyl group W of M is isomorphic to S_3. Let $A - B$ be the Coxeter graph of W. Table 4.2 shows the elements of $D_I^* = D_I^{-1}$ for all $I \subseteq S$.*

Table 4.2: Elements of R^* for the canonical monoid M with $W = S_3$.

$I \subseteq S = \{A, B\}$	$D_I^* = D_I^{-1}$
$\emptyset$	$\{1, A, B, AB, BA, ABA\}$
A	$\{1, B, BA\}$
B	$\{1, A, AB\}$
$\{A, B\}$	$\{1\}$

If M is a canonical monoid with $e \leq f$, for $e, f \in \Lambda \setminus \{0\}$, then $\lambda(e) = \lambda^*(e) \subseteq \lambda^*(f) = \lambda(f)$, by Theorem 3.1.10. Thus $W(e) \leq W(f)$ and so if $[\sigma] \in \tilde{R}(e)$, $[\theta] \in \tilde{R}(f)$, Corollary 4.2.8 now says that $[\sigma] \leq [\theta]$ if and only if $a^{-1}\sigma a \leq \theta$ for some $a \in W(f)$. As with the case for dual canonical monoids, we will soon examine this in further detail for $e = f$ and $e < f$ (in Sections 5.3 and 6.3, respectively). For now, we note that for a canonical monoid, we have fewer calculations to perform than it first seems by Corollary 4.2.8, since $[\sigma] = \{\sigma\}$. We therefore conclude this subsection as we did with the last, with a construction of $\tilde{R}$. Figure 4.3 is the conjugacy poset for M from Example 4.5.1. The notation for the vertices (representing the classes) is the same as in Figure 4.2.
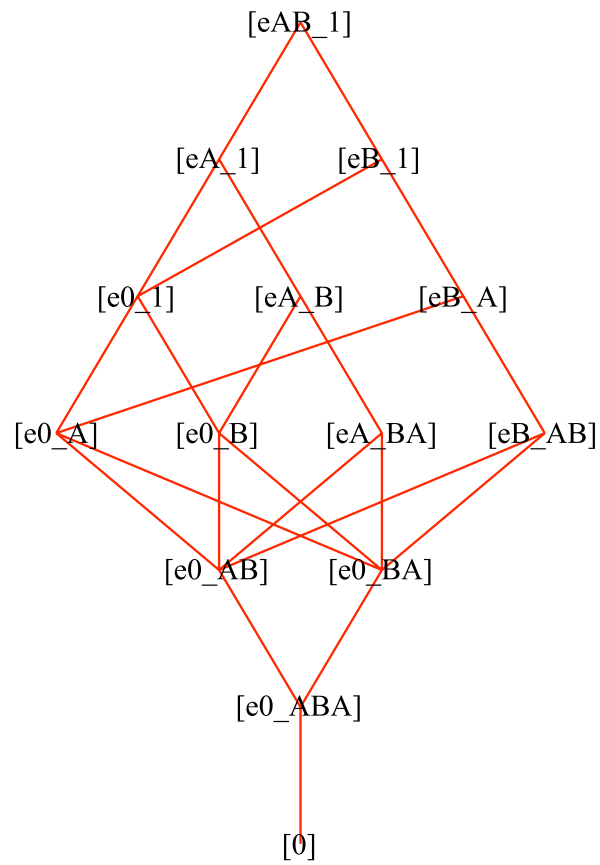
Figure 4.3: $\tilde{R}$ for the canonical monoid with W of Type A2.

Chapter 5

The Structure of $\tilde{R}(e)$ Classes

We begin our more in depth study of the conjugacy order with a description of the order within $\tilde{R}(e)$. This description is the generalization of a result proved in [25] for the case that M is a canonical monoid. Before stating the theorem, we give three lemmas that are used in its proof. The first is from [23] and the second and third are from [25].

Lemma 5.0.2. *Let* $x, y \in D(e)^{-1}$ *and* $w, u \in W(e)$ *such that* $wx \leq uy$. *Then* $w = w_1 * w_2$ *with* $w_1 \leq u$ *and* $w_2 x \leq y$.

Lemma 5.0.3. *Let* $y, y_1 \in D(e)^{-1}$, $x \in W$, *and* $w \in W(e)$ *such that* $yx = wy_1$. *If* $w' < w$, *then* $w'y_1 = yx'$ *for some* $x' < x$. *In particular,* $x'ey \leq xey$ *and* $x'ey \sim w'ey_1$.

Lemma 5.0.4. *Let* $x, y \in D(e)^{-1}$, $u, w \in W(e)$, *such that* $xu = wy$. *Then* $y = xu'$ *for some* $u' \leq u$. *If* $xu = x * u$, *then* $y = x * u'$.

Recall, (3.7), that the parabolic subgroup $W(e) = \{w \in W \mid we = ew\}$ of W can be written as $W(e) = W^*(e) \times W_*(e)$. In the following, it will be necessary to factor $w \in W(e)$ as a product of elements from these subgroups. We write w as $\hat{w}\check{w}$ (or $\check{w}\hat{w}$), where $\hat{w} \in W^*(e)$ and $\check{w} \in W_*(e)$. Note that $\hat{w}$ and $\check{w}$ here are unique and $\hat{w}\check{w} = \hat{w} * \check{w}$.

Theorem 5.0.5. *Let* $e \in \Lambda$ *and* $y, z \in D(e)^{-1}$. *Then the following conditions are equivalent:*

1. $[ey] \leq [ez]$.

2. $\hat{w}z \leq yw$ for some $w \in W(e)$.

3. $\hat{w}zw^{-1} \leq y$ for some $w \in W(e)$.

Proof. $(1. \Rightarrow 2.)$ Suppose $[ey] \leq [ez]$. Then, by Corollary 4.2.8, $w^{-1}eyw \leq ez$ for some $w \in W(e)$. Let $yw = uy_1$ and $u \in W(e)$. Then $w^{-1}eyw = \hat{w}^{-1}e\hat{u}y_1 = \hat{w}^{-1}\hat{u}ey_1 \leq ez$. By Theorem 3.2.1, $\hat{u}^{-1}\hat{w}z \leq y_1$. So

$$\hat{w}z = \hat{u}(\hat{u}^{-1}\hat{w}z) \leq \hat{u} \circ (\hat{u}^{-1}\hat{w}z) \leq \hat{u} \circ y_1 = \hat{u}y_1 \leq uy_1 = yw.$$

$(2. \Rightarrow 3.)$ Suppose $\hat{w}z \leq yw$ for some $w \in W(e)$. Choose w minimal. Now $yw = y * w'$ for $w' \leq w$ and $\hat{w}'z \leq \hat{w}z \leq yw \leq y \circ w = yw'$. Since w is minimal, $w' = w$ and so $yw = y * w$. Thus $\hat{w}z = y_1w_1$ for some $y_1 \leq y$ and $w_1 \leq w$. Since $z \in D(e)^{-1}$,

$$\hat{w}_1z \leq \hat{w}z = y_1w_1 \leq y_1 \circ w_1 \leq y \circ w_1 = y * w_2$$

for some $w_2 \leq w_1$. So $\hat{w}_2z \leq \hat{w}_1z \leq yw_2$. By minimality of w, $w_2 = w$ and so $w_1 = w$. Hence $\hat{w}z = y_1w$ and thus $\hat{w}zw^{-1} = y_1 \leq y$.

$(3. \Rightarrow 2.)$ Suppose $\hat{w}zw^{-1} \leq y$ for some $w \in W(e)$. Then

$$\hat{w}z = \hat{w}zw^{-1}w \leq (\hat{w}zw^{-1}) \circ w \leq y \circ w = yw_1$$

for some $w_1 \leq w$. Since $z \in D(e)^{-1}$, $\hat{w}_1z \leq \hat{w}z \leq yw_1$.

$(2. \Rightarrow 1.)$ Suppose $\hat{w}z \leq yw$ with $w \in W(e)$. Choose w minimal. Let $yw = uy_1$, for $u \in W(e)$ and $y_1 \in D(e)^{-1}$. By Lemma 5.0.4, $y_1 = yv$ for some $v \leq w$. Since $\hat{w}z \leq yw = uy_1$, we see by Lemma 5.0.2 that $\hat{w} = u_1 * v_1$ with $u_1 \leq u$ and $v_1z \leq y_1$. So $\hat{w}z = (u_1v_1)z = u_1 * (v_1z) \leq u_1y_1$. Suppose $u_1 < u$. Since $uy_1 = yw$, we see by Lemma 5.0.3 that $u_1y_1 = yw_1$ for some $w_1 = \hat{w}_1\check{w}_1 < \hat{w}\check{w} = w$, and hence $\hat{w}_1 \leq \hat{w}$. So $\hat{w}_1z \leq \hat{w}z \leq u_1y_1 = yw_1$, contradicting the minimality of w. Hence $u = u_1$, and so $v_1 = u^{-1}\hat{w}$. Since $v_1z \leq y_1$,

$$w^{-1}eyw = (\check{w}\hat{w})^{-1}eyw = \hat{w}^{-1}euy_1 = \hat{w}^{-1}uey_1 \leq ez.$$

by Theorem 3.2.1. That is, $[ey] \leq [ez]$. This completes the proof. $\qquad\square$

5.1 Matrices

Theorem 5.0.5 describes the order in $\tilde{R}(e)$ for a general reductive monoid M. In the case of $M = M_n(k)$, we have the following:

Corollary 5.1.1. *Let* $M = M_n(k)$, *with* $e_i \in \Lambda$ *and* $y, z \in D(e_i)^{-1}$. *Then the following conditions are equivalent:*

1. $[e_i y] \leq [e_i z]$.

2. $\hat{w}z \leq yw$ *for some* $w \in W(e_i)$.

3. $\hat{w}zw^{-1} \leq y$ *for some* $w \in W(e_i)$.

This description is satisfactory, however we wish to examine $\tilde{R}_n(e_i)$ in more detail, in terms of the new vocabulary introduced in the previous chapter.

Let $\sigma \in \mathcal{GJ}_n$. Then $\sigma \sim I_d \oplus N$, where $N = \sum\limits_{i=1}^{k} J_{\alpha_i}(0)$ for some $\alpha \vdash (n-d)$, as in (4.10). Now

$$
\begin{aligned}
\mathrm{rk}(N) &= (\alpha_1 - 1) + (\alpha_2 - 1) + \cdots + (\alpha_k - 1) \\
&= \alpha_1 + \alpha_2 + \cdots + \alpha_k - k \\
&= (n-d) - k
\end{aligned}
$$

and so $\mathrm{rk}(\sigma) = \mathrm{rk}(I_d) + \mathrm{rk}(N) = d + (n-d-k) = n-k$. If $\sigma, \theta \in \mathcal{GJ}(e_i)$, then $\mathrm{rk}(\sigma) = \mathrm{rk}(\theta) = i$. Thus

$$
k = n - \mathrm{rk}(\sigma) = n - \mathrm{rk}(\theta) = n - i
$$

and so we have the following:

Lemma 5.1.2. *If* $[\sigma], [\theta] \in \tilde{R}_n(e_i)$, *then the partitions corresponding to the nilpotent blocks of* σ *and* θ *have the same number of parts,* $n - i$. *Equivalently, the Jordan partitions of* σ *and* θ *have the same number of parts,* $n - i + 1$.

Define $p(n)$ to be the number of partitions of n and $p_k(n)$ to be the number of partitions of n into k parts, as in [33]. Observe that $p_n(n) = p_{n-1}(n) = p_1(n) = 1$ for

$n > 0$ and $p_k(n) = 0$ for $k > n$. Additionally, we follow the convention that $p_0(k) = 0$ for $k > 0$ and $p_0(0) = 1$.

Proposition 5.1.3. *The number of elements in $\tilde{R}_n(e_i)$ is*

$$\left| \tilde{R}_n(e_i) \right| = \sum_{j=k}^{n} p_k(j) = \sum_{j=0}^{n} p_k(j)$$

where $k = n - i$.

Computing $p_k(n)$ is, in general, quite difficult. However, the following recursive formula, as presented in [33], makes the computations easier:

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k). \tag{5.1}$$

Example 5.1.4. *Using Table 5.1, the number of elements of $\tilde{R}_8(e_5)$ is $1 + 1 + 2 + 3 + 4 + 5 = 16$. This is the sum of entries in the row for $k = 8 - 5 = 3$ from column $n = 0$ to column $n = 8$. The Hasse diagram for $\tilde{R}_8(e_5)$ is shown in Figure 5.1. Each vertex in the diagram is denoted by the Jordan partition of the corresponding class. Observe that $\tilde{R}_8(e_5)$ is not graded.*

Table 5.1: $p_k(n)$ for $n \leq 10$.

$k \backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1	1	1
2	0	0	1	1	2	2	3	3	4	4	5
3	0	0	0	1	1	2	3	4	5	7	8
4	0	0	0	0	1	1	2	3	5	6	9
5	0	0	0	0	0	1	1	2	3	5	7
6	0	0	0	0	0	0	1	1	2	3	5
7	0	0	0	0	0	0	0	1	1	2	3
8	0	0	0	0	0	0	0	0	1	1	2
9	0	0	0	0	0	0	0	0	0	1	1
10	0	0	0	0	0	0	0	0	0	0	1

(5;1,1,1)

(4;2,1,1)

(3;3,1,1)

(3;2,2,1)   (2;4,1,1)

(2;3,2,1)   (1;5,1,1)

(0;6,1,1)   (1;4,2,1)

(0;5,2,1)   (2;2,2,2)   (1;3,3,1)
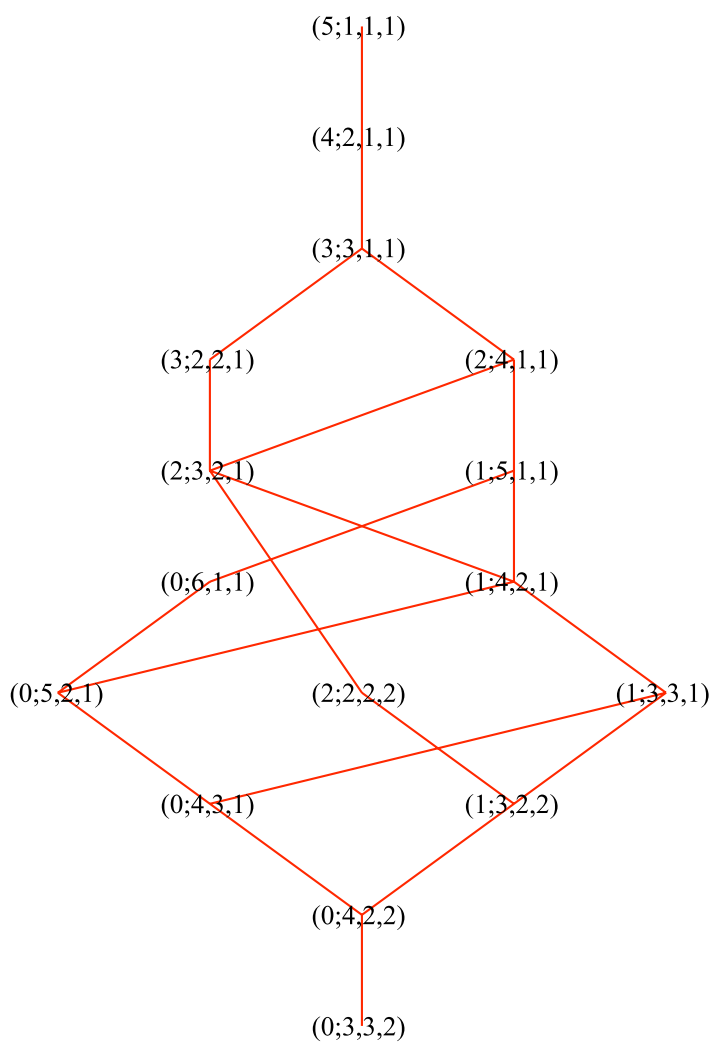
(0;4,3,1)   (1;3,2,2)

(0;4,2,2)

(0;3,3,2)

Figure 5.1: The conjugacy poset of rank 5 elements in R_8, labeled by Jordan partitions.

5.2 Dual Canonical Monoids

Recall that for a dual canonical monoid, $W(e) = W_*(e)$ for all $e \in \Lambda$. By Theorem 5.0.5, we therefore have the following description for the conjugacy order in $\tilde{R}(e)$:

Corollary 5.2.1. *Let $e \in \Lambda$ and $y, z \in D(e)^{-1}$. If $W(e) = W_*(e)$, then the following conditions are equivalent:*

1. *$[ey] \leq [ez]$.*

2. *$z \leq yw$ for some $w \in W(e)$.*

3. *$zw \leq y$ for some $w \in W(e)$.*

Let $w \in W$ with $J \subseteq S$. Then $w = w^j w_j$, where $w_j \in W_J$ and $w^j \in D_J$. Define $P^J : W \to D_J$ by $P^J(w) = w^j$. This map is order-preserving, as shown in [1]. That is, if $w_1, w_2 \in W$ then:

$$w_1 \leq w_2 \Rightarrow w_1^j \leq w_2^j. \tag{5.2}$$

Now, suppose $e \in \Lambda \setminus \{1\}$. Then $\lambda(e) = I$ for some $I \subseteq S$ and, since M is a dual canonical monoid, $\lambda_*(e) = I$ by Theorem 3.1.10. Thus $D^*(e) = D_I^*(I)$ and so $D^*(e) = D(e) \cap D(e)^{-1}$, by Remark 4.2.1. As we have noted, $D^*(e)$ serves as our indexing set for $\tilde{R}(e)$, where

$$\tilde{R}(e) = \bigsqcup_{y \in D^*(e)} [ey]. \tag{5.3}$$

By Corollary 5.2.1, $[ey] \leq [ex]$ if and only if $x \leq yw$ for some $w \in W(e)$. If $x, y \in D^*(e)$, then $x \leq yw$ implies $x \leq y$, by (5.2). On the other hand, if $x \leq y$ then $x \leq yw$, since $y \leq yw$ because $y \in D(e)$. Thus, $[ey] \leq [ex]$ if and only if $x \leq y$.

Let $I \subseteq S$ with $W_I = \langle I \rangle \subseteq W$, as usual, and consider the double quotient $W_I \backslash W / W_I$. Recall that for one-sided quotients of W by W_I, the minimum coset representatives form an important set (either D_I or D_I^{-1}, depending on the side). We follow a similar approach here. That is, given the double coset $W_I x W_I$, choose the minimum $y \in W$ such that $W_I y W_I = W_I x W_I$. This element is unique, and the set of such elements is exactly $D_I \cap D_I^{-1}$, see [1]. There is a partial order on these double

cosets, in terms of the Bruhat-Chevalley order on the described representatives. That is, for $x, y \in D_I \cap D_I^{-1}$:

$$W_I x W_I \leq W_I y W_I \iff x \leq y. \tag{5.4}$$

Hence for a dual canonical monoid, we may describe $\tilde{R}(e)$ in terms of double quotients of W by $W_{\lambda(e)}$.

Theorem 5.2.2. *Let M be a dual canonical monoid with $e \in \Lambda \setminus \{1\}$. Then $\tilde{R}(e)$ is isomorphic to the dual of $W(e) \backslash W / W(e)$. That is,*

$$[ey] \leq [ex] \iff W(e)xW(e) \leq W(e)yW(e),$$

for $x, y \in D^*(e) = D(e) \cap D(e)^{-1}$.

In [36], double quotients of maximal parabolic subgroups are studied. The resulting posets here are generally quite nice. Most we would probably initially consider are graded. This is not true in general though, as John Stembridge has pointed out that $W_I \backslash W / W_I$ is not graded for W of Type E6 with $I = S \setminus \{s\}$, where s is the vertex of degree 3 in the Coxeter graph of E6, [37]. We do not restrict ourselves to maximal parabolic subgroups. As one might expect, the resulting posets are not always so nicely behaved.

Example 5.2.3. *Let M be a dual canonical monoid with Weyl group of type $A4$, with Coxeter graph $(12) - (23) - (34) - (45)$. Note that we may explicitly construct such a monoid following the description of 3.1.11. Choose $e \in \Lambda$ such that $\lambda(e) = \{(12), (34)\}$. Then*

$$
\begin{aligned}
D^*(e) \;=\; & \{1, (23), (45), (23)(45), (2345), (2543), \\
& (13)(24), (13)(245), (13)(254), (25), (13)(25)\}.
\end{aligned}
$$

Figure 5.2 shows $\tilde{R}(e)$. Note that this poset is not graded.
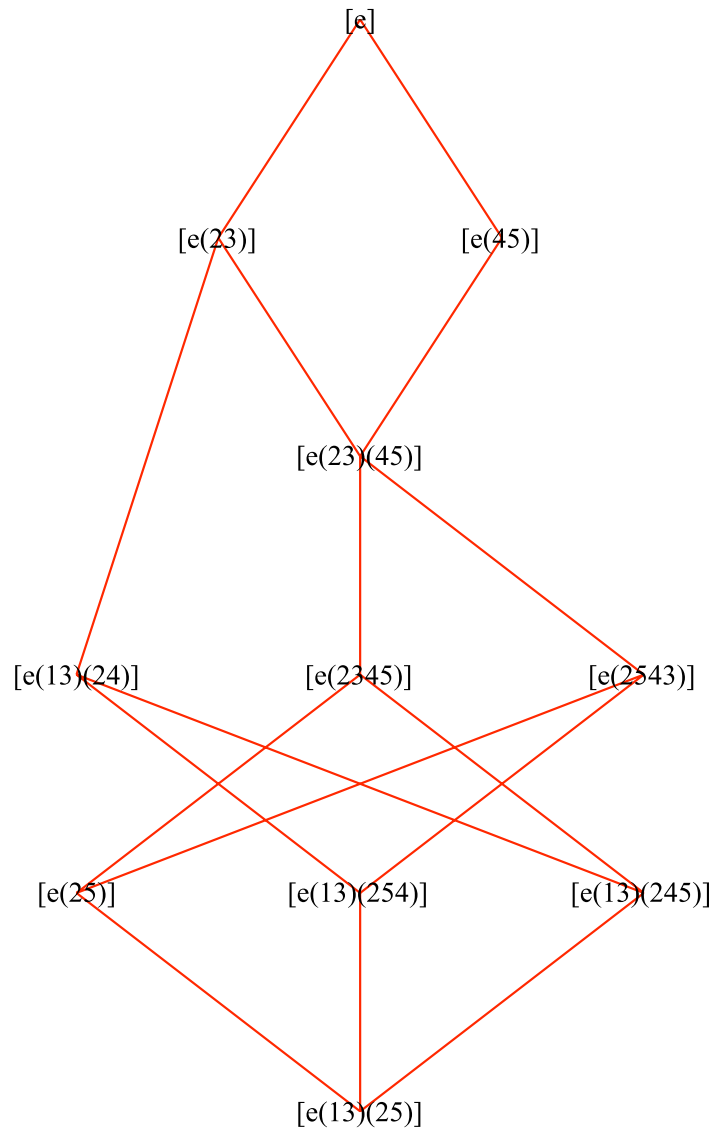
Figure 5.2: $\tilde{R}(e)$ for the dual canonical monoid with $\lambda(e) = \{(12), (34)\}$ for $W = S_5$.

5.3   Canonical Monoids

As expected, our first description of the conjugacy order within an idempotent class is a corollary of Theorem 5.0.5. This result was originally proved in [25] and served as the motivation for the theorem.

Corollary 5.3.1. *Let $e \in \Lambda$ and $y, z \in D(e)^{-1}$. If $W(e) = W^*(e)$, then the following conditions are equivalent:*

1. *$[ey] \leq [ez]$.*

2. *$wz \leq yw$ for some $w \in W(e)$.*

3. *$wzw^{-1} \leq y$ for some $w \in W(e)$.*

As previously noted, $D^*(e) = D(e)^{-1}$ for a canonical monoid, and so there is a one-to-one correspondence between vertices in $\mathcal{GJ}(e)$ under the Bruhat-Chevalley order and $\tilde{R}(e)$ under the conjugacy order. However, unlike the dual canonical case, the conjugacy order here does not coincide with the Bruhat-Chevalley order and so the posets are not isomorphic. In general, $ey \leq ez$ implies $[ey] \leq [ez]$, though not necessarily conversely. The following example, originally from [25], makes this clear.

Example 5.3.2. *Let M be a canonical monoid with Weyl group of type A3, with Coxeter graph $A - B - C$. Let $I = \lambda(e) = \{B\}$. Then $\mathcal{GJ}(e)$ under Bruhat-Chevalley order is isomorphic to a (weak) subposet of $\tilde{R}(e)$ under conjugacy order. The additional relations in $\tilde{R}(e)$ are $[eABC] \leq [eCB]$ and $[eCBA] \leq [eAB]$. Figure 5.3 shows the Hasse diagram for $\tilde{R}(e)$. The additional, non-Bruhat relations are identified by dashed lines.*
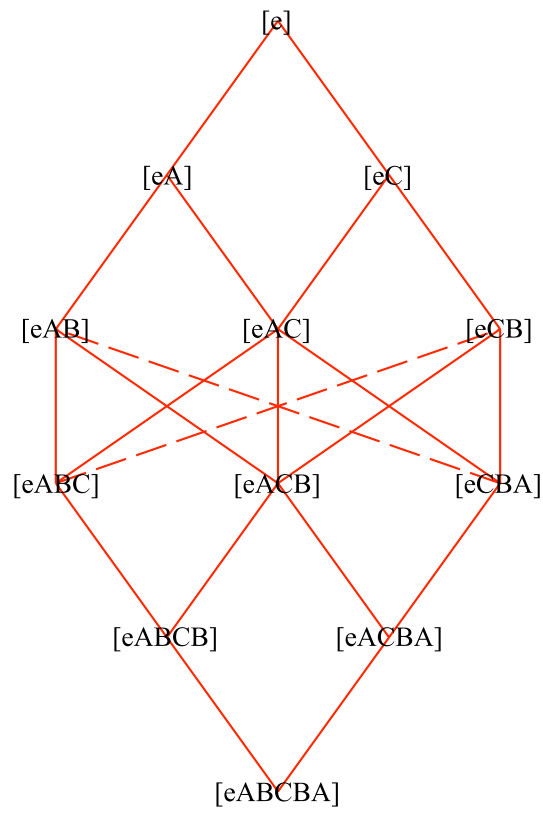
Figure 5.3: $\tilde{R}(e)$ for the canonical monoid with $\lambda(e) = \{B\}$ for $S : A - B - C$.

Chapter 6

Order Between $\tilde{R}(e)$ Classes

One of the goals of this paper is the answer to the following, from [25]:

> *Conjecture 2.13:* Let $e, f \in \Lambda$, $e, < f$. Then there exists a map $p_{e,f} :$ $\tilde{R}(e) \to \tilde{R}(f)$ such that for $\sigma \in \tilde{R}(e)$, $\theta \in \tilde{R}(f)$, $\sigma \leq \theta$ if and only if $p_{e,f}(\sigma) \leq \theta$.

The maps proposed in the conjecture are generalizations of the projection maps defined by (3.13) and satisfy conditions analogous to those in Theorem 3.2.4. We therefore also call these maps *projections*. As $p_{e,f}$ was used for the projection from $R(e) \to R(f)$, we denote the projection map from $\tilde{R}(e)$ to $\tilde{R}(f)$ by $\tilde{p}_{e,f}$. The projection maps for R are a very useful tool. We would like to describe $\tilde{p}_{e,f}$ in terms of $p_{e,f}$ and a representative of $[ey] \in \tilde{R}(e)$, if possible. We will again consider three classes of reductive monoids: matrices, dual canonical monoids, and canonical monoids. For all three, we show that $\tilde{p}_{e,f}$ exists. For dual canonical and canonical monoids, we provide an explicit description for the maps.

6.1 Matrices

As mentioned above, an ideal description of $\tilde{p}_{e,f}$ for $\tilde{R}_n$ would make use of $p_{e,f}$ for R_n. Previously, we described $p_{e,f}$ for the case of $M = M_n(k)$ in terms of the one-line notation for partial permutation matrices (that is, the elements of R_n). This was the

k-insertion algorithm, from Section 3.3. Recall that for $\sigma \in R_n(e_i)$, the algorithm produces $\beta(\sigma)$, which we have shown to equal $p_{e_i,e_{i+1}}(\sigma)$. Our hope then is for a description of $\tilde{p}_{e,f}$ that is similar to β. For $M = M_n(k)$, we therefore denote this proposed map by $\tilde{\beta}$.

The natural first attempt for a definition of $\tilde{\beta}$ is $\tilde{\beta}([\sigma]) = [\beta(\sigma)]$. Unfortunately, the k-insertion algorithm is not constant on conjugacy classes in $\mathcal{GJ}_n$, and so this map is not well defined. The following example illustrates. (Note that when using the one-line notation, we will denote $\beta((a_1 \ \cdots \ a_n))$ simply by $\beta(a_1 \ \cdots \ a_n)$.)

Example 6.1.1. *Let $M = M_4(k)$ and consider $[(0\ 1\ 0\ 0)] \in \tilde{R}_4(e_1)$. We observe that*

$$[(0\ 1\ 0\ 0)] = \{(0\ 1\ 0\ 0)\ ,\ (0\ 0\ 1\ 0)\ ,\ (0\ 0\ 0\ 1)\}$$

however

$$\begin{aligned}
\beta(0\ 1\ 0\ 0) &= (0\ 1\ 0\ 2) \\
\beta(0\ 0\ 1\ 0) &= (0\ 0\ 1\ 2) \\
\beta(0\ 0\ 0\ 1) &= (0\ 0\ 1\ 2)
\end{aligned}$$

and $(0\ 1\ 0\ 2) \not\sim (0\ 0\ 1\ 2)$, hence $[\beta(0\ 1\ 0\ 0)] \neq [\beta(0\ 0\ 0\ 1)]$.

In the previous example, it turns out that $[\beta(0\ 0\ 0\ 1)] = [(0\ 0\ 1\ 2)]$ is the element we want. That is, it is the (unique) least element in $\tilde{R}_4(e_2)$ such that $[(0\ 1\ 0\ 0)] \leq [(0\ 0\ 1\ 2)]$. We note that $(0\ 0\ 0\ 1)$ is the minimum element in the class and that, by Proposition 4.3.5, such elements always exist. Recall that we denoted the minimum element in $[\sigma]$ by σ_m. Our next attempt for a definition of $\tilde{\beta}$ is therefore the class containing σ_m after the k-insertion algorithm. That is, $\tilde{\beta}([\sigma]) = [\beta(\sigma_m)]$. This is a reasonable approach (indeed, it works for the dual canonical case) that does not fall prey to the previous problem, since $\tilde{\beta}([\sigma])$ is well defined. However, it too ultimately fails, as for this definition $\tilde{\beta}$ is not necessarily order-preserving.

Example 6.1.2. *Let $[\sigma], [\theta] \in \tilde{R}_8(e_5)$ with $\mathrm{sig}([\sigma]) = [5, 2, 1, 0, 0, 0, 0, 0]$ and $\mathrm{sig}([\theta]) = [5, 3, 1, 0, 0, 0, 0, 0]$. So $[\sigma] \leq [\theta]$. Using previously described techniques, we observe*

that

$$\sigma_m = (0\ 1\ 0\ 0\ 2\ 3\ 4\ 5)$$

$$\theta_m = (0\ 0\ 1\ 2\ 3\ 0\ 4\ 5)$$

and so, using the k-insertion algorithm,

$$\beta(\sigma_m) = (0\ 1\ 0\ 2\ 3\ 4\ 5\ 6)$$

$$\beta(\theta_m) = (0\ 0\ 1\ 2\ 3\ 4\ 5\ 6).$$

Thus $\operatorname{sig}(\beta(\sigma_m)) = [6,4,2,1,0,0,0,0]$ *and* $\operatorname{sig}(\beta(\theta_m)) = [6,4,2,0,0,0,0,0]$, *and so* $\tilde{\beta}([\sigma]) = [\beta(\sigma_m)] \not\leq [\beta(\theta_m)] = \tilde{\beta}([\theta])$, *even though* $[\sigma] \leq [\theta]$.

To describe $\tilde{\beta} : \tilde{R}_n(e_i) \to \tilde{R}_n(e_{i+1})$, it turns out that we need a different decomposition of $\tilde{R}_n$.

A Decomposition of $\tilde{R}_n$ by Partitions

A natural way to group elements of $\tilde{R}_n$ is by the rank of a class representative. This gives the decomposition

$$\tilde{R}_n = \bigsqcup_{e_i \in \Lambda} \tilde{R}_n(e_i). \tag{6.1}$$

For this decomposition, $[\sigma]$ and $[\theta]$ are in the same class if they have identical first entries in their corresponding rank signatures. We form an alternate decomposition of $\tilde{R}_n$ in terms of the Jordan partitions of elements. We do this by grouping together elements that have the same first entry in their corresponding Jordan partitions. Equivalently, two elements will be in the same class if they have identical final entries in their rank signatures. That is, we define

$$\tilde{R}_n^j = \{[\sigma] \in \tilde{R}_n \mid [\sigma] \longleftrightarrow (j;\alpha)\}$$
$$= \{[\sigma] \in \tilde{R}_n \mid \operatorname{rk}(\sigma^n) = j\}$$

and so we have a new decomposition:

$$\tilde{R}_n = \bigsqcup_{0 \leq j \leq n} \tilde{R}_n^j. \tag{6.2}$$

Suppose $[\sigma], [\theta] \in \tilde{R}_n^j$. Then, by (4.10),

$$\sigma \sim I_d \oplus J_{\alpha_1}(0) \oplus \cdots \oplus J_{\alpha_k}(0)$$

$$\theta \sim I_{d'} \oplus J_{\gamma_1}(0) \oplus \cdots \oplus J_{\gamma_{k'}}(0).$$

Since $\mathrm{rk}(\sigma^n) = \mathrm{rk}(\theta^n) = j$, we have $d = d' = j$. Hence $\alpha, \gamma \in Par(n - j)$. The order given in Section 4.1 is now the usual dominance order on partitions, from Example 2.1.3. That is, $[\sigma] \leq [\theta]$ if

$$\begin{aligned} \alpha_1 &\leq \gamma_1 \\ \alpha_1 + \alpha_2 &\leq \gamma_1 + \gamma_2 \\ \alpha_1 + \alpha_2 + \alpha_3 &\leq \gamma_1 + \gamma_2 + \gamma_3 \\ &\vdots \end{aligned}$$

where α corresponds to the nilpotent blocks of σ, and γ to θ, as usual. This observation, along with results in Example 2.1.6, yields the following:

Proposition 6.1.3. *Let* $\tilde{R}_n^j = \{[\sigma] \in \tilde{R}_n \mid \mathrm{rk}(\sigma^n) = j\}$ *with* $\leq$ *the conjugacy order. Then*

1. $(\tilde{R}_n^j, \leq) \cong (Par(n - j), \trianglelefteq)$.

2. $(\tilde{R}_n^j, \leq)$ *is a lattice.*

3. $(\tilde{R}_n^j, \leq)$ *is self-dual.*

4. $(\tilde{R}_n^j, \leq)$ *is graded if and only if* $n - j \leq 6$.

5. $\left| \tilde{R}_n^j \right| = p(n - j)$.

The classes in this decomposition are crucial in describing projection maps between $\tilde{R}_n(e)$ classes. They are, however, also interesting in their own right. Of particular interest is the set of nilpotent elements of $\tilde{R}_n$, denoted $\tilde{R}_{nil}$, where

$$\tilde{R}_{nil} = \{[\sigma] \in \tilde{R} \mid (\sigma)^n = 0\}. \tag{6.3}$$

For R_n, $\tilde{R}_{nil}$ is just $\tilde{R}_n^0$ in our notation. Figure 6.1 shows the set of nilpotent elements of $\tilde{R}_7$ (that is, $\tilde{R}_7^0$), labeled by rank signature. As noted above, this poset is isomorphic to $(Par(7), \trianglelefteq)$. Note as well that it is not graded.
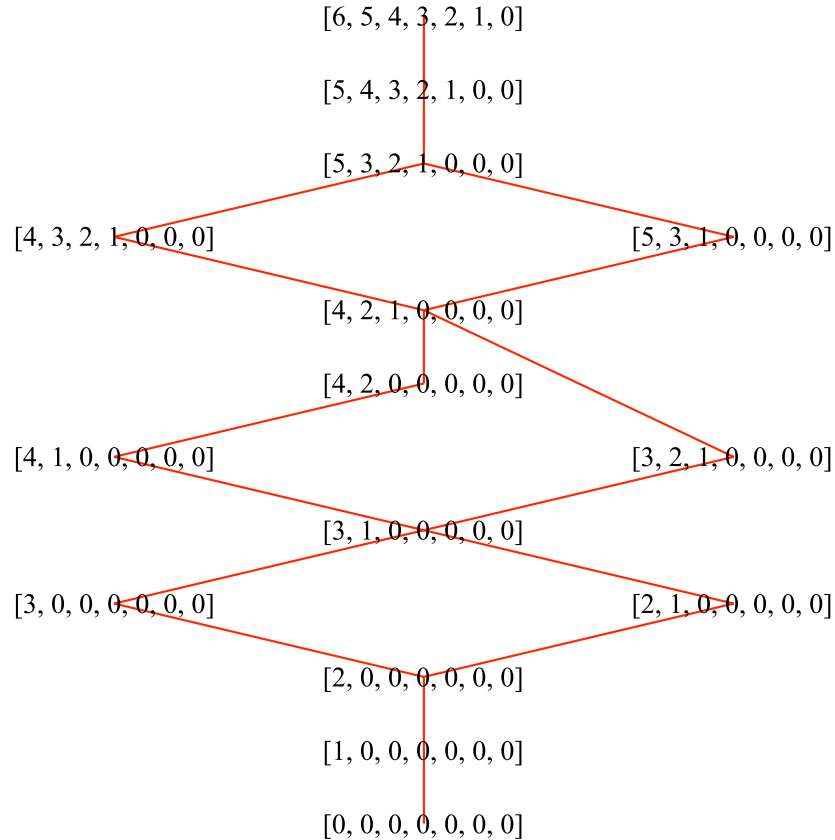

Figure 6.1: The conjugacy poset of nilpotent elements in R_7, labeled by rank signature.

Projection Maps for $\tilde{R}_n(e)$

We now show that Conjecture 2.13 from [25] is true for the case of $M = M_n(k)$. As noted, $\tilde{R}_n^j$ holds the key. Our first description is in terms of rank signatures.

Let $\sigma \in \mathcal{GJ}_n(e_i)$ with $i < n$ and $\text{sig}(\sigma) = [r_1, r_2, \ldots, r_n]$. Consider $\theta \in \mathcal{GJ}_n$ with $\text{sig}(\theta) = [t_1, t_2, \ldots, t_n]$ defined as

$$t_1 = r_1 + 1 \tag{6.4}$$

and, for $1 < t \leq n$,

$$t_i = \min\{x \in \mathbb{Z}_{\geq 0} \mid x \geq r_i, t_{i-1} - x \leq t_{i-2} - t_{i-1}\} \qquad (6.5)$$

where we define $t_0 = n$.

We observe that (6.4) tells us that $\theta \in \mathcal{GJ}_n(e_{i+1})$ and (6.5) tells us that this non-increasing sequence is a rank signature. By construction, $[\theta]$ is the smallest element of $\tilde{R}_n(e_{i+1})$ with $[\sigma] < [\theta]$. That is, $[\theta]$, constructed as we have described, is exactly the projection of $[\sigma]$ from $\tilde{R}_n(e_i)$ to $\tilde{R}_n(e_{i+1})$. We therefore denote this by $\tilde{\beta}([\sigma])$.

Example 6.1.4. *Let $\sigma \in \mathcal{GJ}_7(e_4)$ with* $\operatorname{sig}(\sigma) = [4, 1, 0, 0, 0, 0, 0]$. *Then*

$$
\begin{aligned}
t_1 &= 5 \\
t_2 &= \min\{x \in \mathbb{Z}_{\geq 0} \mid x \geq 1, 5 - x \leq 7 - 5\} \\
&= \min\{x \in \mathbb{Z}_{\geq 0} \mid x \geq 1, 3 \leq x\} \\
&= 3 \\
t_3 &= \min\{x \in \mathbb{Z}_{\geq 0} \mid x \geq 0, 3 - x \leq 5 - 3\} \\
&= \min\{x \in \mathbb{Z}_{\geq 0} \mid x \geq 0, 1 \leq x\} \\
&= 1 \\
t_4 &= \min\{x \in \mathbb{Z}_{\geq 0} \mid x \geq 0, 1 - x \leq 3 - 1\} \\
&= \min\{x \in \mathbb{Z}_{\geq 0} \mid x \geq 0, -1 \leq x\} \\
&= 0
\end{aligned}
$$

and $t_5 = t_6 = t_7 = 0$. Thus $\tilde{\beta}([\sigma]) = [5, 3, 1, 0, 0, 0, 0]$.

So, the projection maps exist and we may describe them in terms of rank signatures. We next provide a description in terms of Jordan partitions.

Suppose $\sigma \in \mathcal{GJ}_n(e_i)$, and σ has exactly one nilpotent Jordan block in its Jordan form. By Lemma 5.1.2, we have $i = n - 1$ and thus $\tilde{\beta}([\sigma]) = [I_n]$. Suppose then that σ has more than one nilpotent Jordan block.

Proposition 6.1.5. *If $\sigma \in \mathcal{GJ}_n$ and the Jordan form for σ has more than one nilpotent Jordan block, then both $[\sigma]$ and $\tilde{\beta}([\sigma])$ are in $\tilde{R}_n^j$, for some $1 \leq j \leq n$.*

Proof. Let $[\sigma]$ and $\tilde{\beta}([\sigma])$ have rank signatures $[r_1, r_2, \ldots, r_n]$ and $[t_1, t_2, \ldots, t_n]$, respectively, with t_i defined as in (6.4), (6.5). By definition, $[\sigma]$ and $\tilde{\beta}([\sigma])$ are in the same $\tilde{R}_n^j$-class if $r_n = t_n$.

If $t_{n-1} \neq t_n$, then we must have $\text{sig}(\tilde{\beta}([\sigma])) = [n-1, n-2, \ldots, 2, 1, 0]$. Since $t_n \geq r_n \geq 0$, we have $t_n = r_n$ and the claim follows. Suppose then that $t_{n-1} = t_n$. By (6.5), t_n is the minimum non-negative integer satisfying $t_n \geq r_n$ and $t_{n-1} - t_n \leq t_{n-2} - t_{n-1}$. Since $t_{n-1} = t_n$, the latter condition is satisfied. Hence t_n is the smallest such that $t_n \geq r_n$. That is, $t_n = r_n$. □

If σ has rank n, then $\sigma = I_n$ and $\tilde{\beta}([I_n]) = [I_n]$. If σ has rank $n-1$, then $\beta(\tilde{[\sigma]}) = [I_n]$, by the remarks preceding the proposition. Otherwise, in studying the projections of elements of $\tilde{R}_n$ we need only examine $\tilde{R}_n^j$, for some j. This amounts to studying the set of partitions of $n - j$, under dominance order. We next show how to find $\tilde{\beta}([\sigma])$ in this lattice.

Let $\alpha \vdash m \leq n$ with $\alpha = [\alpha_1, \ldots, \alpha_k]$, $k > 1$. Define

$$\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_j + 1, \ldots, \alpha_{k-1}, \alpha_k - 1) \tag{6.6}$$

where j is the largest such that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_j + 1 \geq \ldots \geq \alpha_{k-1} \geq \alpha_k - 1$. If $a_k - 1 > 0$, then repeat this process, following the same rules. Eventually, we have a 0 in the last entry. We stop the process here and denote our result by $\tilde{\beta}(\alpha)$. At each step in the process we have a partition of m. What's more, each partition corresponds to an element in $\tilde{R}_n^{(n-m)}$.

Suppose $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{k-1}) \vdash n$ with $\alpha \trianglelefteq \gamma$. Let j be as in (6.6) and suppose

$$\alpha_1 + \alpha_2 + \cdots + \alpha_j + 1 = \gamma_1 + \gamma_2 + \cdots + \gamma_j. \tag{6.7}$$

Then $(\alpha_{j+1}, \ldots, \alpha_k), (\gamma_{j+1}, \ldots, \gamma_{k-1}) \vdash \left(n - \sum_{i=1}^{j} \alpha_i \right)$ and, since $\alpha \trianglelefteq \gamma$, $(\alpha_{j+1}, \ldots, \alpha_k) \trianglelefteq (\gamma_{j+1}, \ldots, \gamma_{k-1})$. The condition on j forces

$$\alpha_{j-1} > \alpha_j = \alpha_{j+1} = \cdots = \alpha_{k-1} \geq \alpha_k$$

and so $(\alpha_{j+1}, \ldots, \alpha_k) = (\alpha_{k-1}, \ldots, \alpha_{k-1}, \alpha_k) \trianglelefteq (\gamma_{j+1}, \ldots, \gamma_{k-1})$. This forces $\gamma_{j+1} > \alpha_{j+1} = \alpha_j$. However, $\alpha \trianglelefteq \gamma$ tells us that $\alpha_1 + \cdots + \alpha_{j-1} \leq \gamma_1 + \cdots + \gamma_{j-1}$ and so $\gamma_j \leq \alpha_j$, by (6.7). But now $\gamma_j < \gamma_{j+1}$, a contradiction, and so we must have $\alpha' \leq \gamma$.

We repeat this process, as necessary. Eventually, we arrive at $\tilde{\beta}(\alpha) \leq \gamma$. Thus for $[\sigma] \in \tilde{R}_n$ with corresponding nilpotent block partition α, $\tilde{\beta}(\alpha)$ corresponds to $\tilde{\beta}([\sigma])$.

Example 6.1.6. *Starting with* $(3, 2, 2)$, *the process above leads us to the projection as follows*

$$(3, 2, 2) \longrightarrow (3, 3, 1) \longrightarrow (4, 3)$$

Figure 6.2 is the Hasse diagram for $\tilde{R}_7^0$, *with vertices labeled by their corresponding nilpotent block partition. By inspection, we see that* $(4, 3)$ *is indeed the smallest pair greater than the triple* $(3, 2, 2)$. *Figure 6.1, from earlier in the section, is the same poset labeled by rank signature. Comparing the figures, we see that* $(3, 2, 2)$ *corresponds to* $[4, 1, 0, 0, 0, 0, 0]$ *and* $(4, 3)$ *corresponds to* $[5, 3, 1, 0, 0, 0, 0]$, *confirming the conclusion of Example 6.1.4.*
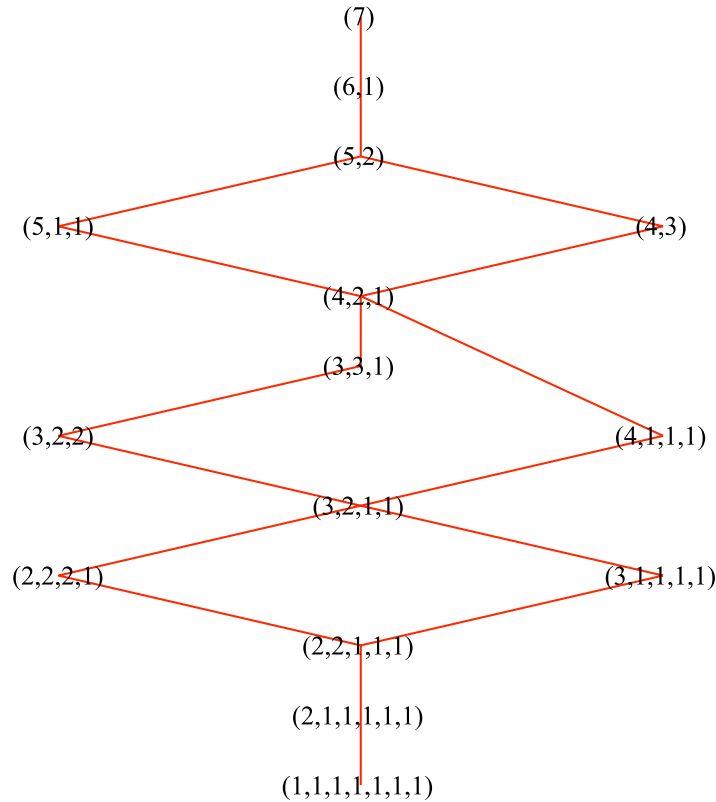

Figure 6.2: The conjugacy poset of nilpotent elements in R_7, labeled by partitions.

6.2   Dual Canonical Monoids

We next examine the order between $\tilde{R}(e)$ classes for dual canonical monoids. It turns out, as hoped, that $\tilde{p}_{e,f}$ may be defined in terms of the projection maps for $W \times W$-orbits, (3.13). For a dual canonical monoid, recall that if $e \leq f$ for $e, f \in \Lambda$, then $\lambda(f) \subseteq \lambda(e)$, and so the projection map from WeW to WfW is one-to-one, by Theorem 3.2.4. Such maps are therefore not class functions and so in using them to define projections between conjugacy classes, we must be careful in our choice of class representative.

Suppose M is a dual canonical monoid. Then $W(e) = W_I = W_*(e)$, by Theorem 3.1.10, and $D^*(e) = D(e) \cap D^{-1}(e)$. Suppose $[\sigma] \in \tilde{R}(e)$ and choose (the unique) $ey \sim \sigma$ such that $y \in D^*(e)$. Let $W_K = W_I \bigcap (y^{-1}W_I y)$ and w_I, w_K be the longest elements in W_I, W_K, respectively.

Proposition 6.2.1. $yw_Kw_I \in D(e)^{-1}$.

Proof. By Proposition 2.5.8, $yw_Kw_I = wy'$ for some $w \in W(e)$ and $y' \in D(e)^{-1}$. Since $ey \sim ey'$, we have $y' = yv$ for some $v \in W_I$. Let $w_K = y^{-1}v_Ky$ and so

$$yw_Kw_I = wy' = wyv. \tag{6.8}$$

Keeping in mind that $y \in D^*(e) = D(e) \cap D(e)^{-1}$ and $yv \in D(e)^{-1}$, we observe the following:

$$
\begin{aligned}
\ell(yw_Kw_I) &= \ell(wyv) \\
\ell(y) + \ell(w_Kw_I) &= \ell(w) + \ell(yv) \\
\ell(y) + \ell(w_Kw_I) &= \ell(w) + \ell(y) + \ell(v) \\
\ell(w_Kw_I) &= \ell(w) + \ell(v)
\end{aligned}
$$

On the other hand, we may rewrite (6.8) as

$$vw_I = y^{-1}w^{-1}yw_K = y^{-1}w^{-1}yy^{-1}v_Ky = y^{-1}w^{-1}v_Ky.$$

Now, since $y^{-1}w^{-1}v_Ky \in W_K$, by definition $y^{-1}w^{-1}v_Ky \leq w_K$. Both elements are in W_I as well, so $v = y^{-1}w^{-1}v_Kyw_I \geq w_Kw_I$, since multiplication on the right by w_I is an antiautomorphism.

Putting everything together, we see that $\ell(w) + \ell(v) = \ell(w_Kw_I) \leq \ell(v)$ and hence $\ell(w) = 0$. Therefore $w = 1$ and we have $yw_Kw_I = wy' = y' \in D(e)^{-1}$. That is, $yw_Kw_I \in D(e)^{-1}$. This completes the proof. $\qquad\square$

This proposition tells us that each conjugacy class has a minimum element. For $[ey]$, we denote this element by ey_m. For $e, f \in \Lambda$, $e \leq f$, we define the projection of $[ey]$ in $\tilde{R}(f)$ as:

$$\tilde{p}_{e,f}([ey]) = [p_{e,f}(ey_m)]. \tag{6.9}$$

The following theorem shows that this map does indeed satisfy the conditions of Conjecture 2.13 from [25].

Theorem 6.2.2. *Let $e, f \in \Lambda$, $e \leq f$. Then*

1. *$\tilde{p}_{e,f} : \tilde{R}(e) \to \tilde{R}(f)$ is order-preserving.*

2. *$[ey] \leq \tilde{p}_{e,f}([ey])$.*

3. *$[ey] \leq [fz]$ if and only if $\tilde{p}_{e,f}([ey]) \leq [fz]$.*

Proof.   1. Let $[ey] \leq [ey']$. Then $w^{-1}ey_mw \leq ey'_m$ for some $w \in W(e)$, by Corollary 4.2.8. Hence $ey_mw \leq ey'_m$ or $ey_mw = eu_1y_1 = ey_1 \leq ey'_m$, where $u_1 \in W(e)$, $y_1 \in D(e)^{-1}$. But $ey_m \sim ey_1$ and so $ey_m \leq ey_1 \leq ey'_m$. Then $p_{e,f}(ey_m) \leq p_{e,f}(ey'_m)$ and hence $[p_{e,f}(ey_m)] \leq [p_{e,f}(ey'_m)]$ or $\tilde{p}_{e,f}([ey]) \leq \tilde{p}_{e,f}([ey'])$.

2. Since $ey_m \leq p_{e,f}(ey_m)$, by Theorem 3.2.4, we have

$$[ey] = [ey_m] \leq [p_{e,f}(ey_m)] = \tilde{p}_{e,f}([ey]).$$

3. Suppose $[ey] \leq [fz]$ and let $\tilde{p}_{e,f}([ey]) = [fy']$. Then $wey_mw^{-1} \leq fz_m$ and $ey_m \leq fy'_m$. But $ey_m \sim wey_mw^{-1} = ey_2$ so $ey_m \leq ey_2$ and hence $ey_m \leq fz_m$. Thus $p_{e,f}(ey_m) \leq fz_m$ from which it follows that $\tilde{p}_{e,f}([ey]) \leq [fz]$.

The other direction follows from the previous part. ☐

We conclude by noting that although the conjugacy order on $\tilde{R}(e)$ corresponds to the Bruhat-Chevalley order on $R^*(e) = \{ey \mid e \in \Lambda, y \in D^*(e)\}$ for M a dual canonical monoid, this is not the case for all of $\tilde{R}$. For example, recall the dual canonical monoid shown in Figure 4.2. Here $[e_A B] < [e_\emptyset ABA]$, since $e_A B \sim e_A BA$ and $p_{e_A, e_\emptyset}(e_A BA) = e_\emptyset ABA$. However, $e_A B \not\leq e_\emptyset ABA$.

6.3 Canonical Monoids

Our focus remains on obtaining a description of the proposed map from Conjecture 2.13 of [25]. Before doing so, we observe a pleasant generalization of Theorem 5.0.5. The proof depends on the fact that for canonical monoids, if $e, f \in \Lambda$ with $e \leq f$ then $W(e) \subseteq W(f)$. The last case in the proof also uses the following lemmas (the second of which is a generalization of Lemma 5.0.4). An analogous result does not hold for general reductive monoids.

Lemma 6.3.1. *Let M be a canonical monoid and $e, f \in \Lambda$ with $e \leq f$. If $x \in W(f)$ and $x = x_1 x_2$ with $x_1 \in W(e)$ and $x_2 \in D(e)^{-1}$, then $x_2 y \in D(e)^{-1}$ for all $y \in D(f)^{-1}$.*

Proof. Let $x \in W(f)$ with $x = x_1 x_2$ for some $x_1 \in W(e)$ and $x_2 \in D(e)^{-1}$ and let y be any element in $D(f)^{-1}$. We first note that since M is a canonical monoid and $e \leq f$, we have $W(e) \subseteq W(f)$ and thus $x_1^{-1} x = x_2 \in W(f)$. Since $y \in D(f)^{-1}$, this means that $\ell(x_2 y) = \ell(x_2) + \ell(y)$. Now, suppose $u \in W(e)$. Then $u x_2 \in W(f)$ and so $\ell(u x_2 y) = \ell(u x_2) + \ell(y)$. On the other hand, $x_2 \in D(e)^{-1}$ and so $\ell(u x_2) = \ell(u) + \ell(x_2)$. Putting everything together, we observe

$$
\begin{aligned}
\ell(u x_2 y) &= \ell(u x_2) + \ell(y) \\
&= \ell(u) + \ell(x_2) + \ell(y) \\
&= \ell(u) + \ell(x_2 y)
\end{aligned}
$$

Since $u \in W(e)$ was arbitrary, this means that $x_2 y \in D(e)^{-1}$, completing the proof.

☐

Lemma 6.3.2. *Let $x, y \in D(e)^{-1}$, $u \in W$, and $w \in W(e)$, such that $xu = wy$. Then $y = xu'$ for some $u' \leq u$. If $xu = x * u$, then $y = x * u'$.*

Proof. The proof is by induction on the length of w. If $\ell(w) = 0$ then $w = 1$ and the result follows. So let $\ell(w) > 0$ and let $w = s * w_1$, $s \in \lambda(e) = I \subseteq S$. Then $\ell(sxu) = \ell(w_1 y) < \ell(wy) = \ell(xu)$. By Theorem 2.5.3 and the Exchange Property, either $sxu = x'u$ for some $x' < x$ or else $sxu = xu'$ for some $u' < u$. In the first case $x = s * x'$, $s \in I$, a contradiction since $x \in D(e)^{-1}$. Hence $w_1 y = sxu = xu'$, $\ell(w_1 y) = \ell(wy) - 1$. If $xu = x * u$, then $\ell(xu') = \ell(xu) - 1$ and hence $xu' = x * u'$. This completes the proof. □

Finally, before stating the theorem, we recall the following descriptions of $x \circ y$, from Lemma 3.2.3:

$$x \circ y = x_1 * y = x * y_1 \text{ for some } x_1 \leq x, y_1 \leq y$$

and

$$
\begin{aligned}
x \circ y &= \max\{xy' \mid y' \leq y\} \\
&= \max\{x'y \mid x' \leq x\} \\
&= \max\{x'y' \mid x' \leq x, y' \leq y\}.
\end{aligned}
$$

Theorem 6.3.3. *Let $e, f \in \Lambda$ with $e \leq f$, $y \in D(e)^{-1}$, and $z \in D(f)^{-1}$. If $W(e) = W^*(e)$ and $W(f) = W^*(f)$, then the following conditions are equivalent:*

1. $[ey] \leq [fz]$.

2. $wz \leq yw$ for some $w \in W(f)$.

3. $wzw^{-1} \leq y$ for some $w \in W(f)$.

Proof. $(1. \Rightarrow 2.)$ By Corollary 4.2.8, $w^{-1}eyw \leq fz$ for some $w \in W(e)W(f) = W(f)$. Let $yw = uy_1$, $u \in W(e)$, and $y_1 \in D(e)^{-1}$. Then $w^{-1}uey_1 \leq fz$. By Theorem 3.2.1, $u^{-1}wz \leq y_1$. So

$$wz = u(u^{-1}wz) \leq u \circ (u^{-1}wz) \leq u \circ y_1 = uy_1 = yw.$$

$(2. \Rightarrow 3.)$ Let $wz \leq yw$ with $w \in W(f)$. Choose w minimal. Then $wz = y_1 w_1$ for some $y_1 \leq y$ and $w_1 \leq w$. Since $z \in D(f)^{-1}$,

$$w_1 z \leq wz = y_1 w_1 \leq y_1 \circ w_1 \leq y \circ w_1 = y * w_2$$

for some $w_2 \leq w_1$. So $w_2 z \leq w_1 z \leq yw_2$. By minimality of w, $w_2 = w$. So $w_1 = w$ and $wz = y_1 w$. So $wzw^{-1} = y_1 \leq y$.

$(3. \Rightarrow 2.)$ Suppose $wzw^{-1} \leq y$ for some $w \in W(f)$. Then

$$wz = (wzw^{-1})w \leq (wzw^{-1}) \circ w \leq y \circ w = yw_1$$

for some $w_1 \leq w$. Since $z \in D(f)^{-1}$, $w_1 z \leq wz \leq yw_1$.

$(2. \Rightarrow 1.)$ Suppose $wz \leq yw$ for some $w \in W(f)$. Choose w minimal. Let $yw = uy_1$, $y_1 \in D(e)^{-1}$, and $u \in W(e)$. By Lemma 6.3.2, $y_1 = yv$ for some $v \leq w$. Since $wz \leq yw = uy_1$, $wz \leq uy_1$. Let $w = w_1 w_2$ with $w_1 \in W(e)$ and $w_2 \in D(e)^{-1}$. Since $w_2 \in W(f)$, we have $wz = w_1 z_1$ where $w_1 \in W(e)$ and $z_1 = w_2 z \in D(e)^{-1}$, by Lemma 6.3.1. Then $w_1 z_1 \leq uy_1$ and, by Lemma 5.0.2, $w_1 = u_1 * v_1$ with $u_1 \leq u$ and $v_1 z_1 \leq y_1$. Thus $w_1 z_1 = u_1 v_1 z_1 \leq u_1 y_1$. Suppose $u_1 < u$. Since $uy_1 = yw$, if $u_1 < u$ then $u_1 y_1 = yw_3$ for some $w_3 < w$, by Lemma 5.0.3. So $w_3 z < wz = w_1 z_1 \leq u_1 y_1 = yw_3$, contradicting the minimality of w. Hence $u = u_1$ and so $u * (v_1 z_1) = uv_1 z_1 = w_1 z_1 = wz \leq uy_1 = u * y_1$. Now, by Lemma 5.0.2, $v_1 z_1 \leq y_1$ and so $u^{-1} w_1 z_1 = u^{-1} wz \leq y_1$. Hence

$$w^{-1} eyw = w^{-1} euy_1 = w^{-1} uey_1 \leq fz$$

and so $[ey] \leq [fz]$, completing the proof. $\qquad \square$

We are close to defining the projection map we desire. We first need two lemmas that will be important in proving that the maps we will define are the projection maps we seek. Both lemmas have to do with a new relation, $\leq_I$.

Suppose $z \in D_I^{-1}$ and $y \in W$. Define

$$z \leq_I y \text{ if } wzw^{-1} \leq y \text{ for some } w \in W_I.$$

Note that by the Theorem 6.3.3,

$$z \leq_I y \iff [e_\emptyset y] \leq [e_I z] \iff wz \leq yw$$

for some $w \in W_I$, where $e_\emptyset, e_I$ are the (unique) idempotents in $\Lambda \setminus \{0\}$ such that $\lambda(e_\emptyset) = \emptyset$ and $\lambda(e_I) = I$.

Lemma 6.3.4. *Let* $y \in D_I^{-1}$, $u \in W_I$, *and* $z \in W$. *Then* $z \leq_I uy$ *if and only if* $z \leq_I y \circ u$.

Proof. Suppose $z \leq_I uy$. Then $wzw^{-1} \leq uy$ for some $w \in W_I$. Choose $u_1 \leq u$ minimal such that $wzw^{-1} \leq u_1 y$. Then, since u_1 is minimal, $wzw^{-1} = u_1 y_1$ for some $y_1 \leq y$. So $wz = u_1 y_1 w$ and thus $y_1 w = w_1 z$, where $w_1 = u_1^{-1} w \in W_I$. We now have

$$
\begin{aligned}
wz &= u_1 y_1 w \\
u_1 w_1 z &= u_1 y_1 u_1 w_1 \\
w_1 z &= y_1 u_1 w_1 \\
w_1 z w_1^{-1} &= y_1 u_1 \\
&\leq y_1 \circ u_1 \\
&\leq y \circ u.
\end{aligned}
$$

Hence $z \leq_I y \circ u$.

Conversely, suppose $z \leq_I y \circ u$. Then $wzw^{-1} \leq y \circ u$ for some $w \in W_I$ and so $wzw^{-1} = y_1 u_1$ for some $y_1 \leq y$ and $u_1 \leq u$. Now

$$
\begin{aligned}
(u_1 w) z (u_1 w)^{-1} &= u_1 wzw^{-1} u_1^{-1} \\
&= u_1 y_1 \\
&\leq u_1 \circ y_1 \\
&\leq u \circ y \\
&= uy
\end{aligned}
$$

and since $u_1 w \in W_I$, $z \leq_I uy$. This completes the proof. $\qquad\square$

Lemma 6.3.5. *Let $W_K = \bigcap_{i \geq 0} (y^{-i} W_I y^i)$. Let $y, z \in D_I^{-1}$ and $u \in W_K$. Then $z \leq_I uy$ if and only if $z \leq_I y$.*

Proof. Suppose $z \leq_I y$. Then $wzw^{-1} \leq y$ for some $w \in W_I$. Since $u \in W_k \subseteq W_I$, $y \leq uy$ and so $wzw^{-1} \leq uy$. That is, $z \leq_I uy$.

Conversely, suppose $z \leq_I uy$. So $wzw^{-1} \leq uy$ for some $w \in W_I$. By a previous theorem, this means $wz \leq uyw$ for some $w \in W_I$. Let $\mathcal{U} \subseteq (W_K, W_I)$ such that $\mathcal{U} = \{(u, w) \mid wz \leq uyw\}$. Define $\ell(u, w) = \ell(u) + \ell(w)$, where $\ell(x)$ for $x \in W$ is the usual length function. Choose $(u, w) \in \mathcal{U}$ such that $\ell(u, w)$ is minimal. Now, $yw \leq y \circ w = yw'$ for some $w' \leq w$. Hence $w'z \leq wz \leq uyw \leq uyw'$. Hence $z \leq_I uy$ and since $\ell(u, w') \leq \ell(u, w)$ and $w' \leq w$, we have $w' = w$. Thus $yw = y * w$. We also have $uy = u * y$, since $u \in W_K \subseteq W_I$, and so $uyw = u * y * w$.

Since $wz \leq u * y * w$, $wz = u'y_1w'$ for some $u' \leq u$, $y_1 \leq y$, and $w' \leq w$. But then $wz \leq u'yw$ and so $z \leq_I u'y$. Hence $\ell(w) + \ell(u') \leq \ell(w) + \ell(u)$ and so $u' = u$. So $wz = uy_1w'$. However, now we have $w'z \leq wz \leq uyw'$. So $z \leq_I uy$ with $\ell(w') + \ell(u) \leq \ell(w) + \ell(u)$, and hence $w' = w$. Thus we have $wz = uy_1w$, for some $y_1 \leq y$.

Let $w_1 = u^{-1}w \in W_I$. Then $w = uw_1$ and $w_1z = y_1w$. But $w = uw_1 \leq u \circ w_1 = u_2 * w_1$ for some $u_2 \leq u$, and so now we have

$$
\begin{aligned}
wz &= uw_1z \\
&\leq (u \circ w_1)z \\
&= u_2w_1z \\
&= u_2y_1w \\
&\leq u_2 \circ (y_1w) \\
&= u_3 * (y_1w) \\
&\leq u_3yw
\end{aligned}
$$

for some $u_3 \leq u_2$. So $z \leq_I u_3y$ and since $\ell(w) + \ell(u_3) \leq \ell(w) + \ell(u)$ with $u_3 \leq u_2 \leq u$, we have $u_3 = u_2 = u$. Since $w = uw_1 \leq u_2 * w_1$, we also know that $w = u * w_1$.

Since $wz = uy_1w$ we have $u^{-1}wz = y_1w$, or $w_1z = y_1uw_1$. But $y_1uw_1 \le y \circ (uw_1) = yu_4w_2$ for some $u_4 \le u$ and $w_2 \le w_1 \le w$. So $w_2z \le yu_4w_2$. Since $u_4 \in W_K$, $u_4 = y^{-1}u_5y$ for some $u_5 \in W_I$. What's more, since for any $k \ge 2$ we have $u_4 = y^{-k}u_6y^k$ for some $w_6 \in W_I$, $u_5 = yu_4y^{-1} = y^{-(k-1)}u_6y^{k-1}$ and so $u_5 \in W_K$.

Putting everything together, we have $w_2z \le u_5yw_2$ and so $z \le_I u_5y$. Since $u_5 \in W_K$, we must have $\ell(u) + \ell(w) \le \ell(u_5) + \ell(w_2)$. But $u_5y = yu_4$ and $\ell(u_5y) = \ell(u_5) + \ell(y)$, since $u_5 \in W_I$, so $\ell(u_5) \le \ell(u_4) \le \ell(u)$. Hence

$$\ell(u) + \ell(w) \le \ell(u_5) + \ell(w_2) \le \ell(u) + \ell(w_2).$$

Since $w_2 \le w_1 \le w$, we conclude that $w_2 = w_1 = w$. Since $w = u * w_1$, we must have $u = 1$.

Thus $wz \le yw$ and so $z \le_I y$, completing the proof. $\square$

Let $e, f \in \Lambda$ with $e \le f$. Let $y \in D(e)^{-1}$ and $y = u_1y_1$ with $u_1 \in W(f)$, $y_1 \in D(f)^{-1}$. Consider $y_1 \circ u_1 = \max\{y_1u_1' \mid u_1' \le u_1\}$. Then $y_1 \circ u_1 = y_1 * u_1'$ for some $u_1' \le u_1$ and $y_1 * u_1' = u_2y_2$ for some $u_2 \in W(f)$, $y_2 \in D(f)^{-1}$. Furthermore, by Lemma 5.0.4, $y_2 = y_1 * v_1$ for some $v_1 \le u_1'$, and hence $\ell(u_1) \ge \ell(u_1') \ge \ell(u_2)$.

We repeat this process. That is, we next consider $y_2 \circ u_2 = y_2 * u_2' = u_3y_3$ where $y_3 = y_2 * v_2 = y_1 * v_1 * v_2$. Hence $\ell(u_1) \ge \ell(u_1') \ge \ell(u_2) \ge \ell(u_2') \ge \ell(u_3) \ge \cdots$. Continuing, we eventually have $\ell(u_j) = \ell(u_{j+1})$ for all $j \in \mathbb{Z}$ greater than m. Thus

$$y_m \circ u_m = y_m * u_m' = u_{m+1}y_{m+1} = u_{m+1}y_m * v_m$$

with

$$\ell(y_m * u_m') = \ell(y_m) + \ell(u_m') = \ell(u_{m+1}) + \ell(y_{m+1})$$

and so $\ell(y_m) = \ell(y_{m+1})$. But now $\ell(y_m) = \ell(y_m * v_m) = \ell(y_m) + \ell(v_m)$. So $\ell(v_m) = 0$ and hence $v_m = 1$ and thus $y_{j+1} = y_j$ for all $j \ge m$. In the following, by y_m we mean the element obtained from u_1 and y_1 as described here.

In the above construction, we note that $y_m \circ u_m = y_mu_m = u_{m+1}y_m$. Thus $u_m = y_m^{-1}u_{m+1}y_m$. Furthermore, $y_m \circ u_{m+1} = y_mu_{m+1} = u_{m+2}y_m$ and so $u_{m+1} = y_m^{-1}u_{m+2}y_m$, from which it follows that $u_m = y_m^{-2}u_{m+2}y_m^2$. This continues on, and so we see that $u_m \in \bigcap_{i \ge 0} (y_m^{-i}W_Iy_m^i)$.

For $e, f \in \Lambda$, $e \leq f$, we define the projection of $[ey]$ in $\tilde{R}(f)$ as:

$$\tilde{p}_{e,f}([ey]) \;=\; [fy_m]$$

where $y = u_1 y_1$ with $u_1 \in W(f)$ and $y_1 \in D(f)^{-1}$ and y_m is as noted.

Theorem 6.3.6. *Let $e, f \in \Lambda$, $e \leq f$. Then*

1. *$[ey] \leq \tilde{p}_{e,f}([ey])$.*

2. *$[ey] \leq [fz]$ if and only if $\tilde{p}_{e,f}([ey]) \leq [fz]$.*

3. *$\tilde{p}_{e,f} : \tilde{R}(e) \to \tilde{R}(f)$ is order-preserving.*

Proof.   1. We note that for $y = u_1 y_1$, $y_1 \circ u_1 = u_2 y_2$ for some $u_2 \in W(f)$ and $y_2 \in D(f)^{-1}$, and in general $y_{j-1} \circ u_{j-1} = u_j y_j$. By Lemma 6.3.5, $y_m \leq_J u_m y_m$, where $J = \lambda(f)$. Now, applying Lemma 6.3.4, we have

$$
\begin{aligned}
y_m &\leq_J & u_m y_m &= y_{m-1} \circ u_{m-1} \\
\Longleftrightarrow \quad y_m &\leq_J & u_{m-1} y_{m-1} &= y_{m-2} \circ u_{m-2} \\
& & \vdots & \\
\Longleftrightarrow \quad y_m &\leq_J & u_2 y_2 &= y_1 \circ u_1 \\
\Longleftrightarrow \quad y_m &\leq_J & y &= u_1 y_1.
\end{aligned}
$$

   Hence $y_m \leq_J y$, and so by Theorem 6.3.3, $[ey] \leq [fy_m] = \tilde{p}_{e,f}([ey])$.

2. Suppose $[ey] \leq [fz]$ and let $\tilde{p}_{e,f}([ey]) = [fy_m]$. By Theorem 6.3.3, $wzw^{-1} \leq y$ for some $w \in W(f)$. That is, $z \leq_J y$, where $J = \lambda(f)$. Using Lemma 6.3.4, we have

$$
\begin{aligned}
z &\leq_J & y &= u_1 y_1 \\
\Longleftrightarrow \quad z &\leq_J & y_1 \circ u_1 &= u_2 y_2 \\
& & \vdots & \\
\Longleftrightarrow \quad z &\leq_J & y_{m-2} \circ u_{m-2} &= u_{m-1} y_{m-1} \\
\Longleftrightarrow \quad z &\leq_J & y_{m-1} \circ u_{m-1} &= u_m y_m
\end{aligned}
$$

and, by Lemma 6.3.5,

$$z \leq_J y_m$$

Hence $z \leq_J y_m$, and so by Theorem 6.3.3, $[fy_m] = \tilde{p}_{e,f}([ey]) \leq [fz]$.

The other direction follows from the previous part.

3. Suppose $[ey] \leq [ey']$. By the first part above, $[ey'] \leq \tilde{p}_{e,f}([ey'])$ and hence $[ey] \leq \tilde{p}_{e,f}([ey'])$. By the second part above, $\tilde{p}_{e,f}([ey]) \leq \tilde{p}_{e,f}([ey'])$.

$\square$

Chapter 7

Conclusion

The conjugacy poset of a reductive monoid is a relatively new object of study. This dissertation has been an attempt towards a better understanding of this poset. We began our analysis by looking at the classes for a fixed idempotent, denoted $\tilde{R}(e)$. Theorem 5.0.5 gave conditions equivalent to the conjugacy order, but strictly in terms of relevant Weyl group elements and the (well known) Bruhat-Chevalley order. In our study of the order between $\tilde{R}(e)$ classes, we followed an approach similar to that of [24] by finding order-preserving maps between the classes with specific properties. These are the projection maps conjectured in [25]. We found that such maps exist for matrices, dual canonical monoids, and canonical monoids, but were unable to obtain a description for the general case. If such maps exist, such a description seems likely to be a combination of the approaches for dual canonical and canonical monoids. The following is a possible description:

Conjecture 7.0.7. *Suppose $e, f \in \Lambda$ with $e \leq f$ and choose $[\sigma] \in \tilde{R}(e)$. Assume $[\sigma]$ has a minimal element, call it ey. Factor y as $u_1 y_1$ with $u_1 \in W(f)$ and $y_1 \in D(f)^{-1}$. Next, consider $y_1 \circ \hat{u}_1$, where $u_1 = \hat{u}_1 \check{u}_1$ with $\hat{u}_1 \in W^*(f)$ and $\check{u}_1 \in W_*(f)$, and rewrite this as $u_2 y_2$. Proceed likewise until eventually obtaining $u_m y_m$, similar to the procedure described in Section 6.3. Then $\tilde{p}_{e,f}([ey]) = [fy_m]$.*

This proposed description coincides with the dual canonical and canonical cases

presented in Chapter 6. Additionally, it satisfies the desired conditions for matrices for computations completed so far, for small n.

Regarding matrices, we would also like to obtain an alternate description. Much effort was spent in Chapter 3 establishing the description of the projection maps for R_n in terms of the one-line notation (this was the "k-insertion algorithm"). We seek a similar description for $\tilde{R}_n$ in the one-line notation, preferably making use of this algorithm. As noted in the examples from Section 6.1, the difficulty here lies in finding the right element on which to apply the algorithm.

Finally, we plan to continue the general study of these posets, making use of the results presented in this dissertation. For matrices and dual canonical monoids, the conjugacy poset is usually not graded, as we have observed in several examples. The case is not as clear with canonical monoids. The examples of $\tilde{R}$ and $\tilde{R}(e)$ presented here are both graded and preliminary reports suggest that this is the case in general. In [24], the projection maps are used to show that for a canonical monoid, the poset $R^* = R \setminus \{0\}$ is Eulerian. We see in Figure 4.3 that $\tilde{R} \setminus \{[0]\}$ is not Eulerian (for example, the interval $[[e_\emptyset BA], [e_A]]$ is not balanced), but other poset properties might be uncovered using techniques similar to those from [24].

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