Abstract

BURDIS, JOSEPH MICHAEL. Object-Image Correspondence of Under Projections.
(Under the direction of Irina A. Kogan.)

In this thesis we consider a problem of object-image correspondence under parallel and central projections from a 3-dimensional space to a plane. The motivation comes from an important problem in computer vision – determining the correspondence between an object and its image obtained by a camera with unknown position and parameters. Defining features of objects and images can often be represented by curves and by finite ordered sets of points. Therefore. we concentrate on providing projection criteria for object-image pairs for curves and finite ordered sets of points.

For parallel projections, a solution to the projection problem for finite ordered sets of points was presented in [2, 1]. We review this solution and we provide a comparison with a novel solution presented in this thesis. An algorithmic solution for the projection problem for curves under parallel and central projection with a large number of unknown parameter appears to be previously unknown.

The main result of the thesis reduces the projection problem under parallel and central projections to a certain variation of the equivalence problem of planar objects under affine and projective groups of transformations. The group-equivalence problem can be solved by adapting known techniques from differential and algebraic invariant theory. This leads to an algorithmic solution to the projection problem for curves, and for finite ordered sets of points, under either parallel or central projections. We implemented this algorithm using the computer algebra system Maple. We describe this implementation, provide some examples and discuss possible further improvements of its efficiency. The method presented here has a potential of being developed into a practically efficient general algorithm for establishing object-image correspondence between real-life objects and their images.

Object-Image Correspondence of Under Projections

by
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Dedication

This thesis is dedicated to my parents, John Burdis and Sherry Kolar, and my wife Wilma Burdis.

Biography

Joseph Michael Burdis was born on August 5, 1982 in Columbia, Maryland to John Burdis and Sherry Kolar. He has an older sister, Loni. Following graduation from high school, Joseph enrolled at the University of Pittsburgh. In 2004, he received a B.S. in Computer Science and Mathematics with a minor in Economics. In 2005, while enrolled in the Ph.D. program at North Carolina State University, he received an M.S. in Mathematics. On June 28, 2008 Joseph married Wilma Jackson.

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Chapter 1

Introduction

In this thesis we introduce criteria and propose algorithms to decide correspondence between objects in 3-dimensional space and planar images. Areas of computer vision in which these algorithms may prove useful include scene recognition, image reconstruction, motion analysis, and object recognition. In fact, the fundamental issue in object recognition is to efficiently decide correspondence between an object and an image [17, pp 254].

In this thesis, we consider objects and images represented by ordered sets of points, and also consider objects and images represented by curves. Using an ordered set of points is useful when an object has identifiable feature points [2, 1]. For example, an airplane's feature points include the front nose, wing tips, and various tail points. Using curves to represent objects and images is useful when representing borders of surfaces and for medical imaging applications. For example, determining when 3-D imaging of vessels (MRI or CT) corresponds to 2-D X-ray imaging can be used to make a correct diagnosis, plan therapy, and control therapy [10].

To provide a solution to the projection problem for curves and the projection problem for finite ordered sets of points, we use a combination of algebraic and differential geometry techniques. These techniques include group actions, differential and algebraic invariants, computer algebra algorithms, and other methods for differential and algebraic geometry.

Since the defining features of many objects can be represented by curves, obtaining an algorithmic solution for the projection problem for curves is essential, but appears to be unknown in the case of projections with a large number of unknowns. We address the projection problem on curves and finite ordered sets of points for two classes of cameras: finite projective cameras and affine cameras.

The set of *finite projective cameras* (also called *finite cameras*) has 11 parameters and corresponds to the set of central projections. The set of *affine cameras* has 8 parameters and corresponds to the set of parallel projections. As discussed in Section 2.2, an affine camera can be obtained as a limit of a finite camera, as the camera center approaches infinity along the perpendicular from the camera center to the image plane. See [14] for more details on camera projections and the related geometry. An affine camera has fewer parameters and provides a good approximation of a finite camera when the distance between the camera and the object is significantly greater than the object depth [14, 2]. The projection problem for curves is formulated as follows:

Problem 1.0.1. *Given a smooth curve C_Γ in $\mathbb{R}^3$ and a smooth curve C_γ in $\mathbb{R}^2$, does there exist a finite or an affine camera that maps C_Γ to C_γ?*

Previous works on related problems include [10], where a solution to Problem 1.0.1 is given for finite cameras with known internal parameters, so the number of free parameters is reduced from 11 to 6 parameters, which correspond to the position and orientation of the camera. The method presented in [10] also uses an additional assumption that a planar curve, C_γ, has at least two points, whose tangent lines coincide. In the current paper we assume that the internal parameters of the camera are unknown. A solution of the projection problem for finite ordered sets of points under affine cameras appeared in [2, 1] and served as an inspiration for this work.

In Chapter 2, after reviewing the geometry of finite and affine cameras, we define actions of direct products of affine and projective groups on the sets of cameras. We use these actions to reduce Problem 1.0.1 for finite and affine cameras to a certain variation of the equivalence problem for *planar* curves under projective and affine transformations, respectively. This leads to one of the main results of this thesis, projection criteria for curves, formulated in Section 4.2. In Section 3.1, we review a solution for the group-equivalence problem, based on differential signature construction [5]. In Section 4.3, we combine our projection criteria and the differential signature construction in order to obtain an algorithm for solving the projection problem. To demonstrate how the algorithms are used we give examples in Section 4.4.

In Chapter 5, we adapt the solution of Problem 1.0.1 to produce a solution to the projection problem for finite ordered sets of points. The projection problem for points is formulated as follows:

Problem 1.0.2. *Given an* ordered *set* $Z = (\mathbf{z}^1, \dots, \mathbf{z}^r)$ *of* r *points in* $\mathbb{R}^3$ *and an* ordered *set* $X = (\mathbf{x}^1, \dots, \mathbf{x}^r)$ *of* r *points in* $\mathbb{R}^2$*, does there exist a finite or an affine camera taking* Z *to* X*?*

In [2, 1] the authors present a solution to Problem 1.0.2 for *affine* projections, without having to find a projection explicitly. In Chapter 5, we adapt the projection criteria to obtain an algorithm for solving Problem 1.0.2 for both *finite* and *affine* projections, and provide examples. Our solution reduces the projection problem for ordered sets of points in $\mathbb{R}^3$ to $\mathbb{R}^2$ to an equivalence problem of ordered sets of points in $\mathbb{R}^2$. In Section 5.5, we compare the complexity of this algorithm to that found in [2, 1].

In Chapter 6, we discuss possible variations of our algorithm based on alternative solutions of the group-equivalence problem, as well as possible adaptations to curves presented by samples of discrete points whose coordinates may be known only approximately. A joint work with Irina Kogan, [4], has served as a foundation of this thesis.

1.1 Projective Space

To define projective space, we define the following equivalence relation $\sim$ on $\mathbb{R}^n \setminus \{0\}$,

$$(x'_1, \ldots, x'_n) \sim (x_1, \ldots, x_n)$$

if there $\exists \lambda_{\neq 0} \in \mathbb{R}$ such that $(x'_1, \ldots, x'_n) = (\lambda\, x_1, \ldots, \lambda\, x_n)$. [7].

Definition 1.1.1. (n)-dimensional projective space*, denoted* $\mathbb{P}^n$*, is the set of equivalence classes of* $\sim$ *on* $\mathbb{R}^{n+1} \setminus \{0\}$*. Therefore,*

$$\mathbb{P}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim .$$

Each non-zero point in $\mathbb{R}^{n+1}$ defines a point in $\mathbb{P}^n$. Also, there are many ways to embed $\mathbb{R}^n$ in $\mathbb{P}^n$. For the purpose of this thesis we will use the following embedding

$$(x_1, \ldots, x_n) \rightarrow \{(\lambda\, x_1, \ldots, \lambda\, x_n, \lambda) | \lambda_{\neq 0} \in \mathbb{R}\} \tag{1.1}$$

and make use of the following notation.

Notation 1.1.2. Square brackets *around matrices (and, in particular, vectors) will be used to denote an equivalence class with respect to multiplication of a matrix by a nonzero scalar. Multiplication of equivalence classes of matrices* A *and* B *of appropriate sizes is well-defined by* $[A]\,[B] := [A\,B]$.

With this notation, a point $(x, y) \in \mathbb{R}^2$ corresponds to a point $[x, y, 1] = [\lambda x, \lambda y, \lambda] \in \mathbb{P}^2$ for all $\lambda \neq 0$, and a point $(z_1, z_2, z_3) \in \mathbb{R}^3$ corresponds to $[z_1, z_2, z_3, 1] \in \mathbb{P}^3$. We will refer to the points in $\mathbb{P}^n$ whose last homogeneous coordinate is zero as *points at infinity.* Embedding (1.1) provides a correspondence between curves in $\mathbb{R}^n$ and curves in $\mathbb{P}^n$. We will use the following notation.

Notation 1.1.3. *Let* $\gamma\colon I_\Gamma \to \mathbb{R}^n$*, where* $I_\gamma \subset \mathbb{R}$ *is an interval, be a smooth parametric curve. Its image in* $\mathbb{R}^n$ *will be denoted by* $C_\gamma := \{\gamma(t) | t \in I_\gamma\}$.

The corresponding set of points in $\mathbb{P}^n$ *(represented in homogeneous coordinates as column* $(n+1)$*-vectors) will be denoted by* $[C_\gamma]$*. For example, if* $\gamma(t) = (x(t), y(t))$ *for* $t \in I_\gamma$*, then* $[C_\gamma] := \{\,[x(t),\, y(t),\, 1]^{\mathrm{tr}}\,|\, t \in I_\gamma\} \subset \mathbb{P}^2$.

If $[P]$ *is a map from* $\mathbb{P}^n$ *to* $\mathbb{P}^m$ *then by* $[P][C_\gamma]$ *we denote the image of the set* $[C_\gamma]$ *under this map.*

1.1.1 The Plüker Embedding

In this section, we introduce Grassmannians and the Plüker embedding which will be used in Section 5.5 to review the work in [2, 1]. The Plüker embedding is not used for the main results of this thesis, so it is not discussed in detail here. For more details see [30].

Definition 1.1.4. *The set of all* m*-dimensional subspaces of* $\mathbb{R}^n$ *is called the* Grassmannian*, denoted* $Grass(m, n)$.

Let $\bigwedge^m \mathbb{R}^n$ be the m-th skew symmetric tensor power of $\mathbb{R}^n$. Then, $\bigwedge^m \mathbb{R}^n$ is a linear space isomorphic to $\mathbb{R}^N$, where $N = C_m^n$. Let $\mathbb{P}(\bigwedge^m \mathbb{R}^n)$ denote the corresponding projective space of dimension $C_m^n - 1$.

Definition 1.1.5. *The* Plüker embedding *is a map from Grassmanian space to projective space defined by*

$$\begin{aligned} Grass(m,n) &\rightarrow \mathbb{P}\left(\bigwedge^m \mathbb{R}^n\right) \\ span\{v_1, \cdots, v_m\} &\mapsto v_1 \wedge \cdots \wedge v_m \end{aligned} \tag{1.2}$$

The image of the Plücker embedding is a projective variety. The vanishing ideal is generated by a system of quadratic polynomials known as the Plücker relation.

Proposition 1.1.6. *Let* $V, W \subset \mathbb{R}^r$ *be subspaces with* $\dim(V) = l < \dim(W) = m$. *Then* $V \subset W$ *if and only if the relative system of Plücker relation polynomials vanish.*

1.2 Geometric group actions on $\mathbb{R}^n$ and $\mathbb{P}^n$

Since group actions and, in particular, actions of the affine and projective groups play a crucial role in our construction, we review the relevant definitions:

Definition 1.2.1. *An* action *of a group* G *on a set* S *is a map* $\Phi\colon G \times S \rightarrow S$ *that satisfies the following two properties:*

1. $\Phi(e, s) = s$, $\forall s \in S$, *where* e *is the identity of the group.*

2. $\Phi(g_1, \Phi(g_2, s)) = \Phi(g_1\, g_2, s)$, $\forall\, s \in S$ *and* $\forall\, g_1, g_2 \in G$.

For $g \in G$ and $s \in S$ we sometimes write $\Phi(g, s) = g\, s$.

Definition 1.2.2. *For a fixed element* $s \in S$ *the set* $O_s = \{g \cdot s | g \in G\} \subset S$ *is called* the orbit of s.

Definition 1.2.3. *An action is called* transitive *if for all* $s_1, s_2 \in S$ *there exists* $g \in G$ *such that* $s_1 = g\, s_2$.

Definition 1.2.4. *For a fixed element* $s \in S$ *the set* $G_s = \{g \in G | gs = s\} \subset G$ *is called the* stabilizer *of* s.

Definition 1.2.5. *Given an action of* G *on* S*, the* global isotropy group,

$$G_S^* = \{g \in G | g \cdot s = s \ for \ all \ s \in S\} \tag{1.3}$$

is the set consisting of the group elements that fix all members of S.

It can be shown that a stabilizer, G_s, and the global isotropy group G_S^* are subgroups of G. In fact, G_S^* is a normal subgroup.

Definition 1.2.6. *A group acts* effectively *if* $G_S^* = \{e\}$.

If the group does not act effectively, i.e. $G_S^* \neq \{e\}$, one can consider the equivalent action on $G \,/\, G_S^*$.

Definition 1.2.7. *A group acts* freely *if every stabilizer subgroup is trivial.*

Definition 1.2.8. *The* projective group, $\mathcal{PGL}(n+1)$*, is a quotient of the general linear group* $\mathcal{G}L(n+1)$*, consisting of* $(n+1)\times(n+1)$ *non-singular matrices, by a 1-dimensional abelian subgroup* λI*, where* $\lambda \neq 0 \in \mathbb{R}$ *and* I *is the identity matrix. Elements of* $\mathcal{PGL}(n+1)$ *are equivalence classes* $[B] = [\lambda B]$*, where* $\lambda \neq 0$ *and* $B \in \mathcal{G}L(n+1)$.

The affine *group* $\mathcal{A}(n)$ *is a subgroup of* $\mathcal{PGL}(n+1)$ *whose elements* $[B]$ *have a representative* $B \in \mathcal{G}L(n+1)$ *whose last row is* $(0,\dots,0,1)$.

The special affine *group* $\mathcal{SA}(n)$ *is a subgroup of* $\mathcal{A}(n)$ *whose elements* $[B]$ *have a representative* $B \in \mathcal{G}L(n+1)$ *with determinant* 1 *and the last row equal to* $(0,\dots,0,1)$.

In homogeneous coordinates the standard action of the projective group $\mathcal{PGL}(n+1)$ on $\mathbb{P}^n$ is defined by:

$$\Phi([B],[z_1,\dots,z_n,z_0]^{\mathrm{tr}}) = [B]\,[z_1,\dots,z_n,z_0]^{\mathrm{tr}}. \tag{1.4}$$

The action (1.4) induces an almost everywhere defined linear-fractional action on $\mathbb{R}^n$. In particular, for $n=2$, $[B] \in \mathcal{PGL}(3)$ we have

$$(x,y) \to \left(\frac{b_{11}\,x + b_{12}\,y + b_{13}}{b_{31}\,x + b_{32}\,y + b_{33}}, \frac{b_{21}\,x + b_{22}\,y + b_{23}}{b_{31}\,x + b_{32}\,y + b_{33}}\right). \tag{1.5}$$

The restriction of (1.4) to $\mathcal{A}(n)$ induces an action on $\mathbb{R}^n$ consisting of compositions of linear transformations and translations. In particular, for $n=2$ and $[B] \in \mathcal{A}(2)$ represented by a matrix B whose last row is $(0,0,0,1)$,

$$(x,y) \to (b_{11}\,x + b_{12}\,y + b_{13},\; b_{21}\,x + b_{22}\,y + b_{23})\,. \tag{1.6}$$

Chapter 2

Cameras

This chapter discusses the geometry of pinhole camera models, and, in particular, how finite and affine cameras are modeled by central and parallel projections respectively.

For real parameters $p_{ij}, i = 1 \ldots 3, j = 1 \ldots 4$, a generic projection maps a point $(z_1, z_2, z_3) \in \mathbb{R}^3$ to a point in the image plane with coordinates

$$(x, y) = \left(\frac{p_{11}\, z_1 + p_{12}\, z_2 + p_{13}\, z_3 + p_{14}}{p_{31}\, z_1 + p_{32}\, z_2 + p_{33}\, z_3 + p_{34}}\, , \frac{p_{21}\, z_1 + p_{22}\, z_2 + p_{23}\, z_3 + p_{24}}{p_{31}\, z_1 + p_{32}\, z_2 + p_{33}\, z_3 + p_{34}} \right) \tag{2.1}$$

A convenient matrix representation of this map is obtained by embedding $\mathbb{R}^n$ into projective space $\mathbb{P}^n$ and utilizing homogeneous coordinates on $\mathbb{P}^n$; the square bracket notation was discussed in 1.1. In homogeneous coordinates, projection (2.1) is given by

$$[x,\, y\, , 1]^{\mathrm{tr}} = [P]\, [z_1,\, z_2,\, z_3,\, 1]^{\mathrm{tr}}, \quad where \quad [P] = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \tag{2.2}$$

is a 3×4 matrix of rank 3.

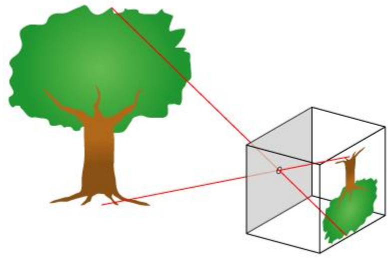

Figure 2.1: Pinhole camera [20].

2.1 Finite cameras

A projection is called *finite* if its center is not at infinity. This corresponds to P having a non-singular left 3×3 submatrix. Finite projections represent the action of taking a picture with a pinhole camera.

Let (z_1, z_2, z_3) be coordinates in $\mathbb{R}^3$, relative to an orthonormal coordinate basis, such that the camera is located at the origin on $\mathbb{R}^3$, and the image plane is passing through the point $P = (0, 0, 1)$ perpendicular to the z_3-axis. Assume the coordinate system on the image plane is provided by the first two coordinate functions on $\mathbb{R}^3$. We call the coordinates on the image plane the x-axis and y-axis. This model corresponds to taking a picture in the direction of the z_3 axis. Then, a point (z_1, z_2, z_3), such that $z_3 \neq 0$, is projected to the point

$$(x, y) = \left(\frac{z_1}{z_3}, \frac{z_2}{z_3} \right). \tag{2.3}$$

Projection (2.3) models the simple pinhole camera illustrated in Figure 2.1 and is a central projection.

Assume the image plane is at some distance f, called the focal length, from the camera center. Figure 2.2 illustrates how this camera projection works in the $z_3\, z_2$-plane. The camera center is at the origin, $C = (0, 0, 0)$, and the image plane is at point $P = (0, 0, f)$

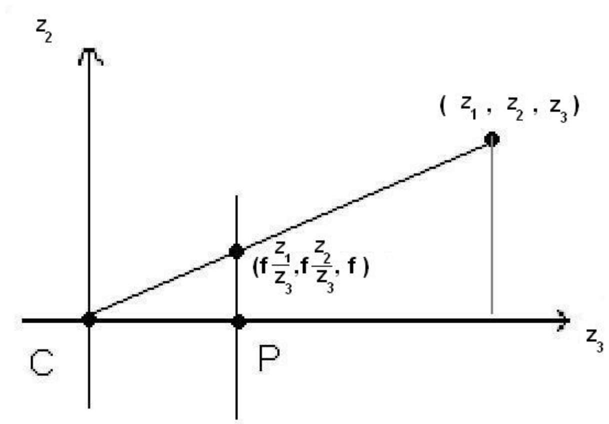

Figure 2.2: $z_3\, z_2$-plane of pinhole camera model.

where f is the focal length. Here a point (z_1, z_2, z_3), such that $z_3 \neq 0$, is projected to a point

$$(x, y) = \left(\frac{f\, z_1}{z_3}, \frac{f\, z_2}{z_3} \right).$$

Or, in homogeneous coordinates,

$$\begin{bmatrix} f\, z_1 \\ f\, z_2 \\ z_3 \end{bmatrix} = [K] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ 1 \end{bmatrix} \qquad where \quad K = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.4}$$

In (2.4), K represents the internal parameters of the camera. In general, we allow for scaling, translation, and skew of the axes on the image plane, and view these changes as internal features of the camera. This creates a total of 5 degrees of freedom on the internal parameters of the camera,

$$K = \begin{pmatrix} scale_x & skew & t_1 \\ 0 & scale_y & t_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5}$$

The focal length, or distance from the camera center to the image plane, is accounted for with the scaling parameters. In general, a projection in homogeneous coordinates by a general camera is decomposed into a product of the following linear maps

$$[P] = [K][R_{z_3}]\,[R_{z_1}]\,[R_{z_2}] \begin{bmatrix} 1 & 0 & 0 & -c_1 \\ 0 & 1 & 0 & -c_2 \\ 0 & 0 & 1 & -c_3 \end{bmatrix}, \tag{2.6}$$

where R_{z_1}, R_{z_2}, *and* R_{z_3} are rotations about the z_1, z_2, *and* z_3 axis. In (2.6) matrix K, with 5 parameters, represents the internal properties of the camera. The rotation matrix, R_{z_3}, with 1 parameter, represents orientation of the camera (vertical, horizontal, or other spin of the camera). Two rotation matrices, R_{z_1} and R_{z_2}, represent the possible directions the camera could point; two degrees of freedom. The last matrix describes the camera center location, $C = (c_1, c_2, c_3)$, that could be anywhere in $\mathbb{R}^3$. To summarize, the projection has 11 degrees of freedom, 5 from internal parameters and 6 corresponding to the group of rigid motions, $SE(3)$, in $\mathbb{R}^3$.

An alternate interpretation arises from the fact that every element in $GL(2)$ can be written as a product of an invertible upper triangular matrix and an orthogonal matrix, $GL(2) = \{Invertable\,Upper\,Triangular\} \times SO(2)$. Therefore, the set of all matrices of the form, $\{K\,R_{z_3}\}$, is equal to $\mathcal{A}(2)$. This leads to interpreting Equation 2.6 as moving the camera center to the origin, choosing the direction to take the picture, and applying an affine change of coordinates on the image plane.

Any 3×4 matrix, whose left 3×3 minor is nonzero, can be decomposed into the product (2.6). This leads to the following definition for a finite projection.

Definition 2.1.1. *A* finite *projection is a* rank 3 *linear map from* $\mathbb{P}^3$ *to* $\mathbb{P}^2$ *that has a representation given by*

$$[P_f] = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}, \quad \textit{where} \quad \det \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \neq 0. \tag{2.7}$$

The set of all finite projections will be denoted, $\mathcal{FP}$. In this thesis, it will be important to refer to the most basic *finite* projection, so we make the following definition.

Definition 2.1.2. *A projection represented by the matrix,*

$$P_f^0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.8}$$

is called the standard finite projection*.*

In homogeneous coordinates, the pinhole camera projection in (2.3) is represented by the standard finite projection. The term *finite* is used because the center of the camera corresponds to a finite point of the projective space.

2.2 Affine cameras

On the contrary to *finite*, an *infinite* camera has its center at an infinite point, $[a, b, c, 0]$, of $\mathbb{P}^3$, and, for reasons discussed in this section, the left 3×3 submatrix of P is singular. An infinite camera is called *affine* when the preimage of the line at infinity in $\mathbb{P}^2$ is the plane at infinity in $\mathbb{P}^3$. In this case, $[P]$ is represented by a matrix whose last row is $(0, 0, 0, 1)$.

An affine camera can be obtained as a limit of the finite camera, when its center moves to infinity and the image is scaled by the corresponding factor. In the projection decomposition, (2.6), let $R := R_{z_3} R_{z_1} R_{z_2}$. Define the principal plane to be the plane through the camera center that is parallel to the image plane. We cannot draw a line from the camera center to these points through the image plane, so the principal plane consists of the set of points mapped to the line at infinity of the image plane. In particular, the principal plane consists of all points that satisfy $[P]\,[z_1, z_2, z_3, 1]^{\mathrm{tr}} = [x, y, 0]^{\mathrm{tr}}$. Notice the third row of P is perpendicular to the principal plane. Consequently, the third row of P is perpendicular to the image plane. This tells us that the picture is taken in the direction given by the third row of P, or equivalently, the third row of R. So using (2.6), moving the camera center in this direction corresponds to the family of projections

$$[P_t] = [K] \begin{bmatrix} \mathbf{r}_{1*} & -\mathbf{r}_{1*} \cdot (\,C + t\,\mathbf{r}_{3*}^{\mathrm{tr}}) \\ \mathbf{r}_{2*} & -\mathbf{r}_{2*} \cdot (\,C + t\,\mathbf{r}_{3*}^{\mathrm{tr}}) \\ \mathbf{r}_{3*} & -\mathbf{r}_{3*} \cdot (\,C + t\,\mathbf{r}_{3*}^{\mathrm{tr}}) \end{bmatrix} \tag{2.9}$$

where $\mathbf{r}_{i*}$ is the row vector representing the i-th row of R, and $C = (c_1, c_2, c_3)$ is the camera center. Note that $\mathbf{r}_{i*}\,\mathbf{r}_{j*} = \delta_{i\,j}$. Define $d_t = -\mathbf{r}_{3*} \cdot C + t$ and scale the image plane by a factor of $\frac{d_t}{d_0}$, where $d_0 = -\mathbf{r}_{3*} \cdot C$, then

$$[\widetilde{P}_t] = \begin{bmatrix} \frac{d_t}{d_0} & 0 & 0 \\ 0 & \frac{d_t}{d_0} & 0 \\ 0 & 0 & 1 \end{bmatrix} [K] \begin{bmatrix} \mathbf{r}_{1*} & -\mathbf{r}_{1*} \cdot C^{\mathrm{tr}} \\ \mathbf{r}_{2*} & -\mathbf{r}_{2*} \cdot C^{\mathrm{tr}} \\ \mathbf{r}_{3*} & d_t \end{bmatrix} = \begin{bmatrix} \frac{d_t}{d_0} & 0 & 0 \\ 0 & \frac{d_t}{d_0} & 0 \\ 0 & 0 & \frac{d_t}{d_0} \end{bmatrix} [K] \begin{bmatrix} \mathbf{r}_{1*} & -\mathbf{r}_{1*} \cdot C^{\mathrm{tr}} \\ \mathbf{r}_{2*} & -\mathbf{r}_{2*} \cdot C^{\mathrm{tr}} \\ \frac{d_0}{d_t}\mathbf{r}_{3*} & d_t \frac{d_0}{d_t} \end{bmatrix}. \tag{2.10}$$

The left most matrix is a member of the identity equivalence class, so,

$$[\widetilde{P}_t] = [K] \begin{bmatrix} \mathbf{r}_{1*} & -\mathbf{r}_{1*} \cdot C^{\mathrm{tr}} \\ \mathbf{r}_{2*} & -\mathbf{r}_{2*} \cdot C^{\mathrm{tr}} \\ \frac{d_0}{d_t}\mathbf{r}_{3*} & d_0 \end{bmatrix}. \tag{2.11}$$

As $t \to \infty$, the limiting result is the projection,

$$[P_\infty] = [K] \begin{bmatrix} \mathbf{r}_{1*} & -\mathbf{r}_{1*} \cdot C^{\mathrm{tr}} \\ \mathbf{r}_{2*} & -\mathbf{r}_{2*} \cdot C^{\mathrm{tr}} \\ 0 & d_0 \end{bmatrix}, \tag{2.12}$$

and this brings us to our definition of the *affine* projection.

Definition 2.2.1. *An* affine *projection (also referred to as a* generalized weak perspective projection*) is a* rank 3 *linear map from* $\mathbb{P}^3$ *to* $\mathbb{P}^2$ *that has a representation given by*

$$[P_a] = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.13}$$

In this thesis, it will be important to refer to the most basic *affine* projection, so we make the following definition.

Definition 2.2.2. *The* standard affine projection *is the orthogonal projection on the* $z_1 z_2$*-plane. It is represented by the matrix*

$$P_a^0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.14}$$

The sets of finite and affine projections are disjoint. Projections that are not included in these two classes are infinite non-affine projections. These are not frequently used in computer vision and are not considered in this thesis.

2.3 Group actions on cameras

2.3.1 Action on finite cameras

A straightforward exercise in matrix multiplication shows that the map $\Phi : (\mathcal{PGL}(3) \times \mathcal{A}(3)) \times \mathcal{FP} \to \mathcal{FP}$ defined by

$$\Phi\big(([A],[B]),\ [P]\big) = [A]\,[P]\,[B^{-1}] \tag{2.15}$$

for $[P] \in \mathcal{FP}$ and $([A],[B]) \in \mathcal{PGL}(3) \times \mathcal{A}(3)$, satisfies Definition 1.2.1 of a group-action.

Proposition 2.3.1. *The action of* $\mathcal{PGL}(3) \times \mathcal{A}(3)$ *on* $\mathcal{FP}$*, defined by (2.15) is transitive.*

Proof. According to Definition 1.2.3 we need to prove that for all $[P_1], [P_2] \in \mathcal{FP}$ there exists $([A],[B]) \in \mathcal{PGL}(3) \times \mathcal{A}(3)$ such that $[A][P_1][B^{-1}] = [P_2]$. It is sufficient to prove that for all $[P] \in \mathcal{FP}$ there exists $([A],[B]) \in \mathcal{PGL}(3) \times \mathcal{A}(3)$ such that $[P] = [A]\,[P_f^0]\,[B]$, where $[P_f^0]$ is the standard finite projection (2.8). A finite projection is given by a 3×4 matrix $P = (p_{ij})_{j=1...4}^{i=1...3}$ whose left 3×3 submatrix is non-singular. Therefore there exist

$c_1, c_2, c_3 \in \mathbb{R}$ such that $p_{*4} = c_1\, p_{*2} + c_2\, p_{*2} + c_3\, p_{*3}$, where p_{*j} denotes the j-th column of the matrix P. We define $A := (p_{ij})^{i=1...3}_{j=1...3}$ to be the left 3×3 submatrix of P and

$$B := \begin{pmatrix} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.16}$$

We observe that $([A], [B]) \in \mathcal{PGL}(3) \times \mathcal{A}(3)$ and $[A][P^0_f][B] = [P]$. $\square$

Corollary 2.3.2. *The set $\mathcal{FP}$ of finite projections is diffeomorphic to the homogeneous space $(\mathcal{PGL}(3) \times \mathcal{A}(3))/H^0_f$, where H^0_f is the 9-dimensional stabilizer of $[P^0_f]$.*

A straightforward computation shows that

$$H^0_f = \left\{ \left([A], \begin{bmatrix} A & \mathbf{0}^{\mathrm{tr}} \\ \mathbf{0} & 1 \end{bmatrix} \right) \right\}, \text{ where } A \in \mathcal{G}L(3). \tag{2.17}$$

Remark 2.3.3. *It follows from the proof of Proposition 2.3.1 that any finite projection is a composition of a translation in $\mathbb{R}^3$ (corresponding to translation of the camera center to the origin), the standard projection (2.8) (pinhole camera), and a projective transformation on the image plane.*

2.3.2 Action on affine cameras

Formula (2.15) with $[P] \in \mathcal{AP}$ and $([A], [B]) \in \mathcal{A}(2) \times \mathcal{A}(3)$ also defines an action of the direct product $\mathcal{A}(2) \times \mathcal{A}(3)$ on the set of affine projections $\mathcal{AP}$.

Proposition 2.3.4. *The action of $\mathcal{A}(2) \times \mathcal{A}(3)$ on $\mathcal{AP}$, defined by (2.15), is transitive.*

Proof. It is sufficient to prove that for all $[P] \in \mathcal{AP}$ there exists $([A],[B]) \in \mathcal{A}(2) \times \mathcal{A}(3)$ such that $[P] = [A]\,[P_a^0]\,[B]$, where P_a^0 is the standard projection (2.14). An affine projection P is given by the matrix

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.18}$$

of rank 3. Therefore there exist $1 \le i < j \le 3$ such that the rank of the submatrix $\begin{pmatrix} p_{1i} & p_{1j} \\ p_{2i} & p_{2j} \end{pmatrix}$ is 2. Then for $1 \le k \le 3$, such that $k \neq i$ and $k \neq j$, there exist c_1, $c_2 \in \mathbb{R}$, such that $\begin{pmatrix} p_{1k} \\ p_{2k} \end{pmatrix} = c_1 \begin{pmatrix} p_{1i} \\ p_{2i} \end{pmatrix} + c_2 \begin{pmatrix} p_{1j} \\ p_{2j} \end{pmatrix}$. We define

$$A := \begin{pmatrix} p_{1i} & p_{1j} & p_{14} \\ p_{2i} & p_{2j} & p_{24} \\ 0 & 0 & 1 \end{pmatrix} \tag{2.19}$$

and define B to be the matrix whose columns are vectors $b_{*i} := (1,0,0,0)^{\mathrm{tr}}$, $b_{*j} := (0,1,0,0)^{\mathrm{tr}}$, $b_{*k} := (c_1, c_2, 1, 0)^{\mathrm{tr}}$, $b_{*4} = (0,0,0,1)^{\mathrm{tr}}$. We observe that $([A],[B]) \in \mathcal{A}(2) \times \mathcal{A}(3)$ and that $[A][P_a^0][B] = [P]$. $\square$

Remark 2.3.5. *Note that there are only three possible values of (i,j,k) in the above proof:*

if $(i,\, j,\, k) = (1, 2, 3)$, *then*

$$B = \begin{pmatrix} 1 & 0 & c_1 & 0 \\ 0 & 1 & c_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \tag{2.20}$$

if $(i,\, j,\, k) = (1, 3, 2)$, *then the corresponding matrix* B *is obtained by interchanging the second and the third column in (2.20);*

if $(i,\, j,\, k) = (2, 3, 1)$, *then the corresponding matrix* B *is obtained by a cyclic shift (by one to the right) of the first three columns in (2.20).*

Corollary 2.3.6. *The set* $\mathcal{AP}$ *of affine projections is diffeomorphic to the homogeneous space* $(\mathcal{A}(2) \times \mathcal{A}(3))/H_a^0$, *where* H_a^0 *is the 10-dimensional stabilizer of* $[P_a^0]$.

A straightforward computation shows that

$$H_a^0 = \left\{ \left(\begin{bmatrix} m_{11} & m_{12} & a_1 \\ m_{21} & m_{22} & a_2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} m_{11} & m_{12} & 0 & a_1 \\ m_{21} & m_{22} & 0 & a_2 \\ m_{31} & m_{32} & m_{33} & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \right\}, \tag{2.21}$$

where $m_{33}\,(m_{11}m_{22} - m_{12}m_{21}) \neq 0$.

Chapter 3

Equivalence Problems on the Plane

In Chapter 4 we will show that the problem of object-image correspondence under affine and finite cameras can be reduced to a problem of equivalence of planar objects under affine and projective group actions. In this chapter we review the existing methods for solving such problems for curves and finite sets of ordered points.

3.1 Group-equivalence problem for curves

A variety of methods exist to solve the group-equivalence problem for curves. We base our algorithm on the differential signature construction described in [5] which originates from Cartan's moving frame method [6]. We consider the possibility of using other methods in Section 6. Using the notation discussed in Section 1.1 we make the following definition.

Definition 3.1.1. *Given an action of a group G on $\mathbb{R}^n$, we say that two curves $\alpha\colon I_\alpha \to \mathbb{R}^n$ and $\beta\colon I_\beta \to \mathbb{R}^n$ are* G-equivalent *if there exists an element $g \in G$ such that $C_\alpha = g\, C_\beta$.*

*We say that two curves have a G-*overlap *if there exists an element $g \in G$ such that curves C_α and $g\,C_\beta$ overlap over a curve segment.*

3.1.1 Differential invariants for planar curves

An action of a group G on $\mathbb{R}^2$ induces an action on parametric curves $\gamma(t) = (x(t), y(t))$. Using the chain rule this action can be *prolonged* to the k-th order jet space of curves denoted by $\mathcal{J}^k$. Variables $x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}, \dots$, which represent the derivatives of x, y with respect to the parameter of orders from 0 to k, serve as coordinate functions on $\mathcal{J}^k$.

Definition 3.1.2. Restriction of a function*, $\mathcal{F}$, on $\mathcal{J}^k$, to a curve, $\gamma(t) = (x(t), y(t))$, $t \in I_\gamma$, is a single-variable function $\mathcal{F}|_\gamma(t) := \mathcal{F}\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{d^2x(t)}{dt^2}, \dots\right)$. A function $\mathcal{F}$ is* invariant under reparameterizations *if $\mathcal{F}|_\gamma(\phi(\tau)) = \mathcal{F}|_{\tilde{\gamma}}(\tau)$, where $\tilde{\gamma}(\tau)$, $\tau \in I_{\tilde{\gamma}}$ is a reparameterization of $\gamma(t)$, i.e. $\gamma(\phi(\tau)) = \tilde{\gamma}(\tau)$ for a smooth map $\phi\colon I_{\tilde{\gamma}} \to I_\gamma$.*

For example, $\dot{x}|_{\tilde{\gamma}}(\tau) = \dot{x}|_\gamma(\phi(\tau))\phi'(\tau)$, and hence $\dot{x}$ is not invariant under reparameterizations, but $\frac{\dot{x}}{\dot{y}}$ is invariant under reparameterizations.

Definition 3.1.3. *A* k-th order differential invariant *is a function on $\mathcal{J}^k$ that depends on k-th order jet variables and is invariant under the prolonged action of G and reparameterizations of curves.*

For example, for the action of the 3-dimensional Euclidean group, consisting of rotations, translations and reflections on the plane, the curvature $\kappa = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$ is (up to a sign) a lowest order differential invariant. The sign of κ changes when a curve is reflected, rotated by π radians or traced in the opposite direction (κ^2 is invariant under the Euclidean group). Higher order differential invariants are obtained by differentiation of curvature with respect to Euclidean arclength $ds = \sqrt{\dot{x}^2 + \dot{y}^2}\,dt$, i. e. $\kappa_s = \frac{d\kappa}{ds} = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}}\frac{d\kappa}{dt}$.

Any other Euclidean differential invariant can be locally expressed as a function of $\kappa, \kappa_s, \kappa_{ss}, \ldots$.

For the majority of Lie group actions on $\mathbb{R}^2$, a lowest order differential invariant appears at order $r-1$ where $r = \dim G$. The group actions on the plane with this property are called *ordinary*. All actions considered in this thesis are ordinary. A lowest order differential invariant for an ordinary action of a group G is called G-curvature, and a lowest order invariant differential form is called infinitesimal G-arclength. Any differential invariant with respect to the G-action can be locally expressed as a functions of G-curvature and its derivatives with respect to G-arclength. Affine and projective curvatures and infinitesimal arclengths are well known, and can be expressed in terms of Euclidean invariants [9, 16]. In particular, $\mathcal{SA}$-curvature μ and infinitesimal $\mathcal{SA}$-arclength $d\alpha$ are expressed in terms of their Euclidean counterparts as follows:

$$\mu = \frac{3\,\kappa\,(\kappa_{ss} + 3\,\kappa^3) - 5\,\kappa_s^2}{9\,\kappa^{8/3}}, \quad d\alpha = \kappa^{1/3} ds. \tag{3.1}$$

$\mathcal{SA}(2)$-curvature has the differential order 4. Any $\mathcal{SA}$-differential invariant can be locally expressed as a function of μ and its derivatives with respect to the $\mathcal{SA}$-arclength: $\mu_\alpha = \frac{d\mu}{d\alpha}$, $\mu_{\alpha\alpha} = \frac{d\mu_\alpha}{d\alpha}$, ... $\mathcal{SA}$-curvature is undefined for straight lines ($\kappa = 0$) and $\frac{d}{d\alpha}$ is undefined at the inflection points of a curve. It is shown, for instance, in [12] that $\mu|_\gamma$ is constant if and only if C_γ is a conic. Moreover, $\mu|_\gamma \equiv 0$ if and only if C_γ is a parabola, $\mu|_\gamma$ is a positive constant if and only if C_γ is an ellipse, and $\mu|_\gamma$ is a negative constant if and only if C_γ is a hyperbola.

By considering the effects of scalings and reflections on $\mathcal{SA}(2)$-invariants we obtain two lowest order $\mathcal{A}(2)$-invariants that are *rational functions* in jet variables:

$$J_a = \frac{(\mu_\alpha)^2}{\mu^3}, \quad K_a = \frac{\mu_{\alpha\alpha}}{3\,\mu^2} + 5. \tag{3.2}$$

Remark 3.1.4. *Adding 5 to $\frac{\mu_{\alpha\alpha}}{3\,\mu^2}$ in the definition of K_a is not necessary, but leads to a more simple expression of $K_a|_\gamma$ for most curves. This leads to a faster computation of the signature curve.*

$\mathcal{PGL}(3)$-curvature η and infinitesimal arclength $d\rho$ are expressed in terms of their $\mathcal{SA}$-counterparts:

$$\eta = \frac{6\mu_{\alpha\alpha\alpha}\mu_\alpha - 7\,\mu_{\alpha\alpha}^2 - 9\mu_\alpha^2\,\mu}{6\mu_\alpha^{8/3}}, \quad d\rho = \mu_\alpha^{1/3} d\alpha. \tag{3.3}$$

The two lowest order *rational* $\mathcal{PGL}(3)$-invariants

$$J_p = \eta^3 \quad \textit{and} \quad K_p = \eta_\rho \tag{3.4}$$

are of differential order 7 and 8, respectively. Explicitly,

$$\eta_\rho = \frac{36\mu_{\alpha\alpha\alpha\alpha}\mu_\alpha^2 - 144\,\mu_{\alpha\alpha\alpha}\,\mu_{\alpha\alpha}\,\mu_\alpha + 36\mu_{\alpha\alpha}\,\mu_\alpha^2\,\mu - 54\mu_\alpha^4 + 112\mu_{\alpha\alpha}^3}{36\mu_\alpha^4}. \tag{3.5}$$

Definition 3.1.5. *A curve γ is called* $\mathcal{A}(2)$-exceptional *if invariants (3.2) are undefined. Equivalently C_γ is (a part of) a straight-line or (a part of) a parabola. In the former case, its Euclidean curvature $\kappa|_\gamma \equiv 0$, while in the latter case only its $\mathcal{SA}$-curvature $\mu|_\gamma \equiv 0$.*

A curve γ is called $\mathcal{PGL}(3)$-exceptional *if invariants (3.4) are undefined. Equivalently, C_γ is (a part of) a conic. In this case $\mu|_\gamma$ is a constant.*

3.1.2 Differential signature for planar curves

Following [5] we will use differential signatures to solve the equivalence problem for curves under a group action.

Definition 3.1.6. *Let J_G and K_G be differential invariants of orders $r-1$ and r, respectively, for an ordinary action of an r-dimensional Lie group G on the plane. A G-*signature *of a non-exceptional parametric curve $\gamma(t) = (x(t), y(t))$, $t \in I_\gamma$, is a parametric curve $\mathcal{S}_\gamma(t) = \big(J_G|_\gamma(t), K_G|_\gamma(t)\big)$.*

Definition 3.1.7. *A parametric curve $\gamma(t) = (x(t), y(t))$, $t \in I_\gamma$, is G-*regular *if for all $t \in I_\gamma$ the signature curve $\mathcal{S}_\gamma(t)$ is defined and has a non-zero tangent.*

Among G-nonregular curves a special role is played by curves with constant signatures: $J_G|_\gamma(t) \equiv j, K_G|_\gamma(t) \equiv k$ for some $j, k \in \mathbb{R}$ and all $t \in I_\gamma$. These curves are symmetric with respect to a one-dimensional subgroup of G. For example, circles and lines have constant Euclidean signatures. A circle is symmetric under rotations about its center and a line is symmetric under translations along itself.

It follows from the definition of invariants that the image $\mathcal{S}_\gamma := \{\mathcal{S}_\gamma(t) | t \in I_\gamma\}$ is invariant under reparametrizations of the curve γ and that the following theorem holds:

Theorem 3.1.8. *If planar curves $\alpha\colon I_\alpha \to \mathbb{R}^2$ and $\beta\colon I_\beta \to \mathbb{R}^2$ are G-equivalent then the images of their signatures coincide: $\mathcal{S}_\alpha = \mathcal{S}_\beta$.*

The full converse of the theorem is not valid. For example, curves $y = \cos(x), x \in [0, 2\pi]$, and $y = \sin(x), x \in [0, 2\pi]$, have the same Euclidean signatures. These curves are not Euclidean-equivalent, but have a Euclidean overlap (see Definition 3.1.1). A variety of counter-examples for the converse of Theorem 3.1.8 that arise due to insufficient degree of smoothness are presented in [21]. The converse of Theorem 3.1.8 for analytic G-regular curves $y = y(x)$, $x \in \mathbb{R}$, that have analytic signatures is stated in Theorem 8.53 of [23]. We state the following partial converse of Theorem 3.1.9,

Theorem 3.1.9. *Let $\alpha\colon I_\alpha \to \mathbb{R}^2$ and $\beta\colon I_\beta \to \mathbb{R}^2$ be sufficiently smooth regular curves. If regular parts of the G-signatures $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ overlap or if α and β have the same constant G-signatures, then the curves α and β G-overlap.*

Remark 3.1.10. *Signature construction reduces the problem of G-equivalence of curves to the problem of deciding whether two parametric curves (that represent the signatures of the given curves) have the same image. If a curve $\gamma(t)$ has a rational parameterization then the implicit equation $\hat{\mathcal{S}}_\gamma(K, J) = 0$ for its signature can be computed by an elimination algorithm as outlined, for instance, in Section 3.3 of [7]. When comparing signatures using their implicit equations, one has to be aware that for $t \in \mathbb{R}$, two non overlapping regular signature curves can have the same implicit equation as shown by Example 8.69 in [23].*

3.1.3 Examples

Example 3.1.11. *This example illustrates that plane curves of the form*

$$\gamma_a(t) = \left(\frac{a\,t}{1 + a^2\,t^3}\,,\ \frac{a\,t^2}{1 + a^2\,t^3}\right),\ t \in \mathbb{R},\ \ for\ \forall a \neq 0 \tag{3.6}$$

are $\mathcal{PGL}(3)$*-equivalent. First, evaluate the invariants given in equation (3.4) on* $\gamma_a(t)$*:*

$$J_p|_{\gamma_a}(t) = \left(-\frac{1}{1350} \frac{(23\, a^8\, t^{12} + 63\, a^6\, t^9 + 80\, a^4\, t^6 + 63\, a^2\, t^3 + 23)^3\ t^3\, a^2}{(1 + a^2\, t^3)^6\, (a^2\, t^3 - 1)^8} \right)^{\frac{1}{3}}, \tag{3.7}$$

$$K_p|_{\gamma_a}(t) = \left(\frac{(23\, a^6\, t^9 + 229\, a^4\, t^6 + 229\, a^2\, t^3 + 23)^3\ (1 + a^2\, t^3)^2}{2700\, (a^2\, t^3 - 1)^{11}} \right)^{\frac{1}{3}}. \tag{3.8}$$

If we reparametrize the curve with $s = t\sqrt[3]{a^2}$*, our invariants are independent of* a*. Therefore for all* $a \neq 0$*, there exists a parametriztion of the curve with the following invariants,*

$$J_p|_{\gamma}(t) = \left(-\frac{1}{1350} \frac{t^3\, (23\, t^{12} + 63\, t^9 + 80\, t^6 + 63\, t^3 + 23)^3}{(t^3 + 1)^6\ (-1 + t^3)^8} \right)^{\frac{1}{3}}, \tag{3.9}$$

$$K_p|_{\gamma}(t) = \left(\frac{(23\, t^9 + 229\, t^6 + 229\, t^3 + 23)^3\, (t^3 + 1)^2}{2700\, (-1 + t^3)^{11}} \right)^{\frac{1}{3}}. \tag{3.10}$$

So, for all $a \neq 0$ *the signature of* $\gamma_a(t)$ *must be the same. Therefore, all curves of the form given by (3.6) are* $\mathcal{PGL}(3)$*-equivalent.*

In the next example we use $\mathcal{A}(2)$-signatures to determine $\mathcal{A}(2)$-equivalence classes of the family of curves, $(x(t), y(t))$, parameterized by third degree polynomials in t.

Example 3.1.12. *Given a curve* $\gamma(t) = (x(t), y(t))$*, assume* $x(t)$ *and* $y(t)$ *are 3rd degree polynomials in* t*. So,*

$$x(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0, \tag{3.11}$$

$$y(t) = b_3 t^3 + b_2 t^2 + b_1 t + b_0. \tag{3.12}$$

Restricting our invariants,

$$\begin{aligned} i_1 = J_a|_\gamma &= \frac{N_1(t)}{D(t)^3}, \\ i_2 = K_a|_\gamma &= \frac{N_2(t)}{D(t)^2}, \end{aligned}$$

we find rational expressions in t *with coefficients depending on* $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3$. *It turns out that the numerator and denominator of* $J_a|_\gamma$ *are* 6*-th degree polynomials in* t *and the numerator and denominator of* $K_a|_\gamma$ *are* 4*-th degree polynomials in* t. *For expressions of* $N_1(t)$, $N_2(t)$ *and* $D(t)$ *see equations (3) of Appendix B.*

When calculating the signature we arrive at the following implicit equation, shown on Figure 3.1.

$$147000\, i_2 - 857500 + 9261\, i_1^2 - 26460\, i_1\, i_2 + 160\, i_2^3 + 10500\, i_2^2 = 0 \tag{3.13}$$

When defined, the invariants for every curve from the family above lie on this curve. It does not mean that every curve can be mapped to each other because the signature could be a point on this curve or the curve itself. We determine the possible equivalance classes of γ *by examining* $N_1(t)$, $N_2(t)$ *and* $D(t)$.

Case 0: $a_3b_2 = b_3a_2$ *and* $a_3b_1 = b_3a_1$. *In this case* $D(t) \equiv 0$ *and* $\gamma(t)$ *is a parabola or a straight line. All straight lines are in the same equivalence class, and all parabolas belong to either the equivalence class* $y = x^2$ *or the equivalence class* $y = -x^2$.

Case 1: *Another possibility is* $N_1(t)$, $N_2(t)$, *and* $D(t) \neq 0$ *are all constants because their coefficients of* t *are all* 0. *The only way this can happen is if* $a_3b_2 = b_3a_2$ *and*

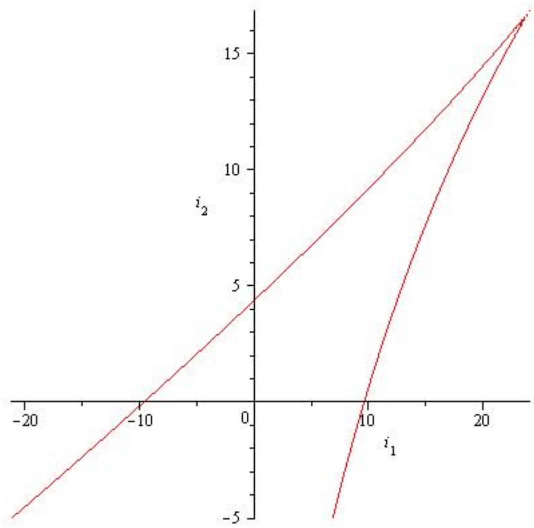

Figure 3.1: Signature given by the implicit equation (3.13)

$a_3b_1 \neq b_3a_1$. *In this case, the invariants are*

$$\begin{aligned} i_1 &= \frac{-64}{5}, \\ i_2 &= \frac{-7}{5}. \end{aligned}$$

Case 2: *The only other way the invariants can be constant is if the numerators and denominators of* i_1 *and* i_2 *are constant multiples of each other. By examining the coefficients, we observe that, when* $D(t)$ *is not constant, it is a multiple of* $N_1(t)$ *if and only if* $3\,(b_3\,a_1 - a_3\,b_1)^2 - 4\,(b_3\,a_2 - a_3\,b_2)\,(b_2\,a_1 - b_1\,a_2) = 0$ *and* $a_3b_2 \neq b_3a_2$. *With these conditions* $D(t)^2$ *divides* $N_2(t)$ *and the invariants are given by*

$$\begin{aligned} i_1 &= 25, \\ i_2 &= \frac{35}{2}. \end{aligned}$$

Case 3: *The last possibility is that the invariants are not constant (depend on t), and satisfy the implicit signature equation 3.13.*

The constant invariants in cases 1 and 2 lie on the signature curve given by the more general case 3, shown on Figure 3.1.

3.2 Group-equivalence problem for ordered sets of points

The group-equivalence problem for ordered point sets is discussed in [22]. For the purpose of this dissertation, we review the relevant results from this paper and other classical results for sets of points on the plane.

Let $\mathbb{R}^2_{\times r} = \mathbb{R}^2 \times \mathbb{R}^2 \cdots \times \mathbb{R}^2$ denote r-fold Cartesian product of the plane. Given a Lie Group G acting on $\mathbb{R}^2$, we consider the joint action of G on $\mathbb{R}^2_{\times r}$ given by

$$g \cdot (\mathbf{x}^1, \cdots, \mathbf{x}^r) = (g \cdot \mathbf{x}^1, \cdots, g \cdot \mathbf{x}^r), \quad g \in G,\ \mathbf{x}^1, \cdots, \mathbf{x}^r \in \mathbb{R}^2. \tag{3.14}$$

Definition 3.2.1. *An r-point* rational joint invariant *is a rational function on $\mathbb{R}^2_{\times r}$ that is invariant under the action defined by 3.14. I.e. $I(\mathbf{x}^1, \cdots, \mathbf{x}^r) = I(g \cdot \mathbf{x}^1, \cdots, g \cdot \mathbf{x}^r)$ for all $g \in G$.*

We use the notation of separating invariants from [25].

Definition 3.2.2. *For a group G acting on $\mathbb{R}^2_{\times r}$, a set S of invariants is called* separating *if there a Zariski open subset $U \subset \mathbb{R}^2_{\times r}$ such that for every $\mathbf{x}$, $\mathbf{y} \in U$ not on the same orbit, there $\exists I \in S$ such that $I(x) \neq I(y)$.*

To define these invariants we use areas of triangles. Given an *ordered* set $X = (\mathbf{x}^1, \ldots, \mathbf{x}^r)$ of r points in $\mathbb{R}^2$ with coordinates $\mathbf{x}^i = (x^i, y^i)$, define

$$A(i,j,k) = \frac{1}{2}\det\begin{vmatrix} x^i & x^j & x^k \\ y^i & y^j & y^k \\ 1 & 1 & 1 \end{vmatrix} \tag{3.15}$$

to be the area of the triangle with vertices $\mathbf{x}^i, \mathbf{x}^j, \mathbf{x}^k$.

Remark 3.2.3. *$A(i,j,k) = 0$ if and only if $\mathbf{x}^i, \mathbf{x}^j, \mathbf{x}^k$ are collinear.*

For the $\mathcal{A}(2)$-action on $\mathbb{R}^2_{\times r}$, the well known generating set of rational joint invariants is given by the following result in [31].

Theorem 3.2.4. *Every joint invariant of $\mathcal{A}(2)$ acting on $\mathbb{R}^2$ is a function of area ratios $\frac{A(i,j,k)}{A(l,m,n)}$.*

Theorem 3.2.5. *On the open subset where the first three points are non-collinear, invariants*

$$B(2,k) = \frac{A(1,2,k)}{A(1,2,3)} \qquad for\ k = 4, \ldots, r \tag{3.16}$$

and

$$B(3,k) = \frac{A(1,3,k)}{A(1,2,3)} \qquad for\ k = 4, \ldots, r \tag{3.17}$$

always separate orbits with respect to the $\mathcal{A}(2)$-action on $\mathbb{R}^2_{\times r}$.

Proof. Let two ordered sets, $X = (\mathbf{x}^1, \ldots, \mathbf{x}^r)$, with $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ non-collinear, and $U = (\mathbf{u}^1, \ldots, \mathbf{u}^r)$, with $\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3$ non-collinear, of r points in $\mathbb{R}^2$ be given. Let the coordinates for X be given by $\mathbf{x}^i = (x^i, y^i)$ and the coordinates for U be given by $\mathbf{u}^i = (u^i, v^i)$. Since

$\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ are non-collinear and $\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3$ are non-collinear, we can find affine transformations A_1 and A_2 that map $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ and $\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3$, respectively, to $(0,0)$, $(0,1)$, $(1,0)$. The only affine transformation that maps $(0,0)$, $(0,1)$, $(1,0)$ to itself is the identity transformation. Invariants (3.16) and (3.17) have the same values for X and $A_1 \cdot X$ and the same values for U and $A_2 \cdot U$. For $k \geq 4$, denote

$$\begin{aligned} A_1 \cdot \mathbf{x}^k &= (\widetilde{x}^k, \widetilde{y}^k), \\ A_2 \cdot \mathbf{u}^k &= (\widetilde{u}^k, \widetilde{v}^k). \end{aligned} \tag{3.18}$$

A straightforward computation shows

$$\begin{aligned} B|_X(2,k) &= \widetilde{x}^k, \\ B|_X(3,k) &= -\widetilde{y}^k, \\ B|_U(2,k) &= \widetilde{u}^k, \\ B|_U(3,k) &= -\widetilde{v}^k. \end{aligned} \tag{3.19}$$

Therefore, $B|_X(2,k) = B|_U(2,k)$ and $B|_X(3,k) = B|_U(3,k)$ if and only if $\widetilde{x}^k = \widetilde{u}^k$ and $\widetilde{y}^k = \widetilde{v}^k$ for all k such that $4 \leq k \leq r$. This can only happen if $X = (A_1^{-1} A_2) \cdot U$. Therefore, for ordered sets of r points, the set of invariants given by (3.16) and (3.17) separate orbits on the open subset where the first 3 points are non-collinear.

□

In [22], using the moving frame method, it is stated that the set given by (3.16) and (3.17) generates the field of $\mathcal{A}(2)$ joint r-point invariants. In [25], the following result is given. Also, see [26] and [28] for more details.

Theorem 3.2.6. *Any separating set of rational invariants generates the field of rational invariants.*

As an example, we provide an algebraic proof for the following special case of this result.

Example 3.2.7. *It has been proved in [31] that area ratios, $\frac{A(i,j,k)}{A(l,m,n)}$, generate affine invariants. We will show how to express such ratios in terms of separating invariants (3.16) and (3.17).*

We need to show $\frac{A(i,j,k)}{A(l,m,n)}$ can be expressed in terms of $B(2,k)$'s and $B(3,k)$'s,

$$\frac{A(i,j,k)}{A(l,m,n)} = \frac{A(i,j,k)}{A(1,2,3)} \frac{A(1,2,3)}{A(l,m,n)}.$$

Therefore, it suffices to express $\frac{A(i,j,k)}{A(1,2,3)}$ in terms of $B(2,\cdot)$'s and $B(3,\cdot)$'s. Without loss of generality assume $i < j < k$. A straightforward computation shows

$$\begin{aligned} A(i,j,k)\,A(1,2,3) = {} & A(1,2,i)\,(A(1,3,j) - A(1,3,k)) \\ & -A(1,2,j)\,(A(1,3,i) - A(1,3,k)) \\ & -A(1,2,k)\,(A(1,3,j) - A(1,3,i)). \end{aligned} \tag{3.20}$$

Dividing both sides by $A(1,2,3)^2$ gives the desired relationship

$$\begin{aligned} \frac{A(i,j,k)}{A(1,2,3)} = {} & B(2,i)\,(B(3,j) - B(3,k)) \\ & -B(2,j)\,(B(3,i) - B(3,k)) \\ & -B(2,k)\,(B(3,j) - B(3,i)). \end{aligned} \tag{3.21}$$

Note that $B(2,1) = B(3,1) = B(2,2) = B(3,3) = 0$ *and* $B(2,3) = -B(3,2) = 1$*, so* $\frac{A(i,j,k)}{A(1,2,3)}$ *will only depend on* $B(2,k)$*'s and* $B(3,k)$*'s with* $k \geq 4$ *as required.*

The action of $\mathcal{PGL}(3)$ on $\mathbb{R}^2$ becomes free on the 4-fold Cartesian product, but the projective joint invariants depend on 5 points. The following theorems were derived in [22] using the method of moving frames.

Theorem 3.2.8. *Let* $\mathcal{PGL}(3)$ *act on* $\mathbb{R}^2_{\times n}$*. A generating set of projective joint invariants is given by the following cross ratios of areas*

$$C(i;j,k,l,n) = \frac{A(i,j,k)\,A(i,l,n)}{A(i,j,l)\,A(i,k,n)}. \tag{3.22}$$

By a simialar technique to that used in Theorem 3.2.5 one can prove:

Theorem 3.2.9. *On an open subset where no* 3 *of the first* 4 *points are collinear, the fundamental area cross ratios:*

$$C(1;2,3,4,k), \qquad k = 5,6,\ldots \tag{3.23}$$

and

$$C(2;1,3,4,k), \qquad k = 5,6,\ldots \tag{3.24}$$

separate orbits with respect to the $\mathcal{PGL}(3)$*-action on* $\mathbb{R}^2_{\times r}$*.*

3.2.1 Examples

In this subsection, we illustrate how Theorems 3.2.5 and 3.2.9 provide a solution to the equivalence problems for ordered sets of points.

Example 3.2.10. *Given three* ordered *sets of points in* $\mathbb{R}^2$,

$$\begin{aligned} X &= (\mathbf{x}^1, \ldots, \mathbf{x}^4) \quad \textit{where} \quad \mathbf{x}^1 = (0,0),\, \mathbf{x}^2 = (0,1),\, \mathbf{x}^3 = (1,1),\ \textit{and}\ \mathbf{x}^4 = (1,0), \\ U &= (\mathbf{u}^1, \ldots, \mathbf{u}^4) \quad \textit{where} \quad \mathbf{u}^1 = (0,1),\, \mathbf{u}^2 = (1,2),\, \mathbf{u}^3 = (3,2),\ \textit{and}\ \mathbf{u}^4 = (2,1), \\ V &= (\mathbf{v}^1, \ldots, \mathbf{v}^4) \quad \textit{where} \quad \mathbf{v}^1 = (0,0),\, \mathbf{v}^2 = (0,1),\, \mathbf{v}^3 = (1,1),\ \textit{and}\ \mathbf{v}^4 = (2,2), \end{aligned}$$

decide which point sets are affinely-equivalent. Using Theorem 3.2.5, we first make the following calculations. For X*,*

$$\begin{aligned} A|_X(1,2,3) &= \det \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -1 \qquad \textit{and} \\ A|_X(1,2,4) &= -1. \end{aligned}$$

So,

$$B|_X(2,4) = \frac{A|_X(1,2,4)}{A|_X(1,2,3)} = 1,$$

and similarly,

$$B|_X(3,4) = 1.$$

For U*,*

$$\begin{aligned} B|_U(2,4) &= 1, \\ B|_U(3,4) &= 1. \end{aligned}$$

For V,

$$\begin{aligned} B|_V(2,4) &= 2, \\ B|_V(3,4) &= 0. \end{aligned}$$

By comparing the values of $B(2,4)$ and $B(3,4)$ for the three ordered sets of points, we determine that X and U are affinely-equivalent to each other, but are not affinely-equivalent to V.

Example 3.2.11. *Given three* ordered *sets of points in $\mathbb{R}^2$,*

$$\begin{aligned} X &= (\mathbf{x}^1, \ldots, \mathbf{x}^5) \\ & \quad \textit{where} \quad \mathbf{x}^1 = (-3,0),\, \mathbf{x}^2 = (2,2),\, \mathbf{x}^3 = (6,1),\, \mathbf{x}^4 = (4,1),\ \textit{and}\ \mathbf{x}^5 = (3,2), \\ U &= (\mathbf{u}^1, \ldots, \mathbf{u}^5) \\ & \quad \textit{where} \quad \mathbf{u}^1 = (-2,-2),\, \mathbf{u}^2 = (\frac{5}{3}, \frac{7}{3}),\, \mathbf{u}^3 = (4, \frac{9}{2}),\, \mathbf{u}^4 = (3, \frac{7}{2}),\ \textit{and}\ \mathbf{u}^5 = (2, \frac{8}{3}), \\ V &= (\mathbf{v}^1, \ldots, \mathbf{v}^5) \\ & \quad \textit{where} \quad \mathbf{v}^1 = (-2,-2),\, \mathbf{v}^2 = (\frac{5}{3}, \frac{7}{3}),\, \mathbf{v}^3 = (4, \frac{9}{2}),\, \mathbf{v}^4 = (2, \frac{7}{2}),\ \textit{and}\ \mathbf{v}^5 = (2, \frac{8}{3}), \end{aligned}$$

decide which point sets are projectively-equivalent. Using Theorem 3.2.9, we first make the following calculations. For X,

$$\begin{aligned} C_X(1;2,3,4,5) &= \frac{26}{27}, \\ C_X(2;1,3,4,5) &= \frac{13}{9}. \end{aligned}$$

For U,

$$\begin{aligned} C_U(1;2,3,4,5) &= \frac{26}{27}, \\ C_U(2;1,3,4,5) &= \frac{13}{9}. \end{aligned}$$

For V,

$$\begin{aligned} C_V(1;2,3,4,5) &= \frac{65}{51}, \\ C_V(2;1,3,4,5) &= \frac{65}{17}. \end{aligned}$$

By comparing the values of $C(1;2,3,4,5)$ and $C(2;1,3,4,5)$ for the three ordered sets of points, we determine that sets X and U are projectively-equivalent to each other, but are not projectively-equivalent to V.

Chapter 4

The Projection Problem for Curves

In this chapter we assume that objects and images are represented by smooth curves in $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. We formulate criteria for the existence of a finite or an affine projection. In Section 4.3 we provide algorithms that use these criteria, and in Section 4.4 we give examples.

4.1 Problem Formulation

First, we restate the problem given in Chapter 1.

Problem 4.1.1. *Given a smooth curve C_Γ in $\mathbb{R}^3$ and a smooth curve C_γ in $\mathbb{R}^2$, does there exist a finite or an affine camera that maps C_Γ to C_γ?*

Using the notation discussed in Section 1.1, we give the following definition.

Definition 4.1.2. *We say that a curve $\Gamma\colon I_\Gamma \to \mathbb{R}^3$* projects onto *$\gamma\colon I_\gamma \to \mathbb{R}^2$ if there exists a 3×4 matrix P of rank 3 such that $[C_\gamma] = [P][C_\Gamma]$.*

In the next section, we show that the projection problem for finite (or affine) cameras can be reduced to the problem of equivalence of planar curves under projective (or affine) actions on the plane. Before stating the projection criteria we make the following simple, but important observation.

Proposition 4.1.3. *(i) If Γ projects onto γ by an* affine projection *then any curve that is $\mathcal{A}(3)$-equivalent to Γ projects onto any curve that is $\mathcal{A}(2)$-equivalent to γ by an* affine projection. *In other words, affine projections are defined on affine equivalence classes of curves.*

(ii) If Γ projects onto γ by a finite projection *then any curve that is $\mathcal{A}(3)$-equivalent to Γ projects onto any curve that is $\mathcal{PGL}(3)$-equivalent to γ by a* finite projection.

Proof. (i) Assume that there exists an affine projection $[P] \in \mathcal{AP}$ such that $[C_\gamma] = [P][C_\Gamma]$. Then for all $(A, B) \in \mathcal{A}(2) \times \mathcal{A}(3)$ we have $[A]\,[C_\gamma] = [A]\,[P]\,[B^{-1}]\,([B]\,[C_\Gamma])$. Since $[A]\,[P]\,[B^{-1}] \in \mathcal{AP}$, the curve $[B][C_\Gamma]$ projects onto $[A][C_\gamma]$. (ii) is proved similarly using Proposition 2.15. □

It is *not true*, in general, that if Γ can be projected onto two planar curves γ_1 and γ_2 by an affine (or a finite) camera, then γ_1 is equivalent to γ_2 by an affine (or a projective) transformation. (See Example 4.4.2 and Example 4.4.5)

4.2 Projection criteria for curves

4.2.1 Projection criterion for finite cameras

Theorem 4.2.1. *A spatial curve $\Gamma(s) = \big(z_1(s), z_2(s), z_3(s)\big)$, $s \in I_\Gamma$, projects onto a planar curve $\gamma(t) = \big(x(t), y(t)\big)$, $t \in I_\gamma$, by* a finite projection *if and only if there exist*

$c_1, c_2, c_3 \in \mathbb{R}$ *such that planar curves* $\gamma(t)$ *and*

$$\epsilon_{c_1,c_2,c_3}(s) = \left(\frac{z_1(s) + c_1}{z_3(s) + c_3}, \frac{z_2(s) + c_2}{z_3(s) + c_3} \right) \tag{4.1}$$

are $\mathcal{PGL}(3)$*-equivalent.*

Proof. ($\Rightarrow$)Assume there exists a finite projection $[P]$ such that $[C_\gamma] = [P]\,[C_\Gamma]$. It was established in the proof of Proposition 2.3.1 that $[P] = [A]\,[P_f^0]\,[B]$ for some $[A] \in \mathcal{PGL}(3)$ and $[B] \in \mathcal{SA}(3)$, where B is given by (2.16) for some $c_1, c_2, c_3 \in \mathbb{R}$, and P_f^0 is the standard finite projection (2.8). Therefore $[C_\gamma] = [A][P_f^0]\,[B]\,[C_\Gamma]$. Since

$$[P_f^0]\,[B][z_1(s), z_2(s), z_3(s),\ 1]^{\text{tr}} = [z_1(s) + c_1, z_2(s) + c_2, z_3(s) + c_3]^{\text{tr}},$$

$[C_\gamma] = [A][C_{\epsilon_{c_1,c_2,c_3}}]$, where ϵ_{c_1,c_2,c_3} is defined by (4.1) ($z_3(s) + c_3$ is non-zero except for possibly a discrete set of values of s). Thus $C_\gamma = A\,C_{\epsilon_{c_1,c_2,c_3}}$ under the $\mathcal{PGL}(3)$-action (1.5).

($\Leftarrow$) To prove the converse direction we assume that there exists $[A] \in \mathcal{PGL}(3)$ and $c_1, c_2, c_3 \in \mathbb{R}$ such that $[C_\gamma] = [A][C_{\epsilon_{c_1,c_2,c_3}}]$, where ϵ_{c_1,c_2,c_3} is defined by (4.1). A direct computation shows that $[C_\Gamma]$ is projected onto $[C_\gamma]$ by the finite projection $[P] = [A]\,[P_f^0]\,[B]$, where B is given by (2.16) and $[P_f^0]$ is the standard finite projection (2.8). $\square$

4.2.2 Projection criteria for affine cameras

Theorem 4.2.2. *A curve* $\Gamma(s) = \big(z_1(s),\, z_2(s), z_3(s)\big)$ *for* $s \in I_\Gamma$ *projects onto* $\gamma(t) = \big(x(t), y(t)\big)$ *for* $t \in I_\gamma$ *if and only if there exist* $c_1, c_2 \in \mathbb{R}$ *and an ordered triple* $(i, j, k) \in$

$\{(1,2,3),\,(1,3,2),\,(2,3,1)\}$ *such that the* planar *curves* $\gamma(t)$ *and*

$$\delta^{ijk}_{c_1,c_2}(s) = \big(z_i(s) + c_1\, z_k(s),\; z_j(s) + c_2\, z_k(s)\big) \text{ for } s \in I_\Gamma \tag{4.2}$$

are $\mathcal{A}(2)$*-equivalent.*

Proof. ($\Rightarrow$)Assume Γ projects onto γ. Then there exists an affine projection $[P] \in \mathcal{AP}$ such that $[C_\gamma] = [P][C_\Gamma]$. Recall that the matrix P is of the form (2.18) and let (i,j,k) be a permutation of numbers $(1,2,3)$ such that $i < j$ and the submatrix of P formed by the i-th and j-th columns has rank 2. As it was established in the proof of Proposition 2.3.4 there exist $[A] \in \mathcal{A}(2)$ and $[B] \in \mathcal{A}(3)$, listed in Remark 2.3.5, such that $[P] = [A]\,[P^0_a]\,[B]$, where $[P^0_a]$ is the standard projection (2.14). Since $[P^0_a][B][C_\Gamma] = [C_{\delta^{ijk}_{c_1c_2}}]$, then $C_\gamma = AC_{\delta^{ijk}_{c_1,c_2}}$ under the $\mathcal{A}(2)$-action(1.6) and the direct statement is proved.

($\Leftarrow$) To prove the converse direction we assume that there exist $[A] \in \mathcal{A}(2)$, two real numbers c_1 and c_2, and a triplet of indices such that $(i,\ j,\ k) \in \{(1,2,3),\ (1,3,2),\ (2,3,1)\}$, such that $[C_\gamma] = [A][C_{\delta^{ijk}_{c_1c_2}}]$, where a planar curve $\delta^{ijk}_{c_1c_2}(s)$ is given by (4.2). Let B be a matrix listed in Remark 2.3.5, corresponding to the (i,j,k)-triple. A direct computation shows that $[C_\Gamma]$ is projected onto $[C_\gamma]$ by the affine projection $[P] = [A][P^0_a][B]$. □

The families of curves $\delta^{ijk}_{c_1c_2}(s)$ given by (4.2) with $(i,\ j,\ k) \in \{(1,2,3),\ (1,3,2),\ (2,3,1)\}$ and $c_1, c_2 \in \mathbb{R}$ have a large overlap. The following corollary eliminates this redundancy and, therefore, is useful for practical computations.

Corollary 4.2.3. *(*Reduced affine projection criteria) *A curve* $\Gamma(s) = \big(z_1(s),\, z_2(s), z_3(s)\big)$ *for* $s \in I_\Gamma$ *projects onto* $\gamma(t) = \big(x(t), y(t)\big)$ *for* $t \in I_\gamma$ *if and only if there exist* $b, c, f \in \mathbb{R}$ *such that the curve* γ *is* $\mathcal{A}(2)$*-equivalent to one of the* *following* planar *curves*

$$\alpha(s) = \big(z_2(s), z_3(s)\big), \tag{4.3}$$

$$\beta_b(s) = \big(\, z_1(s) + b\, z_2(s),\ z_3(s)\big), \tag{4.4}$$

$$\delta_{cf}(s) = \big(z_1(s) + c\, z_3(s),\ z_2(s) + f\, z_3(s)\big). \tag{4.5}$$

Proof. We first prove that for any permutation (i, j, k) of numbers $(1, 2, 3)$ such that $i < j$, and for any $c_1, c_2 \in \mathbb{R}$ a curve $\delta^{ijk}_{c_1,c_2} = \big(z_i(s) + c_1\, z_k(s),\ z_j(s) + c_2\, z_k(s)\big)$ is $\mathcal{A}(2)$-equivalent to one of the curves listed in (4.3)-(4.5).

Obviously, $\delta^{123}_{c_1 c_2} = \delta_{cf}$ with $c = c_1$ and $f = c_2$.

For $\delta^{132}_{c_1 c_2}$, if $c_2 \neq 0$ then $\begin{pmatrix} 1 & -\frac{c_1}{c_2} \\ 0 & \frac{1}{c_2} \end{pmatrix} \begin{pmatrix} z_1 + c_1 z_2 \\ z_3 + c_2 z_2 \end{pmatrix} = \begin{pmatrix} z_1 - \frac{c_1}{c_2} z_3 \\ z_2 + \frac{1}{c_2} z_3 \end{pmatrix}$ and so $\delta^{132}_{c_1 c_2}$ is $\mathcal{A}(2)$-equivalent to δ_{cf} with $c = -\frac{c_1}{c_2}$ and $f = \frac{1}{c_2}$. Otherwise, if $c_2 = 0$, the $\delta^{132}_{c_1 c_2}(s) = \beta_b(s)$ with $b = c_1$.

Similarly for $\delta^{231}_{c_1 c_2}$, if $c_2 \neq 0$ then $\delta^{231}_{c_1 c_2}$ is $\mathcal{A}(2)$-equivalent to δ_{cf} with $c = \frac{1}{c_2}$ and $f = -\frac{c_1}{c_2}$. Otherwise, if $c_2 = 0$, the $\delta^{231}_{c_1 c_2}(s) = (z_2(s) + c_1 z_1(s),\ z_3(s))$. If $c_1 \neq 0$ then $\delta^{231}_{c_1 c_2}$ is $\mathcal{A}(2)$-equivalent to β_b with $b = \frac{1}{c_1}$, otherwise $c_1 = 0$ and $\delta^{231}_{c_1 c_2} = \alpha$.

We can reverse the argument and show that any curve given by (4.3)-(4.5) is $\mathcal{A}(2)$-equivalent to a curve from family (4.2). Then the reduced criteria follows from Theorem 4.2.2. $\square$

4.3 Algorithms

In this section we present algorithms for solving the projection problem based on a combination of the projection criteria of Section 4.2 and the group equivalence criteria of

Section 3.1. We used MAPLE to implement these algorithms. In both algorithms, the main bottleneck lies in symbolically solving polynomial equations over real numbers. Numerical solvers are much faster, but are susceptible to round-off errors. Computing the implicit equation of the signature is also a time consuming computation. For this computation we used Gröbner basis algorithms, but alternative methods such as resultants may greatly improve efficiency.

4.3.1 Projection of curves by finite cameras

We formulate an algorithm that provides a necessary condition for a given spatial curve Γ to project onto a given planar curve γ by a finite camera with unknown position and parameters. For generic curves this also provides sufficient conditions that a segment of Γ can be projected onto a segment of γ.

Algorithm 4.3.1. (FINITE CAMERAS.)
INPUT: a planar curve $\gamma(t) = (x(t), y(t))$, $t \in \mathbb{R}$, *and a spatial curve* $\Gamma(t) = \big(z_1(s), z_2(s), z_3(s)\big)$, $s \in \mathbb{R}$, *with rational parameterizations.*
OUTPUT: YES or NO answer to the question "Is the necessary condition for existence of finite projection $[P]$ *such that* $[C_\gamma] = [P][C_\Gamma]$ *is satisfied?".*

1. *If* $\forall t$: $x''(t)y'(t) - y''(t)x'(t) = 0$ *then*

   - *compute the Euclidean curvature* $\mathcal{K}(s)$ *for the curve* Γ;
   - *if* $\forall s$: $\mathcal{K}(s) = 0$ *then OUTPUT:* YES *and exit the procedure,*

     else

     – *compute the Euclidean torsion* $\mathcal{T}(s)$ *for the curve* Γ;

– *if* $\forall s$*:* $\mathcal{T}(s) = 0$ *then OUTPUT:* YES, *else OUTPUT:* NO

- *exit the procedure;*

else proceed to the next step.

2. *Use (3.1) to evaluate the cube of* $\mathcal{SA}$*-curvature* $\mu^3|_\gamma(t)$*. The result is a rational function of* t*.*

3. *For arbitrary real* c_1, c_2, c_3 *define a curve* $\epsilon_{c_1,c_2,c_3}(s) = \left(\frac{z_1(s)+c_1}{z_3(s)+c_3}, \frac{z_2(s)+c_2}{z_3(s)+c_3}\right)$.

4. *If* $\exists\ m \in \mathbb{R}$ *s. t.* $\forall t \in \mathbb{R}$*:* $\mu|_\gamma(t) = m$ *then*

   - *Use (3.1) to evaluate* $\mu^3|_{\epsilon_{c_1,c_2,c_3}}(s)$*. The result is a rational function of* c_1, c_2, c_3 *and* s*.*

   - *If* $\exists\ a, c_1,\ c_2,\ c_3 \in \mathbb{R}$ *s. t.* $\forall s \in \mathbb{R}$

$$\mu^3|_{\epsilon_{c_1,c_2,c_3}}(s) = a \tag{4.6}$$

   then $OUTPUT$: YES, *else* $OUTPUT$: NO.

   - *Exit the procedure.*

else proceed to the next step.

5. *Evaluate* $\mathcal{PGL}(3)$*-invariants (3.4) on* $\gamma(t)$*. Obtain two rational functions of* t*,* $J_p|_\gamma(t)$ *and* $K_p|_\gamma(t)$*.*

6. *Evaluate* $\mathcal{PGL}(3)$*-invariants (3.4) on* $\epsilon_{c_1,c_2,c_3}(s)$*. Obtain two rational functions* $J_p|_\epsilon(c_1, c_2, c_3, s)$ *and*
   $K_p|_\epsilon(c_1, c_2, c_3, s)$ *of* c_1, c_2, c_3 *and* s*.*

7. *If* $\exists j, k \in \mathbb{R}$ *s. t.* $\forall t \in \mathbb{R}$*:* $J_p|_\gamma(t) = j$ *and* $K_p|_\gamma(t) = k$*, then*

   - *If* $\exists\ c_1,\ c_2,\ c_3 \in \mathbb{R}$ *s. t.* $\forall s \in \mathbb{R}$,

$$J|_\epsilon(c_1, c_2, c_3, s) = j \text{ and } K_p|_\epsilon(c_1, c_2, c_3, s) = k, \tag{4.7}$$

     OUTPUT: YES*, else OUTPUT:* NO.

   - *Exit the procedure.*

   else proceed to the next step.

8. *Compute the implicit equation* $\hat{\mathcal{S}}_\gamma(i_1, i_2) = 0$ *for the* $\mathcal{PGL}(3)$*-signature of* γ *by eliminating* t *from equations* $i_1 = J_p|_\gamma(t)$ *and* $i_2 = K_p|_\gamma(t)$*.*

9. *If* $\exists c_1, c_2, c_3 \in \mathbb{R}$ *s. t.:*

$$\forall s : \hat{\mathcal{S}}_\gamma\left(J_p|_\epsilon(c_1, c_2, c_3,\ s),\ K_p|_\epsilon(c_1, c_2, c_3,\ s)\right) = 0 \tag{4.8}$$

   and

$$J_p|_\epsilon(c_1, c_2, c_3,\ s) \text{ or } K_p|_\epsilon(c_1, c_2, c_3,\ s) \text{ are non-constant} \tag{4.9}$$

   OUTPUT: YES*, else OUTPUT:* NO.

In the first step, we consider the possibility that the given planar curve $\gamma(t)$ is a part of a line. A spatial curve $\Gamma(s)$ can be projected to a part of a straight line if and only if it is a planar curve. To determine whether $\Gamma(s)$ is a planar curve or not, we can use its Euclidean curvature and torsion. If the curvature is zero, then $\Gamma(s)$ belongs to a straight line and the torsion is undefined. If the torsion is defined and identically zero, then $\Gamma(s)$ is a planar curve, but not a part of a line. In both cases $\Gamma(s)$ can be projected to (a part of) a straight line, and the output is "yes". Otherwise, the output is "no". If $\gamma(t)$ is not a part of a line, we proceed to step 2 and compute μ^3 – the cube of the $\mathcal{SA}$-curvature of $\gamma(t)$. We then define a family of planar curves $\epsilon_{c_1,c_2,c_3}(s)$. The goal is to decide if there are some values of the parameters c_1, c_2, c_2 for which $\epsilon_{c_1,c_2,c_3}(s)$ is $\mathcal{PGL}(3)$-equivalent to $\gamma(t)$. In step 4 we consider a possibility that $\gamma(t)$ is a part of a conic, or equivalently its $\mathcal{SA}$-curvature μ is a constant. All conics on the plane are $\mathcal{PGL}(3)$-equivalent, and, therefore, if there exist $a, c_1, c_2, c_3 \in \mathbb{R}$ such that $\mu_{\epsilon_{c_1,c_2,c_3}}(s) = a$ the output is "yes", otherwise it is "no". If we proceed to step 5 then γ is not $\mathcal{PGL}(3)$-exceptional and we can compute its $\mathcal{PGL}(3)$-invariants. In step 7 we address a possibility that these invariants are constant. Otherwise, in the rest of the algorithm, we use the $\mathcal{PGL}(3)$-signature of γ to decide if there are values of the parameters c_1, c_2, c_2 for which $\epsilon_{c_1,c_2,c_3}(s)$ is $\mathcal{PGL}(3)$-equivalent to $\gamma(t)$.

Two well known, but challenging, computational procedures are involved in this algorithm: (i) implicitization in step 8 and (ii) a real quantifier elimination problem that may occur in steps 4, 7, or 9. The implcitization can be achieved by an appropriate Gröbner basis computation, or by procedures based on resultants (see, for instance, [7, 8] and references therein). Problem (ii), in our case, reduces to polynomial solving over real numbers. In the case of finitely many solutions, they can be isolated using, for instance, algorithms presented [24, 18]. In many cases MAPLE is able to solve these equations explicitly.

If the output of Algorithm 4.3.1 is "no" then there is no finite projection of Γ to γ. An output of "yes" provides a strong indication that Γ and γ are related by a finite projection. If c_1, c_2, c_3 can be found explicitly then we can check that the signatures of the curves $\gamma(t)$ and $\epsilon_{c_1,c_2,c_3}(s)$ not only have the same implicit equation, but, in fact, coincide (see Remark 3.1.10). This provides a strong indication that $\gamma(t)$ and $\epsilon_{c_1,c_2,c_3}(s)$ are equivalent under a $\mathcal{PGL}(3)$-transformation (1.5), or at least that these curves have a $\mathcal{PGL}(3)$-overlap. *If we can confirm this equivalence then* Γ *and* γ *are related by a finite camera.*

If $[A] \in \mathcal{PGL}(3)$ maps C_γ to $C_{\epsilon_{c_1,c_2,c_3}}$, where c_1, c_2, c_3 satisfy (4.8) and (4.9) (or (4.7), or (4.6) when appropriate), then $[P] = [A][P_f^0][B]$, where P_f^0 is defined by (2.8) and B is defined by (2.16), projects Γ onto γ. The center of the projection is located at $(-c_1, -c_2, -c_3)$.

4.3.2 Projection of curves by affine cameras

The algorithm for curves under affine cameras is similar to Algorithm 4.3.1. It relies on the reduced projection criterion stated in Corollary 4.2.3. The algorithm requires deciding if a given planar curve γ is $\mathcal{A}(2)$-equivalent to curve (4.3), or at least to one of the curves from 1-parametric family (4.4), or at least to one of the curves from 2-parametric family (4.2). Affine invariants given by (3.2) are used to solve the $\mathcal{A}(2)$-equivalence problem. Although the algorithms for affine cameras require, in general, consideration of three cases, the computations are less demanding due to lower differential order of the invariants and fewer number of parameters. In the following algorithm, $\alpha(s)$, $\beta(b, s)$, and $\delta(c, f, s)$ will be as defined in 4.2.3.

Algorithm 4.3.2. (Affine cameras.)
INPUT: a planar curve $\gamma(t) = (x(t), y(t))$, $t \in \mathbb{R}$, *and a spatial curve* $\Gamma(t) = \big(z_1(s), z_2(s), z_3(s)\big)$, $s \in \mathbb{R}$, *with rational parameterizations.*
OUTPUT: YES or NO answer to the question "Is necessary condition for existence of affine projection $[P]$ *such that* $[C_\gamma] = [P][C_\Gamma]$ *satisfied?".*

1. *If* $\forall t$: $x''(t)y'(t) - y''(t)x'(t) = 0$ *then*

   - *compute the Euclidean curvature* $\mathcal{K}(s)$ *for the curve* Γ;
   - *if* $\forall s$: $\mathcal{K}(s) = 0$ *then OUTPUT:* YES *and exit the procedure,*

     else

     – *compute the Euclidean torsion* $\mathcal{T}(s)$ *for the curve* Γ;

     – *if* $\forall s$: $\mathcal{T}(s) = 0$ *then OUTPUT:* YES, *else OUTPUT:* NO
   - *exit the procedure;*

   else proceed to the next step.

2. *Obtain two rational functions of* t, $J_a|_\gamma(t)$ *and* $K_a|_\gamma(t)$.

3. *If* $\exists\ m \in \mathbb{R}$ *s. t.* $\forall t \in \mathbb{R}$: $K_a|_\gamma(t) = m$ *then*

   a. *If* $\exists\ n \in \mathbb{R}$ *s. t.* $\forall t \in \mathbb{R}$: $J_a|_\gamma(t) = n$ *then*

   – *If* $\forall s \in \mathbb{R}$: $K_a|_\alpha(s) = m$ *AND* $J_a|_\alpha(s) = n$

     OR

     $\exists\ b \in \mathbb{R}$ *s. t.* $\forall s \in \mathbb{R}$: $K_a|_\beta(b, s) = m$ *AND* $J_a|_\beta(b, s) = n$

     OR

     $\exists\ c, f \in \mathbb{R}$ *s. t.* $\forall s \in \mathbb{R}$: $K_a|_\delta(c, f, s) = m$ *AND* $J_a|_\delta(c, f, s) = n$

     then OUTPUT : YES *and exit the procedure.*

   – *Else, OUTPUT* : NO *and exit the procedure.*

   b. *Else, the first invariant is not constant.*

   – *If* $\forall s \in \mathbb{R}$: $K_a|_\alpha(s) = m$ *AND* $\nexists\ n \in \mathbb{R}$ *s. t.* $\forall s \in \mathbb{R}$ $J_a|_\alpha(s) = n$

     OR

$\exists\ b \in \mathbb{R}$ *s. t.* $\forall s \in \mathbb{R}$: $K_a|_\beta(b,s) = m$ *AND* $\nexists\ n \in \mathbb{R}$ *s. t.*
$\forall s \in \mathbb{R}\ J_a|_\beta(b,s) = n$
OR
$\exists\ c, f \in \mathbb{R}$ *s. t.* $\forall s \in \mathbb{R}$: $K_a|_\delta(c,f,s) = m$ *AND* $\nexists\ n \in \mathbb{R}$ *s. t.*
$\forall s \in \mathbb{R}\ J_a|_\delta(c,f,s) = n$
then $OUTPUT$: YES *and exit the procedure.*

– *Else,* $OUTPUT$: NO *and exit the procedure.*

4. *Compute the implicit equation* $\hat{\mathcal{S}}_\gamma(i_1, i_2) = 0$ *for the* $\mathcal{A}(2)$*-signature of* γ *by eliminating* t *from equations* $i_1 = J_a|_\gamma(t)$ *and* $i_2 = K_a|_\gamma(t)$.

5. *Substitute* $J_a|_\alpha(s)$ *and* $K_a|_\alpha(s)$ *for* i_1 *and* i_2*, respectively, into* $\hat{\mathcal{S}}_\gamma(i_1, i_2)$.

6. *If* $\hat{\mathcal{S}}_\gamma(J_a|_\alpha(s), K_a|_\alpha(s)) = 0\ \forall s \in \mathbb{R}$ *and* $\nexists\ m \in \mathbb{R}$ *s. t.* $\forall s \in \mathbb{R}$: $K_a|_\alpha(s) = m$ *then* $OUTPUT$: YES *and exit the procedure.*

7. *If* $\exists\ b \in \mathbb{R}$ *s. t.* $\hat{\mathcal{S}}_\gamma(J_a|_\beta(b,s), K_a|_\beta(b,s)) = 0\ \forall s \in \mathbb{R}$ *and* $\nexists\ m \in \mathbb{R}$ *s. t.* $\forall s \in \mathbb{R}$: $K_a|_\beta(b,s) = m$ *then* $OUTPUT$: YES *and exit the procedure.*

8. *If* $\exists\ c, f \in \mathbb{R}$ *s. t.* $\hat{\mathcal{S}}_\gamma(J_a|_\delta(c,f,s), K_a|_\delta(c,f,s)) = 0\ \forall s \in \mathbb{R}$ *and* $\nexists\ m \in \mathbb{R}$ *s. t.* $\forall s \in \mathbb{R}$: $K_a|_\delta(c,f,s)) = m$ *then* $OUTPUT$: YES *and exit the procedure.*
   else $OUTPUT$: NO *and exit the procedure.*

The algorithm for curves under affine cameras relies on the reduced projection criterion stated in Corollary 4.2.3 and requires deciding if a given planar curve γ is $\mathcal{A}(2)$-equivalent to curve (4.3), or at least to one of the curves from 1-parametric family (4.4), or at least to one of the curves from 2-parametric family (4.2). Affine invariants given by (3.2) are used to solve the $\mathcal{A}(2)$-problem. We first check if our plane curve is affine-exceptional. In step 3, we check if the plane curve has a constant invariant. In the constant invariant situation, the check for correspondence is more simple than the general case.

4.4 Examples

A MAPLE implementation and more examples are posted at `www.math.ncsu.edu/~jmburdis`.

Example 4.4.1. *In order to decide whether the spatial curve*

$$\Gamma(s) = (z_1(s),\, z_2(s),\, z_3(s)) = \left(s^4+1,\, s^2,\, s\right),\ s \in \mathbb{R},$$

can be projected onto $\gamma(t) = (t\,,\, t^4+t^2)\,,\ t \in \mathbb{R}$ *by an affine camera, we start by determining that* γ *is not an* $\mathcal{A}(2)$*-exceptional curve (neither a straight line nor a parabola). The curve* γ *has non-constant* $\mathcal{A}(2)$*-invariants (3.2) that satisfy the following implicit signature equation:*

$$-448\, i_1^2 + (3780\, i_2 - 4375)\, i_1 + 28125\, i_2 + 245\, i_2^3 - 5250\, i_2^2 = 0. \tag{4.10}$$

Following Corollary 4.2.3 we first check whether $\gamma(t)$ *is* $\mathcal{A}(2)$*-equivalent to* $\alpha(s) = (z_2(s),\, z_3(s)) = (s^2,\, s)$*. The answer is no, since* $\alpha(s)$ *is an* $\mathcal{A}(2)$*-exceptional curve (parabola) and* $\gamma(t)$ *is not* $\mathcal{A}(2)$*-exceptional. We next check whether there exists* $b \in \mathbb{R}$ *such that* $\gamma(t)$ *is* $\mathcal{A}(2)$*-equivalent to* $\beta_b(s) = \left(z_1(s) + b\, z_2(s),\, z_3(s)\right) = \left(s^4+1+b\, s^2,\, s\right)$*.*

We evaluate invariants (3.2) on $\beta_b(s)$*:*

$$J_a|_{\beta_b}(s) = \frac{100\, s^2\, (3\, b - 14 s^2)^2}{(b - 14 s^2)^3}, \tag{4.11}$$

$$K_a|_{\beta_b}(s) = \frac{140\, s^2 (2\, s^2 + b)}{(b - 14\, s^2)^2}. \tag{4.12}$$

When $b = 0$ *the invariants are constant:* $J_a|_{\beta_0}(s) \equiv 50/7$ *and* $K_a|_{\beta_0}(s) \equiv 10/7$*, and, therefore,* $\beta_0(s)$ *is not* $\mathcal{A}(2)$*-equivalent to* $\gamma(t)$*. For all* $b \neq 0$ *the invariants (4.11) and (4.12) are non-constant and satisfy the signature equation (4.10). This provides a necessary condition and a strong indication that* $\gamma(t)$ *is* $\mathcal{A}(2)$*-equivalent to* $\beta_b(s)$ *for* $g \neq 0$*. For* $b = 1$ *this* $\mathcal{A}(2)$*-equivalence is obvious, and hence* $\Gamma(s)$ *projects onto* $\gamma(t)$ *by an affine projection.*

Example 4.4.2. *We would like to decide if the spatial curve*

$$\Gamma(s) = (z_1(s)\, z_2(s) z_3(s)) = \left(s^2 + s,\ s^3 - 3\, s^2,\ s^4\right),\ s \in \mathbb{R} \tag{4.13}$$

projects onto any of three given planar curves for $t \in \mathbb{R}$*:*

$$\begin{aligned} \gamma_1(t) &= \left(t^4 + t\,,\ t^2\right), \\ \gamma_2(t) &= \left(t^3 - t\,,\ t^3 + t^2\right), \\ \gamma_3(t) &= \left(t/(1 + t^3),\ t^2/(1 + t^3)\right) \text{ (Folium of Descartes).[19]} \end{aligned}$$

None of the given γ*'s are* $\mathcal{A}(2)$*-exceptional and the implicit equations of their* $\mathcal{A}(2)$*-*

signatures are given, respectively, by:

$$\begin{aligned} (-210 + 75\, i_2)\, i_1 - 2023 + 1190\, i_2 - 175\, i_2^2 &= 0, \\ -857500 + 147000 i_2 + 10500 i_2^2 - 26460 i_1 i_2 + 9261 i_1^2 + 160 i_2^3 &= 0, \\ 10\, i_2 - 49 &= 0. \end{aligned}$$

Following an algorithm based on Corollary 4.2.3 we establish that $\alpha(s) = \big(z_2(s), z_3(s)\big)$ *and* $\beta_b(s) = \big(\, z_1(s) + b\, z_2(s),\, z_3(s)\big)$, *for all* $g \in \mathbb{R}$, *are not* $\mathcal{A}(2)$*-equivalent to either of the* γ*'s. We then establish that* $\delta_{cf}(s) = \big(z_1(s) + c\, z_3(s),\, z_2(s) + f\, z_3(s)\big)$ *is* $\mathcal{A}(2)$*-equivalent to* γ_1 *when* $c = 0$ *and* $f = 1/2$ *and is* $\mathcal{A}(2)$*-equivalent to* γ_2 *when* $c = 0$ *and* $f = 0$, *but there are no real values of* f *and* c *such that* $\delta_{cf}(s)$ *and* γ_3 *are* $\mathcal{A}(2)$*-equivalent.*

We conclude that there are affine projections of $\Gamma(s)$ *onto both* $\gamma_1(t)$ *and* $\gamma_2(t)$, *but not onto* $\gamma_3(t)$.

We note that although $\Gamma(s)$ *affinely projects to both* $\gamma_1(t)$ *and* $\gamma_2(t)$, *the curves* $\gamma_1(t)$ *and* $\gamma_2(t)$ *are not* $\mathcal{A}(2)$*-equivalent because their signatures have different implicit equations.*

Example 4.4.3. *Given the space curve*

$$\Gamma(s) = \begin{pmatrix} z_1(s) \\ z_2(s) \\ z_3(s) \end{pmatrix} = \begin{pmatrix} \left(\frac{1-s^2}{1+s^2}\right)^2 \\ \frac{s}{1+s^2} + \frac{s\,(1-s^2)}{(1+s^2)^2} \\ \frac{1-s^2}{1+s^2} \end{pmatrix} \tag{4.14}$$

and three plane curves

$$
\begin{aligned}
\gamma_1(t) &= \left(\frac{1-t^2}{1+t^2} + \left(\frac{1-t^2}{1+t^2} \right)^2 , \frac{2t}{1+t^2} + \frac{2t\,(1-t^2)}{(1+t^2)^2} \right), \\
\gamma_2(t) &= \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} + \frac{2t\,(1-t^2)}{(1+t^2)^2} \right), \\
\gamma_3(t) &= \left(t^2 , t^4 + t + 1 \right).
\end{aligned}
$$

where γ_1 is called the Cardioid. The signatures for γ_1, γ_2, and γ_3 and given, respectively, by the following implicit equations:

$$
\begin{aligned}
-21296\, i_1^2 + (35640\, i_2 - 188900)\, i_1 + 1960\, i_2^3 & \\
-556875 + 258750\, i_2 - 39300\, i_2^2 &= 0, \\
(*) &= 0, \\
(-210 + 75\, i_2)\, i_1 + 1190\, i_2 - 2023 - 175\, i_2^2 &= 0,
\end{aligned}
$$

where (*) is given by equation (1) of Appendix A. Since the three signatures are different, no two of these three planar curves are affinely-equivalent. Following our algorithm, there exists an affine projection taking $\Gamma(s)$ to γ_1 since $\delta_{cf}(s) = \big(z_1(s) + c\, z_3(s),\; z_2(s) + f\, z_3(s)\big)$ is affinely-equivalent to $\gamma_1(t)$ for $c = 1$ and $f = 0$. Also, there exists an affine projection taking $\Gamma(s)$ to γ_2 since $\alpha(s) = \big(z_2(s), z_3(s)\big)$ is affinely-equivalent to $\gamma_2(t)$. After going through all three cases in Corollary 4.2.3 we determine there does not exist an affine projection taking $\Gamma(s)$ to $\gamma_3(t)$.

Example 4.4.4. *Given the space curve*

$$\Gamma(s) = \begin{pmatrix} z_1(s) \\ z_2(s) \\ z_3(s) \end{pmatrix} = \begin{pmatrix} s \\ s + \frac{1}{s} \\ s^2 + 1 \end{pmatrix} \qquad (4.15)$$

and three plane curves

$$\begin{aligned} \gamma_1(t) &= \left(t\,,\ \frac{t^3 + 2\,t^2 + t + 2}{t}\right), \\ \gamma_2(t) &= \left(\frac{1 + t^2}{t}\,,\ \frac{t^3 + t}{t}\right), \\ \gamma_3(t) &= \left(t^2\,,\ t^4 + t + 1\right). \end{aligned}$$

The signatures for γ_1, γ_2, and γ_3 are given, respectively, by the following implicit equations:

$$\begin{aligned} (810\, i_2^3 - 19750\, i_2 - 5100\, i_2^2 + 85000)\, i_1 + (9025 - 550\, i_2)\, i_1^2 & \\ + i_1^3 - 15625 - 2025\, i_2^4 + 62500\, i_2 - 73750\, i_2^2 + 22500\, i_2^3 &= 0, \\ 1225\, i_1^2 + (31360 - 6300\, i_2)\, i_1 + 188356 + 8100\, i_2^2 - 78120\, i_2 &= 0, \\ (-210 + 75\, i_2)\, i_1 + 1190\, i_2 - 2023 - 175\, i_2^2 &= 0. \end{aligned}$$

Since the three signatures are different, no two of these three planar curves are affinely-equivalent. Following our algorithm, there exists an affine projection taking $\Gamma(s)$ to γ_1 since $\delta_{cf}(s) = \big(z_1(s) + c\, z_3(s),\ z_2(s) + f\, z_3(s)\big)$ is affinely-equivalent to $\gamma_1(t)$ for $c = 0$ and $f = 1$. In fact, it is affinely-equivalent for any $f \neq 0$. Also, there exists an affine *projection taking* $\Gamma(s)$ *to* γ_2 *since* $\alpha(s) = \big(z_2(s), z_3(s)\big)$ *is affinely-equivalent to* $\gamma_2(t)$. *After going through all three cases in Corollary 4.2.3 we determine there does not exist an affine projection taking* $\Gamma(s)$ *to* $\gamma_3(t)$.

Example 4.4.5. *Given the space curve*

$$\Gamma(s) = (z_1(s),\, z_2(s),\, z_3(s)) = \left(s^3,\, s^2,\, s\right),\ s \in \mathbb{R},$$

and three plane curves

$$\begin{aligned}\gamma_1(t) &= \left(t^2,\, t\right),\ t \in \mathbb{R},\\ \gamma_2(t) &= \left(\frac{t^3}{t+1},\, \frac{t^2}{t+1}\right),\ t \in \mathbb{R},\\ \gamma_3(t) &= \left(t,\, t^5\right),\ t \in \mathbb{R},\end{aligned}$$

determine correspondence under finite projections.

Clearly, $\Gamma(s)$ *projects to* $\gamma_1(t)$ *by the standard finite projection*

$$P_f^0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.16}$$

and to $\gamma_2(t)$ *by finite projection*

$$P := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{4.17}$$

Note that curves γ_1 *and* γ_2 *are not* $\mathcal{PGL}(3)$*-equivalent:* γ_1 *parameterizes a parabola* $y = x^2$ *and is* $\mathcal{PGL}(3)$*-exceptional, whereas the* $\mathcal{PGL}(3)$*-invariants for* γ_2 *are*

$$J_p|_{\gamma_2} = \frac{250047}{12800} \text{ and } K_p|_{\gamma_2} = 0.$$

Note that an implicit equation for γ_2 *is* $y^2x + y^3 - x^2 = 0$*. Following Algorithm 4.3.1, to determine if* $\Gamma(s)$ *projects to* $\gamma_3(t)$ *by a finite camera, we first calculate the* $\mathcal{PGL}(3)$*-invariants restricted to* γ_3*.*

$$J_p|_{\gamma_3} = \frac{1029}{128} \text{ and } K_p|_{\gamma_3} = 0.$$

We then restrict the $\mathcal{PGL}(3)$*-invariants to* $\epsilon_{c_1,c_2,c_3}(s) = \left(\frac{s^3+c_1}{s+c_3}, \frac{s^2+c_2}{s+c_3}\right)$, $s \in \mathbb{R}$. $J|_\epsilon(c_1, c_2, c_3, s)$ *and* $K_p|_\epsilon(c_1, c_2, c_3, s)$ *depend on* $s, c_1, c_2,$ *and* c_3*. We need to determine if there exist* $c_1, c_2, c_3 \in \mathbb{R}$ *such that*

$$J|_\epsilon(c_1, c_2, c_3, s) = J_p|_{\gamma_3} \text{ and } K_p|_\epsilon(c_1, c_2, c_3, s) = K_p|_{\gamma_3}. \tag{4.18}$$

By substituting various values of s *into equations (4.18) we conclude that there does not exist* $c_1, c_2, c_3 \in \mathbb{R}$ *that satisfy equations (4.18) for all* s*. Therefore,* $\epsilon_{c_1,c_2,c_3}(s)$ *can not be* $\mathcal{PGL}(3)$*-equivalent to* $\gamma_3(t)$*, and* $\Gamma(s)$ *can not be projected to* $\gamma_3(t)$ *by a finite projection.*

Remark 4.4.6. *For the curves defined in Example 4.4.5, there does not exist an affine projection taking* Γ *to* γ_2*. The implicit equation for the* $\mathcal{A}(2)$*-signature of* γ_2 *is given by equation (2) of Appendix A. By Corollary 4.2.3, if* Γ *projects to* γ_2 *under an affine* *projection then* γ_2 *is affinely-equivalent to one of the following curves,*

$$\begin{aligned}\alpha(s) &= \big(s^2, s\big),\\ \beta_b(s) &= \big(\,s^3 + b\,s^2,\ s\big),\\ \delta_{cf}(s) &= \big(s^3 + c\,s,\ s^2 + f\,s\big).\end{aligned}$$

$\alpha(s)$ *is an affine-exceptional curve, and the signatures for curves of the form of* $\beta_b(s)$ *and* $\delta_{cf}(s)$ *can be found in Example 3.1.12. By comparing the signatures, we observe that none of these curves are affinely-equivalent to* γ_2*, and, therefore, there does not exist an affine projection taking* Γ *to* γ_2*.*

Chapter 5

The Projection Problem for Finite Ordered Sets of Points

In this chapter we assume that objects and images are represented by finite ordered sets of points in $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. In Section 5.2 we introduce the criteria that reduces the projection problem, for finite and affine projections, to a variation of the equivalence problem of two ordered sets of r points in $\mathbb{R}^2$ under the action of $\mathcal{PGL}(3)$ and $\mathcal{A}(2)$ groups, respectively. Separating sets of invariants for sets of r ordered points in $\mathbb{R}^2$ under $\mathcal{A}(2)$-action and under $\mathcal{PGL}(3)$-action are listed in Chapter 3. In the case of finite projections, we obtain a system of polynomial equations on c_1, c_2 and c_3 that have solutions if and only if the given set Z projects to the given set X and the analog of Algorithm 4.3.1 follows. Affine projections are treated in a similar way.

5.1 Problem Formulation

First, we restate the problem given in Chapter 1.

Problem 5.1.1. *Given an* ordered *set* $Z = (\mathbf{z}^1, \dots, \mathbf{z}^r)$ *of* r *points in* $\mathbb{R}^3$*, an* ordered *set* $X = (\mathbf{x}^1, \dots, \mathbf{x}^r)$ *of* r *points in* $\mathbb{R}^2$ *and a class of projections, when does there exist a projection from this class taking* Z *to* X*?*

We use the notation discussed in Section 1.1 and give the following definition.

Definition 5.1.2. *Given an* ordered *set* $Z = (\mathbf{z}^1, \dots, \mathbf{z}^r)$ *of* r *points in* $\mathbb{R}^3$ *with coordinates* $\mathbf{z}^i = (z_1^i, z_2^i, z_3^i)$*,* $i = 1 \dots r$ *and given an* ordered *set* $X = (\mathbf{x}^1, \dots, \mathbf{x}^r)$ *of* r *points in* $\mathbb{R}^2$ *with coordinates* $\mathbf{x}^i = (x^i, y^i)$ *we say* Z *projects to* X *if there exits a* 3×4 *matrix* P *of rank 3 such that*

$$[x^i, y^i, 1]^{tr} = [P]\,[z_1^i,\ z_2^i,\ z_3^i,\ 1]^{tr} \text{ for } i = 1 \dots r. \tag{5.1}$$

Figure 5.1 illustrates that a solution to the projection problem for ordered sets of points does not, however, provide a solution to the discretization of the projection problem for curves. Indeed, if $Z = (\mathbf{z}^1, \dots, \mathbf{z}^m)$ is a discrete sampling of a curve Γ and $X = (\mathbf{x}^1, \dots, \mathbf{x}^m)$ is a discrete sampling of γ, these sets might not be in a correspondence under a projection even when the curves are related by a projection.

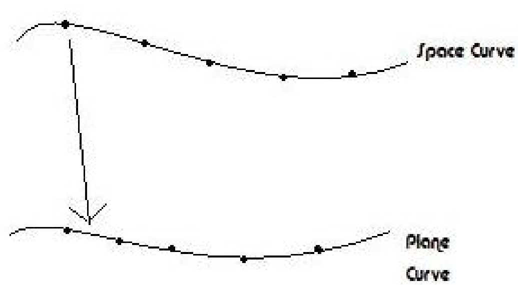

Figure 5.1: Projection problem for curves vs. projection problems for finite ordered sets of points

5.2 Projection criteria

The projection criteria of Section 4.2 can be straightforwardly adapted to ordered sets of points, reducing the projection problem to a variation of the group-equivalence problem for ordered sets of points in $\mathbb{R}^2$. The latter problem can be addressed using a separating set of invariants discussed in Section 3.2.

5.2.1 Projection of ordered point sets by finite cameras

In this section we address Problem 5.1.1 for the finite camera.

Problem 5.2.1. *Given an* ordered *set* $Z = (\mathbf{z}^1, \ldots, \mathbf{z}^r)$ *of* r *points in* $\mathbb{R}^3$ *and an* ordered *set* $X = (\mathbf{x}^1, \ldots, \mathbf{x}^r)$ *of* r *points in* $\mathbb{R}^2$ *when does there exist a finite projection taking* Z *to* X*?*

The finite projection criteria of Theorem 4.2.1 in the case of finite sets of points reformulates as follows:

Theorem 5.2.2. *A given* ordered *set* $Z = (\mathbf{z}^1, \ldots, \mathbf{z}^r)$ *of* r *points in* $\mathbb{R}^3$ *with coordinates* $\mathbf{z}^i = (z_1^i, z_2^i, z_3^i)$, $i = 1 \ldots r$ *projects onto a given* ordered *set* $X = (\mathbf{x}^1, \ldots, \mathbf{x}^r)$ *of* r *points in* $\mathbb{R}^2$ *with coordinates* $\mathbf{x}^i = (x^i, y^i)$ *if and only if there exist* $c_1, c_2, c_3 \in \mathbb{R}$ *and* $[A] \in \mathcal{PGL}(3)$ *such that*

$$[x^i, y^i, 1]^{tr} = [A]\,[z_1^i + c_1,\ z_2^i + c_2,\ z_3^i + c_3]^{tr} \text{ for } i = 1 \ldots r. \tag{5.2}$$

Proof. ($\Rightarrow$)Assume there exists a finite projection $[P]$ such that

$$[x^i, y^i, 1]^{tr} = [P]\,[z_1^i,\ z_2^i,\ z_3^i,\ 1]^{tr} \text{ for } i = 1 \ldots r. \tag{5.3}$$

It was established in the proof of Proposition 2.3.1 that $[P] = [A]\,[P_f^0]\,[B]$ for some $[A] \in \mathcal{PGL}(3)$ and $[B] \in \mathcal{A}(3)$, where B is given by (2.16) for some $c_1, c_2, c_3 \in \mathbb{R}$, and P_f^0 is the standard finite projection (2.8). Therefore $[x^i, y^i, 1]^{tr} = [A]\,[P_f^0]\,[B]\,[z_1^i,\, z_2^i,\, z_3^i,\, 1]^{tr}$ for $i = 1 \ldots r$. Since

$$P_f^0[B][z_1^i,\, z_2^i,\, z_3^i,\, 1]^{\mathrm{tr}} = [z_1^i + c_1,\, z_2^i + c_2,\, z_3^i + c_3]^{\mathrm{tr}} \text{ for } i = 1 \ldots r,$$

$$[x^i, y^i, 1]^{tr} = [A]\,[z_1^i + c_1,\, z_2^i + c_2,\, z_3^i + c_3]^{tr} \text{ for } i = 1 \ldots r. \tag{5.4}$$

($\Leftarrow$) To prove the converse direction we assume that there exists $[A] \in \mathcal{PGL}(3)$ and $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$[x^i, y^i, 1]^{tr} = [A]\,[z_1^i + c_1,\, z_2^i + c_2,\, z_3^i + c_3]^{tr} \text{ for } i = 1 \ldots r. \tag{5.5}$$

A direct computation shows that $[z_1^i,\, z_2^i,\, z_3^i,\, 1]^{tr}$ is projected onto $[x^i, y^i, 1]^{tr}$ by the finite projection $[P] = [A]\,[P_f^0]\,[B]$, where B is given by (2.16) and $[P_f^0]$ is the standard finite projection (2.8). $\square$

We note that the proof of Theorem 5.2.2 is a straightforward adaptation of the proof of Theorem 4.2.1.

Theorem 5.2.2 reduces Problem 5.2.1 to a problem of equivalence of ordered sets of points in the plane under $\mathcal{PGL}(3)$. We use Theorem 3.2.9 to solve this problem. For at least 6 points, $r \geq 6$, this theorem gives us $2\,(r-4)$ separating invariants. We can use the first 6 points to find the 3 unknown parameters $c_1, c_2, c_3 \in \mathbb{R}$. We then have to check $2\,(r-4)$ equations to determine if two sets are in correspondence under a finite projection. Finding $c_1, c_2, c_3 \in \mathbb{R}$ is a straightforward computation.

Example 5.2.3. *Given an* ordered *set of points in* $\mathbb{R}^3$, $Z = (\mathbf{z}^1, \ldots, \mathbf{z}^6)$ *with coordinates*

$$\mathbf{z}^1 = (1,2,-1),\ \mathbf{z}^2 = (0,2,0),\ \mathbf{z}^3 = (2,3,2),\ \mathbf{z}^4 = (3,4,0),\ \mathbf{z}^5 = (1,5,1),\ \textit{and}\ \mathbf{z}^6 = (1,2,3),$$

and given an ordered *set of points in* $\mathbb{R}^2$, $X = (\mathbf{x}^1, \ldots, \mathbf{x}^6)$ *with coordinates*

$$\mathbf{x}^1 = (2,5),\ \mathbf{x}^2 = (\frac{1}{2}, \frac{5}{2}),\ \mathbf{x}^3 = (\frac{3}{4}, \frac{3}{2}),\ \mathbf{x}^4 = (2, \frac{7}{2}),\ \mathbf{x}^5 = (\frac{2}{3}, \frac{8}{3}),\ \textit{and}\ \mathbf{x}^6 = (\frac{2}{5}, 1),$$

does there exists an finite projection taking Z *to* X*?*

Using Theorem 5.2.2 this is equivalent to deciding whether there exist $c_1, c_2, c_3 \in \mathbb{R}$ *and* $[A] \in \mathcal{PGL}(3)$ *such that*

$$[x^i, y^i, 1]^{tr} = [A]\,[z_1^i + c_1,\ z_2^i + c_2,\ z_3^i + c_3]^{tr}\ \textit{for}\ i = 1 \ldots 6. \tag{5.6}$$

Let $\widetilde{Z}_{c_1,c_2,c_3}$ *denote the* ordered *set of* 6 *points in* $\mathbb{R}^2$ *with coordinates* $\widetilde{\mathbf{z}}^i = (\frac{z_1^i + c_1}{z_3^i + c_3}, \frac{z_2^i + c_2}{z_3^i + c_3})$. *Using the fundamental set of invariants from Theorem 3.2.9 we pick the first* 3 *invariants to solve for* $c_1, c_2, c_3 \in \mathbb{R}$.

Equalities,

$$\begin{aligned} C|_X(1;2,3,4,5) &= C|_{\widetilde{Z}_{c_1,c_2,c_3}}(1;2,3,4,5), \\ C|_X(2;1,3,4,5) &= C|_{\widetilde{Z}_{c_1,c_2,c_3}}(2;1,3,4,5), \\ C|_X(1;2,3,4,6) &= C|_{\widetilde{Z}_{c_1,c_2,c_3}}(1;2,3,4,6), \end{aligned} \tag{5.7}$$

lead to equations on $c_1, c_2, c_3 \in \mathbb{R}$*;*

$$-\frac{(-8+c_3-4\,c_2+c_1)\,(-13+c_1+6\,c_3-4\,c_2)}{(2\,c_3+2\,c_1-6-3\,c_2)\,(14+7\,c_1-3\,c_3+2\,c_2)}-\frac{68}{63} = 0, \tag{5.8}$$

$$-\frac{(-8+c_3-4\,c_2+c_1)\,(2\,c_1+7\,c_3-6-3\,c_2)}{5\,((2\,c_3+2\,c_1-6-3\,c_2)\,(-c_3+c_1))}-\frac{17}{45} = 0, \tag{5.9}$$

$$\frac{(2\,(-8+c_3-4\,c_2+c_1))}{2\,c_3+2\,c_1-6-3\,c_2}-\frac{34}{9} = 0, \tag{5.10}$$

whose solution is $c_1 = 1$*,* $c_2 = 3$*, and* $c_3 = 2$*. We substitute these into* $\widetilde{Z}_{c_1,c_2,c_3}$ *to obtain the set of points*

$$\widetilde{\mathbf{z}}^1 = (2,5),\ \widetilde{\mathbf{z}}^2 = (\frac{1}{2},\frac{5}{2}),\ \widetilde{\mathbf{z}}^3 = (\frac{3}{4},\frac{3}{2}),\ \widetilde{\mathbf{z}}^4 = (2,\frac{7}{2}),\ \widetilde{\mathbf{z}}^5 = (\frac{2}{3},\frac{8}{3}),\ \textit{and}\ \widetilde{\mathbf{z}}^6 = (\frac{2}{5},1).$$

We know that the first 3 *invariants are equal for* X *and* $\widetilde{Z}_{c_1,c_2,c_3}$ *by construction. All there is left to do is to decide if the remaining separating invariants are the same for* X *and* $\widetilde{Z}_{c_1,c_2,c_3}$*. Indeed,*

$$C|_X(2;1,3,4,6) = C|_{\widetilde{Z}_{c_1,c_2,c_3}}(2;1,3,4,6) = \frac{731}{171}. \tag{5.11}$$

For $c_1 = 1$*,* $c_2 = 3$*, and* $c_3 = 2$*, by Theorem 3.2.9,* X *is equivalent to* $\widetilde{Z}_{c_1,c_2,c_3}$ *under the action of* $\mathcal{PGL}(3)$*, and by Theorem 5.2.2,* Z *projects onto* X *with a finite projection.*

5.2.2 Projection of ordered point sets by affine cameras

We adapt the affine projection criteria for curves from Theorem 4.2.2 and Corollary 4.2.3 to give a solution to Problem 5.1.1 for affine cameras.

Problem 5.2.4. *Given an* ordered *set* $Z = (\mathbf{z}^1, \ldots, \mathbf{z}^r)$ *of* r *points in* $\mathbb{R}^3$ *and an* ordered *set* $X = (\mathbf{x}^1, \ldots, \mathbf{x}^r)$ *of* r *points in* $\mathbb{R}^2$ *when does there exist an affine projection taking* Z *to* X*?*

For finite sets of ordered points we formulate the following analog of Corollary 4.2.3:

Theorem 5.2.5. *A given* ordered *set* $Z = (\mathbf{z}^1, \ldots, \mathbf{z}^r)$ *of* r *points in* $\mathbb{R}^3$ *with coordinates* $\mathbf{z}^i = (z^i_1, z^i_2, z^i_3)$, $i = 1 \ldots r$ *projects onto a given* ordered *set* $X = (\mathbf{x}^1, \ldots, \mathbf{x}^r)$ *of* r *points in* $\mathbb{R}^2$ *with coordinates* $\mathbf{x}^i = (x^i, y^i)$ *if and only if there exist* $b, c, f \in \mathbb{R}$ *and* $[A] \in \mathcal{A}(2)$ *such that at least one of the following relationships hold*

$$\left[x^i, y^i, 1\right]^{tr} = [A]\,[z^i_2,\, z^i_3,\, 1]^{tr} \qquad \textit{for } i = 1 \ldots r, \tag{5.12}$$

$$\left[x^i, y^i, 1\right]^{tr} = [A]\,[z^i_1 + b\, z^i_2,\, z^i_3,\, 1]^{tr} \qquad \textit{for } i = 1 \ldots r, \textit{ or} \tag{5.13}$$

$$\left[x^i, y^i, 1\right]^{tr} = [A]\,[z^i_1 + c\, z^i_3,\, z^i_2 + f\, z^i_3,\, 1]^{tr} \qquad \textit{for } i = 1 \ldots r. \tag{5.14}$$

Again, the proof of Theorem 5.2.5 is a straightforward adaptation of the proofs of Theorem 4.2.2 and Corollary 4.2.3. Theorem 5.2.5 reduces Problem 5.2.4 to a problem of $\mathcal{A}(2)$-equivalence of ordered sets of points on the plane. Let,

$\widetilde{Z}$ represent the *ordered* set of r points in $\mathbb{R}^2$ with coordinates $\widetilde{\mathbf{z}}^i = (z^i_2,\, z^i_3,)$,

$\widetilde{Z}_b$ represent the *ordered* set of r points in $\mathbb{R}^2$ with coordinates $\widetilde{\mathbf{z}}^i = (z^i_1 + b\, z^i_2,\, z^i_3)$ and

$\widetilde{Z}_{c,f}$ represent the *ordered* set of r points in $\mathbb{R}^2$ with coordinates $\widetilde{\mathbf{z}}^i = (z^i_1 + c\, z^i_3,\, z^i_2 + f\, z^i_3)$.

For at least 5 points, $r \geq 5$, Theorem 3.2.5 gives us $2\,(r - 3)$ separating invariants. We only have to use the first 5 points to solve for the 3 unknown potential parameters. If the sets of r points correspond under an affine projection, then clearly the first 5 points must correspond under the same affine projection. From

$$B|_X(2,4) = B|_{\widetilde{Z}_b}(2,4) \tag{5.15}$$

$$B|_X(3,4) = B|_{\widetilde{Z}_b}(3,4) \tag{5.16}$$

it follows that

$$b = \frac{A_{1,2,4}\,C_{1,3,4} - A_{1,3,4}\,C_{1,2,4}}{A_{1,3,4}\,D_{1,2,4} - A_{1,2,4}\,D_{1,3,4}}. \tag{5.17}$$

Similarly, from

$$B|_X(2,4) = B|_{\widetilde{Z}_{c,f}}(2,4), \tag{5.18}$$

$$B|_X(3,4) = B|_{\widetilde{Z}_{c,f}}(3,4), \tag{5.19}$$

$$B|_X(2,5) = B|_{\widetilde{Z}_{c,f}}(2,5), \tag{5.20}$$

$$B|_X(3,5) = B|_{\widetilde{Z}_{c,f}}(3,5), \tag{5.21}$$

it follows that

$$c = \frac{A_{1,2,5}\,(C_{1,3,5}\,B_{1,2,4} - B_{1,3,5}\,C_{1,2,4}) + A_{1,2,4}\,(C_{1,2,5}\,B_{1,3,5} - B_{1,2,5}\,C_{1,3,5}) - A_{1,3,5}\,(C_{1,2,5}\,B_{1,2,4} - B_{1,2,5}\,C_{1,2,4})}{A_{1,2,5}\,(D_{1,2,4}\,C_{1,3,5} - D_{1,3,5}\,C_{1,2,4}) - A_{1,3,5}\,(D_{1,2,4}\,C_{1,2,5} - D_{1,2,5}\,C_{1,2,4}) + A_{1,2,4}\,(D_{1,3,5}\,C_{1,2,5} - D_{1,2,5}\,C_{1,3,5})},$$

$$f = \frac{A_{1,2,5}\,(D_{1,3,5}\,B_{1,2,4} - D_{1,2,4}\,B_{1,3,5}) - A_{1,3,5}\,(D_{1,2,5}\,B_{1,2,4} - D_{1,2,4}\,B_{1,2,5}) + A_{1,2,4}\,(D_{1,2,5}\,B_{1,3,5} - D_{1,3,5}\,B_{1,2,5})}{A_{1,2,5}\,(D_{1,2,4}\,C_{1,3,5} - D_{1,3,5}\,C_{1,2,4}) - A_{1,3,5}\,(D_{1,2,4}\,C_{1,2,5} - D_{1,2,5}\,C_{1,2,4}) + A_{1,2,4}\,(D_{1,3,5}\,C_{1,2,5} - D_{1,2,5}\,C_{1,3,5})}, \tag{5.22}$$

where

$$A_{i,j,k} = \det \begin{vmatrix} x^i & x^j & x^k \\ y^i & y^j & y^k \\ 1 & 1 & 1 \end{vmatrix}, \qquad B_{i,j,k} = \det \begin{vmatrix} z_1^i & z_1^j & z_1^k \\ z_2^i & z_2^j & z_2^k \\ 1 & 1 & 1 \end{vmatrix},$$

$$C_{i,j,k} = \det \begin{vmatrix} z_1^i & z_1^j & z_1^k \\ z_3^i & z_3^j & z_3^k \\ 1 & 1 & 1 \end{vmatrix}, \qquad D_{i,j,k} = \det \begin{vmatrix} z_2^i & z_2^j & z_2^k \\ z_3^i & z_3^j & z_3^k \\ 1 & 1 & 1 \end{vmatrix}.$$

5.3 Algorithms

In this section we write down the algorithms for using Theorems 5.2.2 and 5.2.5 to decide, for ordered sets of points, the existence of finite and affine projections. We used MAPLE to implement the algorithms. They are less complex than the algorithms for curves, and are linear in the number of equations needed to determine equivalence.

Algorithm 5.3.1. (FINITE CAMERAS.)

INPUT: An ordered set $X = (\mathbf{x}^1, \ldots, \mathbf{x}^r)$ *of* r *points, with no* 3 *of* $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, , \mathbf{x}^4$ *on the same line, in* $\mathbb{R}^2$ *with coordinates* $\mathbf{x}^i = (x^i, y^i)$ *and an ordered set* $Z = (\mathbf{z}^1, \ldots, \mathbf{z}^r)$ *of* r *points in* $\mathbb{R}^3$ *with coordinates* $\mathbf{z}^i = (z_1^i, z_2^i, z_3^i)$.

OUTPUT: YES or NO answer to the question "Does Z *project onto* X *with a finite projection?".*

1. *Define the ordered set* $\widetilde{Z}_{c_1,c_2,c_3} = (\widetilde{z}^1, \ldots, \widetilde{z}^r)$ *of* r *points in* $\mathbb{R}^2$ *to have coordinates* $\widetilde{z}^i = \left(\frac{z_1^i + c_1}{z_3^i + c_3}, \frac{z_2^i + c_2}{z_3^i + c_3}\right)$.

2. *Compute* c_1, c_2, *and* c_3 *using Equations 5.7.*

3. *For every pair of area cross-ratios, $C(1;2,3,4,k)$ and $C(2;1,3,4,k)$ given by Theorem 3.2.9*

   - *Compute $C|_X(1;2,3,4,k)$ and $C|_X(2;1,3,4,k)$.*
   - *Compute $C|_{\widetilde{Z}_{c_1,c_2,c_3}}(1;2,3,4,k)$ and $C|_{\widetilde{Z}_{c_1,c_2,c_3}}(2;1,3,4,k)$.*
   - *If $C|_X(i;j,3,4,k) \neq C|_{\widetilde{Z}_{c_1,c_2,c_3}}(i;j,3,4,k)$ then $OUTPUT$: NO and exit the procedure.*

   If $C|_X(i;j,3,4,k) = C|_{\widetilde{Z}_{c_1,c_2,c_3}}(i;j,3,4,k)$ for every fundamental area cross ratio then $OUTPUT$: YES and exit the procedure.

As in Example 5.2.3, this algorithm uses the first 3 invariants to find c_1, c_2, $c_3 \in \mathbb{R}$. Once found, the ordered set of points $\widetilde{Z}$ does not depend on unknowns, and we use the remaining invariants from Theorem 3.2.9 to determine if $\widetilde{Z}$ is $\mathcal{PGL}(3)$-equivalent to X. If so, there exists a finite projection taking Z to X.

Algorithm 5.3.2. (Affine cameras.)
INPUT: an ordered set, $X = (\mathbf{x}^1, \dots, \mathbf{x}^r)$, with $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ non-collinear, of r points in $\mathbb{R}^2$ with coordinates $\mathbf{x}^i = (x^i, y^i)$ and an ordered set $Z = (\mathbf{z}^1, \dots, \mathbf{z}^r)$ of r points in $\mathbb{R}^3$ with coordinates $\mathbf{z}^i = (z_1^i, z_2^i, z_3^i)$.
OUTPUT: YES or NO answer to the question "Does Z project onto X with an affine projection?".

1. *Define an ordered set $\widetilde{Z} = (\widetilde{z}^1, \dots, \widetilde{z}^r)$ of r points in $\mathbb{R}^2$ with coordinates $\widetilde{z}^i = (z_2^i, z_3^i)$.*

2. *For every fundamental area ratio, $B(i,k)$ given by Theorem 3.2.5 do*

- *Compute* $B|_X(i,k)$*, the fundamental simplex area ratio for the set of points* X*.*
- *Compute* $B|_{\widetilde{Z}}(i,k)$*, the fundamental area ratio for the set of points* $\widetilde{Z}_b$*.*
- *If* $B|_X(i,k) \neq B|_{\widetilde{Z}}(i,k)$ *then exit the loop and proceed to step 3.*

If $B|_X(i,k) = B|_{\widetilde{Z}}(i,k)$ *for every fundamental area ratio then* $OUTPUT$: YES *and exit the procedure.*

3. *Define the ordered set* $\widetilde{Z}_b = (\widetilde{z}^1, \ldots, \widetilde{z}^r)$ *of* r *points in* $\mathbb{R}^2$ *to have coordinates* $\widetilde{z}^i = (z_2^i + b\, z_2^i, z_3^i)$.

4. *Compute* b *using equation (5.17).*

5. *For every fundamental simplex area ratio,* $B(i,k)$ *given by Theorem 3.2.5 do*

   - *Compute* $B|_{\widetilde{Z}_b}(i,k)$*, the fundamental simplex area ratio for the set of points* $\widetilde{Z}$*.*
   - *If* $B|_X(i,k) \neq B|_{\widetilde{Z}_b}(i,k)$ *then exit the loop and proceed to step 6.*

   If $B|_X(i,k) = B|_{\widetilde{Z}_b}(i,k)$ *for every fundamental simplex area ratio then* $OUTPUT$: YES *and exit the procedure.*

6. *Define the ordered set* $\widetilde{Z}_{c,f} = (\widetilde{z}^1, \ldots, \widetilde{z}^r)$ *of* r *points in* $\mathbb{R}^2$ *to have coordinates* $\widetilde{z}^i = (z_1^i + c\, z_3^i, z_2^i + f\, z_3^i,)$.

7. *Compute* c *and* f *using equation (5.22).*

8. *For every fundamental simplex area ratio,* $B(i,k)$ *given by Theorem 3.2.5 do*

- *Compute* $B|_{\widetilde{Z}_{c,f}}(i,k)$*, the fundamental simplex area ratio for the set of points* $\widetilde{Z}_{c,f}$*.*
- *If* $B|_X(i,k) \neq B|_{\widetilde{Z}_{c,f}}(i,k)$ *then* $OUTPUT$: NO *and exit the procedure.*

If $B|_X(i,k) = B|_{\widetilde{Z}_{c,f}}(i,k)$ *for every fundamental simplex area ratio then* $OUTPUT$: YES *and exit the procedure. Else,* $OUTPUT$: NO *and exit the procedure.*

This algorithm uses equations (5.17) and (5.22) to find b, c, $f \in \mathbb{R}$. Once found, the three ordered sets of points from Theorem 5.2.5 no longer depend on unknowns. We use Theorem 3.2.5 to determine if one of these 3 ordered sets of points is $\mathcal{A}(2)$-equivalent to X.

5.4 Examples

To illustrate how the algorithms in Section 5.3 are used we provide the following examples.

Example 5.4.1. *Given an* ordered *set of points in* $\mathbb{R}^3$*,* $Z = (\mathbf{z}^1, \ldots, \mathbf{z}^5)$*, where*

$$\mathbf{z}^1 = (1,2,-1),\ \mathbf{z}^2 = (0,2,0),\ \mathbf{z}^3 = (2,3,2),\ \mathbf{z}^4 = (3,4,0),\ \textit{and}\ \mathbf{z}_5 = (1,5,1),$$

and given an ordered *set of points in* $\mathbb{R}^2$*,* $X = (\mathbf{x}^1, \ldots, \mathbf{x}^5)$*, where*

$$\mathbf{x}^1 = (0,6),\ \mathbf{x}^2 = (0,4),\ \mathbf{x}^3 = (4,2),\ \mathbf{x}^4 = (3,8),\ \textit{and}\ \mathbf{x}^5 = (2,8),$$

does there exists an affine projection taking Z *to* X*?*

Following Algorithm 5.3.2 and using Theorem 3.2.5, we first make the following *calculations:*

$$
\begin{aligned}
B|_X(2,4) &= \frac{3}{4}, \\
B|_X(3,4) &= \frac{5}{2}, \\
B|_X(2,5) &= \frac{1}{2}, \\
B|_X(3,5) &= 2.
\end{aligned}
$$

Defining $\widetilde{Z}$ *as in step 1, we have* $\widetilde{z}^1 = (2,-1)$, $\widetilde{z}^2 = (2,0)$, $\widetilde{z}^3 = (3,2)$, $\widetilde{z}^4 = (4,0)$ *and* $\widetilde{z}_5 = (5,1)$. *Computing*

$$B|_{\widetilde{Z}}(2,4) = 2 \neq \frac{3}{4},$$

we know that $\widetilde{Z}$ *is not affinely equivalent to* X *and proceed to step 3. Define* $\widetilde{Z}_b$ *with coordinates* $\widetilde{z}^1 = (z_1^i + b\, z_2^i, z_3^i)$ *and use equation (5.17) to calculate* b. *Determine that* $b = 0$ *and* $\widetilde{z}^1 = (1,-1)$, $\widetilde{z}^2 = (0,0)$, $\widetilde{z}^3 = (2,2)$, $\widetilde{z}^4 = (3,0)$, *and* $\widetilde{z}_5 = (1,1)$. *Computing*

$$
\begin{aligned}
B|_{\widetilde{Z}_b}(2,4) &= \frac{3}{4}, \\
B|_{\widetilde{Z}_b}(3,4) &= \frac{5}{4} \neq \frac{5}{2},
\end{aligned}
$$

we know that $\widetilde{Z}_b$ *is not affinely equivalent to* X *and proceed to step 6. Define* $\widetilde{Z}_{c,f}$ *with coordinates* $\widetilde{z}^1 = (z_1^i + c\, z_3^i, z_2^i + f\, z_3^i)$ *and use equation (5.22) to calculate* c *and* f. *Determine that* $c = 1$, $f = -1$, *and* $\widetilde{z}^1 = (0,3)$, $\widetilde{z}^2 = (0,2)$, $\widetilde{z}^3 = (4,1)$, $\widetilde{z}^4 = (3,4)$, *and* $\widetilde{z}_5 = (2,4)$.

Computing

$$\begin{aligned}
B|_{\widetilde{Z}_{c,f}}(2,4) &= \frac{3}{4},\\
B|_{\widetilde{Z}_{c,f}}(3,4) &= \frac{5}{2},\\
B|_{\widetilde{Z}_{c,f}}(2,4) &= \frac{1}{2},\\
B|_{\widetilde{Z}_{c,f}}(3,4) &= 2,
\end{aligned}$$

we observe that $\widetilde{Z}_{c,f}$ *is affinely equivalent to* X*, and therefore, there exists an affine projection taking* Z *to* X*.*

Example 5.4.2. *Given an* ordered *set of points in* $\mathbb{R}^3$*,* $Z = (\mathbf{z}^1, \ldots, \mathbf{z}^5)$ *where*

$$z^1 = (1,2,-1),\ z^2 = (0,2,0),\ z^3 = (2,3,2),\ z^4 = (3,4,0),\ \textit{and } z_5 = (1,5,1),$$

and given an ordered *set of points in* $\mathbb{R}^2$*,* $X = (x^1, \ldots, x^5)$

$$\textit{where} \quad x^1 = (-3,0),\ x^2 = (2,2),\ x^3 = (6,1),\ x^4 = (4,1),\ \textit{and } x^5 = (3,2)$$

does there exists an affine projection taking Z *to* X*?*

Following Algorithm 5.3.2 and using Theorem 3.2.5, we first make the following *calculations:*

$$\begin{aligned} B|_X(2,4) &= \frac{9}{13}, \\ B|_X(3,4) &= -\frac{2}{13}, \\ B|_X(2,5) &= \frac{2}{13}, \\ B|_X(3,5) &= -\frac{12}{13}. \end{aligned}$$

Defining $\widetilde{Z}$ *as in step 1, we have* $\widetilde{z}^1 = (2,-1)$, $\widetilde{z}^2 = (2,0)$, $\widetilde{z}^3 = (3,2)$, $\widetilde{z}^4 = (4,0)$ *and* $\widetilde{z}_5 = (5,1)$. *Computing*

$$B|_{\widetilde{Z}}(2,4) = 2 \neq \frac{9}{13},$$

we know that $\widetilde{Z}$ *is not affinely equivalent to* X *and proceed to step 3. Define* $\widetilde{Z}_b$ *with coordinates* $\widetilde{z}^1 = (z_1^i + b\, z_2^i, z_3^i)$ *and use equation (5.17) to calculate* b. *Determine that* $b = -\frac{12}{23}$ *and* $\widetilde{z}^1 = (-\frac{1}{23}, -1)$, $\widetilde{z}^2 = (-\frac{24}{23}, 0)$, $\widetilde{z}^3 = (\frac{10}{23}, 2)$, $\widetilde{z}^4 = (\frac{21}{23}, 0)$, *and* $\widetilde{z}_5 = (-\frac{37}{23}, 1)$. *Computing*

$$B|_{\widetilde{Z}_b}(2,4) \;=\; \frac{1}{2} \neq \frac{9}{13},$$

we know that $\widetilde{Z}_b$ *is not affinely equivalent to* X *and proceed to step 6. Define* $\widetilde{Z}_{c,f}$ *with coordinates* $\widetilde{z}^1 = (z_1^i + c\, z_3^i, z_2^i + f\, z_3^i)$ *and use equation (5.22) to calculate* c *and* f. *Determine that* $c = -\frac{5}{7}$, $f = -\frac{23}{7}$, *and* $\widetilde{z}^1 = (\frac{12}{7}, \frac{37}{7})$, $\widetilde{z}^2 = (0,2)$, $\widetilde{z}^3 = (\frac{4}{7}, -\frac{25}{7})$, $\widetilde{z}^4 = (3,4)$, *and* $\widetilde{z}_5 = (\frac{2}{7}, \frac{12}{7})$. *Computing*

$$B|_{\widetilde{Z}_{c,f}}(2,4) \;=\; \frac{1}{2} \neq \frac{9}{13},$$

we observe that $\widetilde{Z}_{c,f}$ *is not affinely equivalent to* X*, and therefore, there DOES NOT exists an affine projection taking* Z *to* X*.*

Example 5.4.3. *Given an* ordered *set of points in* $\mathbb{R}^3$*,* $Z = (\mathbf{z}^1, \ldots, \mathbf{z}^6)$ *with coordinates*

$$\mathbf{z}^1 = (1, 2, -1),\ \mathbf{z}^2 = (0, 2, 0),\ \mathbf{z}^3 = (2, 3, 2),\ \mathbf{z}^4 = (3, 4, 0),\ \mathbf{z}^5 = (1, 5, 1),\ \textit{and}\ \mathbf{z}^6 = (1, 2, 3),$$

and given an ordered *set of points in* $\mathbb{R}^2$*,* $X = (\mathbf{x}^1, \ldots, \mathbf{x}^6)$ *with coordinates*

$$\mathbf{x}^1 = (2, 5)\,,\ \mathbf{x}^2 = \left(\frac{1}{2}, \frac{5}{2}\right),\ \mathbf{x}^3 = \left(1, \frac{3}{2}\right),\ \mathbf{x}^4 = \left(2, \frac{7}{2}\right),\ \mathbf{x}^5 = \left(\frac{2}{3}, \frac{8}{3}\right),\ \textit{and}\ \mathbf{x}^6 = \left(\frac{2}{5}, 1\right),$$

does there exists an finite projection taking Z *to* X*?*

Using Theorem 5.2.2 this is equivalent to deciding whether there exist $c_1, c_2, c_3 \in \mathbb{R}$ *and* $[A] \in \mathcal{PGL}(3)$ *such that*

$$[x^i, y^i, 1]^{tr} = [A]\,[z_1^i + c_1,\ z_2^i + c_2,\ z_3^i + c_3]^{tr}\ \textit{for}\ i = 1 \ldots 6. \tag{5.23}$$

Let $\widetilde{Z}_{c_1,c_2,c_3}$ *denote the* ordered *set of* 6 *points in* $\mathbb{R}^2$ *with coordinates* $\widetilde{\mathbf{z}}^i = (\frac{z_1^i+c_1}{z_3^i+c_3}, \frac{z_2^i+c_2}{z_3^i+c_3})$*. Using the fundamental set of invariants from Theorem 3.2.9 we pick the first* 3 *invariants to solve for* $c_1, c_2, c_3 \in \mathbb{R}$*.*

Equalities,

$$\begin{aligned} C|_X(1; 2, 3, 4, 5) &= C|_{\widetilde{Z}_{c_1,c_2,c_3}}(1; 2, 3, 4, 5), \\ C|_X(2; 1, 3, 4, 5) &= C|_{\widetilde{Z}_{c_1,c_2,c_3}}(2; 1, 3, 4, 5), \\ C|_X(1; 2, 3, 4, 6) &= C|_{\widetilde{Z}_{c_1,c_2,c_3}}(1; 2, 3, 4, 6), \end{aligned} \tag{5.24}$$

lead to equations on $c_1, c_2, c_3 \in \mathbb{R}$*;*

$$\frac{(8 - c_3 + 4\,c_2 - c_1)\,(-13 + c_1 + 6\,c_3 - 4\,c_2)}{(2\,c_3 + 2\,c_1 - 6 - 3\,c_2)\,(14 + 7\,c_1 - 3\,c_3 + 2\,c_2)} - \frac{22}{21} = 0, \tag{5.25}$$

$$\frac{(8 - c_3 + 4\,c_2 - c_1)\,(2\,c_1 + 7\,c_3 - 6 - 3\,c_2)}{5\,(2\,c_3 + 2\,c_1 - 6 - 3\,c_2)\,(-c_3 + c_1)} - \frac{11}{27} = 0, \tag{5.26}$$

$$\frac{2\,(-8 + c_3 - 4\,c_2 + c_1)}{(2\,c_3 + 2\,c_1 - 6 - 3\,c_2)} - \frac{11}{6} = 0, \tag{5.27}$$

whose solution is $c_1 = -\frac{549}{575}$*,* $c_2 = -\frac{224}{115}$*, and* $c_3 = \frac{504}{575}$*. We substitute these into* $\widetilde{Z}_{c_1,c_2,c_3}$ *to obtain the set of points*

$$\begin{aligned}\widetilde{\mathbf{z}}^1 &= \left(-\frac{26}{71}, -\frac{30}{71}\right), \widetilde{\mathbf{z}}^2 = \left(-\frac{61}{56}, \frac{5}{84}\right), \widetilde{\mathbf{z}}^3 = \left(\frac{601}{1654}, \frac{605}{1654}\right), \widetilde{\mathbf{z}}^4 = \left(\frac{7}{3}, \frac{295}{126}\right),\\ \widetilde{\mathbf{z}}^5 &= \left(\frac{2}{83}, \frac{135}{83}\right), \textit{ and } \widetilde{\mathbf{z}}^6 = \left(\frac{26}{2229}, \frac{10}{743}\right).\end{aligned}$$

By construction, we know that the first 3 *invariants are equal for* X *and* $\widetilde{Z}_{c_1,c_2,c_3}$*. All there is left to do is to decide if the remaining separating invariants are the same for* X *and* $\widetilde{Z}_{c_1,c_2,c_3}$*. This is not the case;*

$$C|_X(2;1,3,4,6) = \frac{273}{153} \neq \frac{731}{171} = C|_{\widetilde{Z}_{c_1,c_2,c_3}}(2;1,3,4,6). \tag{5.28}$$

By Theorem 3.2.9, X *is not equivalent to* $\widetilde{Z}_{c_1,c_2,c_3}$ *under the action of* $\mathcal{PGL}(3)$*, and by Theorem 5.2.2,* Z *DOES NOT project onto* X *with a finite projection.*

5.5 Comparison with previous works

In [2, 1] the authors present a solution to this problem for the class of affine projections, without finding a projection explicitly. These papers refer to affine projections as *Generalized Weak Perspective Projections*. They identify the sets of points, $Z = (\mathbf{z}^1, \dots, \mathbf{z}^r)$, in $\mathbb{R}^3$ and sets of points, $X = (\mathbf{x}^1, \dots, \mathbf{x}^r)$, in $\mathbb{R}^2$ as elements of certain Grassmanian spaces. They use the Plüker embedding to embed Grassmanians into projective spaces, and to explicitly define the algebraic variety that characterizes object-image pairs that can be related by an affine projection. We review the solution given in [2, 1].

Given an *ordered* set $Z = (\mathbf{z}^1, \dots, \mathbf{z}^r)$ of r points in $\mathbb{R}^3$ and an *ordered* set $X = (\mathbf{x}^1, \dots, \mathbf{x}^r)$ of r points in $\mathbb{R}^2$, define

$$M = \begin{pmatrix} z_1^1 & z_1^2 & \dots & z_1^r \\ z_2^1 & z_2^2 & \dots & z_2^r \\ z_3^1 & z_3^2 & \dots & z_3^r \\ 1 & 1 & \dots & 1 \end{pmatrix}, \; N = \begin{pmatrix} x_1^1 & x_1^2 & \dots & x_1^r \\ x_2^1 & x_2^2 & \dots & x_2^r \\ 1 & 1 & \dots & 1 \end{pmatrix}. \tag{5.29}$$

The following proposition shows that the kernels of the above matrices represent the sets Z and X up to affine transformations.

Proposition 5.5.1. *If M_1 and M_2 are two $(n+1) \times r$ matrices with the last row all ones, the* $\ker(M_1) = \ker(M_2)$ *if and only if $M_1 = A\,M_2$ for some $A \in \mathcal{A}(n)$.*

Proof. Clearly, if $M_1 = A\,M_2$ then $\ker(M_1) = \ker(M_2)$. If $\ker(M_1) = \ker(M_2)$, then the row space of M_1 is also the row space of M_2 because the kernel is the orthogonal compliment to the row space. Therefore, the rows of M_2 can be written as linear combinations of rows of M_1, and hence, $M_1 = A\,M_2$ for some $A \in \mathcal{A}(n)$. ∎

From Proposition 5.5.1, the affine equivalence classes of ordered sets of points Z and X can be uniquely represented by the kernels of the matrices they create. The kernels are subspaces of $\mathbb{R}^r$, and can be identified with points in Grassmanian spaces. Since the last row of both M and N consists of all ones, the

$$\ker(M), \ker(N) \subset H^{r-1} := \{(a_1, \ldots, a_r) \in \mathbb{R}^r | \sum_{i=1}^{r} a_i = 0\}. \tag{5.30}$$

H^{r-1} is an $(r-1)$-dimensional space containing both $\ker(M)$ and $\ker(N)$. Assume non-collinearity for the points in $\mathbb{R}^2$ and non-coplanarity for the points in $\mathbb{R}^3$. By this assumption, M and N are full rank. By identifying a basis for H^{r-1}, one can view $\ker(M)$ and $\ker(N)$ as points of $r-4$ and $r-3$ Grassmannians of H^{r-1}, respectively;

$$\ker(M) \in Grass(r-4, H^{r-1}) \textit{ and } \ker(N) \in Grass(r-3, H^{r-1}).$$

As discussed in Section 1.1 the *Plücker embedding* maps elements of a *Grassmannian* to *projective space*. The image of the Plücker embedding is a projective variety. The vanishing ideal is generated by a system of quadratic polynomials known as the Plücker relation.

Proposition 5.5.2. *The ordered set of points, Z, in $\mathbb{R}^3$ projects to the ordered set of points, X, in $\mathbb{R}^2$ with an affine projection if and only if*

$$\ker(M) \subset \ker(N) \subset H^{r-1} \subset \mathbb{R}^r. \tag{5.31}$$

Using this proposition and Plücker relations discussed in Section 1.1, the following projection criteria can be derived.

Theorem 5.5.3. *For Z, X, M and N as above. For $1 \leq i_1 < i_2 < i_3 < i_4 \leq r$ and $1 \leq j_1 < j_2 < j_3 \leq r$ define*

$$m_{i_1,i_2,i_3,i_4} = det \begin{pmatrix} z_1^{i_1} & z_1^{i_2} & z_1^{i_3} & z_1^{i_4} \\ z_2^{i_1} & z_2^{i_2} & z_2^{i_3} & z_2^{i_4} \\ z_3^{i_1} & z_3^{i_2} & z_3^{i_3} & z_3^{i_4} \\ 1 & 1 & 1 & 1 \end{pmatrix} \text{ and } n_{j_1,j_2,j_3} = det \begin{pmatrix} x_1^{j_1} & x_1^{j_2} & x_1^{j_3} \\ x_2^{j_1} & x_2^{j_2} & x_2^{j_3} \\ 1 & 1 & 1 \end{pmatrix}.$$

Then X is an ordered image of Z under an affine projection if and only if

$$\sum_{1 \leq \lambda_1 < \lambda_2 \leq r} \epsilon_{\lambda_1,\lambda_2} m_{\alpha_1,\alpha_2,\lambda_1,\lambda_2} n_{\gamma_1\gamma_2\gamma_3} = 0 \tag{5.32}$$

for all choices of $1 \leq \alpha_1 < \alpha_2 \leq r$ and all choices of $1 \leq \beta_1 < \beta_2 < \ldots < \beta_{r-5} \leq r$ where $1 \leq \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq r$, $\{\lambda_1, \lambda_2, \beta_1, \ldots, \beta_{r-5}, \gamma_1, \gamma_2, \gamma_3\} = \{1, \ldots, r\}$ and $\epsilon_{\lambda_1,\lambda_2}$ is the sign of the permutation of $(\gamma_1, \gamma_2, \gamma_3, \lambda_1, \lambda_2, \beta_1, \ldots, \beta_{r-5})$.

In the case of 5 points, the projection criteria from Theorem 5.5.3 is given by the following system of 10 equations.

Example 5.5.4. *Given an* ordered *set $Z = (\mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^3, \mathbf{z}^4, \mathbf{z}^5)$ of points in $\mathbb{R}^3$ and an* ordered *set $X = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4, \mathbf{x}^5)$ of points in $\mathbb{R}^2$ an affine projection takes Z to X if and only if*

$$\begin{aligned}
m_{1,2,3,4}\, n_{1,2,5} - m_{1,2,3,5}\, n_{1,2,4} + m_{1,2,4,5}\, n_{1,2,3} &= 0, \\
m_{1,2,3,4}\, n_{1,3,5} - m_{1,2,3,5}\, n_{1,3,4} + m_{1,3,4,5}\, n_{1,2,3} &= 0, \\
m_{1,2,3,4}\, n_{1,4,5} - m_{1,2,4,5}\, n_{1,3,4} + m_{1,3,4,5}\, n_{1,2,4} &= 0, \\
m_{1,2,3,5}\, n_{1,4,5} - m_{1,2,4,5}\, n_{1,3,5} + m_{1,3,4,5}\, n_{1,2,5} &= 0,
\end{aligned}$$

$$
\begin{aligned}
m_{1,2,3,4}\, n_{2,3,5} - m_{1,2,3,5}\, n_{2,3,4} + m_{2,3,4,5}\, n_{1,2,3} &= 0, \\
m_{1,2,3,4}\, n_{2,4,5} - m_{1,2,4,5}\, n_{2,3,4} + m_{2,3,4,5}\, n_{1,2,4} &= 0, \\
m_{1,2,3,5}\, n_{2,4,5} - m_{1,2,4,5}\, n_{2,3,5} + m_{2,3,4,5}\, n_{1,2,5} &= 0, \\
m_{1,2,3,4}\, n_{3,4,5} - m_{1,3,4,5}\, n_{2,3,4} + m_{2,3,4,5}\, n_{1,3,4} &= 0, \\
m_{1,2,3,5}\, n_{3,4,5} - m_{1,3,4,5}\, n_{2,3,5} + m_{2,3,4,5}\, n_{1,3,5} &= 0, \\
m_{1,2,4,5}\, n_{3,4,5} - m_{1,3,4,5}\, n_{2,4,5} + m_{2,3,4,5}\, n_{1,4,5} &= 0. \qquad (5.33)
\end{aligned}
$$

For r points system 5.32 consists of $\binom{r}{r-2}\binom{r-2}{r-5} = \frac{r\,(r-1)\ldots(r-4)}{12}$ equations. In comparison, from Theorem 5.2.5 of this thesis only $r\,(r-3)$ equations must be satisfied to establish existence of an affine projection between sets Z and X. For large r the difference is significant (see Table 5.5), and thus the algorithm based on Theorem 5.2.5 may have a practical advantage.

Table 5.1: Comparison of Methods

Number of Equations to Check		
Number of Points	Current Approach	Arnold and Stiller [1]
5	4	10
10	14	2520
100	194	752875200
r	$2\,(r-3)$	$\frac{r\,(r-1)\ldots(r-4)}{12}$

One of the essential contributions of [2, 1] is the definition of an object/image distance between ordered sets of r points in $\mathbb{R}^3$ and $\mathbb{R}^2$ such that the distance is zero if and only if these sets are related by a projection. Obtaining an object-image metric based on the criteria developed in this thesis is one of our future goals.

Chapter 6

Conclusions and Future Work

The projection criteria developed in Chapter 4 for curves and Chapter 5 for points reduce the object-image correspondence problems to variations of group-equivalence problems in $\mathbb{R}^2$. For curves, we use differential signature construction [5] to address the group-equivalence problem. In practical applications, curves are often represented by samples of points. In this case, invariant numerical approximations of differential invariants presented in [5, 3] may be used to obtain signatures. Differential invariants and their approximations are highly sensitive to image perturbations and, therefore, are not practical in many situations. Other types of invariants, such as semi-differential (or joint) invariants [29, 22], integral invariants [27, 13, 11] and moment invariants [15] are less sensitive to image perturbations and may be employed to solve the group-equivalence problem. One future project is to develop variations of Algorithm 4.3.1 and its affine counterpart that are based on alternative solutions of the group-equivalence problem.

For finite ordered sets of points, we use the separating invariants discussed in [22] and Section 3.2 to address the group-equivalence problem on the plane. Our projection criteria do not account for image perturbations or inexact positions of points. In practice we are only given an approximate position of points. A "good" object/image distance provides a tool for deciding whether a given set of points in $\mathbb{R}^2$ is a good approximation of some projection of a given set of points in $\mathbb{R}^3$. In [2, 1], such a distance is given for affine projections of finite ordered sets of points. Defining such object/image distance based on the projection criteria given in this thesis for curves and points under finite and affine projections is an important direction of further research. As pointed out in [2], determining what locally continuous surface could have been projected to a given set of points on the plane also remains open.

Although the projection algorithms presented here may not be immediately applicable to real-life images, we consider this work to be a first step toward the development of more efficient algorithms to determine projection correspondence for curves and other continuous objects. An algorithmic solution to this problem, for classes of projections with large degrees of freedom, does not seem to appear in the literature. We also consider this thesis to be a good addition to the the work done in [2, 1] on projection correspondence of finite ordered sets of points.

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Appendices

APPENDIX A

The missing signature in Example 4.4.3.

$$
\begin{aligned}
&- 5030912\, i_1^7 + (-5940251520\, i_2 + 151925420675) i_1^6 + (30636630990000 + 1861338396750\, i_2 \\
&- 10638397440\, i_2^3 - 38468841600\, i_2^2) i_1^5 + (422436606250\, i_2^3 - 14727073554375\, i_2^2 \\
&- 216957629137500\, i_2 - 217263340800\, i_2^4 + 1112560285921875) i_1^4 + (17624874881250000 \\
&- 39010642600000\, i_2^3 + 1187479883250000\, i_2^2 + 40479436800\, i_2^6 + 1373185440000\, i_2^5 \\
&- 8403282904218750\, i_2 - 3436007880000\, i_2^4) i_1^3 + (34434326738671875\, i_2^2 \\
&- 108767888437500000\, i_2 - 5868698027343750\, i_2^3 + 143469615761718750 + 562755420187500\, i_2^4 \\
&+ 791528760000\, i_2^6 - 303595776000\, i_2^7 - 26046416850000\, i_2^5) i_1^2 + (-14048100000000\, i_2^7 \\
&- 85526942238281250\, i_2^3 + 16256399934375000\, i_2^4 + 192765985000000\, i_2^6 - 2087189006250000\, i_2^5 \\
&- 610997087988281250\, i_2 + 592675576171875000 + 758989440000\, i_2^8 + 294121178789062500\, i_2^2) i_1 \\
&- 330778550390625000\, i_2^3 + 990472686767578125 + 817322075683593750\, i_2^2 \\
&+ 2711566100000000\, i_2^6 - 632491200000\, i_2^9 + 18242928000000\, i_2^8 - 18680146558593750\, i_2^5 \\
&- 1277944475097656250\, i_2 + 92753599658203125\, i_2^4 - 276624382500000\, i_2^7 \\
&= 0 \qquad (1)
\end{aligned}
$$

The affine signature of γ_2 in Remark 4.4.6.

$$
\begin{aligned}
&- 13824\, i_1^5 + (112320\, i_2 - 582525)\, i_1^4 + (-8910000 + 2560\, i_2^3 - 360000\, i_2^2 + 3510000\, i_2)\, i_1^3 \\
&- 1200\,(16\, i_2^2 - 465\, i_2 + 2025)\,(i_2 - 5)^2\, i_1^2 + 24000\,(2\, i_2 - 15)\,(i_2 - 5)^4\, i_1 - 40000\,(i_2 - 5)^6 \\
&= 0 \qquad (2)
\end{aligned}
$$

APPENDIX B

The functions in the numerators and denominators of the invariants in Example 3.1.12.

$$
\begin{aligned}
N_1(t) = {} & 400\,(2\,(b_3\,a_2 - b_2\,a_3)\,t + b_3\,a_1 - b_1\,a_3)^2\,((b_3\,a_2 - b_2\,a_3)^2\,t^2 + ((b_3\,a_2 - b_2\,a_3)(b_3\,a_1 - b_1\,a_3))\,t \\
& - 2\,(b_3\,a_1 - a_3\,b_1)^2 + 3\,(b_3\,a_2 - a_3\,b_2)\,(a_1\,b_2 - a_2\,b_1) \\
N_2(t) = {} & 280\,(b_3\,a_2 - b_2\,a_3)^4\,t^4 + 560(b_3\,a_2 - b_2\,a_3)^3\,(b_3\,a_1 - b_1\,a_3)\,t^3 \\
& - 140\,(b_3\,a_2 - b_2\,a_3)(8\,(a_3\,b_2 - b_3\,a_2)\,(-b_1\,a_2 + b_2\,a_1) + 3\,(-a_3\,b_1 + b_3\,a_1)^2)\,t^2 \\
& + 140\,(b_2\,a_3 - b_3\,a_2)(b_3\,a_1 - b_1\,a_3)\,(5\,(-a_3\,b_1 + b_3\,a_1)^2 + 8\,(a_3\,b_2 - b_3\,a_2)\,(-b_1\,a_2 + b_2\,a_1))\,t \\
& 35\,(-(b_3\,a_1 - a_3\,b_1)^4 - 4\,(b_3\,a_2 - a_3\,b_2)\,(a_1\,b_2 - a_2\,b_1)\,((b_3\,a_1 - a_3\,b_1)^2 - 2\,(b_3\,a_2 - a_3\,b_2)\,(a_1\,b_2 - a_2\,b_1))) \\
D(t) = {} & 4\,(b_3\,a_2 - b_2\,a_3)^2\,t^2 + 4\,(b_3\,a_2 - b_2\,a_3)(b_3\,a_1 - b_1\,a_3)\,t - 5\,(-a_3\,b_1 + b_3\,a_1)^2 + 8\,(a_3\,b_2 - b_3\,a_2)\,(b_1\,a_2 - b_2\,a_1)
\end{aligned}
\tag{3}
$$