

## ABSTRACT

MOJICA, GEMMA FOUST. Preparing Pre-service Elementary Teachers to Teach Mathematics with Learning Trajectories. (Under the direction of Dr. Jere Confrey.)

In the past two decades, research on learning has focused on understanding how students think and how that thinking becomes more sophisticated over time. Some researchers have verified sufficient consistency and robustness in their findings relating to these constructs, which they have articulated in the form of learning trajectories. While an articulation of such constructs has contributed greatly to the knowledge base of how students learn, the field has just begun to explore the extent to which learning trajectories can be integrated into the practice of teaching. Though useful at the level of curriculum, assessment, and standards development, it remains to be shown that learning trajectories can be incorporated into teachers' practice and become a tool to understand students' thinking, for planning instructional activities, for interacting with students during instruction, and for assessing students' understandings. Thus, bringing learning trajectories into the classroom through teacher education is one critical area of knowledge that needs to be investigated. This study addresses to what extent and in what ways can pre-service elementary teachers use a learning trajectory for equipartitioning to build models of student thinking.

Over an eight-week period, within an elementary mathematics methods course, 56 pre-service teachers (PSTs) participated in this design study. Data included the following: video & audio recordings of class meetings, researcher's notes of class meetings and school-based experiences, pre- and post-test data, clinical interviews and analysis of interviews, and other artifacts. Findings from this study indicate that PSTs used an equipartitioning learning

trajectory to 1) deepen their understanding of mathematics and knowledge for teaching mathematics; 2) build more precise and adequate models of student thinking; and 3) incorporate models of student thinking into instructional practices.

Preparing Pre-service Elementary Teachers to Teach Mathematics with Learning Trajectories

by  
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## CHAPTER 1

### INTRODUCTION

One of the major goals of teacher education is to prepare prospective elementary school mathematics teachers to create environments where all students engage in high levels of academic performance. This is a monumental task, and mathematics educators face many challenges in supporting pre-service teachers (PSTs) as they develop the necessary skills to create this type of mathematical environment for students. Often, teachers hold the same mathematical misconceptions as students (Graeber, Tirosh, & Glover, 1989). In addition, PSTs enter teacher education programs with little to no experience in working with students around mathematical ideas. When PSTs initially make sense of students' mathematical understanding, they often use their own reasoning as a lens, unable to distinguish children's thinking from their own. Further, PSTs own mathematical understanding may be weak.

While mathematics educators face many challenges in preparing PSTs to effectively deal with performance-based accountability, some critics argue that many current practices of teaching do not support this challenge and that American schools are not currently organized to engage all students at high levels of performance (Elmore, 2002). Since the effectiveness of many current models for teaching have come into question, mathematics educators need to seek new and innovative ways to prepare PSTs within teacher education programs.

Those that advocate for new models of teaching make recommendations about the components of more effective models. Elmore (2002) advocates that the preparation of teachers should be grounded in student learning, models of effective practice, and well-articulated theories of learning, where issues of pedagogy are based on assessment of student

learning. In accordance with Elmore, Corcoran, Mosher, and Rogat (2009) argue that current reform goals in education cannot be realized for all students unless teachers use formative assessment, understand how domain specific knowledge develops over time, and are able to respond to evidence from formal assessment in light of this knowledge. Further, they indicate new models should develop teacher's ability to continually collect evidence of students' progress and difficulties so that teachers are able to make pedagogical decisions in response to this formative assessment in order to move towards meeting their goals and remediate when necessary. This would suggest that an important goal for mathematics educators would be to help PSTs shift from using their own thinking as a primary lens to understanding students mathematical thinking to relying on collecting and using formative assessment in making sense of student thinking.

Recommendations by proponents of new teaching models would suggest that teachers' ability to assess students' informal knowledge and how this knowledge develops plays an important role in teaching. Students' informal knowledge, when consistent with formal mathematical ideas, can be a foundation to build new knowledge. However, sometimes learners' understanding does not align with formal mathematical knowledge and is resistant to change. Overall, it is important to use students' existing conceptions in teaching, and for students to be deeply engaged and to reflect about their own thinking. Teachers need to have an understanding of the informal knowledge that children bring to the classroom and how their informal knowledge and prior experiences can be leveraged in instruction so that teachers help children in actively connecting their own informal ideas to

more formal mathematical ideas. Teachers also need to have an understanding of where children's informal ideas are situated within theories of learning that develop over time.

While there may be some agreement on components of new teaching models that have the potential to prepare PSTs to create environments where all students engage in high levels of performance in mathematics, it is unclear which mechanisms should be utilized. Teacher educators have an opportunity to provide PSTs with appropriate tools so that they are prepared to deal with performance-based accountability when they enter the classroom. One tool that may have the potential to provide guidance for PSTs as they learn to engage in instructional practices that move toward supporting all children at high levels of achievement is a learning trajectory.

In the past two decades, research on learning has focused on understanding how students think and how that thinking changes and evolves over time. Some researchers have verified sufficient consistency and robustness in their findings relating to these constructs, which they have articulated in the form of learning trajectories (Clements & Sarama, 2004; Confrey, Maloney, Nguyen, Wilson, & Mojica, 2008). While an articulation of such constructs has contributed greatly to the knowledge base of how students learn, the field has just begun to explore the extent to which learning trajectories can be integrated into the practice of teaching. Though useful at the level of curriculum, assessment, and standards development (Corcoran et al., 2009), it remains to be shown that learning trajectories can be incorporated into teachers' practice and become a tool to understand students' thinking. Thus, bringing learning trajectories into the classroom through teacher education is one critical area of knowledge that needs to be investigated.

There are several reasons that learning trajectories have the potential to be a tool in more effective models of teaching. Learning trajectories are grounded in student learning and are well-articulated theories of how this learning develops over time. An important component of some perspectives of a learning trajectory is that instruction impacts student learning and should take into account how children's informal knowledge and prior experiences can be a foundation to build new knowledge. Another key construct of learning trajectories is that pedagogical decisions should be based on formative assessment of student learning.

Research Focus

While many researchers have defined the construct of a learning trajectory and have hypothesized how such a construct might be useful for classroom teachers, few, if any, have begun to explore how learning trajectories can be situated within the dynamics of classroom practice. Furthermore, the field does not yet have evidence of how learning trajectories could have an impact on the preparation of PSTs as they learn to become teachers and learn to create environments where all students engage in high levels of mathematical performance. Thus, this study seeks to gain insight into how PSTs learn to use a learning trajectory as they develop their understanding of student thinking about equipartitioning. Part of the reason to study this is to consider ways to prepare teachers to be ready to anticipate and respond appropriately to the diversity of student approaches and knowledge that they will encounter as they enter the profession. The purpose of this study is therefore to explore whether and to what extent pre-service elementary teachers are able to use a learning trajectory as a tool in learning to teach. Specifically, this study investigates the following research question: To what extent and in what ways can pre-service elementary teachers use a learning trajectory

for equipartitioning to build models of student thinking? Since related questions are only now under investigation across the nation, an exploratory study of a descriptive nature is the best methodology to address this question.

### Organization of Dissertation

Chapter 1 presents an overview of the study, the research focus, and the research question under investigation. Chapter 2 reviews relevant literature focusing on three important aspects of this study: learning trajectories, equipartitioning, and how pre-service teachers learn about student thinking. In Chapter 3, the theoretical basis for the study is discussed and related to the choice of methodology. Chapter 3 concludes with a description of the design of the study, including the participants, context, goals, instructional activities, data collection, and data analysis. Chapter 4 presents the findings of the data analysis, and Chapter 5 discusses the results in relation to the research question. Also addressed are the limitations of the study and implications for research, which will include suggestions for a use model for teacher education, as well as recommendations for future research.

## CHAPTER 2

### REVIEW OF LITERATURE

The purpose of this study is to investigate the extent and ways in which pre-service teachers (PSTs) use a learning trajectory for equipartitioning to build models of student thinking. Chapter 2 includes a review of the relevant literature on research relating to learning trajectories and the construct of equipartitioning. The learning trajectory on which this study rests was developed as part of a larger project developing a set of learning trajectories for rational number reasoning. This research team<sup>1</sup> known as “DELTA”, diagnostic e-learning trajectories approach, conducted a synthesis on the research related to equipartitioning, redefined the term, identified a set of cases, conducted additional clinical interviews and built a learning trajectory. Currently, the team is continuing to revise the trajectory and develop diagnostic assessments for it. As a member of the team, I assisted in these research activities for one year prior to the conduct of this study. In this chapter, I describe the results of the initial work on equipartitioning, which forms the basis of my study.

I begin by reviewing the research on learning trajectories, including the DELTA definition. Next, equipartitioning is defined and relevant literature relating to children’s reasoning of the cases of equipartitioning is reported. Then, I summarize the DELTA findings relating to children’s reasoning of these cases and present the equipartitioning

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<sup>1</sup> The researcher works on the DELTA research team that is led by Dr. Jere Confrey and Dr. Alan Maloney. At the time of this study, graduate students on the team included P. Holt Wilson, Kenny Nguyen, and Marrielle Myers. References to the research team acknowledge that this work is conducted in collaboration with other members of the team and/or is built on Dr. Confrey’s and Dr. Maloney’s work on equipartitioning.

learning trajectory as it was used in this study; it has evolved since then. Afterwards, a review of literature on PSTs' learning about student thinking is presented. Finally, conjectured advantages to using a learning trajectories approach are presented. This chapter concludes with a description of the purpose of this study.

### Learning Trajectories

Learning trajectories have been defined throughout literature in different ways, and different terminology has been used to describe this construct. Simon (1995) uses the terminology *hypothetical learning trajectory*. Brown and Campione (1996) refer to a *developmental corridor*, while Schifter (1998) refer to *big ideas*. Clements and Sarama (2004) use the terminology *learning trajectory*. Catley, Lehrer, and Reiser (2005) refer to the term *learning performances*. Confrey (2006) describes a *conceptual corridor* and *conceptual trajectory*. Later, Confrey's research team defines a *learning trajectory* (Confrey, Maloney, Nguyen, Wilson, & Mojica, 2008). Another term that is widely used throughout literature is *learning progressions* (Corcoran et al., 2009). A common theme among these different terminology is that knowledge progresses from less sophisticated to more sophisticated levels of understanding in a relatively predictable way.

While many researchers define learning trajectories as articulations of potential pathways for the progression of learning as it becomes more sophisticated, Simon (1995) indicates that a *hypothetical learning trajectory*, a teacher's anticipation of the progression of the learning path, provides a rationale for designing instruction, taking into account the learning goal that defines the direction, learning activities, and the teacher's prediction of the potential reasoning and learning of students.

Brown and Campione (1996) suggest that a *developmental corridor* articulates how children's ideas are reexamined with increasing sophistication about specific disciplinary understandings over time. Children do not revisit concepts haphazardly; instead, children's understanding of concepts are refined and influenced by previous experiences and their developing knowledge base.

According to Clements and Sarama (2004), a *learning trajectory* is comprised of a mathematical goal, domain-specific developmental progressions that children advance through, and activities that correspond with these distinct levels of progression. Further, they describe the relationship between the mathematical goal and developmental progression as the "processes involved in the construction of the mathematical goal across several qualitatively distinct structural levels of increasing sophistication, complexity, abstraction, power and generality" (p. 83). Pivotal tasks are designed and sequenced to support students' understanding at a conceptual level or benchmark within the developmental progression. Clements and Sarama specify particular mental structures and patterns of reasoning that illustrate student's thinking at each level of progression.

Catley et al. (2005) suggest learning should be viewed as the process of developing key conceptual structures (Case & Griffin, 1990), or *big ideas* (Schifter, 1998), which coordinate and integrate isolated conceptual components. These *big ideas* are generative. Further, Catley et al. indicate that instruction can be viewed as an orientation towards core ideas that direct teaching and assessment around a minimal number of foundational concepts. They suggest teaching should trace a prospective *developmental corridor* (Brown & Campione, 1996), or a *conceptual corridor* (Confrey, 2006), for learning that spans grades,

as well as ages. Catley et al. indicate that a developmental corridor establishes that central concepts are introduced early in the learner's school experience and are progressively refined, elaborated, and extended.

Building on the work of others in the fields of mathematics education, science education, and the learning sciences, Confrey et al. (2008) define a *learning trajectory*, as:

a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time.

Building from this definition, I propose to view a learning trajectory as a tool that can be utilized by teachers to inform key instructional activities, such as planning, teaching, and assessing. While student understanding cannot be observed directly, learning trajectories seek to identify and describe key items, constructs, and behaviors, which can be observed. From these, I will investigate the extent to which learning trajectories can mediate PSTs' ability to construct models of student thinking.

Confrey identifies several ways that her team's perspective of a learning trajectory (Confrey et al., 2008) differs from others in the field. First, learning trajectories are models of students' likely passageways, which are constructed by researchers. The DELTA team indicates that this pathway is not linear. Students' multiple trajectories are within similar corridors, whose boundaries are set by the mathematical activity. Therefore, learning trajectories are embedded within conceptual corridors (Confrey, 2008). Furthermore, Confrey

(2006) explains that within these corridors, students encounter landmarks and obstacles (see Figure 1). For example, one of the first landmarks that most children encounter at a very early age is the ability to split a collection of discrete objects or to split a continuous whole in half. While there is substantial evidence that constructing a 2-split and a 4-split are among the easiest equipartitioning concepts for children to master, most young children find that attempting to fairly split a whole object between 3 people is quite a challenge (Confrey et al., 2008; Pothier & Sawada, 1983; Sawada & Pothier, 1984). Thus, many children encounter one of their first obstacles when trying to construct a 3-split because of the oddness of 3 and the difficulty of creating a radial cut.

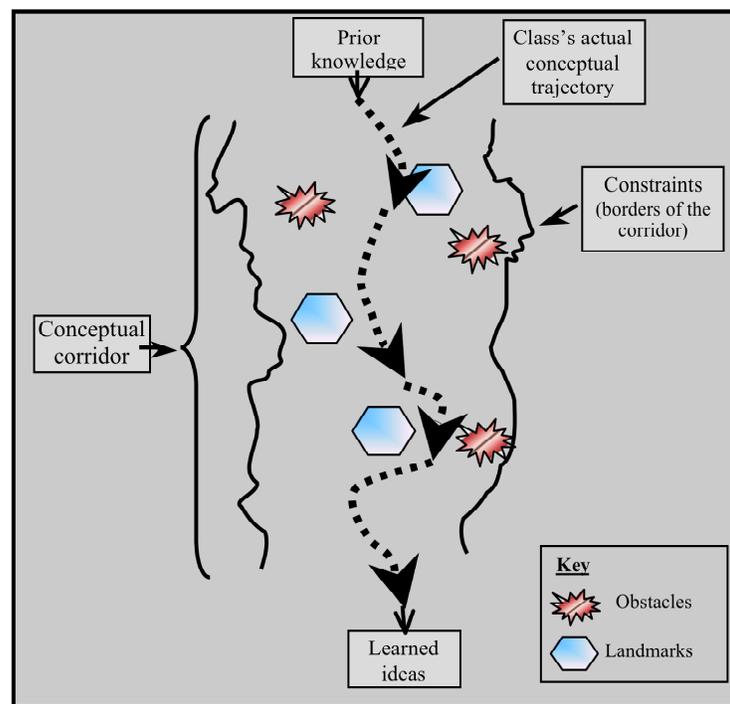


Figure 1. Confrey's (2006) Conceptual Corridor.

Second, a learning trajectory is based on a synthesis of the literature, not simply a review. There is support in the literature for taking this approach to the development of a

learning trajectory. Corcoran et al. (2009) suggests that learning trajectories should be “initially based on ‘systematic’ examinations of the relevant theory and research about how students learn a particular concept” (p. 16) and should provide guidance in the development of the learning trajectory. Third, this perspective recognizes the role of instruction in that activity must be sequenced intentionally to support students’ progress, so these are not “natural progressions” but derived from and fostered by intentional teacher acts. Finally, this view acknowledges the importance of students’ active role in the learning process. There is also an empirical component of this perspective that is based on conducting empirical research using theories of measurement to investigate the validity of the potential pathway. Corcoran et al. advises learning trajectories be substantiated by empirical evidence and that these factors are fundamental in differentiating learning trajectories from other approaches.

Corcoran et al. (2009) define *learning progressions* as “empirically-grounded and testable hypotheses about how students’ understanding of, and ability to use, core scientific concepts and explanations and related scientific practices grow and become more sophisticated over time, with appropriate instruction” (p. 15). Further, Corcoran et al. elaborate on the essential components of learning trajectories:

- Endpoints for learning are defined by concepts and themes that are central to a discipline, such as mathematics.
- Progress variables, which identify dimensions of knowledge, develop over time.
- Levels of progression delineate the development that most children progress towards in acquiring proficiency.

- Learning performances afford the conditions for the development of assessments and activities that locate students' progress.
- Assessments measure the development of students' knowledge of concepts as they evolve over time.

### Equipartitioning

As part of the DELTA research project, which is working to build diagnostic measures that teachers can use in their practice, Confrey et al. (2008) has conducted a synthesis of the literature on equipartitioning and other areas of rational number reasoning. I will first define equipartitioning and then review the literature relating to children's reasoning of equipartitioning.

#### *Equipartitioning Defined*

Confrey et al. (2008) define *equipartitioning* as being indicative of the "cognitive behaviors that have the goal of producing equal-sized groups (from collections) or parts (from continuous wholes) as 'fair shares' for each of a set of individuals." Confrey distinguishes between creating unequal-sized parts, which she refers to as "breaking," "fracturing," "fragmenting," or "segmenting," and creating equal-sized groups or parts of a collection or whole (i.e., equipartitioning) since breaking usually leads to addition and equipartitioning is foundational to division and multiplication. Many researchers refer to *equipartitioning* as *partitioning*. Confrey uses the terminology *equipartitioning* to emphasize the cognitive behavior of creating equal-sized groups or parts, rather than breaking collections or wholes into shares that are not equal in size. Thus, this terminology will be

used, even though other researchers (e.g., Pepper, 1991; Pothier & Sawada, 1983, 1989) use different terms to describe the same construct.

Based on the synthesis work, Confrey et al. (2008) organize children's reasoning of equipartitioning into four cases (i.e., Case A, Case B, Case C, and Case D). Each of these cases will be described. In the symbolic notation that is used to describe the four cases,

- $m$  represents the number of objects or items to be shared.
- $p$  represents the number of people who share object(s) or item(s).
- $f$  represents a fair share.

In Case A, a child is given  $m = pf$  objects and is asked to fairly share the objects among  $p$  people. For example, 4 people are asked to share a dozen pencils fairly. In Case B, a child is given a single continuous object to fairly share among  $p$  people. For example, 6 people are asked to fairly share 1 pizza. In Case C, a student is given  $m$  objects to share among  $p$  people, for  $m > 1$ , and  $p > m$ , so that each person gets a fair share. For example, 3 people share 2 cookies fairly. In Case D, a student is given  $m$  objects to fairly share among  $p$  people, for which  $p$  is not a factor of  $m$ , and  $p < m$ . For example, 3 people share 5 pizzas fairly.

#### *Children's Reasoning about Case A*

Several researchers (Davis & Pepper, 1992; Pepper, 1991; Pepper & Hunting, 1998) have conducted studies investigating the relationship between children's counting abilities and ability to equipartition a collection of discrete objects. In each of these studies, Pepper and her colleagues found that even very young children are quite successful at creating fair shares when asked to equipartition a collection of discrete objects, regardless of their ability to count. In 1991, Pepper conducted two clinical interviews with 75 4 and 5-year-old

preschool children. In one interview, she investigated children's counting abilities. Based on Steffe and Cobb's (1988) types of counters of unit items, Pepper classified 12% of the children as good counters (i.e., counters of motor, verbal, or abstract unit items), 27% as developing counters (i.e., counters of figural unit items), and 61% (i.e., of the children as poor counters.

In the other interview, Pepper (1991) investigated children's sharing strategies by asking children to share 12 biscuits between 2 dolls. Children's methods were classified into one of three categories: i) unsystematic strategies producing unequal shares; ii) unsystematic strategies producing equal shares; and, iii) systematic strategies producing equal shares. In another task, a third doll was introduced, and children were asked to redistribute the biscuits among the 3 dolls. Pepper classified children's behaviors for solving this task into three other categories. Good sharers created equal shares either systematically or by using fewer than four steps. Intermediate sharers also created equal shares by using some type of systematic strategy or by using four to seven steps. Poor sharers created unequal shares or needed to use more than seven steps. Pepper found that the predominant strategy used by children to share the biscuits was a dealing strategy, "a cyclic distribution of discrete objects (regarded as identical) with the same number distributed to each place on each round of the cycle until there are none left" (Davis & Pitkethly, 1990, p. 145). Eighty percent of the all the children used this strategy, regardless of their counting abilities, while some preschoolers made visual comparisons or counted to verify that the shares were indeed fair. A small number of 4 and 5-year-olds even recognized that the behavior of dealing would produce fair shares. Seventy-six person of the children who were identified as poor counters used a systematic method to

solve the first task. Regarding the second task, 74% of all the children were able to solve this task. Pepper concluded that there was no relationship between children's counting abilities and ability to equipartition in relation to the first task and children's ability to deal did not relate directly to their counting skills. Good sharers exhibited a range of counting competence.

In 1992, Davis and Pepper more closely examined the relationship between children's counting skills (i.e., good, developing, or poor) and their performance on the second task from Pepper's (1991) study. In investigating the number of steps (i.e., a single cycle in dealing) that it took the 4 and 5-year-old students to solve the task, the authors found that poor counters solved the task with a mean of 4.429 steps, developing counters solved the task with a mean of 4.929 steps, and good counters solved the task with a mean of 4.375 steps. Overall, more than 65% of all the children were able to complete the task in 4.5 steps or fewer. Forty-two percent of the poor counters completed the task in 4 or fewer steps, 43% of the developing counters did the same, and 50% of the good counters were able to do this, suggesting minimal differences between any types of counters who solved the task in four or fewer steps. Davis and Pepper suggested that these children were mentally able to split a pile of 6 discrete objects in the ratio of 2:1. There were only thirteen instances of counting in all of the clinical interviews, where nine of these instances appeared at the end of a solution for verification.

Based on these results, Davis and Pepper (1992) concluded that developing and poor counters were as efficient, or more efficient, than good counters at solving this redistribution task. The authors concluded that children, without good counting skills, are capable of

solving pre-numeric tasks. Davis and Pepper claim that an ability to operate on patterns of objects might be a developmental mode that is an alternative, or complementary, to counting.

In a follow-up study, Hunting and Pepper (1998) explored how counting and sharing relate to one another and examined strategies that preschool children use to share a collection of discrete items. Using similar tasks to those in Pepper's (1991) original study, Hunting and Pepper interviewed 25 preschool children who, on average, were 5 years and 4 months. In these clinical interviews, children solved the following tasks:

- Task 1: Sharing 12 crackers between 2 dolls.
- Task 2: Introducing a third doll, and sharing 21 cookies among 3 dolls.
- Task 3: Sharing 15 coins among the 3 dolls, using metal money boxes.

As in Pepper's study, children's counting skills and ability to equipartition were determined.

Ninety-two percent of the children were successful in sharing 12 crackers fairly among 2 dolls; children predominately used a systematic dealing strategy. Sixty percent of the children were able to share 21 cookies fairly among 3 dolls. Many of the children used a systematic dealing strategy to distribute the cookies, although counting appeared to play a more significant role in children's strategies. The authors thought that the third task would be the most difficult for children, but 64% were successful at distributing the coins. Children also used systematic dealing strategies to solve this task.

Based on the results of this study, Hunting and Pepper (1998) suggest that methods other than subitizing or counting were used to create equal shares and that dealing competence does not relate directly to counting skills. Children's systematic dealing procedures, involving no use of counting or measurement skills, resulted in groups of

discrete items being equipartitioned. Furthermore, children classified as good sharers exhibited different degrees of counting competence, as in Pepper's (1991) and Davis and Pepper's (1992) study.

The work of Pepper (1991) and her colleagues suggests that young children often use methods other than subitizing or counting to create equal shares, and that children's dealing competence does not relate directly to counting skills. Children's systematic dealing procedures, involving no use of counting or measurement skills, often result in groups of discrete items being equipartitioned successfully. Moreover, children are usually quite successful at equipartitioning whether they are classified as good, developing, or poor counters.

#### *Children's Reasoning about Case B*

In several different studies, Pothier and Sawada (1983, 1989) and Sawada and Pothier (1984) investigated children's ability to equipartition a continuous whole. They articulated a theory about how children reason in relation to equipartitioning and what characterizes each level as children's understanding progresses towards mastery. Later, Pothier and Sawada explored how children verify quality of fair shares in relation to their theory.

In 1983, Pothier and Sawada interviewed 43 K – 3<sup>rd</sup> grade elementary children. In these clinical interviews, they asked children to show how he or she would share a birthday cake so that 2, 4, 3, and 5 children would get the same amount, where older children were sometimes given an additional sequence (i.e., 4, 8, 16; 5, 10, 20; or 3, 6 12). Children conducted this task with circular and rectangular birthday cakes. Based on children's

responses, Pothier and Sawada (1983) proposed a theory describing the development of children's reasoning of equipartitioning.

At Level 1, *Sharing*, children learn to create a line through the center, halving the object or constructing fourths. At the Sharing level, children sometimes *break* the whole, constructing unequal-sized parts. Children performing at Level II, *Algorithmic halving*, have mastered the doubling process, allowing the student to create halves, fourths, eighths, and sixteenths of circles and rectangles. These children sometimes use a halving strategy, even when this strategy is unproductive. At Level III, *Evenness*, children focus on the size and shape of parts, distinguishing between which shares are fair or not fair, based on the equality of the parts. Since equality and evenness are key constructs in determining fairness, this level marks a landmark in children's understanding. This level is also characterized by children's ability to equipartitioning objects for even numbers of people. The next level, Level IV, or *Oddness*, is characterized by children's recognition that algorithmic halving will not always produce fair shares, such as in the case of sharing for 3 or 5. Finally, at the highest level of sophistication, Level V, children create *composition*, using a multiplicative algorithm, to efficiently construct unit fractions.

Work by Empson and Turner (2006) also studied students' understanding of algorithmic, or repeated, halving but extended previous work by examining the relationship between repeated halving and children's multiplicative reasoning. Empson and Turner investigated the role of repeated halving in relation to 30 1<sup>st</sup>, 3<sup>rd</sup>, and 5<sup>th</sup> grade students' multiplicative thinking as they engaged in paper folding tasks. They found that children initially connected the action of folding and the outcome in *non-recursive* ways. Other

children used an *emergent recursive strategy*, such as recursive doubling, understanding that creating a half fold would double the number of partitions. Few children used such a *recursive strategy* to connect the fold and subsequent number of partitions.

In 1984, Sawada and Pothier conducted another study to elaborate on their theory about how children's reasoning about equipartitioning becomes more sophisticated as it progresses through five distinct levels. In this study, Sawada and Pothier extended their previous work by investigating children's reasoning relating to equipartitioning triangular and pentagonal objects. In the first task, children equipartitioned a circular and rectangular cake between 3 and 5 dolls, respectively. Whereas, in the second task, children were asked how they would equipartition a triangular and a pentagonal cookie. If the children did not choose 3 or 5, they were asked to share the cookie for 3 and 5 people. After conducting clinical interviews with 31 children, the authors found that 77% of the children chose to equipartition the triangles and pentagons by 2 and 4. In contrast, only 13% of the children first selected 3 and 5 parts. After allowing the children to reconsider the number of parts in which to equipartition a triangle and a pentagon, only 13% of the children suggested 3 and 5 parts. The authors concluded that children prefer to equipartition objects into an even number of parts (i.e., 2 and 4), rather than an odd number of parts (i.e., 3 and 5), regardless of the geometric shape. Furthermore, the properties of the geometric shapes did not impact children's performance; children relied on parallel and orthogonal cuts, even on triangles and pentagons.

Later work by Pothier and Sawada (1989) investigated the verification processes of children when determining whether shares created from equipartitioning a continuous whole

are fair. The authors conducted over 200 clinical interviews with children in grades K through 6, over the course of several years. Children were asked to fairly share giant cookies of different geometric shapes. Pothier and Sawada found that they could classify children's verification processes into eight categories. Some children verify the equality of parts created from equipartitioning a whole by approximating whether or not the parts appear to be the same size, using a *visual estimation*. Other children believe that specific methods, *or a reference to the technique used*, create fair shares, regardless of the geometric properties of the object being shared. Another verification process used by children is a *compensatory description of the parts*, where children try to compensate for parts that are not the same size and shape. Some children focus on the lengths and widths of the shares, using a *measurement process*. Children also verify the equality of shares by *superimposing the parts* on top of one another. Children who determine the equality of shares based on whether or not the parts are the same size and shape utilize the process of *congruency as a requirement*. Others do not necessarily consider the size and shape of the shares; instead, these children make a *reference to the geometry of the parts*. Yet, other children believe that *the geometry of the whole informs the partitioning*. According to the authors, only one of these processes of verification attends to the whole, rather than the parts that are produced, which may lead to children's difficulties in determining whether shares are equal. The authors suggest that children need opportunities to equipartition different geometric shapes in order to support the recognition of the underlying structural properties. Pothier and Sawada also indicate that children should consider different methods for equipartitioning the same geometric shape.

### *Children's Reasoning about Cases C and D*

Since concepts relating to Cases A and B are more appropriate for K – 2 mathematics instruction than Cases C and D, the literature review of the later cases will be limited. The literature relating to Cases C and D have documented studies that investigated students' strategies in fairly sharing more than one continuous object among various numbers of people (Charles & Nason, 2000; Lamon, 1996; Toluk & Middleton, 2003). Lamon investigated the partitioning strategies of 346 students in grades 4 through 8, where they were asked to share multiple meal servings (e.g., pizzas, cookies, and Chinese dinners) among various numbers of people in eleven different tasks. She found that students used the following strategies: *preserved-pieces*, *mark-all*, *distribution*. Students who used a *preserved-pieces* strategy initially distributed wholes, marked and cut any remaining wholes into the appropriate number of parts, and distributed these parts. For example, to fairly share 4 objects among 3 people, 1 object would be distributed to each person, and the remaining object would be equipartitioned into thirds and distributed to each person. Students using a *mark-all* strategy marked each whole into the appropriate number of partitions, where only the remaining whole would be cut into the appropriate number of parts to be distributed. For instance, when fairly sharing 4 objects among 3 people, students would mark each object to equipartition it into thirds, distribute one object to each person, and cut the final object into thirds and distribute it to each person. Students who employed a *distribution* strategy marked and cut each whole into the appropriate number of parts and distributed them, either economically (e.g., equipartitioning a whole into thirds when sharing among 3 people) or by over-marking (e.g., equipartitioning a whole into sixths when sharing among 3 people).

According to Lamon, as grade level increased, students used preserved-pieces and mark-all strategies, indicating a shift from distributing single units towards the utilization of composite units and an understanding of equivalence.

After reviewing the literature on fair sharing and interviewing 12 3<sup>rd</sup> grade students, Charles and Nason (2000) constructed a taxonomy for classifying students' equipartitioning strategies. In contrast to the strategies used by the students in Lamon's (1996) study, Charles and Nason identified twelve strategies that were classified into three categories: partitive quotient construct, multiplicative, and iterative sharing. Also, unlike Lamon, Charles and Nason focused on students' ability to quantify the fair shares produced by their equipartitioning strategies. While Lamon provided descriptions of students' strategies and concluded that younger students tended to use a distribution strategy that older students abandoned, Charles and Nason categorized strategies into four classes that were ordered in terms of sophistication. Class 4 strategies are the least sophisticated, and Class 1 are the most sophisticated strategies in relation to the ability to abstract the partitive quotient fraction construct. *Class 4* strategies were characterized by the failure to generate fair shares, inaccurate quantification of shares, and no conceptual mapping between the number of people ( $p$ ) and the fraction name of each share ( $1/p^{th}$ ). *Class 3* strategies did generate fair shares; however, the quantification of shares was inaccurate, and there was no conceptual mapping. *Class 2* strategies were characterized in the same way as Class 3, except the quantification of shares were accurate. Finally, *Class 1* strategies were characterized by generating fair shares, the accurate quantification of shares, and a conceptual mapping.

Toluk and Middleton (2003) investigated 4 5<sup>th</sup> grade students' understanding of fractions and division while conducting four parallel teaching experiments. Like Charles and Nason (2000), they found that students knowledge progressed relating to the partitive quotient fraction construct from less sophisticated to more sophisticated understandings. When progressing towards an understanding of fractions as a quotient, Toluk and Middleton found that students first conceptualized that a whole number quotient was always the result of situations involving division. By engaging in equipartitioning, students then progressed to a part-whole understanding of fractions. Next, students were able to generalize about the relationship between a division situation and the fraction representing the quotient. Finally, students were able to conceptualize fraction as a division. According to Toluk and Middleton, explicit connections need to be made in instruction to help students connect these different schemes.

## Equipartitioning Learning Trajectory

Confrey et al. (2008) have articulated a learning trajectory for equipartitioning, which is a foundational component of a learning trajectory for rational number reasoning. Figure 2 illustrates Confrey's (2008) conceptual map of rational number reasoning.

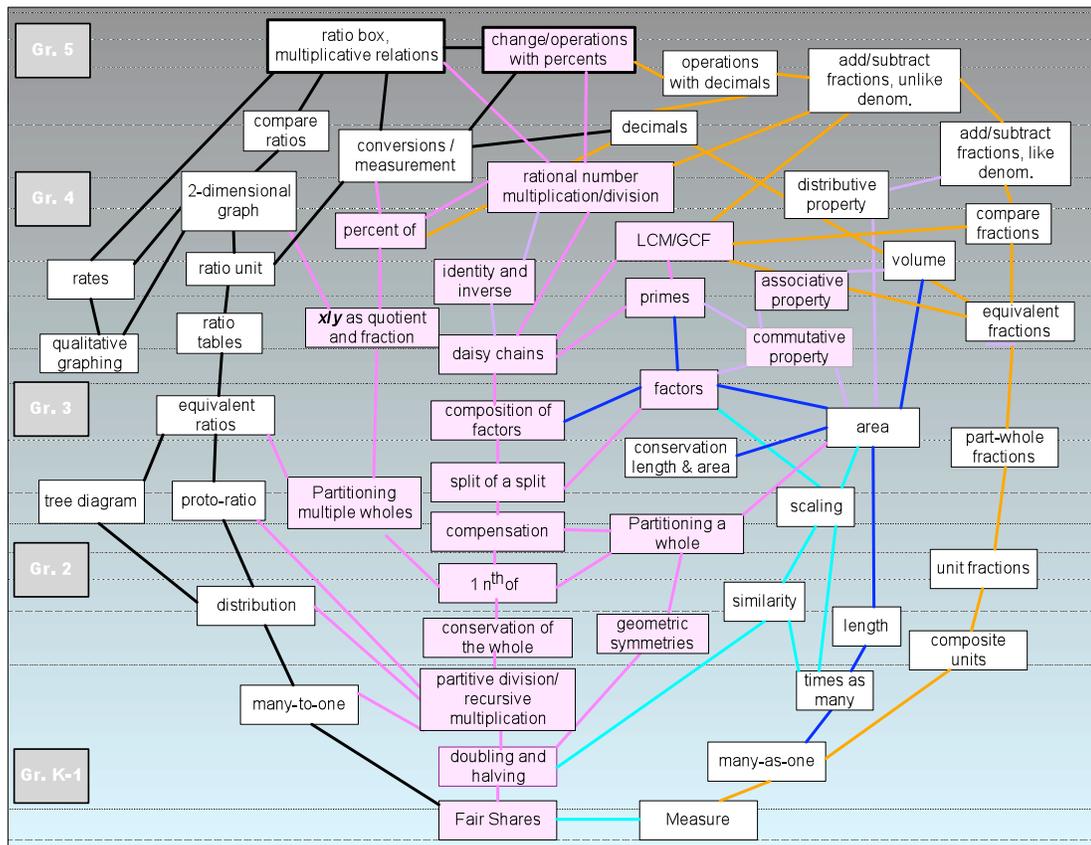


Figure 2. Confrey's (2008) Equipartitioning Learning Trajectory.

While Figure X situates the equipartitioning learning trajectory within the wider domain of rational number reasoning, Table X represents how children's knowledge progresses from less sophisticated understandings to more sophisticated reasoning, accounting for what children understand at different phases in their development of equipartitioning. Based on Confrey's synthesis work, the DELTA team built this learning trajectory for equipartitioning

to describe the behaviors and verbalizations of these levels so that researchers and teachers can conceive of how this knowledge develops in children (Confrey et al., 2008). While the work of the DELTA team has evolved since the time of this study, the learning trajectory in Table 1 was used with the PSTs in this study.

Table 1. DELTA equipartitioning learning trajectory at the time of the study.

Case	Level
Case D	1.8 $m$ objects shared among $p$ people, $m > p$
Case C	1.7 $m$ objects shared among $p$ people, $p > m$
Case B	1.6 Splitting a continuous whole object into odd # of parts ( $n > 3$ )
Case B	1.5 Splitting a continuous whole object among $2n$ people, $n > 2$ , and $2n \neq 2^i$
Case B	1.4 Splitting continuous whole objects into three parts
Case B	1.3 Splitting continuous whole objects into $2^n$ shares, with $n > 1$
Case A	1.2 Dealing discrete items among $p = 3 - 5$ people, with no remainder; $mn$ objects, $n = 3, 4$ , or $5$
Case B	1.1 Partitioning using 2-split (continuous and discrete quantities)

At the least sophisticated level of the equipartitioning learning trajectory is the understanding that both discrete and continuous quantities can be split in half, followed by the ability to deal any number of discrete objects, where there will be no remainders after all the objects are dealt. After children understand the concept of splitting a whole in half, they come to understand how to equipartition a whole by a factor of  $2^n$  (e.g., 4, 8, 16, etc.), which often begins by the understanding that even objects can be repeatedly split in half. The next level of sophistication would be equipartitioning a whole that has an odd number of equal-sized parts. The most sophisticated understanding of equipartitioning is the ability to share fairly when there are more objects than people and more people than objects. Currently the DELTA research team is exploring whether Case C is actually more sophisticated than Case D, or vice versa.

Within each level (e.g., 1.1, 1.2, 1.3, etc.) of the equipartitioning learning trajectory, the DELTA team describes another level of the progression of knowledge: methods, multiple methods, justification, naming, reversibility, and properties. Within a level, the ability to share a group of discrete objects or a continuous whole fairly using more than one method is considered more sophisticated than using a single strategy. The next level of sophistication is the ability to justify how one knows he or she has created a fair share. The ability to understand why a strategy produces fair shares is more sophisticated than performing the behavior by itself. This is followed by the ability to name a fair share (e.g., one-half), which is followed by the knowledge that reassembling the objects in a group or parts of a whole constitutes as much as the original whole. Within naming, the different ways in which shares are named indicate different levels of sophistication. The least sophisticated way to name a share is using a general name (e.g., a group), followed by a count (e.g., 1 piece), followed by a relation (e.g.,  $\frac{1}{2}$ ). The highest level of understanding in the learning trajectory is the knowledge of properties, such as equivalence, composition, compensation, and geometric properties.

### PSTs' Learning about Children's Thinking

One major goal of teacher education programs is to support PSTs in developing an understanding about how children think about mathematics. Several studies provide insight relating to the evolution of PSTs' perceptions of children's reasoning about different mathematical ideas and identify instructional activities that can be used to inform the design of activities within a methods course to support this development (Ambrose, 2004; D'Ambrosio & Campos, 1992; McDonough, Clarke, & Clarke, 2002; Philipp et al., 2007;

Tirosh, 2000). In 1992, D'Ambrosio and Campos conducted a teaching experiment to investigate the extent to which a research experience could increase 5 PSTs' knowledge of children's understanding of fractions. According to D'Ambrosio and Campos, lectures and readings are insufficient methods of instruction in developing PSTs' understanding of the teaching and learning of mathematics. In order to create opportunities for PSTs to change their understanding relating to the teaching and learning, the researchers supported PSTs as they engaged in their own research projects with children and reflected on this experience. During this study, D'Ambrosio and Campos examined changes in PSTs' perceptions of children's thinking about fractions and were particularly interested in how PSTs resolved conflicts relating to these issues.

Throughout the teaching experiment, PSTs continually made hypotheses about children's understanding of fractions as a result of their reflection on these conflicts and their efforts to resolve them. D'Ambrosio and Campos (1992) found that PSTs utilized research reports, which lead to the resolution of conflicts encountered regarding children's thinking. Another finding of this study was that PSTs began to question typical instructional practices of teaching through this experience and were able to conceptualize alternatives to common instructional sequences found in textbooks relating to fractions. Finally, the researchers found that PSTs became more knowledgeable and skilled in gaining insight into children's informal and formal reasoning of fractions.

Tirosh (2000) studied 30 PSTs' understanding of children's mathematical thinking in an elementary education methods course. PSTs were in their second year of a four-year teacher education program. In the methods course, Tirosh designed activities to enhance

PSTs' subject matter knowledge and pedagogical content knowledge of fraction division. While D'Ambrosio and Campos (1992) found that instructional activities focusing on engaging PSTs in research projects with children impacted the ways in which they thought about children's reasoning, Tirosh designed activities that focused on the following: i) children's conceptions and misconceptions; ii) the root of these misconceptions; and, iii) instructional strategies for supporting children as they refine their understanding. Data in this study included a diagnostic questionnaire of items relating to the division of fractions, interviews, and data from implementing the activities in the methods course.

Before the methods course, 83% of PSTs successfully divided fractions on four different fraction problems in one item, and 97% of the PSTs were able to write correct expressions that could be used to solve three different word problems involving fraction division in the second item. Even though Tirosh (2000) found that most of the PSTs were able to divide fractions, they were unable to explain this operation. PSTs who were unsuccessful made the same mistakes that have been attributed to children's errors in literature. Further, the majority of PSTs were not knowledgeable about the sources of children's misconceptions. Only a small number of PSTs recognized that one major misconception held by children is that they inappropriately attribute properties of the division of natural numbers to the division of fractions, such as that the divisor must always be a whole number or that the divisor must be less than the dividend. The majority of PSTs attributed children's misconceptions only to algorithmic mistakes or reading comprehension.

After the methods course, most PSTs were aware of various sources of misconceptions relating to the division of fractions: attribution of properties of division of

natural number properties to the division of fractions, limitation of using a partitive model of division, and other intuitively based mistakes. Tirosh concluded that a central goal for mathematics educators is to analyze PSTs' understanding of students' misconceptions, which is strongly related to PSTs subject-matter knowledge. Furthermore, teacher education programs should address various conceptions and misconceptions and the cognitive processes that may result in different ways of reasoning. Tirosh also proposed that teacher education programs should provide opportunities for PSTs to observe children as they engage in tasks, make inferences about children's responses, and analyze various models of children's thinking. According to Tirosh, "a general theoretical framework related to cognitive processes and sources of misconceptions could support teachers in their attempts to foresee, interpret, explain, and make sense of students' ways of thinking" (p. 23).

Another study exploring PSTs' perceptions of student thinking was conducted by McDonough, Clarke, and Clarke (2002) at two different universities with 140 PSTs. As in the D'Ambrosio and Campos (1992) study, McDonough et al. centered instructional activities around working with students, specifically in conducting interviews. Using a questionnaire, interviews, and a discussion group, McDonough et al. investigated the effectiveness of PSTs use of one-to-one interviews in providing insight into pedagogical content knowledge. During this study, PSTs at one university conducted a one-to-one interview with 1<sup>st</sup> and 2<sup>nd</sup> graders, aged 6 to 7-years-old, on addition and subtraction tasks. PSTs at the other university conducted one interview with elementary school children whose ages ranged from 7 to 10-years-old.

McDonough et al. (2002) found PSTs' understanding was enhanced in several different ways. PSTs became more knowledgeable about the types of strategies that children use, as well as the variety of methods used in relation to their levels of sophistication. PSTs became aware that using one-to-one interviews is a powerful way to gain insight into individual children's thinking, providing PSTs with model questions and tasks. Like D'Ambrosio and Campos (1992), reflection was an important instructional activity that initiated changes in PSTs' perceptions of student thinking. Where D'Ambrosio and Campos found that engaging PSTs in research projects where they worked with children, McDonough et al. found that the interview provided opportunities for PSTs to reflect on appropriate experiences for children.

In accordance with D'Ambrosio and Campos (1992) and McDonough et al. (2002), Ambrose (2004) designed instructional activities to center on working with children. Ambrose explored how an intense experience with elementary school children impacted 15 PSTs views of teaching and beliefs about learning mathematics. PSTs participating in this study volunteered to enroll in an experimental course while they were simultaneously enrolled in mathematics course. Five of the PSTs were Freshmen, 8 were Juniors, and 2 were postgraduates obtaining their credentials for teaching. During this experimental course, pairs of PSTs worked with 6 to 9-year-old children using whole number story problems over three sessions. For four sessions, pairs of PSTs worked with 10-year-old students on fraction concepts. During these sessions, PSTs worked with children on various tasks that were designed to elicit student thinking and focused on problem solving. Data in this study included surveys, interviews, PSTs' written work, and field notes.

Ambrose (2004) found that providing PSTs with intense experiences involving work with children resulted in the change of PSTs beliefs about how children learn. In particular, PSTs initially held few, undifferentiated beliefs. Later, PSTs came to believe that teachers should base the beginning of instruction on listening to children, allow children to have sufficient time to think, and explain mathematical concepts in various ways. PSTs also realized that teaching mathematics is more complicated than they had originally anticipated. PSTs initially believed that teaching involves explaining mathematics to children and continued to hold on to this belief, although they began to believe that it is important for teachers to present multiple solutions. Like D'Ambrosio and Campos (1992), Ambrose advocates for incorporating reflection with the experience of working intensely with children to support PS in examining and refining their beliefs about children's learning.

Within an elementary mathematics methods course, Philipp et al. (2007) investigated whether combining learning of mathematical content knowledge with an emphasis on children's thinking about mathematics would deepen PSTs' knowledge of mathematics and support changes in their beliefs. At the time of the study, 159 PSTs were enrolled in 12 different sections of their first mathematics content course. Graduate students in mathematics, who received instructional support from a senior mathematics educator, taught the sections. Using modified random assignment, PSTs who volunteered to participate in the study were assigned to either one of four treatment groups or a control group. PSTs in the control group were enrolled in the methods course; however, these PSTs did not focus on children's reasoning by analyzing videos or working with children nor did they visit elementary mathematics classrooms. While learning about whole number and rational

number concepts, PSTs assigned to a treatment group concurrently experienced one of the following field experiences:

1. Watching and analyzing videos to learn about children's reasoning of mathematics (CMTE-V),
2. Watching and analyzing videos, along with working with individual children during six problem-solving experiences to learn about children's reasoning of mathematics (CMTE-L),
3. Visiting typical elementary school mathematics classrooms, where teachers were selected because of their proximity to campus (MORE-C),  
or
4. Visiting specially selected mathematics classrooms, where teachers were identified as reform oriented (MORE-S).

In order to explore the extent to which the treatments impacted PSTs' mathematical understanding and beliefs, PSTs completed a web-based beliefs survey and an assessment of mathematical knowledge before and after the study. Philipp et al. (2007) found a larger percentage of CMTE-L PSTs showed significant increases on five of the seven beliefs than MORE-C PSTs, and a larger percentage of CMTE-L PSTs also showed significant increases on one of the seven beliefs than MORE-S. PSTs in the CMTE-L group had significantly higher beliefs scores than those in the control group on four of the seven beliefs. CMTE-V PSTs also scored significantly higher than those in the MORE-C group on five of the seven beliefs and those in the control group on three of the seven beliefs. While no significant differences were found between PSTs who analyzed videos and PSTs who analyzed videos

and worked with children, these PSTs showed the greatest percentage of increase on every belief as compared to PSTs who only visited classrooms. Furthermore, Philipp et al. concluded that PSTs focusing on children's mathematical thinking displayed more sophisticated beliefs about both mathematics and mathematics learning than PSTs who did not focus on children's reasoning of mathematics. Interestingly, PSTs who visited conveniently located classrooms showed little change in their beliefs as compared to PSTs in the other treatment groups and the control group, indicating that PSTs engagement in these classrooms actually hindered the establishment of beliefs that were promoted in the methods course.

Of the PSTs who focused on children's thinking by analyzing videos and/or working with children, 50.7% scored 15 points higher on the post-test measuring content knowledge, while 30.5% of PSTs who were in treatment groups that did not focus on children's reasoning scored 15 more points. Overall, Philip et al. found that there were only modest changes in PSTs content knowledge of mathematics, where PSTs who focused on children's mathematical thinking had a mean change that was only one-fourth of a standard deviation higher than the mean change for the other groups of PSTs.

Philipp et al. (2007) indicated that three factors led to the success of PSTs who focused on children's mathematical reasoning: i) the control of variables that had the potential to confound PSTs; ii) the support of experiences relating to PSTs' future instructional practices; and, iii) the support of guided reflection. Thus, providing PSTs with a structured environment where they had opportunities to interact with children and collectively reflect on these experiences was more likely to support PSTs change in beliefs.

These findings support the work of Ambrose (2004), D'Ambrosio and Campos (1992), and McDonough et al. (2002) in that experiences relating to work with children played an important role in contributing to PSTs shifts in thinking about how students' reason, indicating the important of instructional activities that relate to these experiences. Furthermore, Philipp et al. is in agreement with Ambrose and D'Ambrosio and Campos in that reflection on these experiences, not the experiences alone, is a central tenet in changing PSTs' perceptions about student thinking.

#### Conjectured Advantages of using a Learning Trajectories Approach

Researchers conjecture several advantages to using a learning trajectories approach to teaching. Using a learning trajectories approach should facilitate both vertical and horizontal integration of concepts, like rational number reasoning. In other words, learning trajectories integrate students' knowledge of rational number reasoning concepts across and among grade levels (Confrey et al., 2008). For example, Confrey's synthesis argues that equipartitioning in K-2 is a foundation for multiplication and division, and for fractions, ratios and rates in upper elementary school and middle school. She proposes that this range of concepts can help early elementary teachers (K-2) recognize the importance of their work in students' future school experiences and help upper elementary and middle school teachers know what students bring to instruction in the later grades.

Another conjectured advantage to using learning trajectories is that it can be used as a tool to help teachers integrate different content strands of mathematics (Confrey et al., 2008). For example, an equipartitioning learning trajectory supports teachers understanding of how

number, measurement, and geometry are all related to equipartitioning concepts (Pothier & Sawada, 1983).

According to Corcoran et al. (2009), learning trajectories have the potential to improve student learning by creating more precise standards, influencing the design of coherent curricula, constructing valid assessments, and producing more effective classroom instruction. Designing instruction around central concepts, or “big ideas,” helps teachers align tasks with learning performances (Catley et al., 2005), resulting in greater coherence and alignment between teaching and learning (Catley et al., 2005; Schifter & Fosnot, 1992), which is essential for valid assessment. Furthermore, concentrating on “big ideas supports assessment by suggesting dimensions for assessment (the big ideas) and by orienting assessment toward *clusters of standards* adhering to big ideas, rather than toward standards considered in isolation” (p. 4). Catley et al. indicate that big ideas summarize and convey what is important about a discipline for purposes of teaching, learning, and assessment; learning about “big ideas” motivates further learning.

#### Purpose and Research Questions

In Chapter 2, a review of the literature presents a knowledge base for learning trajectories, equipartitioning, and the ways in which PSTs learn about children’s reasoning. The literature relating to PSTs’ perceptions of student thinking indicates that instructional activities within a methods course have the potential to contribute to shifts in PSTs’ understanding of the ways in which students reason about mathematical ideas (Ambrose, 2004; D’Ambrosio & Campos, 1992; McDonough, Clarke, & Clarke, 2002; Philipp et al., 2007; Tirosh, 2000). While current research on learning trajectories suggests that one

potential benefit of using a learning trajectories approach is improved instruction (Corcoran et al., 2009), very little is known about how learning trajectories can be used by PSTs within the dynamics of a classroom as they learn to teach. Corcoran et al. advocate utilizing learning trajectories in pre-service teacher education to provide a foundation for the pedagogical content knowledge that is required to guide PSTs' decision making relating to instructional practices. The purpose of this study is to investigate how an equipartitioning learning trajectory can be operationalized by an instructor of an elementary mathematics methods course in preparing PSTs to use learning trajectories as a tool in learning to teach.

Specifically, the following research question will be investigated: To what extent and in what ways can pre-service elementary teachers use a learning trajectory for equipartitioning to build models of student thinking? In order to examine this larger research question, the following supporting questions were developed:

- In what ways and to what extent can an equipartitioning learning trajectory can be utilized to deepen their own understanding of the structure of mathematics and knowledge for teaching mathematics?
- In what ways and to what extent can PSTs use an equipartitioning learning trajectory to build stronger and more precise models of student thinking?
- To what extent and in what ways can PSTs utilize a learning trajectory for equipartitioning to incorporate models of student thinking into instructional practices?

It is conjectured that an equipartitioning learning trajectory can be utilized to assist PSTs in improving their ability to compare students' behaviors and verbalizations to their

own to deepen their own understanding of the structure of mathematics and knowledge for teaching mathematics. It is also conjectured that PSTs will be able to use a learning trajectory for equipartitioning as a tool to build stronger and more precise models of student thinking by improving their ability to identify students' behaviors and verbalizations with respect to equipartitioning. Additionally, it is conjectured that PSTs will use an equipartitioning learning trajectory as a tool to incorporate models of student thinking into instructional practices, so that PSTs are better able to: predict diverse student strategies, behaviors, and verbalizations; identify common misconceptions and landmarks relating to equipartitioning; and, inform instructional practices, such as their questioning or in their next steps with children.

## CHAPTER 3

### THEORY AND METHODOLOGY

In Chapter 2, the literature review focused on two key aspects of this study: 1) students' knowledge base of equipartitioning; and, 2) learning trajectories. A review of the literature described how students reason about equipartitioning and outlined the contributions of Confrey's synthesis work on equipartitioning, a foundational component of rational number reasoning in young children (Confrey et al., 2008). The literature review also expounded how Confrey and her DELTA colleagues have verified consistent findings relating to how students' thinking of equipartitioning changes and evolves over time, which they have articulated in the form of an equipartitioning learning trajectory. While this has contributed greatly to the knowledge base of how students learn, the field has just begun to explore the extent to which learning trajectories can be integrated into the practice of teaching or in the preparation of PSTs. A design study was utilized to investigate the extent to which PSTs are able to use an equipartitioning learning trajectory as a tool in building models of student thinking.

In Chapter 3, the conceptual framework of this study is first discussed. Next, I will describe my theoretical orientation and the subsequent methodological implications for this study. Finally, the design of the study is described, including the participants, context, goals, instructional activities, data collection, and data analysis.

#### Conceptual Framework

The conceptual framework for this study is comprised of two perspectives as a basis for understanding how PSTs build models of students' thinking using an equipartitioning

learning trajectory. The first perspective is a framework that describes PSTs' processes for constructing models of students' reasoning (Hollebrands, Lee, & Wilson, 2007). The other component of this conceptual framework, an equipartitioning learning trajectory, provides a lens for connecting PSTs model building of student thinking with key instructional practices. While student understanding cannot be observed directly, learning trajectories seek to identify and describe key tasks, constructs, and behaviors, which can be observed. Thus, in this framework, a learning trajectory is viewed as a *tool* that assists PSTs in building models of student thinking and supports PSTs as they connect models of student thinking with key instructional practices: planning, teaching, and assessment.

According to Cobb and Steffe (1983), students construct models of mathematical concepts, while teachers or researchers build models of students' thinking. Hollebrands et al. (2007) identified four distinct processes that PSTs employ in creating such models: Describing, Comparing, Inferring, and Restructuring.

*Describing* is characterized by PSTs' explicit attention to student's actions and words, written or verbal, in making decisions about student thinking. Figure 3 illustrates that in the process of describing, PSTs analyze ( $T_t$ ) student work ( $S_w$ ), which includes the student's actions and words. In this process of model building, there is no explicit comparison to PSTs' own work ( $T_w$ ), and there is no attempt to make inferences about what the student is thinking ( $S_t$ ) based on the student's work ( $S_w$ ).

When PSTs construct models of student reasoning, by relating the student's work to their own, they are *comparing*, making either explicit or implicit comparisons between their

own work ( $T_w$ ) to that of student's ( $S_w$ ). In Figure 3, the solid line represents an explicit comparison, while the dashed line represents an implicit comparison.

*Inferring* is described as the process of making inferences about student's thinking based on student work. In the process of inferring, PSTs analyze ( $T_t$ ) student work ( $S_w$ ) and build models of student thinking by making inferences about how the student is reasoning ( $T_t$ ), using student work ( $S_w$ ) as evidence.

*Restructuring* is characterized by PSTs' use of models of student thinking in their own practice of teaching. In this case, PSTs analyze ( $T_t$ ) student work ( $S_w$ ) and make inferences about student thinking ( $S_t$ ) based on student work ( $S_w$ ). These models of student thinking are in turn used by PSTs to inform their own thinking ( $T_t$ ), which is used to regulate instructional decisions that are made in planning, teaching, and assessing. Since these behaviors are distinct, they serve as a place to examine the effects of learning trajectories on the model building process of student thinking.

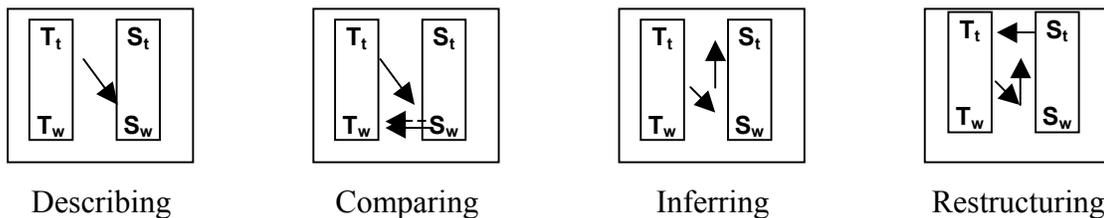


Figure 3. Hollebrands's et al. (2007) Model of PSTs' Processes of Model Construction.

The conceptual framework for this study brings together the behaviors identified by Hollebrands et al. (2007) in the model building process with the DELTA equipartitioning learning trajectory (see Figure 4). I conjecture that the DELTA equipartitioning learning trajectory will inform PSTs as they enact the instructional activities of planning, teaching, and assessing. I argue that the DELTA equipartitioning learning trajectory coordinates: 1)

equipartitioning tasks, 2) teacher’s anticipation of student performance on these tasks, 3) and students’ engagement with equipartitioning tasks, assisting PSTs in understanding how these components interact in a way that is supportive of refining students’ understanding. PSTs create models of student thinking as they engage in planning, teaching, and assessing. The processes described by Hollebrands et al. (2007) provide a way to articulate these models. I conjecture that as PSTs will begin to use the DELTA equipartitioning learning trajectory as a tool to inform their model building process as they engage in instructional activities.

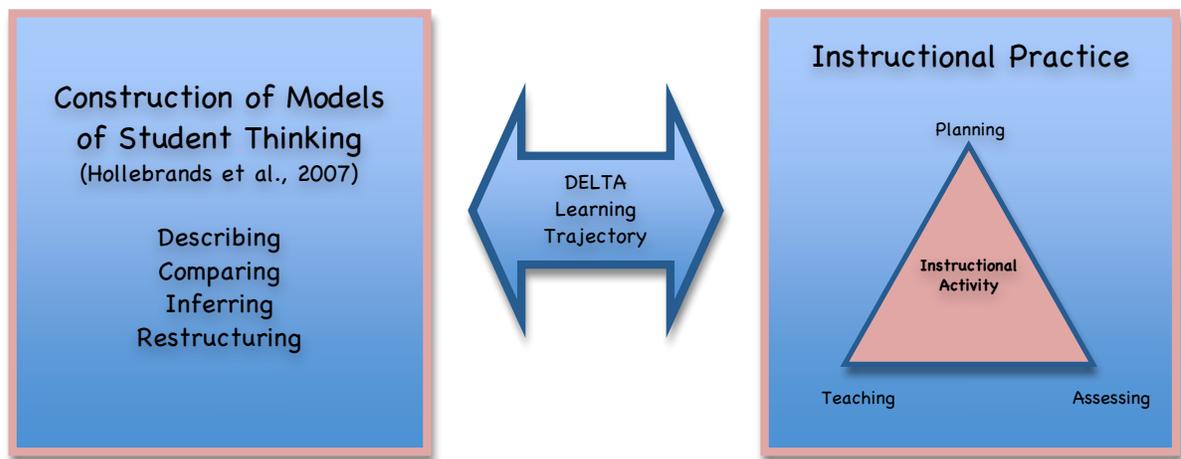


Figure 4. Conceptual Framework.

Specifically, I make the following three conjectures:

- An equipartitioning learning trajectory can be utilized as a tool to assist PSTs in improving their ability to compare students’ behaviors and verbalizations to their own to deepen their own understanding of the structure of mathematics and knowledge for teaching mathematics.
- An equipartitioning learning trajectory can be utilized as a tool to assist PSTs in building robust and precise models of student thinking by

improving their ability to identify students' behaviors and verbalizations with respect to equipartitioning.

- PSTs will be able to use an equipartitioning learning trajectory to incorporate models of student thinking into instructional practices, so that PSTs are better able to: i) predict diverse student strategies, behaviors, and verbalizations; ii) identify common misconceptions and landmarks; and, iii) inform instructional practices.

### Theoretical Perspectives

Two theories of intellectual development, constructivism and socio-cultural perspectives, have heavily influenced the DELTA definition of a learning trajectory and the subsequent development of a learning trajectory for equipartitioning. Thus, both traditions are the basis for using a learning trajectories approach in the design of the study.

#### *Constructivism*

One of the central tenets of constructivism is that individuals actively construct knowledge through their own activity (Confrey, 1990; Confrey & Kazak, 2006; Von Glasersfeld, 1982, 1984, 1989, 1990) and knowledge cannot be transferred directly or wholesale from one individual to another (Simon, 1995; Von Glasersfeld, 1990), as in the case from the teacher to the student (Confrey, 1990). According to Von Glasersfeld (1984), radical constructivism “is *radical* because it breaks with convention and develops a theory of knowledge in which knowledge does not reflect an ‘objective’ ontological reality, but exclusively an ordering and organization of a world constituted by our experience” (p. 24). Further, Simon asserts that “we construct our knowledge of our world from our perceptions

and experiences, which are themselves mediated through our previous knowledge” (p 115), indicating that an objective reality is independent of our way of knowing. Thus, learning is a result of an individual’s cognition adapting to these experiential situations (Confrey, 1990; Confrey & Kazak 2006; Piaget, 1970; Simon, 1995; Von Glasersfeld, 1982, 1984, 1989, 1990) towards viability or fit (Confrey & Kazak, 2006; Von Glasersfeld, 1982, 1984, 1989, 1990).

*Scheme theory.* Individuals’ adaptation to their experiential world can be explained by scheme theory. Piaget (1970) describes a *scheme* as “whatever is repeatable and generalizable in an action” (p. 42). Schemes are a way of organizing actions and thoughts that allow individuals to build mental structures based on their perceptions and experiences with the world (Woolfolk, 1998). As individuals interact with their world, they reuse and modify schemes (Confrey & Kazak, 2006) to create meaning from their experiences (Bee & Boyd, 2004). There are two significant ways in which an individual encounters new ideas, through assimilation and accommodation. When an individual incorporates new actions and ideas with existing schemes, this is called *assimilation*. *Accommodation* involves alerting extant schemes to create a new structure.

According to Von Glasersfeld (1990), abstracting of regularities is the result of assimilation, and the only means to achieve regularity is by discounting certain differences. Furthermore, he claims that schemes cannot be constructed if the individual is unable to isolate situations that lead to positive results. When the individual assimilates new perceptions and experiences with existing schemes, the individual is determining viability or

fit with existing knowledge. Von Glasersfeld (1982) explains the idea of *viability* in the following way:

cognitive structures, i.e., actions schemes, concepts, rules, theories, and laws, are evaluated primarily by the criterion of success, and success must ultimately be understood in terms of the organism's efforts to gain, maintain, and extend its internal equilibrium in the face of perturbations (pp. 5 – 6).

Equilibrium is the state in which an individual's cognitive structures yield expected results (Von Glasersfeld, 1989). When a scheme does not lead to an expected outcome, a cognitive conflict, or perturbation, takes place (Von Glasersfeld, 1989, 1990). A *perturbation* is a disequilibrating experience where one's current understanding does not fit with existing conceptions (Simon, Tzur, Heinz, and Kinzel, 2004). This cognitive conflict triggers intellectual growth (Confrey & Kazak, 2006; Simon et al., 2004). In order to eliminate the perturbation, the individual engages in new activity (Confrey & Kazak, 2006). This may result in a modification of the abstracted pattern or modification of the action (Von Glasersfeld, 1990). When a perturbation leads to an accommodation that creates a new equilibrium, learning takes place (Simon et al., 2004; Von Glasersfeld, 1989).

*Reflective abstraction.* Reflective abstraction explains how concepts are formed, the process in which individuals progress from lower level knowledge to higher level mental structures (Piaget, 1970; Simon et al., 2004). Piaget (1970) describes *physical knowledge* as understanding based on general experience, which is concrete, while *knowledge of mathematical structures* is abstract. Further, mathematical knowledge is drawn from different types of abstraction: simple and reflective abstraction. Piaget (1970) states that *simple*

*abstraction* is the “transposition from one hierarchical level to another” (p. 17). Since simple abstraction is tied to the concrete, knowledge is abstracted from *acting* on an object. Mathematical structures begin to take shape in the *coordination* of action. According to Piaget, *reflective abstraction* is the “mental process of reflection, that is, at the level of thought a reorganization takes place” (p. 18), which is based on *coordinated actions*. Confrey (1990) adds that reflection “functions as the bootstrap by which the mathematician pulls her/himself up in order to stabilize the current construction and to obtain the position from which the next construct can be created” (p. 109), requiring a level of consciousness (Confrey & Kazak, 2006).

#### *Vygotsky’s Socio-cultural Theory*

One of the underlying principles of Vygotsky’s (1978) socio-cultural theory is that one’s interaction with members of his or her own culture shapes cognitive development, especially with those who are more knowledgeable. According to Wertsch (1985), there are several salient themes in Vygotsky’s work, two of which are that tools and signs are important in the development of higher psychological processes and that higher mental functions originate in social processes.

*Speech.* Vygotsky suggests that cultural tools are crucial to cognitive development in that these tools mediate higher mental functions (Vygotsky, 1978). According to Wertsch and Cole (1996), higher psychological functions require “an indirect action, one that takes a bit of material matter used previously and incorporates it as an aspect of action” (p. 251), suggesting that culture as a medium assumes that “the special mental quality of human beings is their need and ability to mediate their actions through artifacts and to arrange for

rediscovery and appropriation of these forms of mediation by subsequent generations” (p. 251). Furthermore, since other members of the culture previously shaped the tool, the individual’s actions benefit from the psychological functions that created this socially and historically situated artifact. Thus, individual processes are mediated by tools, which arise from social processes. In accordance with Vygotsky’s (1978) perspective, the individual’s use of such tools transforms the way the individual thinks and views the world.

For Vygotsky (1978), the most important tool is language. As a result of the human capacity for language, speech functions as a tool in several ways. Language equips children with auxiliary tools that can be used to solve difficult tasks, as well as overcome impulsive action. Speech supports psychological growth by enabling children to plan solutions to problems before the plan is actually executed. Finally, speech allows the child to master his or her own behavior. Thus, the role of speech plays an integral role in the cognitive development of a child (Vygotsky, 1978). The convergence of speech and activity results in both practical and abstract intelligence. When individuals are able to incorporate speech and symbol use with action, the intellect is transformed. Further, speech both accompanies activity and also plays a central role in carrying out activity. Since speech and action are the same psychological function, speech becomes more significant as the complexity of action increases or when solutions are less direct.

Vygotsky (1978) suggests that words shape activity into structure. The child’s actions lead to speech, which is dominated by the activity. Thus, speech becomes a catalyst for new activity, where new relations between words and action emerge. Next, speech guides and

dominates action by providing the child with a function for planning. Finally, the planning function of speech comes into being with language to reflect the external world.

According to Vygotsky (1978), inner speech accounts for all higher levels of thought and guides cognitive development. Speech has roots in social origins. Inner, or private, speech has an interpersonal function and begins to develop in children around age 5; by age 9, it disappears when verbalizations are internalized as inner silent speech. External speech has a communicative or interpersonal function. The transition from egocentric speech that is audible to inner speech, which is silent, is fundamental in the development of cognition. Vygotsky indicates egocentric speech is an intermediate stage that does not reflect egocentric thought. On the contrary, it corresponds with realistic thought, which is used as a form of planning. Vygotsky suggests the child's psychological fields change dramatically when they begin to use this function of speech.

*Internalization.* Wertsch (1985) defines *internalization* as “a process involved in the transformation of social phenomena into psychological phenomena” (p. 63). According to Wertsch and Stone (1985), internalization is the way functions transform from a social to an individual plane of consciousness. For Vygotsky, internalizing external knowledge means transforming thought into a tool for conscious control (Bruner, 1985). Thus, once tools become internalized, the individual becomes aware and more reflective of the function and can use the function to solve problems. Instruction has an important role in the internalization process and in developing students' understanding. In fact, Vygotsky (1986) maintains that instruction comes before development.

*Zone of proximal development.* One aspect of Vygotsky’s work, the *zone of proximal development*, relate directly to the instruction of mathematics and other school subjects. According to Vygotsky (1978), the *zone of proximal development* (ZPD) is “the distance between the actual development level as determined by independent problem solving and the actual level of potential development as determined through problem solving under adult guidance or in collaboration with more practical peers” (p. 86). In other words, a child holds a current, or *actual*, achievement level (see Figure 5). Some problems are completely out of the child’s reach, with or without the help of others. However, there are some problems that the child can solve with the help of adults or more able-bodied peers; this area is known as the ZPD.

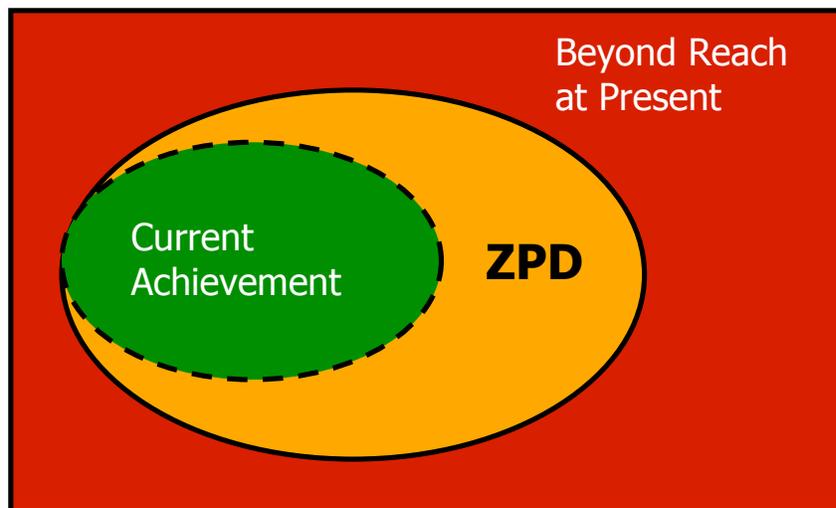


Figure 5. Vygotsky’s Zone of Proximal Development.

According to Norton and D’Ambrosio (2008), the principal supposition of the ZPD is that students will learn with assistance, and this knowledge will subsequently allow students to become independent problems solvers. Vygotsky (1986) states, “With assistance, every child can do more than he can by himself—though only within the limits set by the state of

his development” (p. 187). In order to facilitate learning, the teacher’s role is to present tasks or problems that are within students’ ZPD and assist students as necessary (Norton & D’Ambrosio, 2008). Furthermore, Vygotsky suggests that “instruction must be oriented toward the future, not the past” (p. 189). Teachers should pose tasks and problems that offer challenges for students, but such tasks should not be so difficult that they are unable to be solved without the guidance of a teacher or more capable other.

### *Methodological Implications*

Both traditions (i.e., constructivism and socio-cultural theory) have shaped the DELTA definition of a learning trajectory and the development of an equipartitioning learning trajectory in several ways. The DELTA definition of a learning trajectory (Confrey et al., 2008), as defined in Chapter 2, is as follows:

a researcher-conjectured, empirically-supported description of the ordered network of experiences a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time.

From a constructivist perspective, the DELTA definition is grounded in the belief that individual’s understanding is based on an ordering and organization of the individuals’ experiences. It is through making meaning of these ordered network of experiences that students move from lower to more sophisticated understandings. Thus, students’ activity and experiences are fundamental to the learning process.

Further, there is an underlying belief that students' activity can be predicted and that there are specially designed equipartitioning tasks that students will benefit from as the learning sequence moves towards higher levels of conceptual development through reflection, or reflective abstraction. In the DELTA definition of a learning trajectory, we suggest that concepts develop through successive "refinements of representation, articulation, and reflection." In reflecting, patterns of thought are reused and modified by making adjustments based on comparisons and noticing patterns and invariance as students adapt to experiences with equipartitioning.

From a socio-cultural perspective, the DELTA definition of a learning trajectory emphasizes the importance of speech as students move from informal to more complex conceptions of equipartitioning over time. One of the ways in which learning progresses is through articulation, or speech. Language, as a tool, mediates higher psychological processes, or more complex concepts. Articulation, or speech, also serves the purpose of acting as a tool to help students plan their activity and to serve as a communicative purpose, between the teacher and student. Students' verbalizations as they interact with equipartitioning tasks give us insight into students' understanding of equipartitioning. The DELTA equipartitioning learning trajectory provides a framework for teachers to situate students' verbalizations about equipartitioning within a range of speech, assisting the PST in assessing students' understanding and chronicling how this speech and knowledge evolves over time. Our definition also addresses the idea of internalization. Students gain greater conceptual knowledge of equipartitioning through interactions with teachers, suggesting that interactions based on instruction lead to more complex understandings.

For Vygotsky (1978), instruction precedes development. Fundamental to the DELTA idea of a learning trajectory is that instruction is an intervention that occurs before the most complex understanding develops. Teachers' knowledge of equipartitioning tasks or mathematical activities, tools or materials, and representations influence the ways that students learn. The DELTA definition of a learning trajectory places value on assessment, the design of equipartitioning tasks and instruction, and the importance of the interaction between the teacher and the student. Part of the idea in the ZPD is that students are eventually able to become independent problem solvers based on teachers choosing appropriate tasks and offering the appropriate amount of assistance. Central to the DELTA definition is that appropriate tasks need to be selected and sequenced based on an evaluation of the students' current conceptions of equipartitioning or actual achievement level and that students will benefit from specially designed tasks if the equipartitioning tasks are sequenced appropriately. Teachers' choice of tools and materials, as well as representations, have an impact on the way that students interact with equipartitioning tasks and play an important role in the development of students' understanding.

Several principles from constructivist theory are also the basis for the DELTA equipartitioning learning trajectory. From a constructivist perspective, the idea that children think qualitatively differently than adults and that students' strategies, behaviors, and verbalizations should be valued are important components of the DELTA equipartitioning learning trajectory. Students' activity and experiences are fundamental to the learning process. This activity, along with students' verbalizations, should be the basis of evaluating students' understanding of equipartitioning. It is crucial that teachers have an understanding

of the diversity of students' strategies, behaviors, and verbalizations and that they are able to connect this with cognition. From a socio-cultural perspective, teachers' interactions and instructional decisions regarding sequencing equipartitioning tasks, representations, and media impact student learning, as well.

In turn, using a learning trajectories approach to equipartitioning based on constructivist theory should enlist a method that allows the researcher to investigate PSTs' understanding of building models of student thinking. Therefore, I chose to use a design study to gain insight into PSTs' knowledge. This design is also ideally suited for this study because the implementation of an equipartitioning learning trajectory within an elementary mathematics education methods course is a novel idea. This type of design draws on prior research in equipartitioning and allows me to investigate how instruction can support PSTs as they learn to build models of student thinking and enact key instructional practices.

A crucial component of the design was the creation and implementation of instructional activities, like the extensive use of clinical interviews, another commonly used method among constructivist researchers. According to Hunting (1997), clinical interviews provide teachers access to observing, interpreting, and assessing students' understanding of equipartitioning tasks. Since PSTs generally have limited experiences interacting with students as they engage in equipartitioning tasks, it is important to provide them with opportunities to consider the wide range of student strategies, behaviors, and verbalizations that they will encounter in assessing students' thinking. The use of clinical interviews is one way to provide such an opportunity. In the next section, I will discuss the methods of a design study and the clinical interview method.

## Design Studies

This investigation utilizes the methods of a design study. Researchers who situate themselves in the learning sciences also refer to design studies as *design research* or *design experiments* (Confrey, 2006). Mathematics educators (e.g., Cobb, 2000; Confrey & Lachance, 2000; Simon, 2000; Steffe & Thompson, 2000) have historically referred to this type of design as a *teaching experiment*. I have chosen to use the terminology of *design study* to emphasize the placement within the larger DELTA project for the design of diagnostic measures based on learning trajectories, and will draw on the work of researchers from both the learning sciences and mathematics education.

Design studies refer to a method of research that involves “engineering particular forms of learning and systematically studying those forms of learning with the context defined by the means of supporting them” (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p.9). The duration of design studies typically take place over a significant period of time (Confrey & Lachance, 2000), lasting from a few weeks to an academic school year (Cobb, 2000). Cobb et al. (2003) have identified five features that apply to all design studies: purpose, interventions, iterative design, prospective and reflective analysis, and pragmatism. Each feature of design will be described in further detail below.

According to Steffe & Thompson (2000), the most important goal of a design study is to create a *living model* of students’ mathematical activity. The purpose of a design study is not only to develop theory about the process of learning but also includes the mechanisms that were designed and implemented to support the evolution of learning (Cobb et al., 2003). The design study begins with the identification of learning goals for students. In order to

capture how activity changes as a product of interactions in the classroom environment (Steffe & Thompson, 2000), the researcher conceptualizes how the learning process might come to fruition in the classroom (Cobb, 2000).

The researcher creates a series of interventions, or instructional activities, designed to support the learning goals (Cobb, 2000) that will be implemented within the classroom. The highly interventionist nature supports testing innovative instruction, such as using a learning trajectories approach in a methods course for pre-service elementary teachers (Cobb et al., 2003; Confrey & Lachance, 2000; Confrey, 2006). The researcher then makes conjectures about the trajectory of learning and tests these conjectures within classroom instruction, which evolve throughout the study to achieve the specific learning goals for students (Cobb, 2000; Confrey, 2006). According to Confrey and Lachance (2000), conjectures should have two dimensions: mathematical and pedagogical dimensions. Conjectures must be related to mathematical content; they answer the following question: What should be taught? There should also be a pedagogical aspect related to the content, answering questions like, “How should this be taught?” Therefore, the mechanisms that are designed to support the learning process include practices of teaching and learning, both affordances and constraints of materials, policy, and design-specific norms (Cobb et al., 2003).

The process of design and analysis is cyclic and iterative (Cobb, 2000), allowing the researcher to revise and elaborate on the conjectures throughout the study (Cobb et al. 2003; Confrey & Lachance, 2000; Confrey, 2006). Design studies are both prospective and retrospective (Cobb et al., 2003). From the prospective side, researchers implement interventions with a hypothesized learning process for two reasons: 1) to expose the process

to scrutiny; and, 2) to support the development of alternative trajectories for learning that emerge based on the implementation of the design. On the retrospective side, design studies are conjecture driven. According to Cobb et al. (2003), conjectures become more specific as they are tested. Further, when initial conjectures are refuted, alternative conjectures are generated and tested. This process iterates throughout the study. The final feature of the design study is its pragmatic roots, whose theories can be employed in educational contexts, such as methods courses for pre-service elementary teachers (Cobb et al., 2003).

### The Clinical Interview Method

Hunting (1997) argues that traditional approaches to assessing students are of limited use to teachers in planning effective teaching strategies because these assessments do not get to the heart of students' thinking processes. Further, Hunting suggests that the clinical interview, pioneered by Piaget (1976), can be useful for classroom teachers to assess students' reasoning. Piaget (1976) developed the clinical interview to understand the form and functioning of children's reasoning. He considered contemporary methods at the time of his early work (i.e. standardized testing and pure observation) to be inadequate techniques to study students' thinking, if used alone. Opper (1975) adds that standardized testing is not flexible enough to identify the specificities of intellectual mechanisms. To understand students' mental processes, standardized testing does not allow the teacher or researcher insight into actions and verbalizations that may illuminate students' thinking.

Interviews allow the researcher "to collect and analyze data on mental processes at the level of a subject's authentic ideas and meanings, and to expose hidden structures and processes in the subject's thinking that could not be detected by less open-ended techniques"

(Clement, 2000, p. 547). Interviews are attempts to comprehend a situation from the perspective of the participant, in order to discover the meaning of the participant's experience (deMarrais, 2004; Kvale, 1996; Opper, 1975). Piaget recognized that children's mental processes are different than those of adults. Students can communicate their understanding of mathematics by explaining the meaning of their actions (Hunting, 1997).

#### *Characteristics of the Clinical Interview*

Often, the interview setting is less formal than a standardized testing atmosphere, creating a more relaxed environment for the student (Opper, 1975). Usually, the interviewer has some ideas about the types of possible reasoning that a student will engage in during an interview, which Opper (1975) calls a guiding hypothesis. The student is asked a sequence of questions relating to a set of tasks, or problems, that motivate the student to predict, observe, and explain. Tasks are often introduced in an entertaining way to students (Opper, 1975). Hunting (1997) also recommends using novel tasks in interviews with students. According to Opper (1975), the series of questions may confirm or disconfirm the hypothesis by allowing the researcher to make inferences about the components of the student's reasoning.

Piaget (1975) emphasizes the importance of probing the student's responses to ensure that the answers provided by the student are well-grounded, allowing for the investigation of students' mental processes by asking open-ended questions to draw out and record naturalistic forms of thinking that can not always be documented from tests (Clement, 2000). According to Clement (2000), assessments are usually written from the viewpoint of an adult, like an educator, not accounting for students' alternate conceptions and informal thinking processes. During interviews, the researcher has the option of asking additional

questions that might clarify and extend student thinking (Clement, 2000), which has the potential to provide in-depth knowledge of conceptual understanding (Clement, 2000; deMarrais, 2004).

Although a student's verbal responses often offer insight into his or her thinking about a task or problem, verbalizations are not the only source of data that can be gathered during an interview regarding a student's mental processes (Opper, 1975). Further, students' actions, including the manipulation of concrete objects, are as consequential as verbalizations (Opper, 1975). Other actions, such as gestures and facial expressions, are also of particular importance, especially when interviewing young children. The interviewer can make inferences about the student's reasoning based on both verbalizations and actions (Hunting, 1997).

## Design of the Study

### *Participants*

The participants in this study are 56 pre-service elementary teachers, 6 sophomores and 50 juniors, who are enrolled in a teacher education program at a large university in the southeastern region of the United States. At the time of the study, all PSTs were enrolled in their first mathematics methods course. All PSTs enrolled in the course were invited to participate in this study, and all agreed to take part in this investigation. Prior to the study, PSTs had completed 21 hours of mathematics and science content courses, including Calculus I and Introduction to Statistics, since PSTs are specializing in the teaching of elementary school mathematics and science. In addition to taking content and mathematics

methods courses, PSTs participate in extensive classroom-based experiences. By the time PSTs have graduated, they will have spent 1,000 hours in elementary classrooms.

### *Context of the Course*

Participants were enrolled in one of two sections of a mathematics methods course that was taught within an elementary education department. Twenty-seven PSTs were enrolled in one section, and 29 PSTs were enrolled in the other. The focus of the course was to prepare PSTs to enact instructional practices in early elementary mathematics, such as planning, teaching, and assessing. Another emphasis in the course was on preparing PSTs to teach the strands of number and operations, measurement, geometry, and equipartitioning. This study took place within the unit on equipartitioning.

During the semester long methods course, I was the instructor and acted as a teacher-researcher during the study. There were two components to the methods course: regularly scheduled class meetings and an extensive K-2 classroom-based experience. Each class met twice a week, for 75 minutes on each day. As part of the classroom-based experience, PSTs were placed in a K-2 classroom for the entire semester. In addition to regular class meetings, PSTs spent one day a week in their respective classrooms. PSTs also spent one entire week in their respective K-2 classrooms at the end of September and another full week at the end of October. During these weeklong classroom-based experiences, PSTs did not meet for regular class meetings.

### *Goals of Instructional Activities*

In order to investigate the extent to which PSTs used an equipartitioning learning trajectory in learning to teach, a series of instructional activities were created and implemented within the equipartitioning unit to support the following goals: 1) increase PSTs' own knowledge of equipartitioning and equipartitioning tasks; 2) increase PSTs' knowledge of an equipartitioning learning trajectory; and, 3) support PSTs in utilizing an equipartitioning learning trajectory as they learned to build models of student thinking for teaching. Since PSTs had limited prior experiences interacting with children in a mathematics learning environment, several sub-goals were crucial to the design of the equipartitioning unit: 1) increase PSTs' knowledge of the range of possible student strategies, behaviors, and verbalizations of equipartitioning; 2) connect student strategies, behaviors, and verbalizations to student thinking; 3) use student strategies, behaviors, and verbalizations that they observed to build models of students' understanding of equipartitioning.

An outline of the instructional activities, that were designed and implemented to assist PSTs in using learning trajectories to build models of student thinking and in learning to enact instructional practices can be found in Table 2 which begins on page 61. Since equipartitioning concepts relating to Cases A and B are more appropriate for K-2 mathematics instruction, the treatment of instructional activities relating to Cases C and D were limited. These instructional activities include the following:

- an introduction to the articulation of the equipartitioning learning trajectory;

- engagement with equipartitioning tasks in order to develop PSTs own content knowledge;
- instruction in the conduct of clinical interviews on different types equipartitioning tasks; and
- instruction in the analysis of videos of clinical interviews where students engage in equipartitioning tasks.

Table 2. Outline of instructional activities, outcomes, and data.

Instructional Activities	Outcomes	Data
<p><b>Week 1 – Session 1 (9.15.08)</b></p> <ul style="list-style-type: none"> <li>• Pre-test administration</li> <li>• Brown, Sarama, &amp; Clements (2007) article discussion on LT</li> <li>• Mini-lecture/introduction to LT as a tool (No Equipartitioning LT)</li> </ul>	<ul style="list-style-type: none"> <li>• Identify PSTs’ baseline knowledge of equipartitioning and knowledge for teaching equipartitioning</li> <li>• LT introduced to PSTs as a tool for teaching</li> </ul>	<ul style="list-style-type: none"> <li>• Pre-test</li> <li>• Video (whole class)</li> </ul>
<p><b>Week 1 – Session 2 (9.17.08)</b></p> <ul style="list-style-type: none"> <li>• Introduction to equipartitioning tasks for Cases A and B (No Equipartitioning LT)</li> </ul>	<ul style="list-style-type: none"> <li>• PSTs solve equipartitioning tasks and anticipate K-2 students’ thinking</li> </ul>	<ul style="list-style-type: none"> <li>• Video (whole class)</li> </ul>
<p><b>Week 2 – Sessions 3 – 7 (Week of 9.22.08)</b></p> <ul style="list-style-type: none"> <li>• K-2 classroom-based experience (No Equipartitioning LT)</li> </ul>	<ul style="list-style-type: none"> <li>• PSTs solve equipartitioning tasks and anticipate K-2 students’ thinking</li> <li>• PSTs plan and implement equipartitioning tasks with students in K-2 classrooms</li> <li>• PSTs analyze implementation and build models of students’ thinking without an equipartitioning LT</li> </ul>	<ul style="list-style-type: none"> <li>• Classroom-based Assignment #1</li> <li>• Observations</li> </ul>
<p><b>Week 3 – Session 8 (9.29.08)</b></p> <ul style="list-style-type: none"> <li>• Jacobs, Ambrose, Clement, &amp; Brown (2006) article discussion on teacher-produced videotapes of student interviews</li> <li>• Mini-lecture on clinical interview method</li> <li>• Analysis of clinical interviews</li> </ul>	<ul style="list-style-type: none"> <li>• Clinical interview method introduced as an approach to assess and analyze student thinking</li> </ul>	<ul style="list-style-type: none"> <li>• Video (whole class)</li> <li>• Audio (small group)</li> </ul>
<p><b>Week 3 – Session 9 (10.6.08)</b></p> <ul style="list-style-type: none"> <li>• Pothier &amp; Sawada (1990) article discussion of equipartitioning and equipartitioning tasks</li> <li>• Watanabe (1996) article discussion of a student’s understanding of one-half</li> <li>• Discussion of K-2 classroom-based experience (No Equipartitioning LT)</li> </ul>	<ul style="list-style-type: none"> <li>• PSTs discuss student thinking relating to equipartitioning</li> <li>• PSTs discuss their implementation of equipartitioning tasks with students in K-2 classrooms</li> </ul>	<ul style="list-style-type: none"> <li>• Video (whole class)</li> <li>• Audio (small group)</li> <li>• Transparencies</li> <li>• Group Charts</li> </ul>
<p><b>Week 4 – Session 10 (10.08.08)</b></p> <ul style="list-style-type: none"> <li>• Equipartitioning LT for Cases A &amp; B (PV 1.1 &amp; 1.2)</li> <li>• Analysis of clinical interview clips</li> </ul>	<ul style="list-style-type: none"> <li>• Equipartitioning LT introduced to PSTs</li> <li>• PSTs analyze clinical interview clips and use a LT to build models of students’ thinking</li> </ul>	<ul style="list-style-type: none"> <li>• Video (whole class)</li> <li>• Audio (small group)</li> </ul>

Table 2 Continued

Instructional Activities	Outcomes	Data
<p><b>Week 5 – Session 11 (10.13.08)</b></p> <ul style="list-style-type: none"> <li>• Equipartitioning LT for Case B (PV 1.3, 1.4, 1.5, &amp; 1.6)</li> <li>• Analysis of clinical interview clips</li> </ul>	<ul style="list-style-type: none"> <li>• PSTs analyze clinical interview clips and use a LT to build models of students’ thinking</li> </ul>	<ul style="list-style-type: none"> <li>• Video (whole class)</li> <li>• Audio (small group)</li> </ul>
<p><b>Week 5 – Sessions 12 &amp; 13 (10.15.08 &amp; 10.20.08)</b></p> <ul style="list-style-type: none"> <li>• Introduction to equipartitioning tasks for Cases C and D (No Equipartitioning LT for Cases C and D)</li> <li>• Equipartitioning LT for Cases C and D (PV 1.7 &amp; 1.8)</li> <li>• Analysis of clinical interview clips</li> </ul>	<ul style="list-style-type: none"> <li>• PSTs solve equipartitioning tasks and anticipate K-6 students’ thinking</li> <li>• PSTs analyze clinical interview clips and use a LT to build models of students’ thinking</li> </ul>	<ul style="list-style-type: none"> <li>• Video (whole class)</li> <li>• Audio (small group)</li> </ul>
<p><b>Week 6 – Session 14 (10.22.08)</b></p> <ul style="list-style-type: none"> <li>• Group planning of equipartitioning lessons</li> </ul>	<ul style="list-style-type: none"> <li>• PSTs plan and create K-2 equipartitioning lessons plans in small groups</li> </ul>	<ul style="list-style-type: none"> <li>• Video (whole class)</li> <li>• Audio (small group)</li> </ul>
<p><b>Week 7 – Sessions 15 – 19 (Week of 10.27.08)</b></p> <ul style="list-style-type: none"> <li>• K-2 classroom-based experience (Equipartitioning LT)</li> </ul>	<ul style="list-style-type: none"> <li>• PSTs plan and implement equipartitioning tasks/lessons with students in K-2 classrooms</li> <li>• PSTs analyze implementation and build models of students’ thinking with an equipartitioning LT</li> </ul>	<ul style="list-style-type: none"> <li>• Classroom-based Assignment #2</li> <li>• Observations</li> </ul>
<p><b>Week 8 – Session 20 (11.03.08)</b></p> <ul style="list-style-type: none"> <li>• Individual clinical interview analysis</li> </ul>	<ul style="list-style-type: none"> <li>• PSTs solve equipartitioning tasks and anticipate K-2 students’ thinking</li> <li>• PSTs analyze clinical interview clips conducted by a researcher and build models of students’ thinking without an equipartitioning LT</li> </ul>	<ul style="list-style-type: none"> <li>• Individual Clinical Interview Analysis</li> </ul>
<p><b>Week 8 – Session 21 (11.05.08)</b></p> <ul style="list-style-type: none"> <li>• Post-test administration</li> </ul>	<ul style="list-style-type: none"> <li>• Identify PSTs’ knowledge of equipartitioning and knowledge for teaching equipartitioning</li> </ul>	<ul style="list-style-type: none"> <li>• Post-test</li> </ul>
<p><b>Post-week 8 – Post-session 21</b></p> <ul style="list-style-type: none"> <li>• Clinical interviews</li> </ul>	<ul style="list-style-type: none"> <li>• PSTs plan and conduct equipartitioning clinical interviews with K-6 students</li> <li>• PSTs analyze clinical interviews and build models of students’ thinking with an equipartitioning LT</li> </ul>	<ul style="list-style-type: none"> <li>• Clinical Interview Videos</li> <li>• Clinical Interview Assignment</li> </ul>

### *Design of Instructional Activities*

In this section, the instructional activities are described for supporting the PSTs as they learned to use an equipartitioning learning trajectory to build models of student thinking. For each, I describe what has been previously introduced to the PSTs and make conjectures about what they should have learned relative to my framework for the enactment of the equipartitioning learning trajectory.

*Instructional activities in week 1: Sessions 1 and 2.* During Session 1, the pre-test was administered to obtain a baseline of PSTs' "knowledge of" equipartitioning and "knowledge for" teaching equipartitioning. After the PSTs completed the pre-test, they engaged in the first instructional activity. Its purpose was to: 1) introduce the construct of a learning trajectory to PSTs, and 2) present a learning trajectory as a framework for organizing teachers' instructional activities. The PSTs discussed an article about one Pre-K teacher's use of a learning trajectory in her practice (Brown, Sarama, & Clements, 2007) and focused on the use a learning trajectory as a tool in planning, teaching, and assessing. Students learned 1) components of a learning trajectory based on Clements and Samara's (2004) definition; 2) ways that a learning trajectory helped a Pre-K teacher improve her practice; 3) the role of a learning trajectory in assisting a Pre-K teacher in understanding her students' learning; and, 4) how learning trajectories can be used by classroom teachers as they engage in the practice of teaching. To conclude Session 1, the DELTA definition of a learning trajectory was introduced in relation to a conceptual corridor (Confrey, 2006). Similarities and differences were discussed between the DELTA definition (Confrey et al., 2008) and the definition espoused by Brown et al. (2007), which is based on previous work by Clements and Sarama

(2004). Researcher conjectured advantages of using learning trajectories were presented to PSTs, and the role of both the teacher and student were discussed in this approach to organizing teachers' practice.

In Session 2, PSTs engaged in solving tasks relating to the equipartitioning of discrete objects in a collection (i.e., Case A) and equipartitioning a continuous whole (i.e., Case B). The purpose of this instructional activity was to introduce PSTs to equipartitioning tasks (see Appendix A) in order to increase their own knowledge of equipartitioning and equipartitioning tasks and simultaneously measure PSTs' content knowledge of equipartitioning and their knowledge of how students might solve the same tasks *before* the equipartitioning learning trajectory had been introduced. PSTs were provided various tools to use for solving equipartitioning tasks: counters (Case A), patty paper (Case B), playdough (Case B), and, construction paper and scissors (Case B). In small groups, PSTs were asked to solve the tasks themselves and to anticipate how K-2 students might solve the same tasks. During the subsequent whole class discussion, individual PSTs or pairs of PSTs discussed their solutions with the class.

*Instructional activities in week 2: Sessions 3 – 7.* During the second week of the study, PSTs spent a week in their respective K-2 classrooms and did not attend regularly scheduled class meetings. The elementary education department, prior to this study, had already set these dates. Thus, instructional activities in this study were designed to take advantage of this fieldwork. Most PSTs have limited experience interacting with children in school environments. Therefore the purpose of this instructional activity was for PSTs to 1) increase their knowledge of the range of possible student strategies, behaviors, and

verbalizations of equipartitioning; 2) connect student strategies, behaviors, and verbalizations to student thinking; 3) use student strategies, behaviors, and verbalizations that they observed to build models of students' understanding of equipartitioning. Artifacts collected during fieldwork allowed me to amass evidence about the ways in which PSTs construct models of student thinking *before* they have been fully exposed to an equipartitioning learning trajectory.

While in their classrooms, PSTs first selected equipartitioning tasks (i.e., Cases A and B) that they would implement with K-2 students. Tasks were chosen among those that PSTs had solved during class the previous week (see Appendix A). PSTs were permitted to modify tasks to fit needs of their particular students. PSTs then solved the equipartitioning tasks using their own strategies and made predictions about K-2 students' thinking relating to the same tasks. PSTs also planned how they would implement the tasks with either one K-2 student or a small group of students. After the implementation, PSTs analyzed this experience and built models of students' thinking by responding to questions from Classroom-based Assignment #1 (see Appendix B).

*Instructional activities in week 3: Sessions 8 and 9.* Instructional activities focused on the clinical interview method. The purpose of these instructional activities was to 1) introduce the clinical interview method to PSTs, and 2) present the clinical interview as an approach that teachers can use in the classroom to assess and analyze student thinking. PSTs first discussed an article on using teacher-produced videotapes of student interviews (Jacobs, Ambrose, Clement, & Brown 2006) to assess students' understanding, focusing on:

- components of a problem-solving interview, as described by Jacobs et al. (2006);
- role of the teacher and the role of the student in a problem-solving interview;
- benefits of conducting interviews with individual children;
- effects of providing teachers with opportunities to discuss how they make decisions; and,
- types of prompts as points of discussion for teachers when talking about interviews.

PSTs next received instruction on the origins and characteristics of the clinical interview method, as well as the strengths and weaknesses of the method. Finally, PSTs watched three video exemplars of an eighth grade student engaged in interviews with Dr. Grayson Wheatley on three different rational number reasoning tasks. PSTs were provided with transcripts from the video clips, as well as copies of the student's work from each interview. During this class, PSTs engaged in an analysis of the student's thinking.

At the end of week 3, PSTs discussed several ideas generated from an article by Pothier and Sawada (1990) on equipartitioning tasks in order to increase PSTs' knowledge of equipartitioning tasks for teaching: 1) definition of partitioning (Pothier & Sawada, 1990), which was later compared with Confrey et al.'s (2008) definition of equipartitioning; 2) given a set of equipartitioning tasks, the appropriate sequencing of those tasks with respect to levels of difficulty; 3) benefits and drawbacks of different geometric shapes in equipartitioning tasks (e.g., circle, equilateral triangle, rectangle, parallelogram, etc.); 4)

benefits of using equipartitioning tasks in the classroom; 5) difficulties students encounter when engaged in equipartitioning tasks; and, 6) how different media (i.e., construction paper, playdough, etc.) impact student's understanding when engaged in equipartitioning tasks. Throughout this discussion, PSTs were encouraged to provide specific examples from their first classroom-based experience with K-2 to support their arguments.

In this same session, PSTs also discussed another article (Watanabe, 1996) about a 2<sup>nd</sup> grade student's understanding of one-half to 1) increase PSTs' knowledge of the range of possible student strategies, behaviors, and verbalizations of equipartitioning, particularly relating to one-half; and, 2) connect student strategies, behaviors, and verbalizations to student thinking. This article was also selected because the introduction of the equipartitioning learning trajectory would begin with a 2-split in the next class meeting. The discussion focused on classroom implications for teachers, regarding student's conception of one-half. PSTs also made comparisons between the understanding of the 2<sup>nd</sup> grade student from the article and the knowledge of children that they had worked with in their classroom-based experiences in the previous week. At the conclusion of this session, PSTs discussed their first classroom-based experiences, focusing on the strategies that they observed K-2 students using while engaging in tasks involving a discrete collection and a continuous whole (i.e., Cases A and B). PSTs also discussed what they learned about how students reason and what they found surprising about the first classroom-based experience.

*Instructional activities in week 4: Sessions 10 and 11.* The equipartitioning learning trajectory for Cases A and B were introduced in order to 1) increase PSTs' knowledge of an equipartitioning learning trajectory; 2) support PSTs in utilizing an equipartitioning learning

trajectory as they learned to build models of student thinking for teaching, 3) increase PSTs' knowledge of the range of possible student strategies, behaviors, and verbalizations of equipartitioning; 4) connect student strategies, behaviors, and verbalizations to student thinking; and, 5) use student strategies, behaviors, and verbalizations that they observed to build models of students' understanding of equipartitioning.

Equipartitioning (Confrey et al., 2008) was defined and distinguished from *breaking* or *fracturing*, and examples of the four cases of equipartitioning (i.e., Cases A, B, C, and D) were provided. After presenting Confrey's (2008) conceptual map of rational number reasoning, the most current learning trajectory (i.e., levels 1.1 to 1.8) for equipartitioning was presented. Then, the within-level framework of the learning trajectory (i.e., methods, multiple methods, justification, naming, reversibility, and properties) was discussed.

After this brief overview of the equipartitioning learning trajectory, level 1.1 of the learning trajectory (i.e., equipartitioning using a 2-split for both discrete and continuous quantities) was presented, along with level 1.2 (i.e., dealing discrete items among  $p = 3 - 5$  people, with no remainder;  $mn$  objects,  $n = 3, 4, \text{ or } 5$ ). See Table 3 for all eight levels of the equipartitioning learning trajectory that was presented to PSTs. As the equipartitioning learning trajectory was explicated, I provided video clips of six equipartitioning clinical interviews, conducted by different researchers on the DELTA team, to illustrate examples of the types of student strategies, behaviors, and verbalizations of equipartitioning that teachers would expect to encounter as children engaged in tasks. In small groups, PSTs discussed student strategies, behaviors, and verbalizations that they observed and what this might mean about what the student understood and was followed by a whole class discussion. At the end

of the week, levels 1.3, 1.4, 1.5, and 1.6 (see Table 3) of the equipartitioning learning trajectory were presented in the same approach as with levels 1.1 and 1.2; nine video clips of equipartitioning clinical interviews were shown, discussed, and analyzed for student thinking.

Table 3. DELTA equipartitioning learning trajectory.

Level	
1.8	$m$ objects shared among $p$ people, $m > p$
1.7	$m$ objects shared among $p$ people, $p > m$
1.6	Splitting a continuous whole object into odd # of parts ( $n > 3$ )
1.5	Splitting a continuous whole object among $2n$ people, $n > 2$ , and $2n \neq 2^i$
1.4	Splitting continuous whole objects into three parts
1.3	Splitting continuous whole objects into $2^n$ shares, with $n > 1$
1.2	Dealing discrete items among $p = 3 - 5$ people, with no remainder; $mn$ objects, $n = 3, 4, \text{ or } 5$
1.1	Partitioning using 2-split (continuous and discrete quantities)

*Instructional Activities in Week 5: Sessions 12 and 13.* At the beginning of week 5, PSTs were introduced to equipartitioning tasks for Cases C and D *before* the equipartitioning learning trajectory (i.e., levels 1.7 and 1.8) were introduced. The purpose of this instructional activity was to increase PSTs' own knowledge of equipartitioning and equipartitioning tasks, and to measure PSTs content knowledge of equipartitioning and their knowledge of how students might solve the same tasks *before* the equipartitioning learning trajectory had been introduced. PSTs solved a series of Cases C and D equipartitioning tasks. PSTs were provided with a straight edge, circular construction paper, and scissors for splitting continuous wholes. In small groups, PSTs were asked to solve the tasks themselves and to anticipate how K-6 students might solve the same tasks. During the subsequent whole class discussion, individual PSTs or pairs of PSTs discussed their solutions.

After PSTs engaged in equipartitioning tasks for Cases C and D, the equipartitioning learning trajectory was introduced to PSTs in order to: 1) increase PSTs' knowledge of an equipartitioning learning trajectory; 2) support PSTs in utilizing an equipartitioning learning trajectory as they learned to build models of student thinking for teaching, 3) increase PSTs' knowledge of the range of possible student strategies, behaviors, and verbalizations of equipartitioning; 4) connect student strategies, behaviors, and verbalizations to student thinking; and, 5) use student strategies, behaviors, and verbalizations that they observed to build models of students' understanding of equipartitioning.

Levels 1.7 and 1.8 (see Table 3) of the equipartitioning learning trajectory were presented using the same approach as for Levels 1.1 – 1.6. As the equipartitioning learning trajectory was explained, video clips of equipartitioning clinical interviews were provided to illustrate examples of the types of student strategies, behaviors, and verbalizations of equipartitioning that teachers would expect to encounter as children engaged in equipartitioning tasks for levels 1.7 and 1.8 (i.e., Cases C and D). After watching the clips, small groups discussed student strategies, behaviors, and verbalizations that they observed and what this might mean about what the student understood, which was followed by a whole class discussion.

*Instructional Activities in Weeks 6 and 7: Sessions 14 – 19.* PSTs planned and wrote equipartitioning lessons plans for their K-2 classrooms to: 1) increase their knowledge of the range of possible student strategies, behaviors, and verbalizations of equipartitioning; 2) connect student strategies, behaviors, and verbalizations to student thinking; 3) use student strategies, behaviors, and verbalizations to build models of students' understanding of

equipartitioning as they planned, taught, and assessed. In addition to submitting lesson plans, PSTs were required to provide their own solution for each task in the lesson plan.

During the seventh week of the study, PSTs did not meet for regularly scheduled class meetings spent the entire week in their respective K-2 classrooms. This fieldwork provided opportunities for me to collect evidence about the ways in which PSTs construct models of student thinking *after* PSTs had been exposed to an equipartitioning learning trajectory and allowed me to measure the extent to which PSTs utilized an equipartitioning learning trajectory to build models of student thinking in planning, teaching, and assessing. At this point in the study, PSTs had also had extensive experience with analyzing video exemplars pertaining to equipartitioning tasks to analyze student thinking.

During this weeklong classroom-based experience, PSTs implemented the equipartitioning lesson plans that they planned and wrote in small groups in the previous week. PSTs implemented these equipartitioning lessons with individual children or small groups of children. After PSTs implemented the lessons, they analyzed this experience and built models of students' thinking by responding to questions from Classroom-based Assignment #2 (see Appendix C) and reflecting on their implementation.

*Instructional Activities in Week 8: Sessions 20 and 21.* By the beginning of week 8, PSTs had been fully exposed to the equipartitioning learning trajectory for Cases A through D. Each time PSTs were exposed to the learning trajectory, they engaged in the following instructional activities:

- working with equipartitioning tasks for each case;

- discussing the equipartitioning learning trajectory and where each case is situated;
- viewing video exemplars of the methods, multiple methods, justifications, naming, reversibility, and properties children use while engaging in equipartitioning tasks; and,
- analyzing clinical videos of students engaging in equipartitioning tasks to identify the methods, multiple methods, justifications, naming, reversibility, and properties that children use.

In addition to spending one day of the week in their respective K-2 classrooms while attending regular class meetings in their methods course, PSTs had also spent two full weeks in their K-2 classrooms. PSTs had also been introduced to the method of clinical interview and had engaged in extensive analysis of student thinking from equipartitioning clinical interviews.

During week 8, PSTs engaged in an Individual Clinical Interview Analysis (see Appendix D). The purpose of the individual clinical interview analysis was to collect evidence after PSTs had been fully exposed to the equipartitioning learning trajectory in order to gain insight into the ways in which they construct models of student thinking as they engage in assessing. This took place during a regularly scheduled class meeting; PSTs were given 75 minutes to complete this activity. After PSTs responded to prompts and questions in this activity, they were not allowed to remove or modify previous responses.

PSTs watched three video clips of a K student engaged in five equipartitioning tasks relating to Cases A and B. PSTs were provided with pictures of the student's work. Before

viewing the video clips, PSTs first solved the five equipartitioning tasks that the child solved. Next, PSTs predicted how K-2 students might solve the same tasks. Finally, PSTs watched each clip, one at a time, responding to prompts and questions relating to the K student's strategies, behaviors, and verbalizations.

At the conclusion of the study in week 8, the post-test was administered to PSTs in order to measure PSTs' content and pedagogical content knowledge after all of the instructional activities have been enacted. The post-test was administered during a regular class meeting under the same conditions as the pre-test. PSTs were given 60 minutes to complete the posttest. For the pre-test, half of the PSTs had randomly been assigned to form A and the other half were assigned to form B, a parallel form of A; alternate forms of the assessment were assigned for the post-test.

#### *Study Instruments and Measures*

Instruments and other measures were developed to assess the extent to which PSTs use an equipartitioning learning trajectory and the ways that they use such a tool to build models of student thinking. The following instruments and measures were designed and implemented in this study:

- Pre- and Post-tests;
- Classroom-based Assignment #1;
- Classroom-based Assignment #2;
- Individual Clinical Interview Analysis; and,
- Clinical Interview.

Figure 6 illustrates the placement within the design study of each of the instruments and measures that were created and enacted in this study.

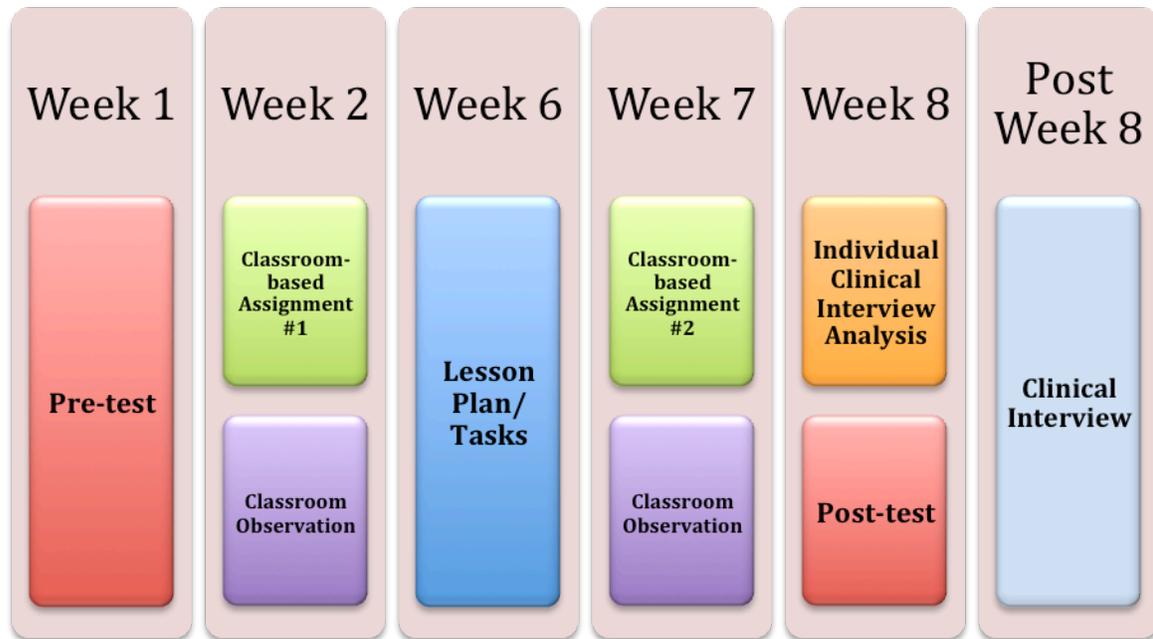


Figure 6. Study Instruments and Measures.

The equipartitioning unit that I created was designed to assess PSTs knowledge and model building processes at different points within the study. PSTs content knowledge and pedagogical content knowledge of equipartitioning was measured both before and after all the instructional activities had been implemented using the pre- and post-test instruments. Classroom-based Assignment #1 was used to assess PSTs model building processes of K-2 children's knowledge of equipartitioning *before* the equipartitioning learning trajectory was introduced, while Classroom-based Assignment #2 was used to assess PSTs model building processes of K-2 children's knowledge of equipartitioning *after* the equipartitioning learning trajectory was presented. The Individual Clinical Interview Analysis was designed to obtain another measure of PSTs model building processes of K-2 children's knowledge of

equipartitioning *after* the equipartitioning learning trajectory had been introduced. Not only had PSTs been exposed to the full equipartitioning learning trajectory, PSTs had engaged in at least two classroom-based experiences with K-2 children and had analyzed numerous clinical interviews of students engaged in equipartitioning tasks. As my final measurement of PSTs' model building processes of K-2 children's knowledge of equipartitioning, PSTs planned and conducted clinical interviews with K-6 students at the conclusion of the study. In Chapter 4, I will discuss the results of the analysis.

#### Methods of Data Collection and Analysis of Data

Data included video and audio recordings of each class meeting, notes of class meetings, field notes of observations made during PSTs' K-2 classroom-based experiences, pre- and post-tests, and artifacts relating to the following: Classroom-based Assignment #1, Classroom-based Assignment #2, individual clinical interview analysis, and, clinical interviews. In addition to artifacts relating to the clinical interview, PSTs also recorded their clinical interviews. These video recordings were also used as data.

#### *Collection of Video and Audio Data*

Each class meeting was video recorded by an undergraduate who operated the digital camcorder from a location in the classroom that captured the whole class. The camera focused on individuals during small group work and whole class discussions. After the first class meeting, I realized that I was not going to capture all the small group discussions using only one camera. Therefore, I began to use ipods to record small group discussions. Throughout the study, the classroom in which PSTs met for regular class meetings was arranged so that PSTs were assembled in small groups. Each group was assigned an ipod.

Each time PSTs engaged in group discussions regarding students' thinking of equipartitioning, discussions were recorded. On-going analysis occurred after class meetings, which included reviewing video recorded class meetings and listening to small group audio in order to determine if the learning goals of the day's lesson had been met. This information was used to determine if plans for the next class meeting would be revised in light of the data. I also took notes while viewing the video data, recording any critical events, so that important small group and/or classroom discussions could be easily identified in the retrospective analysis. Student work was reviewed before the retrospective analysis. The retrospective analysis of the video and audio recordings of class meetings was cumulative. In the retrospective analysis, I focused on the following: 1) evidence that confirmed or disconfirmed the use of an equipartitioning learning trajectory by PSTs to build models of student thinking; and, 2) the ways in which PSTs utilized a learning trajectory for equipartitioning in the model building process. Conjectures and trajectories were recorded and refined based on analysis of the video and audio data, in conjunction with other data.

#### *Analysis of Video and Audio Data*

The Powell, Francisco, & Maher (2003) model for video analysis was used to conduct the retrospective analysis of the video and audio data, which includes the class meetings from the methods course and the clinical interviews. In the first phase of analysis, each of the video recordings of class meetings was viewed. Initial viewing of the video recordings was conducted so that I could become familiar with the data, not to interpret or to make inferences. Then, video recordings were viewed several times so that each meeting could be described. According to Powell et al. (2003), the purpose of viewing video data at the

descriptive phase should be to create a record so that an outside observer would be able to have an objective idea of what occurred in the video. Initial analysis of the small group audio data was conducted in the same way.

Next, critical events were identified (Powell et al., 2003). Verbatim transcripts of critical events were created. The transcripts include all sounds that can be heard. Utterances and noises made by both the PSTs and myself were recorded. Some movements or actions that were visible were also included in the transcripts, along with some representations created by the PSTs or myself. In the second phase of analysis, the video was used as data. Each video was watched many times in order to identify themes and categories, as well as patterns in the data (Strauss & Corbin, 1998). Other data, such as student artifacts, were used in conjunction with the video data. In the third and final phase of video analysis, a storyline and narrative were written (Powell et al., 2003). Merriam (1998) suggests that the end result of qualitative research should be richly descriptive. The storylines and narratives were based on evidence from the video data and other artifacts, such as PSTs work.

#### *Collection and Analysis of Pre- and Post-tests*

The pre- and post-tests were developed by the DELTA research team to assess PSTs content and pedagogical content knowledge of equipartitioning. Many items came from or were modified from Confrey et al.'s (2008) synthesis work or the current DELTA work on equipartitioning and rational number reasoning (Empson & Turner, 2006; Lamon, 1996; Pothier, 1981; Pothier & Sawada, 1989). Instruments were piloted with teachers who were either teaching elementary school at the time or who had previously taught elementary school. The scoring of the piloted instruments informed the revisions of items on the pre- and

post-tests helped determine the amount of time that would be needed for PSTs to complete the assessments.

Rubrics were created to score the assessments so that responses to each item could be categorized into four distinct levels. Initially, these categories were based on the following: 1) student responses from previous studies (Empson & Turner, 2006; Lamon, 1996; Pothier, 1981; Pothier & Sawada, 1989) relating to equipartitioning and rational number reasoning; 2) Confrey's (Confrey & Hotchkiss, 1995) prior work with K-3 children on equipartitioning tasks; and, 3) the DELTA team's current work, including student responses from equipartitioning clinical interviews. Next, several cycles of revisions were made based on my on-going collaboration with another doctoral candidate who used the same pre- and post-test instruments and rubrics in his investigation of how in-service teachers use learning trajectories in their practice (Wilson, 2009). During these cycles of revision, we scored pre-service and in-service pre- and post-tests together. In cases where responses could not be categorized, the rubrics were refined. At the beginning of this process, Wilson scored a small sample of my data, and my scores were compared to his in order to determine if using the rubrics were producing the same scores. Throughout the entire process, when I was unsure about how to score PSTs' responses to items, those responses were scored by Wilson and a score was determined once we reached a consensus. Each PSTs' pre- and post-tests were also scored by an independent researcher using the rubrics. The other scorer is a doctoral candidate who is independent of the DELTA research team. Inter-rater reliability (IRR) was established for scoring the pre- and post-tests between the independent researcher and myself. The IRR was 81%.

After each item was scored, a composite score was determined for each PSTs pre- and post-tests. Gain scores were also determined for each PST. Pre- and post-test data were paired by PSTs and analyzed for differences using the Wilcoxon signed rank test. The Wilcoxon signed rank test was also used to determine if there were significant difference in PSTs gain scores in content knowledge and pedagogical content knowledge of equipartitioning. Since the composite and gain scores were measurements made on an ordinal scale and could not be regarded as samples from a normally distributed population, nonparametric methods were utilized to determine whether or not there were significant differences in PSTs gain scores (Rao, 1998). Descriptive statistics such as the median and standard deviation were determined to describe the distributions of the data.

In addition to scoring pre- and post-test data to determine if there was a significant difference in PSTs content and pedagogical content knowledge of equipartitioning, PSTs' responses were analyzed for evidence of the following: 1) PSTs use of an equipartitioning learning trajectory to build models of student thinking; and, 2) the ways in which PSTs utilized a learning trajectory for equipartitioning in the model building process.

#### *Collection and Analysis of Other Data*

Data collection was designed so that evidence could be collected pertaining to PSTs' classroom-based experiences as students who were learning to use an equipartitioning learning trajectory to teach. Additionally, data collection was designed to purposefully take advantage of students' experiences in K-2 classrooms to gather evidence about how they engaged in teaching and assessing students while using an equipartitioning learning trajectory. Several key types of data were collected in addition to the data that has already

been described: Classroom-based Assignment #1, Classroom-based Assignment #2, Individual Clinical Interview Analysis, and, Clinical Interviews. These data were collected at different places throughout the study so that information about the processes used for model building could be compared at different locations in PSTs' development. Since instructional activities relating to Cases C and D were limited because Cases A and B are more appropriate for K-2 instruction, the analysis of data relating to these cases will also be limited in the analysis of these artifacts.

In order to determine the extent to which and the ways in which PSTs utilized a learning trajectory for equipartitioning in the model building process, I analyzed artifacts that were collected relating to PSTs' Classroom-based Assignment #1, Classroom-based Assignment #2, Individual Clinical Interview Analysis, and, Clinical Interviews. When I initially examined PSTs' Classroom-based Assignment #1 artifacts, I coded PSTs written responses to identify their model building processes: describing, comparing, inferring, and restructuring. After data were coded to identify the model building process, all of the data identified as describing was examined. Based on patterns and themes within data identified as describing, secondary codes were created. The same procedure for analysis was used for each model building process. When findings are presented in Chapter 4, the findings will be organized by these secondary codes, where the model building processes (e.g., describing, comparing, inferring, restructuring) will be highlighted within the categories explicating findings relating to the secondary codes. As these written responses were analyzed line by line, I identified PSTs' arguments within their responses, and coded each argument as described above. Most of the data was therefore coded by a paragraph or multiple

paragraphs, although words or sentences were sometimes coded. Classroom-based Assignment #2, the individual clinical interview analysis, and the clinical interviews were initially coded in the same way. An important part of analyzing subsequent data was comparing each PSTs previous model building processes to the process being analyzed. Thus, Classroom-based Assignment #2 had to be analyzed in light of Classroom-based Assignment #1, and the individual clinical interview analysis had to be analyzed with respect to Classroom-based Assignment #1 and #2. Notes were taken to describe this development and identify changes in modeling building processes, such as restructuring.

To determine how PSTs utilized a learning trajectory for equipartitioning in the model building process, I did a second level of analysis. After all of the artifacts were coded, I analyzed each artifact in order to identify themes and categories within each behavior, as well as looked for patterns in the data, using a constant comparative approach (Strauss & Corbin, 1998). Data within each category were constantly compared. Each category was described and all the occurrences were noted, although I continually searched for new occurrences. Finally, I sought to identify the relationship.

### *Validity and Reliability*

Several steps were taken to address the reliability and validity of the data. In order to address reliability, inter-rater reliability was established for scoring the pre- and post-test (Morse, Swanson, & Kuzel, 2001) data. To contribute to the validity of the study, triangulation strategies, such as using multiple researchers (Merriam, 2002), were utilized. According to Confrey and Lachance (2000), multiple sources of data should also be collected during a design study. Cobb (2000) suggests collecting the following sources of data: video

recordings of class meetings, artifacts (e.g. student work), and researchers' field notes. In this study, I collected multiple sources of data, which included the following: pre- and post-test assessments, video recordings of class meetings, audio recordings of small group discussions during class meetings, video recordings of PSTs engaged in clinical interviews, artifacts (e.g. PSTs' written responses) relating to interactions with children engaging in equipartitioning tasks, observations of PSTs engaged in their interactions with children, and my own notes. Comparing results across multiple sources of data contributed to confirming and disconfirming the findings in the study.

To address content validity, items created for the pre- and post-tests were adapted from Confrey et al.'s (2008) synthesis work or the current DELTA work on equipartitioning and rational number reasoning (Empson & Turner, 2006; Lamon, 1996; Pothier, 1981; Pothier & Sawada, 1989) to ensure that items were assessing knowledge of equipartitioning. Confrey, who is an expert in the field of rational number reasoning, along with her DELTA colleagues, reviewed items for the pre- and post-tests and revised and selected items that would provide the most information about PSTs knowledge of equipartitioning and the teaching of equipartitioning. Another way in which I ensured content validity in this study is by choosing equipartitioning tasks that were grounded in Confrey's on-going work in rational number reasoning. Prior to the study, these tasks had also been piloted in clinical interviews with elementary and middle school students by the DELTA research team. Confrey (Confrey & Hotchkiss, 1995) has also used these tasks with elementary school students in her previous teaching experiments.

## Summary

In this chapter, I discussed the conceptual framework, my theoretical perspectives, the study's design, and the methods of data collection and analysis. In this study, the following research question was explored: To what extent and in what ways can pre-service elementary teachers use a learning trajectory for equipartitioning to build models of student thinking? In order to investigate the process of PSTs' learning to use an equipartitioning learning trajectory in model building, I utilized the methods of a design study to gain insight into PSTs development. Since the DELTA definition of a learning trajectory is built in part on constructivist theory, using a learning trajectories approach relied heavily on the use of clinical interviews in instructional activities in this design study. Multiple sources of data were collected in order to understand how PSTs used an equipartitioning learning trajectory: pre- and post-test assessments, video recordings of class meetings, audio recordings of small group discussions during class meetings, video recordings of PSTs engaged in clinical interviews, artifacts (i.e., Classroom-based Assignment #1, Classroom-based Assignment #2, individual clinical interview analysis, and, clinical interviews) relating to interactions with children engaging in equipartitioning tasks, observations of PSTs engaged in their interactions with children, and my own notes. The results of the data analysis will be presented in Chapter 4.

## CHAPTER 4

### RESULTS OF THE DESIGN STUDY

In Chapter 3, the theory and methodology were presented describing how a design study was conducted to explore the extent to which and ways in which PSTs used an equipartitioning learning trajectory to build models of student thinking during eight weeks of a semester long elementary mathematics methods course. Instructional activities were designed to support the following goals: 1) increase PSTs' own knowledge of equipartitioning and equipartitioning tasks; 2) increase PSTs' knowledge of an equipartitioning learning trajectory; and, 3) support PSTs in utilizing an equipartitioning learning trajectory as they learned to build models of student thinking for teaching. An outline of the activities was presented in Chapter 3 in Table 2 on pages 61 – 62. In Chapter 4, results of the data analysis are presented, followed by a discussion of the findings and how initial conjectures were refined as a result of this analysis.

#### Analysis of Pre- and Post-test Data

The pre-test (see Appendices F and G) was administered prior to the study, and the post-test (see Appendices F and G) was completed at the conclusion of the study. PSTs were randomly assigned to one of two parallel test forms for each assessment. Both assessments were administered during regular class meetings, where PSTs were allotted 60 minutes to complete each assessment and were provided with blank paper. Pre- and post-test data were simultaneously scored after the study had concluded. In order to establish inter-rater reliability (IRR), each PSTs' pre- and post-tests were scored by a researcher independent of the DELTA research team, in addition to myself. The IRR was 81%. As discussed in Chapter

3, nonparametric methods were used in the analysis since composite and gain scores were measurements made on an ordinal scale and could not be regarded as samples from a normally distributed population (Rao, 1998). Additionally, parametric tests were conducted.

#### *Parallel Test Forms*

For the pre-test, half of the PSTs were randomly assigned to form A and the other half were assigned to form B, a parallel form of A; alternate forms of the assessment were assigned for the post-test. In order to confirm that the forms were parallel, a Kruskal-Wallis test was conducted after pre- and post-tests were scored. The Kruskal-Wallis test does not assume that the data are from a normally distributed population. The assumptions of the Kruskal-Wallis test are as follows (Hollander & Wolfe, 1999): i) the observations are independent and identically distributed; ii) the observations are mutually independent; and, iii) the measurement scale should be at least ordinal.

Using the Kruskal-Wallis test, the null hypothesis that the pre-test composite scores for forms A and B are from populations with identical locations (i.e., the mean ranks of pre-test scores for forms A and B are the same) was not rejected ( $H = 1.458$ ,  $df = 1$ ,  $p = 0.227$ ). Likewise, the null hypothesis that the post-test composite scores for forms A and B are from populations with identical locations (i.e., the mean ranks of post-test scores for forms A and B are the same) was not rejected ( $H = 0.204$ ,  $df = 1$ ,  $p = .652$ ). Since gain scores were also computed, a Kruskal-Wallis was conducted on gain scores, as well. The null hypothesis that the gains scores for forms A and B are from populations with identical locations (i.e., the mean ranks of gain scores for forms A and B are the same) was not rejected ( $H = 0.78$ ,  $df =$

1,  $p = 0.375$ ). Therefore, it can be concluded that PSTs' scores on forms A and B did not differ significantly, and the forms are parallel.

*Composite and Gain Scores*

To determine if data from both sections could be combined for further analysis, a Kruskal-Wallis test was conducted to resolve whether the two sections are from identical populations. When the distributions of different samples have dramatically different shapes, the Kruskal-Wallis test leads to inaccurate results. Thus, the shape of the distributions of the two sections of the methods course was compared. Based on the graphical displays in Figures 7 and 8, along with the descriptive statistics, it was concluded that the distributions of the pre-test composite scores from sections 1 and 2, as well as the post-test composite scores from both sections, are the same. Therefore, the Kruskal-Wallis test is an appropriate test to use to determine if the two sections can be treated as one set of data.

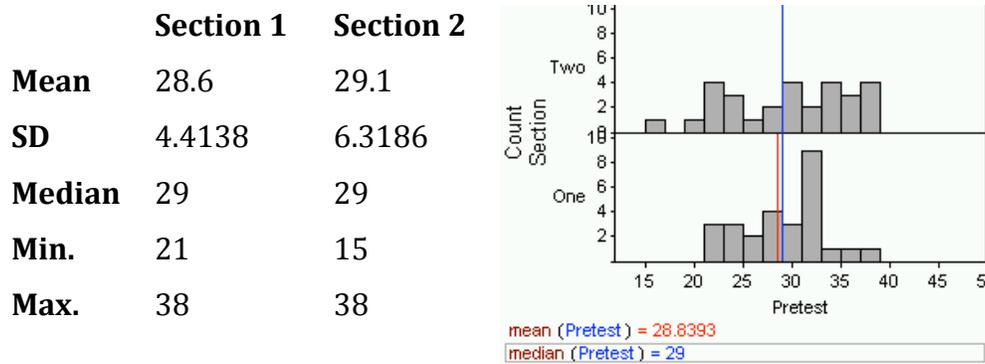


Figure 7. Pre-test composite scores (n = 56) for sections 1 and 2.

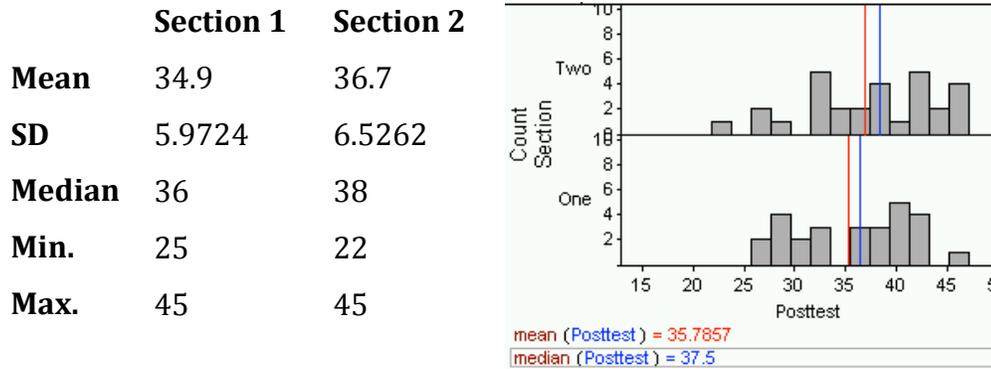


Figure 8. Post-test composite scores (n = 56) for sections 1 and 2.

Using the Kruskal-Wallis test, the null hypothesis that the pre-test composite scores for sections 1 and 2 come from populations with identical locations (i.e., the mean ranks of pre-test scores for sections 1 and 2 are the same) was not rejected ( $H = 0.38$ ,  $df = 1$ ,  $p = 0.538$ ). Similarly, the null hypothesis that the post-test composite scores for sections 1 and 2 come from populations with identical locations (i.e., the mean ranks of post-test scores for sections 1 and 2 are the same) was also not rejected ( $H = 1.23$ ,  $df = 1$ ,  $p = .267$ ). Since gain scores were also computed, a Kruskal-Wallis was conducted on gain scores, as well. The null hypothesis that the gains scores for sections 1 and 2 come from populations with identical locations (i.e., the mean ranks of gain scores for sections 1 and 2 are the same) was not rejected ( $H = 0.38$ ,  $df = 1$ ,  $p = 0.538$ ). Thus, sections 1 and 2 were analyzed as one set of data.

Items from both assessments were scored using rubrics (see Appendices H and I) so that responses for each item could be categorized into four distinct levels (i.e., 3, 2, 1, or 0). Responses classified at the highest level received 3 points; whereas, the lowest level responses were categorized as 0 points. The pre- and post-tests each consisted of 18 items, 12 items assessed content knowledge of equipartitioning and rational number reasoning (i.e.,

items 1 – 12) and 6 items measured pedagogical content knowledge of equipartitioning (i.e., items 13 – 18). Thus, the maximum composite score possible was a 54, while the maximum score in content knowledge and pedagogical content knowledge of equipartitioning were 36 and 18, respectively. Descriptive statistics of PSTs’ composite scores and gain scores were calculated and are described in Table 4. Figure 9 illustrates PSTs’ gain scores in their knowledge of equipartitioning.

Table 4. Descriptive Statistics of PSTs’ Composite and Gain Scores.

	<b>Pre-test</b>	<b>Post-test</b>	<b>Gains</b>
<b>Mean</b>	28.8	35.8	6.9
<b>Standard Deviation</b>	5.4398	6.2747	5.7316
<b>Median</b>	29	37.5	8
<b>IQR</b>	8.5	10	6.5
<b>Range</b>	23	24	28
<b>Minimum</b>	15	22	-7
<b>Maximum</b>	38	46	21

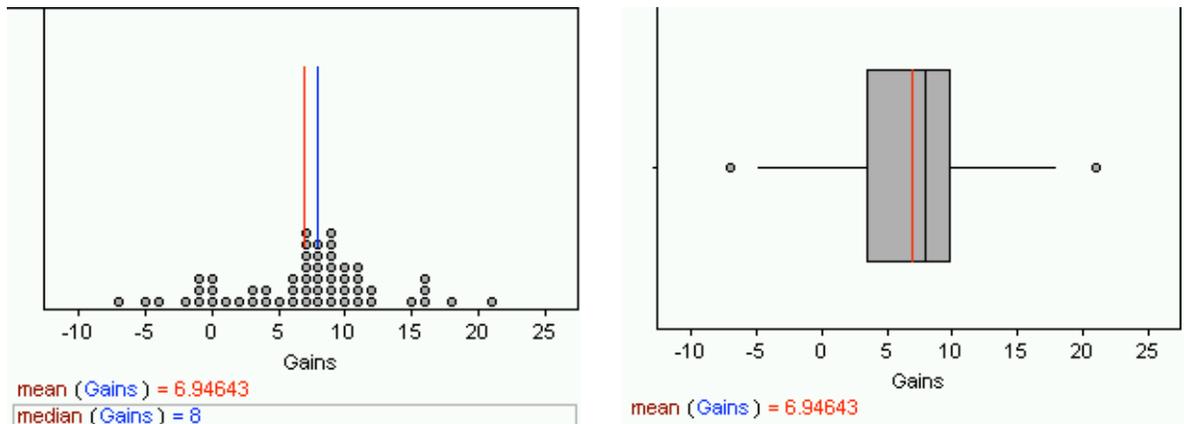


Figure 9. PSTs’ gain scores (n = 56).

Pre-test scores ranged from 15 to 38, and post-test scores ranged from 22 to 46. The median gain score for PSTs was 8, which on average corresponds to a gain of almost 3 more of the highest level responses. Since PSTs could receive either 0, 1, 2, or 3 points for each item, the average gain of 8 points could have other interpretations. For instance, PSTs on

average could have gained 1 more point on 8 items, 2 more points on 4 items, or some other combination.

In order to investigate PSTs’ content knowledge and pedagogical content knowledge of equipartitioning, composite scores and gain scores were also determined (see Figure 10). PSTs’ content knowledge gains ranged from a loss of 6 points to a gain of 12 points, while pedagogical content knowledge ranged from a loss of 4 points to a gain in 12 points. The median gain score for both PSTs’ content and pedagogical content knowledge was 4 points. This could equate to an average gain of a little more than 1 highest level response on the post-test or some other combination of 3, 2, 1, and 0 points.

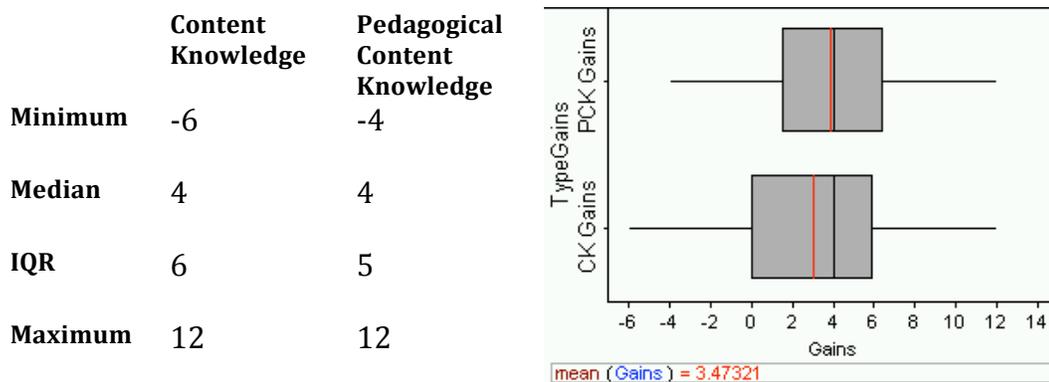


Figure 10. PSTs’ gain scores (n = 56) in content and pedagogical content knowledge.

### *Changes in PSTs’ Equipartitioning Knowledge*

*Pre- and post-test differences.* To determine whether PSTs’ knowledge of equipartitioning had changed significantly, the pre- and post-test scores were paired and a Wilcoxon signed-rank test was conducted. The Wilcoxon signed-rank test, a nonparametric test, is analogous to the paired t-test but does not assume observations are from a normally distributed population. The null hypothesis that the median difference between the paired

pre- and post-test composite scores is zero was rejected ( $W = 56, p < 0.0001$ ); a median gain score of 8 points is significantly greater than zero.

When items assessing content and pedagogy were analyzed, the median gain score was 4 points. The null hypothesis that the median difference between the paired pre- and post-test composite scores in content knowledge ( $W = 226, p < 0.0001$ ) and pedagogical content knowledge ( $W = 90.5, p < 0.0001$ ) is zero was rejected. Both median gain scores of 4 points in content knowledge and pedagogical content knowledge were found to be significantly different than zero.

In addition to conducting a Wilcoxon signed-rank test, a paired t-test was used to determine whether PSTs' knowledge of equipartitioning had changed significantly and if there were significant changes in PSTs' content knowledge and pedagogical content knowledge. The assumptions of the t-test are as follows (Rao, 1998): i) the populations have normal distributions; ii) the population variances are equal but unknown; and, iii) the data are independent random samples from the populations. The null hypothesis for the test was that the mean difference between the paired pre- and post-test scores is zero. In the case of the composite scores, content knowledge scores, and pedagogical content knowledge scores, the null hypothesis was rejected, indicating that post-test scores were significantly higher than pre-test scores in all three instances. The test statistics, degrees of freedom, and p-values are reported in Table 5 below.

Table 5. PSTs' Composite, Content, and Pedagogical Content Knowledge Scores ( $n = 56$ ).

	<b>t Test Statistic</b>	<b>df</b>	<b>p-value</b>
Composite Scores	$t = 9.0694$	55	$p < 0.0001$
Content Knowledge Scores	$t = 5.2446$	55	$p < 0.0001$
Pedagogical Content Knowledge Scores	$t = 8.1535$	55	$p < 0.0001$

While instructional activities in this design study included activities related to increasing PSTs knowledge of Cases A – D, most of the instructional activities centered on Cases A and B since these are the most appropriate concepts for K – 2 children. After analyzing the pre- and post-test data to determine if PSTs’ knowledge had increased significantly, the gain scores were analyzed to determine the size of gains in relation to what PSTs know about the following: i) Cases A and B only; ii) Cases A – D; and, iii) other rational number reasoning concepts. Items in the third category related to one or two of the equipartitioning cases but required PSTs to extend their knowledge beyond equipartitioning to other rational number reasoning concepts. Table 6 indicates how each item was classified and provides an example of each type of item.

Table 6. Item Classification.

Type of Item	Items	Example
Case A/B Items	1, 2, 3, 4, 6, 7, 13, 14, 16, 18	<b>Item 16:</b> Draw a picture of how a student might respond to the following task given the indicated understanding. Provide an explanation: Sharing a round cookie among 3 people if the <i>focus is on the number of pieces</i> .
Case C/D Items	10, 11, 15, 17	<b>Item 17:</b> Circle the task that you anticipate would to be <i>more difficult</i> for K – 2 students. Explain your reasoning. “Sharing 5 cookies between 2 children,” or “Sharing 2 cookies among 5 people.”
Other RNR Items	5 (C/D), 8 (B), 9 (B), 12 (C/D)	<b>Item 8:</b> Mustafa folded a square piece of paper and created 12 equal parts. Describe in steps, in as many ways as you can, how he folded the paper ( adapted from Empson & Turner, 2006).

The percentage of gains in Case A and B items in comparison to the total gains was determined for each PST. The percentage of gains in items relating to Cases A – D to the total gain scores was also determined, as well as the percentage of items relating to other

rational number reasoning concepts to the total gains. As evidenced in Figure 11, most of the gains made by PSTs resulted in gains in knowledge relating to Cases A and B. For 41% of the PSTs, 60% or more of their total gains can be attributed to increased knowledge relating to Cases A and B. Moreover, 18% of PSTs only increased their knowledge in relation to Case A and B items.

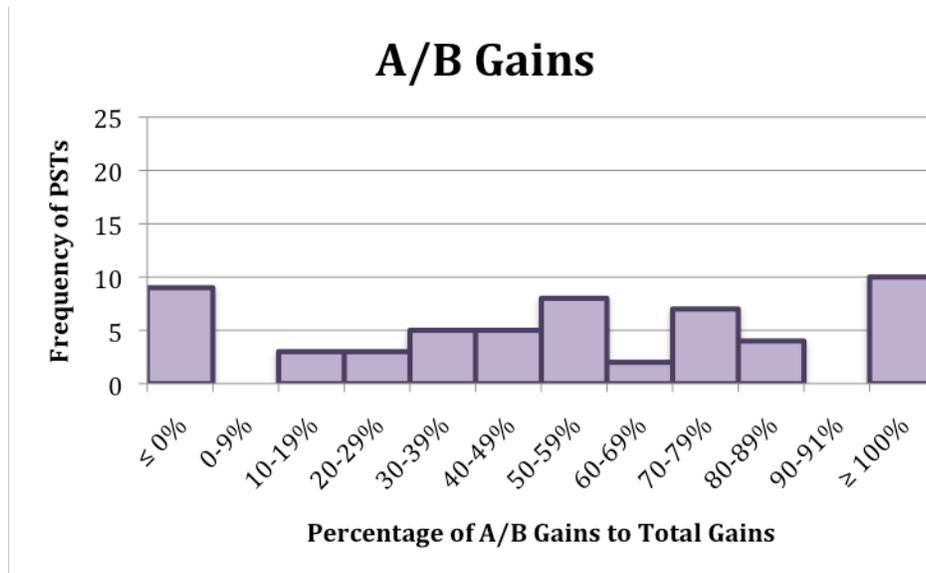


Figure 11. Percentage of PSTs’ Case A and B gains to total gains (n = 56).

Very little of the gains in PSTs’ knowledge can be explained by increased knowledge in relation to Cases C and D. As illustrated in Figure 12, 41% of the PSTs either gained no knowledge relating to Case C and D items or their knowledge actually decreased.

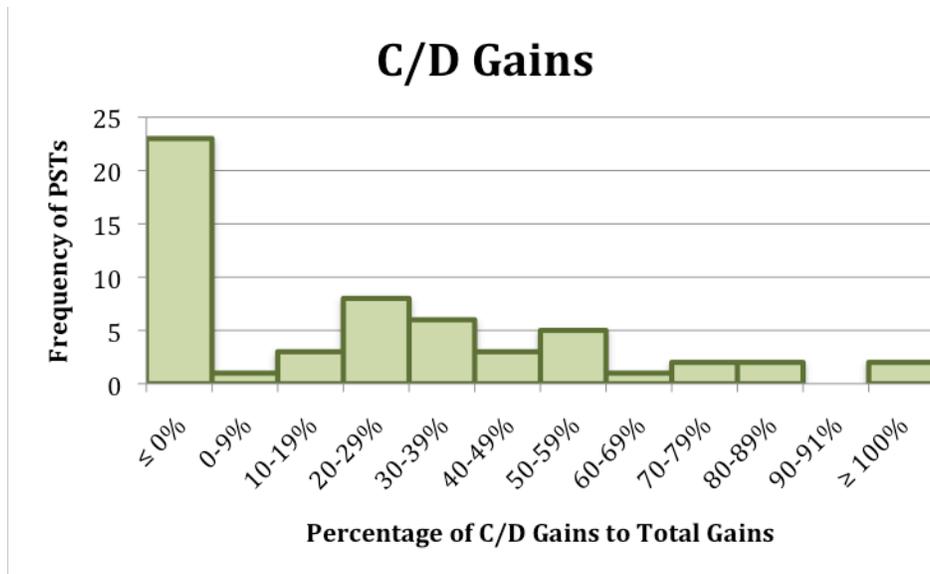


Figure 12. Percentage of PSTs’ Case C and D gains to total gains (n = 56).

Furthermore, it is apparent that almost all of the gains in PSTs’ knowledge can be explained by increased knowledge relating to Cases A – D. As indicated in Figure 13, for 80% of PSTs, 60% or more of PSTs’ gains can be explained by increases in knowledge relating to Cases A – D items. For 18% of the PSTs, all of their gains could be explained by increased knowledge in relation to Cases A – D. Similarly, 23% of the PSTs gained no knowledge in items that extended into other rational number reasoning concepts, as seen in Figure 14. Moreover, 84% of the PSTs gained less than 40% of their total in relation to other rational number reasoning concepts.

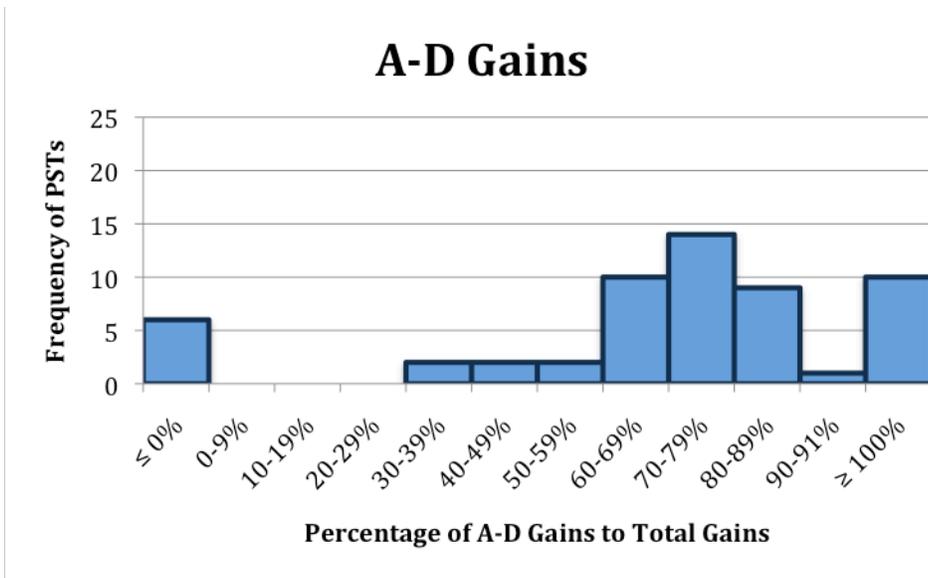


Figure 13. Percentage of PSTs' Case A – D gains to total gains (n = 56).

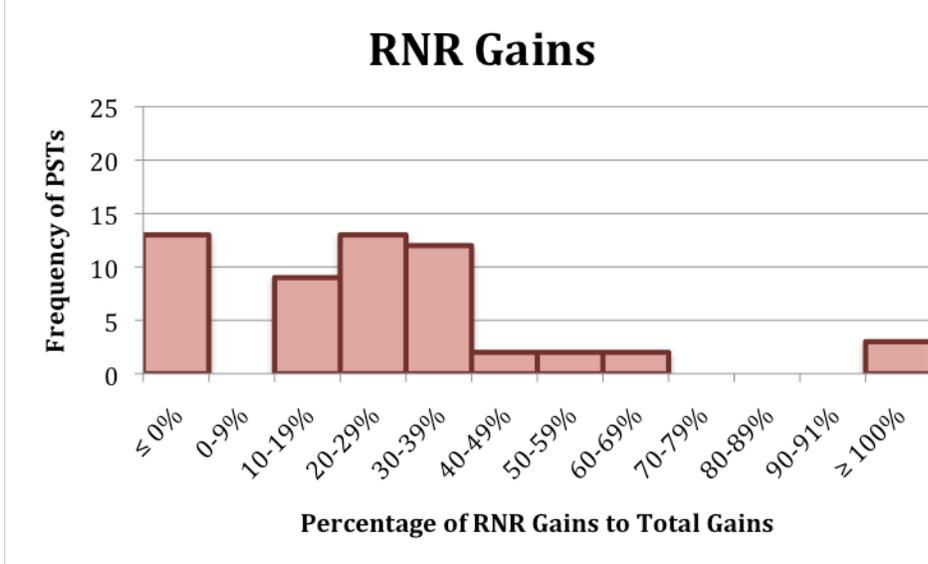


Figure 14. Percentage of PSTs' RNR gains to total gains (n = 56).

While an analysis of pre- and post-test data determined that PSTs' understanding of equipartitioning increased significantly, this information only provided partial insights into PSTs' growth in relation to learning about equipartitioning and the knowledge for teaching equipartitioning. In order to investigate the extent and ways in which PSTs can use a learning

trajectory for equipartitioning to build models of student thinking, further analysis was conducted on selected items to assess content knowledge of equipartitioning (i.e., Item 8) and pedagogical content knowledge (i.e., Item 16). Item 8 was selected because it extended PSTs' knowledge of equipartitioning into rational number reasoning more than any other content item. Likewise, Item 16 was chosen because it extended PSTs' knowledge for teaching more than any other pedagogical item. Each item and scoring rubric will be described, followed by a report on the frequencies of responses and representative examples illustrating PSTs' different levels of understanding.

*Item 8.* Item 8 assessed PSTs' own content knowledge of equipartitioning, which was adapted from Empson and Turner (2006) for forms A and B:

- Mustafa folded a square piece of paper and created 12 equal parts.  
Describe in steps, in as many ways as you can, how he folded the paper.
- Mustafa folded a square piece of paper and created 18 equal parts.  
Describe in steps, in as many ways as you can, how he folded the paper.

Level 3 responses for Item 8 listed four distinct methods for folding the paper to create either 12 (i.e., form A) or 18 (i.e., form B) equal-sized parts. Responses categorized into Level 2 describe three distinct methods, whereas Level 1 responses listed one or two methods for folding the paper. Level 0 responses included one of the following: a) the method of creating  $n - 1$  (i.e., 11 or 17) parallel folds, where  $n$  is the number of folds; b) an incorrect method; or, c) no attempt. Distinct methods for creating 12 equal-sized parts (form A) include some permutation of folding the paper in the following ways: i) half, half, third; ii) fourth, third; iii) half, sixth; and, iv) 11 parallel folds or a permutation involving a 12<sup>th</sup>.

Methods for creating 18 equal-sized parts (form B) include some permutation of folding the paper in the following ways: i) half, third, third; ii) sixth, third; iii) half, ninth; and, iv) 17 parallel folds or a permutation involving an 18<sup>th</sup>. The percentage of PSTs' responses that correspond with each of the four levels is located in Table 7.

Table 7. PSTs' Pre- and Post-test Responses to Item 8 (Forms A and B).

<b>Level</b>	<b>Pre-test</b>	<b>Post-test</b>
<b>3</b>	0%	4%
<b>2</b>	1%	5%
<b>1</b>	13%	55%
<b>0</b>	86%	36%

On the pre- and post-tests, the majority of PSTs were unable to describe more than one method for folding the paper to obtain the correct number of equal-sized regions. The most typical response for Item 8 (form A) was some permutation of folding into a half, a half, and a third; the most common response for Item 8 (form B) was some permutation of folding into a half, a third, and a third. On both assessments, PSTs who did not provide the above strategies often responded with an unproductive strategy that involved folding in half or repeated folding in halves. Almost one-third of the PSTs (32%) did not attempt to respond to Item 8 on the pre-test, while 11% of the PSTs did not attempt Item 8 on the post-test. Very few PSTs utilized the strategy of creating  $n - 1$  (i.e., 11 or 17) parallel folds; however, more PSTs used this strategy on the post-test than the pre-test, sometimes in conjunction with other methods. Although very few PSTs were able to list three or more distinct strategies for folding a piece of paper to create a given number of equal-sized parts, 4% of the PSTs showed a better understanding on the post-test. Almost two-thirds of the PSTs (64%) were able to describe one or two strategies on the post-test, while 86% of the PSTs could not

elucidate even one strategy for folding a piece of paper to create a given number of equal-sized parts on the pre-test. Of all the items from the pre- and post-tests, PSTs performed the poorest on Item 8, scoring significantly lower than any other item. This particular item, more than any other, extends PSTs' knowledge of equipartitioning to other types of rational number reasoning because it involves understanding the multiplicative relationship between the number of folds and the types of folds in relation to the number of equal-sized parts that are produced (i.e., folding in half creates 2 times as many equal-sized parts). PSTs who inappropriately use additive reasoning will be unsuccessful.

*Item 16.* Equipartitioning a continuous object (i.e., Case B) involves coordinating three components: i) using the entire whole; ii) creating equal-sized parts of the whole; and, iii) creating the appropriate number of equal-sized parts of the whole (Confrey, 2008). Item 16 required PSTs to anticipate responses for students who only focus one of these components (form A: component iii; form B: component ii). This item required PSTs to consider how the other two components could be varied to produce an incorrect or incomplete result. Item 16, forms A & B, are stated below:

- Draw a picture of how a student might respond to the following task given the indicated understanding. Provide an explanation: Sharing a round cookie among 3 people if the *focus is on the number of pieces*.
- Draw a picture of how a student might respond to the following task given the indicated understanding. Provide an explanation: Sharing a (rectangular) pan of brownies among 5 people if the *focus is on the size of the pieces*.

Current work by the DELTA research team and earlier work by Confrey (2008) have identified distinct behaviors of students who cannot coordinate creating both equal-sized parts and the appropriate number of equal-sized parts when splitting a continuous whole. The rubric for scoring responses to Item 16 can be found in Appendix H and Table I below.

Table 8. Rubric for Item 16 (Forms A and B).

<b>Level</b>	<b>Response Characteristics (form A)</b>	<b>Response Characteristics (form B)</b>
<b>3</b>	Correct number of pieces (with either unequal-sized pieces, or not using the whole) with complete explanation	Equal-sized pieces (with either the incorrect number of pieces, or not using the whole) with complete explanation
<b>2</b>	Same as Level 3 with incomplete explanation	Same as Level 3 with incomplete explanation
<b>1</b>	Same as Level 3 with no explanation OR Correct number of equal-sized pieces that uses whole with complete explanation	Same as Level 3 with no explanation OR Correct number of equal-sized pieces that uses whole with complete explanation
<b>0</b>	Correct number of equal sized pieces that uses whole with incomplete, unreasonable, or no explanation OR Incorrect response or no attempt	Correct number of equal-sized pieces that uses whole with incomplete, unreasonable, or no explanation OR Incorrect response or no attempt

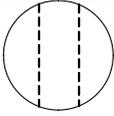
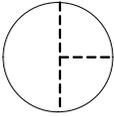
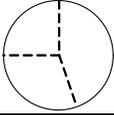
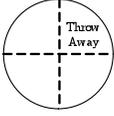
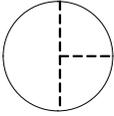
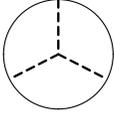
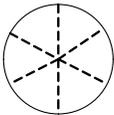
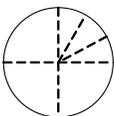
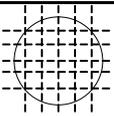
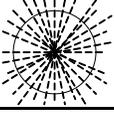
The percentage of PSTs’ responses in each of the four levels is located in Table 9. On the post-test, more PSTs provided Level 3 and 2 responses, 45% in comparison to 16% on the pre-test. Fewer PSTs provided Level 0 responses on the post-test (21%) than on the pre-test (46%).

Table 9. PSTs’ Pre- and Post-test Responses to Item 16 (Form A).

<b>Level</b>	<b>Pre-test</b>	<b>Post-test</b>
<b>3</b>	11%	27%
<b>2</b>	5%	18%
<b>1</b>	38%	34%
<b>0</b>	46%	21%

Examples of PSTs' representative responses to Item 16 can be found in Tables 10 and 12, respectively and were selected to illustrate how various types of predicted student strategies and explanations were coded. These responses are PSTs' prediction of student strategies for that task, not necessarily their own strategy.

Table 10. Representative PSTs' Responses to Item 16 (Form A).

Level	Strategy	Explanations
3		PST1: They will each have 1 piece, even if they are not equal.
		PST2: Kids can visually see making straight cuts that don't overlap – for some reason this comes more natural to them. When size is not important, they will likely do this before they will see making pieces equal in size.
		PST3: If they have not worked with circle 3rds before, it would be hard for them to come up with how to create 3 equal parts, besides how they already know, which is basic cut straight down.
		PST 4: If the focus was on the number of pieces and not fair sizes, the student may make an easy split then just get 2 pieces from one side.
		PST 5: I think most students don't know thirds, so they will cut the cookie in half and then one of those halves in half. It just says 3 pieces not 3 equal pieces.
		PST 6: They will just try to make 3 pieces but not worry about equality.
2		PST7: Throw away piece would be an extra and not used. Only pay attention to the other 3.
		PST8: I think they would attempt to 1 <sup>st</sup> divide the cake in half, and then in half again.
		PST9: If the focus was on number of pieces and not fair shares, the student may make an easy split then just get 2 pieces from one side.
1		PST10: They will understand that they need 3 equal pieces for 3 people.
		PST11: Peace signs is something a child has already seen before and probably knows that there is already 3 even sections.
		PST12: Each receives 1/3 of the pie.
		PST13: The cookie is divided into 3 equal pieces. PST15: Divide the cookie into 1/6's and everyone would get 2 pieces.
		PST14: They could each have 1 1/3 piece if divided into 4 pieces, or they could have four pieces if it is divided into 12, 2 if divided into 6, or 1 if divided into 3.
0		PST15: Each child would get a quarter and 1/12 of the cookie.
		PST16: Simply cut into as many pieces as possible – not even knowing how many. PST17: The student would try to cut the cake into as many pieces of cake as possible.

One common Level 3 response can be characterized in the following way: PSTs indicated that students might create 2 parallel cuts that result in 3 unequal-sized pieces of the cookie (e.g., PST1), explaining that this student strategy would result in creating the appropriate number of pieces that would not be the same size. Another Level 3 response by PSTs suggested students might split the cookie in half, and then half one of the halves, using the same explanation as above (e.g., PST4). Another Level 3 response by PSTs is that a student would use repeated halving to create four equal-sized parts, and then throw away one of the pieces (e.g., PST7). All Level 3 responses provide evidence that these PSTs were able to identify strategies indicative of the common misconceptions that students encounter. PSTs providing a Level 2 response indicated the same strategies as described in Level 3; however, their explanations were incomplete in that they did not explicitly state the child intended to create the appropriate number of equal-sized pieces.

A common Level 1 response suggested a student might use the entire cookie by creating 3 radial cuts, coordinating equal-sized pieces, appropriate number of pieces, and using the entire cookie (e.g., PST10). Even though Item 16 asks PSTs to provide a strategy for students who *focus on the number of appropriate pieces*, these PSTs presumably assumed the student would focus on creating equal-sized pieces, as well. While all of these PSTs were able to identify landmarks that students encounter, there is not evidence that they were able to identify misconceptions on this particular item.

The appropriateness of scoring correct predicted responses (i.e., correct number of equal-sized pieces that use the whole) on Item 16 of the pre-test as a Level 1 response may be

too strict. While these responses meet the criteria for the question, they over-define it.

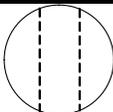
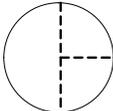
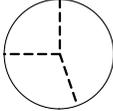
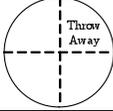
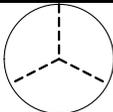
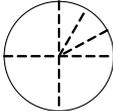
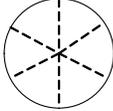
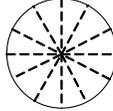
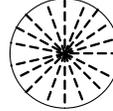
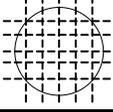
However, it is appropriate for the post-test because the language used in the item was made clear in the context of instruction. These items will be modified in the future to clarify that PSTs should *only focus on the number of pieces or size of the pieces*.

The Level 0 responses provided by PSTs did not show a student strategy that would result in creating the appropriate number of pieces. Almost all of these responses indicated that a student would create as many pieces as possible by making many cuts. For example PST16 broke the cookie into 32 pieces, which is not divisible by 3 without a remainder.

PSTs responses were also examined for strategy use only (see Table 11). PSTs showed a greater awareness of predicting the types of strategies indicative of misconceptions relating to the inability to coordinate all three components of equipartitioning at the conclusion of the study. On the post-test, 67% of PSTs showed evidence of anticipating and adequately interpreting common student strategies from the equipartitioning learning trajectory regarding misconception of splitting a circle into thirds. Only 23% were able to identify such strategies on the pre-test. It should also be noted that none of the PSTs described the student strategy of creating a half and then halving both halves, before the instructional activities. In each type of strategy for identifying common misconceptions, the percentage of PSTs increased on the post-test. Almost 40% identified the strategy of creating 2 parallel cuts resulting in 3 unequal-sized pieces of the cookie, as opposed to 17% on the pre-test. It is evident that PSTs used components of the equipartitioning learning trajectory in their responses relating to students' understanding, particularly in their ability to identify common student misconceptions and landmarks relating to equipartitioning after the learning

trajectory had been fully implemented. To be prepared for the complexity and range of diversity in student knowledge that PSTs will encounter as they enter the profession, drawing such distinctions is key.

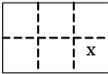
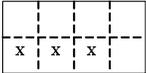
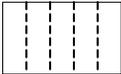
Table 11. PSTs' Predicted Strategies for Item 16 (Form A).

	Pre-test	Post-test
<b>Strategies Identifying Common Misconceptions</b>		
	17%	39%
	3%	14%
	3%	7%
 	0%	7%
<b>Strategies Ignoring Common Misconceptions</b>		
	43%	14%
	3%	3%
  	17%	4%
 	14%	11%

Examples of PSTs' representative responses to Item 16 (Form B) can be found in Tables 11. The examples were chosen to illustrate how various types of predicted student

strategies and explanations were coded. These responses are PSTs' prediction of student strategies for splitting a rectangle into 5 equal-sized parts, not necessarily their own strategy.

Table 12. Representative PSTs' Responses to Item 16 (Form B).

Level	Strategy	Explanations
3		Might pay attention to size but cut 5 strips (creating 6 pieces) because they know they need 5 brownies.  Could cut the brownie 5 times, making 6 pieces.
		It is easier to cut equal pieces if the number needed is even, so a student might cut the brownies in half and then cut it into thirds forming 6 equal pieces – the student will disregard the extra piece.  There are 5 equal pieces, and 1 left over for everyone to share is how they would probably explain this.
		The student might cut part of the cake into 5 equal pieces like they have seen at home and not think about the whole cake.
		A student who may be familiar with brownies might cut them how their parents cut them making smaller pieces 5ths is hard number to sort evenly so a child might draw 8 pieces and say the extra 3 are for seconds.
1		I think that if they know that they must be the same size, then they would make 4 vertical cuts to create 5 equal pieces.  If they're given a ruler and the lines were straight, the pan could be divided into 5 equal parts with vertical lines. They would try to evenly cut the brownies maybe by making 10 brownies and everyone gets 2.
		Cut into thirds. Then, the largest 2 thirds that is bigger than the other one cut in half.
0		There cannot be five equal pieces. They would respond by putting triangles into the rectangle.

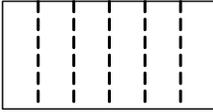
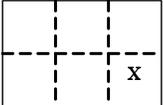
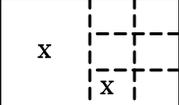
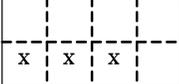
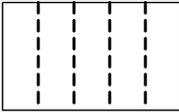
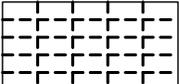
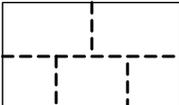
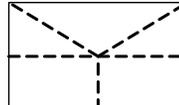
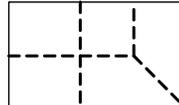
With respect to Level 3, PSTs suggested that students might create both the appropriate number of pieces, as well as equal-sized pieces of brownies, by creating a composition, explaining that extra pieces would not be used or would be saved as leftovers.

Another Level 3 response was that students might create  $n$  cuts, instead of  $n - 1$  cuts, explaining that students who focused on the size of the pieces would create 6 equal-sized pieces, not 5. PSTs' responses at a Level 2 included the strategies from Level 3, without complete explanations.

Almost a third of the PSTs (32%) responded with a student strategy that coordinated size, appropriate number of pieces, and use of the entire cake by creating  $n - 1$  cuts. Even though the task asks PSTs to provide a strategy for students who *focus on the size of the pieces*, these PSTs presumably assume that students might focus on the number of appropriate pieces, as well. The Level 0 response provided in Table X shows a student strategy that focuses on the appropriate number of pieces, but these pieces are not the same size.

PST responses for Item 16 (form B) were also examined for strategy use only (see Table 12). Irrespective of PSTs' explanations, they showed a greater awareness of predicting the types of strategies that students might use who do not coordinate all three components of equipartitioning. As with Item 16 on the other form (i.e., form A), PSTs showed a greater awareness of predicting common student misconceptions from the equipartitioning learning trajectory.

Table 13. PSTs' Predicted Strategies for Item 16 (Form B).

	Pre-test	Post-test
<b>Strategies Identifying Common Misconceptions</b>		
 <i>n</i> cuts	0%	7%
 	7%	21%
 	0%	12%
<b>Strategies Ignoring Common Misconceptions</b>		
 <i>n</i> - 1 cuts	46%	54%
	4%	7%
 	43%	18%
 	0%	3%

Analysis of Other Data

Findings based on the analysis of other data in this study suggest that PSTs used an equipartitioning learning trajectory to:

- restructure their own understanding of mathematics and their knowledge for teaching mathematics, to varying degrees.
- build more precise and adequate models of student thinking by improving their ability to identify students' strategies, behaviors, and verbalizations with respect to equipartitioning.

- incorporate models of student thinking into instructional practices that involved working with individual students, so that PSTs were better able to: i) predict diverse student strategies, behaviors, and verbalizations; ii) identify common misconceptions and landmarks; and, iii) inform instructional practices.

In the remainder of this chapter, results will be presented relating to each finding. The responses of PSTs that were selected as examples in the following discussion are representative of the types of responses of all PSTs whose data was coded in the same way.

### Deepening PSTs' Understanding of Mathematics and Knowledge for Teaching Mathematics

PSTs used an equipartitioning learning trajectory to restructure their own understanding of mathematics and their knowledge for teaching mathematics, to varying degrees. PSTs developed a more sophisticated understanding of the following: language, equipartitioning strategies, mathematical reasoning practices (i.e., representing, naming, identification of the referent unit, justifying), emergent properties (i.e., reassembly), multiplicative reasoning, and children's thinking.

#### *PSTs' Language*

Before the equipartitioning learning trajectory had been introduced, PSTs' language was often informal and vague. PSTs were unable to utilize mathematical language to effectively communicate strategies and behaviors, in relation to equipartitioning and other rational number reasoning concepts. For instance, only one PST, Hailey, used the word

*dealing* to describe this strategy at the beginning of the study by using the metaphor “like dealing a deck of cards” to describe her behavior. Another PST, Alyta, described her strategy for sharing 36 crackers fairly between 2 people by using informal language that did not directly connect to equipartitioning and to her understanding of other rational number reasoning ideas:

We were a little less patient and a little more greedy, and it looked like there were enough for us both to each grab four at a time. So, we did that until there were only four left. And, then we narrowed, we brought it down to grab two at a time.

While Alyta utilized an equipartitioning strategy, *dealing* 4 crackers at a time, her language does not explicitly convey an awareness of using this sophisticated behavior. Further, it is plausible to conclude that Alyta had an understanding that the total number of crackers was divisible by an even number and that dealing 4 at a time and then 2 was a more efficient strategy than dealing less than 4 crackers at a time. Her language does not explicitly communicate that she was aware of the efficiency of her strategy nor does it directly address her more sophisticated understanding of the relationship between the 4 crackers and the total number of unknown crackers, which are ideas that relate equipartitioning to other areas of rational number reasoning. While it is important that PSTs value this type of description in children’s responses, it is important that PSTs have the ability to use these types of descriptions to transition to more formal ideas. This example demonstrates how these ideas can be captured by an equipartitioning learning trajectory to make descriptions more precise and predictive.

Another common way that PSTs used informal language before the equipartitioning learning trajectory was introduced is illustrated in the following example. Laurel described her own strategy for predicting the number of equal-sized parts created from folding a piece of paper in half twice by stating, “I folded it once down the middle to make a *rectangle*, then one more time to make a *square*. I was able to clearly see four separate parts to the piece of paper.” Laurel went on to make an explicit comparison between her own strategies and behaviors to those of a K student in her anticipated student strategy:

I thought that the child would do the same thing that I had done above.

Another solution to this problem would be to fold the paper once, corner to corner making a *triangle* and then fold it once again making a smaller *triangle*.

In both responses, PSTs like Laurel described the type of folds in relation to the shape that is created as a result of the fold, which could be problematic for several reasons. First, shapes created from Laurel’s folds on a square piece of paper would not necessarily be the same for folding a rectangular piece of paper; folding a rectangular piece of paper in half twice would result in four rectangular shaped pieces, not squares. Secondly, by focusing on the type of shape that is created, the essence of the task is somewhat lost, masking the mathematical relationship between creating specific types of folds (i.e., halves, thirds, etc.), the number of folds, and the number of equal-sized parts: folding in half creates two equal-sized parts and folding in half twice creates four equal-sized parts. This is an important relationship that students should come to understand from engaging in equipartitioning tasks like this paper folding task. Finally, describing folds in terms of folding on a vertical, horizontal, or diagonal

line are precise mathematical terms that elementary teachers are expected to know for themselves so that elementary school students can eventually come to know them, as well. In responses like Laurel's, there was no difference in the language that PSTs used to describe their own strategies and behaviors and anticipated strategies and behaviors of K-2 students.

To varying degrees, the equipartitioning learning trajectory provided language for PSTs to communicate strategies, behaviors, and verbalizations, in relation to equipartitioning. Overall, after the equipartitioning learning trajectory had been introduced, most PSTs language became more precise, whether they were communicating their understanding of their own strategies and behaviors, predicted student strategies and behaviors, or models of student thinking.

For example, Ruth elaborated on her representation (see Figure 15) of fairly sharing a rectangular pizza among 4 people, where each person would get 2 pieces of the cake:

As seen below, I figure that in order to get eight equal slices, the rectangle needs two slices on one side and four on the other,  $2 \times 4 = 8$ . Therefore I cut so that I create two equal halves and then equal fourths in the other direction. From the visual with the play dough (shown below) I know that all the pieces are equal in size and my means of cutting is a possible solution.

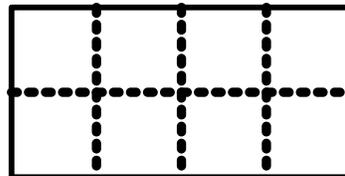
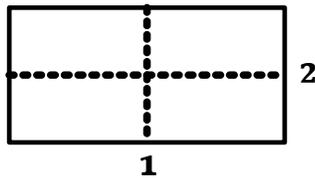


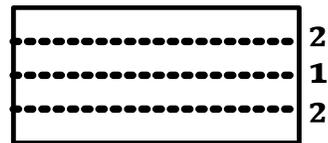
Figure 15. Ruth's representation for sharing a rectangular pizza among 4 people.

Another PST, Chloe, described how she made vertical and horizontal cuts halfway between both sides and made a 2-split and then split each of those halves in half (see Figure 16):

Using a wooden popsicle stick, I divided a rectangular-shaped piece of play-do in half one time vertically and then in half again horizontally. Realizing there was more than one way to complete this task, I divided another rectangular shaped piece of play-do[ugh] in half one time vertically and then in half again vertically. Illustrations of these two solutions can be found on the sheets turned in. Dividing a rectangle either in half one time vertically and then in half again horizontally or in half one time vertically and then in half again vertically will create four rectangles, in which each rectangle would have two parallel lines, equal in length, in both directions.



Chloe's Strategy 1



Chloe's Strategy 2

Figure 16. Chloe's representation for sharing a rectangular pizza between 4 people.

These examples illustrate how PSTs began using equipartitioning as a lens for creating their own models and models of student thinking that described specific strategies and behaviors relating to equipartitioning, focusing coordinating creating equal-sized parts of the whole and creating the appropriate number of equal-sized parts of the whole.

However, as in the cases illustrated above, PSTs usually did not explicitly mention using the entire whole, rather this was implicitly done. While these PSTs did not explicitly state that they used the entire whole, they described a strategy where all of the equal-sized

parts were used, indicating an awareness that PSTs knew they had to use the entire object. PSTs also explicitly used language from the framework of the equipartitioning, such as justification and naming.

While most PSTs began to communicate through more precise language after the equipartitioning learning trajectory had been introduced, some continued to use very informal language to describe strategies and behaviors, without connecting this language to more formal equipartitioning ideas. For example, PSTs' use of the metaphors "hot-dog style," "hamburger-style," and "taco-style" persisted throughout the study to refer to folding an object in half. While metaphors can be a powerful way for teachers to communicate mathematical ideas, some of these PSTs attempted to shift the focus away from mathematics not because they believed it made concepts easier for children but because they themselves found it difficult to communicate their own ideas using mathematical terminology. Some PSTs explicitly expressed their doubt that children were incapable of solving some equipartitioning tasks because either themselves or their classmates struggled with certain tasks. They expressed a concern for trying to make the concepts easier for children.

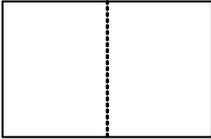
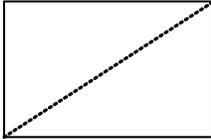
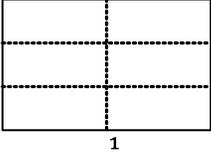
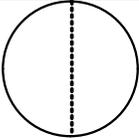
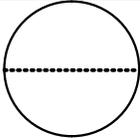
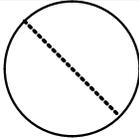
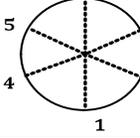
### *PSTs' Equipartitioning Strategies*

Before the equipartitioning learning trajectory had been introduced, PSTs used a single method to solve equipartitioning tasks. When completing the tasks involving sharing a collection, PSTs used three strategies. The majority used division, counting the total number of objects and dividing by the number of people. They were unaware of how more sophisticated rational number reasoning concepts, like division, connect to equipartitioning. Some used methods and behaviors unrelated to equipartitioning. These PSTs "broke" the

original set of objects into piles by using visualization to estimate the size of each group, counted, and adjusted the size of each group to obtain the appropriate outcome. Finally, some used dealing, occasionally dealing two or more objects at a time, but they did not describe their actions as involving one-to-one correspondence or distinguish between the different types of dealing. For example, PSTs did not make distinctions between strategies that involved dealing one object at a time and those that involved simultaneously dealing one object to each group at the same time. Nor did PSTs distinguish between dealing one object at a time and strategies involving dealing more than one object at a time. A few created arrays spontaneously when dealing, but none used the term arrays or noticed the multiplicative relationship that is represented by the vertical and horizontal structure. For example, PSTs did not notice that a  $2 \times 3$  array represents 2 groups of 3 or 3 groups of 2. Many who dealt switched to using division after the total number of objects was known.

As illustrated in Table 14, PSTs used a very limited number of strategies to solve equipartitioning tasks involving fairly sharing a continuous whole before the equipartitioning learning trajectory had been introduced. Further, PSTs used a single method to solve each of these tasks. PSTs almost exclusively used the strategy of making  $n - 1$  cuts for  $n$  people for rectangular and circular objects.

Table 14. A Summary Table of PSTs' Strategies for Splitting a Continuous Whole (Case B).

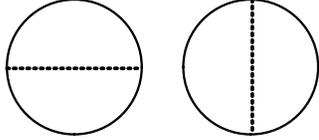
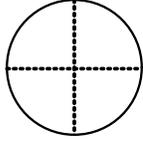
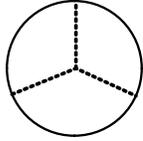
	PSTs' Strategies for Case B Equipartitioning Tasks		
<b>Task 1:</b> Rosa is having a birthday party. She wants to share her birthday cake fairly with Mario. How could she share the cake fairly?			
<b>Task 2:</b> Rosa is having a birthday party. She wants to share her birthday cake fairly with Mario and Emma. How could she share the cake fairly?			
<b>Task 3:</b> Two brothers want to share a mini pizza fairly. How could the brothers share the pizza fairly?			
<b>Task 4:</b> Three sisters want to share a mini pizza fairly. How could the sisters share the pizza fairly?			

To varying degrees, PSTs restructured their own understanding of equipartitioning strategies. After the equipartitioning learning trajectory had been introduced, while some continued to use single strategies, many PSTs used multiple methods to solve Case A and B equipartitioning tasks. These PSTs showed a greater awareness of the different types of reasoning that could be used to successfully solve equipartitioning tasks. By comparing their own thinking to children's reasoning, these PSTs adopted children's strategies from the learning trajectory as their own, in addition to using more sophisticated rational number reasoning strategies (i.e., division) that they used prior to the introduction of the equipartitioning learning trajectory. Further, PSTs began to connect strategies like division and multiplication to equipartitioning.

Before viewing video clips in the Individual Clinical Interview Analysis, PSTs solved the tasks: i) fairly sharing 24 pieces of treasure between 4 pirates; ii) fairly sharing 24 pieces

of treasure between 3 pirates; iii) fairly sharing a circular cake between 2 people; iv) fairly sharing a circular cake between 4 people; and, v) fairly sharing a circular cake between 3 people. A summary of PSTs' strategies and behaviors can be found in Table 14. For the tasks involving fairly sharing a collection of discrete objects (i.e., Case A), 64% of the PSTs in one section used multiple methods to solve one task and 54% used more than one strategy on the other. However, only 23% of PSTs in the other section used multiple strategies to solve one task and 26% of PSTs used multiple methods on the other task. All PSTs solved tasks involving equipartitioning a single continuous object (i.e., Case B) using a single strategy.

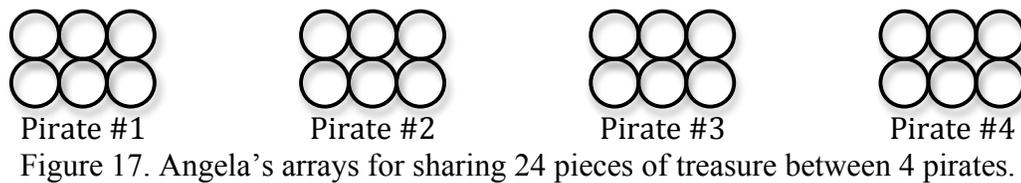
Table 15. PSTs' Individual Clinical Interview Analysis Strategies.

Task	PSTs' Strategies Equipartitioning Tasks	Section 1 (n = 26)	Section 2 (n = 27)		
<b>Task 1:</b> Can you share the pirate treasure fairly among 4 pirates? Show me how you would do this.	<b>USES A SINGLE STRATEGY</b> Deals 1-1 Builds an array Uses division and/or multiplication	4% 4% 69%	4% 14% 14%		
	<b>USES MULTIPLE STRATEGIES</b> Deals 1-1, builds an array Deals 1-1, uses division and/or multiplication Deals more than 1-1, builds an array Deals more than 1-1 at a time, uses division and/or multiplication Deals 1-1, builds an array, then counts for verification Deals 1-1, builds an array, uses division and/or multiplication, then counts for verification	11% 8% 0% 0% 0% 4%	41% 11% 4% 4% 4% 4%		
	<b>Task 2:</b> Can you share the pirate treasure fairly among 3 pirates? Show me how you would do this.	<b>USES A SINGLE STRATEGY</b> Deals 1-1 Deals more than 1-1 Builds an array Uses division and/or multiplication Redistributes	0% 0% 4% 67% 4%	4% 4% 19% 19% 0%	
		<b>USES MULTIPLE STRATEGIES</b> Deals 1-1, builds an array Deals 1-1, uses division and/or multiplication Deals more than 1-1, builds an array Deals more than 1-1 at a time, uses division and/or multiplication Builds an array, uses division and/or multiplication Deals 1-1, builds an array, then counts for verification Deals 1-1, uses division and/or multiplication, redistributes Deals 1-1, builds an array, uses division and/or multiplication, then counts for verification Redistributes, builds an array	0% 4% 0% 0% 11% 0% 4% 7% 0% 4%	7% 4% 0% 4% 31% 4% 0% 0% 0% 4%	
		<b>Task 3:</b> Share the [circular] cake fairly between 2 pirates.		100%	100%
		<b>Task 4:</b> Share the [circular] cake fairly between 4 pirates.		100%	100%
		<b>Task 5:</b> Share the [circular] cake fairly between 3 pirates.		100%	100%

As illustrated in Table 14, the most common combination of strategies was to build arrays and to use division and/or multiplication. For example, Angela, described her own reasoning for sharing the collection of pirate treasure fairly between 4 pirates:

I know that if there are 24 pieces of pirate treasure and there are 4 pirates, I can divide the treasure (24) into the amount of pirates (4). Thus  $24/4 = 6$ .  
There will be 6 pieces of treasure per pirate.

In addition to the above response, Angela also drew four  $3 \times 2$  arrays to represent each pirate's fair share (see Figure 17), showing each received 6 pieces of pirate treasure.



Even after the equipartitioning learning trajectory had been implemented, PSTs used a single strategy to solve Case C and D equipartitioning tasks. For example, Hillary provided the representation (see Figure 18) below and described her solution to task that she gave to a 1<sup>st</sup> grader in the following way, “Yes, you can share 3 cookies fairly among 4 children by cutting the first two cookies in half and cutting the third cookie into four equal parts. Each child's share is  $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ .”

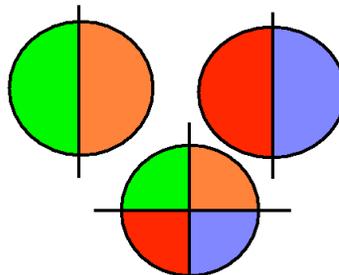


Figure 18. Hillary's representation for fairly sharing 3 cookies among 4 children.

Based on the current work of Confrey and her DELTA research team, this strategy is likely beyond the capabilities of most children in grades K – 2. While teachers should have the knowledge to solve these tasks, Case C and D tasks may be inappropriate for 1<sup>st</sup> graders. Further, Hillary’s response does not recognize the need to have the children first engage in and develop reasoning about Cases A and B that lead to Cases C and D. There is limited evidence that PSTs held such knowledge. Ruth described her solution to fairly sharing 3 pumpkin pies among 4 friends below (see Figure 19):

I cut each pie into fourths, and each friend receives 3 pieces total. Each friend gets a fair share. Each friend gets 3 pieces and since each piece is the same size, they get equal shares. Each friend gets  $\frac{1}{4}$  of one pie or  $\frac{3}{12}$  of the total pieces. There are several other ways to cut the pies - as long as the total number of pieces is a multiple of four, the friends will get equal shares (ex. To the left – cut each pie into eighths).

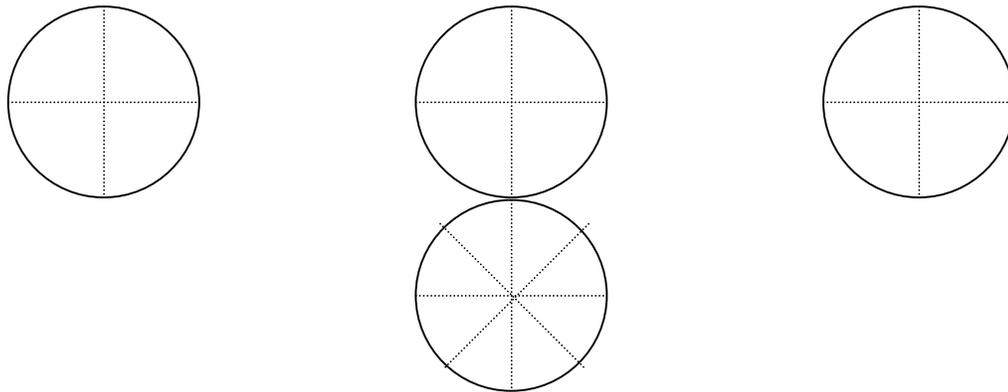


Figure 19. Ruth’s representation for sharing 3 pumpkin pies fairly among 4 friends.

Although Ruth only used a single method, unlike most PSTs, she acknowledged that there are multiple ways to split each pie, focusing on how reasoning about splitting a whole could be used to solve tasks involving splitting multiple wholes. While each friend would receive  $\frac{1}{4}$

of one pie, Ruth's solution is not complete. Ruth does not indicate that she is aware that each friend would also receive  $\frac{3}{4}$  of a pie in total. For teachers to help children develop a more sophisticated knowledge base of equipartitioning, they must recognize the appropriate place in children's school experience to introduce Cases C and D. Also, teachers need to understand how to develop children's reasoning by building on their prior knowledge of reasoning about Cases A and B.

### *Mathematical Reasoning Practices*

Children's participation in classroom activity is influenced, in part, by the ways in which they perceive themselves as part of a community of learners, which manifests as mathematical practices (Bowers, Cobb, & McClain, 1999; Cobb & Bowers, 1999). Confrey, et al. (in progress) identify some of children's mathematical reasoning practices in relation to equipartitioning as *representing*, *naming*, *identification of the referent unit*, and *justifying*.

*Representing.* Many PSTs were unable to identify the mathematical structures within their representations and models (e.g., arrays and compositions) as they described their own strategies and behaviors before the equipartitioning learning trajectory had been introduced. For example, during a whole class discussion, Keisha and Tina built two  $2 \times 9$  arrays to fairly share 36 crackers among 2 people (see Figure 20).

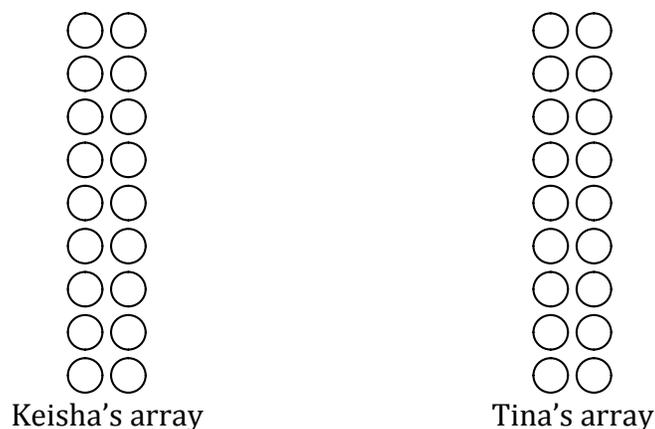


Figure 20. Keisha and Tina's two 2 x 9 arrays.

Keisha and Tina's exchange not only demonstrates how they informally describe going from one-to-one dealing to simultaneous dealing but also illustrates they may not have been aware of the multiplicative relationship within the array that they created.

- Tina:            Ok. We both just grabbed a handful, then, like put 'em down one at a time. Yeah, one for me [lays down a counter on the Elmo.]
- Keisha:        And, then one for you, one for me [lays down a counter across from Tina's counter on the Elmo]. Yeah. One for me [lays down another counter below her first counter as Tina does the same thing on the other side.] And, just keep putting one down [Keisha and Tina continue to build a 1 x 9 array. After both PSTs finish building the 1 x 9 array, they start another column beside the 1 x 9 they already created.]
- Instructor:    So, is that how you lined them up when you did this initially?
- Keisha:        Well, they were right beside each other.
- Instructor:    So, you guys lined them up.
- Keisha:        So, it was kinda like coordinated, like what we're doing now. I put mine in that spot and we keep going.
- Tina:            And we put 'em, like at first, we put 'em down one at a time, but then one of us would have one more than the other. So, we started putting them down at the same time.
- Keisha:        Yeah.
- Instructor:    Ok. So, you guys put them down one at a time at the same time. So, how do you know that you both got the same amount?
- Tina:            Because ...

Keisha: Well, 'cause they look the same, and, like, the patterns the same. And, that was the immediate way, just looking at it. And, then we counted.

Instructor: So, what kind of pattern do you see?

Keisha: [pause] By twos.

Instructor: Uh-huh.

Tina: Then, we count to see if they are the same amount [inaudible]  
...

Instructor: Say that a little bit louder so we can ...

Tina: If they're the same amount of rows, if we have the same amount of rows of two.

Instructor: Do you have the same amount of rows of two?

Tina: Yeah.

Instructor: So, Keisha, I heard you say that you could look at them without, do you mean without counting?

Keisha: Right, without counting.

Instructor: Or, you said that you did count to verify. Is that what you are saying?

Keisha: Yeah, I mean I looked at it, and I was like it looked the same, but then let's just double check and count. I went back afterwards and counted to make sure it was the same ...  
[pause] ... to double check.

It seems that for Keisha, at least, the representation of the array was significant, although she was unable to describe the components and underlying mathematical structure of the arrays by using the expected language of 2 groups of 9 or 9 groups of 2. It was not until they were asked about their pattern that Keisha said, "by twos" and Tina followed up with, "we have the same amount of rows of two." While it seemed like the array itself might have been verification for Keisha that each friend received the same amount of crackers, she was unable to explain how her pattern of "by twos" related to the multiplicative structure in the  $2 \times 9$  array, and it is unclear if she saw this relationship prior to the class discussion. Tina's statement, "Then, we count to see if they are the same amount" may indicate she may not have seen the multiplicative structure in the array until she was questioned about the pattern.

After the equipartitioning learning trajectory had been introduced, many PSTs used arrays and compositions to represent or model fair shares in equipartitioning tasks. These representations were often used to justify or verify that PSTs had produced the appropriate number of equal-sized groups, while exhausting the collection or whole. For example, Cheryl indicated that she dealt two pieces of treasure at a time, representing her solution with an array (see Figure 21):

I would deal 2 at a time until each pirate has the same amount. Because 24 is divisible by 8 it will work to count by 2s. One dealing uses 8 pieces of treasure and I would have to deal to everyone 3 times.

Cheryl indicated that she knew she could deal two pieces of treasure at a time because the total number of pieces of treasure, 24 coins, is divisible by the number of pieces of treasure that had been distributed after the first round of dealing two pieces of treasure to 4 pirates. She knew that she would need to repeat this two more times. PSTs, like Cheryl, showed a greater awareness of the multiplicative relationship within the structure of the array itself, connecting equipartitioning with division and multiplication.

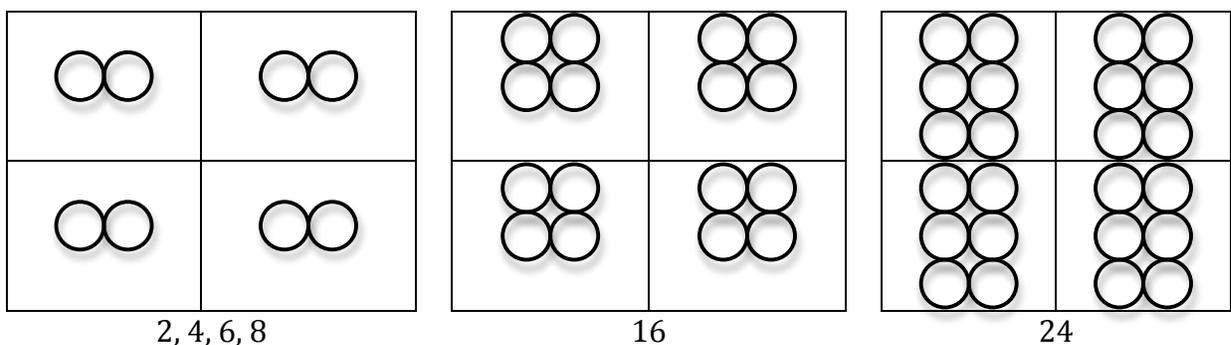


Figure 21. Angela's arrays for sharing 24 pieces of treasure between 4 pirates.

*Naming.* While all PSTs did not attend to naming after the equipartitioning learning trajectory had been introduced, many PSTs focused on naming fair shares as a count (i.e., 4 pieces of treasure) and/or as a relation (i.e.,  $\frac{1}{4}$ ) to a whole in describing their own strategies, behaviors, and verbalizations of equipartitioning tasks.

After the equipartitioning learning trajectory had been implemented, many PSTs were also able to identify the significance of children's verbalizations with respect to their understanding of equipartitioning. These PSTs often distinguished between children naming fair shares as a count or  $1/n^{th}$ , a relation.

These PSTs recognized student behaviors and verbalizations that are indicative of less sophisticated understandings of equipartitioning. For example, Beth, made an inference about a student's understanding in relation to his ability to name, identifying instances where the 5-year-old K student exhibited behaviors that were more sophisticated. Beth indicated that Neil had difficulty naming fair shares and described his performance below:

I think that Neil has difficulty in knowing what to call each share in terms of a part of the whole. Each time that Neil was asked what each pirate's fair share was, he always responded by recounting the number of counters in each group and providing that number as a response. For example, after completing task 1 and being asked what each pirate's fair share was, Neil recounted to make sure that each pirate had three pieces of treasure and then responded "three." He was never able to give a response in terms of what part of the whole amount of treasure each pirate got, such as "each pirate's fair share is one half" or "each pirate's fair share is three out of six."

*Identification of the referent unit.* While there is evidence that most PSTs restructured their own knowledge of equipartitioning in relation to Cases A and B, this was not true with regards to Cases C and D. Even after the equipartitioning learning trajectory had been fully implemented, some PSTs held misconceptions about these cases in relation to identifying the referent unit. Confrey et al. (in progress) emphasize the importance of being able to make distinctions between different referent units: the single whole and the total collection, which may consist of multiple wholes. Furthermore, Confrey et al. (in progress) establish that the inability to identify the referent unit has the potential to mask an important equivalence:  $m/p^{\text{th}}$  of one object, where  $m$  objects are fairly shared among  $p$  people, is equivalent to  $1/p^{\text{th}}$  of  $m$  objects. It is crucial that PSTs are able to coordinate this equivalence for themselves as they are laying the foundation for a more sophisticated understanding of rational number reasoning.

For example, in sharing 2 cookies among 3 people, PSTs often split both cookies in half, giving each person  $\frac{1}{2}$  of 1 cookie, and split  $\frac{1}{2}$  of 1 cookie into thirds. While  $\frac{1}{3}$  of  $\frac{1}{2}$  of 1 cookie is  $\frac{1}{6}$  of a cookie, many PSTs would refer to the third of a half as  $\frac{1}{3}$ , suggesting each person would receive  $\frac{1}{2}$  and  $\frac{1}{3}$  of a cookie.

*Justifying.* Before an equipartitioning learning trajectory had been introduced, PSTs did not justify or verify that their own strategies and behaviors produced equal-sized shares. Further, many were unsure if their strategies could be generalized.

While all PSTs did not focus on justifications, many attended to justifying that their own strategies produced fair shares after the equipartitioning learning trajectory had been introduced. For example, PSTs like Hillary justified that each pirate got a fair share by

stacking the pieces in describing her solution for sharing a circular birthday cake fairly among 2 pirates:

I would find the center of the cake and cut the cake straight down the middle creating two equal parts so that each pirate gets  $\frac{1}{2}$  of the cake. Then, I would stack the halves on top of each other to be sure they were the same size.

Another PST, Angela, provided her justification of fairly sharing a circular cake between 4 pirates:

Each pirate will receive 1 quarter or 1 fourth ( $\frac{1}{4}$ ) of the birthday cake. To solve this task, I cut the cake in half twice because I knew that by cutting in half once, I create 2 equal pieces. Thus,  $2 \cdot 2 = 4$ . Therefore, I cut the cake in half twice creating 4 equal pieces.

Angela used multiplicative reasoning to describe the relationship between cutting something in half and the number of equal-sized parts that are created by the 2-split. These PSTs recognize the importance of being able to justify, which was presented as a more sophisticated understanding than simply carrying out a strategy.

PSTs also began to identify children's justifications in their equipartitioning behaviors and strategies. Some PSTs also focused on children's justifications in their predictions of students' strategies and behavior. This was especially evident for equipartitioning tasks relating to Case A, fairly sharing discrete collections.

#### *Emergent Properties: Reassembly*

Confrey et al. (in progress) have found that children's experience with equipartitioning sometimes acts as a catalyst in developing children's understanding of

emergent properties relating to this construct, such as reassembly and redistribution. Confrey et al. (in progress) explain that the reversal of equipartitioning requires that students have an understanding that reassembly results in reconstituting the whole. After the equipartitioning learning trajectory had been introduced, many PSTs were able to identify children's strategies and behaviors in relation to reassembly.

### *PSTs' Additive and Multiplicative Reasoning*

PSTs restructured their own knowledge of equipartitioning to varying degrees. Before an equipartitioning learning trajectory had been introduced, it is evident that PSTs' knowledge varied. In many cases, PSTs inappropriately used additive thinking instead of multiplicative reasoning in solving equipartitioning tasks. Many PSTs had a weak understanding of equipartitioning and its relation to other rational number reasoning concepts, like multiplication and division. This result was unexpected since all PSTs have taken at least Calculus I prior to this study, and many PSTs had completed mathematics content courses in the mathematics department beyond Calculus I.

Before the equipartitioning learning trajectory had been introduced, all PSTs were able to accurately predict that folding a piece of paper in half would result in 2 equal-sized parts and that folding a piece of paper in half twice would result in 4 equal-sized parts. However, almost all of the PSTs predicted that folding a piece of paper in half three times would result in 6 equal-sized parts, instead of 8. The following excerpt from a whole class discussion illustrates the range of PSTs' understanding in relation to the multiplicative relationship involved in successfully solving this task.

Instructor: What about folding it in half a third time? What were some predictions that you made? [pause] How many equal parts?

Samantha: Eight equal parts.

Instructor: Eight equal parts. [pause] Did anybody make a prediction other than 8? [long pause] Everyone predicted 8 before they did it? [pause] Ok, Alyta.

Alyta: I thought it was gonna be 6.

Instructor: Ok. Thank you for being brave, Alyta. Um, a lot of people in my last class thought 6, too. Explain, like, your reasoning behind why you thought it was gonna be 6.

Alyta: Well, I saw a pattern like 2, 4, and then I thought it was going to be going up by even numbers.

Instructor: Ok. So, if you are going with that line of thinking, if we folded it again, what would be the number of equal parts probably that someone might predict?

Alyta: For folding it four times?

Instructor: Yeah.

Alyta: I said 8.

Instructor: Can someone explain what type of thinking that a student is doing if they would predict 8 for the fourth fold? Ehm-huh.

Samantha: It was doubling the amount or adding ...

Instructor: So, is it doubling?

Samantha: Oh, for the fourth one?

Instructor: Uh-huh.

Samantha: Ok, never mind. Uhm ...

Instructor: But, you said they were doubling, or they're ...

Samantha: Well, I was thinking that when, like, when you started with half, and then when you went to fourths, and then I would think, like, you would add 2 more then you would get 6, and you add two more and you get 8.

Instructor: Ok. So, they're adding.

Samantha: Ehm-huh. Right.

Instructor: Ok. Hillary.

Hillary: I was going to say its like multiplication 'cause if you look at it the first time they're folding one piece of paper in half and getting 2. Then, if you fold it in half again they're getting 4. Two times 2 is 4. So, then, if they fold it in half, again, 4 times 2 is 8.

Almost all PSTs held an additive misconception, like Alyta, which Samantha described by explaining that every time a piece of paper is folded in half, two equal-sized

parts are created. Only a few PSTs, like Hillary, were aware of the existence of a multiplicative relationship. She clearly links folding a piece of paper in half and getting two equal-sized parts with  $2 \times 2 = 4$ , and she connects folding a piece of paper in half twice and creating four equal-sized parts with  $4 \times 2 = 8$ . For the few who were able to make a generalization about the multiplicative relationship between the type and number of folds and the resulting equal-sized parts, they were unsure if this relationship was always true, especially when moving from folding in halves to thirds and combinations of halves and thirds.

After engaging in equipartitioning tasks, some PSTs content knowledge began to increase, indicating that engagement with equipartitioning tasks and class discussions deepened PSTs own understanding of the structure of mathematics by comparing their own strategies and behaviors to other PSTs'. By the end of the study, some PSTs showed evidence that their knowledge went beyond equipartitioning, connecting equipartitioning with other areas of rational number reasoning, using multiplicative reasoning appropriately.

Laney is an example of a PST who unsuccessfully used additive reasoning to solve tasks before the equipartitioning learning trajectory was introduced. In her inference about a 5-year-old K student named Constance, it is evident that Laney restructured her knowledge by showing an awareness of multiplicative relationships:

When asked how many pieces there would be after she folded the paper in half twice, she explained that there would be three pieces. She could not conceptualize that she would have to double the number after two folds, just

as she had done after one fold. She seemed disappointed after counting the pieces to find four pieces instead of three as she had predicted.

Cheryl also restructured her own knowledge of the multiplicative relationship relating to the same task in the following way:

Either way, there is still 8 equal parts because each time I am folding the paper in half, and this increases the number of parts by multiple of 2. Each time, there are not 2 more parts, but double the number from the time before.

Cheryl implicitly compared her own initial additive reasoning to a 3<sup>rd</sup> grader's in making an inference about the 3<sup>rd</sup> grade student's understanding:

She understands that when you fold a paper in half, it multiplies the number of shapes by 2. She made a prediction that when she folded the paper in half once, there would be 2 parts, then twice would give her 4 parts, and then 3 times would give her 8 parts. At first she guessed 6, probably because she was adding each time, but then she changed it to 8 because she said that splitting 4 in half would give her 8 (or folding the 4 parts in half would give her twice as much).

It is important that teachers have the ability to understand when it is appropriate to use additive thinking and situations that are appropriate for utilizing multiplicative thinking in order to help children develop an understanding of the same concept.

#### *PSTs' Knowledge of Children's Thinking*

Even before the equipartitioning learning trajectory had been introduced, PSTs began to restructure their knowledge for teaching equipartitioning by making implicit and explicit

comparisons among their own thinking, their predictions about K-2 students' strategies and behaviors, and K-2 children's strategies and behaviors. For PSTs who restructured their understanding, these comparisons resulted in an awareness of one or both of the following: 1) children are capable of solving equipartitioning tasks; and, 2) children often reason differently than they do.

After the equipartitioning learning trajectory had been introduced, PSTs used an equipartitioning learning trajectory to restructure their knowledge for teaching mathematics, to varying degrees. While PSTs continued to engage in the process of comparing after the equipartitioning learning trajectory had been implemented, PSTs began to compare students' strategies, behaviors, and verbalizations to other types of reasoning besides their own in building models of student thinking. PSTs compared children's performance to strategies, behaviors, and verbalizations from the equipartitioning learning trajectory, grounding their models in theory. PSTs also began to make comparisons between grade levels. For instance, comparing a K student's performance to that of a 1<sup>st</sup> or 2<sup>nd</sup> grader. They also made comparisons within a grade, distinguishing between less and more sophisticated thinking. Some PSTs built models of student thinking by making comparisons about the same child with regard to the student's strategies, behaviors, and verbalizations across different cases (e.g., Cases A and B). Some PSTs began to connect these experiential models based on their experiences with strategies, behaviors, and verbalizations explicated in the equipartitioning learning trajectory, coordinating experiential models with theory from the equipartitioning

Some PSTs compared their own reasoning to children's thinking began to distinguish between their own strategies and behaviors and those that would be appropriate for young

children. These PSTs often recognized that children's use of informal, and sometimes inappropriate, mathematical language did not necessarily indicate that children had a poor understanding of equipartitioning. These PSTs connected children's verbalizations to their own, recognizing how children's verbalizations provide insight into their reasoning, even when children do not use the same mathematical language as PSTs.

This was particularly evident with respect to their conceptions about students' capabilities. Confrey (1995) describes this transformation in point of view in the following way:

one can articulate mathematical voice more fully by reconstructing one's own mathematical perspective to support rather than suppress diversity. Thus, the dialectic one engages in is the pursuit and articulation of student voice, and the articulation and revision of perspective in light of student voice (p.7).

Rather than viewing children's informal ideas and language from a deficit perspective, PSTs began to value children's voice and attempt to readjust their own thinking to accommodate children's behaviors and verbalizations as meaningful. For example, Hillary, made the following inference about Gabe's understanding after interviewing the 6-year-old 1<sup>st</sup> grader:

Based on the interview I would say Gabe understands how to divide things in half very well. He also understands the concept of naming different sized portions of things different names. For example, when he cut the circles in half, he called the halves bowls and when he cut the circles in fourths he called the fourths pizzas. Gabe also knew how to cut into fourths. He also demonstrated that he knew two halves created one whole cookie. He showed

this when he cut the cookies apart and then put them back together on the table.

Based on Gabe's ability to split continuous objects into halves and fourths, as well as his recognition of reassembly (i.e., two halves make a whole), Hillary inferred that Gabe has an understanding of splitting objects in half. She went on to describe her surprise in the way that Gabe named these shares:

I was very surprised when Gabe named his halves bowls and his fourths pizzas. He started doing this after a couple tasks so I was not expecting it. I thought it was very clever and demonstrated that he understands the basic concept of naming but not the actual mathematical names.

Hillary noticed that he consistently used this language for naming halves and fourths. Rather than viewing this unusual terminology as unproductive, she recognized that Gabe has a conceptual understanding of one-half and one-fourth but has not used the relational names that correspond with these concepts (i.e.,  $\frac{1}{2}$ ,  $\frac{1}{4}$ ), pointing out that her next instructional move would be to help him connect naming with the formal mathematical language since he understands that the different sized parts have different names:

I would also work with Gabe on naming his fraction properly. While he understands that different size portions should have different names, he does not use the correct mathematical names. So, I would work with him on naming his fractions as  $\frac{1}{2}$ ,  $\frac{1}{4}$ , etc.

Hillary summarized how her perspective changed in that she could focus on Gabe's reasoning as opposed to whether he named the shares correctly or incorrectly by comparing her anticipations of Gabe's behaviors and verbalizations to what he did in the interview:

From this experience, I learned not to expect a bright student to respond in exactly the same way that I would expect him to. Gabe surprised me because he struggled with things I did not expect him to struggle with and he named things differently from how I expected. I learned that children will connect something unfamiliar with something familiar. For example, Gabe did not understand the concept of one half so he named his one half a 'bowl' because of its shape. This is an important realization because it frees the teacher to appreciate the reasoning behind the naming rather than simply focusing on if the naming is 'technically correct'.

After an interview with a 7-year-old 2<sup>nd</sup> grader named Roger, Tina made the following inference about Roger's understanding of equipartitioning:

Based on this interview, I think that Roger has a solid knowledge of halves and fourths. He was repeatedly able to tell me that he was cutting the cookies in half, and when he would cut that half in half, he would tell me it was one-quarter. He also understood, that when you have two halves, it makes a whole, and when you have two of what he called "halves of a quarter" (eighths) you have one-quarter. When he was asked if four halves makes anything else, he was able to exclaim that it was equal to 2 cookies. There was one instance where he had piles that contained two quarters, and two eighths or "half of a

quarter.” When asked what this pile was a few times, he was able to state that it was “practically three-quarters” showing that he is able to put these together instead of only being able to think about it as individual pieces  
( $2/4+1/8+1/8=3/4$ ).

In this particular example, Tina’s own reasoning and her interpretation of Roger’s understanding is apparent. In the response above, Tina supported her inference that Roger has a “solid” understanding of halves and fourths in describing his ability to reassemble. While Tina described Roger’s language with respect to his ability to reassemble, Roger consistently used this language throughout the interview. Tina recognized that the name that Roger associated with splitting a half in half is “one-quarter.” In several parts of Tina’s response, her own more formal mathematical language is written in parentheses, capturing Roger’s less formal language in quotes. For Tina, splitting one-fourth in half results in an “eighth.” At the same time, Tina recognized that when Roger split one-fourth in half, he called the eighths “halves of a quarter.” Tina also made a distinction between her understanding that  $2/4 + 1/8 + 1/8 = 3/4$  and that when Roger put the  $2/4$ ,  $1/8$ , and  $1/8$  parts together, he thought of the result as “practically three-quarters.” Tina recognized that although Roger did not use the same mathematical terminology for naming shares that she did, Roger’s own language was significant and provided insight to his understanding. Rather than rejecting Roger’s language, Tina acknowledged that there are diverse verbalizations that are indicative of having an understanding of equipartitioning. Tina was able to value Roger’s verbalizations, while at the same time sustain her own.

## Building More Precise and Adequate Models of Student Thinking

Most PSTs used an equipartitioning learning trajectory to build more precise and adequate models of student thinking by improving their ability to identify students' strategies, behaviors, and verbalizations with respect to equipartitioning. Before the equipartitioning learning trajectory was introduced, PSTs' models of student thinking did not take children's strategies, behaviors, and verbalizations relating to equipartitioning into account. Instead, PSTs' models of student thinking were general or evaluative.

PSTs who created general models of student thinking described K-2 students as feeling "eager," "responsive," "happy," "confident," and "embarrassed." Others focused on students' nonmathematical behavior, like being off-task, instead of emphasizing behaviors specifically relating to students' performance on equipartitioning tasks. Bonnie described 2<sup>nd</sup> grade students' understanding in the following way:

I think some of the students were very rude to their classmates when they saw they were having trouble getting the answer. Also, they were disrespectful to me when asked to first wait and raise their hands to respond to the questions being asked. The student's typical trend during this activity was to blurt out the answer. Another thing that surprised me was how distracted the students were when I was trying to explain the task. They were talking off subject, throwing pencils and touching the paper when asked not to.

While it is important that elementary school teachers have an awareness of children's feelings and general behavior, it is crucial that teachers are able to identify behaviors and verbalizations that provide information about students' understanding of equipartitioning so

that this information can be used to inform their pedagogical decisions in order to help children develop mathematical ideas.

Some PSTs created models that were evaluative, where their inferences reduced students' reasoning to being "correct" or "incorrect." For example, Sadie responded, "He also knows a lot about sharing things equally between four people, since he got those tasks correct on his own as well." Alyta stated that, "I think this student can reason very well – she reasoned through the cracker task very well, and knew how many equal parts folding the paper would make." Alyta's inference does mention the action of creating the correct number of equal-sized parts, but there was no attention to specific equipartitioning behaviors. In teacher's work with children, it is important that they are able to distinguish between children's correct and incorrect solutions; however, their knowledge should go further by recognizing that these behaviors and strategies often provide teachers with evidence of children's reasoning. It is important that teachers are able to identify these behaviors, strategies, and verbalizations in order to connect them to cognition and create more robust models of student thinking.

After the equipartitioning learning trajectory had been implemented, PSTs became more sensitive to identifying students' strategies, behaviors, and verbalizations from the equipartitioning learning trajectory. Rather than making inferences that were simply evaluative (i.e., "correct" or "incorrect"), PSTs' inferences became more qualified, creating models that identified and described children's strategies, behaviors, and verbalizations with respect to the coordination of the following: i) using the entire collection or whole; ii)

creating equal-sized parts of the collection or whole; and, iii) creating the appropriate number of equal-sized parts of the collection or whole.

For example, Beth inferred that Neil, a 5-year-old K student, had an understanding of fairly sharing 6 pieces of treasure among 2 pirates, after initially not using the entire collection:

Each person should receive the same number of objects in order to share fairly. At first, Neil did not understand that all of the treasure had to be used, and when he was asked to share six pieces of treasure with two pirates he gave each pirate only one piece. However, after explaining to him that he needed to use all of the treasure, Neil was able to give each pirate three pieces of treasure.

After intervening, Beth recognized that Neil coordinated sharing the entire collection of pirate treasure and dealt equal-sized pieces of treasure so that each pirate had the appropriate amount of treasure. By giving “the same number of objects in order,” Beth indicated that she was aware that Neil dealt the treasure systematically and recognized this behavior as being characteristic of fairly sharing a collection.

Others identified the coordination of two of the components. For example, Sadie inferred that Chuck, a 5-year-old K student, understood equipartitioning and made the following inference about Chuck’s thinking:

I think that Chuck knows a lot. He knows that to be fair, each person needs the same number of pieces. He also always made all of the pieces about the same size, so he knew that the pieces must be equal in size.

Tina described her inference about the understanding of a 7-year-old 2<sup>nd</sup> grader, named Roger:

Where as some students worry about each pile having the same number of pieces, irregardless of size, Roger understood that the piles needed to be equal in regards to their size or the amount of cookie that each pile contained. He understood that each child needed a fair share.

PSTs like Sadie and Tina both recognized that their K-2 students coordinated creating both equal-sized parts and the appropriate number of equal-sized parts. If teachers are to use models of student thinking to inform instructional activities that support students in moving towards more sophisticated understandings, these models should be built on behaviors, strategies, and verbalizations that connect the construct of equipartitioning to students' cognition. The first step in using models in instruction is to identify children's equipartitioning strategies, behaviors, and verbalizations.

Some went beyond identifying and describing children's behavior in relation to coordinating the three components of equipartitioning, making inferences about more or less sophisticated understandings of equipartitioning. For example, Ruth inferred that Steve, an 8-year-old 2<sup>nd</sup> grader, had a "basic" understanding of equipartitioning while sharing 24 goldfish crackers among 2, 3, and 4 people:

Based on the interview, I can conclude that Steve has a very basic understanding of equipartitioning. For the sharing tasks involving the goldfish, Steve was able to equally split all of the fish into equal groups. Steve shared by dealing, which is a more sophisticated manner than just estimating

and never counting or verifying that the groups are equal. He was able to count by twos and even shared some in twos and was able to explain to me how counting by twos is a quicker way than counting by ones. When I asked him to explain how he knew the groups were equal, he counted again to show me that they had the same number of goldfish.

Ruth recognized that Steve's ability to deal out goldfish crackers into equal-sized groups while exhausting the entire collection, as well as his ability to verify equality by counting, is a more sophisticated strategy than *breaking* the collection. Interestingly, in making an inference about Steve's understanding, Ruth made an implicit comparison to her own strategy for solving the same task. Ruth *broke* the objects into the appropriate number of groups and readjusted the objects until each group contained the same number of objects. She recognized that Steve's strategy is more sophisticated than her own.

Ruth went on to make the following inference about Steve's understanding:

Reflecting on Steve's reasoning and methods of sharing, I believe he is still developing his ability to split non-discrete objects such as cakes. With the counters, Steve could count to verify the equality of the shares, but he could do no such thing for the cake sharing. Steve is definitely more advanced in dealing objects and can count by twos and understand why the shares are equal, but when it comes to cutting cakes, he is less sophisticated in his reasoning; his inability to recognize he cut too many pieces, the unevenness of his pieces, and his lack of trying to name the pieces is evidence.

Ruth recognized that Steve had a more sophisticated understanding of fairly sharing a discrete collection in comparison to his knowledge of sharing a continuous whole and that it was important to compare his performance relating to his ability to name, justify, and coordinate the components of equipartitioning in creating a model of his understanding.

Some PSTs also began to identify behaviors and verbalizations from the equipartitioning learning trajectory relating to children's practices of mathematical reasoning and emergent properties in creating models of student thinking. Evidence from the Individual Clinical Interview Analysis indicates PSTs were aware that children might verify the equality of the size of the groups by either dealing, counting, stacking, or building an array, in addition to predicting students' strategies, is significant. Whether PSTs explicitly described the ability to justify as being more sophisticated or whether they simply described children's ability to do so, PSTs were at least more conscious of the ways that children might justify their strategies and were more aware that they should pay attention to these justifications.

Other PSTs began to describe how children's understanding was more or less sophisticated by describing children's reasoning relating to justifications, naming, and reassembly. For example, Laney identified a 1<sup>st</sup> grader's justification and understanding of reassembly while he fairly shared 9 cookies among 4 people:

Harry made four groups of two cookies using the counting-by-two method. He then realized he had one cookie left and tried to hide it under the table until I informed him he had to use all of the cookies. He then performed a two-split on the cookie where he divided it into four equal parts. He gave each group a piece of this cookie and claimed that each person received two and one half

cookies. He justified his answer by saying that when you put two together (two being two-one fourth pieces), it makes one half and when you put two-one half pieces together, you have one whole cookie.

Laney also focused on Harry's understanding of justification, naming, the referent unit, and reassembly by stating:

Harry was right when he stated that putting two-one fourth pieces together makes one half; however, he failed to take the entire cookie into consideration. Again, as discussed in class, it goes back to what you define as the whole, or the unit. In this case, Harry defined one half of the cookie as the whole instead of the entire cookie. If defining each piece as one fourth, each person would really receive two and one fourth cookie.

It is evident that Laney recognized Harry's ability to to successfully share the cookies but that Harry was not accurate in naming the share because to him, a unit was a whole cookie in one instance and half of a whole in another, rather than one-fourth of a whole cookie. To move towards building more accurate models of student thinking, teachers need to go beyond the ability to identify strategies, behaviors, and verbalizations and recognize children's practices of mathematical reasoning and their understanding of emergent properties. Further, if teachers are to refine children's understanding, they must have the ability to distinguish between less sophisticated and more sophisticated reasoning.

## Incorporating Models of Student Thinking into Instructional Practices

To varying degrees, PSTs used an equipartitioning learning trajectory to incorporate models of student thinking into instructional practices that involved working with individual students, so that PSTs were better able to:

- predict diverse student strategies, behaviors, and verbalizations;
- identify common misconceptions and landmarks; and,
- inform instructional practices.

### *Predicting Diverse Student Strategies, Behaviors, and Verbalizations*

Before the learning trajectory had been introduced, PSTs' predictions of children's strategies and behaviors could be characterized in one of the following ways:

- PSTs were unable to predict how children would reason about equipartitioning tasks and were unable to anticipate successful and unsuccessful student strategies, behaviors, and verbalizations, relating to equipartitioning.
- PSTs held a deficit model of student thinking, predicting that K-2 children would be unable to generate successful solutions to equipartitioning tasks.
- PSTs anticipated that children would successfully solve equipartitioning tasks, using the same strategies and behaviors as themselves, like division, without making connections to equipartitioning and multiplication/division.

- PSTs predicted that K-2 children would be “correct” or “incorrect,” without anticipating diverse student strategies, behaviors, and verbalizations.

For example, Joanne, like many PSTs, made comparisons between their own reasoning and students'. In fairly sharing a rectangular birthday cake among 4 people, Joanne stated, “I would make a t in the pizza, to make four equal pieces. This would create four slices of the same size and shape.” Joanne implicitly compared her reasoning to her prediction of how a K-2 student would solve the same task, “I predict that she will be able to do this one. She might cut a t into the cake.” After working with a 2<sup>nd</sup> grader, Joanne made another implicit comparison to the student's thinking, “Kenya was able to create four parts to the pizza. She did it exactly the way that I predicted. She cut straight down the middle twice, making a t to create four identical parts.” Rather than anticipating diverse behaviors and strategies, these PSTs tended to express a desire to see children conform to their own ways of thinking.

Even after PSTs began to work with K-2 children, some PSTs' interactions in the classroom only reinforced a deficit model of students' reasoning about equipartitioning, which is illustrated in Kim's response after working with a 5-year-old K student:

I don't think I really learned anything about how students reason because this was done with such a young student. That is why I think it would be more beneficial with an older more advanced student. I would ask her how she knew her answer was correct and she may have guessed or say, “I don't know.” She could explain a few of her answers but it was only because she knows how to fold in half from classroom activities or that she knows how to

count and see that two groups of eighteen are equal. There was really no “reasoning” going on with a five year old.

Even though the child was able to successfully solve the tasks, Kim ignored the child’s performance. The student’s inability to explain or justify her reasoning reinforced her belief that K-2 could not reason. Early in the study, Kim became angry during class when her belief that elementary school students are incapable of solving equipartitioning tasks was challenged. Work by Confrey and the on-going work of the DELTA research team indicates that even young children are usually quite successful in solving equipartitioning tasks, particularly performing a 2-split. Kim did not consider alternate possibilities to account for the student’s inability to explain her thinking, such as justification is more sophisticated than carrying out a strategy or that K students’ language is still developing.

PSTs, like Cheryl, were influenced by her Site-based Teacher Educator’s (SBTE’s) deficit model of K children’s capabilities:

It also surprised me at first to see that she cut the pizza into fourths and did not use a whole fourth. But, before I started these tasks with my student, I was telling her teacher about them, and the teacher said that at the end of their fourth quarter this year, they will be tested on these subjects. They will be given numbers like 4, 5, 6, and 10 and asked to split them into 2 parts. She said that they will have to explain why 5 cannot be split into 2 equal parts. So, I learned that kindergarteners do not learn to split the whole amount into equal parts. My student will not learn this year how to split the whole pizza between 3 people. I think that sometimes I expect the students in my kindergarten class

to be more advanced than what is normal just because I am not used to kindergarten level.

Others also created deficit models of student thinking, even when children were successful or partially successful. For example, Lisa who was also placed in a K classroom, stated:

If I were to use this task again, I would complete it more towards the end of the year. Before I completed the numerous tasks with my student, my head teacher was explaining that in kindergarten they do not learn about equal parts and halves until the third quarter.

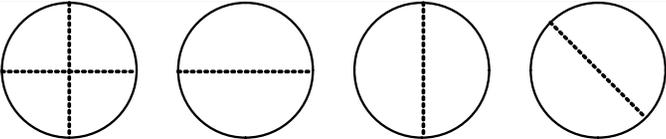
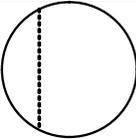
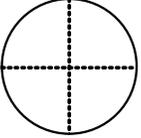
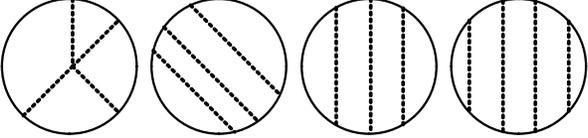
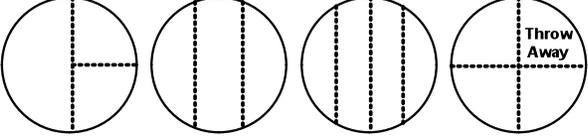
Although the child was successful, her (SBTE) comments trumped the student's performance. Without the use of an equipartitioning learning trajectory, some PSTs' predictions were influenced by their SBTE or the curriculum used in their classroom. These PSTs anticipated deficit models of student thinking, explaining that their SBTEs indicated that children were incapable of solving equipartitioning tasks. Alternatively, some of these PSTs' predicted that children would be unable to reason about equipartitioning because the curriculum would not address these constructs until some future time. Later, when PSTs observed that children had been successful or partially successful, some created deficit models that matched their predictions, indicating that these beliefs impacted the way that PSTs reason about children's understanding.

The equipartitioning learning trajectory assisted PSTs in predicting a range of student strategies, behaviors, and verbalizations. After the equipartitioning learning trajectory had been introduced, PSTs were able to predict both successful and unsuccessful student

strategies, particularly showing tremendous growth in their anticipations of children's unsuccessful strategies and behaviors.

PSTs made predictions about K-2 students' possible strategies and behaviors for solving five tasks that were solved by a K student in the video clips they viewed during the Individual Clinical Interview Analysis. Table 16 summarizes PSTs' strategies, indicating that almost every PST anticipated a strategy and behavior from the equipartitioning learning trajectory. The only strategy that a few PSTs predicted that was not associated with the equipartitioning learning trajectory involved *breaking* a collection into the appropriate number of groups, counting the number of pieces in each group, and adjusting the size of the groups until they had the same number of pieces. These PSTs recognized that this strategy might result in unequal-sized groups. Further, these data suggest that PSTs were able to consider a range of strategies that children might use to solve tasks, predicting both successful and unsuccessful student strategies. This awareness is linked to PSTs' ability to identify misconceptions and landmarks, which will be discussed in the next section. As evidenced in Table 16, PSTs showed a greater recognition of successful student strategies and behaviors for fairly sharing discrete objects and a greater awareness of children's unsuccessful strategies and behaviors for sharing a continuous whole.

Table 16. PSTs Individual Clinical Interview Analysis Predicted K-2 Student Strategies.

	PSTs' Successful Predicted K-2 Strategies	PSTs' Unsuccessful Predicted K-2 Strategies
<b>Task 1: Can you share the pirate treasure fairly among 4 pirates? Show me how you would do this.</b>	Deals 1-1 Deals more than 1-1 at a time Deals 1-1, then counts for verification Builds an array Piles pieces, counts pieces, then moves pieces to form = groups	Piles to form $\neq$ groups
<b>Task 2: Can you share the pirate treasure fairly among 3 pirates? Show me how you would do this.</b>	Deals 1-1 Deals more than 1-1 at a time Deals 1-1, then counts for verification Deal more than 1-1, counts Deals 1-1, then stacks for verification Deals more than 1-1, then stacks for verification Builds an array Piles pieces, counts pieces, then moves pieces to form = groups	Piles to form $\neq$ groups
<b>Task 3: Share the [circular] cake fairly between 2 pirates.</b>	 <p>Each gets 2</p>	 <p>Can't be done</p>
<b>Task 4: Share the [circular] cake fairly between 4 pirates.</b>	 <p>2-split, 2-split</p>	 <p>Can't be done</p> <p><math>n - 1</math>   <math>n - 1</math>   <math>n - 1</math>   <math>n</math></p> <p><math>n</math> is the number of people</p>
<b>Task 5: Share the [circular] cake fairly between 3 pirates.</b>	 <p><math>n</math> radial cuts, where <math>n</math> is the number of people</p>	 <p>Can't be done</p> <p><math>n - 1</math>   <math>n - 1</math>   <math>n</math></p> <p><math>n</math> is the number of people</p>

Further, PSTs predominately predicted that K-2 students would deal the pieces in a collection using a one-to-one correspondence or by dealing more than one piece at a time. Many indicated students might stack or count the pieces to verify each group contained the same number of pieces. Others predicted that students might build an array, instead of dealing. Interestingly, none of the PSTs suggested that children might use division to solve these tasks, even though this was a strategy that many PSTs commonly used in their predictions of K-2 students' solutions before the equipartitioning learning trajectory had been introduced. Further, when solving the same tasks, a large number of PSTs used division to solve the same tasks. This suggests that PSTs had begun to distinguish between strategies that are appropriate for children and those that are appropriate for adults.

Many PSTs were able to predict multiple methods that children might use to solve equipartitioning tasks, indicating that PSTs were better able to predict a range of possible equipartitioning strategies and behaviors, after the equipartitioning learning trajectory had been introduced.

For example, Sadie asked a 5-year-old K student named Chuck to fairly share a circular cake between 2 people. Sadie predicted that a K-2 student would use one of the following six methods to solve this task (see Figure 22):

A child might draw a horizontal line down the middle of the circle [strategy 1], or they might draw a vertical line down the middle [strategy 2]. They might also say the task cannot be done. They might draw a random line down the circle [strategy 3], creating two unequal parts. They could also cut the circle into fourths and give each person two pieces [strategy 4]. A child might

also cut the circle into lots of little pieces, giving each person lots of pieces [strategy 5].

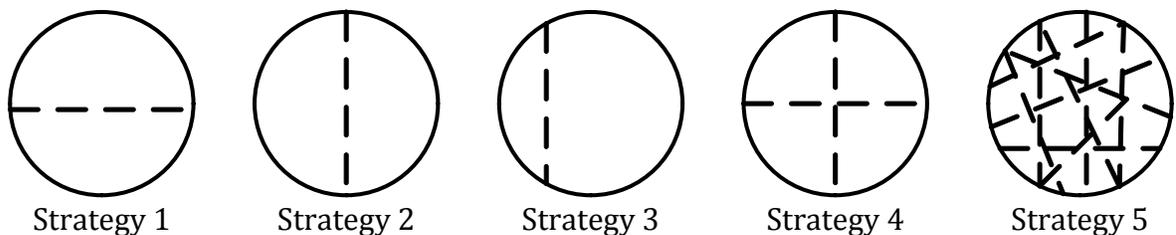


Figure 22. Sadie’s predictions of student strategies and behaviors for sharing a circular cake.

Sadie also asked Chuck to fairly share a rectangular birthday cake among 4 people , anticipating the following strategies (see Figure 23):

A child might correctly make fourths by drawing a vertical line down the middle and a horizontal line down the middle [strategy 1]. They might also say the task cannot be done. They could create four unequal pieces by drawing lines not in the middle [strategy 2]. They could also draw four lines since the cake is shared by four people, creating more than four pieces [strategy 3]. A child could also cut the cake into eight pieces, giving each person two pieces [strategy 4]. They could also cut the cake into lots of little pieces, giving each person some little pieces [strategy 5].

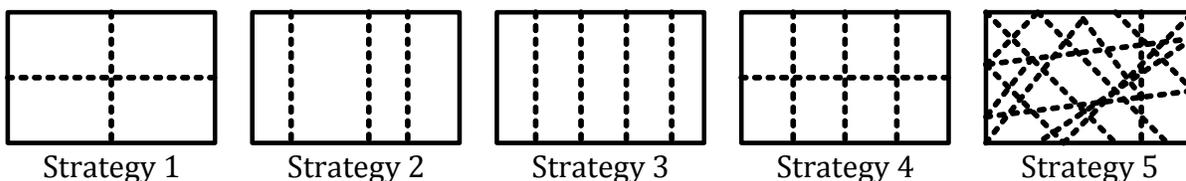


Figure 23. Sadie’s predictions of student strategies and behaviors to share a rectangular cake.

For both tasks, Sadie predicted a range of student strategies and behaviors from the equipartitioning learning trajectory. In the first task, she predicted three strategies that might

lead to success, like cutting a cake in half down the center, and two unsuccessful strategies, like creating two pieces of cake that are not equal-sized. In the second task, Sadie predicted two strategies that would lead to success, creating a 2 x 2 composition, and three unsuccessful strategies, like creating the appropriate number of unequal-sized shares, for sharing a rectangular cake fairly between 4 people. This is significant because PSTs must be prepared to respond to the diverse strategies, behaviors, and verbalizations that they will encounter as they begin to teach.

Some PSTs went even further by describing equipartitioning behaviors and strategies in relation to a range of student strategies and behaviors, distinguishing between less and more sophisticated reasoning. Chloe, in describing her prediction of how K-2 students might fairly share a circular cake among 4 people, responded by stating, “K-2 students on a higher level will solve this problem as on the left. However, lower level students may decide to make four cuts, resulting in five pieces.”

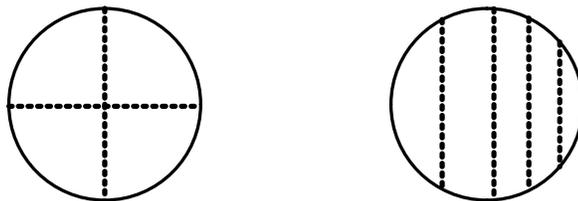


Figure 24. Chloe’s predicted K-2 strategies for sharing a cake among 4 pirates.

Like Chloe, Alyta predicted multiple student strategies and behaviors that were both successful and unsuccessful for the same task. For Alyta and some other PSTs, lower level to higher level understanding was related to grade level. Alyta predicted a K student would cut the cake into 5 unequal-sized pieces by making 4 parallel cuts, while she responded that some K and 1<sup>st</sup> grade students would make 3 parallel cuts on the circular cake creating 4

unequal-sized pieces (see Figure 25). In addition, Alyta drew a diagram of a successful strategy that she predicted K-2 students would use, where students coordinate using the entire cake, the appropriate number of pieces, and the size of the pieces.

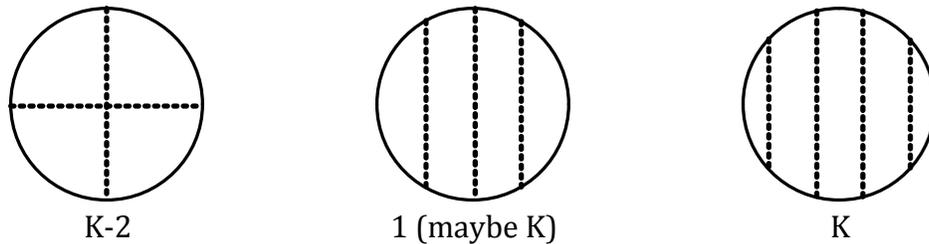


Figure 25. Alyta's predicted K-2 strategies for sharing a cake among 4 pirates.

Being able to predict a range of possible student strategies and behaviors is a necessary skill for teachers

#### *Identifying Common Misconceptions and Landmarks*

After the equipartitioning learning trajectory has been introduced, PSTs were able to identify common equipartitioning misconceptions and landmarks. PSTs recognized that most children would be successful creating a 2-split and 4-split, whereas creating a 3-split would be challenging for many K-2 students. PSTs were able to make predictions about common strategies that students might use if they had difficulty creating a 3-split, like splitting a circle in half and splitting one half in half because a 2-split is primitive and a 3-split involves a radial cut.

In particular, PSTs were better able to identify unsuccessful student strategies and behaviors that children use that are indicative of these misconceptions. Many PSTs explicitly predicted common misconceptions relating to unsuccessfully creating a 3-split. For example, PSTs, like Sandra, anticipated that a 2-split or repeated 2-split is easier for K-2 students than a 3-split:

At this age, the concept of thirds is not yet understood usually so the student may cut it in half. They would see that they have two pieces. They would know that in order to get three, they need one more piece, so they may either cut the cake in half again and discard a piece, or they may cut one of the halves in half making 3 pieces, where one piece is bigger than the other two.

Other PSTs, such as Chloe, anticipated that young children find it difficult to fairly share a circular cake between 3 pirates (see Figure 26) by predicating specific unsuccessful student strategies and behaviors from the equipartitioning learning trajectory:

I doubt that any K-2 students will solve this problem correctly. Again, they may draw three lines, resulting in four pieces. However, they might divide the cake into three unequal parts, as well. Finally I suspect that some K-2 students will either throw one of the fourths away or simply state that the problem cannot be solved.

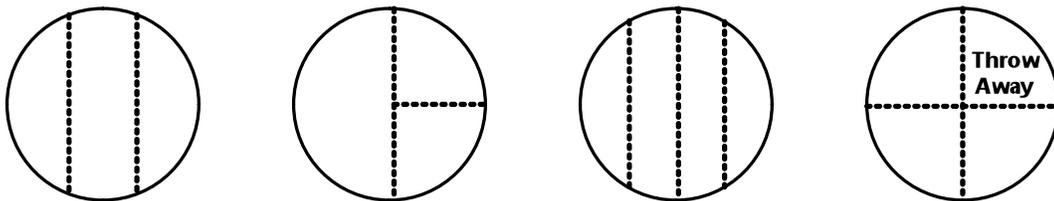


Figure 26. Chloe’s predicted K-2 strategies for sharing a cake among 3 pirates.

In Sadie’s example from the previous section, she identified both common misconceptions and landmarks regarding equipartitioning, predicting that students might create the appropriate number of unequal-sized parts (i.e., strategy 3) for sharing a circular cake. She anticipated that children might create  $n$  cuts, instead of  $n - 1$  cuts for sharing a rectangular cake (i.e., strategy 3), making an inappropriate number of equal-sized pieces.

Sadie predicted that children might create compositions for both tasks, recognizing children might reason that each person could get more than one piece of cake (i.e., strategy 4). This strategy is a very important landmark in children's understanding of equipartitioning. It is crucial that teachers have the ability to not only predict successful student strategies and behaviors but also unsuccessful strategies and behaviors. Teachers must be able to identify specific strategies and behaviors that indicate the difficulties students encounter, as well as behaviors and strategies that mark students' progress.

### *Informing Instructional Practices*

After an equipartitioning learning trajectory had been introduced, PSTs used an equipartitioning learning trajectory to inform their questioning and other pedagogical decisions, when interacting with individual students. In some cases these questions and pedagogical decisions were appropriate, while in other situations they were not.

While PSTs used an equipartitioning learning trajectory to incorporate models of student thinking into instructional practices that involved working with individual students, instructional activities that focused on individual student thinking and work with clinical interviews did not translate into successful planning for an entire class for most PSTs, even after the equipartitioning learning trajectory had been fully implemented. PSTs needed more support in using an equipartitioning learning trajectory and progress variable for writing and implementing lesson plans for whole class instruction. It is conjectured that an absence of curricular materials amenable to the equipartitioning learning trajectory was also a factor in PSTs inability to plan and implement whole class lessons. While it is encouraging that most PSTs were able to incorporate more precise and robust models of student thinking in practice

that involved working with an individual child or small group of children, an important part of teacher education is preparing PSTs to teach an entire class.

To varying degrees, PSTs were able to use an equipartitioning learning trajectory to incorporate models of student thinking into instructional practices to inform their next hypothetical steps with students. PSTs proposed asking a variety of appropriate hypothetical questions after watching a video clip of a 5-year-old K student named Emma fairly sharing a collection of pirate treasure among 3 pirates. See Appendix J for a complete synopsis of the video clips from the Individual Clinical Interview Analysis. At the beginning of the clip, Emma had already built four  $2 \times 3$  arrays to fairly share the treasure for 4 pirates, naming each share 6 pieces of treasure. To share for 3 pirates, Emma put the treasure back together in one pile, built a  $2 \times 4$  array, and counted the pieces of treasure out loud. As she built two more  $2 \times 4$  arrays, Emma said she was using “8, ” and “8 is the magic number.” At the end, the interviewer asked Emma how she knew that 8 was the “magic number.” Emma provided the following explanation:

Ok. Last time it was 6. Now you just added 2 more [points to first row of coins in one of the  $2 \times 4$  arrays]. ‘Cause he had six [as she points to the place on the table where the 4<sup>th</sup> pirate’s treasure was in previously sharing for 3 pirates]. I added two more to each one [points to the first row of coins in each  $2 \times 4$  array], which makes 6.

Some PSTs wanted to ask general questions relating to this particular behavior to learn more about how Emma reasoned. For example, PSTs, like Keisha, suggested asking Emma to, “Explain your reasoning.” Other PSTs, like Chloe, wanted to ask Emma to justify

her strategy, “Does each pirate have a fair share? How do you know this?” Based on their observations, these PSTs were able to identify specific equipartitioning behaviors and verbalizations were able to create a question that could potentially provide more insight into Emma’s thinking.

Others wanted to ask specific questions to find out more about Emma’s reasoning. Deb specifically wanted to know, “What is your reasoning in going from 4 to 3 pirates?” Angela, Claire, Sandra, and Tina wanted to ask Emma, “How did you know 8 is the magic number?” or “How did you know that they each got 8?” Similarly, Angela, Hillary, Kim, and Sadie would ask, “How did you know each got 2 more?” or “Why did you take one group away and give two pieces to the remaining?” It is plausible that Emma redistributed the pieces of treasure from the 4<sup>th</sup> pirate’s share, distributing two pieces to each of the other 3 pirates. These PSTs noticed this behavior and were aware that they needed more information about Emma’s actions to create a model of her thinking.

Other PSTs thought that knowing more about why Emma pushed all the treasure back together would help inform their model of Emma’s understanding. For example, Bonnie and Sadie responded, “Why did you put the coins back in a pile?” Lisa would ask, “Why did you begin task 2 over?” These types of questions are significant because they have the potential to clarify whether Emma knew she was going to redistribute the 4<sup>th</sup> pirate’s share before she piled the coins back together.

There is limited evidence that PSTs suggested asking inappropriate questions or responded with unrelated tasks. For example, Laney responded that her next instructional step would be to have Emma share the pirate treasure fairly among 5 pirates. This

instructional task is inappropriate because 24 is not divisible by 5, without a remainder. Laney's suggestion would transform this Case B task into a Case D. Even if it were possible to split a coin into 5 equal-sized pieces, a 5-split is usually challenging for K students.

### Revised Conjectures

In this section, the initial conjectures and revised conjectures will be discussed. First, the original conjecture of the study will be stated. Then, a table illustrating how this conjecture changed will be presented. Finally, each refined conjecture will be presented.

#### *Conjecture 1*

*Initial.* One of the initial conjectures of the study was that an equipartitioning learning trajectory could be utilized as a tool to assist PSTs in improving their ability to compare students' behaviors and verbalizations to their own to deepen their own understanding of the structure of mathematics and knowledge for teaching mathematics.

Table 17. PSTs' Conceptual Development of Conjecture 1.

CONJECTURE 1	Pre-test	S2	CB1	CB2	ICIA	Post-test	CI
PSTs were unable to utilize mathematical language to effectively communicate their equipartitioning strategies and behaviors.	X	X	X				
PSTs use precise language with equipartitioning as lens for communicating their own strategies and those of students.				X	X	X	X
PSTs used a single method to solve equipartitioning tasks.	X	X (A/B)	X (A/B)	X (C/D)			X (C/D)
Some PSTs used multiple strategies to solve equipartitioning tasks.			X some (A/B)	X (A/B)	X (A/B)	X	X (A/B)
Some PSTs used strategies unrelated to equipartitioning, like breaking, and do not connect these to equipartitioning.		X	X				
PSTs content knowledge of equipartitioning begins to increase.		X					
PSTs content knowledge of equipartitioning increased significantly (statistically).						X	
PSTs inappropriately used additive reasoning to solve equipartitioning tasks.	X	X					
PSTs appropriately used multiplicative reasoning to solve equipartitioning tasks.			X some	X	X	X	X
Some PSTs held misconceptions of Cases C and D.				X		X	X
PSTs make connections between equipartitioning and other areas of rational number reasoning, like multiplication and division.				X some	X	X	X
PSTs were unable to identify the mathematical structures within their representations and models (e.g., arrays and compositions) as they described their own strategies and behaviors.		X	X				
PSTs were unable to justify.		X					
PSTs justify and name shares.					X		X (A/B)
PSTs recognize that children often reason differently than they do about equipartitioning tasks.			X				
PSTs adopt children's strategies as own.					X		X
PSTs value student's voice and perspective.						X	X

*Revised.* PSTs use an equipartitioning learning trajectory to restructure their own understanding of mathematics and their knowledge for teaching mathematics. To varying degrees, this knowledge goes beyond equipartitioning, connecting it with other areas of rational number reasoning. An equipartitioning learning trajectory assists PSTs in comparing students' models of thinking to their own, allowing PSTs to distinguish appropriate strategies for students and themselves. An equipartitioning learning trajectory supported the awareness of diverse student strategies that were often adopted by PSTs themselves.

*Conjecture 2*

*Initial.* Another conjecture of this study was that an equipartitioning learning trajectory could be utilized as a tool to assist PSTs in building robust and precise models of student thinking by improving their ability to identify students' behaviors and verbalizations with respect to equipartitioning.

Table 18. PSTs' Conceptual Development of Conjecture 2.

CONJECTURE 2	Pre-test	S2	CB1	CB2	ICIA	Post-test	CI
PSTs were unable to create models.		X	X				
PSTs' models of student thinking were general descriptions of children's feelings and nonmathematical behaviors that do not focus on children's strategies, behaviors, and verbalizations with respect to equipartitioning.			X				
PSTs' models of student thinking are evaluative inferences (e.g., "correct" or "incorrect") not identifying students' strategies, behaviors, and verbalizations with respect to equipartitioning.			X				
PSTs were able to identify and describe behaviors and verbalizations relating to children's strategies for equipartitioning, using equipartitioning as a lens.				X	X	X	X
Some PSTs were able to identify children's strategies, behaviors, and verbalizations in relation to justification, naming, and reassembly.				X	X	X	X

*Revised.* PSTs use an equipartitioning learning trajectory in building models of student thinking that are focused and qualified. They are more sensitive to students' strategies, behaviors, and verbalizations in relation to equipartitioning. PSTs are able to make connections between the strategies, behaviors, and verbalizations that they notice through observation to the equipartitioning learning trajectory, which coordinated behaviors and verbalizations with student thinking.

### *Conjecture 3*

*Initial.* Finally, a third conjecture of this study was that PSTs would be able to use an equipartitioning learning trajectory to incorporate models of student thinking into instructional practices, so that teachers are better able to:

- predict diverse student strategies, behaviors, and verbalizations;
- identify common misconceptions and landmarks; and
- inform instructional practices.

Table 19. PSTs' Conceptual Development of Conjecture 3.

<b>CONJECTURE 3</b>	<b>Pre-test</b>	<b>S2</b>	<b>CB1</b>	<b>CB2</b>	<b>ICIA</b>	<b>Post-test</b>	<b>CI</b>
PSTs were unable to predict successful and unsuccessful student strategies, behaviors, and verbalizations, relating to equipartitioning.		X					
PSTs held a deficit model of student thinking, predicting that K-2 children would not be able to generate successful solutions to equipartitioning tasks.		X	X		X (few)		X (few)
PSTs anticipated that children would successfully solve equipartitioning tasks, using the same strategies and behaviors as themselves.			X				
PSTs predicted that K-2 children would be “correct” or “incorrect,” without anticipating diverse student strategies, behaviors, and verbalizations.			X				
PSTs were able to use an equipartitioning learning trajectory to incorporate models of student thinking into instructional practices to inform their next hypothetical steps with students.					X		
PSTs anticipated a range of student strategies and behaviors from the equipartitioning learning trajectory that included both successful and unsuccessful student performance.					X	X	X
PSTs were able to predict multiple student strategies.					X	X	X
PSTs identified both common misconceptions and landmarks relating to equipartitioning.					X	X	X

*Revised.* PSTs use an equipartitioning learning trajectory to incorporate models of student thinking into instructional practices that involve working with individual students. The equipartitioning learning trajectory assists PSTs in predicting a range of diverse student strategies, behaviors, and verbalizations and in identifying common misconceptions and landmarks. When interacting with individual students, PSTs use an equipartitioning learning trajectory to inform their questioning and other pedagogical decisions. PSTs need support in using an equipartitioning learning trajectory for writing and implementing lesson plans for whole class instruction and need curricular materials amenable to the equipartitioning learning trajectory.

## CHAPTER 5

### DISCUSSION AND CONCLUSIONS

In Chapter 4, results from the data analysis were presented relating to the ways in which PSTs used an equipartitioning learning trajectory to build models of student thinking. Chapter 5 discusses how these results relate to the research question, addresses the limitations of this study, and discusses implications for future research, which will include suggestions for a use model for teacher education.

#### An Answer to the Research Question

This study examined the following research question: To what extent and in what ways can pre-service elementary teachers use a learning trajectory for equipartitioning to build models of student thinking? Using the Hollebrands et al. (2007) categories of *describing*, *comparing*, *inferring*, and *restructuring*, instances were identified relating to PSTs' use of each model building process at different places in the design study in relation to the equipartitioning learning trajectory. Within each process, themes emerged in relation to PSTs' knowledge of equipartitioning and the learning trajectory. These themes were described, as well as changes to PSTs' knowledge of equipartitioning and the learning trajectory. To varying degrees, PSTs used an equipartitioning learning trajectory to:

- deepen their understanding of the structure of mathematics and knowledge for teaching mathematics;
- build more precise and adequate models of student thinking; and,
- incorporate models of student thinking into instructional practices.

*Deepening PSTs' Understanding of Mathematics and Knowledge for Teaching Mathematics*

PSTs used an equipartitioning learning trajectory to restructure their own understanding of mathematics and their knowledge for teaching mathematics, to varying degrees. Like PSTs in the Ambrose (2004) study, PSTs increased their understanding by using multiple methods, rather than a single method, to solve Case A and B equipartitioning tasks. While Ambrose's PSTs began to believe that it is important for teachers to present multiple solutions, PSTs in this study went a bit further by adopting children's strategies from the equipartitioning learning trajectory in relation to Cases A and B as their own. These PSTs did not just see the significance in using multiple strategies but specifically valued the use of children's strategies in addition to their own initial methods. It is important that teachers have the ability to distinguish between their own strategies and behaviors and those that would be appropriate for young children.

While some PSTs showed evidence that their knowledge went beyond equipartitioning, connecting equipartitioning with other areas of rational number reasoning by using multiplicative reasoning appropriately, others continued to use additive thinking inappropriately, even at the conclusion of the study. Even though PSTs' gain scores significantly increased after the instructional activities, most PSTs scored very low on items from the post-test that extended PSTs' thinking beyond equipartitioning tasks to other areas of rational number reasoning in comparison to items relating to Cases A – D on the assessment. Moreover, most of PSTs' increases in knowledge can be attributed to increased knowledge in relation to Cases A and B rather than Cases C and D. In investigating the relationship between children's multiplicative reasoning and repeated halving through paper

folding tasks, Empson and Turner (2006) found that children initially connected the action of folding and the outcome in *non-recursive* ways. Few children used such a *recursive* strategy to connect the fold and subsequent number of partitions. On the pre-test, 76% of PSTs used non-recursive strategies that were nonproductive, often repeating a half fold when there was an odd factor in the number of partitions that needed to be created. On the post-test, only 29% of PSTs utilized non-recursive strategies in the paper folding tasks. The majority of the PSTs needed concrete materials for the paper folding item and only provided one or two methods. Like the children in the Empson and Turner study, very few PSTs used recursive strategies. Similarly, Sawada and Pothier (1984) found that elementary school children prefer to equipartition objects into an even number of parts (i.e., 2 and 4), rather than an odd number of parts (i.e. 3 and 5), regardless of the geometric shape. Some PSTs held the same misconceptions as young children. While some PSTs began to increase their understanding of other rational number reasoning concepts, this suggests that other learning trajectories relating to rational number reasoning may be necessary to support PSTs in restructuring their knowledge of rational number reasoning concepts that go beyond equipartitioning.

PSTs not only increased their own knowledge of equipartitioning after the trajectory was introduced, they became more knowledgeable about the types of strategies that children use, as did PSTs in the McDonough, Clarke, and Clarke's (2002) study. In addition to increasing their understanding of the various methods that children use in equipartitioning, PSTs were able to identify children's strategies and behaviors in relation to their ability to name, represent, and justify. They also connected these strategies and behaviors to emergent properties, such as reassembly and redistribution.

In the Hollebrands et al. (2007) study, PSTs comparisons most often resulted in PSTs creating student models that were similar to their own thinking. However, while PSTs initially did this in this study, PSTs comparisons evolved into seeing student models as different than their own. For some PSTs, student strategies and behaviors were adopted by PSTs, while other distinguished between strategies and behaviors for themselves and strategies and behaviors for children. The equipartitioning learning trajectory helped PSTs organize these distinctions and become sensitive to recognizing that students' work is valuable and appropriate along a path of development, rather than being simply correct or incorrect, especially when it differed from their own reasoning. PSTs in this study made many other types of comparisons that have already been discussed.

While Hollebrands et al. (2007) propose that PSTs build models of student thinking using the processes of describing, comparing, inferring, and restructuring, I found that the process of restructuring is very different than the other three. In fact, describing, comparing and, and inferring often led PSTs to restructure their own knowledge of mathematics and knowledge for teaching equipartitioning, especially in the case of comparing. For instance, the comparisons between PSTs' predictions of K-2 students' strategies, behaviors, and verbalizations to students' actual performance often led PSTs to restructure their understanding of how children reason about equipartitioning. These comparisons and others acted as a mechanism for restructuring. I hypothesize that the equipartitioning learning trajectory assisted PSTs in coordinating the processes of describing, comparing, and inferring and that it is the *coordination* of these processes that result in restructuring one's knowledge of mathematics and the knowledge for teaching mathematics. In addition, I view the ability to

*identify* equipartitioning behaviors and verbalizations as an important process in building models. Before PSTs can describe student behaviors and verbalizations, engage in comparing, or make inferences, PSTs must first identify behaviors and verbalizations that related to specific constructs.

### *Building More Precise and Adequate Models of Student Thinking*

Most PSTs used an equipartitioning learning trajectory to build more precise and adequate models of student thinking by improving their ability to identify students' strategies, behaviors, and verbalizations with respect to equipartitioning. Without the use of an equipartitioning learning trajectory, other factors often influenced the ways that PSTs created models of students' thinking. In these cases, the SBTE or curriculum was usually a strong influence on PSTs' models. Previous work by Philipp et al. (2007) indicate that PSTs placement in elementary mathematics classrooms hindered the establishment of beliefs that were promoted in a methods course. Without the use of an equipartitioning learning trajectory, some PSTs in this study also held beliefs that were contradicted in the methods course that can be linked to experiences in their elementary school classrooms. Some SBTEs explicitly told PSTs that children would not be able to solve equipartitioning tasks or advised PSTs to make modifications to turn equipartitioning tasks into counting tasks. The underdevelopment of equipartitioning in the previous state standards and curricular materials that emphasize counting in the early grades may have influenced SBTE's beliefs about what children can and cannot do, which impacted PSTs beliefs about students' capabilities. Some of the SBTEs were still implementing the previous standard course of study, not the revised version that included equipartitioning.

After the learning trajectory for equipartitioning was introduced, PSTs began to build models based on this trajectory. PSTs need a framework or a tool like an equipartitioning learning trajectory to help them identify strategies and behaviors. The equipartitioning learning trajectory has the potential to help PSTs ground their models in theory and student performance, rather than relying solely on information provided by their SBTEs. Perhaps, this knowledge will be one of the factors that influence these PSTs as they become classroom teachers.

Like Philipp et al. (2007), there is evidence that focusing on children's thinking about mathematics, through the use of video, deepened PSTs' knowledge of mathematics and supported changes in their beliefs. While Ambrose (2004) and McDonough, Clarke, and Clarke's (2002) found that experiences that included working with children changed PSTs' understanding of children's thinking, Philipp et al. study found that PSTs who analyzed videos and who analyzed videos and worked with children, showed significant changes in their beliefs as compared to PSTs who just worked with children. The results of this study were similar and suggest that working with students alone is not enough to prepare PSTs to teach. It was not until after the learning trajectory was introduced with an intentional focus on student thinking that a shift in PSTs thinking occurred. In accordance with Philipp et al., PSTs need to consider student thinking within a structured environment where they have the opportunity to reflect on their experiences and student thinking. The equipartitioning learning trajectory has the potential to be used as a tool to inform and guide PSTs' reflection.

### *Incorporating Models of Student Thinking into Instructional Practices*

To varying degrees, PSTs used an equipartitioning learning trajectory to incorporate models of student thinking into instructional practices that involved working with individual students, so that PSTs were better able to:

- predict diverse student strategies, behaviors, and verbalizations;
- identify common misconceptions and landmarks; and,
- inform instructional practices.

McDonough, Clarke, and Clarke's (2002) work with PSTs in using interviews resulted in PSTs becoming more knowledgeable about the types of strategies that children use.

Likewise, PSTs in this study became more knowledgeable about the different types of strategies that children use in equipartitioning. In addition, PSTs in this study began to understand the levels of sophistication in relation to various methods after the equipartitioning learning trajectory had been introduced. This suggests that while clinical interviews play an important role in increasing PSTs' knowledge of student strategies, the equipartitioning learning trajectory has the potential to be used as a tool to distinguish between more and less sophisticated understandings.

Like Tirosh (2000), focusing explicitly on children's misconceptions during instruction raised PSTs' awareness of common difficulties that students encounter on the way to developing more sophisticated understandings. The learning trajectory was more than a tool to help PSTs identify these misconceptions; it has the potential to assist teachers in identifying more and less sophisticated understanding in relation to major benchmarks and misconceptions.

## Limitations

This study has some limitations. All of the participants were enrolled in an elementary education program where PSTs specialized in teaching either mathematics or science, taking a considerable amount of coursework in mathematics and/or science content before their participation in this methods course. Also, as part of the requirements in this mathematics methods course, PSTs spent an extensive amount of time in K-2 classrooms. Since these program components are unique, these findings cannot be generalized to a wider population of PSTs.

Although I collected and analyzed multiple sources of data (i.e., pre- and post-test assessments, video recordings of class meetings, audio recordings of small group discussions during class meetings, video recordings of PSTs engaged in clinical interviews, artifacts, my own field notes) during this study, I was both the teacher and researcher. Since I implemented, collected, and analyzed the data, my own theoretical perspectives and prior experiences influenced the results. In order to convey to the reader what these potential biases and assumptions might be, I described my own theoretical perspective in detail in Chapter 3. In order to minimize this bias, I used triangulation to confirm or disconfirm themes that emerged from the analysis of the data.

## Implications

One implication of this study is that mathematics courses alone are inadequate for preparing PSTs to refine the mathematical knowledge that is needed to teach elementary school mathematics. All of the PSTs in this study had completed at least Calculus I but still showed weaknesses in their understanding of rational number reasoning at the

beginning of the study. Previous work by Ma (1999) suggests that more exposure to advanced mathematics in high school and college does not necessarily lead to a more robust understanding of elementary school mathematics. Given that PSTs had completed an extensive number of high-level mathematics courses in the mathematics department prior to this study, it is evident that mathematics methods courses that focus on both content and knowledge for teaching mathematics support PSTs in refining their content knowledge of mathematics. Further, this study shows that including an emphasis on equipartitioning greatly increased PSTs knowledge of mathematics, which will improve their preparation for teaching mathematics at the elementary school level.

Another implication of this study is that PSTs need additional support in using a learning trajectory to plan and implement whole class lessons on equipartitioning. While focusing instructional activities on clinical interviews and individual student thinking supported PSTs use of a learning trajectory in instructional practices related to working one-on-one with students or small groups of students, these instructional activities were not enough to help PSTs build on this knowledge to prepare and teach a whole class lesson. I conjecture that part of this additional support could come in the form of curricular materials, if they existed. Current elementary mathematics textbooks or other curricular materials do not put emphasis on the development of equipartitioning. I hypothesize that the absence of coherent curricular materials for equipartitioning contributed to PSTs inability to plan and teach equipartitioning lessons. The development of these materials could be used to prepare PSTs as they learn to teach a whole class.

D'Ambrosio and Campos (1992) found that PSTs began to question typical instructional practices of teaching through using research in their work with students. Further, PSTs were able to conceptualize alternatives to common instructional sequences found in textbooks. This suggests that research on students' thinking relating to mathematical concepts should be an important component in teacher education and may support PSTs as they learn to teach a whole class. While the learning trajectory was built on Confrey's synthesis work of equipartitioning and the on-going empirical work by the DELTA research group, the research findings are embedded within the learning trajectory itself. The underlying research may not have been apparent to PSTs in the representation of the learning trajectory that was presented in this study. This implies that research may need to be made explicit to PSTs within a methods course to help them connect the literature and the learning trajectory.

In addition to making research explicit in using a learning trajectories approach, the use of instructional materials, such as the diagnostic assessments that are currently being developed by the DELTA research group, may help prepare PSTs as they learn about whole class instruction in the absence of adequate curricular materials. Diagnostic assessments could provide instructional materials for PSTs to assess student thinking. This feedback could be used by PSTs to inform the planning of whole class instruction in conjunction with the learning trajectory. Currently, the best methods for implementing diagnostic assessments, along with a learning trajectory, are unclear. In turn, these research findings could be used to inform the ways in which diagnostic assessments could be refined to make them accessible and usable to teachers. This suggests that there may be value in conducting research on

learning trajectories and diagnostic assessments while simultaneously conducting research on teacher education.

### *Implications for a Use Model in Teacher Education*

Finally, findings from this study have implications for a use model for drawing on a learning trajectories approach in teacher education. A use model describes the components of using a learning trajectories approach within instruction in a mathematics methods course. At a minimum, a use model for preparing PSTs to use a learning trajectory as they learn to engage in the practice of teaching should include the following components: exploration of equipartitioning tasks, introduction to the learning trajectory as a tool for coordinating student performance with cognition, use of video exemplars of clinical interviews and analysis of clinical interviews that focuses on student thinking, instruction on the conduct of clinical interviews, experience working with K-2 students on equipartitioning tasks and clinical interviews, structured reflection on these experiences, and explicit instruction on using curriculum.

Before an equipartitioning learning trajectory has been introduced, PSTs' should explore equipartitioning tasks so that they can begin to consider the mathematical underpinnings of the tasks. Explicit connections should be made between equipartitioning and other rational number reasoning constructs since some PSTs do not have a robust understanding of rational number reasoning.

The equipartitioning learning trajectory should be introduced as a tool to coordinate cognition with student strategies, behaviors, and verbalizations. Instructional design (i.e.,

tasks, instructional sequence, media/tools/representations) should be connected to the learning trajectory and student thinking.

The use of video exemplars of clinical interviews and analysis of clinical interviews is crucial. The extensive use of the interviews and analysis of student thinking should be used when introducing the components of the equipartitioning learning trajectory and progress variable. These resources not only provide concrete examples of authentic student performance and provide opportunities to discuss student thinking, but they also provide opportunities for PSTs to realize K-2 children's capabilities. Since many PSTs initially underestimate children's ability, which is often reinforced by their SBTEs, PSTs own beliefs about students' capabilities are challenged. Like in the McDonough, Clarke, and Clarke (2002) study, clinical interviews and PSTs subsequent reflection were powerful tools to gain insight into children's thinking, which may be a catalyst for initiating change in how PSTs understand children's reasoning.

Since most PSTs usually have limited prior experience with children, they need opportunities to work with K-2 students on equipartitioning tasks and clinical interviews. PSTs should be asked to solve equipartitioning tasks themselves and make predictions about how they think children will solve these tasks. Many PSTs begin to restructure their own knowledge of equipartitioning and knowledge for equipartitioning by making comparisons between their own reasoning and children's thinking, as well as comparing their anticipations of children's strategies, behaviors, and verbalizations with actual student performance. As students begin to restructure their understanding of how children reason, it may also be valuable to include a discussion that explicitly addresses deficit models of student thinking.

A curricular component should be added to assist PSTs with using a learning trajectory for lesson planning and teaching a whole class. Instead of initially requiring PSTs to write and implement lesson plans, a better place to start may be to engage PSTs in a curriculum analysis on equipartitioning. The curriculum analysis could not only explore the different ways that equipartitioning is developed in different materials, but it could include instruction on modifying curricular materials with a focus on tasks, task sequencing, and the potential impact in using different materials and tools (i.e., playdough, counters, paper, etc.). In addition, to focusing on tasks and the sequencing of tasks, instructional activities should highlight which tasks (i.e., Cases A, B, C, or D) are developmentally appropriate for children at specific grade levels. Given that few curricular materials exist that adequately address equipartitioning, it is highly likely that PSTs will need to engage in this type of practice when they begin student teaching and/or enter their first year in the classroom. Regardless of the type of curricular piece that is implemented, there needs to be an explicit link between the importance of focusing on individual student thinking and its relationship to teaching equipartitioning for an entire class.

#### Future Research

In this study, I explored the extent to which pre-service elementary teachers use a learning trajectory for equipartitioning to build models of student thinking. At the time of my study, the DELTA research team had only completed the first version of the equipartitioning learning trajectory. Future research should explore how PSTs' engage in the model building process using learning trajectories for other areas of rational number reasoning, like multiplication and division or length, area, and volume. Further, how would this model

building process be influence by *only* using learning trajectories for rational number reasoning as an approach to organizing a mathematics methods course for elementary teachers? In other words, what would PSTs model building process look like if an instructor used learning trajectories for equipartitioning, multiplication and division, length, area, and volume, ratio, etc. in preparing pre-service teachers?

One important finding of this study was that PSTs were unprepared to plan and teach whole class lessons relating to equipartitioning concepts. The instructional activities and instructional activities in this study did not support PSTs ability to use their knowledge of individual students to consider the collective, or whole class. What mechanism and/or support do pre-service teachers need in order to use a learning trajectory to organize whole class instruction? Perhaps, one piece of this puzzle is relates to assessing student understanding of the whole class. Since the completion of this study, the DELTA research team has moved their work forward in developing diagnostic assessments for elementary classrooms. It would be worth exploring how diagnostic assessments and learning trajectories can be utilized in conjunction to prepare PSTs as they learn to teach.

Finally, future research should focus on using learning trajectories in teacher preparation in a broader sense. Learning trajectories have the potential to be used to organize professional development for in-service teachers (ISTs). Studies should investigate what ISTs model building process would look like. Research should also focus on how PSTs' use of a learning trajectory in the model building process compares to ISTs' model building process.

## Conclusions

This study contributes to the knowledge base of how PSTs build models of student thinking using an equipartitioning learning trajectory as they learn to engage in the practice of teaching. According to Corcoran et al. (2009), in order to ensure that all students succeed, the norms of practice should move towards a model where teachers are

continually seeking evidence on whether the students are on track to learning what they need to if they are to reach the goals, along with tracking indicators of what problems they might be having, and then for making pedagogical responses to that evidence designed to keep their students on track, or to get them back on track, moving toward meeting the goals (p. 8).

Further, Corcoran et al. specify that teachers need to understand how students' learning of specific concepts, like equipartitioning, develops over time and that teachers need to be familiar with appropriate ways to respond to students' progress or difficulties. Findings from this study indicate that an equipartitioning learning trajectory has the potential to be used as a tool to prepare teachers to shift towards the model advocated by Corcoran et al. An equipartitioning learning trajectory can be used as a tool to coordinate:

- students' behaviors, strategies, and verbalizations with cognition;
- various models of student thinking; and,
- models of student thinking with instructional practices.

First, an equipartitioning learning trajectory enables PSTs to identify and describe specific equipartitioning behaviors, strategies, and verbalizations, allowing PSTs to coordinate the evidence they have gathered through describing behaviors, strategies, and verbalizations with

cognitive processes related to equipartitioning in the model building process. Second, PSTs engage in comparing their own reasoning to student thinking in constructing models of how students' reason. PSTs also build models of student thinking by making comparisons between their predictions of students' behaviors, strategies, and verbalizations with actual student performance, as well as making comparisons between students at different grade levels and students at different levels of understanding. An equipartitioning learning trajectory supports PSTs in coordinating these different comparisons by articulating how reasoning about equipartitioning develops over time, so that PSTs can identify where specific models of student thinking are located along this continuum. Finally, an equipartitioning learning trajectory supports the understanding of one model in relation to other models, so that these models are useful for PSTs in planning, teaching, and assessing. When various models of student thinking are coordinated with cognition and one another, PSTs can use evidence gathered relating to student thinking to inform pedagogical decisions in relation to students' progress towards equipartitioning landmarks and difficulties with common obstacles.

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## APPENDICES

## Appendix A

### **Equipartitioning Tasks**

#### **Sharing Tasks**

- Two friends want to share a bag of crackers. Open your bag. How would you share the crackers fairly between the two friends?
- Three friends want to share a bag of crackers. Open your bag. How would you share the crackers fairly between the three friends?
- Four friends want to share a bag of crackers. Open your bag. How would you share the crackers fairly between the four friends?

#### **Birthday Party Tasks**

- Rosa is having a birthday party. She wants to share her birthday cake fairly with Mario. How could she share the cake fairly?
- Rosa is having a birthday party. She wants to share her birthday cake fairly with Mario and Emma. How could she share the cake fairly?
- Rosa is having a birthday party. She wants to share her birthday cake fairly with Mario, Emma, and Kwami. How could she share the cake?
- If Rosa's mother cut the birthday cake into 8 pieces of the same size, how could she cut the cake?
- If Rosa's father cut the birthday cake into 12 pieces of the same size, how could he cut the cake?
- If Rosa's birthday cake were cut into 18 pieces of the same size, how many people could have a piece of cake?

#### **Pizza Party Tasks**

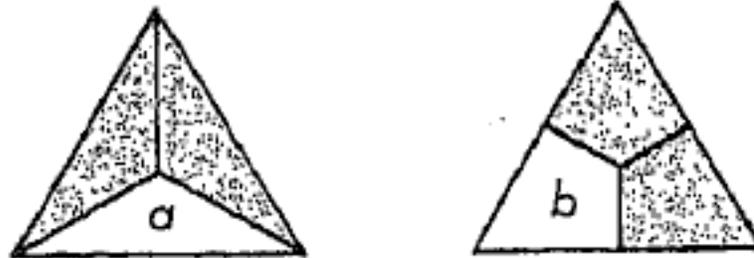
- Two brothers want to share a mini pizza fairly. How could the brothers share the pizza fairly?
- Three sisters want to share a mini pizza fairly. How could the sisters share the pizza fairly?

- A family of four wants to share a pizza fairly. How could the family share the pizza fairly?

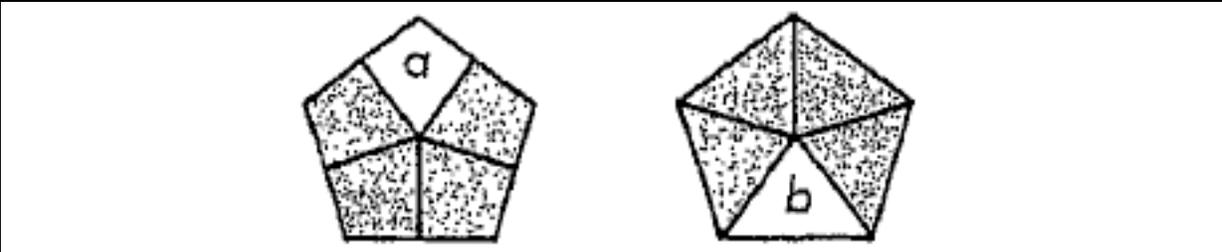
### Paper Folding Tasks

(adapted from Empson & Turner, 2006)

- Before you fold the paper, make a prediction. Record your prediction. Fold this piece of paper in half one time, how many equal parts will you create?
- Before you fold the paper, make a prediction. Record your prediction. Fold this piece of paper in half two times, how many equal parts will you create?
- Before you fold the paper, make a prediction. Record your prediction. Fold this piece of paper in half three times, how many equal parts will you create?
- Before you fold the paper, make a prediction. Record your prediction. Fold this piece of paper in half four times, how many equal parts will you create?
- Fold this piece of paper in half, in thirds, and in thirds, again. How many equal parts will you create?
- I folded a piece of paper and created 24 equal parts. How could I have folded the paper?
- I folded a piece of paper and got nine equal parts. How could I have folded the paper?
- Camilla folded a piece of paper into three equal parts, then eight equal parts. Derrick folded his piece of paper into six equal parts. If he wants to make exactly as many parts as Camilla, how many parts should he fold his paper into next?

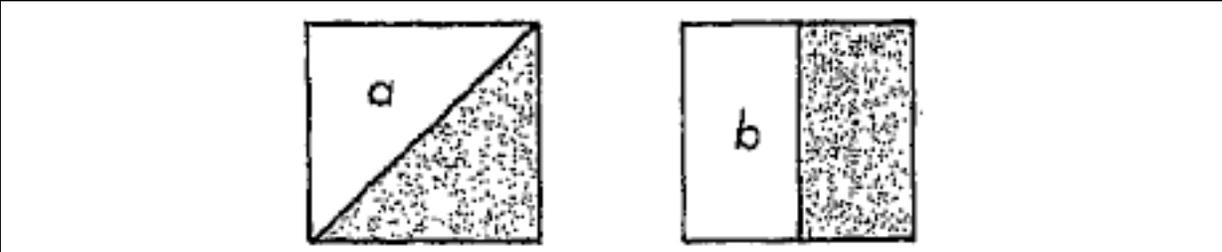


Are parts  $a$  and  $b$  equal in area, or is one greater than the other? Show that your answer is true.



Are parts  $a$  and  $b$  equal in area, or is one greater than the other? Show that your answer is true.

\*Adapted from Pothier and Sawada (1990)



Are parts  $a$  and  $b$  equal in area, or is one greater than the other? Show that your answer is true.



Are parts  $a$  and  $b$  equal in area, or is one greater than the other? Show that your answer is true.

\*Adapted from Pothier and Sawada (1990)

**Classroom-based Assignment #1**

**SELECT TASKS**

During your field experience, you will implement two tasks from each group, with one student. Below is a list of different types of tasks that were discussed in class. Select two tasks from each group (8 tasks total):

- Paper Folding Tasks
- Sharing Tasks
- Birthday Party Tasks
- Pizza Party Tasks

**COMPLETE PARTS I, II, AND III *BEFORE* YOU IMPLEMENT THE TASKS WITH A STUDENT.**

**Part I:**

- State each task.

**Part II:**

For each task, do the following:

- Provide your solution(s).
- Explain how you solved each task.
  - Be sure to include any drawings or other representations that support your explanation.
- Give a justification for your solution(s).
  - Be sure to include any drawings or other representations that support your justification(s).

**Part III:**

For each task, do the following:

- Provide any solution(s) that you anticipate students might give.
- Explain how you think students will solve each task.
  - Be sure to include any drawings or other representations that support your explanation.
- Give justifications that you anticipate students might provide to support their solution(s).
  - Be sure to include any drawings or other representations that support your anticipation of student justification(s).

**IMPLEMENT TASKS**

During your field experience, implement two tasks from each group, with one student.

**COMPLETE PARTS IV and V *AFTER* YOU IMPLEMENT THE TASKS WITH A STUDENT.**

**Part IV:**

For each task, do the following:

- Provide the solution(s) that the student gave.

- Explain how the student solved each task.
  - Be sure to include any drawings or other representations that support your explanation.
- Give any justifications that the student gave to support his or her solution(s).
  - Be sure to include any drawings or other representations that support your description of student justification(s).

**Part V:**

1. Based on your interaction with the student, what do you think the student knows? Provide evidence to support your claim?
2. What would you do next with this student and why?
3. Did anything regarding the student's actions or verbalizations surprise you? Explain.
4. What were the advantages and disadvantages of using the different types of manipulatives (i.e., playdough, paper, counters, etc.)?
5. Why did you select the tasks that you implemented?
6. If you were to use this task again, what would you do differently? What would you repeat?
7. What did you learn about how students reason from this experience?

**Classroom-based Assignment #2**

<p><b>Part I: Design a Lesson Plan</b></p> <p>In small groups, create a lesson plan on equipartitioning that is appropriate for children in grades K-2. Use the Elementary Education Lesson Plan form (NCSU Lesson Plan Sheet) that is available at the department website (<a href="http://ced.ncsu.edu/elementaryed/forms.php">http://ced.ncsu.edu/elementaryed/forms.php</a>). In your lesson plan, be sure to include any tasks that would be implemented in your lesson. You can submit <i>one</i> lesson plan per group. Along with the lesson plan, submit a summary of each group meeting or planning session.</p>
<p><b>Part II: Tasks</b></p> <p>For each task in your lesson plan, provide your solution to the equipartitioning task.</p>
<p><b>Part III: Implement Lesson</b></p> <p>Communicate with your teacher and make arrangements to implement the lesson with one student, or a small group of students. Implement this lesson with a student or small group of students during your field experience.</p>
<p><b>Part IV: Reflection</b></p> <ol style="list-style-type: none"><li>1. Provide a short description of the student(s) you worked with. Be sure to include their age, grade, gender, and race.</li><li>2. Based on your interaction with the student(s), what do you think the student(s) knows? What do you think the student(s) had difficulty with? Provide specific evidence to support your claims?</li><li>3. What would you do next with this student(s) and why?</li><li>4. Did anything regarding the student's actions or verbalizations surprise you? Explain.</li><li>5. What were the advantages and disadvantages of using the materials you chose (i.e., playdough, paper, counters, etc.)?</li><li>6. Why did you select the tasks that you implemented?</li><li>7. If you were to use this lesson again, what would you do differently? What would you repeat?</li><li>8. What did you learn about how students reason from this experience?</li></ol>

### Individual Clinical Interview Analysis

#### **TASKS**

A child is given a bag containing 24 pieces of pirate treasure. The child is not told how many pieces of treasure are in the bag. The teacher asks the child:

**Task 1:** Can you share the pirate treasure fairly among 4 pirates? Show me how you would do this.

**Task 2:** Can you share the pirate treasure fairly among 3 pirates? Show me how you would do this.

A child is given a circular birthday cake and is asked to do the following tasks:

**Task 3:** Share the cake fairly between 2 pirates.

**Task 4:** Share the cake fairly between 4 pirates.

**Task 5:** Share the cake fairly between 3 pirates.

***Before you watch the video clips, answer the following questions:***

1. Explain your solution for Task 1.
2. Explain your solution for Task 2.
3. Explain your solution for Task 3.
4. Explain your solution for Task 4.
5. Explain your solution for Task 5.
6. What type of responses would you expect to see from K-2 students for Task 1?
7. What type of responses would you expect to see from K-2 students for Task 2?
8. What type of responses would you expect to see from K-2 students for Task 3?
9. What type of responses would you expect to see from K-2 students for Task 4?
10. What type of responses would you expect to see from K-2 students for Task 5?

***Watch video clip #1, and respond to the following questions:***

Emma is a 5-years-old kindergartener who attends a public school in Wake County. In the following video clip, Emma has just shared 24 pieces of pirate treasure fairly between 4 pirates. Watch video clip #1, and respond to the questions below:

11. The Interviewer asks Emma, “So, how do you know that they each get the same amount?” Emma responds by saying, “Last time it was six, now you just added two more cause he had six. And, you would add two more to each one which makes eight.” What can you infer about Emma’s reasoning based on her response in the video clip.
12. What does Emma understand? Explain.
13. What does Emma have difficulty with? Explain.
14. If you were the teacher, is there anything you would want to ask Emma based on what you have observed? Explain.

***Watch video clip #2, and respond to the following questions:***

15. Interpret Emma’s justification for arguing that the pirate’s in this task have the same share.
16. What does Emma understand? Explain.
17. What does Emma have difficulty with? Explain.

***Watch video clip #3, and respond to the following questions:***

18. What does Emma understand? Explain.
19. What does Emma have difficulty with? Explain.

### Clinical Interview Assignment

**Part I: Create Interview Tasks and Questions**

Create a sequence of interview tasks and questions that you will use to conduct an individual interview with a child to assess his or her understanding of equipartitioning. State the tasks and questions you plan to use in the interview. Along with the tasks and questions, include your own solutions to the tasks.

**Part II: Conduct the Interview**

Conduct the individual interview with the child. Videorecord the interview. Video and other equipment can be checked out of the Learning Resource Library on the 4<sup>th</sup> floor of Poe (<http://ced.ncsu.edu/medctr/index.php>). See the following website for information about checking out equipment: <http://ced.ncsu.edu/medctr/equipment.php>.

**Part III: Reflection**

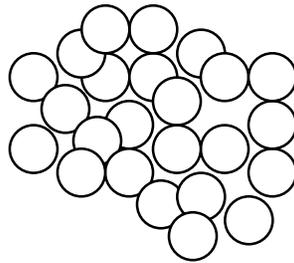
After watching your videorecording, answer the following questions:

1. Provide a short description of the child you interviewed. Be sure to include his or her age, grade, gender, and race.
2. Based on the interview, what do you think the child knows? What do you think the child had difficulty with? Provide specific evidence to support your claims?
3. What would you do next with this child and why?
4. Did anything regarding the child's actions or verbalizations surprise you? Explain.
5. What were the advantages and disadvantages of using the media you chose (i.e., playdough, paper, counters, etc.)?
6. Why did you select the tasks that you implemented?
7. If you were to do this interview again, what would you do differently? What would you repeat?
8. What did you learn about how children reason from this experience?

**Pre-/Post-test Form A**

Respond as completely as possible to the questions below. The questions vary from elementary to more difficult. Work as many of the problems as you can. Please show all work, and circle your answer.

1. Three pirates found this treasure and want to share it fairly.
  - a. Draw a line from *each* coin to the pirates' treasure chests.



b. What *mathematical* name(s) would you give to each pirate's share?

c. Answer(s): \_\_\_\_\_

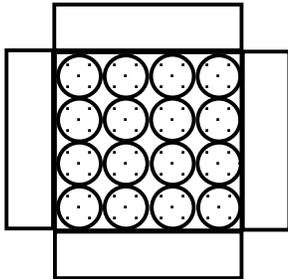
2. In general, if  $n$  objects are shared among  $q$  people, what is each person's share?

Answer: \_\_\_\_\_

3. After fairly sharing an entire deck of Old Maid cards, Erin, Marrielle, and Kenny each have 17 cards. How many cards are in the whole deck of Old Maid cards? Show your work.

Answer: \_\_\_\_\_

4. Below is a box of caramels. Four children fairly share 9 boxes of caramels. What is each child's share? Explain your approach. (from Pothier, 1981)



Answer:

\_\_\_\_\_

Explanation:

For the comparison below, indicate if the tasks are mathematically equivalent and explain your reasoning. (from Lamon, 1996)

*“Three teachers share 4 pepperoni pizzas of the same size. What is each teacher’s share?”*

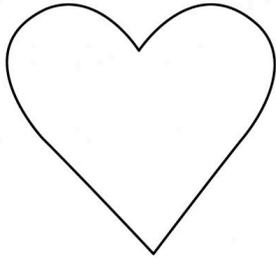
- and -

*“Three teachers all like cheese, mushroom, sausage, and pepperoni pizzas equally. They share one cheese, one mushroom, one sausage, and one pepperoni pizza among themselves, where the pizzas are the same size. What is each teacher’s share?”*

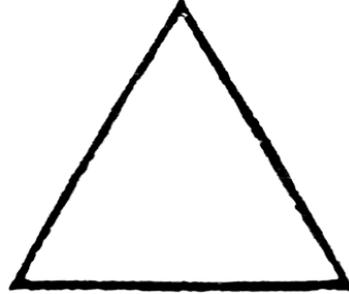
5. Are the two problems mathematically equivalent? \_\_\_\_\_  
Explanation:

6. Below are different shaped cakes. For each one, a) draw a line where you would cut the cake to share it fairly among the number of people indicated and b) shade one share of the cake.(from Pothier, 1989)

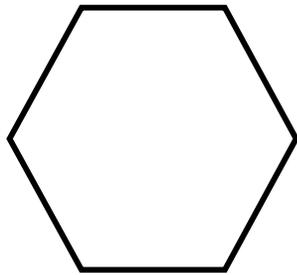
Among 2 people



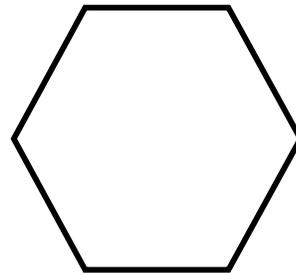
Among 3 people



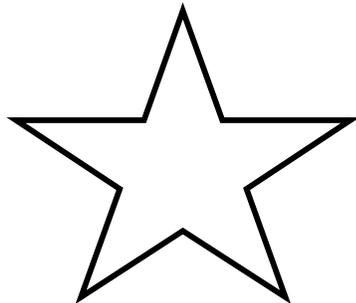
Among 3 people



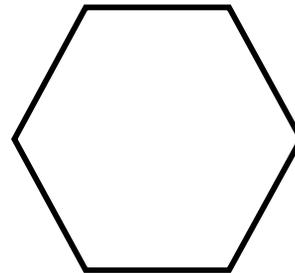
Among 4 people



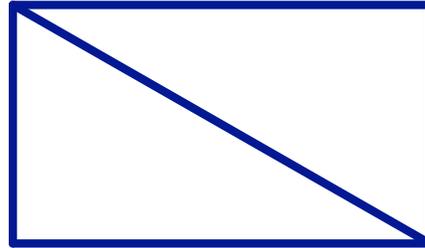
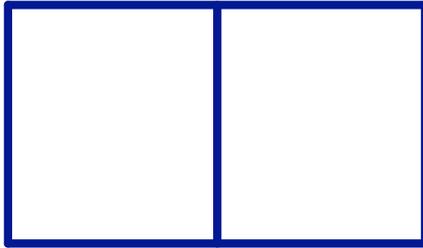
Among 5 people



Among 6 people



7. Yvonne and Pedro each have a rectangular piece of construction paper of the same size. Both children cut their pieces in half. Yvonne's paper is on the left, and Pedro's paper is on the right. Both Yvonne and Pedro keep one piece and trade the other piece. Do Yvonne and Pedro have less than, more than, or the same amount of paper as they did before they traded? Circle your response and explain your reasoning.



Circle:            less than                    the same as                    more than  
cannot tell

Explanation:

8. Mustafa folded a square piece of paper and created 12 equal parts. Describe in steps, in as many ways as you can, how he folded the paper. (from Empson & Turner, 2006)

<u>Method One</u>	<u>Method Two</u>	<u>Method Three</u>	<u>Method Four</u>
•	•	•	•
•	•	•	•
•	•	•	•
•	•	•	•

9. Jeni folded a rectangular piece of paper in half four times. How many equal parts did she create? (from Empson & Turner, 2006)

Answer: \_\_\_\_\_

10. Sweet potato pies of the same size are served as dessert for Thanksgiving dinner. There are 6 adults at the adult table and 4 children at the children's table. Assume the pies are cut so that the pieces are of equal size and all of the pies are used. For a - c, indicate whether an adult's piece is larger, a child's piece is larger, or if both an adult and a child get the same sized piece of pie. Explain your reasoning.

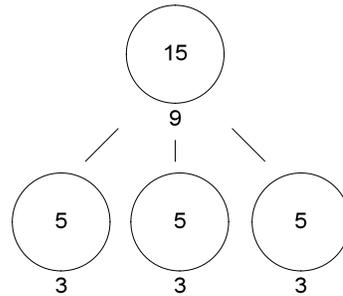
<p>a.</p> <p>Adult table 1 pie for 6 adults</p> <p>Child table 1 pie for 4 children</p>	<p>An adult's piece _____ a child's piece.</p> <p>Circle: <i>is larger than</i>   <i>is smaller than</i>   <i>is the same size as</i></p> <p>Explanation:</p>
<p>b.</p> <p>Adult table 2 pies for 6 adults</p> <p>Child table 1 pie for 4 children</p>	<p>An adult's piece _____ a child's piece.</p> <p>Circle: <i>is larger than</i>   <i>is smaller than</i>   <i>is the same size as</i></p> <p>Explanation:</p>
<p>c.</p> <p>Adult table 3 pies for 6 adults</p> <p>Child table 2 pies for 4 children</p>	<p>An adult's piece _____ a child's piece.</p> <p>Circle: <i>is larger than</i>   <i>is smaller than</i>   <i>is the same size as</i></p> <p>Explanation:</p>

11. The next year, sweet potato pies of the same size are served again for Thanksgiving dinner. This year, there are 7 adults at the adult table and 5 children at the children's table. There are 4 pies at the adult table and 2 pies at the child table. Assume the pies are cut so that the pieces are of equal size and all of the pies are used. Indicate whether an adult's piece is larger, a child's piece is larger, or if both an adult and a child get the same size piece of pie. Explain your reasoning.

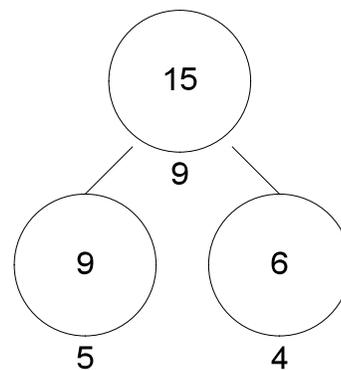
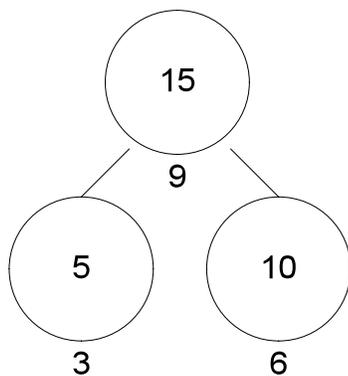
Circle: "An adult's piece *is larger than* | *is smaller than* | *the same size* as a child's piece."

Explanation:

12. Nine people are invited to a birthday party. At the party, 15 pizzas are served. Since there are no tables that sit 9 people, one way for people to share the pizzas equally among themselves is for 5 pizzas to be served to 3 tables with 3 people at each. This can be represented with the diagram below, where the number of pizzas at a table is inside the circle, and the number of people at the table is below the circle:



Below are two other ways to arrange pizzas and people at tables. Do both configurations ensure that all 9 people at the party get a fair share of pizza? Explain your reasoning.



Explanation:

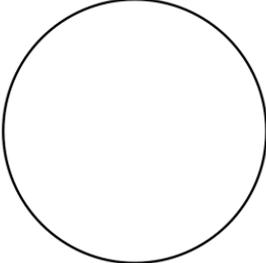
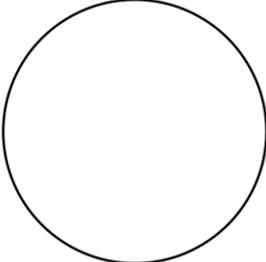
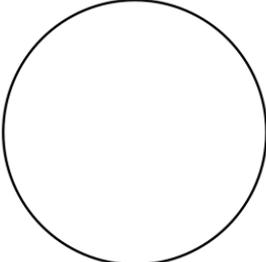
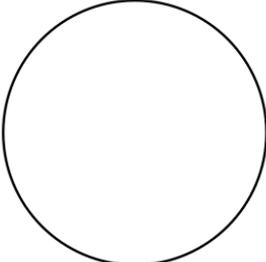
13. Below are pans of brownies to be shared among 4 people.
- Draw a variety of strategies *that you anticipate students might use* to solve the task, and shade one share.

- b. Describe in words the ways that they may cut to find the solution.
- c. Indicate the level of sophistication of the strategy as  
I. Unsophisticated, II. Intermediate, III. Sophisticated

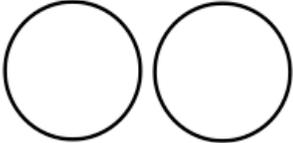
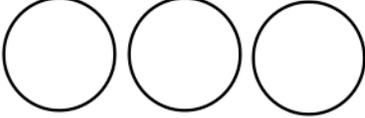
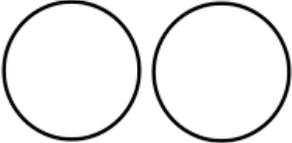
Drawing	Description	Level of Sophistication
		
		
		
		
		
		
		

14. Below are round birthday cakes to be shared among 6 people.

- a. Draw a variety of strategies *that you anticipate students might use* to solve the task, and shade one share.
- b. Describe in words the ways that they may cut to find the solution.
- c. Indicate whether the strategy will yield a correct or incorrect solution.

Drawing	Description	Correct / Incorrect
		
		
		
		

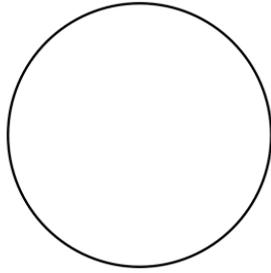
15. Below are three tasks involving numbers of cookies to be shared among various numbers of children. Order the tasks below to indicate the level of difficulty for students, from 1 to 3, where 1 is the *least difficult* and 3 is the *most difficult*. Provide an explanation for your rankings.

Task 1	Task 2	Task 3
 <p>Among 3 children</p>	 <p>Among 5 children</p>	 <p>Among 4 children</p>
Ranking: _____	Ranking: _____	Ranking: _____

Explanation:

16. Draw a picture of how a student might respond to the following task given the indicated understanding. Provide an explanation.

Sharing a round cookie among 3 people if the *focus is on the number of pieces*.



Explanation:

17. For each of the following pairs of tasks, circle the task that you anticipate would be *more difficult* for K-2 students. Explain your reasoning.

a.                    “Sharing 5 cookies  
                          between 2 children”

                          “Sharing 2 cookies  
                          among 5 people”

Explanation:

b.                    “Sharing a round birthday cake  
                          3 people”

                          “Sharing a round birthday cake  
                          among 4 people”

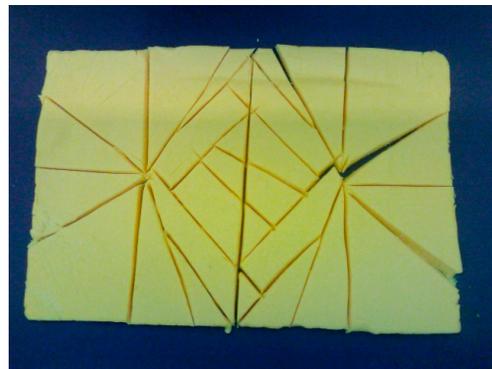
Explanation:

c.                    “Sharing a rectangular birthday cake  
                          among 8 people”

                          “Sharing a rectangular birthday cake  
                          among 3 people”

Explanation:

18. Lizzie was asked to share a rectangular birthday cake between two people. Here is a photograph of her work.

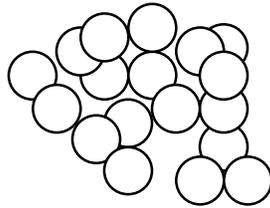




**Pre-/Post-test Form B**

Respond as completely as possible to the questions below. The questions vary from elementary to more difficult. Work as many of the problems as you can. Please show all work, and circle your answer.

1. Three pirates found this treasure and want to share it fairly.
  - a. Draw a line from *each* coin to the pirates' treasure chests.



- b. What *mathematical* name(s) would you give to each pirate's share?

Answer(s): \_\_\_\_\_

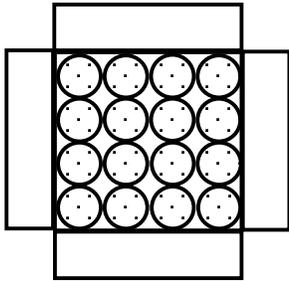
2. In general, if  $n$  objects are shared among  $q$  people, what is each person's share?

Answer: \_\_\_\_\_

3. After fairly sharing an entire deck of Uno cards, Erin, Pedro, Keisha, Gwynn, Marrielle, and Kenny each have 18 cards. How many cards are in the whole deck of Uno cards? Show your work.

Answer: \_\_\_\_\_

4. Below is a box of caramels. Two children fairly share 5 boxes of caramels. What is each child's share? Explain your approach. (adapted from Pothier, 1981)



Answer:

\_\_\_\_\_

Explanation:

For the comparison below, indicate if the tasks are mathematically equivalent and explain your reasoning. (adapted from Lamon, 1996)

*“Six children share four cookies of the same size. What is each child's share?”*

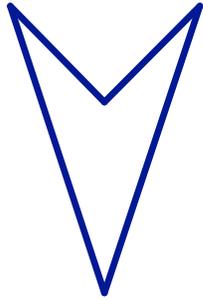
- and -

*“Six children all like pork, chicken, beef, and vegetarian Chinese dinners equally. They share one pork, one chicken, one beef, and one vegetarian dinner among themselves, where the dinners are the same size. What is each child's share?”*

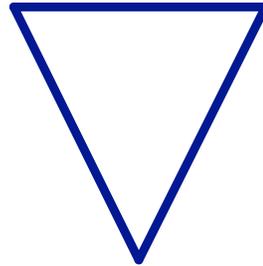
5. Are the two problems mathematically equivalent? \_\_\_\_\_  
Explanation:

6. Below are different shaped cakes. For each one, a) draw a line where you would cut the cake to share it fairly among the number of people indicated and b) shade one share of the cake. (adapted from Pothier, 1989)

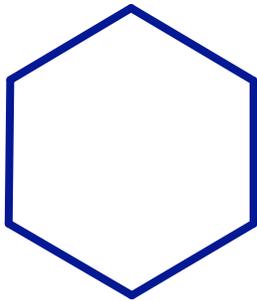
Among 2 people



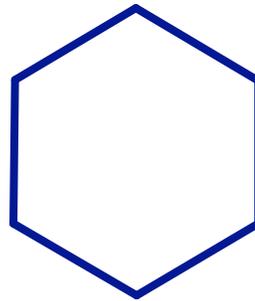
Among 3 people



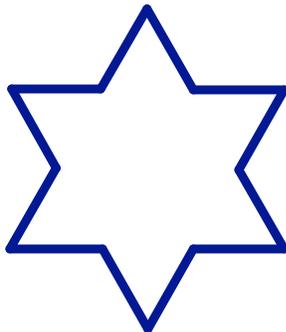
Among 3 people



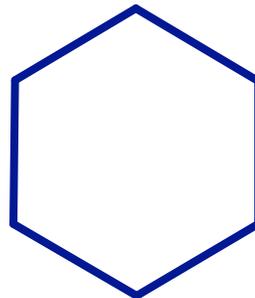
Among 4 people



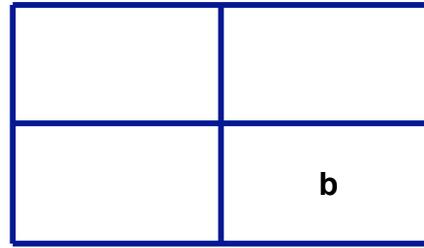
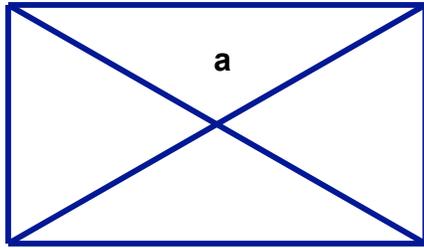
Among 6 people



Among 6 people



7. Two rectangles of the same size were marked in half and marked in half again. Are parts  $a$  and  $b$  the same amount, or is one greater than the other? Circle your response and explain your reasoning. (adapted from Pothier & Sawada, 1990)



Circle:       $a < b$        $a = b$        $a > b$       cannot tell

Explanation:

8. Mustafa folded a square piece of paper and created 18 equal parts. Describe in steps, in as many ways as you can, how he folded the paper. (adapted from Empson & Turner, 2006)

<u>Method One</u>	<u>Method Two</u>	<u>Method Three</u>	<u>Method Four</u>
•	•	•	•
•	•	•	•
•	•	•	•
•	•	•	•

9. Jeni folded a rectangular piece of paper in half three times. How many equal parts did she create? (adapted from Empson & Turner, 2006)

Answer: \_\_\_\_\_

10. Sweet potato pies of the same size are served as dessert for Thanksgiving dinner. There are 8 adults at the adult table and 6 children at the children’s table. Assume the pies are cut so that the pieces are of equal size and all of the pies are used. For a - c, indicate whether an adult’s piece is larger, a child’s piece is larger, or if both an adult and a child get the same sized piece of pie. Explain your reasoning.

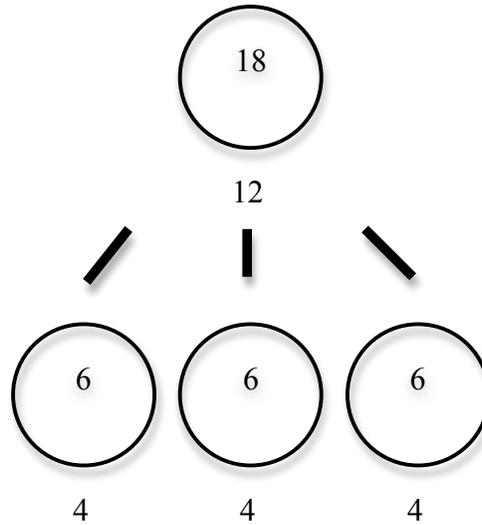
<p>a.</p> <p>Adult table 1 pie for 8 adults</p> <p>Child table 1 pie for 6 children</p>	<p>An adult’s piece _____ a child’s piece.</p> <p>Circle: <i>is larger than</i>   <i>is smaller than</i>   <i>is the same size as</i></p> <p>Explanation:</p>
<p>b.</p> <p>Adult table 2 pies for 8 adults</p> <p>Child table 1 pie for 6 children</p>	<p>An adult’s piece _____ a child’s piece.</p> <p>Circle: <i>is larger than</i>   <i>is smaller than</i>   <i>is the same size as</i></p> <p>Explanation:</p>
<p>c.</p> <p>Adult table 4 pies for 8 adults</p> <p>Child table 3 pies for 6 children</p>	<p>An adult’s piece _____ a child’s piece.</p> <p>Circle: <i>is larger than</i>   <i>is smaller than</i>   <i>is the same size as</i></p> <p>Explanation:</p>

11. The next year, sweet potato pies of the same size are served again for Thanksgiving dinner. This year, there are 8 adults at the adult table and 7 children at the children’s table. There are 3 pies at the adult table and 4 pies at the child table. Assume the pies are cut so that the pieces are of equal size and all of the pies are used. Indicate whether an adult’s piece is larger, a child’s piece is larger, or if both an adult and a child get the same size piece of pie. Explain your reasoning.

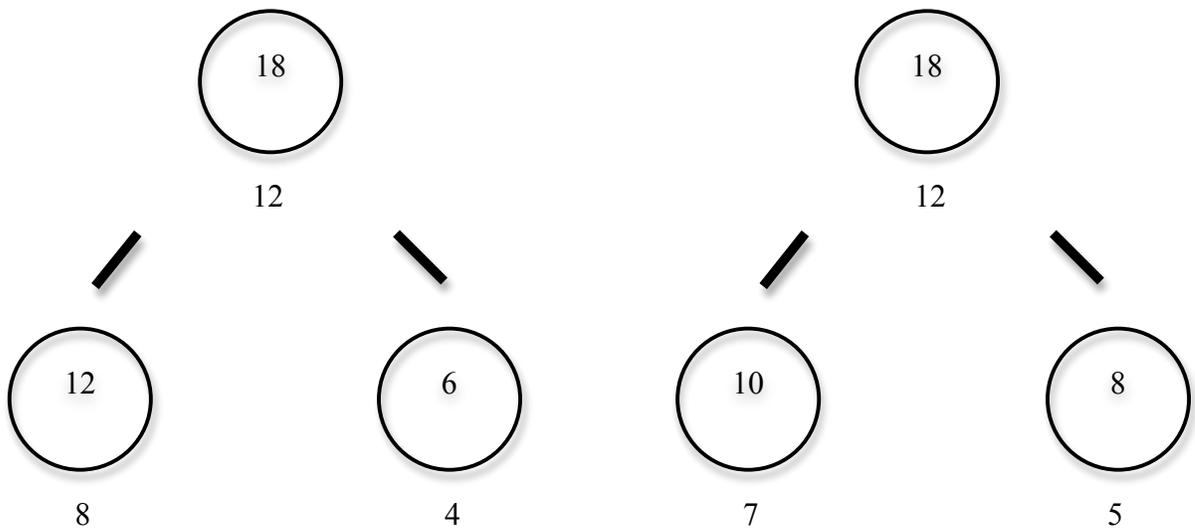
Circle: “An adult’s piece *is larger than* | *is smaller than* | *the same size* as a child’s piece.”

Explanation:

12. Twelve people are invited to a birthday party. At the party, 18 pizzas are served. Since there are no tables that sit 12 people, one way for people to share the pizzas equally among themselves is for 6 pizzas to be served to 3 tables with 4 people at each. This can be represented with the diagram below, where the number of pizzas at a table is inside the circle, and the number of people at the table is below the circle:



Below are two other ways to arrange pizzas and people at tables. Do both configurations ensure that all 12 people at the party get a fair share of pizza? Explain your reasoning.

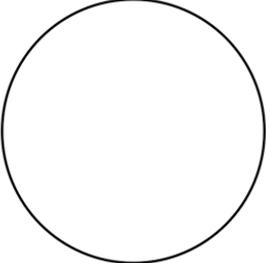
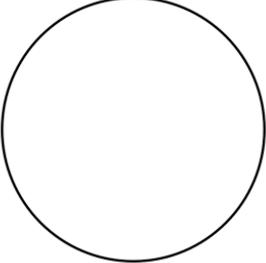
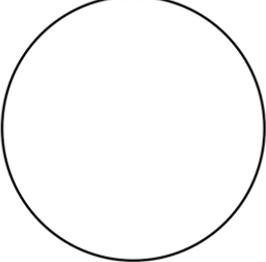
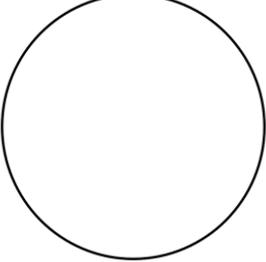


Are they the same? \_\_\_\_\_

Explanation:

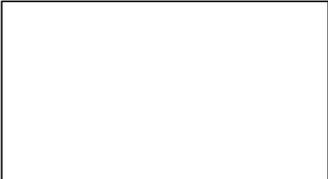
13. Below are round birthday cakes to be shared among 6 people.

- a. Draw a variety of strategies *that you anticipate students might use* to solve the task, and shade one share.
- b. Describe in words the ways that they may cut to find the solution.
- c. Indicate the level of sophistication of the strategy as  
I. Unsophisticated, II. Intermediate, III. Sophisticated

Drawing	Description	Level of Sophistication
		
		
		
		

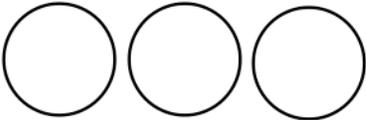
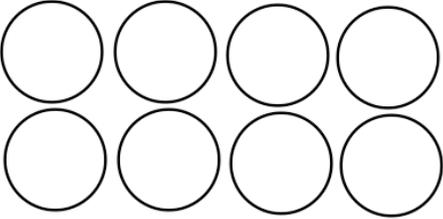
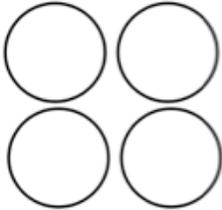
14. Below are pans of brownies to be shared among 4 people.

- a. Draw a variety of strategies *that you anticipate students might use* to solve the task, and shade one share.
- b. Describe in words the ways that they may cut to find the solution.
- c. Indicate whether the strategy will yield a correct or incorrect solution.

Drawing	Description	Correct / Incorrect
		
		
		
		
		
		

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15. Below are three tasks involving numbers of cookies to be shared among various numbers of children. Order the tasks below to indicate the level of difficulty for students, from 1 to 3, where 1 is the *least difficult* and 3 is the *most difficult*. Provide an explanation for your rankings.

Task 1	Task 2	Task 3
 Among 2 children	 Among 7 children	 Among 3 children
Ranking: _____	Ranking: _____	Ranking: _____

Explanation:

16. Draw a picture of how a student might respond to the following task given the indicated understanding. Provide an explanation.

Sharing a pan of brownies among five people if the *focus is on the size of the pieces*.

Explanation:

17. For each of the following pairs of tasks, circle the task that you anticipate would be *more difficult* for K-2 students. Explain your reasoning.

a. *“Sharing 7 cookies  
between 2 children”*

*“Sharing 2 cookies  
among 7 people”*

Explanation:

b. *“Sharing a round birthday cake  
among 8 people”*

*“Sharing a round birthday cake  
among 3 people”*

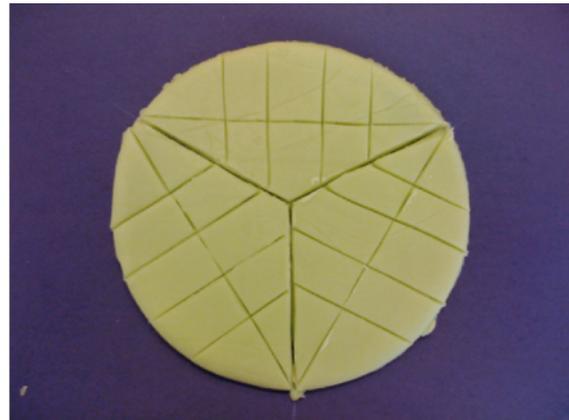
Explanation:

c. *“Sharing a rectangular birthday cake  
among 3 people”*

*“Sharing a rectangular birthday cake  
among 8 people”*

Explanation:

18. Lodge was asked to share a circular birthday cake between three people. Here is a photograph of his work.



a. What might Lodge understand about fair sharing? Refer to the photograph to justify your claim.

b. What is unclear about Lodge's understanding about fair sharing? Refer to the photograph to justify your claim.

c. What question or task might you pose next to Lodge? Explain your reasoning.

Appendix H

**Teacher Pre/Post-test Rubric (Form A)**

1 – Naming of outcome of discrete sharing

Response to 1a is correct if:

- Lines are drawn from each coin or from groups of coins to the chest  
OR
- The number of coins (8) that each pirate gets is explicit.

Correct responses to 1b are categorized as below:

- a. Count – 8 coins or numerical expressions (e.g.  $24 \div 3$ ).
- b. Ratio – 8 coins per chest, 16 coins per 2 chests, 24 coins per 3 chests; 8 coins to each pirate.
- c. Fraction –  $8/24$ ,  $1/3$ , other equivalent fractions, or word equivalents (e.g. “one third”).
- d. Operator –  $8/24$  of the coins,  $1/3$  of the coins.
- e. General mathematical name – thirds, fraction, quotient, fair share, equal portions, equipartitions, partitions, portions, equal groups, equal distribution.

Points	Description
3	1a correct AND includes at least three categories from the 1b list
2	1a correct AND includes two categories from the 1b list
1	1a correct AND includes one category from the 1b list
0	1a correct or incorrect AND no attempt or superfluous names OR 1a incorrect OR 1a and 1b incorrect OR No response

2 – Quotient construct

Correct Responses:

- “ $n/q$ ” OR
- “ $n \div q$ ” OR
- “ $n$  divided by  $q$ ” OR
- “ $1$  qth of  $n$ ” OR
- “ $n$  one-qths”

Points	Description
3	Correct response

2	Specific case used as explanation with some mention of generality (e.g., “1/qth” without a reference to the unit or $n \div q = 1/3$ or “ $n = 24, q = 3, 24/3 = 8$ ”) Note: If includes a general statement and then makes a specific case as an example, score as 3.
1	Specific case used as explanation with no mention of generality (e.g., “8” or “1/3”) OR Specific case used that is incorrect with some mention of generality (e.g., “8/3” with reference to “n/q”)
0	No mention of generality or specific case used OR Specific case used with incorrect generality OR Superfluous answer OR Incorrect Response OR No response

### 3 – Reversibility of discrete equipartitioning

Correct response: 51 cards or 51

Points	Description
3	Correct response or expression equivalent to the correct response
2	Incorrect response by computational error OR Correct response with incomplete explanation
1	Incorrect response with incomplete explanation
0	Incorrect answer with unreasonable explanation OR No response

### 4 – Dealing composite units and splitting remainder

Correct responses:

- “36 caramels”
- “2 and  $\frac{1}{4}$  boxes”
- “ $\frac{9}{4}$  boxes”
- “2 boxes and 4 caramels”

Note: does not have to include units for credit

Complete explanations:

- Numerical expressions and computations, simplified or unsimplified, or a written explanations of these computations
- Described dealing and splitting the last box (e.g., each child gets 2 boxes and the box that is left over is split into fourths)
- Splitting and multiplying (e.g., each child gets 4 from each box and  $9 \times 4 = 36$ )
- Diagrams or models

Incomplete explanations:

- Calculations without a clear solution indicated

Points	Description
3	Correct response AND complete explanation
2	Correct response AND incomplete or no explanation OR Incorrect response but complete explanation (e.g. computational error)
1	Incorrect response with incomplete explanation
0	Incorrect response with unreasonable explanation OR No response

#### 5 – Equivalence of tasks – Case D

Correct response:

Yes – both result in  $\frac{4}{3}$  or  $1 \frac{1}{3}$

Complete explanation:

- The first problem can be modeled mathematically as  $4 \div 3$  AND the second problem can be modeled as  $\frac{1}{3} + \dots + \frac{1}{3}$  or  $4(\frac{1}{3})$   
OR
- Uses an area model to illustrate this argument  
OR
- Both involve three children receiving  $\frac{1}{3}$  of the total  
OR
- Explains that pizza type is an extraneous variable

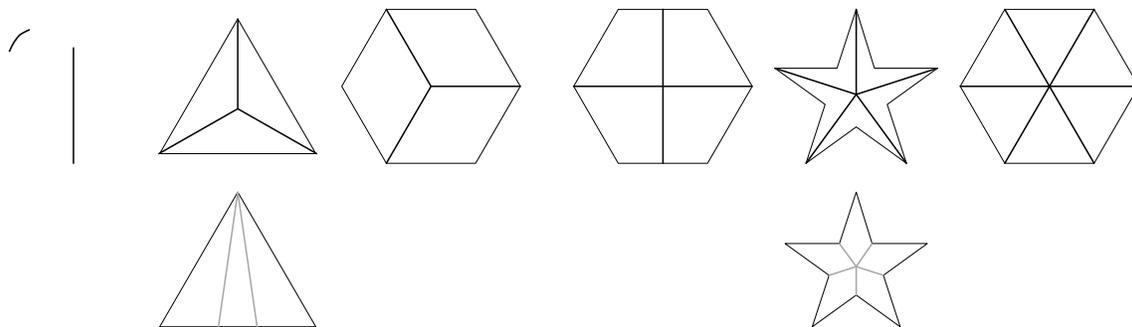
Incomplete explanation:

- Both result in  $\frac{4}{3}$  or  $1 \frac{1}{3}$   
OR
- Both involve sharing 4 things among 3 people  
OR
- Both involve the same values/units; may use an area model to illustrate this argument

Points	Description
3	Correct response AND complete explanation
2	Correct response AND incomplete or no explanation OR Correct response and ignores extraneous variable OR Responds “not equivalent” with explanation that the second problem demands 1/3 of each pizza OR Responds “not equivalent” with a complete explanation (e.g. recognized distinction between the two but wrote responded no) Note: If responds “Yes and no” with complete explanations, give 2.
1	Responds “not equivalent” with incomplete explanation OR Responds “equivalent” but to something other than 4/3 or 1 1/3 OR Correct response with unreasonable or no explanation
0	Responds “not equivalent” with unreasonable or no explanation OR No response

### 6 – Equipartitioning irregular shapes

Correct equipartitioning:



Note: Compositions of these are also correct responses (e.g. equipartitioning the hexagon into 6 equilateral triangles and shading two to split for 3)

Points	Description
3	6 correct AND all shaded
2	5 correct AND all shaded

	OR 6 correct AND not shaded
1	3 or 4 correct AND all shaded OR 5 correct AND not shaded
0	0, 1 or 2 correct AND all shaded OR 3 or 4 correct AND not shaded OR No response

Note: If > 50% of correct responses are shaded, then count as shaded.

Note: Rotations of the radial segments that preserve angle measure are also correct.

### 7 – Equivalence of equal-sized pieces

Correct response: “the same as”

Incorrect responses: “less than”, “greater than”, or “cannot tell”

Complete explanation:

- Explains that two halves of the same size unit are equivalent  
OR
- Uses an area argument (e.g., Uses text and/or diagrams/models to decompose and recompose to show the parts are equal in area)

Incomplete explanation:

- Does not make reference to the idea that the rectangles are the same size unit

Points	Description
3	Correct response AND complete explanation
2	Correct response AND incomplete or no explanation
1	Incorrect response with complete explanation OR Correct response with unreasonable explanation
0	Incorrect response with unreasonable or no explanation OR No response

### 8 – Predicting composition of splits

Correct methods:

I: Permutation of half, half, third

II: Permutation of fourth, third

III: Permutation of half, sixth

IV: 11 parallel folds or some permutation involving a 12th

Points	Description
3	Lists method from all 4 categories
2	Lists method from 3 categories
1	Lists method from 2 categories OR Lists 1 method from categories I, II, and III
0	Lists 1 method from categories IV only OR No response

Notes:

- If response lists the results in sequential order but does not describe the folds, then do not award credit (E.g. “fold in thirds, then sixths, then 12ths”)
- Subtract one point for the inclusion of an incorrect method.
- Diagrams of I-IV are acceptable

9 – Repeated 2-splits

Correct response: 16 regions

Points	Description
3	Correct response
1	8 regions
0	A number of regions other than 16 or 8 OR No response

10 – Compensation

Correct responses:

- “is smaller than”
- “is larger than”
- “is the same as”

Complete explanations:

- One-sixth < one-fourth  
OR  
Uses an area model to illustrate this argument with text or symbols  
OR  
Some argument involving compensation

- b. Two-sixths  $>$  one-fourth or one-third  $>$  one-fourth  
 OR  
 Uses an area model to illustrate this argument with text or symbols  
 OR  
 Some argument involving compensation
- c. Three-sixths = two-fourths or one-half = one-half  
 OR  
 Uses an area model to illustrate this argument with text or symbols

Incomplete explanations:

- Implied comparisons of fractions that do not use text or symbols (e.g. an area model or fraction notation with out words or “ $>$   $<$  =” symbols)

Points	Description
3	Correct response and complete explanations for A, B, and C
2	Correct response and complete explanations for A & B, A & C, or B & C OR Correct response and incomplete explanations to A, B, and/or C
1	Correct response and complete explanations for A, B, or C OR Correct response and incomplete explanations for A & B, A & C, or B & C OR Correct response and unreasonable or no explanations for A, B, and C
0	Correct response and incomplete explanations for A, B, or C OR Correct response and unreasonable or no explanations for A & B, A & C, or B & C OR Answers all incorrectly OR No response

## 11 – Compensation

Correct response: “is larger than”\*

Complete explanation:

- Four-sevenths  $>$  two-fifths or  $20/35 > 14/35$   
 OR
- Uses an area model to illustrate this argument with text or symbols  
 OR
- Some argument involving compensation (e.g., there are twice as many pies for the adults, but only 2 more people; therefore, the adults pieces are larger or the children’s pieces are

smaller)

OR

- Some argument involving benchmark fractions (e.g. four-sevenths is larger than a half but two-fifths is less than a half)

OR

\*Note – if respondents create a ratio of people per pie, the relation reverses and yields seven-fourths < five-halves or  $1\frac{3}{4} < 2\frac{1}{2}$

\*\*Note – To receive full credit for a compensation argument, the respondent must describe the relationship between the number of adults or children in relation to the size of the pieces, otherwise score as “incomplete”

Incomplete explanations:

- Implied comparisons of fractions that do not use text or symbols (e.g. an area model or fraction notation with out words or “> < =” symbols

Points	Description
3	Correct response and complete explanation
2	Correct response with incomplete or no explanation OR No response but complete explanation
1	Responds “is the same as” by using the additive misconception (e.g. the difference between 4 and 2 is 2 which is the same as the distance between 7 and 5) OR Responds “is smaller than” with complete explanation (e.g. seven-fourths < five-halves or $1\frac{3}{4} < 2\frac{1}{2}$ )
0	Responds “is smaller than” with explanation other than additive misconception or no explanation OR Responds “the same as” with no explanation OR Correct response with unreasonable explanation OR No response

12 – Compensation

Correct response: No

Complete explanations:

- Explains that in the left diagram people at both tables get the same amount of pizza, and the right diagrams shows that people at both tables do not get the same amount of pizza (i.e., people at the table of 5 get more pizza than those sitting at the table of 4 or vice

versa)

OR

- Explains that in the left diagram people at both tables get the same amount of pizza, and the right diagrams shows that people at both tables do not get the same amount of pizza using symbolic representations:  $1.67 \neq 1.8 \neq 1.5$ ,  $1 \frac{2}{3} \neq 1 \frac{4}{5} \neq 1 \frac{1}{2}$ ,  $\frac{36}{20} \neq \frac{30}{20}$ ,  $\frac{3}{5} \neq \frac{5}{9} \neq \frac{4}{6}$ ,  $.6 \neq .55 \neq .667$ ,  $1.67(5) = 8.33 \neq 9$  and  $1.67(4) = 6.67 \neq 6$ .

OR

- Use a combination of diagrams/models and text to illustrate the argument(s) above.

Incomplete Explanation:

- Explains that the shares determined in the right diagram are smaller or are different than the left diagram or vice versa while not explicitly addressing the other.

Points	Description
3	Correct response with complete explanation OR Responds “yes” for the left diagram with complete explanation and “no” for the right diagram with complete explanation
2	Correct response with incomplete explanation OR Incorrect response by computational error with complete explanation OR No response with complete explanation
1	Correct response with no explanation OR Incorrect response with complete explanation for only one diagram OR No response with incomplete explanation
0	Incorrect response with incomplete, unreasonable, or no explanation OR Correct response with unreasonable explanation OR No response

Scoring notes:

- If respondent uses ratios of people per pizza correctly, these count as complete explanations.

13 – Knowledge of variety and sophistication of strategies

Levels

I: Unsophisticated

- Cannot be done
- Breaking with no attention to creating equal-sized pieces and correct number of equal-sized pieces
- Creating correct number of pieces but of unequal size without composition
- Creating the wrong number of equal-sized pieces
- Failure to exhaust the whole
- $n$  or  $n - 1$  parallel cuts
- Use of landmark fractions and then dividing the remainder because cannot do the split

II: Intermediate

- Must exhaust whole
- A composition of cuts to create all congruent pieces
- Incorrect compositions

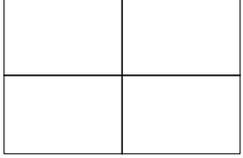
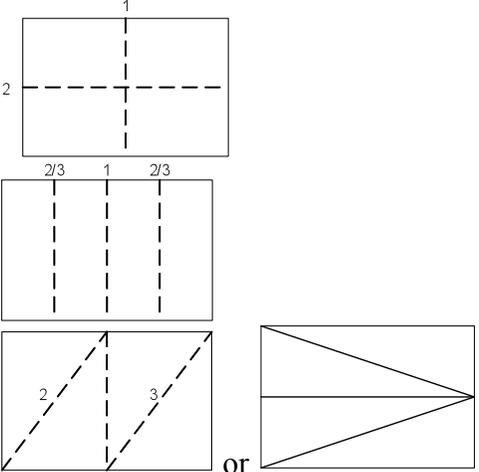
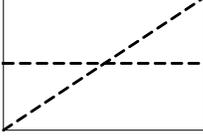
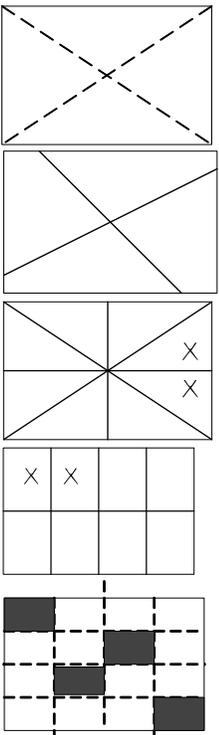
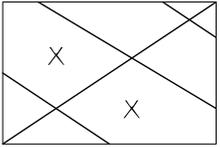
III: Sophisticated

- Must exhaust whole
- A composition of cuts to create incongruent pieces
- Correct or incorrect use of equivalence (e.g. creating 8 congruent pieces and giving 2 per person)

Notes:

- Strategies that simply change the orientation of the cuts or similar strategy using equivalence should be counted only once (e.g. three horizontal parallel cuts and three vertical parallel cuts count as only one strategy; a  $2 \times 8$  array with 2 regions shaded and a  $2 \times 16$  array with 4 regions shaded when sharing among four)
- Unless specifically labeled or described as a composition, count a strategy as parallel cuts.
- Repeated examples with the distinction of measuring should only count once (i.e. if three strategies are repeated but say ‘actually measure to find the center’ count all three of these only once).

Level	Correct	Incorrect
I		<p>“Cannot be done”</p> <p>Breaks</p>

	 <p>Creating 4 “boxes” like drawing 4 squares of cake</p>	
<p>II</p>	 <p>or</p>	
<p>III</p>	 <p>All uses of equivalence are III</p>	

Locate the number of distinct strategies on the left and the number of levels of sophistication represented across the top. The corresponding number in the table is the raw score for the item. Next, consider the notes below the table to finalize the item's score.

Score	3 Levels	2 Levels	1 Level
4+ strategies	3	2	2
3 strategies	3	2	2
2 strategies	--	2	1
1 strategy	--	--	0

#### Scoring notes

- Subtract one point if a correct strategy is listed as incorrect
- Subtract one point for incorrect levels of sophistication or no indicated levels
- Subtract one point for  $\leq 50\%$  of the strategies have complete descriptions. Descriptions may be written or numbers but specific (e.g. 'cut into fourths' is specific, '1/4ths' is not)
- Score as 0 for No Response

#### 14 – Knowledge of variety and correct/incorrect strategies

##### Levels

##### I: Unsophisticated –

- Cannot be done
- Breaking with no attention to creating equal-sized pieces and correct number of equal-sized pieces
- Creating correct number of pieces but of unequal size without composition
- Creating the wrong number of equal-sized pieces
- Failure to exhaust the whole
- $n$  or  $n - 1$  parallel cuts
- Sequential radial cuts
- Use of landmark fractions and then dividing the remainder because cannot do the indicated split

##### II: Intermediate

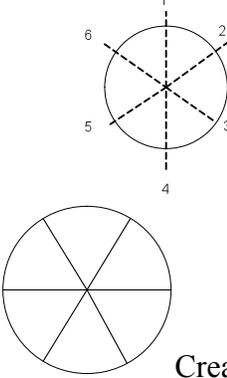
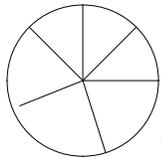
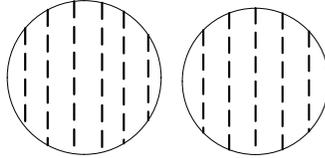
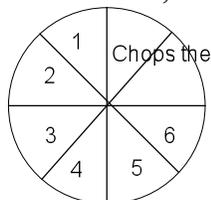
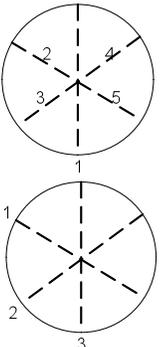
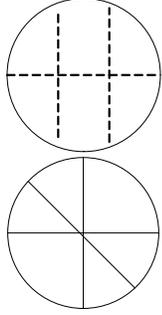
- Must exhaust whole
- A composition of cuts to create all congruent pieces
- Incorrect compositions

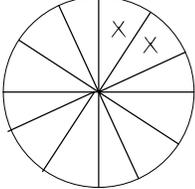
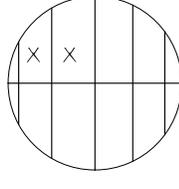
##### III: Sophisticated

- Must exhaust whole
- Correct or incorrect use of equivalence (e.g. creating 12 congruent pieces and giving 2 per person)

Notes:

- Strategies that simply change the orientation of the cuts should be counted only once (e.g. 5 horizontal parallel cuts and 5 vertical parallel cuts count as only one strategy)
- Unless specifically labeled or described as a composition, count a strategy as parallel cuts.
- Repeated examples with the distinction of measuring should only count once (i.e. if three strategies are repeated but say ‘actually measure to find the center’ count all three of these only once).
- Landmark strategy is different from a composition (e.g. a 3-split on a 2-split) because with a composition, still attending to each piece, whereas with landmarks, the actions focus on remaining piece after distribution.

Level	Correct	Incorrect
I	 <p data-bbox="349 1029 917 1102">Creating 6 sectors like drawing out 6 slices of cake</p>	<p data-bbox="982 672 1218 703">“Cannot be done”</p>  <p data-bbox="1136 829 1234 871">Breaks</p>  <p data-bbox="982 1029 1315 1060">Parallel cuts, either 6 or 5</p>  <p data-bbox="1226 1239 1380 1270">Landmarks</p>
II		

III		
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Points	Description
3	Indicates 2 correct strategies and 2 incorrect strategies OR Indicates 3 correct strategies and 1 incorrect strategy
2	Indicates 2 correct strategies and 1 incorrect strategy OR Indicates 1 correct strategies and 2 incorrect strategy OR Indicates 1 correct strategy and 3 incorrect strategies OR Indicates 3 correct strategies and no incorrect strategies OR Indicates 3 incorrect strategies OR Indicates 4 incorrect strategies
1	Indicates 2 correct strategies and no incorrect strategies OR Indicates no correct strategies and 2 incorrect strategies OR Indicates 1 correct strategy and 1 incorrect strategy
0	Indicates 1 correct strategy OR 1 incorrect strategy OR No response

Scoring notes:

- Subtract one point for incorrect or no labeling of correct/incorrect strategies
- Subtract one point for  $\leq 50\%$  of the strategies have complete descriptions. Descriptions may be written or numbers but specific (e.g. 'cut into fourths' is specific, '1/4ths' is not)

15 – Knowledge of sophistication

Correct response:

- 1) Task 3 is the *least difficult*,
- 2) Task 1 is the *next difficult*, and
- 3) Task 2 is the *most difficult*

Complete explanation:

- Explains that a split of a 2-split is easiest for students and a 3-split is easier than a 5-split, or vice versa (i.e. must relate all three of the splits, not just the evens versus the odds)
- Explains that even splits are usually easier than odd splits; therefore, a 2-split is the easiest and splitting into thirds is easier than splitting into sevenths or vice versa

Incomplete explanation:

- Correct area models with no verbal explanations
- Fails to address how pertinent observations relate to one another
- Explains that even splits are usually easier than odd splits with no reference to how each split relates to the others

Points	Description
3	Correct response with complete explanation
2	Correct response with incomplete explanation
1	Correct response for exactly one of a, b, or c with complete explanation OR Correct response with unreasonable or no explanation OR Incorrect response with reasonable explanation
0	Incorrect response with unreasonable or no explanation OR Correct response for exactly one of a, b, or c with incomplete, unreasonable, or no explanation OR No response

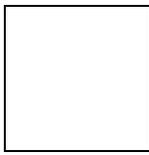
Notes:

- Explanations about the number of objects being divided do not count.
- Implicit attention to smaller odd numbers counts as incomplete (e.g. “sharing for 3 is less divisions than for 5”)

16 – Variety of strategies and knowledge of students’ thinking

Correct responses include:

I.



or



or a configuration where 3 pieces (or a multiple of 3 pieces) are not the same size

OR

II.



OR

III. Chops and deals pieces

Complete explanations:

- Students unable to make radial cuts and/or difficulty with 3-splits
- Student not using the whole or any other correct composition where student does not use the whole
- Student must fairly deal after creating many pieces or if does not create a multiple of 3 pieces, states that it cannot be done.
- Student finds it easier to half and then halve one-half to get 3 pieces and pieces are not the same size
- Student cuts 3 unequal pieces but ensures that each person gets a piece (these may be radial cuts)

\*Note: The focus should be on creating 3 pieces that are unequal in size OR creating 3 pieces that are equal in size without using the whole object to be counted as a “correct response.” Strategies that create 3 equal sized pieces should be scored as a “1” if the respondent mentions creating both 3 pieces and that the pieces are the same size. If both of these conditions are not mentioned for strategies creating 3 equal sized pieces, score as a “0.”

Points	Description
3	Correct response from I – III and complete explanation
2	Correct response with incomplete explanation
1	Correct response with explanation that does not mention number of pieces OR Correct response with no explanation OR *Correctly splits circle with complete explanation
0	Incorrect response OR *Correctly splits circle with incomplete, unreasonable, or no explanation

	OR No response
--	-------------------

17 – Knowledge of sophistication

Correct response:

- a. Circles “2 among 5”
- b. Circles “among 3”
- c. Circles “among 3”

Complete explanations:

- a. States that 2-splits are easier or 5-splits are harder  
OR  
Dealing is easier  
OR  
Use symmetry since 2 is an even number
- b. States radial cuts are harder  
OR  
3-splits are harder  
OR  
Repeated halving is easier  
OR  
Use symmetry since 4 is an even number
- c. States radial cuts are harder  
OR  
3-splits are harder  
OR  
Repeated halving is easier  
OR  
Use symmetry since 8 is an even number

Note:

- Number of cuts is superfluous
- For Gemma’s, if there is a distinction of case C being less sophisticated than case D, score as complete explanation.

Points	Description
3	3 correct responses and complete explanation
2	2 correct responses with complete explanations OR 3 correct responses less than 3 complete explanations OR 2 or 3 incorrect responses with 2 or 3 complete explanations (it

	may be possible that the student circled the easier tasks)
1	1 correct response with a complete explanation OR 2 correct responses less than 2 complete explanations
0	3 incorrect responses with unreasonable or no explanations OR 1 correct response with incomplete, unreasonable, or no explanation OR No response

### 18 – Knowledge of sophistication

Correct responses with complete explanations:

- I. Number of pieces – explains that the child created the same number of pieces on both sides of the 2-split (explicit comparison of the number of pieces on both sides).
- II. Size of pieces – explains that the student created two equal parts with a 2-split and notes the symmetric nature of corresponding pieces.
- III. Exhaustion of whole – explains that the student used the whole cake.
- IV. Allocation – explains that it is unclear how the student will allocate the pieces of cake (i.e. indicating what each person’s share is)
- V. Related question – includes a question or task that relates to their uncertainties about what they student knows or does not know about fair sharing.

Points	Description
3	Response addresses 4 or 5 points from I, II, III, IV, & V
2	Response addresses points from 3 of I, II, III, IV, & V
1	Response addresses points from 2 of I, II, III, IV, & V
0	Response addresses points from 0 or 1 of I, II, III, IV, & V OR No response

Notes:

- Demonstrations in response to the next task do not answer the question and should not be counted as addressing IV.

Appendix I

**Teacher Pre/Post-test Rubric (Form B)**

1 – Naming of outcome of discrete sharing

Response to 1a is correct if:

- Lines are drawn from each coin or from groups of coins to the chest  
OR
- The number of coins (6) that each pirate gets is explicit.

Correct responses to 1b are categorized as below:

- a. Count – 6 coins or numerical expressions (e.g.  $18 \div 3$ ).
- b. Ratio – 6 coins per chest, 12 coins per 2 chests, 18 coins per 3 chests; 6 coins to each pirate.
- c. Fraction –  $6/18$ ,  $1/3$ , other equivalent fractions, or word equivalents (e.g. “one third”).
- d. Operator –  $6/18$  of the coins,  $1/3$  of the coins.
- e. General mathematical name – thirds, fraction, quotient, fair share, equal portions, equipartitions, partitions, portions, equal groups, equal distribution.

Points	Description
3	1a correct AND includes at least three categories from the 1b list
2	1a correct AND includes two categories from the 1b list
1	1a correct AND includes one category from the 1b list
0	1a correct or incorrect AND no attempt or superfluous names OR 1a incorrect OR 1a and 1b incorrect OR No response

2 – Quotient construct

Correct Responses:

- “ $n/q$ ” OR
- “ $n \div q$ ” OR
- “ $n$  divided by  $q$ ” OR
- “ $1$  qth of  $n$ ” OR
- “ $n$  one-qths”

Points	Description
--------	-------------

3	Correct response
2	Specific case used as explanation with some mention of generality (e.g., “1/qth” without a reference to the unit or $n \div q = 1/3$ or “n = 18, q = 3, $18/3 = 6$ ”) Note: If includes a general statement and then makes a specific case as an example, score as 3.
1	Specific case used as explanation with no mention of generality (e.g. “6” or “1/3”) OR Specific case used that is incorrect with some mention of generality (e.g., “6/3” with reference to “n/q”)
0	No mention of generality or specific case used OR Specific case used with incorrect generality OR Superfluous answer OR Incorrect Response OR No response

### 3 – Reversibility of discrete equipartitioning

Correct response: 108 cards or 108

Points	Description
3	Correct response or expression equivalent to the correct response
2	Incorrect response by computational error OR Correct response with incomplete explanation
1	Incorrect response with incomplete explanation
0	Incorrect answer with unreasonable explanation OR No response

### 4 – Dealing composite units and splitting remainder

Correct responses:

- “40 caramels”
- “2 and ½ boxes”
- “5/2 boxes”
- “2 boxes and 8 caramels”

Note: does not have to include units for credit

Complete explanations:

- Numerical expressions and computations, simplified or unsimplified, or a written explanations of these computations
- Described dealing and splitting the last box (e.g., each child gets 2 boxes and the box that is left over is split into halves)
- Splitting and multiplying (e.g., each child gets 8 from each box and  $8 \times 5 = 40$ )
- Diagrams or models

Incomplete explanations:

- Calculations without a clear solution indicated

Points	Description
3	Correct response AND complete explanation
2	Correct response AND incomplete or no explanation OR Incorrect response but complete explanation (e.g. computational error)
1	Incorrect response with incomplete explanation
0	Incorrect response with unreasonable explanation OR No response

#### 5 – Equivalence of tasks – Case C

Correct response:

Yes – both result in  $4/6$  or  $2/3$

Complete explanation:

- The first problem can be modeled mathematically as  $4 \div 6$  AND the second problem can be modeled as  $1/6 + \dots + 1/6$  or  $4(1/6)$   
OR
- Uses an area model to illustrate this argument  
OR
- Both involve four children receiving  $1/6$  of the total  
OR
- Explains that pizza type is an extraneous variable

Incomplete explanation:

- Both result in  $4/6$  or  $2/3$   
OR
- Both involve sharing 6 things among 4 people

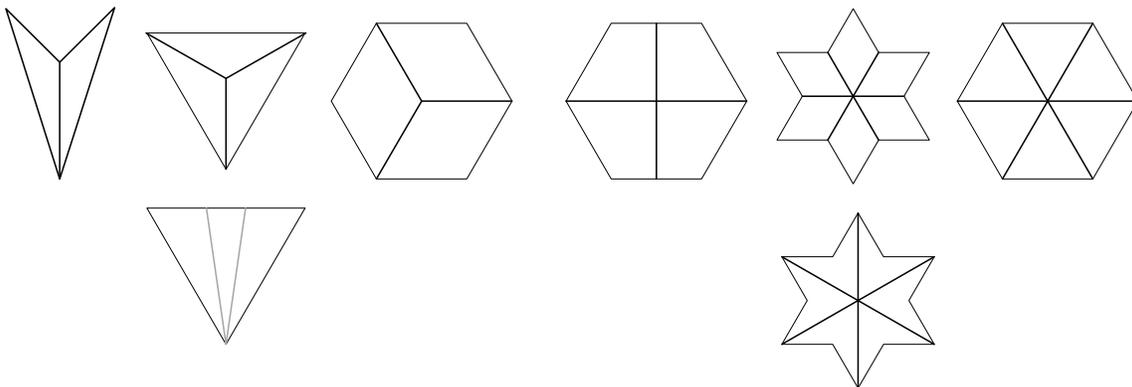
OR

- Both involve the same values/units; may use an area model to illustrate this argument

Points	Description
3	Correct response AND complete explanation
2	Correct response AND incomplete or no explanation OR Correct response and ignores extraneous variable OR Responds “not equivalent” with explanation that the second problem demands $\frac{1}{6}$ of each pizza OR Responds “not equivalent” with a complete explanation (e.g. recognized distinction between the two but wrote responded no) Note: If responds “Yes and no” with complete explanations, give 2.
1	Responds “not equivalent” with incomplete explanation OR Responds “equivalent” but to something other than $\frac{2}{3}$ OR Correct response with unreasonable or no explanation
0	Responds “not equivalent” with unreasonable or no explanation OR No response

### 6 – Equipartitioning irregular shapes

Correct equipartitioning:



Note: Compositions of these are also correct responses (e.g. equipartitioning the hexagon into 6 equilateral triangles and shading two to split for 3)

Points	Description
3	6 correct AND all shaded
2	5 correct AND all shaded OR 6 correct AND not shaded
1	3 or 4 correct AND all shaded OR 5 correct AND not shaded
0	0, 1 or 2 correct AND all shaded OR 3 or 4 correct AND not shaded OR No response

Note: If > 50% of correct responses are shaded, then count as shaded.

Note: Rotations of the radial segments that preserve angle measure are also correct.

#### 7 – Equivalence of equal-sized pieces

Correct response: “ $a = b$ ”

Incorrect responses: “ $a < b$ ”, “ $a > b$ ”, or “cannot tell”

Complete explanation:

- Explains that two fourths of the same size unit are equivalent  
OR
- Uses an area argument (e.g., Uses text and/or diagrams/models to decompose and recompose to show the parts are equal in area)

Incomplete explanation:

- Does not make reference to the idea that the rectangles are the same size unit

Points	Description
3	Correct response AND complete explanation
2	Correct response AND incomplete or no explanation
1	Incorrect response with complete explanation OR Correct response with unreasonable explanation
0	Incorrect response with unreasonable or no explanation OR No response

8 – Predicting composition of splits

Correct methods:

I: Permutation of half, third, third

II: Permutation of sixth, third

III: Permutation of half, ninth

IV: 17 parallel folds or some permutation involving an 18th

Points	Description
3	Lists method from all 4 categories
2	Lists method from 3 categories
1	Lists method from 2 categories OR Lists 1 method from categories I, II, and III
0	Lists 1 method from categories IV only OR No response

Note:

- If response lists the results in sequential order but does not describe the folds, then do not award credit (E.g. “fold in thirds, then ninths, then 18ths”)
- Subtract one point for the inclusion of an incorrect method.
- Diagrams of I-IV are acceptable

9 – Repeated 2-splits

Correct response: 8 regions

Points	Description
3	Correct response
1	6 regions
0	A number of regions other than 8 or 6 OR No response

10 – Compensation

Correct responses:

- d. “is smaller than”
- e. “is larger than”
- f. “is the same as”

Complete explanations:

a. One-eighth < one-sixth

OR

Uses an area model to illustrate this argument with text or symbols

OR

Some argument involving compensation

b. Two-eighths > one-sixth or one-fourth > one-sixth

OR

Uses an area model to illustrate this argument with text or symbols

OR

Some argument involving compensation

c. four-eighths = three-sixths or one-half = one-half

OR

Uses an area model to illustrate this argument with text or symbols

Incomplete explanations:

- Implied comparisons of fractions that do not use text or symbols (e.g. an area model or fraction notation with out words or “> < =” symbols

Points	Description
3	Correct response and complete explanations for A, B, and C
2	Correct response and complete explanations for A & B, A & C, or B & C OR Correct response and incomplete explanations to A, B, and/or C
1	Correct response and complete explanations for A, B, or C OR Correct response and incomplete explanations for A & B, A & C, or B & C OR Correct response and unreasonable or no explanations for A, B, and C
0	Correct response and incomplete explanations for A, B, or C OR Correct response and unreasonable or no explanations for A & B, A & C, or B & C OR Answers all incorrectly OR No response

11 – Compensation

Correct response: “is smaller than”

Complete explanation:

- Four-sevenths > three-eighths or  $32/56 > 21/56$   
OR
- Uses an area model to illustrate this argument with text or symbols  
OR
- Some argument involving compensation (e.g., there are less children and more pies; therefore, their pieces are larger or vice versa)  
OR
- Some argument involving benchmark fractions (e.g., four-sevenths is greater than one-half but three-eighths is less than one-half)  
OR

\*Note – if respondents create a ratio of people per pie, the relation reverses and yields eight-thirds > seven-fourths or  $2 \frac{1}{4} > 1 \frac{3}{7}$

\*\*Note – To receive full credit for a compensation argument, the respondent must describe the relationship between the number of adults or children in relation to the size of the pieces, otherwise score as “incomplete”

Incomplete explanations:

- Implied comparisons of fractions that do not use text or symbols (e.g. an area model or fraction notation with out words or “> < =” symbols

Points	Description
3	Correct response and complete explanation
2	Correct response with incomplete or no explanation OR No response but complete explanation
1	Responds “is the same as” by using the additive misconception (e.g., the difference between 3 and 4 is 1, which is the same as the difference between 8 and 7) OR Incorrect response with complete explanation (e.g., eight-thirds > seven-fourths or $2 \frac{1}{4} > 1 \frac{3}{7}$ )
0	Responds “is larger than” with explanation other than additive misconception or no explanation OR Responds “the same as” with no explanation OR Correct response with unreasonable explanation OR No response

12 – Compensation

Correct response: No

Complete explanations:

- Explains that in the left diagram people at both tables get the same amount of pizza, and the right diagrams shows that people at both tables do not get the same amount of pizza (i.e., people at the table of 7 get more pizza than those sitting at the table of 5 or vice versa)  
OR
- Explains that in the left diagram people at both tables get the same amount of pizza, and the right diagrams shows that people at both tables do not get the same amount of pizza using symbolic representations:  $1.5 \neq 1.43 \neq 1.6$ ,  $1 \frac{1}{2} \neq 1 \frac{3}{7} \neq 1 \frac{3}{5}$ ,  $\frac{50}{35} \neq \frac{56}{35}$ ,  $\frac{2}{3} \neq \frac{7}{10} \neq \frac{5}{8}$ ,  $.667 \neq .7 \neq .625$ ,  $1.5(7) = 10.5 \neq 10$  and  $1.5(5) = 7.5 \neq 8$ .  
OR
- Use a combination of diagrams/models and text to illustrate the argument(s) above.

Incomplete Explanation:

- Explains that the shares determined in the right diagram are smaller or are different than the left diagram or vice versa while not explicitly addressing the other.

Points	Description
3	Correct response with complete explanation OR Responds “yes” for the left diagram with complete explanation and “no” for the right diagram with complete explanation
2	Correct response with incomplete explanation OR Incorrect response by computational error with complete explanation OR No response with complete explanation
1	Correct response with no explanation OR Incorrect response with complete explanation for only one diagram OR No response but incomplete explanation
0	Incorrect response with incomplete, unreasonable, or no explanation OR Correct response with unreasonable explanation OR No response

Scoring notes:

- If respondent uses ratios of people per pizza correctly, these count as complete explanations.

### 13 – Knowledge of variety and sophistication of strategies

#### Levels

##### I: Unsophisticated

- Cannot be done
- Breaking with no attention to creating equal-sized pieces and correct number of equal-sized pieces
- Creating correct number of pieces but of unequal size without composition
- Creating the wrong number of equal-sized pieces
- Failure to exhaust the whole
- $n$  or  $n - 1$  parallel cuts
- Use of landmark fractions and then dividing the remainder because cannot do the split

##### II: Intermediate

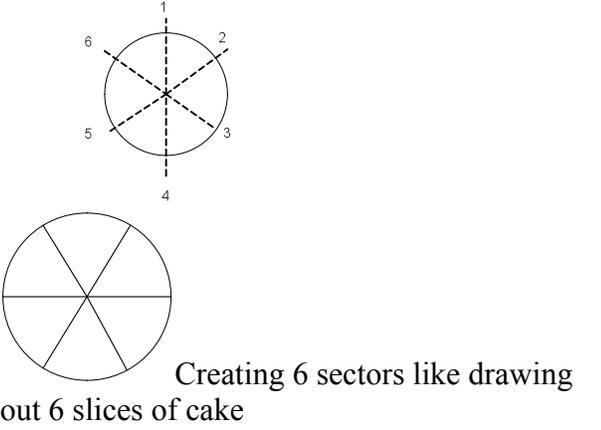
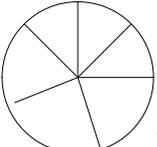
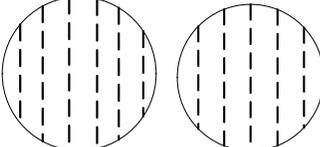
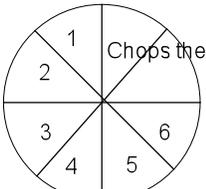
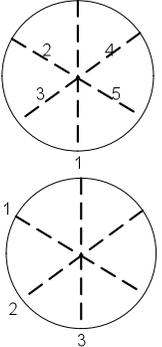
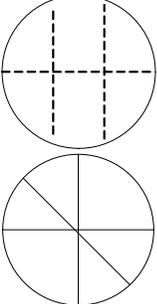
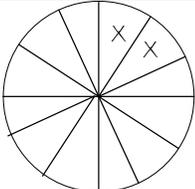
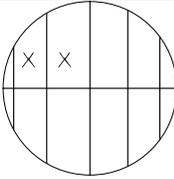
- Must exhaust whole
- A composition of cuts to create all congruent pieces
- Incorrect compositions

##### III: Sophisticated

- Must exhaust whole
- A composition of cuts to create incongruent pieces
- Correct or incorrect use of equivalence (e.g. creating 8 congruent pieces and giving 2 per person)

#### Notes:

- Strategies that simply change the orientation of the cuts or similar strategy using equivalence should be counted only once (e.g. three horizontal parallel cuts and three vertical parallel cuts count as only one strategy; a  $2 \times 8$  array with 2 regions shaded and a  $2 \times 16$  array with 4 regions shaded when sharing among four)
- Unless specifically labeled or described as a composition, count a strategy as parallel cuts.
- Repeated examples with the distinction of measuring should only count once (i.e. if three strategies are repeated but say ‘actually measure to find the center’ count all three of these only once).

Level	Correct	Incorrect
I	 <p>Creating 6 sectors like drawing out 6 slices of cake</p>	<p>“Cannot be done”</p>  <p>Breaks</p>  <p>Parallel cuts, either 6 or 5</p>  <p>Landmarks</p>
II		
III		

Locate the number of distinct strategies on the left and the number of levels of sophistication represented across the top. The corresponding number in the table is the raw score for the item. Next, consider the notes below the table to finalize the item’s score.

Score	3 Levels	2 Levels	1 Level
4+ strategies	3	2	2

3 strategies	3	2	2
2 strategies	--	2	1
1 strategy	--	--	0

#### Scoring notes

- Subtract one point if a correct strategy is listed as incorrect
- Subtract one point for incorrect levels of sophistication or no indicated levels
- Subtract one point for  $\leq 50\%$  of the strategies have complete descriptions. Descriptions may be written or numbers but specific (e.g. 'cut into fourths' is specific, '1/4ths' is not)
- Score as 0 for No Response

#### 14 – Knowledge of variety and correct/incorrect strategies

##### Levels

##### I: Unsophisticated –

- Cannot be done
- Breaking with no attention to creating equal-sized pieces and correct number of equal-sized pieces
- Creating correct number of pieces but of unequal size without composition
- Creating the wrong number of equal-sized pieces
- Failure to exhaust the whole
- $n$  or  $n - 1$  parallel cuts
- Sequential radial cuts
- Use of landmark fractions and then dividing the remainder because cannot do the indicated split

##### II: Intermediate

- Must exhaust whole
- A composition of cuts to create all congruent pieces
- Incorrect compositions

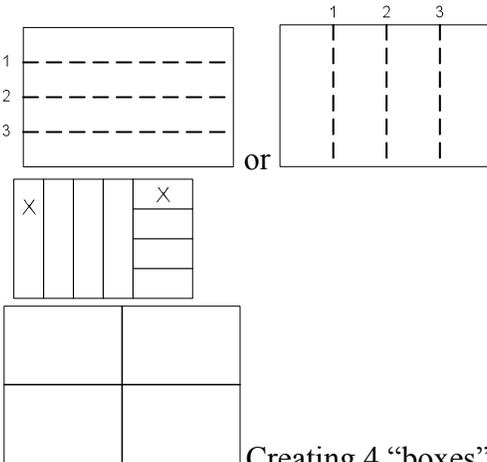
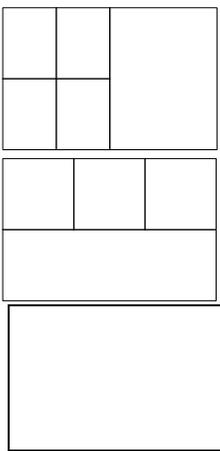
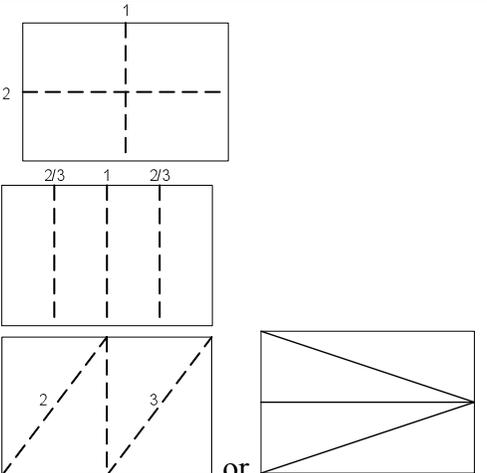
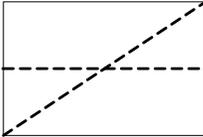
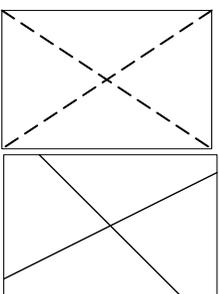
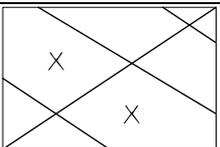
##### III: Sophisticated

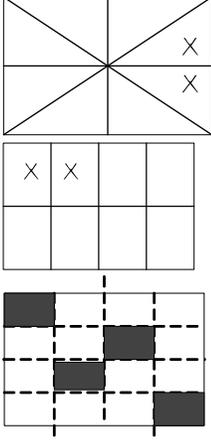
- Must exhaust whole
- Correct or incorrect use of equivalence (e.g. creating 12 congruent pieces and giving 2 per person)

##### Notes:

- Strategies that simply change the orientation of the cuts should be counted only once (e.g. 5 horizontal parallel cuts and 5 vertical parallel cuts count as only one strategy)
- Unless specifically labeled or described as a composition, count a strategy as parallel cuts.

- Repeated examples with the distinction of measuring should only count once (i.e. if three strategies are repeated but say ‘actually measure to find the center’ count all three of these only once).
- Landmark strategy is different from a composition (e.g. a 3-split on a 2-split) because with a composition, still attending to each piece, whereas with landmarks, the actions focus on remaining piece after distribution.

Level	Correct	Incorrect
I	 <p>1 2 3</p> <p>1 2 3</p> <p>X X</p> <p>Creating 4 “boxes” like drawing 4 squares of cake</p>	<p>“Cannot be done”</p> <p>Breaks</p> 
II	 <p>1</p> <p>2</p> <p>2/3 1 2/3</p> <p>2 3</p> <p>or</p>	
III		

		
	All uses of equivalence are III	

Points	Description
3	Indicates at least 3 correct strategies and at least 2 incorrect strategies
2	Indicates at least 3 correct strategies and 1 incorrect strategy OR Indicates at least 4 strategies without labeling correct/incorrect
1	Indicates at least 3 strategies that are all correct OR Indicates at least 3 strategies without labeling correct/incorrect OR Indicates fewer than 3 correct strategies and at least 2 incorrect strategies
0	Indicates fewer than 3 correct strategies and at least 1 incorrect strategy OR Indicates fewer than 3 correct strategies OR Indicates strategies that are all incorrect

Scoring notes:

- Subtract one point for incorrect or no labeling of correct/incorrect strategies
- Subtract one point for  $\leq 50\%$  of the strategies have complete descriptions. Descriptions may be written or numbers but specific (e.g. ‘cut into fourths’ is specific, ‘1/4ths’ is not)

15 – Knowledge of sophistication

Correct response:

- 1) Task 1 is the *least difficult*,
- 2) Task 3 is the *next difficult*, and
- 3) Task 2 is the *most difficult*

Complete explanation:

- Explains that a split of a 2-split is easiest for students and a 3-split is easier than a 7-split, or vice versa (i.e., must relate all three of the splits not just the evens versus the odds)
- Explains that even splits are usually easier than odd splits; therefore, a 2-split is the easiest and splitting into thirds is easier than splitting into sevenths or vice versa

Incomplete explanation:

- Correct area models with no verbal explanations
- Fails to address how pertinent observations relate to one another
- Explains that even splits are usually easier than odd splits with no reference to how each split relates to the others

Points	Description
3	Correct response with complete response
2	Correct response with incomplete explanation
1	Correct response for exactly one of a, b, or c with complete explanation OR Correct response with unreasonable or no explanation OR Incorrect response with reasonable explanation
0	Incorrect response with unreasonable or no explanation OR Correct response for exactly one of a, b, or c with incomplete, unreasonable, or no explanation OR No response

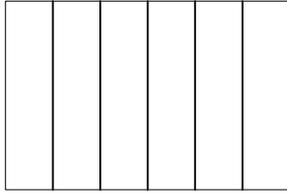
Notes:

- Explanations about the number of objects being divided do not count.
- Implicit attention to smaller odd numbers counts as incomplete (e.g. “sharing for 3 is less divisions than for 7”)

16 – Variety of strategies and knowledge of students’ thinking

Correct responses include:

I.



(five parallel splits) OR

II.



OR

1	2	5	
3	4		

III. Chops into many equal sized pieces and deals pieces OR

Complete explanations:

- Student makes 5 parallel cut of same width and doesn't know to make 4
- Student not using the whole or any other correct composition where student does not use the whole
- Student deals after creating many pieces and does not attend to dealing fairly
- Student uses landmarks and then chops and deals the left over pieces without attending to fairness

\*Note: The focus should be on creating 5 pieces that are equal in size without using the whole object to be counted as a "correct response." Strategies that create 5 equal sized pieces should be scored as a "1" if the respondent mentions creating both 5 pieces and that the pieces are the same size. If both of these conditions are not mentioned for strategies creating 5 equal sized pieces, score as a "0."

Points	Description
3	Correct response from I – III and complete explanation
2	Correct response with incomplete explanation
1	Correct response with explanation that does not mention size of pieces OR Correct response with no explanation OR Correctly splits circle with complete explanation
0	Incorrect response OR Correctly splits circle with incomplete, unreasonable, or no explanation OR No response

17 – Knowledge of sophistication

Correct response:

- a. Circles “2 among 7”
- b. Circles “among 3”
- c. Circles “among 3”

Complete explanations:

- a. States that 2-splits are easier or 7-splits are harder  
OR  
Dealing is easier  
OR  
Use symmetry since 2 is an even number
- b. States radial cuts are harder  
OR  
3-splits are harder  
OR  
Repeated halving is easier  
OR  
Use symmetry since 8 is an even number
- c. States radial cuts are harder  
OR  
3-splits are harder  
OR  
Repeated halving is easier  
OR  
Use symmetry since 8 is an even number

Note:

- Number of cuts is superfluous
- For Gemma’s, if there is a distinction of case C being less sophisticated than case D, score as complete explanation.

Points	Description
3	3 correct responses and complete explanation
2	2 correct responses with complete explanations OR 3 correct responses less than 3 complete explanations OR 2 or 3 incorrect responses with 2 or 3 complete explanations (it may be possible that the student circled the easier tasks)
1	1 correct response with a complete explanation OR 2 correct responses less than 2 complete explanations

0	3 incorrect responses with unreasonable or no explanations OR 1 correct response with incomplete, unreasonable, or no explanation OR No response
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18 – Knowledge of sophistication

Correct responses with complete explanations:

- VI. Number of pieces – explains that the child created the same number of pieces in each sector of the 3-split (explicit comparison of the number of pieces).
- VII. Size of pieces – explains that the student created three equal parts with a 3-split and notes the similarity of the corresponding pieces in each sector
- VIII. Exhaustion of whole – explains that the student used the whole cake.
- IX. Allocation – explains that it is unclear how the student will allocate the pieces of cake (i.e. indicating what each person’s share is)
- X. Related question – includes a question or task that relates to their uncertainties about what they student knows or does not know about fair sharing.

Points	Description
3	Response addresses 4 or 5 points from I, II, III, IV, & V
2	Response addresses points from 3 of I, II, III, IV, & V
1	Response addresses points from 2 of I, II, III, IV, & V
0	Response addresses points from 0 or 1 of I, II, III, IV, & V OR No response

Notes:

- Demonstrations in response to the next task do not answer the question and should not be counted as addressing IV.

## Appendix J – Individual Video Analysis Tasks and Video Clip Synopsis

Video Clip 1	Video Clip 2	Video Clip 3
<p>Task 1: Can you share the pirate treasure fairly among 4 pirates? Show me how you would do this.</p> <p>Task 2: Can you share the pirate treasure fairly among 3 pirates? Show me how you would do this.</p>	<p>Task 3: Share the [circular] cake fairly between 2 pirates.</p>	<p>Task 4: Share the [circular] cake fairly between 4 pirates.</p> <p>Task 5: Share the [circular] cake fairly between 3 pirates.</p>
Video Synopsis	Video Synopsis	Video Synopsis
<p>As the video clip begins, Emma has already built four 2 x 3 arrays. She tells the interviewer that all the pirates have 6. She counts the pieces of treasure, and she says, “6 out of 24 coins.” When asked how much of the total treasure did each pirate get, she says, “\$6” then says, “6¢.”</p> <p>The interviewer tells Emma that one of the pirates leaves and the 3 remaining pirates want to share the treasure fairly. Emma pushes all the coins together in one pile. She builds a 2 x 4 array and counts the coins “1, 2, 3, 4, 5, 6, 7, 8.” As she builds two more 2 x 4 arrays, Emma says she is using “8,” that “8 is the magic number.” She gives the following explanation to the interviewer: “Ok. Last time it was 6. Now you just added 2 more [points to first row of coins in one of the 2 x 4 arrays]. ‘Cause he had six [as she points to the place on the table where the 4<sup>th</sup> pirate’s treasure was]. I added two more to each one [points to the first row of coins in each 2 x 4 array] which makes 6.” Emma counts, “1, 2, 3, 4, 5, 6,” as she points to the first row of coins in each 2 x 4 array. She says, “So, I thought that 8 was the magic number.” She names the pirate’s share “a group,” and says that each pirate gets 8¢ of the total treasure.</p>	<p>Emma draws a line down the middle of a construction paper circle and says “it’s halves, 1, 2.” The interviewer asks her how she knows it’s fair. Using a new construction paper circle, Emma creates a dialogue between two pirates,</p> <p>“This is the pirate. They’re like, ‘Do you want to share this cake?’ ‘Ok.’</p> <p>And, they’re like, ‘how will we split it?’</p> <p>‘We could cut it like that [folds circle into two unequal-sized pieces], and I would have the big and you could have the tiny.’</p> <p>‘No, that won’t work. We need the same amount so that it will be fair.’</p> <p>‘Ok, why don’t we cut it in the middle?’</p> <p>‘Ok’ [folds in half].</p> <p>She uses this half to redraw the original line on the first circle, which she calls 1/2. She says each pirate gets, “half of a whole.”</p> <p>When asked if she can do it another way, she draws a line on a new construction paper circle that she calls a “dag-nol.” She uses the other circle that she had already folded into a half to re-draw the line.</p>	<p>Emma uses the folded half to make a fourth, by folding the half in half a second time. Emma traces the fourths on a new construction paper circle with the fourth she created by folding. She says, “we’re going to call each one, 1/4.” She says, “1/4, 2/4, 3/4, 4/4,” as she points to each fourth. She calls each share, “1/4.” Then, Emma drew 2 “diagonal” lines to show another way, and calls each share, “1/4.”</p> <p>When the interviewer asks Emma if she could share fairly for 3, she says she cannot.</p> <p>Emma tries to fold one of the circles into three equal-sized parts, but she is unsuccessful.</p> <p>Emma says there is not a way to share for 3. She draws a line on a new circle, points to the parts, and says “1, 2.” Emma draws another line that is parallel to the first line and says, “that is more as you think [as she points to the space between the two parallel lines].” She calls the pieces, “1/3s, 1/3.”</p> <div data-bbox="1166 1497 1308 1640" data-label="Image"> </div> <p>Emma’s representation.</p>