# ABSTRACT

LAKHANI, CHIRAG MANMOHAN. Geometric Invariant Theory Compactification of Quintic Threefolds. (Under the direction of Amassa Fauntleroy).

Quintic threefolds are some of the simplest examples of Calabi-Yau varieties. An interesting relationship, discovered by string theorists, is that every Calabi-Yau variety Y has a mirror Calabi-Yau variety $\overline{Y}$. In fact mirror symmetry is a relationship which relates complex structure moduli space of Y to the complexified Kähler moduli of its mirror $\overline{Y}$ . The purpose of this dissertation is to describe the complex structure moduli space from the point of view of geometric invariant theory (GIT).

The GIT compactification of quintic threefolds consists of adding certain singular quintic threefolds to the space of smooth quintic threefolds. An explicit description of the allowed singularities for this moduli space will be described. The description of allowed singularities is arrived at by a combinatorial procedure described by Mukai [15]. His method can be used to find the maximal semistable families of the moduli space. These maximal semistable families give a description of the possible singularities which can occur in the moduli space. The boundary structure of the compactification is also described in this dissertation.

Geometric Invariant Theory Compactification of Quintic Threefolds

by
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# DEDICATION

To my parents, whose support and love helped get me to this point.

# BIOGRAPHY

Chirag Lakhani was born and grew up in North Carolina. He has remained in that state ever since then. He attended North Carolina State University as an undergraduate and couldn't get enough so he remained there as a graduate student as well. He hopes to eventually escape the gravitational pull of North Carolina. In his free time he enjoys the outdoors and running.

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# Introduction

Quintic threefolds are a class of projective varieties which occupy a special place in algebraic geometry. They are some of the simplest examples of Calabi-Yau varieties. Calabi-Yau varieties have received a great deal of attention in the last 30 years because they give the right geometric conditions for some superstring compactifications [4]. A surprising geometric relationship found for Calabi-Yau varieties, due to string theory, is a phenomenon called mirror symmetry. Mirror symmetry, in its original form, states that a Calabi-Yau variety $Y$ has a mirror variety $\overline{Y}$ where the complex structure moduli space of $Y$ is the same as the complexified Kähler moduli space of $\overline{Y}$. This was explicitly calculated in [5] for the case of quintic threefolds. The purpose of this dissertation is to give an explicit description of the space of complex structure moduli space for Calabi-Yau quintic threefolds using geometric invariant theory (GIT).

Complex structures are inherently an analytic structure on quintic threefolds. When a quintic threefold is viewed as a manifold, there is an underlying real smooth manifold structure. The difference between a real smooth manifold and a complex manifold is that a complex manifold has holomorphic mappings rather than just smooth mappings. An underlying even dimensional real smooth manifold may have many different complex structures that can be imposed on it. The set of different complex structures for a real smooth manifold is the complex structure moduli space.

A quintic threefold hypersurface $X$ is the zero set of a degree 5 homogeneous polynomial $F \in \mathbb{C}[x_0, x_1, x_2, x_3, x_4]$. In this case, all of the complex structures of $X$ correspond to deforming the coefficients of the polynomial $F$ [9]. The advantage of this correspondence is that complex structures of quintic threefolds correspond to degree 5 polynomials $F \in \mathbb{C}[x_0, x_1, x_2, x_3, x_4]$. Two polynomials $F$ and $\overline{F}$ represent the same complex structure if there is a coordinate transformation which transforms one polynomial into the other. This coordinate transformation on quintic threefolds is given by a $SL(5, \mathbb{C})$ action on the set of degree 5 polynomials. The classification of complex structures reduces to classifying polynomials $F$ up to a $SL(5, \mathbb{C})$ coordinate transformation. GIT is a natural tool that can be used to study this problem.

GIT is a tool in algebraic geometry used to construct quotient varieties. If a group $G$ acts on an algebraic variety $X$ then, using GIT, one can find the quotient variety $X//G$. The quotient $X//G$ is meant to be a variety whose points correspond to unique $G$-orbits of $X$. Unfortunately, this does not always occur. In GIT there can be points in $X//G$ which represent multiple $G$-orbits. This means that multiple $G$-orbits may map to the same point in $X//G$. Despite this mapping it is still possible to find a unique $G$-orbit which represents a point in $X//G$. This unique orbit is called the minimal orbit and its purpose is to give a representative orbit in $X$ which corresponds to a point in $X//G$.

In the preface of [16] the original developer, David Mumford, states that one of his main intentions in developing GIT was so that it can be used to create moduli spaces. Moduli space is a general term that means a space which classifies a class of geometric objects up to some equivalence relation. In the setting of GIT, the variety $X$ would represent the parameter space of all objects which are to be classified. The group $G$ represents the equivalence relations between objects in this parameter space, so the quotient $X//G$ parameterizes objects up to equivalence relation. This approach to constructing moduli spaces has some advantages and disadvantages but it is still a useful method for constructing moduli spaces. This approach has been used to classify certain low degree hypersurfaces in certain $\mathbb{P}^n$ [1, 12, 19, 25]. The classification, in all of these cases, is given in terms of the singularities which may arise on the hypersurfaces. The set of smooth hypersurfaces represent nice orbits in the GIT problem. The quotient variety of the set of smooth hypersurfaces is an open variety. By allowing certain singular hypersurfaces this moduli space can be compactified. One of the goals of this dissertation is to describe which singularities need to be allowed in order to compactify the moduli space of quintic threefolds. The parameter spaces of hypersurfaces have very nice properties which can be used to describe their moduli spaces. A combinatorial method for finding the set of maximal semistable families is explained in Mukai's book [15] and applied to the case of cubic fourfolds by Laza [12]. This set of maximal semistable families are used to classify the allowable singular hypersurfaces in the moduli space. The combinatorial methods of Mukai and Laza have been used in this dissertation to find the moduli space of quintic threefolds.

The dissertation is organized as follow. Chapter 2 is an introduction to the geometry of hypersurfaces. A description of the types of singularities which arise in this dissertation will be given. Chapter 3 is an introduction to GIT. The properties of quotient varieties, the various definitions of stability are given, and a detailed explanation of minimal orbits is given. Chapter 4 gives a description of the semistable locus. The combinatorial approach of Mukai and Laza is explained and applied to quintic threefolds. A description of singularities which arise in the semistable locus will be given. Lastly, a partial description of the stable locus is also given. Chapter 5 describes the minimal orbits for the GIT problem. This chapter essentially describes the structure of the boundary components of the GIT compactification and what the most degenerate points are in the GIT compactification.

# Chapter 2

# Geometry of Quintic Threefolds

Quintic threefolds are a class of projective varieties which occupy a special place in algebraic geometry. They are some of the simplest examples of Calabi-Yau varieties. Calabi-Yau varieties have received a great deal of attention in the last 30 years because they give the right geometric conditions for some superstring compactifications [4]. A surprising geometric relationship found for Calabi-Yau varieties, due to string theory, is a phenomenon called mirror symmetry. Mirror symmetry, in it's original form, states that a Calabi-Yau variety $Y$ has a mirror variety $\overline{Y}$ where the complex structure moduli of $Y$ is the same as the complexified Kähler moduli of $\overline{Y}$. This was explicitly calculated in [5] for the case of quintic threefolds. The purpose of this dissertation is to give an explicit description of the complex structure moduli for Calabi-Yau quintic threefolds using GIT.

## 2.1 Geometry of Hypersurfaces

**All varieties will be over the complex numbers $\mathbb{C}$.**

A quintic threefold is a hypersurfaces in $\mathbb{P}^4$ which is defined as the zero locus of a degree 5 homogeneous polynomial $F \in \mathbb{C}[x_0, x_1, x_2, x_3, x_4]$. Hypersurfaces in $\mathbb{P}^n$ are easy to describe since they are only the zero locus of a single homogeneous polynomial.

**Definition 2.1.1** (Degree $d$ hypersurface in $\mathbb{P}^n$)**.** A algebraic variety $X$ is a *degree* *d hypersurface* in $\mathbb{P}^n$ if it is the zero locus of a homogeneous degree $d$ polynomial $F \in \mathbb{C}[x_0, x_1, \ldots, x_n]$. The ideal of $X$ is generated by $F$.

The space $\mathbb{P}^n$ is defined as $\mathbb{P}^n := (\mathbb{C}^{n+1} \setminus [0, 0, \ldots, 0])/\sim$ where the equivalence relation is given by $x \sim y$ if $x = [a_0, a_1, \ldots, a_n]$ and $y = [\lambda a_0, \lambda a_1, \ldots, \lambda a_n]$ for $\lambda \in \mathbb{C}^*$. Since $F \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ is homogeneous it respects the equivalence relation of $\mathbb{P}^n$, so it can be thought of as a hypersurface in $\mathbb{C}^{n+1}$ or it descends to a hypersurface in $\mathbb{P}^n$. Since hypersurfaces are described by homogeneous polynomials then the geometric characteristics of a hypersurface will be stated in terms of conditions on $F$.

*Remark* 2.1.2. We will interchangeably use the fact that a hypersurface $X$ is also given as the zero set of a homogeneous polynomial $F$. If a homogeneous polynomial $F$ is described as a threefold then it means that the threefold is the zero locus of $F$.

### 2.1.3 Singularities of Hypersurfaces

The study of the different types of singularities which arise in quintic threefolds will be a very important aspect of studying the GIT compactification. One goal will be to classify the types of singularities which will be allowed in the GIT compactification. The idea of the GIT compactification is that given the space of smooth quintic threefolds, singular hypersurfaces need to be included in order to compactify the space of quintic threefolds. In the case of cubic threefolds [1] only hypersurfaces with at worst $A_n$ or $D_4$ isolated singularities need to be included in the GIT compactification. In the case quintic threefolds there will be hypersurfaces which have isolated and non-isolated singularities which must be included in order to compactify the space. This section will give an overview of the singularity terminology which will be used throughout the dissertation.

Given a homogeneous polynomial $F$, the set of singular points of $F$ can be defined as the set of points which cause all partial derivatives to vanish. This set of points is defined by the Jacobian ideal of $F$.

**Definition 2.1.4.** Given a homogeneous polynomial $F \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ the *singular locus* of $F$ is the zero set of the Jacobian ideal $\langle F, \frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \rangle$.

If a hypersurface $F$ of degree $d$ is singular at an isolated point then by a coordinate transformation the singular point can be moved to $[1 : 0 : \cdots : 0] \in \mathbb{P}^n$. The homogeneous polynomial can be expanded in terms powers of $x_0$ as follows:

$$F(x_0, x_1, \ldots, x_n) = x_0^{d-1} f_1(x_1, \ldots, x_n) + \ldots + f_d(x_1, \ldots, x_n), \tag{2.1}$$

where $f_i(x_1, \ldots, x_n)$ is a homogeneous polynomial of degree $i$ in $x_1, \ldots, x_n$. In order for $[1 : 0 : \ldots : 0]$ to be a point in $F$, the monomial $x_0^d$ can not be included. The polynomial $F$ is singular at $[1 : 0 : \ldots : 0]$ if the the highest power of $x_0$ is $x_0^{d-2}$. This is the case because the partial derivatives will not vanish at $[1 : 0 : \ldots : 0]$ if the power $x_0^{d-1}$ is present in the polynomial and hence will not be in the Jacobian ideal.

The classification of the singularities, such as the point $[1 : 0 : \ldots : 0]$, involve understanding the tangent cone of the singularity. In geometry, the tangent space is the set of all tangent vectors at a point. In general the dimension of the tangent space at a point is the same as the dimension of the space. At a singularity this correspondence breaks down because there are tangent vectors which are not uniquely defined at a singular point. The tangent cone can be thought of as a generalization of the tangent space which provides some information about the singularity.

**Definition 2.1.5** (Tangent cone at $F$)**.** Let $[1 : 0 : \ldots : 0]$ be a point of a degree $d$ polynomial $F$. The *tangent cone* of $[1 : 0 : \ldots : 0]$ at $F$ is the zero locus of the lowest degree polynomial $(f_i)$ in the following decomposition. The polynomial $F$ has the decomposition

$$F := x_0^{d-i} f_i(x_1, \ldots, x_n) + x_0^{d-i-1} f_{i+1}(x_1, \ldots, x_n) + \ldots + f_d(x_1, \ldots, x_n). \tag{2.2}$$

The degree of the tangent cone is $i$.

If $[1 : 0 : \ldots : 0]$ is non-singular then the tangent cone is the tangent space $f_1$ of $F$. If $[1 : 0 : \ldots : 0]$ is singular then the tangent cone $f_i$ will have $i \geq 2$. One geometric interpretation of the tangent cone at $[1 : 0 : \ldots : 0]$ is that it represents the set of lines through $[1 : 0 : \ldots : 0]$ whose intersection multiplicity with $[1 : 0 : \ldots : 0]$ is greater than or equal to $i$ [2]. Intersection multiplicity is used to count the degeneracy of intersections of two varieties in complimentary dimensions.

**Definition 2.1.6** (Intersection Multiplicity c.f. [2] p.103)**.** For two projective varieties $X, Y \in \mathbb{P}^n$ where dim $X$ + dim $Y = n$, let $I(X)$ and $I(Y)$ be the homogeneous ideals defining $X$ and $Y$. For a point of intersection $p \in X \cap Y$ the *intersection multiplicity* of $p$ is

$$m_p(X, Y) := dim_{\mathbb{C}}\big(\mathcal{O}_{\mathbb{P}^n,p}/(I(X) + I(Y))\big), \tag{2.3}$$

where $dim_{\mathbb{C}}$ means dimension as a complex vector space and $\mathcal{O}_{\mathbb{P}^n,p}$ is the structure sheaf localized at the point $p$.

This definition is used to generalize the familiar notion that a curve of degree $d$ and curve of degree $e$ intersect in $de$ points.

**Theorem 2.1.7** (Bezout's theorem [2] ch.4)**.** *For two curves $C_1, C_2 \in \mathbb{P}^2$ of degree $d$ and degree $e$ respectively, if they intersection in finite points then following relation holds,*

$$\sum_{p \in C_1 \cap C_2} m_p(C_1, C_2) = de. \tag{2.4}$$

**Theorem 2.1.8** (Generalized Bezout's Theorem [2] ch.4)**.** *For two Cohen-Macaulay projective varieties $X, Y \in \mathbb{P}^n$ where dim $X$ + dim $Y$=$n$ and having degree $d$ and degree $e$ respectively, the following relation holds,*

$$\sum_{p \in X \cap X} m_p(X, Y) = de. \tag{2.5}$$

Bezout's theorem says that when counting intersection points, if intersection multiplicities are included, the classical notion of two curve intersecting in $de$ points still holds. In this context, intersection multiplicities gives a method of understanding intersections which are more degenerate than just transversal intersections. The points of intersection of a line with a degree $d$ hypersurface will sum up to $d$ when intersection multiplicity is taken into account.

The geometric interpretation of the tangent cone gives a method for classifying singularities. The first coarse classification of singularities that arise in quintic threefolds is to classify singularties by describing their tangent cones. The higher the degree of the tangent cone the more complicated the singularity. Unfortunately classifying tangent cones is not enough to completely describe the singularities that arise in quintic threefolds. The decomposition (2.2) can also give some finer data about intersection multiplicities. As stated earlier, the zero locus of $f_i$ give the set of points whose lines with $[1 : 0 : \ldots : 0]$ which intersect $F$ . The iintersection multiplicity of this line with $F$ is least $i$. The decomposition (2.2) gives a filtration for higher multiplicity intersections with $[1 : 0 : \ldots : 0]$. The set of lines which have intersection multiplicity at least $i + 1$ is given by the zero locus of both $f_i$ and $f_{i+1}$.

**Proposition 2.1.9** ( [2] p.110)**.** *Given a degree $d$ polynomial $F$ with $[1 : 0 : \ldots : 0]$ as its singular point, degree $i$ tangent cone, and decomposition of the form (2.2), the set of lines whose intersection multiplicity with $[1 : 0 : \ldots : 0]$ is at least $r$ is given by the zero locus of $f_i = f_{i+1} = \ldots = f_{i+r}$ where $i \leq r \leq d - i - 1$.*

This proposition will be important when classifying singularities of quintic threefolds in chapter 4. The classification of singularities will include characterizing the tangent cone to the singularity as well it's higher order intersection multiplicities. When restricting to the case of polynomials in $\mathbb{P}^4$, a polynomial is of the form $F \in \mathbb{C}[x_0, x_1, x_2, x_3, x_4]$. The degree of the tangent cone and decomposition (2.2) for a polynomial $F$ also give information about the maximum multiplicity of the the ideal $\langle x_1, x_2, x_3, x_4\rangle$ which contains $F$.

**Definition 2.1.10.** Given the a homogeneous polynomial $F$ with singular point $[1 : 0 : 0 : 0 : 0]$, the singular point is a

1. *double point* if the degree of the tangent cone is 2 i.e. $F \in \langle x_1, x_2, x_3, x_4\rangle^2$,

2. *triple point* if the degree of the tangent cone is 3 i.e. $F \in \langle x_1, x_2, x_3, x_4\rangle^3$,

3. and a *quadruple point* if the degree of the tangent cone is 4 i.e. $F \in \langle x_1, x_2, x_3, x_4 \rangle^4$.

The definitions double point, triple point, and quadruple point can be extended to non-isolated singularities. If $F$ has a non-isolated singularity such as a line or plane then, by a change of coordinates, it can be mapped to lines and planes in standard coordinates. We will focus only on lines and planes since they are the singularities which arise in the GIT quotient of quintic threefolds. A line can be represented by coordinates $[a : b : 0 : 0 : 0] \in \mathbb{P}^4$ the ideal for this line is $\langle x_2, x_3, x_3 \rangle$. A plane can be representated by the coordinates $[a : b : c : 0 : 0] \in \mathbb{P}^4$ the ideal for this plane is $\langle x_3, x_4 \rangle$. Singular lines and planes, similar to the case of points, can be defined using their ideals.

**Definition 2.1.11.** Given the a homogeneous polynomial $F$ with singular line $[a : b : 0 : 0 : 0]$, the singular line is a

1. *double line* if $F \in \langle x_2, x_3, x_4 \rangle^2$,
2. *triple line* if $F \in \langle x_2, x_3, x_4 \rangle^3$,
3. and *quadruple line* if $F \in \langle x_2, x_3, x_4 \rangle^4$.

**Definition 2.1.12.** Given the a homogeneous polynomial $F$ with singular plane $[a : b : c : 0 : 0]$, the singular plane is a

1. *double plane* if $F \in \langle x_3, x_4 \rangle^2$,
2. *triple plane* if $F \in \langle x_3, x_4 \rangle^3$,
3. and a *quadruple plane* if $F \in \langle x_3, x_4 \rangle^4$.

*Remark* 2.1.13. In algebraic geometry if a hypersurface is contained in an ideal $F \in \langle x_{a_1}, x_{a_2}, \ldots, x_{a_r} \rangle^d$ then it is known that the zero locus of $\langle x_{a_1}, x_{a_2}, \ldots, x_{a_r} \rangle^d$ denoted $V(\langle x_{a_1}, x_{a_2}, \ldots, x_{a_r} \rangle^d)$ is contained in the zero locus of $F$ denoted $V(F)$ i.e. $V(\langle x_{a_1}, x_{a_2}, \ldots, x_{a_r} \rangle^d) \subseteq V(F)$. We will use these notions interchangeably. The statement "X contains the ideal $\langle x_{a_1}, x_{a_2}, \ldots, x_{a_r} \rangle^d$" means $X$ contains the zero locus of the ideal.

These singularities will give a complete classification of singularities which arise in the GIT compactification of quintic threefolds.

## 2.2 Parameter Space of Quintic Threefolds

In $\mathbb{P}^4$ there are $\binom{9}{5} = 126$ degree 5 variables. An arbitrary quintic threefold $F$ can be written in the following form

$$F(x_0, x_1, x_2, x_3, x_4) := \sum_{i+j+k+l+m=5} a_{ijklm} x_0^i x_1^j x_2^k x_3^l x_4^m. \tag{2.6}$$

The 126 coefficients $a_{ijklm}$ can be thought as coordinates of a 126 dimensional vector space which represent the polynomial $F$. As a representation this vector space is denoted $Sym^5(\mathbb{C}^5)$. Since the hypersurfaces are in projective space then $F$ and $\lambda F$ represent the same polynomial therefore $Sym^5(\mathbb{C}^5)$ needs to be projectivized.

**Definition 2.2.1.** The *parameter space of quintic threefolds* is $\mathbb{P}(Sym^5(\mathbb{C}^5))$

The vector space $Sym^5(\mathbb{C}^5)$ is an $SL(5, \mathbb{C})$-reprsentation, which comes from the natural action of $SL(5, \mathbb{C})$ on $\mathbb{P}^4$. The GIT compactification, as discussed in the next chapter, corresponds to quotienting $\mathbb{P}(Sym^5(\mathbb{C}^5))$ by $SL(5, \mathbb{C})$.

## 2.3 Calabi-Yau Condition

As stated earlier, quintic threefolds are some of the simplest example of Calabi-Yau varieties. If a Calabi-Yau variety is a manifold then there are many equivalent definitions of a Calabi-Yau manifold.

**Definition 2.3.1.** A smooth and compact Kähler manifold $X$ is *Calabi-Yau* if it has the following equivalent properties:

1. the canonical line bundle $\omega_X$ (volume form) is trivial i.e. $\omega_X \cong \mathcal{O}_X$,

2. the first Chern class $c_1(X)$ vanishes,

3. the holonomy group of $X$ is $SU(n)$,

4. there exists a Kähler metric on $X$ with vanishing Ricci curvature.

The various equivalences between these statements have been proven such as in Yau [24]. One of the more difficult equivalences to be proved was (2) $\Rightarrow$ (4). This was the conjecture proposed by Calabi and proven by Yau [24]. For the case of quintic threefolds the adjunction formula can be used to show that $\omega_X$ is trivial.

**Proposition 2.3.2** (Adjunction formula [18] p.40)**.** *Let $X \subseteq Y$ be a non-singular hypersurface. Then $\omega_X = (\omega_Y \otimes \mathcal{O}_Y(X))|_X$, where $\mathcal{O}_Y(X)$ is the line bundle associated to the hypersurface X.*

If $X$ is a degree $d$ hypersurface and $Y = \mathbb{P}^4$ then $\omega_Y \cong \mathcal{O}_{\mathbb{P}^4}(-5)$ and $\mathcal{O}_Y(X) \cong \mathcal{O}_{\mathbb{P}^4}(d)$. By the adjunction formula

$$\omega_X = (\mathcal{O}_{\mathbb{P}^4}(-5) \otimes \mathcal{O}_{\mathbb{P}^4}(d))|_X = (\mathcal{O}_{\mathbb{P}^4}(d-5))|_X. \tag{2.7}$$

For the quintic threefold $\omega_X = (\mathcal{O}_{\mathbb{P}^4}(0))|_X = \mathcal{O}_X$, so $X$ satisfies the condition to be a Calabi-Yau manifold. The above result is for smooth quintic hypersurface but the GIT compactification will also include singular quintic Calabi-Yau varieties.

# Chapter 3

# Geometric Invariant Theory

In this section we will describe the theory of geometric invariant theory. Broadly speaking, given a group $G$ acting on a variety $X$, geometric invariant theory gives a method of constructing a quotient variety $X//G$. The variety $X//G$, as the quotient of the parameter space of quintic threefolds, is supposed to parameterize $G$-orbits of points on $X$. In the general setting, there are bad orbits which must not be included in the construction of $X//G$ and there may be several orbits which map to same point in $X//G$. In order to give a precise description of these caveats the notion of semistable, stable and unstable points will be introduced. In this chapter various definitions of quotients, various definitions of stability, and how quotients are formed will all be discussed.

## 3.1 Reductive Groups

Geometric invariant theory is best understood in the case of a reductive linear algebraic group $G$ acting on a variety $X$. Reductive group actions on varieties have finite generation properties which allow one to construct quotient varieties. For general group actions these finite generation properties do not always hold. There has been some recent progress, made by Doran and Kirwan, on GIT for arbitrary groups but there are still some open questions [8, 10].

**Definition 3.1.1.** A *linear algebraic group* $G$ over $\mathbb{C}$ is a closed subgroup of $GL(n, \mathbb{C})$.

**Definition 3.1.2.** A *reductive linear algebraic group* is a linear algebraic group whose radical (connected component of the maximal normal solvable subgroup which contains the identity) is diaganolizable.

One of the advantages of using reductive linear algebraic groups is that their representations are completely reducible. This property was very important in Mumford's development of GIT.

**Definition 3.1.3** (c.f. [11] p.98 )**.** If $G$ is a linear algebraic group with a rational representation $(\rho, V)$ then the representation is *completely reducible* if for every $G$-invariant subspace $W \subseteq V$ there exists a $G$-invariant subspace $U$ where $V = W \oplus U$.

**Proposition 3.1.4** (c.f. [11] p.98)**.** *Every rational representation of a reductive linear algebraic group is completely reducible.*

Reductive linear algebraic groups also have the nice property that if such a group $G$ acts on a ring $A$, which is finitely generated over $\mathbb{C}$, then the ring of invariants, $A^G$, is finitely generated. This is important in GIT because the ring of invariants $A^G$ will be the ring used to construct the quotient space.

**Theorem 3.1.5** (Nagata's Theorem)**.** *[ [6] p.41] If $G$ is a reductive linear algebraic group which acts on a ring $A$ then the ring of invariants $A^G$, from this action, is finitely generated.*

Nagata's Theorem gives a natural way to map an affine variety $X = \mathrm{Spec(A)}$ with a linear reductive group action $G$ into a $G$-invariant space. Since $A^G \subseteq A$ on the level of rings, this inclusion reverses to give

$$\Phi : X \to \mathrm{Spec(A^G)}. \tag{3.1}$$

Since $A^G$ is finitely generated by Nagata's theorem, the quotient $\mathrm{Spec(A^G)}$ is an affine variety.

This induced map $\Phi$ is a dominant morphism (the image $\Phi(\mathrm{X})$ is dense in $\mathrm{Spec}(\mathrm{A}^{\mathrm{G}})$.

Nagata's theorem is also important when looking at quotients of projective varieties. In the case of a projective variety $\mathrm{X} = \mathrm{Proj}(\mathrm{A})$ the inclusion $A^G \subseteq A$ generates a map $\Phi : X = \mathrm{Proj}(\mathrm{A}) \to \mathrm{Proj}(\mathrm{A}^{\mathrm{G}})$. This map is only a rational map. In order to get a morphism for this map, we need to restrict to a subset of points on $X$ called the semistable points ($X^{ss}$) which will map into $\mathrm{Proj}(\mathrm{A}^{\mathrm{G}})$. This will be explained further in the next sections.

## 3.2 Notions of Quotients

**For the remainder of this dissertation we will assume all groups are reductive linear algebraic groups unless otherwise specified.**

In GIT there are various types of quotients. Following Lakshmibai [11] we will give describe three standard types of quotients: the categorical quotient, good quotient, and geometric quotient. We will also describe the relationships between these different notions of quotients . In the previous section it was shown that some notion of a quotient space can be constructed by taking Spec or Proj of the ring of invariants. In the next sections we will describe precisely how Spec and Proj are related to categorical, good, and geometric quotients.

**Definition 3.2.1** (Categorical quotient [11] p.102)**.** Given a group action of $G$ on a variety $X$, a *categorical quotient* is a variety $Y$ and a $G$-invariant morphism

$$\Phi : X \to Y \tag{3.2}$$

with the following universal property: for any other $G$-invariant morphism $\tilde{\Phi} : X \to \tilde{Y}$ there is a unique morphism $\tau : Y \to \tilde{Y}$ such that

$$
\begin{tikzcd}
X \arrow[rr, "\tilde{\Phi}"] \arrow[dr, "\Phi"'] & & \tilde{Y} \\
& Y \arrow[ur, "\tau"'] &
\end{tikzcd}
$$

commutes. Denote the categorical quotient $Y := X//G$. It is unique up to isomorphism.

The universal property shows that all $G$-invariant morphisms must factor through $\Phi$ therefore a categorical quotient $X//G$ for a $G$-action on $X$ is unique. A more restrictive class of categorical quotients are good quotients. In this dissertation we will more concerned with good quotients.

**Definition 3.2.2** (Good quotient [11] p.103))**.** Given a group $G$ acting on a variety $X$, a *good quotient* of the $G$-action of $X$ is a variety $Y$ and a $G$-invariant morphism $\Phi : X \to Y$ with the following properties:

1. $\Phi$ is surjective;

2. For any open subset $U \subseteq Y$, $\Phi^{-1}(U) \subseteq X$ is open and affine if and only if $U \subseteq Y$ is open and affine;

3. for any open subset $U \subseteq Y$ the homomorphism of rings;

$$\Phi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\Phi^{-1}(U)) \tag{3.3}$$

   is an isomorphism onto $\mathcal{O}(\Phi^{-1}(U))^G$;

4. if $W \subseteq X$ is $G$-stable and closed then $\Phi(W)$ is closed;

5. if $W_1$ and $W_2$ are two disjoint $G$-stable closed subsets of $X$, then $\Phi(W_1) \cap \Phi(W_2) = \emptyset$.

By abuse of notation the good quotient will also be denoted $Y := X//G$.

From the definitions it is not obvious that good quotients satisfy the universal mapping property (3.2.1) in the categorical quotient definition. Since good quotients behave well locally it can be shown that they satisfy the universal mapping property.

**Proposition 3.2.3** (c.f. [11] p.104)**.** *A good quotient is a categorical quotient.*

*Remark* 3.2.4. In the case of a $G$-action on an affine variety $X = \mathrm{Spec(A)}$ the quotient $X//G = \mathrm{Spec(A^G)}$ from (3.1) is a good quotient.

**Definition 3.2.5** (Orbit Space)**.** Given a group $G$ acting on a variety $X$, a variety $Y$ and $G$-invariant morphism $\Phi : X \to Y$ is an *orbit space* if each $y \in Y$ corresponds to a unique $G$-orbit in $X$ i.e. $\Phi^{-1}(x)$ is a $G$-orbit.

**Definition 3.2.6** (Geometric quotient [11] p.105)**.** A good quotient (Y,Φ) which also an orbit space is called a *geometric quotient*. We will denote the geometric quotient as $X/G$

A geometric quotient is an ideal type of quotient in GIT. Along with the properties of a good quotient, a geometric quotient parameterizes precisely the $G$-orbits of $X$. This is not the case for a good quotient, where it is possible for two different orbits to map to the same point in the good quotient.

**Definition 3.2.7.** For a $G$-action on $X$ if $x \in X$ then $Gx := \{gx|g \in G\}$ is the $G$-orbit of $x$. The space $\overline{Gx}$ is the *closure of* $Gx$ in $X$.

**Proposition 3.2.8** (c.f. [11] p. 104)**.** *For a $G$-action on a variety $X$ if $(X//G, \Phi)$ is a good quotient then $\Phi(x_1) = \Phi(x_2)$ if and only if $\overline{Gx_1} \cap \overline{Gx_2} \neq \emptyset$.*

This proposition shows that if the closure of two orbits intersect then they are map to the same point in the good quotient. It shows that there can be multiple orbits which can be mapped to the same point in the good quotient. This phenomenon only occurs if a $G$-orbit is not closed. If the $G$-orbit is closed then the above proposition is vacuous. Even if multiple orbits map to the same point in the good quotient, there is still a closed orbit which represents this point as the next lemma shows.

**Lemma 3.2.9** (Closed orbit lemma [3, 11])**.** *For a $G$-action on a variety $X$ each $G$-orbit $Gx$ is a smooth variety which is open in its closure. The boundary of the $G$-orbit, $\overline{Gx} \setminus Gx$, is a union of strictly smaller dimensional $G$-orbits. The orbits of minimal dimension are closed.*

**Definition 3.2.10.** For a $G$-action on $X$, a *minimal orbit* is a closed $G$-orbit of smallest dimension in $\overline{Gx}$.

If a $G$-orbit is closed then it is a minimal orbit because there are no other orbits in it's closure. In this dissertation we will be concerned with finding minimal orbits in the closure of non-closed orbits. The minimal orbit will be considered a degeneration of the larger non-closed orbit.

**Definition 3.2.11.** Given a $G$-action on $X$ and a $x \in X$, a *degeneration* of $Gx$ is a minimal orbit in $\overline{Gx}$

## 3.3 Stability and Semistability for Projective Varieties

In the case of affine varieties it is easy to describe quotient spaces. The map (3.1) is a morphism of varieties from $X$ to a quotient. If $X = \mathrm{Proj(A)}$ is a projective variety then the corresponding quotient

$$\Phi : X := \mathrm{Proj(A)} \dashrightarrow \mathrm{X//G} := \mathrm{Proj(A^G)} \tag{3.4}$$

is not, in general, a morphism of varieties. It is a rational map. In order to make (3.4) into a morphism of varieties one must restrict to an open subset of $X$ called the semistable points ($X^{ss}$). There are multiple equivalent definitions for semistable and stable points. We will give the more geometric definitions because they are the most relevant in this dissertation. The definition of stability and semistability for a $G$-action on a projective variety $X$ involves a choice of a representation.

**Definition 3.3.1** (Linearization of G-action)**.** Given a $G$-action on a projective variety $X$, a *linearization of the $G$-action* is a representation

$$\rho : G \to GL(V) \tag{3.5}$$

where $X \subseteq \mathbb{P}(V)$ is closed in $\mathbb{P}(V)$ and the $G$-action on $X$ is induced by the representation given by (3.5).

The extra data that a linearization provides is an embedding of $X$ into some projective space $\mathbb{P}(V)$ and the extension of the $G$-action of $X$ onto this ambient projective space. In the case of quintic threefolds the linear action is canonically defined. As shown in chapter 2, the space of quintic threefolds is $\mathbb{P}(Sym^5(\mathbb{C}^5))$. The linearization of $SL(5, \mathbb{C})$-action on $\mathbb{P}(Sym^5(\mathbb{C}^5)$ is naturally defined in this situation.

**Definition 3.3.2.** The *linearization of the $SL(5, \mathbb{C})$ action on* $\mathbb{P}(Sym^5(\mathbb{C}^5))$ is given by the $5th$ symmetric power ($Sym^5$) representation of the standard $SL(5, \mathbb{C})$ representation on $\mathbb{C}^5$.

Given a linearization, a point $x \in X \subseteq \mathbb{P}(V)$ can be viewed as an element in $V$ up to projective equivalence. This correspondence gives a precise definition for semistability.

**Definition 3.3.3** (Semistable)**.** Given a $G$-action on $X$ with a linearization $V$, a point $x \in X$ is *semistable* if there is a representative $\hat{x} \in V$ of $x$ where $0 \notin \overline{G\hat{x}}$ in $V$. The set of semistable points in $X$ will be denoted $X^{ss}$.

**Definition 3.3.4** (Unstable)**.** Given a $G$-action on $X$ with a linearization $V$, a point $x \in X$ is *unstable* if it is not semistable. The set of unstable points in $X$ will be denoted $X^{us}$.

From the previous discussions about good quotients it was explained that closures of orbits were identified in the quotient. If a point is unstable then it can be identified with the 0 orbit in the quotient. Since we are working in projective space there is no 0 point in the projective variety. Therefore unstable points do not make sense for good quotients of projective varieties. They will be precisely the points which are thrown away in order to make (3.4) into a morphism of varieties. To define a geometric quotient the set of semistable points must be further restricted to stable points.

**Definition 3.3.5** (Stable)**.** Given a $G$-action on $X$ with a linearization $V$, a point $x \in X$ is *stable* if there is a representative $\hat{x} \in V$ of $x$ such that:

1. $0 \notin \overline{G\hat{x}}$;
2. the orbit $G\hat{x}$ is closed;
3. the isotropy group, $G_x$, of $\hat{x}$ is finite.

The set of stable points in $X$ will be denoted $X^s$. Note: $X^s \subseteq X^{ss}$ due to condition 1.

**Definition 3.3.6** (Strictly semistable)**.** Given a $G$-action on $X$ with a linearization $V$, a point $x \in X$ is *strictly semistable* if it is semistable but not stable.

These definitions give precisely the conditions needed to determine the geometric quotient and good quotient for a $G$-action on $X$.

**Theorem 3.3.7** (c.f. [11, 17] )**.** *Let there be a $G$-action on $X = \mathrm{Proj(A)}$ with a linearziation $X \subseteq \mathbb{P}(V)$. The set of semistable points $X^{ss}$ have a good quotient*

$$\Phi : X^{ss} \to X//G := X^{ss}//G = \mathrm{Proj(A^G)}. \tag{3.6}$$

*The good quotient $X//G$ is a projective variety and $\Phi$ is surjective.*

**Theorem 3.3.8** (c.f. [11, 17] )**.** *Let there be a $G$-action on $X = \mathrm{Proj(A)}$ with a linearziation $X \subseteq \mathbb{P}(V)$. The set of stable points $X^s$ have a geometric quotient*

$$\Phi : X^s \to X^s/G. \tag{3.7}$$

*The geometric quotient $X^s/G$ is open (quasiprojective) in $X//G$.*

*Remark* 3.3.9. It is also important to note that these definitions of stable, semistable, and unstable depend on the choice of linearization. A different choice of linearization may change the set of semistable points. A very interesting connection to varying the choice of linearization and birational transformations of quotients is given in the papers of Thaddeus [22] and Dolgachev-Hu [7]

In the case of projective varieties the geometric quotient $X^s/G$ is the orbit space which represents all of the "nice orbits" in the GIT quotient. The good quotient $X//G$ gives a compactification of $X^s/G$. The boundary $X//G \setminus X^s/G$ correspond to the most degenerate orbits in the GIT quotient.

## 3.4 Hilbert-Mumford Criterion

Using only the definitions it is, in general, very difficult to determine precisely which points are semistable and stable. One of Mumford's innovative ideas in GIT was to provide a numerical method for determining which points are semistable and stable. The criterion uses the set of one parameter subgroups (1-PS) of $G$.

**Definition 3.4.1** (One-parameter subgroup (1-PS))**.** Given a group $G$ a *1-PS* $\lambda$ is a homomorphism

$$\lambda : \mathbb{C}^* \to G. \tag{3.8}$$

A 1-PS can be thought of as a function $\lambda(t)$ where $t \in \mathbb{C}^*$.

Given a linearized $G$-action on $X$ and a point $x \in X$ with representative $\hat{x} \in V$, the Hilbert-Mumford criterion gives a method for determining whether $x$ is semistable or stable using the set of 1-PS of $G$. Given a 1-PS $\lambda$ the vector space $V$ can be decomposed into weight spaces

$$V = \bigoplus_{i \in \mathbb{Z}} W_i \tag{3.9}$$

where $\lambda(t)w_i = t^i w_i$ for $t \in \mathbb{C}^*$ and $w_i \in W_i$.

The resulting (3.9) decomposes the point $\hat{x}$ into

$$\hat{x} = \sum_i w_i. \tag{3.10}$$

The decomposition of the point $\hat{x}$ using a 1-PS $\lambda$ is needed to define Mumford's numerical function.

**Definition 3.4.2** (Mumford's numerical function)**.** Given a $G$-action on $X$ with a linearization $V$, a point $x \in X$, and a 1-PS $\lambda$, *Mumford's numerical function* is

$$\mu(x, \lambda) = -min\{i, w_i \neq 0\}. \tag{3.11}$$

The vectors $w_i$ come from the decomposition (3.10) associated to the 1-PS $\lambda$.

A 1-PS $\lambda$ has a diagonalized action on the vector space $V$. This decomposes the point $\hat{x}$ into a sum of weight vectors. The Hilbert-Mumford numerical function's value on $\hat{x}$ is the negative of the lowest weight of that point with respect to the diaganolization. The Hilbert-Mumford criterion states that $x \in X$ is semistable or stable precisely when $x$ satisfies certain properties of the numerical function.

**Theorem 3.4.3** (Hilbert-Mumford criterion [11,16,17])**.** *Given a $G$-action on $X$ with a linearization $V$, a point $x \in X$ is*

1. *semistable if and only if $\mu(x, \lambda) \geq 0$ for all 1-PS $\lambda$ in $G$;*

2. *stable if and only if $\mu(x, \lambda) > 0$ for all 1-PS $\lambda$ in $G$.*

The numerical function must be nonnegative or positive for all 1-PS $\lambda$. One way of thinking of a semistable or stable point is that it has some positive and some negative weights since the criterion must be satisfied by all $\lambda$ and its inverse $\lambda^{-1}$. One difficulty of Mumford's criterion is that the numerical function must be checked for all 1-PS $\lambda$ of $G$. There is one property of the numerical function that allows one to restrict the number of 1-PS $\lambda$ that need to be checked.

**Lemma 3.4.4** ( [11])**.** *For a point $x \in X$ and a 1-PS $\lambda$*

$$\mu(x, \lambda) = \mu(gx, g\lambda g^{-1}) \tag{3.12}$$

*for all $g \in G$.*

This property is useful if one is concerned with classification of $G$-orbits rather than just one particular point. One interpretation is that all 1-PS $\lambda$ can be conjugated to $g\lambda g^{-1}$ while still saying in the same $G$-orbit. Despite the ability to restrict the class of 1-PS $\lambda$ it is still quite cumbersome to calculate the stability or semistability for a point $x \in X$. By taking the negation of the Hilbert-Mumford criterion, the criteria for determining whether a point is unstable or strictly semistable points results from the existence of a 1-PS $\lambda$ which causes the Hilbert-Mumford criterion to fail.

**Proposition 3.4.5.** *Given a $G$-action on a variety $X$ with linearization $V$, a point $x \in X$ is*

1. *unstable if and only if there exists a 1-PS $\lambda$ such that $\mu(x, \lambda) < 0$;*
2. *strictly semistable if and only it is not unstable and there exists a 1-PS $\lambda$ such that $\mu(x, \lambda) = 0$.*

In the case of quintic threefolds the existence of such a 1-PS $\lambda$ can be determined using linear programming which will be explained in the next chapter.

## 3.5 Minimal Orbits

Proposition (3.4.5) shows that a strictly semistable point $x \in X$ has a 1-PS $\lambda$ where $\mu(x, \lambda) = 0$. This means that given the weight decomposition (3.10) the point $\hat{x}$ will have all nonnegative weights.

**Definition 3.5.1** (Destabilizing 1-PS)**.** Given a $G$-action on $X$ with linearization $V$, a strictly semistable point $x \in X$ has a *destabilizing 1-PS* $\lambda$ if $\mu(x, \lambda) = 0$.

Since the weight decomposition of $x$, induced by a destabilizing 1-PS $\lambda$, has all nonnegative weights then one can take the closure of the $\lambda$ orbit of $x$.

**Definition 3.5.2** (Degeneration of Strictly Semistable Point)**.** Given a strictly semistable point $x$ with destabilizing 1-PS $\lambda$ the *degeneration of* $x$ is

$$\overline{\lambda x} := \lim_{t \to 0} \lambda(t) x \tag{3.13}$$

From previous sections it was discussed that closure of orbits map to the same point in a good quotient. If it is known that a point $x \in X$ is strictly semistable and $\lambda$ is its destabilizing 1-PS then the point $x$ and $\overline{\lambda x}$ will map to the same point in the geometric quotient. For the GIT compactification, the minimal orbits give unique representative of points in the boundary $X//G \setminus X^s/G$. Minimal orbits can be systematically found using degenerations. In the weight space decomposition of a strictly semistable point x the degeneration $\overline{\lambda x}$ causes all weight vectors which have non-zero weights to vanish. When determining minimal orbits this significantly reduces the complexity of the problem since only the weight vectors which have weight 0 need to be analyzed.

**Proposition 3.5.3.** *Given a strictly semistable point $x \in X$ with destabilizing 1-PS $\lambda$, the degeneration $\overline{\lambda x}$ is invariant under $\lambda$.*

*Proof.* Let $x$ be a strictly semistable point with destabilizing 1-PS $\lambda$. Using the decomposition (3.10) the action of $\lambda$ on $\hat{x}$ is of the form

$$\lambda(t)\hat{x} = \sum_{i \in \mathbb{Z}} t^i w_i. \tag{3.14}$$

Since $\mu(x, \lambda) = 0$ then all $i \geq 0$. Then in $\overline{\lambda \hat{x}}$ the $t^i$ go to zero unless $i = 0$. The only weight vectors of $\overline{\lambda \hat{x}}$ which are still left correspond to vectors with weight 0. Therefore, they are invariant under $\lambda$.

$\square$

In many cases the point $\overline{\lambda \hat{x}}$ may represent a closed $G$-orbit and hence a minimal orbit. It is possible that even $\overline{\lambda \hat{x}}$ does not represent a closed $G$-orbit so the problem would be to find a 1-PS $\lambda$ which destabilizes $\overline{\lambda \hat{x}}$. In Chapter 5 this will be done systematically to find the set of all minimal orbits in the case of quintic threefolds. An important theorem which will be used to find these minimal orbits will be Vinberg's theorem which will also be discussed in Chapter 5.

## 3.6 Quotient Construction for Quintic Threefolds

As stated in Chapter 2, the parameter space for quintic threefolds is the space $\mathbb{P}(Sym^5(\mathbb{C}^5))$. The natural action of $SL(5,\mathbb{C})$ on the coordinates of $\mathbb{P}^4$ induces a representation on the linearzation $V = Sym^5(\mathbb{C}^5)$. The GIT compactification of interest in this dissertation is the quotient $\mathbb{P}(Sym^5(\mathbb{C}^5))//SL(5,\mathbb{C})$. To give an explicit description of the quotient $\mathbb{P}(Sym^5(\mathbb{C}^5))//SL(5,\mathbb{C})$ the problem of determining which points are semistable and stable will analyzed. The minimal orbits of this compactification will also be studied in Chapter 5.

# Chapter 4

# Semistable Locus

This section will describe the semistable locus of the GIT compactification. In particular, using the Hilbert-Mumford numerical criterion, we will explicitly descibe the strictly semistable points of the GIT compactification. The advantage of using the Hilbert-Mumford criterion is that a strictly semistable or unstable point is detected by showing the existence a 1-PS which causes the criterion to fail. By restricting to a certain family of 1-PS one can use the Hilbert-Mumford criterion to find the maximal strictly semistable families of hypersurfaces. One can then study the singularities of these families using the principle of bad flags. These results can, in turn, be used to give a partial description of hypersurfaces in the stable locus.

## 4.1 Hilbert-Mumford Criterion for Hypersurfaces

Given the linearization $V = Sym^5(\mathbb{C}^5)$, a point $v \in V$ represents a quintic hypersurface. A modification of the Hilbert-Mumford criterion explicitely determines the set of semistable points.

**Proposition 4.1.1.** *Given a $G$-action on a variety $X$ with linearization $V$, a point $v \in V$ is*

1. *unstable if and only if there exists a 1-PS $\lambda$ such that $\mu(v, \lambda) < 0$,*

2. *and strictly semistable if and only it is not unstable and there exists a 1-PS* $\lambda$ *such that* $\mu(v, \lambda) = 0$*.*

This modification of the Hilbert-Mumford criterion shows that determining whether $v$ is semistable or unstable amounts to finding a 1-PS which satisfies the right conditions on the numerical function.

**Definition 4.1.2.** A destabilizing 1-PS $\lambda : \mathbb{C}^* \to SL(5, \mathbb{C})$ of a strictly semistable (unstable) point $v$, is a 1-PS such that $\mu(v, \lambda) = 0$ ($\mu(v, \lambda) < 0$).

The $G$-equivariant property of the numerical function $\mu$, namely,

$$\mu(gv, g\lambda g^{-1}) = \mu(v, \lambda), \tag{4.1}$$

can be used to restrict the class of 1-PS.

If the standard torus $T \subseteq SL(5, \mathbb{C})$ is fixed then a 1-PS $\overline{\lambda} : \mathbb{C}^* \to T$ acts on $x_0^i x_1^j x_2^k x_3^l x_4^m$ by the following action:

$$\overline{\lambda}(t) x_0^i x_1^j x_2^k x_3^l x_4^m = t^{ai+bj+ck+dl+em} x_0^i x_1^j x_2^k x_3^l x_4^m. \tag{4.2}$$

Every such $\overline{\lambda}$ can be represented by a vector $\langle a, b, c, d, e \rangle$, which are the weights of the $\mathbb{C}^*$-action on each degree 1 monomial. One simplyfying assumption that can be made is that every 1-PS $\lambda$ can be conjugated to a standard 1-PS.

**Proposition 4.1.3.** *Given a 1-PS* $\lambda : \mathbb{C}^* \to SL(5, \mathbb{C})$ *it can be conjugated* ($g\lambda g^{-1}$) *to a 1-PS* $\overline{\lambda} : \mathbb{C}^* \to T$ *represented by the vector* $\langle a, b, c, d, e \rangle$ *where* $a \geq b \geq c \geq d \geq e$ *and* $a + b + c + d + e = 0$*. This class of 1-PS will be called a normalized 1-PS.*

Since we are interested in finding strictly semistable hypersurfaces, up to a $SL(5, \mathbb{C})$ coordinate transformation, the $G$-equivariance of the Hilbert-Mumford numerical function allows us to restrict to a normalized 1-PS given by the proposition 4.1.3. By restricting to normalized 1-PS there is an ordering that can be put on the monomials which determine a poset structure on the set of quintic monomials. We will represent a monomial $x_0^i x_1^j x_2^k x_3^l x_4^l$ by the vector $[i, j, k, l, m]$.

**Definition 4.1.4.** For a monomial $[i, j, k, l, m]$ if there is a normalized 1-PS $\lambda = \langle a, b, c, d, e \rangle$ which acts on the monomial via the diagonal action shown above then, by the definition of the numerical function,

$$\mu([i, j, k, l, m], \langle a, b, c, d, e \rangle) = ia + jb + kc + ld + me \tag{4.3}$$

i.e. the dot product of the vectors.

**Definition 4.1.5.** The set of monomials $[i, j, k, l, m]$ has a *partial order* given by $[i_0, i_1, i_2, i_3, i_4] \geq [j_0, j_1, j_2, j_3, j_4]$ if

$$\mu([i_0, i_1, i_2, i_3, i_4], \lambda) \geq \mu([j_0, j_1, j_2, j_3, j_4], \lambda) \tag{4.4}$$

for all normalized 1-PS $\lambda$ of Prop. 4.1.3.

The following lemma gives a numerical method for determining the poset structure on monomials.

**Lemma 4.1.6** (c.f. [15] p.225)**.** *For two monomials* $[i_0, i_1, i_2, i_3, i_4]$ *and* $[j_0, j_1, j_2, j_3, j_4]$

$$[i_0, i_1, i_2, i_3, i_4] \geq [j_0, j_1, j_2, j_3, j_4] \iff \begin{cases} i_0 \geq j_0 \\ i_0 + i_1 \geq j_0 + j_1 \\ i_0 + i_1 + i_2 \geq j_0 + j_1 + j_2 \\ i_0 + i_1 + i_2 + i_3 \geq j_0 + j_1 + j_2 + j_3 \end{cases}$$

This criterion is useful because one can directly check whether two monomials are related in the poset. Using Maple, the above criterion can be used to find all partial order relationships between monomials [14]. Stembridge's poset package for Maple [21] is used to find the minimal covering relationships for these monomials and therefore create the poset for quintic monomials. The script for this entire procedure is given in Appendix A.

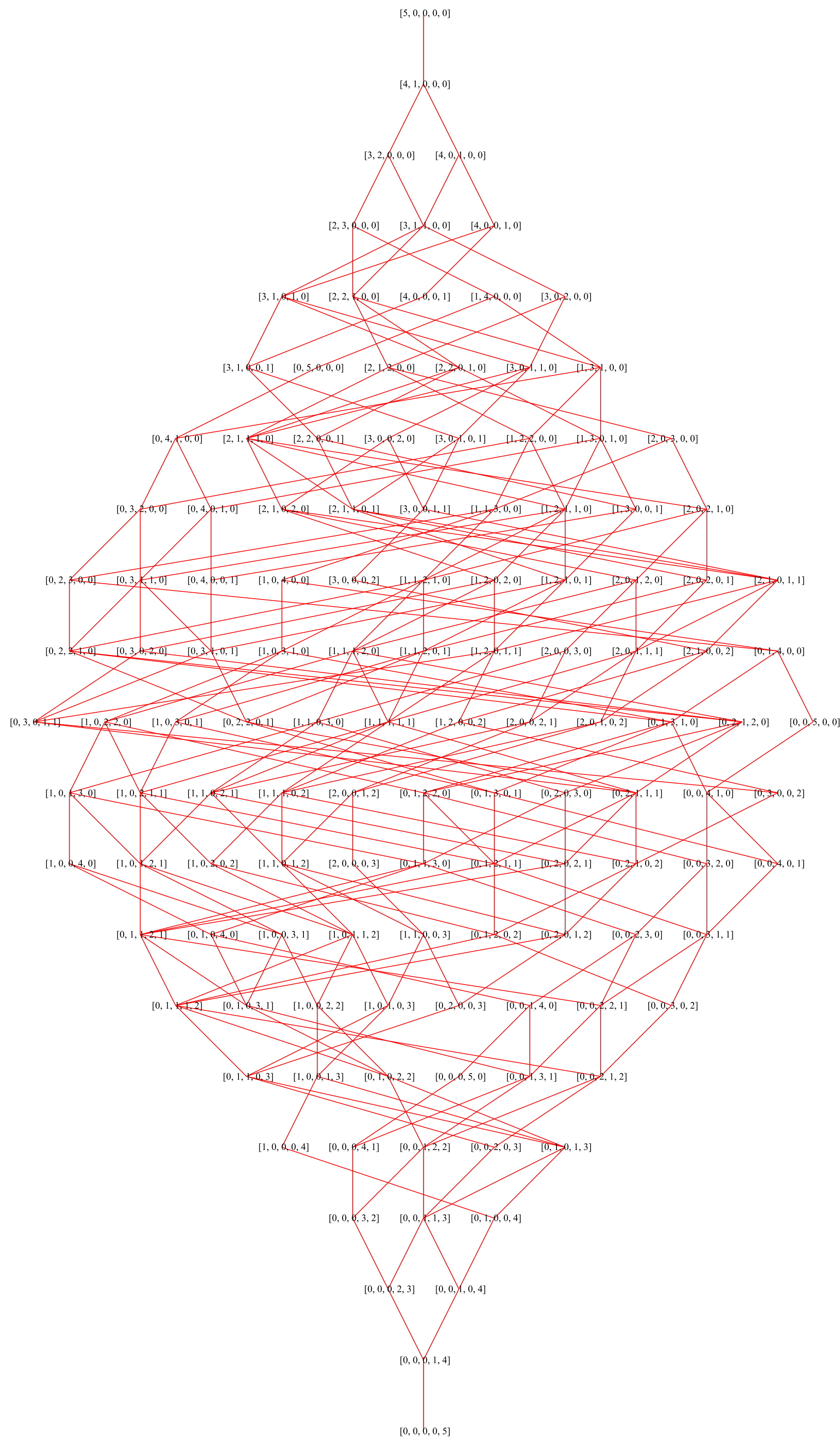


Figure 4.1: Poset structure of quintic monomials

## 4.2 Combinatorics of Maximal Semistable Families

In this section we will show how the poset structure can be used to determine the set of maximal semistable families. The restriction to normalized 1-PS from Prop 4.1.3 allows us to use linear programming to find these maximal semistable families. We can precisely define these maximal stable families using terminology from posets.

**Definition 4.2.1** (c.f. [20] section 3.1)**.** Given a poset $P$ with a partial ordering $\geq$, a subset $I \subseteq P$ is called an *ideal* if $\forall x, y \in P$ where $x \in I$ and $x \geq y$, then $y \in I$.

**Definition 4.2.2.** For a monomial $[i, j, k, l, m]$ define the set

$$I([i, j, k, l, m]) := \{[a, b, c, d, e] \mid [i, j, k, l, m] \geq [a, b, c, d, e]\}\,. \tag{4.5}$$

From the definition $I([i, j, k, l, m])$ is an ideal of the poset.

The set $I([i, j, k, l, m])$ represent all vectors below $[i, j, k, l, m]$ in the poset. The set of polynomials associated to $I([i, j, k, l, m])$ is denoted $P(I([i, j, k, l, m]))$.

**Definition 4.2.3.** The set $P(I([i, j, k, l, m]))$ represents all polynomials of the form

$$\sum_{a+b+c+d+e=5} r_{abcde} x_0^a x_1^b x_2^c x_3^d x_4^e \tag{4.6}$$

where $[a, b, c, d, e] \in I([i, j, k, l, m])$.

**Lemma 4.2.4.** *For a polynomial $F \in P(I([i, j, k, l, m]))$ and normalized 1-PS $\lambda$*

$$\mu(F, \lambda) = \mu([i, j, k, l, m], \lambda). \tag{4.7}$$

*Proof.* Since $\lambda$ acts diagonally on the set of monomials in $P(I([i, j, k, l, m])$ then, by the definition of $\mu$, $\mu(F, \lambda)$ is the highest weight of all the monomials in this set. By definition of the poset structure the highest weight occurs at the monomial $[i, j, k, l, m]$, therefore (4.7) holds. $\square$

**Definition 4.2.5.** Given a normalized 1-PS $\lambda$ let

$$M(\lambda) := \{[i, j, k, l, m] \mid \mu([i, j, k, l, m], \lambda) \leq 0\}\,. \tag{4.8}$$

A **maximal semistable family** is precisely a set $M(\lambda)$ where there exists no $\overline{\lambda}$ where $M(\lambda) \subseteq M(\overline{\lambda})$.

Due to the poset structure, there is a set of maximal monomials which define each set $M(\lambda)$. This property is useful because the maximal monomials can be used to describe the general form of the equation for some family $M(\lambda)$.

**Lemma 4.2.6.** *For a fixed 1-PS $\lambda$, the family $M(\lambda)$ has a set of monomials $[i_0^1, i_1^1, i_2^1, i_3^1, i_4^1]$, $[i_0^2, i_1^2, i_2^2, i_3^2, i_4^2]$, ... $[i_0^r, i_1^r, i_2^r, i_3^r, i_4^r]$ where*

$$M(\lambda) = I([i_0^1, i_1^1, i_2^1, i_3^1, i_4^1]) \cup I([i_0^2, i_1^2, i_2^2, i_3^2, i_4^2]) \ldots \cup I([i_0^r, i_1^r, i_2^r, i_3^r, i_4^r]). \tag{4.9}$$

*The maximal monomials are the set of monomials $[i, j, k, l, m] \in M(\lambda)$ where there are no other monomials $[a, b, c, d, e]$ such that $I([i, j, k, l, m]) \subseteq I([a, b, c, d, e]) \subseteq M(\lambda)$.*

*Proof.* Given a monomial $[i, j, k, l, m] \in M(\lambda)$, by the definition of the ordering relation, every element $[a, b, c, d, e] \in I([i, j, k, l, m])$ has $\mu([i, j, k, l, m], \lambda) \geq \mu([a, b, c, d, e], \lambda)$. Therefore $I([i, j, k, l, m]) \subseteq M(\lambda)$. Since there are a finite number of monomials $[i, j, k, l, m]$ in $M(\lambda)$ there are a finite set of $I([i, j, k, l, m]) \subseteq M(\lambda)$. So $M(\lambda)$ is of the form (4.9). $\square$

Since the value of $\mu$ is the largest at the top of the poset we can use this fact to determine the maximal semistable families. Starting with the topmost monomial of the poset ($[5, 0, 0, 0, 0]$) and working down the poset one can use linear programming to determine whether there is a $\lambda$ where a specific monomial $[i, j, k, l, m]$ satisfies $\mu([i, j, k, l, m], \lambda) \leq 0$. The implementation of the linear programming program is given in Appendix B.

Using this procedure it is determined that the monomials $[3, 0, 0, 2, 0]$, $[4, 0, 0, 0, 1]$, $[2, 0, 3, 0, 0]$, and $[1, 4, 0, 0, 0]$ are the maximal monomials which have $\mu \leq 0$.

Table 4.1: Strictly Semistable Families SS1 - SS4

| *Family* | *Destabilizing 1-PS Subgroup* | *Maximal Monomial* |
|---|---|---|
| SS1 | $\langle 2, 2, 2, -3, -3 \rangle$ | $[3, 0, 0, 2, 0]$ |
| SS2 | $\langle 1, 1, 1, 1, -4 \rangle$ | $[4, 0, 0, 0, 1]$ |
| SS3 | $\langle 3, 3, -2, -2, -2 \rangle$ | $[2, 0, 3, 0, 0]$ |
| SS4 | $\langle 4, -1, -1, -1, -1 \rangle$ | $[1, 4, 0, 0, 0]$ |

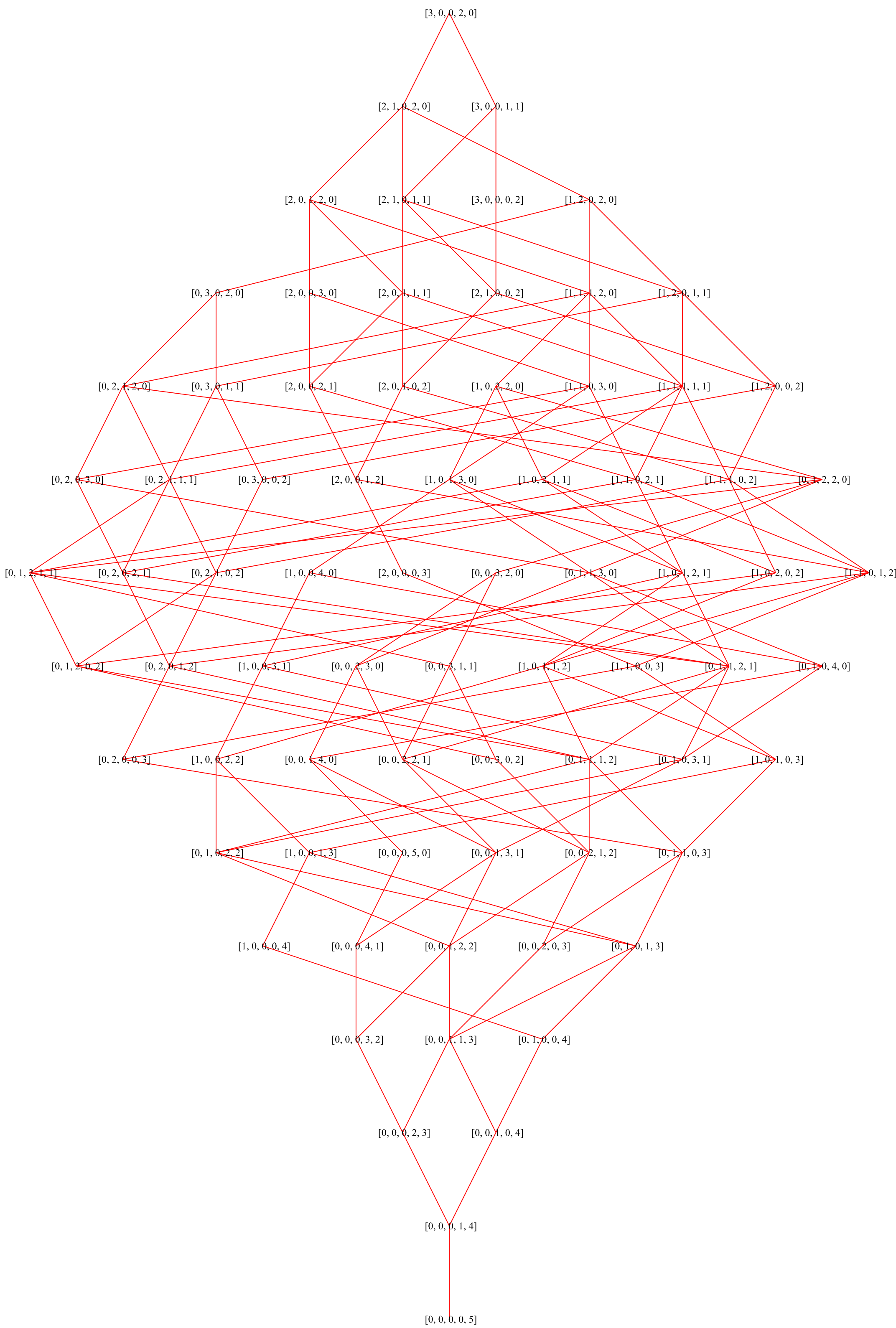


Figure 4.2: Poset structure of family SS1

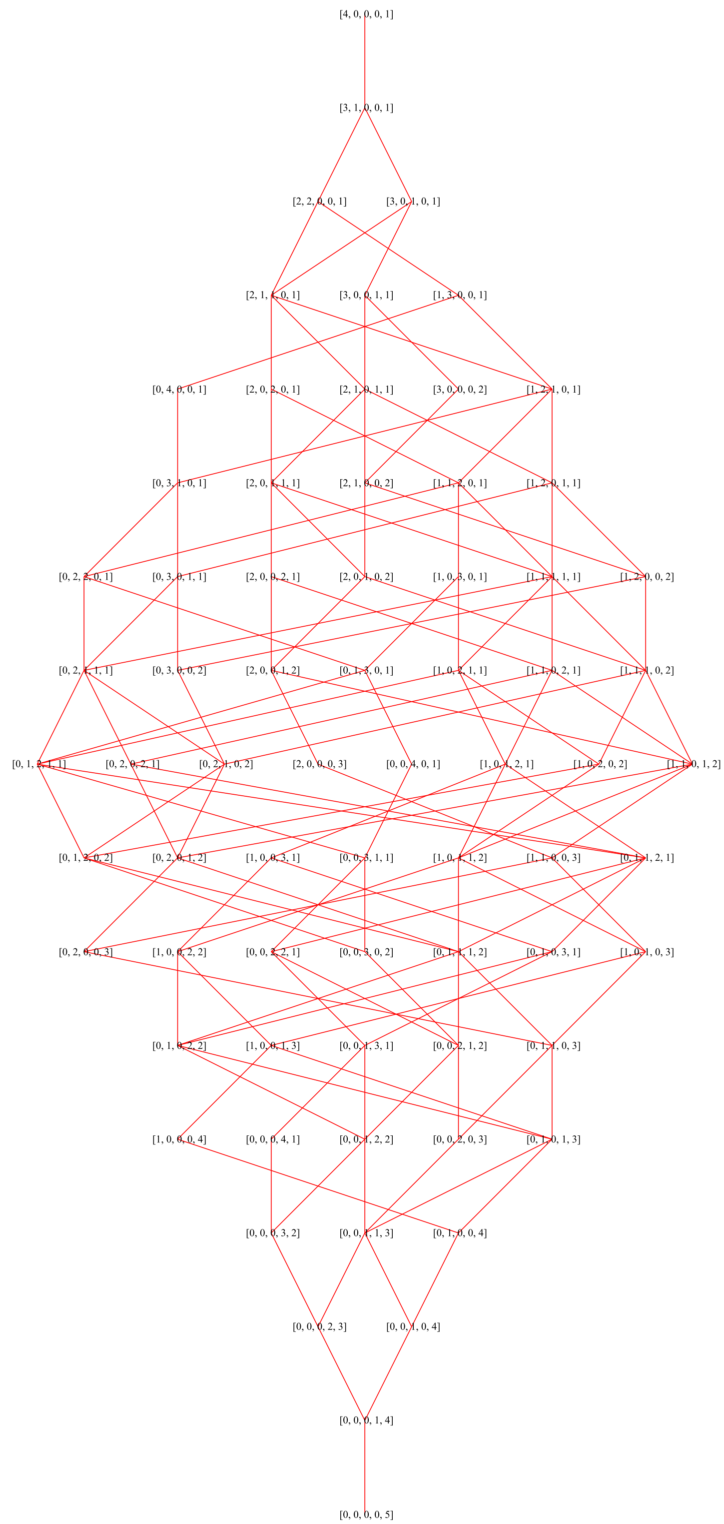


Figure 4.3: Poset structure of family SS2

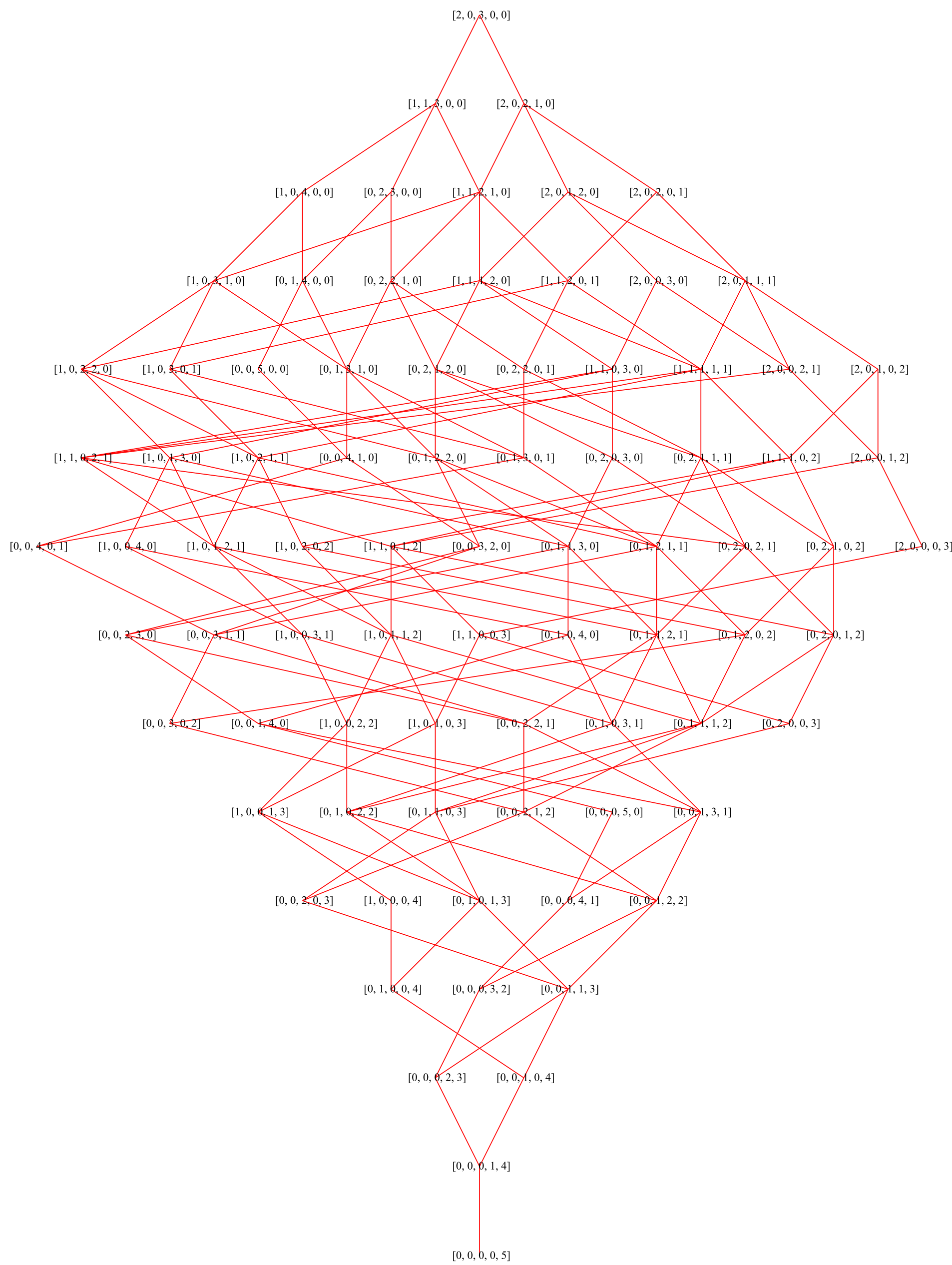


Figure 4.4: Poset structure of family SS3

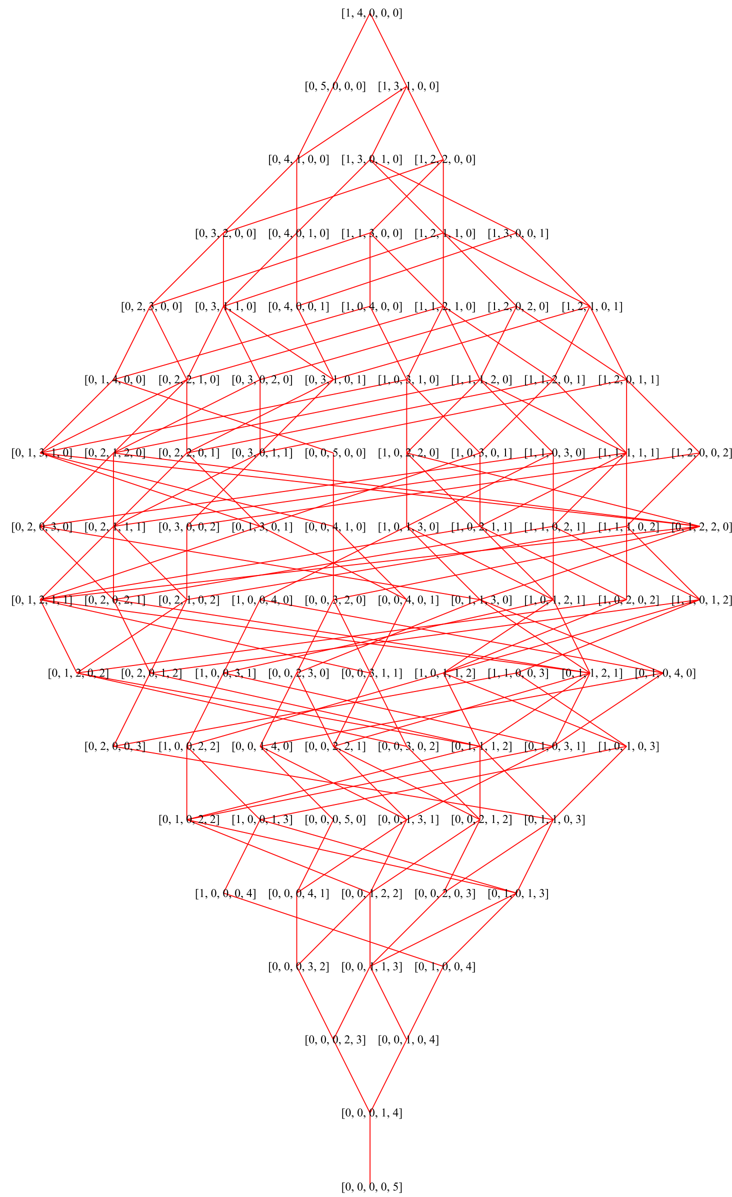

Figure 4.5: Poset structure of family SS4

There are other maximal families which can be found by finding monomials in the above families which have a common destabilizing 1-PS $\lambda$. Starting with the monomials at the top of families $SS1 - SS4$ and working down it can be found that the only other maximal semistable families are the following:

Table 4.2: Strictly Semistable Families SS5 - SS7

| *Family* | *Destabilizing 1-PS Subgroup* | *Maximal Monomial* |
|---|---|---|
| SS5 | $\langle 1, 0, 0, 0, -1 \rangle$ | $[0, 5, 0, 0, 0], [1, 3, 0, 0, 1], [2, 1, 0, 0, 2]$ |
| SS6 | $\langle 4, 4, -1, -1, -6 \rangle$ | $[1, 0, 4, 0, 0], [3, 0, 0, 0, 2], [2, 0, 2, 0, 1]$ |
| SS7 | $\langle 6, 1, 1, -4, -4 \rangle$ | $[0, 4, 0, 1, 0][1, 2, 0, 2, 0][2, 0, 0, 3, 0].$ |

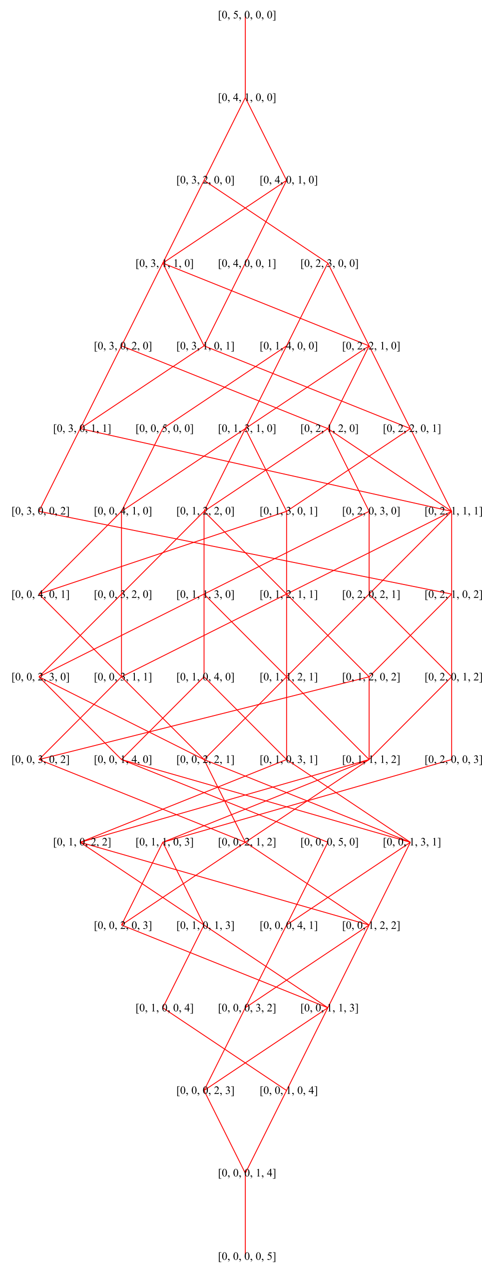


Figure 4.6: Poset structure of family SS5 I

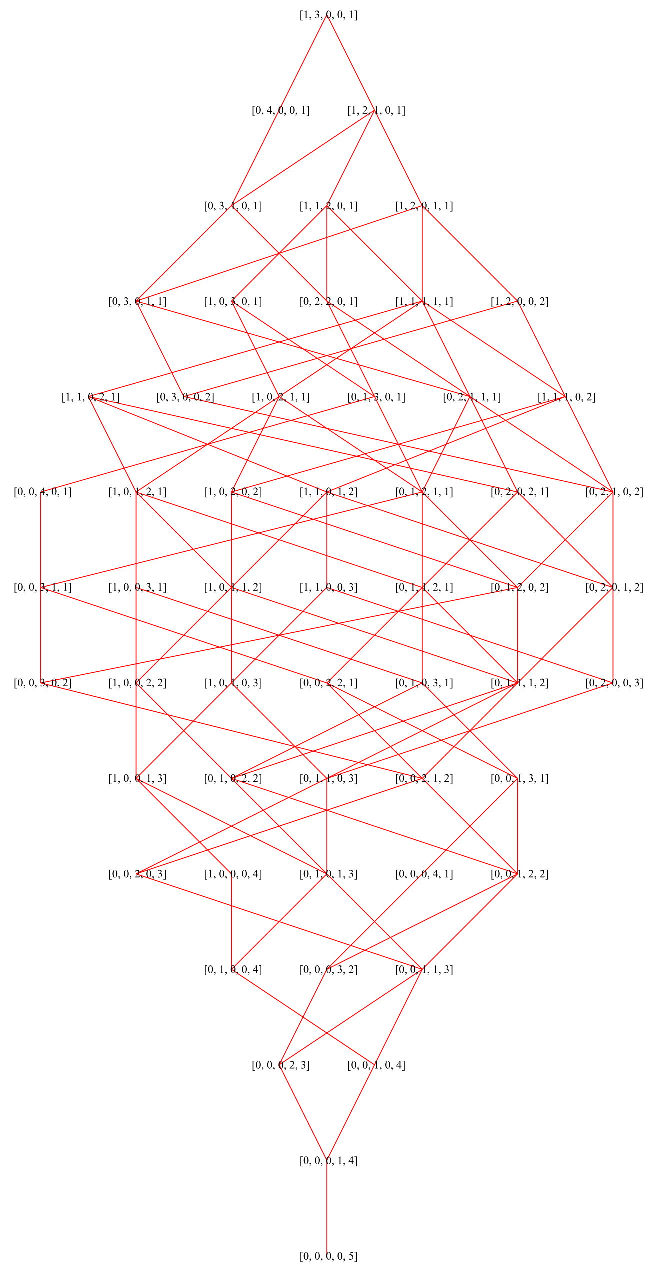


Figure 4.7: Poset structure of family SS5 II

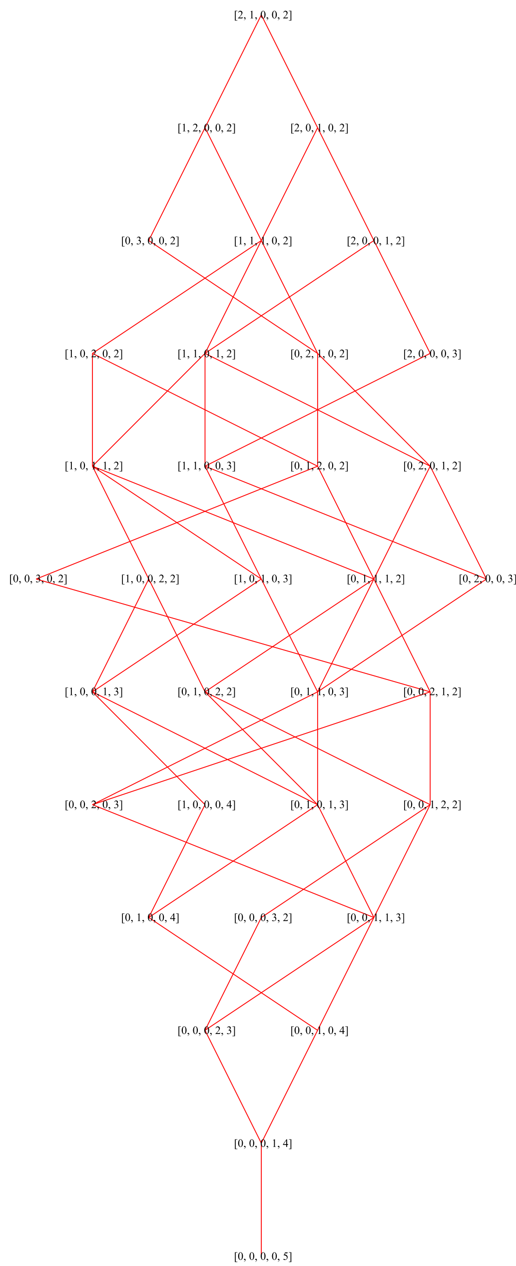


Figure 4.8: Poset structure of family SS5 III

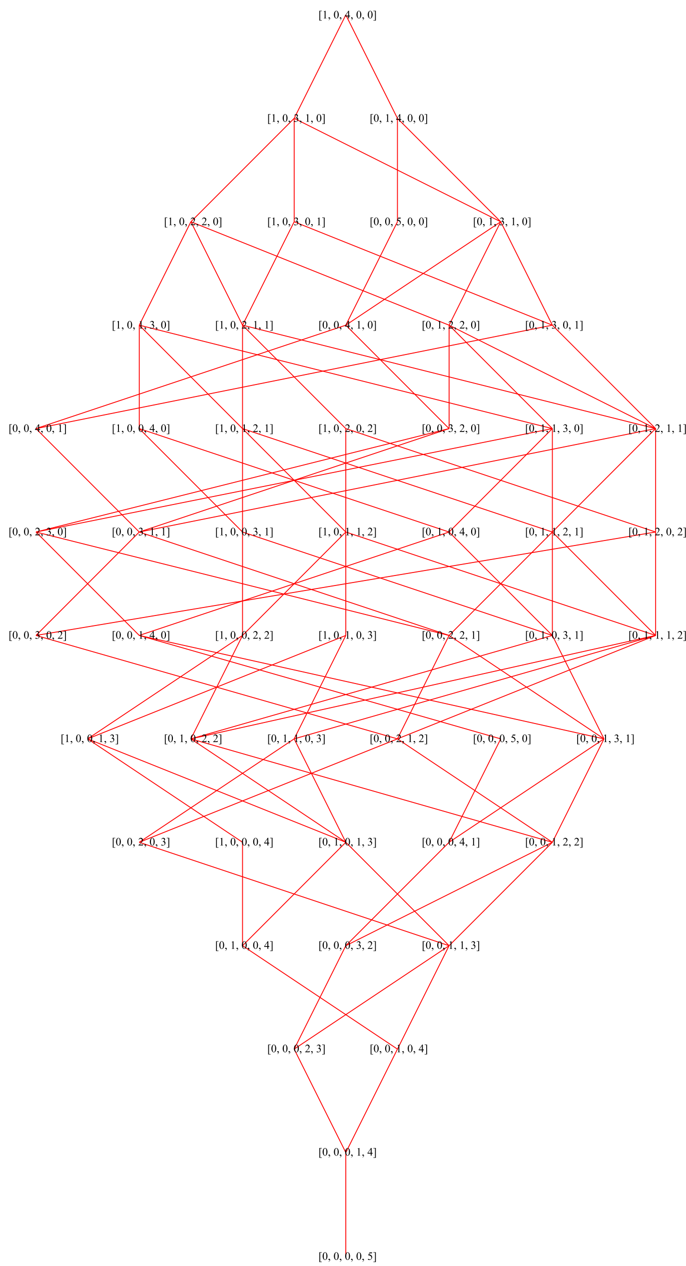


Figure 4.9: Poset structure of family SS6 I

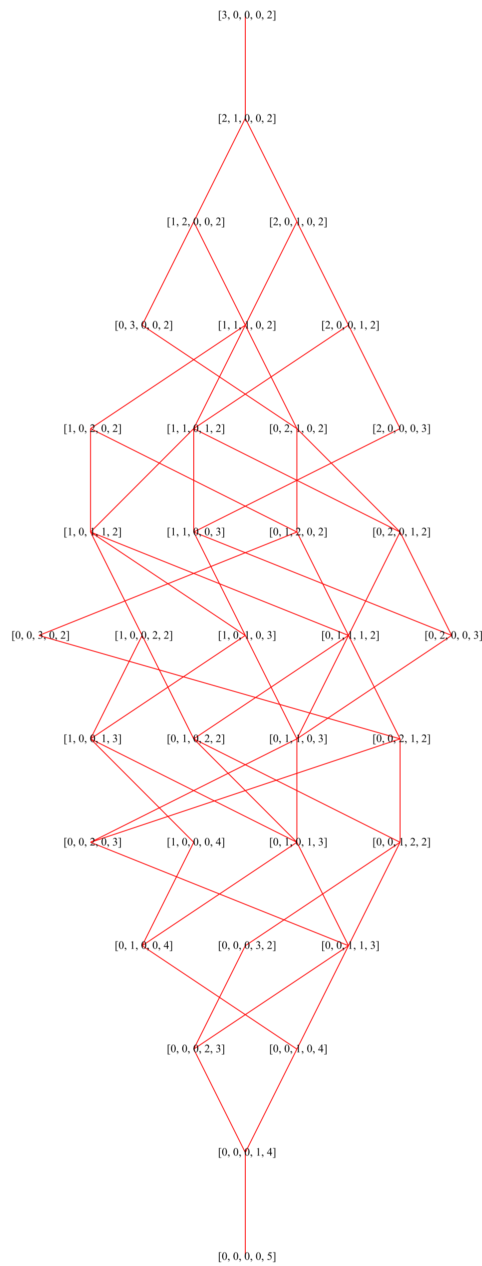


Figure 4.10: Poset structure of family SS6 II

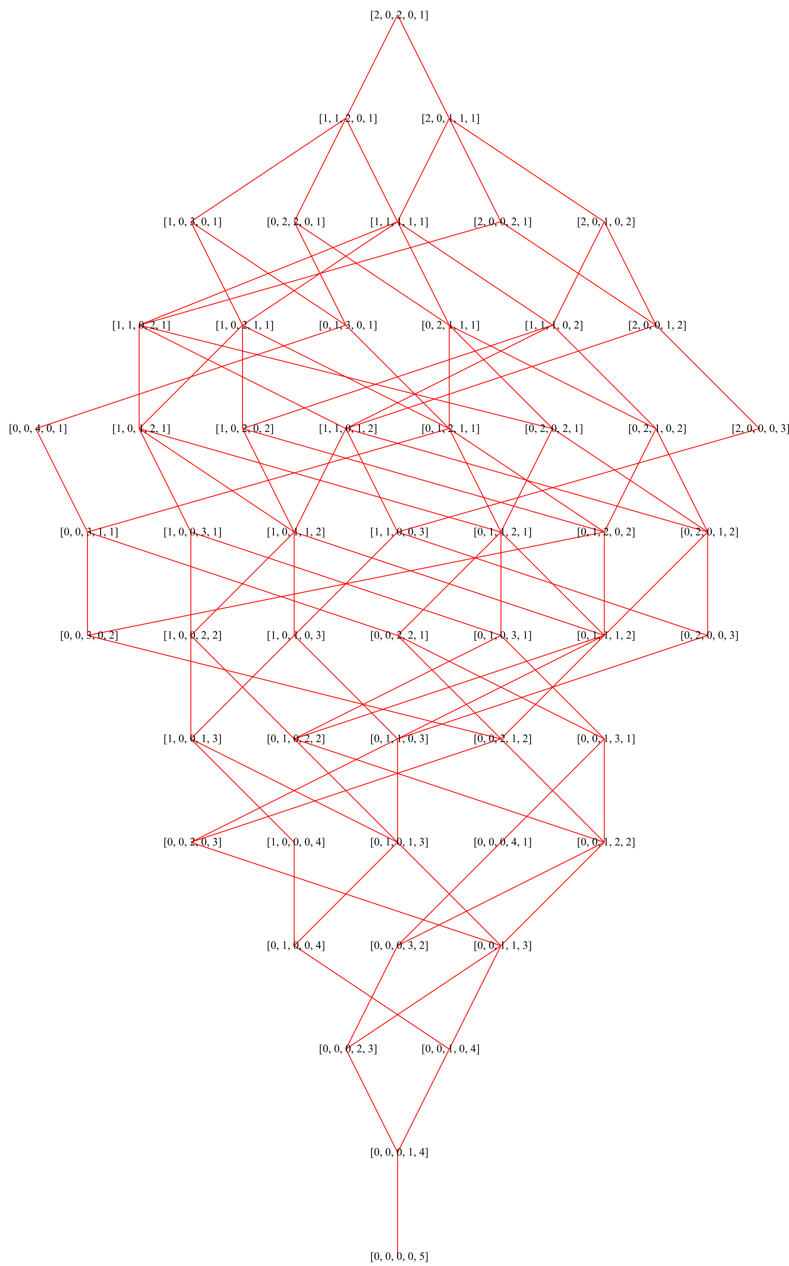


Figure 4.11: Poset structure of family SS6 III

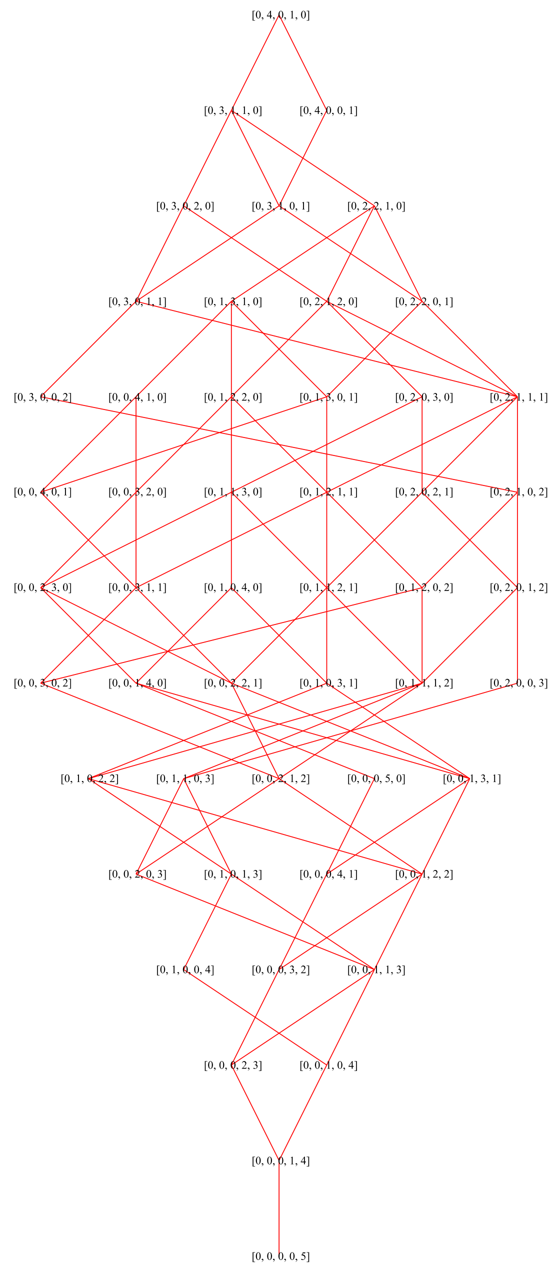


Figure 4.12: Poset structure of family SS7 I

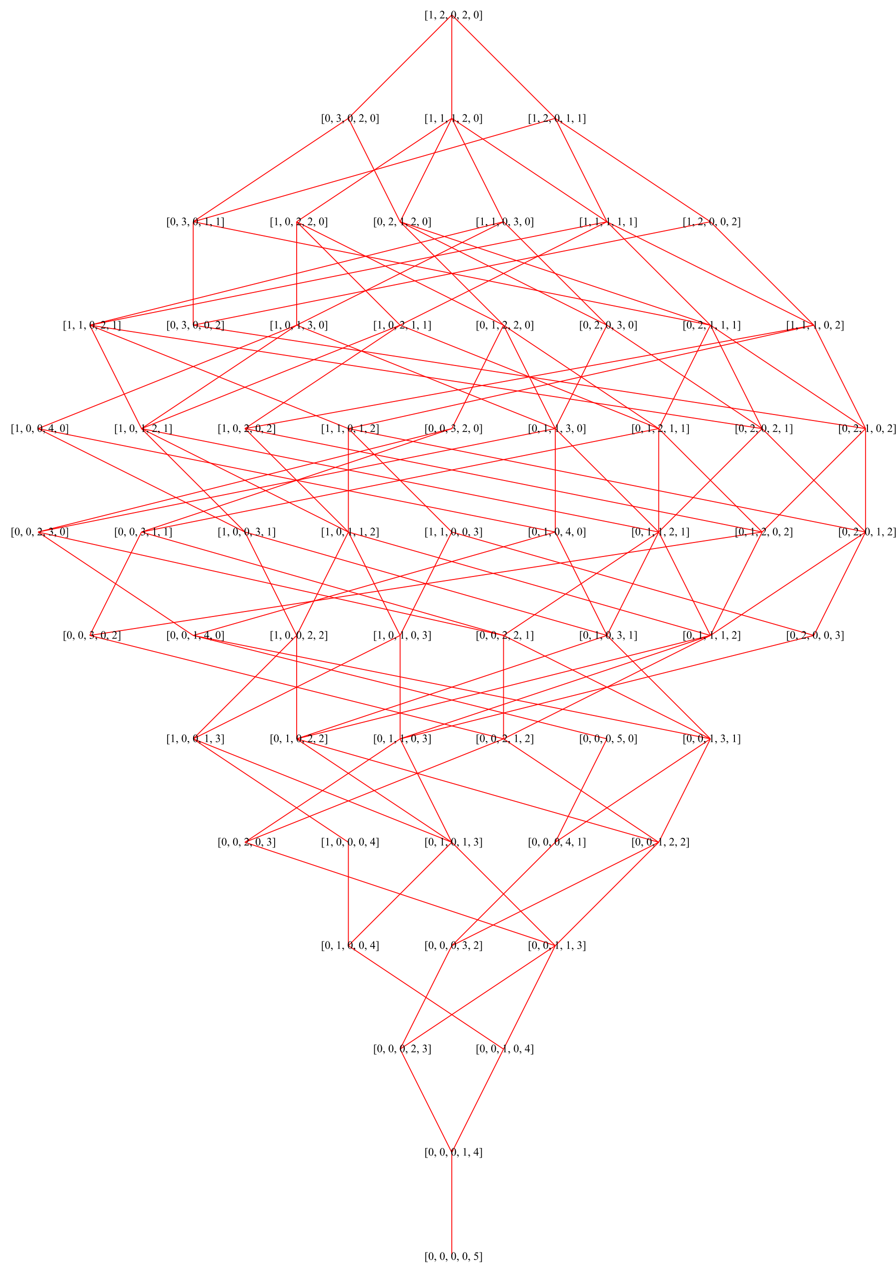


Figure 4.13: Poset structure of family SS7 II

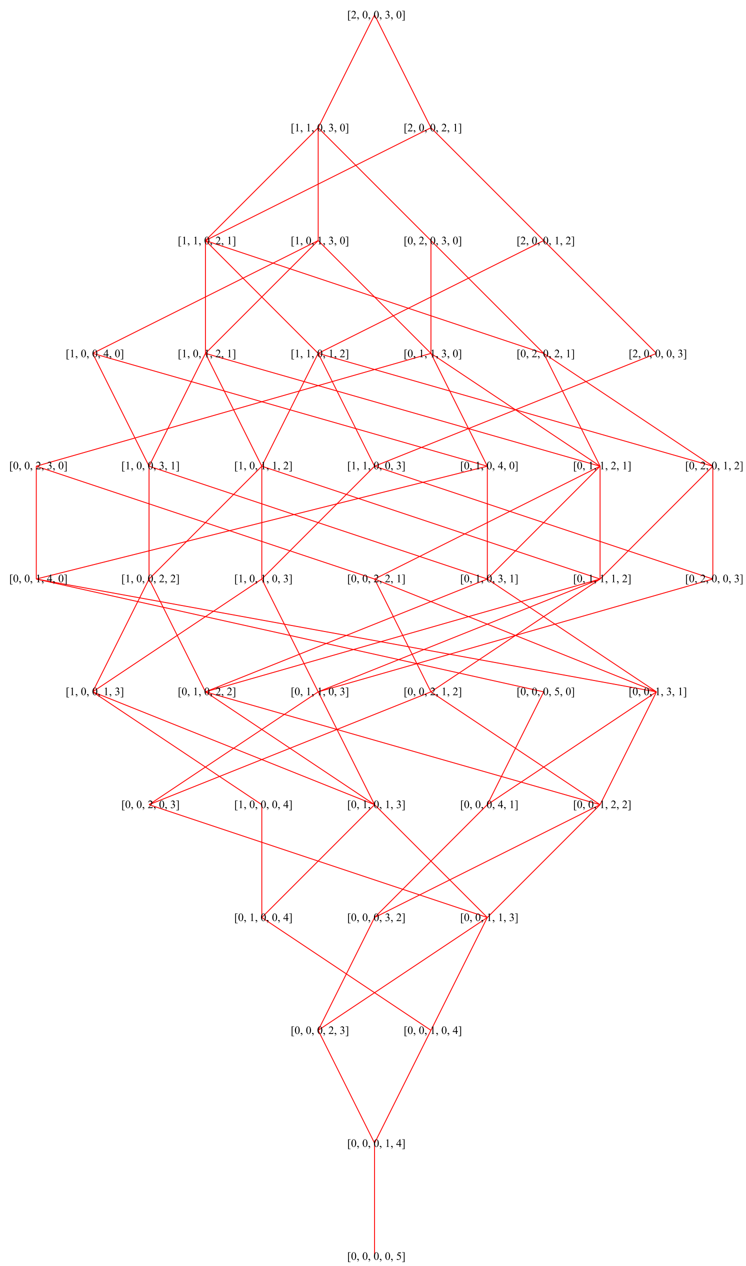


Figure 4.14: Poset structure of family SS7 III

From the combinatorial procedure above, all of the maximal semistable families are determined. The maximal semistable families classify all hypersurfaces, up to coordinate transformation, which are strictly semistable and unstable. The maximal monomials in each family can be used to give the general polynomial form for each family.

*Remark* 4.2.7. The polynomials will be written in the following forms

| | |
|---|---|
| $q_a(x_p \dots x_r)$ | polynomial of degree $a$ in the variables $x_p \dots x_r$ |
| $q_{a,b}(x_p \dots x_r \parallel x_t \dots x_s)$ | linear combination of monomials of degree $a$ in $x_p \dots x_r$ and degree $b$ in $x_t \dots x_s$ |

**Proposition 4.2.8.** *$Y$ is strictly semistable or unstable if it is equivalent, via coordinate transformation, to a hypersurfaces in one of the following families:*

Table 4.3: Strictly Semistable Families SS1 - SS7

| *Family* | *Destabilizing 1-PS Subgroup* | *Maximal Monomial* |
|---|---|---|
| SS1 | $\langle 2,2,2,-3,-3\rangle$ | $[3,0,0,2,0]$ |
| $q_{3,2}(x_0,x_1,x_2 \parallel x_3,x_4) + q_{2.3}(x_0,x_1,x_2 \parallel x_3,x_4) + q_{1,4}(x_0,x_1,x_2 \parallel x_3,x_4) + q_5(x_3,x_4)$ | | |
| SS2 | $\langle 1,1,1,1,-4\rangle$ | $[4,0,0,0,1]$ |
| $x_4 q_4(x_0,x_1,x_2,x_3,x_4)$ | | |
| SS3 | $\langle 3,3,-2,-2,-2\rangle$ | $[2,0,3,0,0]$ |
| $q_{2,3}(x_0,x_1 \parallel x_2,x_3,x_4) + q_{1,4}(x_0,x_1 \parallel x_2,x_3,x_4) + q_5(x_2,x_3,x_4)$ | | |
| SS4 | $\langle 4,-1,-1,-1,-1\rangle$ | $[1,4,0,0,0]$ |
| $x_0 q_4(x_1,x_2,x_3,x_4) + q_5(x_1,x_2,x_3,x_4)$ | | |
| SS5 | $\langle 1,0,0,0,-1\rangle$ | $[0,5,0,0,0], [1,3,0,0,1], [2,1,0,0,2]$ |
| $x_0^2(x_4^2 q_1(x_1,x_2,x_3,x_4)) + x_0 x_4 q_3(x_1,x_2,x_3,x_4) + q_5(x_1,x_2,x_3,x_4)$ | | |
| SS6 | $\langle 4,4,-1,-1,-6\rangle$ | $[1,0,4,0,0], [3,0,0,0,2], [2,0,2,0,1]$ |
| $x_4^2 q_3(x_0,x_1) + x_4 q_{2,2}(x_0,x_1 \parallel x_2,x_3,x_4) + q_{1,4}(x_0,x_1 \parallel x_2,x_3,x_4) + q_5(x_2,x_3,x_4)$ | | |
| SS7 | $\langle 6,1,1,-4,-4\rangle$ | $[0,4,0,1,0][1,2,0,2,0][2,0,0,3,0]$ |
| $x_0^2 q_3(x_3,x_4) + x_0(q_{2,2}(x_1,x_2 \parallel x_3,x_4) + q_{1,3}(x_1,x_2 \parallel x_3,x_4) + q_4(x_3,x_4)) + q_{4,1}(x_1,x_2 \parallel x_3,x_4) + q_{3,2}(x_1,x_2 \parallel x_3,x_4) + q_{2,3}(x_1,x_2 \parallel x_3,x_4) + q_{1,4}(x_1,x_2 \parallel x_3,x_4) + q_5(x_3,x_4)$ . | | |

## 4.3 Unstable Families

The previous section gives a complete characterization of all strictly semistable and unstable quintic threefolds. In the GIT quotient the unstable threefolds are removed to produce the GIT compactification. In order to understand the GIT compactfication one must distinguish between unstable and strictly semistable threefolds. A direct approach would be to determine all maximal unstable families i.e. all maximal families where $\mu < 0$ as opposed to $\mu \leq 0$. This approach produces a large number of unstable families which makes it very tedious and not very illustrative. Another approach would be to find and characterize the minimal orbits as is done in Chapter 5. The idea of the minimal orbit approach is that each maximal semistable family will degenerate, via the destabilizing 1-PS $\lambda$, to a smaller family of hypersurfaces. It is then much easier to distinguish between unstable and strictly semistable hypersurfaces in the smaller family. See Chapter 5 for a complete characterization of strictly semistable and unstable orbits.

## 4.4 Bad Flags

Having found the maximal semistable families, one can give a geometric description by characterizing the types of singularities found on a generic member of one of these families. The existence of a destabilizing 1-PS $\lambda$ gives rise to a "bad flag" of the vector spaces $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \cong \mathbb{C}^5$. A general principal given by Mumford ( [16] p.48) is that these "bad flags" pick out the singularities which cause the family to become semistable or unstable.

Using the approach given by Laza ( [12] p.7) it can be shown that a 1-PS $\lambda : \mathbb{C}^* \to T$ gives a weight decomposition of $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) = \oplus_{i=0}^{5} W_i$ based on the eigenvalues of $\lambda$ acting on $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$.

**Definition 4.4.1.** For a 1-PS $\lambda = \langle a, b, c, d, e \rangle$ let $m_i$ be a subset of $\{a, b, c, d, e\}$ which have the same weights and let $n_i$ be the weight.

$$W_{m_i} := \bigoplus_{i \text{ where } W_i \text{ has eigenvalue } n_i} W_i \tag{4.10}$$

The standard flag is given by

$$\begin{aligned} \emptyset \subseteq F_1 = (x_1 = x_2 = x_3 = x_4 = 0) \subseteq F_2 = (x_2 = x_3 = x_4 = 0) \subseteq \\ F_3 = (x_3 = x_4 = 0) \subseteq F_4 = (x_4 = 0) \subseteq \mathbb{P}^4 \end{aligned} \tag{4.11}$$

**Definition 4.4.2.** Given a 1-PS $\lambda = \langle a, b, c, d, e\rangle$ let $m_1$, $m_2 \ldots$, $m_s$ represent the collection of common weights of $\lambda$. Let $m_i$ be ordered by increasing value of weights (i.e. $m_1$ has lowest weight). The *associated flag for* $\lambda$ is

$$F_\lambda : \quad \emptyset \subseteq F_{m_s} := \bigoplus_{i=1}^{s} W_{m_i} \subset F_{m_{s-1}} := \bigoplus_{i=1}^{s-1} W_{m_i} \subset \ldots \subseteq F_{m_1} := W_{m_1} \subseteq \mathbb{P}^4. \quad (4.12)$$

This is a subflag of the standard flag (4.12).

For the maximal destabilizing families $SS1$-$SS7$ the associated "bad flags" are

Table 4.4: Destabilizing Flags of SS1-SS7

| *Family* | *Destabilizing 1-PS Subgroup* | *Flag* $F_\lambda$ |
|---|---|---|
| SS1 | $\langle 2,2,2,-3,-3\rangle$ | $\emptyset \subseteq (x_3 = x_4 = 0) \subseteq \mathbb{P}^4$ |
| SS2 | $\langle 1,1,1,1,-4\rangle$ | $\emptyset \subseteq (x_4 = 0) \subseteq \mathbb{P}^4$ |
| SS3 | $\langle 3,3,-2,-2,-2\rangle$ | $\emptyset \subseteq (x_2 = x_3 = x_4 = 0) \subseteq \mathbb{P}^4$ |
| SS4 | $\langle 4,-1,-1,-1,-1\rangle$ | $\emptyset \subseteq (x_1 = x_2 = x_3 = x_4 = 0) \subseteq \mathbb{P}^4$ |
| SS5 | $\langle 1,0,0,0,-1\rangle$ | $\emptyset \subseteq (x_1 = x_2 = x_3 = x_4 = 0) \subseteq (x_4 = 0) \subseteq \mathbb{P}^4$ |
| SS6 | $\langle 4,4,-1,-1,-6\rangle$ | $\emptyset \subseteq (x_2 = x_3 = x_4 = 0) \subseteq (x_4 = 0) \subseteq \mathbb{P}^4$ |
| SS7 | $\langle 6,1,1,-4,-4\rangle$ | $\emptyset \subseteq (x_1 = x_2 = x_3 = x_4 = 0) \subseteq (x_3 = x_4 = 0) \subseteq \mathbb{P}^4.$ |

## 4.5 Geometric Interpretation of Maximal Semistable Families

In order to determine the singularities of the families $SS1$-$SS7$ we can intersect the general form of the equation with the flag arising from its destabilizing 1-PS. This will give some description of the types of singularities which occur in each family. A precise description of each such family is given in the propositions below.

**Proposition 4.5.1.** *A hypersurface $Y$ is of type SS1 if and only if $Y$ contains a double plane.*

*Proof.* Let $Y$ be of type $S1$ then it is equivalent, via a coordinate transformation, to the hypersurface

$$\begin{aligned} q_{3,2}(x_0, x_1, x_2 \parallel x_3, x_4) + q_{2.3}(x_0, x_1, x_2 \parallel x_3, x_4) \\ + q_{1,4}(x_0, x_1, x_2 \parallel x_3, x_4) + q_5(x_3, x_4). \end{aligned} \tag{4.13}$$

This hypersurface contains the ideal $\langle x_3, x_4 \rangle^2$ which is a double plane in $\mathbb{P}^4$.

Let $Y$ be a hypersurface which contains a double plane. By a coordinate transformation we can assume the double plane is $\langle x_3, x_4 \rangle^2$. The most general equation which contains the ideal $\langle x_3, x_4 \rangle^2$ is (4.13).

□

**Proposition 4.5.2.** *A hypersurface $Y$ is of type SS2 if and only if $Y$ is a reducible variety, where a hyperplane is one of the components. In particular, the singularity is the intersection of the hyperplane with the other component which is generically a degree 4 surface.*

*Proof.* Let $Y$ be of type $S2$ then it is equivalent, via a coordinate transformation, to the hypersurface

$$x_4 q_4(x_0, x_1, x_2, x_3, x_4) \tag{4.14}$$

This hypersurface has the hyperplane $\langle x_4 \rangle$ as a component.

Let $Y$ be a reducible hypersurface where a hyperplane is a component. The polynomial $f \in \mathbb{C}[x_0, x_1, x_2, x_3, x_4]$ defining $Y$ can be factored into $f = gh$, where $h$ is a degree 1 polynomial. By a coordinate transformation we can map the hyperplane defining $h$ to $x_4$. Without loss of generality $f = x_4 h$. Since since $f$ is of degree 5 then by neccesity $h$ is of degree 4 therefore $f$ is of the form (4.14).

□

**Proposition 4.5.3.** *A hypersurface $Y$ is of type SS3 if and only if $Y$ contains a triple line.*

*Proof.* Let $Y$ be of type $SS3$ then it is equivalent, via a coordinate transformation, to the hypersurface

$$q_{2,3}(x_0, x_1 \parallel x_2, x_3, x_4) + q_{1,4}(x_0, x_1 \parallel x_2, x_3, x_4) + q_5(x_2, x_3, x_4) \tag{4.15}$$

This hypersurface contains the ideal $\langle x_2, x_3, x_4\rangle^3$ which is a triple line in $\mathbb{P}^4$.

Let $Y$ be a hypersurface which contains a triple line. By a coordinate transformation, we can assume the triple line is $\langle x_2, x_3, x_4\rangle^3$. The most general equation which contains the ideal $\langle x_2, x_3, x_4\rangle^3$ is (4.15).

□

**Proposition 4.5.4.** *A hypersurface $Y$ is of type $SS4$ if and only if $Y$ contains a quadruple point.*

*Proof.* Let $Y$ be of type $SS4$ then it is equivalent, via a coordinate transformation, to the hypersurface

$$x_0 q_4(x_1, x_2, x_3, x_4) + q_5(x_1, x_2, x_3, x_4) \tag{4.16}$$

This hypersurface contains the ideal $\langle x_1, x_2, x_3, x_4\rangle^4$ which is a quadruple point in $\mathbb{P}^4$.

Let $Y$ be a hypersurface which contains a quadruple point. By a coordinate transformation we can assume the quadruple point is $\langle x_1, x_2, x_3, x_4\rangle^4$. The most general equation which contains the ideal $\langle x_1, x_2, x_3, x_4\rangle^4$ is (4.16).

□

**Proposition 4.5.5.** *A hypersurface $Y$ is of type $SS5$ if and only if $Y$ has a triple point $p$ with the following properties:*

*i) the tangent cone of $p$ is the union of a double plane and another hyperplane;*

*ii) the line connecting a point in the double plane with the triple point has intersection multiplicty 5 with the hypersurface.*

*Proof.* Let $Y$ be of type $SS5$ then it is equivalent, via a coordinate transformation, to the hypersurface

$$x_0^2\Bigg(x_4^2q_1(x_1,x_2,x_3,x_4)\Bigg)+x_0x_4q_3(x_1,x_2,x_3,x_4)+q_5(x_1,x_2,x_3,x_4) \tag{4.17}$$

This hypersurface contains the triple point $\langle x_1,x_2,x_3,x_4\rangle^3$. The tangent cone is the hypersurface defined by

$$x_4^2q_1(x_1,x_2,x_3,x_4) \tag{4.18}$$

which is the union of a double hyperplane $\langle x_4\rangle^2$ and another general hyperplane $q_1(x_1,x_2,x_3,x_4)$. The points whose lines passing through the triple point which have intersection multiplictity 5 with the hypersurface, is the locus of $\langle x_4^2q_1(x_1,x_2,x_3,x_4)\rangle$ and $\langle x_4q_3(x_1,x_2,x_3,x_4)\rangle$. Since $x_4$ is a component of both terms then a line emanating from the hyperplane $\langle x_4\rangle$ to the triple point will have multiplicity 5.

Let $Y$ be a hypersurface which contains a triple point. By a coordinate transformation we can assume the triple point is $\langle x_1,x_2,x_3,x_4\rangle^3$. The most general equation which contains the ideal $\langle x_1,x_2,x_3,x_4\rangle^3$ is

$$x_0^2\Bigg(q_3(x_1,x_2,x_3,x_4)\Bigg)+x_0\Bigg(q_4(x_1,x_2,x_3,x_4)\Bigg)+q_5(x_1,x_2,x_3,x_4). \tag{4.19}$$

If the tangent cone is the union of a double plane and another hyperplane then

$$q_3(x_1,x_2,x_3,x_4)=f^2g \tag{4.20}$$

where $f$ and $g$ are linear forms. By a coordinate transformation which keeps the triple point fixed we can map the hyperplane $f$ to $x_4$. So without loss of generality

$$q_3(x_1,x_2,x_3,x_4)=x_4^2g. \tag{4.21}$$

If a general line from the hyperplane $\langle x_4 \rangle$ to the triple point has multiplicity 5 then

$$x_4 = q_4 = 0. \tag{4.22}$$

This occurs only if $q_4$ has $x_4$ as a component so

$$q_4 = x_4 q_3 \tag{4.23}$$

which is precisely of the form (4.17).

$\square$

**Proposition 4.5.6.** *A hypersurface $Y$ is of type $SS6$ if and only if $Y$ has a double line $L$ where every point $p \in L$ has the following properties:*

*i) the tangent cone of each point $p \in L$ is a double plane $P_p$;*

*ii) each point $p \in L$ has the same double plane tangent cone i.e. $P_p = P$ for some double plane $P$;*

*iii) the line connecting the point on the tangent cone $P_p$ and a point $p \in L$ has intersection multiplicty 4 with the hypersurface.*

*Proof.* Let $Y$ be of type $SS6$ then it is equivalent, via a coordinate transformation, to the hypersurface

$$\begin{gathered}\Big(q_3(x_0, x_1)x_4^2\Big) + \Big(x_4 q_{2,2}(x_0, x_1 \parallel x_2, x_3, x_4)\Big) \\ + q_{1,4}(x_0, x_1 \parallel x_2, x_3, x_4) + q_5(x_2, x_3, x_4)\end{gathered} \tag{4.24}$$

This hypersurface contains the double line $\langle x_2, x_3, x_4 \rangle^2$. For any point $[\lambda : \nu : 0 : 0 : 0]$ of the double line the tangent cone is the same double plane given by $\langle x_4 \rangle^2$. The points which have intersection multiplictity 4 with the double line are the locus of

$\langle x_4\rangle^2$ and $\langle x_4 q_{2,2}(\lambda,\nu \parallel x_2,x_3,x_4)\rangle$. Since $x_4$ is a component of both terms then the line emanating from the hyperplane $\langle x_4\rangle$ to any point of the double line will have multiplicity 4.

Let $Y$ be a hypersurface which contains a double line. By a coordinate transformation we can assume the double line is $\langle x_2,x_3,x_4\rangle^2$. The most general equation which contains the ideal $\langle x_2,x_3,x_4\rangle^2$ is

$$q_{3,2}(x_0,x_1 \parallel x_2,x_3,x_4) + q_{2,3}(x_0,x_1 \parallel x_2,x_3,x_4) + q_{1,4}(x_0,x_1 \parallel x_2,x_3,x_4) + q_5(x_2,x_3,x_4) \tag{4.25}$$

If the tangent cone at every point on the double line is the same double plane then

$$q_{3,2}(x_0,x_1 \parallel x_2,x_3,x_4) = q_3(x_0,x_1) f(x_2,x_3,x_4)^2 \tag{4.26}$$

where $f$ is a linear form. By a coordinate transformation, which keeps the double line fixed, the hyperplane $f$ is mapped to $x_4^2$. So without loss of generality,

$$q_{3,2}(x_0,x_1 \parallel x_2,x_3,x_4) = q_3(x_0,x_1)x_4^2. \tag{4.27}$$

If the line going from the hyperplane $\langle x_4\rangle$ to any point of the double line has multiplicity 4 then

$$x_4 = q_{2,3}(\lambda,\nu \parallel x_2,x_3,x_4) = 0. \tag{4.28}$$

This occurs only if $q_{2,3}(x_0,x_1 \parallel x_2,x_3,x_4)$ has $x_4$ as a component so

$$q_{2,3} = x_4 q_{2,2}(x_0,x_1 \parallel x_2,x_3,x_4) \tag{4.29}$$

which is precisely of the form (4.24).

□

**Proposition 4.5.7.** *A hypersurface $Y$ is of type $SS7$ if and only if $Y$ contains a triple point $p$ and a plane $P$, where $p \in P$ has the following properties:*

*i) the tangent cone of $p$ contains a triple plane of $P$;*

*ii) the singular locus of $Y$, when restricted to $P$, is the intersection of two quartic curves $q_1$ and $q_2$;*

*iii) the point $p$ is a quadruple point of $q_1$ and $q_2$.*

*Proof.* Let $Y$ be of type $SS7$ then it is equivalent, via a coordinate transformation, to the hypersurface

$$\begin{aligned}
&x_0^2 q_3(x_3, x_4) + x_0\Big(q_{2,2}(x_1, x_2 \parallel x_3, x_4) + q_{1,3}(x_1, x_2 \parallel x_3, x_4) + q_4(x_3, x_4)\Big) \\
&+\Big(q_{4,1}(x_1, x_2 \parallel x_3, x_4) + q_{3,2}(x_1, x_2 \parallel x_3, x_4) + q_{2,3}(x_1, x_2 \parallel x_3, x_4) \\
&+q_{1,4}(x_1, x_2 \parallel x_3, x_4) + q_5(x_3, x_4)\Big)
\end{aligned} \tag{4.30}$$

This hypersurface contains the triple point $p$ given by the ideal $\langle x_1, x_2, x_3, x_4\rangle^3$ and a plane $P$ given by $\langle x_3, x_4\rangle$. The tangent cone is the hypersurface defined by $q_3(x_3, x_4)$ which which contains the triple plane $\langle x_3, x_4\rangle^3$ of $P$. When the differential of Y is restricted to the plane $\langle x_3, x_4\rangle$ the only non-trivial contribution comes from the term

$$q_{4,1}(x_1, x_2 \parallel x_3, x_4) = q_4(x_1, x_2)x_3 + \tilde{q}_4(x_1, x_2)x_4. \tag{4.31}$$

The differential, when restricted to the plane, is zero when

$$q_4(x_1, x_2) = \tilde{q}_4(x_1, x_2) = 0. \tag{4.32}$$

Therefore, the plane contains two quartic curves $q_4(x_1, x_2)$ and $\tilde{q}_4(x_1, x_2)$ which contain $p$ as the quadruple point.

Let $Y$ be a hypersurface which contains a triple point $p$ and a plane $P$, where $p \in P$. By a coordinate transformation we can assume the triple point is $\langle x_1, x_2, x_3, x_4\rangle^3$

and the plane is $\langle x_3, x_4\rangle$. The most general equation which contains the ideal $\langle x_1, x_2, x_3, x_4\rangle^3$ and $\langle x_3, x_4\rangle$ is

$$\begin{aligned}
&x_0^2\Big(q_{2,1}(x_1, x_2 \parallel x_3, x_4) + q_{1,2}(x_1, x_2 \parallel x_3, x_4) + q_3(x_3, x_4)\Big)\\
&+x_0\Big(q_{3,1}(x_1, x_2 \parallel x_3, x_4) + q_{2,2}(x_1, x_2 \parallel x_3, x_4) + q_{1,3}(x_1, x_2 \parallel x_3, x_4) + q_4(x_3, x_4)\Big)\\
&+\Big(q_{4,1}(x_1, x_2 \parallel x_3, x_4) + q_{3,2}(x_1, x_2 \parallel x_3, x_4) + q_{2,3}(x_1, x_2 \parallel x_3, x_4)\\
&+q_{1,4}(x_1, x_2 \parallel x_3, x_4) + q_5(x_3, x_4)\Big)
\end{aligned} \tag{4.33}$$

If the tangent cone contains the triple plane of $P$ then it contains the ideal $\langle x_3, x_4\rangle^3$. Then the coeffecients of the $x_0^2$ term of (4.33) contains only the $q_3(x_3, x_4)$ term. The differential of (4.33), when restricted to the plane $\langle x_3, x_4\rangle$, contains the equations of the form $x_0q_3(x_1, x_2) + q_4(x_1, x_2)$ and $x_0\tilde{q}_3(x_1, x_2) + \tilde{q}_4(x_1, x_2)$. If the singular locus of $Y$ in the plane is the intersection of two quartic curves then

$$x_0q_3(x_1, x_2) + q_4(x_1, x_2) = x_0\tilde{q}_3(x_1, x_2) + \tilde{q}_4(x_1, x_2) = 0. \tag{4.34}$$

So $x_0q_3(x_1, x_2) + q_4(x_1, x_2)$ and $x_0\tilde{q}_3(x_1, x_2) + \tilde{q}_4(x_1, x_2)$ are the quartic curves. If $p$ is a quadruple point of both quartic curves then $q_3$ and $\tilde{q}_3$ are 0, so $Y$ is of the form (4.30).

□

## 4.6 Stable Locus

In this section we will describe the stable locus of the GIT compactification. The stable locus represents all of the closed orbits in the moduli space. In general it is not known what types of singularities would be allowed in the stable locus. The strategy for describing the stable locus, in this dissertation, is to classify the singularities which arise on the boundary of the moduli space. This gives some description of the singularities which arise in the stable locus. Ideally, the stable locus would only be smooth hypersurfaces and the boundary would include hypersurfaces with singularities. As shown in [1, 12] even in the case of cubic threefolds and cubic fourfolds this is not the case. As the degree and dimension of hypersurfaces increases more singularities will be included in the stable locus. In [16] there is a general proposition which states that a smooth hypersurface will always be stable.

**Proposition 4.6.1** ( [16] Prop. 4.2)**.** *A smooth hypersurface $F$ in $\mathbb{P}^n$ with degree $\geq 2$ is a stable hypersurface.*

A complete classification of all possible singularities in the stable locus has not been found. Using the results of the previous section a partial list of singularities can be determined.

**Proposition 4.6.2.** *If $X$ is a quintic threefold with at worst a double point then it is stable.*

*Proof.* Suppose $X$ is not stable, then it is strictly semistable or unstable. So it belongs to one of the families SS1 - SS7, but $X$ does not satisfy the singularity criteria for any of these families. Hence, it is stable. ☐

**Proposition 4.6.3.** *If $X$ is a quintic threefold with at worst a triple point whose tangent cone is an irreducible cubic surface and $X$ does not contain a plane then it is stable.*

*Proof.* Suppose $X$ is not stable, then it is strictly semistable or unstable. So it belongs to one of the families SS1 - SS7. The only families which have at worst a triple point are families SS5 and SS7. Since the tangent cone of $X$ is irreducible then it is not in SS5. Since $X$ does not contain a plane it is not in SS7, therefore it belong to neither family. Hence, it is stable. ☐

**Proposition 4.6.4.** *If $X$ is a quintic threefold with at worst a double line whose tangent cone at each point on the line is irreducible then it is stable.*

*Proof.* Suppose $X$ is not stable, then it is strictly semistable or unstable. So it belongs to one of the families SS1 - SS7. SS6 is the only family which has at worst a double line as a singularity. Since the tangent cone of $X$ at each point is irreducible then it is not in SS6. Hence, it is stable. ☐

These four classes of hypersurfaces give the most generic classes of hypersurfaces which are stable. There are also other classes which degenerate tangent cones which may still not fit into one of the classes SS1-SS7 but a complete classification is still unknown.

# Chapter 5

# Minimal Orbits

The classification of closed orbits in the GIT compactification is important for understanding the structure of the moduli space. Since, in the strictly semistable locus, multiple orbits map to the same point in the quotient space it is important to have a method for determining a unique representatives in the quotient space. The closed orbit lemma (3.2.9) shows that minimal orbits give such representatives. Classification of minimal orbits becomes a tractable problem due to Luna's criterion and knowledge of a destabilizing 1-PS for each maximal semistable family.

## 5.1 Degenerations

As stated in Chapter 3, strictly semistable points do not neccesarily have closed $G$ orbits. For a point $x \in X^{ss} \setminus X^s$, the closure $\overline{Gx}$ and the orbit $Gx$ maps to the same point in the GIT quotient. The purpose of minimal orbits is to find a unique representative for the orbit $Gx$. The presence of a destabilizing 1-PS $\lambda$ is used as a first approximation in determining the minimal orbit inside $\overline{Gx}$. Given a 1-PS $\lambda : \mathbb{C}^* \to SL(5, \mathbb{C})$, the closure of $\lambda x$ is defined as follows:

$$\overline{\lambda x} := \lim_{t \to 0} \lambda(t) x \tag{5.1}$$

where $t \in \mathbb{C}^*$. The closure $\overline{\lambda x} \subseteq \overline{Gx}$ is invariant under the action $\lambda$. The advantage of this process is that it reduces the large family of strictly semistable families to smaller set of invariant families. There are two levels of degenerations which occur in quintic threefolds. The first set of minimal orbits are $MO$-$(A-D)$. This set of minimal orbits degenerate further to $MO2$-$(I-X)$.

Table 5.1: Corresponding Degenerations of SS1 - SS7

| *Family* | *Destabilizing 1-PS Subgroup* | *Degeneration* |
|---|---|---|
| SS1 | $\langle 2,2,2,-3,-3\rangle$ | *MO-A* |
| SS2 | $\langle 1,1,1,1,-4\rangle$ | *MO-D* |
| SS3 | $\langle 3,3,-2,-2,-2\rangle$ | *MO-A* |
| SS4 | $\langle 4,-1,-1,-1,-1\rangle$ | *MO-D* |
| SS5 | $\langle 1,0,0,0,-1\rangle$ | *MO-B* |
| SS6 | $\langle 4,4,-1,-1,-6\rangle$ | *MO-C* |
| SS7 | $\langle 6,1,1,-4,-4\rangle$ | *MO-C* |

Table 5.2: First Level of Minimal Orbits MO-A - MO-D

<table>
<tr><th>Family</th><th>Invariant 1-PS Subgroup (H)</th><th>Centralizer of H ($Z_G(H)$)</th></tr>
<tr><td>MO-A</td><td>$\langle 3,3,-2,-2,-2\rangle$</td><td>$SL(2,\mathbb{C})\times SL(3,\mathbb{C})$</td></tr>
<tr><td colspan="3">$q_{2,3}(x_0,x_1\parallel x_2,x_3,x_4)$</td></tr>
<tr><td>MO-B</td><td>$\langle 1,0,0,0,-1\rangle$</td><td>$\mathbb{C}^*\times SL(3,\mathbb{C})\times\mathbb{C}^*$</td></tr>
<tr><td colspan="3">$q_5(x_1,x_2,x_3)+x_0x_4q_3(x_1,x_2,x_3)+x_0^2x_4^2q_1(x_1,x_2,x_3)$</td></tr>
<tr><td>MO-C</td><td>$\langle 4,4,-1,-1,-6\rangle$</td><td>$SL(2,\mathbb{C})\times SL(2,\mathbb{C})\times\mathbb{C}^*$</td></tr>
<tr><td colspan="3">$q_{1,4}(x_0,x_1\parallel x_2,x_3)+x_4q_{2,2}(x_0,x_1\parallel x_2,x_3)+x_4^2q_3(x_0,x_1)$</td></tr>
<tr><td>MO-D</td><td>$\langle 4,-1,-1,-1,-1\rangle$</td><td>$\mathbb{C}^*\times SL(4,\mathbb{C})$</td></tr>
<tr><td colspan="3">$x_0q_4(x_1,x_2,x_3,x_4)$</td></tr>
</table>

Table 5.3: Second Level of Minimal Orbits SS2I - SS2X

| *Family* | *Invariant 1-PS Subgroup (H)* | *Centralizer of H ($Z_G(H)$)* |
|---|---|---|
| MO2-I | $\langle 6, 0, 2, -3, -5\rangle$ | $\mathbb{C}^{*5}$ |
| $x_0^2(x_2x_4^2) + x_0x_1(x_2x_3x_4) + x_1^2(x_2x_3^2)$ | | |
| MO2-II | $\langle 4, 2, -1, -2, -3\rangle$ | $\mathbb{C}^{*5}$ |
| $x_0^2(x_3x_4^2) + x_0x_1(x_3^3 + x_2x_3x_4) + x_1^2(x_2^2x_3)$ | | |
| MO2-III | $\langle 4, 2, 0, -2, -4\rangle$ | $\mathbb{C}^{*5}$ |
| $x_0^2(x_2x_4^2 + x_3^2x_4) + x_0x_1(x_3^3 + x_2x_3x_4) + x_1^2(x_2x_3^2 + x_2^2x_4)$ | | |
| MO2-IV | $\langle 4, 2, -1, -1, -5\rangle$ | $\mathbb{C}^{*2} \times SL(2, \mathbb{C}) \times \mathbb{C}^*$ |
| $x_0x_4(x_1q_2(x_2, x_3))$ | | |
| MO2-V | $\langle 5, 3, -1, -2, -7\rangle$ | $\mathbb{C}^{*5}$ |
| $x_0x_4(x_1x_2x_3)$ | | |
| MO2-VI | $\langle 2, 1, 0, -1, -2\rangle$ | $\mathbb{C}^{*5}$ |
| $(x_1x_2^3x_3 + x_1^2x_2x_3^2) + x_0x_4(x_1x_2x_3)$ | | |
| MO2-VII | $\langle 5, 3, 0, -2, -6\rangle$ | $\mathbb{C}^{*5}$ |
| $x_4x_0^2x_3^2 + x_4x_0x_1x_2x_3 + x_4x_1^2x_2^2$ | | |
| MO2-VIII | $\langle 4, 2, -2, -2, -2\rangle$ | $\mathbb{C}^{*2} \times SL(3, \mathbb{C})$ |
| $x_0x_1q_3(x_2, x_3, x_4)$ | | |
| MO2-IX | $\langle 4, 0, 0, -2, -2\rangle$ | $\mathbb{C}^* \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ |
| $x_0q_{2,2}(x_1, x_2 \parallel x_3, x_4)$ | | |
| MO2-X | $\langle 4, 0, -1, -1, -2\rangle$ | $\mathbb{C}^{*2} \times SL(2, \mathbb{C}) \times \mathbb{C}^*$ |
| $x_0(q_4(x_2, x_3) + x_1q_2(x_2, x_3)x_4 + x_1^2x_4^2)$ | | |

*Remark* 5.1.1. The normalized 1-PS $\lambda$ is represented by a vector of the form $\langle a_0, a_1, a_2, a_3, a_4\rangle$. The action of this $\lambda(t)$ on the variables $x_i$ is given by $\lambda(t)x_i = t^{a_i}x_i$. So the action on a monomial is of the form
$\lambda(t)x_0^{b_0}x_1^{b_1}x_2^{b_2}x_3^{b_3}x_4^{b_4} = t^{a_0b_0+a_1b_1+a_2b_2+a_3b_3+a_4b_4}x_0^{b_0}x_1^{b_1}x_2^{b_2}x_3^{b_3}x_4^{b_4}$. Which can be thought of the dot product of $\langle a_0, a_1, a_2, a_3, a_4\rangle$ and $\langle b_0, b_1, b_2, b_3, b_4\rangle$.

**Proposition 5.1.2.** *If $Y$ is of type $SS1$ then it degenerates, via the 1-PS $\lambda =$*

$\langle 2,2,2,-3,-3\rangle$*, to a hypersurface of type MO-A.*

*Proof.* Let the 1-PS $\lambda = \langle 2,2,2,-3,-3\rangle$ act on the monomials in the family $x = q_{3,2}(x_0,x_1,x_2 \parallel x_3,x_4) + q_{2.3}(x_0,x_1,x_2 \parallel x_3,x_4) + q_{1,4}(x_0,x_1,x_2 \parallel x_3,x_4) + q_5(x_3,x_4)$. The closure $\overline{\lambda x}$ represents the monomials which are invariant under the action of $\lambda$. These are precisely the monomials equivalent, up to a coordinate transformation, to *MO-A*. ☐

**Proposition 5.1.3.** *If $Y$ is of type $SS2$ then it degenerates, via the 1-PS $\lambda = \langle 1,1,1,1,-4\rangle$, to a hypersurface of type MO-D.*

*Proof.* Let the 1-PS $\lambda = \langle 1,1,1,1,-4\rangle$ act on the monomials in the family $x = x_4q_4(x_0,x_1,x_2,x_3,x_4)$. The closure $\overline{\lambda x}$ represents the monomials which are invariant under the action of $\lambda$. These are precisely the monomials equivalent, up to a coordinate transformation, to *MO-D*. ☐

**Proposition 5.1.4.** *If $Y$ is of type $SS3$ then it degenerates, via the 1-PS $\lambda = \langle 3,3,-2,-2,-2\rangle$, to a hypersurface of type MO-A.*

*Proof.* Let the 1-PS $\lambda = \langle 3,3,-2,-2,-2\rangle$ act on the monomials in the family $x = q_{2,3}(x_0,x_1 \parallel x_2,x_3,x_4) + q_{1,4}(x_0,x_1 \parallel x_2,x_3,x_4) + q_5(x_2,x_3,x_4)$. The closure $\overline{\lambda x}$ represents the monomials which are invariant under the action of $\lambda$. These are precisely the monomials equivalent, up to a coordinate transformation, to *MO-A*. ☐

**Proposition 5.1.5.** *If $Y$ is of type $SS4$ then it degenerates, via the 1-PS $\lambda = \langle 4,-1,-1,-1,-1\rangle$, to a hypersurface of type MO-D.*

*Proof.* Let the 1-PS $\lambda = \langle 4,-1,-1,-1,-1\rangle$ act on the monomials in the family $x = x_0q_4(x_1,x_2,x_3,x_4)+q_5(x_1,x_2,x_3,x_4)$. The closure $\overline{\lambda x}$ represents the monomials which are invariant under the action of $\lambda$. These are precisely the monomials equivalent, up to a coordinate transformation, to *MO-D*. ☐

**Proposition 5.1.6.** *If $Y$ is of type $SS5$ then it degenerates, via the 1-PS $\lambda = \langle 1,0,0,0,-1\rangle$, to a hypersurface of type MO-B.*

*Proof.* Let the 1-PS $\lambda = \langle 1, 0, 0, 0, -1\rangle$ act on the monomials in the family $x = x_0^2(x_4^2 q_1(x_1, x_2, x_3, x_4)) + x_0 x_4 q_3(x_1, x_2, x_3, x_4) + q_5(x_1, x_2, x_3, x_4)$. The closure $\overline{\lambda x}$ represents the monomials which are invariant under the action of $\lambda$. These are precisely the monomials equivalent, up to a coordinate transformation, to $MO$-$B$. □

**Proposition 5.1.7.** *If $Y$ is of type $SS6$ then it degenerates, via the 1-PS $\lambda = \langle 4, 4, -1, -1, -6\rangle$, to a hypersurface of type $MO$-$C$.*

*Proof.* Let the 1-PS $\lambda = \langle 4, 4, -1, -1, -6\rangle$ act on the monomials in the family $x = x_4^2 q_3(x_0, x_1) + x_4 q_{2,2}(x_0, x_1 \parallel x_2, x_3, x_4) + q_{1,4}(x_0, x_1 \parallel x_2, x_3, x_4) + q_5(x_2, x_3, x_4)$. The closure $\overline{\lambda x}$ represents the monomials which are invariant under the action of $\lambda$. These are precisely the monomials equivalent, up to a coordinate transformation, to $MO$-$C$. □

**Proposition 5.1.8.** *If $Y$ is of type $SS7$ then it degenerates, via the 1-PS $\lambda = \langle 6, 1, 1, -4, -4\rangle$, to a hypersurface of type $MO$-$C$.*

*Proof.* Let the 1-PS $\lambda = \langle 6, 1, 1, -4, -4\rangle$ act on the monomials in the family $x = x_0^2 q_3(x_3, x_4) + x_0(q_{2,2}(x_1, x_2 \parallel x_3, x_4) + q_{1,3}(x_1, x_2 \parallel x_3, x_4) + q_4(x_3, x_4)) + (q_{4,1}(x_1, x_2 \parallel x_3, x_4) + q_{3,2}(x_1, x_2 \parallel x_3, x_4) + q_{2,3}(x_1, x_2 \parallel x_3, x_4) + q_{1,4}(x_1, x_2 \parallel x_3, x_4) + q_5(x_3, x_4))$. The closure $\overline{\lambda x}$ represents the monomials which are invariant under the action of $\lambda$. These are precisely the monomials equivalent, up to a coordinate transformation, to $MO$-$C$. □

## 5.2 Luna's Criterion

The ability to degenerate the large families $SS$(1-7) into much smaller invariant families $MO$-($A$-$D$) makes the problem of finding minimal closed orbits much more tractable. Generically, a hypersurface in the families $MO$-($A$-$D$) will be closed and thus minimal. To explicitly determine which elements in $MO$-($A$-$D$) are closed and which elements further degenerate one can use Luna's criterion.

Luna's criterion is used when there is an affine $G$-variety $X$ and a point $x \in X$ which has a non-finite stabilizer $H \subseteq G$. If $X^H$ is the set of points in $X$ which are

$H$-invariant and $N_G(H)$ is the normalizer of $H$ in $G$ then there is a natural action of $N_G(H)$ on $X^H$. Luna's criterion reduces the problem of determining whether $Gx$ is closed in $X$ to whether $N_G(H)$ is closed in $X^H$.

**Proposition 5.2.1** (Luna's Criterion). *[13, 23]*

*Let $X$ be an affine variety with a $G$-action and $x \in X$ a point stabilized by a subgroup $H \subseteq G$. Then the orbit $Gx$ is closed in $X$ if and only if the orbit $N_G(H)x$ is closed in $X^H$.*

*Remark* 5.2.2 ( [23]). In the case where $H$ is reductive and connected $N_G(H) = H \cdot Z_G(H)$, where $Z_G(H)$ is the centralizer of $H$ in $G$. Since $N_G(H)$ acts on $X^H$ we can quotient out by $H$. Thus, we can study the action of $Z_G(H)$, instead of $N_G(H)$, on $X^H$.

The case of quintic threefolds consists of an $SL(5, \mathbb{C})$ action on the projective variety $\mathbb{P}(V)$, where $V = Sym^5(\mathbb{C}^5)$. $V$ is the linearization of the $SL(5, \mathbb{C})$ on $\mathbb{P}(V)$. The closed orbits of points in the linearization $V$ correspond to closed orbits of points in $\mathbb{P}(V)$. The correspondence between a projective variety and it's linearization allows us to apply Luna's criterion to $V$. Given a point from one of the families $MO$-($A$-$D$) the stabilizer subgroup is the invariant $1 - PS$ i.e. $H = \lambda$. The following lemma reduces the problem of finding minimal orbits to finding stable points in the $Z_G(H)$-action on $V^H$.

**Lemma 5.2.3.** *Let $v \in V$ be a point with stabilizer $H$ i.e. in $V^H$. If $v \in V$ is stable with respect to the $Z_G(H)$-action on the $H$-invariant space $V^H$ then the orbit $Gv$ is closed.*

*Proof.* Let $v \in V^H$ be stable with respect to the $Z_G(H)$-action on $V^H$. By the definition of stable point, the orbit $Z_G(H)v$ is closed. By Luna's criterion $Gv$ is closed. □

Stable points can be found by using the Hilbert-Mumford criterion for the $Z_G(H)$-action on $V^H$. If a point $v \in V^H$ is a point from $MO$-($A$-$D$) and is strictly semistable with respect to the $Z_G(H)$-action, then there is a corresponding destablizing 1-PS $\lambda$. This destabilizing 1-PS further degenerates $v$ with a different stabilizer $H'$ and it must be repeated to determine closed orbits for those points.

## 5.3 First Level of Minimal Orbits

In this section we will apply the Hilbert-Mumford criterion to $Z_G(H)$-actions on $V^H$. We explicitly determine the semistable and unstable points for each minimal orbit to show which points are closed, which are semistable and degenerate further, and which are unstable. We will explictly describe this procedure for the case of $MO$-$A$. Similar modifications can be made for all other cases.

### 5.3.1 Minimal Orbit A

In the case of $MO$-$A$ the centralizer $Z_G(H) = \mathbb{C}^2 \times SL(3, \mathbb{C}) \subseteq SL(5, \mathbb{C})$ acts on polynomials in $MO$-$A$ which are of the form $q_{2,3}(x_0, x_1 \parallel x_2, x_3, x_4)$. The minimal orbit can be written as

$$x_0^2 s_3(x_2, x_3, x_4) + x_0 x_1 t_3(x_2, x_3, x_4) + x_1^2 u_3(x_2, x_3, x_4), \tag{5.2}$$

where $s_3(x_2, x_3, x_4)$, $t_3(x_2, x_3, x_4)$, $u_3(x_2, x_3, x_4)$ are degree 3 polynomials in the variables $x_2$, $x_3$, and $x_4$. The polynomials which represent closed orbits in this family are stable with respect to the $\mathbb{C}^2 \times SL(3, \mathbb{C})$ action on (5.2). The polynomials which further degenerate are semistable with respect to the $\mathbb{C}^2 \times SL(3, \mathbb{C})$ action. The unstable points are unstable with respect to the $\mathbb{C}^2 \times SL(3, \mathbb{C})$ action. The set of semistable and unstable points can be found by modifying the techniques in Chapter 4 which involved classifying $G$-orbits by using (3.12) to find which polynomials which satisfy the Hilbert-Mumford criterion for the set of normalized 1-PS $\lambda$. A normalized 1-PS in $\mathbb{C}^2 \times SL(3, \mathbb{C})$ is of the following form:

$$\begin{pmatrix} t & 0 & 0 & \cdots & 0 \\ 0 & t^{-1} & 0 & \cdots & 0 \\ 0 & 0 & t^a & 0 & 0 \\ \vdots & \vdots & 0 & t^b & 0 \\ 0 & 0 & 0 & 0 & t^c \end{pmatrix} \tag{5.3}$$

where $a+b+c=0$ and $a \geq b \geq c$. The normalization restriction of the $SL(3,\mathbb{C})$ block gives an ordering of the degree 3 monomials in the variables $x_2, x_3, x_4$. The weights of $x_0^2$, $x_0x_1$, and $x_2^2$ are 2,0, and $-2$ respectively. The set of maximal strictly semistable polynomials $F$ are those where $\mu(F,\lambda) \leq 0$ and the maximal unstable families have $\mu(F,\lambda) < 0$. A general polynomial in (5.2) will be semistable if $s_3$, $t_3$, and $u_3$ have at most weights $-2$, 0, and 2, with respect to the action (5.3). These weights are neccesary in order balance the weights arising from $x_0^2$, $x_0x_1$, $x_1^2$ so that $\mu$ is less than zero. Similary, the weights for a polynomial in (5.2) the weights for $s_3$, $t_3$, and $u_3$ are at most $-3$, $-1$, and 1 for the polynomial to be unstable. The calculation below gives the set of semistable and unstable families. A similar method can be used for all other minimal orbits in order to explicitly calculate the set of semistable and unstable families.

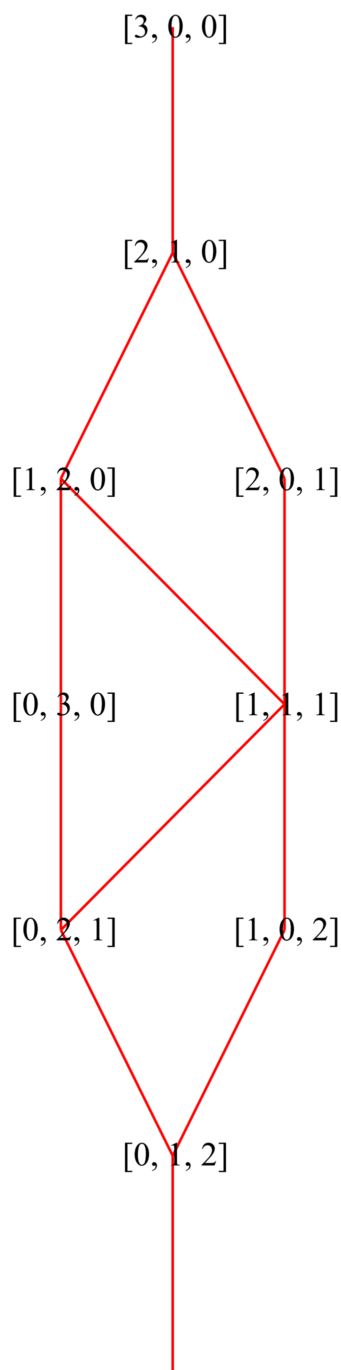


Figure 5.1: Poset structure of degree 3 monomials

**Semistable Families**

**Family 1** $\langle 3,3,-2,-2,-2\rangle$

***Invariant Family*** $q_{2,3}(x_0, x_1 \parallel x_2, x_3, x_4)$

(SS1-A) $x_0^2\bigg(q_3(x_3,x_4)+q_1(x_2,x_3)x_4^2+x_4^3\bigg)+x_0x_1\bigg(q_3(x_3,x_4)+x_2x_3x_4+q_1(x_2,x_3)x_4^2+x_3x_4^2+x_3^2x_4+x_4^3\bigg)+x_1^2\bigg(x_2q_2(x_3,x_4)+q_3(x_3,x_4)\bigg)$

(a) *1-PS* : $\langle 1,-1,4,-1,-3\rangle$

(b) *Degeneration* $(MO2-I)$ : $x_0^2\bigg(x_2x_4^2\bigg)+x_0x_1\bigg(x_2x_3x_4\bigg)+x_1^2\bigg(x_2x_3^2\bigg)$

(SS2-A) $x_0^2\bigg(q_3(x_3,x_4)\bigg)+x_0x_1\bigg(x_2q_2(x_3,x_4)+q_3(x_3,x_4)\bigg)+x_1^2\bigg(x_2q_2(x_3,x_4)+q_3(x_3,x_4)\bigg)$

(a) 1-PS: $\langle 1,-1,2,-1,-1\rangle$

(b) *Degeneration* $(MO2-IV)$ : $x_0x_1\bigg(x_2q_2(x_3,x_4)\bigg)$

(SS3-A) $x_0^2\bigg(q_1(x_2,x_3)x_4^2+x_4^3\bigg)+x_0x_1\bigg(q_2(x_2,x_3)x_4+q_1(x_2,x_3)x_4^2+x_4^3\bigg)+x_1^2\bigg(q_2(x_2,x_3)x_4+q_1(x_2,x_3)x_4^2+x_4^3\bigg)$

(a) 1-PS: $\langle 1,-1,1,1,-2\rangle$

(b) *Degeneration* $(MO2-IV)$ : $x_0x_1\bigg(q_2(x_2,x_3)x_4\bigg)$

(SS4-A) $x_0^2\bigg(x_3x_4^2+x_4^3\bigg)+x_0x_1\bigg(q_3(x_3,x_4)+x_2x_3x_4+q_1(x_2,x_3)x_4^2+x_3x_4^2+x_3^2x_4+x_4^3\bigg)+x_1^2\bigg(x_2^2q_1(x_3,x_4)+x_2q_2(x_3,x_4)+q_3(x_3,x_4)\bigg)$

(a) 1-PS: $\langle 1,-1,1,0,-1\rangle$

(b) *Degeneration* $(MO2-II): x_0^2\Big(x_3x_4^2\Big)+x_0x_1\Big(x_3^3+x_2x_3x_4\Big)+x_1^2\Big(x_2^2x_3\Big)$

(SS5-A) $x_0^2\Big(q_1(x_2,x_3)x_4^2+x_3^2x_4+x_4^3\Big)+x_0x_1\Big(q_3(x_3,x_4)+x_2x_3x_4+q_1(x_2,x_3)x_4^2+x_3x_4^2+x_3^2x_4+x_4^3\Big)+x_1^2\Big(x_2q_2(x_3,x_4)+q_3(x_3,x_4)+q_2(x_2,x_3)x_4+x_4^3\Big)$

(a) 1-PS: $\langle 1,-1,2,0,-2\rangle$

(b) *Degeneration* $(M02-IV): x_0^2\Big(x_2x_4^2+x_3^2x_4\Big)+x_0x_1\Big(x_3^3+x_2x_3x_4\Big)+x_1^2\Big(x_2x_3^2+x_2^2x_4\Big)$

**Unstable Families**

(US1-A) $x_0^2(q_3(x_3,x_4)+x_2x_4^2)+x_0x_1(q_3(x_3,x_4)+x_2x_4^2)+x_1^2(q_3(x_3,x_4)+x_2x_3x_4+x_2x_4^2)$

(US2-A) $x_0^2(q_3(x_3,x_4))+x_0x_1(q_3(x_3,x_4))+x_1^2(q_3(x_3,x_4)+x_2q_2(x_3,x_4))$

(US3-A) $x_0^2(x_2x_4^2+x_3^2x_4+x_3x_4^2+x_4^3)+x_0x_1(x_2x_4^2+x_3^2x_4+x_3x_4^2+x_4^3)+x_1^2(q_3(x_3,x_4)+x_2x_3x_4+x_2x_4^2)$

(US4-A) $x_0^2(x_2x_4^2+x_3x_4^2+x_4^3)+x_0x_1(x_2x_4^2+x_3^2x_4+x_3x_4^2+x_4^3)+x_1^2(q_2(x_2,x_3)x_4+q_1(x_2,x_3)x_4^2+x_4^3)$

(US5-A) $x_0^2(x_4^3)+x_0x_1(q_1(x_2,x_3)x_4^2+x_3^2x_4+x_4^3)+x_1^2(q_3(x_3,x_4)+x_2^2x_4+x_2x_3x_4+x_2x_4^2)$

(US6-A) $x_0^2(x_3x_4^2+x_4^3)+x_0x_1(q_1(x_2,x_3)x_4^2+x_3^2x_4+x_4^3)+x_1^2(q_3(x_3,x_4)+x_2x_3x_4+x_2x_4^2)$

(US7-A) $x_0^2(x_3x_4^2+x_4^3)+x_0x_1(q_1(x_2,x_3)x_4^2+x_3^2x_4+x_4^3)+x_1^2(q_2(x_2,x_3)x_4+q_1(x_2,x_3)x_4^2+x_4^3)$

(US8-A) $x_0^2(x_3^2x_4+x_3x_4^2+x_4^3)+x_0x_1(q_1(x_2,x_3)x_4^2+x_3^2x_4+x_4^3)+x_1^2(x_2q_2(x_3,x_4)+q_3(x_3,x_4))$

(US9-A) $x_0^2(x_3x_4^2+x_4^3)+x_0x_1(q_1(x_2,x_3)x_4^2+x_3^2x_4+x_4^3)+x_1^2(x_2q_2(x_3,x_4)+q_3(x_3,x_4))$

(US10-A) $x_0^2(x_4^3)+x_0x_1(q_1(x_2,x_3)x_4^2+x_3^2x_4+x_4^3)+x_1^2(x_2q_2(x_3,x_4)+q_3(x_3,x_4))$

(US11-A) $x_0^2(x_3x_4^2+x_4^3)+x_0x_1(q_1(x_2,x_3)x_4^2+x_3^2x_4+x_4^3)+x_1^2(x_2x_3x_4+x_2x_4^2+q_3(x_3,x_4))$

(US12-A) $x_0^2(x_3x_4^2 + x_4^3) + x_0x_1(q_1(x_2, x_3)x_4^2 + x_3^2x_4 + x_4^3) + x_1^2(q_2(x_2, x_3)x_4 + q_1(x_2, x_3)x_4^2 + x_4^3)$

(US13-A) $x_0^2(x_4^3) + x_0x_1(q_1(x_2, x_3)x_4^2 + x_3^2x_4 + x_4^3) + x_1^2(q_2(x_2, x_3)x_4 + q_1(x_2, x_3)x_4^2 + x_4^3 + x_3^3)$

(US14-A) $x_0^2(x_4^3 + x_3^2x_4) + x_0x_1(x_4^3 + x_3^2x_4) + x_1^2(x_2q_2(x_3, x_4) + q_3(x_3, x_4))$

(US15-A) $x_0^2(x_4^3) + x_0x_1(x_4^3 + x_3^2x_4) + x_1^2(x_2q_2(x_3, x_4) + q_3(x_3, x_4) + x_2^3x_4)$

(US16-A) $x_0^2(x_3^2x_4 + x_4^3) + x_0x_1(x_4^3 + x_3^2x_4) + x_1^2(q_3(x_3, x_4) + x_2x_3x_4 + x_2x_4^2)$

(US17-A) $x_0^2(x_3^2x_4 + x_4^3) + x_0x_1(x_4^3 + x_3^2x_4) + x_1^2(q_2(x_2, x_3)x_4 + q_1(x_2, x_3)x_4 + x_4^3)$

(US18-A) $x_0^2(x_4^3) + x_0x_1(x_4^3 + x_3^2x_4) + x_1^2(q_3(x_3, x_4) + x_2x_3x_4 + x_2x_4^2 + x_3^3)$

**Proposition 5.3.2.** *Let $Y$ be, up to a coordinate transformation of the form $q_{2,3}(x_0, x_1 \parallel x_2, x_3, x_4)$.*

1. *If $Y$ belongs to one of the families US1-A - US18-A then $Y$ is unstable and does not represent a closed orbit.*
2. *If $Y$ is of type SS1-A then the orbit is not closed and it degenerates to MO2-I.*
3. *If $Y$ is of type SS2-A then the orbit is not closed and it degenerates to MO2-IV.*
4. *If $Y$ is of type SS3-A then the orbit is not closed and it degenerates to MO2-IV.*
5. *If $Y$ is of type SS4-A then the orbit is not closed and it degenerates to MO2-I.*
6. *If $Y$ is of type SS5-A then the orbit is not closed and it degenerates to M02-IV.*

*Otherwise, $Y$ is a closed orbit.*

### 5.3.3 Minimal Orbit B

**Semi-Stable Families**

**Family 2** $\langle 1,0,0,0,-1\rangle$

***Invariant Family:*** $q_5(x_1,x_2,x_3)+x_0q_3(x_1,x_2,x_3)x_4+x_0^2q_1(x_1,x_2,x_3)x_4^2$

(SS1-B) $$\Bigg(x_1q_4(x_2,x_3)+q_5(x_2,x_3)\Bigg)+x_0x_4\Bigg(x_1q_2(x_2,x_3)+q_3(x_2,x_3)\Bigg)+ x_0^2x_4^2\Bigg(q_1(x_2,x_3)\Bigg)$$

(a) 1-PS: $\langle 1,2,-1,-1,-1\rangle$

(b) *Degeneration* $(MO2-IV):x_0x_4\Bigg(x_1q_2(x_2,x_3)\Bigg)$

(SS2-B) $$\Bigg(x_1q_4(x_2,x_3)+q_5(x_2,x_3)+q_2(x_1,x_2)x_3^2\Bigg)+ x_0x_4\Bigg(q_3(x_2,x_3)+x_1x_2x_3+q_1(x_1,x_2)x_3^2\Bigg)+x_0^2x_4^2\Bigg(q_1(x_2,x_3)\Bigg)$$

(a) 1-PS: $\langle 1,3,-1,-2,-1\rangle$

(b) *Degeneration* $(MO2-V):x_0x_4\Bigg(x_1x_2x_3\Bigg)$

(SS3-B) $$\Bigg(q_3(x_1,x_2)x_3^2+q_2(x_1,x_2)x_3^3+q_1(x_1,x_2)x_3^4+x_3^5\Bigg) +x_0x_4\Bigg(q_2(x_1,x_2)x_3+q_1(x_1,x_2)x_3^2+x_3^3\Bigg)+x_0^2x_4^2\Bigg(x_3\Bigg)$$

(a) 1-PS: $\langle 1,1,1,-2,-1\rangle$

(b) *Degeneration* $(MO2-IV):x_0x_4\Bigg(q_2(x_1,x_2)x_3)\Bigg)$

(SS4-B) $$\Bigg(q_5(x_2,x_3)+x_1x_2^3x_3+x_1x_2^2x_3^2+x_1x_2x_3^3+x_1x_3^4+x_1^2x_2x_3^2+q_2(x_1,x_2)x_3^3\Bigg)+ x_0x_4\Bigg(q_3(x_2,x_3)+x_1x_2x_3+q_1(x_1,x_2)x_3^2\Bigg)+x_0^2x_4^2\Bigg(q_1(x_x,x_3)\Bigg)$$

(a) 1-PS: $\langle 1,1,0,-1,-1\rangle$

(b) *Degeneration* $(MO2-VI):\left(x_1x_2^3x_3+x_1^2x_2x_3^2\right)+x_0x_4\left(x_1x_2x_3\right)$

**Unstable Families**

(US1-B) $(x_1q_4(x_2,x_3)+x_1^2x_3)+x_0x_4(q_3(x_2,x_3)+x_1x_3^2)+x_0^2x_4^2(x_2+x_3)$

(US2-B) $(q_5(x_2,x_3)+x_1x_2^3x_3+x_1x_2^2x_3^2+x_1^2x_3^3+x_1x_2x_3^3+x_1x_3^4)+x_0x_4(q_1(x_1,x_2)x_3^2+x_2^2x_3+x_3^3)+x_0^2x_4^2(x_3)$

(US3-B) $(x_1^2x_2x_3^2+x_1x_2^2x_3^2+x_2^4x_3+x_2^3x_3^2+x_2^2x_3^3+q_2(x_1,x_2)x_3^3+q_1(x_1,x_2)x_3^4+x_3^5)+x_0x_4(q_1(x_1,x_2)x_3^2+x_2^2x_3+x_3^3)+x_0^2x_4^2(x_3)$

(US4-B) $(q_3(x_1,x_2)x_3^2+q_2(x_1,x_2)x_3^3+q_1(x_1,x_2)x_3^4+x_4^5)+x_0x_4(q_1(x_1,x_2)x_3^2+x_2^2x_3+x_3^3)+x_0^2x_4^2(x_3)$

**Proposition 5.3.4.** *Let $Y$ be, up to a coordinate transformation of the form $q_5(x_1,x_2,x_3)+x_0q_3(x_1,x_2,x_3)x_4+x_0^2q_1(x_1,x_2,x_3)x_4^2$.*

1. *If $Y$ belongs to one of the families US1-B - US4-B then $Y$ is unstable and does not represent a closed orbit.*

2. *If $Y$ is of type SS1-B then the orbit is not closed and it degenerates to MO2-IV.*

3. *If $Y$ is of type SS2-B then the orbit is not closed and it degenerates to MO2-V.*

4. *If $Y$ is of type SS3-B then the orbit is not closed and it degenerates to MO2-IV.*

5. *If $Y$ is of type SS4-B then the orbit is not closed and it degenerates to MO2-VI.*

*Otherwise, $Y$ is a closed orbit.*

### 5.3.5 Minimal Orbit C

**Semi-Stable Family**

**Family 3** $\langle 4, 4, -1, -1, -6\rangle$

***Invariant Family:*** $q_{1,4}(x_0, x_1 \parallel x_2, x_3) + q_3(x_0, x_1)x_4^2 + q_{2,2}(x_0, x_1 \parallel x_2, x_3)x_4$

$$\text{(SS1-C)} \quad \Big(x_0(x_2x_3^3 + x_3^4) + x_1(x_2^2x_3^2 + x_2x_3^3 + x_3^4)\Big) + x_4^2\Big(x_0x_1^2 + x_1^3\Big) + x_4\Big(x_0^2(x_3^2) + x_0x_1(x_2x_3 + x_3^2) + x_1^2(x_2^2 + x_2x_3 + x_3^2)\Big)$$

(a) 1-PS: $\langle 1, -1, 1, -1, 0\rangle$

(b) *Degeneration* $(MO2 - VII) : x_4x_0^2x_3^2 + x_4x_0x_1x_2x_3 + x_4x_1^2x_2^2$

**Unstable Family**

$$\text{(US1-C)} \quad x_0(x_2x_3^3+x_3^4)+x_1(x_2^2x_3^2+x_2x_3^3+x_3^3)+x_4^2(x_0x_1^2+x_1^3)+x_4x_0x_1(x_3^2)+x_4x_1^2(x_2x_3+x_3^2)$$

**Proposition 5.3.6.** *Let $Y$ be, up to a coordinate transformation of the form* $q_{1,4}(x_0, x_1 \parallel x_2, x_3) + q_3(x_0, x_1)x_4^2 + q_{2,2}(x_0, x_1 \parallel x_2, x_3)x_4$.

1. *If $Y$ belongs to the family US1-C then $Y$ is unstable and does not represent a closed orbit.*

2. *If $Y$ is of type SS1-C then the orbit is not closed and it degenerates to MO2-VII.*

*Otherwise, $Y$ is a closed orbit.*

### 5.3.7 Minimal Orbit D

**Semistable Family**

**Family 4** $\langle 4, -1, -1, -1, -1\rangle$ ***Invariant Family:*** $x_0 q_4(x_1, x_2, x_3, x_4)$

(SS1-D) $x_0\Big(x_1 q_3(x_2, x_3, x_4) + q_4(x_2, x_3, x_4)\Big)$

(a) 1-PS: $\langle 0, 3, -1, -1, -1\rangle$

(b) *Degeneration* $(MO2 - VIII) : x_0 x_1 q_3(x_2, x_3, x_4)$

(SS2-D) $x_0\Big(q_3(x_1, x_2, x_3)x_4 + q_2(x_1, x_2, x_3)x_4^2 + q_1(x_1, x_2, x_3)x_4^3 + x_4^4\Big)$

(a) 1-PS: $\langle 0, 1, 1, 1, -3\rangle$

(b) *Degeneration* $(MO2 - VII) : x_0 x_4 q_3(x_1, x_2, x_3)$

(SS3-D) $x_0\Big(q_{2,2}(x_1, x_2 \parallel x_3, x_4) + q_{1,3}(x_1, x_2 \parallel x_3, x_4) + q_4(x_3, x_4)\Big)$

(a) 1-PS: $\langle 0, 1, 1, -1, -1\rangle$

(b) *Degeneration* $(MO2 - IX) : x_0 q_{2,2}(x_1, x_2 \parallel x_3, x_4)$

(SS4-D) $x_0\Big(q_4(x_2, x_3, x_4) + x_1 q_2(x_2, x_3)x_4 + q_2(x_1, x_2, x_3)x_4^2 + q_1(x_1, x_2, x_3)x_4^3\Big)$

(a) 1-PS: $\langle 0, 1, 0, 0, -1\rangle$

(b) *Degeneration* $(MO2 - X) : x_0\Big(q_4(x_2, x_3) + x_1 q_2(x_2, x_3)x_4 + x_1^2 x_4^2\Big)$

**Unstable Family**

(US1-D) $q_4(x_2, x_3, x_4) + q_{1,3}(x_1, x_2 \parallel x_3, x_4) + q_4(x_3, x_4) + x_1 x_2 x_4^2$

(US2-D) $x_2^2 x_3 x_4 + q_{1,3}(x_1, x_2 \parallel x_3, x_4) + q_4(x_3, x_4) + q_2(x_1, x_2, x_3)x_4^2 + q_1(x_1, x_2, x_3)x_4^3$

(US3-D) $x_2q_3(x_3,x_4)+q_4(x_3,x_4)+q_3(x_2,x_3)x_4+q_2(x_2,x_3)x_4^2+q_1(x_2,x_3)x_4^3+$
$q_1(x_1,x_2)x_3^2x_4+q_2(x_1,x_2,x_3)x_4^2+q_1(x_1,x_2,x_3)x_4^3$

**Proposition 5.3.8.** *Let $Y$ be, up to a coordinate transformation of the form $x_0q_4(x_1,x_2,x_3,x_4)$.*

1. *If $Y$ belongs to one of the families US1-D - US3-D then $Y$ is unstable and does not represent a closed orbit.*
2. *If $Y$ is of type SS1-D then the orbit is not closed and it degenerates to MO2-VII.*
3. *If $Y$ is of type SS2-D then the orbit is not closed and it degenerates to MO2-VIII.*
4. *If $Y$ is of type SS3-D then the orbit is not closed and it degenerates to MO2-IX.*
5. *If $Y$ is of type SS4-D then the orbit is not closed and it degenerates to MO2-X.*

*Otherwise, $Y$ is a closed orbit.*

## 5.4 Second Level of Minimal Orbits

### 5.4.1 MO2-I

**Semistable Family**

SS1-I $x_0^2\left(x_2x_4^2\right)+x_0x_1\left(x_2x_3x_4\right)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1,1,-1,0,-1\rangle$

SS2-I $x_0x_1\left(x_2x_3x_4\right)+x_1^2\left(x_2x_3^2\right)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1,-1,0,0,0\rangle$

**Unstable Family**

US1-I $x_0^2 x_2 x_4^2$

US2-I $x_1^2 x_2 x_3^2$

**Proposition 5.4.2.** *Let $Y$ be, up to a coordinate transformation, of the form $x_0^2(x_2x_4^2) + x_0x_1(x_2x_3x_4) + x_1^2(x_2x_3^2)$.*

1. *If $Y$ belongs to one of the families US1-I - US2-I then $Y$ is unstable and does not represent a closed orbit.*
2. *If $Y$ is of type SS1-I then the orbit is not closed and it degenerates to MO2-V.*
3. *If $Y$ is of type SS2-I then the orbit is not closed and it degenerates to MO2-V.*

*Otherwise, $Y$ is a closed orbit.*

### 5.4.3 MO2-II

**Semistable Family**

SS1-II $x_0^2\left(x_3x_4^2\right) + x_0x_1\left(x_3^3 + x_2x_3x_4\right)$

(a) *Degeneration* $(MO2 - V) : x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, 1, 0, -1, -1\rangle$

SS2-II $x_0x_1\left(x_3^3 + x_2x_3x_4\right) + x_1^2\left(x_2^2x_3\right)$

(a) *Degeneration* $(MO2 - V) : x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, 0, 0, -1, 0\rangle$

SS3-II $x_0x_1\left(x_2x_3x_4\right) + x_1^2\left(x_2^2x_3\right)$

(a) *Degeneration* $(MO2 - V) : x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, -1, 0, 0, 0\rangle$

SS4-II $x_0^2\Bigg(x_3x_4^2\Bigg) + x_0x_1\Bigg(x_2x_3x_4\Bigg)$

(a) *Degeneration* $(MO2 - V) : x_0x_1x_2x_3x_4$

(b) **Destabilizing 1-PS** $\langle 1, 1, 0, -1, -1\rangle$

SS5-II $x_0x_1\Bigg(x_3^3 + x_2x_3x_4\Bigg)$

(a) *Degeneration* $(MO2 - V) : x_0x_1x_2x_3x_4$

(b) **Destabilizing 1-PS** $\langle 1, -2, 0, 0, 1\rangle$

SS6-II $x_0x_1\Bigg(x_2x_3x_4\Bigg)$

(a) *Degeneration* $(MO2 - V) : x_0x_1x_2x_3x_4$

(b) **1-PS** No 1-PS

**Unstable Family**

US1-II $x_0^2x_3x_4^2 + x_0x_1x_3^2$

US2-II $x_0x_1x_3^2 + x_1^2x_2^2x_3$

US3-II $x_0x_1x_3^2$

US4-II $x_0^2x_3x_4^2$

US5-II $x_1^2x_2^2x_3$

**Proposition 5.4.4.** *Let $Y$ be, up to a coordinate transformation, of the form* $x_0^2(x_3x_4^2) + x_0x_1(x_3^3 + x_2x_3x_4) + x_1^2(x_2^2x_3)$ .

1. *If $Y$ belongs to one of the families US1-II - US5-II then $Y$ is unstable and does not represent a closed orbit.*

2. *If $Y$ is of type SS1-II then the orbit is not closed and it degenerates to MO2-V.*

3. *If $Y$ is of type SS2-II then the orbit is not closed and it degenerates to MO2-V.*

4. *If $Y$ is of type SS3-II then the orbit is not closed and it degenerates to MO2-V.*

5. *If $Y$ is of type SS4-III then the orbit is not closed and it degenerates to MO2-V.*

6. *If $Y$ is of type SS5-II then the orbit is not closed and it degenerates to MO2-V.*

7. *If $Y$ is of type SS6-II then the orbit is not closed and it degenerates to MO2-V.*

*Otherwise, $Y$ is a closed orbit.*

### 5.4.5 MO2-III

**Semistable Family**

SS1-III $x_0^2\left(x_2x_4^2 + x_3^2x_4\right) + x_0x_1\left(x_3^3 + x_2x_3x_4\right)$

(a) *Degeneration* $(MO2 - V) : x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, 3, 3, -4, -3\rangle$

SS2-III $x_0x_1\left(x_3^3 + x_2x_3x_4\right) + x_1^2\left(x_2x_3^2 + x_2^2x_4\right)$

(a) *Degeneration* $(MO2 - V) : x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, -8, 1, 2, 4\rangle$

SS3-III $x_0^2\left(x_2x_4^2 + x_3^2x_4\right) + x_0x_1\left(x_2x_3x_4\right)$

(a) *Degeneration* $(MO2 - V) : x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, 2, 0, 0, -3\rangle$

SS4-III $x_0x_1\left(x_2x_3x_4\right) + x_1^2\left(x_2x_3^2 + x_2^2x_4\right)$

(a) *Degeneration* $(MO2 - V) : x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, -1, 0, 0, 0\rangle$

SS5-III $x_0^2\left(x_3^2x_4\right) + x_0x_1\left(x_3^3 + x_2x_3x_4\right) + x_1^2\left(x_2x_3^2\right)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, 1, 2, -7, 3\rangle$

SS6-III $x_0^2\left(x_3^2x_4\right) + x_0x_1\left(x_2x_3x_4\right) + x_1^2\left(x_2x_3^2\right)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, 0, 0, -2, 1\rangle$

SS7-III $x_0^2\left(x_2x_4^2\right) + x_0x_1\left(x_2x_3x_4\right) + x_1^2\left(x_2^2x_4\right)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, 0, -1, 1, -1\rangle$

SS8-III $x_0^2\left(x_2x_4^2\right) + x_0x_1\left(x_3^3 + x_2x_3x_4\right)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, 1, 1, -1, -2\rangle$

SS9-III $x_0^2\left(x_3^2x_4\right) + x_0x_1\left(x_3^3 + x_2x_3x_4\right)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, 0, 0, -2, 1\rangle$

SS10-III $x_0x_1\left(x_3^3 + x_2x_3x_4\right) + x_1^2\left(x_2^2x_4\right)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, -2, 0, 0, 1\rangle$

SS11-III $x_0x_1\left(x_3^3 + x_2x_3x_4\right) + x_1^2\left(x_2x_3^2\right)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1,-2,0,0,1\rangle$

SS12-III $x_0x_1\Bigg(x_2x_3x_4\Bigg)+x_1^2\Bigg(x_2^2x_4\Bigg)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1,0,0,0,-1\rangle$

SS13-III $x_0x_1\Bigg(x_2x_3x_4\Bigg)+x_1^2\Bigg(x_2x_3^2\Bigg)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1,-1,0,0,0\rangle$

SS14-III $x_0^2\Bigg(x_2x_4^2\Bigg)+x_0x_1\Bigg(x_2x_3x_4\Bigg)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1,1,-1,0,-1\rangle$

SS15-III $x_0^2\Bigg(x_3^2x_4\Bigg)+x_0x_1\Bigg(x_2x_3x_4\Bigg)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1,2,0,0,-3\rangle$

SS16-III $x_0x_1\Bigg(x_3^3+x_2x_3x_4\Bigg)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1,-2,0,0,1\rangle$

SS17-III $x_0x_1\Bigg(x_2x_3x_4\Bigg)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** No 1-PS

**Unstable Family**

US1-III $x_0^2\left(x_2x_4^2+x_3^2x_4\right)+x_0x_1\left(x_3^3\right)$

US2-III $x_0x_1\left(x_3^3+\right)+x_1^2\left(x_2x_3^2+x_2^2x_4\right)$

US3-III $x_0^2\left(x_3^2x_4\right)+x_0x_1\left(x_3^3\right)+x_1^2\left(x_2x_3^2\right)$

US4-III $x_0^2\left(x_2x_4^2\right)+x_0x_1\left(x_3^3\right)+x_1^2\left(x_2^2x_4\right)$

US5-III $x_0^2\left(x_2x_4^2+x_3^2x_4\right)$

US6-III $x_1^2\left(x_2x_3^2+x_2^2x_4\right)$

US7-III $x_0^2\left(x_3^2x_4\right)+x_1^2\left(x_2x_3^2\right)$

US8-III $x_0^2\left(x_2x_4^2\right)+x_1^2\left(x_2^2x_4\right)$

US9-III $x_0^2\left(x_2x_4^2\right)+x_0x_1\left(x_3^3\right)$

US10-III $x_0x_1\left(x_3^3+\right)+x_1^2\left(x_2^2x_4\right)$

US11-III $x_0^2\left(x_3^2x_4\right)+x_0x_1\left(x_3^3\right)$

US12-III $x_0x_1\left(x_3^3+\right)+x_1^2\left(x_2x_3^2\right)$

US13-III $x_0^2\left(x_2x_4^2\right)$

US14-III $x_0^2\left(x_3^2x_4\right)$

US15-III $x_0x_1\left(x_3^3\right)$

US16-III $x_1^2\left(x_2x_3^2\right)$

US17-III $x_1^2\left(x_2^2x_4\right)$

**Proposition 5.4.6.** *Let $Y$ be, up to a coordinate transformation, of the form* $x_0^2(x_2x_4^2 + x_3^2x_4) + x_0x_1(x_3^3 + x_2x_3x_4) + x_1^2(x_2x_3^2 + x_2^2x_4)$ .

1. *If $Y$ belongs to one of the families US1-III - US17-III then $Y$ is unstable and does not represent a closed orbit.*
2. *If $Y$ is of type SS1-III then the orbit is not closed and it degenerates to MO2-V.*
3. *If $Y$ is of type SS2-III then the orbit is not closed and it degenerates to MO2-V.*
4. *If $Y$ is of type SS3-III then the orbit is not closed and it degenerates to MO2-V.*
5. *If $Y$ is of type SS4-III then the orbit is not closed and it degenerates to MO2-V.*
6. *If $Y$ is of type SS5-III then the orbit is not closed and it degenerates to MO2-V.*
7. *If $Y$ is of type SS6-III then the orbit is not closed and it degenerates to MO2-V.*
8. *If $Y$ is of type SS7-III then the orbit is not closed and it degenerates to MO2-V.*
9. *If $Y$ is of type SS8-III then the orbit is not closed and it degenerates to MO2-V.*
10. *If $Y$ is of type SS9-III then the orbit is not closed and it degenerates to MO2-V.*
11. *If $Y$ is of type SS10-III then the orbit is not closed and it degenerates to MO2-V.*
12. *If $Y$ is of type SS11-III then the orbit is not closed and it degenerates to MO2-V.*
13. *If $Y$ is of type SS12-III then the orbit is not closed and it degenerates to MO2-V.*
14. *If $Y$ is of type SS13-III then the orbit is not closed and it degenerates to MO2-V.*
15. *If $Y$ is of type SS14-III then the orbit is not closed and it degenerates to MO2-V.*

16. *If $Y$ is of type SS15-III then the orbit is not closed and it degenerates to MO2-V.*

17. *If $Y$ is of type SS16-III then the orbit is not closed and it degenerates to MO2-V.*

18. *If $Y$ is of type SS17-III then the orbit is not closed and it degenerates to MO2-V.*

*Otherwise, $Y$ is a closed orbit.*

### 5.4.7 MO2-IV

**Semistable Family**

SS1-IV $x_0x_4\left(x_1x_2x_3 + x_1x_3^2\right)$

(a) *Degeneration* $(MO2 - V) : x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, 0, 1, 0, -2\rangle$

**Unstable Family**

US1-IV $x_0x_4x_1x_3^2$

**Proposition 5.4.8.** *Let $Y$ be, up to a coordinate transformation, of the form $x_0x_4(x_1q_2(x_2, x_3))$ .*

1. *If $Y$ belongs to one of the families US1-IV then $Y$ is unstable and does not represent a closed orbit.*

2. *If $Y$ is of type SS1-IV then the orbit is not closed and it degenerates to MO2-V.*

*Otherwise, $Y$ is a closed orbit.*

### 5.4.9 MO2-V

**Semistable Family**

SS1-V $x_0x_4x_2x_3x_1$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** No 1-PS

**Unstable Family**

1. NONE

**Proposition 5.4.10.** *Let $Y$ be, up to a coordinate transformation, of the form $x_0x_4(x_1x_2x_3)$ then it is a closed orbit.*

### 5.4.11 MO2-V

**Semistable Family**

SS1-VI $\left(x_1x_2^3x_3 + x_1^2x_2x_3^2\right) + x_0x_4\left(x_1x_2x_3\right)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, 0, -1, 0, 0\rangle$

SS2-VI $\left(x_1x_2^3x_3\right) + x_0x_4\left(x_1x_2x_3\right)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **Destabilizing 1-PS** $\langle 1, 0, -1, 0, 0\rangle$

SS3-VI $\left(x_1^2x_2x_3^2\right) + x_0x_4\left(x_1x_2x_3\right)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **Destabilizing 1-PS** $\langle 1, 0, -1, 0, 0\rangle$

SS4-VI $x_0x_4\left(x_1x_2x_3\right)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **Destabilizing 1-PS** No 1-PS

**Unstable Family**

US1-VI $\left(x_1x_2^3x_3 + x_1^2x_2x_3^2\right)$

US2-VI $\left(x_1x_2^3x_3\right)$

US3-VI $\left(x_1^2x_2x_3^2\right)$

**Proposition 5.4.12.** *Let* $Y$ *be, up to a coordinate transformation of the form* $(x_1x_2^3x_3 + x_1^2x_2x_3^2) + x_0x_4(x_1x_2x_3)$ *.*

1. *If* $Y$ *belongs to one of the families US1-VI - US3-VI then* $Y$ *is unstable and does not represent a closed orbit.*
2. *If* $Y$ *is of type SS1-VI then the orbit is not closed and it degenerates to MO2-V.*
3. *If* $Y$ *is of type SS2-VI then the orbit is not closed and it degenerates to MO2-V.*
4. *If* $Y$ *is of type SS3-VI then the orbit is not closed and it degenerates to MO2-V.*
5. *If* $Y$ *is of type SS4-VI then the orbit is not closed and it degenerates to MO2-V.*

*Otherwise,* $Y$ *is a closed orbit.*

### 5.4.13 MO2-VII

**Semistable Family**

SS1-VII $x_4x_0^2x_3^2 + x_4x_0x_1x_2x_3$

(a) *Degeneration* $(MO2 - V) : x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, 2, 0, 0, 0\rangle$

SS2-VII $x_4x_0x_1x_2x_3 + x_4x_1^2x_2^2$

(a) *Degeneration* $(MO2 - V) : x_0x_1x_2x_3x_4$

(b) **1-PS** $\langle 1, 0, 0, 0, -1\rangle$

SS3-VII $x_4x_0x_1x_2x_3$

(a) *Degeneration* $(MO2 - V) : x_0x_1x_2x_3x_4$

(b) **Destabilizing 1-PS** No 1-PS

**Unstable Family**

US1-VII $x_4x_1^2x_2^2$

US2-VII $x_4x_0^2x_3^2$

**Proposition 5.4.14.** *Let* $Y$ *be, up to a coordinate transformation, of the form* $x_4x_0^2x_3^2 + x_4x_0x_1x_2x_3 + x_4x_1^2x_2^2$ .

1. *If* $Y$ *belongs to one of the families US1-VII - US2-VII then* $Y$ *is unstable and does not represent a closed orbit.*
2. *If* $Y$ *is of type SS1-VII then the orbit is not closed and it degenerates to MO2-V.*
3. *If* $Y$ *is of type SS2-VII then the orbit is not closed and it degenerates to MO2-V.*
4. *If* $Y$ *is of type SS3-VII then the orbit is not closed and it degenerates to MO2-V.*

*Otherwise,* $Y$ *is a closed orbit.*

### 5.4.15 MO2-VIII

**Semistable Family**

SS1-VIII $x_0x_1\bigg(x_2q_2(x_3, x_4) + q_3(x_3, x_4)\bigg)$

(a) *Degeneration* $(MO2 - IV) : x_0x_1\bigg(x_2q_2(x_3, x_4)\bigg)$

(b) **1-PS** $\langle 1, -1, 2, -1, -1\rangle$

SS2-VIII $x_0x_1\left(q_2(x_2,x_3)x_4+q_1(x_2,x_3)x_4^2+x_4^3\right)$

(a) *Degeneration* $(MO2-IV): x_0x_1\left(q_2(x_2,x_3)x_4\right)$

(b) **Destabilizing 1-PS** $\langle 1,-1,1,1,-2\rangle$

**Unstable Family**

US1-VIII $x_0x_1\left(q_3(x_3,x_4)+x_2x_4^2\right)$

**Proposition 5.4.16.** *Let $Y$ be, up to a coordinate transformation of the form $x_0x_1q_3(x_2,x_3,x_4)$ .*

1. *If $Y$ belongs to one of the families US1-VIII then $Y$ is unstable and does not represent a closed orbit.*

2. *If $Y$ is of type SS1-VIII then the orbit is not closed and it degenerates to MO2-IV.*

3. *If $Y$ is of type SS2-VIII then the orbit is not closed and it degenerates to MO2-IV.*

*Otherwise, $Y$ is a closed orbit.*

### 5.4.17 MO2-IX

**Semistable Family**

SS1-IX $x_0\left(x_1x_2+x_2^2 \parallel x_3x_4+x_4^2\right)$

(a) *Degeneration* $(MO2-V): x_0x_1x_2x_3x_4$

(b) **Destabilizing 1-PS** $\langle 0,1,-1,1,-1\rangle$

**Unstable Family**

US1-IX $x_0\left(x_2^2 \parallel x_3x_4 + x_4^2\right)$

US2-IX $x_0\left(x_1x_2 + x_2^2 \parallel x_4^2\right)$

US3-IX $x_0\left(x_2^2 \parallel x_4^2\right)$

**Proposition 5.4.18.** *Let $Y$ be, up to a coordinate transformation of the form $x_0q_{2,2}(x_1, x_2 \parallel x_3, x_4)$.*

1. *If $Y$ belongs to one of the families US1-IX - US3-IX then $Y$ is unstable and does not represent a closed orbit.*
2. *If $Y$ is of type SS1-IX then the orbit is not closed and it degenerates to MO2-V.*

*Otherwise, $Y$ is a closed orbit.*

### 5.4.19 MO2-X

**Semistable Family**

SS1-X $x_0\left(q_4(x_2, x_3) + x_1\left(x_2x_3 + x_3^2\right)x_4 + x_1^2x_4^2\right)$

(a) *Degeneration* $(MO2 - V) : x_0x_1\left(x_2x_3x_4\right)$

(b) **Destabilizing 1-PS** $\langle a, b, 1, -1, c\rangle$

SS2-X $x_0\left(q_4(x_2, x_3) + x_1\left(x_2x_3 + x_2^2\right)x_4 + x_1^2x_4^2\right)$

(a) *Degeneration* $(MO2 - V) : x_0x_1\left(x_2x_3x_4\right)$

(b) **Destabilizing 1-PS** $\langle a, b, 1, -1, c\rangle$

**Unstable Family**

US1-X $x_0\left(x_2^2x_3^2 + x_2x_3^3 + x_3^4 + x_1x_3^2x_4 + x_1^2x_4^2\right)$

US2-X $x_0\left(x_2x_3^3 + x_3^4 + x_1(x_2x_3 + x_3^2)x_4 + x_1^2x_4^2\right)$

**Proposition 5.4.20.** *Let $Y$ be, up to a coordinate transformation of the form $x_0(q_4(x_2, x_3) + x_1q_2(x_2, x_3)x_4 + x_1^2x_4^2)$ .*

1. *If $Y$ belongs to one of the families US1-X - US2-X then $Y$ is unstable and does not represent a closed orbit.*

2. *If $Y$ is of type SS1-X then the orbit is not closed and it degenerates to MO2-V.*

3. *If $Y$ is of type SS2-X then the orbit is not closed and it degenerates to MO2-V.*

*Otherwise, $Y$ is a closed orbit.*

The most degenerate point in the GIT compactification is the normal crossing singularities hypersurface $x_0x_1x_2x_3x_4$. The following flow chart gives a pictorial description of the various degenerations which have been explained in this chapter. It also shows that the most degenerate point of this compactification is the normal crossing hypersurface.

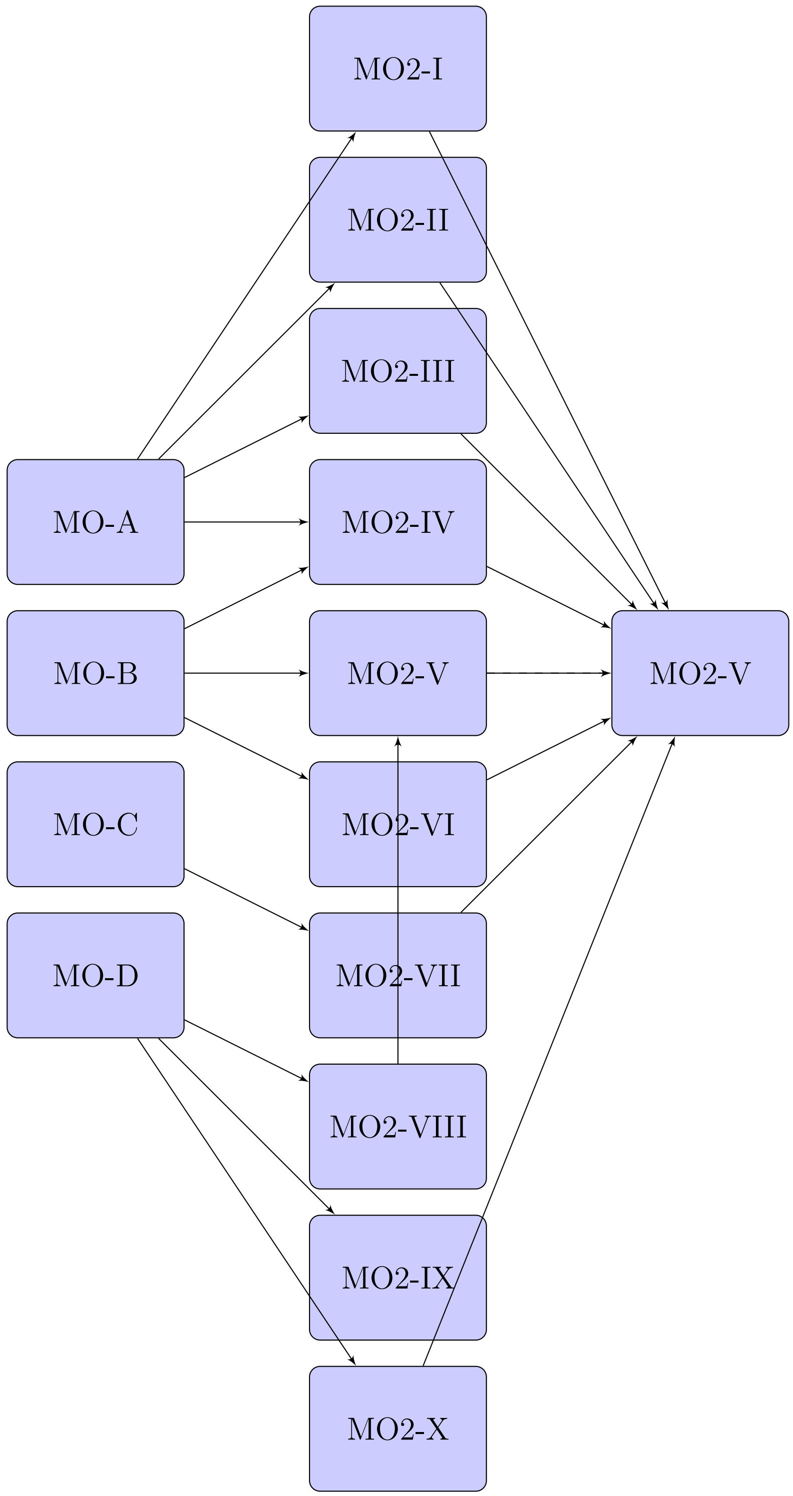


Figure 5.2: Boundary Stratification of the Moduli Space

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# Appendix

# Appendix A

# Code for Poset Structure

```
>  withposets();
>  poset := proc (n) local P, z1, z2, z3, z4, z5, z6, x0, x1, x2,
x3, x4, x5, y1, y2, y3, y4, y5, y6;
>  P := NULL;
>  for z1 from 0 to n do
>  for z2 from 0 to n do
>  for z3 from 0 to n do
>  for z4 from 0 to n do
>  for z5 from 0 to n do
>  for y1 from 0 to n do
>  for y2 from 0 to n do
>  for y3 from 0 to n do
>  for y4 from 0 to n do
>  for y5 from 0 to n do
```

```
>  if z1+z2+z3+z4+z5 = n and y1+y2+y3+y4+y5 = n and z1 <= y1 and
z1+z2 <= y1+y2 and z1+z2+z3 <= y1+y2+y3 and z1+z2+z3+z4 <=
y1+y2+y3+y4 and x0^z1*x1^z2*x2^z3*x3^z4*x4^z5 <>
x0^y1*x1^y2*x2^y3*x3^y4*x4^y5 then
>  P := P, [[z1, z2, z3, z4, z5], [y1, y2, y3, y4, y5]]
>  else P := P
>  end if
>  end do
>  end do
>  end do
>  end do
>  end do
>  end do
>  end do
>  end do
>  end do
>  end do;
>  P := covers({P})
>  end;
```

# Appendix B

# Sample Linear Programming Calculation

```
> with(Optimization);
> with(LinearAlgebra);
> with(VectorCalculus);
> with(ListTools);
> v1:=[a,b,c,d,e];
> v2:=[4,1,0,0,0];
> mon:=DotProduct(v1,v2);
> constraints:={mon<=0,a+b+c+d+e=0,a>=1,a>=b,b>=c,c>=d,d>=e};
> LPSolve(1,constraints,assume=integer);
```