ABSTRACT

GAFFNEY, JONATHAN DAVID HOUSLEY. Possibilities: A Framework for Modeling Students' Deductive Reasoning in Physics. (Under the direction of Ruth Chabay.)

Students often make errors when trying to solve qualitative or conceptual physics problems, and while many successful instructional interventions have been generated to prevent such errors, the process of deduction that students use when solving physics problems has not been thoroughly studied. In an effort to better understand that reasoning process, I have developed a new framework, which is based on the mental models framework in psychology championed by P. N. Johnson-Laird. My new framework models how students search possibility space when thinking about conceptual physics problems and suggests that errors arise from failing to flesh out all possibilities. It further suggests that instructional interventions should focus on making apparent those possibilities, as well as all physical consequences those possibilities would incur.

The *possibilities framework* emerged from the analysis of data from a unique research project specifically invented for the purpose of understanding how students use deductive reasoning. In the *selection task*, participants were given a physics problem along with three written possible solutions with the goal of identifying which one of the three possible solutions was correct. Each participant was also asked to identify the errors in the incorrect solutions. For the study presented in this dissertation, participants not only performed the selection task individually on four problems, but they were also placed into groups of two or three and asked to discuss with each other the reasoning they used in making their choices and attempt to reach a consensus about which solution was correct. Finally, those groups were asked to work together to perform the selection task on three new problems.

The possibilities framework appropriately models the reasoning that students use, and it makes useful predictions about potentially helpful instructional interventions. The study reported in this dissertation emphasizes the useful insight the possibilities framework provides. For example, this framework allows us to detect subtle differences in students' reasoning errors, even when those errors result in the same final answer. It also illuminates how simply mentioning overlooked quantities can instigate new lines of student reasoning. It allows us to better understand how well-known psychological biases, such as the belief bias, affect the reasoning process by preventing reasoners

from fleshing out all of the possibilities. The possibilities framework also allows us to track student discussions about physics, revealing the need for all parties in communication to use the same set of possibilities in the conversations to facilitate successful understanding. The framework also suggests some of the influences that affect how reasoners choose between possible solutions to a given problem.

This new framework for understanding how students reason when solving conceptual physics problems opens the door to a significant field of research. The framework itself needs to be further tested and developed, but it provides substantial suggestions for instructional interventions. If we hope to improve student reasoning in physics, the possibilities framework suggests that we are perhaps best served by teaching students how to fully flesh out the possibilities in every situation. This implies that we need to ensure students have a deep understanding of all of the implied possibilities afforded by the fundamental principles that are the cornerstones of the models we teach in physics classes.

Possibilities: A Framework for Modeling Students' Deductive Reasoning in Physics

by
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DEDICATION

To my students, who teach me how to teach.

BIOGRAPHY

Jon Gaffney was born in 1980 in the outer suburban sprawl of Pittsburgh, Pennsylvania. Raised to push the limits of his mathematical and scientific abilities, he grew up expecting to become a scientist. Upon graduation from high school in 1997, he matriculated to Bethany College in the northern panhandle of West Virginia, where he developed a desire to understand the human condition. After working for a year in a civil engineering firm, he decided to attend the University of Pittsburgh in August 2002 to pursue an advanced degree in physics. After a short time as a research assistant in high energy particle physics, he discovered Physics Education Research.

At Pitt, Dr. Singh provided guidance and some small PER projects that familiarized Jon with the field. Wanting more teaching experience, he began teaching whatever sections of physics he could get ahold of at Pitt, and he also gained employment at Duquesne University as an adjunct instructor. However, Jon soon realized that obtaining a doctorate from Pitt with a specialization in PER would not be possible, and he transferred to NC State to complete his degree under the advisement of Dr. Chabay. While at NC State, he obtained training with the SCALE-UP pedagogy under Dr. Beichner and even traveled to other colleges as a consultant to help them create their own SCALE-UP classes. He was awarded a STEM Education Fellowship, which allowed him to focus on a number of projects including this dissertation and an exploration of student expectations and experiences in SCALE-UP classes.

The project in this dissertation blends Jon's desire to understand how people think and behave with his love of physics and mathematics. Finding this project, and making it all come together in a sensible way, was a long and often painful process. However, Jon believes it was worthwhile and hopes that you, the reader, will think so as well.

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With a grateful heart, I wish to acknowledge those whose contributions to my life have led me here. My committee at NC State has been very supportive of this project, even if no one was quite sure where it would lead. Dr. Beichner has been an exceptional teaching mentor, while Dr. Chabay has provided much-needed direction and advice as we honed in on the meaning in the data. Dr. Risley and Dr. Sherwood have graciously contributed their time to ensure that this project was as good as it could be. The entire Physics Education Research group at NC State has been a positive and energetic community and a pure joy to work with. I will miss the camaraderie and support that my graduate student colleagues and I shared. Evan Richards has been at my side virtually throughout my doctoral work, and he has been a source of support, insight, and inspiration to me. Shawn Weatherford has been a rock-solid teaching partner and an excellent colleague to volley ideas with. Mary Bridget Kustusch and Brandon Lunk were both very patient assistants during the long and arduous process of nailing down coding schemes and establishing inter-rater reliability. Jeff Polak and Meghan West, as well as other more transient members of the group, have also helped in many ways both visible and invisible on various projects, and I'm certainly appreciative for all of their help. Finally, Jenny Allen has been an incredible administrative assistant for the group, going far beyond her call of duty to help me with every little detail of my day-to-day relationship with the University, and I would certainly have endured much more hardship without her.

I am thankful for the education I received at the University of Pittsburgh, where I learned more than I ever thought possible. Specifically, in the physics department, Dr. Duncan, Dr. Boyanovsky, Dr. Shepard, and Dr. Singh have each taken time to help me learn. In addition to the physics faculty, I would be remiss in omitting the science education department, and Dr. Cartier in particular, who pushed my limits and helped me understand the social science involved in a true study of student learning. My graduate colleagues at Pitt had a role in my success, and I'd especially like to thank Mark Hartz, Benjamin Brown, Andrew Mason, and Brian Cherinka for commiserating with me both during my residence there and afterwards. I would also like to thank Dr. Davies at Duquesne University for taking a chance and hiring me, and the rest of the faculty and staff (especially Dr. Hilger) for being supportive, giving me freedom when I wanted it and guidance when I needed it.

My college education was full of shifting paradigms and awakenings, mostly thanks to a devoted

faculty. I would like to thank Dr. Clothier and Dr. Sawtarie for providing my introductory physics education, Dr. Meyers and Dr. Becker for teaching me philosophy and how to ask questions, and the late and dearly missed "Doc" Allison for being an exemplary professor and mentor not only in mathematics but also in life.

Finally, I am grateful for my loving parents, who would have been proud of me no matter what I chose to do, and for my brother Paul who always helps me see the world in new and challenging ways. I thank all of my friends, especially Gregory Smith, his wife Jessica, and their daughter (and my goddaughter) Carolyn, who are essentially part of my own family. Most of all, I wish to thank my wife Amy, who has persevered through all eight long years of my time in graduate school and beyond. Without her selfless love and devotion, I never would have made it through. She even proofread this whole dissertation!

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Deduction in Physics

1.1 Introduction

Deduction, the form of reasoning where the conclusion is guaranteed to be true if the premises are true, is a very important tool for the enterprise of physics. It is generally used in situations where one wishes to apply a model of a system to make predictions about a particular situation, such as predicting the location of an object some time in the future given its initial position and momentum and the net impulse acting on it. In physics instruction, therefore, deduction appears in problem-solving and conceptual reasoning, where the answers to that problem are precisely defined from fundamental physics principles and given information.

Unfortunately, students repeatedly commit errors while solving physics problems that require deductive reasoning. Many struggle with knowing when deduction is appropriate or when it can be trusted, and some students resort to attempting to determine whether some reasoning "makes sense" to guide them in their physical understanding. This vague and often incorrect understanding about reasoning generally leads to confusion. For instance, a participant (who was given the pseudonym "Bob") in a pilot study said the following when he was trying to decide whether a particular solution to a physics problem was correct:

> I don't know, this – this one makes sense. Now whether it's right or not, that's, I mean,
> stuff can make sense and just be totally wrong, which is kind of one of the things about

physics that kind of blows your mind.

Meanwhile, Ana and Sally, in a discussion during the full "selection task" research study discussed at length in this dissertation, made this observation:

Ana: When I first saw this problem, I was like oh yeah, it's like, the answer is something really weird, it's not like what you would think it would be, because I know I got it wrong.

Sally: Yeah, I think a lot of physics problems in general are kind of like that.

For Bob, as well as for Ana and Sally, solving problems in physics is something mysterious. Like most students, they do not use formal logic to help them solve such problems in spite of instructional efforts that promote logical reasoning. Instead, in this dissertation I will propose that one way to understand their reasoning is by modeling how they are considering possibilities. Then, by investigating how reasoners flesh out and eliminate possibilities, we can gain a different perspective on how they perform their deductive reasoning, which will provide insight as to the errors they make and suggest potential interventions to correct such errors.

1.1.1 Distinction between logic and deduction

One major source of confusion for the reader could be that colloquially the terms "deduction" and "formal logic" are used interchangeably. However, the words have specific meanings that are very precise throughout this dissertation. The terms "deduction" and "deductive reasoning" will refer to any reasoning that uses the given information to generate a conclusion that is guaranteed to be true if the premises are true. Specifically, students will be assumed to use deductive reasoning unless they provide reason to believe otherwise (say, by appealing to memory, a "gut feeling," or analogy). In those non-deductive cases, I will generally mention that the reasoning is non-deductive and move on.

Formal logic is a specific approach to solving deductive problems, by analyzing the *form* of the premises, rather than their content. Logic is a powerful tool for an expert trained in using it, as it can reduce confusion and errors by focusing the reasoner simply on the structure of the premises that are used. However, as we will see throughout this dissertation, students in introductory physics classes

very seldom use formal logic when solving physics problems, although they are often attempting to generate a conclusion that is guaranteed to be true if the premises are true. For this reason, I may state that a particular reasoner is using deduction but not formal logic. The implication from this observation is that if we are to learn how to modify student reasoning, we should first understand *how* students attempt to deduce.

For further information regarding deduction, see Section 2.2; for more about logic, see Section 2.2.3. There are two main theories that address how people reason deductively (see Section 2.6), and I will base my framework for understanding student reasoning in physics on the mental models framework proposed by P. N. Johnson-Laird. Formal logic can be represented in my framework, as seen in Section 3.3.

1.1.2 Understanding the problem

At this point, it may not be obvious where the problem lies. We have considerable evidence that students often make errors when trying to solve physics problems that require deduction. Because much instruction revolves around using logically connected premises in the form of physics premises and definitions with given information, one may conjecture that students fail to solve deductive problems because they are either incapable of using logic in general, or simply that they do not know how to apply logical reasoning to physics problems. However, this conclusion suggests two things: that logic is the only valid method for solving physics problems and that students are somehow deficient in their ability to reason, because they are unable to use logic flawlessly when solving physics problems.

However, there are grounds for doubting that errors when solving deductive physics problems stem from an inability to use logic. For example, consider Charlie, who was a participant in the first pilot study in this dissertation. He correctly completed a non-physical logic problem (although it is not clear that he used logic to do so), indicating that he has the requisite deductive reasoning skills to solve physics problems. However, when asked to determine the direction of the net force acting on Tarzan at the bottom of his swing, he generates this explanation:

> Acceleration, as I found out in the previous problem, is toward the center (he indicates
> the direction up). But the force being felt on Tarzan, being the place he's at, is going to

push out as the centripetal force, so pushing in this direction (he draws an arrow down).

Charlie has proven that can reason deductively, and yet he makes an error here by allowing the direction of the net force to be in a direction other than the direction of the acceleration. Why is this? It is fruitless to debate whether or not he *could* apply formal logic to situations like these, as we can simply conclude that Charlie did not do so. But, rather than think about what Charlie did not do, we are more interested in what he actually did. One way of thinking about his reasoning is that he has not fully considered the implications for choosing the direction of the force to be down. That is, he has not realized that it is impossible for the direction of the net force to not be in the same direction as the direction of the acceleration.

As Charlie demonstrates, one explanation for errors in reasoning that does not resort to assessing whether or not reasoners can (or do) use formal logic is that errors may arise from failing to flesh out all relevant possibilities for the problem under consideration. Indeed, to consider all possibilities requires a significant amount of effort: the fundamental principles and definitions that are used in solving such problems allow numerous possibilities, and they are often used in long chains that add to the total number of possibilities that need to be considered for the problem. Thus, to flesh out all of the possibilities for a given problem may prove to be taxing to one's working memory, leading to the willful or accidental neglect of information through the use of heuristics or simplifications.

The question should therefore not be, "why do students not use formal logic when solving physics problems?" but rather, "what is the reasoning that the students are *actually* using?" Because of this focus, I will not attempt to represent student reasoning in terms of formal logic except to demonstrate that it is insufficient for creating an effective model for deduction.

1.1.3 Implications for physics instruction

Clearly, because Charlie is capable of deduction but his reasoning is error-prone, we would like to generate instructional interventions that would allow him to recognize and correct such errors as they occur, or even to prevent making them in the first place. Efforts to teach him logic would be ill-advised, as it is clear that he is not using that method of reasoning when solving this problem. When designing an intervention, it is crucial to understand what the students are actually doing.

Indeed, if we can in fact model students' deductive reasoning by focusing on the possibilities they are (and are not) considering, this implies that we need to train students how to use this form of reasoning more effectively. Such training may involve instruction regarding the nature of deduction in terms of possibilities, specifically that, unlike most thinking that is done in everyday life, all possibilities need to be considered because precision is of utmost importance in physics. It may also involve teaching students how to flesh out those possibilities, considering "impossible" values or combinations of values for quantities that are to obey a relationship that is very precise.

In whatever shape such interventions occur, it is important to note that they are specifically not intended to change the fundamental method of reasoning used by students: we must understand how they are reasoning and teach them how to reason correctly within that framework. This perspective, of course, carries a very strong assumption, which is this: while many students may not be able to use logic, all students who would enroll in an introductory level physics course at the collegiate level are capable of performing deduction. In addition to data that shows such reasoning (for example, some pilot-study participants like Charlie solved complicated logic problems even though they didn't necessarily use logic to do so), there is plenty of evidence in the literature, some of which suggests that we only need to learn how to "tap in" to students' ability to reason for it to become apparent (see, for example Lawson, 1993).

All of this is to say that it is important to understand how students attempt to perform deduction when solving physics problems, when they make such attempts, because by doing so we will have motivation for creating instructional interventions to improve those attempts, without engaging in the overwhelmingly difficult challenge of making students use logical reasoning.

1.2 Research Questions

Therefore, the question is: "How do students reason about physics problems that require deduction?"

Specifically, can an alternative framework be more effective than formal logical reasoning at representing how students solve deductive physics problems? How can we represent students' attempts to reason with this framework, and what can doing so tell us about how they do deduction? How do students decide between possible solutions to a physics problem?

To answer these questions, I have created the possibilities framework, which draws heavily from the mental models framework developed by Johnson-Laird (2001) to describe how people perform deductive reasoning. This framework casts student deduction into a search within possibility space rather than an attempt to use the form or structure of physics principles and given information. As such, the context and content within the information they are given play a role in their reasoning, as do other non-deductive sources of information such as intuition and real-world experience. Because students take those things into consideration while reasoning, a framework that seeks to understand student reasoning must do so as well.

Throughout this dissertation, I will provide a variety of evidence that indicates that this framework can indeed be used to represent and model student reasoning better than formal logic can. Thinking about student deduction in terms of possibilities allows us to cast student errors in terms of failing to identify available possibilities and failing to eliminate impossibilities that result. I propose that errors often occur because reasoners inadvertently simplify the problem by neglecting relevant physical quantities and possible values for those quantities. We can also identify the role that biases like the belief bias, which prevents reasoners from fully investigating the search space, play in the reasoning process. It is especially important to not hastily assume that student errors in deduction are identical from one student to another, even when such errors result in the same erroneous answer to a problem; by using a fine grain size in exploring student reasoning, the possibilities framework allows us to identify subtle differences that may otherwise be overlooked. Using this framework, we will view student reasoning as deep attempts at sense-making rather than doomed exercises in failing to use logical tools.

1.3 Physics Terminology

In this dissertation, I will consistently use some terms to refer to specific concepts in physics, many of which resulted from the textbook *Matter and Interactions* (Chabay & Sherwood, 2007b). Because these terms may not be familiar to all readers, I present a brief discussion of some of them and their definitions throughout this dissertation.

In physics problems, the **system** is the object (or group of objects) that is under consideration.

One may be interested in, for example, the position of a baseball some time after it has been hit into the air by a batter. In this case, the baseball can be chosen to be the system. The **surroundings** are all of the other objects in the universe that influence the system. Most objects in the universe usually have little or no effect and can be ignored, such as the effect of Neptune on the baseball in the air. However, Earth, which exerts a gravitational force on the ball, and the air, which exerts a force on the ball as well, are in the surroundings and cannot be neglected.

The **initial state** of a system is the set of properties for that system at some point in time before any influence occurs. For example, the initial state of a baseball may be that it has a mass of 145 grams, a speed of zero meters per second, a roughly spherical shape, a temperature of twenty Kelvin, and so forth. The **final state** of a system is the set of properties for that same system at some later point in time, after the surroundings have acted on the system. For example, the final state of the baseball may be that it has a mass of 145 grams but a velocity of $< 0, 10, 3 >$ meters per second, and so forth. The choices one makes for the system, surroundings, initial state, and final state for a problem are arbitrary, in that there is no such thing as a physically invalid choice. However, some choices are better than others in that some may allow one to calculate a final answer for a given problem while others may not.

In *Matter and Interactions*, much is made of three fundamental principles in modern mechanics. These three principles are fundamental because they hold true in all situations. They are listed below.

1. The **momentum principle**, or Newton's second law, is:

$$\vec{p}_{final} = \vec{p}_{initial} + \vec{F}_{net}\Delta t$$

That is, the final momentum of a system is equal to the initial momentum of that system plus the impulse applied to that system from the surroundings. This relationship is appropriate when values of Δt are small enough that $\vec{F}_{net}$ can be considered constant. Impulse is $\vec{F}_{net}\Delta t$, or the net force applied to a system times the duration over which it is applied. The definition of momentum is $\vec{p} = m\vec{v}\gamma$, where $\gamma = 1/\sqrt{1 - (\frac{v}{c})^2}$, although whenever an object is not moving with a speed greater than about 10% the speed of light (where the speed of light is about 3×10^8 meters per second), γ is approximately equal to 1, and the definition of momentum is

approximated to $\vec{p} = \vec{v}$. When discussing the momentum principle, I will often refer to the net force being the sum of all the other forces acting on the system. The fact that the vector sum of all of these individual forces is equal to the net force I call the **superposition principle of forces**, or just the **superposition principle** for short. That is, the superposition principle of forces is that $\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + ...$, and it is just a consequence of vector addition.

2. The **energy principle**, or the first law of thermodynamics, states:

$$\Delta E_{system} = W_{surroundings} + other\ transfers$$

However, throughout this dissertation, we will only consider situations where there are no other transfers of energy, so $\Delta E_{system} = W_{surroundings}$. Note that to use the energy principle, one may need to identify the different forms of energy present within the system.

3. The third fundamental principle is the **angular momentum principle**, which is:

$$\vec{L}_{final} = \vec{L}_{initial} + \vec{\tau}_{net} \Delta t$$

The vector $\vec{L}$ is angular momentum. Translational angular momentum (that is, for an object moving relative to some fixed point A) is calculated by $\vec{L}_{trans,\ A} = \vec{r}_{CM,\ A} \times \vec{P}_{total}$ where $\vec{r}_{CM,\ A}$ is the position of the center of mass of the moving object with respect to A and $\vec{P}_{total}$ is the total momentum of the moving object. Rotational angular momentum (that is, for an object spinning around an axis through its center) is $\vec{L}_{rot} = \vec{r}_{1,\ CM} \times \vec{p}_1 + \vec{r}_{2,\ CM} \times \vec{p}_2 + ...$, where $\vec{r}_{2,\ CM}$ represents the displacement vector from the center of mass of the object to piece 1 of the object. The sum of the translational and rotational angular momentum gives the total angular momentum for the system, which must obey the angular momentum principle.

Finally, the concept of **reciprocity** is that for certain interactions between objects, the force exerted by object B on object A is equal in magnitude and opposite in direction to the force exerted by object A on object B. This is also known as Newton's Third Law. The gravitational force, electric force, and contact force between two objects (which is an interatomic electric force) are characterized by reciprocity. In this dissertation, one problem that was used in the full study ("Two Blocks," see Section 6.1) could be solved by taking advantage of the reciprocal nature of the forces between two blocks that were in contact with one another.

1.4   Dissertation Overview

Armed with knowledge of the problem and research questions, and with clarified terminology, I will make a case throughout this dissertation for how deductive reasoning can be thought of as searching for possibilities. Below is the outline for how I will do so.

Chapter 2 is a review of the literature in Physics Education Research and related fields, specifically noting the problem-solving and modeling literature in physics education research. Noticeably absent from the literature is a model of student deduction in physics, although two very general models are provided for deduction at large. I present these two different frameworks for deduction and show that the mental models framework by Johnson-Laird (2001) has the most potential for being relevant to the domain of physics.

Influenced by the models of deduction in the literature, I present my own model for addressing student deduction in physics in Chapter 3. This model, called the possibilities framework, was the result of the research described throughout the dissertation. I describe its structure, formally laying out a procedure for using a graphical method for representing the deduction that a reasoner is doing. I show how conclusions reached by formal logic can be replicated with this framework. Additionally, I present a full example from my research of a group of students solving a physics problem, fully represented in terms of the possibilities framework.

Chapters 4 through 7 present the research that led to the generation of the possibilities framework. In Chapter 4 are the pilot studies that led to the development of the full "selection task study" that is the heart of the research presented in this dissertation. The methodology for that full "selection task study" is presented in Chapter 5. Chapter 6 then presents the results from the groups participating in the "selection task study," which are largely responsible for the generation of the possibilities framework. In Chapter 7, the possibilities framework describes the how participants in their individual problem sessions chose solutions from three possibilities by focusing on rejecting possible solutions.

Finally, Chapter 8 presents conclusions from the study and identifies possible future projects that could test this framework.

In the appendix is the collection of materials from the selection task study: the equation sheet,

scripts and consent forms, and problems and solutions that were used in the study. Also included are two of the individual student transcripts from a single problem.

2

Review of the Literature

2.1 Introduction

One goal of introductory courses in physics is to improve students' ability to solve physics problems. Because of this goal, there have been many studies designed to ascertain the distinctions between novices and experts in physics, or between successful and unsuccessful novice problem-solvers. Based largely on these findings, many reasonable models of mental structures and reasoning processes have been developed within the physics education research community. While such models have merit and some practical implications, they rarely (if ever) explicitly mention a fundamental aspect of problem-solving in physics: the use of deductive reasoning to draw conclusions. As a result, how students perform deductions in physics contexts has not been explicitly studied, although what we know from the existing literature about students' knowledge structures and reasoning ability has played an influential role in the development of curricula.

Johnson-Laird (2006) proposes a mental models framework that focuses on the consideration of available possibilities. Such a framework would serve the physics education community well, as it provides a predictive and testable description of how students reason. The purpose of this review is to show that not only does deduction need to be explicitly addressed in the physics education community, but also that this mental models framework is supported by existing literature and provides the best opportunity to understand how students reason during tasks where deduction is

required.

In this review, we will investigate the existing literature in physics education concerning problem-solving and models of student reasoning, establishing that while the literature does not explicitly consider the role of deduction, it is supportive of the mental models framework. Then we will consider the nebulous term "qualitative reasoning" and how that may be connected with deduction, including considerations the literature has given to why it is so difficult for students to "reason qualitatively" in physics classes and what curricular developments have been created to try to remedy this. Next, we will consider frameworks for dealing with deduction and argue that the mental models framework is the most useful for considering how people reason deductively in physics. Finally, we will provide a brief summary of studies of deduction in other fields related to physics and how the mental models framework has successfully addressed those results.

2.2 Deductive Reasoning

Before we can consider the role that deductive reasoning plays in physics, we must properly define deduction and explicitly state what is meant by that term. In addition, while it is apparent that students generally do not use formal logic when they solve physics problems (see, for example, Lawson, 1992; Johnson-Laird & Byrne, 1991; Reif & Larkin, 1991), we will provide a brief overview of some relevant terminology and operations. Doing so will make references to deduction throughout this review more transparent.

2.2.1 What is deduction?

Deduction is a form of reasoning whereby a conclusion that is generated is *guaranteed to be true* if the premises of the argument are true (for example, see Blumberg, 1976). Additionally, all of the information that is required to draw a conclusion is present in the problem statement. As such, some have argued that no new information is actually contained in the conclusion (e.g., Evans, 2005).

Deduction requires a defined starting point, which are the premises of the argument, and clear knowledge of a goal, which is argument's conclusion. However, other forms of reasoning also require these features, notably *abduction*, which is a form of induction. Abduction, where the conclusion is

strongly implied but not required by the premises, is the method of reasoning that Sherlock Holmes uses to solve cases, and it is often mistaken for deduction (Johnson-Laird & Byrne, 1991).

Because abduction does not require a guarantee of validity, it is a form of induction. Inductive reasoning sacrifices some of the guaranteed validity of the argument for plausibility. Inductive techniques such as extrapolation, abduction, and reasoning by analogy are useful but allow false conclusions to be drawn, even when the premises are true and the arguments valid. In scientific enterprises, many forms of reasoning are used. However, it is possible to generate tasks that require deduction. Such tasks are of specific interest, and of specific interest is what students do on such tasks.

2.2.2 Limits on the possible

We can scarcely discuss deductive reasoning without first discussing possibilities. To do so, we will take a brief venture into the philosophical so that we can return to the physical; this short digression is necessary to define some terms that we will use. By *possibilities* we literally mean what might be or might have been. Conceivability is not required for something to be possible; just because early humans were unable to conceive of the possibility that the earth was a sphere did not mean it was impossible for the earth to be shaped thusly (Bradley & Swartz, 1979). This distinction is very important; in a sense there exists a space made up of possibilities for a given situation, a subset of which are conceivable. We can only reason with the subset we can conceive, so to some extent our goal in reasoning is to flush out as many possibilities as possible, by being open-minded to alternatives. As Johnson-Laird (2006) claims, one of the ways we can improve our reasoning is to "[improve our] ability to think of all possibilities compatible with [the premises]" (p. 288).

The premises we include in a given situation control the space of possibilities. For example, a world in which a dropped rubber ball bounces off of cement higher than its initial release is impossible if the laws of classical mechanics are included as premises, but philosophically possible if such laws are not included. In physics, we are specifically concerned with physical possibilities and therefore will always include as premises the physical laws of the domain of interest. Choosing these physical laws effectively collapses the space of real possibilities. Recall the distinction between conceivability and possibility: one may imagine that there are more possibilities than there really

are (and thus maintain a contradiction in the conceptualization of the situation) or that there are fewer possibilities, or even that these possibilities are different (see Figure 2.1).

For example, "it's possible that a ball is experiencing a non-zero change in momentum without a net force being applied to it," indicates that a conceived possibility is not a real possibility. Similarly, "it doesn't matter whether there is a net force on the ball or not; that doesn't tell me anything about the change in momentum of the ball," indicates that the space of conceived possibilities includes both real physical possibilities and possibilities that are not physical. When the space of conceived possibilities is the same as the space of real possibilities, we may see expert-like reasoning.

Later, we will discuss a consequence of not recognizing the difference in type of possibilities when we consider the difference between everyday and scientific reasoning (e.g., Reif & Larkin, 1991). Right now, however, we will turn to an overview of logic, which is a language developed to describe the set of available possibilities for a given situation.
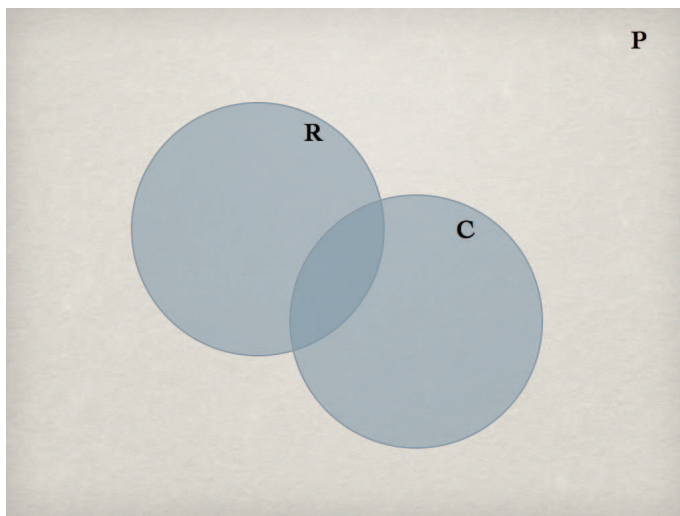
Figure 2.1: Generic possibility space P, with real physical possibility space R and a conceived possibility space C. Possibilities may be present in only R, only C, both C and R, or in neither space.

2.2.3   A brief overview of formal deductive logic

An argument is a sequence of statements and a claim. The claim is the conclusion of the argument, while the sequence of statements forms the premises. Further, in a valid deductive argument, if

the premises are true then the conclusion must also be true (Blumberg, 1976). Arguments that are claimed to be deductive but do not fit this requirement are invalid (and therefore must be rejected). However, it is possible to have valid deductive arguments with false premises; in this case, the conclusion may be either true or false, but the argument as a whole is said to be unsound. In logic, it is the structure of the argument, rather than its content, that determines whether it is deductive (and if so, whether it is valid).

Logic and language are closely entwined, and it is through the imprecision or misuse of language that many fallacies of reasoning emerge (Blumberg, 1976). Logic itself is precise and defines a clear set of rules for evaluating the validity of deductive arguments. While there are many such rules, we will look at just a few of the most relevant. We will focus on sentence logic, since that form of scientific argumentation is ubiquitous in introductory physics classes. Sentence logic considers whole sentences with the connectives between them, and "constructs a theory of inference for just those arguments whose validity depends solely on whole sentences and how these are combined into compound sentences" (Blumberg, 1976, p. 117).

For example, consider this argument from Chabay and Sherwood (2007a): "Because the force on the proton acts in the *same* direction as the displacement of the proton, the kinetic energy of the proton increases" (p. 555). Here, the conclusion is that the kinetic energy of the proton increases and the premises are that the force acts in the same direction as the displacement and the implied premise that if the force is in the same direction as displacement for a proton, then the kinetic energy of the proton is increased (this is actually a compiled premise, using both the definition of work and the energy principle). Is this argument valid? To answer this question, we will list a few relevant rules and recast this argument in formal terms. In the following valid inferences, p and q are statements in the sentences.

modus ponens: IF p then q AND p, therefore q.

modus tollens: IF p then q AND *not-q*, therefore *not-p*.

Hypothetical syllogism: IF p then q AND IF q then r THEN IF p then r.

Disjunctive syllogism: IF p OR q, AND *not-p* THEN q.

Double negation: p is interchangeable with *not-not-p*.

Transposition: IF p then q is interchangeable with IF *not-q* then *not-p*.

There are also fallacies in reasoning, which are not acceptable rules for determining validity. The two most popular are:

Affirming the consequent: IF p then q AND q, therefore p.

Denying the antecedent: IF p then q AND *not-p*, therefore *not-q*.

Let us return to our example and recast it in formal logic language: IF the force is in the same direction as the displacement for a proton THEN the kinetic energy of the proton is increased AND the force is in the same direction as the displacement for the proton, therefore the kinetic energy of the proton is increased. Or, IF p then q AND p, therefore q. The argument is true by *modus ponens*.

As noted earlier, this analysis is not typically done by students. Our inclusion of the language and rules here was to expose the formal deductive way of analyzing scientific arguments. In later discussions of frameworks for deduction (see Section 2.6), we will take formal logic to be the authority for validity.

2.3   Reasoning in Problem-Solving

Research in how people solve physics problems has been a major thrust in physics education since its youth. Throughout the 1970s and 1980s, a major goal of research in cognitive science has been to determine how experts solve problems in a domain, how novices solve those same problems, and how to model both extremes and the transition from one to the other (Maloney, 1994; Hsu, Brewe, Foster, & Harper, 2004). As Reif (1995) stated, the goal of instruction is to determine the initial state of the students, the desired final state, and the operator that will transform them from their initial state to the desired state. Consequently, problem-solving research has attempted to characterize those initial and final states while suggesting possible transformation operators.

Physics problems were especially relevant for cognitive scientists because such problems require

extensive knowledge to solve. As such, they provide researchers with detailed information about what kinds of knowledge are needed for efficient problem-solving (Chi & Glaser, 1985). As a result of many experiments, a number of distinctions between novices and experts became evident and were frequently verified. Another motivation in the problem-solving literature concerns how novices who were successful at transitioning to expert-like behavior differ from those who were not. Below, we will investigate both distinctions.

2.3.1 Differences between novices and experts

Cognitive scientists noted distinctions between experts (generally, graduate students or professors) and novices (generally, first-year students in an introductory college physics course, who may or may not have taken a high school physics course), usually through numerous case studies. In these studies, a small number of novices and/or experts were usually asked to solve standard textbook problems while thinking aloud, using the protocol supported by Ericsson and Simon (1993). Exceptions to the protocol analysis are noted where applicable. These studies have found that experts chunk relevant data, use a forward-reasoning approach, rely on intermediate physical descriptions and develop full plans, categorize problems into few categories organized around fundamental principles, and exhibit a few other distinctions from novices. Overall, these studies seem to indicate that experts use "qualitative reasoning" or "physical intuition."

Experts chunk relevant data

One of the biggest limitations that we all must face is a limited memory capacity. Specifically, we can only use a few pieces of information at once. A seminal work by Miller (1956) claimed that we were limited to only about seven pieces of information, and that to be productive with so few pieces of data, we developed the ability to "chunk" information, thereby combining related pieces of information in a way that allowed us to remember and use it more efficiently.

In the follow-up development of his theory of working memory by Baddeley (1997), this fundamental feature remained. Working memory consists of a central control mechanism for processing information that we can hold for a very short period of time in two domains: the visuo-spacial and the phonological loop. While we can process information from these two domains in parallel, we are

still limited by a small memory capacity, and the chunking of information significantly aids us in problem-solving.

One way of coping with the limitations of working memory can be seen in how experts utilize diagrams (J. Larkin & Simon, 1987; J. Larkin, 1989). J. Larkin explains the relative success of problem-solvers who create diagrams by claiming that diagrams are a form of "external memory" that is subject to being searched as efficiently as internal memory without the limitation of space (1989). Additionally, such diagrams (and schematic displays of information in general) can be organized in ways that promote efficient computational processes. For example, they may group together related information, utilize physical space to represent information about that group without requiring labels, and allow for easy perceptual inferences (1987). We will see later how diagrams contribute to experts' creation of intermediate physical representations.

Another solution to the limitations of working memory is that experts have chunked information that's specific to their domain of expertise. For example, Newell and Simon (1972) studied how novices and experts in chess recalled the position of various chess pieces on a board. When the pieces were positioned in sensible positions that are likely to happen in a game of chess, the experts were much more proficient at recalling the positions of the pieces. However, when the pieces were distributed randomly, experts performed no better than those who were not chess players. This study eloquently demonstrates that the ability to chunk information is strictly limited to the domain of expertise: having a familiar context is not sufficient for efficient chunking; the information must be connected by some fundamental relationship that is domain-specific.

J. Larkin (1979b) and J. H. Larkin, McDermott, Simon, and Simon (1980) saw similar results within the domain of physics problem solving while developing computer simulations of novice and expert behavior. J. Larkin (1979b) concluded that experts perform a qualitative analysis, which organized the information given in the problem and known from the domain in chunks organized around central physics principles. Similarly, J. H. Larkin et al. (1980) identified the experts' use of chunking by noting that the experts had access to a greater knowledge bank than did the novices. The knowledge was not only present, it was organized meaningfully and easily accessed. In a sense, an expert was more than just a novice with a list of relevant equations in working memory (such a list would be too large for working memory to handle); the important domain information is chunked

by fundamental principles. The understanding that experts chunk knowledge around fundamental principles is very robust and aligns well with other landmark results in the field (see, for example, Chi, Glaser, & Rees, 1982).

Experts work forward; novices use means-end approaches

Another major distinction between experts and novices is how they approach solving problems. While experts generally "work forward," applying fundamental physics principles to a problem and reasoning forward deductively to apply them to a given problem, novices apply a "means-end" strategy. In a means-end strategy, the initial state of the problem and the goal state of the problem are considered simultaneously and the problem-solving approach attempts to bring those two states closer together by performing operations to reduce their difference. Because of the intense cognitive power that is required to use such an approach, Sweller (1988) noted that rather than novices learning anything general about the domain of the problem, they learn about specific features of the problem they're working on, making it difficult to transfer understanding to a novel similar problem.

Bhaskar and Simon (1977) identified the approach in novice problem-solving by studying a single novice on engineering thermodynamics problems. Then, Simon and Simon (1978) studied two subjects, a more- and a less experienced solver, who worked one-dimensional kinematics problems and were able to model their approaches with production-system computer models. J. H. Larkin et al. (1980) also generated production-system models that modeled the different approaches made by experts and novices. Later, Dufresne, Gerace, Hardiman, and Mestre (1992) drew upon these results and others (e.g., J. Larkin, 1983) to develop an instructional intervention, HAT (Heirarchical Analysis Tool), which constrained novices to perform a forward-reasoning approach by answering computerized questions that began with broad principles and then followed a sequence that became more problem-specific. At the end of the questioning process, the computer tool would provide the student with an equation that could be used to solve the problem. Dufresne et al. saw a significant improvement in problem solving when so constrained, but did not explicitly test whether the students adapted a forward-reasoning approach on their own after the intervention.

Voss, Tyler, and Yengo (1983) proposed that the experts were able to use forward reasoning strategies because the problems they were solving were similar to other problems that they had seen

before, and that they were able to work through a sequence of steps without having to evaluate whether they were successfully solving the problem. However, novices may take time after each step in their solution to assess whether what they have done has in fact brought them closer to the desired solution.

Priest and Lindsay (1992) re-examined the claim that novices use a backward-reasoning or means-end approach to solve physics problems. They took a production-model approach, identifying two main strategies for controlling inferences: forward chaining ("what conclusions can be reached from what is known?") and backward chaining ("what is necessary to know to reach a particular goal?"). In their study of 76 participants ranging from competent novices (those who had passed an introductory physics course) to experts (postgraduate students), they saw that there was no significant difference in approach direction: both groups used forward inferences. It should be noted that the study did not use protocol data, but rather analyzed the written work that the participants generated. However, Priest and Lindsay also found that experts generated complete plans (see below) much more frequently than did novices. Therefore, if "forward reasoning" is defined as "creating a plan beginning with given information and then following it to find the goal," then experts alone do this. However, if "forward reasoning" is defined merely in terms of stringing together equations that proceed from the given information to the goal quantity, apparently novices are also able to do this. Because of this distinction, we will take "forward reasoning" to have the former definition. It should be made clear that in this study, while the novices chained forward inferences together, they were unable to have much success at solving the problems compared to experts.

Experts rely on intermediate physical descriptions and develop plans

Frequently, studies demonstrate that while novices try to solve a problem immediately by assigning a mathematical representation for the problem (often colloquially described as "hunting for equations"), experts take the time to develop a "physical representation" of the problem and to plan out an approach for solving such a problem (see, for example J. Larkin & Reif, 1979). Even among novices, a similar distinction can be seen: good problem-solvers generally spend more time to perform these actions when beginning a problem, but they are able to completely solve problems more rapidly than poor problem-solvers, who attempt to solve such problems without the planning

and translation of the problem (Finegold & Mass, 1985). This initial effort by experts (and good problem-solvers) clearly allows them to use their memory more efficiently and support their preferred forward-reasoning approach for solving the problem.

Before describing the physical representations that experts generate, we should note evidence that experts do in fact differ from novices with respect to their planning of solutions. For example, Priest and Lindsay (1992) point out that experts planned out their solutions significantly more thoroughly. Furthermore, there was a very strong correlation between frequency of production of a plan and the number of problems correctly solved in their experiment.

Regarding problem representations, a two-participant (one novice and one expert) case study by J. Larkin and Reif (1979) showed that the expert's problem-solving approach consisted of translating the original problem into a qualitative physical representation that related the situation to relevant physical principles. This physical representation was then translated to mathematics (that is, relevant equations) which were then manipulated to reveal the correct answer. However, the novice attempted to solve the problems by directly obtaining relevant equations and then solving these without attempting to qualitatively attribute physical meaning to them.

J. Larkin (1979a) studied experts as they solved problems, suggesting that we should teach their practices to students. She noted that experts routinely perform qualitative analyses and generate physical representations such as diagrams before starting work with equations. She proposed that doing so reduced the chance of error because of the ease of comparing the intermediate step to both the original problem statement and the eventual answer, and also that doing so provided a guide and global representation of the problem. By instructing novices to generate physical representations when solving, she saw definite improvements in performance.

She later compared the physical representations that experts create with naive representations that novices create. While naive representations focus on visible, familiar entities, physical representations focus on principles that only have meaning within the context of formal physics, like velocity or force. Unlike naive representations, physical representations are useful for drawing inferences and therefore guide the solver to identify relationships and solve the formal physics problem. In fact, she found that on a particular hard problem, experts were unable to create a correct solution until

they developed an adequate physical representation of the problem. Novices, failing to create physical representations for even relatively easy problems, simply manipulated equations and generally followed a means-end approach (J. Larkin, 1983).

J. Larkin and Simon (1987) identified when diagrams are particularly effective as physical representations, noting that they localize information, involve intentionally minimal labeling, and enhance perceptual inferences with cues. For example, free body diagrams in physics localize information by restricting it to the forces acting on one body; they require labels that identify only the real-world agent of the force; the size and direction of the arrows on the diagram provide a visible cue that allows for easy inference: here, that the net force is in a certain direction. Experts create diagrams such as these, greatly enhancing their success in solving problems. Likely, this enhancement is because they reduce strain on working memory.

Based on some preliminary results about novice and expert generation of representations while problem-solving, Anzai and Yokoyama (1984) describe the process for transforming novices' models to being more expertlike. They extrapolate the mental models (see Section 2.4.3) from the representations that the solvers create when working on a problem. While they saw that getting novices to shift their models may be quite difficult, they saw that the institution of proper attentional cues facilitated such a shift. They point out that we can take advantage of our knowledge that novices and experts alike search for cues for solving problems and that we should ensure that such cues point to the physical structure of the problem, to aide in the development of an appropriate representation.

To conclude this section, note the claim that Van Heuvelen (1991) made regarding such representations:

> Instead of thinking of the problem as an effort to determine some unknown quantity, we might instead encourage students to think of the problem statement as describing a physical process... the objective is to have students represent that process or event in ways that lead to qualitative and quantitative understanding (pp. 891-892).

He is arguing for an increase in our attention to students' physical representations, noting that a main distinction between experts and novices is the generation of such a representation. In addition, he describes the "qualitative reasoning" that experts use when solving physics problems, something that we will address again in the summary of this section. Van Heuvelen also relates the generation

of a physical representation to the use of Feynman diagrams in particle physics, a particularly useful example of how complex problems are represented graphically, fulfilling the requirements that J. Larkin and Simon (1987) proposed for useful physical representations.

Experts categorize problems into few categories based on fundamental principles

Two major categorizations studies (Chi, Feltovich, & Glaser, 1981; Chi et al., 1982) have shown that when experts are presented physics problems and told to organize them, they will do so according to the fundamental physical principles that underlie the problems rather than according to surface features, which is how novices categorize such problems. These studies align well with the foundational findings of J. Larkin (1979b), which found that experts tend to generate principles in clusters when solving problems (and that they generated physical representations) while novices generate principles largely at random. These studies have been used as supporting evidence that experts arrange their knowledge about physics in hierarchical structures that center around fundamental principles. As such, they have played a particularly important role in the development of reasoning models that we will investigate in Section 2.4.

In the former study, Chi et al. (1981) asked sixteen students (eight experts and eight novices) to sort twenty-four physics problems. While both groups generated about the same number of categories, the novices were quicker and tended to organize the problems around literal features of the problems, grouping for example all of the "inclined plane" or "spring" problems together. The experts, on the other hand, organized by principle, grouping into categories like "second law" or "energy principles." It should be noted, however, that participants in both groups were equally likely to generate an "angular motion" category.

To follow up and test the results from their first study, Chi et al. (1981) then created a set of problems that had striking surface similarity. For example, they presented two different problems that used an Atwood's Machine apparatus. In this case, experts sorted by deep physical principle while novices focused on surface features. Intermediate solvers sorted according to both features, indicating that such a transition is gradual; this finding provides supporting evidence for the proposition that experts slowly generate a hierarchy of knowledge organized by principles. Chi et al. then argued for a difference in the problem schema between novices and experts, according to the

information processing theory (see Section 2.4.1).

Chi et al. (1982) contained more studies in addition to those published in 1981. In one such study, experts and novices were asked to sort forty problems and then given the opportunity to subdivide or combine if possible. What they found was that novices tended to generate more categories, subdividing into categories that contained only one or two problems. In another study within that chapter, experts were found to be better judges of the difficulty of physics problems than novices. As a result of such studies, Chi et al. conclude that experts bring to the problems that they solve a good deal of procedural knowledge; that is knowledge about "how a knowledge structure can be manipulated, the condition under which it is applicable, etc." (p. 72), whereas novices would be unable to utilize the factual knowledge that they have because of a lack of procedural knowledge. Furthermore,

> If expertise in learning is the ability for representing and solving school problems, then for a less intelligent learner, a problem representation may be in close correspondence with the literal details of a problem, while for a more intelligent learner, the representation contains, in addition, inferences and abstractions derived from knowledge structures acquired in past experiences (p.71).

It seems, therefore, that the sorting of problems is closely linked to the problem representations discussed earlier. As such, Chi et al. argue for a procedural knowledge emphasis in teaching.

Given the landmark nature of these studies of organization, it is not without its share of detractors. Veldhuis (1990) reproduced and extended the categorization study (Chi et al., 1981) with a larger sample size and found that while experts do in fact sort according to deep structure, his novices used a combination of deep structure and surface features in their classification. Additionally, De Jong and Ferguson-Hessler (1986) argue that perhaps the salient feature of the Chi et al. (1981) and Chi et al. (1982) studies was not so much that the knowledge structures of novices and experts are different, but that experts have developed successful problem schema that are organized around physical principles. The key issue that De Jong and Ferguson-Hessler are raising here is in the interpretation of the results of the previous studies, rather than the results themselves. In Section 2.4.1, we will see how problem schema play an important role in the distinction between good and poor problem solvers.

Additional distinctions

In addition to the major categories of differences between experts and novices, some studies have indicated some other ways experts and novices differ. Experts tend to be goal-oriented, generating subgoals and breaking problems down into smaller, manageable tasks (Catrambone, 1998). When studying worked examples, they focus on the purpose of each individual subgoal and from noting the purpose and procedure for each one are able to organize their own problem schema to be productive when solving similar problems.

Additionally, when experts use mathematical equations, they focus on the result of the calculation rather than on the process of combining or using the equations (J. H. Larkin et al., 1980). This focus is likely thanks to the planning that experts engage prior to the use of mathematical equations; since they have already processed the meaning of the problem, the equations are merely tools to them.

Also, when experts generate physical models of situations, they are able to analyze the applicability of their models with appropriate analogies (Schultz & Lochhead, 1991). Doing so helps them solve problems because they have some basis for judging whether their model will be effective; novices often have difficulty in picking a single model for solving a problem (Anzai & Yokoyama, 1984).

These additional differences between experts and novices are minor but worth noting, as they are all in agreement with the major results noted above. Furthermore, they are in agreement with the claim that what separates novices from experts is that experts use some form of "qualitative reasoning" and "physical intuition." The meaning of those phrases will be explored below.

Summary: experts use "qualitative reasoning" and "physical intuition"

In short, one way of summarizing the difference between novices and experts is by claiming that experts do "qualitative reasoning" (e.g., Van Heuvelen, 1991; McMillan & Swadener, 1991) and that they have a sense of "physical intuition" (e.g., J. H. Larkin et al., 1980; Singh, 2002) that novices do not yet have. Such a summary aligns well with the major findings of this section; namely, experts chunk data, work forward, create physical representations and plan before solving problems, and categorize problems based on physics principles. However, claiming that experts have physical intuition or do qualitative reasoning simply renames the distinction; we need to investigate exactly

what is meant by those terms.

Simon and Simon (1978) mention that "physical intuition" is quite important in understanding how people solve physics problems. They explain that when people encounter a problem, they create a node-and-link representation of that problem, generating what amounts to a problem schemata that gets stored in memory, so the next time they see a problem of that nature, they are easily able to access the schemata and solve the problem. Thus, they develop a physical intuition about that type of problem. Simon & Simon emphasize the causal connections between the nodes in such a representation.

J. H. Larkin et al. (1980) also try to understand what is meant by the term, "physical intuition." They claim that it has been used colloquially to refer to the phenomenon that good physical intuition allows one to just "see" the proper principle to use or "know" that a certain term in an equation is small enough to be ignored. The thrust of their article is to try to understand what physical intuition actually means, and how we can acquire it. Taking an information-processing and production-model approach, they claim that the indexing of experts' knowledge and the formation of complex problem schemata constitute a substantial portion of that physical intuition, agreeing with the claims of Simon and Simon (1978).

Twenty-two years later, Singh (2002) claimed that "physical intuition is elusive – it is difficult to define, cherished by those who possess it, and difficult to convey to others" (p. 1103). Furthermore, her investigation was how experts in physics attempted to solve a novel problem, claiming that such a problem was outside the scope of the experts' physical intuition and that they would process information like novices when their well-indexed memory of similar problems failed to provide an obvious solution path. Sure enough, experts were unable to solve the problem (although their problem-solving approaches were superior to those of novices), and Singh claimed that the defining characteristic in physical intuition is experience. From experience, experts construct and adapt their knowledge structure; Singh argues that we explicitly teach novices heuristics for solving problems to help them build intuition.

While physical intuition can supply experts with the knowledge of which fundamental physics principle to use when solving a problem, it does not directly address how experts reason through problems. However, we have seen studies reporting that they create physical representations of

problems and that they work forward from their chosen general principles. Van Heuvelen (1991) addresses these actions directly by claiming that experts engage in "qualitative reasoning" while novices do not. In fact, he claims that students often "leave introductory courses unable to reason qualitatively about physical processes" (p. 891). Because students learn by doing qualitative reasoning, and we can more easily assess how well they understand physics by asking questions that require it (McDermott, 2001), many instructional reforms have been developed to address this problem and to encourage students to learn how to reason qualitatively about physical processes (see Section 2.5.3), and have had some success in improving the situation. However, what is the process that "qualitative reasoning" refers to?

While Van Heuvelen (1991) does not provide a satisfactory definition, he provides an example. He discusses how he asks students to translate the words of the problem into a pictoral representation and then into a physical representation before generating a mathematical representation. Often, he asks qualitative questions of the students after they generate the diagrams and before they solve the problem. One example he provides is to ask the students considering a problem about a parachutist whose chute did not opening landing in the snow "how the magnitude of the average upward normal force of the snow on the parachutist compares to the magnitude of her downward weight as she sinks into the snow" (p. 892). Van Heuvelen claims that the same students who struggle with this problem routinely solve quantitative questions using the mathematical form of Newton's Second Law. Qualitative reasoning thus clearly stands in opposition to mathematical manipulation of equations, and yet it often involves comparing the magnitude (or direction) of one quantity to another.

Questions like the one posed by Van Heuvelen (1991) can be easily answered by experts, who use physical intuition to identify the appropriate physics principle to use (in this case, Newton's Second Law) and then use "forward reasoning" to focus on the relevant pieces of Newton's Second Law (here, that the parachutist is accelerating upward). Finally, an expert will use deduction to state the conclusion that because the parachutist is accelerating upward, the net force must be upward. Therefore, the force acting on the parachutist in the up direction (the normal force due to the snow) must be greater than the force acting on the parachutist in the down direction (gravity due to Earth). So is qualitative reasoning really an application of deductive reasoning? In Section 2.5 we will discuss this question in more detail and show the role that deduction plays in problem-solving.

2.3.2 Differences between good and poor novice problem-solvers

In the previous section, the differences between novices and experts were examined. However, an additional distinction is worth an investigation: among novices (who lack physical intuition), what do successful problem-solvers do that unsuccessful problem-solvers don't do? In the literature, it is conventional to make the distinction between "good solvers" and "poor solvers" based on performance on a specific problem-solving task. In the following studies, a small number of novice physics students were given a series of tasks and then grouped for analysis by their success in solving a set of problems. Good solvers exhibited many behaviors that were "expert-like": for example, they planned their solutions, used more principle-based reasoning, focused on procedural knowledge, and imposed a structure on text that was not explicitly present (Chi, Bassok, Lewis, Reimann, & Glaser, 1989; Ferguson-Hessler & De Jong, 1990; Finegold & Mass, 1985). These actions are indicative of the process of constructing an efficient knowledge structure and developing problem schemata for future use. Therefore, a careful look at these studies is important to understand how expertise is generated.

Good solvers demonstrate expert-like behaviors when solving problems

Basing their investigation in the work of Polya (1945), Finegold and Mass (1985) looked at the differences between good and poor physics problem solvers in high school (twelfth year) physics students. Teachers chose to include students in the study who either excelled in problem-solving or were struggling with problems but demonstrated that they had the necessary knowledge to do the problems. However, by choosing to include in the study only students whose scores were above 60% in the class, Finegold & Mass intentionally attempted to hold declarative knowledge constant. They found that good solvers translated the problems more accurately, planned their solutions more fully, took less time to complete the problems even while spending more time on the translation and planning stages of problem-solving, and used "physical reasoning" more than poor problem solvers.

Physical reasoning here refers to a very specific instance in three of the four problems that they evaluated (it was not tested in the fourth problem). In problem one, which concerned relative resistances of two wires of different thicknesses, student responses were coded as containing physical reasoning when they explicitly made use of "R1/R2," the ratio of the resistances of the wires (p.64).

Further details are not provided for what is meant by physical reasoning.

Finegold and Mass (1985) suggest that if poor solvers explicitly emulate the good solvers, their problem-solving abilities should improve. Specifically, they urge instruction to include help in generating translations and plans, much like the expert/novice studies previously described.

Good solvers study differently than poor solvers

In a study of ten university students, Ferguson-Hessler and De Jong (1990) found that good performers in physics applied more deep processing and focused on more procedural and situational knowledge than poor performers, when the students were asked to study a ten page physics text.

Ferguson-Hessler and De Jong (1990) point out that deep processing relates to attempts to grasp the meaning of the information, relating it to prior knowledge, while surface processing focuses on the information with the goal of reproducing it. From this, they created a coding scheme consisting of three major types of study processes: superficial processing (reading the text, comparing symbols, etc.), integrating (bringing structure to their new knowledge), and connecting (relating what they just read to their previous knowledge) and another consisting of four major types of knowledge: situational, declarative, procedural, and strategic.

Poor performers did more superficial processing, employing a process that Ferguson-Hessler and De Jong (1990) dub "taking for granted" (p. 47), where students assume that a certain procedure is followed in the text but do not reproduce it to verify it. While good performers do not seem to do significantly more integrating, a certain subset of those processes are more often attributed to them. For instance, good performers confront text with other arguments present in the text, doubting it or generating alternatives. Good performers also more frequently performed a derivation independently of the text. Finally, good performers said that "everything is clear" three times less frequently than poor performers; this corresponds with other results from worked example literature that claim that good problem-solvers question the text more and generate more "negative monitoring" statements (of the type "I don't get why this is the case") and fewer "positive monitoring" statements (of the type "yes, that's clear") than do poor problem-solvers (e.g., Chi et al., 1989; Pirolli & Recker, 1994; Stark, Mandl, Gruber, & Renkl, 2002).

Additionally, good solvers in this study were found to focus on procedural knowledge more

frequently than poor solvers. While poor solvers seemed to focus on declarative knowledge more and good solvers seemed to also focus on situational knowledge more, neither of those results were significant.

In summary, when good performers study, they are attempting to modify their existing knowledge to include what they're reading by focusing on the procedural information in the text for efficient storage and generation of that knowledge.

Good solvers generate more self-explanations when reading worked examples

Students clearly learn differently from worked examples than they do from studying a text, and two papers analyzing the same study sought to identify how novice good and poor problem-solvers differed in how they used the worked examples (Chi et al., 1989; Chi & VanLehn, 1991).

Chi et al. (1989) discussed explanations that students generated when studying worked examples and thinking aloud in terms of their utility. These self-explanations took on one of four forms: either they refined and expanded the conditions for taking an action, extrapolate the conditions for making an action, provide a goal for actions, or explained the meaning of quantitative explanations. Chi & VanLehn point out that self-explanations differ from elaborations because the former refer to the learning of new material.

These two studies account for the difference between good and poor problem-solvers by the differences in the number and content of the self-explanations of the participants, noting that both groups demonstrated a declarative comprehension of the material being tested (Chi et al., 1989; Chi & VanLehn, 1991). Corresponding to the increased number of protocols and self-explanations generated by the good solvers, those students spent significantly more time on the examples than the poor solvers. Good solvers generated significantly more explanations of physics than did the poor solvers, and as in the study by Ferguson-Hessler and De Jong (1990) they generated more monitoring statements that corresponded to comprehension failure and fewer that corresponded to understanding.

Perhaps most striking, however, was how good students used examples to support their attempts at problem solving: when working on isomorphic problems, the good solvers immediately went to specific lines in the examples, rereading those lines and not starting from the beginning, whereas the

poor solvers often started rereading the examples at the very first line. The specificity of the search that the good solvers performed in contrast to the general nature of the poor solvers' attempting to understand what the examples did indicated that the good solvers already generated a plan for solving the problem, whereas the poor solvers attempted to find an answer to directly map to their current problem (Chi et al., 1989).

Good solvers seemed to generate more and different knowledge while studying the worked examples than they did when reading the text in preparation for the problems (Chi & VanLehn, 1991), probably because the text itself did not explicitly show how physics principles can be applied to problems. They model the students' knowledge in terms of information-processing "constituent knowledge pieces" that take the form of conditionals, upon which the students apparently perform *modus ponens* when solving problems. Further, Chi & VanLehn claim that the students probably did not perform explicit deductions from principles they learned in the text, but rather learned aspects of those principles directly from the worked examples. Their support is dubious (and they admit that they are speculating), and they infer that the only likely way to form such knowledge pieces is to encounter a trigger to that precise piece in the worked example. The role of mental models (Johnson-Laird, 2006) will provide an alternate explanation to this, which does not contradict the findings or speculations that Chi & VanLehn have demonstrated but rather provide a mechanism for those constituent knowledge pieces to be added either through deduction from fundamental physics principles or as they are encountered in the worked examples.

Summary: good solvers are building expertise

As we continue to see, novices who are successful in learning physics do not focus on the surface features but rather on the fundamental physics principles when studying text and worked examples, preparing themselves by creating physical intuition and learning how to reason qualitatively from those principles. In this way, they are generating expertise, constructing their knowledge hierarchically around fundamental principles rather than simply attempting to reproduce declarative knowledge or map solutions from one problem to another. As a result of such studies of novice learning and of expert/novice differences, a number of models of student reasoning have been developed in Physics Education Research.

2.4   Models of Student Reasoning

Various models of how students reason about physics have been proposed in the literature. Among them, there are numerous similarities and some important distinctions. In the following discussion, the models that have proposed are divided into four categories: information processing models, network models, mental models, and descriptions of conceptual change. In many ways, these categories complement each other; the empirical data that is generated by studying protocols of people solving problems can be described with different models. However, considering the larger features of these models in these clusters will serve us well, as we will be able to focus on the salient features of the models that have been proposed and be able to discern the role of deduction in each of them.

2.4.1   Information processing

Information processing theory basically treats humans as if they were information processing systems, capable of receiving sensory inputs, processing those, storing objects in memory, retrieving objects from memory, and producing outputs (Newell & Simon, 1972). Information processing systems were the paradigm for much of the problem-solving research discussed in Section 2.3; certainly throughout the 1970s and 1980s, and even into the 1990s it was the dominant framework. It is still a useful framework for considering how people solve problems, so it is imperative that we investigate it more deeply.

Information processing systems focus on performance

Information processing systems (IPS) do not attempt to model the internal processes of a system; that is, when an IPS is used to describe how novices or experts solve physics problems, it does not claim that the people perform the task *how* the model describes. Rather, the IPS attempts to create a model that can produce the same output as a novice or expert when provided with the same input. These models usually take the form of computer programs and can be usefully applied to provide a description of how people process task-oriented symbolic information (Newell & Simon, 1972).

As Newell and Simon (1972) further state, "If performance is not well understood, it is somewhat premature to study learning... it is our judgment that in the present state of the art, the study of

performance must be given precedence, even if the strategy is not costless." In so doing, Newell and Simon provide an argument for why physics education research has proceeded as it has, by first utilizing IPS theory to describe the performance of people on problem-solving tasks before attempting to describe learning. Later in this review, we will see descriptions and models of learning under the heading "conceptual change." These would not have been possible without an early focus on performance.

By omitting variables like motor skills and aspects of perception, along with motivation and personality variables, the information processing approach treats humans as production systems that follow specific production codes and are deeply rational. As such, it is deeply connected with artificial intelligence and integrated with the content that it is attempting to explain. Newell and Simon (1972) claim that a "good information processing theory of a good human chess player can play good chess" and thus infers that a good information processing theory of a good physics problem solver can solve physics problems well.

Indeed, the goal of much research that has been to do just that by performing case studies and using protocol analysis to get at the heart of the performance of the problem-solvers. A number of such programs has been created. STUDENT converted physics principles into equations, and ISAAC used built-in schema to translate a problem first to a physical representation and then constructed equations from that (J. H. Larkin et al., 1980). These and other computer models, such as KD, a knowledge-development expert-like model (J. Larkin, 1980), ABLE and MORE ABLE (J. Larkin, 1981), and FERMI (J. H. Larkin, Reif, Carbonell, & Gugliotta, 1988), have generated insights that are strongly supported by studies of novices and experts solving problems (Maloney, 1994). EUREKA(Elio & Scharf, 1990) was even able to model the shift from novice-like to expert-like behavior, moving from means-end to forward reasoning by constantly evolving its problem schemas.

Let us now investigate with some depth the important components of the specific IPS that applies to physics problem solving, in an effort to understand how it explains the performance of people with varying levels of expertise in the discipline.

Production systems

Production systems are the external representation of an IPS, the way that we as outsiders describe the system itself. The production system, or program, models the IPS in the sense that it provides the same output given the same input; it is not intended to represent how the system itself processes the information (Newell & Simon, 1972). For example, consider a thermostat that is set to turn the furnace on at 70 degrees and turn it off at 72 degrees. The thermostat's internal system itself may be a bimetalic strip that bends slightly to connect a circuit when the temperature is low and breaks the circuit when the temperature gets high enough. However, we can model this with a production system, or a program, that lists conditional statements referring to the performance of the thermostat, such as "Observe the temperature. If the temperature is below 70 degrees, then turn the furnace on" (Newell & Simon, 1972, p. 31).

These production systems are key to physics education research, as they have developed important programs like the ones listed previously. They've also appeared to interpret protocol data. Chi et al. (1981) describe specific instances of this, explaining how doing so profoundly enhances the difference between novice and expert performance. Chi et al. further explain that while experts' production rules contain "explicit solution methods, such as 'use F = MA,' 'sum all the forces to 0,'" (p. 139), novices' rules do not contain explicit solution procedures, focusing instead on attempts to find specific unknowns, like the mass (without clear direction for how that is to be done). These productions are consistent with earlier results that experts reason forward while novices use means-end approaches. Furthermore, this research supports the argument that experts' performances are more closely aligned with well-defined deductive approaches. Rather than focusing on unknowns, experts focus on actions that correspond to employing fundamental physics principles in a deductive manner, to arrive at the solution to the problem.

The role of hierarchical knowledge structures

By showing that performance on recall and problem-solving tasks was improved by presenting the information with the same organizational approach as the task itself, Eylon and Reif (1984) argued that we generate hierarchical knowledge structures when we learn. Additionally, the evidence that low-ability students benefited least from matching the presentation method with the task implies

that successful students learn how to organize their knowledge hierarchically, while unsuccessful students leave that knowledge in heterarchical (that is, single-layer) structures. As Eylon & Reif imply, the nature of the hierarchy depends on the task; that is, if students expect a historical task, they will learn to organize their knowledge chronologically, whereas a causal task will promote a causal organization.

Similarly, students need to learn to connect the concepts and principles that are relevant with the relationships between them (Robertson, 1990). These links seem to be deductive in nature; how does knowledge about one concept trigger the use of a related concept? Such questions imply a network structure of knowledge; we will explore network structures in Section 2.4.2.

Such studies seem to show that for people to be modeled as information processing machines, the information that is contained in their memories must be indexible and easily accessed. A hierarchical structure is therefore a fundamental component of any information processing system that models problem-solving. How such structures are organized and accessed is difficult to ascertain from the theory, however. Newell & Simon state that,

> the question remains of how the particular problem space used by a problem solver is determined. The studies in this book do not shed much light on the determining mechanisms. They do make it clear, however, that the general intelligence of the problem solver plus the knowledge in relevant domains is often sufficient to predict what problem space he will use (p.790).

In Section 2.4.3 we will encounter a model that has the potential to provide a mechanism for how we traverse this space.

The role of problem schemata

Prior knowledge is represented by a schema, a structure that specifies relations between various objects. A major purpose of schemata is to "construct interpretations of new situations" (Chi & Glaser, 1985, p. 241). For example, we have schemata that we invoke when we are at restaurants; they help us know when to order (e.g., at a counter or at the table), how to sit (e.g., to choose our own booth or to be seated by a host), and how to pay (e.g., to pay the server or a cashier; to pay

before or after receiving food). If we are familiar with a restaurant, we simply invoke the schema for that restaurant; if we are at a new restaurant, we must learn from perceptual cues how to modify existing schemata to create a schema for the new restaurant. For example, we may notice that there is no cashier station and thus predict that we will have to pay the server. Similarly, we construct problem schemata from problems that we have experienced in the past. When we are faced with a similar problem, we may adapt an existing schema or create a new one.

De Jong and Ferguson-Hessler (1986) claim that attempting to describe the process of solution is quite different from the attempts that researchers have made to understand their structure of knowledge. Further, they argue that an organization of knowledge by adequate problem schemata is quite efficient and can lead to high scores on exams. Problem-solving would therefore consist of identifying relevant features of a problem, which allows access to the relevant knowledge to solve it, much like identifying relevant features of a restaurant leads one to pick out (or modify and use) the appropriate schema.

It follows then that problem schemata are not the same as the hierarchical knowledge structure that was described previously. Ferguson-Hessler and De Jong (1987) describe how students who are learning about electromagnetism first create problem schemata around formulas like Coulomb's Law, formulas that appear many levels down on a true (expert-like) hierarchical mapping of the knowledge structure of electromagnetism. However, as more and more knowledge is learned, more abstraction is possible, and the elements that are present in the schemata form elements in the hierarchical knowledge structure. Ferguson-Hessler & De Jong argue that problem schemata is distinct from the knowledge structure, and that both are important for expertise. Moreover, it may be easier to teach students first how to construct a problem schema, for example when they are learning about electromagnetism.

The role of categories in information processing systems

The major premise of studies such as the ones carried out by Chi et al. (1981) is that the internal representation that experts create of problems is significantly different than that of novices. Furthermore, as people become more experienced, their knowledge base becomes more rich and organized according to "problem type" or category. Chi et al. (1981) point out that both "physical intuition"

as described by Simon and Simon (1978) and "qualitative knowledge" play a role in this organization. Novak and Araya (1980) have proposed that categorizing problems by type allows access to associated information in memory; arranging the knowledge base around problem schemata allows one to successfully solve problems by first classifying that problem into a category for solving it.

Chi et al. (1981) claim that there is evidence in the literature to suggest that experts do in fact solve problems this way, by representing them by category first. They use this evidence to support their studies on the role of categorization, including an investigation into what knowledge categorical representations cue in experts. Because novices categorize problems based on features like "inclined plane," Chi et al. claim that novices' problem schemata are very well-developed around the concept of "inclined plane" while experts' are very well-developed around physics principles, using procedural knowledge about their applicability. Thus, they claim that knowledge is common to both groups, but that it's organized differently, resulting in different production systems. All of this follows very well from the assumptions the authors make, and the use of the Chi et al. (1981) paper is to provide an exemplar from the literature; much of the problem-solving literature from this era makes similar claims about novices and experts or is otherwise built upon this work (for example, J. Larkin, 1983; Elio & Scharf, 1990). Computer programs generated in this era do very well at emulating performance, but does that mean that they represent the actual structure of expert and novice mental structures?

diSessa and Sherin (1998) present a compelling discussion about concepts and conceptual change, which will be revisited later when conceptual change models are discussed. For now, it will serve us to look at how concepts are defined, and the role that categories might play in knowledge structures. To frame the discussion, two quotes from this article are absolutely relevant. First, "just because we, as researchers, can name a particular cognitive task... this does not tell us anything directly or obviously about how that task is accomplished" (diSessa & Sherin, 1998, p. 1160). In other words, the creation of production systems may replicate the outputs that people generate, but that does not indicate that the model is indicative of the actual mental structure. In fact, this claim doesn't oppose the original claims of information processing theory; the goal of information processing theory is to model the system's output, not to figure out its internal structure (Newell & Simon, 1972).

The second relevant quote is, "Numbers are very different from dogs" (diSessa & Sherin, 1998,

p. 1161). If we are to use concepts to represent a wide range of items, for example, numbers and dogs or inclined planes and forces, we need to be very careful at how we do so. For example, if we wish to represent concepts themselves as categories, as studies such as those in Chi et al. (1981) are wont to do, we must recognize that a "classical" view of categories as being defined by a list of features has been rejected by more recent literature (diSessa & Sherin, 1998). If we turn to a prototype model of categories, where a category is represented by a prototype (Rosch, 2002), we note that it is relatively easy to imagine that such a prototype exists for a bird or dog, but what is a prototype for a number? By analogy, we can imagine a prototype for "inclined plane," but "force" is a bit more difficult.

Indeed, Lakoff (1987) claims that treating categories based on prototypes undermines the actual mental structures that we have. To treat inclined planes and forces similarly, we have to take a step back from the tight connection between categories and mental representations. While using categorization studies to propose distinctions between expert and novice performance is one thing; implying that such studies say something specific about their mental structures, even in terms of problem schema, is pushing the limits of information processing theory. In fact, Chi (1997) has stated that her analysis procedures deviate from the protocol analysis of the information processing school. While the mechanics of her technique are similar to protocol analysis as described by Newell and Simon (1972) and Ericsson and Simon (1993), the foci of the analyses are different. While protocol analysis is meant to uncover the process of the problem-solving task, the qualitative method described by Chi seeks to uncover the knowledge structures of the participants. Through her analysis, she establishes that the structure of experts' and novices' knowledge is indeed quite different. We will now turn to some models that reflect this difference by treating the knowledge structures as networks of some sort.

2.4.2   Network models

One class of models attempts to explain how we solve problems in physics by modeling our knowledge as a network, made up of *nodes* and *links*. Such network models may be generic, defining nodes as "concepts" and links as "associations," (e.g., Thagard, 1992), or they may be quite specific, proposing specific nodes and links. Network models may be useful for describing how we use analogies (e.g.,

Holyoak & Thagard, 1989), how we think about new problems (e.g., Hammer, 2000), and how we learn and adapt (e.g., diSessa & Sherin, 1998). In this section, we will explore the structure and benefits of network models by first understanding what is meant by nodes and links, then by looking at two specific examples of networks in physics education literature: coordination classes and resource theory.

Nodes

In most network models, nodes refer to concepts such as "bird" or "inclined plane" (e.g., Thagard, 1992); however, the role that nodes have is actually more general than just as a nominative. Nodes could instead be actions or principles, corresponding to the purpose of the network model. The nodes function as the constituents of the theory: if people are thought of as reasoning with concepts, then nodes are concepts. If they are thought of as behaving in a probabilistic (or deterministic) way based on stimuli and previous actions, then the actions themselves could be the nodes. In physics education research, three particular kinds of nodes are worth mentioning: facets, p-prims, and resources. The first two will be dealt with here; the third will be addressed with resource theory in whole.

Rather than attempting a fuller theoretical description of student reasoning, Minstrell (1992) proposes a "practical, classroom-based approach" (p. 112). He notes that students use constituents of knowledge or reasoning when attempting to solve problems, and he calls these pieces *facets*. These facets, which are recorded in the students' own words, may relate to content, strategy, or generic reasoning.

Minstrell (1992) mentions that, "horizontal motion prevents things from falling as rapidly" (p. 112) and "more ... means more ..." (p. 112) are both possible facets. The key feature of facets is that they are useful. They may not be appropriate in all situations, but they are useful. As such, it is improper to give them titles like "misconceptions;" it is in their associations that they are misused, not in the simple fact that students have them. As such, the goal of instruction should be strengthening links between facets that are appropriate while weakening links between facets that are inappropriate. One particular example of doing this is found in a very popular description of teaching about the normal force that a table can impart on a book (Minstrell, 1982). In this study, Minstrell focused on the productive facets that the students already had, showing them that the hand

provides a normal force to the book, so it's reasonable to predict that the table also does. By using a laser to show a slight movement by the table when objects are placed on it, many of the students are convinced: their facets have been connected with the desired outcomes while connections with undesired outcomes have been weakened.

A somewhat different proposition for the nodes of the network structure can be found in phenomenological primitives, or p-prims for short (diSessa, 1993). Although p-prims grew out of an attempt to explain the perceived "knowledge in pieces" (diSessa, 1993, p. 111) that novices have as opposed to the connected knowledge of experts, they eventually became a viable candidate for nodes (diSessa & Sherin, 1998).

P-prims are small (consisting of very little structure), possibly behavioral, knowledge structures that serve important roles in explaining physical phenomena. They are "ready schema in terms of which one sees and explains the world" (diSessa, 1993, p. 111), and they are primitive in two ways: they are self-explanatory and atomic. In a sense, p-prims are the final explanation; after a p-prim is given as a reason the only response to "why?" becomes "because it is, that's all." P-prims are activated in situations when they are perceived to be important; for example, the p-prim, "closer means hotter" may be invoked when considering how far to stand from a fire, or it may be used incorrectly to try to explain the seasons of the year.

In some ways, p-prims are similar to facets, but they are more general, less frequently referring to explicitly physics-related contexts. They are also more precisely defined, with a complete theoretical (rather than pragmatic) framework. However, either may be used as nodes in a network structure that is used to model student thinking. Later, in our discussion of coordination classes, we will see a specific example of how p-prims can be used in such a model.

Links

Links are generally less-developed than nodes. Many sources seem to disregard them entirely, or treat them merely as a sort of association without providing much explanation. Some models consider that links may be causal (e.g., Gentner & Gentner, 1983; Wittmann, 2006), possibly a characteristic inspired from the causal links present in the production models in information processing systems. Other models associate the nature of links with neural activity (Redish, 2004), a feature of resource

theory that we will discuss in more depth below.

Thagard (1992), who takes the nodes of his structure to be concepts, fleshes out five different types of links. *Kind* links indicate that one concept is a kind of another, *instance* links indicate that a particular object is an instance of a concept (for example, this baseball is an instance of a ball), *rule* links express general but not necessarily universal relations (for example, new baseballs are white with red lacing), *property* links indicate that specific objects have certain properties (such as, this baseball is tan with red lacing), and *part* links indicate a part of a whole (for example, lacing is part of a baseball). From these five types of links, which Thagard points out could all be represented with formal logic in first-order predicate calculus, we can create full maps of concepts, which can then be used to describe conceptual change.

Because of the nature of the nodes for the relevant models in physics education, these five types of links are probably not appropriate. It may be that the relevant links are neurologically associative, or that they are causal. It is also possible that we could conceive of a network structure with deductive links. For now, we will consider two specific theories in some detail, coordination classes and resource theory.

Coordination classes

In an attempt to understand conceptual change, diSessa and Sherin (1998) realized that they needed to understand what is meant by the term "concept." Finding the literature lacking, they sought to create their own definition for at least one subset of things under the "concept" umbrella. For something to be an appropriate model of a concept, diSessa & Sherin warn that it must have clarity and have an empirical focus (that is, there should be an articulation of what it means to "know a concept"), it must appreciably describe a wide variety of concepts (for example, both numbers and dogs), it should provide for conceptual change in a way that describes the operation and change of concepts, and it should describe students' thinking as that thinking changes, not just in terms of the before and after states that much of conceptual change research has focused on. Further, diSessa & Sherin warn that we not "make the mistake of presuming there is a simple alignment between expert concepts... and the technical vocabulary of a domain" (p. 1167), something that can be difficult when we expect that experts simply follow the philosophically idealized approach of a domain (for

example, are expert concepts identical to the philosophical concepts of "momentum principle" and "work"?).

Coordination classes are proposed as a specific type of concept that is particularly important in science learning, fulfilling the requirements for concepts (diSessa & Sherin, 1998). The goal for coordination classes is to describe what it means to get information from the world. This performance-based criterion stands in contrast to the classification-based criteria that are associated with placing objects into categories (such as deciding whether "penguin" fits into "bird"). The work that scientific concepts do, claim diSessa & Sherin, is to gain information about the world; therefore, one has to both select attention and integrate observations into their knowledge about the world. Coordination classes thus coordinate in two ways: through invariance, by which is meant that the knowledge that accomplishes a readout must be consistent and coherent, and integration.

diSessa and Sherin (1998) provide a definition for a specific structure that allows for this coordination. A causal net is what relates observations with "plausible or necessary determinations" (p. 1174). In a sense, the causal net takes the place of a theory in theory-based accounts of categories. When different people are faced with the same problem and perform differently, we can assume that information in their causal net is different.

Physical quantities could be considered coordination classes, and causal nets can be created about tasks such as "determining force" (diSessa & Sherin, 1998). For example, there is a causal net corresponding to Newton's Second Law; if you want to determine a force, you could first look and measure change in momentum. Equations are an important part of the causal net, but it will not suffice to equate equations with causal nets.

P-prims, while not coordination classes by themselves, may be nodes of some sort and may certainly be a part of the causal nets. In fact, diSessa and Sherin (1998) claim that the set of them form the first causal net, providing a naive sense of mechanisms. To illustrate the way in which physics principles, p-prims, and causal nets all interact, diSessa and Sherin provide examples of a novice working through problems. While experts create causal nets around physics principles such as Newton's Second Law, novices may create causal nets around p-prims, allowing "beliefs" such as "balance implies stability" to override physics principles. It is not that students don't understand the principles; rather, in the featured example, the participant carefully decides that the fundamental

principle simply isn't appropriate in a specific situation where the p-prim seems most relevant. In many ways, this conflict and rejection is similar to what is seen in psychology with respect to belief bias. In Section 2.6.2, we will discuss similar studies and discuss another model for describing it.

In summary, (diSessa & Sherin, 1998) provide an empirically useful class of concepts: those that help us learn about the world. By doing so, they have provided us with a unit that is appropriate for discussing conceptual change (see Section 2.4.4) and which account in some general sense for novice/expert differences. The development of coordination classes also paved the way for a larger grain-sized network model: resource theory.

Resource theory

If we consider peoples' mental structures as relatively stable, strongly held, and resistant to change, which research on student misconceptions often does, then we have difficulty explaining contextual sensitivities of their reasoning and have to struggle to maintain a constructivist view of learning (Smith, diSessa, & Roschelle, 1994). If instead, we choose to consider student reasoning as manifold, that is, consisting of patterns of activation rather than pieces of compiled knowledge, then we can explain more empirical results, such as why equivalent question posed in different contexts yield different results (e.g., in the context of FCI or exam scores, Steinberg & Sabella, 1997). Resource theory states that people have a set of available resources that are activated in problem-solving situations and are used differently depending on context (Hammer, 2000). These resources are linked together in some sort of network, so that the activation of one resource may trigger or inhibit the activation of another.

Hammer (2000) describes two types of resources: conceptual and epistemic. Conceptual resources may refer to knowledge of the physical world or physics principles; the coordination classes that diSessa & Sherin describe may be appropriate as conceptual resources. Similarly, facets (Minstrell, 1992) or "raw intuitions" (Elby, 2001) can be productively thought of as conceptual resources: regardless of the language used, Hammer argues that we need to learn how to connect these resources with the productions we want, using other tools than confrontation.

Epistemic resources address how people understand the nature of knowledge and how it is obtained Hammer (2000). Examples of epistemic resources include, "knowledge is invented" or "knowledge is passed to me from someone with authority." These resources help guide not only which conceptual resources trigger but also how the resource network is modified. One particular type of epistemological construct to consider is the "epistemic anchor," which serve as targets for metaphors or bridging analogies. For example, Hammer notes that the comparison between physical exertion and mental exertion can have the epistemological anchor in the context of physical exercise, because persistence and improvement are quite noticeable in that context.

So if resources are the nodes of the network structure, what constitute the links? Redish (2004) describes the theoretical framework as starting from various hypotheses, and one of these is that "all cognition takes place as a result of the functioning of neurons in the individual's brain" (p. 5). From this, we can draw on many principles of neuroscience that describe how neurons connect and fire. Notably, this includes the principle that "learning appears to be associated with the growth of connections (synapses) between neurons" (p. 6).

To connect these neural-level principles with behavioral observations, Redish (2004) imports the concepts of activation, association, and enhancement/inhibition. Because the evidence shows that memory is "associated," in the sense that the activation of one element in memory may lead to activate others (see, for example, Baddeley, 1997), resources are thought to be associated in the same way, including phenomena such as priming. Redish addresses the extent to which resources are linked by considering three separate axes: robustness, degree of compilation, and level of integration. However, he does not, in a sense, consider how resources are linked together. For example, how is the coordination class for "force" linked to that of "motion?" Similarly, Sabella and Redish (2007) mention that "lines connecting nodes represent links or associations among facts or rules" (p. 1019), following the conventions and descriptions established by Marshall (1995), but provide little beyond classifying such associations by the strength of their connections. In an attempt to model conceptual change with resource graphs, Wittmann (2006) notes that his description is also limited: "defining links (and finding appropriate observational tools to define them) is a major area of research when discussing causal nets and readout strategies" (p. 020105-5).

Resource theory has at least one definite instructional implication: as a result of taking it seriously, we have to understand student performances as deeply rooted in the context for the task; the activation of particular resources depends on the environment in which the task is situated (Hammer, Elby, Scherr, & Redish, 2005). The instructional task partly becomes teaching students how to activate appropriate epistemological resources to guide which conceptual resources then get acivated. Certainly, there is merit to this approach; how can we expect good performance from people who do not understand the "nature of the game?" Additionally, framing student performance and questions within resource theory will allow us to understand them better when listening to them; rather than repeating lectures to them, we need to meet them where they are (Redish, 2004).

However, this is the point where resource theory alone, useful as it is, reaches a limit. Without an adequate and pragmatic descriptor of the links, we are left with the difficulty of how to strengthen or build them. Is it possible that a different perspective could shed a light on how resources are linked to each other? With this goal in mind, we now turn to a somewhat different approach at modeling people's thinking: mental models.

2.4.3 Mental models

First, we must be very precise in what we mean by "mental models." There are many distinct occurrences of this phrase in the literature, and it means different things when used by different people. Rather than addressing each instance individually, we will look at three and claim that they form close to a basis set from which other descriptions of "mental models" are derived.

Functional models, not cognitive models

The conceptualization of a task plays a vital role in the completion of the task itself. Norman (1983) states that as a result of functioning in our environment, "people form internal, mental models of themselves and of the things with which they are interacting" (p. 7). To explain mental models, Norman describes four specific entities: a target system (the actual system in the environment), the conceptual model (invented and designed by scientists or other professionals), the mental model of the target system (individual, functional models), and the scientist's conceptualization of the mental model (a model of a model).

Mental models have features corresponding to these general observations: they are incomplete; they are difficult to fully run; they are unstable, easily losing details; they are vague and allow confusion of the target systems; they tend to be unscientific in the sense that people may include superstitions and patterns of behavior that are not needed; and they are parsimonious, as people often prefer to do physical labor rather than a little bit of mental planning that would render such actions unnecessary (Norman, 1983).

By way of example, consider a personal experience of mine: sometimes my car refuses to start. As a result, I have formed a mental model about how the car functions, but this model is incomplete and allows superstitions. Not only do I not know the details about how starters or electrical systems or internal combustion engines combine to turn a spark into locomotion, but I find myself performing tasks unnecessarily in a routine to "coax" the car into starting. For example, I try putting the car into neutral, then I take the key out, and then I pat the dashboard of the car. Clearly, patting the car does not aid in allowing the car to run, so why do I do it? Mental models, according to Norman (1983), provide for my actions. Rather than holding a true conceptual model of the target system, my mental model functions enough for me to keep it (i.e., my car almost always eventually starts).

In fact, true conceptual models are philosophical creations that exist in textbooks and treatises. As instructors, the best we can hope for is mental models to be created from conceptual models rather than experience. However, for this to happen, the conceptual model that is taught needs to be learnable, functional, and useable (Norman, 1983). Even when the conceptual model meets these three requirements, the task ahead is daunting, but it needs to meet these qualifications at a minimum to have any hope of success.

Norman (1983) has shown a distinction between the different systems, urging us not to confuse mental models with conceptual models; we cannot pretend that after instruction a student has acquired a textbook knowledge of a subject as if we were able to just give it to them. This distinction has been referenced again and again throughout the literature, and the general description of mental models provided in this chapter has served as a foundation for a number of more complex theories.

Redish (1994) built upon the description of mental models provided by Norman (1983), claiming that "they consist of propositions, images, rules of procedure, and statements as to when and how they are used" (p. 797) and further adding to Norman's description that they may contain

contradictory elements.

Further developing the idea that mental models may contain contradictory elements, one theory claims that people can hold more than one mental model about physics simultaneously, and which one is used depends on the exact specifications and surface features of the problem (Bao, Hogg, & Zollman, 2002; Bao & Redish, 2006). Bao et al. claim that only a finite number of mental models exist for any given concept (for example, they cite three for the relationship between force and motion). They then represent a student's mental state as a superposition of these models. Performing a measurement (that is, asking them a specific question) collapses the mental state into a single mental model for that question. Asking that same student a different question may collapse the mental state of that individual into a different model; therefore, students are treated as either having consistent or inconsistent (that is, consisting of more than one mental model) representations. The prescription for finding a student's mental state is by asking numerous versions of the same question, varying surface features. Then, a probabilistic model is developed, and that can be used to predict performance (though only probabilistically).

The process of identifying common student models, designing multiple-choice instruments to characterize the student responses as vectors in model space, creating a density matrix for the class, and using the eigenvectors and eigenvalues of this density matrix to express the level of confusion of the class (Bao & Redish, 2006) has its advantages. Namely, we can have a descriptor of the actual models the students use when answering physics questions, allowing us to disentangle the confused-sounding explanations they provide. Such a matrix also provides a reasonable alternative to the assumption that probes of student understanding will provide essentially a true value, by allowing for more than one complete but competing mental model to be held simultaneously by students. However, without the ability to predict with precision the responses of the students or class, and without providing ways to generate the appropriate Hamiltonian to move the density matrix into a pure model state, this theory seems to fall short. Moreover, it attempts to provide a description for initial and final states rather than for the process, something that we need to turn to other definitions of mental models to obtain.

Mental structures that represent target systems

With an eye toward modeling the process of conceptual change, Vosniadou (1994) used mental models as representations of target systems. Specifically, she wanted to describe how such representations change as one's understanding of a domain develops. Pointing out a distinction between enriching and revising mental structures, she claims that "misconceptions are viewed as students' attempts to interpret scientific information within an existing framework theory that contains information contradictory to the scientific view" (p. 46).

Mental models, for Vosniadou (1994), maintain the structure of the thing they represent. For example, a mental model of the earth might be spherical, round like a pancake, or round like a fishbowl. As such, the way to access students' mental models for such things as the shape of the earth is to ask them to draw them. Misconceptions, such as "the earth is round like a pancake" can be explained by the attempt to reconcile "the earth is round" with the earlier mental model of the earth as flat (since most empirical evidence doesn't require a spherical earth, and it appears that the earth is flat). In fact, such misconceptions arise because people often respond differently to different questions, and if such events are not segregated by time or environment, people become confused and mix them. Note the contrast to the model proposed by (Bao & Redish, 2006), where the mixing is simply due to our observation of the robust and fixed states.

Vosniadou (1994) specifically claims that mental models are spontaneously generated and that they are connected, tied to a consistent structure. This description contrasts both with work by Bao and Redish (2006) and work by diSessa (1993), but for different reasons. The former was already mentioned, but the latter is due to the fragmentary nature of novice knowledge that diSessa maintains. Note that later work by diSessa and Sherin (1998) no longer claims such fragmentary nature but instead describes causal networks of knowledge, even for novices.

What Vosniadou (1994) affords us is the ability to have "snapshots" of how misconceptions (to use her term) arise in instruction, allowing us to see the effects of confusion and instruction. The assumption is that such mental models are applied in some way to specific problems to arrive at solutions. But how exactly are they applied? And how detailed are these models, especially with respect to situations that cannot be drawn, such as the relationship between force and motion? Vosniadou notes that novices are unaware "of the hypothetical status of the presuppositions and

beliefs which constrain the way they interpret new information" (p. 67). Indeed, this claim may be true, but can we consider a description of mental models that explicitly accounts for these presuppositions and beliefs? To address this, we now turn a description of mental models that gives us a more complete model of how people reason.

How we perform deductions

The motivation behind the work of Johnson-Laird (2006) is to provide an explanation for how we reason in general; his mental models theory is applicable to everything from how we determine spatial relationships to how we know where to look for lost keys. However, for the sake of this review, we will look specifically at one special form of reasoning for which mental models are especially powerful: deduction.

Johnson-Laird (2006) proposes that we generate iconic, rather than pictorial, mental models whenever we attempt to make a deduction. These mental models correspond to possibilities that exist and are created from the content and context of language and experience. As we attempt to reason with more and more mental models, our working memory becomes overwhelmed and we make errors, explaining a large number of biases known from the psychological literature (for example, see Johnson-Laird & Byrne, 2002). In fact, Johnson-Laird (2006) claims that even considering three distinct possibilities simultaneously in our heads leads us into trouble.

So how does he propose that we reason? Consider his explanation of how we reason about premises that have a relation between them using mental models. We start a new model for each referent and then combine them according to their relationship. Then, we verify that a possibility (proposition) exists that combines the premises. If so, we search for a counterexample to refute the proposition; if not, we search for an example to make the proposition true (Johnson-Laird, 2006). In other words, we consider possibilities and then verify our findings by searching for counterexamples. Notice that in order to search for counterexamples, we must have some understanding of the meanings of the premises; if we reasoned by pure logic, we would not be able to perform such a search.

Mental models, according to Johnson-Laird (2006), are quite different from those proposed by Norman (1983) or Vosniadou (1994). Rather than describing a "target system" and an internal

representation of that target, Johnson-Laird concerns himself with how we perform the actual reasoning task. According to these mental models, we become more expert-like as our ability to reason within a specific field improves. In other words, as our reasoning within a field improves, we search more efficiently for counterexamples and improve in our ability to reason with possibilities. As such, mental models according to Johnson-Laird support a more fluid interpretation of conceptual change and learning without rejecting other models, such as network descriptions of connected knowledge. Later, we will investigate the mental models framework in more depth (see Section 2.6). For now, however, we merely present it as a third possibility for mental models and the only proposed model of reasoning that presents a testable explanation for an actual process of reasoning.

2.4.4 Conceptual change

Descriptions of conceptual change often have roots in existing frameworks, such as the ones described above. Additionally, a number of such frameworks provide their own description of conceptual change. However, other studies merely attempted to explain how we undergo conceptual change without explicitly requiring a prior framework. In this section, we will survey some explanations of conceptual change and conclude that qualitative reasoning is required for this transition.

Elicit, confront, resolve

One major thrust in conceptual change has derived from Kuhnian models from the history and philosophy of science. According to T. S. Kuhn (1962/1970), progress in science has come in the form of scientific revolutions. Paradigms develop in science, and research is often done within such a paradigm. When this paradigm needs to be changed, or when the central tenants of the philosophy need to be changed, a scientific revolution comes. Such a change can only come about after a long period of time, and only when the new theory proves to be more powerful than the old. This process, dubbed "accommodation" by Posner, Strike, Hewson, and Gertzog (1982), requires a dissatisfaction with existing concepts and that the proposed replacement be intelligible, plausible, and suggest a research roadmap. Only if these conditions are met will accommodation be possible. Even if it is plausible, it may take some time for the paradigm to shift; Lawson (2000a) provides some examples from the biology literature.

Taking the assumption that individuals can (and do) undergo conceptual change in the same way as the scientific community at large, Posner et al. (1982) provide a detailed mechanism for teaching conceptual change that formed the groundwork for what became known colloquially as "elicit, confront, resolve." Briefly, this procedure for conceptual change is described by making students explicitly state their beliefs (or initial paradigm), forcing them to confront a situation in which that belief fails to account for observed phenomena, and then providing them with a resolution in the form of an alternative paradigm. Presumably, this procedure would allow the accommodation that would be required for meaningful conceptual change. Redish (1994) further points out that "the clearer the prediction and the stronger the conflict, the better the effect (p.801)." An example of such an accommodation that Redish uses is when teachers give an exam and see poor performance by their students and recognize the conflict with their mental model of teaching as telling. To respond to this, they need to adapt a new model, resolving the conflict.

Quite a few curricula have adapted "elicit, confront, resolve" approaches. For example, McDermott and Shaffer (2002) focus on "the development of important physical concepts and scientific reasoning skills" and use a very directed approach to elicit conceptions, confront them, and resolve them with alternative (more scientifically correct) conceptualizations. This curriculum, and others, will be discussed in more detail in Section 2.5.3.

Unfortunately, the "elicit, confront, resolve" framework faces some difficult problems. First, the assumption that individuals progress in the same way as the scientific communities through paradigms is questionable and without clear foundation. Furthermore, this framework aligns itself with misconceptions research; that is, it encourages researchers to ask questions such as, "what are the incorrect beliefs that students hold and how do we fix them?" Indeed, Eylon and Linn (1988) not only provide a comprehensive (as of 1988) overview of some major misconceptions in science, but also discuss how instruction follows, by emphasizing contradictions.

This type of research not only devalues the initial conceptions of the students (after all, these conceptions were formed because they proved to be useful in some context) but also treats conceptions as things that can be confronted and replaced with better ones in a sort of unitary process. However, as Smith et al. (1994) point out, this sort of research is in stark contrast to constructivism. They ask "how misconceptions that (a) interfere with learning, (b) must be replaced, and (c) resist

instruction can also play the role of useful prior knowledge that supports students' learning" (pp. 123-124). Accordingly, they challenge the view that confrontation and replacement are appropriate and valuable for learning and teaching.

Finally, the "elicit, confront, resolve" paradigm assumes that people will act according to logic, or appeal to pure deduction when comparing models and resolving contradictions. However, as Eylon and Linn (1988) point out, students are hesitant to reconcile such contradictions as we would like. For example, Gunstone and White (1980) show that students who believe that wooden balls will fall slower than metal balls will be more likely to report upon watching a demonstration that the wooden balls do in fact hit the ground later. Clearly, even providing empirical evidence is often not enough to inspire a conceptual change, possibly because students do not value formal logic in many contexts (e.g., Reif & Larkin, 1991).

If this framework for instruction is not going to prove constructive, and some research indicates that it might not be (for example, changing the surface features or presenting an isomorphic problem soon after instruction within this framework often reveals that the actual "misconception" was not replaced after all), what other descriptions of conceptual change might we utilize?

Network approaches to conceptual change

Rather than envisioning misconceptions as being isolated fragments that can be replaced with a correct concept, network theories of consider conceptual change as either a process of linking fragmented nodes in an appropriate way (e.g., diSessa, 1993) or as rearranging the links in the network to change the pattern of activation (e.g., Wittmann, 2006). Both of these approaches take the individual resources as fixed but hold that the relationships between them can change, for example by activating or inhibiting the neural pathways connecting them (as in Redish, 2004). This model allows for more fluidity in students' use of concepts, explaining some of the data that shows that students seem to use different concepts when asked questions with different surface features (as seen from Bao et al., 2002).

diSessa (1993) treats novice processes as based on a fragmentary understanding of the world; learning something requires organizing these fragments into the coherent network of experts. diSessa (1993) thus argues that p-prims are not themselves wrong and in need of replacement, but rather

that we need to connect the p-prims in a way that activates them appropriately. For example, we want students to activate "closer is hotter" for roasting marshmallows but not for explaining the role of the seasons. We can instead choose to understand all people as reasoning from a coherent network structure while maintaining the centrality of p-prims, as diSessa and Sherin (1998) do. The challenge for educators is then to restructure the network from being organized around p-prims to being organized around physical principles. This restructuring is accomplished primarily by strengthening the links to physical principles or by creating different patterns of activation.

One example of how to describe conceptual change of coordination classes uses resource graphs to represent shifts and modifications in reasoning (Wittmann, 2006). He represents a number of distinct kinds of change and describes them with resource graphs. *Incremental change*, the addition or deletion of a resource in a resource graph, refers to including a new resource by enhancing the links to it (or removing a resource by inhibiting those resources). For example, the addition of "conservation of energy" when considering what happens when balls bounce is a form of incremental change. *Cascade change*, or the process of one change triggering further changes in the structure (either slowly or quickly), can be thought of as a substantial number of incremental changes occurring in relatively near proximity and often fairly quickly. In some ways it may appear like an "ah-ha" moment, as a set of resources that was previously inaccessible suddenly appears. For example, a statement of the form, "oh, so if we consider the momentum principle, the net force is applied to the left which means the final speed needs to be greater than the initial speed" for some specific situation.

A third type of conceptual change described by Wittmann (2006) is called *wholesale change*. This type of change requires a breaking of some of the original links that made up the network into tightly connected resources, which are then rebuilt into a (presumably stronger) network. This change may occur through incremental or cascade processes (or other processes). For example, a characterization of a student's model of motion may change from being consistent with the "impetus model" to being consistent with the "Newtonian model" (e.g., Clement, 1982; Halloun & Hestenes, 1985). This description is similar to transforming a matrix of student models in model analysis according to Bao et al. (2002).

Model analysis suggests the fourth type of conceptual change by having students' models represented in matrices that are not diagonalized; that is, Bao et al. (2002) saw that students often call upon more than one model in situations that were not separated by time. Wittmann (2006) explains this ability to hold two models simultaneously as *dual construction*. In dual construction, a single network is transformed into two networks that are simultaneously activated by a resource. Numerous reformed curricula in physics education research utilize dual constructions to require students to choose between models (such as in McDermott & Shaffer, 2002). Dual construction also presents an alternative interpretation to "elicit, confront, resolve" frameworks of conceptual change.

Wittmann (2006) describes two more types of conceptual change: *analogical reasoning* and *differentiation*. The latter of these refers to breaking down a conflated and over-connected network into numerous separate concepts. For example, differentiation may mean recognizing that certain resources related to bouncing balls need to be activated (for example, energy-based resources as opposed to momentum-based resources) in some situations but not in others.

The punch-line here is to recognize that each of the six descriptions of conceptual change, all of which align well with established literature and provide an interpretation of learning that is consistent with both recorded observations and resource theory, can also be interpreted with mental models according to Johnson-Laird (2006). Specifically, incremental change (as well as cascade change) can be associated with the revealing or elimination of possibilities; for example, recognizing that conservation of energy is applicable to problems involving bouncing balls means the inclusion of a new searchable possibility space that was previously inaccessible. If, prior to learning that conservation of energy can describe how balls bounce, one always interpreted bouncing based purely on experience, one may not understand how racquetballs heat up when they are bounced. By opening a channel of possibilities (here, through the universality of the energy conservation principle), conceptual shifts are promoted.

Similarly, wholesale change and dual construction can both be easily described by using mental models. Because conclusions are built on the fly from premises that are available (and the availability of the premises may be modeled with resource theory), wholesale changes occur when a new set of possibilities is realized, promoting some premises while inhibiting others. Similarly, dual construction theory can emerge from situations where the reasoner believes that more than one possibility remains

and therefore the premises do not uniquely define a space. Differentiation can be described in a similar way, as the elimination of possibilities results in a contradiction and a differentiation of the premises into two or more different situations. Analogical reasoning is less well defined. It may be that mental models can describe analogies in a fruitful manner similar to their use for other forms of reasoning, but such work is not yet well-established (Johnson-Laird, 1989, 2005).

The advantage in using the mental models framework to describe conceptual change in conjunction with established frameworks is that the mental models framework is predictive in a way that other network theories are not. While resource theory (or network theories in general) can provide some substantive grounding for how to understand and interpret students' actions, they do not provide us with a clear road map for interacting with them. The use of mental models suggest that we look at the space of possibilities that the students are considering and understand how they traverse that space. Adopting a mental models framework for considering conceptual change maintains agreement with the established research while affording us the ability to probe more deeply into students' reasoning.

2.5   Qualitative Reasoning and Deduction

If we want students to use "qualitative reasoning" in introductory physics classes, we need to understand what that means. While it may have different interpretations in various situations, one interpretation is that qualitative reasoning actually refers to deductive processes. In fact, some curricula in physics education research have already made this assumption and built that into tested reforms, which we will explore in some detail. Furthermore, we will discuss the question of qualitative reasoning from a different viewpoint, comparing the domain of everyday reasoning with that of physics classes.

2.5.1   What is qualitative reasoning?

Qualitative reasoning as it is discussed in physics education is often fairly ill-defined (e.g., Van Heuvelen, 1991), as noted previously (see Section 2.3). The definitions afforded us in the literature allow us to discern that deduction is one important component of qualitative reasoning, but are otherwise

quite vague.

However, qualitative reasoning has been an important study in artificial intelligence fields for quite some time, as they note that "reasoning about, and solving problems in, the physical world is one of the most fundamental capabilities of human intelligence and a fundamental subject for AI" (Bredeweg & Struss, 2003, p. 13). Specifically, while fields like science are able to develop precise and formal models about the world (and computers have no difficulty modeling these formal models), one of the more challenging tasks is understanding how humans use more commonsense reasoning to interact with the real world on a daily basis. How is it that we perceive, analyze, and adapt to the world around us?

The thrust into Artificial Intelligence research began with the Naive Physics Manifesto (Hayes, 1978), a pioneering work that set out to describe physics in everyday, commonsense language. Since then, AI has realized the difficulty in understanding all of the processes that we use to do something as simple as function in a world in spite of our incomplete knowledge of it. One way that we reason with incomplete knowledge, according to (Kuipers, 1994), which is well-aligned with the literature reviewed earlier (see Section 2.4.3), is that we build models to help us reason. We put as much information as we have into various models, store these models, pick an appropriate model for a specific situation, and then run it to make explicit some fact about the world that is important to us. This process of "making explicit" relevant facts is in fact deduction at its most fundamental. Computers have no difficulty in following logic, but people often do. In fact, it is precisely because humans think so differently from formal machine language that it is so difficult to model human reasoning in that way. Indeed, rarely do people need to be precise in their everyday lives, so being somewhat sloppy in their adaptation and application of models is acceptable; understanding the limits of this fuzzy logic is a great challenge to those studying artificial intelligence and no less of one to us.

2.5.2 Everyday versus scientific reasoning

Lawson (2000b) claims that we all use hypothetico-deductive reasoning in everyday life. Hypothetico-deductive reasoning is a process whereby we generate a hypothesis from the available data and then attempt to find evidence to disprove that hypothesis. Finding none, we accept the hypothesis for the

time being and can use it to reason about similar situations. To a rough approximation, this is what scientists do, and we all act in this way everyday. Repeating his position later (Lawson, 2005), he acknowledges that he's generalizing from this own experiences, and being a scientist himself, that his data is not scientifically valid. Still, he claims that hypothetico-deductive reasoning is our natural method of learning and believes that providing formal training for tasks supports this claim (e.g., Lawson, 1993).

Is this really how people reason? Reif and Larkin (1991) argue quite a different perspective. Because the goals of everyday life are quite different from the goals of scientific work, people use different cognitive skills to meet their needs. In everyday life, where the goal is simply to "lead a good life," people need only a rough understanding of the world and seek only adequate explanations for most events. It is only somewhat important the conclusion necessarily follows from the premises, and usually a plausibility argument is sufficient. Because of this, the reasoning method that can be used is a form of induction, amassing a large database of information, models, and schemas, and applying these to situations that appear similar: a thoroughly inductive approach. For example, you may know that when your toaster begins to smoke it's time to get a new one through experience and therefore reason that because your computer is smoking it's time to get a new computer. This argument is inductive rather than deductive (which would require a thorough investigation of the computer by taking it apart, testing connections with a voltmeter, etc.) and is therefore not guaranteed to be valid. However, it is obviously good enough for most of us most of the time, and arguments of this nature are commonplace in everyday life.

However, in a scientific domain, the goal is to "precisely model the world." Therefore, optimal prediction and explanation is required, and the validity of an argument is of paramount importance. Because of the emphasis on validity, hypothetico-deductive reasoning is used by scientists. One difficulty in instruction is that without clear indication of how they are supposed to think, students may bring their inductive, schema-matching reasoning techniques into the science classroom (Reif & Larkin, 1991). Domain cognition, or understanding that the reasoning one uses is dependent upon the reference frame and goals of the domain, is specifically what Reif and Larkin call for us to teach. The importance of domain cognition is obvious; some of the reasons and explanations we require in science classes may seem strange to students, who are not used to reasoning deductively in many

situations. For example, as Hammer and Elby (2003) pointed out, it would seem strange (if not dysfunctional) after ordering fish for dinner to go back and study the flaws in ordering pasta.

Indeed, a student's epistemology plays a fundamental role in their behaviors in class. Interviews and observations by Hammer (1994) established a useful framework for how students understand the domain of physics. Specifically, Hammer described three dimensions for characterizing students' beliefs: structure of physics knowledge as a collection of pieces or coherent system, content of physics knowledge as formula-based or concept-based, and learning physics as receiving or constructing knowledge. Not surprisingly, many introductory students hold beliefs that are decidedly not expert-like about the nature of physics. Multiple-choice assessments such as the Maryland Physics Expectations (MPEX) Survey (Redish, Saul, & Steinberg, 1998) and the Colorado Attitudes about Science Survey (Adams et al., 2006), based to some degree on the epistemological characterizations done by Hammer, have indicated not only that students' beliefs about science and learning are not what expert scientists' are, but that introductory physics courses do little to modify those beliefs (and often harm them).

Similarly, Tobias (1990) observed six graduate students and a professor, who were not experts in science, as they enrolled in an introductory physics course and recorded their experiences. The participants felt they were engaged in training rather than education and that the courses were low on theories but high on facts. In short, they didn't seem to understand the enterprise that was so obvious to the instructors of the courses, that in science we use hypothetico-deductive processes and value precision in reasoning. The participants in this study were frustrated in the physics courses because they were unaware of the importance and role of deduction and were therefore resentful of the emphasis on problem-solving rather than discourse about the functioning of the world.

In short, the bulk of the literature disagrees with Lawson (2000b); while people clearly have the ability to make deductions, in everyday situations their reasoning is more appropriately classified as inductive (matching scenarios based on familiarity), and they will not reason differently unless they understand that the goal of the scientific enterprise requires them to. This use of everyday reasoning then leads to consequences that we have already discussed, such as poor problem-solvers studying examples in an apparent attempt to build a database of exemplars, while good problem-solvers study those same examples to try to learn how they were done (Chi et al., 1989).

Clearly, then, simply providing students with conceptual questions in hopes that they will use them to enhance their physical understanding is unlikely to succeed. In fact, Sabella and Redish (2007) point out that drawing attention to conceptual questions that activate relevant resources from experts may actually inhibit resources in students that we want to activate because their knowledge structures are simply different from that of the experts. Further, if students are unsure of how to integrate this knowledge appropriately, they are likely to find themselves more confused than helped by such questions. To be successful, then, curricula must not only provide opportunities to develop deductive reasoning abilities, but they must also convey to the students the need to do so.

2.5.3   Reformed curricula and qualitative reasoning

Because of the push to encourage and develop students' use of qualitative reasoning to foster a deep understanding of physics, a number of reform curricula have been developed. These curricula explicitly call upon the students to perform deductions, although such reasoning is generally taken for granted as something that the students are able and willing to do (or in fact that they have no choice but to do). Such curricula are generally effective, and part of the reason for their effectiveness is the requirement that the students use deductive reasoning.

For example, curricula that focus on modeling (e.g., Goldberg & Bendall, 1995; Otero, Johnson, & Goldberg, 1999), following work by Hestenes (1987) require critical evaluation of the models that are generated, a process that requires deduction, often by pointing out contradictions or situations that allow precisely zero real possibilities. Interactive Lecture Demonstrations (Sokoloff & Thornton, 1997) involve eliciting predictions from students and then taking data, calling for the students to make inferences from their models, and comparing those inferences with actual data. The Matter and Interactions curriculum (Chabay & Sherwood, 2004, 2006) relies heavily not only on deductive arguments to motivate modern accepted physical models but also attempts to teach such methods to students. There are many other examples as well, but we will take a detailed look at just two, one that is very well-known and one that explicitly claims to improve scientific abilities.

University of Washington tutorials

The University of Washington group has generated a large library of physics tutorials (McDermott & Shaffer, 2002) that address major misconceptions in introductory physics. Firmly entrenched in the "elicit, confront, resolve" framework, these tutorials follow the same general approach: present a problem and ask the students to predict what would happen in such a situation, then present a tutorial that explicitly reveals the predictions made by both a correct physics model and popular incorrect models of that situation. Then, by performing a small experiment, students see that the correct model is upheld and therefore the other models are rejected. This argument hinges on deduction, namely that different models generate different predictions, and that such predictions are mutually exclusive.

One example of how this procedure works involves electricity, where a number of student difficulties are realized, such as the belief that the direction and order of elements matter and that current is used up in the circuit (McDermott & Shaffer, 1992). To address these issues, Shaffer and McDermott (1992) generated curricular materials. Students are first introduced to complete circuits by building circuits with just a bulb, battery, and piece of wire. Then they compare the brightness of a single bulb in a circuit with two in parallel and presumably deduce that because the equal brightness of the bulbs in parallel means that the current is not "used up" in a circuit. Similar observations continue, with the conclusions getting more and more complex. For these conclusions to have any meaning, they need to be explicitly made. Specifically, if students are to understand that current is not "used up" in a circuit, they need to make the explicit prediction that *if the current is used up in a circuit, then two bulbs in parallel will have different brightness.* By *modus tollens*, the observation that two bulbs in parallel have the same brightness leads to the conclusion that current is not used up in the circuit.

Such interventions have been shown to have some positive effect, but they do rely heavily on formal deductive reasoning by the students. *Modus tollens* is known to be fairly complicated to do (see Section 2.6). They also rely on identical bulbs and that the students do not claim that "they look the same to me, but there could be a slight difference that I can't observe," a situation that invokes additional possibilities.

Another favorite strategy of tutorials is to present different arguments from fictional students

and have the actual students explain whether they agree or disagree with the statements. While the authors apparently expect students to use deduction to argue in favor of or against these arguments (and the teachers can engage students in such arguments), there is no explicit research showing that students do use such arguments, or that they value them or even find them appropriate. That students understand the role of deductive arguments in science is taken as an assumption, one that is not clearly supported especially in light of the discussion in Section 2.5.2.

Investigative Science Learning Environment (ISLE)

Rather than assuming that students use certain forms of reasoning in their physics class, the Investigative Science Learning Environment (ISLE) was developed to teach students how to use scientific abilities (Etkina, Van Heuvelen, et al., 2006). Scientific abilities, according to Etkina, Van Heuvelen, et al. (2006), include many important procedures, processes, and methods that scientists use, including multiple representations, the ability to test qualitative relationships, the ability to modify a qualitative explanation, the ability to design experiments and to collect and analyze data, the ability to evaluate predictions and outcomes from experiments, and the ability to communicate with others. At least two of these abilities, testing qualitative relationships and evaluating predictions and outcomes, have obvious deductive components.

Specifically, ISLE emphasizes that experiments need to be designed to rule out, rather than support, hypotheses (Etkina, Van Heuvelen, et al., 2006; Etkina, Murthy, & Zou, 2006). As in previously examined curricula, students look critically at the implications of their (and other) models, making predictions and running experiments that may rule them out. Where ISLE differs is that such actions are explicitly examined in the curriculum rather than hidden; because students are engaged in the actual creation of experiments that have the potential to refute certain models, they must critically engage in the scientific enterprise and use hypothetico-deductive reasoning.

Additionally, when evaluating claims and experimental predictions, students engage in a process that follows a procedure characterized thus: "If (general hypothesis) and (auxiliary assumptions) then (expected result) and/but (compare actual result to expected result), therefore (conclusion)" (Etkina, Van Heuvelen, et al., 2006, p. 020103-7). This process is extended from simply referring to experiments and is taken to refer to problem-solving as well, and is used as the framework

for teaching students how to evaluate work (both others' and their own). As such, supervisory and integrated evaluation tasks are scaffolded throughout the semester to teach students a proper approach to critically evaluate work and correct it. The procedure for the supervisory task follows a procedure of finding a nearby special-case solution that has a simple answer, stating the answer to that case with reasoning, recalculating the other solution for the special case, comparing your conceptual answer to this recalculation, and concluding about the validity of the other solution. While this technique is well-aligned with expert approaches to evaluating solutions, it isn't as purely deductive as the framework suggests. Instead, there is a sort of inductive, heuristic-based feel about this approach, as internal checks for validity are not explicitly done. In other words, students are not trained to look for mistakes in deduction but rather, perhaps, for plausibility.

The research that has been done regarding ISLE has focused on how effective it is in helping students use scientific abilities in appropriate situations, and they show that it is somewhat effective, although the success varies by objective. Students transferred their ability to create and run an experiment in a biological (rather than physical) domain with moderate success, although there is little to compare the transfer rates to (Etkina, Van Heuvelen, et al., 2006). Additionally, students who did evaluation tasks benefited on problems that had utilized the same physical principles, but their problem-solving abilities did not transfer to other principles. Research has not been done on the actions that students were doing while performing evaluation tasks or engaging in experimental design; such research would be useful at informing how students reason on tasks where deduction is required.

2.6   Frameworks for Deduction

Because people do not use formal logic to solve problems, especially in everyday life, we need to ask the question, "how do they reason"? Two rival theories attempt to answer that question, and after investigating them both, we will argue for why the mental models theory is a better framework for this particular study by showing evidence of its strength both in its explanation and prediction of biases. We will further support the mental models framework with a neurological study using fMRI technology.

2.6.1 A tale of two theories

Since the late 1980s, two major types of theories have emerged from psychology to try to explain how people make deductive inferences. One is mental-logic, which actually has two distinct theories: one championed by Braine & O'Brien contends that people have rules, or inference schemata, in their minds which they use to perform deductions (Braine & O'Brien, 1998) and the other, ANDS (A Natural Deduction System) by Rips (1983) grew into another full theory (Rips, 1994) that shared some important qualities with the theory developed by Braine & O'Brien while disagreeing in other important respects.

In mental models, on the other hand, Johnson-Laird and his colleagues argue that people create and run provisional mental models, combine them, and form conclusions in this way (e.g., Johnson-Laird & Byrne, 1991). The contentions between the two sides have been often heated; for example, one particular issue of *Psychological Review* in 1994 printed three articles specifically geared at presenting and arguing both sides of the issue (O'Brien, Braine, & Yang, 1994; Bonatti, 1994; Johnson-Laird, Byrne, & Schaeken, 1994). Both theories are able to call upon evidence and are able to provide counter-evidence to refute claims made by proponents of the opposing theory, and so it would be a disservice not to look at them both in some detail.

Rule-based theories of deduction

In an attempt to model human deduction that resembles the simultaneous information-processing approaches to problem-solving in general, Rips (1983) developed a computer model that he nick-named ANDS. This system had a memory structure and inference routines to mimic deduction by holding true to a central assumption: "deductive reasoning consists in the application of mental inference rules to the premises and conclusion of an argument" (p. 40). Notice that these inference rules are not the same as formal logic; the rules, when executed in a certain order, must be able to mimic formal logic (at least in certain situations), and Rips provides a table displaying such a mapping. As in problem-solving tasks, empirical evidence for the generated model was derived from protocol analysis. These tasks confirmed ANDS's memory organization and provided a reasonable fit to its attempts; however, ANDS falls short in explaining certain phenomena such as invalid inferences.

ANDS provided a starting place for the modeling rule-based inference schema that people used to deduce. Rips (1994) continued developing his theory of mental proofs, eventually claiming this assumption:

> when people confront a problem that calls for deduction they attempt to solve it by generating in working memory a set of sentences linking the premises or givens of the problem to the conclusion or solution. Each link in this network embodies an inference rule... which the individual recognizes as intuitively sound. Taken together, this network of sentences then provides a bridge between the premises and the conclusion that explains why the conclusion follows (p.103).

Rips (1994) provides a list of some inference rules, such as "modus ponens" and "condition-alization" (p.85). These rules are production codes for performing a specific list of operations. While acknowledging that people sometimes fail in their attempts to apply these rules correctly, Rips nonetheless claims that human deduction basically falls out of a marriage of natural-deduction systems in logic with the subgoaling present in problem-solving in artificial intelligence.

While Braine and O'Brien (1998) agree that natural logics in the sense of rule-based inferences are at the core of human deduction, they present a somewhat different theory. For them, mental logic (the name of their theory of human deduction) consists of three parts: the set of inference schemas, a reasoning program that consists of both direct and indirect reasoning strategies, and a set of pragmatic principles that account for judgments that cannot be explained by the first two parts alone. Braine and O'Brien contend that this pragmatic part is not merely an afterthought, but rather is an argument that mental logic itself is embedded in a pragmatic architecture. Namely, the claim is that reasoning takes root in propositions, which refer to states of affairs that cannot exist outside of pragmatic activities such as "setting goals and understanding goals set by others" (p. 26). Thus, mental logic is more than reasoning with symbols; it is about "applying sound inference procedures to propositions assumed true to infer propositions that inherit that truth" (p. 27).

The inference schema to which O'Brien et al. (1994) refer specify how connectives like "if" and "or" and negation are used when reasoning by providing a set of appropriate deductive steps that are available to people to use, much like problem-solving schemata provide a system for solving

novel (but still relatively familiar) problems. A list of such schema are provided by Braine (1990, pp. 140-141). Like the list built by Rips (1994), modus ponens is included as a schema.

The reasoning program models how people select the appropriate schema (O'Brien et al., 1994), and it consists of two parts: a direct reasoning routine and a set of strategies. The direct reasoning routine allows for some problems to be solved immediately and routinely, while the set of strategies tackles new and difficult reasoning challenges, much like information processing experiments in problem-solving showed that experts have both "physical intuition" (to help them solve routine problems) and could use "qualitative reasoning" (to help them with new, challenging problems).

Such a rules-based theory is advantageous not only because it is familiar in form, but also because it allows one to make predictions that seem sensical, such as that the difficulty of a problem should be a function of the number of steps (and the difficulty of those steps) required to solve it by applying the schema rules of mental logic (Braine, Reiser, & Rumain, 1984). Additionally, O'Brien et al. (1994) claim that it is parsimonious and reasonable, while the mental models theory is implausible and fails to explain certain results. We now turn to that theory to present its arguments.

The mental models theory of deduction

As opposed to rule-based theories, which claim that people analyze statements in terms of their form and use of rules of inference to reach a conclusion in a chain of deductions that resembles (but is not) a formal proof, semantic models postulate that people build models of situations, explicitly accounting for possibilities, and use these to develop a conclusion (e.g., Johnson-Laird et al., 1994). For example, consider the example given in the previous citation: given the premises "either the battery is dead or there is a short in the circuit" and "the battery is not dead," we generate the following two mental models:

s

d

where s stands for "short circuit" and d stands for "dead battery." Each line denotes a model, indicating separate possibilities. In the top model, where a short circuit is possible, nothing is said about the state of the battery; in the bottom model, where a dead battery is possible, nothing is said about the state of the circuit. Upon mention of the second premise, that the battery is not dead,

the second model is eliminated and we are left with only one possibility and a conclusion: there is a short in the circuit.

Conditionals are handled in somewhat the same manner, but evidence regarding the difficulty of *modus tollens* and the frequency of denial of the antecedent and affirmation of the consequent have led to a vastly developed framework to handle not only the core, philosophical definition of conditionals but also the semantic and pragmatic modulations that conditionals undergo in various situations, which modify their meaning (Johnson-Laird & Byrne, 2002). In short, according to mental models theory we interpret a conditional, "if A then C" in the following way:

>   A   C
>
>   ...

where the ellipse indicates a mental footnote that there are further mental models that *could* be generated. If we were to fully flesh out these models, they would take this appearance:

>   A   C
>
>   not-A   C
>
>   not-A   not-C

With these models, we would be able to respond appropriately to any follow-up premise, such as "not-C" or "A." If, however, we do not fully flesh out the models, we would be unable to reduce or expand the number of possibilities that we can conceive of appropriately. For example, if we have only generated one model and receive the follow-up premise, "C," we may err into affirming the consequent and generating the conclusion, "A."

Consider one more example, which combines three premises: "if A then C; if C then not-D; D." The models for the first premise are given above; the models for the second premise are:

>   C   not-D
>
>   not-C   not-D
>
>   not-C   D

Combining these two sets of models to eliminate C, we get:

>   A   not-D
>
>   not-A   not-D
>
>   not-A   D

And therefore, D implies not-A, because not-A is the only possibility remaining. Fully processing this conditional requires fleshing out all of the models, even those that are initially represented by mental footnotes.

However, people tend to forget the mental footnotes they generate (Barres & Johnson-Laird, 2003), which makes them err in ways similar to the example above. Additionally, both semantic and pragmatic information can influence which mental models are considered and which are not (Johnson-Laird & Byrne, 2002). Semantic meaning refers to the meanings of the antecedent and the consequent; if they are related as in the conditional, "if it's a game, then it's not soccer," then that meaning controls which models are expanded (here, inhibiting the expansion of the model that holds both "not-game" and "soccer." Pragmatic meaning refers to the context in which the phrase was uttered or appears in reading; in a sense, this is similar to the idea that the framing of a task affects which resources are activated (Hammer et al., 2005). For example, pragmatic meaning allows us to reconcile arguments that take a form of "if A then C; if B then not-C; A and B," such as when dealing with conflicting evidence about someone's reliability. Notice that such contradictions are not explicit in physical models, but may be present in a person's understanding and present a formidable obstacle for that person to resolve.

Note that the mental models framework has been simulated with computer programs, which are able to emulate various levels of performance ranging from flawless deduction to very erroneous deductions (such as situations where mental footnotes are all forgotten), with reasonable success (e.g., Johnson-Laird, 2006; Johnson-Laird & Byrne, 1991; Johnson-Laird, Byrne, & Schaeken, 1992).

2.6.2   Support for mental models

Next, we need to take a look at how deduction has been handled in the psychology literature, where it has been the subject of interest. Studies of people reasoning from syllogisms, from conditionals, and on sorting tasks have led to revealing results exposing common difficulties in reasoning. The studies that follow are by no means exhaustive, but they are sufficient to provide a survey of the relevant and useful issues that mental models are able to address. Then we will look at a result from neuroscience that supports the mental model theory.

Difficulty of *modus tollens*

Ubiquitous in the literature is the claim that *modus tollens* is significantly harder to perform than *modus ponens*. In fact, according to Evans (1993), people are slower and less accurate when performing *modus tollens*, using it correctly only about 70% of the time. Both rule theories and mental models theory provide explanations for this phenomenon. Rule theories claim that while *modus ponens* is hardwired in our brain, *modus tollens* requires a multi-step process including *reductio ad absurdum*, which postulates that p is possible and thus results in a contradiction (Evans, 2005). Because it requires more steps than *modus ponens*, it is more error-prone.

The mental models theory claims that we represent only a single mental model for "if p then q" explicitly:

>   p   q
>
>   ...

Expanding to include the other two models requires time and effort (explaining the results that people are slower and less accurate than when fleshing out additional models is not required), and doing so requires management of three mental models rather than one, using up additional resources in working memory. This framework allows us to make a prediction about an experiment that could distinguish between the rule theory explanation and the mental models theory: if, instead of proposing the conditional statement, a biconditional ("if and only if p then q") is proposed, mental models theory predicts that both possible models will be elicited:

>   p   q
>
>   not-p   not-q

This prediction means that biconditionals should slow the response time and reduce accuracy for *modus ponens* while simultaneously improve accuracy and speed for *modus tollens*. In fact, this is what happens (Johnson-Laird et al., 1992).

Belief bias

Before looking at belief bias proper, we should take a moment to investigate a related effect: soundness versus validity. Recall that sound arguments are valid arguments with true premises; usually in psychology studies, participants are asked to indicate whether arguments are valid. In fact, they

often consider the truth of the premises in addition to the validity of the arguments (Thompson, 1996). The two effects, the validity of the argument and the truth of the premises, did not have an interaction, which implies that participants did not likely evaluate the premises first and then decide whether to use deduction on the argument. Thompson provides two possible explanations: either people replace unbelievable premises with believable ones and perform deductions on those, as Markovits and Vachon (1989) claim occurs with young children, or that a filtering mechanism exists to screen premises for believability, similar to a screening process proposed by Newstead, Pollard, Evans, and Allen (1992) for screening for believable conclusions, a proposed mechanism for explaining belief bias. In either case, the phenomenon is that people often reject conclusions not because they are invalid but because they are based on unbelievable premises.

The belief bias proper is a related claim: people often reject conclusions because they are unbelievable (or accept them because they are believable), regardless of the validity of the argument. Again, there are two effects: validity and believability (this time in the conclusion of the argument); unlike in the previous case, these two effects generally interact in the belief bias. In three separate experiments regarding categorical syllogisms, consistently large effects of belief bias were noted (Evans, Barston, & Pollard, 1983). Through analysis of protocol data, they found that while subjects generally accept invalid but believable conclusions, they only sometimes accept valid but unbelievable conclusions. Furthermore, when explaining their acceptance of valid arguments, participants focused on the premises but while explaining their acceptance of believable conclusions, participants focused on extraneous information. The results also showed a within-subject rather than between-subject conflict, implying that people reason differently in different situations.

Evans, Handley, and Harper (2001) claim that unbelievable conclusions help suppress logical fallacies; while untrue conclusions may be the result of either invalid arguments or invalid premises, if the premises are true and the conclusion false then the argument must have been invalid. Reasoning in this way, it seems more likely that an untrue conclusion is the result of an invalid argument. Based on studies like these, (Evans, 2005) actually calls this effect a "belief debias."

However, when provided with two premises that cannot be judged either true or false and an unbelievable conclusion, people are likely to reason that the argument must be invalid. For example, (Newstead et al., 1992) used syllogisms with nonsense words to elicit belief bias; while resulting

in clearly absurd conclusions, the arguments themselves were valid. It is possible that they were rejected outright because arguments with false conclusions must be unsound, and the bias already noted regarding soundness and validity comes into play (Evans, 2005).

The belief bias aligns with research that shows that people can reason differently in different situations, a result seen time and again in physics education research, and it is explained differently: students hold simultaneous models (Bao et al., 2002), students suffer from misconceptions when trying to accommodate information (Vosniadou, 1994), or students perform differently depending on the framing of the problem (Hammer et al., 2005) among others. Mental model theory can describe belief bias by noting that certain possibilities may not be considered. That is, if a certain conclusion is seen as unbelievable, that model may be eliminated unduly from the set of possibilities. If, for instance, the following models are available:

    a   b
               b   c
    not-a   not-c

but b is a completely unbelievable happening, those models will be eliminated and the conclusion "not-a and not-c" will be reached, perhaps too quickly.

Belief bias is extremely relevant to physics because of the number of counterintuitive or surprising conclusions that are drawn at the end of a long chain of reasoning. If an argument finds only a single physical possibility and that possibility is unbelievable, a student may refuse to conceive of it, choosing instead to believe that there must have been a flaw in the argument.

The effects of content and context

Wason (1968) created the following popular reasoning task: a participant is shown four cards on a table that have either numbers or letters showing (for example, the four cards might show K, D, 2, and 3). The participant is informed that each card has a letter on one side and a number on the other. The participant is further told a rule applicable only for those four cards (for example, "if there is a K on one side of the card, then there is a 2 on the other side"). The participant's task is to select which of the cards need to be flipped over (thereby exposing the other side) to disprove the rule. While the logically correct choice is "K" and "3," most participants choose to flip either

just the "K" or the "K" and the "2." A number of variations and possible explanations for this result have developed over the years, and both rule-based theories and mental models theory provide explanations.

The first explanation was that participants were trying to confirm, rather than disprove, the rule (Wason & Shapiro, 1971; Wason & Johnson-Laird, 1972). This explanation was soon abandoned, as Evans (1973) showed that when participants were provided with a rule with negations (such as, "all cards with no K on one side have a 2 on the other"), participants still selected the same two cards. As a result, Evans (1973) proposed a matching bias; people chose "K" and "2" because these were explicitly mentioned. Johnson-Laird and Byrne (1991) have shown that this can be explained using mental models.

Instead of providing abstract letters and numbers on the cards, (Griggs & Cox, 1982) repeated the study with a more realistic situation is given such as "four people are sitting around a table in a bar, and each card in front of you is a representation of a person. Each card contains information about that person's age on one side of the card and the beverage (a specific type of cola or beer) they're drinking on the other. Which card or cards do you need to flip over to verify that no one is illegally drinking alcohol?" In this case, participants overwhelmingly chose the correct two cards. Even if the participants had never experienced the real-world situation represented in such tasks, rules that express permission or obligation are usually sufficient for the participants to succeed (Evans, 2005). Cheng and Holyoak (1985) argue that people have "permission schema" with production rules that activate upon entering a bar (or similar situation); these schema are activated in this adaptation of the card task, and proper selections are then made. Later, they added a similar "obligation schema" (Holyoak & Cheng, 1995).

However, Johnson-Laird and Byrne (1991) list five specific modifications that can be made to the Wason selection task that result in a significant change in performance: change the form of the rule, the content of the rule, the context of the rule, the content of the cards, or change the task (that is, instruct participants to test violations of the rule). Clearly, the content and context of a task affects our performance, a result that has recently been noted, often as an argument for the difficulty of transfer and recognition that people work very much within certain contexts (Redish, 2004; Hammer et al., 2005). Again, with mental models theory, these effects can be explained by

the enhancement or inhibition of specific models.

Suppression of valid inferences

Logical fallacies can be suppressed by providing additional information (Johnson-Laird & Byrne, 1991). For example, the premise, "if she meets her friend, then she will go to the play," encourages *affirming the consequent*. However, the addition of another premise such as, "if she meets her brother, then she will go to a play," inhibits the fallacy. Markovits (1989) conducted such an experiment, embedding conditionals in paragraphs that provide alternative explanations for the observed phenomena. If those alternatives are accessible to the reasoner, illogical conclusions are not drawn as frequently. As before, making alternatives apparent aids reasoning, as explained by mental models theory. It should be noted that in everyday life, such inferences may be suppressed more than in a laboratory, as participants may assume that there is no additional information in controlled experiments. Again, the framing of the task plays a significant role in how that task is accomplished.

Byrne (1989) followed these suppressions of fallacies with similar experiment to show that valid inferences can also be suppressed. Providing additional information such as, "if she has enough money, she will go to the play," prevents participants from performing correct *modus ponens* on the original statement about meeting her friend. While these two statements had the same form as the pair that suppressed the affirmation of the consequent, they had different meaning. Therefore, semantic information played an important role in the conclusion that was drawn, further arguing for mental models and against rule-based theories.

Note that in physics, we often provide multiple pieces of information phrased as conditionals. How the students combine numerous statements that refer to the same conclusion in a physics classroom is not well understood; they are capable of suppressing both fallacies and inferences.

Support from neuroscience

In addition to psychology experiments that support mental models, there is a significant functional MRI study that addresses the question of where deductions take place in the brain (Knauff, Mulack, Kassubek, Salih, & Greenlee, 2002). Three competing theories of how we perform deductions yield

three different predictions: rule-based theories claim language-based proofs, mental model theory claims construction and manipulation of spatially organized material, and imagery theories claim the use of visual mental images. To test these theories, twelve healthy right-handed males were given both conditional reasoning and relational reasoning problems and the blood oxygenation level dependent (BOLD) response was measured.

Activation was seen in the bilateral occipito-parietal network and prefrontal cortical areas, which are associated with spatial reasoning; however, activation was not seen in the primary visual cortex, which would be associated with the visual mental image theory (Knauff et al., 2002). However, other studies of mental imagery do not necessarily show activation in the primary visual cortex, and so the relationship between the observed results and the visual mental image theory is not clear. Additionally, activity in the middle frontal gyrus implies that reasoners are actively manipulating and inspecting the information in working memory. Thus, Knauff et al. conclude that their evidence appears to corroborate the mental model theory, but that rule-based accounts of reasoning are explicitly not supported.

2.7 Related Studies About Reasoning in Science

While no studies in physics education research have specifically focused on how students use deduction when solving problems, some studies in physics or related fields have taken a careful look at how students reason. In this section, we will take a brief look at some frameworks and experiments that have shed some light on how people reason in science.

2.7.1 Reasoning and discourse

Rather than using scientific reasoning as an assumed starting point, from which curricular developments may take place, D. Kuhn (1993) argues that scientific thinking should be treated like an endpoint that comes from a complex process of intellectual development, and that scientific reasoning should be explicitly taught. The way in which it is taught is by conceiving of science as a social activity, and recognizing that science cannot be divorced from controversy or argument. In short, science is argument; as legal advocacy models require explicit justification for claims, so does

science. By asking the question of how people justify their claims in everyday or informal reasoning situations, D. Kuhn claims that we can find and develop scientific reasoning (including deduction).

Using this framework, D. Kuhn (1993) analyzed everyday dialogic arguments about topics like crime, where there are two different positions and both of them cannot be correct. Sometimes people use evidence (in the scientific sense) to defend their claims, while others do not. When those who are unable to generate evidence are asked for an alternative reason, many are unable to come up with one. About half of the participants are able to specify what evidence, if provided, would disprove their stance. Taken together these sorts of observations indicate that people need to "distance themselves from their own beliefs to a sufficient degree to be able to evaluate them, as objects of cognition" (D. Kuhn, 1993, p. 331). Moreover, one of the major challenges for learners is to coordinate their existing theories with the new evidence they themselves generate.

D. Kuhn (1993) concludes that any theory that treats the development of scientific thinking as only relevant within the domain of study loses explanatory power, because the development of children through adulthood with respect to scientific thinking is gradual and must be repeated over and over again in many different contexts. Not only must they learn appropriate reasoning strategies, they also must learn how and where and why to use them. D. Kuhn thus seems to agree in some sense with both the mental models framework (which does not treat the process of reasoning that people use in physics class as separate from the process they use in a grocery store) and the claim by Reif and Larkin (1991) that the goals of the enterprise help determine what conclusions and supporting evidence are appropriate. She adds, however, a plea to link scientific thinking with everyday thinking, encouraging instructors to be explicit about the power that is unique to scientific thinking and to relate that to the students' own lives.

One related effort in science education has been research into mechanistic reasoning; that is, whether and how students use reasoning that not only describes causes for phenomena but particular ways in which such phenomena occur. Mechanistic reasoning has suffered from a sort of, "you know it when you see it" interpretation for some time, so Russ, Scherr, Hammer, and Mikeska (2008) developed a coding scheme to more precisely identify the features that are present when people use mechanistic reasoning so that such reasoning is more evident in argument and dialog. Their goal was to analyze reasoning as mechanistic, not debating its correctness in a given situation, separating

their research from other work in education research that focuses on concepts.

The coding scheme they propose consists of a hierarchy of seven codes: "describing the target phenomenon, identifying setup conditions, identifying entities, identifying activities, properties of entities, organization of entities, and chaining" (Russ et al., 2008, p. 512). The first listed are the lowest order and provide little evidence of mechanistic reasoning, while higher levels cannot occur in isolation (without at least some of the lower ones) and indicate significant evidence for mechanistic reasoning. In addition, they identify analogy and animated models as codes; these are not hierarchical but may be used in numerous ways. This particular organization for coding presents a unique opportunity to analyze arguments from children through adults without attempting to classify them according to concepts or other mental constructs.

Further research into students' use of scientific argument is actively being done by Atkins (2008). Basing her work in the theoretical constructs proposed by Toulmin, Rieke, and Janick (1979) and Sfard (2007), Atkins focuses on how students use evidence and mechanisms in their arguments. This analysis relies on a formal reasoning, as it depends on warrants and grounds being taken together as premises to yield the claim. Warrants are specifically premises that refer to general phenomena, while the grounds refer specifically to the problem at hand. Warrants are often implied, and so they often play a hidden role in argumentation. As such, two different rules for arguments are often used, ones focusing on the evidence (grounds) and ones focusing on the warrants (mechanism). These may be used differently by different students, leading to an impasse in the conversation. Therefore, making the rules of the argument (that is, what warrants and grounds we are using) explicit is important for discussions to have any meaning.

Similarly, a thesis by Hollabaugh (1995) utilizes the Toulmin et al. (1979) framework to analyze how cooperative groups set up a physical description of challenging physics problems. He found that in most cases, all members of the groups contributed to the construction of an argument, consisting of a claim, grounds, warrants, and backing. By so doing, Hollabaugh insinuates, the arguments that are constructed by the group are understood by all members and are more than just an argument made by the strongest member; this perspective provides a description of how cooperative groups can construct superior understanding and performance on problem-solving tasks. Furthermore, Hollabaugh's work implies that the nature of constructing physical descriptions of problems in groups

entails some amount of negotiating the role of warrants and grounds to develop claims about the problems at hand.

Voss et al. (1983) also used a Toulmin et al. (1979) framework to analyze arguments made by experts and novices in social science tackling the question of how to survive an impending agricultural crisis in Russia. Voss et al. point out that this framework is especially well suited for ill-defined problems, where the problem statement or the goal statement is not made explicit, and where there is no agreed-upon correct answer. In such situations, the ability to make a claim and support it with warrants and grounds is paramount. Prior to 1983, the framework had not been used in cognitive science studies of physics problem solving because it didn't adequately describe the techniques that were being used by experts or novices; a full justification of individual steps in reasoning is not usually done for standard physics problems. The Toulmin et al. (1979) framework is especially relevant for considering what constitutes grounds, warrants, or backings for claims about physical descriptions people use. However, the framework does not explain how the grounds and warrants are combined to form a claim; the assumption remains that some form of logical reasoning is used to combine them. Therefore, while the Toulmin et al. (1979) framework is well-suited for describing attempts to solve open problems such as one might encounter in social science or cutting-edge physics or ill-defined problems such as one might encounter in certain introductory physics pedagogical reform efforts, it does not adequately provide a description of how deductions are done on physics tasks that can be solved through the use of deduction alone, where the question at hand is how that deduction is accomplished.

2.7.2 Hypothetico-deductive reasoning in science

Lawson (1978, 1987) is perhaps most well known for his creation of a test of scientific reasoning, which was initially billed as a way to classify the formal reasoning abilities of students as formal operational, concrete operational, or transitional. Lawson (1992) has since questioned that applicability of such tests, proposing instead that we consider using his test to classify students as "intuitive" or "reflective" thinkers, according to whether they reason from mental representations of specific situations, or whether they tend to be more reflective, searching for alternative explanations. However, the scientific reasoning test continues to be used in its original intent, even by Lawson himself, for

example in a study that described types of scientific concepts that exist and showed that a significant relationship exists between conceptual knowledge and the developmental level found by the test, indicating that declarative knowledge acquisition and concept construction are correlated with high scores on the exam (Lawson, Alkhoury, Benford, Clark, & Falconer, 2000).

Over the years, Lawson has continued to research scientific reasoning, from developmental studies (Lawson, 1993) to questioning whether formal logic plays a role in reasoning (Lawson, 1991) to the role of hypothetico-deduction and induction in scientific inquiry (Lawson, 2000a, 2003, 2005) and the role of hypotheses in reasoning (Lawson, 2005). We will briefly investigate his results and discuss the relevance to deductive reasoning in physics.

Developmental studies

Using a task he developed, called the Mellinark task, Lawson (1993) showed that concept acquisition was strongly age-dependent. In the original Mellinark task (Lawson et al., 1991), high-school students were shown a set of diagrams and told they were Mellinarks and another set that were not Mellinarks and then asked to pick out which of a third set were Mellinarks. The poor performance on this task was startling, and Lawson et al. interpreted the results by saying that the students who failed had not yet reached a sufficiently high reasoning ability, according to his scientific reasoning test.

In the follow-up task, Lawson (1993) introduced training to address his original hypothesis; namely, if the training succeeds, then the failure of the students on the task cannot be attributed to a lack of deductive reasoning skills, which are unlikely to be developed in fifteen minutes. As a result of the training, nearly all children older than seven years old were successful, but no children younger than seven were. The age of seven years is significant; during this time, a major growth spurt of the frontal lobes, the home of working memory and higher-order functions, occurs. Lawson goes on to suggest that network models of conceptual changes may help explain the data, and he concludes that because two types of reasoning are needed to perform well on the Mellinark task (hypothetico-deductive and inductive-deductive), that many high school students are not skilled in hypothetico-deductive reasoning while virtually all children at age seven are skilled in inductive-deductive reasoning. Specifically, it seems that high school students have particular difficulty when

their first hypothesis about what strategy is required for the task is wrong; in such a situation, fully hypothetico-deductive reasoning is required. The training regimen provided a way for such students to reject incorrect hypotheses and reformulate them.

The role of formal reasoning

To argue against the idea that when adults have reached a certain stage of development, namely the formal operational stage, that they reason with abstract logic, (Lawson, 1991) proposes an alternate theory, if only to show that alternate theories can be proposed that would better fit the data. The alternate hypothesis theory states that rather than using logic, people use context and their awareness of other possible hypotheses. To test this hypothesis, he generated a set of seven conditionals and asked a group of 922 students to draw one-step conclusions from them (each participant responded to only one conditional). His data showed that the context and content of the conditional, rather than the form, determined the number of possible one-step conclusions the student drew, results that upheld the alternate hypothesis theory, a result that was also used to argue against the use of formal reasoning stages for classification of students on his exam (Lawson, 1992). It should be noted how well the result of this study corroborate the mental models theory; the possibilities listed by the students reflected the suppressions expected by the mental models framework.

The role of hypothetico-deduction and induction in science

A survey of biology textbooks by Gibbs and Lawson (1992) showed that the role of deduction was significantly downplayed; they did not instruct students what deduction is, what it's for, or how to do it. Instead, the textbooks focused on the acquisition of facts. Spurred by this, Lawson sought to make explicit the role that various forms of reasoning, specifically hypothetico-deduction and induction, are relevant in scientific endeavors like biology. Some arguments have been historical (e.g., Lawson, 2000a) and focus on the hypothetico-deductive arguments that have led to major advancements in science. Others emphasize that many arguments that scientists make involve forming hypotheses, drawing conclusions, and then testing whether the predicted outcome actually occurs. If it does occur, then the hypothesis is supported (possibly true), whereas if it fails to occur,

then the hypothesis is rejected (if the experiment was done correctly). Lawson (1993) has hinted that the difficult part of the hypothetico-deductive process for students is knowing when to reject a hypothesis and formulate a new one.

Lawson (2005) showed that people process information by using hypothetico-deductive reasoning and that therefore science instruction should provide opportunities to students to develop and test complex and abstract hypotheses, a sentiment that Etkina, Murthy, and Zou (2006) has used for curricular reform. Again using a form of the Mellinark task, Lawson (2005) tested whether students would match patterns or whether they would identify critical features and generalize from them. In this task, students were divided into two groups, both of which received a set of Mellinarks and a set of possible Mellinarks. The difference was that in one group, the set of Mellinarks was a set of five identical pictures; in the other group, it was only one. Because the number of possible Mellinarks selected by the students was suppressed in the first case, Lawson (2005) reasoned that people use hypothetico-deductive processes (specifically using quasi-formal reasoning of the type "IF being upside down allows an object to be a Mellinark AND I look at all of the Mellinarks in my set, THEN at lease one of them should be upside down. BUT none of them are upside down, so that upside down creature is probably not a Mellinark. Therefore, people use hypothetico-deductive rather than inductive reasoning to make this decision.

It is interesting to reconsider the results of the Lawson (2005) experiment in terms of mental models. Repeated exposure to the same single mental model seemed to inhibit people from forming additional possible (allowed) models by extrapolating the shape or features of Mellinarks from the set when the set consisted of five instances of the same object. Not having a field of counterexamples to select from, one is left thinking that perhaps only that single shape is permitted. In fact, the mental models framework can explain this phenomenon without claiming that formal logic of the form suggested by Lawson is used.

Relationship to deduction in physics

Perhaps the most striking conclusion about the work done by Lawson is that while it doesn't present much that is surprising or orthogonal to the mental models framework, its fundamental philosophy does not seem to be consistent. Sometimes, Lawson (1991, 1992) seems to argue that the scientific

reasoning is not a universal ability, nor is it appropriate to classify people according to their formal reasoning ability; other times, he makes sweeping claims about hypothetico-deductive reasoning being a resource that students either have or don't (e.g., Lawson, 1993). Overall, however, Lawson has done much to shed light on the relevance of deduction in the teaching and learning of science and to encourage its inclusion in science curricula (e.g., Gibbs & Lawson, 1992; Lawson, 2000a, 2005).

2.8 Literature Related to the Development of the Selection Task

The task that formed the core of this dissertation was motivated by the literature in a number of ways. The use of worked examples to enhance learning has been studied extensively, and they are known to be useful ways to generate student self-explanations, which could be used to track students' reasoning. Additionally, while very little has been studied about how students learn from incorrect solutions, incorrect solutions have been used in pedagogies for some time now. Ideally, worked incorrect solutions should generate substantial student self-explanations.

2.8.1 Worked examples and self-explanations

Worked examples have been shown to help students learn how to solve problems in many contexts. Specific studies have shown how successful students learn from the worked examples. For example, successful students generate more negative-monitoring statements, where they question the actions of the worked example by indicating their own lack of understanding (Chi et al., 1989; Pirolli & Recker, 1994; Stark et al., 2002). Successful students also tend to generate more self-explanations (Chi et al., 1989; Chi & VanLehn, 1991). Self-explanations refer to actions taken by learners that indicate an attempt to generate their own explanation for a piece of the worked example that does not have accompanying reasoning. In other words, successful students attempt to explain steps in the worked out solution. This extra cognitive effort reveals itself when they are permitted to use worked examples on isomorphic problems. Rather than starting on the first line of the example as poor performers did, good performers went immediately to the relevant line of the example, using it to help solve the problem (Chi et al., 1989).

Thanks to such research, worked examples have been structured to emphasize features, like encouraging students to generate self-explanations, to help all students learn effectively from them. Atkinson, Derry, Renkl, and Wortham (2000) summarize many such features. For example, they emphasize the need to combine various representations (such as pictures, text, and equations) in a way that encourages self-explanations without overwhelming them cognitively through split attention.

Building on these results, computer aides such as the SE (Self Explanation)-Coach (Conati & VanLehn, 2000) have attempted to guide students to more effectively use worked examples. While these have seen some success, the results are mixed, with students getting distracted by the prompts or otherwise ignoring them. Additionally, the scaffolding provided by the SE-Coach may require too much effort on the part of the learner, especially after some proficiency of the subject matter has been achieved.

Taking the need for self-explanations in a different direction, Catrambone (1998) split probability examples based on subgoals and provided labels for those subgoals. Because of the lack of a link to the context and because it required more processing by the learners, he found that abstract labels were more helpful for transfer. Similarly, Stark et al. (2002) found that using incomplete solutions allowed students with more prior knowledge to learn more than those with less prior knowledge.

In much the same way, Große and Renkl (2007), in a study about basic probability, found that incorrect solutions benefited students with more prior knowledge without helping those who had limited prior knowledge. In their study, they created a 3 x 2-factor design, with type of solution combined with the presence of prompts for self-explanations. All 118 participants were initially given correct solutions to study. Some of the participants were then given one of two types of incorrect solutions: either the error was highlighted or it was not. Half of the participants were then asked to provide self-explanations without prompts and the other half with prompts. The type of prompts for highlighted errors were of the type, "why is the indicated step not correct?" while the prompts for the solutions without highlighted errors prompted the participants to discuss which step was incorrect and why, what the correct solution might look like, and a problem for which the solution would actually be correct. They found that for both near and far transfer, neither the presence of prompts nor the type of solution provided a significant effect; however, as with incomplete solutions, both variables interacted with prior knowledge.

In the discussion provided by Große and Renkl (2007), potential explanations are provided for why students with low prior knowledge might not benefit from the use of incorrect solutions: they might be incapable of generating useful self-explanations, they might experience very high cognitive demands, and they might be incapable of learning from incorrect solutions simply because they do not know why the solution is incorrect. The authors further claimed that participants with low prior knowledge might have had fewer experiences with the type of error and therefore may not be able to recognize or explain why it was wrong. To address this question, they explored the nature of self-explanations provided by the participants. This second experiment found that participants given both correct and incorrect solutions verbalized more than those who were only given correct solutions to study, indicating that incorrect solutions motivated more extensive thinking. Most notably, incorrect solutions provided more impasses (indicators of a struggle to understand the current situation) while reducing the number of principle-based explanations. Additionally, when learning with incorrect solutions, learners often "show spontaneous elaborations concerning the errors" (p. 626) which should help learning; however, any benefit from this effect was in opposition to the reduction in principle-based explanations, resulting in an overall wash in performance. In short, therefore, the authors found no positive overall effects in using both correct and incorrect solutions, although the self-explanations their participants generated were intriguing. It is with these results in mind that we designed our particular study exploring correct and incorrect solutions.

2.9   Pedagogical Reforms Using Incorrect Solutions

Regardless of the lack of literature clearly supporting the use of incorrect solutions in physics, a few instances of their use in classroom teaching have developed. These instances are usually parts of larger programs, and as such, it is difficult to disentangle the specific effect of those incorrect solutions. Nonetheless, intuition and anecdotal evidence have supported their development and use; therefore, it is relevant to summarize some of the forms in which they have appeared.

TIPERs (Tasks Inspired by Physics Education Research) are tasks that have been developed for electrostatics and magnetism and consist of a wide variety of activities that are meant to be used together to supplement standard physics instruction (Hieggelke, Maloney, Kanim, & O'Kuma,

2006). One specific task, called the "What, if anything, is Wrong Task (WWT)," requires students to analyze a situation to determine whether anything is wrong. If the student believes it is correct, he or she then must explain why the situation works as described. If it is incorrect, then he or she must not only identify the error but also explain how to correct it. This task seems to be mostly conceptual in nature rather than based on standard physics problems, and many of the activities have a form like, "A student says..." This activity cites its origin from a description of similar tasks given to students, which showed that even honors students have the same sorts of difficulty with concepts in physics as students in lower-level courses (Peters, 1982). As such, these tasks are not literally incorrectly solved physics problems but rather instances of conceptual checking.

Reif and Scott (1999) developed computer programs called PALs (Personal Assistants for Learning), which take advantage of a Reciprocal Teaching strategy, based on research done by Palinscar and Brown (1984) on how to improve elementary reading comprehension. During the reciprocal learning instruction, the student decides the approach to solving the problem and provides instructions for the tutor, which then acts accordingly. However, the tutor often makes student-like mistakes, which the student then must recognize and correct. This approach is intended to highlight common student errors and engage the learner in correcting them before he or she makes them.

ISLE (Investigative Science Learning Environment), developed at Rutgers (Etkina, Van Heuvelen, et al., 2006), is an innovative pedagogy that combines lectures with laboratories to provide students with a set of scientific skills. One particular skill that the pedagogy develops is the "ability to evaluate experimental predictions and outcomes, conceptual claims, problem solutions, and models." In short, this is the ability to recognize mistakes and correct them. When students perform the "supervisory task," they work through a prescribed method to check whether a solution provided is correct or incorrect. The student essentially works the problem, checking the limiting cases, and judging whether or not the given solution is correct based on that work.

Harper (2001) describes an activity where students are presented a fully worked physics problem solution and they are asked to determine whether there are errors in it, and if so where they are. These problems, called "WRONG" problems, are used in problem-solving sessions, along with other unique kinds of problems like "Jeopardy" problems, where students are given a solution and are asked to write a problem statement for that solution (Van Heuvelen & Maloney, 1999). While these

developments seem to be beneficial, published research has only provided minimal evidence of their success.

To supplement pedagogical reforms that use incorrect solutions, Yerushalmi and Polingher (2006) have provided a guide for developing appropriate incorrect solutions for classroom use. They warn against using student-generated incorrect solutions in favor of instructor-designed solutions that emphasize the specific error to be learned.

2.10   Conclusion

We have seen from the literature that people do not use abstract, formal logic in their reasoning. Therefore, we need to model what they actually do when problem-solving so that we can learn how to teach them. One likely candidate that may prove helpful is the mental models framework generated by Johnson-Laird, which has not yet been tested or used to explain how students in physics make deductions. Such a framework is compatible with available data and models but stands to make further predictions, namely that explicit instruction that helps students find and evaluate physical possibilities should improve their ability to reason deductively to solve problems.

3

Possibilities Framework

In this chapter, I will discuss a new framework that can be used to describe how students reason while solving physics problems that require deduction. This framework is adapted heavily from the mental models framework laid out by Johnson-Laird (2006), and it developed as a result of the investigations in this dissertation. It is presented here because having a framework helps significantly in describing and connecting these studies. The students' reasoning in all studies fits into the framework well enough to justify its use throughout this dissertation.

According to the possibilities framework, reasoners consider possibilities, eliminating impossibilities, until he or she believes that only one possibility remains at which point the reasoner draws a conclusion. By tracking these possibilities, we will be able to describe that person's process of reasoning. We can identify the physical quantities the reasoner believes are relevant to the solution of a problem, together with the relationship between those quantities and possible values for those quantities. This chapter will describe the structure of the diagrams we use in the possibilities framework, provide examples of how the framework describes the reasoning process, and set the stage for understanding how students perform deduction.

3.1 The Structure of the Possibilities Framework

When solving physics problems, reasoners can hold one or more sets of possibilities in working memory at a given time. Each set consists of physical quantities, held together by some sort of relationship, and a collection of possible values for those quantities. When reasoners deduce, they reduce the number of possibilities in each set until there are either no possibilities remaining in the set, at which point that set is rejected, or until there is only one possibility remaining in that set. When there is only one possibility remaining, the reasoner should conclude that the remaining possibility is the solution to the deduction.

However, because there are many things that interfere with proper deduction, reasoners may either fail to identify all of the possible sets of physical quantities that could be relevant to the problem or fail to identify all of the possible values for the quantities within a given set. Reasoners may also not eliminate all of the possibilities that they could, leaving too many possibilities in a given set or too many sets. Then, because there is no further information from the problem, the reasoner will either conclude that he or she doesn't know the answer to the deduction, or he or she will make a non-deductive guess.

This section will demonstrate the structure of the diagrams that can be used to track reasoning. These diagrams are called "possibility sets." Each possibility set is represented as a table. The top row in the table holds a relationship for the physical quantities. The second row holds those quantities themselves, and each row below that holds possible values for those quantities. The order of the physical quantities (columns) does not matter; however, the order of the rows of values does have a meaning that will be discussed in Section 3.2. Figure 3.1 shows the template for a possibility set.

Relationship between quantities			
quantity 1	quantity 2	quantity 3	...
value 1.1	value 2.1	value 3.1	...
value 1.2	value 2.2	value 3.2	...
value 1.3	value 2.3	value 3.3	...
...			

Figure 3.1: The basic setup for a possibility set

This representation of possibility sets is best suited to physics problems with a qualitative or conceptual component. For example, consider the following problem:

> A supermarket employee pushes a palette of oranges across the stockroom floor in the $+x$ direction. The floor applies 150 N of force to the palette in the $-x$ direction. Is the magnitude of force that the employee needs to apply to the palette greater than, less than, or equal to 150 N to keep it moving at a constant speed? You may assume that there is no air resistance.

Many students struggle with problems like this, stating that one has to apply slightly more force than the floor to keep the palette moving at a constant speed. Understanding the role that the momentum principle plays in this situation is crucial to solving this problem correctly. Because the final momentum of the crate needs to be equal to the initial momentum of the crate, there must be zero net force applied to the crate, so one can conclude that the two forces need to be equal. When we explore the momentum principle in terms of the possibilities that are afforded by the momentum principle in terms of which quantities are equal to zero, we see that there are eight possibilities (see Figure 3.2), but not all of them are physically legitimate, and others do not apply to this problem.

$\vec{p}_{final} = \vec{p}_{initial} + \vec{F}_{net}\Delta t$		
$\vec{p}_{final}$	$\vec{p}_{initial}$	$\vec{F}_{net}\Delta t$
$\vec{0}$	$\vec{0}$	$\vec{0}$
$\vec{0}$	$\vec{0}$	$\neq \vec{0}$
$\vec{0}$	$\neq \vec{0}$	$\vec{0}$
$\vec{0}$	$\neq \vec{0}$	$\neq \vec{0}$
$\neq \vec{0}$	$\vec{0}$	$\vec{0}$
$\neq \vec{0}$	$\vec{0}$	$\neq \vec{0}$
$\neq \vec{0}$	$\neq \vec{0}$	$\vec{0}$
$\neq \vec{0}$	$\neq \vec{0}$	$\neq \vec{0}$

Figure 3.2: A set of possibilities corresponding to whether terms in the momentum principle are zero or non-zero

When we eliminate all rows that are not legal or applicable, we are left with two possibilities: that either the net impulse is equal to zero or it is not (see Figure 3.3).

$\vec{p}_{final} = \vec{p}_{initial} + \vec{F}_{net}\Delta t$		
$\vec{p}_{final}$	$\vec{p}_{initial}$	$\vec{F}_{net}\Delta t$
$\vec{0}$	$\vec{0}$	$\vec{0}$
$\vec{0}$	$\vec{0}$	$\neq \vec{0}$
$\vec{0}$	$\neq \vec{0}$	$\vec{0}$
$\vec{0}$	$\neq \vec{0}$	$\neq \vec{0}$
$\neq \vec{0}$	$\vec{0}$	$\vec{0}$
$\neq \vec{0}$	$\vec{0}$	$\neq \vec{0}$
$\neq \vec{0}$	$\neq \vec{0}$	$\vec{0}$
$\neq \vec{0}$	$\neq \vec{0}$	$\neq \vec{0}$

Figure 3.3: All possibilities that are not legal or applicable to this problem have been shaded gray, indicating that they are eliminated.

Because Figure 3.3 still does not reduce the possibility set to only one allowed possibility, we must carefully flesh out the initial and the final momenta. In fact, it makes sense to think of the momentum principle as $\Delta \vec{p} = \vec{F}_{net}\Delta t$ rather than in terms of the initial and final momenta. In that way we see that there is only one possibility that cannot be eliminated for this problem (see Figure 3.4).

$\Delta \vec{p} = \vec{F}_{net}\Delta t$	
$\Delta \vec{p}$	$\vec{F}_{net}\Delta t$
$\vec{0}$	$\vec{0}$
$\vec{0}$	$\neq \vec{0}$
$\neq \vec{0}$	$\vec{0}$
$\neq \vec{0}$	$\neq \vec{0}$

Figure 3.4: When we also include information comparing the initial and final momenta, only one possibility remains in the set.

Of course, one also needs to use the superposition principle (that is, the sum of all forces acting on the crate) and the definition of momentum correctly to solve this problem, but those can also be represented by possibilities sets. The next section provides a detailed analysis on how to use these possibility sets to track the reasoning of a hypothetical student for this example.

3.2 Basic Rules for Using Possibility Sets

Possibility sets are intended for use in tracking deductive reasoning. This section discusses how to generate these sets.

1. In the top row of the possibility set, write the simplest *correct* physical relationship that contains some quantities that the reasoner is using and would generate the correct answer to the problem (or step of the problem). However, do not include a relationship if it both includes at least one quantity that the reasoner did not at least implicitly use and does not contain at least one quantity that the reasoner did use. For example, if a reasoner discusses the x-component of momentum and the x-component of velocity but does not discuss gamma or mass, the appropriate relationship to use is the definition of momentum (which should contain gamma only if the object's speed is near that of the speed of light). However, if the reasoner talks about the x-component of momentum, mass, rest energy, and the temperature of the object in trying to ascertain the answer to a single step of the problem, do not include a relationship in the top line of the possibility set. When the top row of a possibility set is left blank, that indicates that the reasoner was not using a correct physics relationship.

2. In the next row, write each quantity that the reasoner actually used in addition to all quantities from the relationship in the top row. The order in which these are written left to right does not matter, although it is conventional to write the quantities in the relationship as they appear in that relationship before adding the other quantities the reasoner used. The presence of a quantity that does not belong to the relationship is indicative of an error in using a superfluous quantity in the reasoning.

3. In order of appearance in the reasoning, list all possible values for the quantities that appear in the set. After a value for a quantity is established, write that value in every subsequent row. Each row indicates a collection of values that appear together (e.g., if a reasoner says that "p_x could be zero if m_x is zero" then denote that by placing a zero beneath p_x and m_x both). If a quantity is not mentioned for a specific instance, leave the space beneath that quantity blank. An entirely blank column beneath a quantity indicates that a necessary quantity was

neglected in the reasoning. If the reasoner explicitly eliminates a possibility, indicate that by shading the entire row dark gray.

4. Add all "implied negations." An implied negation is an indication that certain possibilities were eliminated without them being explicitly mentioned. For example, the statement, "X must be zero" carries the implied negation that "X can not be (not zero)." Similarly, "X cannot be zero" carries the implied negation that "X must be (not zero)." However, the statement that "X can be zero" carries no implied negations. Each implied negation should be listed out as a row of values on the possibility set immediately beneath the actual statement made by the reasoner. However, the value that is the implied negation should be indicated as such by placing square brackets around it. Note that when there are implied negations, either the reasoner's statement or the implication must be an eliminated row of possibilities; indicate that by shading that row dark gray.

5. Indicate all implied values by placing parentheses around them. An implied value is one that has not been explicitly stated but which has been otherwise indicated. For example, the statement "To prevent the object from moving, a force needs to oppose the applied force" has the superposition principle as the relationship, and the quantity $\vec{F}_{net}$ is listed. The value $\vec{0}$ is not explicitly provided for the net force, but it is implied by the statement. This gets represented in the possibility set by writing "$(\vec{0})$" in the column beneath $\vec{F}_{net}$.

6. Remove all rows that carry repeated information. For example, a row that contains a zero under $\vec{F}_{net}$ and a blank space under $d\vec{p}$ should be eliminated if there is another row that contains a zero under both of those quantities.

7. If there are any possible rows of values that are not listed within the possibility set, then add a row with only the text, "Other possibilities not considered." Otherwise, do not add an additional row to the set.

8. If the result from one possibility set is later used in another possibility set, indicate a connection from one set to the next with doubled arrows pointing from the original set into the other. Additionally, shade the cell that contains that value in both sets the same color (in this

dissertation, the first connected value is colored pink, the second blue, and the third green).

3.2.1 A hypothetical example

To see how these rules play out, consider an example. A hypothetical reasoner provides the following solution to the problem from Section 3.1:

> Because the palette is moving at constant speed, the final momentum of the palette needs to be equal to its initial momentum. Then, according to the momentum principle, the net force acting on the palette can't be nonzero, because then the palette's momentum would change. Therefore, the force applied by the employee must be equal to the friction applied by the floor.

This reasoning consists of three steps. First, the reasoner relates speed to momentum by using the definition of momentum. Second, the reasoner applies the momentum principle to determine that the net force on the palette of oranges is zero. Third, the reasoner uses the superposition principle to conclude that the forces are equal. Each of these steps corresponds to a possibility set, a representation of which can be constructed by following the eight rules listed in Section 3.2, as shown below.

> The final momentum of the palette needs to be equal to its initial momentum. Then, according to the momentum principle, the net force acting on the palette can't be nonzero, because then the palette's momentum would change.

First, we need to identify the relationship that is being used. Here, it is the momentum principle. Even if the reasoner had not explicitly mentioned it, we would start with that principle because it is the simplest relationship that provides the correct answer and uses the reasoner's quantities. Figure 3.5 demonstrates the first rule applied to this reasoning.

$$\boxed{\vec{p}_{final} = \vec{p}_{initial} + \vec{F}_{net}\Delta t}$$

Figure 3.5: Rule 1: Establishing the relationship.

Next, we need to include all of the physical quantities that are in that relationship as well as all that are explicitly mentioned by the reasoner. This rule is demonstrated in Figure 3.6. Note that while the reasoner does not explicitly mention Δt, it needs to be included because that term appears in the relationship we chose for this possibility set. Typically, these quantities are listed in the order that they appear in the relationship for the sake of clarity.

$\vec{p}_{final} = \vec{p}_{initial} + \vec{F}_{net}\Delta t$			
$\vec{p}_{final}$	$\vec{p}_{initial}$	$\vec{F}_{net}$	Δt

Figure 3.6: Rule 2: Listing the quantities in the reasoning.

Now, we list the possible values for these quantities provided in order of reasoning, leaving blank any instance where a value is not provided. This rule is demonstrated in Figure 3.7. First, the reasoner establishes that the initial and final momenta need to be equal to each other. Next, the reasoner states that the net force on the palette could not be nonzero. Third, the reasoner provides a way for the net force on the palette to be nonzero, but that possibility is eliminated because it violates earlier reasoning (that the initial and final momenta need to be equal). Because the reasoner never mentions anything about Δt, that column is left entirely blank.

$\vec{p}_{final} = \vec{p}_{initial} + \vec{F}_{net}\Delta t$			
$\vec{p}_{final}$	$\vec{p}_{initial}$	$\vec{F}_{net}$	Δt
$\vec{p}$	$\vec{p}$		
$\vec{p}$	$\vec{p}$	$\neq \vec{0}$	
$\neq \vec{p}$	$\vec{p}$	$\neq \vec{0}$	

Figure 3.7: Rule 3: Listing the values for the quantities in chronological order and indicating explicit eliminations.

Notice that in Figure 3.7, we indicate that two quantities being considered are the same by assigning each of them a variable. In this case, we assigned the value "$\vec{p}$" to both the initial and the final momenta to indicate that while the reasoner has not indicated a numerical value for either one, the reasoner has indicated that they must be the same. Also, notice that the new information "builds" upon the old: after it had been established that the momenta were equal, that information was kept in subsequent rows.

At this point, we need to add in the implied negations directly beneath each statement that created them. The implied quantity is presented within square brackets (see Figure 3.8). In this case, that means adding in the rejection of the momenta not being equal. We also need to create a row corresponding to the implied negation of the statement that the net force could not be nonzero; in other words, that the net force is zero. The final row, because it was eliminated for violating earlier reasoning, does not afford an implied negation. In a sense, it is an explicit negation of the previous possibility: the only way for the net force to be nonzero is if the initial and final momenta were different, but this is not possible. Explicitly eliminated possibilities that are stated for reason of justification or explanation do not invoke implicit negations.

$\vec{p}_{final} = \vec{p}_{initial} + \vec{F}_{net}\Delta t$			
$\vec{p}_{final}$	$\vec{p}_{initial}$	$\vec{F}_{net}$	Δt
$\vec{p}$	$\vec{p}$		
$[\neq \vec{p}]$	$\vec{p}$		
$\vec{p}$	$\vec{p}$	$\neq \vec{0}$	
$\vec{p}$	$\vec{p}$	$[\vec{0}]$	
$\neq \vec{p}$	$\vec{p}$	$\neq \vec{0}$	

Figure 3.8: Rule 4: Including the implied negations.

Now we need to indicate the implied quantities in this solution. In this case, the value of Δt being greater than zero is implied by the fact that time is implicitly running forwards in the problem. Had the reasoner explicitly stated something about the value of Δt, that information would have been included in rule #3. We highlight implied values by placing parentheses around them as they are added to each row. This is demonstrated in Figure 3.9.

Two of the rows in Figure 3.9 contain repeated information. The top row repeats some of the information presented in the fourth row, which is the implied statement that the net force needs to be zero. Additionally, the second row repeats some of the information present in the bottom row: if the initial and final momenta are not equal, then the net force acting on the palette must be nonzero. Once those rows are removed from the set, the result is Figure 3.10.

Note that in Figure 3.10, there are no possibilities that the reasoner has not discussed, when we consider that the value of Δt is implicitly set as being greater than zero. Because there are no

$\vec{p}_{final} = \vec{p}_{initial} + \vec{F}_{net}\Delta t$			
$\vec{p}_{final}$	$\vec{p}_{initial}$	$\vec{F}_{net}$	Δt
$\vec{p}$	$\vec{p}$		(>0)
$[\neq \vec{p}]$	$\vec{p}$		(>0)
$\vec{p}$	$\vec{p}$	$\neq \vec{0}$	(>0)
$\vec{p}$	$\vec{p}$	$[\vec{0}]$	(>0)
$\neq \vec{p}$	$\vec{p}$	$\neq \vec{0}$	(>0)

Figure 3.9: Rule 5: Indicating implied values.

$\vec{p}_{final} = \vec{p}_{initial} + \vec{F}_{net}\Delta t$			
$\vec{p}_{final}$	$\vec{p}_{initial}$	$\vec{F}_{net}$	Δt
$\vec{p}$	$\vec{p}$	$\neq \vec{0}$	(>0)
$\vec{p}$	$\vec{p}$	$[\vec{0}]$	(>0)
$\neq \vec{p}$	$\vec{p}$	$\neq \vec{0}$	(>0)

Figure 3.10: Rule 6: Removing repeated information.

possibilities missing from this set, we do not need to indicate that possibilities are not included in the set. In other words, the possibility set in Figure 3.10 exhausts all of the possibilities. If this were not the case, we would need to add a row with the text, "Other possibilities not considered," to the bottom of the set, unshaded. The possibility set shown in Figure 3.10 is therefore the complete set for describing this step of reasoning, and it does not need to be modified for rule 7.

However, the reasoning analyzed in Figure 3.10 was only one of three steps in a chain of reasoning. Not only does it use the result of previous reasoning, a conclusion drawn in this reasoning is used in future reasoning. We must therefore apply rule 8 to link the entire chain of reasoning together. Rather than explicitly demonstrating the above process for the first and third steps in the reasoning, I will simply display the complete reasoning chain in Figure 3.11.

The first set in Figure 3.11, which relates a constant speed to a constant momentum, contains information that helps us conclude that there were possibilities that were not considered. Specifically, possible values for the quantities "change in mass" and "change in direction of the velocity" were not considered. This omission led the reasoner to draw a conclusion based on the single possibility present in the set although additional rows were possible, indicated by the bottom row of the set, which

$m\vec{v} = \vec{p}$			
Δm	$\Delta|\vec{v}|$	$\Delta \hat{v}$	$\Delta \vec{p}$
	0		$\vec{0}$
	0		$[\neq \vec{0}]$
Other possibilities not considered			

$$\downarrow\downarrow$$

$\vec{p}_{final} = \vec{p}_{initial} + \vec{F}_{net}\Delta t$			
$\vec{p}_{final}$	$\vec{p}_{initial}$	$\vec{F}_{net}$	Δt
$\vec{p}$	$\vec{p}$	$\neq \vec{0}$	(> 0)
$\vec{p}$	$\vec{p}$	$[\vec{0}]$	(> 0)
$\neq \vec{p}$	$\vec{p}$	$\neq \vec{0}$	(> 0)

$$\downarrow\downarrow$$

$\vec{F}_{net} = \vec{F}_{friction} + \vec{F}_{employee}$		
$\vec{F}_{net}$	$\vec{F}_{friction}$	$\vec{F}_{employee}$
$\vec{0}$	$\vec{F}$	$-\vec{F}$
$\vec{0}$	$\vec{F}$	$[\neq -\vec{F}]$

Figure 3.11: Rule 8: Connecting reasoning steps with arrows and color-codes representing the information that is carried from one set to another.

states that there were "other possibilities not considered." This process of drawing a conclusion from incomplete information is an error of omission in reasoning, which throughout this dissertation is discussed as neglecting or failing to consider a quantity. In this case (as is often true with this type of problem), those omissions were relatively benign as the conclusion that the reasoner drew would likely be regarded as correct, in spite of failing to flesh out all of the possibilities. A different reasoner, however, may have suggested that if oranges were being added to or removed from the palette as it was being pushed, the amount of force the employee needed to apply would *not* be 150 N in the $+x$ direction. That reasoning would have a different possibility set representation than what is shown in 3.11.

Notice that the result from the first possibility set in Figure 3.11 did not import cleanly into the second possibility set. The reasoner provided more information than was necessary; had the reasoner simply stated that the momentum had not changed, meaning the net force have been zero, the transferred information would have appeared more clean. Also, it is atypical to include an implicit negation for imported information, but it was done in this case for demonstration purposes.

Suppositions

Sometimes, reasoners make suppositions. For example, if a reasoner said, "If the mass of the palette did not change, then with no change in the speed of the palette, we would know that the momentum of the palette could not have changed," that would be represented that by generating a row corresponding to the supposed value, as in Figure 3.12. Notice that the supposition implies the possibility that the mass of the palette might have changed, but that no information is given about that possibility. The implied negation is still represented, but not rejected because the reasoner did not reject this possibility.

In the previous statement, the reasoner neither concluded that the mass of the palette changed, nor did the reasoner reject that possibility. Imagine that the reasoner's supposition is followed up with the statement, "But of course the mass of the palette could not have changed." In this case, the supposition is accepted, but the implied negation is now rejected. Figure 3.13 is a representation of this reasoning.

$m\vec{v} = \vec{p}$			
Δm	$\Delta |\vec{v}|$	$\Delta \hat{v}$	$\Delta \vec{p}$
0	0		$\vec{0}$
0	0		$\vec{0}$
$[\neq 0]$			
Other possibilities not considered			

Figure 3.12: A possibility set for a reasoner making a supposition.

$m\vec{v} = \vec{p}$			
Δm	$\Delta |\vec{v}|$	$\Delta \hat{v}$	$\Delta \vec{p}$
0	0		$\vec{0}$
0	0		$\vec{0}$
$[\neq 0]$			
Other possibilities not considered			

Figure 3.13: A possibility set for a reasoner accepting a supposition and drawing a conclusion.

3.3 Formal Logic in the Possibilities Framework

In this section, we demonstrate how the possibilities framework can replicate the results of formal logical reasoning. The ability to reproduce the results of formal logic reasoning was one of the features of mental models (Johnson-Laird, 2006) that was absolutely vital to maintain in the domain of physics. Specifically, the possibilities framework can describe any conclusions that are also described by formal logical reasoning. However, the possibilities framework is substantially different from formal logic in that it more closely describes student deduction and the errors that students make. Errors such as those from logical fallacies can be described in this framework as resulting from failing to consider all of the relevant possibilities rather than failing to follow a logical rule correctly. For this reason, we consider a physical situation and all of the possibilities within that situation to demonstrate that by considering different possibilities one can draw different conclusions, corresponding to both correct logical moves and logical fallacies.

3.3.1 Possibilities

The momentum principle in physics states that $F_{net} = d\vec{p}/dt$. In other words, the net force is equal to the derivative of momentum with respect to time. While this precisely defines the relationship between the net force and the change in momentum with respect to time, questions are often asked only about the directions of these two quantities. For example, consider a situation where the change in momentum of an object is observed to be in the direction "up." What are the possible directions for the net force? Figure 3.14 displays some possibilities.

$\vec{F}_{net} = d\vec{p}/dt$	
$d\hat{p}/dt$	$\hat{F}_{net}$
up	up
up	down
up	left
up	right
Other possibilities not considered	

Figure 3.14: Possibilities for the direction of the net force, given that the change in direction is up.

Thanks to the given information, the direction of the change in momentum is constrained to be "up." Because of the momentum principle, we must reject all possible directions of the net force except the direction that is the same as the change in momentum, "up." That is, all of the infinite other directions: "down," "left," "right," and so forth, all must be rejected. At some point, the reasoner can simply reject all of the possibilities that do not have the direction of the net force and the direction of the change of momentum in the same direction. Coming to this conclusion requires some effort, as it means accepting that there is no way that the direction of the net force can not be a different direction than that of the change in momentum. This development is represented in Figure 3.15.

If we were never given any information regarding the direction of the change in momentum, we could also list all of the possible directions for that as well. We could again collapse those possibilities around a single direction like "up," giving us four possibilities, which completely cover the possibility space for this problem. These four possibilities are listed in Figure 3.16.

The momentum principle allows two of the possibilities: either the direction of the net force is

$\vec{F}_{net} = d\vec{p}/dt$	
$d\hat{p}/dt$	$\hat{F}_{net}$
up	up
up	not up

Figure 3.15: The collapse of the possibilities of the net force.

$\vec{F}_{net} = d\vec{p}/dt$	
$d\hat{p}/dt$	$\hat{F}_{net}$
up	up
up	not up
not up	up
not up	not up

Figure 3.16: The possibility space for directions in the momentum principle, collapsed around "up."

up and the direction of the change in momentum is up, or the direction of the net force is not up and the direction of the change in momentum is not up. Notice that some information is lost (for example, that it is impossible for the direction of the net force to be to the left and the direction of the change in momentum to be down). However, this set of possibilities is sufficient for reasoning when the direction "up" is given in one of the premises. With this as a starting point, we can explore both proper logical reasoning and logical fallacies in terms of possibilities.

3.3.2   Modus ponens

The most effortless logical conclusion to draw is *modus ponens*. This has the form: "If A, then B. Given A, therefore B." If we consider the possibility space demonstrated in Figure 3.16, together with the premise, "The direction of the change in momentum is up," and the momentum principle, it is very easy to draw the conclusion, "Therefore, the direction of the net force is also up." In fact, to draw this conclusion, we only need to consider the first row of possibilities, making *modus ponens* very easy to use: we never need to fully identify all of the possibilities. This abbreviated version of our possibility set is demonstrated in Figure 3.17, below.

Regardless of the context, people tend to be very capable of drawing the conclusions described

$\vec{F}_{net} = d\vec{p}/dt$	
$d\hat{p}/dt$	$\hat{F}_{net}$
up	up
Other possibilities not considered	

Figure 3.17: Relevant possibilities for drawing a *modus ponens* conclusion.

by *modus ponens*. In physics, however, when the explicit statement, "If the direction of the change in momentum is up, then the direction of the net force is up" is not provided in lieu of the more abstract momentum principle, it may be more difficult to reason about direction because there are so many possibilities. Therefore, to know that the direction of the change in momentum being up *requires* the direction of the net force to be up, one must use Figure 3.15. Failing to do so can lead to erroneous conclusions which correspond to this possibilities framework but not to formal logical reasoning. An example of such an error can be seen in the first pilot study in this dissertation, when some participants did precisely this by providing a direction for the net force that was different from the direction of the change in momentum.

3.3.3 Modus tollens

The other basic logical conclusion one can draw is relatively effortful. *Modus tollens* has the form: "If A, then B. Given not-B, therefore not-A." If we consider the possibility space demonstrated in Figure 3.16, together with the premise, "the direction of the net force is not up," then we can draw the conclusion that the direction of the change in momentum is not up. Just as in the *modus ponens* case, only one row of possibilities (in this case, the bottom row) is really necessary to draw the conclusion. However, also as with the *modus ponens* case, it is important to recognize that it is impossible for the direction of the net force to be "not up" while the direction of the change in momentum is "up." The set of possibilities for drawing a *modus tollens* conclusion is given in figure 3.18.

$\vec{F}_{net} = d\vec{p}/dt$	
$d\hat{p}/dt$	$\hat{F}_{net}$
up	not up
not up	not up

Figure 3.18: Relevant possibilities for drawing a *modus tollens* conclusion.

3.3.4 Biconditionals and conditionals

The momentum principle, which has been used in the previous examples, is known as a biconditional. In logic, biconditionals have the form, "If and only if A, then B." The same techniques of *modus ponens* and *modus tollens* can both be used on biconditionals, but it is impossible to draw the logical fallacies of denial of the antecedent or affirmation of the consequent, which will be investigated in Section 2.2.3. Fully fleshing out and understanding all of the permissible possibilities for a biconditional can lead to rapid, error-free conclusions. Our representation of all of the possibilities entitled to this statement was given in Figure 3.16.

Conditionals have the form, "If A, then B." In contrast to biconditionals, conditional statements can yield logical fallacies and are more difficult to reason with. Our previous example, the momentum principle, is a biconditional. A conditional based on the momentum principle is, "If the net force acting on an object is $\vec{0}$ N, then its speed does not change." The four possibilities for this situation are listed in Figure 3.19.

$\vec{F}_{net} = \Delta\vec{p}/\Delta t$			
$\vec{F}_{net}$	$\Delta|\vec{p}|$	$\Delta\hat{p}$	Δt
$\vec{0}$	0		$\neq 0$
$\vec{0}$	$\neq 0$		$\neq 0$
$\neq \vec{0}$	0		$\neq 0$
$\neq \vec{0}$	$\neq 0$		$\neq 0$

Figure 3.19: The four possibilities for the conditional statement "If the net force acting on an object is $\vec{0}$ N, then its speed does not change."

Note that there are now three rows of allowed possibilities rather than only two. This happens

because there is information not conveyed; namely, either the direction or the magnitude of the momentum must change if there is a nonzero net force acting on the object. However, by only attending to the change of the speed of the object, a conditional statement emerges. Such conditionals are substantially more difficult to deduce from, as revealed by the two logical fallacies affirmation of the consequent and denial of the antecedent.

3.3.5 Affirmation of the consequent

The logical fallacy "affirmation of the consequent" has the form, "If A, then B. Given B, therefore A." It is a fallacy because the conclusion "A" is not guaranteed by the presence of B. This fallacy is insidious because it is so easy to overvalue the possibility that A and B both occur simultaneously (whenever there is A, there is B) and "forget" that B may occur independent of the presence of A.

For our example, it is possible for the object to be moving and experiencing either a zero or non-zero net force (in the former case its momentum must be constant), and it is also possible for an object to be stationary and not experience a net force. Affirmation of the consequent can occur when all of these possibilities are not considered. If only the last row of possibilities is considered, namely that the magnitude of the momentum is not zero and the magnitude of the net force is not zero, one may be tempted to conclude that given a nonzero velocity that there must be a net force on the object. This affirmation of the consequent is shown in Figure 3.20.

$\vec{F}_{net} = \Delta(\vec{p})/\Delta t$		
$\vec{F}_{net}$	$\vec{p}$	Δt
$\neq \vec{0}$	$\neq \vec{0}$	$\neq 0$
Other possibilities not considered		

Figure 3.20: The possibility set corresponding to the affirmation of the consequent.

By only considering one possibility, it is easy to neglect the other possibility, namely that it is possible for the magnitude of the net force to be zero while the magnitude of the velocity is not zero. By neglecting this possibility, one may draw the conclusion that objects in motion must be experiencing a net force, which is physically not correct.

3.3.6 Denial of the antecedent

The logical fallacy, "denial of the antecedent," has the form, "If A, then B. Given not-A, therefore not-B." It is a fallacy because A can be absent while B is present. This fallacy also comes about by failing to consider all possibilities, but it is a different possibility that is neglected. In the example about net force and momentum, if only the rejected possibility of the net force being not zero and the momentum being zero (the third possibility row in Figure 3.19) is considered, without considering the possibility that the net force might be not zero while the magnitude of the velocity is not zero, one may draw the incorrect conclusion. In that case, one denies the antecedent and draws the physically erroneous conclusion that an object that is experiencing a net force of zero must not be moving. This fallacy is demonstrated in Figure 3.21.

$\vec{F}_{net} = \Delta(\vec{p})/\Delta t$		
$\vec{F}_{net}$	$\vec{p}$	Δt
$\neq \vec{0}$	$\vec{0}$	$\neq 0$
$\neq \vec{0}$	$\neq \vec{0}$	$\neq 0$
Other possibilities not considered		

Figure 3.21: The possibility set corresponding to the denial of the antecedent.

Note that in this case, the possibility row corresponding to *modus ponens* (and the affirmation of the consequent) remains in the possibility set. The fallacy of the denial of the antecedent requires somewhat more effort than affirming the consequent because one needs to consider more possibilities to draw the conclusion. However, it is still easier to draw a conclusion based on the denial of the antecedent than *modus tollens*, because there are still some possibilities being neglected.

As can be seen, both of these fallacies are the result of failing to flesh out all of the possibilities for a given situation. This aligns well with their prevalence in people who attempt to use logic; it is simply easier to commit a fallacy than to draw the correct solution because of the extra effort that is required to think of all possibilities. In physics, where there are literally infinite possibilities, it is important for the reasoner to reduce this space of possibilities. Doing so by failing to consider some of them, however, will result in conclusions that are the same as those reached by logical fallacies.

3.3.7   An example with possibilities and logic

To provide an example of how someone might reason with both possibilities and logic, let us consider a physical scenario. Imagine that during a physics lab, a group of students observes a fan cart moving at a constant speed along a track and wants to know whether there is a net force acting on it. I will provide two fictitious explanations that can help demonstrate that the same result can be reached by reasoning with logic and possibilities.

Formal logic explanation

The formal logic reasoning may proceed by *modus tollens*: "The momentum principle states that $F_{net} = d\vec{p}/dt$. Therefore, if there is a net force acting on the cart, then the cart's momentum will be changing. However, because the cart's momentum is not changing, we can conclude that there is no net force acting on the cart." There is also a formal explanation that proceeds by the logical fallacy of "affirming the consequent": "The momentum principle states that $F_{net} = d\vec{p}/dt$. Therefore, if there is no net force acting on the cart, then the cart's momentum will not be changing. And since the cart's momentum is not changing, we can conclude that there is no net force acting on the cart." In this case, the logical fallacy yields the correct conclusion because the momentum principle is a biconditional rather than a conditional.

Here is an example of how an attempt at logical reasoning could yield an incorrect result from a fallacy: "The momentum principle states that $F_{net} = d\vec{p}/dt$. Therefore, if there is a net force acting on the cart, then the cart will be moving. Then, because the cart is moving, we can conclude that there is a net force acting on the cart." In this case, the "affirmation of the consequent" results in an error because the intermediate conditional the reasoner made, while technically correct, was a conditional rather than a biconditional like the original momentum principle was.

Possibilities explanation

The possibilities afforded by this situation are shown in Figure 3.22.

Someone reasoning correctly using possibilities might say, "there could either be a non-zero net force acting on the cart or the net force could be zero. The momentum principle is $F_{net} = d\vec{p}/dt$. Therefore, we can determine if there is a net force on the cart by observing whether its momentum

$\vec{F}_{net} = d\vec{p}/dt$	
$|F_{net}|$	$|dp/dt|$
0	0
0	$\neq 0$
$\neq 0$	0
not 0	not 0

Figure 3.22: The possibility set relating a fan cart's motion and net force.

changes. The only way its momentum can change is if there is a net force, and the only way its momentum can stay the same is if the net force is non-zero. Because the cart's momentum stayed the same, we can conclude that the net force on the cart must have been zero."

In other words, someone who reasons with possibilities will seek to uncover a relationship between the relevant quantities (here, the change in momentum of the cart and the net force on the cart are related by the momentum principle), just like someone using formal logic. However, someone using formal logic will then need to cast that relationship into a conditional or biconditional statement and use the *form* of that statement to reason while someone using possibilities will then try to figure out all of the different possibilities that result from that relationship. As mentioned before, failing to flesh out all of the possibilities can result in flaws in reasoning that are the same as logical fallacies, while succeeding in identifying all possibilities yields correct reasoning that is the same as that reached by logic.

3.3.8 Summary of possibilities and logic

In summary, all of the conclusions that can be reached by logic can be described by the possibilities framework. However, the possibilities framework also demonstrates how reasoners can commit errors in physics even when the only logical move required is the very simple *modus ponens*. Also, the use of possibilities describes how logical fallacies occur. Because the framework can be used to replicate the results of formal logic, we lose nothing by using it. What we gain is an opportunity to use a description that is capable of modeling more closely how the students actually use deduction in physics.

3.4   An Example from the Data

To demonstrate how these diagrams can be used to track possibilities, consider the reasoning of a group of three graduate students (Fay, Marco, and Omar) solving a physics problem. Below are segments of dialogue followed by diagrams of the possibilities they are indicating. For this example, we will treat the group as a whole as the unit of analysis.

The text of the physics problem reads as follows:

> Two heavy blocks, with massless ropes attached to them, are sitting at rest on a table. Block A sits on block B, which is in contact with the table, as pictured below. Amir pulls on the rope attached to block A, applying 20 N of force. Barbara pulls on the rope attached to block B, applying 7 N of force. While they are pulling, neither block moves. What direction is the friction force on block B due to the table?

A diagram of the situation was provided to the participants as well (see Figure 3.23).
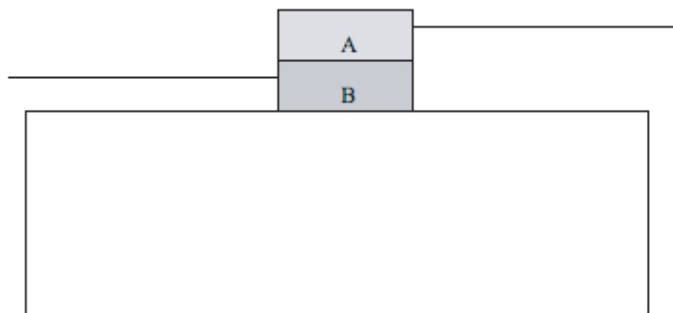
Figure 3.23: The given diagram for the "Two Blocks" problem.

As the problem was being read, Fay drew diagrams on the whiteboard. After reading the problem, the group began discussing it.

> **Fay**: So this is the interface that we are interested in (Fay points to her drawing, indicating the interface between block B and the table), but there is no motion of the blocks.

Marco: Yup. We just need to write the two force diagrams.

Fay: So we have A (drawing a free body diagram of block A on the white board), here's gravity, force due to block B, 20 N, and the force of friction due to block B, sub, sub B.

By drawing block A and the forces acting on it, the group has indicated the relevant quantities and explicitly mentions some of the possibilities, which we denote with Figure 3.24. For some of the forces, the considered possibilities are limited to the direction of the force, which we represent with arrows in the appropriate direction. While the group has not explicitly indicated the relationship between the physical quantities involved, the correct physical relationship for this situation that uses the quantities they mention is the superposition principle; therefore, that relationship is indicated on the group's possibility set. Throughout this problem, this group indicated that there is no motion of the blocks. A proper analysis of this statement would include the relationship between the blocks remaining at rest and there being no net force on them. To simplify this example, I have interpreted the statement "there is no motion of the blocks" as an implication that there is no net force on the either block, and represent the quantity "$\vec{0}$" with parentheses to indicate this. For more of an explanation as to why I feel confident with this interpretation, see Section 6.1.2.

$\vec{F}_{net,onA} = \vec{F}_{gravity} + \vec{F}_{B,normal} + \vec{F}_{Amir} + \vec{F}_{B,friction}$				
$\vec{F}_{net,onA}$	$\vec{F}_{gravity}$	$\vec{F}_{B,normal}$	$\vec{F}_{Amir}$	$\vec{F}_{B,friction}$
$(\vec{0})$	↓	↑	20 N →	←
$(\vec{0})$	↓	↑	20 N →	$[not \leftarrow]$
Other possibilities not considered				

Figure 3.24: Possibilities for forces acting on block A.

Some values for the vector forces contain both magnitude and direction while others include only direction. Some information is missing because only information that is explicitly mentioned, and the implied negation of that information, is contained in possibility sets. The grayed-out row in Figure 3.24 corresponds to the implied negation of Fay's diagram: she draws the friction force due to block B on block A to the left, indicating that she is eliminating all other possible directions for that force. Contrast this with how she deals with the friction force on block B due to friction in Figure 3.26.

At this point, Fay went on to list the forces on block B, as she drew a free body diagram on the whiteboard for block B.

> **Fay**: F table, FG, force of friction due to A, 7 N, and then we had a Ff question, Ff question (indicating two possible friction forces between the table and block B, one in the $+x$ direction and the other in the $-x$ direction).

To help visualize this situation, Fay has been writing on a whiteboard. At this point in the discussion, the whiteboard looked like Figure 3.25.
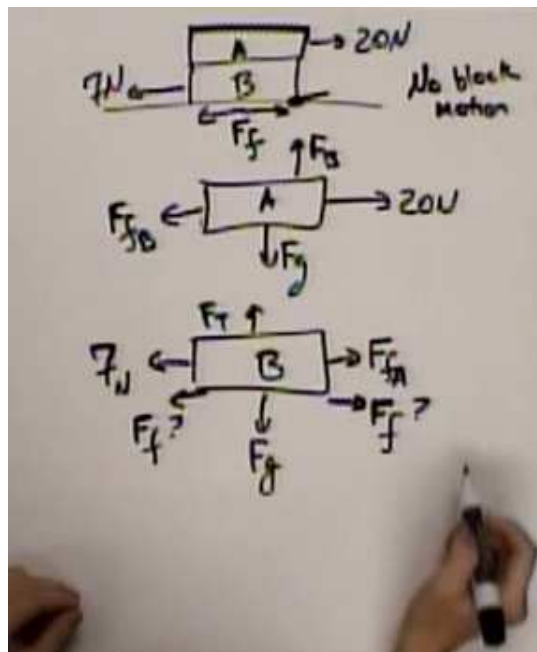
Figure 3.25: Screen capture of the whiteboard showing the two possible directions for the force of the table on block B.

The list of forces provided by Fay indicates a set of possibilities for block B, as represented in Figure 3.26. Particularly worth noticing is that this group explicitly mentioned two possibilities for the direction of the friction force due to the table on block B. Because there are two possible directions listed for the friction force due to block B on block A, there is no implied negation.

Next, the group related the two possibility sets together by considering the relationship between block A and block B.

$\vec{F}_{net,onB} = \vec{F}_{gravity} + \vec{F}_{table,normal} + \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table,friction}$					
$\vec{F}_{net,onB}$	$\vec{F}_{gravity}$	$\vec{F}_{table,normal}$	$\vec{F}_{Barbara}$	$\vec{F}_{AonB}$	$\vec{F}_{table,friction}$
$(\vec{0})$	$\downarrow$	$\uparrow$	$7\,\text{N} \leftarrow$	$\rightarrow$	$\leftarrow$
$(\vec{0})$	$\downarrow$	$\uparrow$	$7\,\text{N} \leftarrow$	$\rightarrow$	$\rightarrow$
Other possibilities not considered					

Figure 3.26: Possibilities for forces acting on block B.

Fay: This (pointing to the arrow indicating the force of friction on block B due to block A) is going to be equal to Ff B (the force of friction on block A due to block B), right? So –

Omar: Hmm, it could also be –

Fay: Between the two blocks, can we say that their, this force is the same?

Omar: Yeah, it doesn't move...

Marco to **Fay**: Why did you write Ff A equals Ff B?

Fay: This being a force of friction applied by block A on it, and block A can't apply more to block B than B is applying to A.

Marco: Right, right, right. Okay.

Fay: So, I am just tying these together. Because –

Marco: Oh I see, ...

Fay: We will say oh, so this force needs to be equal to that one compared to this one, and see which way the force,

Marco: And this one is the label for force of friction due to the table.

Fay: This is, yes.

In this segment, the group decided that reciprocity (or Newton's Third Law; see Section 1.3) applies to the friction forces between block A and block B. In other words, they agreed that the magnitude of the friction force on block A due to block B has to be equal to the friction force on block B due to block A. Figure 3.27 represents this reasoning.

Reciprocity: $\vec{F}_{AonB} = -\vec{F}_{BonA}$	
$\vec{F}_{AonB}$	$-\vec{F}_{BonA}$
$|\vec{F}| \to$	$|\vec{F}| \leftarrow$

Figure 3.27: The possibility set for reciprocity between the two blocks.

The possibility set in Figure 3.27 contains only a single possibility; one of the quantities in the set is imported from a previous deduction (in this case, that term is $\vec{F}_{BonA}$), while the other is defined within the set by the relationship (in this case, $\vec{F}_{AonB}$). Because there is only a single possible value for $\vec{F}_{AonB}$, that value can be imported into another possibility set. Thus, we have the chain of reasoning that is shown in Figure 3.28.

$\vec{F}_{net,onA} = \vec{F}_{gravity} + \vec{F}_{B,normal} + \vec{F}_{Amir} + \vec{F}_{B,friction}$				
$\vec{F}_{net,onA}$	$\vec{F}_{gravity}$	$\vec{F}_{B,normal}$	$\vec{F}_{Amir}$	$\vec{F}_{B,friction}$
$(\vec{0})$	$\downarrow$	$\uparrow$	20 N $\to$	$|\vec{F}| \leftarrow$
$(\vec{0})$	$\downarrow$	$\uparrow$	20 N $\to$	$[\neq |\vec{F}| \leftarrow]$

$\Downarrow$

Reciprocity: $\vec{F}_{AonB} = -\vec{F}_{BonA}$	
$\vec{F}_{BonA}$	$\vec{F}_{AonB}$
$|\vec{F}| \leftarrow$	$|\vec{F}| \to$
20 N $\leftarrow$	$[\neq |\vec{F}| \to]$

$\Downarrow$

$\vec{F}_{net,onB} = \vec{F}_{gravity} + \vec{F}_{table,normal} + \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table,friction}$					
$\vec{F}_{net,onB}$	$\vec{F}_{gravity}$	$\vec{F}_{table,normal}$	$\vec{F}_{Barbara}$	$\vec{F}_{AonB}$	$\vec{F}_{table,friction}$
$(\vec{0})$	$\downarrow$	$\uparrow$	7 N $\leftarrow$	$|\vec{F}| \to$	$\leftarrow$
$(\vec{0})$	$\downarrow$	$\uparrow$	7 N $\leftarrow$	$|\vec{F}| \to$	$\to$
Other possibilities not considered					

Figure 3.28: The chain of reasoning connecting block A with block B.

At this point, there are still too many possibilities to draw any conclusions regarding the forces on either block. The group found more possibilities to eliminate, by sequestering all quantities that referred to the vertical direction.

Fay: So, $Ff\ T$, $Ff\ T$ (pointing to the two possible friction forces due to the table on block B). And we can assume that there is no normal, the block is not moving up or down, so these all sum out (referring to the vertical forces).

The group made a mental distinction between the forces in the x direction and the forces in the y direction. The y-direction forces are all completely determined by the momentum principle, and they are a tautology which needs no further consideration. Figure 3.29 shows that there is only one possibility for the y-components of the forces, which does not inform the solution of the problem.

$dp_y/dt = (F_{gravity,\ y,\ onA} + F_{B,\ y})$			$dp_y/dt = (F_{gravity,\ y,\ onB} + F_{table,\ y})$		
dp_y/dt	$F_{gravity,\ y}$	$F_{B,\ y}$	dp_y/dt	$F_{gravity,\ y,\ onB}$	$F_{table,\ y}$
(0 N)	$|F|\downarrow$	$|F|\uparrow$	(0 N)	$|F|\downarrow$	$|F|\uparrow$
0 N	$|F|\downarrow$	$[\neq |F|\uparrow]$	0 N	$|F|\downarrow$	$[\neq |F|\uparrow]$

Figure 3.29: Possibility sets for the y-components of the forces on the two blocks.

Eliminating the y-components of the forces significantly reduces the possibility space under consideration, making the group's set of possibilities much simpler, as shown in Figure 3.30.

The group was still allowing for the possibility that the frictional force due to the table on block B could be either to the left or to the right. So, Marco explicitly talked about how to determine the direction of that force.

Marco: So what we have to figure out is if this is (pointing to something on the white board) bigger or less than 7 N. Okay.

The group indicated that they were using the momentum principle for the blocks, combined with the superposition principle, and that they are considering the possible values of X that would determine the direction of the friction on block B due to the table. Specifically, if the friction force from block A is greater in magnitude than 7 N, the friction force due to the table must be to the right, but if the friction force from block A is less in magnitude than 7 N, the friction force due to the table on block B must be to the left. Note that the group did not explicitly consider what would happen if the magnitude of the friction force due to block A were exactly 7 N, so that possibility is not explicitly represented. The updated possibility set for block B is shown in Figure 3.31.

$F_{net,x,onA} = F_{Amir,\ x} + F_{BonA,\ x}$		
$F_{net,x,onA}$	$F_{Amir,\ x}$	$F_{BonA,\ x}$
(0 N)	20 N →	$|F| \leftarrow$
(0 N)	20 N →	$[\neq |F| \leftarrow]$

↓↓

Reciprocity: $\vec{F}_{AonB} = -\vec{F}_{BonA}$	
$\vec{F}_{BonA}$	$\vec{F}_{AonB}$
$|\vec{F}| \leftarrow$	$|\vec{F}| \rightarrow$
$|\vec{F}| \leftarrow$	$[\neq |\vec{F}| \rightarrow]$

↓↓

$F_{net,x,onB} = F_{Barbara,\ x} + F_{AonB,\ x} + F_{table,\ x}$			
$F_{net,x,onB}$	$F_{Barbara,\ x}$	$F_{AonB,\ x}$	$F_{table,\ x}$
(0 N)	7 N ←	$|F| \rightarrow$	←
(0 N)	7 N ←	$|F| \rightarrow$	→
Other possibilities not considered			

Figure 3.30: Possibility sets for the two blocks with the forces in the y direction removed.

$dp_x/dt = F_{Barbara,\ x} + F_{A,\ x} + F_{table,\ x}$			
dp_x/dt	$F_{Barbara}$	$F_{A,\ x}$	$F_{table,\ x}$
(0 N)	7 N ←	< 7 N →	←
(0 N)	7 N ←	> 7 N →	→
Other possibilities not considered			

Figure 3.31: The explicit possibility set for block B. Note that the group did not explicitly consider the possibility of $F_A = 0$.

The next step for the group was to determine whether the magnitude of the friction force on block B due to block A was greater or less than 7 N. They did this by determining the magnitude of the friction force on block A due to block B, since they had already indicated that they knew that to be the same as the magnitude of the friction force on block B due to block A.

Fay: And we know that this block is not moving.

Marco: Right. So, this (pointing to the friction force on block A due to block B) is 20 N.

By using the momentum principle, they eliminated every possibility except that the magnitude of the friction on block A due to block B is 20 N. This is shown in Figure 3.32.

$F_{net,x,onA} = F_{Amir,\ x} + F_{BonA,\ x}$		
$F_{net,x,onA}$	$F_{Amir,\ x}$	$F_{BonA,\ x}$
(0 N)	20 N $\rightarrow$	20 N $\leftarrow$
(0 N)	20 N $\rightarrow$	[$\neq$ 20 N $\leftarrow$]

Figure 3.32: Only one possibility remains for block A.

Finally, the group added the only remaining possible value of the friction force on block B due to block A into their reasoning about block B. This allowed them to immediately assess which of the two possible directions for the friction force due to the table is correct.

Fay: (Writes "= 20 N" on the whiteboard beside the friction on block B due to block A) Which means that the force of the table must be applying in this direction (gesturing on the white board to the left).

Without difficulty, they indicated the correct answer to the problem. This final step is noted in Figure 3.33.

3.5 Summary

According to the possibilities framework, deductive reasoning in physics progresses by first identifying the key relationship and the quantities present in that relationship. Then, one must consider

$F_{net,x,onB} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$			
$F_{net,x,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{AonB}$	$\vec{F}_{table}$
(0 N)	7 N $\leftarrow$	20 N $\rightarrow$	$\leftarrow$
(0 N)	7 N $\leftarrow$	20 N $\rightarrow$	$\rightarrow$

Figure 3.33: Only one possibility remains for block B, and this determines the final answer to the problem.

possible values for those quantities both individually and with respect to the other quantities in the relationship. By doing so, one can eliminate impossible values and combinations of values. Whatever possibilities remain provide the answer to the question. Sometimes, this process needs to be repeated numerous times, once for each step in a deductive chain of reasoning. Each step corresponds to a set of possibilities, and usually a value needs to be transferred from one step to the next. This entire process of reasoning can be represented diagrammatically, and the rules for using diagrams to demonstrate reasoning within the possibilities framework were presented in this chapter.

The possibilities framework replicates the results from formal abstract logic; however, with the diagrammatic representation presented here, we can also probe more deeply into the reasoning that is being used to solve problems. This framework is most useful when investigating the reasoning used in problems that are qualitative, as the values for quantities in those problems can be grouped (for example, into "zero" and "non-zero"), making it easier to flesh out all of the possibilities.

4

Pilot Studies

4.1   Pilot Study 1

From anecdotal evidence, students struggle to follow long chains of deduction, even when the reasoning is presented very clearly and aided by polling questions. Our first pilot study was an attempt to investigate this phenomenon.

4.1.1   Goals and methodology

We had two distinct goals with this pilot study. The first goal was to try to identify the specific difficulties students had when they tried to follow long chains of deduction. We simulated a polling-question session where an interviewer gave participants multiple-choice questions that stepped them through a chain of deduction. The questions were asked one at a time, and whenever the participant gave an incorrect response, the correct answer was provided with a physical explanation. After each question, the interviewer stacked the sheet of paper with the question on it off to the side, within reach of the participant. For this study, we used a think-aloud protocol (Ericsson & Simon, 1993), asking the students to say everything they were thinking as they worked through each question. We told the participants we were investigating reasoning, but not specifically what our research questions were.

The second goal was to determine whether students were simply not capable of long chains of

deductive reasoning. To probe this, we asked the participants to solve two logic problems. These were given at the end of the study, and the participants were again asked to say everything they were thinking as they worked through the problems.

4.1.2 Participants

A total of seven students were recruited from summer session classes to take part in this study. The participants ranged in their experience with physics, but they had all taken some calculus-based physics at the college level (some were currently enrolled in the first semester course, while others had previously taken it). Some had taken a reformed-curriculum (Matter and Interactions) course, while others had experience with a traditional curriculum. We constructed two different versions of the questions, one for the students enrolled in the reformed curriculum and the other for the students enrolled in traditional sections.

4.1.3 Problem

One of the problems, which was about Tarzan swinging on a vine, proved to be particularly valuable for understanding how the participants were reasoning. We therefore focused our analysis on their solutions to this problem, which consisted of five separate multiple-choice questions that were presented one at a time. Each of the five questions repeated the diagram of Tarzan at the bottom of his swing on a vine, which is shown in Figure 4.1.
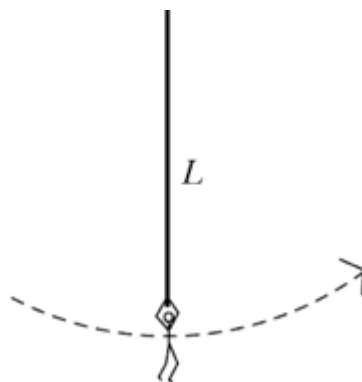
Figure 4.1: Tarzan swinging on a vine, at the bottom point of his trajectory.

Below is the reformed curriculum (Matter and Interactions) version of those five questions as presented to the participants:

1. Tarzan swings from a vine. At the bottom of the swing, what is the direction of his $d\vec{p}/dt$?

   The participants were given the direction wheel shown in Figure 4.2 to choose a direction from.
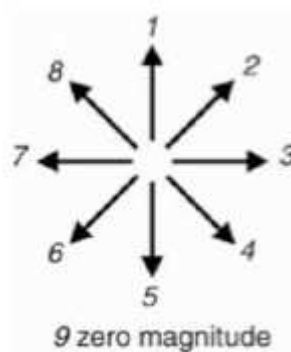
Figure 4.2: The generic direction wheel from which participants chose a direction.

2. Tarzan swings from a vine. At the bottom of the swing, what is the direction of the net force acting on Tarzan?

   The participants were again given the direction wheel in Figure 4.2.

3. Tarzan swings from a vine. At the bottom of the swing, what objects exert forces on Tarzan (neglecting air resistance)?

   The following options were given to the participants:

   (a) Earth, vine

   (b) centrifugal force only

   (c) Earth and centrifugal force

   (d) Earth, vine, centrifugal force

4. At the bottom of the swing, how does the magnitude of the force on Tarzan by the vine compare to the magnitude of the force on Tarzan by the Earth?

   The following options were given to the participants:

(a) $F_{vine} > F_{Earth}$

(b) $F_{vine} = F_{Earth}$

(c) $F_{vine} < F_{Earth}$

(d) Not enough info.

5. Tarzan's mass: 100 kg. Length of vine: 5 m. Tarzan's speed: 13 m/s. What is the tension in the vine at this instant?

   The following options were given to the participants:

   (a) 980 N

   (b) 3380 N

   (c) 2400 N

   (d) 4360 N

4.1.4   Results

Of the seven participants, none got every question related to Tarzan correct. Their performance varied, but certain patterns emerged. For example, only two participants made explicit that they drew the conclusion of the second question from the explanation of the first question. There were also two incidents where a participant drew a correct deductive conclusion from a previous explanation but then rejected it without saying why, instead selecting a different response. Additionally, no participant ever explicitly grabbed a previous question to help in answering the current question. We had not suggested that the participants do so, but neither did we discourage it.

   In our initial analysis of this problem, we classified the participants into three groups based on how they solved the problem. Two participants seemed to reset after each question, treating each as if it were entirely separate from the others rather than part of a chain of deduction. A different pair of participants seemed to recognize that the questions were linked together as in a chain, but still chose to use reasoning other than deduction to answer them. The final three participants drew some deductive conclusions when answering the questions, but they still failed to be consistently successful throughout the problem. All three groups demonstrated significant difficulties in following the chain

of reasoning; even when they solved the final question successfully, the reasoning the participants used did not invoke prior questions.

Each participant was given logic problems to solve as part of this project. Six of the seven participants were given two logic problems, and the seventh participant was only given one of the two in the interest of time. Four of the participants found the correct answer to exactly one of the logic problems. Each of the groups from the analysis based on the participant's work on the Tarzan problem had at least one member solve a logic problem correctly. This indicates that many students do in fact have the ability to solve deduction problems, although they do not necessarily apply that ability to physics problems.

4.1.5   Example: Tina

Tina had previously completed a semester of traditional calculus-based physics, so the traditional version of the problem was used for her, which meant that she was asked about Tarzan's acceleration rather than Tarzan's $d\vec{p}/dt$. Like other participants in this study, she was not particularly vocal while solving these problems. However, what she does say can be cast into the possibilities framework that was described in Chapter 2. She was classified into the "transition" group, because she considers a valid result but then rejects it. Her progression through the Tarzan problem is demonstrated below, separated into the individual multiple-choice questions that make up the problem:

> **Tina**: Tarzan swings from a vine. At the bottom of the swing, what is the direction of his acceleration? At the bottom of his swing... it's 5 (she selects number 5, which is down), oh, no that's gravity. There is no acceleration; it's 9. (she crosses off her earlier selection and selects number 9, which is no acceleration.)

We do not have enough information to be sure as to her reason for selecting the option that there is no acceleration. One conjecture is that she first imagines that gravity is a necessary quantity, but then considers the fact that the vine is also applying a force to Tarzan. Figure 4.3 represents the framework for this conjecture.

After the interviewer explains that the acceleration is straight up at that location, Tina is given the next question. She continues:

$\vec{F}_{gravity} + \vec{F}_{vine} = m\vec{a}$			
$\vec{F}_{gravity}$	$\vec{F}_{vine}$	m	$\vec{a}$
↓			↓
↓	(↑)		$\vec{0}$
Other possibilities not considered			

Figure 4.3: A potential possibility set for Tina thinking about the first question in the Tarzan sequence.

> **Tina**: At the bottom of the swing, what is the direction of the net force? Let's see, it has gravitational force, I do remember that one, going down that way (she draws an arrow down near Tarzan). What's the net force? This is 5 (she selects number 5, which is down).

In this case, she seems to only be considering one force, the gravitational force. However, we can not conclude that she is not also considering the force of the vine. If she is considering the force of the vine, she is treating it as having a smaller magnitude than the force of gravity. Notice that her response, "down," cannot follow by *modus ponens* from the premise that was given to her that the interviewer that the acceleration is actually straight up. Neither does "down" follow by *modus ponens* from Tina's early response that the acceleration was zero in magnitude. The fact that she answered "down" is a strong indicator that she was not using logic to answer this question. Instead, she must have been considering the possibility that the net force could be directed down, maybe because she considered the gravitational force but no other forces.

After the interviewer explains that the net force is straight up at that location, Tina is given the next question. She continues:

> **Tina**: What objects exert a force? Um, well, he's going in a circle, so no centripetal force, so 4 (she selects number 4, which is 'Earth, vine, and centrifugal force.')

This quote indicates that she is believes all three of these forces are relevant for solving the problem. She does not provide any information about the relationship between these quantities, nor does she give us any reason to believe that this set of quantities is related in any way to any previous reasoning she has done. Figure 4.4 indicates this incomplete set of possibilities below.

$\vec{F}_{gravity}$	$\vec{F}_{vine}$	$\vec{F}_{centrifugal}$

Figure 4.4: Tina's incomplete possibility set for the third question in the Tarzan sequence.

After the interviewer explains that only the earth and the vine apply forces to Tarzan, Tina is given the next question. She continues:

> **Tina**: (Reading) "How does the magnitude of the force on Tarzan by the vine compare to the magnitude of the force on Tarzan by the earth?"
>
> **Interviewer**: (After a long pause by Tina) What are you thinking?
>
> **Tina**: It's a lot easier with numbers, that's what I'm thinking. Um, if the net force is up then the vine might be more than the earth, but for some reason I think that... (she selects number 2, that the force of the vine is equal to the force of the earth).

Tina is considering two different sets of possibilities. In one set, the only possibility is the one that follows logically from the information and the situation that is given. This possibility set is given in Figure 4.5.

$\vec{F}_{net} = \vec{F}_{gravity} + \vec{F}_{vine}$		
$\vec{F}_{net}$	$\vec{F}_{gravity}$	$\vec{F}_{vine}$
↑	$|\vec{F}| \downarrow$	$> |\vec{F}| \uparrow$
↑	$|\vec{F}| \downarrow$	$[\leq |\vec{F}| \uparrow]$

Figure 4.5: The correct possibility set for the magnitudes and directions of the forces acting on Tarzan's vine.

However, because she both acknowledges this possibility and does not accept it, we suspect that there is another possibility set that she must be considering. Unfortunately, we do not know much about the set she used, except that it allows the possibility that the magnitude of the gravitational force is equal to the magnitude of the force of the vine, something the prior possibility set does not. Therefore, Tina did not use the relationship shown in Figure 4.5 *if she was assigning the correct values to the quantities in the relationship.*

The alternative is that Tina may have been assigning an incorrect value to the net force, setting it to zero. In this case, she did use the relationship in Figure 4.5, but she kept some possibilities in that set corresponding to setting the value of $\vec{F}_{net}$ to $\vec{0}$. In this case, her possibility set may have looked like Figure 4.6, where she eliminated the possibilities corresponding to the net force being up, rather than the possibilities corresponding to the net force being zero. If so, she did not use as a premise the conclusion from question #2, which informed her that the net force should have been up.

$\vec{F}_{net} = \vec{F}_{gravity} + \vec{F}_{vine}$		
$\vec{F}_{net}$	$\vec{F}_{gravity}$	$\vec{F}_{vine}$
$\vec{0}$	$|\vec{F}| \downarrow$	$|\vec{F}| \uparrow$
$\vec{0}$	$|\vec{F}| \downarrow$	$[\neq |\vec{F}| \uparrow]$

Figure 4.6: A potential possibility set for Tina thinking about the fourth question in the Tarzan sequence.

Unfortunately, we simply do not have enough information to identify what possibilities she was considering when answering this problem. However, we can be fairly confident in noting that because she rejected the conclusion she ought to have reached with formal logic that she did not use formal logic to answer this question. This lack of information was part of the motivation behind more detailed future studies that eventually culminated in the main research project that forms the base of this dissertation. To form a more complete picture of how student deduction occurs, we need to probe more deeply into the reasoning that students are using.

After the interviewer explains that the magnitude of the force of the vine on Tarzan is greater than the magnitude of the force of the earth on Tarzan, Tina is given the next question. She continues:

Tina: More Tarzan! Tarzan's mass: 100 kg. Length of the vine 5 m. Tarzan's speed. What is the tension of the vine? I don't remember the tension equation. [What is the tension] in the vine at this instant? I'll see if I can play with the numbers, but it'll be a waste of time because I don't remember (pause). I've got a speed and a distance. Maybe (inaudible, gets a calculator and begins punching numbers). Multiplying them

all together. For some reason, I think mass, speed, and distance is right but it's not (she multiplies 100, 5, and 13 together and gets an answer that's not listed as a choice). I don't remember the tension equation, so I guess I'll just guess (she selects number 3, 2400 N).

She indicates that she cannot remember the equation for tension. In other words, she seems to believe that there is a certain possibility set that is appropriate for solving this problem that she does not have access to. She cannot remember it, and she does not attempt to rebuild it, even though she has seen it in various forms throughout this problem. Instead, she tries something on a whim, which turns out to be incorrect. Figure 4.7 shows her attempted set of possibilities.

m	l_{vine}	$|\vec{v}|$	$|\vec{F}_{tension}|$
100 kg	5 m	13 m/s	6500 N

Figure 4.7: Tina's possibility set for the final question in the Tarzan sequence.

The fact that she is just multiplying the values of these quantities together is represented by the lack of a relationship in the possibility set. When the conclusion Tina draws from this set is not one of the options in the multiple-choice question, she rejects the set entirely and resorts to guessing.

This example with Tina has demonstrated how her thinking was not formally logical, although it did invoke elements of deductive reasoning. However, we had only sparse data on the actual reasoning she was doing, and we used these gaps to motivate our future research.

4.1.6 Conclusions from pilot study 1

None of the participants used formal deductive reasoning in this pilot study. Although some were able to draw some single-step *modus ponens* conclusions, there were no instances of participants following the chain of reasoning in the Tarzan problem through to the end. However, some of the participants were able to solve a logic problem that was completely divorced from any Physics context. This indicates that students have the ability to use deductive reasoning, but they do not use formal logic while solving physics problems.

In addition, two different participants rejected conclusions that they reached via proper deductive reasoning. In neither case did we know the reason for the rejection. We were also unable to determine whether other participants were trying to decide between different modes of reasoning because there was so little verbalization.

4.2 Pilot Study 2

In our second pilot study, we wanted to more deeply address the question of what interferes with proper deductive reasoning. We wanted to build off of the results from the first pilot study, which indicated that participants did not use formal logic when solving problems. Was this because they were incapable of using logic, or was something interfering with their ability to do so in physics?

4.2.1 Goals and methodology

This pilot study had three distinct parts, each of which was designed to address a separate question. The first goal was to determine whether the physics context was necessarily obfuscating the participants' ability to perform deductive reasoning. To address this question, we generated two isomorphic problems, one of which had a physical context and the other of which did not. In both cases, all of the information required to solve the problem was provided in the problem itself; that is, the participants did not need to remember anything about physics in order to work out the problem. The two versions of the problem are provided below.

1. In our world, positively charged objects repel other positively charged objects but are attracted to neutral or negatively charged objects. Similarly, negatively charged objects repel other negatively charged objects but are attracted to neutral or positively charged objects.

   Imagine that you walked into a room and saw five small plastic balls hanging far from each other by strings from the ceiling. A sign informs you that two of them are positively charged, two are negatively charged, and one is neutral. You notice that you are able to pick up the plastic balls by their strings and move them around.

   Using only the five plastic balls, is it possible to determine which one is neutral? If so, explain how. If not, explain why not.

Is it possible to determine which two are positively charged? If so, explain how. If not, explain why not.

2. At Cliqueshire University, it is well known that members of fraternities won't party with members of other fraternities but will party with any sorority member or unaffiliated individual. Similarly, members of sororities won't party with members of other sororities but will party with any fraternity member or unaffiliated individual.

You are seated in front of a computer, and you are in an instant-messaging conversation with five students from CU. Their usernames (given to them by the university) are Student A, Student B, Student C, Student D, and Student E. You know that two are members of different fraternities, two are members of different sororities, and one is unaffiliated, but you don't know any student's gender.

Asking the students only whether they would party with a certain individual (questions of the type, "Student A, would you party with Student B?"), can you determine who the unaffiliated student is? If so, explain how. If not, explain why not.

Can you determine which of the students are members of a fraternity? If so, explain how. If not, explain why not.

One version of this problem was presented to the participant at the beginning of the interview, while the other was presented after a number of physics problems were given. The order in which the two problems were given was randomized.

The second goal was to again follow the participants' reasoning as they solved physics problems. However, instead of being chains of multiple-choice questions, the problems took one of two forms. In the first form, a physics question was asked directly. In the second form, at least one alternate question was asked before the main question in an attempt to prompt the participant to think about the salient features of the problem. For example, two versions of a problem about a bungee jumper were used for this study, as shown below.

1. A bungee jumper has jumped from a bridge, using a cord with an unstretched length of 80 meters. At the bottom of the jump, indicated by the diagram below, compare the force of the

cord and the force of gravity on the jumper. Which is larger, or are they equal? The picture below is not to scale (the diagram of the bungee jumper is shown in Figure 4.8).
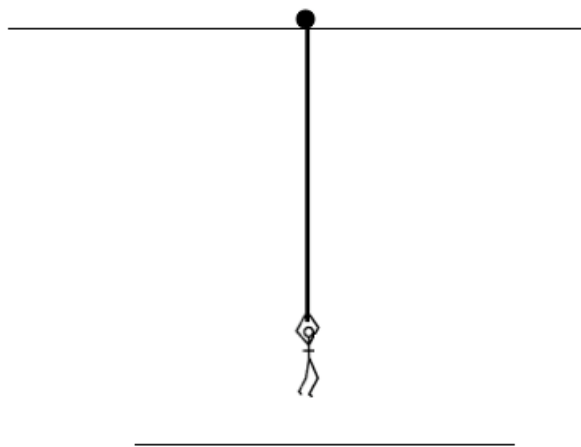
Figure 4.8: A bungee jumper hanging on the bungee cord. The context of the problem determines whether the jumper is at rest or at the bottom of her jump.

2. A bungee jumper is holding on to a bungee cord with an unstretched length of 80 m. The picture below is not to scale.

   (a) If the diagram depicts a situation in which the bungee jumper is hanging motionless for a minute, compare the force of the cord and the force of gravity on the jumper. Which is larger, or are they equal?

   (b) If instead the diagram depicts a situation where the bungee jumper had leapt from a bridge and is at the bottom of her jump, compare the force of the cord and the force of gravity on the jumper. Which is larger, or are they equal?

As with the first pair of problems, the version that was given to the participant was randomized.

The third goal of this pilot study was to try to determine what affects students' ability to perform formal logical reasoning. To do this, we created two sets of conditional statements. Below each statement were five options for possible conclusions one could draw from those statements. These five options consisted of two statements reflecting valid logic: *modus ponens* and *modus tollens*, two statements reflecting fallacies: the affirmation of the consequent and the denial of the antecedent

(see Section 2.2.3), as well as a choice "none of these." The participants were given the following instructions regarding this task:

> Consider each of the following 'If' statement to be true. Below each statement are some
> student-drawn conclusions. Circle each student-drawn conclusion that can be logically
> drawn from the statement. If none of the conclusions can be logically drawn, select 'none
> of these.' Remember: the term 'net force on an object' means the vector sum of all forces
> acting on an object.

Each set of eight conditional statements ranged from the very abstract, "If Condition A is true, then Condition B is true" to statements with physics content that are taught as bi-conditionals rather than conditionals, such as "If an object is experiencing a net force upward, then that object has a $d\vec{p}/dt$ that is directed upward." By asking all of these questions, we hoped to identify whether the content and context of the logical statements had an effect on the participants' use of logic. The two different sets were randomized among the participants.

4.2.2 Participants

Five participants, who were recruited from the second semester Matter and Interactions (reformed curriculum) calculus-based introductory physics course, took part in this study. One of these five did not participate in the logic part of the interview.

4.2.3 Results

Physics context

Of the five participants in this study, only one participant got neither of the isomorphic context questions correct. Three of the remaining participants got both questions correct, with two of those explicitly identifying that the problems were isomorphic. Both of those participants had received the version with the physics context first. These results indicate that the additional context of the physics versions was actually helpful; the students were not only able to solve those problems correctly but the two who were given that version first also may have used that problem as a target for comparison with the later version. The performance of the final participant on this task,

Cory, is further evidence that far from preventing successful deduction, the physical context actually encouraged it. Cory was presented with the non-physics version first, and although he struggled with the problem for about 12 minutes, he was unable to solve it correctly. Below are some excerpts from his interview.

> **Cory**: If you ask X (that is, the one whose affiliation we don't know) will he party with F and he answers no then it has to be fraternity. Yeah. It's as simple as that.
>
> **Interviewer**: (After Cory thinks for a moment) You have five students, A, B, C, D, and E. One of them is unaffiliated, but you don't know which one. Can you determine which student is unaffiliated?
>
> **Cory**: Um, yeah, because he would party with anybody. Yeah. Let's assume that - um - an unaffiliated - and it doesn't say anything about an unaffiliated in that direction - it says if the fraternity, whether or not a fraternity or sorority will party with an unaffiliated which is they will - it doesn't say anything about the unaffiliated partying with them, so, you assume, yes or no, it could be either way. There's not enough information, I don't think.

Cory gives an indication that he is thinking about the correct set of possibilities; namely, that the fraternity and sorority members will party with the unaffiliated student, but not whether the unaffiliated student will party with any of them because he does not assume that "partying" is reciprocal (that is, just because Student A may party with Student B, that does not mean that Student B will party with Student A).

The interviewer then points to the second part of the question and asks whether he could distinguish fraternity members from sorority members.

> **Cory**: Yeah. You should be able to. Because when they're asked whether they'll party with a certain person there's only - if they say yes it's two cases - they're either - if it's a fraternity and you ask the question and they say yes, um, the party... um, I'm not... I can't really figure out the way to explain this.

Cory struggles with this problem for a long time, and in the end he gives incomplete and somewhat incorrect answers. However, when he is asked the isomorphic question about small plastic balls

carrying charge, he seems to think much more clearly, as shown below.

> **Cory**: It's just like that other problem... We got (he draws 5 circles on whiteboard). Hm. Two positive, two negative, one neutral. Um. So the neutral one would attract everything if you moved it closer. Um. Yeah. Yeah, that's easy. You just move the neutral one toward all of them and see what happens. You move - you try each one and the one that attracts everything is the neutral one.

Cory clearly indicates how to detect the neutral ball. He must assume that electrical attractiveness is reciprocal in a way that "partying" is not. Clearly the physics context is affecting how he thinks about and solves the problem.

> **Cory**: (After reading the second part of the problem)"Um. Hm? No. It's not. Because they could either be the negative - the positive and the negative – the two positive and the two negative charges will act exactly the same. So, that's a no."

Cory only spends four minutes on this second problem, which he identifies as being "just like" the earlier problem. However, in this case, he has no difficulty in solving it correctly. The interviewer asked him about this.

> **Interviewer**: So you noticed that that last one was just like the first one.
>
> **Cory**: But much easier.
>
> **Interviewer**: Why was it easier?
>
> **Cory**: Cause it was more, if, I don't know, it was more simplified. Uh, when I think of humans, things get complicated, I don't know, but if I, that's a very simplistic way of viewing it, that if I applied that to the other problem then it would've been much easier.

Cory acknowledges that both problems could have been solved the same way. Why then was it easier to solve the problem with the physics context? According to Cory, it is because there is more that goes into understanding human behavior than simple rules, like the physical rules given in the problem statements. Cory's performance is evidence that even when presented with presumably all of the information required to solve the problem, students can think outside the constraints of

that problem, bringing in additional information where they feel it is necessary. In other words, the physical context can actually help one solve the problem. However, it also means that in situations where students might view the physics as "complicated," as Cory viewed the human behavior, it may be a challenge for students to distinguish between the deductive reasoning that is required to solve the problem and other forms of thinking such as intuition.

Reasoning about physics problems

The participants exhibited plenty of difficulty in solving the physics problems included in the study. Some of the problems were too difficult for some of the students to fully attempt, while other ones generated some interesting insight into the students' reasoning. For example, one participant who solved the second (two-part) version of the bungee problem included earlier in this dissertation clearly indicates the same sort of difficulty in distinguishing between sets of possibilities that we noticed in Bob and Tina in the previous pilot study. Below is a description of Travis's work on this problem together with the possibilities that he is considering.

> **Travis**, after reading part one, says: "Well, I think the bungee cord, depending on the material, might have some sort of force associated with it. If, and, it's definitely holding the bungee jumper where he's at. So I think that gravity, um, gravity would be pointing down on the jumper and the force of the bungee cord is holding him up, so I believe they're equal for part A."

Travis is indicating that there should be two forces acting on the jumper, which should be equal. Presumably he is making this conclusion because the jumper is at rest, as indicated by his phrase: "holding the bungee jumper where he's at." Travis only considers one possibility, which makes this conclusion easy for him to draw. His possibility set is shown in Figure 4.9.

> **Travis**: (After reading part two) In this case, when somebody jumps off a bridge with a bungee cord, they go down, and this is gonna - the bungee cord is gonna um stretch, and it's gonna be more than 80 m, which is the unstretched length. And at that point, the cord will go up and then maybe back down a little and then back up some so I believe that the force on the, um, the force of the bungee cord would be greater, is greater.

$\vec{F}_{gravity} + \vec{F}_{vine} = d\vec{p}/dt$		
$\vec{F}_{gravity}$	$\vec{F}_{vine}$	$d\vec{p}/dt$
$|\vec{F}| \downarrow$	$|\vec{F}| \uparrow$	$\vec{0}$
$|\vec{F}| \downarrow$	$[\neq |\vec{F}| \uparrow]$	$\vec{0}$
Other possibilities not considered		

Figure 4.9: Travis's possibility set for a bungee jumper sitting at rest at the bottom of the cord.

Travis is completely correct in identifying the possibility that the bungee cord has to pull up on the jumper to try to bring him back to the unstretched length of the cord. This reasoning is demonstrated in Figure 4.10.

$\vec{F}_{gravity} + \vec{F}_{vine} = d\vec{p}/dt$		
$\vec{F}_{gravity}$	$\vec{F}_{vine}$	$d\vec{p}/dt$
$|\vec{F}| \downarrow$	$> |\vec{F}| \uparrow$	$\uparrow$
$|\vec{F}| \downarrow$	$[\leq |\vec{F}| \uparrow]$	$\uparrow$
Other possibilities not considered		

Figure 4.10: Travis's possibility set for the bungee jumper at the bottom of the jump.

However, he is not done fleshing out possibilities. Travis continues thinking about the problem and considers that maybe it's possible that the net force acting on the bungee jumper is actually $\vec{0}$.

> **Travis**: But at the same time, at that split second when she's at the bottom and she's paused, they might technically be equal, so I think maybe a little bit more information would help for me to make a confident answer, because I would essentially guess between one or the other, and either one that I guessed I'd feel kinda confident in based on the wording of this question, but I kind of can see how it can be both ways because for that split second it's technically, it seems like this top question to a degree. But, um, so I could go with bungee cord has a greater force or they're equal; I'm kinda swayed both ways (see Figure 4.11.

Now Travis is considering a second possibility, that the second part of the problem resembles the first part during the split second where she's paused at the bottom. This could manifest itself in
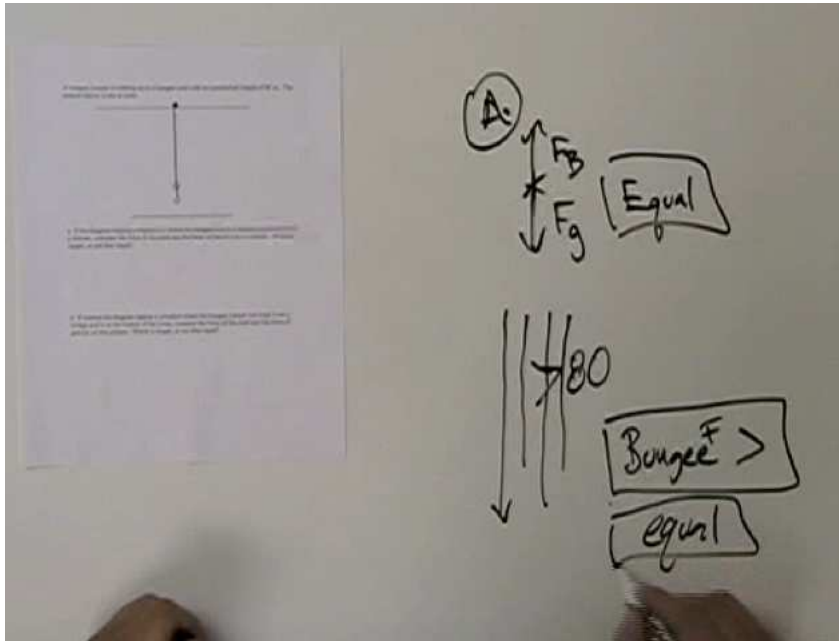
Figure 4.11: An overhead picture of Travis working on the bungee jumper problem.

one of two ways: either he is considering another possibility that fits within the set represented by Figure 4.10, or he is thinking of an entirely new set of possibilities corresponding to a different (and incorrect) relationship between an object being at rest and the net force on that object.

First, let us consider the representation for Travis fleshing out more possibilities within his existing possibility set by suggesting a new possible value for the quantity "$\vec{F}_{net}$." These additional possibilities are shown in Figure 4.12.

$\vec{F}_{gravity} + \vec{F}_{vine} = d\vec{p}/dt$		
$\vec{F}_{gravity}$	$\vec{F}_{vine}$	$d\vec{p}/dt$
$|\vec{F}| \downarrow$	$> \ |\vec{F}| \uparrow$	$\uparrow$
$|\vec{F}| \downarrow$	$[\leq |\vec{F}| \uparrow]$	$\uparrow$
$|\vec{F}| \downarrow$	$|\vec{F}| \uparrow$	$\vec{0}$
$|\vec{F}| \downarrow$	$[\neq |\vec{F}| \uparrow]$	$\vec{0}$
Other possibilities not considered		

Figure 4.12: Travis's possibilities for the bungee jumper at the bottom of the jump.

If, indeed, he has fleshed out this additional possibility, the he could identify very easily which of the two possibilities is the correct one by identifying whether the change in the momentum on the jumper is zero.

However, instead of simply fleshing out more possibilities for the set in Figure 4.10, he may have actually generated a second possibility set that does not use a valid physical relationship. Figure 4.13 is a representation of that interpretation.

$\vec{F}_{gravity}$	$\vec{F}_{vine}$	$\vec{v}$
$|\vec{F}| \downarrow$	$|\vec{F}| \uparrow$	$\vec{0}$
$|\vec{F}| \downarrow$	$[\neq |\vec{F}| \uparrow]$	$\vec{0}$
Other possibilities not considered		

Figure 4.13: Travis's possible alternative possibility set for the bungee jumper at the bottom of the jump.

If Travis did in fact create a second possibility set like that in Figure 4.13, he would have no easy way to choose between them: the relationships that they call out are different, and if Travis does not recognize that one is appropriate while the other is not, then he has not way of knowing which to choose. The fact that he did not attempt to determine which of the two possibilities for the direction of the force of the vine was correct implies that perhaps he did create a second set. Of course, we do not have enough information to know for sure.

In fact, the appearance of this second possibility set here raises an interesting question: what was he really cuing on in the first part of this problem? Perhaps the set of possibilities presented in Figure 4.9 was not completely correct; perhaps instead his initial set of possibilities was much like what he indicated on the second part of this problem, where the relationship connecting the variables is the *motion* of the object rather than the *change in that motion*. We have to be careful not to read too much into what a participant says; the example from this pilot study indicates how difficult it is to correctly interpret a participant's explanation. Furthermore, Travis's explanation is more evidence that for students, proper deduction is only one mode of reasoning, and it is not always the preferred method.

Formal logical reasoning

The four participants demonstrated varying levels of logical reasoning use in the portion of the study devoted to formal logic. To be considered a "logical response," the participant needed to provide both the *modus ponens* and *modus tollens* responses without giving either of the logical fallacies. One participant gave the logical response on five of the seven questions she answered, while another participant gave zero logical responses. Varying the content (that is, what the particular objects being reasoned about were) and context (that is, what the domain for the question was: abstract, everyday, or physical) both seemed to affect the participants' responses. The statement that was a conditional presentation of the momentum principle yielded bi-conditional responses from two of the participants, indicating that they were using their knowledge of physics to answer the question rather than an abstraction of the question.

Another finding from the literature was replicated as well; the participants chose the option corresponding to *modus ponens* 28 times in 30 chances, while they only chose the option corresponding to *modus tollens* 18 times in 30 chances. This pattern is to be expected because of the presumed difficulty of *modus tollens*, especially when compared to *modus ponens*.

When the participants reasoned through each question, they noticeably considered each option by itself, rather than treating the entire problem as a whole. In other words, they were not "abstracting out" the form of the questions, instead allowing the content and context of those questions to influence their reasoning. This, combined with the lack of any definite patterns in responses from any of the participants, speaks to the lack of logical reasoning when solving these problems.

4.2.4 Conclusions from pilot study 2

Our second pilot study gave further evidence that students, while capable of deducing, do not use formal logical reasoning when solving physics problems. Instead, they take into account many other factors, which allow them to consider alternative sets of possibilities in addition to those obtained through pure deduction (such as with Travis).

Sometimes, providing a physical context can help participants understand and solve deductive problems, especially when the participant believes that all of the relevant and necessary information

for solving the problem is provided with the problem. However, the physics problems that students were asked to solve in this study were very difficult, meaning that the participants gave little useful data that could be used to track how they used possibilities in their deduction. Therefore, while this study was useful in continuing to establish the difficulties that students have in solving physics problems, it could not address the overrarching question of how they actually do so.

4.3   Pilot Study 3

To continue to address the question of how students deduce, we created another pilot study that took the form of a short survey about the momentum principle. The survey consisted of two problems, both of which were designed to better understand how students use possibilities when performing deductions in physics.

4.3.1   Goals and methodology

This survey was generated to address two specific questions. First, what do students in the first semester physics course identify as physical possibilities? Specifically, are these identifications in accordance with or in contradiction with those afforded by the momentum principle? To test this question, we created four versions of a question that asked about a ball's momentum and either its change in momentum or net force (which are equivalent to each other according to the momentum principle). We also generated four additional questions that were equivalent to the first four except that instead of "During some period of time Δt," they said, "...initially...". Version one of the question is written below as an example.

> Please carefully consider the following statements. Beside each one, please write "P" if you believe that it describes a "possible" situation or "N" if you believe it describes one that is "not possible."

> 1. During some period of time Δt, a ball has a constant momentum of <2,3,-4.5>kg·m/s and experiences a rate of change in momentum of <3,-2,8>kg·m/s/s.

2. During some period of time Δt, a ball has a constant momentum of <0,0,0>kg·m/s and experiences a rate of change in momentum of <3,-2,8>kg·m/s/s.

3. During some period of time Δt, a ball has a constant momentum of <2 3,-4.5>kg·m/s and experiences a rate of change in momentum of <0,0,0>kg·m/s/s.

4. During some period of time Δt, a ball has a constant momentum of <0,0,0>kg·m/s and experiences a rate of change in momentum of <0,0,0>kg·m/s/s.

The second research question that this survey was designed to address is how providing the relevant physical principle affects the possibilities that students view as viable. Four versions of the question below were created.

A ball was in motion for a long period of time. Sometime during this period of time, it experienced a net force for some finite time interval Δt. While this force was acting, the ball's speed did not change.

In two of the four versions, the word "speed" was changed to "velocity." Each version asked the participants on the survey to "list as many possible explanations for the behavior" as they can. Additionally, two versions (one with "speed" and one with "velocity") added the following text: "HINT: make sure you consider the momentum principle."

A fifth version of the question was also included. This version is shown below.

Consider a situation in which a ball's momentum changes over some small period of time. Below, list as many of the other characteristics of the ball and its motion that may have changed over that period of time as you can think of.

This version of the question was created with the intention of providing a baseline. Our hypothesis was that most participants would not consider the ball's mass, but that by encouraging them to consider the momentum principle that they would be more likely to think of it.

4.3.2 Participants

This survey was administered during the last five minutes of an introductory Matter and Interactions calculus-based physics lab during the course of a week partway through a semester, and it was taken

by over 400 students. The surveys were alternated between students, and all versions of the surveys were given in all sections of the lab.

4.3.3 Results

Identification of physical possibilities

Results on the first question showed that students struggle to identify the appropriate physical possibilities for given situations. For both the questions that had the word "initially" in them and those that did not, about half of the participants gave an incorrect set of possibilities and impossibilities. More specifically, only 72 out of 163 students responded correctly to the "initially" versions of the questions. For the versions that did not use the word "initially," 125 responses out of 242 were correct, making a total of only 197 correct out of 405 responses (less than 50%) to the questions.

One example of students' struggles in identifying correct possibilities can be seen by looking more closely at the example that was given earlier. The first of the four parts to this question reads: "During some period of time Δt, a ball has a constant momentum of <2,3,-4.5>kg·m/s and experiences a rate of change in momentum of <3,-2,8>kg·m/s/s." For this part, 16 out of the 41 responses given indicated that this situation was possible, although it is impossible for an object to have both a constant momentum and a non-zero rate of change of momentum.

While we can not be sure what led to this large number of incorrect responses on these problems, the unambiguous conclusion that can be drawn from this survey is that students are confused when asked about physical possibilities. Perhaps if they are attempting to use these possibilities to reason, as the literature has suggested (Johnson-Laird, 2001), that could go some distance to helping us understand whence their errors emanate. Improper conceptions of physical possibilities cannot help but lead to incorrect physical deductions.

Effect of physical principles

Responses to the second question were varied and sometimes included multiple responses from individuals. On all versions, some participants suggested that the net force could have been zero,

which is a deductive possibility. Some also suggested non-deductive possibilities such as the time interval being very small, the mass of the ball being very large, or another force acting on the ball in addition to the net force. The inclusion of these non-deductive possibilities seems to indicate that the students are not interpreting this question as a deductive question, meaning that they are more willing to use non-deductive reasoning to answer it.

As expected, the versions that asked about speed yielded a number of correct deductive responses that the net force could have been perpendicular to the object's motion (49 out of over 200 possibilities listed by the participants). Whether the momentum principle was provided did not affect the frequency of that response. Participants also provided that possibility on the version of the question that asked about the ball's velocity changing, but at a suppressed rate (only 28 out of over 200 possibilities listed).

Only a few participants mentioned that the ball's mass changing could be an explanation for the behavior. This possibility was only listed seven times total for the "speed" versions, and the hint to provide the momentum principle did not affect that response. However, for the "velocity" version, no participant suggested that the mass could have changed without the hint, but 10 suggested the mass changing possibility when the hint was provided. This suggests that providing the momentum principle aided them in thinking about relevant physical quantities, changing at least some of their possibility sets from that shown in Figure 4.14.

$\vec{F}_{net}$	$\Delta\vec{v}_{ball}$
$\neq \vec{0}$	$\vec{0}$
$\neq \vec{0}$	$\neq \vec{0}$
Other possibilities not considered	

Figure 4.14: The possibility set for net force affecting velocity rather than momentum.

The more correct model, which some of the students may have obtained by thinking of the momentum principle, is shown in Figure 4.15.

Only a small proportion of the responses to all versions of this question were correct deductive solutions. Nonetheless, this question was fruitful in generating a wide variety of different ideas

$\vec{F}_{net} = (m\Delta\vec{v})_{ball} + (\vec{v}\Delta m)_{ball}$		
$\vec{F}_{net}$	$(m\Delta\vec{v})_{ball}$	$(\vec{v}\Delta m)_{ball}$
$\neq \vec{0}$	$\vec{0}$	0
$\neq \vec{0}$	$\neq \vec{0}$	0
$\neq \vec{0}$	0	$\neq 0$
$\neq \vec{0}$	$\neq \vec{0}$	$\neq 0$
Other possibilities not considered		

Figure 4.15: The possibility set for net force affecting momentum, which includes mass as a relevant physical quantity.

about possibilities. Because of this fertility, we chose to continue to pursue this question in the full study later described in this dissertation (see Section 6.2). We learned from asking the question at this stage that we needed to specify that the ball was undergoing a *non-zero* net force, to prevent students from responding with the correct but trivial response that $\vec{F}_{net}$ must equal $\vec{0}$.

4.3.4 Results

The third pilot study continued to affirm that students do not use formal logical reasoning. It would have been possible for them to have abstracted from the questions and answered about possibilities through the use of logic; however, the results on the survey questions seem to contradict that. Because student performance in identifying possibilities was so poor, we were unable to make any conjectures as to how they reason on those kinds of problems. Our discouraging data suggests that research needs to be done to understand how students reason about physical possibilities.

The second question sought some insight into how that reasoning happened and whether the addition of a fundamental principle would help that reasoning. The results from that question suggested that sometimes providing the fundamental principle can aid deductive reasoning by revealing additional relevant physical quantities that reform the set of physical possibilities they are considering. However, students did not always interpret such problems as deductive in nature. Because of these results, we decided to continue to investigate this particular question.

4.4   Conclusions from the Pilot Studies

The pilot studies, taken together, both postulate suggestions about student reasoning and reveal interesting questions for further investigation. When solving physics problems that require deduction, students apparently use varying levels of reasoning but do not use formal logic in any case. The fail to use logic not because they are incapable of deduction but rather that the form of deductive reasoning that students use is not formal logic. Therefore, to better understand how the students solve deductive physics problems, which must include a description of how they choose between alternate solutions, we needed to develop a framework, which ended up being the possibilities framework presented in Chapter 2. To help in the development of this framework, we needed a more complete study with more student vocalization to provide a deeper insight into their reasoning process. This study is described in the next chapter.

5

Study Design and Methodology

5.1 Selection Task Pilot

In each data collection session, individual participants were presented with physics problems, one at a time. Along with each problem, they were given three possible solutions. One of those solutions was a correctly worked solution to the problem. The other two solutions contained at least one error apiece. In one case an incorrect solution had the correct final answer, but in all other cases the incorrect solution contained an incorrect final answer. Participants were given the following instructions:

> I will ask you to figure out which solution is correct and find the errors in the other, in-correct solutions. I have intentionally picked these problems and solutions to be difficult. So don't worry if you struggle with this task in any way. I am interested in how you identify the correct solution and find the errors, so I need to use challenging problems and solutions. I just ask that you try your best. If you are not certain which solution is correct, then please narrow down which might be the correct solution as much as you can. Similarly, if you are not sure that you have found all errors, just try your best to find as many as possible. At the end of the session, I will be happy to explain the answers to you. I just can't explain the answers in between the problems. We can move onto the next problem and solutions set at any time; just let me know when you're done working

with each problem.

Participants were given access to a calculator, reference sheet, whiteboard, and marker. Following the protocol analysis of Ericsson and Simon (1993), participants were also told to think aloud:

> Please talk constantly from the first moment you see the problem and solutions to the final moment you are going over the problem and solutions. You should not plan ahead what you are going to say or explain to me what you are saying. Just act as if you are alone in this room talking through the problem to yourself. It is very important that you keep talking, so if you are silent for a period of time, then I will remind you to keep talking. We will go through some warm-up exercises to get you used to the process, and I will give you suggestions.

After they completed their set of problems, the participants were asked clarification questions regarding how they solved the problems or determined the correct solution or errors in the incorrect solutions. This questioning was done at the end of each session so as not to disturb their reasoning during the session. After all of the researcher's questions were answered, the correct solution to each of the problems was discussed with the participants.

5.1.1 Problems

We generated a large number of physics problems and solutions for the pilot study. Participants were presented with a subset of these one at a time and allowed to take as long as they needed for each one. Their interviews were limited in length to approximately 90 minutes, so most participants did not see all of the problems. Additionally, problems were continually being revised or added or removed from the pool, meaning that different participants were given different sets of problems. Because of the relatively uncontrolled environment, data from the pilot participants was not included in the main study. However, it was used to help create coding schemes, to revise the problems, and to inform the study design of the main project.

5.1.2 Participants

Participants were recruited from a pool of students who had recently completed a second-semester calculus-based physics course that used the Matter and Interactions (Chabay & Sherwood, 2007a, 2007b) curriculum. I sent an email to select top-performing students from a SCALE-UP section of the course in which I was the teaching assistant. Six of those students, all male, agreed to participate in this pilot study. A different researcher (Evan Richards), who had no teaching responsibilities for those students, conducted the data collection sessions and shared the data after the collection was complete.

5.2 Three-Part Task Design

As a result of the pilot study, some modifications were made for the full study. For example, some of the participants in the pilot spoke softly or infrequently, and this made it very difficult to identify the reasoning they were using. To encourage more vocalization by the participants, we decided to expand the study significantly.

The full study built around the selection task consisted of three parts. The first part was similar to the pilot study, in that each participant was given physics problems one at a time and asked to identify both the correct solution and the errors in the incorrect solutions. Unlike the pilot study, there were exactly four problems, which were given in a specific order (these are located in the appendix). Participants were also limited to fifteen minutes per problem, a modification that was necessary due to the widely varying time that the pilot participants needed to complete the task. The participants were once again given access to an equation sheet (which is located in the appendix), a graphing calculator, and a whiteboard with marker. In addition, to help clarify the participants' final answers, they were given a red pen and told to make a check mark on the solution they believed was correct. They were also told that they would receive the problems back for the second part of the study and encouraged to make notes on the solutions to help remind themselves of their reasoning and decisions.

In the second part, three participants who had just completed the first part were brought together into a group. Each participant was told to present one of the problems he or she had already solved,

the solution he or she chose, and the reason for that solution. After that presentation, the other participants were allowed to share their work with the group if they chose to. As a group, they were asked to try to come to a consensus as to the correct solution to the problem (but they were not required to do so). The group was given a total of ten minutes per problem to do this.

In the third part, the group was given new problems with solutions and told to select the correct solution and identify the errors in the incorrect solutions, just as they had been asked to do on the first part. As a group, they were given fifteen minutes per problem, and they were given three different problems to solve. Each of these problems are included in the appendix. There was a fourth problem on reserve as well, which some groups also solved. However, the data from that problem were not analyzed, so this problem is not included in the appendix.

All three parts of this study occurred sequentially, although they were separated by breaks. During the first part, three different sessions were conducted with different researchers in different rooms. Data collection was held on Saturdays or in the evening during the week, and one or two groups participated per day. Due to students failing to show up for sessions, we sometimes had pairs of students in the group parts rather than triples. In one case, only one student showed up and he was asked the "part three" questions in addition to his individual questions.

In no situation were the participants given the correct solution to any of the problems until the entire session had been completed. The participants were also asked to keep the problems a secret from their peers as we continued to conduct the study.

5.2.1 Physics problems

Four physics problems, all tested and modified from the pilot study, were given in the first part of the task. The problems all involved only material that would be covered in the first semester physics course using the Matter and Interactions (Chabay & Sherwood, 2007b) curriculum. The order of the problems and the solutions within the problems were decided upon before the study began and the order was kept the same for each participant. Problems A and D, the first and fourth that the participants were given, required a qualitative answer, and the three final answers given by the possible solutions spanned the space of possible responses. In addition, within those two problems, each solution began from a different fundamental physical principle. Problems B and C, the second

and third given to the participants, were quantitative in nature and each used only one fundamental physical principle over the three solutions.

In the second part of the study, participants in their groups were asked to present their reasoning and answers to the problems in the study. The order in which the problems were discussed was: D, A, and B. In some cases, the group was given the opportunity to go over problem C if it provided a volunteer to do so and if they had proceeded rapidly through the other problems. If a group had only two participants, they were given the opportunity to discuss either problem B or C in addition to problems D and A if they wanted to. In the first group studied (Fay, Marco, and Omar), the participants discussed the problems in a different order (A, C, D), and we decided to change that for all future groups. The participant who was to present each problem was selected at random. Participants were given the opportunity to defer to a different problem, but only one participant took advantage of that opportunity.

In the third part of the study, three additional problems were given to the participants in a specific order. The first group that participated (Fay, Marco, and Omar) received a different second problem than the other groups. After this first group, we decided that the problem was too difficult and removed it from the study. Again, this problem was not analyzed in this dissertation and so is not included in the appendix.

Below, the inspirations for the problems for both the individual sessions (Table 5.1) and the group sessions (Table 5.2) are listed by their letter and nickname. Note that most of the problems originated from the Matter and Interactions curriculum, unless otherwise noted. It is also worth pointing out that the questions were created by the researchers specifically for this task and are not meant to be pedagogically valuable. That is, the problems were constructed to probe the particular research questions of the investigators and are not intended as classroom problems or activities.

Table 5.1: The inspirations for each of the problems in the individual sessions.

Label	Nickname	Inspiration
Individual A	"Close the Door"	Polling Question
Individual B	"Swinging Mass"	Exam Question
Individual C	"Fission"	Textbook Example and In-Lab Problem-Solving Activity
Individual D	"Two Pucks"	Lecture Demo and Video

Table 5.2: The inspirations for each of the problems in the group sessions.

Label	Nickname	Inspiration
Group A	"Ball in Motion"	Self-Designed
Group B	"Space Probe"	Homework Question
Group C	"Two Blocks"	Tutorials in Introductory Physics (McDermott & Shaffer, 2002)

5.2.2 Participants

Participants were recruited in two phases. First, we recruited six first- or second-year physics graduate students at the beginning of the fall 2008 semester from personal invitation to participate in the study. Two were female, one of whom was the only second-year graduate student in our study, and four were male. We ran two evening sessions to accommodate these participants, and they helped to provide feedback about the clarity of the problems and instructions in addition to taking part in the study. Problem D was revised slightly after the first group of participants indicated difficulty in understanding the diagram. Additionally, the interviewers' scripts were changed slightly after the first session in response to clarification questions that the participants asked.

In the second phase, undergraduates were recruited from lecture sections of second-semester calculus-based physics courses that used the Matter and Interactions curriculum. They were told that previous completion of the first semester Matter and Interactions course was required to participate in the study. We received many requests for participation in the study, and the requests were honored on a first-come, first-serve basis. We initially scheduled three Saturdays, with two groups each day: one in the morning and one in the afternoon. However, due to participants failing to attend morning sessions, we added an additional session on a fourth Saturday in the afternoon. In the end, seventeen undergraduates participated in seven sessions. There were four groups of three, two pairs, and a single participant in the seven sessions. Unfortunately, one individual session was not video or audio recorded . Additionally, one participant's data were not analyzed because he mentioned that he had not taken a Matter and Interactions course previously. His group, a pair, was also removed from the analysis for this reason.

Of the remaining fifteen undergraduates, five were female and ten male. All participants chose a pseudonym upon arrival at the session and were given a name tag with that pseudonym to wear

during the session to prevent anyone's accidental use of their real name.

Participants were also informed that they would be audio and video recorded during the entire session and asked if they would allow their data to be used in presentations. This permission was requested at the conclusion of the study, after they had already received their financial compensation for participating. One participant requested that her identity be protected by masking her likeness and voice, and the remaining participants gave their permission to use the video and audio as we had recorded it. All consent forms are attached in the appendix.

5.2.3 Research environment

This research took place in the Qualitative Education Research Laboratory at North Carolina State University. This laboratory contains two rooms that are suitable for data collection sessions with individual students and one room that is large enough to accommodate group data collection. All three rooms were used for the individual data collection and the larger room was used for the group sessions. Each small room contains a hard-wired, table-top microphone and an overhead pan-tilt-zoom camera that are both connected to a computer for immediate processing and storage of the data. The large room contains four such overhead cameras and microphones; for the data collection in this study, three overhead cameras were used to provide different visual angles, and two microphones were used for redundancy. All of the data was directly captured and stored on a computer. During the individual sessions, the researcher conducting the session controlled the video camera. During the group session, other researchers controlled the cameras from the observation room attached to the three data collection rooms.

5.3 Analysis Methods

5.3.1 Group data

The group sessions were conducted with very little interruption by the researcher, who presented the problems to the participants one after another without giving feedback. He occasionally clarified to the participants what they were to do during the session and ensured that the conversation was finished before moving on to the next problem. At the end of some of the sessions, the researcher

asked clarification questions. In one instance, the session with Eduard, the researcher conducted longer, more thorough discussions because Eduard did not participate in the group. This data is included in this section as well, because this intervention occurred after Eduard first attempted to solve the problems, and it also hints at how such interventions may cause reasoners to develop different possibility sets that can help their reasoning on those problems.

Each of the sessions was transcribed and annotated with gestures and indications regarding work done on the whiteboard or paper. The transcripts were then segmented into units that contained conversation about a single concept or solution. Some such discussions were very short, such as this discussion in the Two Blocks problem between Hugo and Teddy:

Hugo: I think that B is going to be pulled this way (gestures to the left).

Teddy: Uh-huh.

Hugo: There's gotta be friction this way (gestures to the right).

Teddy: Yeah.

Others were quite long, with many turns between the speakers, such as the following discussion between Ike, Christobal, and Dolly:

Christobal: She's pulling 20 N of force that way (gesturing with his right thumb pointing to the right), and she is applying seven N of force that way (gesturing with his left thumb pointing to the left), and so friction needs to be –

Ike: (Having picked up the black marker and begun drawing on the white board while Christobal was talking) So, uh, so that means B is applying a force on block A of 20 this way (drawing an arrow on the white board) –

Christobal: Of seven.

Ike: 20, because it's got to balance out (pointing to the problem statement).

Christobal: It's seven.

Dolly: (Apparently supporting Ike) Because they're not moving.

Christobal: It's seven. The frictional force will provide the rest, so the frictional force has to be in the same direction as the force on block B, of 13.

Dolly: Yeah.

Ike: Well, the frictional force that block B is providing to block A is 20. So block B is experiencing a force of 20 due to friction from A. Now, Barbara is also (in unison with Dolly) pulling with seven –

Dolly: Which is 27, which means you need a frictional force in the other direction.

Ike: Yeah, so F net is, or F friction is this way with a net force of 27. (draws an arrow on the white board).

For each such discussion, the segments were coded according to the possibilities framework laid out in chapter 3.

After the initial development of the possibilities framework, a group meeting consisting of active members of the Physics Education Research and Development Group was held. During this meeting, group members were trained to use this framework for coding such segments. Through discussion, many aspects of the framework were refined and clarified. Following this meeting, a session with an individual coder was held to establish reliability. After training, she coded the presented solutions to the Two Blocks problem and the Ball in Motion problem. She also coded a total of four groups' transcripts, one for the Ball in Motion problem and three for the Two Blocks problem. Agreement with between this coder and myself was very good, with only one instance of major disagreement that resulted in a change in the segmentation. After discussion, all discrepancies between the coders were resolved so that both agreed on the possibilities set that was depicted.

To convey a measure of our agreement, the number of times the two coders agreed on a specific piece of data (relationship, quantity, or value) was counted along with the total number of pieces of data for that problem. By counting in this way, we agreed on 86% of the coding for the final two transcripts that were coded. Worth noting, the second coder made a systematic error in writing "←" instead of "≠→," even though she indicated that she was thinking "not right," and representing that with a left arrow. Two of these errors were included in the reported value. However, most of the error came from a single segment, where we chose substantially different ways to represent the same reasoning. My representation was preferable, as it conveyed additional data; however, the second coder did not contradict my coding in any way, and we reached an agreement as to how to

best represent the data after a short discussion.

5.3.2   Individual data

During the individual data collection sessions, participants were asked to think aloud, and the researchers only intervened to remind participants to keep talking. This protocol was strictly enforced to permit an analysis of the individuals' natural thinking process (Ericsson & Simon, 1993). As such, the data in these sessions were not nearly as rich as that generated during the group sessions; the spoken reasoning was often incomplete, with many missing steps. We did not attempt to analyze the individual data by coding it according to the possibilities framework, but instead generated analysis that supported the notion that deductive reasoning progresses through the elimination of possibilities, and that errors in deduction can be described through the possibilities framework.

Therefore, the interviews were transcribed and annotated with gestures and indications of work done on the whiteboards or paper. The transcripts were then segmented, but into generally smaller segments than for the group sessions. Each segment corresponded to a topical chain. That is, as long as a participant continued talking about the same object (which may be, e.g., a physics principle, a presented solution, or how difficult the problem was), it was included in the same segment.

Segment coding

The coding progressed through three passes. First, the segments were coded as a *solution claim* if they referred to the correctness of a particular presented solution as a whole. Such claims may range from "based on what I've seen so far, solution three seems to be the best" to "Solution two is definitely wrong." Statements that indicate an indecision or confusion about the solutions were coded as a *solution claim*, as long as that indecision or confusion is about the solutions themselves and not about the physics. An inability to reject (or select) a solution was also coded as a *solution claim* as long as the inability to choose is explicit. Finally, implied claims about the solution, such as instances where a participant indicates dissatisfaction about a particular solution and never returns to it, were also included as *solution claims*. The following are examples of *solution claims*:

- **Arthur**: "Okay, so it's, it's this one (pointing to solution three)."

- **Earl**: "Because this is what I'm thinking. It starts with $EF = EI$ plus work, but work is zero because the system, this whole thing is the system, so work cancels out. So this, solution one is looking right right about now."
- **Sally**: "For the y-components again need to have equal magnitudes in order to have the pucks pulled at the same speed. It seems like there's nothing obviously wrong."

A different researcher coded a set of segments as to whether they were *solution claims* or not. That researcher's codes agreed with mine over 90% of the time, yielding a Cohen's Kappa of .85.

Cohen's Kappa is a measure of reliability that, unlike simple agreement, accounts for accidental agreements between coders due to chance. Because of this, the value it provides is always less than or equal to the value given by simple agreement. Note that Cohen's Kappa could be a negative number, which is indicative of coders agreeing less than predicted by chance, and its maximum value is 1.0. Generally, a Kappa value of .8 is considered to be very good and is a reasonable target number for studies such as this. For more information about Cohen's Kappa, including detailed calculation notes and information on how to use that measure of reliability for various situations, see Banerjee, Capozzoli, McSweeney, and Sinha (1999).

The second pass was to determine whether the *solution claims* were rejections, selections, or non-actions. Rejection statements indicate a preference for or choice of a particular solution to be incorrect. The following statement by **Otto** is a rejection: "The initial state in solution 2 is wrong, so that can't be, so that can't be right." Selection statements, on the other hand, indicate a preference for or choice of a particular solution to be correct. The following statement by **Sally** is an example of a selection: "If I was given these three and told to choose on a test, I would also probably say this one still (pointing to number three)." In either case, the statement need not be definitive. Non-actions indicated a consideration of more than one possible solution without making a statement of judgment. In essence, any statement about the solutions that is unable to select, reject, or indicate a clear preference for one over another is a non-action. A different researcher coded a set of segments as to which of these three types of statements the *solution claims* were. That researcher's codes agreed with mine over 90% of the time, yielding a Cohen's Kappa of .89.

The final pass was to code the *rejection claims* as to the reason for the participant rejecting the solution. There are five categories of reasons for rejection. The *answer* code refers to situations

where the participant rejected the solution based solely on the final answer it states, not based on anything in its approach. The *relationship* code indicates a rejection based on the solution's choice of fundamental principle as inappropriate for the problem. The *term* code refers to the participant claiming that the solution contains either a superfluous term or lacks a necessary physical quantity. Rejections based on the quantities in the chosen initial or final states of the problem are *term* rejections. The *value* code refers to rejections based on the value for some quantity being incorrect at some time. This may be because of bad units, a poor substitution (e.g., setting angular speed equal to V/R), or the inappropriate choice of initial and final states. Finally, the *other* code refers to segments that are either too short or vague to provide any indication as to the reason for the participant rejecting the solution, or it is an indication that the participant is rejecting the solution because he or she cannot understand it. Another researcher coded a set of segments as to which of the five categories the segment belonged. That researcher's codes agreed with mine over 80% of the time, yielding a Cohen's Kappa of .79. However, a short discussion revealed that one of the segments was completely misread by the other coder without having the video clip for context. Upon its removal from the coding, Cohen's Kappa was .82.

Figure 5.1 shows diagrammatically how the segments were coded. The number in parentheses beside each code indicates the number of segments that were coded into that category. Of the 207 rejection codes, only 170 of them were unique and thus further coded. The remaining 37 rejection codes were simply instances of the participant reiterating that a solution was incorrect. Table 5.3 summarizes each pass of the coding and includes the reliability data. Because of the inherent difficulty in categorizing the reasoning given by the participants in a study using a think-aloud protocol, I believe that the level of agreement for each pass is acceptable.

Table 5.3: Summary including reliability for the three coding passes of the individual data.

Coding Pass	Description	Cohen's Kappa
Solution	Is the segment evaluating a written solution?	.85
Type	Is the segment a rejection or selection of a written solution?	.89
Reason	What is the participant's reason for rejection?	.79
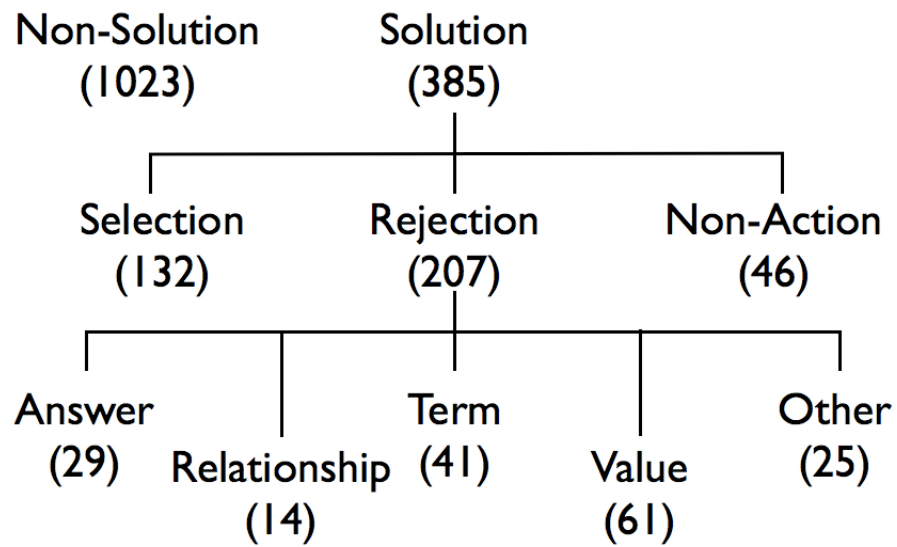
Figure 5.1: The three coding passes for the individual data and the number of segments included in each category.

6

Results from the Group Sessions

In this chapter is a presentation and discussion of the results from four problems that were discussed in the group sessions. The first two problems, "Two Blocks" and "Ball in Motion," were given to the participants to solve in groups of 2 or 3 (or, in the case of Eduard, by himself). For these problems, every group will be discussed. The latter pair of problems, "Two Pucks" and "Close the Door," were given to the participants in the individual sessions. The participants then discussed those problems in the group session. For this second pair of problems, only highlights will be discussed. These four problems are discussed in this chapter because they were the qualitative problems in this study (that is, they did not require numerical final answers), thus making them easier to represent with the possibilities framework. Note, however, that two of the problems ("Two Blocks" and "Two Pucks") provided numbers in the problem statement and used them for intermediate steps.

In all four problems described in this chapter, we will see how the possibilities framework can describe the reasoning that the participants used. Specifically, this framework can help us identify distinctions between errors that reasoners make, gain some insight into how written solutions may affect reasoning by helping reasoners identify relevant quantities in the problem, and describe how participants choose solutions and debate those choices when there is a disagreement. This list is not exhaustive, and through the numerous transcripts and explanations provided below, we see a wide range of implications provided by the possibilities framework. Some of these are discussed in depth at the end of each section, and then a summary is provided at the end of the chapter.

6.1 The "Two Blocks" Problem

The "Two Blocks" problem is included in the appendix, but it is repeated here for continuity. This was the third problem that the groups were given.

> Two heavy blocks, with massless ropes attached to them, are sitting at rest on a table. Block A sits on block B, which is in contact with the table, as pictured below. Amir
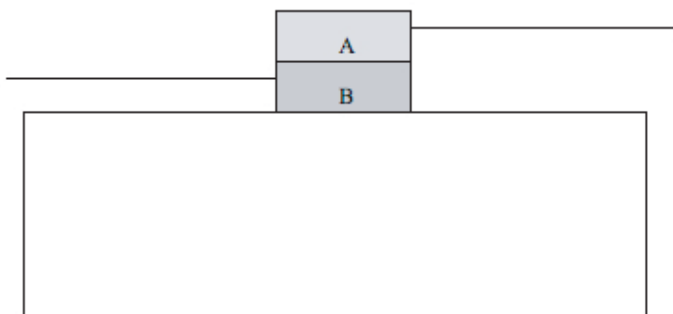
Figure 6.1: Given diagram for the "Two Blocks" problem.

> pulls on the rope attached to block A, applying 20 N of force. Barbara pulls on the rope attached to block B, applying 7 N of force. While they are pulling, neither block moves. What direction is the friction force on block B due to the table?

Many of the participants in the study attempted to solve this problem before looking at the solutions, as this problem must have seemed to be quite simple. However, there were subtleties that made it more difficult than it immediately seemed to many of the participants, leading to a number of interesting results. For example, the written solutions often had an effect on the participants' thinking, by encouraging them to consider (or stop considering) particular quantities. Additionally, the participants' attempts to solve the problems by themselves provided data on their problem-solving process, including a number of errors that, while giving the same result, can be shown with the possibilities framework to have quite different causes.

In this section, I first present the three given solutions in terms of the possibilities framework. Then I demonstrate each group's choice for the correct solution to this problem by analyzing the participants' transcripts. Usually, the entire group is considered as the unit of analysis, although in some particular instances it is clear that there were at least two distinct sets of possibilities being presented; in these instances, those sets are identified as belonging to the participant(s) who promoted them. After presenting all of the groups' discussions, I highlight a few specific ideas that are emphasized by using the possibilities framework to analyze this problem: consideration of whether the mass of the blocks is important and the distinction between similar errors regarding the interface between the two blocks. Finally, I summarize the results from this problem and indicate their usefulness.

6.1.1   The given solutions

Each of the written solutions that the participants had to choose from can be represented with the possibilities framework. In this section, I present those solutions and their representations. Then, when participants indicated that they completely agreed with a presented solution without adding further information, I refer to these depictions.

"Two Blocks" solution #1

The first written solution reads as follows:

> Because Barbara is pulling on block B, she is applying a force to the left. According to the momentum principle, the net force acting on an object is equal to that object's change in momentum. Therefore, in order to keep block B from experiencing a change in momentum, a friction force needs to be equal and opposite to the force that Barbara is applying. Therefore, the friction force on block B due to the table is to the right.

In solution #1, there is no discussion about block A. Instead, the solution takes block B as the system and indicates that the only forces acting on it are due to the pull from Barbara and the friction from the table. As such, this solution comes to an erroneous conclusion: the friction force due to the table needs to oppose the pulling force. However, this conclusion is based upon the implementation of

the momentum principle. Therefore, we need two possibility sets: one for the momentum principle and one for the superposition principle as it applies to the forces on block B. Note that the net force being zero is *implied* by the momentum principle, although that information is used in the superposition principle. The representation for this problem in the possibilities framework is given in Figure 6.2.

$\vec{F}_{net}\Delta t = \Delta\vec{p}$		
$\Delta\vec{p}$	Δt	$\vec{F}_{net}$
$\vec{0}$	(>0)	$(\vec{0})$
$\vec{0}$	(>0)	$[\neq (\vec{0})]$

$\rightarrow\rightarrow$

$\vec{F}_{net,onB} = \vec{F}_{Barb} + \vec{F}_{AonB} + \vec{F}_{table}$			
$\vec{F}_{net}$	$\vec{F}_{Barb}$	$\vec{F}_{AonB}$	$\vec{F}_{table}$
$(\vec{0})$	$\vec{F}, \leftarrow$		$-\vec{F}, (\rightarrow)$
$(\vec{0})$	$\vec{F}, \leftarrow$		$[\neq -\vec{F}]$
Other possibilities not considered			

Figure 6.2: The possibility set representation of solution #1 of the "Two Blocks" problem.

In this solution, the obvious error is that the reasoner did not consider that there was a friction force on block B due to block A, which is evident due to the blank column beneath that physical quantity. Because the reasoner did not explicitly mention that the net force needed to be zero but used that information implicitly in the superposition principle, the value is represented as being within parentheses. However, there was no implication that block A was even considered in the problem; hence no entry appears below that quantity. However, although the quantity "Δt" is not discussed either, it is implied that Δt is positive (see Section 3.2.1, which explains why that implication is included in this possibility set).

"Two Blocks" solution #2

The second written solution reads as follows:

> Barbara is applying a 7 N force to the left on block B, and block A is applying a force of 20 N to the right on block B because block B applies a 20 N friction force on block A to keep A's momentum from changing. According to the momentum principle, the net force acting on an object is equal to that object's change in momentum. So, to keep block B from experiencing a change in momentum, the combination of these two forces plus the friction due to the table needs to be zero. Therefore, the friction force due to

the table on block B is to the left.

The second solution is much more detailed, including all of the relevant information and yielding the correct answer. To demonstrate the progression of the solution, I use a total of five separate possibility sets, grouped into three stages representing the steps of reasoning (see Figure 6.3). The first two sets use the superposition principle and momentum principle together as they apply to block A. The next possibility set shows the reciprocity of the forces between the two blocks, importing the value for the force on block A due to block B from the previous stage. Then, the final step has the two possibility sets that use the value for the force on block B due to block A and apply both the momentum principle and superposition principle to block B. Worth noting, the net force regarding block A is only discussed implicitly just as happened in solution #1. The net force is explicitly discussed in solution #2 for block B.

The written solution describes the forces on block A before mentioning the momentum principle. However, the representation in Figure 6.3 presents the momentum principle first because it influences the reasoning about the superposition principle rather than the other way around.

Other than not being explicit about the net force on block A and not explicitly discussing "Δt," this solution brings to light all of the possibilities and discusses them correctly. Being explicit about possibilities helps this solution arrive at the correct answer, that the force of the friction due to the table on block B is to the right. Notice that the depth of the reasoning on this problem requires quite an extensive representation with possibility sets. There are plenty of opportunities to make a mistake when attempting to develop this chain of deductive reasoning on one's own, and the solution is also very difficult to follow. Indeed, most of the participants only attended to the answer that this solution gave rather than trying to follow its involved reasoning chain.

"Two Blocks" solution #3

The third written solution reads as follows:

> The momentum principle states that an object's change in momentum is equal to the
> net force acting on that object. But since the block isn't moving, the friction force on
> block B due to the table is zero. Because friction forces oppose motion, there can only

$\vec{F}_{net,onA}\Delta t = \Delta\vec{p}$		
$\Delta\vec{p}$	Δt	$\vec{F}_{net}$
$\vec{0}$	(>0)	$(\vec{0})$
$\vec{0}$	(>0)	$[\neq(\vec{0})]$

$\rightarrow\rightarrow$

$\vec{F}_{net,onA} = \vec{F}_{Amir} + \vec{F}_{BonA}$		
$\vec{F}_{net}$	$\vec{F}_{Amir}$	$\vec{F}_{BonA}$
$(\vec{0})$	20 N $\rightarrow$	20 N $\leftarrow$
$(\vec{0})$	20 N $\rightarrow$	$\neq$(20 N $\leftarrow$)

$\downarrow\downarrow$

Reciprocity: $\vec{F}_{AonB} = -\vec{F}_{BonA}$	
$\vec{F}_{BonA}$	$\vec{F}_{AonB}$
20 N $\leftarrow$	20 N $\rightarrow$
20 N $\leftarrow$	$\neq$ (20 N $\rightarrow$)

$\downarrow\downarrow$

$\vec{F}_{net}\Delta t = \Delta\vec{p}$		
$\Delta\vec{p}$	Δt	$\vec{F}_{net}$
$\vec{0}$	(>0)	$\vec{0}$
$\vec{0}$	(>0)	$[\neq\vec{0}]$

$\rightarrow\rightarrow$

$\vec{F}_{net,onB} = \vec{F}_{Barb} + \vec{F}_{AonB} + \vec{F}_{table}$			
$\vec{F}_{net}$	$\vec{F}_{Barb}$	$\vec{F}_{AonB}$	$\vec{F}_{table}$
$\vec{0}$	7 N $\leftarrow$	20 N $\rightarrow$	$\leftarrow$
$\vec{0}$	7 N $\leftarrow$	20 N $\rightarrow$	not $\leftarrow$
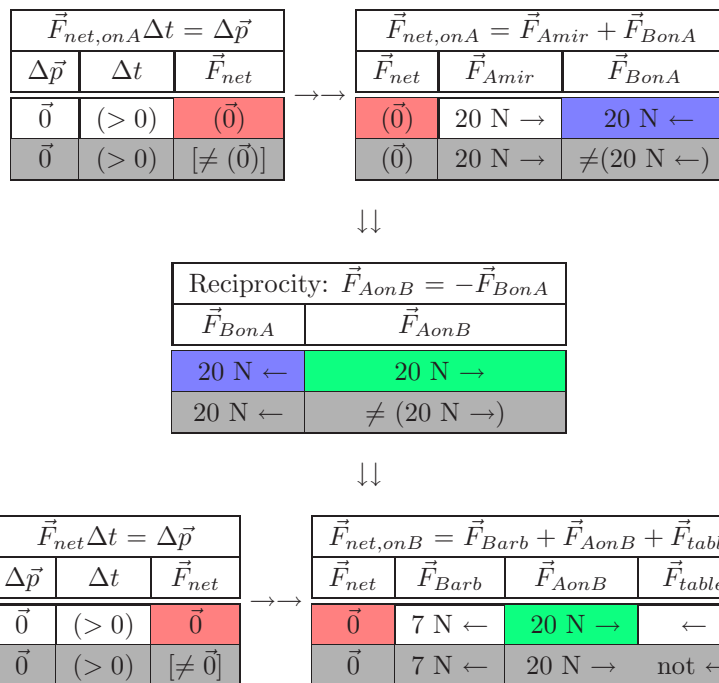
Figure 6.3: The possibility set representation of solution #2 of the "Two Blocks" problem.

be a friction force if the block were moving.

In contrast to the depth of the correct solution, this solution is incredibly simplistic, reducing the problem to an inappropriate implementation of a heuristic that itself is not always correct. Indeed, while this solution mentions the momentum principle, it does not actually use it at all in its reasoning. A representation of this solution is given in Figure 6.4.

$\vec{F}_{net}\Delta t = \Delta\vec{p}$		
$\Delta\vec{p}$	Δt	$\vec{F}_{net}$
	(>0)	
Other possibilities not considered		

direction of motion	$\hat{F}_{table}$
$\hat{k}$	$-\hat{k}$
$\hat{k}$	$\neq -\hat{k}$
0	0
Other possibilities not considered	
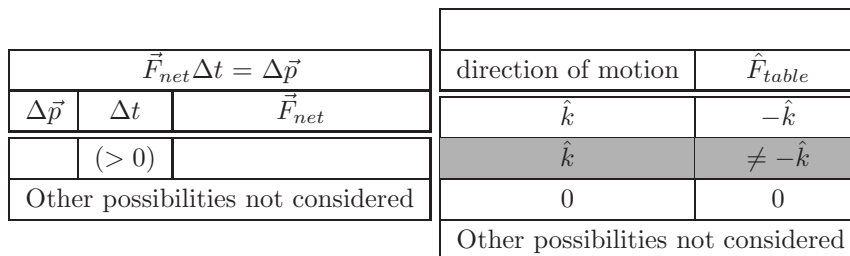
Figure 6.4: The possibility set representation of solution #3 of the "Two Blocks" problem.

Not only is there no connection between the two possibility sets, but the first possibility set does not contain any explicit possibilities, and the second possibility set does not have any physically correct relationship connecting the information present in the set. This blank space at the top of the second possibility set is indicative of an error in establishing the correct relationship between the physical quantities: no physically correct relationship can represent what the reasoner has done without including both quantities that the reasoner did not include **and** excluding at least one quantity that the reasoner did include.

6.1.2 Notes about participant solutions on the "Two Blocks" problem

The participants did not always provide full explanations in their discussions within their groups. Therefore, there may be omissions in their possibility sets. Additionally, although they were tasked with choosing between these solutions, they did not always properly understand them. In instances where their interpretation of a solution differs from what is shown in Figures 6.2 through 6.4, the participants' interpretation will be demonstrated.

Unless a participant explicitly discussed the momentum principle in more detailed terms than did the written solutions, I will omit the possibility set that includes information relating an implied or explicit net force of $\vec{0}$ from my description of the reasoning. It is worth mentioning that most of the participants provided as evidence that the "block was not moving." I interpret this to mean, for this problem, that the net force on that block is $\vec{0}$. This conclusion may technically be reached from an application of the momentum principle in the following manner: because $\vec{F}_{net}\Delta t = \vec{p}_f - \vec{p}_i$ and $\vec{p}_i = \vec{0}$, if the block is not moving and Δt is greater than zero, then the net force must be zero, as displayed in the possibilities framework in Figure 6.5. I have no evidence that the participants are in fact reasoning this way; however, for my purposes the reason for their equating "$\vec{F}_{net} = \vec{0}$" with "no motion" is irrelevant, as in many cases they use the terms interchangeably on this problem, something that does not have an effect on the process of deduction in this case.

6.1.3 An ideal solution (Fay, Marco, and Omar)

At the time of the session, Fay was a second-year graduate student while Marco and Omar were both first-year graduate students. None of them were familiar with the Matter and Interactions

$\vec{F}_{net}\Delta t = \vec{p}_f - \vec{p}_i$			
$\vec{p}_f$	$\vec{p}_i$	Δt	$\vec{F}_{net}$
$\vec{0}$	$\vec{0}$	(> 0)	$\vec{0}$
$\vec{0}$	$\vec{0}$	(> 0)	$\neq \vec{0}$

Figure 6.5: Relating "no motion" with "net force is equal to zero" for the "Two Blocks" problem.

curriculum, although they, like the other participants, were given equation sheets as an aide (the equation sheets are located in the Appendix). A detailed account of their reasoning is provided as an example for the framework in Section 3.4. I present the transcript here as well as the completed possibility sets from that example, before discussing the rest of the transcript.

Marco: I like this problem.

Fay: I like this problem.

Marco: Okay, three blocks with massless ropes attached to them are sitting at rest on the table. Block A sits on block B as shown, which is in contact with the table. Amir Pulls on the rope attached to block A, applying 20 N of force, that might be worth writing, Barbara pulls on the rope attached to block B, applying 7 N of force. While they are pulling, neither block moves. What direction is the friction force on block B due to the table?

Omar: Okay (nods as if he understands the problem).

Fay: So this is the interface that we are interested in (indicates the contact between block B and the table), but there is no motion of the blocks.

Marco: Yup. we just need to write the two force diagrams.

Fay: So that we have (drawing a free body diagram for block A) A, here's gravity, force due to block B, 20 N, and the force of friction due to block B, sub, sub B. (Drawing a free body diagram for block B) F table, FG, force of friction due to A, 7 N, and then we had a Ff question, Ff question (indicating two possible friction forces between the table and block B, one in the $+x$-direction and the other in the $-x$-direction, see Figure 6.6).
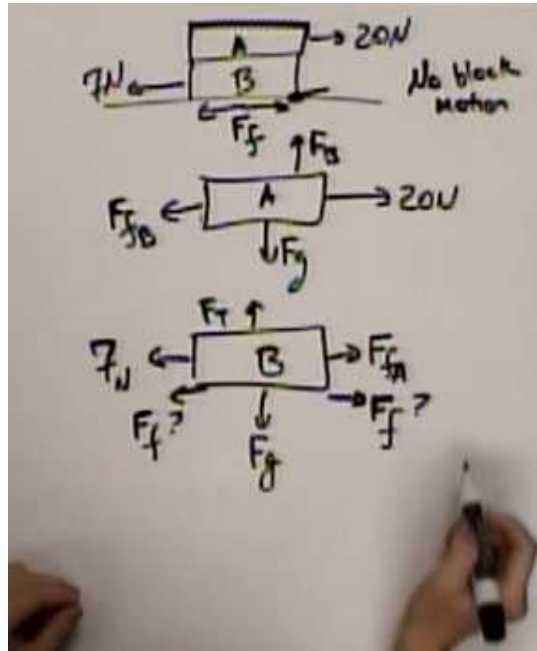
Figure 6.6: Screen capture of Fay, Marco, and Omar's whiteboard showing the two possible directions for the force of the table on block B.

As can be seen from the dialogue and Figure 6.6, this group was explicitly considering two possible directions for the force of the table on block B. The possibility set for the forces acting on block B is represented in Figure 6.7.

$\vec{F}_{net,onB} = \vec{F}_{gravity} + \vec{F}_{normal} + \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$					
$\vec{F}_{net,onB}$	$\vec{F}_{gravity}$	$\vec{F}_{normal}$	$\vec{F}_{Barbara}$	$\vec{F}_{AonB}$	$\vec{F}_{table}$
$\vec{0}$ N	↓	↑	7 N ←	→	←
$\vec{0}$ N	↓	↑	7 N ←	→	→
Other possibilities not considered					

Figure 6.7: Fay, Marco, and Omar's possibility set for block B.

Fay: This (pointing to the arrow indicating the force of friction on block B due to block A) is going to be equal to Ff B (the force of friction on block A due to block B), right? So

Omar: Hmm, it could also be –

Fay: Between the two blocks, can we say that their, this force is the same (pause)

Omar: Yeah, A and B doesn't move, (inaudible) friction...

Marco: (To Fay) Why did you write Ff A equals Ff B?

Fay: This being a force of friction applied by block A on it, and block A can't apply more to block B than B is applying to A.

Marco: Right, right, right. Okay.

Fay: So, I am just tying these together. Because –

Marco: Oh I see, and this one –

Fay: What we'll see is that we'll then say that this force needs to be equal to that one, compare it to this one, and see which way the force –

Marco: And, and this one is the label for force of friction due to the table.

Fay: This is, yes. So, Ff T, Ff T (indicating on the diagram that there are two possible directions for the force of the table on block B) and we can assume that there is no normal, the block is not moving up or down, so these all sum out (referring to the vertical forces).

Marco: So what we have to figure out is if this is (pointing to something on the white board) bigger or less than 7 N. Okay.

Fay: And we know that this block is not moving.

Marco: Right. So, this (pointing to the friction force on block A due to block B) is 20 N.

Fay: (Writes "= 20 N" on the whiteboard beside the friction on block B due to block A). Which means that the force of the table must be applying in this direction (gesturing on the white board to the left; crosstalk from other participants who are saying the same thing).

The transcript to this point demonstrates the development of the possibility sets that were used to solve for the correct solution to the "Two Blocks" problem. These possibility sets are shown in Figure 6.8.

$F_{net,\ x,\ onA} = F_{Amir,\ x} + F_{BonA,\ x}$		
$F_{net,x,onA}$	$\vec{F}_{Amir,\ x}$	$\vec{F}_{BonA,\ x}$
(0 N)	20 N $\rightarrow$	20 N $\leftarrow$
(0 N)	20 N $\rightarrow$	[$\neq$ 20 N $\leftarrow$]

$\Downarrow$

Reciprocity: $\vec{F}_{AonB} = -\vec{F}_{BonA}$	
$\vec{F}_{BonA}$	$\vec{F}_{AonB}$
20 N $\leftarrow$	20 N $\rightarrow$
20 N $\leftarrow$	[$\neq$20 N $\rightarrow$]

$\Downarrow$

$F_{net,\ x,\ onB} = F_{Barbara,\ x} + F_{AonB,\ x} + F_{table,\ x}$			
$F_{net,x,onB}$	$F_{Barbara,\ x}$	$F_{AonB,\ x}$	$F_{table,\ x}$
(0 N)	7 N $\leftarrow$	20 N $\rightarrow$	$\leftarrow$
(0 N)	7 N $\leftarrow$	20 N $\rightarrow$	$\rightarrow$

Figure 6.8: Fay, Marco, and Omar's final possibility sets for the "Two Blocks" problem.

The group explicitly considered the two possibilities that the friction due to the table on block B could either be to the left or to the right. They did not explicitly consider that it could be zero, but their final statement above indicates that they had eliminated all possibilities except for the friction due to the table being to the left. It is worth noting that the process of applying the last piece of information, namely that Amir was pulling block A to the right but that the block was not moving, resulted in a solution very quickly and the group was not explicit about each step of the reasoning. However, the detailed reasoning before that point was what permitted the creation of such detailed possibility sets.

Notice how similar to solution #2 this reasoning is. The group had not looked at that solution, and yet they generated possibility sets that were remarkably similar. However, a major difference can be seen in the third possibility set in Figure 6.8, and that is the explicit inclusion of the possibility that the friction force on block B due to the table could be to the right. This group being explicit about this possibility allowed them to eliminate it later on, noting that what would allow that possibility to be true did not occur. This work by Fay, Marco, and Omar is an ideal, expert solution

because they were explicit about their chain of reasoning and drew a correct conclusion.

> **Fay**: Let's see what they say.
>
> **Marco**: (Begins reading from solution #1) because Barbara is pulling on block B, she is applying a force to the left, is that correct? (Other participants affirm). Okay, according to the momentum principle, the net force acting on the object is equal to that object's change in momentum. (Under his breath:) This is already going to be wrong. (Other participants laugh). In order to keep block B from experiencing a change in momentum, a friction force needs to be equal and opposite to the force that Barbara is applying. Therefore the friction force on block B must be to the [right] –
>
> **Fay**: That would be true if block A did not exist. (Other participants agree).
>
> **Marco**: Okay. Well, (inaudible), I'll get to it (referring to writing what is incorrect on that solution).

Not only did they reject solution #1, but they also identified exactly what the error was: the reasoner in solution #1 neglected block A. Their ability to point this out indicates the depth to which they considered and understood this problem. They had already fleshed out all of the quantities that were required for the superposition principle to hold for the forces acting on block B, and they realized immediately that solution #1 was failing to include the friction force that acted on block B.

> **Marco**: (Now reading from solution #2) Barbara is applying a 7 N force to the left on block B, and block A is applying a 20 N force on block B, because block B applies a 20 N force on block A to keep block A's momentum from changing. According to the momentum principle, the net force acting on an object, blah blah blah, so to keep block B from experiencing a change in momentum, the combination of these two forces plus the friction due to the table needs to be zero. Therefore the friction due to the table on block B is to the left.
>
> **Fay**: This sounds like our solution. (other participants affirm, Marco indicates the group's selection of that solution).

Without difficulty, the group identified solution #2 as being virtually the same as their reasoning and selected it as being correct.

> **Fay**: And just out of curiosity, let's see what solution #3 is, along with (to researcher:) the group interview [script] that you are not supposed to give us.
>
> **Researcher**: Thank you.
>
> **Fay**: (Reading from solution #3:) the momentum principle states that an object's change in momentum is equal to the net force, but the block isn't moving, so therefore the friction force between block B and the table is zero. (Makes buzzer sound, indicating that she believes the solution is incorrect).
>
> **Marco**: (To Fay's buzzer noise:) That was great. (Unison agreement that solution #3 is incorrect).
>
> **Fay**: (Waving arms around, sarcastic:) there can't be a friction force if the block isn't moving! static friction doesn't exist!

The group also recognized the absurdity of the third solution, eliminating that out of hand. They displayed no difficulty in progressing through all three solutions, applying their possibility sets to the problem.

6.1.4   The single-block system solution (Christobal, Dolly, and Ike)

Christobal, Dolly, and Ike were all first-year graduate students who had not been trained with the Matter and Interactions curriculum. Like Fay, Marco, and Omar, this group reached the correct solution to the problem. However, they did so in quite a different way. At first, separate possibility sets were proposed by Christobal and Ike. Soon, however, the group decided that Christobal's reasoning was appropriate. Although solution #2 actually used different reasoning, it resulted in the same answer as Christobal, and the group selected that as the correct solution for this problem.

> **Christobal**: (Reading the problem statement) "Two heavy blocks with massless ropes attached to them are sitting at rest on a table. Block A sits on block B, which is in

contact with the table is pictured below. Amir Pulls on the rope attached to block A, applying 20 N of force. Barbara pulls on the rope... (trails off)."

Ike: Hmm (silence).

Christobal: Barbara pulls the rope attached to block B, applying 7 N of force. While they are pulling neither block moves.

Ike: Okay. This is fun. Okay.

Dolly: Okay?

Ike: So let's look at [solution] number one.

Christobal: (Reading solution #1) "Because Barbara is pulling on block B, she is applying a force to the left. According to the momentum principle, the net force acting on an object is equal to that objects change in momentum. Therefore in order to keep block B from experiencing a change in momentum, a friction force needs to be equal and opposite to the force that Barbara is applying. Therefore the friction for some block be due to the table is to the right." She [Amir] is pulling 20 N of force that way (gesturing with his right thumb pointing to the right), and she [Barbara] is applying seven N of force that way (gesturing with his left thumb pointing to the left), and so friction needs to be –

Ike: (Drawing arrows on the white board). So, uh, so that means B is applying a force on block A of 20 this way (to the left).

Christobal: of seven.

Ike: 20 (pointing to the problem statement), because it's got to balance out.

Christobal: It's seven.

Dolly: (Apparently supporting Ike) Because they're not moving.

Christobal: It's seven. The frictional force will provide the rest, so the frictional force has to be in the same direction as the force on block B, of 13.

Dolly: Yeah.

Ike: Well, the frictional force that block B is providing to block A is 20. So block B is experiencing a force of 20 due to friction from A. Now, Barbara is also (in unison with Dolly) pulling with seven

Dolly: Which is 27, which means you need a frictional force in the other direction.

Ike: Yeah, so F net is, or F friction is this way (to the right) with a net force of 27 (see Figure 6.9). Okay, so.

Dolly: Does that make sense though?

Ike: Well it makes sense to me, so let's go back and look to see if all of their (solution #1's) handwaving vectors were correct.
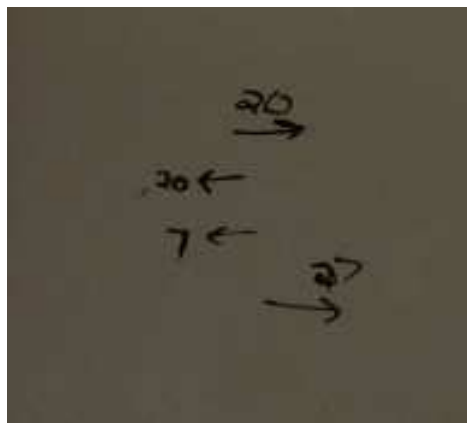
Figure 6.9: Ike's incorrect direction arrows on the whiteboard.

Ike and Christobal disagreed about whether block B applied 7 or 20 Newtons of force to block A. This is a symptom of a larger disagreement: what they were choosing as their system. Christobal, as we will soon see, chose the two blocks together as the system, meaning that the 7 N force was actually a transmission of Barbara's force. Ike, on the other hand, treated the blocks separately as did Fay, Marco, and Omar. Why, then, did he reach a different answer than they? To understand this, consider his possibility sets as displayed in Figure 6.10. As usual, note that the entire group seemed to implicitly assert that the net force was equal to zero on both blocks, not even considering other possibilities.

$\vec{F}_{net,onA} = \vec{F}_{Amir} + \vec{F}_{BonA}$		
$\vec{F}_{net,onA}$	$\vec{F}_{Amir}$	$\vec{F}_{BonA}$
(0 N)	20 N $\rightarrow$	20 N $\leftarrow$
(0 N)	20 N $\rightarrow$	$[\neq(20\text{ N }\leftarrow)]$

$\downarrow\downarrow$

Reciprocity: $\vec{F}_{AonB} = -\vec{F}_{BonA}$			
$|\vec{F}_{BonA}|$	$\hat{F}_{BonA}$	$|\vec{F}_{AonB}|$	$\hat{F}_{AonB}$
20 N	$\leftarrow$	20 N	
20 N	$\leftarrow$	$[\neq 20\text{ N}]$	
Other possibilities not considered			

$\downarrow\downarrow$

$\vec{F}_{net,onB} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$			
$\vec{F}_{net,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{AonB}$	$\vec{F}_{table}$
$(\vec{0}\text{ N})$	7 N $\leftarrow$	20 N ($\leftarrow$)	27 N $\rightarrow$
$(\vec{0}\text{ N})$	7 N $\leftarrow$	20 N ($\leftarrow$)	$[\neq 27\text{ N }\rightarrow]$

Figure 6.10: Ike's initial possibility sets for the "Two Blocks" problem.

The error is located in the second and third possibility sets; when Ike applied reciprocity in the second set, he did not explicitly include the direction of the force of block A on block B. Because he did not explicitly include this information, that information was lost when it was carried into the third possibility set. This lost information led Dolly and Ike to sum the force due to Barbara and the force due to block A as if they were in the same direction, as shown in Figure 6.9. Note that this error seems to be most likely a matter of losing a piece of information in working memory thanks to not being explicit rather than any sort of situated misconception. The structure of Ike's statements and his status as a graduate student both attest to the fact that he was probably quite familiar with Newton's Third Law as it applies to this type of problem, although his imprecise gestures and lack of specificity did not confirm this assertion. Still, the most likely interpretation of this error is that by not stating explicitly the direction of the reciprocal force, that lacking piece of information permitted Dolly and Ike to allow a wrong possibility, which was accepted quickly before recognizing that there was another possibility that they did not consider corresponding to the 20 N force of block A being in the other direction on block B. Christobal seemed very disconcerted at this point, and as soon as he got the chance, he expressed his own possibility sets.

> **Ike**: (Reading solution #1) "...Therefore the friction force on block B due to the table is to the right." So that at least agrees in direction.
>
> **Dolly**: what do you think, Christobal?
>
> **Christobal**: I don't agree with that.
>
> **Ike**: You don't agree with this (pointing to the whiteboard, symbolically implying his own explanation) or with that (pointing at solution #1).
>
> **Christobal**: With this (pointing to the whiteboard).
>
> **Ike**: Okay.
>
> **Christobal**: Because, so block A and B aren't moving with respect to each other, right?
>
> **Ike**: (Erasing something from the whiteboard) Yeah, that's probably not right.
>
> **Christobal**: And so we apply basically we can just, we are not dealing with angular momentum, so we can just assume that it is pulling at the center of mass, right? We are

taking 20 Newtons of force that way (points to the right), 7 N of force that way (points the left), it's not moving, therefore we need, uh, 13 N of force that way (pointing to the left).

Christobal described how he had been viewing the problem, taking the two blocks together as one system. Choosing such a system would be allowed because there was no angular momentum acting on this system, so the forces could be thought of as applying to the center of mass of the system. Christobal's possibility set is demonstrated in Figure 6.11.

$\vec{F}_{net} = \vec{F}_{Barbara} + \vec{F}_{Amir} + \vec{F}_{table}$			
$\vec{F}_{net}$	$\vec{F}_{Barbara}$	$\vec{F}_{Amir}$	$\vec{F}_{table}$
$(\vec{0}$ N$)$	7 N $\leftarrow$	20 N $\rightarrow$	13 N $\leftarrow$
$(\vec{0}$ N$)$	7 N $\leftarrow$	20 N $\rightarrow$	$[\neq$13 N $\leftarrow]$
Other possibilities not considered			

Figure 6.11: Christobal's initial possibility set for the "Two Blocks" problem.

Christobal's solution was elegant in that it did not even need to account for the friction between the blocks; it avoided his groupmates' error entirely. However, the fact that that he chose a different system than both Ike and solution #2 means that his relationship and relevant quantities were different as well. This decision led to some difficulties in his understanding of solution #2, as well as in troubleshooting the solution attempt that Ike and Dolly made. In the following section of the transcript, the group together attempted this troubleshooting. Merely trying to uncover what mistakes they made is a sophisticated maneuver, especially since Ike and Dolly had changed their minds to agree with Christobal's reasoning.

Christobal: Now why does your line of reasoning not work?

Ike: Probably because of –

Dolly: (Pointing to the interface between the two blocks) Because of this interface?

Ike: Yeah, probably because of the AB interface is –

Dolly: Doesn't have anything to do with the table and the entire block.

Ike: (Crosses out his earlier solution) Okay, crossed out.

Christobal: Yeah because basically you are assuming that on block B there is no net force from A.

Dolly: Right.

Ike: (Drawing on the white board) Yeah so, at this interface (between the blocks), F equals zero, and then we are left with seven. (To himself, regarding the drawing) That looks like an E.

Christobal: See, this is why, yeah, because we have, so at this interface (drawing an arrow to the right on the white board, above a line indicating the interface between the blocks) we have here F, 20, and we need something over here total (drawing an arrow to the left, below the line) equal to 20 down here, well, that's going to come from her seven (draws an arrow below that, pointing to the left and writes "7") and from the F friction, uh, 13 (writes "$F_f = 13$").

Dolly: That makes more sense.

Ike: Okay

The group's attempts to troubleshoot initially focused on the interface between the two blocks. However, Ike and Dolly could not figure out where they made their mistake. In an effort to identify the location of the error, Christobal explained his reasoning a second time. That second explanation is represented by somewhat more detailed possibility sets than before, as seen in Figure 6.12.

$\vec{F}_{net} = \vec{F}_{Amir} + \vec{F}_{other}$				$\vec{F}_{other} = \vec{F}_{Barbara} + \vec{F}_{table}$		
$\vec{F}_{net}$	$\vec{F}_{Amir}$	$\vec{F}_{other}$		$\vec{F}_{other}$	$\vec{F}_{Barbara}$	$\vec{F}_{table}$
$(\vec{0}\ N)$	20 N →	20 N ←	$\rightarrow\rightarrow$	20 N ←	7 N ←	13 N ←
$(\vec{0}\ N)$	20 N →	$[\neq(20\ N\leftarrow)]$		20 N ←	7 N ←	$[\neq 13\ N\leftarrow]$

Figure 6.12: Christobal's second explanation for the "Two Blocks" problem.

Specifically note that while he explained his solution differently, Christobal did not actually address the interface between the two blocks or indicate in any way that he was considering the two blocks as separate systems. Instead, he merely restated his earlier solution in a more explicit manner, indicating an intermediate "$\vec{F}_{other}$" that he then depicted as being the sum of the forces due

to Barbara and the table. Unfortunately, this second explanation did not actually identify the error that Ike and Dolly made when first trying to solve the problem. Nonetheless, Ike and Dolly both concurred that Christobal's reasoning was correct, and the group moved on. When they encountered solution #2, which did not use the same possibility sets as Christobal (recall that the possibility sets for solution #2 are given in Figure 6.3), the group had difficulty understanding it.

Christobal: But we have one more solution?

Dolly: We have a couple

Ike: Uh, two, we didn't look at solution #2.

Dolly: We should mark this (solution #1) we should mark solution #1 with a red pen on their explanation.

Christobal: Oh wait, we don't know if she is quite yet, (reading solution #1) "because Barbara pulling block B, she is applying a force to the left. According to the momentum principle, the net force acting on an object is equal to that objects change in momentum. Therefore in order to keep block B from experiencing a change in momentum, a friction force needs to be equal and opposite to the force that Barbara [applies] –"

Ike: Which is not correct.

Christobal: That is not correct.

Dolly: Right. Not correct.

Christobal: And solution #2, solution #2 is saying, oh this is all, so (solution #1 says) the frictional force on block B due to the table is to the right, but it is actually to the left.

Ike: Yes.

Christobal: So that's wrong.

Ike: Right.

Christobal: Okay, (reading solution #2) "Barbara is applying 7 N of force to the left on block B and block A is applying a force of 20 N to the right on block B because

block B applies a 20 N frictional force on block A to keep block A's momentum from changing. According to the momentum principle, the net force acting on an object is equal to that objects change in momentum. So to keep block B from experiencing a change in momentum, the combination of these two forces plus the friction force due to the table needs to be zero. Therefore the friction force due to the table on block B is to the left." Well, that's true.

Ike: Yeah, so the final words are true. But,

Dolly: But is the reasoning actually reasonable?

Ike: Yeah, but that reasoning was kind of convoluted, for me at least.

Christobal: (Reading solution #3) States that "an object's change in momentum is equal to the net force acting on that object. But since the block isn't moving, the friction force on block B due to the table is zero. Because friction forces oppose motion, there can only be a friction force if the block were moving." No.

Ike: Oh well.

Christobal: There is something called the coefficient of static friction.

Dolly: Yeah.

Christobal: So it has to be solution #2.

Ike: (crosstalk) It looks like the convoluted reasoning is correct.

Christobal: Well, it makes sense. Because she is applying 20 N of force on block A to keep, so basically what they are doing, –

Dolly: Is what you said over there (pointing to the white board).

Ike: I admit that while you were reading, I was kind of zoning out, trying to figure out how that (pointing to the solution statement) applied to this (pointing to the work on the whiteboard).

In this section, the group struggled with understanding the solution that they knew by the process of elimination to be correct. After all, it gave the same answer as Christobal's reasoning,

which the group agreed was correct. However, the group still found the reasoning of the second solution "convoluted." This difficulty in understanding solution #2 is an example of how difficult it is to communicate between two people (or a group of people and a written solution) that have different possibility sets. Even though Christobal fully understood one solution to the problem, in this next segment he misrepresented solution #2 in explaining it to the rest of his group. Again, this misrepresentation was not likely due to misconceptions or any deeply-held beliefs but rather because the written explanation was simply not congruent with Christobal's explanation, meaning that there was no way for him to fit it in other than trying to force it into his possibility sets, which was precisely what he did.

> **Christobal**: Yeah, well I mean, I think the way she (solution #2) is looking at it is that she is looking at it, because here is block A (draws a block with an arrow to the right), we got 20, here is block B, we got 20 here (drawing an arrow to the left on the top of block B), well, Barbara's providing seven (draws an arrow to the left on block B) –
>
> **Ike**: Okay –
>
> **Christobal**: And that means that this has to provide 13 (draws an arrow to the left on block B).
>
> **Ike**: Okay. That will work.
>
> **Christobal**: Which is kind of what I did, except I wasn't quite so convoluted.
>
> **Dolly**: Yeah, I think yours is a little clearer. But it is solution #2.

In this final piece of the transcript, Christobal repeated the possibility sets in Figure 6.12, except he attributed those sets to the written solution #2, which is an inappropriate description of that solution. The implication from this occurrence is that it is possible that when someone creates his or her own possibility sets to answer a problem, (whether those sets are correct or not) and then either listens to or reads an explanation that uses different possibility sets, the communication of that alternate solution may ultimately fail due to the misunderstandings that arise from trying to interpret one solution within the terms of an incongruent possibility set. In the group consisting of Arthur, Otto, and Walter, this occurred with dramatic effect, as two group members debated possible solutions to this problem.

6.1.5   A lively debate (Arthur, Otto, and Walter)

The group with Arthur, Otto, and Walter is particularly interesting, as the members of the group discussed a physics problem with the intention of convincing each other. Initially, Otto and Walter solved the problem just as Christobal worked it out in Section 6.1.4. However, Otto soon changed his mind, which began a long discussion between him and Walter about how to solve the problem. As long as the two participants maintained different possibility sets, they were unable to communicate in any meaningful capacity; however, when Walter finally understood Otto's possibility sets and reasons within those sets, he was able to point out the deficiencies in Otto's reasoning. Although Otto was eventually forced to concede his error in reasoning, he found himself unable to accept the possibilities that Walter proposes, resulting in them agreeing to disagree.

In the first segment of the transcript, Otto and Walter initially agreed.

Otto: (Reading problem statement) "Two heavy blocks with massless ropes attached to them are sitting on a table at rest. block A sits on block B, which is in contact with the table, as pictured below. Amir pulls on a rope attached to block A, applying 20 N of force. Barbara pulls on the rope attached to block B..."

Walter: What direction is the – okay. So, let's draw it like, so just draw it like, okay, so we are going to have block A, we're going to have a square block, block A, and then have 20 N of force going, arrowed (gestures to the right while Otto draws the blocks and forces) right there, 20 N

Otto: And block B right below it?

Walter: Yeah, and this [problem] seems a lot easier than the last one, 7 N.

Otto: Smaller arrow for 7 N?

Walter: Yeah, so won't we need 13 N going that way (gesturing to the left) of friction? Wait –

Otto: Yeah, we would need an extra 13 (draws an arrow to the left starting on block A, see Figure 6.13), an extra 13 N of friction, I think I spelled that right –

Walter: But it is, it is friction, but, wait, what, what is it –

Otto: That was just –

Walter: (Reading the problem statement) "What is the direction of friction force on block **B** due to the table?" Ah-ha! So –

Otto: So that doesn't matter (scratches out his left arrow on block A).

Walter: That's, that's not the right one. It's what, it's what B has to do.

Otto: Yeah, B, in order for it to counteract the 20 N –

Walter: It's 13 N by the table, still.

Otto: Yeah, it's still in the same direction (he draws a table and an arrow to the left.

Walter: Yeah, and it's still 13 because the table has to –

Otto: Because it has to cancel out the 20 N going this direction (gesturing with his right thumb to the right).

Walter: Yep, otherwise –

Otto: To keep neither to move.
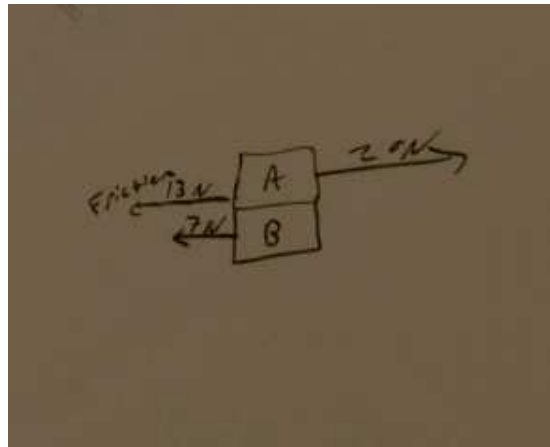
Figure 6.13: Otto and Walter's initial single-block solution.

The possibility set that Otto and Walter agreed to hold at this point (shown in Figure 6.14) is the correct single-block system solution that Christobal also demonstrated in Figure 6.11. Initially, Otto drew the friction arrow as if it acted on block A, but after Walter read the problem statement

again, Otto scribbled it out and drew another arrow that clearly represented the interface between the table and block B. The language they used ("counteract" and "cancel") indicates that they were balancing the vector forces to get a result of 0 N acting on the blocks so that there would be no motion.

$\vec{F}_{net} = \vec{F}_{Barbara} + \vec{F}_{Amir} + \vec{F}_{table}$			
F_{net}	$\vec{F}_{Barbara}$	$\vec{F}_{Amir}$	$\vec{F}_{table}$
(0 N)	7 N ←	20 N →	13 N ←
(0 N)	7 N ←	20 N →	[≠13 N ←]
Other possibilities not considered			

Figure 6.14: The initial possibility set for Otto and Walter on the "Two Blocks" problem.

Next, the group looked at the different written solutions. Otto began to doubt his initial reasoning, but Walter temporarily convinced him that the reasoning they had used was correct.

Walter: Let's see, let's quickly go through the answers and let's see which one agrees with us.

Otto: (Reading solution #1) "Because Barbara is pulling on block B, she is applying a force to the left." Yes.

Walter: Hey, this (solution #1) says, "therefore the friction force on block B due to the table is to the right." Wrong. Slayed.

Otto: Wait, friction – friction would slow an object down, not go with it to make it more.

Walter: Yeah, but they are not moving.

Otto: Yeah, I know that.

Walter: But, well this is static friction so it goes like, opposite the way, oh I see what you're saying.

Otto: Yeah, because it [the friction from the table] would fight against hers (draws an arrow to the right at the table/block B interface) and give more to A, that's what I was thinking.

Walter: No, but, listen to what it says. What is the direction of the friction force on block B?

Otto: On block B against the table, and friction fights against, –

Walter: Yeah, because.

Otto: If it's pulling this way, it is sliding this way (gesturing directions with the marker on the white board), which means the table is trying to –

Arthur: Push it the other way.

Walter: The, the table is feeling 13 to the right, but the (in unison with Otto) block is feeling 13 to the left (by himself) from the table.

Otto: That doesn't, that doesn't make right.

Walter: No, that's right. The table is getting 13 to the right from the block B,

Otto: All right.

Walter: But block B is getting 13 to the left, because it has to all equal out.

Otto proposed that friction should oppose the pulling force; that it would be impossible for the friction to go in the same direction as Barbara's force. Walter started to disagree but found himself temporarily baffled. Figure 6.15 is Otto's possibility set for this situation. Note that no relationship is provided in this possibility set. At this point in the dialogue, there is not enough evidence to determine whether Otto was actually considering the full superposition principle for block B, or whether he was simply trying to apply a rule such as "friction always opposes a pulling force," which is what is displayed in Figure 6.15.

$\hat{F}_{pulling}$	$\hat{F}_{friction}$
$\hat{k}$	$-\hat{k}$
$\hat{k}$	$\neq -\hat{k}$
Other possibilities not considered	

Figure 6.15: Otto's reasoning that the friction must oppose the pulling force.

At first, Walter seemed to understand Otto's reasoning, but then his explanation to Otto indicates that perhaps he did not. When Otto argued that the friction from the table would give more force to block A, Walter apparently interpreted that as a confusion about whether the problem is about the friction force on the table or block B. Consequently, his explanation to Otto clarified the law of reciprocity as it applied to the table and block B. However, the next segment of the transcript reveals that his explanation did not address Otto's concerns.

> **Otto**: (Reading solution #2) "... block A... because block B applies a 20 N frictional force on block A, to keep A's momentum from changing. According to the momentum principle, the net force acting on an object is equal to that object's change in momentum, the combination of these two forces plus the friction due to the table needs to be zero. Therefore" – oh, no, well one we're wrong with this (on the white board, scratches out the value of the arrow to the left at the interface between the table and block B). B applies a friction to A, in that direction (gesturing with the marker) of 20 N. Which means the table applies to the left (but he indicates an arrow to the right, see Figure 6.16) 7 N to keep it from, to keep the momentum from changing.
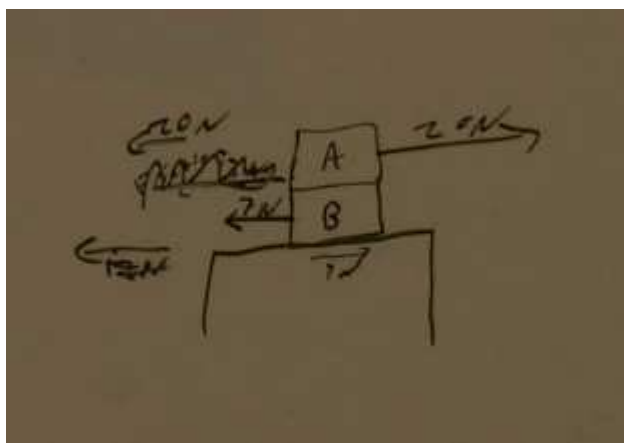
Figure 6.16: The whiteboard after Otto labeled the friction force due to the table as 7 N to the right.

Something about solution #2 resonated with Otto. Possibly because he was already thinking of friction opposing the pulling force, he misinterpreted solution #2. He correctly noted the 20 N

force that block B applies to block A, but then he used that to argue by analogy that the table must oppose Barbara's 7 N applied force in the same way that block B opposed Amir's 20 N applied force. There is no indication that he was thinking about reciprocity in any fashion. Instead, he dealt with the two blocks separately, and therefore held two possibility sets that did not carry a deductive connection (and are therefore not connected with doubled arrows). This reasoning by analogy is represented by Figure 6.17.

$\vec{F}_{net,onA} = \vec{F}_{Amir} + \vec{F}_{BonA}$		
$\vec{F}_{net,onA}$	$\vec{F}_{Amir}$	$\vec{F}_{BonA}$
(0 N)	20 N →	20 N ←
(0 N)	20 N →	[≠20 N ←]
Other possibilities not considered		

$\vec{F}_{net,onB} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$			
$\vec{F}_{net,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{AonB}$	$\vec{F}_{table}$
(0 N)	7 N ←		7 N →
(0 N)	7 N ←		[≠ 7 N →]
Other possibilities not considered			

Figure 6.17: Otto's treatment of the two blocks individually.

Note that Otto said the word "left" while he gestured to the right, indicating that he did not correctly understand solution #2. Indeed, although his transcript hints that he initially agreed with that solution, he found himself arguing against it. Walter was quick to disagree with Otto, though at first both participants just tried to explain their own reasoning, which did not provide any traction in the discussion.

Walter: No, I, I thought –

Otto: They're not joined.

Walter: I thought, I thought it was, well no, they are sitting on top of each other, and (Otto affirms) friction is perfect, so you can assume that friction is perfect, okay, so Amir pulls a block, applying 20 N of force –

Otto: Which means block B has to apply fiction, (correcting himself) friction in the opposite direction (indicating to the left) (Arthur agrees) to A to counteract the 20. (Walter acknowledges). Which means that the table has to apply a friction in the opposite direction of the force of being pulled to make it, to keep it from moving.

Walter: I see what you are saying but I, I disagree.

Arthur: So, B is seven this way (pointing to the right) and 20 that way (pointing to the left) [apparently trying to enumerate the forces acting on B]? (meanwhile Otto keeps saying "because")

Walter: No, B, B is definitely seven this way (pointing to the left). That is 100 [percent certain]– because when someone is pulling on the rope they are applying 7 N of force, so that is correct (pointing to Barbara's force arrow) regardless. Because they tell you that.

Otto: (Writes out the momentum principle on the white board)... equals Δp. And if the moment – (circles the change in momentum part) if this is zero, Δt can never be zero, we need force to equal to zero (indicates this on the white board). The only way we can get the force to equal to zero is if we have two conflicting forces, not two forces going in the same direction (indicating with both of his hands to the left).

Walter: But they are going in different directions, look at it (shows the problem statement to Otto).

Otto: Yes, they are –

Walter: And, and you need to balance out the forces, to make the, you got to balance out just the, just the system between A and B and count the surroundings as the table.

Otto tried to justify his reasoning as shown in Figure 6.17, while Walter tried to promote his own ideas, still represented by Figure 6.14. However, Otto did explicitly bring the momentum principle into the discussion. His very explicit reasoning about how the net force would need to be zero is shown in Figure 6.18. Note that he was also explicit that the only way to get zero net force would be to have two conflicting forces. He was still apparently unaware of the force on block B due to block A.

The group then took a short detour in their discussion to eliminate solution #3.

Otto: (Reading solution #3) "The momentum principle states that an object's change in momentum is equal to the net force acting on that object. But since the block isn't moving, the friction force on block B due to the table is zero." That's not, no, that can't –

$\vec{F}_{net}\Delta t = \Delta\vec{p}$		
$\vec{F}_{net}$	Δt	$\Delta\vec{p}$
$\vec{0}$	0	$\vec{0}$
	$\neq 0$	
$[\neq \vec{0}]$	0	$\vec{0}$
Other possibilities not considered		

$\rightarrow\rightarrow$

$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2$				
$\vec{F}_{net}$	$|\vec{F}_1|$	$\hat{F}_1$	$|\vec{F}_2|$	$\hat{F}_2$
$\vec{0}$		$\hat{k}$		$-\hat{k}$
$\vec{0}$		$\hat{k}$		$[\neq -\hat{k}]$
$\vec{0}$		$\hat{k}$		$\hat{k}$
$\vec{0}$		$\leftarrow$		$\leftarrow$
Other possibilities not considered				

Figure 6.18: Otto's use of the momentum principle to justify the conflicting forces.

Arthur: No (agreeing).

Walter: The table has no friction now? (Laughs) Okay, that's, that's wrong.

Next, the group re-established their positions.

Otto: So what's the difference between these two (looking at solutions #1 and #2)? Oh, it's just left and right. So,

Walter: Yeah, and I think the, I'm going to say the other way, the direction of the, the direction of the friction force on block B due to the table, I believe, is to the left (simultaneously, Otto says "right"). And you believe it is to the right (talking to Otto).

Arthur: I believe it is to the right, too.

Otto: Yeah, to the right.

Walter: I believe it's to the left.

Arthur: Because if (Crosstalk as Otto begins to speak over top of him) if it's being pulled this way (indicating the left), then there needs to be pressure in the other direction (indicating to the right with his other hand), so –

Otto: (Crosstalk with Arthur) so you, so you think it's (pointing to the white board), oh, okay so we go with solution #1 (indicating himself and Arthur), and you (indicating Walter) go solution #2.

Walter: I go solution #2. I mean, we got some time, we can, uh, argue this out, so –

Otto: Yeah.

Researcher: If you want a new white board, you can just let me know and I will get you one.

Walter: Oh, no, we're not gonna –

Otto: Nah, we'll be, we'll be, we have –

Walter: We have enough room.

The group decided to discuss the problem enough to figure out who was right. They began with Otto prompting Walter to explain his position about taking the two blocks in the same system once again.

Otto: So, B does have friction on A. Correct? Because they are sitting on each other, B has to have friction on A (Walter agrees), which if we do this all as a big system (indicating the whole diagram on the whiteboard), including A and B in the same system –

Walter: That's, that's what I was doing, so just call it, call it block – Yeah, so, we've got block AB or something, and then we have this table (drawing on the white board),

Otto: But we don't have block AB, we have block[s] A and B.

Walter: Well, I am going to assume it is block AB, because it is block A and B. You know what I mean?

Otto: All right.

Walter: It is like a super death block or something, I really don't know.

Otto: All right (chuckles).

Walter: And then there is definitely, and the total net is 13 in this direction (drawing an arrow to the right).

Otto: 20.

Walter: No, because 20 this way (to the right), 7 this way (to the left), 13. So it is getting pulled with 13 N in this direction (to the right), now it is saying how much is the table going to have to push against that? And I think to counter out that force, to stop it from moving, the friction from the table is going to have to point 13 to the left.

Much like Christobal found another way to explain his single-block system, Walter broke down his simple explanation into two parts: considering the superposition principle as if there were no friction, and then adding in the appropriate friction force to result in a net force of zero. This reasoning is shown in Figure 6.19.

$\vec{F}_{pulling} = \vec{F}_{Amir} + \vec{F}_{Barbara}$				$\vec{F}_{net} = \vec{F}_{pulling} + \vec{F}_{table}$		
$\vec{F}_{pulling}$	$\vec{F}_{Amir}$	$\vec{F}_{Barbara}$		$\vec{F}_{net}$	$\vec{F}_{pulling}$	$\vec{F}_{table}$
13 N →	20 N →	7 N ←	→→	(0 N)	13 N →	13 N ←
[≠ 13 N →]	20 N →	7 N ←		(0 N)	13 N →	[≠ 13 N ←]

Figure 6.19: Walter's second explanation for the "Two Blocks" problem.

However, Otto and Arthur continued to disagree. Arthur reminded the group that there would be friction between the blocks, and that statement leads to more discussion.

Otto: But you are adding –

Arthur: But there's two, there's two friction forces in B (pointing to the white board).

Otto: Yeah, there is two frictions [sic].

Walter: There is a friction force inside of here (pointing to the blocks on the whiteboard), I know what you are saying there.

Otto and **Arthur** (Using different words and cross talking, but saying the same thing): But there are two,

Arthur: One on top and one on the bottom.

Otto: Two on top and one on the bottom.

Walter: *One* on top, one on bottom.

Otto: One on – yeah, whatever, the ratio is the same either way (Walter affirms), we get, using the f–, using that there is two, without friction we know that there is two forces, going each in a different direction (gesturing with his thumbs) (Walter affirms), if there is two frictions, then shouldn't they also be going in two different directions (gesturing by moving his hands in opposite directions)? If the friction of A, of B on A, on, of B on

A, is going in that direction (indicating to the left), wouldn't the friction of the table on B go in the opposite direction of the way it is being pulled? To stop it? Because if it went in the same direction, if the table went in the same direction as it is being pulled, B would move.

Walter: If the table went in the same direction –

Otto: Direction as B – and it doesn't, it goes in the opposite direction at which B is being pulled.

Walter: Yes.

Otto: And if B is being pulled to the left, it [friction] has to go to the right.

Arthur: Yeah.

Otto: In order to stop B from moving.

Otto built off of Arthur's reminder that there would be friction between the two blocks to reaffirm his own reasoning. In short, he claimed that the friction force would oppose the pulling forces to keep each of the two blocks from moving. Apparently, he knew that there would need to be a pair of friction forces at the interface between the blocks. However, he only indicated a single friction force, which he explicitly called the force of the friction due to the table, opposing the pull on block B. In short, his set of possibilities did not change at all since he first laid them out in Figure 6.17.

It was at this point that Walter seemed to finally figure out what Otto was talking about, and specifically that Otto was failing to consider a very important force.

Walter: Yes, but, but B is also being pulled by A.

Otto: It's not asking that though, that doesn't matter.

Walter: But yes it does, because it is still part of the question.

Arthur: (crosstalk with Walter) But this part of B is canceling out that A.

Otto: No, B, A, A doesn't –

Walter: But A is still pulling on B, that's what I am saying.

Otto: Okay, I see what you are saying, you are saying that B, A also has a friction on B.

Walter: Yeah.

Walter temporarily abandoned his own solution to the problem to point out that there was a deficiency in Otto's. In fact, Walter was able to determine that Otto had failed to consider that the friction force of block A on block B has an important effect. This recognition was a very significant event that shaped the remainder of the dialogue. At first, Otto claimed that the friction force in question was not relevant to the solution of the problem, indicating that perhaps he was willingly neglecting it in his solution. However, he eventually decided to try to work out the problem with that quantity taken into account.

Otto: Yeah, okay fine. Then this (the friction due to the table) doesn't equal seven, this equals an unknown value, and then the friction on A equals another value.

Walter: But *neither* of the blocks move. A does not like fall off on top.

Otto: No, neither of the blocks move (writing on the whiteboard). So that means y of n, so that just means, ack (fixing what he wrote), which means y plus x must equal seven (writes "$y + x = 7$" on the whiteboard, see Figure 6.20), right?

Walter: (Inaudible)... this too much. I don't, oh, oh I see what you are saying.

Otto: Yeah, x plus y has to equal seven, has to, no has to equal the inverse of seven (he draws a minus sign before the "$(y + x)$" on the whiteboard).

Walter: How can I get across what I'm saying? – just seven in the other direction.

Otto: So if seven is going in the right, seven is going into, (inaudible).

Walter: Left.

Otto: So, which means we have to have an inverse of seven, which means we have to have the inverse of left.

Walter: Oh, ah,

Otto: Which means it's going to the right.
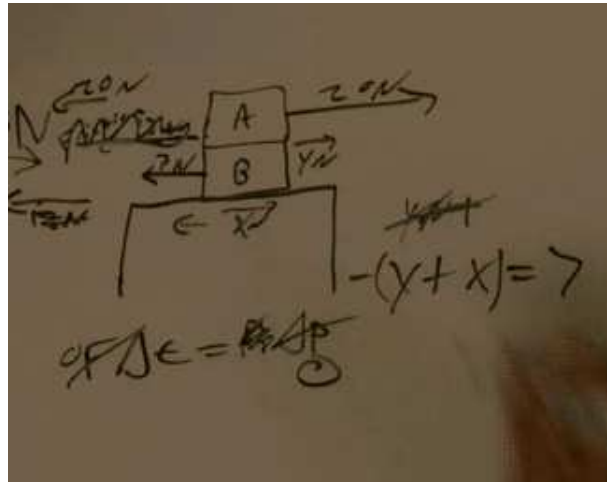
Figure 6.20: The whiteboard after Otto labeled the forces on block B "x N," "y N," and "7 N," and (correctly) set up an equation to solve for x.

Otto indicated that the sum of the magnitude of the frictional forces due to block A and the table should equal the 7 N of Barbara's pulling force. He did this by replacing the old "7 N" label that referred to the friction force by the table on block B with "x N" and adding an arrow for the friction force due to block A and labeling it, "y N," as shown in Figure 6.20. At this point, Otto was considering all of the relevant quantities in the relationship, as shown in Figure 6.21. However, some blank spaces remained, and Otto was still indicating that the friction force due to the table was to the right.

According to what information he has given, Otto should not have been able to draw a legal deduction; however, he continued to claim that the friction force due to the table on block B was to the right. In this instance, his representation of "$-(y + x)$" indicates that he chose the direction of that friction to be the same as the direction of the friction due to block A on block B (that is, to the right; this is in conflict with the direction for the arrow on Figure 6.20). However, careful mathematics would give him a negative value for x, which would have indicated to him that the friction force due to the table would in fact need to be to the left, as the arrow was originally drawn on the whiteboard. Thanks to Walter, Otto finally held the skeleton of the correct reasoning. Otto was finally allowing the correct possibilities, although he had yet to mention the law of reciprocity as it applied to the two blocks.

$\vec{F}_{net,onA} = \vec{F}_{Amir} + \vec{F}_{BonA}$		
$F_{net,x,onA}$	$\vec{F}_{Amir}$	$\vec{F}_{BonA}$
(0 N)	20 N →	
Other possibilities not considered		

↓↓

$\vec{F}_{BonA}$	$\vec{F}_{AonB}$
	Y →
Other possibilities not considered	

↓↓

$\vec{F}_{net,onB} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$			
$F_{net,x,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{AonB}$	$\vec{F}_{table}$
(0 N)	7 N ←	Y →	X →
Other possibilities not considered			

Figure 6.21: Otto's reasoning after discussion with Walter.

Walter: I see what you are doing, and I don't know how to explain what I am doing but I still disagree with you, but I finally get what you are saying now. Yeah, that makes a lot more sense now, because before I was like what? How can you think that? And now – I thought you were going all crazy dyslexic, which happens to me sometimes, but I see what you're saying now, and I'm still going to disagree with you on it, and I am still going to go with mine because, I mean, I might be oversimplifying this a bit but I prefer to go simplistic than trying to go super complex, because that normally tends to work out better for me at least. I mean, I am just, I guess I'm just, in that (gestures to the white board and what he previously worked out), just kind of saying, hmm, well, assuming that they are not moving, and neither of them are moving relative to each other, that means the forces, so I just regarded A and B separately, and ignored the table, like my first thought through, and I was like, what will have to happen, it's like, well this is now having to apply force, in other words, I did it in like two steps in my head. I just did A and B –

Otto: Well, –

Walter: And then I added A and B to the table, so that might not be the right way to do it, but –

Walter failed to follow up on his previous insight, instead returning to how he originally solved the problem, as shown in Figure 6.14, by choosing the two blocks as a single system. Otto responded by steering the conversation back to the two-block system.

Otto: Yeah, but if instead of using diagrams, you actually try to solve it using, with the momentum principle, we get F of Barbara, we know that, we know that moment, friction is going to go against it, is going to go against whatever force you give. Right?

Walter: Yes.

Otto: Force of, force of friction (writing the momentum principle on the white board), Δt, which if we are doing, this is a static problem, t, time never changes, we'll just assume this is one. Get rid of it. Equals the change in p, which we know that p doesn't change, so that it was zero. So in other words, the force is going to have to cancel out.

Walter: I am thinking about what you just said. Give me a sec. Okay, so the force is –

Otto: And we know that, that, this (referring to the $F_{net}\Delta t$ part of the problem) has to be a product where when it is multiplied by t, it equals zero. t doesn't matter, because we are not being asked to find what is the time in which, blah blah blah, t doesn't matter. So what we need to [meaning, "what we"] get, is we need to get F equals to p, and if p equals zero, we need to get F to equal to zero as well. And if we take each block as a separate system, we know that there has to be 20 N of force against F [meaning F_A, see below] to equal zero.

Walter: Yes, well, against –

Otto: In the opposite direction of F.

Walter: F of the top one, A, just call it A. FA.

Otto: FA minus (writing on the whiteboard) –

Walter: Needs to be 20 in this direction. There needs to be 20 in this direction on A (to the left) which (crosstalk with Otto) has to be equal and opposite to B,

Otto: (crosstalk) Which has to be equal and opposite,

Walter: Which would be in that direction (indicating to the right).

Otto: It is, no it is opposite out of B by the fact that B applies 20 N to A.

Walter: Yes. It does. But, but.

Otto: In the opposite direction of the way it's going.

Walter: B applies 20 N to A, and because it applies 20 N in that way, B, A must be applying 20 N to that direction (drawing an arrow to the right on block B) to cancel out the force.

Otto: Okay?

Walter: That's the way I'm thinking of it.

Otto: Yeah, that *is* the way you are thinking of it, but that doesn't cancel it out all the way.

Walter: Well no, because then we have –

Otto: (Crosstalk) the friction of the table –

Walter: 13 total that way, going to the right.

Otto: Or else –

Walter: We are going a little bit of a loop here.

Otto returned to the momentum principle to justify that the net force acting on each block would need to be zero. He then began to fill in some of the blanks in his possibility sets by setting the force of block B on block A equal to 20 N to the left. However, neither he nor Walter correctly accounted for reciprocity. Walter, by drawing the arrow to the right on block B and indicating its magnitude of 20 N immediately after discussing the force on block A due to block B, probably understood it correctly, but he did not explain it properly to Otto, leading to some confusion. An extrapolation of Walter's set of possibilities from his statements is shown in Figure 6.22, although it is unclear

exactly how much of that reasoning Otto was able to follow or agree with at this moment in the discussion. Note that the 0 N net force was not implied, as it came directly from Otto's explanation.

$\vec{F}_{net,onA} = \vec{F}_{Amir} + \vec{F}_{BonA}$		
$F_{net,onA}$	$\vec{F}_{Amir}$	$\vec{F}_{BonA}$
0 N	20 N $\rightarrow$	20 N $\leftarrow$
0 N	20 N $\rightarrow$	$[\neq(20$ N $\leftarrow)]$

$\downarrow\downarrow$

Reciprocity: $\vec{F}_{AonB} = -\vec{F}_{BonA}$	
$\vec{F}_{BonA}$	$\vec{F}_{AonB}$
20 N $\leftarrow$	20 N $\rightarrow$
20 N $\leftarrow$	$[\neq 20$ N $\rightarrow]$

$\downarrow\downarrow$

$\vec{F}_{net,onB} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$			
$F_{net,x,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{AonB}$	$\vec{F}_{table}$
(0 N)	7 N $\leftarrow$	20 N $\rightarrow$	13 N $\leftarrow$
(0 N)	7 N $\leftarrow$	20 N $\rightarrow$	$[\neq 13$ N $\leftarrow]$

Figure 6.22: Walter's second form of correct reasoning for the "Two Blocks" problem.

After thinking for a little while, Otto agreed with Walter, although it is not clear whether he completely understood the entirety of the possibility sets that Walter promoted, especially because Walter did not explicitly mention reciprocity in his explanation.

> **Otto**: If (points to something on the white board and pauses). I think he's right. Because if there is also friction, but that doesn't make physical sense, if there is a friction going along with the force (Arthur affirms), friction should always go against the force that's being pulled.
>
> **Walter**: Yes.
>
> **Otto**: Physically speaking, that is the answer It should be. So, it should be to the right.
>
> **Walter**: Physically speaking, I think it is to the left. We can –
>
> **Otto**: Physically speaking –

Walter: We can agree to disagree on this one.

Otto: Physically speaking, and this is a physics problem (Arthur chuckles), it would be to the right, if it was a magic table, it would be to the right (but gestures to the left) –

Walter: Well, I think physically – (now addressing Otto's inconsistency) I mean, you mean, you just said "to the right" twice, but,

Walter: But which one was the one where it was going to the –

Otto: That way (pointed to the left). I said "right" but I pointed to the left

Walter: I know, that was very skilled. (To Arthur) so you agree with [solution #]1? Or something?

Otto: Yeah, we agree with [solution #]1.

Walter: We can just agree to disagree on this one too. (Arthur hands Otto a red pen). I'm going with [solution #]2.

The data indicate that Otto finally understood the mathematics behind what Walter had been trying to convince him, something that may have been aided by Walter switching from his original reasoning about the problem as being a single-block system to Otto's reasoning about the problem as containing each block as a unique system. However, almost as quickly as Otto acknowledged that Walter may have been correct, he snapped back to his earlier reasoning that friction would need to oppose the force rather than go along with it. He had previously expressed this idea in Figure 6.15. Unfortunately, we don't have any more data than this; we do not know exactly what Otto was thinking or, more importantly, why he suddenly snapped back to an incorrect set of possibilities, blocking all of the possibilities that Walter had revealed to him. One possible explanation is that the possibility set that Otto held in Figure 6.15 is very simple and easy to hold in working memory, while the correct explanation (Figure 6.22) is incredibly complicated and provides many opportunities to make mistakes. It is indeed easier to hold the smaller possibility set in working memory, and it may be that one common error that reasoners make is oversimplifying the problem by excluding possibilities inappropriately. This sort of error is a recurring theme throughout the analysis; many of the participants' mistakes can be attributed to such improper simplification.

6.1.6   Missing reciprocity (Ana and Sally)

Ana and Sally were not explicit in their discussion about the Two Blocks problem. Unfortunately, they often trailed off after they began a chain of reasoning, leaving ideas only half-completed if at all. However, we can still identify some clues as to their reasoning by keying on some particular things that they focused on. In this first segment of the transcript, Sally pointed out an idea that the masses of the two blocks should be different.

> **Ana**: Ok, so it's (reading the problem statement) "two heavy blocks with massless ropes attached to them are sitting at rest on a table. Block A sits on block B, which is in contact with the table as pictured below. Amir pulls on the rope attached to block A, applying 20 N of force. Barbara pulls on the rope attached to block B, applying 7 N of force. While they are pulling, neither block moves. What is the direction of the friction force on block B due to the table?"
>
> **Sally**: Okay, so they're not moving.
>
> **Ana**: Yeah.
>
> **Sally**: So their forces have to be, I guess, equal to the friction, right? (Ana affirms) But in the opposite direction (Ana affirms). So this is being pulled in this direction (gesturing with her hand, but the direction is unclear), you know.
>
> **Ana**: Yeah and that's in the other, opposite direction.
>
> **Sally**: So, um, they're pulling with different forces, so that, I would imagine, the masses would be a little different because, I guess, the friction is gonna be the same, you know it's just one.
>
> **Ana**: (Crosstalk over the end of Sally's previous statement) Wait, say that again?
>
> **Sally**: Friction is just a thing, like a single thing, right?
>
> **Ana**: Yeah.
>
> **Sally**: So, the forces are different and then friction is a single thing. In order for them to not move, I don't know. I guess there's friction going on here (points to the diagram in the problem statement) but also here.

Ana: Yeah, right.

Sally: So, since it's 7 N, which is less than the 20, I guess that is because of the table and the block, it's (she trails off).

Sally reasoned that to keep the blocks from moving, they should have different masses. Her justification that "friction is gonna be the same" is difficult to decipher given the context, especially since she only identified a single friction and did not indicate what that friction "is going to be the same" as. However, the statement that the masses should be different is interesting, as the friction force is often taught as being some coefficient of friction times the normal force, which in this case would be equal in magnitude to the weight of the object. Therefore, the friction force might be thought of as being proportional to the weight of the objects, although it is unclear how this is consistent with her statement that friction is a "single thing" and her pointing out two places where friction was present. Additionally, her statement that the forces by Amir and Barbara would need to be equal to the friction is unclear; it may refer to each of those forces separately or summed, and it may refer to either a single friction or two or more friction forces.

In any case, we can list what little bit of information Sally did provide in the previous segment. Namely, she was thinking about certain quantities. Figure 6.23 displays this incomplete reasoning.

$\vec{F}_{net}$	$\vec{F}_{Amir}$	$\vec{F}_{Barbara}$	$\vec{F}_{friction1}$	$\vec{F}_{friction2}$	m_A	m_B
(0 N)	$20N \rightarrow$	$7N \leftarrow$	$\vec{F}_1$	$\vec{F}_2$	m_1	$m_2 \neq m_1$
(0 N)	$20N \rightarrow$	$7N \leftarrow$	$\vec{F}_1$	$\vec{F}_2$	m_1	$m_2 = m_1$
Other possibilities not considered						

Figure 6.23: Sally's initial (incomplete) possibility set for the "Two Blocks" problem.

At this point, Sally and Ana turned to the written solutions that they were given. They quietly dropped the idea that mass has anything to do with the solution to this problem, as that idea did not end up being fruitful.

Sally: So, (reading from solution #1) she's applying a force to the left, but actually, "yeah, ok, according to the momentum principle, the net force acting on an object is equal to that object's change in momentum." $F_{net}\Delta t$, right?

Ana: Yep.

Sally: (Reading from solution #1) "Therefore, in order to keep block B from experiencing a change in momentum, the friction force needs to be equal and opposite to the force that Barbara is applying, therefore the friction force on block B due to the table is to the right." See, I think those two friction forces are, yeah, I don't know, let's look at the other ones.

Ana: (Reading from solution #2) "Barbara is applying a 7 N force to the left of block B and block A is applying 20 N to the right on block B because block B applies a 20 N friction force on block A to keep A's momentum from changing. According to the momentum principle, the net force acting on an object is equal to the change in momentum, so to keep block B from experiencing a change in momentum, the combination of these two forces plus the friction due to the table needs to be zero. Therefore, the friction force due to the table on block B is to the left."

Sally: (Reading from solution #2) "... keep block B from experiencing a change in momentum, the combination of these two forces plus the friction due to the table needs to be zero." There's friction in between the two blocks, I don't know if it's negligible but there's friction between it.

Ana: No, I mean, I think you have to take that into account though, right?

Sally: I think so.

Ana: Because otherwise, I mean block B couldn't move on the table but block A would slide off of block B. They're saying neither one moves, right?

Sally: Right.

After reading through the first two solutions, Sally suggested that there would be friction between the two blocks. Ana immediately picked up on this and asserted that the friction between the blocks that was mentioned in solution #2 did need to be considered. In fact, she even suggested something that could happen if there were no friction between the blocks. Figure 6.24 demonstrates the possibility sets that Ana and Sally held at this point.

$\vec{F}_{net,onA} = \vec{F}_{Amir} + \vec{F}_{BonA}$		
$\vec{F}_{net,onA}$	$\vec{F}_{Amir}$	$\vec{F}_{friction\ between\ blocks}$
(0 N)	20 N $\rightarrow$	$\neq$ 0 N
($\neq$ 0 N)	20 N $\rightarrow$	0 N
[(0 N)]	20 N $\rightarrow$	0 N
Other possibilities not considered		

$\vec{F}_{net,onB} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$			
$\vec{F}_{net,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{friction\ between\ blocks}$	$\vec{F}_{table}$
(0 N)	7 N $\leftarrow$	0 N	
[(0 N)]	7 N $\leftarrow$	0 N	
Other possibilities not considered			

Figure 6.24: Sally and Ana's reasoning about the two blocks does not acknowledge a relationship between them even though they recognized that the friction between the blocks should be important.

While they discussed the friction between the blocks, they were apparently unaware that two separate frictions acted on the two separate blocks. Further, they did not indicate a relationship between the two blocks.

> **Sally**: (Reading solution #3) "The momentum principle states that an object's change in momentum is equal to the net force acting on that object. But since the block isn't moving, the friction force on block B due to the table is zero. Because friction forces oppose motion, there can only be a friction force if the block were moving."
>
> **Ana**: Oh. I see what they're saying.
>
> **Sally**: I don't know (she reads inaudibly). I mean, these forces can be really small and they can all be just canceling out.
>
> **Ana**: Yeah, that's true.
>
> **Sally**: I don't know.

Sally read solution #3, and Ana indicated that she understood it. They both seemed baffled by the problem and continued to suggest possibilities, and Sally suggested that maybe the forces could cancel out because they were small. This may appear not to make sense, but it could indicate that

she was trying to provide a mechanism for solution #3 to be correct.

> **Sally**: Okay, so [block] B is [pulled] with a bigger force than A. So in order for it, for them to stay on top of each other,
>
> **Ana**: No, A is getting 20 N of force.
>
> **Sally**: Oh, it is?
>
> **Ana**: Yeah, A is bigger than B.
>
> **Sally**: So that means the friction force must be going to the left there (points to the problem statement), right?
>
> **Ana**: Yeah.
>
> **Sally**: To slow it down?
>
> **Ana**: Yeah.
>
> **Sally**: The friction force is going to the left there, but this is being pulled to the left, so maybe it's going to the right.
>
> **Ana**: Oh, yeah.
>
> **Sally**: Which would be solution #1.
>
> **Ana**: Yeah.

Finally, Sally provided a complete chain of reasoning. She committed the error that was developing in Figure 6.24 by treating the friction between the blocks as a single entity rather than as a pair of forces that would follow the law of reciprocity. This resulted in the erroneous conclusion that the friction force due to the table on block B would be to the right. The transcript itself could also indicate that Sally was completely ignoring the friction between the blocks; indeed, the result of "to the right" could have been reached if she reasoned about the blocks analogically; that is, a friction force would be needed to prevent block A from moving due to Amir's force, so a friction force due to the table must be needed to prevent block B from moving due to Barbara's force. That reasoning, which Otto used in Figure 6.17, could fit in this situation but for Sally's earlier statement that the friction between the blocks seemed like an important quantity. Sally's reasoning at this point was

unclear, although the interpretation that is most likely – that Sally treated the friction between the blocks as a single entity – is demonstrated in Figure 6.25.

$\vec{F}_{net,onA} = \vec{F}_{Amir} + \vec{F}_{BonA}$		
$\vec{F}_{net,onA}$	$\vec{F}_{Amir}$	$\vec{F}_{friction\ between\ blocks}$
0 N	20 N $\rightarrow$	$\leftarrow$
0 N	20 N $\rightarrow$	$[\neq(\leftarrow)]$

$$\Downarrow$$

$\vec{F}_{net,onB} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$			
$\vec{F}_{net,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{friction\ between\ blocks}$	$\vec{F}_{table}$
(0 N)	7 N $\leftarrow$	$\leftarrow$	$\rightarrow$
(0 N)	7 N $\leftarrow$	$\leftarrow$	$[\neq\ \rightarrow]$
Other possibilities not considered			

Figure 6.25: Sally's complete reasoning for the "Two Blocks" problem.

In contrast to Ike and Dolly, who forgot to flip the direction of the friction force on block B due to block A, Sally was apparently not even thinking about reciprocity. Instead, she treated the friction between the blocks as if it were a single force that acted on both blocks.

Next, Ana indicated that she might be questioning Sally's reasoning, but she never vocalized her disagreement with it, and in the end the two participants agreed.

Ana: Wait, (possibly reading; unclear) The friction force on block B due to the table is going to the right. So if that's going to the left (gesturing with her left hand), that's going to the right (gesturing with her right hand, looks at Sally).

Sally: Yeah, I think it might be tw– [solution #]1, I'm not sure though.

Ana: Yeah, that makes sense. So it's not two, then, I guess (crosstalk with below), cause that one.

Sally: (Crosstalk with above) I mean, as far as we think.

Ana: Yeah, and then solution #3 –

Sally: Is just the definition.

Ana: I don't, I don't think [solution #]3.

Sally: Yeah.

Ana: I think it's just [solution #]1 is that okay?

Sally: Yeah.

If Ana did recognize an error with Sally's reasoning, she didn't mention it. Instead, she agreed, and the group selected the incorrect solution to the problem. Worth noting, the one feature that these participants repeatedly mentioned was the friction between the blocks. It is notable, therefore, that the solution they chose did not mention that friction at all. In fact, only solution #2 mentioned the friction between the blocks, and they rejected that one for no clear reason.

6.1.7 Searching for relevant quantities (Claudette, Earl, and Gustov)

Unlike Sally and Ana, who took notice of the friction force but did not choose the solution that uses it, Claudette, Earl, and Gustov struggled for the entire allotted 15 minutes trying to identify the quantities that were necessary before recognizing the importance of the friction force. Once he decided that this friction force was important, Gustov simply chose the solution with that quantity in it, solution #2. One unique aspect of this group is that they distributed the solutions at the beginning, with each participant taking some responsibility for presenting one of the solutions to the group. Therefore, sometimes members of this group referred to solutions as "mine" and "yours."

The group began by trying to understand the problem.

> **Claudette**: (Reading problem statement) "Two heavy blocks of massless ropes attached to them are sitting at rest on a table. Block A sits on block B which is in contact with the table as pictured below. Amir pulls on the rope attached to block A, applying 20 N of force. Barbara pulls on the rope attached to block B applying 7 N of force. While they are pulling neither block moves. What direction is the friction force on block B due to the table?" And, here is the solutions. (Hands a solution to Earl).

> **Earl**: Thanks.

> **Gustov**: So they are massless ropes. Sitting at rest, okay, so block A sits on block B in contact with the table as pictured. So if they are at rest, so here they are, their p initial

is going to be zero, right? (Earl and Claudette indicate agreement). All right. Now, Amir pulls on the rope attached to block A, so that's going to be, going to be, we'll say in the positive direction. Okay. So this is positive –

Claudette: How many Newtons of force is that?

Gustov: It's 20 – Barbara pulls on the rope attached to B. So this is going to be in the negative x. It's going to be 7 N. While they are pulling neither block moves. So, if they don't move, then would you say p final be 0 too? Because their mass times their velocity is not changing. (Reading from the problem statement) While both blocks are pulled, "while they are pulling, neither block moves. What is the direction of friction on block B due to the table?"

Earl: So, they're both pulling on the blocks. But yet the blocks do not move, so they stay in that picture?

Gustov: Yep.

The group did not yet draw any conclusions, but Gustov repeated back much of the important information in the problem. He also hinted at the momentum principle by identifying that the initial and final momenta of the blocks were both zero. However, the group did not yet distinguished any of the relationships. Therefore, an initial description of this group's possibility set (shown in Figure 6.26 shows that their reasoning to this point was very incomplete.

$\vec{F}_{Amir}$	$\vec{F}_{Barbara}$	$\vec{p}_{initial}$	$\vec{p}_{final}$
$20N, +\hat{x}$	$7N, -\hat{x}$	$\vec{0}$	$\vec{0}$
Other possibilities not considered			

Figure 6.26: The initial (incomplete) possibility set for Claudette, Earl, and Gustov for the Two Blocks problem.

In Figure 6.26, no implied negation is provided because the group did not yet draw any conclusions with the information in the possibility set. However, they did translate the phrase "no motion" as the blocks having both initial and final momenta of zero, which put them in a position to think about the momentum principle later.

Claudette: Well see, like, the only thing is like, A would have to be –

Gustov: Real heavy.

Claudette: Yeah, so.

Gustov: So there would be a force coming down. There would be like a force probably going down, it has to be on block B. It has to be. (Crosstalk with Claudette)

Claudette: And then block B is going to be pushing back up on block A.

Earl: Yeah, for them not. (Crosstalk with Gustov)

Gustov: For the table.

Earl: For them not to be moving.

Gustov: If the table is going to be pushing up –

Claudette: On B –

Gustov: So the force, so the force of A is going down, the force of B is going which way?

Claudette: It's up.

Gustov: The force of B would be going up?

Claudette: And down.

Gustov: Yeah.

Claudette: Because, because it's got a force from the table.

Gustov: Gravity, yeah, gravity is going down.

Claudette: And then a force pushing up on A.

Gustov: So force A, force, A would also have a force on B going up, right? Or B would have a force on A going up, I mean.

Claudette: Yeah, B would have a force going up and down.

Next, the group tried to figure out the forces that blocks A and B applied to each other. At first, they seemed to agree that in order to keep the blocks from moving, block A would need to

	$\vec{F}_{weight\ of\ block}$	$\vec{p}_{B,initial}$	$\vec{p}_{B,final}$
$W \downarrow$, large	$\vec{0}$	$\vec{0}$
$\neq (W \downarrow$, large$)$	$\vec{0}$	$\vec{0}$
Other possibilities not considered		

$$\downarrow\downarrow$$

Reciprocity: $\vec{F}_{AonB} = -\vec{F}_{BonA}$		Reciprocity: $\vec{F}_{B\ on\ table} = -\vec{F}_{table\ on\ B}$	
$\vec{F}_{AonB}$	$\vec{F}_{BonA}$	$\vec{F}_{B\ on\ table}$	$\vec{F}_{table\ on\ B}$
$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$
$\downarrow$	$[\neq\uparrow]$	$\downarrow$	$[\neq\uparrow]$

Figure 6.27: The implications of a massive block A, as explored by Claudette, Earl, and Gustov.

be "real heavy," which would keep block B in place. However, this would have ramifications due to reciprocity, which the group then explored.

Figure 6.27 represents an expansive list of possibilities that Claudette, Earl, and Gustov explored. They seemed determined to flesh out all of the possibilities that they could before drawing any conclusions. For whatever reason, at this point in the session they chose to begin looking at the written solutions that they had been given for this problem.

Gustov: So but if they're not moving then P final would have to be zero, right?

Claudette: Mm-hmm.

Gustov: All right. Neither blocks would move. All right, so let's see, what are these solutions talking about?

Claudette: All right, so solution #3 says, "Momentum principle states that an object's change of momentum is equal to the net force acting on that object. But since the block isn't moving, the friction force on the block B due to the table is zero. Because friction forces op-, oppose motion, there can only be a frictional force if the block were moving.

Gustov: Yeah, that makes sense.

Earl: And mine (Solution #1) says that "because Barbara is pulling on block B, she is applying a force to the left. Okay. And according to the momentum principle, the

net force acting on the object is equal to that object's mome–, change in momentum. Therefore in order to keep block B from experiencing a change in momentum, a frictional force need to be equal and opposite to the force that Barbara is applying, therefore the friction force on block B is due to, due to the table, to the right."

Gustov: All right, so that one is going to the right. So this is friction to the right. (Earl indicates agreement).

Claudette: I don't think that would be the only reason why because they are saying that the only reason why B isn't moving is because the friction force of the table but that's not necessarily true because A could be really heavy.

Earl: (to Claudette) Yeah.

Gustov: So this one is saying that (inaudible muttering) and this one has to be saying (inaudible muttering).

Earl: So it would have to be A because, so – (cut off by Gustov)

Gustov: (Reading solution #3) "So because the frictional force opposed motion, there can only be a friction force if the blocks, if the block will move." Okay so this (solution #2) is saying left, that (solution #1) is saying right, and this (solution #3) is saying none (comparing all three solutions). (Claudette indicates agreement). All right. So,

Earl: Okay, so what's happening, is block A being pulled? Or is it being held steady?

Claudette: Block A is being pulled.

Gustov: He (Amir) is pulling A, he pulls A.

Claudette: Yeah, with 20 N.

Gustov: And she (Barbara) pulls B. –

Claudette: Force, and I am feeling like A has got to be pretty heavy. He's pulling with 20 N and he is not moving it.

In the previous segment, the group mostly tried to understand the solutions that they were given. However, Claudette felt that block A should be heavy, as the group discussed earlier. In fact, she

argued that there were two possible explanations for why block B did not move: either the friction due to the table or the weight of block A could explain that phenomenon. This reasoning is shown in Figure 6.28.

$\vec{F}_{weight\ of\ A}$	$\vec{F}_{friction\ of\ table}$	$\vec{p}_{B,initial}$	$\vec{p}_{B,final}$
$W_A \downarrow$, large		$\vec{0}$	$\vec{0}$
	$\neq \vec{0}$	$\vec{0}$	$\vec{0}$
Other possibilities not considered			

Figure 6.28: Two possibilities, given by Claudette, that could explain why block B is not moving.

In fact, she pushed that the first possibility should be accounted for explicitly. She maintained this position for a while longer in the group's discussion and brought it up again later.

> **Earl**: Yeah, A. Yeah. But at the same time, his force could be taking away some, no his force could be increased cause the other force is going in the other direction, but mine sounds about right, "because F net is equal to the change in momentum and since the change in momentum is zero, F net is equal to zero."

> **Gustov**: (Reading solution #3) ..."Equals F net, but since the block isn't moving, the friction force on block B due to the table is zero."

> **Earl**: So.

> **Gustov**: (Reading solution #3) "PF equals PI, the friction force on block B due to the table is zero. Because friction forces oppose motion, there can only be frictional forces if the block is moving."

> **Earl**: That's true.

> **Gustov**: So, force, looking for a force, there is no change so this (the change in momentum) is zero, this is zero equals F net again.

> **Earl**: Mm-hmm. What's our net, F net? We have the force of gravity on A and B, we have the table pushing up, we have the block B pushing up on block A, block A pushing down on block B.

Claudette: You got 20 N and then 7 N because of your, them pulling –

Earl: In the left and right direction.

Claudette: Mm-hmm, so basically it's going to be positive 13. In the right direction.

Gustov recognized that the momentum principle appeared over and over in the solutions; specifically, that there was no change in momentum, which means that "zero equals F net." However, Earl then responded by asking what F net was; either he was asking what all of the forces were so that the group could sum them to zero, or else he was confused by the conclusions reached by the solutions regarding the momentum principle. In either case, Earl began listing off the forces that acted on the blocks, and Claudette helped. Claudette then apparently summed the two pulling forces that were acting on the blocks and indicated that the sum of those forces would be 13 N in the "right" direction. Whether she believed that this would be the **net** force acting on the blocks is unclear, thanks to Claudette's vague use of the pronoun "it." This listing of the forces by Earl and Claudette can be represented by a possibility set, which is shown in Figure 6.29. Of course, there is no relationship indicated between these forces because Claudette did not explicitly choose a system; these forces each act on a different object.

$\vec{F}_{gravity\ on\ A}$	$\vec{F}_{gravity\ on\ B}$	$\vec{F}_{table}$	$\vec{F}_{BonA}$	$\vec{F}_{AonB}$	$\vec{F}_{Amir}$	$\vec{F}_{Barbara}$
($\downarrow$)	($\downarrow$)	$\uparrow$	$\uparrow$	$\downarrow$	20 N $\rightarrow$	7 N $\leftarrow$
Other possibilities not considered							

Figure 6.29: Claudette and Earl's list of the forces that are acting in the "Two Blocks" problem.

As Claudette finished her statement, Gustov talked over her and changed the subject.

Gustov: So would you say, yeah, would you say that you would have to, that, hold on, let me see. Would you need to.

Earl: It's like glue basically, it's like, you know how each thing has something to hold it, like, some amount of force before it starts getting pushed. Like, if I were to push hard enough, it's not going to move, it will have to overcome friction in order for it to move. So they pull it just enough for it not to overcome friction and cause the block to move.

Gustov: So these, if the, the formula equals initial –

Earl tried to explain why the blocks were still stationary by using an analogy to glue: friction force holds the blocks in place and Amir and Barbara do not pull hard enough to overcome that force to cause the blocks to move. This explanation was largely ignored by the rest of the group. Gustov seemed to be working by himself, trying to figure out the problem.

Claudette: Well, your mg, like, I feel like the force that we're going to have to worry about will be mg –

Earl: Yeah, because the mass –

Claudette: Because the mass is going to be.

Gustov: And mg it's going to be going down –

Claudette: And it's going to be a lot. –

Gustov: Which none of these say.

Earl: Do they have the mass of these two blocks?

Gustov: No. But could we, could we find those? We really don't know the masses.

Earl: Not really. But, think about it, mass times gravity of this block, mass times gravity of this block in the x-direction, that's going to be, in the positive direction, it's going to be the sum of these two things.

Gustov: No wouldn't, wouldn't the force on the B be zero because it is, the table is pushing up and the, and A is pushing down so they would, and it is sitting still so there is no force.

Earl: What I'm trying to say –

Gustov: So there would only be force on A?

Claudette: Not necessarily, because the table is going to be pushing up more than B is pushing down just because A is on top of B.

Gustov: Yeah. That is why it is staying there.

Earl: Yeah, so it would be the mass times gravity of A, plus the mass times gravity of B, and that's going to be equal to the force that this [the table] is pushing up on [the blocks], because we have two blocks on top of, so, so this would be zero.

Gustov: (Reading off the conclusions of the solutions) So these are going to the left, these are going to the right, and this is going to the right (Earl indicates agreement).

Claudette: The only thing that I would, like I, I agree with three partially, but I feel like that in the y-direction, your F net would be zero just because neither one of them is going in anywhere, but in the x-direction, I feel like that something should be happening because one is going 7 and the other one is 20 so.

Gustov: But that's, that's just going to be, so this mass could be like .001 and this could be like 1000 or something like that (Earl indicates agreement). It could be anything. So it would take more force to move it (presumably the greater mass) than, than that would. So could it possibly be going to the, so they are both pulling, they are both pulling this way (gestures with his hands in opposite directions), something, so the friction, so if B it is getting pulled that way (gesturing with hands).

Claudette: Because we know that the force in the y is going to be zero, and z is zero. The only thing at issue is the x.

Quite a bit happened in the previous segment of the transcript. Claudette wanted to focus once again on the weight of the blocks, which led to the group realizing that there was no information in the solutions or problem about their masses. After some discussion, they determined that there was "no force" on block B. Moreover, Claudette herself decided that focusing on the vertical forces was not useful, as the sum of all forces in that direction should just be zero. Her reasoning was similar to that of Fay, Marco, and Omar when they determined that the forces did not need consideration. It served the same purpose for that group as it did here: it significantly reduced the forces that needed to be considered. Instead of all of the forces of one block pushing down on another, Claudette suggested that the group look at the x-direction, where there were two forces of different strengths acting on the blocks. However, only Amir and Barbara's pulling forces had been considered to this point.

Also in this segment, Gustov indicated that the masses of the blocks could be different, which he claimed could affect how easy it would be to pull them. This argument was reminiscent of Sally's initial suggestion that the masses needed to be different because the forces were different (see Figure 6.23). However, here Gustov only suggested that they **could** be different, not that they necessarily were. However, after Claudette stated that only the forces in the x-direction needed to be considered, there was no more conversation about the mass of the blocks. That quantity simply dropped out of their discussion, probably because it was no longer productive.

> **Gustov**: (Mumbling) Friction, so B is going, B is going to try to slide to the left, and A is going to try to slide to the right. So if you move it like that, it's going to, you are going to go that way (bottom hand sliding slightly to the left; upper hand staying still), so the force would have to be going to be that way, (indicating the left) right?
>
> **Earl**: Okay (not indicating agreement).
>
> **Gustov**: Because –
>
> **Earl**: Okay. The force, okay, she is pulling this way (gestures "left" with his finger), friction is going this way (gestures "right" with his finger), so it's like the neutron and electron thing?
>
> **Gustov**: So it would be going opposite? So this thing is going this way (left).
>
> **Earl**: So the force, so some kind of force has to be going this way (right) in equal –
>
> **Gustov**: To make it not move?
>
> **Earl**: Mm-hmm.

Gustov and Earl discussed the direction of the friction force, which they concluded should be opposite the pulling force. Specifically, Earl stated that because Barbara was pulling to the left, the friction force would need to be to the right. He did not indicate whether this was the friction due to the table, but that seemed to be implied. This reasoning was very similar to Otto's reasoning in Figure 6.15, except it was only used for block B in this case. As of this point in the transcript, Block A had been overlooked. The reasoning for friction opposing Barbara's pulling force is provided in Figure 6.30.

$\hat{F}_{Barbara\ pulling}$	$\hat{F}_{friction}$
←	→
←	$\neq\rightarrow$
Other possibilities not considered	

Figure 6.30: Gustov and Earl's reasoning that set the table's friction opposite the pulling force.

Now armed with a possible answer, the group turned back to the solutions to see if they could find a match.

Gustov: So let's see what this one (solution #1) says, so that would be pointing to the right then? (Earl indicates agreement). (Reading solution #1) "So because Barbara is pulling on block B, she is applying a force to the left. According to the momentum principle, the net force acting on an object equal to the opposite. So the net force, so you have Δp equals F net. (He writes this on solution #1) Therefore, in order for the block to remain experiencing, therefore in order for to keep the block from experiencing a change in momentum, the friction force needs to be equal and opposite to the force that Barbara is applying."

Earl: But not really equal and opposite because.

Gustov: Yeah, it would, cause it works out to be zero.

Earl: It would, OK, but (inaudible).

Gustov: If it wasn't, if it wasn't, it wouldn't equal zero.

Earl: That block at the bottom, yeah, if there was no block at the top. Okay we have a force here going in the opposite direction, but it would be this force plus this force (apparently indicating the two pulling forces), wouldn't it?

Gustov: But it is still, it is still going to be, it didn't give us any numbers, but we still know that it would have to be equal and opposite –

Earl: Yeah it does, it does –

Gustov: And not move, right? It does not move so wouldn't it have to be equal and opposite?

Earl: Mm-hmm. I was just saying that, don't forget that we have a force that's going in this direction (to the right) that is attached to this block (block A) as well, so that is applying a force in the opposite direction as well. So we can't get all of the force through friction alone. So,

Gustov read aloud solution #1, which supports the reasoning that he and Earl had just demonstrated in Figure 6.30. Not surprisingly, Gustov agreed with this solution, but then Earl brought up the fact that block A had not been considered in this reasoning. He claimed that the force of the pull attached to block A should have some effect on block B, although he was not particularly clear about exactly what that effect should be. This claim shifted the conversation from one in which friction opposed the pulling force to one in which the group was once again thinking about the momentum principle and the superposition principle. Earl's reasoning suggests the possibility set in Figure 6.31. Note that the conversation continued because he uncovered a new possibility, which prevented the group from coming to a conclusion too quickly.

$F_{net,x,onB} = \vec{F}_{Barbara} + \vec{F}_{other} + \vec{F}_{table}$			
$F_{net,x,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{other}$	$\vec{F}_{table}$
(0 N)	7 N $\leftarrow$		
Other possibilities not considered			

Figure 6.31: Earl's introduction of some new possibilities as a result of introducing a new quantity.

At this point, Claudette asked for clarification regarding the goal of the problem, which led into a discussion about the quantity they were actually looking for.

Claudette: Well, I think they're, they said the net frictional force?

Earl: So it says –

Gustov: (Reading, not attending to Claudette or Earl) "Therefore the frictional on the"

Earl: Ah, Now that does have to be equal and opposite. The net frictional for[ce].

Gustov: (Still reading) "According to the momentum principle, the net force acting on the object is equal to the object's change in momentum, so yeah, so that's (pointing to what he wrote) the momentum principle."

Claudette: What's this question say? It says – (reading from the problem statement) "What direction is the frictional force on block B due to the table?"

Earl: Oh, it says the direction?

Gustov: Due to the table.

Earl: Due, oh, [to] the table.

Claudette: So it's gotta be in the right.

Earl: That's right.

Claudette: Yeah, if it said net frictional force, then you'd have to take into account A, but really, I don't even think that really block A or anything to do with block A.

Gustov: Yeah, I think that they are just trying to –

Claudette: Yeah, they are just trying to mess with you, like get you to think about it. Because your system is not even going to be, it's just going to be B, and then your surroundings are going to be A and the table. (Gustov and Earl indicate agreement).

Earl had previously hinted that block A needed to be considered when thinking about the friction force on block B. However, Claudette clarified that the force that they needed to find was the frictional force on block B due to the table, and Earl retracted his earlier idea. Because of Claudette's clarification, Earl and Claudette stated that perhaps the problem was "messing with" them, and that all of the information about block A would be unnecessary. They considered neglecting block A, and they temporarily held possibility sets like those in Figure 6.32.

In addition, Claudette said that block B should be the system while block A should be in the surroundings. In this situation, for her that would mean that block A should be neglected rather than contributing a force on the system, which it did.

Next, the group read solution #2, which contains information about the friction between the blocks. Keying off the new information, the group reconsidered their reasoning yet again.

$\vec{F}_{net} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$			
$\vec{F}_{net}$	$\vec{F}_{Barbara}$	$\vec{F}_{AonB}$	$\vec{F}_{table}$
(0 N)	←		→
(0 N)	←		≠→

Figure 6.32: Earl and Claudette's neglect of block A.

Gustov: Let's just see what this one (Solution #2) says real quick. "Barbara is applying seven that way on block B, she is applying that on block, hold on, Barbara is applying a net force to the left on block B" hold on, on A, force to the right on block B. So this one is also saying "and block A is applying a force of 20 to the right."

Claudette: Well, you are going to have –

Gustov: On block B –

Claudette: Frictional force here too (between blocks) because they are rubbing together (Earl and Gustov indicate agreement).

Earl: So it's sum of frictional forces.

Gustov: (Reading solution #2) "Because of block, because block B applies a 20 N frictional force on the block, block, on block A to keep block A's momentum from changing," Barbara is applying a force to the left on block B. And block A is applying a force to the right on block B. So that's saying that it is getting a force that way, this is block B. See, one to the right *and* the left (draws two forces acting on block B, one to left and one to right).

Claudette: Well, for this, if it has a –

Gustov: This is 7, and this is 20, okay (writes "7" and "20" for the two arrows on solution #2).

Claudette: If it's having a frictional force in this direction (left) of 20 N since it's pulling this way (right), the frictional force would have to be that way (left), and the table would have to do the same amount of frictional force in this direction (to the right) for it not to move, right? So wouldn't the table also have to have a frictional force of 20 N going

in that direction to the right?

Gustov: Yeah, that sounds good.

Claudette: Because we know that it'll have to be equal and opposite (Earl and Gustov indicate agreement), so the frictional force, if x in this direction (right) is 20, and you have to be in this direction (left) on B, and the table would also have to be down here (right). So.

Claudette's reasoning was similar to Otto's, in that she might have been reasoning by analogy; that is, by noting that block B would need to apply a frictional force of 20 N to the left on block A, one can analogically reason that the table needs to apply a friction force to the right on block B to keep it from moving as a result of Barbara's pull. This reasoning was precisely what Otto did in Figure 6.17. However, the forces didn't quite add up for Claudette, as shown in Figure 6.33, and it's not precisely clear what her reasoning was.

$\vec{F}_{net,onA} = \vec{F}_{Amir} + \vec{F}_{BonA}$			$\vec{F}_{net,onB} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$			
$\vec{F}_{net,onA}$	$\vec{F}_{Amir}$	$\vec{F}_{BonA}$	$\vec{F}_{net,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{AonB}$	$\vec{F}_{table}$
(0 N)	20 N →	20 N ←	(0 N)	7 N ←		20 N →
(0 N)	20 N →	[≠20 N ←]	(0 N)	7 N ←		[≠ 20 N →]
Other possibilities not considered			Other possibilities not considered			

Figure 6.33: Implications from Claudette treating the blocks separately.

There was a mistake somewhere in Claudette's math; "20 N to the right" on block B does not work out mathematically. However, Earl liked this reasoning and attempted to work through it himself, which yielded different results.

Earl: That's making, that's making sense. Because I am seeing the two, like what you said. Okay. He, he is pulling the block in this direction (right), the frictional force is going to be going in this direction (left), which is going to be added to the force that she is pulling it too, as well. But the table is still going to be going in this direction (right), but it is going to be the sum.

Claudette: But it's going to be greater, so it's going to be like 27 instead of just –

Earl: Whatever –

Claudette: 20.

As opposed to Claudette, who seemed to be treating the blocks separately, Earl explicitly used the fact that the blocks were in contact to link them together, with Claudette's help. He reasoned that because there was a friction force to the left on block A, the friction force must have acted to the left on block B as well. Notice particularly that he never mentioned reciprocity; he did not forget to flip a direction. His treatment of the problem as if it contained a single friction force was just like one possibility for Sally's reasoning, as shown in Figure 6.25. A representation of Earl's reasoning, treating the friction between the blocks as a single force is shown in Figure 6.34.

$\vec{F}_{net,onA} = \vec{F}_{Amir} + \vec{F}_{BonA}$		
$\vec{F}_{net,onA}$	$\vec{F}_{Amir}$	$\vec{F}_{friction\ between\ blocks}$
0 N	20 N $\rightarrow$	(20 N) $\leftarrow$
0 N	20 N $\rightarrow$	$[\neq (20\ \mathrm{N}) \leftarrow]$

$$\Downarrow$$

$\vec{F}_{net,onB} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$			
$\vec{F}_{net,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{friction\ between\ blocks}$	$\vec{F}_{table}$
(0 N)	7 N $\leftarrow$	(20 N) $\leftarrow$	27 N $\rightarrow$
(0 N)	7 N $\leftarrow$	(20 N) $\leftarrow$	$[\neq 27\ \mathrm{N} \rightarrow]$
Other possibilities not considered			

Figure 6.34: Implications of Earl treating the friction between the blocks as a single force.

This group was approaching the correct reasoning, but they were still missing the very important reasoning step with reciprocity.

Gustov then suggested a completely different approach as the group reached the end of its time limit. Like Christobal and Walter, he suggested considering both blocks together as a single system and using that to calculate the friction force on block B.

Gustov: I think that we would probably have to take all of the table, the block, both blocks in as the system. We are going to have to take this whole thing into the system.

Claudette: Well, it just wants to know what is the force on block B though.

Earl: Yeah.

Gustov: But, but, but it is part of this whole system.

Earl: Yeah, so it is going into the, that's, that's the force going in this direction (right) on block B, and the force going in this direction (left) on block B. So I mean.

Claudette: Which one is the greater though? You have to sum them together.

Claudette and Earl continued with Gustov's idea that the two blocks could be considered as a single system. They brought up the idea that if considered together, the blocks would have two forces applied on them, one to the left and one to the right. To figure out which one was greater (and therefore the direction of the friction force), the two forces would have to be summed. This idea is completely correct, if the two forces they referenced were Amir and Barbara pulling on the blocks. This incomplete reasoning is shown in Figure 6.35, where the two forces acting on the big block are simply noted as $\vec{F}_1$ and $\vec{F}_2$.

$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_{table}$			
F_{net}	$\vec{F}_1$	$\vec{F}_2$	$\vec{F}_{table}$
(0 N)	$\leftarrow$	$\rightarrow$	
Other possibilities not considered			

Figure 6.35: Claudette, Earl, and Gustov's incomplete attempt to use a single-block solution.

Unfortunately, the group was not explicit about what those forces actually were, so we can not be sure exactly what Earl was referring to or how close they were to reasoning correctly using the single-block approach. Instead, the group was pressured by time to make a quick decision and they wrapped up, trying to make an informed choice between the solutions they looked at.

Gustov: So can we throw this one (solution #3) out? So do we, do we think that it is going to go right or left now? (Earl and Claudette indicate agreement). All right.

Earl: Due to the table, it's going to be going to the right to the, um, to the block,

Claudette: It'll be to the left. (Earl Indicates agreement). So, I would say right just because the table is goin – (Earl indicates agreement).

Gustov: But that's just, that's not even taking into account solution that, this one is not even taking into account like, the frictional force of A on this block, on block B.

Earl: But it's –

Gustov: But this one here is (comparing solutions).

Earl: But it, oh well.

Gustov: (Reading solution #2) So it says that, it says that block A is applying to the right on block B because block B applies a 20 N force on block A to keep the block's momentum from changing.

Earl: So you want to say that it is, um, solution #1? Or solution #2? Which one?

Gustov: I don't know.

Earl: Me either. Okay.

Claudette: Let's just go with, do [solution #]1. I don't know.

Researcher: All right the 15 minutes are up. Does the group want to make a statement of consensus or do you want to say you're undecided or what?

Gustov: I kind of want to go with [solution #]2 because it is not taking into, it is taking into account for the forces on this one, I really don't know, I just think, I just think it would need to be included. (Earl agrees).

Claudette: Okay, well then, do [solution #]2. (Earl agrees)

Gustov: You're sure? (Claudette and Earl both agree).

In the end, the group was unable to decide which solution was best (note that they eliminated solution #3 very easily). Claudette suggested solution #1, but Gustov chose solution #2 for the group because it included the friction between the blocks, which solution #1 did not. The group came very close to performing the correct reasoning on their own, but time and again mistakes prevented them from reaching the proper conclusion. On the other hand, the members of this group were remarkably open to considering new possibilities, something that did not often happen with the participants in this study. Being open to possibilities allowed them to narrow their focus to the

x-direction and identify the importance of the friction between the two blocks, and it prevented them from coming to a rapid, incorrect conclusion in a number of instances throughout the discussion.

6.1.8 Friction opposes the pulling force (Hugo, Josephine, and Teddy)

Hugo, Josephine, and Teddy were remarkably quiet participants. They did not discuss very much during their group session, and as such it is quite difficult to make many detailed possibility sets to model their reasoning. However, with some prompts from the researcher, they did provide some evidence as to how they solved the problem. First, however, they made judgments about the worked solutions with some very interesting comments.

> **Hugo**: (After reading the problem, but before looking at the solutions) I think that B is going to be pulled this way (gestures to the left).
>
> **Teddy**: Yeah.
>
> **Hugo**: There's gotta be friction this way (gestures to the right). (Now seems to have moved problem statement aside).
>
> **Teddy**: (Looking at solution #1) To the right, that one. (More quiet group processing).
>
> **Teddy**: That makes perfect sense. (Hugo indicates agreement). Equal and opposite (Hugo indicates agreement). Direction. (Seems to be reading).
>
> **Hugo**: Yeah maybe.
>
> **Teddy**: I don't think so. Friction is never going to be in the direction of motion. (Josephine indicates agreement). (To Hugo) do you think it's [solution #]2? (Hugo shakes his head)

Teddy stated that friction would never be in the direction of motion, a heuristic that is modeled in Figure 6.36.

Note that there is no relationship shown in this possibility set because there is no physically correct way to relate the direction of motion to the direction of the friction force. Nonetheless, this possibility set played a significant role in Teddy's reasoning on this problem.

$\hat{v}$	$\hat{F}_{friction}$
$\hat{k}$	$\hat{k}$
$\hat{k}$	$[\neq \hat{k}]$
Other possibilities not considered	

Figure 6.36: Teddy's possibility set for the direction of motion and the direction of friction.

Hugo: (Shakes his head, then reads solution #3) I don't think that's right. If there is no friction force then it would move.

Teddy: But there is friction even if it's not moving, right? Or is there not?

Josephine: I don't know.

The group continued to reason about friction. While Josephine was uncertain, Teddy and Hugo explicitly listed possibilities relating motion and friction. In this case, the claim they made had nothing to do with direction, relating motion only to whether or not there was a friction force. This possibility set is provided in Figure 6.37.

$|\vec{v}|$	$|\vec{F}_{friction}|$
not 0	0
[0]	0
0	not 0
Other possibilities not considered	

Figure 6.37: Teddy and Hugo's explicit list of possibilities regarding friction and motion.

The statements by Teddy and Hugo rejected the possibility that there could be no motion and no friction force for this problem. Therefore, they rejected solution #3, which states exactly that. Teddy's elaboration on that statement merely reinforces this point (the implied negation of Teddy's statement is that it would be impossible to have both no motion and no friction force, which Hugo's statement already implied).

Teddy: (Inaudible) solution (inaudible) in here?

Teddy: Yeah, I don't think that it's [solution #]3. I think that it's [solution #]1.

Josephine: Here you go (hands marker to Teddy, who marks solution #1 correct).

Teddy: Cause that would be like saying if something was sitting on something like that (using hands to simulate an inclined plane and placing red pen on top of hand), it wouldn't have any friction, like if it would move, if it would say that it has no friction because it's not moving (Hugo indicates agreement).

Researcher: So, could you, um, tell me how you ended up choosing solution #1 on this one?

Teddy: Because it has to have friction to keep it from moving.

Hugo: So it's like getting pulled to the left, and there's got to be a friction force to the right to stop it. If it was to the left, then it would move to the left.

After some prompting by the researcher, Teddy stated that friction would be required to keep the blocks from moving, with an elaboration by Hugo. Their reasoning for this problem is given in Figure 6.38. This model of their possibility set interprets "moving" as "non-zero net force," as has been done for other groups as well.

$\vec{F}_{net} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{friction}$					
$\vec{F}_{net}$	$|\vec{F}_{Barbara}|$	$\hat{F}_{Barbara}$	$\vec{F}_{AonB}$	$|\vec{F}_{friction}|$	$\hat{F}_{friction}$
0		$\leftarrow$			$\rightarrow$
0		$\leftarrow$			$[\neq\rightarrow]$
$\leftarrow$		$\leftarrow$			$\leftarrow$
$[\neq\leftarrow]$		$\leftarrow$			$\leftarrow$
Other possibilities not considered					

Figure 6.38: A possibility set representation of Teddy and Hugo solving the "Two Blocks" problem.

Teddy and Hugo presented possibilities, including extra information about what would happen if they had gotten the other result. Fay, Marco, and Omar solved the problem similarly, by identifying an alternate possibility and figuring out what would reveal that such a possibility were true. However, Teddy and Hugo drew the wrong implication because they were clearly neglecting many important quantities, namely the friction force acting on block B due to block A and the magnitudes of all of

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the forces. This oversimplification of the problem certainly made it easier for the participants, but it also led to an incorrect solution.

> **Researcher**: Okay. And you told me what was wrong with solution #3. Could you tell me what was wrong with solution #2?

> **Teddy**: [Solution #]2, it says it's going in the direction of the, the net f-, the force that she is applying and that's never, that's never in the same direction as the force. Does that make sense? (Addressing the researcher). I don't know if I explained that very well. Like since she is pulling this way, the friction force isn't going to be in the same direction.

> **Josephine**: It always has to like oppose it.

> **Teddy**: Yeah.

Again prompted by the researcher, the group continued to elaborate on their possibility set, adding even more rows to it, including a generalization by Josephine. Figure 6.39 shows the more complete set.

$\vec{F}_{net} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{friction}$					
$\vec{F}_{net}$	$|\vec{F}_{Barbara}|$	$\hat{F}_{Barbara}$	$\vec{F}_{AonB}$	$|\vec{F}_{friction}|$	$\hat{F}_{friction}$
0		$\leftarrow$			$\rightarrow$
0		$\leftarrow$			$[\neq \rightarrow]$
$\leftarrow$		$\leftarrow$			$\leftarrow$
$[\neq \leftarrow]$		$\leftarrow$			$\leftarrow$
		$\leftarrow$			$[\neq \leftarrow]$
		$\hat{k}$			$-\hat{k}$
		$\hat{k}$			$[\neq -\hat{k}]$
Other possibilities not considered					

Figure 6.39: The more elaborate possibility set for Teddy, Hugo, and Josephine solving the "Two Blocks" problem.

In the final lines of dialogue, this group no longer even referenced the blocks not moving. Moreover, the generalization by Josephine (which builds off similar statements by Teddy) implied that it would always be true that a friction force must oppose the pulling force. This generalization is

represented by the bottom two rows in Figure 6.39. Indeed, it may be useful to represent this specific generalization by creating an entirely new possibility set that is much simpler and has many fewer blank spaces. This possibility set is shown in Figure 6.40.

$\hat{F}_{pulling}$	$\hat{F}_{friction}$
$\hat{k}$	$-\hat{k}$
$\hat{k}$	$[\neq -\hat{k}]$
Other possibilities not considered	

Figure 6.40: An alternate possibility set for Teddy, Hugo, and Josephine solving the "Two Blocks" problem.

Unfortunately, because of the limited dialogue, we cannot know exactly what possibility sets this group were considering. All we know for sure is that they chose the wrong solution, and the reason for doing so was somehow related to their premise that the direction of the friction force needed to oppose the force that Barbara was applying.

6.1.9 Difficulty fleshing out possibilities (Gaston, Igor, and Winfred)

Although it was short, the group session with Gaston, Igor, and Winfred was interesting. While this group quickly asserted the incorrect solution and did not waver from it, Winfred continued to mull over possibilities, explicitly listing potential conflicts to what seemed like a simple solution. This group's efforts were quite different from the acquiescence that Hugo, Jospehine, and Teddy displayed once they decided on the answer to the problem. However, when Winfred misinterpreted solution #2, his group did not question it. Despite Winfred's attempts to more fully explore the problem, viable alternatives to solution #1 never really solidified.

Unlike many sessions, this one began with Winfred drawing a large representation of the problem on the whiteboard, which was used throughout the session as a focal point for discussion. This explicit representation greatly facilitated not only their sense-making but also their communicative ability, as their gestures were relatively easy to follow.

Winfred: (Draws a diagram representing the blocks with arrows indicating the given

forces on the whiteboard).

Igor: (Reading) "What direction is the friction?"

Winfred: I mean.

Gaston: Wouldn't it be that way (to the right, see Figure 6.41)?

Igor: Well, there is a table right here (draws a table beneath blocks on whiteboard). So there's a table right here, so that's applying its own friction.

Winfred: Oh yeah.

Igor: Up. And if you're pulling it that way (left) then there would be a friction going that way (right).

Gaston: Yeah.

Igor: It's going this way (left), so there is a force that way (right). (Igor draws two arrows at the interface between the blocks: to the left on block A's side and to the right on block B's side, see Figure 6.42.)
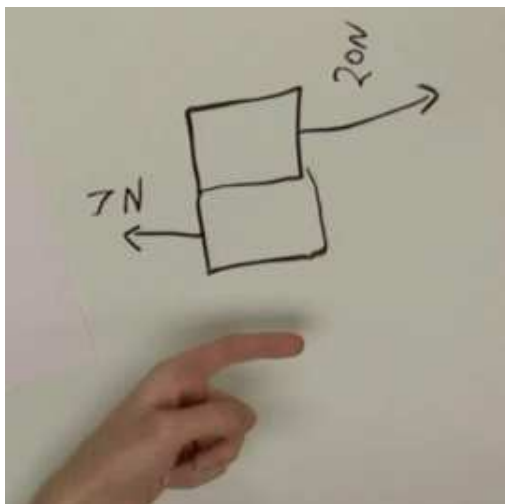
Figure 6.41: An overhead picture showing Gaston indicating the friction force due to the table on block B as to the right by pointing with his finger.

Gaston quickly suggested that the friction needs to be to the right (as shown in Figure 6.41, although he did not provide any explanation why. Igor then fleshed out the actors that contribute

forces to the blocks, as shown in Figure 6.43. Igor made it clear that he, at least, was thinking about friction as a force that would oppose the pulling force. However, as we see by Figure 6.42, Igor drew a total of four arrows although he did not mention what all of them represent.
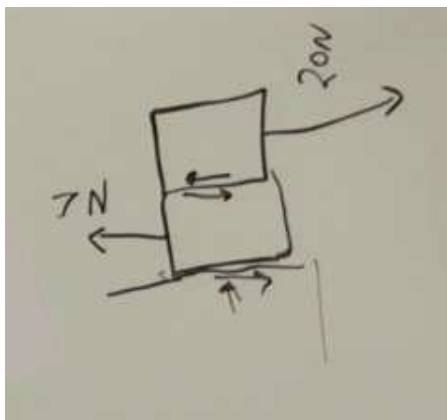
Figure 6.42: The whiteboard drawing by Igor of four unlabeled arrows on and around the two blocks. Some were drawn to refer to frictions that opposed a pulling force.

Especially remarkable about Igor's reasoning is that it was actually proper *modus ponens*: "If you're pulling [left], then there would be a friction [to the right]. It's going [left], so there is a force [to the right]." This logic makes his possibilities particularly transparent. One may argue that this instance is evidence that he was using formal logic; however, even single-step occurrences of *modus ponens*, the simplest form of logic, were incredibly rare in this investigation.

$\vec{F}_{pulling}$	$\vec{F}_{friction}$
$\leftarrow$	$\rightarrow$
$\leftarrow$	$[\neq\rightarrow]$
Other possibilities not considered	

Figure 6.43: Igor's proposition that the friction force must oppose the pulling force.

Winfred: Yeah, I mean it would have to be this way (right).

Gaston: Yeah, I don't see how it could be any other way.

Winfred: Well. I mean, I guess if the frictional force was only like. Uh, never mind, because this would have to be 13 N this way (left) for the frictional force (between the blocks, indicating the A side). And this one (indicating the friction due to the table) would have to be like 7 N (right).

Gaston and Winfred both agreed with Igor's idea that friction would need to oppose the pulling force. When Gaston noted that he could not think of any alternatives, Winfred (possibly interpreting Gaston's comment as a challenge) tried to postulate an alternate explanation. He didn't believe in his own suggestion enough to fully mention it, but the hints that he provided are worth our attention. It is almost as if Winfred were playing around with the forces, trying to imagine how the force due to the table could be anything other than to the right. However, his conclusion was that the force due to the table would be 7 N the right, and there was no clear relationship indicated in his reasoning that made use of the 13 N to the left on block A. This incomplete chain of reasoning is shown in Figure 6.44.

$\vec{F}_{net}$	$\vec{F}_{Amir}$	$\vec{F}_{Barbara}$	$\vec{F}_{between\ blocks}$	$\vec{F}_{table}$
	20 N →	7 N ←	13 N ←	7 N →
	20 N →	7 N ←	13 N ←	[≠ 7N →]
Other possibilities not considered				

Figure 6.44: Possible values for forces, as considered by Gaston.

Had Gaston vocalized his reasoning more fully, we might be able to interpret how his claim about the 13 N to the left at the interface of the blocks was related to his claim that the friction force due to the table needed to be 7 N to the right. One possibility is that they are not actually related; that he was trying to work out how both could be possible, and his "never mind" indicated that they didn't both fit together.

The group moved on and began looking at the written solutions to the problem.

Gaston: (Reading solution #1) Barbara's pulling on block B (inaudible, trails off).

Winfred: I mean.

Gaston: That (unclear which solution, but probably solution #1) makes sense. (After Gaston reads solution #3) Is that true? There can only be friction if that's moving?

Igor: No.

Winfred: I don't think so, because you have to apply a force to overcome a frictional force.

Igor: Friction is the resistance.

Gaston: I'm thinking solution #1 is it.

Winfred: Let's see, solution #2 (inaudible). I mean, this (solution #2) doesn't take into account the friction between the two blocks. It's just saying that these two blocks were, if like this was, area right here (indicating the interface between the blocks) was like frictionless, then it (friction due to the table) would have to be that way (left) to oppose this (Amir's pulling force to the right).

Igor: (Reading) "The momentum –"

Winfred: (Interrupting Igor) So solution #2 is definitely, definitely wrong.

The group quickly agreed with solution #1 and then debated solution #3 before moving on to solution #2, which Winfred misinterpreted. In stark contrast to Winfred's statement that solution #2 did not take into consideration the friction between the blocks, it was the only solution that actually did take that quantity into account. How then did Winfred come to this erroneous interpretation? One possible explanation is that he had already conceived of a way to get the result that the friction due to the table was to the left, and to get that result he imagined that there was no friction between the blocks. Note that choosing the two blocks to be one system means that the friction between the blocks is disregarded, but *not* set to zero! Did Winfred confuse setting a quantity to zero and neglecting that quantity? The distinction between these two interpretations is emphasized by looking at the possibility sets for the two cases. In Figure 6.45, we see a correct one-block solution to the problem (note that this is the same as Figure 6.11). In contrast, Figure 6.46 represents Winfred's incorrect interpretation of solution #2.

So, given his misinterpretation of solution #2, it was reasonable for him to reject it; he seemed to know that the friction force between the blocks could not be zero, or else the blocks would move

$\vec{F}_{net} = \vec{F}_{Barbara} + \vec{F}_{Amir} + \vec{F}_{table}$			
$\vec{F}_{net}$	$\vec{F}_{Barbara}$	$\vec{F}_{Amir}$	$\vec{F}_{table}$
$\vec{0}$ N	7 N ←	20 N →	13 N ←
$\vec{0}$ N	7 N ←	20 N →	[≠13 N ←]

Figure 6.45: The correct one-block system solution to the "Two Blocks" problem.

$\vec{F}_{net,onB} = \vec{F}_{Barbara} + \vec{F}_{table} + \vec{F}_{AonB}$				
$\vec{F}_{net,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{Amir}$	$\vec{F}_{table}$	$\vec{F}_{AonB}$
$\vec{0}$ N	7 N ←	20 N →	13 N ←	$(\vec{0}$ N$)$
$\vec{0}$ N	7 N ←	20 N →	[≠ 13 N ←]	$(\vec{0}$ N$)$

Figure 6.46: Winfred's misinterpretation of solution #2 to the "Two Blocks" problem.

with respect to each other (although he never explicitly said this). However, he did not realize that the choice of the system as including both blocks would make it so that the friction between the blocks would be irrelevant. More directly, he failed to see that solution #2 never tried to reason with that choice of system. The implication therefore is that Winfred projected his own interpretation of and previous thoughts about the problem onto this solution.

Winfred then quickly shifted to looking at solution #3, stating that while there could not be zero friction between the blocks, maybe there was zero friction on the table.

Winfred: But see, I think that it (solution #3) could be true.

Igor: Well, you know, it doesn't have to move, friction can keep it from moving.

Winfred: Yeah.

Igor: I mean, you just need to apply a force larger than the force of the friction to make it move.

Winfred: Yeah.

Igor: Friction is there.

Winfred: Yeah because if, depending on which way you pull, the frictional force, it just acts in the opposite direction.

Gaston: Yeah.

Winfred: Is basically what it's saying because there is a frictional force here (indicating two points of contact between block B and the table, one on the left and one on the right). And here, and either way you're going to pull, you're going to have to oppose the same frictional force.

Igor tried to oppose Winfred's idea that the friction between block B and the table could be zero. Winfred apparently agreed for the time being, reiterating Igor's earlier reasoning that the friction force would oppose the pulling force by the very nature of friction (see Figure 6.43). Then Winfred went back to thinking about possibilities, about whether it would be *possible* for the friction force between the table and block B to not be to the right, and if so, what needed to be true for that to be the case.

Winfred: Look, the real question is to address here is like this part (indicating the interface between the blocks), determining like. (Pause) Because I mean if the, if the frictional force was higher than 20 N here (circling the entire interface between the blocks) then this, pulling this (indicating Amir's force) could pull both blocks which would mean that this (the friction between the table and block B) would have to go that way (to the left). That the frictional force would have to go that way (left). Because if like, if this, if this was frictional force between these two was big enough to stick them together –

Igor: Well you would also have to account for, uh, it pulling down.

Winfred: Yeah.

Igor: But that, that would happen.

Winfred: Let's see, I am now, I am now not sure.

Gaston: Yeah.

Winfred: Because this friction, this table could be frictionless and [then] the blocks still not move if the frictional force between these two (blocks) is 13.

Apparently, Winfred was reasoning through how it could be possible for the friction force between the table and block B to not be to the right. However, he was struggling to include the proper

quantities, and as a result his reasoning was incomplete and unconvincing. This reasoning, such as it is, can be represented by Figure 6.47.

$\vec{F}_{net,onB} = \vec{F}_{Barbara} + \vec{F}_{table} + \vec{F}_{AonB}$				
$\vec{F}_{net,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{Amir}$	$\vec{F}_{table}$	$\vec{F}_{friction\ between\ blocks}$
$\vec{0}$ N	7 N $\leftarrow$	20 N $\rightarrow$	$\leftarrow$	> 20 N
$\vec{0}$ N	7 N $\leftarrow$	20 N $\rightarrow$	$\leftarrow$	$[\leq 20$ N$]$
$\vec{0}$ N	7 N $\leftarrow$	20 N $\rightarrow$	$\vec{0}$ N	13 N
Other possibilities not considered				

Figure 6.47: Winfred's incomplete possibility set for the "Two Blocks" problem.

If Winfred were able to isolate the quantities that were relevant for this situation, and if he were explicit about the direction of the friction force on block B due to block A, he may have stood a good chance of figuring out this problem. As it was, too many possibilities remained in his possibility sets for him to be able to draw a conclusion. Note that Igor stated that "you would also have to account for it pulling down" which might be a suggestion that the weight of block A could hold block B in place. Nothing else was mentioned to provide clues as to Igor's meaning, he may have been attributing the blocks' lack of motion to their weight. In any respect, Igor's attempts at eliminating Winfred's possibilities were not directly successful (that is, Igor does directly argue against Winfred's half-formed ideas), although they did help the group eliminate solution #3.

Igor: Well, if the table were frictionless and this thing, uh, there is, this thing (indicating with two fingers that he means both blocks, together) would still move that way (to the right), but they wouldn't move off the top of each other since this force (Barbara's) is less than that force (Amir's).

Winfred: Are you sure that they wouldn't? Because if you pull 20 N here but the frictional force –

Igor: It would just keep them together.

Winfred: Oh yeah, that's true, that's true, so it could be true.

Igor: So that makes the friction here (interface between table and block B) has to be equal to that (the friction between the blocks) to keep them together, but not moving.

Igor pointed out that for the given information, if there were no friction between the table and block B, the blocks together would move to the right because the magnitude of force from Amir was greater than the magnitude of force supplied by Barbara. Figure 6.48 represents this reasoning.

$\vec{F}_{net} = \vec{F}_{Barbara} + \vec{F}_{Amir} + \vec{F}_{table}$			
$\vec{F}_{net}$	$\vec{F}_{Barbara}$	$\vec{F}_{Amir}$	$\vec{F}_{table}$
$\rightarrow$	7 N $\leftarrow$	20 N $\rightarrow$	$\vec{0}$ N
$[\neq\rightarrow]$	7 N $\leftarrow$	20 N $\rightarrow$	$\vec{0}$ N

Figure 6.48: Igor's reasoning that the absence of friction between block B and the table would mean that both blocks should move to the right.

This reasoning is correct, and Winfred agreed without debate. All that remained at this point was for the group to officially select a solution and reject the other two.

Winfred: Yeah. Yeah. That makes sense. So. Solution #1.

Igor: Yeah.

Winfred: So the question here is like –

Igor: That makes them –

Winfred: I'm going to say that this one (solution #3) is wrong.

Igor: Because there was friction on the table.

Winfred: Yeah. And this one (solution #2) is like, um, the thing wrong with that one is that they didn't take into account this frictional force between the two blocks, but, you can't really mark that anywhere.

Igor: So I guess solution #1.

Winfred: All right.

Throughout the discussion, Winfred struggled to flesh out enough possibilities to really get a handle on the problem. However, he was not ever successful in isolating the quantities that were relevant for a consistent choice of system. This, combined with the fact that he continually treated the friction between the blocks as a single force, doomed him to incomplete reasoning. However,

it should be noted that the potential for understanding all of the possibilities in this problem was clearly present in Winfred's attempts. He was not using formal logic, but rather he was trying to explore possibility space, hunting for a way to make the friction due to the table on block B not to the right. Unfortunately for him, this hunt was unsuccessful on this problem. Nonetheless, his search hints at the potential effectiveness of guiding students' hunts for possibilities in physics instruction.

6.1.10   Is block A important? (Eduard)

Although he was not a part of a group, Eduard was quite explicit about his reasoning. When solving this problem, he considered the weight of the blocks twice, but in the end decided that the friction on block B due to the table would need to oppose the pulling force on block B, and that block A would not play a role in this problem. When the researcher intervened, Eduard began to consider new possibilities. The reasoning that resulted from this discussion with the researcher suggests how certain methods of intervention may be useful.

> **Eduard**: (Reading the problem statement) "Block A sits on Block B. By applying 20 N of force. 7 N of force. Neither block moves. What direction is the friction force on block B due to the table?" Okay, so if neither block moves then they can't be pulling with a weight, with a force greater than the block, they can only be pulling up until it's equal to.

First, Eduard thought about the weight of the blocks. He suggested that the pulling force on a block needs to be less than the weight of that block, reasoning that is represented in Figure 6.49.

$\vec{F}_{net}$	$|\vec{F}_{weight}|$	$|\vec{F}_{pulling}|$
(0 N)	W	$> W$
(0 N)	W	$\leq W$
Other possibilities not considered		

Figure 6.49: Eduard's reasoning about the weight of the blocks.

Notice that the possibility sets reflecting his reasoning do not include a relationship, because no valid physical relationship explains his reasoning. However, it is possible that he was trying to

indicate that heavier objects tend to have more friction than lighter ones, a heuristic that is the result of everyday life and is reflected by the mathematical definition of the friction force as the product of the normal force (which is directly related to the object's weight) and some coefficient of friction.

> **Eduard**: The friction on block B due to the table – so yeah, the friction (draws the blocks on the whiteboard). I don't think it even matters that block A is up here. That's not going to affect the friction from the table. This is being pulled that way (draws an arrow to the left on block B), and it is not moving, [so] friction has to go the opposite direction (draws an arrow to the right on the table) to make it the net – force, I guess, I guess friction is, yeah friction is a force, so in order for F net to equal zero, or not move, then it has to be equal and opposite (writing $F_{net} = 0$ and $F_2 + F_1 = 0$ on whiteboard, see Figure 6.50).
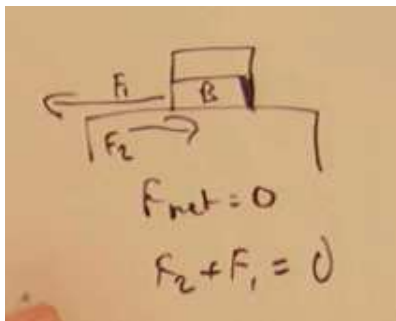
Figure 6.50: The whiteboard after Eduard wrote that the net force should equal zero, meaning that the sum of the two forces equals zero. Note that he did not include vector information.

Eduard quietly put aside the idea that the weights of the blocks would matter. He choose to dismiss block A for some unknown reason, possibly because by doing so he substantially simplified the problem; there is no obvious way in which block A relates to block B, so he simplified the problem by not considering block A at all. Earl and Claudette made a similar move themselves (see Figure 6.32), although Eduard developed this set of possibilities much more quickly than they did. Eduard's reasoning is represented by Figure 6.51.

$\vec{F}_{net} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$			
$\vec{F}_{net}$	$\vec{F}_{Barbara}$	$\vec{F}_{AonB}$	$\vec{F}_{table}$
$\vec{0}$ N	$\leftarrow$		$\rightarrow$
$\vec{0}$ N	$\leftarrow$		$[\neq\rightarrow]$
Other possibilities not considered			

Figure 6.51: Eduard's reasoning about the "Two Blocks" problem.

Eduard was more explicit than most participants, indicating both that he was making the net force on block B zero and that he was ignoring block A, meaning that he was not taking into account the frictional force that block B would apply to block A. Otherwise, his reasoning was correct.

> **Eduard**: Yeah, so which one says that? (Begins flipping through solutions) to the right, to the left. (Reading from solution #3) "There won't be a friction force if the block isn't moving." Hmm.
>
> **Researcher**: What are you thinking about?
>
> **Eduard**: Um, well I'm thinking about how friction affects the system, because I said earlier that – like if you have got a weight that's 10 kg (he draws a block on the whiteboard and labels it "10 kg"), and you pull with a force of, let's see, nine – I don't remember the units, I think it is kilograms per meter I think that's what Newtons is. But if you don't pull with a force that at least 10 kg, I don't think it's going to – it's not going to move. Actually, no. The force on the block doesn't matter, because the force is in the y-direction, so – yeah, force is in the negative y, because that's being acted on by the Earth, and they are pulling to the left, so yeah, so the only thing that matters here is what's in the x plane, so F of x has to equal zero. So that means the direction of friction is to the right. So, yeah I'm going to say solution #1 for that one.

Eduard returned to his earlier ideas about the weight of the blocks. At this point, he tried to come up with an explicit example with numbers. However, in doing so he had a sudden insight: the weight of the blocks was in the y-direction while the friction force was in the x-direction. This insight allowed him to disregard his concern about the weight of the blocks, which was a quantity that he tried unsuccessfully to include in his earlier reasoning about the superposition principle. Having

eliminated the physical quantity "weight" from his possibility sets, he had no difficulty selecting solution #1, which he believed to be the correct solution to the problem.

> **Eduard**: And (reading solution #2) "the net force acting on an object is equal to that object's change in momentum. The combination of these two forces plus the friction (trails off)." I think even if you do pull it, it does change its momentum even if it doesn't move. Actually no because mass times velocity, if it doesn't move it doesn't have velocity and mass times velocity, if velocity is zero, then your momentum is going to be zero. Yeah, I'm gonna – I'm gonna say it's [solution #]1. Yeah.

Eduard did consider solution #2, but after he had already selected solution #1. In thinking about it, he focused on possibilities regarding motion and momentum, first suggesting that it would be possible for an object to have a nonzero momentum but a zero velocity and then promptly rejecting that possibility. This possibility set is shown in Figure 6.52.

$\vec{p} = \vec{v} \times m$		
$\Delta\vec{p}$	$\Delta\vec{v}$	m
$\neq \vec{0}$	$\neq \vec{0}$	
$\vec{0}$	$\vec{0}$	
Other possibilities not considered		

Figure 6.52: Eduard's reasoning about momentum and motion.

At this point, Eduard continued with his session by moving on to the next problem. After Eduard solved all of the problems, the researcher returned to this problem to ask him some questions.

> **Researcher**: All right, let's look at this heavy blocks problem. Okay, so the two heavy blocks. You said that Barbara is pulling to the left so therefore there needs to be a friction force to the right. What if I propose that you think of the two blocks together as one block and that Amir pulls with 20 N to the right and Barbara pulls with 7 N to the left. How would the friction between the table and the big block be?
>
> **Eduard**: It would be dependent on the weights. I didn't really think about, I just said this one's on the bottom (pointing to block B in the problem statement), is the only

one that is touching the table, forget this one (referring to block A), it doesn't matter. But yeah now that I think about it, it's also applying – block A is also applying friction to block B, but block B is the only one that's receiving friction from the table. But I mean if you count the whole thing as a block, I don't know. I don't know about that. I was pretty confident that the time when I was solving it out, that oh yeah it's in this direction (indicating to the right).

Researcher: Now, is it safe to say that now that you're thinking about it a little bit more, you are not as confident?

Eduard: Not quite as confident, no. Because block A does create friction on block B, but not – I think if you count the entire block is one thing then it doesn't.

At first, possibly triggered by the researcher referring to the blocks as "heavy," Eduard claimed that the friction between the table and the block would depend on the weights of the blocks. Soon after, though, Eduard realized that block A would apply friction on block B, accounting for the relationship between the blocks that was not present in his reasoning earlier when he neglected block A. While he did not explore how this additional friction force would affect his solution to the problem, he did indicate that his confidence about his previous solution was wavering. The researcher did not choose to discuss the friction between the two blocks any farther, instead directing Eduard to consider the what would happen if there were a single block instead of two blocks on the table.

Researcher: Well let me, let me get a new white board here. And let me actually propose it as a separate question, where we have one block sitting on a table, and we have one force in that direction and one force in this direction (drawing on white board), and that is 20 N, and this is 7 N, and I ask what is the friction force between the table and the block. In that case, what would your answer be?

Eduard: I would say that the friction force is in this direction (drawing an arrow to the left).

Researcher: And why?

Eduard: It would have to be 13 N, because the block is not moving.

Researcher: Okay. All right, and so answer me if I am correct in asking this question, that you used essentially the momentum principle here, then that force times Δt equals momentum, blah blah blah, and because this is a statics problem, there is no net force because it's not moving there is no net force, so the force on the side has the equal force on that side.

Eduard: Yeah.

Researcher: Okay.

Eduard: It's just like those problems we've had on tests Where it's like this is hanging from a spring, there is a string holding this, and the force of the earth and everything, but I mean there is no forces – the force of the table here (drawing an arrow up) equals the force of the earth (drawing an arrow down), and so the only plane we're worrying about is the x plane.

When the researcher asked Eduard what the solution to the problem would be if it were a single block instead of two blocks, Eduard did not hesitate to provide the correct answer, based on the correct use of the momentum principle. This solution was identical to Christobal's reasoning as well as the first bit of reasoning that Walter and Otto agreed upon. This reasoning is shown in Figure 6.53.

$\vec{F}_{net} = \vec{F}_{Barbara} + \vec{F}_{Amir} + \vec{F}_{table}$			
F_{net}	$\vec{F}_{Barbara}$	$\vec{F}_{Amir}$	$\vec{F}_{table}$
0 N	7 N $\leftarrow$	20 N $\rightarrow$	13 N $\leftarrow$
0 N	7 N $\leftarrow$	20 N $\rightarrow$	[$\neq$13 N $\leftarrow$]
Other possibilities not considered			

Figure 6.53: Eduard's possibility set for considering a single-block version of the "Two Blocks" problem.

Eduard's answers to the researcher's questions indicate that he was solving for the friction force by considering the momentum principle and the net force acting on the block. The researcher then prompted Eduard to articulate the difference between two situations: when there are two separate blocks versus when there is just a single block.

Researcher: Okay. And so, maybe can you propose how that (pointing to the white board) situation is different from this situation (bringing the original problem closer and placing it on the white board) then.

Eduard: Just the fact that there is two separate blocks here (draws a line halfway through a block drawn on the white board) Will, I mean, you throw in a factor like that, I mean that will throw most students off. But I mean, it's pulling the top block and the bottom block separately, they are not, I mean see this is one block and it's pulling in two separate forces. Because students will look at this (the original problem) and they'll say, "well that this block is going to slide off (indicating block A) and this block is going to slide under (indicating block B)" even though they're not moving. But I mean then, obviously if she's pulling this block up here, then there has to be friction forces going on between these two in both directions (draws arrows in both directions at the new interface drawn on the block on the white board).

Researcher: So instead of, so maybe, so let's think about this. In order to keep them from moving maybe we can think about them as having really coarse sandpaper between them. So now we're pulling, and so there is this really coarse sandpaper that is keeping them from sliding. is that starting to get closer to that, does that make it closer to the big block, or does it still keep it as two separate in that case?

Eduard: That would make the system more like one big block. Just depending on how strong the friction force is. If the friction force can withstand 20 N in this direction and 7 N in that direction, then yeah, it's safe to analyze the system as one block. But the fact that they didn't give us any friction values or anything, didn't tell us anything, you know. They just asked the direction of the force of the table.

Eduard struggled to express precisely what he was thinking, but he stated that if the friction between the two blocks could support the 20 N thanks to Amir and the 7 N thanks to Barbara without failing, then the situation would be just like the single block. However, in the case of the actual given problem, information about those friction values was not provided, and so he did not provide a clear decision as to whether it would be appropriate to analyze this problem in that way.

This idea about the friction force being "strong enough" indicates that he was not considering the friction force between the two blocks as something that could be defined by the constraints of the problem; that is, that the friction force on block A due to block B could be calculated by using the superposition principle and the momentum principle on block A. The reasoning represented by Figure 6.3, which is the correct solution to the problem, was still not held by Eduard. However, it is clear that each inner component was present; he very clearly explained that he could use both principles to solve physics problems. What was missing was a distinction between the physical system of two blocks (wherein people could pull hard enough to cause them to slide with respect to one another) and the mathematically defined system (where the people are pulling but the blocks could not slide because of the information provided in the problem). Perhaps further intervention could have brought Eduard to find that distinction, but at this point the researcher abandoned this topic and began asking questions about other problems.

6.1.11 Discussion: considering relevant quantities in possibility sets

At least four participants or groups (Eduard; Ana and Sally; Claudette, Earl, and Gustov; and Fay, Marco, and Omar) talked about the mass or weight of the blocks in their discussions (or in the case of Eduard, in his think-aloud session). Igor may have also hinted at the mass of the blocks. In some cases, the participants stated that the mass of the blocks, as an alternative (or perhaps in addition) to friction, could prevent the blocks from moving. In contrast to such groups, Fay, Marco, and Omar considered the weight of the blocks solely as a force in the y-direction and did not argue that it would keep the blocks in place. In any respect, by considering "mass" or "weight," another quantity of relevance was added to the reasoners' working memories, significantly increasing the number possibilities under consideration.

In every case, the reasoners eventually dropped all consideration (or at least discussion) of the mass of the blocks. In some cases, such as with Claudette, the participants explicitly demonstrated the insight that the mass of the blocks contributed to their weight, which was a force in the y-direction. Since all of the forces in the y-direction were known, and they resulted in zero net force, some of the participants were able to substantially reduce the possibility space in which they were searching by eliminating not only the mass (or weight) of the blocks as a relevant quantity, but

also by focusing exclusively on the x-direction. In the case(s) where the consideration of the mass simply faded away without an explicit rejection (such as with Ana and Sally, and possibly Igor), the implication is that the mass idea was not actually fruitful; that is, it did not help in finding a solution to the problem at all. As such, the quantity was simply not worth carrying around in working memory (or the analog to "working memory" for a group of participants), and it simply was no longer discussed.

Mass or weight should be contrasted with other quantities, such as the friction between block A and block B, which were pervasive throughout groups' discussions once they were suggested. The existence of the three written solutions may have been a driving factor in what was kept and what was dropped, as Fay, Marco, and Omar were the only participants to explicitly drop the weight of the blocks before reading the solutions. The other groups may have realized that since none of the solutions mentioned the mass or weight of the blocks, it was irrelevant for this problem. Additionally, solution #2 mentioned the friction between the blocks, which may have triggered the importance of that quantity for a couple of groups (Ana and Sally as well as Claudette, Earl, and Gustov). However, note that solution #2 was also rejected by Winfred because he thought that solution did **not** attribute a non-zero value to the friction between the blocks.

What this implies is that providing written solutions to problems may be useful in helping reasoners focus on the relevant quantities for solving the problem. However, the solutions may not be correctly interpreted by those reasoners. Further consideration needs to be given to what interferes with the correct understanding of a written solution (or verbal explanation), but all signs from representing this problem with possibility sets points to the idea that reasoners are likely to interpret solutions in terms of the possibilities they are currently considering. This reality makes successful communication between people with different possibility sets particularly difficult to accomplish; only Walter showed any success achieving this feat, although his groupmate Otto did not accept it (possibly thanks to a verbal miscue by Walter). Even Christobal, a graduate student, was unable to correctly interpret solution #2, which used a two-block solution rather than Christobal's single-block solution. Apparently, successful communication between participants with different possibility sets requires a careful negotiation of which possibility space is going to be explored.

In the case of the selection task, the participants were informed that one of the three written

solutions was correct. Since none of those solutions contained information about the mass of the blocks, the solutions implicitly communicated that the mass of the blocks was not relevant. Given the difficulty in communicating ideas across possibility sets, this may be a hopeful sign and a possible benefit from tasks of this sort, or indeed of worked examples in general. Future studies should investigate whether written solutions could be successful at communicating to students the relevant physical quantities in a problem.

6.1.12 Discussion: errors at the interface between the blocks

The "Two Blocks" problem proved to be especially fruitful for understanding the reasoning that participants were using because it motivated most of the participants to attempt to solve the problem on their own before or while checking the proposed solutions. These attempts revealed a wide range of errors and insights, which have been described above in their respective sections. Many of these errors occurred at the interface between the blocks, and certain patterns emerged indicating that different participants made different errors, although some of the errors were committed by more than one participant. The goal here is not to create a taxonomy of the errors that are possible when considering the interface between the blocks for this solution, but rather to demonstrate how the possibilities framework can facilitate the identification of errors, even when those errors all result in the same incorrect answer. Below, different errors that the participants made are repeated, along with suggestions for possible interventions.

Forgetting to flip the reciprocal force direction

Ike hinted that he remembered that according to reciprocity, the friction force from block A on block B would be equal to in magnitude but opposite in direction to the friction force from block B on block A; however, in applying reciprocity, he incorrectly forgot to mention the direction of the reciprocal force, which Dolly then used with an incorrect direction attached to it. This possibility set originally appeared as Figure 6.10 and is repeated here as Figure 6.54.

The possibility sets in Figure 6.54 demonstrate how many items were in working memory that needed to be considered. Because working memory is limited in scope, it is especially important to use external modes of information storage. Therefore, this error is the result of not being explicit

$\vec{F}_{net,onA} = \vec{F}_{Amir} + \vec{F}_{BonA}$		
$\vec{F}_{net,onA}$	$\vec{F}_{Amir}$	$\vec{F}_{BonA}$
(0 N)	20 N $\rightarrow$	20 N $\leftarrow$
(0 N)	20 N $\rightarrow$	$[\neq(20\text{ N }\leftarrow)]$

$\Downarrow$

Reciprocity: $\vec{F}_{AonB} = -\vec{F}_{BonA}$			
$|\vec{F}_{BonA}|$	$\hat{F}_{BonA}$	$|\vec{F}_{AonB}|$	$\hat{F}_{AonB}$
20 N	$\leftarrow$	20 N	
20 N	$\leftarrow$	$[\neq 20\text{ N}]$	
Other possibilities not considered			

$\Downarrow$

$\vec{F}_{net,onB} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$			
$\vec{F}_{net,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{AonB}$	$\vec{F}_{table}$
$(\vec{0}\text{ N})$	7 N $\leftarrow$	20 N ($\leftarrow$)	27 N $\rightarrow$
$(\vec{0}\text{ N})$	7 N $\leftarrow$	20 N ($\leftarrow$)	$[\neq 27\text{ N }\rightarrow]$

Figure 6.54: Error #1: Forgetting to flip the direction of the reciprocal force.

about each step of reasoning, specifically by not drawing a sufficient diagram. The diagram that Ike and Dolly used was very limited in scope, and Ike did not draw the arrows for the forces he described as he reasoned with them. In this case, a suitable intervention could be to ask the participants to repeat their reasoning, explicitly drawing arrows on the diagram during each step.

Not invoking reciprocity

Sally (see Figure 6.25) and Earl (with Claudette's help, see Figure 6.34) both carefully worked through the situation for block A, but then treated the resulting force from block B on block A as if it were the only friction at the interface, rather than one of a reciprocal pair. This error led to an incorrect application of the same force to block B as block A, summing the force on block B with Barbara's pulling force. This mistake culminated in the incorrect result that block B would experience a friction force of 27 N to the right due to the table. The difference between this error and the previous one (where the direction of the reciprocal force was not explicitly mentioned) is that in this case reciprocity was never even suggested. The reasoners simply did not consider it, treating the friction between the blocks as a single force. This reasoning is represented by Figure 6.55.

$\vec{F}_{net,onA} = \vec{F}_{Amir} + \vec{F}_{BonA}$		
$\vec{F}_{net,onA}$	$\vec{F}_{Amir}$	$\vec{F}_{friction\ between\ blocks}$
0 N	20 N $\rightarrow$	(20 N) $\leftarrow$
0 N	20 N $\rightarrow$	$[\neq (20\ N) \leftarrow]$

$$\downarrow\downarrow$$

$\vec{F}_{net,onB} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$			
$\vec{F}_{net,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{friction\ between\ blocks}$	$\vec{F}_{table}$
(0 N)	7 N $\leftarrow$	(20 N) $\leftarrow$	27 N $\rightarrow$
(0 N)	7 N $\leftarrow$	(20 N) $\leftarrow$	$[\neq 27\ N \rightarrow]$
Other possibilities not considered			

Figure 6.55: Error #2: Not invoking reciprocity for the friction force between the blocks

Here, the error is a missing possibility set corresponding to the law of reciprocity, which was acceptable for the reasoner because of incomplete notation about the forces acting on the blocks.

Any intervention would need to inform or remind the reasoner of this law and encourage them to be explicit about both the actors and the recipients of the forces. By clarifying that it is impossible to include as a relevant quantity any force that is not acting on the system, and by invoking the law of reciprocity, all but the correct possibilities should be eliminated from consideration.

Neglecting block A

One way of simplifying the problem and its strain on working memory is to either implicitly ignore or explicitly neglect block A. If a reasoner is able to convince himself or herself that only block B is relevant and therefore can exclude all possibilities related to block A, the "Two Blocks" problem becomes much simpler. Eduard explicitly neglected block A in Figure 6.51, and that possibility set is repeated here as Figure 6.56.

$\vec{F}_{net} = \vec{F}_{Barbara} + \vec{F}_{AonB} + \vec{F}_{table}$			
$\vec{F}_{net}$	$\vec{F}_{Barbara}$	$\vec{F}_{AonB}$	$\vec{F}_{table}$
0 N	$\leftarrow$		$\rightarrow$
0 N	$\leftarrow$		$[\neq\rightarrow]$

Figure 6.56: Error #3: Neglecting block A when reasoning about block B.

In this case, an intervention in the form of a discussion was actually performed. In that intervention, the researcher asked Eduard to consider a situation where the two blocks were replaced with a single, larger block. This discussion helped Eduard recognize that the friction between the two blocks was actually a relevant quantity. While Eduard did not propose a completely correct chain of reasoning, it did move his possibility sets in the correct direction by revealing to him that he needed to consider more than just block B. A proper intervention should pursue this realization in more depth, perhaps by asking how it could be possible to keep block B from moving while Barbara was applying a 7 N force to the left on a frictionless table. Again, by fleshing out possibilities, an intervention should encourage the reasoner to consider more possibilities, specifically those that include block A.

Friction opposes the pulling force (analogical reasoning)

While Otto's reasoning in Figure 6.17 and Claudette's reasoning in Figure 6.33 may appear to share a possibility set with Eduard, their reasoning was in fact very different. While Otto and Claudette did consider block A, they only did so for analogical purposes. That is, they reasoned that because block B applied a friction force on block A to the left, by analogy the friction force due to the table on block B would have to be to the right to oppose Barbara's pulling force. Similarly, Hugo, Josephine, and Teddy indicated that friction should alway oppose the pulling force, as shown in Figure 6.40. However, that group did not explicitly use that reasoning for block A (although the definitiveness with which they stated the rule they used hints strongly that they could have also applied it to block A). This generic error, which greatly simplified this problem, is represented in Figure 6.57.

$\hat{F}_{pulling}$	$\hat{F}_{friction}$
$\hat{k}$	$-\hat{k}$
$\hat{k}$	$[\neq -\hat{k}]$
Other possibilities not considered	

Figure 6.57: Error #4: Following the rule that friction opposes an ill-defined "pulling force".

Reasoners committing this error would eliminate nearly every possibility from their search space by reducing the number of relevant quantities to just a pulling force and the friction force of interest. However, one of the big problems with this reasoning is that the "pulling force" is not well-defined. To address this problem, a possible intervention could be to suggest a situation in which there are two pulling forces on the same block and then relating that situation to this one, which has a pulling force and another friction force acting on block B. Such an intervention may encourage the reasoner to consider more quantities and therefore more possibilities in their reasoning.

Other flesh-out errors

In addition to these errors, which conform to mistakes that are easy to identify, there was at least one instance where the reasoner stated that the friction between the blocks was a relevant and important

quantity, and yet it was unclear to the reasoner exactly how to deal with it. Gaston provided the clearest example of this class of errors which are broadly termed "flesh-out errors," as shown in Figure 6.47, which is repeated here in Figure 6.58. Apparently, he was possibly treating the friction as a single force, but unlike the reasoners who committed only that error, he did not explicitly treat the blocks separately and therefore hesitated, uncertain how to account for it.

$\vec{F}_{net,onB} = \vec{F}_{Barbara} + \vec{F}_{table} + \vec{F}_{AonB}$				
$\vec{F}_{net,onB}$	$\vec{F}_{Barbara}$	$\vec{F}_{Amir}$	$\vec{F}_{table}$	$\vec{F}_{friction\ between\ blocks}$
$\vec{0}$ N	7 N $\leftarrow$	20 N $\rightarrow$	$\leftarrow$	> 20 N
$\vec{0}$ N	7 N $\leftarrow$	20 N $\rightarrow$	$\leftarrow$	$[\leq 20$ N$]$
$\vec{0}$ N	7 N $\leftarrow$	20 N $\rightarrow$	$\vec{0}$ N	13 N
Other possibilities not considered				

Figure 6.58: One instance of Error #5: Errors in fleshing out the possibilities

This error did not itself lead the reasoner to conclude that the direction of the friction force due to the table on block B was to the right. Instead, the reasoning that Gaston provided here was not solid enough to cause him to revoke his previous conclusion (which was most likely a result of Error #4, applying the "rule" that the friction force opposes the pulling force). Also unlike the other errors, where reasoners reduced the space of possibilities that they were considering or, in the case of the first error, committed an error due to the limitations of working memory, this error for Gaston was the result of trying to identify *more* possibilities. That distinction sets this error apart. Fay, Marco, and Omar, in solving this problem, also sought possibilities; however, in the case of that graduate-level group, they had clearly identified the relationships that were relevant for the problem and the role that the friction between the blocks played in those relationships. Gaston, on the other hand, was exploring and trying to establish possibilities without having first identified any of those things. That process led to discovering possibilities to which he did not have particularly strong attachment and that he did not completely flesh out. One possible intervention in this case would be encouragement for the activity, while specifically directing the reasoner to attend to the relevant quantities and the relationship between those quantities. Such an intervention could help the reasoner focus on what would make the friction force due to the table zero or point in either direction, much as Fay, Marco, and Omar did correctly.

Summary: errors at the interface between the blocks

Five distinct errors for the "Two Blocks" problem could be identified by using the possibilities framework. Each of these errors prevented the reasoner from drawing the correct conclusion that the friction force due to the table on block B should be to the left, but they also each carry different instructional implications. For example, the intervention that one would use in the case of someone treating friction as a single force is different than the intervention for someone completely neglecting block A. Thinking about student errors in terms of the possibilities they allow or reject is very helpful; it makes clear that the first of the five errors listed above is a mistake in working memory due to not being explicit, while the next three are all improper reduction of the possibility space (possibly to make the problem easier to think about), and the final error is an instance of someone actually *trying* to expand possibility space but not doing so correctly. Instruction should be tailored to addressing the behaviors that the reasoners use, especially encouraging them to allot full consideration to all possibilities.

6.1.13 Summary

The participants' responses to the "Two Blocks" problem, when considered in terms of the possibilities that they were considering, convey a host of information. For example, they indicate the quantities that the participant believed to be relevant and hint at how those quantities may be reinforced through discussion or written solutions. The responses also provide valuable information about the kinds of errors the participants made. While there were five different types of errors participants made with specific regards to the interface between the two blocks, sometimes more than one participant committed the same error, and it is likely that students solving the "Two Blocks" problem would make errors that fall into those categories. While the goal was not to create a list of the possible errors that reasoners could make for this problem, knowing about the reasons for the participants' errors is especially important because it informs the possible instructional interventions that teachers could implement to help students efficiently explore more possibilities.

6.2 The "Ball In Motion" Problem

Each group session began with the "Ball in Motion" problem, which is included in the appendix and repeated here for continuity.

> A ball was in motion for a long period of time. Sometime during this period of time, it experienced a non-zero net force for some finite time interval Δt. While this force was acting, the ball's velocity did not change. Explain how this situation is possible.

This problem is unique in many respects. For one, questions that ask students to "explain how this situation is possible," which directly require students to explore the possibilities that are afforded by the information given in a problem, rarely occur in physics courses, if at all. Another reason this problem is unique is that it does not have a simple answer. Indeed, the answer to this problem depends, as we will see, on one's choice of system and assumptions about the nature of the objects in physics problems. These unique aspects were intentionally considered as strengths during the creation of this problem; the goal was to make this problem unique enough to encourage the participants in the study to verbalize their ideas and reasoning. I did not evaluate whether the participants got this problem "correct," although one of the given solutions is clearly incorrect, and the choosing of that solution may indicate a number of reasoning errors, as we will see.

Below are the three solutions that the participants were given and the possibility sets that represent them. Then, the transcripts as provided for each of the groups of participants along with descriptions of their reasoning framed within the possibilities framework. Some patterns of participants' reasoning were similar to those seen during the "Two Blocks" problem, while other interesting features emerged because of this particular problem. For example, the participants differed in the kinds of reasoning they used to choose between the possible solutions, their use of the momentum principle to solve this problem, and their handling of the possibility that the mass of the ball could change.

6.2.1 The given solutions

"Ball in Motion" solution #1

The first written solution reads as follows:

> The momentum principle states that $\vec{F}_{net}\Delta t = \vec{p}_f - \vec{p}_i$. Because a non-zero net force acted, $\vec{p}_f - \vec{p}_i$ must be non-zero. However, since the velocity didn't change, the ball's momentum didn't change. Therefore, we have a contradiction, and the situation is not possible.

Solution #1 claims that there is no possible solution to this problem, because there is a contradiction between two statements: one that says that the momentum of the ball must have changed and the other which says that the momentum of the ball did not change. This reasoning is represented by a pair of sequential possibility sets where all of the rows are grayed-out in the second set; that is, every possibility is eliminated. Figure 6.59 represents this reasoning.

$\vec{F}_{net}\Delta t = \vec{p}_f - \vec{p}_i$				$\vec{p} = m\vec{v}$		
$\vec{F}_{net}$	Δt	$\vec{p}_f - \vec{p}_i$	$\rightarrow\rightarrow$	$\Delta\vec{p}$	Δm	$\Delta\vec{v}$
$\neq \vec{0}$	(> 0)	$\neq \vec{0}$		$\neq \vec{0}$		$\vec{0}$
$\neq \vec{0}$	(> 0)	$[\vec{0}]$		Other possibilities not considered		

Figure 6.59: The possibility set representation of solution #1 of the "Ball in Motion" problem.

In the second possibility set, the relationship that is utilized is the definition of momentum, which in this case is just $\vec{p} = m\vec{v}$ and does not include γ (as a matter of fact, some of the participants indicated that they were thinking of relativity, but this was usually to reflect upon how the mass of the ball could have changed; no participant explicitly mentioned γ). Formally, the correct physical relationship for this problem is $\Delta\vec{p} = \vec{v}\Delta m + m\Delta\vec{v}$, which considers the change in momentum as well as the change in mass and is shown in Figure 6.60.

However, as Figure 6.60 demonstrates, representing the chain rule here is particularly bulky and obfuscates the simple point that the change in the mass of the ball is not being considered. And, by not considering this change in mass, the solution overlooks some possibilities and concludes that there is a contradiction and no possible solution. Because no interesting information is gained through

$\Delta\vec{p} = \vec{v}\Delta m + m\Delta\vec{v}$				
$\Delta\vec{p}$	$\vec{v}$	Δm	m	$\Delta\vec{v}$
$\neq\vec{0}$	$(\neq\vec{0})$		$(\neq 0)$	$\vec{0}$
Other possibilities not considered				

Figure 6.60: The explicit possibility set for the chain rule applied to the definition of momentum for solution #1 of the "Ball in Motion" problem.

the use of the formal relationship, throughout this section, I will continue to use the definition of momentum and imply the chain rule form of that definition, omitting the two implied quantities of "mass" and "velocity," which no participant ever suggested could possibly be non-zero.

"Ball in Motion" solution #2

The second written solution reads as follows:

> Because the velocity didn't change, the ball's momentum must not have changed either. However, a non-zero net force acted on the ball. Therefore, some additional force must have opposed the net force to allow the ball's velocity to remain constant. For example, air resistance or gravity could have opposed the net force.

This second solution essentially subverts the definition of "net force," expanding the possibilities afforded by the first solution in a physically illegal (but potentially attractive) way. As can be seen in Figure 6.61, solution #2 flips the order of the reasoning when compared to solution #1. It first uses the definition of momentum, not considering the possibility that mass could change, to draw a conclusion about the change in momentum. Then it applies that result to the momentum principle.

$\vec{p} = m\vec{v}$			$\vec{F}_{net}\Delta t = \Delta\vec{p}$			
$\Delta\vec{p}$	Δm	$\Delta\vec{v}$	$\vec{F}_{net}$	Δt	$\Delta\vec{p}$	$\vec{F}_{other}$
$\vec{0}$		$\vec{0}$	$\vec{F},\neq\vec{0}$	(>0)	$\vec{0}$	$-\vec{F}$
$[\neq\vec{0}]$		$\vec{0}$	$\vec{F},\neq\vec{0}$	(>0)	$\vec{0}$	$\neq-\vec{F}$
Other possibilities not considered			Other possibilities not considered			

$\rightarrow\rightarrow$

Figure 6.61: The possibility set representation of solution #2 of the "Ball in Motion" problem.

The result of the reasoning in solution #2 is that an additional force is created to cancel out the net force that is acting on the ball, essentially redefining the entity to which the "net force" in the problem referred. When viewed through the lens of physics, which fully defines its terminology, redefining a quantity so as to prevent an impossibility is thoroughly inappropriate. Such a redefinition is an attempt to escape from the parameters of the problem as defined. However, because some reasoners may consider that the definition of the words "net force" is negotiable, this solution provides such reasoners with an attractive option.

"Ball in Motion" solution #3

The third written solution reads as follows:

> The momentum principle states that $\vec{F}_{net}\Delta t = \vec{p}_f - \vec{p}_i = m_f\vec{v}_f - m_i\vec{v}_i$. Because the net force and Δt were non-zero, $m_f\vec{v}_f - m_i\vec{v}_i$ must be non-zero. If $m_f = m_i = m$, then $m(\vec{v}_f - \vec{v}_i) = 0$ because $\vec{v}_f = \vec{v}_i$. Therefore, m_f must be different from m_i; the mass of the ball must have changed while the net force was acting on it.

Solution #3 is another attempt to expand the possibility space that is defined in solution #1. Solution #3 takes an explicitly mathematical approach, first stating the momentum principle and then showing how the only way the momentum could have changed (which is necessary for the momentum principle to hold), the mass of the ball must have changed. This reasoning is demonstrated in Figure 6.62.

$\vec{F}_{net}\Delta t = \vec{p}_f - \vec{p}_i$		
$\vec{F}_{net}$	Δt	$\vec{p}_f - \vec{p}_i$
$\neq \vec{0}$	(>0)	$\neq \vec{0}$
$\neq \vec{0}$	(>0)	$[\vec{0}]$
Other possibilities not considered		

$\rightarrow\rightarrow$

$\vec{p} = m\vec{v}$				
$\Delta\vec{p}$	m_f	m_i	$\vec{v}_f$	$\vec{v}_i$
$\neq \vec{0}$	M	M	V	V
$\neq \vec{0}$	M_2	$M_1, \neq M_2$	V	V

Figure 6.62: The possibility set representation of solution #3 of the "Ball in Motion" problem.

The reasoning in Solution #3 explicitly permits only one possibility. This possibility is permissible by both the momentum principle and the definition of momentum. When using purely deduction for this problem, these are the only relevant relationships; whether a ball's mass can

physically change is not part of the endeavor of using deductive reasoning on a model. Thus, solution #3 is the only correct *deductive* solution to this problem, although if one uses reasoning other than pure deduction (as is often expected in physics), solution #1 could stand. Solution #2 is not correct.

6.2.2   Using only mathematics (Eduard)

Eduard's reasoning about this problem was interesting because while he considered the physical system by imagining what could happen, he was able to think of the possibility that the mass could have changed. He then explained this by using the momentum principle and definition of momentum, much like solution #3. As before, Eduard was asked to say aloud what he was thinking as he worked through this problem because he was not part of a group. Also, the researcher debriefed Eduard regarding this problem after he had attempted every problem. This discussion is included in this section as well.

> **Eduard**: Okay, so. (Reading) "The ball is in motion, during this period of time it experienced a nonzero net force at some finite time interval T, the velocity didn't change. Okay." Okay, I am trying to think of ways that force can act upon a ball and not change its velocity. Solution #1 says that it's not possible. And my original idea was that it kept the same momentum.
>
> **Researcher**: So what are you thinking about right now?
>
> **Eduard**: I'm just thinking of ways that could happen, like I don't know, like, if you threw a ball up in the air and it is experiencing a downward force from the earth, but then (he draws a ball in the air, and provides a downward arrow representing the gravitational force) but it says net force, so I was thinking then like if it then experienced a net force, [if] it experienced an upward force for some reason (drawing an upward arrow beside the downward arrow), that would make the net force zero (he crosses off the arrows), so – objects in motion will stay in motion unless acted on by another force.

At this point, Eduard was visualizing an example, trying to think of a possible way that the net force could be non-zero while making the velocity change. He thought of a ball that he threw into

the air. That ball would have a downward force from the earth, but adding an appropriately-sized upward force on the ball would make a zero net force. His repeating Newton's First Law is of dubious importance; it is unclear whether he was repeating it to show that he thought that adding an appropriate upward force (so that the net force would be zero) will not affect the ball's motion, or whether the only way to affect the ball's motion would be by applying that upward force. He simply did not provide enough information here to create an accurate portrayal of his reasoning. However, at this point he was silent for 11 seconds as he re-read the problem statement. When he began speaking again, he came up with the idea that the mass of the ball could have changed. Note that he had still not looked at the written solutions that were provided with the problem.

> **Eduard**: The only way I could see it happening is it some kind of mass was added to the ball to change the – keep the momentum, the momentum will be different but the mass [probably means velocity] will be the same. Uh, MF equals... (trails off). Yeah, that's what I just said (pointing to solution #3), mass on the ball must have changed while the net force is acting on it, yeah, yeah that's my answer. Yeah, what's wrong with this one (solution #1) is that it doesn't take into account that change in mass, so that one is wrong. And this is wrong (pointing to solution #2) because your velocity can stay the same and your momentum can change if you – yes, that's simple enough, I don't think I need to elaborate on that.

To be precise, Eduard specified that some mass had to have been added to the ball, which is only one of two possibilities regarding changing mass; it could have just as easily decreased. He may have eliminated the possibility that the mass of the ball could have decreased because he felt it was physically impossible, but we have no evidence of that. He may just as easily have only pointed out the first of the possibilities that he considered. Figure 6.63 represents his reasoning.

Note that his verbalization only implies that he considered the momentum principle, but it does seem as though he was using it to draw the conclusion that the momentum of the ball had to change. When the researcher returned to this problem for a discussion at the end of his session, Eduard clarified how the momentum principle justifies this conclusion.

> **Eduard**: (After Researcher asked if he had a reason for picking solution #3 over solution

$\vec{F}_{net}\Delta t = \vec{p}_f - \vec{p}_i$				$\vec{p} = m\vec{v}$		
$\vec{F}_{net}$	Δt	$\vec{p}_f - \vec{p}_i$		$\Delta\vec{p}$	Δm	$\Delta\vec{v}$
$\neq \vec{0}$	(>0)	$(\neq \vec{0})$	$\rightarrow\rightarrow$	$\neq \vec{0}$	$+$	$\vec{0}$
$\neq \vec{0}$	(>0)	$[(\vec{0})]$		$\neq \vec{0}$	$[not\ +]$	$\vec{0}$
Other possibilities not considered				Other possibilities not considered		

Figure 6.63: Eduard's suggestion that the mass of the ball must increase.

#1, that it was possible for the ball's mass to change) Um, yeah, I mean I could give an example, not necessarily with legitimate numbers but uh, same thing like with the ball of clay and the door (points to a drawing he made on the whiteboard of that situation). Like if the door was travelling in this direction, and the clay was travelling in this direction, and it's, I mean I guess, I don't know, hang on. Better example. (Reading) "The ball was in motion for a long period of time." I guess the force, let's see. (Writes and says) P final equals P initial plus F net Δt, so that means V times M final, final, V initial M initial plus F net Δt, that would mean that uh since velocity stays the same, I can get rid of initial here (erases subscripts on the momenta). So yeah. So basically, the change in mass would just have to be proportional to the amount that this is put in (circles "$F_{net}\Delta t$" on whiteboard, see Figure 6.64), if that makes sense to you.
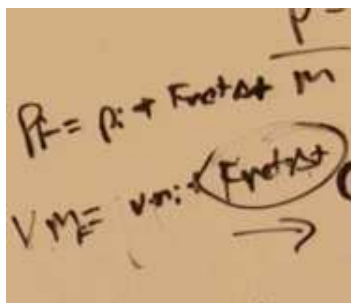
Figure 6.64: Eduard's whiteboard showing the momentum principle (without vectors) and indicating that the mass must change.

Initially, Eduard tried to relate this problem to an earlier problem that he solved (which involved choosing between a lump of clay and a rubber ball to close a door). After that attempt failed, he

decided to turn to the momentum principle and wrote it out correctly (although without vector symbols) on the whiteboard, even indicating that the change in mass would need to be proportional to "$F_{net}\Delta t$," an insight that is correct with the proportionality constant being $1/v$. In other words, Eduard derived the relationship that for the velocity to stay constant, the impulse would be proportional to the change in mass, something that even solution #3 did not propose. While it is mathematically correct, this statement does not explain why he did not reject the idea that the mass of the ball could have changed. Indications from how Eduard solved the problem show that he did not use purely mathematical reasoning, and that he was actually thinking about physical situations. To follow up on how the two were related, the researcher asked whether he was explicitly considering an example as he was reasoning through this problem.

> **Researcher**: Sure, sure. Um, when you were doing this problem, did you try to think of an example? Or did you, did you, were you happy with what was written with the momentum principle?
>
> **Eduard**: Actually, I just kind of thought about it on my own. Like, a comet was travelling and it's desintegrating as it goes. Um, I mean of course its speed is going to, if it didn't lose any weight, its speed would increase as it reached the earth's surface, whereas if it lost the right amount of weight as it's falling towards the earth's surface, then its velocity can potentially be the same because the force acted on it by the earth could remain proportional to it, so, like I mean if it was 5 kg, and it was at a distance from the earth's surface where it was, hold on, I don't know, as it gets closer then the force gets stronger cause it's MGH, so yeah, as long as it's got the same proportions it's possible.

The fact that he did not mention the asteroid during his think-aloud protocol of this problem is somewhat disconcerting, since that indicates either that he was not saying everything he was thinking while he was working on this problem (which would not be surprising, especially given the 11-second silence right before he announced his answer to the problem) or that he was thinking of the asteroid at some point between when he solved the problem and when he responded to this question, and he was reporting false data. We can not be sure which it is, but we can treat his

explanation as if it were given to a group member at the moment it is being said and assume that it represents his reasoning at this moment in the session. As such, we can represent that reasoning, as seen in Figure 6.65.

$\vec{F}_{net}\Delta t = \vec{p}_f - \vec{p}_i$				$\vec{p} = m\vec{v}$		
$\vec{F}_{net}$	Δt	$\vec{p}_f - \vec{p}_i$		$\Delta\vec{p}$	Δm	$\Delta\vec{v}$
$\neq \vec{0}$	(> 0)	$(\neq \vec{0})$	→→	$\neq \vec{0}$	$not -$	$+$
$\neq \vec{0}$		$[(\vec{0})]$		$\neq \vec{0}$	$not -$	$not +$
Other possibilities not considered				$\neq \vec{0}$	$-$	0
				Other possibilities not considered		

Figure 6.65: Eduard's reasoning about a comet whose mass could change.

Interestingly, he solved his own example incorrectly and states that the comet [sic] needed to lose mass in order to maintain its velocity. Physically, if an object experienced a force in the direction of its motion, it would need to *gain* mass to continue moving at a constant velocity. The example Eduard chose to discuss is particularly difficult because it invokes images of meteors breaking apart and experiencing both wind resistance (which is proportional to the square of the meteor's velocity) and gravitational force, in a complicated system. Nonetheless, the general idea that he was trying to convey is absolutely appropriate for this problem: if an object's mass changed proportionally to the impulse it experiences, then it should experience constant velocity. Eduard chose to apply this reasoning to a comet rather than to a ball, which one does not normally think of as having a variable mass (see, e.g., the pilot study discussed in Section 4.3).

6.2.3   Belief bias and the unchanging mass (Claudette, Earl, and Gustov)

This group explicitly used the momentum principle in solving this problem, even calling it out upon first reading the problem statement. In contrast to Eduard, who cited the momentum principle as well, this group explicitly rejected the notion that the mass of the ball could change. After this rejection, they struggled for a while trying to determine whether solution #2 was possible before finally deciding that solution #1 was correct.

Gustov: (Reading) "A ball in motion for a long period of time, a ball was in motion

for a long period of time. Sometime during this period of time it experienced a nonzero net force for some finite time interval, Δt." So it sounds like we are dealing with the momentum principle. (Reading) "While this force was acting, the ball's velocity did not change. Explain this, how the situation is possible." So while this force was acting, the ball's velocity did not change.

Earl: Let me get one of the solutions.

Gustov: So, while this force was acting, the ball's velocity did not change. So let's see here.

Claudette: Well, if you want to read solution #1, and then.

Gustov: Yeah, we just need to compare the differences and see what.

Earl: Yeah, this one (probably solution #1?) has F net Δt is equal to PF minus PI.

Gustov: So the momentum principle.

Claudette: So yeah, they're using the momentum principle.

Gustov: Yeah so,

Earl: Here it said –

Gustov: Are these using the momentum principle as well? (Pointing to other solutions) –

Claudette: They (solution #2) just said that, um, because the velocity didn't change, the ball's momentum must not have changed either, however.

Earl: Yeah, that's right.

Claudette: Yeah, but they (solution #2) said that however a nonzero net force acted on the ball therefore some additional force must have opposed the net force to allow the ball's velocity to remain constant. For example air resistance or gravity could have opposed the net force.

Earl: Well, right here it (solution #1) says that, um, I do agree with this, it says that at time Δt, simply because, because a nonzero net force acting upon, PF minus PI,

no, PF minus PI must be nonzero, however since the velocity didn't change, the ball's momentum didn't change. Therefore we have a contradiction and the solution is not possible. The momentum is equal to mass times velocity, right? (Gustov and Claudette indicate agreement). Now, the velocity didn't change, and the mass certainly didn't change.

Gustov: No (Agreeing).

First, Gustov read the problem for the group, and he immediately suggested that the momentum principle was important in solving the problem. As they did in the "Two Blocks" problem, they distributed the solutions and described them to the group as a whole. Earl went first, reading solution #1 and noting that it used the momentum principle. Gustov was interested in the differences between the solutions, so they tried to identify whether other solutions used the momentum principle as well. Claudette stated that solution #2 did, and she summarized that solution for the other participants in the group, who temporarily ignored her. Earl liked solution #1 and promoted it by explaining that the mass of the ball could not change, and so there is no way that the momentum could change without the velocity changing. By doing so, he fleshed out solution #1 (see Figure 6.59), which overlooked the mass of the ball. Figure 6.66 shows that Earl actually supplied information that was omitted in that solution – that the mass of the ball could not change.

$\vec{F}_{net}\Delta t = \vec{p}_f - \vec{p}_i$				$\vec{p} = m\vec{v}$		
$\vec{F}_{net}$	Δt	$\vec{p}_f - \vec{p}_i$		$\Delta\vec{p}$	Δm	$\Delta\vec{v}$
$\neq \vec{0}$	(>0)	$\neq \vec{0}$	$\rightarrow\rightarrow$	$\neq \vec{0}$	0	$\vec{0}$
$\neq \vec{0}$	(>0)	$[\vec{0}]$		Other possibilities not considered		

Figure 6.66: Earl's claim that the mass of the ball could not change.

There is no implicit negation in this case, because Earl stated a premise as a fact rather than providing a conclusion that depended on other premises. This decision to attribute a value to a physical quantity that was assigned no value in the problem statement or through a calculation or deduction is called "belief bias." A "belief bias" occurs when one allows ideas about the truth of the conclusion to affect his or her reasoning about the premises (see Section 2.6.2). As a reminder, the term "belief bias" is somewhat misleading: by stating that Earl used belief bias in this problem

to set the change in the ball's mass to zero, I am not claiming that Earl believed that it would be impossible for any ball (or object for that matter) to change its mass in every situation; rather, I claim that he decided that the mass of the ball in this problem did not change. By so deciding, he indicated a bias that affected the deductions he drew. Biases such as these are not "bad" or even necessarily "errors;" they are merely a method of reducing the possibility space that one is dealing with in a problem. In other words, "belief bias" is one heuristic that reasoners use to help solve problems. However, this bias can interfere with proper deduction, as seen from the fact that it prevented Earl from fleshing out every possibility for the definition of momentum.

Note the distinction between solution #1 and Earl's solution: in solution #1, all of the possibilities were not fleshed out because one of the physical quantities (the change in mass) is not considered. In principle, the hypothetical reasoner behind solution #1 could realize this and easily flesh out the other possibilities regarding the definition of momentum (and doing so would result in solution #3). However, Earl *was* considering that physical quantity, but by explicitly setting the value he apparently prevented himself (and perhaps the group as a whole) from considering the other possible values for that quantity. Predictably, this group rejected solution #3 when they read it, because that solution did not eliminate the possibility that the mass of the ball could change. However, before getting to solution #3, the group faced a different obstacle to overcome: understanding solution #2 and the net force applied to the ball.

Earl: PF minus PI is equal to F net. Now if F net ch-

Claudette: Well, well, okay, it (solution #2) says that there is an additional force, but it (seemingly referring to the problem statement) doesn't say that it was a net force, so like, this person said, you know, there could have been an additional force, but –

Earl: Not an F –

Claudette: A force acting back on it.

Earl: Could be the same.

Claudette: The same, so F net would be zero.

Earl: Yes. That's true. But they didn't say that it was an increase in the net force, it just said a force.

Claudette: (Crosstalk) It (again, seemingly referring to the problem statement) just says a force applied.

Earl: And, F, yeah, that makes sense.

Apparently, Claudette thought that the problem statement said that while the ball was in motion, a force acted on it, not a "**net** force." This misunderstanding may have led her to develop possibility sets that are reminiscent of those in solution #2. Her reasoning is represented by Figure 6.67.

$\vec{p} = m\vec{v}$				$(\vec{F}_{applied} + \vec{F}_{other})\Delta t = \Delta\vec{p}$			
$\Delta\vec{p}$	Δm	$\Delta\vec{v}$		$\vec{F}_{applied}$	$\vec{F}_{other}$	Δt	$\Delta\vec{p}$
$(\vec{0})$	(0)	$\vec{0}$	→→	$\vec{F}, \neq \vec{0}$	$-\vec{F}$	(>0)	$\vec{0}$
$[(\neq \vec{0})]$	(0)	$\vec{0}$		$\vec{F}, \neq \vec{0}$	$\neq -\vec{F}$	(>0)	$\vec{0}$
Other possibilities not considered				Other possibilities not considered			

Figure 6.67: Claudette's reasoning, which includes an additional force.

Essentially, Claudette considered the possibility that there were two forces acting on the ball that summed to make the net force zero. She was confused about what information was provided in the problem statement, and she acted as though it just said that there was some initial force on the ball. Such a scenario (although not what was actually written in the problem) would not eliminate the possibility suggested by solution #2. Temporarily holding on to solution #2, the group turned to the final written solution.

Earl: (To Gustov) What you got?

Gustov: Let me see, that this is the one that you think it is? (Taking solution #2 from Claudette).

Claudette: Well I, you want me, do you want me to read [solution #]3 out loud too?

Earl: Yeah, go ahead and read [solution #]3 out loud.

Claudette: (Reading, while Gustov reads solution #2 silently in the background) "The momentum principle states that F net Δt equals P final minus P initial, which equals M final V final minus M initial V initial. Because the net force and Δt were nonzero, M final V final minus M initial V initial must be nonzero. If MF-, if M final equals

M initial, which equals M, then M quantity V final minus V initial equals zero because V final equals V initial. Therefore M final must be different from M initial, the mass of the ball must have changed while the net force is acting on it." I don't think that's possible.

Earl: You can't really change the mass.

Claudette: Yeah.

Earl: You could change weight. But you can't change mass.

Gustov: So that one (solution #3) says that it changes the mass?

Claudette: The mass, yes.

Gustov: So you wouldn't need, that one is wrong.

Claudette: Yeah, that one is wrong.

As expected, the group rejected solution #3 very quickly. This rejection was not due to an error in the reasoning in the solution but instead due to its consideration of a possibility that the group had already dismissed. Obviously, the belief bias can interfere with the deductive process, preventing reasoners from ever considering a valid argument. This interference has powerful implications, which will be discussed in Section 6.2.12. The group next turned to the remaining two solutions and tried to figure out which one was correct.

Earl: So we got one that is possible, we got one that is, one that says the solution is not possible, so it is between [solution #]1 and [solution #]2.

Gustov: (Reading solution #2) "Because the velocity did not change..."

Claudette: I wouldn't say that it (the situation) is not possible.

Gustov: (Reading solution #2) "Because the velocity, the velocity did not change... however a nonzero net force acted on the block, therefore some additional force must have opposed the net force, which will allow the ball's velocity to remain constant (inaudible)." Well this one –

Earl: That one does, that one does sound –

Gustov: Doesn't say that it is in like –

Earl: It didn't say an increase of net force, it just said a force (Claudette indicates agreement).

Gustov: And it didn't say that was like, uh, in like outer space or something, when there's no gravity, it doesn't say that here so. (Reading solution #1) "The momentum principle... because a nonzero net force acted, it must be nonzero." Okay. "Because a nonzero net force acted, it must be nonzero." Yeah. "However since the velocity did not change, the ball's momentum didn't change. Since the, (inaudible), since the ball's velocity," so, though momentum is mass times velocity, right? (Claudette and Earl indicate agreement).

Claudette: But your mass didn't change.

Earl: They didn't say it changed.

Gustov: Yeah. So however since the velocity, so this is saying the velocity didn't change (Claudette indicates agreement), so this is saying that the velocity didn't change, then the momentum would stay the same (Earl indicates agreement). The ball's momentum didn't change. Yeah. Therefore, we have a contradiction and the situation is not possible, and the situation is not possible.

Gustov: Okay, so, so the ball's momentum didn't change. So does it say that it changes?

Earl: It doesn't.

Gustov: Because this is nonzero, the net force acted, so this is saying MV initial, or no, MV final minus MV initial must be nonzero. So it has to equal those two. And if the velocity didn't change. So these stayed the same, so do these, so this would be zero, so yeah, this would have to be zero, so this is saying that F net Δt equals zero and they're saying that, so, and then they're saying that F net, F net was a finite number, and there was some time. So this is nonzero, so there is no way that zero can equal zero, so I think that it is [solution #]1.

Gustov led the group carefully through solution #1, fleshing it out even more fully. After some discussion, Gustov indicated his support for that solution. His reasoning, a slight variation on the possibility sets Earl demonstrated, is shown in Figure 6.68.

$\vec{F}_{net}\Delta t = \vec{p}_f - \vec{p}_i$		
$\vec{F}_{net}$	Δt	$\vec{p}_f - \vec{p}_i$
$\neq \vec{0}$	$\neq 0$	$\neq \vec{0}$
$\neq \vec{0}$	$\neq 0$	$[\vec{0}]$
Other possibilities not considered		

$\rightarrow\rightarrow$

$\vec{p} = m\vec{v}$		
$\Delta\vec{p}$	Δm	$\Delta\vec{v}$
$\neq \vec{0}$	0	$\vec{0}$
Other possibilities not considered		

Figure 6.68: Gustov's fully fleshed-out solution, with the belief bias that the ball's mass does not change.

Only one hurdle remained to the group agreeing on this solution, and that was Claudette, who mentioned in the previous segment of transcript that she still thought the situation was possible. She was attracted to solution #2 because she believed the problem statement referred to a force rather than a *net* force. In the final segment from this problem, the group pointed out her error.

Claudette: But, but did they say that F net, or did they say some force?

Gustov: Yeah, they say F net, net force, F net.

Claudette: Oh, there was a nonzero net force?

Gustov: Yeah.

Claudette: Okay. 'Cause if they said there was a nonzero *force* –

Earl: Yeah.

Claudette: All right, well then.

Gustov: Because these two numbers do not equal zero, so that is just saying there is a contradiction because they're saying that this didn't change, so if these to say the same, so if the mass is one and the velocity is two, and then they didn't change. (Earl indicates agreement). You're still going to come out, that still equals zero. (Claudette and Earl indicate agreement).

Earl: But it says that there is a nonzero net force.

Gustov: Nonzero meaning that it is one or two (Earl indicating agreement) or something, so when there is, there is a time because it says that there is some finite time number, so there is some time interval, so this is not zero. The only way that this can be zero if one of these was zero. So [solution] #1?

Claudette: Sounds good.

Gustov: All right.

Upon clarification that the problem statement in fact mentioned a nonzero *net* force, Claudette eliminated the possibility that her sets had allowed and became willing to agree with Gustov, who once again stepped through solution #1. Claudette's refined reasoning is shown in Figure 6.69.

$\vec{p} = m\vec{v}$		
$\Delta\vec{p}$	Δm	$\Delta\vec{v}$
$(\vec{0})$	(0)	$\vec{0}$
$[(\neq\vec{0})]$	(0)	$\vec{0}$
Other possibilities not considered		

$\rightarrow\rightarrow$

$\vec{F}_{net}\Delta t = \Delta\vec{p}$			
$\vec{F}_{net}$	$\vec{F}_{other}$	Δt	$\Delta\vec{p}$
$\vec{F}, \neq\vec{0}$	$-\vec{F}$	(>0)	$\vec{0}$

Figure 6.69: Claudette's reasoning, removing the earlier possibility of an additional force acting on the ball.

The group agreed to select solution #1, which followed from valid deduction after their implementation of the belief bias that the mass of the ball could not change.

6.2.4 Looking for a mechanism (Arthur, Otto, and Walter)

While Claudette, Earl, and Gustov chose solution #1 by applying their belief bias to the problem through the use of the momentum principle and the definition of momentum (see Section 6.2.3), Arthur, Otto, and Walter never explicitly drew upon the use of the momentum principle or the definition of momentum to choose that solution. Instead, they eliminated the other two possible solutions and, still trying to come up with a possible explanation for the problem, chose the solution that argued for the situation being impossible. Their discussion revolved around trying to uncover a physical mechanism that would cause the mass of the ball to change, rather than around the mathematical relationships that would permit such a thing. This focus is especially apparent in

Otto's misrepresentation of solution #3 as suggesting that some force *caused* the mass to change.

> **Otto**: (Reading problem statement) "A ball was in motion for a long period of time. Sometime during this period of time, it experienced a nonzero net force for some infinite [sic] time period Δt. While this force was acting, the ball's velocity did not change. Explain how this situation is possible.
>
> **Walter**: What? (Silently re-reads problem statement) (silence)
>
> **Arthur**: It's another force in the opposite direction?
>
> **Walter**: No.
>
> **Otto**: That's what I would –
>
> **Arthur**: A nonzero –
>
> **Walter**: The ball has no mass?
>
> **Otto**: Well we can –
>
> **Arthur**: The ball does not exist (chuckles) –

After reading the problem statement, Arthur and Walter brainstormed some ideas. Arthur's reasoning was suggestive of that found in solution #2 (see Figure 6.61), although he had not yet seen any of the solutions and was just trying to identify possibilities. Walter rejected Arthur's suggestion immediately without providing any explanation why. He then proposed his own suggestion, which is represented in Figure 6.70.

m	$\Delta \vec{v}$
0	$\vec{0}$
Other possibilities not considered	

Figure 6.70: Walter's suggestion that the ball has no mass.

Of course, it is unclear how seriously the group (or even Walter himself) took Walter's suggestion that the ball has no mass, as he said it in a half-joking manner and Arthur followed up with a response that was clearly not intended to be taken seriously. Indeed, Walter did not provide any

sort of justification for his statement, only that the ball might not experience a change in velocity if it had no mass. If this were in fact a real suggestion, we would have no clue as to how he reached it. In any respect, the group quickly decided to move on and began looking at the written solutions.

> **Otto**: Well we can start looking at the solutions, and then narrow them down from that.
>
> **Arthur**: Yeah that would be best.
>
> **Otto**: (Reading solution #1) "The momentum principle states that F net Δt equals PF minus PI. Because a nonzero net force acted, PF minus PI must be nonzero. However since the velocity did not change, the ball's momentum did not change. Therefore we have a contradiction, and the situation is not possible."
>
> **Walter**: I still say the mass doesn't exist.
>
> **Otto**: (Reading solution #2) "Because the velocity didn't change, the ball's momentum must not have changed either. However a nonzero net force acted on the ball. Therefore some additional force must have opposed the net force to allow the ball's velocity to remain constant. For example, a resistance or gravity could have opposed the net force."
>
> **Walter**: Uh, No. It means, it's net force.
>
> **Otto**: If net force equals zero, it should include everything, all of the forces.
>
> **Walter**: It includes air resistance, yeah (Gestures drawing a huge X over the solution).
>
> **Otto**: (To Arthur) Do you agree with that? (Arthur affirms).

Otto read through the first two solutions. After the first one, the only response was Walter repeating his half-joking statement about the ball's mass being zero, reiterating the possibility in Figure 6.70. However, after the second solution is read, Walter immediately rejected it and was met with Otto's agreement. Otto then explicitly asked Arthur whether he agreed with that rejection (possibly because his suggestion earlier was precisely this solution). Arthur did agree, moving the group to exclude the quantity "another force" from their possibility sets.

> **Otto**: (Reading solution #3) "The momentum principle states that F net Δt equals PF minus PI equals $MF\,VF$ minus $MI\,VI$"–

Walter: Big calculation stuff.

Otto: (Reading) "Because the net force and Δt were nonzero, $MF\ VF$ - $MI\ VI$ must be nonzero. If $MF = MI = M$, then M equals that, equals zero because VF equals VI. Therefore, MF must be di–, must be different from MI; the mass of the ball must have changed while the net force was acting on it"

Walter: Okay, that was a lot of stuff (inaudible). Okay, you can look at this one (hands solution #1 to Arthur). (Walter looks at solution #3 with Otto) Okay, the momentum principle states that (trails off) (silence, with everybody looking at the solutions).

Otto: Velocity changed, this one is basically saying that the force caused the mass of the ball to change.

Arthur: So –

Walter: That is an interesting solution. But can it cause it to change?

Otto: I'm not thinking, I'm trying to think of a force that would cause the mass of anything to change. Unless you compressed it and made it smaller.

Arthur: It would still have the same mass, though, you can't get rid of the mass.

Otto: You can't –

Arthur: You can break it into something else but –

Otto: You can break it into two different things but but you can't get rid of it.

Arthur: Yeah that's –

Walter: Yeah, I am going with solution #1.

Otto: Yeah, because.

Walter: That [solution #3] sounds like interesting but, I am assuming you are going to be including all of the pieces at the end, (other participants affirm) –

Otto read solution #3 for the group, who were intrigued by it. However, Otto then misinterpreted the solution, attributing to it the claim that it was the force acting on the ball that caused the mass of the ball to change. This interpretation fit with the rest of the discussion among the group in

this session, much of which was about whether a force could cause a ball's mass to change, and if so, how. They clearly started looking for a *physical mechanism*, which solution #3 was careful not to require. So, instead of using the momentum principle and definition of momentum to solve this problem, the group approached this problem by thinking about the mass itself and whether it would be possible to eliminate all of the pieces.

Walter suggested that he was assuming that all of the pieces of the ball will be accounted for at the end, and because of that, he thought that it would be impossible for the mass of the ball to change (or perhaps, for a force to cause the ball's mass to change). This reasoning could indicate a sophisticated choice of system; if one defines the system (the "ball") as including every initial molecule of the ball, so that the final system is all of those particles regardless of where they are located, it would be impossible (by definition) for the ball's mass to change. Then, because the ball's mass could not change, it would be impossible for some nonzero net force acting on the ball to not cause its velocity to change. Figure 6.71 is a representation of this sophisticated reasoning, although it seems unlikely, given the evidence present in this transcript, that Walter actually adopted these possibility sets.

$\vec{F}_{net}\Delta t = \vec{p}_f - \vec{p}_i$				$\vec{p} = m\vec{v}$		
$\vec{F}_{net}$	Δt	$\vec{p}_f - \vec{p}_i$		$\Delta\vec{p}$	Δm	$\Delta\vec{v}$
$\neq \vec{0}$	(>0)	$\neq \vec{0}$	$\rightarrow\rightarrow$	$\neq \vec{0}$	0	$\vec{0}$
$\neq \vec{0}$	(>0)	$[\vec{0}]$		$\neq \vec{0}$	$\neq 0$	
Other possibilities not considered				Other possibilities not considered		

Figure 6.71: Possibility sets associated with selecting a system in which all of the mass is included.

Again, we have no indication that this group considered the momentum principle or definition of momentum at all. The group may have simply used an amalgam of imagination, intuition, and other reasoning to draw their conclusions. As the conversation continued, that explanation of their conclusions became more likely.

> **Otto**: I don't think a force can act on anything without any mass either so what you were saying about the mass not existing can't –
>
> **Walter**: Another thing was, like, I am surprised that none of them said anything about

compression, because I thought one of them would be like, oh guess what the ball, it's like all of the force was used to compress the mass –

Otto: Yeah, if the ball wasn't moving to begin with, and it was just pushing down against something –

Walter: No, but even then, that would, wouldn't that move the center of mass?

Otto: Well, yes I guess that's true.

Walter: But, oh well.

Otto: Solution #1 (taps the solution with his hand).

Walter: Solution #1.

Otto: That's a problem for another day.

Walter: There we go.

Arthur: Okay.

Otto: We'll go with solution #1.

Walter: Solution #1, yo. And then we have to find what's wrong with these other two.

Otto: But we have already –

Walter: Should we write it down?

Otto: Go for it, if you want to.

Walter: Okay, okay, let me see –

Otto: Because they have a recording of the way we did it (gesturing to the microphones on the table), so we don't absolutely have to write it down.

Walter: Yeah, that's true. (Reading solution #2) "Because the velocity didn't change, the ball's momentum must not have changed either." (Trails off) No, yeah, I guess you are right we don't have to write this down (caps the pen), I guess they already did it. I am just going to circle air resistance or gravity, and then have something, net force, like duh. And, uh, (Reading solution#3) "The momentum principle states" that some stuff happens, and some, the mass must've changed, and it is like, but aren't you taking

– I don't feel like trying to explain that one. That one I feel like would be confusing to explain, and I think we have already explained that one in words (Otto affirms), I don't want to try to write that down, man. Seems like tricky and difficult and hard.

The group had eliminated both solution #2 and solution #3, so they selected solution #1, the only solution remaining. However, before they committed to that selection, they continued to search for ways this situation could be possible. They talked about compression before deciding, "that is a problem for another day," which may have been a reference to this continued hunt for possibilities. Satisfied that they completed the task, they chose solution #1 by default. They never created an argument in favor of it, instead only providing evidence that they thought the other options were incorrect. Even when Walter considered writing down the errors that they found in the other solutions, he did not use mathematical or deductive arguments but instead referred back to the group's failed search for a mechanism.

As further evidence for this group not using the momentum principle or definition of momentum, consider the follow-up discussion that the researcher had with the group after they had completed all of their tasks for the session.

Researcher: As a group, you said that you did not like solution #3 because the momentum principle, something along these lines, and make sure you correct me if I mischaracterize it, the momentum principle takes into account all of the mass of the object (Walter affirms), um, so, uh, so what exactly do you mean by that with regard to this solution?

Walter: Well, assuming the mass of the object decreased, unless we're assuming that the mass was turned into energy, which I don't, which I cannot imagine a non-net (probably means non-zero net) force, because I guess we are assuming some kind of directional force, whether it was electrical net, or like, you know, some kind of like, directional force, rather than some magic force that somehow turns particles and to energy, because you can't really lose mass without losing momentum. And even then, if you changed them into energy, odds are it would move it.

Otto: In a sense, we are saying that there is no force you can apply to the ball that

would make the ball lose mass,

Walter: That we –

Otto: That we can think, that we know is physically possible,

Walter: That we can think of, or (crosstalk) that we have learned in our physics classes.

Otto: Or with, yeah, (crosstalk with Walter), with our two years of physics classes, it might be one we don't know. There may be one further on, in later physics we would know, but based on the knowledge we have learned so far, we don't know of any.

Walter: Yeah, what he said. I agree with that.

The researcher asked the group a follow-up question about why they had eliminated solution #3. He framed this question in terms of the momentum principle, but the group (beginning with Walter) reframed it in terms of the types of forces that could be applied and whether they could cause the mass to turn into energy. The closest the group came to mentioning the momentum principle is when Walter said that "you can't really lose mass without losing momentum," which is represented by the following Figure 6.72. His next statement contained the phrase "odds are it would move it," a very non-deductive statement that clearly did not appeal to any mathematical or physical relationship.

$\vec{p} = m\vec{v}$			
$\Delta \hat{p}$	$|\Delta \vec{p}|$	Δm	$\Delta \vec{v}$
	< 0	< 0	
	$not \ < 0$	< 0	
Other possibilities not considered			

Figure 6.72: The Possibility Set representation of Walter's statement that "you can't really lose mass without losing momentum."

Because the possibilities framework is meant to handle deductive rather than mechanistic relationships, most of the conversation between Arthur, Otto, and Walter cannot be represented with possibility sets. However, considering the conversation within the realm of possibilities is still illuminating. In the follow-up discussion, the group implied that they concluded that the situation was impossible *because they could not imagine a mechanism that would make it happen.* They conceded

that there may have been some mechanism that they had not yet learned, but the simple fact that they exhausted their list of possible mechanisms without discovering one that was satisfactory was sufficient to reject this situation. There is abundant evidence, therefore, that Arthur, Otto, and Walter were not performing deduction when they solved this problem, hinting that students are not always cued into thinking about deduction when they solve physics problems. Sometimes, as in this case, they may think about causal mechanism rather than fundamental principles, focusing on the former in instances where the latter is required.

6.2.5   Difficulty visualizing changing mass (Christobal, Dolly, and Ike)

Christobal, Dolly, and Ike decided very quickly that solution #3 was correct. However, when pushed, Christobal revealed that he wasn't particularly satisfied with that solution because he was having trouble visualizing a way to make the mass of the ball change while applying a net force without changing its velocity. An interesting discussion resulted between the three group members as they tried to come up with a physically viable way of causing this to happen. However, in stark contrast to Arthur, Otto, and Walter, this group did not list mechanisms but rather drew a very simple schematic to discuss possibilities. Christobal, Dolly, and Ike eventually tired of discussing the problem and agreed to solution #3, even though they had not completely resolved their initial difficulties with the problem.

> **Christobal**: (Reading the problem statement) "...the ball's velocity did not change. A ball is in motion for a long period of time and sometime during that period of time, it experienced a nonzero net force for some finite time interval Δt. While this force was acting, the ball's velocity did not change. Explain how the situation is possible."
>
> **Ike**: Oh, that is fun. So, now that we have read the problem let's look at the solutions, yes!
>
> **Christobal**: (Reading from solution #1) "The momentum principle states that F net Δt equals momentum final minus momentum initial. Because a nonzero net force acted, (stumbling over words), F final minus, momentum final minus momentum initial must be nonzero. However since the velocity did not change, the ball's momentum did not

change. Therefore we have a contradiction and the situation is not possible."

Ike: Uh, I am going to say that if the textbook says that something happens, and you say it didn't, you are probably wrong. So, I think that is wrong.

Christobal: (Reading solution #2) "Because the velocity did not change, the ball's momentum must not have changed either. However a nonzero net force acted on the ball. Therefore, some additional force must have opposed the net force to allow the ball's velocity to remain constant. For example a resistance or gravity could have opposed the net force." Well, it says nonzero net force, so I don't think you understand the concept (in unison with **Dolly**:) of net force.

Christobal and Ike quickly rejected the first two solutions. Ike rejected solution #1 because it contradicted the problem statement; that is, by stating that a certain situation *did* happen, saying that the situation did not happen was no longer allowed. Christobal rejected solution #2 because it misused the term "net force." The group did not look at solution #2 again, and neither did they give much more thought to solution #1.

Christobal: (Reading solution #3) "The momentum principle states that F net Δt is equal to P final minus P initial, MV, M final V final minus M initial V initial. Because the net force and Δt were nonzero, M final V final minus M initial minus V initial must be nonzero. If M final equals M initial is equal to M, then M change in velocity is equal to zero because V final is equal to V initial. Therefore, MF, M final must be different from M initial. The mass of the ball must've changed while the net force is acting on it."

Ike: That would be my guess.

Dolly: Special relativistic. I mean, that is the most reasonable explanation if you want to go into it, there is an object moving close to the speed of light that was acted on and couldn't go any faster. Therefore it all went into the mass.

Christobal: It could go a *little* faster.

Ike: I was actually thinking something like, you push the ball and a little bit falls off

(Dolly laughs), yeah. Okay, but are we agreed that solution #3 is the correct one because solution #1 says nuh-uh, you are lying, and solution #2 says nuh-uh, you are lying?

Dolly: (Crosstalk over last few words) I would have no problem with solution #1 as well.

Christobal: Yeah, sure.

Ike: (To Dolly) Wait, what was your other problem?

Dolly: No, no no no, I think solution #2 I didn't like, but that is different than solution #1. But I could go with solution #3. I mean (pause) –

Christobal: Sure, I like solution #3.

For a moment, it looked as though the group was going to be satisfied selecting solution #3 and moving on to the next problem. Dolly, like Omar in the other graduate student group (see Section 6.2.6), first mentioned that perhaps relativity could cause the mass of the ball to change, but after a brief discussion this possibility was dropped in favor of Ike's simpler suggestion that a piece fell off of the ball as it was being pushed. Figure 6.73 is a representation of this suggestion.

$\vec{F}_{net}\Delta t = \Delta(m\vec{v})$			
$\vec{F}_{net}$	Δt	Δm	$\Delta \vec{v}$
$\neq \vec{0}$	(>0)	<0	$\vec{0}$
Other possibilities not considered			

Figure 6.73: Ike's description of a piece of the ball falling off as it is being pushed.

For simplicity, the definition of momentum was combined with the momentum principle for Figure 6.73, which is used as the basis for future representations of the discussion that takes place between these three participants.

Dolly: (Crosstalk over previous statement) I can think, I can think of a situation where solution #3 would be true.

Christobal: I have a hard time visualizing solution #3 happening, because –

Ike: Well, let's say it's a very weak ball.

Christobal: Like, the momentum, the velocity, the uh, mass that you lose is going to have to have momentum, and so it would basically have to be ejected out with some sort of velocity, so that its momentum cancels the net force. well, isn't that another force? Like just ejecting – (making gestures of particles flying off).

Ike: Not if it's ejected from two sides (using first finger in each hand to point one up in one down).

Christobal: Well then there is no net force to oppose the net force.

Dolly: Yeah.

Ike: (Reading from the problem statement) "A nonzero net force, so you push from behind and the ball shoots off pieces from the sides (gesturing top and bottom ejection of particles)."

Christobal: Exactly, but that ball shooting something out from the side is going to have a force (pointing with his left hand into the air).

Ike: Both sides, *both* sides (gesturing with fingers up and down).

Dolly: Then they cancel out.

Ike: If I push you from the front and the back, you are being squeezed but not, the net force on your body is zero.

Christobal: (Reading from problem statement) "Nonzero net force."

Ike: Right, so let us say that you're also being pushed from below.

Christobal: Well, okay, so suppose you had force on the ball right?(Makes a fist, which he pushes away from himself) So the ball shoots off mass. It has to shoot off mass in the direction opposite of your force, right?

Ike: No it doesn't.

Dolly: Not necessarily.

Ike: (Draws a ball, which is experiencing a force pushing it to the right and is ejecting mass up and down, on the whiteboard, see Figure 6.74).
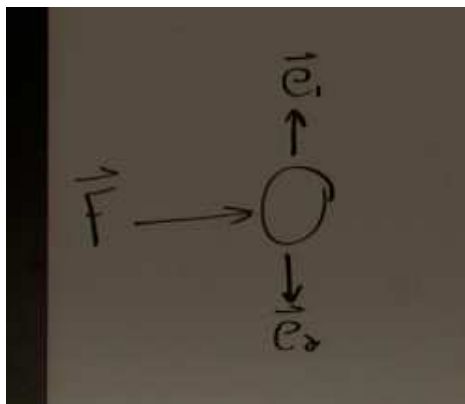
Figure 6.74: The whiteboard showing Ike's picture demonstrating how two masses could shoot from either side of the ball.

Dolly stated that she could think of a way that this could happen, but Christobal said that he was having a hard time actually visualizing this situation. Christobal's inability to visualize the situation prompted a discussion that began with gestures and eventually culminated in Ike's drawing of a picture of the situation that he was considering on the whiteboard, because neither participant was successful in conveying their ideas to the other.

Christobal argued that it was a matter of semantics, that a particle flying off from the ball would impart some force to that ball as it was being ejected, and that by ejecting a mass opposite the applied force, it would essentially apply another force on the object, making the net force zero. Christobal probably meant that the mass would need to be ejected from the ball in the *same* direction as the force, meaning that the ejected mass would apply a force to oppose the applied force. It later became clear that Christobal did in fact mean this. His reasoning (assuming he meant "that it would have to shoot off mass in the same direction as the net force,") is shown in Figure 6.75.

Ike responded by proposing that two pieces fly off of the ball in opposite directions, creating a situation where even if the ejected masses were to apply forces to the ball, those forces wouldn't cancel the provided net force. He drew a picture on the whiteboard, which is shown in Figure 6.74. His reasoning had changed slightly since this problem began, and his new reasoning is represented in Figure 6.76.

$\vec{F}_{net}\Delta t = \Delta(m\vec{v})$					
$\vec{F}_{net}$	Δt	Δm	$\Delta\vec{v}$	$\hat{F}_{applied}$	$\hat{F}_{mass\ piece}$
$\vec{0}$	(>0)	$-$	$\vec{0}$	$\hat{k}$	$-\hat{k}$
$\vec{0}$	(>0)	$-$	$\vec{0}$	$\hat{k}$	$[\neq -\hat{k}]$
Other possibilities not considered					

Figure 6.75: Christobal's argument that the force of the ejected mass must oppose the direction of the applied force, which would make the net force zero.

$\vec{F}_{net}\Delta t = \Delta(m\vec{v})$							
$|\vec{F}_{net}|$	$\hat{F}_{net}$	Δt	Δm	$\Delta\vec{v}$	$\hat{F}_{applied}$	$\hat{F}_{mass\ piece\ 1}$	$\hat{F}_{mass\ piece\ 2}$
$|\neq\vec{0}|$	$\hat{k}$	(>0)	$-$	$\vec{0}$	$\hat{k}$	$\hat{j}, \neq\hat{k}, \neq -\hat{k}$	$-\hat{j}$
Other possibilities not considered							

Figure 6.76: Ike's proposal of two masses leaving the ball in opposite directions while the ball is being pushed.

Ike included the directions of the forces that were generated by the two equal pieces of mass breaking off from the ball, explicitly summing them with the applied force to show that there was still a non-zero net force acting on the ball. One could show that this situation is in fact deductively possible given the superposition principle for the forces, the momentum principle, the definition of momentum, and the chain rule. Because $\vec{F}_{net}\Delta t = \Delta m\vec{v} = m\Delta\vec{v} + \vec{v}\Delta m$, and $\Delta\vec{v} = 0$, as long as $\vec{F}_{net}$ is parallel or antiparallel to the direction of the object's motion, it is possible for a change in mass to completely account for the impulse. Christobal did not see it this way, and he challenged Ike's explanation.

Christobal: Well, okay, but that is not going to work.

Ike: That, however gives you a nonzero net force –

Dolly: What about a situation –

Ike: And it allows you to eject mass as needed.

Christobal: But that velocity's going to change (points to the white board).

Ike: Why?

Christobal: Because this F is acting at some time, right? It is acting on the whole system, we can say –

Ike: Are you suggesting that the other students calling the problem liars are correct?

Christobal: No, no, I mean this problem (still pointing to the white board) is that the velocity would change. If we had the mass ejected in this direction (draws an arrow on the white board in the opposite direction as the net force, then the ball's velocity would not change.

Dolly: But then you wouldn't, then you wouldn't have a positive net force.

Christobal: Well,

Ike: Wait, why does the ball's velocity need to change if these two are just ejected (pointing at the two arrows from the top and bottom of the ball on the white board)?

Christobal: Because –

Dolly: Because your net force in that direction (indicating the direction of the initial net force) is still increasing the velocity in that direction (Christobal affirms).

Christobal: Remember, directions are independent. And so –

Dolly: Especially if you're losing mass

Christobal: But I mean, would you say ejecting this (indicating the ball ejecting a mass straight ahead in the direction of its motion) –

Ike: Okay, okay, fine.

Christobal: It's going to create a force on the ball to oppose the net force. It's just semantics, I would say solution #3.

Ike: Okay so yeah your, your suggestion of the ball just kind of spits out something, that's fine.

Dolly: (Crosstalk over Ike) Solution #3 is the most reasonable, but I am also okay with lying.

Christobal: I don't really *like* that idea (referring to Ike's comment).

Christobal opposed Ike's idea, claiming that the directional nature of the force must affect the velocity. This error is interesting, as it indicates a very subtle oversight, represented in Figure 6.77.

$\vec{F}_{net}\Delta t = m\Delta\vec{v} + \vec{v}\Delta m$						
$|\vec{F}_{net}|$	$\hat{F}_{net}$	Δt	m	$\Delta\vec{v}$	$\vec{v}$	Δm
$\neq \vec{0}$	$\hat{k}$	(> 0)		$\neq \hat{k}$		< 0
$\neq \vec{0}$	$\hat{k}$	(> 0)		$\neq \hat{k}$		< 0
Other possibilities not considered						

Figure 6.77: Christobal's claim that the vector nature of the net force requires the object's velocity to change.

Because Christobal was not thinking about every term in the chain rule, he did not consider the possibility that the vector nature of the net force could be "absorbed" into $\vec{v}$, rather than it being forced into $\Delta\vec{v}$. This treatment is, of course, only true in the situation described earlier, where the net force is either parallel or antiparallel to the object's motion. Otherwise, the vector nature of the net force *does* require that the object's velocity change.

Christobal then once again argued the reasoning that he proposed in Figure 6.75, this time on the whiteboard by indicating a chunk of the mass flying off from the moving ball in the same direction as the force. Dolly claimed that Christobal's suggestion wouldn't work either, because it would create a situation where there would be zero net force acting on the ball. Christobal essentially shrugged, believing it to be a matter of semantics, of the terms being poorly defined. In fact, the issue is *not* one of semantics; Christobal was actually describing a situation that would not be physically correct in this situation. However, Ike apparently understood what Christobal was getting at and was happy to agree with him, especially since they had both decided to select solution #3. Notably, solution #3 just stated that it would be possible for the ball's mass to change; it did not require an explanation of how such an event occurred. Still, Dolly wanted to consider one more idea, which she tried to sell to Christobal and Ike. While the idea was insightful, she was largely brushed aside, as the other participants in the group wanted to move on from this problem.

> **Dolly**: What about a situation in which you have a cart that's moving (using her hand gesture a cart moving along the bottom of the white board), Right? And –

Ike: Then it's not a ball that has been moving for a long time, and (inaudible)

Dolly: Nevermind, that is not going to work either.

Christobal: Well, I mean you would have to eject mass off, but wouldn't you say –

Dolly: (Crosstalk) No no no I was thinking you would add mass to it or you eject mass from it –

Christobal: Yeah, you could add mass to it, you know –

Dolly: Then you are increasing the force in one direction –

Christobal: But you know it's not going to be different, I mean it's difficult to visualize (inaudible crosstalk with Dolly), because I would say that this MV is going to be some sort of force –

Dolly: I'm just saying that if you change the M instead of the V, but you can also be changing like the weight of the object –

Christobal: That's okay.

Dolly: And therefore, you are changing the mass of it without changing the velocity

Christobal: I believe you, let's let, let's be done with this one, it's making me nauseous.

Dolly: Number three is fine, okay. Solution #3?

Ike: Solution #3.

Dolly suggested that the group consider some variation of a problem that is often discussed in introductory physics classes about a cart on a track that is moving along but is either losing sand through a hole in the bottom or else picking up rain as it travels along. Dolly suggested that this example could be helpful to consider in this situation. While she was unable to vocalize her entire idea, the analogical situation of a cart being pushed on a track in the rain may be useful for understanding this problem, and it would have provided a valid possibility, as shown in Figure 6.78.

The other group members did not want to consider this example, and Dolly was not given the opportunity to flesh it out. It is unclear how much of this possibility set she succeeded in fleshing out during this discussion. It was unfortunate that Christobal and Ike pushed this idea aside, as it was a perfectly good example that might have generated a productive discussion.

$(\vec{F}_{applied} + \vec{F}_{rain} + \vec{F}_{track})\Delta t = \Delta(m\vec{v})$					
Δt	Δm	$\Delta \vec{v}$	$\hat{F}_{applied}$	$\hat{F}_{rain}$	$\hat{F}_{track}$
(> 0)	> 0	$\vec{0}$	$+\hat{x}$	$-\hat{y}$	$+\hat{y}$
Other possibilities not considered					

Figure 6.78: A cart being pushed on a track in the rain as an analogy to the "Ball in Motion" problem.

6.2.6 Choosing between solutions #1: a sudden insight (Fay, Marco, and Omar)

The session with Fay, Marco, and Omar is one of four for the "Ball in Motion" problem that were resolved not by identifying the correct solution, but instead by choosing the solution that seemed "best," based on some criterion that was unique for each group. The various reasons the participants provided, either explicitly or implicitly, for choosing between these solutions are discussed in more depth in Section 6.2.10.

The graduate group of Fay, Marco, and Omar exhibited a striking contrast to both Eduard, who thought of the possibility that the mass of the ball could change on his own, and the group of Claudette, Earl, and Gustov, who explicitly rejected the idea that the ball's mass could change. Instead, Fay, Marco, and Omar did not consider the possibility that the mass of the ball could change until they read solution #3. Then they became animated, suddenly aware that there was a possibility that they had not considered. The fact that the written solution itself focused their attention on a physical quantity that they had not previously considered is further evidence that written solutions may help reasoners consider quantities that they had neglected, a conclusion that was discussed in Section 6.1.11. However, while solution #3 provided a possible solution to the problem, Fay, Marco, and Omar then had to choose among the three possibilities, none of which the group seemed able to conclusively rule out. The solution they chose was in their opinion the "best" solution, rather than the "correct" solution.

> **Marco**: (Reading problem statement) "A ball was in motion for a long period of time. Sometime during this period of time, it experienced a nonzero net force for some finite time interval Δt. While this force was acting, the ball's velocity did not change. Explain

how the situation is possible."

Fay: So we have a ball in motion (she draws a ball on whiteboard, and writes "F_{net}" and "Δt"). At some point there was a force net. We don't know what direction the force net was in –

Omar: We don't even know what the motion of the ball was.

Fay: The ball was in motion. In some direction (Marco and Omar make gestures and crosstalk, indicating agreement that they don't know how the ball is moving). Let's just assume that the ball is moving. Let us call this (drawing an arrow from the ball and writing "$\hat{b}$" on the whiteboard)

Marco: The direction of motion.

Fay: Yeah. b-hat, for ball hat. So, the ball is moving in ball -hat [direction]. I'm sliding this (the whiteboard) – force net was applied at some point and the question is the velocity does not change?

Marco: (Reading problem statement) "While the force was acting, the ball's velocity did not change."

Fay: Okay.

Omar: (Inaudible muttering) ... is moving.

Fay: So, does that not imply that the force could not have been in the direction of the ball? Because if, let's model it as force due to gravity, just because that is easy. If the ball was in motion downward at a constant velocity, happily, then [if] gravity was turned on, that would have accelerated the ball. Right? However, if the ball was in motion this way (indicating horizontal direction on the white board), then it would have been given a velocity in the horizontal [sic, meaning and indicating on whiteboard vertical] –

Marco: (Rejecting Fay's suggestion) And the velocity would be greater (Omar agrees).

After reading the problem statement and before looking at the different solutions, Fay, Marco, and Omar tried to come up with their own possible explanations for the problem. The group did

not consider the possibility that the mass could change; indeed, they did not consider that possibility until they read solution #3. Because this group was not taught the Matter and Interactions curriculum, they were probably using the form of Newton's Second Law that is usually taught in introductory physics classes: $\vec{F}_{net} = m\vec{a}$. This relationship is reflected in Figure 6.79, which represents the initial stage of this group's reasoning.

Figure 6.79: Fay, Marco, and Omar's reasoning, which used the "$\vec{F}_{net} = m\vec{a}$" form of Newton's Second Law.

For the rest of this discussion, I do not report the second possibility set in Figure 6.79, which relates $\vec{a}$ to $\vec{v}$. By considering this problem with this form of Newton's Second Law, this group was tacitly associating a non-zero acceleration with a changing velocity. Therefore, only the former possibility set, which relates acceleration to net force, is salient for this group.

Fay's first attempt was to suggest that the force couldn't be in the same direction as the ball's motion. However, Marco responded that even if the net force were not in the same direction as the motion, the ball's velocity would increase. Figure 6.80 represents this reasoning.

Figure 6.80: Fay's suggestion of a force perpendicular to the direction of the motion, which the group rejected.

Fay's first attempt did not result in any success. This failure is easily seen from the model of her possibility set: she invoked a quantity ($\hat{v}$) that is not present in the relationship (Newton's Second Law). Therefore, varying that quantity should have no effect on the possibilities that the relationship proposes. Indeed, the group found that when they considered different values of $\hat{v}$, they did not see different conclusions. Note that the group was overly specific about the value of the acceleration; it would have been more precise to say that the velocity must have changed as a result of the net force, rather than saying that it increased.

> **Fay**: So? How could it have not changed? Where, where we are assuming that it is a
> nonzero force, so we can't just say the force was zero, hence it had no effect. That would
> be nice.

Fay lamented that the one trivial possibility, $\vec{F}_{net} = 0$, was not possible because the problem statement expressly forbade it. This additional eliminated possibility is shown in the group's new possibility set in Figure 6.81.

$\vec{F}_{net} = m\vec{a}$		
$\vec{F}_{net}$	m	$\vec{a}$
$\neq \vec{0}$		$\neq \vec{0}$
$\neq \vec{0}$		$[\vec{0}]$
$\vec{0}$		$\vec{0}$
Other possibilities not considered		

Figure 6.81: The possibility that the net force could be zero was rejected by the premises of the problem.

Marco then suggested yet another impossibility, which was essentially a restatement of Fay's earlier proposal that the net force could be perpendicular to the direction of motion.

> **Marco**: Well, even if it is just circular motion, the direction would change and we would
> say the velocity is changing (Omar agrees).
>
> **Fay**: The speed would be constant, the velocity would change.

The group clarified that even if the ball were moving in uniform circular motion, while the speed of the ball would be constant, its velocity would change because the direction of the ball would

change. This suggestion was nothing new, as it was a restatement of Fay's earlier claim from Figure 6.80.

Marco: There is another force that was acting that they did not tell us about.

Fay: So the ball was in motion by another force.

Omar: Initially?

Marco: No, they just did not tell us all of the information. During Δt, another nonzero net force was applied that canceled out –

Fay: Well,

Omar: Yeah (he nods).

Marco: (Laughs).

Fay: So we are saying that while this (indicating the net force arrow she had drawn on the whiteboard) is acting, this was also acting (draws another arrow on the whiteboard).

Marco suggested that the problem statement was omitting some information, namely that there was an additional net force acting on the ball. Solution #2 was essentially that very solution, but Marco frames his suggestion as if the creator of the problem were deliberately hiding information from them. Of course, the presence of the word "net" means that all of the forces acting on the ball were been accounted for, but Marco negotiated that meaning to provide a possibility. This reasoning is presented in Figure 6.82.

$\vec{F}_{net} = m\vec{a}$			
$\vec{F}_{net}$	$\vec{F}_{other}$	m	$\vec{a}$
$\vec{F}, \neq \vec{0}$	$-\vec{F}$		$\vec{0}$
$\vec{F}, \neq \vec{0}$	$[\neq -\vec{F}]$		$[\vec{0}]$
$\vec{0}$			$\vec{0}$
Other possibilities not considered			

Figure 6.82: Marco's proposal of a force in addition to the net force.

The group liked this proposal; it seemed a reasonable suggestion to them. Armed with a possibility, they investigated the written solutions.

Fay: All right, let us just start looking at what the solutions are.

Marco: Solution #1. (Reading) "The momentum principle states that F net Δt –"

Fay: Pause for one second (to the researcher:) do we get equation sheets?

Researcher: Yes you do.

Fay: Because I do not know your momentum principle.

Researcher: Yes you do [get equation sheets], but I need to get them. Maybe one of them will get it (points off-camera toward the observation room, where other researchers are watching and video-recording the session).

Fay: Maybe one of the mystery behind the glass people will get it.

Marco: This one is cool (pointing to and reading from solution #1) because [solution #1 says that] "F net Δt is equal to change in momentum. Because a nonzero net force acted, the change in momentum must be nonzero. However, since the velocity did not change, the ball's momentum did not change. Therefore, we have a contradiction, and the situation is not possible." That is one possibility (Fay, who had been taking notes on the whiteboard, and Omar agree). If, if we have been given all of the information, right, this would be acceptable.

Marco seemed somewhat relieved that one of the written solutions proposed that the situation was impossible, given how difficult it was for the group to come up with a possibility in the first place. However, he tacked a caveat on to his appreciation of the solution: he would only like it if all of the information had been given in the problem statement. Thus the philosophical question about whether the problem statement was "complete" or not gathered some steam. The group continued to include this caveat throughout their discussion. The debate is not trivial. In a sense, Marco was wondering whether the enterprise itself was purely deductive; that is, he may have been wondering whether it was possible that there were some other information that needed to be induced, rather than deduced, about the problem. Of course, the problem was designed to be purely deductive, but Marco seemed to resist drawing the conclusion that it was so.

Also, during this segment of the session, Fay realized that she was not familiar with the version of the momentum principle presented in the solutions. The researcher had forgotten to give this

group (which was the first to participate in this study) a reference sheet, which explains why Fay requested it. For the remainder of the session, it was unclear which form of the momentum principle the group was using for their reasoning, but I assume that they adopted the standard form from Matter and Interactions, $\vec{F}_{net}\Delta t = \Delta\vec{p}$.

> **Fay**: (To Researcher, who handed the group a reference sheet) Thank you.

> **Marco**: Now [solution #]2.(Reading) "Because the velocity didn't change, the ball's momentum must not have changed either –"

> **Fay**: Which is the same thing [as solution #1 had said].

> **Marco**: We agree, yup. (Reading solution #2) "However, a nonzero net force acted on the ball. Therefore, some (now in unison with Fay and Omar, both apparently very much in agreement with the solution) additional force must have opposed... in order for the velocity to remain constant. For example, air resistance or gravity could have opposed the net force."

> **Fay**: Okay, so far I like that more (Omar agrees), since they do ask us "how is this possible?," the answer of "it's not possible" kind of ignores the question.

> **Marco**: This is somewhat philosophical though, because –

> **Fay**: Yeah

> **Marco**: There is two things. It's either this has all of the information (pointing to problem statement) that can be had or it's not. You know –

> **Fay**: Yeah, is the problem complete? Is the problem not complete?

> **Marco**: Yeah.

Not surprisingly, solution #2 resonated with this group, as it repeated to them the suggestion that they made just seconds before. Fay became excited about the solution and read along with Marco, as Omar smiled and agreed as well. Solution #2 said exactly what they had been thinking. So they turned back to solution #1 and reconsidered it. Marco again stated that solution #1 could still be correct, if all of the information was actually given in the problem. Fay replied with the question that summarized the group's concern: was all of the information given in the problem or

not? Rather than decide the answer to that question at this point in their discussion, the group instead turned its attention to the final solution.

> **Marco**: So, solution #3 (Reading) "The momentum principle states that F net Δt is change in momentum –"
>
> **Fay**: (Pointing to the white board) Same as before, yeah –
>
> **Marco**: And then they write the definitions in. "Because the Net force and Δt were nonzero, the change in momentum must be nonzero. If mass final equals mass initial equals M, then M –" oooh (followed soon with Fay indicating interest as well). "Then $M\,VF$ minus VI is equal to zero because" blah blah blah, "therefore MF must be (now in unison with Fay, who is reading over his shoulder) different from MI"
>
> **Fay**: (Reading) "The mass of the ball must have changed while the net force was acting on it." That's clever because that way it kind of, it sticks in the original problem, so let me just –
>
> **Marco**: It's true. It's possible.
>
> **Fay**: I think that is. We –
>
> **Omar**: We have not thought about relativistic motion.
>
> **Fay**: I don't think we're allowed to think relativistic (laughs). But, yeah, if the mass, then velocity changed. Since velocity didn't change, mass must have changed.

The group's reaction to this solution was very different than their reaction to solution #2. Where in reading solution #2 they had shown cheer and happiness, here they were almost stunned, as if their eyes had just been opened to a possibility that they had not previously considered. Fay noted that solution #3 did not require any information other than what was given in the problem, unlike solution #2. The new possibility sets for the group became similar to those represented in Figure 6.62, although the group had not explicitly eliminated the possibility suggested by Marco yet. However, the possibility that the mass might have changed required a completely new pair of possibility sets based on the Matter and Interactions form of the momentum principle, because $\vec{F}_{net} = m\vec{a}$ does not account for a changing mass. It is therefore not surprising that the group

previously did not consider the possibility that the mass might have changed. This aspect of the group's reasoning is represented in Figure 6.83.

$\vec{F}_{net}\Delta t = \vec{p}_f - \vec{p}_i$		
$\vec{F}_{net}$	Δt	$\vec{p}_f - \vec{p}_i$
$\neq \vec{0}$	(>0)	$(\neq \vec{0})$
$\neq \vec{0}$	(>0)	$[(\vec{0})]$
Other possibilities not considered		

$\rightarrow\rightarrow$

$\vec{p} = m\vec{v}$		
$\Delta\vec{p}$	Δm	$\Delta\vec{v}$
$\neq\vec{0}$	0	$\neq\vec{0}$
$\neq\vec{0}$	$\neq 0$	$\vec{0}$
$\neq\vec{0}$	$[0]$	$\vec{0}$
Other possibilities not considered		

Figure 6.83: Fay, Marco, and Omar's possibility sets after realizing that the mass of the ball could have changed.

Fay explicitly listed the possibilities regarding the definition of momentum: either the mass or the velocity must have changed for the momentum to change. Since the velocity didn't change, the mass must have. Figure 6.83 displays this reasoning. After having established that solution #3 is possible, the group reconsidered all of the solutions to decide which of them was correct, since they were told that only one was.

Fay: I like that (solution #3) more than –

Marco: And this does not require us to exit the problem in any way.

Omar: Yeah, we don't need to assume anything.

Fay: Well, it requires us to exit, that we usually assume in a physical sense the ball is constant. That the ball stays being a ball. If we consider it to be an asteroid, it could have melted. The force could have pulled parts off. So that is the least copout of all of the answers.

Marco: Still, the other mass, I mean there's still information –

Fay: Missing.

Marco: (Pointing to solution #3) This still states that there's information missing the problem in the problem because –

Fay: I think that all three are appropriate. The first one is the poorest (Marco and

Omar agree), that just says no, obviously it is equal, no. The second one says you didn't

tell us about a force. The third one says he didn't tell us about mass.

Marco: Yeah, I agree solution #3 is the most plausible –

Omar: Most sensible.

Marco: Let's go with that.

Fay: I agree.

Omar: Yeah, that's fine.

Fay pointed out that solution #3 required them to imagine that the ball's mass changed, which is not usually a considered physical possibility. She replaced "ball" with "asteroid" and demonstrated that the concept of some object changing mass is not unreasonable. The fact that she did this, and that no one challenged her by reminding her that the problem was about a ball rather than an asteroid, is somewhat interesting in that it demonstrates how this group managed, like Eduard, to focus on the mathematical representations rather than the physical possibilities of the situation.

Marco returned to his argument that some information was not present in the problem statement, and Fay acknowledged this by stating that all three of the solutions were actually possible; however, as a group they prefered solution #3. This idea that any of the three could be a possible explanation to the problem can be explained in terms of their possibility sets. They were able to form possibility sets for each of the three solutions, and these sets were different. That is, each set contained a different relationship and/or quantities that distinguished it from the others. Without making a judgment as to the appropriate quantities to include in the problem (something that Marco was apparently deliberately avoiding by claiming that not all of the information was given in the problem statement), they were unable to rule any of those sets out definitively, and hence they had to choose between them at the end of their session. How participants chose between possible solutions on this problem is investigated more thoroughly later (see Section 6.2.10). However, for the time being I suggest that this group may have chosen solution #3 because it was, not quite "sensible" or "plausible," but rather because it was more "deductive" than the others, and it is likely that this resonated with their intuition about what a physics solution should look like. Of course, there is no way of knowing for sure why they chose this solution, only that they entertained the possibility sets

suggested by it and acted as though reading it generated sudden awareness among them.

6.2.7   Choosing between solutions #2: "more logical" (Hugo, Josephine, and Teddy)

Just as in the "Two Blocks" problem, Hugo, Josephine, and Teddy barely spoke while solving this problem. What they did say, however, provides us a glimpse at what they were thinking. In this problem, the group decided that the situation was possible, eliminating solution #1. Then they set to deciding between the remaining two solutions, choosing solution #3 because they found it to be "more logical" than solution #2.

Teddy: All right.

Hugo: All right.

Teddy: (To Josephine) Can you see it (the problem statement)?

Josephine: No.

Teddy: Well, I'll read it out loud if you want. (Reading problem statement) "A ball is in motion for a long period of time, sometime during this period of time it experienced a nonzero net force for some finite time interval, Δt. While this force was acting, the ball's velocity did not change, explain how this situation is possible." Okay. um. (Long, quiet pause)

Hugo: It could be in a circle.

Teddy: Yeah that's what I was thinking. It is in a circle.

Hugo and Teddy proposed a possible explanation for this situation: the ball could have been moving in a circle. This suggestion is essentially claiming that the force was perpendicular to the direction of the object's motion, something that Fay, Marco, and Omar initially suggested before they reasoned that even such a scenario would change the object's velocity (see Figure 6.80). Hugo and Teddy did not investigate this idea any further, moving on instead to the written solutions. They did not mention this "circular motion" idea again, perhaps because none of the written solutions proposed it.

Teddy: Let's look at the solutions. (Reads solution #1 inaudibly).

Hugo: (Reading from solution #1) "The situation is not possible."

Teddy: Okay. Let's see the other one. (The group then silently reads solution #2 and then moves on to solution #3).

Hugo: (Reading from solution #3) "Mass of the ball must have changed."

Teddy: That's a possibility too.

Hugo: Yeah.

Teddy: The mass changed.

Hugo: They added some mass.

While reading the three solutions, the group entertained each of them as a possibility. They then attended to each one in turn.

Teddy: So, do you all agree that solution #1 probably is out?

Hugo: Yeah.

Teddy: Okay. (Group returns to silent reading). (Probably prompted by solution #2) I think that it would slow down the velocity, wouldn't it? If there is like air resistance or gravity. If it opposed the net force, would it not slow it down?

Hugo: Yeah, I like [solution] #3.

Josephine: Yeah.

Teddy: [Solution #]3 makes most sense. Because of the mass changes but you –

Hugo: Yeah just part of it falls off.

Teddy: Yeah.

After this segment of the dialogue, the group chose solution #3 and moved on to the next problem. They provided only a few pieces of data as to how they came to this conclusion. The group never referenced the momentum principle or the definition of momentum in their explanations. Instead, this group was trying to imagine what would happen if solution #2 were correct. For them, this

would mean an additional force to oppose the net force, which would necessarily slow down the object's motion because it opposed the net force. This reasoning is demonstrated in Figure 6.84. It is important to remember that since Teddy did not explicitly mention the momentum principle, it is not known whether he was aware that his reasoning implied such possibilities.

$\vec{F}_{net}\Delta t = \Delta\vec{p}$						
$|\vec{F}_{net}|$	$\hat{F}_{net}$	$|\vec{F}_{other}|$	$\hat{F}_{other}$	Δt	$\Delta\hat{p}$	$|\Delta\vec{p}|$
$\neq|\vec{0}|$	$\hat{k}$	$\neq|\vec{0}|$	$-\hat{k}$	(>0)		<0
$\neq|\vec{0}|$	$\hat{k}$	$\neq|\vec{0}|$	$-\hat{k}$	(>0)		$[not<0]$
Other possibilities not considered						

Figure 6.84: Teddy's claim that a force opposite the net force would have slowed down the ball.

The second possibility set, which represents the the relationship between $\Delta\vec{p}$ and $\Delta\vec{v}$, is not shown in Figure 6.84. This omission is because the mistakes in reasoning can already be seen in the possibility set above. First, placing an additional force with the net force is erroneous, but because solution #2 did precisely that, it is possible that Teddy was just treating it in much the same way as that solution, subverting the definition of the word "net" and combining that original force with the new force to create a real total or net force that applied to the momentum principle. Taking that assumption, it becomes obvious that their mistake was one of assigning an illegal value to $\Delta\vec{p}$, a value that did not follow from the relationship indicated. The value "<0" under the column "$|\Delta\vec{p}|$" indicates that the object slows down, although it does not carry any information about the direction of the ball's motion. While a *net* force would change an object's momentum, the allowed row of possibilities was far too specific, and the rejected row was far too broad. Indeed, many of the rejected scenarios would actually be possible on physical grounds (e.g., a ball falling at terminal velocity does not experience a change in momentum in the direction opposite its motion).

In other words, Teddy's statement was too specific, excluding too many possibilities. Even if we allowed the "additional force" to be in the model, his error was one of assigning a value that is too specific to one of the physical quantities, the change in the object's speed, implying a negation that excludes physical possibilities. The possibilities framework makes this error apparent, and it even suggests that producing examples to contradict Teddy's statement (specifically those that contradict

his implied negation) may have been a useful instructional intervention, as those examples may have encouraged him to consider more possibilities than he had originally.

It is also worth pointing out that Hugo suggested that the change in mass of the ball was due to a piece of the ball falling off, which opposed his original statement that the ball gained some mass. He pointed out both possibilities for the changing mass, but he did not explain whether he believed that one was possible and the other not, or whether he was just listing alternatives.

After they solved all of the problems, the researcher asked this group for clarification as to how they solved it. Hugo confirmed the reasoning that the group exhibited in the transcript above, adding validity to the data.

Researcher: The first one [problem] you did was this, the ball in motion, and you chose [solution] #3. Could you tell me why you chose [solution] #3 for this one? Do you remember? I'll get this out of your way (the researcher removes the timer from the table).

Hugo: (To Researcher) Why we chose [solution] #3?

Researcher: Yeah.

Hugo: Just because it seems more logical that it could lose mass, like part of it could break off, or, probably in this situation it would break off, more so than, um, that this would happen because we thought that if air resistance or gravity was factored in the velocity would change. In this one, we just thought that this situation was possible because this situation (pointing to solution #3) we think is possible, so.

Hugo reiterated that part of the ball could break off, which he posed as a possibility rather than a requirement (although it is unclear whether he thought that the ball must have lost mass in some fashion). However, the way he phrased this explanation is curious. First, rather than saying that a force opposing the net force would slow down the object's motion, he made a more general statement that the ball's velocity would change. Perhaps this statement indicates a somewhat broader (although still too restrictive) set of possibilities than was displayed in Figure 6.84. Even more remarkable is that he claimed that the explanation of the ball losing mass was "more logical" than the explanation of the ball experiencing another force without its velocity changing. Exactly

what he, or the group as a whole, was thinking is unclear. One possibility is simply that they could not imagine any possibilities where air resistance or gravity acted on an object whose velocity did not change and therefore they chose the solution they could imagine.

6.2.8 Choosing between solutions #3: "equal and opposite" (Ana and Sally)

Ana and Sally considered both solution #2 and solution #3. Something about solution #2 resonated with the phrase "equal and opposite" for Sally, making her prefer that solution. Ana, on the other hand, apparently preferred solution #3 but was unable to convince Sally. In the end, the group decided that solution #2 was the better solution because it was "more open-ended."

Sally: Okay, so it's moving for a long time.

Ana: (Reading problem statement) "Sometime during this period, it experiences a nonzero net force for some finite time interval. While this force was acting, the ball's velocity did not change. Explain how the situation is possible." (Reads problem statement again) "A ball is in motion for a long period of time. Sometime during this period, it experiences a nonzero net force for some finite time interval. While this force was acting, the ball's velocity did not change. Explain how the situation is possible." Okay.

Sally: I guess equal and opposite if that's an option. (Reading solution #1) "Momentum principle. Because a nonzero net force acted, P final minus P initial must be nonzero. However, since the velocity did not change, the ball's momentum did not change. Therefore, we have a contradiction and the situation is not possible. Okay."

Ana: (Reading solution #2) "Because velocity did not change, the ball's momentum must not have changed either. However, a nonzero net force acted on the ball. Therefore, some additional force must have opposed the net force to allow the ball's velocity to remain constant. For example, air resistance or gravity could have opposed the net force."

Sally: I think that's the better choice of the two so far.

Ana: Yeah.

Sally: And then (reading solution #3:) "momentum principle states that F net Δt equals P final minus initial. Because the net force and Δt were nonzero, $M1$ $V1$ must be, or (correcting herself) momentum final minus initial must be nonzero. If the masses are equal, then the final, or (correcting) then the momentum (echoed by Ana) is equal, because final velocity must equal negative [sic] velocity. Therefore, the final mass must be different from the initial mass. The mass of the ball must have changed while the net force was acting on it." I mean –

Ana: Okay, so I don't think it's [solution #]1, right?

Sally: Yeah. (Ana puts solution on top of the microphone) Wait, you've got to watch out for this thing (moves the solution off the mic).

Ana: Oh, sorry.

Immediately upon hearing the problem statement, Sally suggested the option of "equal and opposite." She did not specify exactly what she meant, but that phrase is usually associated with forces. She may have been committing the error in solution #2, which treated the net force acting on the ball as if it were an applied force that could be added to an equal and opposite force, resulting in a net force of zero acting on the ball. Repeatedly while considering this problem, Sally returned to solution #2, probably because that solution most closely reflected her reasoning.

Ana and Sally dismissed solution #1, leaving two possibilities to choose between. They turned their attention to those next.

Sally: Um, (Reading from problem statement) "did not change. Explain how this situation is possible." Well, I mean I guess technically they're both somewhat possible situations. It doesn't say anything about the ball changing, (Ana affirms) so that could have happened.

Ana: Yeah, something could have like, got on the ball.

Sally: Yeah.

Ana: And that's definitely, like, possible.

Sally: And it just says "a force."

Ana: (Reading solution #2) "The ball's momentum is not changing,

however a nonzero..."

Sally: I wonder if this math (in solution #3) is right. I mean it looks right but.

Ana: (Apparently still reading solution #2, inaudible).

Sally: (Apparently reading solution #3) "Initial, mv..."

Ana: Okay, so here (solution #2) they're talking about it being like air resistance or something, so –

Sally: It would still be opposite and equal so, like, for every force there's an opposite and equal –

Ana: Wouldn't you think that the ball's velocity would change if air resistance was acting on it?

Sally: Not if the force is equal and opposite to air resistance. I don't know.

Ana: I don't know either.

Sally: It's just one or the other I think.

Ana: I'm leaning towards [solution #]3 (chuckles). I don't know.

Sally: Yeah. I'm not sure either.

Ana: It could be, it could be either –

Sally: Yeah, this is one of the Newton's Laws I think is [solution #]2 and then [solution #]3 is – (stops). (Reading problem statement) "must have changed while the net force was acting on it" Hm. I don't know. At first, I was just, I didn't think mass could change, but I guess something could stick on it technically or something.

Ana: Yeah,

Sally: So you wanna say [solution #]2, or [solution #]3?

Ana: I don't know.

Sally claimed that both solutions were "technically" possible, and Ana even provided a mechanism for solution #3 to be correct. Sally also admitted to initially thinking that the mass of the ball couldn't have changed, but she apparently accepted Ana's mechanism of something sticking to the ball. However, Sally also revealed her suspicions about the math in solution #3. The math in that solution was quite complicated and provided plenty of opportunities for error. Perhaps Sally suspected that there was a good chance that the math on that solution was wrong. She continued to return to this suspicion throughout the discussion.

Ana stated that the ball's velocity would change if air resistance acted on it, suggesting a possibility set that was similar to Teddy's reasoning in Figure 6.84. However, Sally replied with the counter-example that reduced the impact of Ana's reasoning by contradicting and therefore removing Ana's implied negation. This counter-example created the set of possibilities that is represented in Figure 6.85.

$\vec{F}_{net}\Delta t = \Delta\vec{p}$					
$|\vec{F}_{net}|$	$\hat{F}_{net}$	$|\vec{F}_{other}|$	$\hat{F}_{other}$	Δt	$\Delta\vec{p}$
$\neq |\vec{0}|$	$\hat{k}$	$\neq |\vec{0}|$	$-\hat{k}$	(>0)	$\neq \vec{0}$
$|\vec{F},\neq\vec{0}|$	$\hat{k}$	$|\vec{F}|$	$-\hat{k}$	(>0)	$\vec{0}$
Other possibilities not considered					

$\rightarrow\rightarrow$

$\vec{p} = m\vec{v}$		
$\Delta\vec{p}$	Δm	$\Delta\vec{v}$
$\vec{0}$		$\vec{0}$
Other possibilities not considered		

Figure 6.85: Sally's claim that an object's velocity wouldn't change if forces on it were equal and opposite.

Neither participant committed to a solution, although Sally seemed to side with solution #2 and Ana with solution #3. The two participants did not approach any deep confrontations or discussions (as opposed to Otto and Walter, who definitely did!), but they did make an effort to come to a consensus.

> **Sally**: I don't know if them just putting math there (on solution #3) is trying to trick us, you know what I mean?

> **Ana**: Ok, well, so, (Reading solution #3) "F net Δt is P final minus P initial," yeah that's fine, and then, ok, "because the net force and Δt were nonzero, final momentum minus initial must be (with Sally) nonzero."

Sally: Which makes sense. Well,

Ana: Wait.

Sally: It's moving at the same velocity, but the masses are going to change?

Ana: Well, the mass is going to change, well [it] *has* to change in that situation, that's why it – their formulas are right I guess.

Sally: I don't know. I kind of think it's [solution #]2.

Ana: Yeah (chuckle).

Sally: I think there's something weird about the math [in solution #3], I mean it seems right but –

Ana: (Reading solution #2) "Could have opposed the net force" well, "because velocity [did not] changed, the ball's momentum must not have changed either, however, a nonzero net force..." Ok, let's just go with [solution #]2, is that okay?

Sally: Yeah. I think it's more open-ended.

Ana: Yeah.

To address Sally's concern that there was a mistake somewhere in solution #3's math, Ana began working through it carefully. Sally started to try to understand the situation, stating that the force did not change the velocity, but it did change mass. Ana reiterated that point with emphasis ("[it] *has* to change in that situation") while adding that the formulas were correct. However, Sally reverted back to her agreement with solution #2. Ana did not push the issue, allowing an agreement with solution #2 because it was "more open ended." What "open ended" means is discussed in more detail in Section 6.2.10, although apparently Sally believed that solution #2 was more correct than solution #3 for some reason that was not completely obvious. We only know that she continued to think that there was something weird about the math in solution #3, but she did not deeply investigate it. She was content to make her decision based on the information she already had, which left solution #2 as "more open-ended" for her.

6.2.9 Choosing between solutions #4: "most likely" (Gaston, Igor, and Winfred)

In much the same way that Sally and Ana picked between solutions #2 and #3, Gaston, Igor, and Winfred chose between those same two solutions. While Winfred thought about the possibility that the mass could change before even seeing the solutions, the group became attracted to the idea that a force could oppose the net force, and they spent some time thinking about the possibilities. Like some of the other groups, Gaston, Igor, and Winfred did not appeal to the momentum principle or definition of momentum, instead trying to reason through the situation by constructing physical examples and using those in a non-deductive way. At the end, they chose solution #2 because it was "more likely," a reason that is surprisingly similar to the one that Ana and Sally provided.

> **Winfred**: (After the group reads the problem statement silently) Well, I think that, I think that it is going to be possible but somehow the mass of the ball has to change.
>
> **Igor**: Yeah, that's what I thought.
>
> **Winfred**: Like if something got stuck onto the ball.
>
> **Gaston**: Well if it's a constant force then wouldn't the velocity stay the same? Like, if I applied a constant force to this pen like the velocity wouldn't change (He pushes a pen across the table).
>
> **Winfred**: Yeah.
>
> **Gaston**: But I would still be having a nonzero net force.
>
> **Winfred**: Yeah, that's true too. But I think the fact that –
>
> **Igor**: You're doing this for some time period.
>
> **Winfred**: Well let's look at the other solutions first.

Winfred mentioned that the mass of the ball had to change before any of the group looked at the written solutions. This comment was surprising, as not many reasoners automatically think of that possibility (see, for example, Section 4.3). Winfred may have actually been thinking about the momentum principle and the definition of momentum as in solution #3 (see Figure 6.62).

Unfortunately, he did not ever provide any more detail about the ball's mass changing, so we cannot provide a detailed possibility set for him.

Gaston then proposed his own reasoning, which did not follow from the momentum principle but was accepted by the group nonetheless. He stated that applying a constant force to an object should cause its velocity to remain constant, providing an example by rolling a pen across the table. This reasoning is represented in Figure 6.86.

$\vec{F}_{applied}$	$\Delta \vec{v}$
Constant, $(\neq \vec{0})$	$\vec{0}$
Constant, $(\neq \vec{0})$	$[\neq \vec{0}]$
Other possibilities not considered	

Figure 6.86: Gaston's reasoning, that a constant force would result in a constant velocity.

With these two ideas from Gaston and Winfred, the group turned to the solutions.

Winfred: (Reading solution #2) "Because velocity didn't change, the ball's momentum has not changed either. There was a nonzero net force acting on the ball, therefore some additional [force] – must have opposed the net force." Okay that makes sense too. (inaudible) I think that's legit because like if you had a, if you had like if you're pushing at a constant force here (pushing pen on table towards a calculator that is held up at an angle) and had to go up this hill (referring to the calculator as the hill) you would push with a greater force to maintain the same velocity. Or if you like, you're going down ice and then you had to go past dirt or something.

Igor: But wouldn't the net force change?

Winfred: Well yeah you, you apply, it says you apply a nonzero force for a short, for some certain time but the velocity doesn't change, so it doesn't, the force just needs to change with the velocity though.

Gaston: Well solution #2 seems to make sense.

Winfred related solution #2 to a pair of physical examples: moving along a level ground and then having to go up a hill, and moving along ice and then having to go through dirt. Those two examples both represent an initial force applied to the object and an additional force (gravity or friction) being "turned on" at some point. The result of that additional force is that the pushing force would need to be increased to allow the object to continue to move at a constant velocity. What Winfred has done is blend the possibilities allowed by solution #2 with those suggested by Gaston. This blend yields an initial state, before the additional force is applied, where there is no net force by the momentum principle, although Winfred assigned one anyway. This reasoning is represented by Figure 6.87.

$\vec{F}_{push,before}$	$\Delta\vec{v}$
$\vec{F}, \neq \vec{0}$	$\vec{0}$
Other possibilities not considered	

$|\vec{F}_{push,after}|$	$\hat{F}_{push,after}$	$|\vec{F}_{other}|$	$\hat{F}_{other}$	$\Delta\vec{v}$
$> \vec{F}$	$\hat{k}$	$\neq \vec{0}$	$-\hat{k}$	$\vec{0}$
$\leq \vec{F}$	$\hat{k}$	$\neq \vec{0}$	$-\hat{k}$	$\vec{0}$
Other possibilities not considered				

Figure 6.87: Winfred's reasoning, which includes before and after states that both associate a constant force with a constant velocity.

Winfred's inclusion of an increased applied force was in response to Igor questioning that the net force acting on the object would need to change. This reasoning was incorrect mostly due to it not being properly fleshed out. Winfred was having trouble explaining it to his group because he had not fully identified all of the possibilities associated with it. Igor tried to understand by asking whether it would be possible for the net force to stay the same if another force acted on it, implying that if the net force changed it would require the object's motion to change. Winfred was apparently thinking of the situation in this problem as an action-response system, where the initial actor could provide more of a push to compensate for the additional force, preventing the net force from changing. Note that this is actually quite different from solution #2, which used the additional force as a mechanism to make the net force on the object change to zero so that the object's velocity could remain constant. Winfred was instead building on Gaston's incorrect reasoning from earlier, implying that the only way an object could resist moving would be by experiencing an *unchanging* net force.

The group accepted this reasoning enough to move on to the third solution.

> **Winfred**: (After the group reads solution #3 silently) So I think that this is possible, but –
>
> **Igor**: I don't think that it's very likely.
>
> **Winfred**: Yeah, I mean –
>
> **Igor**: It is possible, the probability of happening this way is low –

Like Sally and Ana, this group acknowledged that solution #3 was *possible*, but here Igor made the additional distinction that the *probability* of it happening would be low. This mention of probability is curious, because the group was merely tasked with identifying what was possible, not what was most probable.

> **Winfred**: Like, cause like I think if you had like a ball and you threw like a piece of gum at it or something, there would be a force, and say the gum stuck to it, it would have a greater velocity but then where does it say that? Does it say that it like, once it hits that velocity it maintains the same velocity?
>
> **Igor**: Well, there is no friction or like, you know, or like drag or anything. There is no friction acting on it. So. This, this would be assumed that it's a vacuum, there would be no –
>
> **Winfred**: Okay so these are two choices, gum or hill. Or, something sticking, not gum necessarily (trails off).
>
> **Gaston**: I'm, I'm thinking that it is solution #2.
>
> **Winfred**: I think that it's [solution #]2 because if you have, if you did this (change the mass, like solution #3), you'd have to take into account the initial velocity of the gum, the initial velocity of the object hitting it, and the object sticking to it and the translational, the translational of all that to maintain (inaudible). Also – I don't know.
>
> **Igor**: How would that, if it hits the ball how would it be, how would the F net – ? Never mind.

Winfred: Cause yeah, like I mean if you're pushing it, if you like pushed it back here and it was moving with constant velocity.

The group pointed out how difficult it would be for something to stick to the ball without the impact changing its velocity. They considered the possibility set corresponding to changing the ball's mass, although they only thought about the possibility that the mass could increase, which reduces the possibility space afforded to it. Additionally, they added quantities into the possibility set that do not belong, such as the duration of the applied non-zero net force, to further reduce that space. Note that they did not actually eliminate all of the possibilities, but instead they pointed out how difficult it would be for the exact right situation to arise where the ball's mass could change. This whittling down of the possibility space in solution #3 culminated in Igor repeating his statement that he found solution #2 more likely.

Igor: And [solution #]2 seems the most likely.

Winfred: Yeah, I guess that it probably, solution #2, seems the best. If it gave more information, you know, maybe we could figure it out because then if they would have a hill here and you're going to push it with a constant velocity, or push it with another force up to the top of the hill, the velocity is going to change here. Whereas if you hit it with something that's going to stick to it, once you hit it with something that's going to stick to it it's going to stay at that, that speed now. But since it doesn't give any information on what happens after you apply the force then this (solution #2) is the most likely situation that is going to happen.

Igor: Yeah, [solution] #2 is likeliest.

Winfred: Yeah. I guess it–

Gaston: Mark [solution #]2? (Others agree, then inaudible muttering as the group decides to mark what's wrong with the other solutions)

Winfred: Solution #1, this is, it's possible, and then we'll just say that this (solution #3) is like, um, unlikely.

What did this group mean when they said that one solution was more likely than another, especially when they admitted that both were possible? The answer to this question may lie in the way they tried to reason about solution #3. By emphasizing the level of difficulty that would be required for the ball to actually change its mass, the perceived number of possibilities was diminished. Solution #2 may have seemed to indicate a larger possibility space. In reality, all of the items in solution #2's possibility space were actually impossible, making that a null set. However, without probing too deeply, one could be tricked into thinking that there were more "ways" to get solution #2 to happen than solution #3. We have dubbed this method of choosing between solutions *entropic selection*, which is discussed in more depth in Section 6.2.10.

6.2.10 Discussion: choosing between possible solutions

The "Ball in Motion" problem, when solved deductively, should result in a single answer that is defined by the momentum principle and the definition of momentum: the only possible way that the situation is possible. If the reasoner assumes that the ball's mass cannot change, then the situation is impossible. When solving this problem deductively, one should not have to choose between different possible solutions, and yet we see that many of the groups did resort to choosing between them, believing for some reason that more than one of them was possible. A telltale sign of a group choosing between solutions rather than selecting the one they believe is correct is the use of a "more" or "most" qualifier: e.g., the chosen solution is "more likely" or "most plausible."

In this section, I explore in detail those choices, investigating how participants made those selections on this problem, and how the possibilities framework can inform those decisions. First, I look at the contrast between two groups who both selected solution #3 even though they felt that other solutions were also possible. Then I describe entropic selection, which is the process of choosing the option that has the largest apparent possibility space available, or the most possible opportunities for occurring.

A strong contrast

On the surface, the process that Fay, Marco, and Omar followed in solving this problem in Section 6.2.6 may seem similar to the process that Hugo, Jospehine, and Teddy followed in Section 6.2.7.

Neither group initially thought of the mass of the ball changing, and indicated upon reading it that they felt it was possible. Both groups also had given a different possible explanation for the motion of the ball before reaching solution #3: in the case of Fay, Marco, and Omar, it was that there was an additional force opposing the net force (essentially solution #2), and in the case of Hugo, Josephine, and Teddy, it was that the ball was traveling in a circle. At the end of each group's discussion, they reported similar reasons for choosing solution #3: in Fay, Marco, and Omar's group, solution #3 was viewed as "more sensible" and "more plausible," while in Hugo, Josephine, and Teddy's group, solution #3 was described as "more logical." It is at this point, however, that the similarities cease.

Fay, Marco, and Omar tentatively held on to each of the three written solutions as representing possible solutions to the problem. For them (and especially Marco), it was the information that was *not* provided in the problem statement that determined which of the solutions was correct. If the problem statement provided information about a second force that could act on the ball, then solution #2 would be correct. If it did not provide the information that the mass of the ball could not change, then solution #3 would be correct. If it provided all of the information, then solution #1 would be correct. Perhaps rather than this list of cases, the group was thinking in terms of what assumptions they should make when solving the problem. Should they assume that "net" truly means all of the forces? Should they assume that the mass of a ball could change? The explanation that Fay, Marco, and Omar provided was thorough, addressing the written solutions as if they were possibilities, reasoning about them not by using logic but instead by trying to identify what would be necessary for each possibility to be true. This process is very similar to how this group reasoned through the "Two Blocks" problem in Section 6.1.3: they identified the possibilities and determined what would be required for each to be true. The conclusion they reached in this problem is that because they did not know what assumptions they are allowed to make, they identified the solution that was, as Fay puts it, "the least copout of all of the answers." Yes, doing so required them to assume that the mass of a ball could change, but they managed to replace the word "ball" with the word "asteroid," allowing them to consider what the solution would be if it were about an object for which change of mass were more salient.

In short, Fay, Marco, and Omar reasoned through the problem by considering possible explanations of the situation and applying first the momentum principle and then the definition of momentum, evaluating the possibilities that survived that analysis in terms of the physical assumptions that were required to allow those possibilities.

Hugo, Josephine, and Teddy, on the other hand, did not provide much insight into their reasoning. While they suggested their own possibility, that the ball could move in a circle, they did not discuss it at all. They did not indicate that they ever considered the momentum principle or the definition of momentum in any of their reasoning. Teddy claimed that if a force opposed the net force, as solution #2 proposed, then the ball would have slowed down. Like Fay, Marco, and Omar, they considered solution #2 as a possibility, but they did not try to figure out how to make it correct. Instead they provided a plausible but incorrect argument that reduced the possibilities allowed by that solution to zero. In fact, the group seemed to eliminate solution #2, but they still used comparative indicators when indicating their final answer: solution #3 made the "most sense" according to Teddy and was "more logical" than solution #2 according to Hugo.

We have very little data to explain why Hugo, Josephine, and Teddy used the comparatives that indicated that they were keeping solution #2 around rather than eliminating it. But, the fact that they did indicate a comparison between the solutions hints that perhaps while they "thought that if air resistance or gravity were factored in the velocity would change," they allowed for their set of possibilities regarding solution #2 to include possibilities that they did not consider. When comparing the space of possibilities that they considered when thinking about solution #2 to the space of possibilities that they considered for solution #3 (ways in which a piece could break off the ball), the possibility space corresponding to solution #3 may have seemed bigger, thereby resulting in the conclusion that it was "more logical" than solution #2. We cannot say for certain that the group reasoned this way, but it fits with the pattern of entropic selection that other undergraduate groups used to choose between solutions.

The distinction between the groups of Fay, Marco, and Omar and Hugo, Josephine, and Teddy is important because it emphasizes that while on the surface, both groups chose between solutions, the reasoning behind why the graduate group did so is quite sophisticated and definitely not entropic. However, the undergraduate group did not indicate any use of deductive reasoning from fundamental

physics principles and apparently chose a solution based on intuition. The ways in which these groups differed in their reasoning are easily seen through the lens of the possibilities framework.

Entropic selection

The law of entropy states that for physical systems with large numbers of particles, the energy in that system is distributed so that it ends up in the macrostate that contains the most microstates. I use the law of entropy here analogically to describe how reasoners chose between different possibility sets in this study. Possibility sets, as we have seen, allow some possibilities while disallowing others. It is possible that a reasoner would select the possibility set that affords the most possibilities, regardless of the quality of those possibilities. In other words, some reasoners may assign a rough value to the size of the possibility space that is associated with a particular possibility set: if many possible situations are enumerated, that value would be large; if many possible situations are eliminated, that value would be low. Some value might even be assigned for sets where all possibilities that have been thought of are eliminated, but the reasoner acknowledges that there are other possibilities that have not been considered. Then, those reasoners would compares these self-assigned values, choosing the possibility set corresponding to the largest.

For example, consider two situations where there is significant evidence that this is how the participants reached their decision. First, Gaston, Igor, and Winfred in Section 6.2.9 solved the "Ball in Motion" problem by considering each of the three solutions. Throughout their discussion of the three solutions, they did not invoke the momentum principle or the definition of momentum, although they attempted to generate physical examples of the solutions. From the examples the group provided, we can infer that they misinterpreted solution #2, perhaps because of a statement that Gaston made that a constant force would result in the ball moving at a constant velocity. The examples that Winfred provided to support solution #2 related to an additional force acting on an object, requiring compensation from the first force to maintain the constant force required for a constant velocity. On the other hand, the example the group generated to support solution #3 was that of another object colliding with and sticking to the ball. As the group evaluated the two solutions, they did so in terms of these examples.

Specifically, when Gaston, Igor, and Winfred were thinking about the object sticking to the ball,

they reported all of the difficulties in making that situation happen perfectly; Winfred said, "you'd have to take into account the initial velocity of the gum, the initial velocity of the object hitting it, and the object sticking to it and the translational, the translational of all that," which indicates that it would take considerable effort to make this happen. By contrast, possibilities were wide open for what would happen after the additional force occurred, so solution #2 would be more likely: "But since it doesn't give any information on what happens after you apply the force then this (solution #2) is the most likely situation that is going to happen."

The parameters of solution #3 were very well-defined for Gaston, Igor, and Winfred: the mass could only change by a collision that increased the mass of the ball, and even then the situation would need to be a very special, one-in-a-million sort of setup that appropriately accounted for all of the relevant physical quantities. On the other hand, solution #2 was more nebulous; there could be any number of forces that could oppose the net force, and any number of things could happen after that force is applied. Because this group had no reason to believe that one set was completely empty, they chose against the one that was more sparsely populated. Of course, pure deduction would prohibit this strategy because there were live possibilities in the set that was eliminated. However, the nature of the task was to choose the correct solution, and this group used a strategy that they believed would give them the best chance of being correct.

Ana and Sally used a sort of entropic selection as well in Section 6.2.8. In that case, Sally liked solution #2 because it was more "open-ended" than solution #3. After they had completed all of their problems, the researcher asked her specifically what she meant by that.

Researcher: When you were looking at this problem right here, which is the ball in motion –

Sally: I really –

Researcher: I'll phrase it for you – the ball in motion for a long period of time and you had this –

Sally: Equal and opposite stuff.

Researcher: Yeah. And you were trying to choose between solutions #2 and #3, and at the very end, Sally, you said, "Solution #2 is more open-ended so yeah," and then

you chose that. Do you remember what you meant by –

Sally: More open-ended?

Researcher: Solution #2 being more open-ended, mm-hmm.

Sally: I think that it just leaves more room for, I think with questions worded like these, that there's a lot of room for little things that can –

Ana: Be taken into account for –

Sally: Yeah, and I think by more open-ended I meant that, you know, for every force there is an equal and opposite to keep it, you know? Am I saying that right?

Ana: Yeah, I mean I, to me you are.

Sally: That's what I was thinking about. Was, you know we hear this a lot, (Reading from solution #2) there's some additional force that must have opposed it. It's more open-ended than saying that the mass had to change because I was thinking how we said that something must have stuck on it, I don't know how else the mass could have changed unless it was, something that deteriorated or something, I don't know. But I guess that's what I thought, I guess with the definition, that's just a simple definition and, I'm not sure.

Sally's response was that she could only think of one way the mass of the ball could have changed: something could have stuck to it. While this response indicated one more way that the mass of the ball could have changed than she had initially thought (she had initially indicated that she couldn't see how it could change), it was only one possibility for that set. On the other hand, the statement that there was an additional force acting on the ball was more "open-ended" to Sally. She did not elaborate on that point, but it is likely that she meant that there were more possible "other forces" than there were ways to make the ball's mass change. In fact, two possible "other forces" were listed on solution #2 itself: "gravity or air resistance could have acted." Furthermore, Sally being skeptical of the math on solution #3 was much like Winfred pointing out the sheer number of ways an attempt to change the mass would fail. For Sally, there were many opportunities in the mathematics to slip and make an error, and there was only one possible correct way of doing it. On

the other hand, the phrase "equal and opposite" was vague enough to allow many more possibilities. Unfortunately, all of those possibilities turn out to be wrong, while the single possibility afforded by solution #3 is correct.

These two examples of groups selecting the solution that they thought provided the most possibilities suggest one way reasoners may decide between options, especially when they find themselves unable to deductively eliminate one of them. Such reasoners may pick the possibility set that allows the most space for possibilities. This heuristic, while perhaps effective in everyday life, is not appropriate for solving deductive physics problems. Indeed, neither of the groups for which we have evidence of entropic selection used the momentum principle at all (Ana and Sally briefly tried to follow the reasoning on solution #3 before eventually moving on only to claim that something about it didn't seem correct). Entropic selection therefore may be an alternative that is available for reasoners who do not use proper deduction – or, more specifically, who do not actively flesh out and evaluate every possibility for a problem.

6.2.11 Discussion: using the momentum principle

Only four of the eight groups (Eduard; Claudette, Earl, and Gustov; Cristobal, Dolly, and Ike; and Fay, Marco, and Omar) showed any evidence of using the momentum principle when solving this problem, in spite of it being a fundamental part of each of the written solutions. Of those four groups, three chose solution #3 and one chose solution #1 because they indicated a belief that the mass of the ball could not change. In other words, the groups that used the momentum principle used deductive arguments and drew deductively valid conclusions. Only one of those four groups (Fay, Marco, and Omar) indicated that they were choosing between the solutions rather than selecting the single correct solution, and as discussed in Section 6.2.10, they did so was probably because of deep reasoning about the assumptions they were allowed to make for the problem.

So, if the four groups who showed evidence of using the momentum principle produced reasoning that could be represented and followed with the possibilities framework because it was deductive in nature, what did the other groups do? They focused on mechanistic reasoning, trying to identify for themselves whether it would be possible to reach the conclusions that the solutions reached through mechanistic means rather than by trying to follow the solutions. It is not that these were

the only groups to consider mechanism; indeed every group but one explicitly tried to think about how the mass of the ball could have changed (the one exception is Claudette, Earl, and Gustov; see Sections 6.2.3 and 6.2.12). The difference is that the groups that used the momentum principle identified a possible mechanism and moved on because a single mechanism is all that is required to show a possibility, whereas the other groups tended to be unsatisfied with a single mechanism and often compared the mechanisms between the different solutions, resulting in at least two cases with entropic selection, which was discussed in 6.2.10.

It is worth pointing out that the other two groups who did not show any evidence of using the momentum principle (Arthur, Otto, and Walter in Section 6.2.4; Hugo, Josephine, and Teddy in Section 6.2.7) may have also used entropic selection in reaching their conclusions, although there is not sufficient data to be confident in this. Arthur, Otto, and Walter attempted to arrive at a possible explanation for how the situation would be possible but do not succeed. Because they ruled that solution #2 was wrong, they chose between solutions #1 and #3. Because they could imagine no way to get the mass of the ball to change, they treated that possibility set as if it only contained possibilities that they were unable to consider given their limited experience with physics. To them, they felt it was more likely that there were no other, unconsidered, possibilities. As a result, they chose solution #1. Similarly, Hugo, Josephine, and Teddy may have believed that solution #3 had more possible ways of happening through a piece breaking off of the mass than solution #2, especially since they said that any additional force acting on the ball would necessarily slow it down.

A table displaying the results from this section in terms of the evidence we have that the groups used the momentum principle and whether they used entropic selection is shown in Table 6.1.

What Table 6.1 emphasizes is that for every participating group in the study, using the momentum principle was associated with deductive reasoning rather than entropic selection. This result is reminiscent of what was seen in the earlier pilot study regarding this problem. The participants who were reminded to use the momentum principle in that case were more likely to indicate that the mass changing was a possible solution to this problem (see Section 4.3). Similarly, the reasoning was qualitatively different between the groups that used the momentum principle and those that did not.

The fact that participants did not all use the momentum principle is interesting. In the "Two

Table 6.1: A summary of the evidence for groups' use of the momentum principle and whether they used entropic selection on the "Ball in Motion" problem.

Participants	Section	Evidence of Momentum Principle	Entropic Selection?
Eduard	6.2.2	Yes	No
Claudette, Earl, and Gustov	6.2.3	Yes	No
Arthur, Otto, and Walter	6.2.4	No	Unclear
Christobal, Dolly, and Ike	6.2.5	Yes	No
Fay, Marco, and Omar	6.2.6	Yes	No
Hugo, Josephine, and Teddy	6.2.7	No	Unclear
Ana and Sally	6.2.8	No	Likely
Gustov, Igor, and Winfred	6.2.9	No	Likely

Blocks" problem, there were indications that the quantities present in the given solutions helped the participants focus on which quantities they should include in their reasoning. However, they did this through omission or a unique instance; participants did not keep the mass of the blocks in their possibility sets because none of the possible solutions contained that quantity, but some groups were made aware of the friction at the interface between the blocks because it appeared in one solution (see Section 6.1.11). In the "Ball in Motion" problem, we have at least one instance where the unique solution did in fact make a group aware of a quantity that they had not considered (see Section 6.2.6), which strengthens the earlier argument.

One might postulate that the presence of a quantity or relationship in each solution would have the opposite effect of a quantity or relationship being absent from each solution: it should encourage participants to use that quantity or relationship. However, we do not have any evidence that this happened. Indeed, fully half of the groups showed no evidence of using the momentum principle even though it was explicitly mentioned on each solution. Why the groups did not use that principle, and whether participants would be more likely to use the momentum principle on this problem if they were explicitly prompted to do so are questions for further investigation. These questions would be worth investigating, as the quality of the deductive reasoning of participants in this study who used the momentum principle was substantially better than the deductive reasoning of those who did not use the momentum principle.

6.2.12   Discussion: belief bias and the changing mass

In Section 6.2.3, Claudette, Earl, and Gustov solved the Ball in Motion problem by considering the momentum principle and the definition of momentum, eliminating two possibilities and selecting solution #1, claiming that the situation was impossible. This group was unique in that respect; no other group followed this procedure (Arthur, Otto, and Walter, who also chose solution #1, did not show any evidence that choosing solution #1 was the result of implementing the momentum principle). The one aspect of their reasoning that was particularly remarkable was that they were the only group not to try to consider a mechanism how the mass of the ball could have changed. Instead, Earl matter-of-factly stated that "the mass certainly didn't change," and the rest of the group agreed. Gustov and Claudette each later indicated the same thing, that the mass of the ball could not have changed. The group never provided a reason why or defended this; it was simply an assumption that surfaced and remained unquestioned, even in light of a solution that explicitly challenged it.

The discussion between the participants in Section 6.2.3 presents an ideal case of belief bias, which was briefly described in that section (recall the definition of belief bias from Section 2.6.2: the believability of a conclusion affects which premises are used). The decision to use a specific premise due to a belief bias is not motivated by previous conclusions. Instead, it results from the believability of the conclusion (in this case, whether it's believable that a ball's mass could change). The premise then keeps that same value throughout the consideration of the problem, affecting the reasoner's thinking for that problem. Within the dimensions of the problem, the belief bias may be robust, resisting even explicit challenges. The possibilities framework is particularly well suited to identify situations of belief bias such as this. The possibilities framework can also represent biases, as was done in Figure 6.66, and distinguish between two apparently similar phenomena: belief bias and overlooking a relevant quantity.

This study, and especially the "Ball in Motion" problem, hints that belief bias is much more difficult to address than an overlooked quantity. Usually an intervention can be constructed to focus reasoners on an overlooked quantity, allowing them to flesh out the new realm of possibilities that are now available as a result. However, belief bias may require an intervention that does more than simply identify that quantity and provide an alternate value for it; Claudette, Earl, and Gustov

quickly reject solution #3 *because* it provides a different value than theirs (by stating that the mass of the ball could change). Moreover, the logical argument that was provided in that solution was completely ignored, as the group focused instead on the answer. This implies that even a well-constructed argument that results in a new value for a quantity that has already been assigned one by the reasoner might be completely disregarded! This potential for reasoners to disregard solutions is similar to the hints that appeared in both this problem and the "Two Blocks" problem that participants would sometimes misinterpret solutions (both written and spoken) when the reasoners tried to understand them without first setting aside their own possibility sets.

The possibilities framework can distinguish between errors, and it can identify instances of belief bias as opposed to instances of overlooked quantities. Much more research needs to be done to see whether the possibilities framework can be used to develop interventions that can address those issues, especially the particularly robust instances of belief bias within a problem. Ideally, reasoners performing deduction in physics should only use premises that originate in fundamental principles and definitions, earlier deductions, or given information in the problem. Perhaps the required intervention for belief bias will find its roots in developing ways to enforce this requirement.

6.2.13 Summary

For the Ball in Motion problem, participants who explicitly used the momentum principle exhibited reasoning that was more clearly deductive than those who did not by completely eliminating from consideration potential solutions that corresponded to empty possibility sets. Of the participants who did not use the momentum principle, at least some provided substantial evidence that they were choosing the solution that they thought had the largest possibility space, or the most possible ways of being correct. Additionally, the participating group that exhibited a belief bias provides hints as to the difficulties associated with changing the possibility sets of reasoners who assign a value to a physical quantity rather than simply overlooking that quantity. The possibilities framework could potentially help in addressing these issues, but much more research needs to be done to establish exactly how it could do so.

6.3   The "Two Pucks" Problem

For six of the seven groups, the "Two Pucks" problem was the first that the participants discussed, although it was the last problem they solved in their individual sessions. Unlike the other groups, Fay, Marco, and Omar discussed this problem last. The text of the "Two Pucks" problem is included in the appendix, but it is repeated here for continuity.

> Two identical pucks (mass $m = 0.15$ kg, radius $R = 0.05$ m, and moment of inertia $I = 1.875 \times 10^{-4}$ kg·m^2) are sitting at rest on a sheet of ice (neglect friction due to the interactions between the puck and ice). One puck has a thread (of negligible mass) attached to its center, while the other puck has a thread (also of negligible mass) wrapped around it. The threads are pulled by a constant force (of magnitudes F_1 and F_2 respectively) for 4 seconds at the free end of the thread (shown by a small circle below), at which time both pucks have a speed of 1.6 m/s. Over the 4 seconds, a length, L, of thread unwinds from the second disk.

> Top view of pucks sliding on ice (looking down on the ice from a location on the $+y$ axis). *Not drawn to scale*

> You find that the distance, d, is 3.2 m and the length, L, is 6.4 m. In order for the pucks to be traveling at the same speed after the 4 seconds, will the second puck need to be pulled with a stronger force? That is, will F_2 need to be larger than F_1?

This problem had the fewest number of correct responses of any of the problems that the participants solved individually, with only 6 of the 21 choosing the correct solution. In a sense, then, this problem proved to be quite difficult for the participants. An application of the possibilities

framework to the group discussions about this problem provides some insight into the difficulties the participants had when solving this problem, and it also helps us characterize their errors. Below are the written solutions that the participants were given to choose from.

6.3.1 The given solutions

Each of the written solutions that the participants were provided for the "Two Pucks" problem can be represented with the possibilities framework. This problem, as well as the "Close the Door" problem, requires the reasoner to compare two situations and explain which one better meets some criterion. To represent that, I provide parallel possibility sets below, labeling each one with the system to which it refers. Also, the heavy quantitative emphasis of the reasoning in this problem means that sometimes the considered value from a quantity is be represented as an expression rather than a direction or a numerical value.

"Two Pucks" solution #1

The first written solution reads as follows:

> For the <u>first</u> puck (thread attached at the center of the puck):
>
> System: Puck, Earth
>
> Surroundings: Thread
>
> Energy Principle: $E_f = E_i + W$
>
> $K_f + U_f = K_i + U_i + W$
>
> $\frac{1}{2}mv_f^2 + mgh_f = \frac{1}{2}m\cancel{v_i^2} + mgh_i + W$
>
> $\frac{1}{2}mv^2 + mgh_f - mgh_i = F_1 d$
>
> $\frac{1}{2}mv^2 + mg(\cancel{h_f - h_i}) = F_1 d$
>
> $F_1 = \dfrac{\frac{1}{2}mv_f^2}{d} = \dfrac{\frac{1}{2}\left(0.15\ \text{kg}\right)\left(1.6\ \frac{\text{m}}{\text{s}}\right)^2}{3.2\ \text{m}} = 0.06\ \text{N}$

For the <u>second</u> puck (thread wrapped around the puck):

System: Puck

Surroundings: Thread

Energy Principle: $E_f = E_i + W$

$K_{trans,f} + K_{rot,f} - K_{trans,i} - K_{rot,i} = W$

$$\frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2 = \frac{1}{2}m\cancel{v_i^2} + \frac{1}{2}I\cancel{\omega_i^2} + F_2(L+d)$$

$$F_2 = \frac{\frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2}{L+d} = \frac{\frac{1}{2}mv_f^2 + \frac{1}{2}I\left(\frac{v_f}{R}\right)^2}{L+d}$$

$$F_2 = \frac{\frac{1}{2}(0.15\ \text{kg})\left(1.6\ \frac{\text{m}}{\text{s}}\right)^2 + \frac{1}{2}(1.875 \times 10^{-4}\ \text{kg} \cdot \text{m}^2)\left(\frac{1.6\ \frac{\text{m}}{\text{s}}}{0.05\ \text{m}}\right)^2}{(6.4+3.2)\ \text{m}} = 0.03\ \text{N}$$

The first puck requires a larger force (0.06 N as opposed to only 0.03 N) to reach the same speed as the second puck.

Solution #1 applies the energy principle to the two pucks, solving each of them explicitly to determine the value of the force required for each one. Then it compares those values and concludes that the first puck requires a greater force to reach the same speed as the second puck. The possibility set for the first puck is shown in Figure 6.88.

$\vec{F} \cdot \vec{d} = (K_f - K_i)$			
$|\vec{F}|$	$|\vec{d}|$	$K_f - K_i$	$U_f - U_i$
.06 N	3.2 m	$\frac{1}{2}mv_f^2$	0
$[\neq .06\ \text{N}]$	3.2 m	$\frac{1}{2}mv_f^2$	0

Figure 6.88: The first solution of the "Two Pucks" problem uses the energy principle to solve for the force on the first puck.

The reasoning in Figure 6.88 is correct (even though it includes a superfluous term for the potential energy in the system, which it appropriately sets to zero), and it results in the correct

conclusion, that the force applied to the first puck is .06 N. However, solution #1 makes an error when trying to apply the energy principle to the second puck, as seen in Figure 6.89.

$\vec{F} \cdot \vec{d} = (K_{rot,f} - K_{rot,i}) + (K_{trans,f} - K_{trans,i})$			
$|\vec{F}|$	$|\vec{d}|$	$K_{rot,f} - K_{rot,i}$	$K_{trans,f} - K_{trans,i}$
.03 N	9.6 m	$\frac{1}{2}mv_f^2$	$(\frac{1}{2})I(v_f/r)^2$
$[\neq .03 \text{ N}]$	9.6 m	$\frac{1}{2}mv_f^2$	$(\frac{1}{2})I(v_f/r)^2$

Figure 6.89: The first solution of the "Two Pucks" problem uses the energy principle to solve (incorrectly) for the force on the second puck.

While solution #1 correctly solves for the distance over which the force is applied, by summing the length of the thread and the distance the puck moved, it makes a mistake in assigning the value "$(\frac{1}{2})I(v_f/r)^2$" to the change in the rotational energy of the puck. The substitution of "$\omega = (v_f/r)$" is only appropriate if the object is rolling without slipping. In this case, the puck is not, and the correct substitution should have been "$\omega = (2v_f/r)$," which would have given the correct result: the force on the second puck should have been .06 N.

"Two Pucks" solution #2

The second written solution reads as follows:

> For the <u>second</u> puck (thread wrapped around the puck):
>
> System: Puck
>
> Surroundings: Thread
>
> Angular Momentum Principle: $\vec{L_f} - \vec{L_i} = \vec{\tau}_{net}\Delta t$
>
> $\vec{L_f} - \cancel{\vec{L_i}} = \vec{\tau}_{net}\Delta t$
>
> $\vec{L}_{trans,f} + \vec{L}_{rot,f} = \vec{\tau}_{net}\Delta t$
>
> $\vec{r} \times \vec{p}_f + I\vec{\omega}_f = \vec{r} \times \vec{F}_2\Delta t$

$$\left\langle 0, r_z p_{f_x}^R, 0 \right\rangle + I \left\langle 0, \omega_f, 0 \right\rangle = \left\langle 0, r_z F_2^R, 0 \right\rangle \Delta t$$

(The puck will spin counter-clockwise as the thread unwinds, so by the Right Hand Rule $\vec{\omega}_f$ is in the y-direction)

For the y-components: $R p_{f_x} + I\omega_f = R F_2 \Delta t$

$R m v_{f_x}^{v_f} + I \left(\dfrac{v_f}{R} \right) = R F_2 \Delta t$, $v_{f_x} = v_f$ since the pucks travel in the x-direction.

$$F_2 = \frac{R m v_f + I \left(\dfrac{v_f}{R} \right)}{R \Delta t}$$

$$F_2 = \frac{(0.05 \text{ m})(0.15 \text{ kg})\left(1.6 \ \dfrac{\text{m}}{\text{s}}\right) + \left(1.875 \times 10^{-4} \text{ kg} \cdot \text{m}^2\right) \left(\dfrac{1.6 \ \frac{\text{m}}{\text{s}}}{0.05 \text{ m}}\right)}{(0.05 \text{ m})(4 \text{ sec})}$$

$$F_2 = 0.09 \text{ N}$$

For the <u>first</u> puck (thread attached at the center of the puck):

System: Puck

Surroundings: Thread

Angular Momentum Principle: $\vec{L}_f - \vec{L}_i = \vec{\tau}_{net} \Delta t$

$$\vec{L}_{trans,f} - \vec{L}_{trans,i} = \vec{\tau}_{net} \Delta t$$

$$\vec{r} \times \vec{p}_f = \vec{\tau}_{net} \Delta t$$

$$\left\langle 0, r_z p_{f_x}^R, 0 \right\rangle = \left\langle 0, r_z F_1^R, 0 \right\rangle \Delta t$$

For the y-components: $R p_{f_x} = R F_1 \Delta t$

$m v_{f_x}^{v_f} = F_1 \Delta t$

$$(0.15 \text{ kg}) \left(1.6 \ \frac{\text{m}}{\text{s}} \right) = 0.24 \text{ kg} \cdot \frac{\text{m}}{\text{s}} = F_1 \Delta t$$

$$F_1 = \frac{0.24 \text{ kg} \cdot \frac{\text{m}}{\text{s}}}{\Delta t} = \frac{0.24 \text{ kg} \cdot \frac{\text{m}}{\text{s}}}{4 \text{ sec}} = 0.06 \text{ N}$$

The second puck requires a larger force (0.09 N as opposed to only 0.06 N) to reach the same speed as the first puck.

Solution #2 uses the angular momentum principle on the two pucks, solving each of them explicitly to determine the value of the force required for each one. Then the solution compares those values and concludes that the second puck requires a greater force to reach the same speed as the first puck. Even though the solution solves for the force applied to the second puck first, let us first consider the possibility set for the first puck in Figure 6.90. Notice that while the solution makes overtures about using the angular momentum principle on this puck, the actual reasoning reduces to the linear momentum principle. As such, the simplest correct physical representation of this reasoning is actually that principle, as noted in 6.90.

$\vec{F}_{net} \Delta t = m \Delta \vec{v}$			
$|\vec{F}_{net}|$	Δt	m	$|\Delta \vec{v}|$
.06 N	4 s	.15 kg	1.6 m/s
[$\neq$ .06 N]	4 s	.15 kg	1.6 m/s

Figure 6.90: The second solution of the "Two Pucks" problem uses the linear momentum principle to solve for the force on the first puck.

This reasoning provides the correct result; the force on the first puck is .06 N. However, solution #2 makes an error when it applies the angular momentum principle to this problem. To simplify the representation, only the reasoning that uses the y-component of the angular momentum is presented in Figure 6.91.

As before, there is an error in that this solution substitutes the wrong value for ω, making $\Delta L_{rot,y}$ and therefore the magnitude of the force applied to the puck incorrect.

$F_x(R sin\ x)\Delta t = \Delta L_{trans,y} + \Delta L_{rot,y}$				
F_x	$(R sin\ x)$	Δt	$\Delta L_{trans,y}$	$\Delta L_{rot,y}$
.09 N	.05 m	4 s	$Rp_{f,x}$	$I(v_f/r)$
$[\neq .09\ N]$	.05 m	4 s	$Rp_{f,x}$	$I(v_f/r)$

Figure 6.91: The second solution of the "Two Pucks" problem uses the angular momentum principle to (incorrectly) solve for the force on the second puck.

"Two Pucks" solution #3

The third written solution reads as follows:

> For the <u>first</u> puck (thread attached at the center of the puck):
>
> System: Puck
>
> Surroundings: Thread, Earth, ice sheet
>
> Momentum Principle: $\vec{p}_f = \vec{p}_i + \vec{F}_{net}\Delta t$
>
> $\vec{F}_{net} = \langle F_1, F_{ice} - F_{Earth}, 0 \rangle$
>
> $\langle p_f, 0, 0 \rangle = \langle \cancel{p_i}, 0, 0 \rangle + \langle F_1, F_{ice} - F_{Earth}, 0 \rangle \Delta t$
>
> $\langle mv_f, 0, 0 \rangle = \langle F_1, F_{ice} - F_{Earth}, 0 \rangle \Delta t$

For the y-components:	For the x-components:
$0 = (F_{ice} - F_{Earth})\Delta t$	$mv_f = F_1 \Delta t$
$F_{ice} - F_{Earth} = 0$	$F_1 = \dfrac{mv_f}{\Delta t}$
$F_{ice} = F_{Earth}$	$F_1 = \dfrac{(0.15\ \text{kg})\left(1.6\ \dfrac{\text{m}}{\text{s}}\right)}{4\ \text{sec}} = 0.06\ \text{N}$

> For the <u>second</u> puck (thread wrapped around the puck):
>
> System: Puck

Surroundings: Thread, Earth, ice sheet

Momentum Principle: $\vec{p}_f = \vec{p}_i + \vec{F}_{net}\Delta t$

$\langle p_f, 0, 0\rangle = \langle \cancel{p_i}, 0, 0\rangle + \langle F_2, F_{ice} - F_{Earth}, 0\rangle \Delta t$

$\langle mv_f, 0, 0\rangle = \langle F_2, F_{ice} - F_{Earth}, 0\rangle \Delta t$

For the x-components:	For the y-components:
$mv_f = F_2\Delta t$	$0 = (F_{ice} - F_{Earth})\Delta t$
$(0.15 \text{ kg})\left(1.6\ \dfrac{\text{m}}{\text{s}}\right) = F_2\Delta t$	$F_{ice} - F_{Earth} = 0$
$F_2\Delta t = 0.24\ \text{kg}\cdot\dfrac{\text{m}}{\text{s}}$	$F_{ice} = F_{Earth}$
$F_2 = \dfrac{0.24\ \text{kg}\cdot\dfrac{\text{m}}{\text{s}}}{\Delta t}$	
$F_2 = \dfrac{0.24\ \text{kg}\cdot\dfrac{\text{m}}{\text{s}}}{4\ \text{sec}} = 0.06\ \text{N}$	

Both forces need to have equal magnitudes (0.06 N) in order for the pucks to have the same speed at 4 seconds.

This solution, although it appears very simple, is actually correct throughout. Note that while it includes information about the y-components of the force to be thorough, it handles those correctly, reporting the relatively unimportant conclusion that the force on the ice in the y-direction needs to have the same magnitude as the gravitational force on the puck in the y-direction. This piece of reasoning, while correct, is not necessary to solve the problem. Therefore, I represent only the possibility sets for the reasoning in the x-direction. Like the other two solutions, solution #3 solves for the force applied to the first puck correctly. And, unlike solution #2, solution #3 makes no secret that it uses the linear momentum principle to do this. This reasoning is shown in Figure 6.92.

Solution #3 then repeats this exact same reasoning for the second puck, even though it varies the appearance of the solution slightly. To emphasize that this solution applies the same reasoning

to the second puck, I present it in Figure 6.93, which is identical to Figure 6.92.

$F_x \Delta t = m(v_f - m_i)$				
$|\vec{F}_{net}|$	Δt	m	v_f	v_i
.06 N	4 s	.15 kg	1.6 m/s	0 m/s
[$\neq$ .06 N]	4 s	.15 kg	1.6 m/s	0 m/s

Figure 6.92: The third solution of the "Two Pucks" problem uses the linear momentum principle in the x-direction to solve for the force on the first puck.

$F_x \Delta t = m(v_f - m_i)$				
$|\vec{F}_{net}|$	Δt	m	v_f	v_i
.06 N	4 s	.15 kg	1.6 m/s	0 m/s
[$\neq$ .06 N]	4 s	.15 kg	1.6 m/s	0 m/s

Figure 6.93: The third solution of the "Two Pucks" problem uses the linear momentum principle in the x-direction to solve for the force on the second puck.

The three solutions each start from a different fundamental principle, which is used for both pucks in that solution. Solution #1 used the energy principle, solution #2 used the angular momentum principle (even though the reasoning for the first puck could be reduced to the linear momentum principle), and solution #3 used the linear momentum principle. Because there were three different starting places, some of the participants in their individual sessions demonstrated a preference for one principle over another in solving this problem. Also, the three solutions contain different quantities; most notably, solution #3 does not contain any information about the rotation of the pucks. Many of the participants chose (or rejected) a solution based on the quantities it contained. For more information about the reasons participants had for rejecting solutions, see Chapter 7.

Many of the participants' reasons for rejection were reinforced during the discussions in the two groups presented below as case studies. The participants provided further justification to their peers for their reasoning and tried to come to a consensus in these discussions. After exploring these two group sessions, we take a broad look at the major reasons participants seemed to have such a hard time using deductive reasoning on this problem. It is worth pointing out that when participants discussed their reasoning in these first group sessions, the reasoning they reported using

did not contradict their think-aloud data from their individual sessions; furthermore, because the participants were not merely asked to report their reasoning but also to continue to use it, their statements in the discussions are analyzed just as if the participants were first facing the problem during the group session itself.

6.3.2 Unable to choose (Christobal and Ike)

The group session that involved Christobal, Dolly, and Ike was filled with deep reasoning and discussion. As a graduate group, these participants were able to use many tools for reasoning. With these tools, they found and were able to thoroughly discuss the errors in solutions #1 and #2. However, they were unable to recognize and choose the correct solution (solution #3), because they were convinced that the correct solution needed to carry information that referred to the rotation of the second puck. This blindness to the correct solution is a particularly intriguing case of belief bias, considering the depth of reasoning that the participants used.

To explore this group session, I first briefly discuss the individual sessions of both Christobal and Ike. Christobal eventually chose solution #1, while Ike did not actually choose a solution in his individual session – the only time in the entire study someone was unable to make a selection. Instead of providing their entire transcripts, I summarize their reasoning and present segments of their reasoning to support the summary. Their entire transcripts can be read in the appendix.

Christobal's individual session

After reading the problem, Christobal began thinking about the situation and immediately noted the different distances over which the two forces act.

> So, let's see, what essentially do you know? Four seconds, length L of thread, okay, and looks like, and D must be the distance that the, okay so D is the distance that it [the puck] moves, and L is the length that the, L is the length that the thread comes out. Okay, that makes sense. So basically we have one force acting on a distance D, where D is basically going to be, the amount that the force is acting on, the other one is acting on a distance L plus D. Okay, that seems to make sense.

Christobal fleshed out the given information appropriately; he identified that in the case of the non-rotating puck, the force acts over the 3.2 meters the puck moves, while in the case of the rotating puck, it acts over 3.2 + 6.4, or 9.6 meters.

He then began looking over the solutions and noted that all three gave the same result for the non-rotating puck. Next, he looked at the first solution's attempts to solve for the force on the rotating puck.

> The thread wrapped around the puck, so we have the energy, oh where does it say, okay it gives you the length of L, 0.15 kg times 1.6 m/s squared plus $\frac{1}{2} \times 1.875 \times 10^{-4}$ kg/m second, so that's the moment of inertia, 1.875 lovely, 1.6 m/s zero, 1.6 m/s divided by 0.05 m, one half $I\omega$ squared, where is our, well I do think you messed up there my good friend. This right here should be ω (picks up a red pen and makes a correction on solution #1) which is not the translational, but rather the velocity coming around that and so, sorry nope, no points for you.

He discovered the error in solution #1, that the substitution "$\omega = v_t/r$" is wrong, and he then moved on to the next solution. He scanned solution #2 and did not make note of any errors in that solution. Then, he scanned solution #3, where he discovered something he didn't like.

> **Christobal**: Second puck, $F2$ F minus, okay I don't know why you're bothering to do that, 0.05 kg, $F2$ (inaudible muttering) I am afraid and that is where you lose me. Um,
>
> **Researcher**: Just remember to keep on talking what you're thinking.
>
> **Christobal**: Yes. So I'm afraid you (solution #3) make no sense at all. 0.15 kg times 1.6 m/s is equal to $F2$ times Δt. Well, yes, but there should be another term in there. What about the angular momentum? And you are not even bothering with that, are you? Well, this equation is wrong because it doesn't include angular momentum. So, that makes sense. That's where you went wrong. Kilogram meter per second, you're not even dealing with angular momentum so, okay.

Christobal examined the possibilities that solution #3 affords and found that it did not include the angular momentum quantity in its analysis. We do not know much about the possibility set that

Christobal held for this problem, but we do know that his set contained an angular momentum term. The fact that solution #3 did not include such a term in its analysis was apparently sufficient grounds for rejection. As noted before, many of the other participants also rejected solution #3 for the same reason. In fact, each of the six graduate students in their individual sessions required a rotational term in a solution to the problem, whether that was rotational energy, rotational momentum, or torque. After making this rejection, Christobal did not again look at solution #3.

Realizing that he had eliminated two of the three solutions, Christobal was prepared to pick solution #2. However, he decided to first verify that solution before accepting it. In this second examination of the problem, he found something else to disagree with.

> And so let's see this. See right here (pointing to a line about halfway down the first page of solution #2), we are going to say that momentum – but see here we are adding momentum and angular momentum, R times momentum, well that is fine, but it is not actually R times momentum, it is R cross momentum, and this momentum is in the same direction and so here we don't have that (crosses off something in solution #2) because R cross MVF is equal to zero. that's our issue there, lovely. Well, I am glad we figured that one out.

Possibly because he was unfamiliar with the terminology that is used in the problem, Christobal interpreted the written term "L_{trans}" as linear momentum, and stated that it could not be added to angular momentum. Then, he looked more closely at that term and recognized that it was $\vec{r} \times \vec{p}_f$. At this point, he made a very curious mistake: he claimed that $\vec{r}$ was in the same direction as $\vec{p}$. One possible representation of part of his possibility set for this error is shown in Figure 6.94.

$\hat{F}_{pull}$	$\hat{r}$	$\hat{p}$
$\rightarrow$	$\rightarrow$	$\rightarrow$
$\rightarrow$	$[\neq \rightarrow]$	$\rightarrow$
Other possibilities not considered		

Figure 6.94: One possibility set that Christobal may have held, which contains the direction of the position vector in the same direction as the pulling force.

As Figure 6.94 shows, Christobal might have interpreted the position vector not as the vector pointing from the point of contact between the thread and the puck and the center of the puck, but rather as the vector pointing from the original position of the puck to the later position of the puck – which would be the same direction as the pulling force and the same direction as the momentum of the puck. Of course, this is just one possible explanation, and we can not be sure what he was thinking, only that he did make this error, which resurfaced later in his group session with Ike.

Finally, Christobal re-examined solution #1 and decided that the mistake he originally pointed out for that solution was not really an error after all.

> Ah, here is something I can work with (solution #1)! Okay, see you are saying that the puck is going to rotate at the same speed at which it goes. I believe that now. So we can say that ω is equal to V Translational over R, not that, okay.

Without providing a reason, he revoked his previous (correct) identification of an error on solution #1, actually drawing a line through the notes he had previously made on the solution. He selected solution #1 and prepared himself for the group session.

Ike's individual session

Meanwhile, in a different interview room at approximately the same time, Ike was also struggling with the "Two Pucks" problem. Like Christobal, he thought about the problem for a moment before moving on to the solutions. However, Ike started with a staggering statement. He immediately identified the key to this problem, but he did not realize it and let it slip away.

> And, my confusion was that both of these wind up with the same linear displacement after four seconds, but apparently you are assuming that they do and that the force is, well, constant force and if they both have the same speed after four seconds they should have both been experiencing the same acceleration, which means they both would've traveled the same distance. Okay, that makes sense.

Ike's chain of reasoning brought him through realizing that the same speed after four seconds implies the same acceleration for the two pucks, which means that they must have traveled the same distance. This reasoning was absolutely correct and is represented by Figure 6.95.

$\vec{a} = \Delta\vec{v}/\Delta t$		
$\vec{a}_1/\vec{a}_2$	$\Delta\vec{v}_1/\Delta\vec{v}_2$	$\Delta t_1/\Delta t_2$
1	1	1
$[\neq 1]$	1	1

→→

$\vec{a} = \vec{v}_i t + (\frac{1}{2})xt^2$			
$\vec{a}_1/\vec{a}_2$	$\vec{v}_i$	t_1/t_2	x_1/x_2
1	$\vec{0}$	1	1
1	$\vec{0}$	1	$[\neq 1]$

Figure 6.95: Ike's reasoning that relates the change in the pucks' velocities to the distance they traveled.

However, what Ike missed was that if the pucks had the same acceleration, they would have also experienced the same force. That reasoning would be represented by Figure 6.96.

$\vec{a} = \Delta\vec{v}/\Delta t$		
$\vec{a}_1/\vec{a}_2$	$\Delta\vec{v}_1/\Delta\vec{v}_2$	$\Delta t_1/\Delta t_2$
1	1	1
$[\neq 1]$	1	1

→→

$\vec{F}_{net} = m\vec{a}$		
$\vec{F}_{net,1}/\vec{F}_{net,2}$	m_1/m_2	$\vec{a}_1/\vec{a}_2$
1	1	1
1	1	$[\neq 1]$

Figure 6.96: The conclusion Ike did not draw: the pucks must have experienced the same net force.

In other words, Ike was one possibility set away from cracking this problem. It is of course impossible to even speculate why he did not associate the acceleration of the pucks with the net force acting on them. All we can say is that something inhibited this association such that he did not generate (or at least he did not attend to) the second possibility set in Figure 6.96.

At this point, Ike began looking over the solutions. He rejected solution #1 for the same reason that Christobal originally found: solution #1 made an inappropriate substitution for ω.

> And, okay, $F2$ is equal to L plus D, they say because – and I would imagine the reasoning is that you've acted, you've pulled this one out for 6.4 m with that force in addition to pulling the puck itself with that same force, the 3.2 m translationally. So what they have done is that they have set this up so that the, or they say the initial speeds are zero so the initial energies are zero. So, the final force is equal to, the final force is equal to and they say the final kinetic energy and here they make a substitution that the final, rotational kinetic energy is based on ω, which is V final over R, and I don't see where they are allowed to say that the puck is rotating. Okay, so I'm going to go ahead and, sorry let's try that again. I don't – the puck is clearly rotating, I don't see where they

get to claim that ω is VF over R. In fact, I see nothing in here that would suggest that, so, let's go ahead there.

He moved on to solution #3 and, just like Christobal, he rejected this solution because it did not consider a term – in this case, Ike claimed that term to be "torque."

For the second puck, again momentum principle. No initial momentum, MVF equals that, it's looking suspiciously like the first puck, and indeed they neglect torque.

Turning to solution #2, he had eliminated the other two solutions and could have just accepted this solution as correct. However, like Christobal, he decided to read it over to verify that it was correct. He verified its solution for the first puck and moved on to the second puck, where he made the agonizing realization that solution #2 made the same inappropriate substitution for ω as did solution #1. This realization caused a conflict within Ike, preventing him from drawing a conclusion as the fifteen minutes he was given to complete the task expired.

Okay, still don't know what ω is, and once again they assume that – once again they assume that ω is linear speed over radius, and that would be fine if the puck were rolling (silence). And, yeah, we are looking down on the ice, so the puck shouldn't be rolling. Okay, well, that makes two of these solutions that have assumed that; that's one and two. This one (solution #3) I feel very comfortable saying is wrong, because – okay, why did I say that? – oh, yeah, solution #3 didn't include rotation (silence). Of course, if two solutions made the same mistake then I guess the third solution would be the right one, but let's look at it. Now, why would the – why would this individual feel free to ignore rotation? The easy solution is that they (solution #3) are wrong. I like the easy solution. The quickest solution.

Ike was completely correct in recognizing that two of the solutions reached the incorrect result by implicitly making an inappropriate assumption that the puck was rolling without slipping, and he even acknowledged that because solutions #1 and #2 are wrong, solution #3 must have been correct, but he simply could not wrap his head around this possibility. For Ike, it was simply *not possible* that the correct solution did not contain any information about the rotation of the puck.

Time expired, and he gave up. After a short break, he and Christobal discussed this problem in the group setting.

The group session

Christobal, Dolly, and Ike were brought together for a group discussion. They were given ten minutes to talk about this problem to try to reach a consensus as to which of the written solutions was correct. Ike was randomly selected to start the conversation by presenting his solution to the problem, but he invoked his right to pass. Dolly was then selected to begin the conversation. She did so by reviewing her process for solving the problem.

> **Dolly**: This is a fun little problem. Classical, relearning classical in 15 minutes doing a problem is not my strong suit, so I, well let's see, so I read the problem, then I went through the solutions. I marked solution #1 as the correct solution, partially because it looks right, and partially because I like dealing with energy more than I like dealing with angular momentum, and so I did not spend a whole lot of time looking at solution #2, so if anyone has an argument for solution #2, go ahead. Um (clears throat), I (chuckles), I do not like solution #3 because student did not include any rotational stuff in the solution, and I didn't feel that that was right at all, although I do feel that they did the first half of the problem correctly. Um, and then solution #2, I am actually really not sure about, but the energy one looked right, so I marked the energy one as right and I don't know if that is like a full presentation or if you want me to work it out on the board, but it's –

In short, she eliminated solution #3 for the same reason as both Christobal and Ike: there was no rotational term in it. Then, she was uncomfortable with solution #2 and angular momentum, so she picked solution #1 which she felt was plausible. She did not feel confident in her choice, and Ike was quick to question her regarding "intuition," which for him was that the answer in solution #2 should have been preferable.

> **Ike**: Can I ask –
>
> **Dolly**: Go for it.

Ike: What you are thinking, uh, my intuition says that the second puck would require a larger force because you'd have to add (Dolly says "ah!" and points her finger at Ike) translational and rotational energy.

Dolly: I, I agree with that, that was what my intuition said, and that would be I believe just the difference between solution #1 and solution #2, and I was not very into doing, or looking through solution #2 all that much, so I still stuck with solution #1.

Dolly agreed with Ike, but stated that she didn't go through solution #2 to see if it made any errors. Christobal picked up on this and contributed to the discussion the errors he found with that solution.

Christobal: I looked through solution #2, and I found their mistakes.

Ike: Oh, really? What was that?

Dolly: Okay.

Christobal: Well, they added, they multiplied R times MV, as the –

Dolly: As R cross P.

Christobal: Yeah, and then they divided by $R\Delta t$, well, I don't know where they got that, you cannot add those things up.

Dolly: They, wait, oh you mean for units? Units didn't work out (crosstalk with Ike)

Ike: Oh, that's right, yeah, I remember why I didn't like that one. Because VF over– they used VF over R, and there was no indication that –

Christobal: Well, you can use VF over R,

Ike: Where does it say that the puck is rolling?

Dolly: Well,

Christobal: Well, it is, it would, you can consider it rolling because it's unwinding, and so it makes sense that it would be unwinding at the same rate, but the problem is that they are adding, they are trying to transform –

Dolly: Would it?

Ike: Hmm, okay.

Christobal: Angular momentum.

Ike: No, wait a second. No, I don't believe that at all.

Dolly: Yeah (Apparently agreeing with Ike)

Ike: No, no no no.

Dolly: I don't know if you can assume that too –

Ike: (To Christobal) You are going to have to convince me of that. There is a white board and two markers.

Christobal: Well, the problem (solution) #1 assumes the same thing.

Dolly: Well, problem, problem (solution) #1 assumed the energy.

Ike: (Crosstalk over Dolly) Exactly. And I did not pick either one of them to be correct.

Christobal: Yeah, I liked, I liked problem #1 – I mean solution #1.

Ike: Well, both of them actually in point of fact said that ω over, or, yeah ω is VF over R.

Dolly: Yup, mhmm.

Ike: And problem 3, or solution #3, is clearly wrong, so both of these have something I dislike.

This segment of dialogue explicitly set out the conflict that Ike had been facing at the end of his individual session: one of the solutions should be correct, but solutions #1 and #2 both committed a physical error, substituting an incorrect value for a quantity, while solution #3 omitted a quantity that each group member believes should be there. The remedy for this quagmire is that either solution #3 was correct, or else the substitution of v_f/r for ω was actually okay, in which case there should have been a different error for one of the other two solutions. Ike did not feel comfortable doing either, but Christobal preferred the latter alternative and argued for it, trying to convince Ike that substituting v_f/r for ω would have been reasonable.

Christobal: Well, I think that VF over R makes sense.

Ike: Okay.

Christobal: Because, you're, well think of it this way, so you are pulling at this point D (pointing to the problem statement), which means that if you think about the, translational momentum that's just pulling from the center, if you think about this momentum (apparently pointing to the second puck), it is pulling from the side opposite, and so this point right here (pointing to problem statement) is actually not going to move until –

Ike: Ah, see, that, that is where I had an issue. I saw absolutely nothing wrong, and I clearly did not have ice and string to try this on my own, I saw nothing wrong with it spinning faster than it was actually translating.

Christobal: Well, I mean, that is, it is possible that it would spin faster than it actually translates.

Ike: Right, so I was totally uncertain as to where they got that V over R from. but since they both did it, and [solution] three is clearly wrong, I spent 15 minutes dithering.

Dolly: So, you didn't actually like solution #2?

Ike: Or solution #1 for that matter. (Dolly laughs)

Now Ike assigned an actual physical meaning to the substitution for ω. The explicit discussion about possibilities proved to be particularly enlightening. Ike proposed the possibility set in Figure 6.97, with which Christobal agreed.

$\omega = (\Delta\theta/\Delta t)/r$	
ω	$\Delta\theta/\Delta x$
$\Delta v/r$	1
$\neq \Delta v/r$	> 1
Other possibilities not considered	

Figure 6.97: Omega can not be defined by v_f/r if it is possible that the puck spins faster than it rotates.

This set of possibilities very clearly did not contain only one entry. In order to enact a substitution of one value for another, that value must be precisely defined as the only possibility, meaning that

there should only be one entry in the table. The explicit acknowledgment of the multiple entries should favor Ike, as he correctly reasoned that the substitution would not be permitted if there existed other possibilities.

So, Christobal turned to the other strategy, which was pointing out a different possible error in solution #2.

> **Christobal**: Well, solution #2 I did not like because they are adding angular momentum, and, uh –
>
> **Dolly**: Rotational momentum
>
> **Christobal**: Rotational momentum, I mean –
>
> **Dolly**: Yeah
>
> **Christobal**: I mean, yes, you can do that if your Rs are the same, but their Rs are not the same, because if you look here (pointing to the problem statement?), R cross, this term right here, they are saying R times MVF, well it should actually be R **cross** MVF, and if you look at the diagram (turns the diagram toward Dolly), R cross MVF is going to be zero because it is in the same direction (inaudible).
>
> **Ike**: That's the second puck.
>
> **Dolly**: Yeah
>
> **Ike**: They start with the second puck and go to the first.
>
> **Christobal**: Well, actually, that's fine. If, I'm sorry, that is the first puck, anyway (indicating he had meant the second puck, which is the first part of the solution), R cross MVF here this is going
>
> **Dolly**: It's going to be perpendicular.
>
> **Ike**: It *is* perpendicular.
>
> **Dolly**: Yeah, so it actually should be fine.
>
> **Christobal**: But you still can't add them like that.
>
> **Ike**: Sure you can. You have got a radius times a, you've got a length times a momentum.

Christobal: (Muttering, crosstalk, sounds like) Length, radius, no? Times R, divided by R –

Ike: Yeah, the units work out.

Dolly: The units work out fine? So,

Ike: so are we about to spend five more minutes dithering as I did?

Dolly: No, I think we are all – (laughs)

Christobal: Well, let me think now...

Christobal repeated the error he found for solution #2, and Dolly and Ike pointed out quickly that Christobal's suggestion was not an actual error. Again, the group returned to the same impasse as before, and Ike quipped about it, asking if the group was going to continue to dither. They turned their attention to solution #3 briefly, and they definitively concluded that, if nothing else, that solution was definitely *not* correct.

Ike: Can we,

Dolly: Let's see,

Ike: [Can we] clearly rule out solution #3 as incorrect?

Dolly: Yes.

Ike: Anyone opposed?

Christobal: Yeah, you're okay [to eliminate that solution].

Dolly: We can rule out solution #3.

Ike: (Emphatically turns solution #3 over and slaps it down on the table).

Having eliminated the correct solution, and possessing the experience and ability to identify physics errors in the other two problems, they painted themselves into a corner and resorted to a discussion about what would be more intuitive in this situation. Resorting to intuition was a turning point for this group, who up until this point had argued within physical principles.

Dolly: And I agree with you in term – intuitively, specifically, that it doesn't, it makes sense that you would have to pull harder on the second [puck] –

Christobal: (Looks up at Dolly) No! That doesn't make sense. You'd have to, you'd have to pull harder on the, on the one that isn't connected here (points to a diagram).

Dolly: Wait, which one, which one do you think you have to pull harder on? (Christobal points to the diagram again) the first one?

Christobal: Yeah.

Ike: Okay. Now you *really* have to draw it out for me.

Dolly: Yeah. I would intuitively say the second one, but –

Christobal: Well, I think for the second one you would have to put in more energy, but you are going to do it over a larger distance. (Long pause)

Ike: Hmm. (chuckles).

Dolly: (Very softly, apparently repeating Christobal) More energy over a longer distance.

Christobal: Well, think of it this way.

Dolly: Say it, say it again, yeah.

Christobal: Suppose I pulled this right here (takes a marker and places it on its side and tries to gesture pulling it), right? (Dolly affirms) would it take more energy for me to, (apparently correcting himself) more *force*, for me to pull like this (gestures with the marker, but is hidden by Dolly's head) or like this (again he gestures but it's not in the camera view)?

Dolly: I don't know. I don't actually remember. Or can think through right now.

Christobal: Well, I think it would be definitely, I think definitely solution #1 is correct.

Christobal held that intuitively it should be easier to pull a puck that spins. He made an interesting slip of the tongue when he asked about the energy required in each situation, as that *would* be different in the two situations, because of the different distances over which the force is acting. It is worth noting that the argument that Christobal made harks back to the very

first elements of his reasoning about this problem. He apparently kept those with him, calling his argument with those points "intuition." Apparently, at least for this instance, his reasoning was not fleeting in that conclusions that he made early on were kept close in working memory.

Ike: So, it appears to be more than just quantum where I have no intuition.

Christobal: (To Ike) Well, what did you say, did you say solution #1 was correct?

Ike: I said I liked neither of them.

Christobal: Oh, so you said solution #2 was correct?

Ike: Uh, no actually I was going to mark them both wrong because of them did the VF over R thing.

Christobal: Well I think the VF over R thing isn't a terrible, isn't a terrible idea.

Ike: Also, my intuition said that puck one should require less force.

Christobal: Puck one should require less force? See that doesn't make any sense, no.

Dolly: (Simultaneously with previous statement) See, that's what my intuition says too.

Christobal: No, puck one is going to require more force.

Dolly: Even when you are putting backspin on puck two?

Ike: And the more I think about it, you can't say that, I'm still not convinced that the final angular speed is VF over R, so there's no way of telling how much force we need.

Dolly: I still think puck two needs more force.

Christobal: (To Dolly, asking her to repeat) Sorry?

Dolly: I still think puck two needs more force.

Christobal: Well, it needs more, more energy, but it needs less force over more distance. (Pause)

Dolly: I guess what's weird is that the pucks themselves are moving the same distance. You are, when you say more, less force over a larger distance –

Ike: Remember that –

Dolly: Tell me.

Ike: See here, your force is the length of a string you have unraveled in the second one.

Dolly: Okay.

Ike: So it is actually 9.6 m. Wait, wait wait wait, no it is not. Oh well (apparently indicating uncertainty) let's hope this is not on the qual[ifying exam in graduate school].

Dolly: (Laughs).

The group continued to get nowhere by trying to appeal to their intuition. That they were unsure whether the distance under consideration when reasoning about this problem should be 3.2 or 9.6 meters is not surprising; because that quantity is superfluous to the correct reasoning, it could have any assigned value and not affect the calculation. Consider this debate within the possibilities framework. The correct relationship for this problem is to consider the momentum principle on puck 2. However, the participants added the quantity "distance over which the force acts," or d, for short. This reasoning is represented by the possibility set in Figure 6.98.

$F_x \Delta t = m(v_f - m_i)$						
$|\vec{F}_{net}|$	Δt	m	v_f	v_i	d	$\vec{L}$
	4 s	.15 kg	1.6 m/s	0 m/s	3.2 m	
	4 s	.15 kg	1.6 m/s	0 m/s	9.6 m	
Other possibilities not considered						

Figure 6.98: Christobal, Ike, and Dolly's reasoning, which added superfluous physical quantities related to puck 2.

These additional terms interfered with their ability to see the relationship that would hold in this case, meaning they could not figure out the correct result for the net force acting on puck two. There were at least two possibilities in this set, so no valid deduction would be possible for this situation.

So, Christobal restated his position and once again tried to find a unique error in solution #2 that he could point to so that the group would accept the substitution for ω. The group briefly chased a red herring, but again their physics skill was solid enough to resolve the issue by using the right hand rule correctly.

Christobal: So, well, no, I mean, I think the first one is correct. What does the second one do wrong? $R \ M \ VF$ Plus $I \ VF$ over R –

Dolly: It sets change in momentum equal to change in momentum. (Pause). Unless you are supposed to subtract the rotational energy instead of adding it.

Christobal: Oh, that could be their mistake.

Ike: Wait, what?

Dolly: If you are supposed to subtract the rotational momentum instead of adding it?

Christobal: Well, let's think about the right-hand rule. So we have (begins gesturing with his right hand) R cross P, so that is going to be positive –

Dolly: I can't think of a situation. If you are supposed to subtract it in this situation, I cannot think of a situation in which you would add it in. That's the thing.

Ike: Here is where it would've been nice to have sat in on a class where they covered how angular momentum could be translational.

Dolly: (Laughing) That kept bothering me too, the notation.

Christobal: Well, yeah. $I\omega \ F$. So it's spinning counterclockwise, which means its ω is in the positive y (other participants agree)

Dolly: And the whole thing is moving.

Christobal: And if you do R, which is this way (gesturing with his right hand) Cross F, which is this way, we get another positive number.

Dolly: Okay.

Dolly suggested subtracting the rotational energy (but she actually said the "rotational angular momentum") rather than adding it, but she could not imagine a possible situation where one would actually do such a thing. Christobal used the right hand rule to affirm that the solution did not in fact make an error there, and that they were treating the vectors correctly.

With time running out, the group tried to reach a consensus, although it was clear that they were uncomfortable with the given solutions.

Christobal: And it is the change in momentum over time –

Ike: Which we are allowed to say, because the change itself is constant.

Christobal: (Crosstalk over Ike) or it's the change in angular momentum divided by the R times the time, so if we get rid of that –

Ike: See, it seems reasonable enough.

Dolly: It does, I would be willing to go for it.

Ike: (Crosstalk over Dolly) I just, I just didn't like the VF over R, which apparently, if I had a choice between bad and worse –

Christobal: (Crosstalk over Ike) Well, it appears in both solutions.

Dolly: Right. And you know one of them has to be right.

Ike: They **say** one of them has to be right. If we believe the people running the experiment. I guess we should.

Dolly: (Crosstalk over Ike) We can come to the conclusion that none of them are right.

Christobal: Well the – (timer goes off) –

Once again, Christobal brought up that the inappropriate substitution occurred in both solutions, and Dolly said that one must be correct. Ike proposed that perhaps there was no correct solution. The fact that he suggested that their task may have allowed a situation where no solution was correct before reconsidering the correctness of solution #3 reinforces the strength with which these participants eliminated that solution.

From this discussion, it is apparent that these participants were certainly not novices in physics. They repeatedly used physical concepts and identified an error that none of the other groups (including the other graduate group) discussed. However, they also had a belief bias in this problem that required a quantity to be in the solution. When this quantity did not appear in one solution, that solution was so completely eliminated as to prevent them from even conceiving that that problem might be correct, a clean resolution to the quandary they found themselves in. Thinking in terms of the possibilities framework, if they had carefully fleshed out the quantities that were actually

required to solve this problem correctly, they might have made the connection in Figure 6.96 that Ike was so close to making at the very beginning of his session.

6.3.3 Different foci (Ana and Sally)

For a different exposure to how participants reasoned, consider how Ana and Sally discussed the "Two Pucks" problem. Ana focused on the solution set-up in her individual session, rejecting solutions because of the relationships (fundamental principles) or quantities included in the solution. Sally, on the other hand, focused on the answer in each of the solutions, trying to determine which one was correct. In their discussion, Ana revealed that she had not even looked at the answer to solution #3, having rejected it for other reasons. Sally proposed that perhaps this was a sort of trick question, and the way to get it right would be to pick the solution that generates the counterintuitive answer, which the group did.

Ana was randomly chosen to present her reasoning for selecting solution #2, and she did so.

> **Ana**: I said it was solution #2, and I said that because, okay, for, for solution #3 they were saying, like okay, the system was a puck, and the surroundings were including like the thread, the Earth, and ice sheet, and I just did not – I didn't think the surroundings were right, I don't think they need to include the earth and the ice sheet. And they are using the momentum principle, which I don't think it's the right principle to be using in this problem, so then I narrowed it down to either [solution #]1 or [solution #]2. And said solution #1 used energy principle, and solution #2 uses the angular momentum principle, and all I remember is I remember K rotational and K trans – K trans and K rotational, and then that made sense to me, and then angular momentum, that also, I think that made little more sense to me using any other momentum principle instead of the energy principle, but I can't, like, really explain why, it just made more sense to me. And so, I mean their final answer is that they came up with that second puck would require a larger force to reach the same speed as the first puck, and because I like visualized it (gestures with her hand pulling the puck), like doing it, like because the second puck as the thread around it, like if you pull it, you would need to pull it harder

then if it was attached to the center, and so that is how I came up with that, that is why I picked solution #2.

Ana first rejected solution #3 for two reasons. First, it included the earth and the ice sheet in the surroundings, which means that both of them applied a force to the pucks. This rejection of solution #3 because it mentioned the ice sheet and actually accounts for it was not unique; in four of the seven groups, at least one participant either proposed rejecting the solution for that reason or reported doing so. Second, Ana rejected solution #3 because it used the wrong principle, the linear momentum principle, rather than the angular momentum principle. She later rejected solution #1 for a similar reason; she did not think that the energy principle was the correct one to use, either. She felt that the problem should have been solved using the angular momentum principle, as in solution #2, and so she discarded the other possibility sets (Figures 6.89 and 6.93) because they used the wrong relationship. Ana also confirmed that solution #2 should have been correct by checking the answer it got with her intuition, which was that the second puck should have received a greater force because that puck was spinning.

After a pause, the researcher reminded them of the task, and they began to discuss the problem, with Sally trying to confirm her understanding of Ana's explanation.

> **Researcher**: All right, now (points to Sally) now you reveal what you did, and you can opt to explain it or not, and then as a group try to reach a consensus as to the correct answer.
>
> **Sally**: Okay. This is interesting. (Ana laughs) Okay. So, it was talking about the forces, how we needed to pull them (referring to pucks). So here is solution #1, [solution #]2, and I said it was solution #3 –
>
> **Ana**: Yeah.
>
> **Sally**: Because, yeah, okay, so (sigh). You said that – (to Researcher) are we allowed to talk to each other?
>
> **Researcher**: Oh yeah (Gestures finger-pointing back and forth between the two), definitely.

Sally: So you said, you ruled out number one first automatically, right?

Ana: Yeah because like the kind of it just didn't –

Sally: That's what I agreed too –

Ana: Like the surroundings that they were –

Sally: I also thought that they were, it didn't make sense to pull it slower, or with the, apply a smaller force to the one wrapped around it, the final one for solution #1 says that, the one wrapped around it is applying a smaller force.

Ana: Oh wait, wait wait wait, no no no no no, I – solution #3 I eliminated at first –

Sally: (Crosstalk) First.

Ana: Because they're using –

Sally: (Crosstalk) This is good.

Ana: [Because they're using] the momentum principle and the surroundings for the earth and the ice sheet and they said "don't worry about friction."

Sally: Yeah

Ana: So why do you (solution #3) care about the ice sheet?

Sally: I think that –

Ana: The momentum principle for that, just I don't ever remember it, because we've done similar problems, I just remember using either angular momentum or energy.

Sally: Yeah.

Ana: Like, K rotational and –

Sally: Yeah, I think that K rotational and K trans are –

Ana: Yeah.

Sally: Like you can solve it definitely, rota– you have to keep those in mind. I don't really remember exactly why I chose this one (solution #3). I was just thinking that – I mean, as far as systems and surroundings go, I think you can pick whatever you want and just –

Ana: Yeah.

Sally: Do your variables, like, according to what you pick.

Sally revealed that she had chosen solution #3, and that she eliminated solution #1 because it was counterintuitive in much the same way that Ana argued that solution #2 was intuitive: "it didn't make sense" for their conclusion to be correct. Then, after some confusion, Ana reiterated that she didn't like the choice of system in solution #3, nor did she like the fundamental relationship that solution used. Sally conceded that "K rotational and K trans are... you have to keep those in mind," but she stated that one could pick whatever system and surroundings one wants, as that just informs how one works out the problem.

Let's take a deeper look at Sally's last two statements one at a time. First, if she meant what she said to Ana that the rotational energy needed to be kept in mind, was she arguing for including them in the possibility set? It seems that this is the proper interpretation of that exchange, although she chooses solution #3, which did not include those terms. In fact, she even rejected solution #1, which did. Given her tone and manner of response to Sally, it was possible that her statement was a social move to value Ana's input. The possible presence of such social moves complicates any attempt to understand the meaning behind this statement.

The second of Sally's statements, that one could pick whatever system and surroundings one wants, is very interesting; this was a sophisticated statement corresponding with the fact that the correct relationship for the problem would be true regardless of the particular quantities present in the possibility set. That is, there is no requirement in possibility sets, or in deductive reasoning in physics, to make any one specific choice of system. Thus, Sally's statement dismissed one of Ana's reasons for rejecting that solution, but it did not address her other problem with that solution, that the principle it chose was inappropriate.

Sally: Um, I think mainly what I looked at, cause I didn't remember exactly how to do the problems, to be a hundred percent sure –

Ana: I mean it made sense what, like, what they were doing –

Sally: (Nods). The outcome –

Both: Yeah.

Sally: I just looked at the outcomes for this one, basically. Cause I wasn't exactly sure. So this (solution #1) says that the one attached to the center has to apply a greater force.

Ana: Yeah that's –

Sally: Than the one wrapped around it.

Ana: Yeah that's what I picked (Although it seems that she is confused about which solution Sally is referring to).

Sally: Which I think makes sense, but I'm not sure if it's actually the right one.

Ana: Yeah, I mean solution #1 also –

Sally: I think solution #1 is wrong.

Ana: Really? I mean like –

Sally: It says that –

Ana: I get, like, K trans and K rotational, I get how like –

Sally: Yeah –

Ana: They're applying that but I don't think the final answer is right.

Sally: Yeah, that's what I was saying; that's why I ruled it out, which, I don't know.

Ana: But then wait, for solution #3 what are they saying that...

Sally: They're saying they're equal.

Ana: Oh. Oh, I can see why you'd pick that. I don't even think I looked at that. I just looked at the system and surroundings and the principle they were using, and I just eliminated it. I didn't even see what their final answer was.

Rather than addressing the question of what principle was being used, Sally explained how she solved the problem, by looking at the answers reached by the solutions. Ana admitted to not having seen that the answer that solution #3 gave was that the two pucks received equal force, and she seemed willing to discuss it. Next, Sally and Ana eliminated solution #1 and tried to understand solution #2.

Sally: See these are tricky (laughs). So we agree that [solution] #1 is probably not right, just bec– just based on –

Ana: Yeah –

Sally: The final, to pull it with a force smaller, so we're looking at [solutions] #2 and #3. (Now looking at solution #2) See, I didn't really remember any of this translational –

Ana: I don't, yeah, I don't understand –

Sally: T, T net is torque, right?

Ana: Yeah.

Sally: So the net torque. Is there an equation for torque (looking at equation sheet), or just that principle?

Ana: Well, I don't even like understand all the work they're doing in this second one. I don't know what they're doing is right, cause I just – I never even understood this.

Neither Ana nor Sally were comfortable with solution #2, partly because of an unfamiliar term: the translational angular momentum that bothered Christobal as well. However, unlike Christobal, Ana and Sally both took a physics course that explicitly used this quantity, so they had at least seen it before. Nonetheless, if they never really learned how to use it, then they may still have been confused by it: how could someone add a quantity they didn't understand to their set of possibilities?

Now, Sally proposed another way to solve this problem: perhaps they should have been thinking of this situation as a sort of "trick question."

Sally: I think that this is one of those cases where you think that you would need to pull it harder, but really you don't.

Ana: But really it's equal.

Sally: Yeah.

Ana: Yeah, I think so (laughing).

Clearly, Sally and Ana either did not understand or did not buy into the enterprise, which is that deductive problems in physics hold no relationship to any imagination or intuition; instead, the correct solution to this problem should follow strictly from the application of fundamental principles and definitions.

Sally: I – I think that also even though these surroundings are different (pointing solution #3), like I think that's how the book explained it, too, like with the y, they did the ice over the earth just to show that like they're gonna cancel out and be zero. Or the same.

Ana: Equal each other.

Sally: Yeah, equal and opposite or whatever.

Ana: Yeah, I see.

Sally: (Inaudible).

Ana: Um,

Sally: I don't know,

Sally returned to the issue of system and surroundings with regards to solution #3 to point out how that solution was apparently just being explicit in taking into account the force of the ice, but neither of the participants showed signs of being convinced.

Ana: See, the way that the puck is like spinning, they take that into account –

Sally: See I think that it probably needs to be taken into account, but but I just a remember this stuff enough to –

Ana: Yeah –

Sally: Say that it is right or not. I don't know. (To the researcher) Do we have to agree?

Researcher: You do not have to reach consensus. I would like you to try to reach a consensus, but you do not have to.

Sally: Okay. Yeah, I understand.

Ana: Um,

Sally: I think they are equal, I am not sure why but –

Ana: Yeah, after, I think I (laughs).

Sally: Yeah, I think I remember it –

Ana: Something like – because I know we have done this problem in physics.

Sally: I think – I'm –

Ana: It was like a clicker question –

Sally: I hope I am not wrong, and trying to convince you to change your idea, but I think sometimes –

Ana: I think cause it's, I think it's trying to trick you –

Sally: I know!

Ana: Because this (solution #2) has liked a full-blown, like all of this work out, and then this (solution #3) is just very simple.

Sally: I think I remember it being equal.

Ana: Yeah.

Sally: And it kind of makes sense, also –

Ana: Yeah.

Sally: Because by the time it unravels,

Ana: No no no I think I, yeah I think you are actually right. I think I remember – cause I know what the problem – when I first saw this problem, I was like oh yeah, it's like, the answer is something really weird, it's not like what you would think it would be, because I know I got it wrong.

Sally: Yeah, I think a lot of physics problems in general are kind of like that.

Ana: Okay, I agree with –

Sally: [Solution #]3?

Ana: [Solution #]3.

Ana and Sally wrapped up the discussion by first looking at the spinning of the puck and debating whether it should be taken into account. They quickly abandoned that line and focused on whether they were remembering the answer to this problem correctly based on when they had encountered it in class. As it turns out, they likely *had* seen this problem in class, as many of the sections of the first semester physics course were shown a video demonstration of the two pucks receiving the same force and moving at the same speed even though one was rotating. Eventually, Ana and Sally agreed that this problem was like a "lot of physics problems in general" where the answer was "not what you would think it would be."

Ana and Sally discussed the principles that the solutions used and the quantities they included in their reasoning, along with whether the choice of the system and surroundings for a solution should matter, but they ultimately chose the solution that gave them the answer they thought they remembered. They were happy with this decision because it was counterintuitive, like they believed so many problems in physics are. For them, this reasoning may represent how they believed they were supposed to reason about physics problems.

6.3.4   What makes the "Two Pucks" problem hard?

As the previous case studies demonstrate, the "Two Pucks" problem is challenging; even sophisticated graduate students who are able to reason through the solutions enough to determine errors often fail to see the solution to this problem. What is it about this problem that makes it so difficult to realize that it's both possible and necessary that the pucks obey the fundamental momentum principle? To answer this question, consider these two case studies and the specific issues that came up in these sessions as well as in the other five group sessions in terms of the possibilities framework.

Specific issues

Most of the reasons that participants cited for rejecting or selecting solutions when working through this problem in their groups revolved around the relationships that the solutions used, the quantities in the solutions, and the final answers that the solutions reached. For example, four of the groups brought up the fact that the problem stated to "neglect the ice," and thus perceived solution #3 negatively even though that solution only considered the y-component of the force of the ice, and

applied that to establish that it was equal to the gravitational force on the pucks – an unnecessary, but not incorrect – step. Similarly, two groups mentioned that anything regarding the y-direction was unnecessary, leading in one case to the incorrect thinking that a torque in the y-direction could not affect motion in the x-direction.

Three groups stated that some accounting of the rotation of puck 2 was required in the correct solution, including the first case study group of Christobal, Dolly, and Ike. Indeed, the rotation of the second puck is a particularly salient feature of the problem, and it is what distinguishes puck 1 from puck 2. However, as Figure 6.93 demonstrates, it is a quantity that is unnecessary to solve the problem correctly. Walter was the only participant in the group setting to suggest that rotation might be a superfluous quantity (although, curiously, that group selected solution #2 rather than solution #3).

None of the groups provided a compelling deductive reason for selecting solution #3 and rejecting the other two solutions. Many focused on the principles that were being used; however, the problem was over-specified and could have been solved from a careful application of any of the three principles. The group of Christobal, Ike, and Dolly was the only group to even find proper errors in solutions #1 and #2. The other groups relied on heuristics, sometimes claiming that the problem was best solved with a certain principle (such as the angular momentum principle) or that it should not be solved using a principle, as Ana claimed in the second case study above.

Particularly interesting were the two errors that graduate students "found" in the solutions to allow them to reject a solution they did not like. Christobal's error solving the cross product was discussed in the first case study, but Marco also pointed out a phantom error in the use of the dot product in solution #1. Where Christobal had argued that $\vec{r}$ was parallel to $\vec{p}$ so that $\vec{r} \times \vec{p}$ was zero, Marco argued that the displacement vector was perpendicular to the force, making $\vec{F} \cdot \vec{d}$ zero. This error, unlike Christobal's, happened during in the group session to remedy a conflict that arose. Christobal was not forced to face the conflict that Ike faced in his session of having no possibly correct solution by finding this unique error in solution #2, allowing him to accept the substitution for ω. In Marco's case, this invented error allowed him to resolve how he had used the method in solution #2 to get the answer that solution #1 found (which the group convinced him was counterintuitive). He reasoned that he had been tricked when looking at solution #1, and that

he missed that the dot product should really have been zero, which explains why solution #1 arrived at that incorrect answer.

Groups also frequently relied on intuition or memory to guide them to the answer. At least four of the groups indicated that their final selection was at least partially based in intuition or memory from class, and much of the time in the groups was spent discussing whether they remembered the problem from class or not (as in the second case study).

Possible explanations

The possibilities framework can provide an explanation as to why these errors arose. First, note that the problem was over-specified. Creating it was actually quite challenging, as we needed to ensure that solving it via each principle yielded the same solution. We realized that to ensure its physical consistency, the length of the thread pulled in the second puck needed to be twice the distance the pucks translated because of the physical relationship $\omega = (2v_t)/r$ that holds in this situation. This information about the amount of thread is only relevant for an energy principle solution to this problem, although other groups wanted to use it regardless of the relationship (see for example Figure 6.98). Indeed, the suggestion is that reasoners may believe that each quantity mentioned in the problem statement needs to be accounted for in the solution of the problem. As a result, they may end up using an incorrect relationship (or at least have difficulty discerning the correct one), as apparently occurred with Christobal, Dolly, and Ike.

A second hinderance to solving this problem deductively is that the most salient feature of the problem is completely irrelevant for the correct solution of this problem. Much like in Sherlock Holmes mysteries, where some of the evidence may appear to be very important on the surface but turn out to be a red herring, this problem proposes a difference between the two pucks in that one rotates while the other doesn't. However, a proper representation of the possibility space would indicate that no quantity like "speed of rotation" or "ease of rotation" appears in the correct relationship, revealing the information about rotation as the distraction it is. Indeed, Sherlock Holmes solved his mysteries by identifying possibilities and eliminating everything that proved to be impossible; so too should reasoners be explicit about possibilities. Reasoners should carefully represent the information that they were given and determine what quantities they need to consider

to calculate the net force applied to each puck. Either they would need the distance over which the force acted and the amount of energy change for each puck, or they would need the time the force acted and the change in each puck's momentum. In fact, there is enough information to solve this particular problem either way, but one needs to be extremely careful in laying out each relationship and quantity.

Finally, it is particularly difficult to lay out the possibilities afforded by a situation if one allows imagination or intuition to take over. Even the belief that many physics problems simply have answers that are counterintuitive considerably constrains the possibility space, making certain possible solutions simply unfathomable. The possibilities framework continues to reinforce that reasoners should be aware of the reasoning they are doing and the implications of their heuristics. Performing deductive reasoning correctly is very effortful, especially when it requires keeping track of many different quantities. As a result, it is quite possible that many reasoners attempt to simplify the problem by applying heuristics – that is, they require that a problem contain specific quantities or a particular relationship. Doing so reduces the possibility space and alleviates some of the burden on working memory by eliminating all possibilities that do not fit within those heuristic constraints. Unfortunately, this sort of reasoning is deadly when solving physics problems, which require careful deductive reasoning all the way through to the conclusion.

6.4   The "Close the Door" Problem

For six of the seven groups, the "Close the Door" problem was the second encountered in the group discussion session, although it was the first problem they solved in their individual sessions. Unlike the other groups, Fay, Marco, and Omar discussed this problem first. The text of the "Close the Door" problem is included in the appendix, but it is repeated here for continuity.

> A door stands open, and you want to shut it by throwing something at it. You could either throw a lump of clay or a rubber ball at the door, both of which have the same mass. You know that the rubber ball will bounce back, while the lump of clay will stick to the door. Which should you pick?

6.4.1   The given solutions

"Close the Door" solution #1

The first written solution reads as follows:

> When the rubber ball contacts the door, it will bounce back with almost the same speed as before the collision, while the clay will stick to the door. This is because the rubber ball's collision is nearly elastic.

> $$K_{f,ball} + K_{f,door} = K_{i,ball} + K_{i,door}$$
> $$\tfrac{1}{2}mv_{f,ball}^2 + K_{f,door} = \tfrac{1}{2}mv_{i,ball}^2$$

> Since the ball leaves with just about as much speed as it came in with, the door gains very little kinetic energy. However, since the clay stops when it hits the door, all of that kinetic energy can be transferred to the door:

> $$K_{f,clay+door} = K_{i,clay} + K_{i,door}$$
> $$K_{f,clay+door} = \tfrac{1}{2}mv_{i,clay}^2$$

> If both the ball and the clay hit the door with the same speed, the clay will transfer more kinetic energy to the door. Therefore the clay is the better choice.

Solution #1 uses the energy principle on both the rubber ball and the lump of clay to solve for the change in kinetic energy of the door. However, that solution neglects the change in the internal energy of the clay (which we know must be non-zero because it deforms upon collision and sticks to the door). We can represent this reasoning with two separate possibility sets, one for the rubber ball (Figure 6.99) and one for the lump of clay (Figure 6.100).

Figure 6.100 demonstrates the reasoning that results from not including the change of the internal energy of the clay: the erroneous conclusion is that the final kinetic energy of the door is the same as the initial kinetic energy of the clay. In fact, the final kinetic energy of the door is the same as the initial kinetic energy of the clay minus the internal energy change of the clay, which can in general

$K_{i,ball} + K_{i,door} = K_{f,ball} + K_{f,door}$			
$K_{i,ball}$	$K_{i,door}$	$K_{f,ball}$	$K_{f,door}$
$\frac{1}{2}mv^2$	0	slightly less than $\frac{1}{2}mv^2$	≈ 0
$\frac{1}{2}mv^2$	0	slightly less than $\frac{1}{2}mv^2$	[not ≈ 0]

Figure 6.99: The first solution of the "Close the Door" problem using the energy principle with the rubber ball to solve for the door's change of kinetic energy.

$K_{i,clay} + K_{i,door} = K_{f,clay} + K_{f,door} + \Delta E_{internal,clay}$				
$K_{i,clay}$	$K_{i,door}$	$K_{f,clay}$	$K_{f,door}$	$\Delta E_{internal,clay}$
$\frac{1}{2}mv^2$	0	0	$\frac{1}{2}mv^2$	
$\frac{1}{2}mv^2$	0	0	$[\neq \frac{1}{2}mv^2]$	
Other possibilities not considered				

Figure 6.100: The first solution of the "Close the Door" problem using the energy principle with the clay ball to solve for the door's change of kinetic energy.

be very substantial (indeed, one can imagine a situation where the clay splats against a metal door, which does not gain any kinetic energy at all!). This correct reasoning is shown as Figure 6.101.

$K_{i,clay} + K_{i,door} = K_{f,clay} + K_{f,door} + \Delta E_{internal,clay}$				
$K_{i,clay}$	$K_{i,door}$	$K_{f,clay}$	$K_{f,door}$	$\Delta E_{internal,clay}$
$\frac{1}{2}mv^2$	0	0	$\frac{1}{2}mv^2 - E$	$E, \neq 0$
$\frac{1}{2}mv^2$	0	0	$[\neq \frac{1}{2}mv^2 - E]$	$E, \neq 0$

Figure 6.101: A correct solution for the "Close the Door" problem using the energy principle with the clay ball to solve for the door's change of kinetic energy.

Of course, without knowing exactly how much the internal energy of the clay changed (or, indeed, the exact amount that the rubber ball's kinetic energy changed upon contacting the door), there is no way to compare the final kinetic energy of the door in the two cases. Therefore, the energy principle, while attractive, is actually a dead-end; there is no way to solve this problem by using it because vital information is missing from the problem statement.

"Close the Door" solution #2

The second written solution reads as follows:

If we pick the ball and door as the system, $\vec{\tau}_{net} = \langle 0,0,0 \rangle$ N·m. Also, since angular momentum is defined around a point, let's pick the hinge of the door to be that point.

For the ball:

$$\vec{L}_{i,trans,ball} + \vec{L}_{i,rot,door} = \vec{L}_{f,trans,ball} + \vec{L}_{f,rot,door}$$
$$\Delta \vec{L}_{rot,door} = \vec{L}_{i,trans,ball} - \vec{L}_{f,trans,ball}$$

And $\vec{L}_{trans} = \vec{r} \times \vec{p}$. Assume the ball initially travels along the $+x$-axis and collides at a 90-degree angle with a door that is initially at rest but free to rotate about the y-axis. In this case, the ball's initial angular momentum is in the $+y$-direction. After the collision, the ball is traveling in the opposite direction so its final angular momentum is in the $-y$-direction.

$$\Delta \vec{L}_{rot,door} = \langle 0, rp_{i,ball}, 0 \rangle - \langle 0, -rp_{f,ball}, 0 \rangle$$
$$|\Delta \vec{L}_{door}| = |r(p_{i,ball} + p_{f,ball})|$$

And, because the final speed of the rubber ball is nearly as much as its initial speed,

$$|\Delta \vec{L}_{door}| \approx |2rmv_{i,ball}|$$

However, for the clay and the door as the system,

$$\vec{L}_{i,trans,clay} + \vec{L}_{i,rot,door} = \vec{L}_{f,trans,clay+door}$$
But, $m_{clay} << m_{door}$

$$\Delta \vec{L}_{rot,door} \approx \vec{L}_{i,trans,clay}$$
$$|\Delta \vec{L}_{door}| \approx |rp_{i,clay}| \approx |rmv_{i,clay}|$$

Since the mass of the clay is equal to the mass of the rubber ball, the door's angu-
lar momentum will change more if the ball hits it with some speed than if the clay hits
it with that same speed. Therefore, the ball is the better choice.

Solution #2 uses the angular momentum principle to account for the motion of the door. This
solution assigns both the lump of clay and the rubber ball translational angular momentum and
then treats the collisions separately: the rubber ball is treated as having a nearly completely elastic
collision with the door, while the lump of clay is treated as having a completely inelastic collision
with the door. Figure 6.102 shows the reasoning for the rubber ball, while Figure 6.103 shows the
reasoning for the lump of clay.

$\vec{L}_{i,ball} + \vec{L}_{i,door} = \vec{L}_{f,ball} + \vec{L}_{f,door}$			
$\vec{L}_{i,ball}$	$\vec{L}_{i,door}$	$\vec{L}_{f,ball}$	$\vec{L}_{f,door}$
$< 0, rp_{i,ball}, 0 >$	$< 0, 0, 0 >$	$\approx < 0, -rp_{i,ball}, 0 >$	$\approx < 0, 2rp_{i,ball}, 0 >$
$< 0, rp_{i,ball}, 0 >$	$< 0, 0, 0 >$	$\approx < 0, -rp_{i,ball}, 0 >$	[not $\approx < 0, 2rp_{i,ball}, 0 >$]

Figure 6.102: The second solution of the "Close the Door" problem using the angular momentum
principle with the rubber ball to solve for the door's change of angular momentum.

$\vec{L}_{i,clay} + \vec{L}_{i,door} = \vec{L}_{f,door}$		
$\vec{L}_{i,clay}$	$\vec{L}_{i,door}$	$\vec{L}_{f,door}$
$< 0, rp_{i,clay}, 0 >$	$< 0, 0, 0 >$	$\approx < 0, rp_{i,clay}, 0 >$
$< 0, rp_{i,clay}, 0 >$	$< 0, 0, 0 >$	[not $\approx < 0, rp_{i,clay}, 0 >$]

Figure 6.103: The second solution of the "Close the Door" problem using the angular momentum
principle with the clay ball to solve for the door's change of angular momentum.

Both parts of solution #2 consider possibilities that result from reasonable assumptions (that
the rubber ball bounces back from the door with almost as much speed as it had before contacting
the door and the mass of the door is much greater than the mass of the clay), and a comparison
between the final angular momentum of the door in each case yields the correct answer, that the
rubber ball is about twice as effective as the lump of clay.

Many possibilities are excluded by the solution assuming that the clay and ball were thrown with
the same velocity (and by the problem statement, which indicated that they had the same mass).

An alternate formulation of the solution could show that the initial momentum of the clay would need to be approximately double the initial momentum of the rubber ball to achieve the same final angular momentum of the door (assuming that both hit the door the same perpendicular distance from the hinge); therefore, by the momentum principle one would need to apply approximately twice the initial force to the clay versus the ball to close the door. This reasoning could also be explained in terms of the possible ways to vary the quantities in the angular momentum principle to yield the same final angular momentum of the door. None of the participants fully explored these lines of reasoning, but it was not uncommon for the participants to indicate that they were eliminating possibilities by stating that they were assuming that the objects were thrown with the same initial velocity, a constraint not imposed upon them by the problem statement.

"Close the Door" solution #3

The third written solution reads as follows:

> Let us consider the momentum of the two objects.

> Because $\vec{p} = m\vec{v}$, and the masses are equal, if the objects have the equal velocities then they will have equal momenta. Since both the clay and the rubber ball will provide the same amount of momentum to the door if they're thrown with the same initial speed, it doesn't matter which one you pick.

Solution #3 improperly uses the momentum principle to assert that in the case where the initial momenta of the two objects are the same, they will impart the same momentum to the door. This solution only considers the initial states of the objects without thinking about their final states. This reasoning can be indicated by showing blank spaces in the possibility set for this situation, as shown in Figure 6.104.

Because solution #3 does not consider the final momentum of the ball or clay, it does not use the momentum principle properly. This reasoning yields the incorrect conclusion that the composition of the object thrown does not matter.

$\vec{p}_{i,object} + \vec{p}_{i,door} = \vec{p}_{f,object} + \vec{p}_{f,door}$			
$\vec{p}_{i,object}$	$\vec{p}_{i,door}$	$\vec{p}_{f,object}$	$\vec{p}_{f,door}$
$m\vec{v}_{i,object}$	$\vec{0}$		$m\vec{v}_{i,object}$
$m\vec{v}_{i,object}$	$\vec{0}$		$[\neq m\vec{v}_{i,object}]$
Other possibilities not considered			

Figure 6.104: The third solution of the "Close the Door" problem using the momentum principle incorrectly to solve for the door's final momentum.

Each of the solutions use a different fundamental principle. However, solutions #1 and #3 each neglect a term in their attempts to solve the problem, resulting in an erroneous conclusion in both cases. As in the "Two Pucks" problem, some of the participants demonstrated a preference for one principle over another and used that to choose which solution was correct, rather than searching for the deductive errors in the solutions. Also, many participants chose or rejected a solution based on the quantity that it contained (for example, rejecting solution #3 because it did not consider the objects' final momenta). For more information about the reasons participants cited for rejecting solutions in this problem, see Section 7. Below, some highlights from the individual sessions and group discussions about this problem highlight some of the more interesting cases of reasoning that demonstrate and reinforce the use of possibilities.

6.4.2   An ideal solution (Christobal, Dolly, and Ike)

Christobal was selected to present his solution for this problem to his group, which consisted solely of graduate students at the start of their first year. The explanation he provided was solid, and it explained precisely the shortcomings in solution #1 and solution #3.

> **Christobal**: So, the door problem, was pretty easy, um, (pause) when a rubber ball comes in contact with the door, you want to shut the door, and you want to shut it using either a rubber ball which is going to bounce, or a clay ball, glob of clay that isn't going to bounce, and solution #1 was doing kinetic energy. I said it was wrong because it ignored the energy that, that the clay takes to mold around the collision, which, which is not zero, and so that's where, basically their whole argument rests around the fact that

the clay isn't going to move. Well, that doesn't make any sense. To have an inelastic collation, there has to be some sort of deformation to the objects that takes into account the missing excess of energy so that you can have a situation where energy isn't conserved and momentum is. So that one was dumb (he places the solution on the table). Solution #3 I said was wrong because that one is not taking into account the fact that afterwards you have some momentum to deal with, and so they were obviously mistaken. And solution #2 I said was right, um, if the ball bounces off at about the same speed as it hit the door with, which makes sense if, um, the door is massive compared to the ball, in that case the momentum transfer is going to be two times the initial momentum of the ball, and so it is going to be much better at closing the door.

Christobal eliminated solution #1 because it neglected the deformation of the clay, and he eliminated solution #3 because it didn't account for each object's momentum after the collision. He stated that inelastic collisions would be impossible without some energy going into the deformation of the colliding objects, and this knowledge allowed him to eliminate the possibility that the clay could "transfer all of its kinetic energy to the door."

Dolly: So you said solution #2?

Christobal: Yeah.

Dolly: I did not look at solution #2. I only looked at [solutions #]1 and 3.

Ike: Yeah, I had, I, I said let's start with [solution] #3, let's go to [solution] #1, by process of elimination, I get solution #2.

Dolly: Yeah, but I didn't eliminate [solution] #1 that quickly, so. I didn't have a chance to look at [solution] #2, or didn't like angular momentum again.

Ike: This one is fun. I just know instinctively that I should be using momentum. But the kinetic energy looks so appealing. Christobal, could you explain to me why exactly, the kinetic energy approach is wrong, or Dolly –

Christobal: Is that the one you said was right?

Ike: Oh, no no no, solution #2 is totally right.

Christobal: Oh, okay.

Ike: But the kinetic energy approach is so appealing.

Christobal: It's not, it's not appealing because you don't know what the kinetic energy of deformation – the energy of deformation of the ball, of the clay is, so you don't have any way to even start it.

Ike: Well, clay aside, um, if the ball leaves with the same speed it entered with, the kinetic energy should be the same, so the door should get no energy (during the statement, Dolly is crosstalking and gesturing in agreement).

Christobal: Well, it is approximately that. Remember it's a V squared term in kinetic energy and so a very massive door moving very slowly it going to have very little kinetic energy (participants are agreeing).

Dolly: Regardless of which one –

Ike: Okay, that will work for me.

Researcher: Are you all happy?

Ike: Yeah, we're happy with [solution] #2.

Dolly had not fully considered solution #2 because it was so complicated, and Ike asked Christobal to explain how the energy principle solution, which seems so appealing, could be incorrect. Christobal simply reiterated his possibility set, which is represented by Figure 6.101. When asked to consider the assumption that was being made about the rubber ball bouncing back with nearly as much speed as it hit the door with, Christobal relied on the definition of kinetic energy to demonstrate that it was possible for the door to have very little energy while still moving (albeit quite slowly). Interestingly, he was not pressed on this point or asked to demonstrate precisely how that related to the door's momentum. It was enough for the group that he simply indicated the *possibility* that the door could still close after the rubber ball hit it. Despite his imprecise reasoning about this particular point, Christobal provided the correct reasoning for eliminating both solution #1 and solution #3, and the group agreed with him enough to continue with the session without much discussion.

6.4.3   Internal conflict (Marco)

Unlike Christobal, Marco (who was also a graduate student beginning his first year), got hung up on solution #1. He found the solution very attractive and immediately resonated with it. Only after much thought and a discussion with his group did he switch to solution #2. In order to do so, he cited an error in the solution he initially agreed with that was not sufficient for explaining what was incorrect on solution #1. Marco's whole process of debating which solution was correct was quite interesting, as he was unable to find an error in solution #2 during his individual session. The internal conflict Marco struggled with was a decision between a solution his gut liked and one that did not have any logical flaws.

Marco's individual session

Marco began by reading the problem statement and identifying quantities that he believed should be in a correct solution.

> **Marco**: My approach will be simple. I'm going to read the problem first, and then I'm going to think about it a little bit, and then go look at the solutions, I'm going to sort the solutions by length. In fact I'm doing that right now. I'm going to read the shortest solution first, then the longer solutions later, and then – that's the way we're going to do it. So (reading the problem statement), a door stands open and you want to shut it by throwing something at, so I'm drawing a picture – well, I can draw it here, so we have a door and it's open (drawing with red pen on the problem statement). You want to throw something at it at some velocity so that the momentum is transferred in a way that the door shuts. Great. So, momentum, I'm looking for solutions that have momentum in them. Great. You could either throw a lump of clay –ah (gestures with finger pointing up, as if he suddenly realized something)– or a rubber ball at the door. Choices: lump of clay or a rubber ball (underlines words on problem statement). Both of which have the same M mass, okay, but they have different elasticities. That is the, so we're looking, perhaps, for something that will take into account the elasticities. You know that the rubber ball will bounce back, while the lump of clay will stick to the door,

uh huh. which should you pick? So, I assume the correct problem will have something about inelastic versus elastic collisions, where the lump of clay sticking is an inelastic collision, and the rubber ball is an elastic collision, and there's different equations for those, and the correct – the correct solution will use that appropriately.

He then began looking at the possible solutions, specifically trying to identify one that appropriately accounted for the difference between the elasticities in the two materials. According to Marco, the correct solution should have accounted for the rubber ball's elastic collision and the clay's inelastic collision.

> **Marco**: Okay, so, (starts looking at shortest solution, which is solution #3) let us consider the momentum of the two objects. Okay, so far we're clear. Because momentum is blah blah blah, and the masses are equal, if the objects have the equal velocities than they will have equal momentum. Yes. (Reading solution #3) "Since both the clay and the rubber ball will provide the same momentum to" – no! – this line right here (marks solution #3 with red pen), "since both the clay and the rubber ball will provide the same amount of momentum to the door if they're thrown at the same initial speed, it doesn't matter which one you pick," is actually incorrect. The amount of momentum that is transferred from one of the objects to the other object depends on the elasticity of the object. So and then, you know, anyway, so this statement is clearly false. So this solution is out. I'll write false there.

Because it didn't account for elasticity, solution #3 was promptly eliminated. Marco then turned his attention to solution #1, which immediately appealed to him.

> **Marco**: Now, this other solution, solution #1, says, "when the rubber ball contacts the door, it will bounce back with almost the same speed is before the collision, while the clay will stick to the door." (Marco sighs). Sure, I will accept that – I will accept that as a viable. Why? Because the rubber ball is nearly perfectly elastic, sure, and it will bounce back with almost the same speed as before, sure, that leaves space for it to bounce back with less speed as before. "This is because the rubber balls collision is nearly elastic,"

great. They gave it with the explanation I found, great. So, they have the equations of elasticity, by showing energy is conserved – and what is that other bizarre equation? – oh, I see, so they have an equation stating conservation of energy, then they plugged in the velocities of the ball and – that's it, that's all they did. And, the initial kinetic energy of the door is zero, so they took that out. So these equations are correct. They state conservation of energy. In an elastic collision energy is conserved, I think I can just say that I know that and I believe that. Now, (reading from solution #1) "since the ball leaves with just about as much speed as it came in with, the door gains very little kinetic energy." I like that argument, sure. "The door gained very little kinetic energy," yeah, that makes sense. "However since the clay stops when it hits the door, all of the kinetic energy can be –" yup yup, this is it (slaps the problem). "If both the ball and the clay hit the door with the same speed, the clay would transfer more kinetic energy to the door. Therefore the clay is the better choice." Now um, this is, this is interesting, because this one of the situations where now I feel that this might not be the correct answer. It's forcing me to check over the argument, however, the argument is pretty good. The momentum is conserved in both cases, well, what we have to figure out is how much, what's the momentum difference – you know, whatever momentum is gained or lost by the incoming object will be gained or lost by the door, so if it's true, all of the momentum will be transferred – ah, here's the thing, here's the thing that may be tricky, oh, the masses are the same, okay, there's something that hasn't been taken into account here, but that makes me happy. Both objects come in with the same mass, so they, both objects when they hit the door, in the case of the clay all of the momentum of the clay – this is correct – transfers to the door. Now, the final velocity of the door depends on the mass of the door, and if the mass of the door is much greater than the mass of the – than the mass of the rubber, the velocity change may not be very much; however, the, the other object wouldn't have any chance of giving it more momentum – the maximum momentum that the object can give is the same, because the masses are the same, so the most momentum that will be transferred will be done in the case where the velocity changes the most for the incoming object, so I believe solution #1 is correct,

and I shouldn't even need to see solution #2. However, I will entertain solution #2 for

kicks and giggles (Marco writes the word "correct" on solution #1).

In explaining why he believed solution #1 was correct, Marco talked not about the energy principle that the solution used, but instead about momentum, which he pointed out in his initial review of the problem as something that the correct solution should have discussed. However, solution #1 never mentioned the conservation of momentum that Marco attributes to that problem. In a sense, Marco confused some aspects of momentum with energy, and he never thought about the possibility that there could be a change of the internal energy of the clay. One of the aspects of momentum and energy that he confused was the scalar nature of energy with the vector nature of momentum. As such, he claimed that "the most momentum that will be transferred will be done in the case where the velocity changes the most for the incoming object," attributing the maximum momentum change to an object that would stop rather than an object that would reverse direction. This reasoning implies that he erroneously assumed that the final momentum could not have a negative component and is represented by Figure 6.105.

$p_{x,i,object} + p_{x,i,door} = p_{x,f,object} + p_{x,f,door}$			
$p_{x,i,object}$	$p_{x,i,door}$	$p_{x,f,object}$	$p_{x,f,door}$
$mv_{x,i,object}$	0	$\geq 0, \leq mv_{x,i,object}$	$mv_{x,f,door}$
$mv_{x,i,object}$	0	(< 0)	$mv_{x,f,door}$

Figure 6.105: Marco's implicit assumption that an object could not transfer more momentum than it carried to another object.

It is likely that Marco confused kinetic energy, which could not be negative, with the component of momentum that he was implicitly discussing, which could have been. He did not consider the possibility that the door could have gained a greater magnitude of momentum than the ball initially had as the result of a collision. Instead, he said that the clay would transfer as much momentum as possible by stopping when it hit the door, and the clay therefore would be the correct choice for closing the door. Marco continued to fail to see this possibility, and his reading of solution #2 began as an effort to figure out where it made a mistake.

Marco: (Reading solution #2) "If we pick the ball and the door as the system, blah

blah" – there is a vector with torque, great. Also "since the angular momentum is
defined around a point, let's pick the hinge of the door to be that point." Okay, okay
I see what they're trying to do. Um, so, for the ball, so they're writing an equation
of conservation of angular momentum. Fine. However, I don't believe that's the easy
way to do it. If we're going to do that, we should do angular, um, conservation of,
angular kinetic energy versus linear kinetic – angular momentum, angular, rotational
energy versus linear velocity – however, I will continue with their plot. Okay, for the
ball, the incoming – I am going to avoid trying to figure out this equation briefly – and
I'm going to continue reading the argument, "assume the ball initially travels along the
x-axis and collides at a 90 degree angle with a door that is initially at rest but free to
rotate" – Details are trying to be complicated – "in this case, the ball's initial angular
momentum is in the y-direction." Uh-huh. "After the collision" – and all of this is just
being picky – "after the collision, the ball is traveling in the opposite direction so its final
Angular momentum is in the –" yeah. This doesn't say anything about comparing – I'm
looking for the part where they're comparing the rubber ball with the clay. Okay, here
we have a line that may be useful, "because the final speed of the rubber ball is nearly as
much as its initial speed, something, however for the clay and the door as the system,"
so there's the comparison, this is the line where I'm looking for the comparison. Good,
I found where they are comparing. Now, in the conclusion, they say, "since the mass
of the clay is equal to the mass of the rubber ball, the door's angular momentum will
change more if the ball hits it with more – with some speed and if the clay hits it with
that same speed (reading slows, and the participant puts his hands on his head as if he
is having a conflict), therefore, the ball is the better choice." Okay, the door's angular
momentum – so, this is their conclusion, this conclusion is different from the one I think
is correct, so I should be able to throw this out; however, now there is something wrong
in their reasoning which I need to find. Here, there is something, no. Okay, so now I
have to figure out what their equations actually mean. They have an equation right after,
"because the final speed of the rubber ball is nearly as much as its initial speed," the
change in momentum – the change in angular momentum of the door is about equal to

two $R\,M\,VI$ ball. Now, I do know where they came [up] with this because the change in velocity is $2[V]I$, I think, the change in the velocity for the ball is $2[V]I$, because velocity final minus velocity initial, the minus cancel out and gives you the two VI, now the R – VR, VR, R, ωR equals V, where did that R come from? VR, weird, oh well, I see I see, right, R cross P, MVR ball's, okay so that's believable – this equation is believable, the R came from the R cross P definition of, angular momentum, so $2MR$, so this is correct. The change on the door with the incoming ball is that, but that's also true for – oh, that's not true actually, this is true for the ball, however, for the clay and the door as the system, LI trans clay LI rot of clay – there's a lot of subscripts, okay, so in the second equation it says that the initial angular momentum of the clay plus the initial momentum of the door is equal to the final momentum, yeah of course, the – to the final momentum of the clay and the door. The initial momentum of the door is zero, the initial, uh, how did they do this? They say that – so they, they skipped a couple steps and so now I am forced to figure out what the steps are. This is great (sarcasm). Oh no, no it's not that difficult. All they did on the next line is they moved the initial that's on the left-hand side to the right-hand side to create the Δ, or the change in rotational momentum in the door. And that's about equal to the initial kinetic energy (inaudible) the initial momentum of the clay. And, they are saying that the initial momentum of the clay does not have the two. I see, I see I see I see (makes large gesture with hands and then places them on his head). So that's the difference. One of them has the two. The other one doesn't. Well now, I have to consider this a little more carefully. I have two minutes to go. Was there a mistake – was there a space in reasoning in the previous one?

Marco spent over six minutes trying to understand solution #2, and he realized that it was using conservation of angular momentum to show that the rubber ball would impart about twice as much angular momentum on the door as would the lump of clay. He acknowledged that the factor of two, which came from the rubber ball switching direction, would be the only difference between the situations. So, because he still thought that solution #2 should yield the wrong conclusion, he used the last two minutes of his individual session to try to understand whether that factor of two

made sense. Recall that this factor of two would only be there if one allowed the possibility that the x-component of the momentum of the ball could be negative after the collision, which Marco had been disallowing (see Figure 6.105).

> **Marco**: Why is there not a two here (pointing to the clay part of solution #2)? Because it comes and it sticks, this is a fishy fishy argument. Let's see (points toward the beginning of solution #2). L rot, L rot, the change in momentum, so I'm having a problem finding the hole in this, I'm having a problem finding the hole in this; however, I feel fairly confident that there is not a hole in [solution] #1. And I haven't had time to, look at, the other equations at the very beginning. I should trust this solution (#2) more because they look like they put more effort into their pickiness; however, I don't know. That doesn't seem – I don't have a good reason to switch to this solution, because solution #1 goes more along the line of what my brainstorming was when I first started. If this was a test, I would pick the solution that I already thought was correct, which is simply stating conservation of energy in both cases, let's see, (reading from solution #1) "the clay will transfer more kinetic" – well that's such a weird, I see, this, I see, some of the verbiage in this solution is wrong. Because you don't speak, usually, you don't speak in this way. You don't say that the clay transfers more kinetic energy to the door. But, do do do do do, what should I do? Well, with 30 seconds left, I will stick to my solution and the next time I have an opportunity to look at this, I will look at it a little more carefully. Some of the details in here made perfect sense (pointing to solution #2), and if, I wouldn't be entirely surprised if solution #2 turned out to be correct. However, I haven't had time to look all the details, so for the time being I stick to [solution] #1 (Marco slaps the white board, and begins putting solutions away).

The best Marco could do was to say that there not being a "2" in front of the change in the angular momentum of the clay was "fishy," but he also admitted that he could not find any errors in the argument that solution #2 provided. Marco allowed that solution #2 could have been correct, but he stayed with his gut instinct that solution #1 was correct even though the wording it used caused Marco some discomfort.

This internal conflict that Marco displayed here indicates two things. First, he did not completely interpret solution #1 correctly. He applied his own understanding, which was the absolutely true assertion that momentum should be conserved in this problem, to a conservation of energy approach to solve the problem. This dissonance apparently led to a confusion between energy and momentum, preventing him from allowing for the possibility of a negative final x-component of momentum. Second, once Marco chose a correct solution, it was very difficult for him to change his mind, even when faced with a solution that contained no errors that he could find. In a way, he bet that his gut instinct would be more likely to be correct than his ability to diagnose a possible solution. While he was sophisticated enough to provide a thorough analysis of solution #2, that only left him with a conflict he could not comfortably resolve. Had he been a less sophisticated student, he would have likely eliminated solution #2 without first generating this conflict.

Marco took this conflict without resolution into his group discussion, where he hoped to realize a fresh perspective from his peers.

Fay, Marco, and Omar's group discussion

Omar was randomly chosen to begin the discussion on this problem. He explained why solutions #1 and #3 were incorrect, and he also explained why he felt that solution #2 was correct.

> **Omar**: (Reading the problem statement) "A door stands open and you want to shut it by throwing something, whether it is the rubber ball or the lump of clay." According to me, I got solution #2 as the right one. I will first go through why solution #1 and [solution #]3 is not correct according to me. In solution #3, we take, only we consider the momentum but the, I feel that since the door has to close and we are considering the rotational motion of the door, but I feel that it is essential for us to consider the angular momentum of the door which is not being considered in solution #3. Here, we are only talking about the momentum being transferred to the door, and since the velocity and the masses are the same, therefore same momentum is transferred and it will just close. I don't think that is right. Solution #1, we are talking about the energy consideration, how energy is transferred from the ball to the door or the clay to the door. And here also we haven't considered the angular momentum, which is, which would be essential for

looking at this problem. The energy consideration would have been fine if there was just the translational motion, instead of the door there would have been just a block to move enough, in a linear distance, but here since the door is being considered, the translational motion – the rotational motion is important, and that therefore we are looking at the angular momentum, and the translational and rotational angular momentum of the door is being conserved and that's how we see solution #2, and therefore the ball would be a better choice as solution #2 has given it. What do you think?

Omar felt that a correct solution to this problem would involve the angular momentum of the door. He claimed that the energy principle would have been sufficient for solving the problem had it been a linear motion problem, and that the error solution #1 makes is neglecting rotation. While considering the rotational energy would have been more appropriate, it would not be sufficient for fixing the deeper error in solution #1, which was that the energy principle consideration did not account for the change of the internal energy of the clay. However, Marco accepted Omar's complaints about solution #1 at face value as he sought resolution to his conflict.

Marco: Yeah. Okay, wait wait, I actually made a mistake because I initially chose solution #1 as correct, but I ran out of time when I was looking through solution #2 because I looked at solution #2 last. And the mistake in solution #1 in my opinion is that they don't write that kinetic energy of the clay with the door as a –

Omar: Total system.

Marco: As a rotational system, with $I\,\omega$ squared (Omar is nodding), so that is clearly wrong. Um, But I couldn't check that the factor, the two versus one in the change of rotational energy –

Fay: (Pointing to the solution) this two?

Marco: Yeah, yeah, [to check] that that was actually correct, so I was like well I can't check it, but now that I have a chance to look at it again and actually my head has been working on it, it is pretty obvious that the two from the rubber ball comes because the rubber ball's speed is nearly as much as the initial –

Omar: Switches direction.

Fay: Bounces.

Marco: So great, and then in the second one, the change in momentum of the object coming in is just the MR, is where that goes. So yeah, there is, that's it. Solution #2.

Marco explained that he had been unable to check whether that factor of "2" in front of the angular momentum for the rubber ball made sense, but he reasoned through it here. Omar and Fay both agreed with him that it came from the fact that the rubber ball switched direction. So, Marco was now able to accept solution #2 as being correct. However, he needed an error for solution #1, so he used the plausible explanation that Omar provided about it not considering the angular energy of the door. Of course, even if it had done so, it would have still been incorrect because it did not account for the change of the internal energy of the clay.

The remainder of the transcript in this problem concerns Fay's understanding of the problem, given that she misread part of solution #2 as she attempted to solve it. Because it does not contain any more data regarding Marco's struggle, I omit it.

Marco's conflict was resolved in the group session because he received support on two fronts: Omar provided a believable explanation that could possibly explain where solution #1 had gone wrong, and Omar and Fay both confirmed that the calculation of the angular momentum for the rubber and clay was correct, which was what Marco had suspected. With this support beyond what he was able to do in the short amount of time he worked on this problem by himself, he was happy to change his decision to solution #2. The implication from Marco's struggle was that such support is useful when someone has to compare two very different possibility sets (in this case, the solutions used two different fundamental principles). All possibilities must be eliminated from one, and a possibility must be revealed in the other. In this case, because the calculations for the clay and rubber were confirmed in solution #2, a possibility could not be eliminated there. Meanwhile, Marco and Omar could eliminate all possibilities in solution #1 by claiming that it did not include a necessary quantity – the rotation of the door. However, they did not recognize that the relevant quantity to include in that solution was the change in internal energy of the clay. Coming to such a conclusion would have required a more extensive fleshing out of the possibilities regarding that

solution than they undertook.

6.4.4 Given solutions change possibilities (Sally)

On this problem, Sally again demonstrated that providing possible solutions to a problem could help shed some light on how to solve the problem, not by emulating the correct problem-solving method, but simply by including a term that the reasoner had previously overlooked. As in the other cases in this dissertation, the participant (here, Sally) spent some considerable time and effort trying to understand and solve the problem. When she investigated the possible solutions, she realized that there was a quantity that she had not fully considered (namely, the elasticity of the collision between the rubber or the clay and the door) and used that to make her decision about which solution was correct.

The group discussion that followed from this individual effort confirmed that Sally did solve the problem this way, and it also revealed that Ana solved the problem simply by considering the final answers provided by each of the solutions. Remarkably, exactly the opposite happened when these two solved in the "Two Pucks" problem, where Ana focused on the approach and Sally focused on the answers. This flip in approach shows that reasoners are not necessarily consistent from one problem to the next with how they try to solve the problem.

Sally's individual session

First, Sally tried to solve the problem on her own, possibly because she did not completely understand the task that she was being asked to perform. She then looked at the possible solutions with only a few minutes remaining and drew a conclusion as the time expired.

> **Sally**: So, (reading problem statement) "A door stands open, and you want to shut it by throwing something at it. You could either throw a lump of clay or a rubber ball at the door both of which have the same mass. You know that the rubber ball will bounce back while the lump of clay will stick to the door. Which should you pick?" So looking at these equations. (Gesturing to reference sheet). Momentum equals F net Δt. I think that it's probably best one to use. So I guess what I'll do first is draw a picture. (Drawing on problem statement) have the door here.

Researcher: Okay. Now, again if you would like to write anything it would help us out if you use the white board.

Sally: Sorry.

Researcher: It's okay.

Sally: Don't draw on this white paper?

Researcher: You're certainly welcome to, it's just, it's just easier for us to see if you draw it out on the white board.

Sally: Okay. Yeah. Sorry.

Researcher: That's okay. Not a problem.

Sally: Okay so, um, here is – this line will be the door (drawing on whiteboard). You're going to be standing here (gesturing to and drawing on whiteboard). And I guess in case one will be the lump of clay and it's going to go here (continues to draw on white board). This is clay. And this will be the rubber ball. And here we have that it will bounce back. I guess (inaudible) will stand here (drawing person on the white board). Oops (modifying drawing of person on white board). Not that it matters. Okay so you either want to throw – "they both have the same mass. You know that the rubber ball will bounce back which should you pick?" (Looking at whiteboard) Okay so with this picture you can see that masses are going to be the same, so you need an equation with mass so that it will cancel out. Um, $M1$ equals $M2$ (writing on whiteboard) this'll be one, this'll be two (labeling clay as one, labeling rubber ball as two). Um, um, the equation sheet, let's see here: (reading off equation sheet) "momentum equals F net Δt. I'm having trouble remembering some of these. Um, So I guess what we can do is F_{net} equals (writing out on whiteboard, see Figure 6.106, then looking at reference sheet), hm, I'm wondering if it even matters since one's going to be bouncing back regardless, or both have the same mass, um, hm, this is difficult. Um (Continues to look at reference sheet). Um, I'm just reading over the formula sheets to see if I've. Okay so momentum is mass times velocity, um, let's see here. I want to show that when the rubber ball bounces back I think that it's going to cancel it out, but I think regardless that force

might still be zero. Um, let's see here. (Continues to look at reference sheet). I think that – I think that there is a difference but I don't know how to show it and, or prove it with formulas. (Looking at reference sheet, eliminating possible equations to use). Um, hm, it's not really a multi-particle system because I'm treating each of these as their own individual systems and seeing which one I think is going to be better. So as far as these formulas (indicates part of the reference sheet), I probably will not use them, um, unless it's a center of mass problem. Let's see. I'm pretty sure that it's not that. Um...

Researcher: Please keep talking.

Sally: Yeah, I'm not really thinking much. L translational. I'm trying to remember what these are: L translational, um, because that could possibly (long pause).
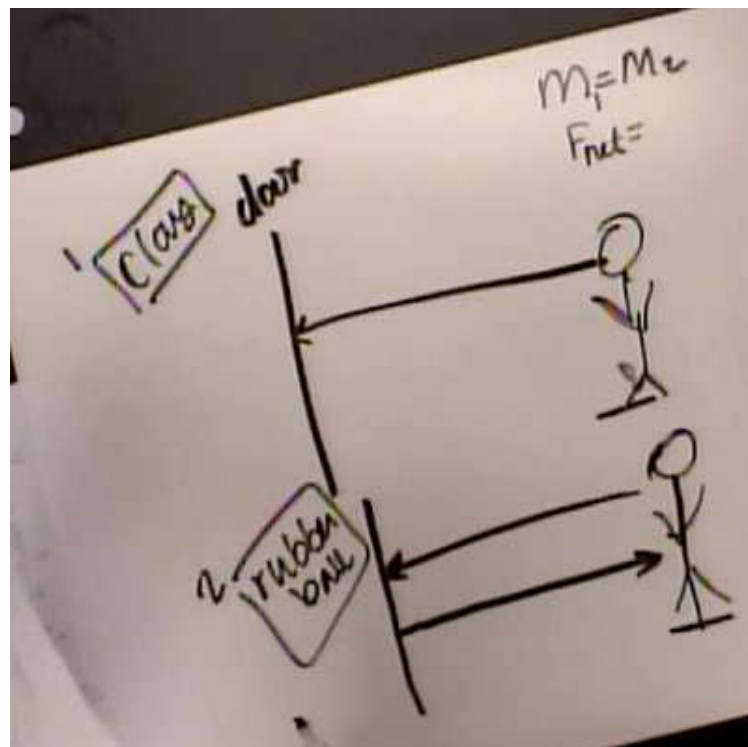
Figure 6.106: The whiteboard showing Sally's drawing of the situation before she began to hunt for an equation.

Sally began by reading out the solution and drawing a picture (see Figure 6.106) to represent the problem on the whiteboard. She distinguished between the two situations: the rubber ball bounced while the clay did not. After drawing the picture, she then indicated that the masses were the same and then hunted for an equations that she liked. After being prompted to continue to speak, she said that she was not really thinking at this point. Indeed, she was apparently searching for a relationship that contained the information that was relevant to the problem, namely one that included "mass so that it will cancel out." Eventually, she found the momentum principle, and she began reasoning from that.

> **Sally**: F net equals DP over DT. Um, I'm trying to think. Momentum equals F net Δt, and that. Okay so, let's see that for this one Δp equals $F_{net}\Delta t$ (writing momentum principle on whiteboard). So p final, this needs to be p final minus p initial (pointing to the Δp written on the white board). So p initial plus $F_{net}\Delta t$ (rewrites momentum principle on whiteboard while substituting in p final minus p initial for Δp, with initial momentum on other side of the equation – but no vector signs). Um, So the final here (pointing to final momentum written on whiteboard), the final momentum is going to be zero and the initial is going to be some V – or some momentum that's going to have, be equal for this one (she is writing these expressions out by her drawing of the clay, and she is gesturing to both the clay and the rubber ball cases when talking about the initial momentum), cause I'm assuming that they're gonna be thrown at the same, um, momentum or speed I guess. Plus $F_{net}\Delta t$. For this case (now gesturing to rubber ball case on whiteboard), Δp equals $F_{net}\Delta t$ (writing on white board). p final equals p initial plus $F_{net}\Delta t$. p Final in this case though should equal p initial, which we said is going to be the same as p_1 (she had labeled the initial momentum for the clay case as p_1, – again no vector notation). So p equals p (she has labeled both the initial and final momenta to as p_1 for the rubber ball case). Plus $F_{net}\Delta t$. Um, the time that it's going to take for this (gesturing to clay case) to happen should be one half the time as this (now gesturing to rubber ball case), because of Newton's laws if we are assuming that nothing is acting on it. Um, since their masses are the same it's going to reach this wall at the same amount of time but since this one's (pointing to rubber ball case) going to go back, it's going

to go back to the same distance. So here we should say that zero equals F net; the Δt is kind of arbitrary at the same time (continues to write on white board). Um, So I'm just going to kind of disregard the Δt. I'm not sure if I can do that actually, but. So I would say that you would want to pick the clay because, see – I don't know now (on the white board she has the net force magnitude for the clay case has been equal to negative p_1, while for the rubber ball case she has set net force magnitude equal to zero, see Figure 6.107), these are confusing. I would just say that since they have the same masses, regardless of what they do after the door, they're going to be going to the door with the same force. So unless it's some really high-level physics that I would say that – (pause) that they're probably going to do the same thing. Because time isn't given, their masses are the same so the force at the – is actually going to be applied on the door, it should be the same. Force equals mass times acceleration. I would think that they would be the same. I don't think that this is really valid (gesturing an X over the rubber ball case written out on the whiteboard), I'm not sure, that is even on the right track at all. So I'm going to say that, just thinking about real-world situations, I'm disregarding these formulas that, that they're both going to be the same (Sally looks at researcher).

By writing out all of her work, Sally demonstrated that she was only considering the magnitudes of the momenta as she solved the problem with the momentum principle. Because she did not include direction in her reasoning, she neglected the possibility that the final momentum of the rubber ball could be in the opposite direction as its initial momentum even though she had drawn it that way on the whiteboard. This reasoning, shown in Figures 6.108 and 6.109, resulted in the initial erroneous conclusion that only the clay would apply a non-zero force to the door. Notice that this error was similar to Marco's confusion about the maximum amount of momentum that the door could obtain from a collision with an object; in that situation, Marco also did not account for the possibilities that arose because of the vector nature of momentum.

However, after thinking about why the clay should be better for a short time, Sally claimed that whatever happened after the collision with the door would have been irrelevant, thereby eliminating any possible solution that accounted for that information. She managed to convince herself that her approach had been incorrect and that both objects should affect the door the same, as she rejected
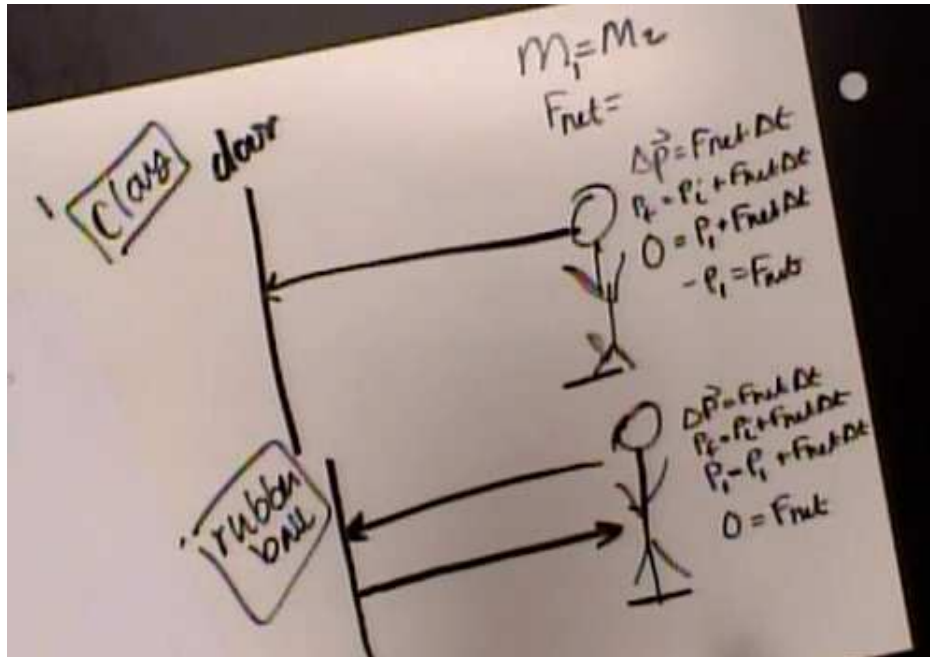
Figure 6.107: The whiteboard showing Sally's solution for the net force on the door due to both the clay and the rubber ball.

$\vec{p}_{f,ball} = \vec{p}_{i,ball} + \vec{F}_{net,\ on\ door}\Delta t$				
$|\vec{p}_{f,ball}|$	$\hat{p}_{f,ball}$	$|\vec{p}_{i,ball}|$	$\hat{p}_{i,ball}$	$\vec{F}_{net,\ on\ door}\Delta t$
p_1		p_1		0
p_1		p_1		$[\neq 0]$
Other possibilities not considered				

Figure 6.108: Sally's possibility set for the rubber ball, which did not include vector information for its momentum.

$\vec{p}_{f,clay} = \vec{p}_{i,clay} + \vec{F}_{net,\ on\ door}\Delta t$				
$|\vec{p}_{f,clay}|$	$\hat{p}_{f,clay}$	$|\vec{p}_{i,clay}|$	$\hat{p}_{i,clay}$	$\vec{F}_{net,\ on\ door}\Delta t$
0		p_1		$-p_1$
p_1		p_1		$[\neq -p_1]$
Other possibilities not considered				

Figure 6.109: Sally's possibility set for the clay ball, which did not include the vector information for its momentum, yielding a negative net force on the door.

her own calculation of the net force on the door as a result of its collision with the rubber ball by gesturing an "X" over that part of the picture (see Figure 6.107). She mentioned $\vec{F} = m\vec{a}$, a form of Newton's second law that is not taught in Matter and Interactions, possibly to support her decision: if one were to misinterpret the acceleration in that equation as being the acceleration of the clay or ball rather than the door, one could erroneously conclude that both objects would apply the same net force to the door. After drawing this conclusion, Sally stopped talking and looked at the researcher, who reminded her of the task.

> **Researcher**: Okay, just as a reminder that the, uh, task is to, um, identify which solution do you think is correct and which, ah, what are the errors in the other solutions.

> **Sally**: Now is when I really, I could have just looked – could I have just looked at these the whole time?

> **Researcher**: You're certainly welcome to tackle this task however you like.

> **Sally**: Okay. I'm just reading these, um, (has fanned out all three solutions). I'm realizing how much I've kind of forgotten. (Reading through solution #1). Okay, so now I see my mistakes. It's coming back, (reading from solution #1) "Since the ball leaves with just as much speed as it comes in with, the door gained very little kinetic energy however since the clay stops when it hits the door, all of that kinetic energy can be transferred to the door." So, "if both the ball and the clay hit the door at the same speed, the clay will transfer more kinetic energy to the door." (apparently re-reading), "The ball and the clay hit the door with the same speed, the clay will transfer more kinetic energy." "Therefore the clay is the better choice." (Reading solution #2) "If you pick the ball and the door as the system" T net. "Also since angular momentum is defined around a point, let's pick the hinge of the door to be that point." As L trans equals R cross P, "assume the ball initially travels across the x-axis." I'm just reading these in my head. I can understand them better. Um, "free to rotate about the y-axis. In this case the ball's initial angular momentum" (inaudible). (Reading solution #3) "Because P equals MV, the masses are equal, the objects have equal velocities when they will, and will have equal momentum. Since both the clay and the rubber ball will

provide the same amount momentum to the door if they are thrown at the same initial speed, it doesn't matter which you pick." Let's see. Just before the collision. Solution #1. It says, because the rubber's collision is nearly elastic. I think solution is [#]1, #1 is correct, um, because I think you do have to consider that the collision is elastic. With just about as much speed as it came with, the door gained very little kinetic energy. KF door equals one half MV. So here we see that (pointing to solution #1) this equation, however since the clay stops when it hits the door. So just mark which one I think it is? (Sally selects solution #1).

Sally looked through the three solutions, picking solution #1 as the time runs out. Remarkably, she did not choose solution #3, which was essentially the answer she had provided when reasoning on her own. But rather than picking that solution, she chose solution #1 because it included information that she deemed important and relevant; specifically, it accounted for the elastic collision between the rubber ball and the door, while the clay stopped when it hits the door. This information manifested itself in the objects' final speeds; the clay's final speed was zero while the ball's final speed was not. As in numerous other instances, she did not deeply assess the strength of the argument provided by solution #1, but she did identify that it included an important quantity. For Sally, clearly accounting for the final speeds of the objects was sufficient for the selection of the solution. To put it more precisely, the absence of a necessary quantity was sufficient for dismissal of a solution: she eliminated solution #3, which neglected the final speeds of the ball and clay.

Ana and Sally's group discussion

In the group session between Ana and Sally, Sally was randomly chosen to explain her choice for the correct solution to this problem, and she did so.

Sally: Okay, so I said it was solution #1.

Ana: Which one was this again?

Sally: This is, you have the, the rubber –

Ana: Ball, ok the ball or the clay.

Sally: The ball bouncing off. So at first I tried to do this with the momentum principle, without looking at these solutions, that's how I thought I was going to go about doing all of these, by trying to actually solve them and then seeing which one was closest.

Ana: Yeah, yeah I had no idea how to like start it, so I just looked at it –

Sally: But then, yeah, I realized after a while that I didn't remember enough to do that. Um, which at first I thought that since it was going backwards, I was thinking, you know, it would bounce off with equal and opposite, that something would cancel out when you are doing the math, and that it would make one better than the other. And then I started thinking that, you know, they're going to hit the door with the same force, so that it wouldn't really matter, but then when I actually started looking at these [solutions], I, I honestly don't know for sure which one I thought, but... I also remembered elastic equa– uh, collisions –

Ana: Yeah.

Sally: And that is why I chose this one (solution #1) , because I think it is the only one that accounts for the elastic, and I'm pretty sure that this is an equation that, or a problem that deals with elastic versus inelastic equations, and that – I mean, it may be opposite, that the other one would be a better choice because it's inelastic, but... I am not sure if you can solve these this, these two ways, but this (solution #2) is angular momentum, I don't think that is the way you would want to go about solving it, whether it is right or wrong. Um, but I guess since we are looking for the right one, um, (reading solution #2:) "since the mass of the clay" (trails off, silence). I still think it's this one (solution #1), just because of the elastic collisions.

Sally's explanation reinforced what she said during the think-aloud session. She chose solution #1 because it accounted for the elastic collision of the rubber ball, although she did not eliminate solution #2 like she did solution #3. She indicated that she didn't think that the angular momentum approach was appropriate. From this evidence we can infer that Sally chose between possibility sets that carried two different relationships (energy and angular momentum) rather than rejecting a possibility set based on the quantities that were listed within it.

Ana responded by speaking briefly about her reasoning, which was quite different from Sally's, although it yielded the same result.

Ana: Okay, I will show you what I put. Yeah, I put, I put [solution #]1 too.

Sally: Okay. (Ana laughs.) Just because of the collision?

Ana: No, because –

Sally: That might be the tricking thing though, is that they –

Ana: No, okay, like, so the third one, the answer is just dumb, like there is, like I don't think that's right (emphatically places it off to the side). That's how I eliminated it (chuckles). And so, okay, for the first and second one, I totally, like, the collision yeah, I looked at the elastic collision, but I didn't remember what that was or what it meant, and the second one, I kind of understood what they were doing, but it just – like,

Sally: Was really advanced.

Ana: Yeah, it just seems like –

Sally: Yeah, it was kind of overwhelming –

Ana: Yeah, like but I just looked at the final answer, so solution #1 says that the clay would be the better choice, and solution #2, the ball would be the better choice and like I kind of like, you would think, well at least I (places her hand on her chest) I thought about it this way, like I pretended like throwing like a ball and throwing a clay, and like I would think that the ball would do a better job than the clay, but like I somehow kind of remember that it's the clay, the weird answer was –

Sally: (Inaudible, crosstalk) ...Or something.

Ana: Yeah.

Sally: Yeah, I think the main thing to think about is a collision in this problem, not necessarily using the torque of the door –

Ana: Because they are using like, the hinge of the door to be the point, like I don't –

Sally: But at the same time, I guess I,

Ana: I mean I can see, yeah –

Sally: I can see what you're doing –

Ana: I can see what they're doing, but – I think that clay is the answer, so that's how I picked solution #1 instead of solution #2, because you would think it would be the ball but it's not. And they said the clay will transfer more kinetic energy to the door and that kind of makes sense –

Sally: Yeah, I think because of the collisions, it is solution #1. I don't know the actual answer should be. So let's – do want to agree on solution #1?

Ana: Yeah.

Ana visualized performing this experiment, closing the door with the clay and with the ball. Her gut instinct was that the ball should have been more effective, but she thought she remembered that the answer to this problem when they had seen it in class was surprising, so she chose the clay. Compare this discussion with the one that Sally and Ana had in Section 6.3.3, where they discussed the "Two Pucks" problem. In that case, they described what they thought about physics problems: that "the answer is something really weird, it's not like what you would think it would be." This belief about the nature of physics problems quite possibly influenced Ana's reasoning on this problem. However, in that discussion about the "Two Pucks" problem, Ana revealed that she was focused on the choices that the solutions made for the system and surroundings and hardly looked at the answers in those solutions, while Sally focused on the answers and identified one of them as being the answer from class. In the "Close the Door" problem, Ana indicated that she focused on the answers while Sally focused on the approach – a complete reversal from the other problem. This reversal hints that not only are there variations between how different reasoners solve problems, but there are also variations in how a single reasoner solves different problems.

This conversation between Ana and Sally confirmed that Sally's reasoning was affected by the inclusion of the three possible solutions. Specifically, the solutions reminded her that the nature of the collision between the door and the other object was an important feature that needed to be accounted for in the problem solutions. However, Sally did not then produce an in-depth analysis of the problems, instead simply choosing one that obviously accounted for this information. Sally's

performance implies that written solutions may help participants identify relevant quantities, but for instructional interventions involving such revelations, more detailed and structured investigations of possibilities that result from such an identification would probably be beneficial.

6.4.5   Misrepresenting solutions (Arthur, Otto, and Walter)

One of the benefits of the possibilities framework is that it helps to describe situations where there is a conflict or otherwise missed communication about physics problems. In the first group session between Arthur, Otto, and Walter, the group struggled to understand each other and reach a consensus about the correct solution to the "Close the Door" problem. Walter had originally selected solution #2, while Arthur and Otto both chose solution #1. As both sides attempted to explain their reasoning and convince the others that they were correct, they continued to misrepresent the solutions and failed to be precise and explicit about the possibilities that they were considering. This vagueness in their explanations doomed the discussion, as the conflict was never resolved. The members of the group were forced to settle for agreeing to disagree at the end of the discussion.

> **Walter**: Okay, which one did I pick? I believe, oh, apparently I picked solution #2, because solution #2 says that the ball would impact more speed on the door, which is what I agree with, because than the clay, I don't see how the clay could give you more momentum than the ball would, but it is not equal because, well just like when an astronaut throws a wrench backwards, and he gets pushed, he gets, you get more momentum when the ball bounces back off of it, because while the kinetic energy is similar, they're like, that's the magnitude of kinetic energy that has hardly changed, but the kinetic energy becomes negative, so you have to take that into account, so since it is bouncing off of it, you're going to get twice as much force approximately. What did you guys put?

Walter's explanation was all over the place. He talked about "impacting more speed to the door" and that "kinetic energy is similar," and he tried to link an analogy to an astronaut throwing a wrench. He even stated that while the magnitude of the kinetic energy had hardly changed, the kinetic energy became negative when the ball hit the door, meaning that approximately twice as

much force should have been imparted to the door. Of course, kinetic energy can never be negative, and it is also not related to force in the way Walter represented. Indeed, he quite possibly meant "momentum" when he said "kinetic energy." However, because he said "momentum," we represent that in Figure 6.110, which shows the correct relationship along with the distinct quantities that Walter mentioned. Notice that he did not once mention "angular momentum" or attempt to connect his reasoning with the solution he chose.

$\vec{L}_{i,ball} + \vec{L}_{i,door} = \vec{L}_{f,ball} + \vec{L}_{f,door}$			
speed	*momentum*	*kinetic energy*	*force*

Figure 6.110: The physical quantities Walter mentioned along with the correct relationship copied from solution #2 for the "Close the Door" problem.

Otto then explained that he chose solution #1 because the clay sticking to the door would mean that the clay could affect the door, while the ball bouncing back from the door would not affect the door in any way.

> **Otto**: I had, uh, selected solution #1, mainly because we actually for our homework for [physics course number] 205 had the exact same problem, and I remember that throwing the lump of clay adds to the mass of the door, making the kinetic, final kinetic energy more, pushing the door against, it goes against, it moves in the wrong direction than you want it to, but you still get more of a movement from the mass of clay than you do, than you will with the ball, because the ball bounces back and it doesn't change the door in any way, where the clay would stick to the door and change it. Change the way it moves.
>
> **Arthur**: Okay. I chose [solution #] 1 too.
>
> **Walter**: For the same reason?
>
> **Arthur**: Same reason.

At first glance, we might think that Otto presented the reasoning from solution #1 (see Figures 6.99 and 6.100) in a nutshell, but a closer investigation reveals that he actually predicted what should happen based on his intuition rather than the momentum principle. Otto reasoned that the

clay would have increased the mass of the door, and since kinetic energy is proportional to mass, the door's final kinetic energy would be more if the clay hit it than if the rubber ball hit it but bounced back. He conceded that the door would travel somewhat slowly as a result, but since he thought the ball would have had essentially no effect, he figured that a little bit of motion would be better than none. Figure 6.111 represents Otto's reasoning about the lump of clay. Note that he did not use the energy principle; instead, he claimed a result based on what he thought should happen (e.g., the door would have gained mass and a small amount of speed). His reasoning with the rubber ball was simply what was provided by the solution: the ball's speed would not change, so the door would gain no energy.

$K_{i,clay} + K_{i,door} = K_{f,clay} + K_{f,door} + \Delta E_{internal,clay}$				
$K_{i,clay}$	$K_{i,door}$	$K_{f,clay}$	$K_{f,door}$	$\Delta E_{internal,clay}$
	0		$\frac{1}{2}(m_{clay} + m_{door})(v_{f,door})^2$	
$\frac{1}{2}mv^2$	0	0	$[\neq \frac{1}{2}(m_{clay} + m_{door})(v_{f,door})^2]$	
Other possibilities not considered				

Figure 6.111: Otto's reasoning, that the clay ball should have increased the mass of the door and caused it to move, thereby giving it some kinetic energy.

To this point in the session, neither Otto nor Walter had represented his solution choice very clearly. They now embarked on a discussion to try to understand each others' reasoning.

Walter: Okey-dokey. I mean, I am going to disagree with you guys on this one, unless you can really convince me, because I am pretty confident it is [solution] #2, for the reasons I explained earlier, but the reason why I don't like the clay hitting it is while it does add mass to the door, it is going to add hardly any mass to the door, and –

Arthur: Yeah, but all of its, uh, momentum is then transferred –

Otto: To the door through the clay.

Walter: Yes, but you get even more momentum transferred to the door from the ball, because the ball is bouncing back in the negative direction. Because, because it is getting even more, because while it might get all of the, because while the ball hits it and bounces off, it will gain more momentum, it's, it's hard to describe. It's almost like if you are

playing, oh man, how to describe this? (Silence while Walter thinks) I don't know, I'll, I'll think –

Arthur: I'm trying to remember the way it was set up, because most of the problems we did, was a Ferris wheel, had balls on the outside, you drop a ball and it sticks, and it spins –

Walter: Yeah, well, I'm not trying to remember solutions, I am just trying to, like, reason my way through it. And I'm just like, I don't see how, I mean while it will add more mass, and the, but it's like, it won't add more momentum to the door or speed because it just, I, and since I'm just saying, since it's bouncing off, you get more energy, I mean, I can't, you've already heard what I said, it just bounces off, it gets more energy because it's bouncing now in the other direction and because force total, what is that? One of Newton's Laws, the one that says like –

Otto: Opposite and equal reaction –

Walter: Yeah, opposite and equal reaction, that's the one I'm trying to, like, that's the one I'm thinking of.

Otto: Yes, but it would, it's not saying that the door won't move if you throw the ball, it is saying that they will both move, just which one will move more? Will the one with the clay move more, the door more? Or [will the] the ball move the door more?

Walter: Yes, and I think the ball will move it more, and it you guys think the clay will move it more so...

Otto: If the clay moves the door. The ball moves the door.

Walter: Yes

Otto: But the momentum – but the weight of the door stays the same, and momentum is basically mass times velocity –

Arthur: Yeah, so if you increase –

Otto: Momentum is mass times velocity.

Walter: Yes, it is mass times velocity but the change in momentum is going to be the change in velocity.

Otto: Well it's, P final is bigger than P initial, the door doesn't move,

In this passage, there were two threads of conversation of interest. First, Walter asserted that the change in mass would be minimal, and that the change in momentum would be a result of a change in velocity of the door, implying that the object that caused the greater change in the door's velocity would be the better choice. Meanwhile, Otto continued to argue his point that momentum consists of both mass and velocity, and that by adding to the mass of the door, the clay would have increased the ball's momentum more. Interestingly, he admitted that both objects would have made the door move, in opposition to his earlier statement that the rubber ball should not have any effect on the door.

The second thread is that Walter was trying to explain how an object that bounces back would give more momentum to the door than an object that stopped when it contacted the door. Arthur argued, in much the same way that Marco and Sally reasoned, that because the clay would have stopped when it hit the door, all of its momentum would have transferred to the door, implying that there could be no possible situation wherein the momentum transferred to the door would be greater than the initial magnitude of the momentum of the object colliding with it. Walter tried to explain that the ball could provide more momentum, but he did not succeed.

In a sense, Arthur's possibility set was similar to Marco's (in Figure 6.105) and Sally's (in Figures 6.108 and 6.109), but Arthur's was much less fleshed out. He had not explicitly rejected the idea that the magnitude of the momentum of the door could be greater than the magnitude of the momentum of the object that collided with it; he simply had not considered it. This possibility set is shown in Figure 6.112.

Walter, as we see in the next segment of dialogue, seemed to have a set of possibilities that were about a single component of the momentum (for example, the x-component). He used terms like "negative" to describe the momentum, and he gave only a numerical speed for the velocity instead of explicitly indicating a direction. His set of possibilities at this point in the discussion is shown in Figure 6.113. Notice that Walter's reasoning was considerably different from Arthur's, even though both participants used the momentum principle.

$\vec{p}_{f,object} = \vec{p}_{i,object} + \Delta p_{door}$				
$|\vec{p}_{f,object}|$	$\hat{p}_{f,object}$	$|\vec{p}_{i,object}|$	$\hat{p}_{i,object}$	$|\Delta p_{door}|$
0		p		p
$(\neq 0)$		p		$< p$
Other possibilities not considered				

Figure 6.112: A possibility set that demonstrates how Arthur overlooked the possibility that the door could gain a greater magnitude of momentum than the object that collided with it initially had.

$p_{x,i,object} + p_{x,i,door} = p_{x,f,object} + p_{x,f,door}$			
$p_{x,i,object}$	$p_{x,i,door}$	$p_{x,f,object}$	$p_{x,f,door}$
p	0	0	p
p	0	$-p$	$2p$
Other possibilities not considered			

Figure 6.113: Walter's reasoning about one component of the momentum.

Walter tried unsuccessfully to point out that Arthur was neglecting the vector nature of the momentum and thereby overlooking some possibilities. All group members realized that the discussion was going nowhere without writing down their ideas on the whiteboard, and Walter and Otto grabbed for the two markers almost simultaneously. They then tried to solve the problem by using temporary numbers and the momentum principle.

> **Walter**: I was like, you, you start writing it down because this is about to get like, bad if we keep discussing this [without writing it down].

> **Otto**: All right, for the, for the ball, the door (drawing on the white board) would equal, the door's final change in momentum – no F (erasing something from the whiteboard, Walter says "don't worry about it") – would just be PF, right? (Walter affirms) because there is no P [initial, apparently], which would equal the mass of the original door times the velocity after the ball hits it, right?

> **Walter**: Yes, but what is going to be the new velocity? Because, because how you are going to calculate the velocity of the door is going to be, it will, how about this? How about we calculate the momentum of the door based off what the momentum of the ball

was? Or, the rubber clay thing? ...so you calculate...

Otto: So, do you want to give them a momentum and then solve it piece by piece?

Walter: Well, actually, I just want, and yes, I guess we might as well do that, because I was like, well, okay it would probably be better if you did it because you're at a better angle, but you can just write down like, like, like okay, let's assume that for convenience that the ball is moving at 10 m/s, an absurdly fast rate, and the ball is one kilogram, just to make things, and the clay thing too (meanwhile, Otto is affirming and writing on the white board).

Otto: Okay so, for the, for the masses,

Walter: 1 kg.

Otto: 1 kg, and then the velocity is –

Walter: 10 m/s, which is absurdly fast, but who the heck cares?

Otto: Yeah. And so for – (crosstalk over Walter below) problem two set the –

Walter: Meters *per* second I think (correcting where Otto had written "ms" on the white board) – whatever, it's hard for me to do the slash, there we go –

Otto: We get that –

Walter: So then, the momentum of the, of the clay and the ball, it is, the *initial* momentum, momentum initial of the clay and the ball, so (picks up the white board marker and begins writing) PI is equal to –

Otto: (Inaudible) Well, also this (solution #2) deals with torque and stuff, though, and the door does not spin, it just moves side to side.

Walter: I mean, technically it would have torque, but that's – I just want to show, let's just go with this and pretend that it just slides for now and, um –

Otto: Well, P initial of which, the P initial of the ball would be 10 as well as the clay (see Figure 6.114).

Walter: And then, but the finals are going to be different, because, because the P final for the ball is going to be approximately –

Otto: P final wouldn't change, it goes off with the same amount of (gesturing a ball bouncing off the door) –

Walter: No! It's negative, that is what I'm saying. It's negative, because velocity is a vector.

Arthur: (Crosstalk) Momentum is not a direction –

Otto: (Crosstalk) Velocity can't, velocity can't, yeah, velocity can't be negative –

Walter: Yes it can, velocity is the one that can be negative, it is, speed is the one that can't be negative. (silence) You can't –

Arthur: A velocity vector can be negative.

Otto: Yeah,

Walter: Yeah, that, that is what velocity is. Velocity is a vector.

Otto: Say, let's say that they threw it in a directly horizontal way, that be, that would be (writes $< -10, 0, 0 >$ on whiteboard) –

Walter: Velocity can only be a vector, that is the definition of a velocity compared to speed.

Otto: Well, if P final was negative, that means change in P for the ball is zero, yeah (writes "$p_f = 0$" on whiteboard).

Walter: No, the change would be 20.

Otto: Yeah, because PF equals PI, or, how did I figure this out –

Arthur: Conservation of momentum.

Walter: P, P, P,... well total PF is equal to total PI.

**Otto**: I think it even says that the ball doesn't change in momentum.

Walter: No, the ball doesn't change in momentum, magnitude of momentum, but its momentum changes –

Otto: Doesn't it? So, yeah that's what we're figuring out because we are using magnitude of momentum.

Walter: Well, I thought that the whole system didn't change momentum.

Otto: No, the door will move either way, so that means something changes.

Walter: Well, the entire system has the same momentum, the momentum of the total system never changes and never splits up. I mean –
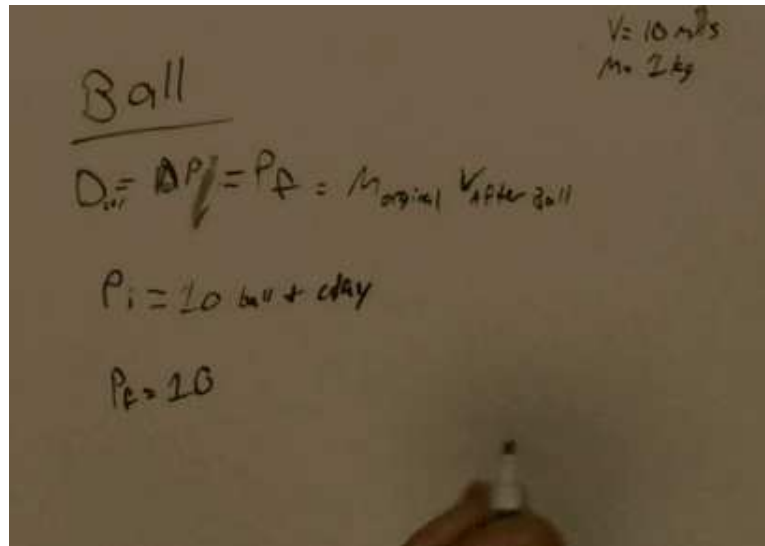
Figure 6.114: The whiteboard showing Otto's attempt to solve the problem with hypothetical numbers.

With the help of the whiteboard, the group got to the heart of the matter but nonetheless failed to resolve the issue. The conflict revolved around whether the final momentum of the rubber ball could have been negative. Walter insisted that it could, and that because of the final momentum being negative, it would have caused twice the change in momentum to the door as the clay because of conservation of momentum. However, Otto and Arthur argued that it would not be possible for momentum to be negative. There are two confounds in the discussion. First, because momentum is a vector, "negative" and "positive" have no meaning, so Otto's statement that "velocity can't be negative" is technically correct. However, Walter apparently meant the x-component of momentum, which textitcould be negative or positive, although he never clarified this point.

The second confound is that Walter tried to explain that the magnitude of the ball's momentum would not have changed, but he did not clarify how that would be different from the momentum

itself. Indeed, Otto thought they were already talking about the magnitude of momentum. This confusion was not resolved even when Otto wrote the initial vector momentum of the ball in 3-space, because he never completed the problem with that notation.

How could Walter have explained his reasoning better? The possibilities framework suggests that if he truly understood the possibility set that Arthur and Otto carried, Walter would have realized that they were overlooking the directional nature of the momentum (namely, $\hat{p}_f$ and $\hat{p}_i$). He could then have constructed an intervention whereby he repeated the possibility that they were envisioning, with the complete vector nature emphasized, thereby fleshing out values for the quantities that they were neglecting. Then, he could have demonstrated other possibilities (say, an initial vector of $< -10, 0, 0 >$ and a final vector of $< 0, 10, 0 >$), which would have allowed Otto and Arthur to realize that they did in fact need to flesh out all of the information about the vector nature of momentum. By staying within his own possibility set, however, Walter was unable to help his peers realize their errors. As we saw with this group in the "Two Blocks" problem (see Section 6.1.5), it is vital for everyone to agree on the same possibility sets when discussing reasoning. Otherwise, there are frequent opportunities for missed communication. Of course, there is no guarantee that even had Walter switched possibility sets that he would have convinced Otto or Arthur; all we really know is that Walter did not succeed in changing the reasoning of the other members of his group.

Otto tried once more to solve the problem using the momentum principle as the time expired, but Walter shut down all further attempts at coming to a consensus.

> **Otto**: (Again writing on the white board) So we have P total equals P of ball or clay plus P, P final, and P final of door is equal to P initial of ball slash clay plus P initial of door, which that is zero (setting P initial of the door to zero), and so the two different equations we will get is: P of ball plus P final of door plus P final of, equals PI of ball,

> **Walter**: P final of the ball plus P final of the door, I see what you are saying, but it is like, but I am like, but you are wrong.

> **Otto**: Well, P final of clay plus P final of door is equal to P initial clay, correct?

> **Walter**: Let me see. P final of –

> **Otto**: Of the clay plus P final of the door equals P initial of clay plus the door, because

right here, this is set to zero (pointing where he set P initial of the door to zero).

Walter: No, okay, that is totally true (agreeing with setting the initial momentum of the door to zero).

Otto: And if the clay sticks to the door, the clay initially stops moving and just becomes P final –

Walter: Well, the clay is moving with the door.

Otto: P final clay plus door equals P initial clay (see Figure 6.115).

Walter: The clay is moving with the door, yes, but (sighs), I mean –

Otto: Where did the problem [statement] go (looking through his papers)?

Walter: I still, I still, I am just going to disagree with you because I still disagree but I don't think we are going to be able to solve anything –

Otto: So you just want to say two to, two, two for solution #1 (Arthur affirms), and –

Walter: One for solution #2. I think its solution #2, yeah. It's solution #2. let's go with that. Watch it be solution #3 or something like –

Otto: No, solution #3 doesn't make much sense because you know that one is going to be better than the other at least, and it says that both of them are equal.

Walter: Oh, yeah, yeah, yeah solution #3 is wrong (Arthur affirms).

Otto: So we can at least agree that we know that solution #3 is wrong.

The group happily agreed that solution #3 was incorrect, but they confirmed their disagreement on which solution was actually the correct choice. Otto demonstrated that he understood the momentum principle, but he was still not clear about the vector nature of momentum and continued to disregard it.

This group discussion reaffirmed how important it is to understand the possibility sets that someone else is using, and to communicate within the realm of those sets in a way that encourages them to flesh out of possibilities within those sets. Neither Walter nor Otto were successful in influencing the other's reasoning, and the discussion gained very little ground. This failure stands in contrast
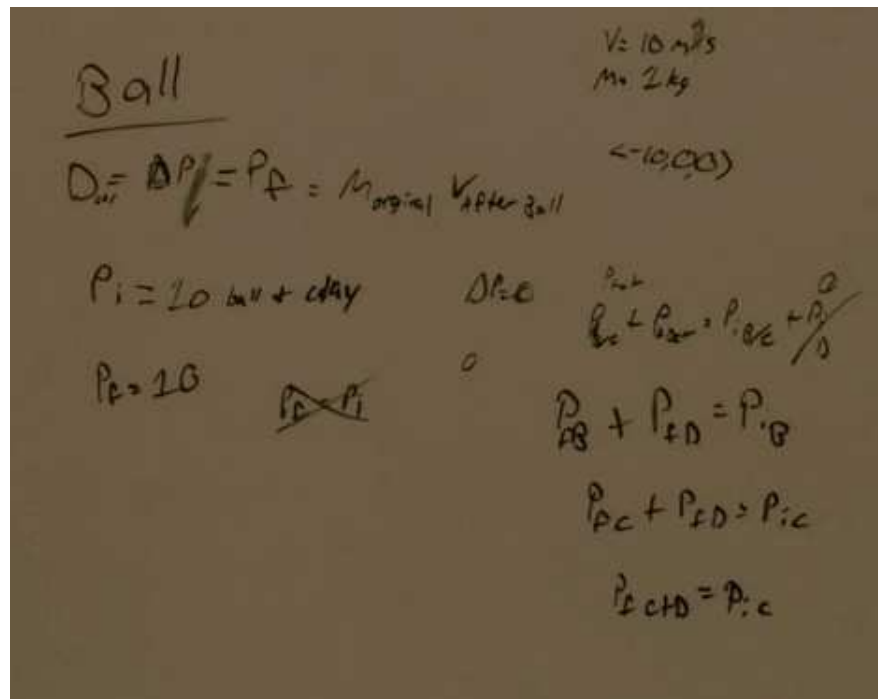
Figure 6.115: The whiteboard showing Otto's second attempt to solve the problem, but this time for the clay (in the lower right corner).

with this group's discussion about the "Two Blocks" problem (see Section 6.1.5), where Walter successfully convinced Otto of his reasoning (although Otto later rejected the entire argument) because Walter was eventually able to change his possibility set to conform with Otto's.

6.4.6 Discussion

These group discussions about the "Close the Door" problem reinforce the same themes that have been identified in previous sections. First, both the constructed solutions to the problem and a correct solution, as explained by Christobal, could be understood in terms of the possibilities framework.

Second, Marco demonstrated an internal conflict when he encountered both a tempting solution and one that was error-free. Although he could not find any errors with the second solution, he still rejected it because he could not conceive of the possibility that the momentum of the ball could be negative. When he realized that possibility, he accepted the second solution but was forced to rely on a plausible (but insufficient) reason why his initial solution could be correct. This example demonstrated yet again how difficult it is to switch from one possibility set, once chosen, to another. Marco's resistance of solution #2 also hints that even logically perfect arguments are not necessarily accepted by reasoners – even very experienced ones – especially when other particularly attractive alternatives exist.

Third, Sally demonstrated a method of selection based on recognizing a quantity that needed to be in the correct solution and choosing a solution that explicitly accounted for it, rather than fleshing out completely how that solution utilized that quantity. This selection again showed how proposed written solutions could be used to help participants identify quantities that they had previously overlooked (a theme that arose in both the "Two Blocks" and "Ball in Motion" problems), while reinforcing that even after recognizing these quantities, the participants did not necessarily thoroughly follow the reasoning of the proposed solutions.

Fourth, the discussion between Walter, Otto, and Arthur reinforced how difficult communication is between reasoners who hold different possibility sets. The possibilities framework indicates that a working understanding of another's set of possibilities is a necessary (although not sufficient) requirement for significantly influencing reasoning.

These themes continue to arise in reasoners' discussions about qualitative and semi-qualitative problems. Clearly, their reasoning can be represented with the possibilities framework, and doing so has definite appreciable advantages for understanding how they are thinking and how one might affect it.

6.5   Using the Possibilities Framework

Throughout this chapter, we have explored how the possibilities framework can describe how the participants in this study reasoned through some of the problems provided for this study. In this section, I present a summary of some of the insights that arose because of such a description. Particularly, the possibilities framework describes not only how the participants completed the task, but also how they solved (or did not solve) certain problems on their own, committed errors, chose between possible solutions, and even discussed and debated with each other when conflicts arose. This summary leads to some implications for how the possibilities framework could be used in more general terms to understand and influence reasoning on deductive physics problems. Specifically, I discuss how the possibilities framework provides a new perspective on what makes problems difficult. Also, the possibilities framework suggests possible interventions for different situations such as neglected quantities and the belief bias; moreover, it provides a way of tracking the effectiveness of such interventions to see if they are successful.

6.5.1   How the possibilities framework describes these results

The possibilities framework describes how participants reasoned when trying to identify the correct solution out of three alternatives for the four problems discussed in this chapter. The first two of these problems, "Two Blocks" and "Ball in Motion," which were solved by groups as a whole, allowed us to create possibility sets that reflected the reasoning that was happening as it occurred. The other two problems could still be viewed with a perspective afforded us by the possibilities framework, although the considerably reduced amount of information prevented the creation of many actual possibility set representations.

On the "Two Blocks" problem, groups often attempted to determine the answer before or without reading the possible solutions. However, in doing so they made a number of mistakes. The possibilities framework allowed us to determine no fewer than five distinct kinds of errors that could be made. Roughly, they corresponded to erring in working memory, omitting an important step (corresponding to a possibility set) in the reasoning process, neglecting relevant quantities, following a heuristic rather than performing deduction, and erring while fleshing out possibilities. These errors are not listed to form a catalogue of general errors that occur for all different problems, but to demonstrate that many different errors can yield similar results and therefore should not be inappropriately lumped together with the intention of using a single intervention to correct all of them at once. Indeed, an instructional intervention that would help one participant be more clear in working memory is unlikely to aid someone who is following a heuristic. As such, we see that the possibilities framework provides an important service in allowing for a distinction between similar errors that can be made.

We also saw that participants chose between solutions differently, depending on their reasoning. Many participants eliminated possible solutions for various reasons ranging from their unhappiness with a particular fundamental principle to their realization that the solution allowed no physical possibilities. With the possibilities framework, we can distinguish between those selections. Additionally, the framework describes how sometimes participants who were unable to eliminate all possibility sets (or, in the case of this task, solutions) used a form of entropic choosing to make the decision. When using an entropic choosing process, the participant made an estimate of the size of the possibility space afforded by two or more possibility sets and chose the one that seemed larger. In a sense, the participant chose the solution that he or she felt was more probable or more likely. However, we saw that participants who reasoned from a fundamental principle, thereby using some degree of deduction, did not ever use entropic choosing methods for selecting the correct solution. Entropic selection being non-deductive implies that instruction should discourage this process of choosing between possible solutions to a problem.

One of the more fascinating uses of the possibilities framework has been the ability to describe and track the reasoning that was discussed when there was a disagreement within a group. Arthur, Otto, and Walter provided two particularly interesting opportunities to do so, in the "Two Blocks"

problem and in the "Close the Door" problem. In both cases, Otto and Walter both struggled to convince the other of their points, and it is likely that this was because they were both using different possibility sets for their reasoning. In other words, Walter noticed possibilities that Otto did not, and Otto simply could not modify his reasoning because he could not access those possibilities. Eventually in the "Two Blocks" problem, Walter realized Otto's set of possibilities and used that set to explain a viable solution to the problem. After doing so, Otto realized the new possibilities but then suddenly rejected them, in favor of a heuristic-based solution. The implication from these discussions is that to influence reasoning, one must be certain that everyone in the conversation is using the same set of possibilities (or, in the case of an intervention that the intervention uses the same set of possibilities as the student).

In short, by looking at how the participants solved and talked about these four problems within the possibilities framework, we gain a perspective on how the participants reasoned about these problems. That the possibilities framework was useful for these four problems implies that it can be used in many other situations, both for understanding student reasoning in general on deductive problems and for generating interventions for improving that reasoning.

6.5.2   Implications for the possibilities framework

One advantage of the possibilities framework is that it can provide a description of what makes a deductive physics problem difficult. Possible reasons for a problem being hard include there being too many possibilities to easily account for in working memory, required quantities being inaccessible or otherwise not salient, or numerous possibility sets (that is, deductive steps) being required to reach the answer. Participants in the study seemed to try to reduce the possibility space of the problems they were working on as much as possible. In some cases, such as with Fay, Marco, and Omar solving the Two Blocks problem, this reduction of possibility space and working memory load meant listing out all possibilities and eliminating ones that proved impossible. However, at other times this reduction meant either insisting that a certain quantity must be present or simply neglecting important quantities. Therefore, problems with information that refers to irrelevant features of the problem (or problems that fail to mention information about a necessary quantity) are often particularly difficult for students because such irrelevant information makes the task of identifying

the quantities that need to be in their possibility sets more difficult.

To combat difficult problems, this study has provided some hints that the possibilities framework could also provide insight for generating interventions. For example, one apparent benefit of this task has been participants recognizing relevant quantities that they had previously overlooked. Apparently, just reading a solution that involves that quantity may alert the reasoner that they had been neglecting that quantity. Noticing a new quantity may cause the reasoner to rethink the problem, fleshing out possible values for that new quantity.

Interventions may also be developed that combat the use of entropic choosing. That method of choosing between possible solutions to a problem is the result of being unable to reduce the possibility space to the point where only one possibility remains. Apparently, interventions that encourage the elimination of possible solutions should help reasoners avoid the temptation of judging one solution to be "more likely" to be correct than another, in favor of promoting the deductive process as one in which solutions are either correct or incorrect, and that its status can be completely determined.

The possibilities framework has also emphasized the role of belief bias in solving problems, as seen in the problems in this chapter, particularly in the "Ball in Motion" problem, where the changing mass of a ball was seen as impossible to at least one group. By eliminating this possibility, that group also eliminated the solution that allowed it as a possibility, rather than entertaining it as a possible solution even temporarily to evaluate the argument in that solution. However, this reaction was quite different than the reaction of participants who had overlooked relevant quantities. In the latter case, participants were open to thinking about the new quantities and the values that those quantities could have. We might conjecture that the same could be said for ranges of values that quantities could have: if, instead of claiming that a particular value could not be negative (say, as Otto did when discussing the "Close the Door" problem), that a reasoner simply had not considered it, that reasoner would be more open to considering that possibility because it would have to be fleshed out, rather than it already being eliminated. The intervention that is suggested for these two situations is apparently very different; suggesting a possibility to someone who has already considered but rejected it may have little to no effect, while it would prompt a new solution to the problem in the case where it had not yet been fleshed out. This distinction that can be made using the possibilities framework, and it could have a significant impact on the instruction that is

designed.

Finally, the possibilities framework is a structure that could be used to track how effective instruction or interventions have been. Specifically, this could be done by recording and analyzing students' solutions of physics problems in much the same way as this chapter, by listing the possibilities that the students discussed. By analyzing the possibilities that participants considered (and neglected), one can revise instruction to account for new possibilities that students omit, or showing impossible ones that they clung to. It is worth noting that the possibilities framework is most effective for analyzing qualitative problems, or at least problems with a qualitative aspect. Such problems permit a wider range of possibilities that can be analyzed without resorting to arithmetic relationships and calculations.

7

Results from Individual Sessions

In the individual sessions, each participant performed the selection task for four problems, within a 15 minute time-limit for each one. Problem A was nicknamed as the "Close the Door" problem and was discussed at length in Section 6.4. Problem B was a quantitative problem that asked for the stiffness of a spring that is swung in a circle in outer space, and will be called "Swinging Mass" for this discussion. Problem C was another quantitative problem, which asked for the distance between two atomic nuclei immediately after a uranium nucleus fissions, and will be called "Fission" for this discussion. Finally, Problem D was nicknamed the "Two Pucks" problem and discussed in Section 6.3.

When participants completed the selection task in their individual sessions, they were asked to say aloud everything they were thinking. While this generated some good information about what actions they were taking, it generally did not result in enough depth for a complete understanding of the reasoning they were using. A few exceptions have been included in previous sections of this dissertation (see, for example, Sections 6.3.2, 6.4.3, and 6.4.4).

The data that were collected from the individual sessions did reveal how participants chose between possible solutions. The nature of the task was to choose the correct solution and to identify the errors in the incorrect solutions, and participants completed this task in numerous ways. Sometimes, a participant examined the solutions to the problem and then picked the one they thought

was most correct. In other cases, the participant apparently matched the possible solutions to memory, intuition, or to their own work from solving the problem during the session. However, in many cases, participants did identify errors in one or more of the solutions and on the basis of those errors rejected those solutions. Not surprisingly, the reasons they cited for those rejection errors fall into categories that reflect the sorts of errors they themselves committed when they solved problems like the "Two Blocks" problem in groups.

The following section will more deeply probe what is meant by textitrejection and explore the prominence of that code. Then, the various reasons for rejection the participants cited will be related to the reasoning errors that the groups of participants made. By that comparison, I will demonstrate how the Possibilities Framework can provide insight as to how individual participants completed the selection task.

7.1   About Rejection and Selection

Segments were coded as a "rejection" whenever the participant expressed that a particular solution was wrong, either by pointing out a specific error or by expressing distaste for it (see Section 5.3.2). There were a total of 170 unique instances of rejection over the four problems and 21 participants, which is an average of about one per incorrect solution, although in a few instances a participant did not reject any of the solutions for a given problem, while at other times a participant rejected a single solution multiple times.

Rejections tended to be final. Regardless of the reason for the rejection, after deciding that a solution was wrong, participants very rarely changed their minds. In fact, only four participants (two graduates and two undergraduates) ever rejected a solution and later selected it, overturning a total of six unique (less than 4% of the 170 total) rejection codes.

For all participants, rejection was apparently an integral part of the problem-solving procedure for this task. Each participant rejected possible solutions, although the number of unique rejections varied between the participants. For example, Sally only made three unique statements of rejection, while Walter made 14, identifying numerous errors in the written solutions. Table 7.1 lists the number of unique rejections per participant. On average, each participant made eight unique rejection

claims, again corresponding to approximately one per incorrect written solution they were given. Both graduates and undergraduates averaged eight; there was no significant difference between them in this case.

Table 7.1: The number of unique rejection claims per participant.

Participant	Number of unique rejection claims
Christobal	12
Dolly	5
Fay	10
Ike	9
Marco	8
Omar	6
Ana	7
Arthur	7
Claudette	7
Earl	9
Eduard	8
Gaston	6
Hanna	9
Hugo	8
Igor	10
Josephine	6
Otto	7
Sally	3
Teddy	8
Walter	14
Winfred	11

Participants also used approximately the same number of unique rejection claims on each problem. Indeed, when the types of solution claims are considered for each problem (selection, rejection, and non-action), a *chi-squared* test reveals that there is no significant difference in the proportion of the three claims on any of the four problems. Since the total number of unique claims on each problem ranged from 67 to 83, this relative uniformity is reflected by the number of unique rejection claims for each problem, shown on Table 7.2.

While it may seem on the surface that the participants completed the task the same way for

Table 7.2: The number of unique rejection claims per problem.

Problem	Number of unique rejection claims
Problem A ("Close the Door")	45
Problem B ("Swinging Mass")	40
Problem C ("Fission")	48
Problem D ("Two Pucks")	37

each of the problems, it is important to note that the reasons for rejecting possible solutions have not been identified in this data, and it is precisely those reasons that both shed some light on the differences in the problems and link the participants' use of rejection claims with the Possibilities Framework.

7.2 Reasons for Rejection

As mentioned in the methods chapter (see Section 5.3.2), five unique reasons for rejecting possible solutions emerged. Each of those reasons corresponded to an error that the participant claimed to find in that solution. Those errors can be related to the errors that the participants made themselves when solving the group problems. In this section, each code will be explored and compared to the errors that the participants made on the "Two Blocks" problem.

7.2.1 "Answer"

Rejection claims that are classified as "Answer" reject a possible solution on the grounds that the answer it obtains is not correct. Below are three examples of "Answer" segments.

- **Gaston**: "(Reading solution #1) 'The first puck requires a larger force to reach the same speed as the second puck.' I'm pretty sure that's not true (Gaston places solution #1 off to the side)."

- **Hanna**: "Solution #2, um, it is too complicated (chuckles). It doesn't really, and I think – I don't think the ball is a good choice, because I tried that once and it didn't work. Clay works, because it transfers the kinetic energy into the door, because it just stays there."

- **Otto**: "I remember doing, uh, an experiment like this in physics lab, where they weren't, weren't the same, so that eliminates solution #3. They shouldn't, they shouldn't equal. So that's, that eliminates this one."

Segments within this category are indicative of the participant not using deductive reasoning when thinking about that solution. Specifically, the answer rejections tend to come from intuition (as in the first example) or memory (as in the second and third examples), which may or may not be applicable to the particular problem being considered. Alternatively, a participant might reject a solution because that participant actually worked out the problem and reached a different final answer than did the proposed solution. However, rather than trying to understand the solution and ascertain whether that solution solved the problem correctly or not, the participant simply states that the answers are different and dismisses the proposed solution for that reason.

In any case, the participant is not attempting to understand the solution's reasoning, and is thereby not even participating in the endeavor of deduction when considering that proposed solution. As such, we can not model the participant's reasoning with the Possibilities Framework.

7.2.2   "Relationship"

When a participant rejects a solution because of an overarching approach or principle that was used, that is tantamount to rejecting the entire possibility set proposed by that solution because the participant rejects the relationship that the relevant quantities in the problem must take. Two examples of participants rejecting a solution because of a relationship are listed below.

- **Ana**: "For the second puck, puck, system, thread, earth, ice sheet. Momentum principle. So they're just saying F is .24 over Delta T. Okay, I don't think that this is, this does not make any sense. I don't think that they're using the right principle."
- **Christobal**: "Well, (places the three solutions out so that they are each visible) if we start with solution number one, the mistake here is the problem would be more appropriately solved using momentum-based physics."

To identify such an error, a participant must have a conception of a "correct" or "preferable" way to solve a particular problem. In physics, any correct relationship can be used to start a problem. However, that attempt to solve the problem may fail because there is not enough information, and

the attempt to solve the problem results in an inconclusive result.

One way to understand the task participants were given is to consider each solution as being a unique possibility set. As such, each problem consisted of three sets, only one of which correctly afforded a single possibility. Some participants chose to eliminate one or more of those sets by identifying the relationship drawn out in that solution and evaluating whether or not it was appropriate, rather than exploring the solution to identify a deductive error. According to the Possibilities Framework, only the possibility set(s) that have the an appropriate relationship are ever created in the participant's mind, and all others are relegated to the realm of the impossible. "Relationship" rejections eliminate an entire possibility set as a whole.

7.2.3   "Term"

The "Term" category for rejections refers to situations where a participant rejects a solution because that solution either contains a quantity that is deemed inappropriate or does not contain a quantity that is deemed necessary. One subset of this category is the choice of system and surroundings that the participant chose, as the choice of system helps define the actual terms that are included in the solution. Below are three examples of "Term" codes, the third of which is an example of a rejection because of the solution's choice of system.

- **Arthur**: "The second puck wrapped around. Is it taking into account? Yes, it is taking into account the earth and the force of the ice so, so it says to neglect it. You still have $y - components$. Maybe I'm just looking at the 3-D diagram wrong or something. But that solution three is definitely wrong because we are supposed to neglect the ice, the friction due to the ice."

- **Claudette**: "And, [solution] #1 doesn't, like we don't have, I don't believe we have, like I don't think we have a velocity so I don't know how you can really use – the velocity in that."

- **Walter**: "Okay is there anything significant in the surroundings? I'm pretty sure. System, let's see. (Looking at all three solutions) The block, spring. Hmm, I'm pretty sure that the system will be the spring, part of the, part of the system will be the spring because it says what is the stiffness of the spring, so I am pretty sure that I can immediately

discount solution #2."

The rejections based on the quantities that are present or absent in the solution are very reminiscent of some particular errors that the groups made. Specifically, situations where a participant neglected a particular quantity (e.g., the friction from block A on block B in the "Two Blocks" problem) or insisted a quantity be present in the solution (e.g., angular momentum in the "Two Pucks" problem) are easy to identify when represented diagrammatically within the Possibilities Framework. In the former case, a quantity has no information below it, leading to a situation where possibilities are not fleshed out. In the latter case, relationships that do not contain the requisite quantity are often overlooked in favor of ones that do, and this often results in an incorrect application of that relationship.

Rejections based on the quantities included in a potential solution may be quite powerful, as they can identify oversights such as the absence of the deformation energy in the clay solution of the "Close the Door" problem. However, they can also be dangerous. If a participant believes that a quantity must be included in a solution, as happened in the "Two Pucks" problem, the participant may incorrectly determine that sufficient grounds exist for rejecting that solution.

7.2.4 "Value"

When a participant claims that a possible solution errs by assigning an incorrect value to a quantity by an incorrect calculation, substitution, or inappropriate choice for an initial or final state, that participant has made a "Value" rejection. Below are three examples, the third of which refers to an inappropriate initial state choice. This is a "Value" rejection because the quantity under consideration (e.g., separation between atomic nuclei) is assigned an inappropriate value for the situation being considered.

- **Omar**: "Okay this is not right, this is not right. The potential energy here has been calculated using the charges for palladium and uranium nuclei, whereas the uranium nuclei has already disappeared and there exist only two palladium nuclei, so this is not correct."

- **Fay**: "Because the ball – because while the change is twice of it, that doesn't mean the energy transferred to the door is therefore greater, not equal to energy transferred to

door."

- **Otto**: "The initial state in solution #2 is wrong, so that can't be, so that can't be right."

"Value" rejections are based on the individual cells within the diagrammatic representation of the Possibilities Framework that contain information that is particular for a quantity given the relationship in the set. Because of this, they correspond to errors wherein possibilities are removed and/or added to a set. For example, a simple mistake like Ike's in the "Two Pucks" problem eliminated the possibility that the force by the table on block B could be to the left while it added the possibility that it could be to the right. Unlike "Term" or "Relationship" rejections, which eliminate entire sets of possibilities, a "Value" rejection eliminates a single possibility within that set. Because "Value" rejections only eliminate a single possibility rather than an entire set, often the attempt to solve the problem was appropriate and "on the right track," except for the error itself.

7.2.5 "Other"

Finally, a number of solution rejections did not contain enough information to classify them as one of the other four categories. The rejection claims that were categorized as "Other" were either prohibitively vague or were the result of a participant claiming to some degree that they did not understand the attempted solution (or that it was different from what they had learned in class). Below are three examples of rejections from this category.

- **Ana**: "And this one (solution #3) is saying since both the clay and the rubber ball will provide the same amount of momentum to the door if they're thrown with the same initial speed, it doesn't matter which one you pick. Hmm, I don't think this one is right, because it just doesn't seem logical. What they're saying doesn't make sense to me."

- **Winfred**: "I couldn't say what's wrong with this one, but I know that I've never seen a problem solved like this before. I remember seeing this (pointing to the analytical solution of a spring mass system given in solution #3), but I can't remember seeing us ever using, taking the derivative of this in class so because of the fact that I never learned how to do it this way, I'm going to say that this is the wrong way. But I could be wrong, just never have learned it."

- **Hanna**: "Well, this one (solution #3) they didn't add enough stuff."

Unfortunately, we can not know for sure what the participants were thinking, and the "Other" rejections are instances where what they said was not particularly helpful in identifying the reasoning they were using. In some instances, the participants may have actually been rejecting a solution because of one of the other reasons listed above, but there is no way of knowing which one. In other instances, the participants may have been suggesting that they were using a form of entropic selection (see Section 6.2.10), although there remains insufficient evidence to draw a definitive conclusion.

7.2.6   Differences between participants

The rejection claims that each participant made were sorted into these five categories pertaining to the reason for the rejection. The number of claims per reason per participant are listed in Table 7.3.

The population of participants is approximately homogenous; that is, the participants did not significantly vary from one another in the reasons they cited when rejecting possible solutions. A *chi-squared* test reveals a χ^2 value of 99.8, corresponding to $p = 0.066$. Much of the variance came from Ana's focus on the relationship between the quantities and Claudette's focus on the appropriate quantities to include in a solution. It is worth noting that these deviations are within expected norms, and there is no appreciable difference between participants.

However, if the six graduate students (Christobal, Dolly, Fay, Ike, Marco, and Omar) are grouped and compared to the 15 undergraduates, a pattern begins to emerge. Table 7.4 shows these results.

While the undergraduates seem to balance their reasons between "answer," "term," and "value," the graduate students seemed to emphasize the "value" reason. They proportionally cited "value" more frequently than did undergraduates ($Z = 2.3$, $p < .05$, two-tailed). In other words, the graduate students seemed more likely to accept the possibility set that was described and presented to them. They worked within that framework to identify an error in generating a specific value for a quantity, rather than discounting potential solutions because of their approach. This difference is apparent on Problem C ("Fission"), where all six graduate students discovered the error that the potential energy between two atomic nuclei was being calculated with the parent nucleus and one of the daughters, instead of the pair of daughters. All six eliminated that possible solution for that reason, but only six of the 15 undergraduates rejected that solution for that reason.

Table 7.3: The number of rejection claims for each reason per participant.

Participant	Answer	Relationship	Term	Value	Other
Christobal	1	1	3	7	0
Dolly	0	0	2	2	1
Fay	1	1	2	5	1
Ike	1	2	2	4	0
Marco	1	0	1	3	3
Omar	0	0	1	4	1
Ana	0	3	1	1	2
Arthur	0	0	1	3	3
Claudette	1	1	5	0	0
Earl	2	1	2	2	2
Eduard	3	0	0	3	2
Gaston	3	1	0	2	0
Hanna	3	1	0	3	2
Hugo	0	1	4	3	0
Igor	2	0	5	3	0
Josephine	1	0	1	3	1
Otto	3	0	1	2	1
Sally	0	0	1	2	0
Teddy	3	1	0	1	3
Walter	2	1	7	3	1
Winfred	2	0	2	6	1

Table 7.4: The number of rejection claims for each reason for graduates and undergraduates.

Student Rank	Answer	Relationship	Term	Value	Other
Graduates	4	4	11	25	6
Undergraduates	25	10	30	36	19

Additionally, the graduate students seemed to appeal to the answer of the solution less frequently than the undergraduates, although this was not statistically significant ($Z = 1.7$, $p = .088$, two-tailed). However, because no distinction was made in the coding between situations where participants worked out the problem and matched a solution to that effort and situations where they seemed to either intuitively guess the answer or remember it from class, it is not exactly clear what a comparison of "answer" codes might tell us.

Overall, it is clear that while both graduates and undergraduates used the process of rejecting solutions to complete this task, they differed substantially in how exactly they rejected the possible solutions. It should be pointed out that there were more variables at play between these two groups than simply their aggregate education level. The graduate students had each been taught introductory physics through a traditional curriculum and had not learned physics from the Matter and Interactions curriculum that the undergraduates were using (although some graduate students may have been reading the textbook to prepare for their teaching assistant assignments). Additionally, the problems chosen for this task were either similar or exactly the same as problems the undergraduates received in their lecture, lab, or homework during their previous semester in introductory physics. As such, they occasionally referenced their experiences and sometimes even remembered the answer from class and used that memory to choose the solution they believed to be correct. One instance of this occurring was on Problem D ("Two Pucks"). Five of the six graduate students rejected the correct solution because it did not contain angular momentum (the sixth rejected the correct solution because it "looked silly"), while only six of the 15 undergraduates rejected the correct solution for a similar reason. However, six undergraduates explicitly mentioned that they remembered this problem from class, and each of those six chose the correct solution (one of those six did not state that he remembered the answer from class, only that he remembered having seen it). Those were the only six correct responses to that problem in this study.

7.2.7 Differences between problems

Not only were there differences between graduates and undergraduates on this task, but there were also differences in the reasons for rejecting possible solutions by problem. The number of claims per reason per problem are listed in Table 7.5.

Table 7.5: The number of unique rejection claims per problem.

Participant	Answer	Relationship	Term	Value	Other
Problem A ("Close the Door")	18	5	12	4	6
Problem B ("Swinging Mass")	0	3	9	14	14
Problem C ("Fission")	2	0	7	37	2
Problem D ("Two Pucks")	9	6	13	6	3

The problems were very different from one another in terms of the types of rejection claims that were made by participants who were solving them ($\chi^2 = 89.5$, $p < .0001$). Just looking at Table 7.5, we can immediately become aware of some numbers that stand out. For example, there are a greatly disproportionate number of answer statements about Problem A, while there are none for Problem B. In fact, all but two of the answer statements appeared in the two problems that had qualitative answers. Problems B and C both required a numerical answer, and it is therefore not surprising that participants did not reject solutions on those two problems because of proposed answers. Interpreted with the Possibilities Framework, this means that participants focused their attention in trying to understand the reasoning on problems B and C; indeed, many of the solutions on those two problems were rejected because participants spotted an error in generating a value for a particular quantity given the relationship of the possibilities set. This method of rejection was severely reduced on Problems A and D.

No solutions were rejected on Problem C for using an inappropriate relationship. This result is not surprising, because all three possible solutions begin the same way by using the energy principle. All solutions in Problem B used the same principle as well (the momentum principle in this case), but they did not all use the same form of the principle, leading to a few statements that the relationship (in this case referring to the form of the momentum principle being used) was inappropriate for this particular problem.

Different problems had different responses by the participants. The problems with qualitative answers yielded more answer-based responses, while problems that used the same principle for each problem tended to focus the participants to make more claims about the quantities used in the reasoning and the values of those quantities. This finding suggests that on selection tasks like this one, participants may guess an answer to a problem with a qualitative answer even when there are

potential worked solutions to that problem, thereby circumventing the deductive reasoning that is modeled in those solutions. On the other hand, problems with possible solutions that only vary in the details of what quantities are included or how values for those quantities are obtained may help focus students on the relevance of those levels of deduction and thereby encourage reasoning with those features of possibility sets. However, these implications are mere speculation, and further research needs to be done to determine if, by modifying the type of problem for this task, one can modify the reasoning of the participants.

8

Conclusions

Throughout this dissertation, we have seen how participants solved physics problems that required deduction. Time and again, their attempts at reasoning have been modeled within the Possibilities Framework. Being able to do so has given us insight into the reasoning that students are actually using and the implications thereof, giving us a potential foundation on which to build instructional interventions that meet the students where they are rather than trying to force them into using a form of reasoning that is foreign to them. Such interventions will need to teach students how to more fully flesh out the possibilities that are afforded by problems, especially those that are more qualitative in nature. However, there is much yet that needs to be done. While the Possibilities Framework fits for the problems discussed in this dissertation, it needs to be applied to more situations to establish its versatility. Moreover, if it is to be a truly useful framework, the Possibilities Framework needs to be tested through the instructional interventions that it suggests.

8.1   Review of Research Questions and Findings

As a way of returning to the key points from this dissertation, let us review the research questions proposed in the introduction. The major question, "how do students reason about physics problems that require deduction?" can broken into smaller questions about this particular study and addressed by way of implication from those results.

8.1.1 Can an alternative framework be more effective than formal logical reasoning at representing how students solve deductive physics problems?

This dissertation presented the Possibilities Framework, which uses a model for reasoning based on the notion that reasoners deduce by considering and eliminating possibilities, rather than by using formal logical rules. This framework is strongly supported by the literature, as shown in Chapter 2. It is an effective framework for modeling the reasoning that students do when solving qualitative physics problems, as demonstrated by the pilot studies and the group selection-task sessions. For example, on the "Two Blocks" problem, Fay, Marco, and Omar explicitly demonstrated that they were considering two different possibilities and reasoned about which one of those possibilities was correct (see Section 6.1.3). Additionally, the individual selection-task sessions resulted in evidence that was not opposed the the Possibilities Framework and could be interpreted within it.

On the other hand, formal logical reasoning was repeatedly demonstrated to be ineffective for modeling the participants' reasoning. Even in the pilot study, participants did not use formal reasoning to solve deduction problems, even those that were so-called "logic problems." Additionally, the isolated cases of reasoning that could have been demonstrated as following logical rules could also be demonstrated through the Possibilities Framework, which provides far more information. Throughout the study, participants sometimes used reasoning that was explicitly not deductive, as they appealed to memory, intuition, analogy, and other forms of reasoning. While the Possibilities Framework does not yet account for such reasoning, attempts to model such reasoning in terms of formal logic completely fail.

8.1.2 How can we represent students' attempts to reason with the Possibilities Framework, and what can doing so tell us about how they do deduction?

Chapter 3, which described the Possibilities Framework, explicitly proposed an answer to the first part of this question. In that chapter is a set of eight rules to follow in generating a graphical representation of the reasoning that someone is using. That graphical representation allows us

to identify when a reasoner is not using an appropriate relationship between the quantities in a problem, when the quantities being considered are incorrect, and most importantly, when a reasoner has neglected to flesh out all of the possibilities afforded by the problem.

Accordingly, this representation facilitates the identification of errors in reasoning that are associated with failing to consider all of the possibilities, or by reducing the possibility space inappropriately. I was able to hypothesize that because of the stress on working memory, reasoners often attempt to reduce the possibility space by whatever means necessary. This may mean using belief bias to set the value (or range of values) for a quantity, or it may mean neglecting quantities altogether, whether intentionally or otherwise.

Errors that reasoners make are often unique to that reasoner but may follow patterns; for example, in the "Two Blocks" problem, many participants reasoned (incorrectly) that the friction force due to the table on block B was to the right, although the manner in which that error occurred was different for different groups. The Possibilities Framework allows the identification of the errors that the participants were making and suggests possible interventions for those errors. Indeed, the strength of the Possibilities Framework is that it can accommodate the precise pinpointing of errors in reasoning and allow instructors to target interventions to specifically address those errors.

Also, the Possibilities Framework provides a meaningful interpretation of what happens when participants suddenly become aware of a possibility that they had not previously considered. For example, when Fay, Marco, and Omar realized that the mass of the ball could change in the "Ball in Motion" problem (see Section 6.2.6), they rapidly flesh out the new possibilities that include the mass of the ball in their possibility set. One strength of the different written solutions is that they sometimes helped participants recognize quantities or values that they had not considered before, thus affecting their reasoning by allowing them to flesh out more possibilities. Worth noting, the intervention that the Possibilities Framework suggests for an accidentally neglected quantity was significantly different from one for belief bias: for a neglected quantity it should be sufficient to simply alert the reasoner of the neglect, whereas belief bias may require a much deeper intervention because the participant is aware of the quantity but has already decided what value it should have.

8.1.3 How do students decide between possible solutions to a physics problem?

The selection task study was precisely designed to answer this question: not only were the participants asked to choose the correct solution to physics problems individually, but they were also asked to explain their reasoning to peers and then perform the selection task within the group. By doing so, participants revealed much about how they chose what they believed to be the correct solution to these particular physics problems. Rejection of possible solutions played a key role, and in the group sessions negotiating whether a solution deserved being eliminated was a common feature. Participants cited numerous reasons for rejecting solutions, but those reasons could generally be summarized by using the Possibilities Framework as a guide: participants rejected solutions because they disagreed with the answer, choice of central relationship (or principle) used, quantities that appeared in the solution, or values for those quantities. The type of problem and given solutions influenced the reason for rejection, as qualitative problems tended to prompt more answer-based rejections while problems with solutions that only featured one central relationship tended to prompt more rejections based on the quantities used and the values obtained for those quantities.

In the group sessions, enough information was obtained in many situations to explicitly draw out the possibility sets of the reasoners. This evidence revealed the process of the rejection of a possible solution by either explicitly eliminating or overlooking the possibilities within a set. In many cases within the group sessions, especially on the "Two Blocks" problem, participants simply worked the problem out and matched the answer of the solutions to their response. Rarely did participants attempt to follow the exact reasoning of the solutions, instead reverse-engineering them to determine a plausible manner by which the given answer could be reached. For example, when Christobal obtained the correct answer in the "Two Blocks" problem by considering the two blocks together as a single block, he misinterpreted the correct written solution in terms of his possibility set, rather than in terms of the reasoning used on that solution (see Section 6.1.4). This mischaracterization implies that reasoners may have difficulty understanding a solution correctly after already deciding on the answer to the problem, regardless of how valid the given solution's logic is.

Indeed, that difficulty in changing one's possibility set is supported by the discussions in the group

sessions. When a disagreement arose, participants found it difficult to explain to one another why one solution was better than another. By deeply investigating the disagreement between Otto and Walter as they worked through the "Two Blocks" problem (and then again later as they discussed the "Close the Door" problem), we saw that Otto only began to understand Walter when Walter shifted his possibility set to be the same as Otto's. Then, Walter was able to flesh out all of the possibilities for Otto, who after a moment's consideration acknowledged that the reasoning was correct but immediately thereupon reverted to a different, much simpler set of possibilities regarding a heuristic about friction. Apparently, communicating ideas across possibility sets is very difficult to do, and that implies that an instructor must be careful to understand the reasoning of students, rather than simply repeating the correct reasoning to them.

In situations where participants did not commit to a particular answer, they debated the merits of numerous possibility sets, trying to choose between them. In some situations, the participants did not eliminate potential solutions, but instead non-deductively chose between them by claiming that one was "more" correct than another. One method participants used was referred to as entropic choosing in this dissertation. When choosing entropically, the participant selected the solution that seemed to have the greatest available possibility space, regardless of whether a deductively correct answer was present in that space. This was indicated by participants stating that a solution was more "open-ended" or "likely" than another. The extent to which participants reasoned this way is debatable because it is not easily determined from the data; more often than not, vague language is associated with entropic selection, making it difficult to pinpoint. However, more investigation could be done to determine when reasoners are likely to resort to that method of selection between possible solutions.

8.1.4 How do students reason about physics problems that require deduction?

From the previous discussion, we see that the participants in this study reasoned in a wide variety of ways. That reasoning could be described through the Possibilities Framework, and doing so suggested possible interventions for bringing student reasoning closer to that of expert reasoners in physics. The implications of this study are that students reason about physics problems that require

deduction in many ways, often by thinking about the possibilities allowed by the premises of the problem. However, they often overlook some possibilities, and this leads to errors in reasoning. As such, instructors should help students consider all possibilities that result from deductive physics problems, rather than simply repeating a correct solution that uses formal logic.

8.2 Implications and Future Work

The development of the Possibilities Framework discussed in this dissertation paves the way for a considerable variety of future work. Most importantly, predictions of the Possibilities Framework need to be tested. For example, this framework predicts that presenting only all of the relevant information should considerably improve reasoners' performance on deductive physics problems. Varying the information provided, either by adding irrelevant information to the problems or by removing necessary information, should increase the number of errors that reasoners make.

Even more important is the prediction that the Possibilities Framework presents an alternative way of learning physics that can be supplemental to existing methods. If reasoners are given experience and training regarding how to explore possibilities and perhaps even list them graphically in a manner similar to the graphical representation provided in this dissertation, they should make fewer errors related to failing to flesh out possibilities. This instruction can be done not only generally by teaching different reasoning strategies such as a graphical analysis method (as in Van Der Henst, Yang, & Johnson-Laird, 2002), but also specifically for particular physics concepts. For example, the average velocity of an object must be in the same direction as that object's displacement. One way to teach this material may involve explicitly listing possible directions for an object's average velocity and displacement and eliminating impossible combinations (see Figure 8.1). Doing so may drive the point home, rather than allowing it to be a minor, unfleshed consequence of a definition.

Indeed, because student reasoning can be better modeled with the Possibilities Framework than with formal logical reasoning, it follows that this should be reflected in the dialogue that occurs in the classroom. Rather than trying to convince students of physical conclusions through formal logic, the preferred language may in fact be that of possibilities. It may be that Socratic dialogue can incorporate questions about possibilities to meaningfully convey the deductive reasoning that

$\vec{v}_{average} = \Delta\vec{x}/\Delta t$	
direction of $\vec{v}_{average}$	*direction of $\Delta\vec{x}$*
$\downarrow$	$\uparrow$
$\leftarrow$	$\uparrow$
$\uparrow$	$\uparrow$
$< 4, -13, 6 >$	$< 7, -2, -1 >$
...	

Figure 8.1: Explicitly listing possibilities may be an effective teaching strategy.

is desired of our students. Teaching methodology based on the Possibilities Framework should be explored and tested to ascertain whether this approach for understanding student reasoning is productive.

Furthermore, the Possibilities Framework itself should be tested by creating sample tasks and predicting how participants should reason in terms of this framework. By asking participants to explicitly list what is possible in certain situations (e.g., Johnson-Laird & Byrne, 2002), we can ascertain errors in fully understanding physics principles and concepts. Studies can also be designed around the frequency of errors made when solving problems (e.g., Johnson-Laird et al., 1994), which would help us understand how different presentations of problems (say, by modifying the mentioned quantities) affected the effectiveness of participants' reasoning. Of course, further analysis of student discourse could provide more depth to the suggestions that were mentioned in this dissertation. These tests should be conducted to determine whether the Possibilities Framework that is presented here is a viable way of understanding student reasoning, or whether we need to explore different possibilities for how students reason.

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Appendices

Appendix A: Consent Forms

Informed Consent Form

North Carolina State University
INFORMED CONSENT FORM for RESEARCH

Title of Study The importance of alternatives: What physics students do when comparing worked examples.

Principal Investigators Evan Richards and Jon Gaffney

Faculty Sponsor Dr. Ruth Chabay

What are some general things you should know about research studies?

You are being asked to take part in a research study. Your participation in this study is voluntary. You have the right to be a part of this study, to choose not to participate or to stop participating at any time. The purpose of research studies is to gain a better understanding of a certain topic or issue. You are not guaranteed any personal benefits from being in a study. Research studies also may pose risks to those that participate. In this consent form you will find specific details about the research in which you being asked to participate in. If you do not understand something in this form it is your right to ask the researcher for clarification or more information. A copy of this consent form will be provided to you. If at any time you have questions about your participation, do not hesitate to contact the researcher(s) named above.

What is the purpose of this study?

This research is intended to understand how students select a correct solution to a given physics problem. ***This research has no connection whatsoever to any course you may currently be enrolled in.***

What will happen if you take part in the study?

If you agree to participate in this study, you will be asked to meet a researcher at 224 Riddick Hall. The researcher will take you to an interview room (either Riddick 214, 216, or 222). You will be given physics problems to study, and you will be provided with either several possible solutions

to each problem or a single solution. If you are given multiple solutions, only one will be correct. You will be asked to identify the correct solution and identify the errors in the other solutions. For each problem, you will have a maximum of 15 minutes to complete this task.

You will be asked to say your thoughts aloud. Warm-up exercises will be provided as practice for saying your thoughts aloud. You will be provided with a brief physics information sheet and a calculator for reference during the session.

At the conclusion of this session, you will participate in an interview with a group of other participants who, like you, have just completed an individual interview. One member of the group, selected at random, will be asked to present his or her decision and justification to a problem from the individual interview. After this presentation, the other group members will reveal how they had responded in the interview (and may provide justification if they wish). The group will then have a few minutes (a time limit will be given) to agree on a particular solution as being correct. Agreement of all members is desired, but not required. This process will be repeated so that each member of the group presents exactly one problem from the individual interviews.

At the onset of the group interview, you will be given a small token, which we will refer to as a "pass chip." If you are selected to present a problem that you do not wish to present, you may return your pass chip in lieu of presenting that problem. You will therefore be allowed to pass exactly once if you wish. If you retain your pass chip, you will gain no additional benefits or compensation.

After each group member has presented a problem and the group has discussed each one, you will be given a short break.

Upon returning, the group will be given new physics examples to study. You will be provided with several solutions for each example. Only one of the solutions will be correct. The other solutions will contain error(s). The group's task will be the same as your original task in your individual session (pick the correct solution and identify errors in the incorrect solutions).

Finally, a researcher will present the correct answers to you as a group, and you will have an opportunity to discuss with the researcher the physics involved with the problems.

The entire session will be audio and video recorded.

Risks

You may be concerned if you are not certain that you have accurately identified the correct solution or if you have identified all of the errors in the incorrect solutions, or if you are required to explain a solution to your peers that you are not positive about. You might also feel frustrated if you become confused by the problem solutions you are given, or justifications that other members of your group are presenting. These are normal reactions when someone is performing a challenging task, and we intentionally selected the problems and solutions to be challenging to analyze. Such a level of difficulty is necessary for us to gain a better understanding of how you identify the correct solution and find errors in the incorrect solutions. You should not be concerned if you are not certain which solution is the correct one or if you are not certain that all errors have been identified. You should also not be concerned if you are unable to understand or agree with your group members, as challenging problems are often difficult to discuss and understand.

You may feel embarrassed if you are unsure of your justification to a particular problem. However, you should bear in mind that all of the participants are in a similar position, having considered the same (quite challenging) problems. You will also have the opportunity to pass once on a problem, so that you can avoid presenting a solution you feel particularly uncomfortable about.

Since the session will be video and audio recorded, your voice and likeness will be in the video and audio data. The researchers will take reasonable measures to protect the data for educational research and development purposes only. The researchers will never use your real name to identify you; instead, they will use a pseudonym.

After the session, you will have the option to allow the researchers to show video clips from this session at research conferences. You are not required to grant permission to do this, and this will not be done without your explicit permission.

Benefits

You will receive practice in using key concepts from introductory physics. Additionally, you will have the opportunity to engage in dialogue with peers about the physics problems that you have been asked to solve. This may provide some practice for your interactions in class, and talking with other physics students may help your reasoning in other physics-related situations. You will also

have the opportunity to discuss the physics problems in this session with the researcher.

Confidentiality

Your real name will not be used in any oral or written reports. Instead, a pseudonym will be associated with the data.

Compensation

A compensation of $15.00/hour will be paid on completion of the session.

What if you have questions about this study?

If you have questions at any time about the study or the procedures, you may contact the researchers, Evan Richards and Jon Gaffney, at 224A Riddick Hall, or [919-513-7214].

What if you have questions about your rights as a research participant?

If you feel you have not been treated according to the descriptions in this form, or your rights as a participant in research have been violated during the course of this project, you may contact Deb Paxton, Regulatory Compliance Administrator, Box 7514, NCSU Campus (919/515-4514), or Joe Rabiega, IRB Coordinator, Box 7514, NCSU Campus (919/515-7515).

Consent To Participate

"I have read and understand the above information. I have received a copy of this form, and I am 18 years old or older. I agree to be videotaped, and I agree to participate in this study with the understanding that I may withdraw at any time."

Subject's signature _____ Date _____

Investigator's signature _____ Date _____

Video Use Consent Form

We would like your permission to show clips from your session at research conferences. The purpose of showing clips is to enhance presentations. Showing such videos will engage colleagues in dialogue about the research focus. Giving your consent allows the following minimal risk: someone may recognize you or possibly form an opinion about you. Without penalty, you may elect to have your likeness obscured or decline to give us permission to show video clips from your session.

"I understand that I may decline permission to allow video data from my session to be presented at conferences or similar scholarly public venues without penalty. I also understand that the researchers will never use my real name, even if I allow the use of video clips in public presentations. Regardless of my choice below, the researchers may still retain and analyze the video taken of this session."

Subject's signature _____ Date _____

Please initial below beside the appropriate choice, indicating your decision on the use of the video.

_____ 1. "Yes, I give permission to use video taken from these interview sessions in public presentations of the results of this study as it is. I understand that a pseudonym will be used in conjunction with my video, and that my identity will never be revealed by the researchers."

_____ 2. "I give permission to use video taken from these interview sessions in public presentations of the results of this study, provided my likeness (both my face and voice) is obscured. I understand that a pseudonym will be used in conjunction with my video, and that my identity will never be revealed by the researchers."

_____ 3. "No, I do not give my permission to use video taken from these interview sessions in public presentations of the results of this study."

Appendix B: Interviewer Scripts

Script for Conducting Each Individual Session

Welcome to my research study, and thank you for participating! I greatly appreciate your help. [*Ensure that the first part of the consent form is signed and dated.*]

Please note the video camera [*point out video camera*] that will record the session. Unless you look up at it, your face will not be seen. This is the microphone; it is very sensitive, so please be careful not to brush against it. Only NC State researchers will see the videotape unless you give me permission to show the tape to others. Once the session is done, you can fill out the last part of the consent form specifying how the video can be used.

If you have a cell phone here, please turn it off.

I am now going to begin the video recording.

During this study, I will give you physics problems to look at. For each problem that I give you, I will also give you three written solutions. Only one of the solutions will be correct. The other solutions will have errors. Your task is to figure out which solution is correct and to find the errors in the incorrect solutions. I have intentionally picked these problems to be difficult. So dont worry if you struggle with this task in any way. Because I am interested in how you identify the correct solution and find the errors, I need to use challenging problems and solutions otherwise we wouldnt get much useful information from this study. I just ask that you try your best. If you are not certain which solution is correct, then indicate as well as you can which one might be the correct solution. If you are not sure that you have found all errors, just try to find as many as possible.

There are others also being interviewed right now. Once these interviews have finished, we will bring all of you together to work as a team to discuss your decisions. During that session, you will be asked to present your decision and reasoning behind it to the other members of your group for one of the problems you will see in this interview. You will have one opportunity to "pass" if you are assigned a problem you do not want to present, so you should bear that in mind if there is one problem in particular that you would rather not present. I want to emphasize again that these problems are challenging, so it is OK if you simply need to take a guess and present that as justification to the group. All we ask is that you try your best.

At the end of the group session today, a researcher will be happy to explain the answers of all of the problems in this interview as well as the problems in the group interview. I cant explain the answers in between the problems or at the end of this interview.

During this study, I am interested in what you are thinking about. So, I will ask you to think aloud as you work through the problems and go over the solutions.

- By this I mean that I would like you to say everything that you are thinking, talking constantly, from the time you first see the problem and solutions until you are finished with the problem and solutions.

- You should not plan ahead what you are going to say or explain to me what you are saying.

- Act as if you are alone in this room talking through the problem to yourself.

- It is very important that you keep talking, so if you are silent for a period of time, then I will remind you to keep talking.

We will go through some warm-up exercises to get you used to the process, and I will give you suggestions to help you improve. Then I will describe your task for this session in more detail before we begin. Any questions?

[*Provide think-aloud training*]

Very good job on the think-aloud training! Remember that the most important thing for you to do is to tell us everything you're thinking. You cant tell us too much!

Here is a red pen that you can use to mark-up the solutions [gesture to red pen] and to mark the corner of the correct solution [*gesture to the square on one of the solutions as an example — KEEP MOST OF IT COVERED*]. Here is a reference sheet that you can use at any time [*gesture to the reference sheet*]. You can also use this calculator at any time [*gesture to the calculator*]. If you want to write anything, please use the whiteboard [*gesture to the whiteboard and marker*]. Please don't erase anything; just cross out your work. We have extra whiteboards [*gesture to whiteboards*]; we will give you a clean whiteboard whenever you want one.

You will have up to 15 minutes for each problem to determine which solution is correct and the errors in the other solutions. This timer will show you how much time is left [*motion to timer*]. We can move onto the next problem and solutions set at any time before that 15 minutes is up; just let

me know when you're done working with each problem and we can go on.

As a reminder, we are asking you to determine which solution is correct and what is wrong with the other solutions. Do you have any questions about your task during this session?

Ok. Here's your first problem [*present the participant the first problem, with the three possible solutions*].

[*Conduct session*]

Thank you for all of your help so far. In a moment, we are going to take a short break before the next part of this session. I'm collecting all of your materials from this interview. You will have access to these again in the group session.

As a reminder, we are going to go over these problems after everything is done today, so if you have any questions about the physics, we will address those at the end. Because of your upcoming group session, I ask that during this break you please not talk about these problems with the other participants in the study.

Ok, if you'll come with me, I'll show you where the restroom is and where the refreshments are. [*Lead the participant out. Take down the "Study in session" signs*].

[*Collect all of the materials from the interview. Place the participant's work in the manila folder with the appropriate pseudonym WRITTEN ON IT (we will be giving that back to the participant in the group interview)*].

Script for Conducting the First Part of Each Group Session

Thank you for participating in this group session! We all greatly appreciate your help.

Please note the video cameras [point out video cameras] that will record the session. Only NC State researchers will see the videotape unless you give me permission to show the tape to others. Once the session is done, you can fill out the last part of the consent form specifying how the video can be used.

If you have a cell phone here, please turn it off. I am now going to begin the video recording.

This group session will consist of two parts, with a short break in between. Right now I will describe what we'll do before the break. I'm handing you the materials from your individual sessions [*hand out folders with their work*]. Now, as a group are going to discuss your decisions from the individual sessions.

I will pick a problem and roll this die [*show die*] to pick one of you to present the problem, your choice for the correct solution, and your reasoning for that choice. Again, remember that these problems were intentionally made to be difficult, so it's ok to be uncertain; we just ask that you do the best you can.

While one of you is presenting, I'd like to ask that the others of you hide your solutions. Then, after the presentation, the others of you will reveal how you had chosen and provide your reasoning only if you want. As a group, you will have up to 10 minutes total to discuss the problem and attempt to reach a consensus. While a consensus is desired, it is not required. We will move on to the next problem after either a consensus is reached or the time is used up. I'll start the timer when you begin your explanation.

Additionally, I'm handing each of you a "pass chip" [*hand out tokens*]. This chip entitles you to pass once on any problem when you are asked to present. If you want to pass, simply return the chip to me. Do not tell me why you are passing. I will then roll the die again, and someone else will present the problem. You will gain no additional compensation if you hold on to your chip.

Please note the markers and whiteboard, the reference sheet, and the calculator. You may use these if you wish. Do you have any questions?

Ok. The first problem is Problem D, the two pucks. [*roll the die. The problems follow thusly: Door (A), Block Spinning (B), Fission (C).*]

Script for Conducting the Second Part of Each Group Sessions

Welcome back from the break! I hope you're refreshed and ready to go. We appreciate your help once again!

This is a reminder to please turn off your cell phone if you turned it on earlier.

I am now going to begin the video recording.

This session will run very similarly to your individual session. As a group, you will be given a problem and three possible solutions. As before, your task will be to choose the correct solution and to find errors in the incorrect solutions. You may approach the task however you wish.

You will have 15 minutes for each problem [*gesture to the timer*]. Please try to reach a consensus. Again, while consensus is desired, it is not required. After this session, a researcher will go over the solutions to all of the problems you have seen today and answer any questions you have about the physics.

Please try to keep your work within the "dots" on the table [*point out dots*], writing on the whiteboard whenever possible. Doing so will help us be sure that we capture what you're doing on the overhead camera.

Also, as before you may use the markers, calculator, and reference sheet. Please mark the correct solution with the red pen.

Do you have any questions?

Ok. Here is the first problem [*present the group with the first problem, with the three possible solutions*].

Exit Script

<u>Researcher from group task</u>: Thank you for all of your help today. Hopefully you have had a pleasant experience and maybe even learned something. As a reminder, we would all appreciate it very much if you did not discuss these problems with anyone else, as we need to be able to run this sort of interview again with new students, and it will hurt our research if you tell others about these problems.

In a moment, [*insert name of other researcher*] is going to take you to another room to discuss the problems you've seen today with you. You should feel free to ask any questions about the physics at that time. At this time, I'm going to hand you off to [*insert name of other researcher*], who will take care of a couple items of business. Again, thank you and have a great semester!

<u>Other researcher</u>: First of all, I'd like to give each of you compensation for your time today [*distribute envelopes with money and the forms indicating that they have received their money*]. Please sign this form to indicate that you have received your payment, so that we can be sure to get reimbursed for it.

Also, I'd like for you to consider whether you'd allow us as researchers to use the video and audio data we've taken today at conferences and other meetings where we will be interacting with fellow researchers and teachers. Please consider your options on this consent form carefully and choose as you wish [*distribute video consent forms*]. Please note that there are TWO decisions you are making: the first consent form is for your individual session. The second is for the group session.

[*Once the forms are handed in*] Thank you very much. If you'd like to come with me, we can discuss the solutions to the problems you have been discussing today [*take participants to the conference room to discuss the problems*].

Appendix C: Individual Session Problems and Solutions

Problem: A

A door stands open, and you want to shut it by throwing something at it. You could either throw a lump of clay or a rubber ball at the door, both of which have the same mass. You know that the rubber ball will bounce back, while the lump of clay will stick to the door. Which should you pick?

Solution #1

When the rubber ball contacts the door, it will bounce back with almost the same speed as before the collision, while the clay will stick to the door. This is because the rubber ball's collision is nearly elastic.

$$K_{f,ball} + K_{f,door} = K_{i,ball} + K_{i,door}$$
$$\tfrac{1}{2}mv_{f,ball}^2 + K_{f,door} = \tfrac{1}{2}mv_{i,ball}^2$$

Since the ball leaves with just about as much speed as it came in with, the door gains very little kinetic energy. However, since the clay stops when it hits the door, all of that kinetic energy can be transferred to the door:

$$K_{f,clay+door} = K_{i,clay} + K_{i,door}$$
$$K_{f,clay+door} = \tfrac{1}{2}mv_{i,clay}^2$$

If both the ball and the clay hit the door with the same speed, the clay will transfer more kinetic energy to the door. Therefore the clay is the better choice.

Solution #2

If we pick the ball and door as the system, $\vec{\tau}_{net} = \langle 0, 0, 0 \rangle$ N·m. Also, since angular momentum is defined around a point, let's pick the hinge of the door to be that point.

For the ball:

$$\vec{L}_{i,trans,ball} + \vec{L}_{i,rot,door} = \vec{L}_{f,trans,ball} + \vec{L}_{f,rot,door}$$
$$\Delta\vec{L}_{rot,door} = \vec{L}_{i,trans,ball} - \vec{L}_{f,trans,ball}$$

And $\vec{L}_{trans} = \vec{r} \times \vec{p}$. Assume the ball initially travels along the +x-axis and collides at a 90-degree angle with a door that is initially at rest but free to rotate about the y-axis. In this case, the ball's initial angular momentum is in the +y direction. After the collision, the ball is traveling in the opposite direction so its final angular momentum is in the -y direction.

$$\Delta\vec{L}_{rot,door} = \langle 0, rp_{i,ball}, 0 \rangle - \langle 0, -rp_{f,ball}, 0 \rangle$$
$$|\Delta\vec{L}_{door}| = |r(p_{i,ball} + p_{f,ball})|$$

And, because the final speed of the rubber ball is nearly as much as its initial speed,

$$|\Delta\vec{L}_{door}| \approx |2rmv_{i,ball}|$$

However, for the clay and the door as the system,

$$\vec{L}_{i,trans,clay} + \vec{L}_{i,rot,door} = \vec{L}_{f,trans,clay+door}$$
But, $m_{clay} \ll m_{door}$

$$\Delta\vec{L}_{rot,door} \approx \vec{L}_{i,trans,clay}$$
$$|\Delta\vec{L}_{door}| \approx |rp_{i,clay}| \approx |rmv_{i,clay}|$$

Since the mass of the clay is equal to the mass of the rubber ball, the door's angular momentum will change more if the ball hits it with some speed than if the clay hits it with that same speed. Therefore, the ball is the better choice.
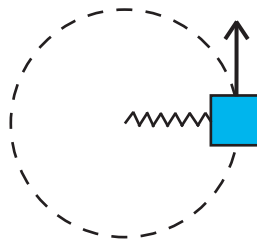
Solution #3

Let us consider the momentum of the two objects.

Because $\vec{p} = m\vec{v}$, and the masses are equal, if the objects have the equal velocities then they will have equal momenta. Since both the clay and the rubber ball will provide the same amount of momentum to the door if they're thrown with the same initial speed, it doesn't matter which one you pick.

Problem: B

A 0.5 kg block of wood is attached to one end of a spring with negligible mass. While performing an experiment far away from other objects, an astronaut holds the other end of the spring and swings the block in a circle with a constant speed. She finds that the block swings around with a period T of 1.5 seconds and that the length of the spring has stretched to a total length of 0.6 meters. The spring has a relaxed length of 0.25 m.

What is the stiffness of the spring?

Solution #1

System: Block, spring

Surroundings: Nothing significant

Momentum Principle: $\vec{p}_f = \vec{p}_i + \vec{F}_{net}\Delta t$

For the radial component (the perpendicular component to the motion): $mv = (k_s s)\, T$

$$k_s = \frac{m\cancel{v}^{\,\omega r}}{sT} = \frac{m\cancel{\omega r}^{\,\frac{2\pi}{T}}}{sT} = \frac{2\pi m r}{sT^2}$$

$$k_s = \frac{2\pi\,(0.5\ \text{kg})\,(0.6\ \text{m})}{(0.6\ \text{m} - 0.25\ \text{m})\,(1.5\ \text{s})^2}$$

$$k_s = 2.39\ \frac{\text{N}}{\text{m}}$$

Solution #2

System: Block

Surroundings: Spring

Momentum Principle: $\dfrac{d\vec{p}}{dt} = \vec{F}_{net}$

Since the block is moving in a circle at a constant speed, only the perpendicular component (to the motion) of $\dfrac{d\vec{p}}{dt}$ is non-zero. The textbook showed that this component is equal to $p\dfrac{v}{r}$ (recall the "kissing" circle).

$$\left| \dfrac{d\vec{p}}{dt} \right| = p\dfrac{v}{r} = p\omega = mv\omega = m\left(r\omega\right)\omega = mr\omega^2, \text{ where } \omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{1.5 \text{ s}} = 4.19 \ \dfrac{\text{rad}}{\text{sec}}$$

$$\left| \dfrac{d\vec{p}}{dt} \right| = \left| \vec{F}_{net} \right| \qquad \overset{mr\omega^2}{\nearrow} \qquad \overset{\left|\vec{F}_{spring}\right| = k_s s}{\nearrow}$$

$$mr\omega^2 = k_s s$$

$$k_s = \dfrac{mr\omega^2}{s}$$

$$k_s = \dfrac{(0.5 \text{ kg})\,(0.6 \text{ m})\left(4.19 \ \dfrac{\text{rad}}{\text{sec}}\right)^2}{0.6 \text{ m} - 0.25 \text{ m}}$$

$$k_s = 15.05 \ \dfrac{\text{N}}{\text{m}}$$

Solution #3

System: Block

Surroundings: Spring

Momentum Principle: $\dfrac{d\vec{p}}{dt} = \vec{F}_{net}$

For the radial component (the perpendicular component to the motion): $\left| \dfrac{d\vec{p}}{dt} \right| = \left| \vec{F}_{spring} \right| = k_s s$

The analytical solution of a spring mass system is: $A \cos(\omega t)$

Substitute the analytical solution into the Momentum Principle:

$$\left| \dfrac{dp}{dt} \right| \approx \left| \dfrac{d(m\overset{\frac{ds}{dt}}{v})}{dt} \right| = \left| m\dfrac{d}{dt}\dfrac{ds}{dt} \right| = \left| m\dfrac{d}{dt}\dfrac{d[A\cos(\omega t)]}{dt} \right| = \left| -\omega m A \dfrac{d}{dt}(\sin(\omega t)) \right| = \left| -\omega^2 m A \cos(\omega t) \right|$$

$$\left| \dfrac{dp}{dt} \right| \approx \left| -\omega^2 m A \overset{s}{\cos(\omega t)} \right| = k_s s, \text{ or } k_s = \omega^2 m$$

$$\omega = \sqrt{\dfrac{k_s}{m}}, \text{ where } \omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{1.5 \text{ s}} = 4.19 \ \dfrac{\text{rad}}{\text{sec}}$$

$$k_s = \omega^2 m = \left(4.19 \ \dfrac{\text{rad}}{\text{sec}} \right)^2 (0.5 \text{ kg})$$

$$k_s = 8.78 \ \dfrac{\text{N}}{\text{m}}$$

Problem: C

A uranium nucleus (U-236) may undergo a fission process resulting in two palladium nuclei (Pd-118). Each Pd-118 nucleus has a charge of $46e$ and a rest mass of 117.894 u, where $e = 1.6 \times 10^{-19}$ C and u $= 1.6603 \times 10^{-27}$ kg. U-236 has a charge of $92e$ and a rest mass of 235.996 u. (There may also be some free neutrons after the fission process, but neglect those for this problem).
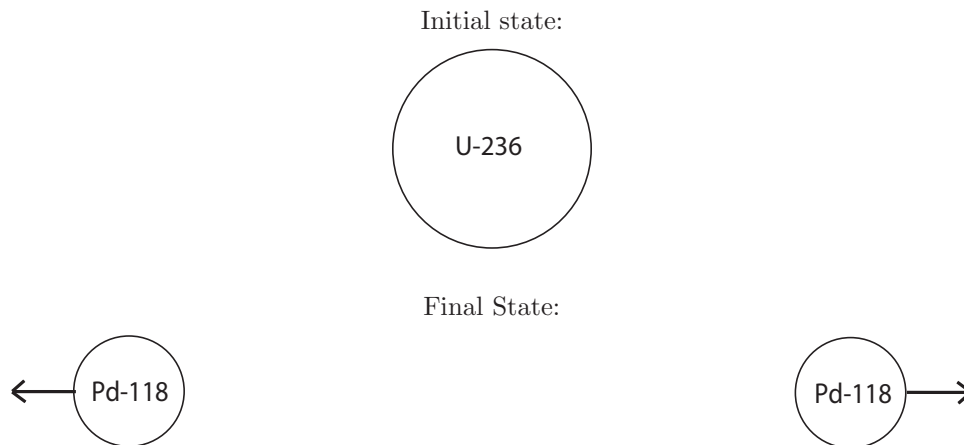
What is the distance between the Pd-118 nuclei centers just after the fission process (when they are initially at rest)?

No figures in solutions drawn to scale

Solution #1

System: All particles

Surroundings: Nothing significant

Initial state:

U-236

Final State:

$\leftarrow$ Pd-118   Pd-118 $\rightarrow$

Energy Principle: $E_f = E_i + W$

$$E_{rest,f} + U_f = E_{rest,i} + \cancel{W}$$

$$\frac{1}{4\pi\epsilon_0} \frac{(Q_{Pd-118})(Q_{U-236})}{r} = E_{rest,i} - E_{rest,f}$$

$$\left(9 \times 10^9 \ \frac{N \cdot m^2}{C^2}\right) \frac{\left(46 \times 1.6 \times 10^{-19} \ C\right)\left(92 \times 1.6 \times 10^{-19} \ C\right)}{r} = m_{U-236}c^2 - 2m_{Pd-118}c^2$$

$$r = \frac{9.8 \times 10^{-25} \ N \cdot m^2}{m_{U-236}c^2 - 2m_{Pd-118}c^2}$$

$$r = \frac{9.8 \times 10^{-25} \ N \cdot m^2}{(235.996 \times 1.6603 \times 10^{-27} \ kg)\left(3 \times 10^8 \ \frac{m}{s}\right)^2 - 2(117.894 \times 1.6603 \times 10^{-27})\left(3 \times 10^8 \ \frac{m}{s}\right)^2}$$

$$r = 5.553 \times 10^{-17} \ m$$

Solution #2

System: All particles

Surroundings: Nothing significant

Initial state:

Pd-118   Pd-118

Final State:

← Pd-118   Pd-118 →
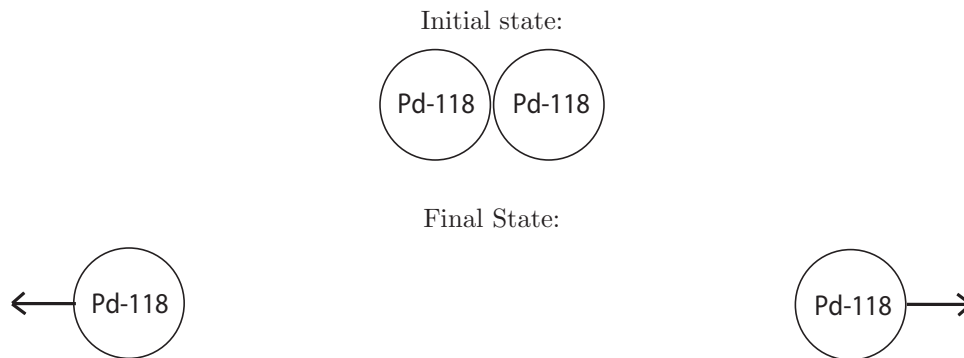
Energy Principle: $\Delta E = W$

$$E_f - E_i = \cancel{W}$$

$$E_f = E_i$$

$$E_{rest,f} + \cancel{U_f}^{\approx 0} = E_{rest,i} + U_i$$

$$\frac{1}{4\pi\epsilon_0}\frac{(Q_{Pd-118})^2}{r} = m_{U-236}c^2 - 2m_{Pd-118}c^2$$

$$r = \left(9 \times 10^9 \ \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}\right)\frac{\left(46 \times 1.6 \times 10^{-19} \ \text{C}\right)^2}{(235.996 \times 1.6603 \times 10^{-27} \ \text{kg})\left(3 \times 10^8 \ \frac{\text{m}}{\text{s}}\right)^2 - 2\left(117.894 \times 1.6603 \times 10^{-27}\right)\left(3 \times 10^8 \ \frac{\text{m}}{\text{s}}\right)^2}$$

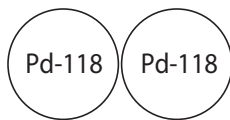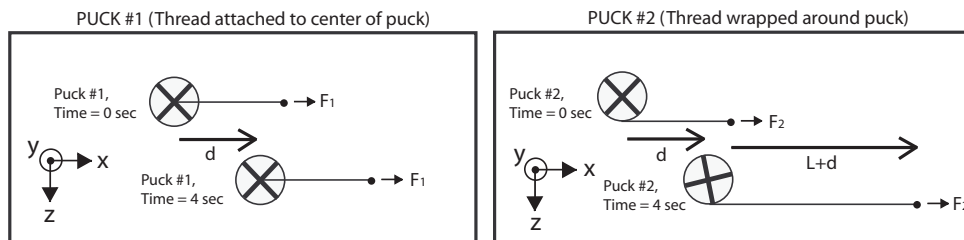$$r = 1.569 \times 10^{-14} \ \text{m}$$

Solution #3

System: All particles

Surroundings: Nothing significant

Initial state:

U-236

Final State:

Pd-118   Pd-118
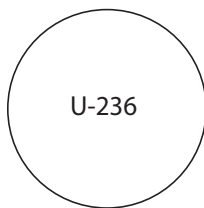
Energy Principle: $\Delta E = W$

$$\cancel{\Delta K} + \Delta U + \Delta E_{rest} = \cancel{W}$$

$$U_f - \cancel{U_i} + E_{rest,f} - E_{rest,i} = 0$$

$$\frac{1}{4\pi\epsilon_0} \frac{(Q_{Pd-118})^2}{r} + 2m_{Pd-118}c^2 - m_{U-236}c^2 = 0$$

$$r = \frac{1}{4\pi\epsilon_0} \frac{(Q_{Pd-118})^2}{m_{U-236}c^2 - 2m_{Pd-118}c^2}$$

$$r = \left(9 \times 10^9 \ \frac{N \cdot m^2}{C^2}\right) \frac{\left(46 \times 1.6 \times 10^{-19} \ C\right)^2}{(235.996 \times 1.6603 \times 10^{-27} \ kg)\left(3 \times 10^8 \ \frac{m}{s}\right)^2 - 2\left(117.894 \times 1.6603 \times 10^{-27}\right)\left(3 \times 10^8 \ \frac{m}{s}\right)^2}$$

$$r = 1.569 \times 10^{-14} \ m$$

Problem: D

Two identical pucks (mass $m = 0.15$ kg, radius $R = 0.05$ m, and moment of inertia $I = 1.875 \times 10^{-4}$ kg$\cdot$m^2) are sitting at rest on a sheet of ice (neglect friction due to the interactions between the puck and ice). One puck has a thread (of negligible mass) attached to its center, while the other puck has a thread (also of negligible mass) wrapped around it. The threads are pulled by a constant force (of magnitudes F_1 and F_2 respectively) for 4 seconds at the free end of the thread (shown by a small circle below), at which time both pucks have a speed of 1.6 m/s. Over the 4 seconds, a length, L, of thread unwinds from the second disk.

PUCK #1 (Thread attached to center of puck)   PUCK #2 (Thread wrapped around puck)

Top view of pucks sliding on ice (looking down on the ice from a location on the $+$y axis). *Not drawn to scale*

You find that the distance, d, is 3.2 m and the length, L, is 6.4 m. In order for the pucks to be traveling at the same speed after the 4 seconds, will the second puck need to be pulled with a stronger force? That is, will F_2 need to be larger than F_1?

Solution #1

For the <u>first</u> puck (thread attached at the center of the puck):

System: Puck, Earth

Surroundings: Thread

Energy Principle: $E_f = E_i + W$

$K_f + U_f = K_i + U_i + W$

$$\frac{1}{2}mv_f^2 + mgh_f = \frac{1}{2}m\cancel{v_i^2} + mgh_i + W$$

$$\frac{1}{2}mv^2 + mgh_f - mgh_i = F_1 d$$

$$\frac{1}{2}mv^2 + mg\cancel{(h_f - h_i)} = F_1 d$$

$$F_1 = \frac{\dfrac{1}{2}mv_f^2}{d} = \frac{\dfrac{1}{2}(0.15 \text{ kg})\left(1.6 \ \dfrac{\text{m}}{\text{s}}\right)^2}{3.2 \text{ m}} = 0.06 \text{ N}$$

For the <u>second</u> puck (thread wrapped around the puck):

System: Puck

Surroundings: Thread

Energy Principle: $E_f = E_i + W$

$K_{trans,f} + K_{rot,f} - K_{trans,i} - K_{rot,i} = W$

$$\frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2 = \frac{1}{2}m\cancel{v_i^2} + \frac{1}{2}I\cancel{\omega_i^2} + F_2(L+d)$$

$$F_2 = \frac{\dfrac{1}{2}mv_f^2 + \dfrac{1}{2}I\omega_f^2}{L+d} = \frac{\dfrac{1}{2}mv_f^2 + \dfrac{1}{2}I\left(\dfrac{v_f}{R}\right)^2}{L+d}$$

$$F_2 = \frac{\dfrac{1}{2}(0.15 \text{ kg})\left(1.6 \ \dfrac{\text{m}}{\text{s}}\right)^2 + \dfrac{1}{2}\left(1.875 \times 10^{-4} \text{ kg} \cdot \text{m}^2\right)\left(\dfrac{1.6 \ \frac{\text{m}}{\text{s}}}{0.05 \text{ m}}\right)^2}{(6.4 + 3.2) \text{ m}} = 0.03 \text{ N}$$

The first puck requires a larger force (0.06 N as opposed to only 0.03 N) to reach the same speed as the second puck.

Solution #2

For the <u>second</u> puck (thread wrapped around the puck):

System: Puck

Surroundings: Thread

Angular Momentum Principle: $\vec{L}_f - \vec{L}_i = \vec{\tau}_{net}\Delta t$

$\vec{L}_f - \cancel{\vec{L}_i} = \vec{\tau}_{net}\Delta t$

$\vec{L}_{trans,f} + \vec{L}_{rot,f} = \vec{\tau}_{net}\Delta t$

$\vec{r} \times \vec{p}_f + I\vec{\omega}_f = \vec{r} \times \vec{F}_2\Delta t$

$\left\langle 0, \cancel{r_z p}^R_{f_x}, 0 \right\rangle + I\left\langle 0, \omega_f, 0 \right\rangle = \left\langle 0, \cancel{r_z F}^R_2, 0 \right\rangle \Delta t$ (The puck will spin counter-clockwise as the thread unwinds, so by the Right Hand Rule $\vec{\omega}_f$ is in the y direction)

For the y-components: $Rp_{f_x} + I\omega_f = RF_2\Delta t$

$Rm\cancel{v_{f_x}}^{v_f} + I\left(\dfrac{v_f}{R}\right) = RF_2\Delta t$, $v_{f_x} = v_f$ since the pucks travel in the x direction.

$F_2 = \dfrac{Rmv_f + I\left(\dfrac{v_f}{R}\right)}{R\Delta t}$

$F_2 = \dfrac{(0.05 \text{ m})(0.15 \text{ kg})\left(1.6 \ \dfrac{m}{s}\right) + \left(1.875 \times 10^{-4} \text{ kg}\cdot m^2\right)\left(\dfrac{1.6 \ \frac{m}{s}}{0.05 \text{ m}}\right)}{(0.05 \text{ m})(4 \text{ sec})}$

$F_2 = 0.09 \text{ N}$

For the <u>first</u> puck (thread attached at the center of the puck):

System: Puck

Surroundings: Thread

Angular Momentum Principle: $\vec{L}_f - \vec{L}_i = \vec{\tau}_{net}\Delta t$

$\vec{L}_{trans,f} - \cancel{\vec{L}_{trans,i}} = \vec{\tau}_{net}\Delta t$

$\vec{r} \times \vec{p}_f = \vec{\tau}_{net}\Delta t$

$\left\langle 0, \cancel{r_z}p_{f_x}^R, 0 \right\rangle = \left\langle 0, \cancel{r_z}F_1^R, 0 \right\rangle \Delta t$

For the y-components: $\cancel{R}p_{f_x} = \cancel{R}F_1\Delta t$

$m\cancel{v_{f_x}}^{v_f} = F_1\Delta t$

$(0.15 \text{ kg}) \left(1.6 \ \frac{\text{m}}{\text{s}} \right) = 0.24 \ \text{kg} \cdot \frac{\text{m}}{\text{s}} = F_1\Delta t$

$F_1 = \dfrac{0.24 \ \text{kg} \cdot \frac{\text{m}}{\text{s}}}{\Delta t} = \dfrac{0.24 \ \text{kg} \cdot \frac{\text{m}}{\text{s}}}{4 \ \text{sec}} = 0.06 \ \text{N}$

The second puck requires a larger force (0.09 N as opposed to only 0.06 N) to reach the same speed as the first puck.

Solution #3

For the <u>first</u> puck (thread attached at the center of the puck):

System: Puck

Surroundings: Thread, Earth, ice sheet

Momentum Principle: $\vec{p}_f = \vec{p}_i + \vec{F}_{net}\Delta t$

$$\vec{F}_{net} = \langle F_1, F_{ice} - F_{Earth}, 0 \rangle$$

$$\langle p_f, 0, 0 \rangle = \langle \cancel{p_i}, 0, 0 \rangle + \langle F_1, F_{ice} - F_{Earth}, 0 \rangle \Delta t$$

$$\langle mv_f, 0, 0 \rangle = \langle F_1, F_{ice} - F_{Earth}, 0 \rangle \Delta t$$

For the y-components:	For the x-components:
$0 = (F_{ice} - F_{Earth})\Delta t$	$mv_f = F_1 \Delta t$
$F_{ice} - F_{Earth} = 0$	$F_1 = \dfrac{mv_f}{\Delta t}$
$F_{ice} = F_{Earth}$	$F_1 = \dfrac{(0.15 \text{ kg})\left(1.6 \, \dfrac{m}{s}\right)}{4 \text{ sec}} = 0.06 \text{ N}$

For the <u>second</u> puck (thread wrapped around the puck):

System: Puck

Surroundings: Thread, Earth, ice sheet

Momentum Principle: $\vec{p}_f = \vec{p}_i + \vec{F}_{net}\Delta t$

$$\langle p_f, 0, 0 \rangle = \langle \cancel{p_i}, 0, 0 \rangle + \langle F_2, F_{ice} - F_{Earth}, 0 \rangle \Delta t$$

$$\langle mv_f, 0, 0 \rangle = \langle F_2, F_{ice} - F_{Earth}, 0 \rangle \Delta t$$

For the x-components:	For the y-components:
$mv_f = F_2 \Delta t$	$0 = (F_{ice} - F_{Earth}) \Delta t$
$(0.15 \text{ kg}) \left(1.6 \ \frac{\text{m}}{\text{s}}\right) = F_2 \Delta t$	$F_{ice} - F_{Earth} = 0$
$F_2 \Delta t = 0.24 \ \text{kg} \cdot \frac{\text{m}}{\text{s}}$	$F_{ice} = F_{Earth}$
$F_2 = \dfrac{0.24 \ \text{kg} \cdot \frac{\text{m}}{\text{s}}}{\Delta t}$	
$F_2 = \dfrac{0.24 \ \text{kg} \cdot \frac{\text{m}}{\text{s}}}{4 \ \text{sec}} = 0.06 \ \text{N}$	

Both forces need to have equal magnitudes (0.06 N) in order for the pucks to have the same speed at 4 seconds.

Appendix D: Group Session Problems and Solutions

Group Problem: A

A ball was in motion for a long period of time. Sometime during this period of time, it experienced a non-zero net force for some finite time interval Δt. While this force was acting, the ball's velocity did not change. Explain how this situation is possible.

Solution #1

The momentum principle states that $\vec{F}_{net}\Delta t = \vec{p}_f - \vec{p}_i$. Because a non-zero net force acted, $\vec{p}_f - \vec{p}_i$ must be non-zero. However, since the velocity didn't change, the ball's momentum didn't change. Therefore, we have a contradiction, and the situation is not possible.

Solution #2

Because the velocity didn't change, the ball's momentum must not have changed either. However, a non-zero net force acted on the ball. Therefore, some additional force must have opposed the net force to allow the ball's velocity to remain constant. For example, air resistance or gravity could have opposed the net force.

Solution #3

The momentum principle states that $\vec{F}_{net}\Delta t = \vec{p}_f - \vec{p}_i = m_f\vec{v}_f - m_i\vec{v}_i$. Because the net force and Δt were non-zero, $m_f\vec{v}_f - m_i\vec{v}_i$ must be non-zero. If $m_f = m_i = m$, then $m(\vec{v}_f - \vec{v}_i) = 0$ because $\vec{v}_f = \vec{v}_i$. Therefore, m_f must be different from m_i; the mass of the ball must have changed while the net force was acting on it.

Group Problem: B

A spherical space probe (mass $M = 100$ kg, radius $R = 4$ m) collides with a small rock (mass $m = 2$ kg) as shown on figure 1. Initially the rock has a velocity of $\vec{v_2} = \langle -80, 0, 0 \rangle$ m/s, and the probe has a velocity of $\vec{v_1} = \langle 2, 0, 0 \rangle$ m/s.

After the collision, the rock has a velocity of $\vec{v_3} = \langle -67, 11.8, 0 \rangle$ m/s (a speed ≈ 68 m/s), and the probe has an angular velocity of $\langle 0, 0, -5.8 \rangle$ rad/s.

What is the angular velocity, $\vec{\omega}_1$ of the probe before the collision?
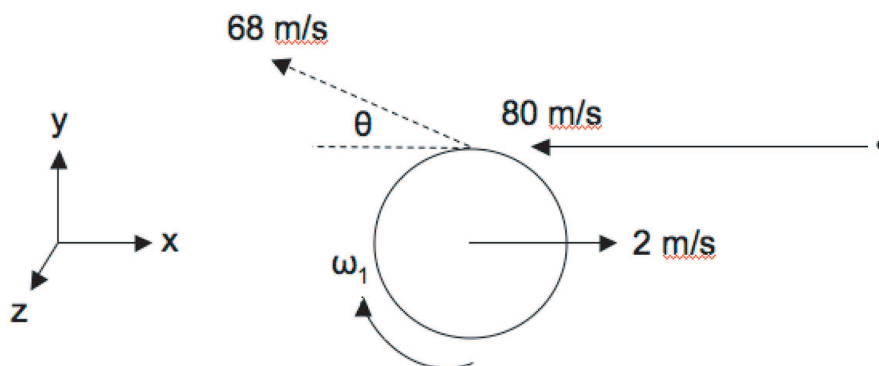

Figure 1 - Probe and rock collision. *Not drawn to scale*

Solution #1

System: Space probe, rock

Surroundings: Nothing significant

Initial state: Just prior to collision

Final state: Just after collision

Angular Momentum Principle: $\vec{L}_f = \vec{L}_i + \vec{\tau}_{net}\Delta t$

$$\vec{L}_{rot,f} = \vec{L}_{rot,i}$$

Since $\vec{L}_{rot} = I\vec{\omega}$, the moments of inertia (I) need to be determined:

$$I_{rock} = \frac{2}{5}mR^2 = \frac{2}{5}(2 \text{ kg})(4 \text{ m})^2 = 12.8 \text{ kg} \cdot \text{m}^2$$

$$I_{probe} = \frac{2}{5}MR^2 = \frac{2}{5}(100 \text{ kg})(4 \text{ m})^2 = 640 \text{ kg} \cdot \text{m}^2$$

Returning to $\vec{L}_{rot,f} = \vec{L}_{rot,i}$: $(I\vec{\omega}_f)_{rock} + (I\vec{\omega}_f)_{probe} = (I\vec{\omega}_i)_{rock} + (I\vec{\omega}_i)_{probe}$

Since the rock exerts a counter-clockwise torque on the probe during the collision, $\vec{\omega}_{rock}$ is in the

$+z$ direction by the Right Hand Rule. Also, the given angular velocity of the probe is only in the

$-z$ direction. Thus, only the z components of the angular momenta are non-zero.

For the z components: $(I\omega_3)_{rock} + (I\omega_f)_{probe} = (I\omega_2)_{rock} + (I\omega_1)_{probe}$

$$\left(12.8 \text{ kg} \cdot \text{m}^2\right)\frac{v_3}{R} + \left(640 \text{ kg} \cdot \text{m}^2\right)\left(-5.8 \ \frac{\text{rad}}{\text{s}}\right) = \left(12.8 \text{ kg} \cdot \text{m}^2\right)\frac{v_2}{R} + \left(640 \text{ kg} \cdot \text{m}^2\right)(\omega_1)$$

$$\left(12.8 \text{ kg} \cdot \text{m}^2\right)\frac{68 \ \frac{\text{m}}{\text{s}}}{4 \text{ m}} - 3712 \ \frac{\text{kg} \cdot \text{m}^2}{\text{s}} = \left(12.8 \text{ kg} \cdot \text{m}^2\right)\frac{80 \ \frac{\text{m}}{\text{s}}}{4 \text{ m}} + \left(640 \text{ kg} \cdot \text{m}^2\right)\omega_1$$

$$-3750.4 \ \frac{\text{kg} \cdot \text{m}^2}{\text{s}} = \left(640 \text{ kg} \cdot \text{m}^2\right)\omega_1$$

$$\omega_1 = -5.84 \ \frac{\text{rad}}{\text{s}}$$

$$\vec{\omega}_1 = \langle 0, 0, -5.84 \rangle \ \frac{\text{rad}}{\text{s}}$$

Solution #2

System: Space probe, rock

Surroundings: Nothing significant

Initial state: Just prior to collision

Final state: Just after collision

Momentum Principle: $\vec{p}_f = \vec{p}_i + \cancel{\vec{F}_{net}}\Delta t$

$$m \left\langle v_{3_x}, v_{3_y}, 0 \right\rangle + M \left\langle v_{f_x}, v_{f_y}, 0 \right\rangle = m \left\langle v_{2,x}, 0, 0 \right\rangle + M \left\langle v_{1_x}, 0, 0 \right\rangle$$

For the x-components:	For the y-components:
$mv_{3_x} + Mv_{f_x} = mv_{2_x} + Mv_{1_x}$	$mv_{3_y} + Mv_{f_y} = 0$
$Mv_{f_x} = -mv_{3_x} + mv_{2_x} + Mv_{1_x}$	$Mv_{f_y} = -mv_{3_y}$
$v_{f_x} = \dfrac{m\left(-v_{3_x} + v_{2_x}\right) + Mv_{1_x}}{M}$	$v_{f_y} = \dfrac{-mv_{3_y}}{M}$
$v_{f_x} = \dfrac{(2 \text{ kg})\left[-\left(-67\ \frac{\text{m}}{\text{s}}\right) + 80\ \frac{\text{m}}{\text{s}}\right] + (100 \text{ kg})\left(2\ \frac{\text{m}}{\text{s}}\right)}{100 \text{ kg}}$	$v_{f_y} = -\dfrac{(2 \text{ kg})\left(11.8\ \frac{\text{m}}{\text{s}}\right)}{100 \text{ kg}}$
$v_{f_x} = 1.74\ \dfrac{\text{m}}{\text{s}}$	$v_{f_y} = -0.24\ \dfrac{\text{m}}{\text{s}}$

$$v_f = \sqrt{v_x^2 + v_y^2} = \sqrt{(1.74)^2 + (-0.24)^2} = 1.76\ \frac{\text{m}}{\text{s}}$$

System: Space probe

Surroundings: Rock

Initial state: Just prior to collision

Final state: Just after collision

Angular Momentum Principle: $\vec{L}_f = \vec{L}_i + \cancel{\vec{\tau}_{net}}\Delta t$

$$\vec{L}_{trans,f} + \vec{L}_{rot,f} = \vec{L}_{trans,i} + \vec{L}_{rot,i}$$

$$\left(\vec{r} \times \vec{p}_f\right)_{rock} + \left(\vec{r} \times \vec{p}_f\right)_{probe} + \left(I\vec{\omega}_f\right)_{probe} = \left(\vec{r} \times \vec{p}_i\right)_{rock} + \left(\vec{r} \times \vec{p}_i\right)_{probe} + \left(I\vec{\omega}_i\right)_{probe}$$

$$\langle 0,0,-r_y p_{3_x}\rangle + \langle 0,0,-r_y p_{f_x}\rangle + \left[\frac{2}{5}MR^2\right]\left\langle 0,0,\omega_f\right\rangle \overset{\frac{v_f}{R}}{=} \langle 0,0,-r_y p_{2_x}\rangle + \langle 0,0,-r_y p_{1_x}\rangle + \left[\frac{2}{5}MR^2\right]\langle 0,0,\omega_1\rangle$$

Since the collision occurs at the top of the probe, the distance, r_y, is equal to the radius of the probe, R.

$$\langle 0,0,-Rp_{3_x}\rangle + \langle 0,0,-Rp_{f_x}\rangle + \left[\frac{2}{5}MR^2\right]\left\langle 0,0,\omega_f\right\rangle \overset{\frac{v_f}{R}}{=} \langle 0,0,-Rp_{2_x}\rangle + \langle 0,0,-Rp_{1_x}\rangle + \left[\frac{2}{5}MR^2\right]\langle 0,0,\omega_1\rangle$$

For the z components:

$$[-Rmv_{3_x} - RMv_{f_x}] + \left[\frac{2}{5}MR^2\right]\frac{v_f}{R} = [-Rmv_{2_x} - RMv_{1_x}] + \left[\frac{2}{5}MR^2\right]\omega_1$$

$$\left[\frac{2}{5}MR^2\right]\omega_1 = -\left[-Rmv_{2_x} - RMv_{1_x}\right] + \left[-Rmv_{3_x} - RMv_{f_x}\right] + \left[\frac{2}{5}MR\right]v_f$$

$$\omega_1 = \frac{-[-mv_{2_x} - Mv_{1_x}] + [-mv_{3_x} - Mv_{f_x}] + \left[\frac{2}{5}M\right]v_f}{\frac{2}{5}MR}$$

$$\omega_1 = \frac{[(2 \text{ kg})\left(-80 \text{ }\frac{m}{s}\right) + (100 \text{ kg})\left(2 \text{ }\frac{m}{s}\right)] + \left[-(2 \text{ kg})\left(-67 \text{ }\frac{m}{s}\right) - (100 \text{ kg})\left(1.74 \text{ }\frac{m}{s}\right)\right] + \left[\frac{2}{5}(100 \text{ ksg})\right]\left(1.76 \text{ }\frac{m}{s}\right)}{\frac{2}{5}(100 \text{ kg})(4 \text{ m})}$$

$$\omega_1 = 0.45 \text{ }\frac{rad}{s}, \text{ so } \vec{\omega}_1 = \langle 0,0,0.45\rangle \text{ }\frac{rad}{s}$$

Solution #3

System: Space probe, rock

Surroundings: Nothing significant

Initial state: Just prior to collision

Final state: Just after collision

Angular Momentum Principle: $\vec{L}_f = \vec{L}_i + \vec{\tau}_{net}\Delta t$

$$\vec{L}_{trans,f} + \vec{L}_{rot,f} = \vec{L}_{trans,i} + \vec{L}_{rot,i}$$

$$(\vec{r} \times \vec{p}_f)_{rock} + (I\vec{\omega}_f)_{probe} = (\vec{r} \times \vec{p}_i)_{rock} + (I\vec{\omega}_i)_{probe}$$

$$\langle 0, 0, -r_y p_{3_x} \rangle + \left(\frac{2}{5}MR^2\right)\langle 0, 0, \omega_f \rangle = \langle 0, 0, -r_y p_{2_x} \rangle + \left(\frac{2}{5}MR^2\right)\langle 0, 0, \omega_1 \rangle$$

Since the collision occurs at the top of the probe, the distance, r_y, is equal to the radius of the probe, R.

$$\langle 0, 0, -R p_{3_x} \rangle + \left(\frac{2}{5}MR^2\right)\langle 0, 0, \omega_f \rangle = \langle 0, 0, -R p_{2_x} \rangle + \left(\frac{2}{5}MR^2\right)\langle 0, 0, \omega_1 \rangle$$

For the z components: $-R p_{3_x} + \left(\frac{2}{5}MR^2\right)\omega_f = -R p_{2_x} + \left(\frac{2}{5}MR^2\right)\omega_1$

$$-\cancel{R}mv_{3_x} + \left(\frac{2}{5}MR\cancel{^2}\right)\omega_f = -\cancel{R}mv_{2_x} + \left(\frac{2}{5}MR\cancel{^2}\right)\omega_1$$

$$\left(\frac{2}{5}MR\right)\omega_1 = mv_{2_x} - mv_{3_x} + \left(\frac{2}{5}MR\right)\omega_f = (2 \text{ kg})\left(-80 \ \frac{m}{s}\right) - (2 \text{ kg})\left(-67 \ \frac{m}{s}\right) + \left[\frac{2}{5}(100 \text{ kg})(4 \text{ m})\right]\left(-5.8 \ \frac{rad}{s}\right)$$
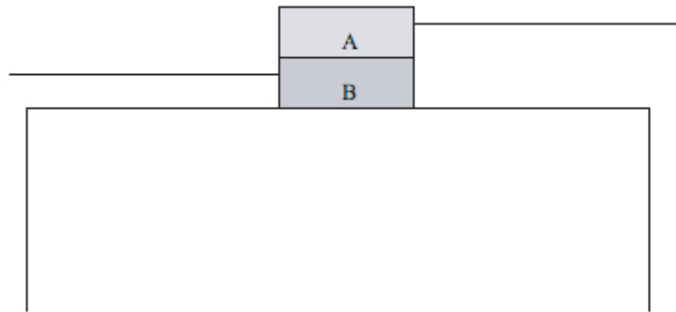
$$\left(\frac{2}{5}MR\right)\omega_1 = -954.07 \text{ kg} \cdot \frac{m}{s}$$

$$\omega_1 = \frac{-954.07 \text{ kg} \cdot \frac{m}{s}}{\frac{2}{5}(100 \text{ kg})(4 \text{ m})} = -5.96 \ \frac{rad}{s}$$

$$\vec{\omega}_1 = \langle 0, 0, -5.96 \rangle \ \frac{rad}{s}$$

Group Problem: C

Two heavy blocks, with massless ropes attached to them, are sitting at rest on a table. Block A sits on block B, which is in contact with the table, as pictured below:

Amir pulls on the rope attached to block A, applying 20 N of force. Barbara pulls on the rope attached to block B, applying 7 N of force. While they are pulling, neither block moves. What direction is the friction force on block B due to the table?

Solution #1

Because Barbara is pulling on block B, she is applying a force to the left. According to the momentum principle, the net force acting on an object is equal to that object's change in momentum. Therefore, in order to keep block B from experiencing a change in momentum, a friction force needs to be equal and opposite to the force that Barbara is applying. Therefore, the friction force on block B due to the table is to the right.

Solution #2

Barbara is applying a 7 N force to the left on block B, and block A is applying a force of 20 N to the right on block B because block B applies a 20 N friction force on block A to keep A's momentum from changing. According to the momentum principle, the net force acting on an object is equal to that object's change in momentum. So, to keep block B from experiencing a change in momentum, the combination of these two forces plus the friction due to the table needs to be zero. Therefore, the friction force due to the table on block B is to the left.

Solution #3

The momentum principle states that an object's change in momentum is equal to the net force acting on that object. But since the block isn't moving, the friction force on block B due to the table is zero. Because friction forces oppose motion, there can only be a friction force if the block were moving.

Appendix E: Reference Sheet

Fundamental principles:

$$\Delta \vec{p} = \vec{F}_{net} \Delta t \qquad \text{or} \qquad \frac{d\vec{p}}{dt} = \vec{F}_{net}$$

$$\Delta E = W_{surr} + Q$$

$$\Delta \vec{L}_A = \vec{\tau}_{net,A} \Delta t \qquad \text{or} \qquad \frac{d\vec{L}_A}{dt} = \vec{\tau}_{net,A}$$

Cross Product: $\vec{A} \times \vec{B} = \langle (A_y B_z - A_z B_y), (A_z B_x - A_x B_z), (A_x B_y - A_y B_x) \rangle$

$\vec{L}_{trans,A} = \vec{r}_A \times \vec{p}$ (point particle) $\qquad\qquad \vec{\tau}_A = \vec{r}_A \times \vec{F}$

Multiparticle systems:

$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots}{m_1 + m_2 + \dots} \qquad \vec{P}_{tot} \approx M_{tot} \vec{v}_{cm} \; (v << c) \qquad K_{tot} = K_{trans} + K_{rel}$$

$$K_{rel} = K_{rot} + K_{vib} \qquad K_{trans} \approx \frac{1}{2} M_{tot} v_{cm}^2 \; (v << c) \qquad I = m_1 r_{1\perp}^2 + m_2 r_{2\perp}^2 + \dots$$

$$K_{rot} = \frac{L_{rot}^2}{2I} = \frac{1}{2} I \omega^2$$

$$\vec{L}_{trans,A} = \vec{r}_{cm,A} \times \vec{P}_{tot} \qquad \vec{L}_{rot} = I \vec{\omega} \qquad \vec{L}_A = \vec{L}_{trans,A} + \vec{L}_{rot}$$

Other physical quantities:

$$\gamma \equiv \frac{1}{\sqrt{1 - \left(\frac{|\vec{v}|}{c}\right)^2}}$$

$$E^2 - (pc)^2 = \left(mc^2\right)^2$$

$$\vec{F}_{grav} = -G\frac{m_1 m_2}{|\vec{r}|^2}\hat{r}$$

$$U_{grav} = -G\frac{m_1 m_2}{|\vec{r}|}$$

$$\left|\vec{F}_{grav}\right| \approx mg \text{ near Earth's surface}$$

$$\Delta U_{grav} \approx mg\Delta y \text{ near Earth's surface}$$

$$\vec{F}_{elec} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}|^2}\hat{r}$$

$$U_{elec} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}|}$$

$$\left|\vec{F}_{spring}\right| = k_s s \text{ opposite to the stretch}$$

$$U_{spring} = \frac{1}{2}k_s s^2 \text{ for ideal spring}$$

$$U_i \approx \frac{1}{2}k_{si}s^2 - E_M \text{ approx. interatomic pot. energy}$$

$$\Delta E_{thermal} = mC\Delta T$$

$$E_N = -\frac{13.6\text{eV}}{N^2} \text{ where } N = 1, 2, 3\ldots \text{ (Hydrogen atom energy levels)}$$

$$E_N = N\hbar\omega_0 + E_0 \text{ where } N = 0, 1, 2\ldots \text{ and } \omega_0 = \sqrt{\frac{k_{si}}{m_a}} \text{ (Quantized oscillator energy levels)}$$

$$\left|\left(\frac{d\vec{p}}{dt}\right)_{\perp}\right| = \frac{pv}{R} \approx \frac{mv^2}{R} \text{ (if } v << c\text{) where } R = \text{radius of kissing circle}$$

$$\omega = \frac{2\pi}{T}$$

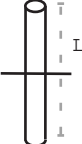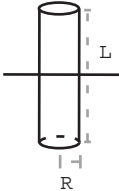$$x = A\cos\omega t$$

$$\omega = \sqrt{\frac{k_s}{m}}$$

$$Y = \frac{F/A}{\Delta L/L} \text{ (macro)}$$

$$Y = \frac{k_{si}}{d} \text{ (micro)}$$

$$\text{speed of sound } v = d\sqrt{\frac{k_{si}}{m_a}}$$

$$\hat{f} = \langle\cos\theta_x, \cos\theta_y, \cos\theta_z\rangle \text{ unit vector from angles}$$

Moment of intertia for rotation about indicated axis

$I = \dfrac{2}{5}MR^2$	$I = \dfrac{1}{2}MR^2$	$I = \dfrac{1}{12}ML^2$	$I = \dfrac{1}{12}ML^2 + \dfrac{1}{4}MR^2$

$$\Omega = \frac{(q + N - 1)!}{q!\,(N - 1)!} \qquad\qquad S \equiv k \ln \Omega \qquad\qquad \frac{1}{T} \equiv \frac{\partial S}{\partial E}$$

$$\Delta S = \frac{Q}{T} \ \text{(small } Q) \qquad\qquad \text{prob}(E) \propto \Omega\,(E)\, e^{-\frac{E}{kT}}$$

Constant	Symbol	Approximate Value
Speed of light	c	3×10^8 m/s
Gravitational constant	G	6.7×10^{-11} N $\cdot$ m^2/kg^2
Approx. grav field near Earth's surface	g	9.8 N/kg
Electron mass	m_e	9×10^{-31} kg
Proton mass	m_p	1.7×10^{-27} kg
Neutron mass	m_n	1.7×10^{-27} kg
Electric constant	$\dfrac{1}{4\pi\epsilon_0}$	9×10^9 N $\cdot$ m^2/C^2
Proton charge	e	1.6×10^{-19} C
Electron volt	1 eV	1.6×10^{-19} J
Avogadro's number	N_A	6.02×10^{23} atoms/mol
Plank's constant	h	6.6×10^{-34} joule $\cdot$ second
hbar $= \dfrac{h}{2\pi}$	$\hbar$	1.05×10^{-34} joule $\cdot$ second
specific heat capacity of water	C	4.2 J/kg
Boltzmann constant	k	1.38×10^{-23} J/K

milli	m	1×10^{-3}		kilo	K	1×10^3
micro	μ	1×10^{-6}		mega	M	1×10^6
nano	n	1×10^{-9}		giga	G	1×10^9
pico	p	1×10^{-12}		tera	T	1×10^{12}

Appendix F: Sample Transcripts

In this appendix are attached the individual session transcripts for Christobal and Ike solving the "Two Pucks" problem (see Section 6.3.2.

Christobal, Two Pucks

Christobal: Two identical pucks, masses radius, moment of inertia, are sitting on a sheet of ice, Neglect frictions. (Reading problem statement) "One puck has a thread of negligible mass attached to its center, while the other puck has a thread also negligible mass, wrapped around it. the threads are pulled by a constant force for four seconds at the free end of the thread, at which time both pucks have a speed of 1.6 m/s. Over the four seconds, a length L of thread unwinds from the second disc." Let's see, there, okay, "you find that the distance D is 3.2 m and the length L is 6.4 m." So, let's see, what essentially do you know? Four seconds, length L of thread, okay, and looks like, and D must be the distance that the, okay so D is the distance that it moves, and L is the length that the, L is the length that the thread comes out. Okay, that makes sense. So basically we have one force acting on a distance D, where D is basically going to be, the amount that the force is acting on, the other one is acting on a distance L plus D. okay, that seems to make sense.

(Christobal places the problem statement off to the side and begins looking at solution #1). So for the first puck, we have the energy is equal to, okay we have the energy principle KF plus UF is equal to KI plus UI plus W, well we don't have to worry about, we don't have to worry about (pause) any, (long pause). Okay I think they have the coordinate system wrong. Oh, HF minus HI, that's going to be zero, okay good for you. So we have one half MV squared over D, so we assume that the force is constant; that's fine, that's a good assumption, one half times is 0.15 kg, lovely, 1.6 m/s squared over 3.2 m is 0.06 N, good for you. Okay, that one is correct (places solution one off to the side; apparently referring to only the first puck's part in solution one). (Christobal looks at solution #2) 0.09 N, yeah you betch ya, it is, oh, that would be the second puck, okay I believe it, and the first puck (silence while he looks at solution #3), well, okay, okay (Sees that all three solutions give the same answer for puck one, and places solutions back in a stack with solution #1 on top), well, I agree with you so far solution #1, let's see how well you do with the second puck. The thread wrapped around the puck, so we have the energy, oh where does it say, okay it gives you the length of L. 0.15 kg × 1.6 m/s squared plus one half × 1.875 × 10 to the -4 kg/m second, so that's the moment of inertia, 1.875 lovely, 1.6 m/s zero, 1.6 m/s divided by 0.05 m, one half $I\omega$ squared, where is our, well I do think you messed up there my good friend. This right here should be ω (Christobal makes a correction on solution #1) which is not the translational, but rather the

velocity coming around that and so, sorry nope, no points for you (he places solution #1 off to the side).

(Christobal begins looking at solution #2). Okay, for the second puck we can see that – lovely, they're using angular momentum, you don't even need to worry about all of that crap but it's good that you will. $F2$ is going to be R MVF plus IVF over R, well, V over R is ω, that's fine, um, over R times ΔT, so you are using the fact that we're pulling it at a constant force for four seconds, that's fine, because the integral of $F \cdot DT$ is going to be P, and the integral of $F \cdot DX$ is equal to E, oh, this is no dot, that's just multiplication. For the first puck, well you are doing angular momentum again, boy do you like angular momentum. You know that that is not the easiest way to solve things. The y-components, you don't have to worry about them, so we have 0.15 kg times 1.6 m/s then we divide that by four seconds to get 0.06 N. "The second puck require, puck will require a larger force," yes I believe you.

And solution #3, the thread attached to the center of the puck, you are basically saying, well the – we don't need to worry about the ice – F, okay, so good for you, you found the normal force. $F1$ MVF over ΔT, yeah you got that right, good for you. Second puck, $F2$ F minus, okay I don't know why you're bothering to do that, 0.05 kg, $F2$ (inaudible muttering) I am afraid and that is where you lose me. Um,

Researcher: Just remember to keep on talking what you're thinking.

Christobal: Yes, so I'm afraid you (solution #3) make no sense at all. 0.15 kg times 1.6 m/s is equal to $F2$ times ΔT. Well, yes, but there should be another term in there. What about the angular momentum? And you are not even bothering with that, are you? Well, this equation is wrong (he indicates an equation on solution #3) because it doesn't include angular momentum. So, that makes sense. That's where you went wrong. Kilogram meter per second, you're not even dealing with angular momentum so, okay. And solution number two, let's just make sure that you are right. Um, $F2$ is equal to 0.09 N. Yes, that's wise. Um, it's going to be, let's see here, you (he points to solution #1) use the wrong velocity. Here, he's (he points to solution #2) doing that right. Wait, now you are using, oh, actually this one's, this one's (he points to a line in solution #2) pretty good. Right here we are okay. And so let's see this. See right here (he points to solution #2), we are going to say that momentum – but see here we are adding momentum and angular

momentum, R times momentum, well that is fine, but it is not actually R times momentum, it is R cross momentum, and this momentum is in the same direction and so here we don't have that (he crosses off something in solution #2) because R cross MVF is equal to zero. that's our issue there, lovely. Well, I am glad we figured that one out.

(Christobal looks back at solution #1). Ah, here is something I can work with! Okay, see you are saying that the puck is going to rotate at the same speed at which it goes. I believe that now. So we can say that ω is equal to V translational over R, not that, okay (he crosses off what he had previously written on solution #1). So, then we are saying that we are going to say that the force is equal to the difference in energy divided by the, length over which the force is applied, yes, that's fine because the force, the difference in energy is equal to $F \cdot DX$ (Draws an integral on solution number one), or in – we don't have to worry about integrating here, so we can just say that ΔE is equal to FX, lovely. And, the reason you went right and solution number two did not is that you cannot add angular and rotational momentum, or angular momentum and linear momentum. Okay, I think I am done.

Ike, Two Pucks

Ike: Oh, okay. (Reading problem statement) "Two identical pucks are sitting at rest on a sheet of ice." Great, we get to neglect friction. One puck has a thread, while the other one has a thread. Oh, one has a thread attached to its center while the other has a thread wrapped around it. "The threads are pulled by constant force for four seconds at the free end of the thread at which time both pucks have a speed of 1.6 m/s." Hmm. Over four seconds – oh, so constant force, different magnitudes. "Over four seconds, a length L of thread unwinds from the second disc." Great. "You find that, you find that the distance D –" oh, no, "the distance D is 3.2 m and the length L is 6.4 m." Oh, okay. "In order for the pucks to be traveling at the same speed after the four seconds, will the second puck need to be pulled with a stronger force? That is, will $F2$ need to be larger than $F1$?" And, my confusion was that both of these wind up with the same linear displacement after four seconds, but apparently you are assuming that they do and that the force is, well, constant force and if they both have the same speed after four seconds they should have both been experiencing the same acceleration, which means they both would've traveled the same distance. Okay, that makes

sense. And, that was two minutes (Ike places the problem statement off to the side and spreads the three solutions in front of himself). Wow, okay.

For the first puck, energy principle, solution #1. Um, so they set it up so that, they set it up so that initial energy plus work is final energy, okay. They wound up removing the potential energy from the equation anyway, and saying that initial velocity is zero, so final kinetic energy is force $F1$ times the distance over which the force acted, which is $F1$ D, so $F1$ is the final kinetic energy over the distance acted upon. (To Researcher) Am I free to assume that students do calculator stuff correctly?

Researcher: Um, all I can tell you is there are some errors or error in two of the solutions and the third solution is correct.

Ike: And that's – this one at least is not hard to plug in (he picks up and uses a calculator). Okay the calculator is fine on this one (the first part of the solution #1). Okay. For the second puck, thread wrapped around the puck, the energy principle again, and this time they don't include potential energy, but we have to include translational, translational and rotational. Great. So, let's look at what they do here. What's the moment of inertia? (He looks through the reference sheet). First puck, that's one half MR squared. Okay. And, okay, $F2$ is equal to L plus D, they say because – and I would imagine the reasoning is that you've acted, you've pulled this one out for 6.4 m with that force in addition to pulling the puck itself with that same force, the 3.2 m translationally. So what they have done is that they have set this up so that the, or they say the initial speeds are zero so the initial energies are zero. So, the final force is equal to, the final force is equal to and they say the final kinetic energy and here they make a substitution that the final, rotational kinetic energy is based on ω, which is V final over R, and I don't see where they are allowed to say that the puck is rotating. Okay, so I'm going to go ahead and, sorry let's try that again. I don't – the puck is clearly rotating, I don't see where they get to claim that ω is VF over R. In fact, I see nothing in here that would suggest that, so, let's go ahead there. (He places an X on solution #1).

Let's look at solution #, I don't know, 3. That looks nice. For the first puck, momentum principle says that momentum, change in momentum is $Fnet\Delta T$, great. $Fnet$ in the x-direction is $F1$; $Fnet$ in the y-direction is normal minus gravity, thank you, $Fnet$ in the z-direction is zero. Okay, they go on to continue the calculations in the y-direction, and in the x-direction, that one

(that is, the x-direction) is more interesting. PF Δ, PF equals $F1$ ΔT, so MVF, since it started from zero I presume, equal – MVF equals $F\Delta T$, so F equals $MVF\Delta T$, which is .06 N, okay. Which is what the first person got. For the second puck, again momentum principle. No initial momentum, MVF equals that, it's looking suspiciously like the first puck, and indeed they neglect torque. (He places an X on solution #3)

Which leads us by process of elimination to solution #2. Hmm, (sigh), so, angular momentum principle. "For the second puck," okay, well, let's look at that one later. For the first puck, angular momentum principle, okay, so, net torque is equal, er net angular – change in angular momentum equals torque times ΔT, so the translational angular momentum final, since it started out with none, is equal to net torque times ΔT. So, R cross P Final is $\tau\Delta T$. And, hmm, well, that's interesting. So, R P final, oh, they used that, they actually did the cross product, so we have the radius in z-direction, is that even right? (He picks up the reference sheet) where is my (inaudible)? There we go. Okay, all of the y-components are zero, so the only thing you're left with is, well, that stuff in the y-comp – the y direction, which is (he puts the problem aside and picks up a white board marker and begins writing to calculate the cross product). So we have RX, RZ, PX, PZ, it seems like there should be more terms in here because this would be – oh, PZ is zero. That's why. Okay, satisfied, so that's fine and RZ here is, well, R, so what we wind up with is momentum final in the x-direction equals $F1\Delta T$. Could have been quicker, but hey. Which means that MV final is $F1\Delta T$, which gets us to $F1$ equals .06 N. Okay. Now for the second puck. Same initial steps, except now we have translational and rotational angular momentum, which equals torque times ΔT. Okay, so, R cross PF plus $I\omega F$ equals R, equals the torque R cross $F2$, times ΔT. Okay, R cross PF, RX times, no RZ times PX, sounds good, and RZ is just R, plus $I\omega F$, hmm. $I\omega F$, so that equals R times $F2\Delta T$, great. And, I am running out of time. For the y-components, RP plus I omega equals RF delta T. Okay, still don't know what omega is, and once again they assume that – once again they assume that omega is linear speed over radius, and that would be fine if the puck were rolling (silence). And, yeah, we are looking down on the ice, so the puck shouldn't be rolling. Okay, well, that makes two of these solutions that have assumed that. That's one and two (he points to solution #1 and #2). This one (solution #3) I feel very comfortable saying is wrong, because – okay, why did I say that? – oh, yeah, solution #3 didn't include rotation (silence). Of course, if two solutions

made the same mistake then I guess the third solution would be the right one, but let's look at it. Now, why would the – why would this individual feel free to ignore rotation? The easy solution is that they (solution #3) are wrong. I like the easy solution. The quickest solution. (The timer goes off, and Ike does not choose a solution).