ABSTRACT

GROVER, AMANDEEP. Time Optimal and Minimum Effort Control of Time-invariant Systems. (Under the direction of Prof. Kazufumi Ito.)

In this thesis, optimal control problems for linear time-invariant systems are studied. Specifically, the minimum norm problem, time optimal control problem and minimum effort problem are studied and solutions to these problems are described. These problems have applications in many areas, for example, quantum systems, electric circuits, fuel regulators, tumor control therapy, spacecraft control and equitable resource allocation.

To solve the minimum norm problem, we use the Lagrange multiplier approach to obtain the necessary optimality system which is then solved using the control grammian to find the optimal L^2 minimum norm solution. We also derive the optimality system for the time optimal and minimum effort problems in terms of L^1 and L^∞ norms. The optimality systems obtained are nonlinear and non-differentiable, which results in bang-bang type optimal control. To overcome this difficulty, a regularized problem is formulated, which is then discretized. Then the semi-smooth Newton method is applied to solve the semi-smooth optimality system. The initialization of the Newton method is achieved by solving the L^2 minimum norm problem. Then we use the continuation method based on sensitivity analysis to solve the problem with varying parameters.

We also study mixed problems of the above three types. The necessary optimality and numerical simulation procedure is extended naturally to solve these problems.

Finally, algorithmic procedure to solve the above mentioned problems is described, and corresponding numerical results are presented.

Time Optimal and Minimum Effort Control of Time-invariant Systems

by
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DEDICATION

To my family for their incessant love and support

BIOGRAPHY

The author was born and brought up in the north western part of India. She then went to south India for her undergraduate education in Mechanical Engineering from the Indian Institute of Technology Madras. She also obtained a Masters degree there in the same department. To diversify her education, she then pursued a Master of Science in Operations Research at the North Carolina State University.

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Chapter 1

Introduction

In this thesis, we study three types of optimal control problems for linear time invariant systems. Optimal control deals with the problem of finding a control law for a given system such that certain optimality criteria are achieved. The following optimal control problem (P) is considered.

$$\min \int_0^T f^0(t, u(t), x(t))dt + h(x(T)) \tag{1.1}$$

subject to

$$\frac{dx}{dt} = f(t, u(t), x(t)),\ x(0) = x_0,\ t \in [0, T], \tag{1.2}$$

target state constraint $\Phi(x(T)) = 0$ and control constraint $u(t) \in U_{ad}$, $U_{ad} = \{u(t) \in U\}$, where $U \subset \Re$. Here the state variable $x(t) \in \Re^n$, running cost $f^0 : \Re \times \Re^n \to \Re$, terminal cost $h : \Re^n \to \Re$, dynamic constraint $f : \Re \times \Re^n \to \Re$ and $\Phi : \Re^n \to \Re^m$. We assume that functions f, g, h and Φ are continuously differentiable functions.

We here consider two cases, the first with a fixed time horizon, that is, the final time $T > 0$ is fixed. In the second case, the final time $T = \tau > 0$ is unknown and the cost functional is minimized over $(u \in U_{ad}, \tau > 0)$. The time optimal problem, described in the next section, belongs to the second case.

The control problem (P) has applications in many areas, for example, quantum systems, electric circuits, fuel regulators, tumor control therapy, spacecraft control and equitable resource allocation. Some of the applications of this problem are described in [2], [4] and [5]. We give some examples of the applications in the next section as we describe three particular types of control problems that we study in this thesis.

1.1 Overview of minimum norm, time optimal and minimum effort control problems

We specifically attempt to solve three types of control problems, the minimum norm control problem, time optimal control problem and minimum effort control problem.

1.1.1 L^2-minimum norm problem

The minimum norm problem is of importance in a variety of fields, including mimimizing the fuel consumption in aircraft engines and chemical reactors.

The L^2 minimum norm problem involves minimizing the cost functional

$$\int_0^T \frac{1}{2}|u(t)|^2 dt,$$

over a fixed time horizon T, subject to

$$\frac{dx(t)}{dt} = f(u(t), x(t)),\ t \in [0, T],$$

with initial state constraint $x(0) = x_0$ and target state constraint $\Phi(x(T)) = 0$.

Because of the importance of minimum norm problems, they have been studied extensively in [3] and [13]. Algorithms based on results from linear programming in [3] and iterative procedures based on the steepest descent method for constrained optimization problems in [13] have been proposed to solve these problems. A solution is proposed in [8] using penalty functions and the gradient method.

We use the Lagrange multiplier approach to obtain the necessary optimality system which is linear and solved by finding the control grammian.

1.1.2 Time optimal control problem

Due to their practical relevance and inherent structural difficulties, time optimal control problem has received considerable amount of attention for decades. This type of problem arises in a wide variety of fields, such as engineering and material sciences. To name some of these problems in application, they are time optimal control on the Kepler equation for the orbital transfer of satellite, time optimal control on quantum spin system, and time optimal control of flexible space structures.

The time optimal control problem is to choose a control in such a way that a dynamical system reaches a target state or manifold from a given initial state in minimum time. The problem can be stated as

$$\begin{aligned} \min \tau &= \int_0^\tau dt \\ \text{subject to } \frac{dx(t)}{dt} &= f(u(t), x(t)),\ t \in [0, \tau], \end{aligned} \tag{1.3}$$

with initial state constraint $x(0) = x_0$, target state constraint $\Phi(x(\tau)) = 0$ and control constraint $|u(t)| \leq \gamma$.

The time optimal control problem was first introduced by Krasovskii, who derived a variational formulation but didn't give any computational methods to numerically solve the problem. Traditional approaches for solving time optimal problems are mainly grouped into direct and indirect methods. Indirect methods are based on multiple shooting techniques to solve the two point boundary value problem. Equipped with a good initial guess for all unknowns, particularly the number and location of the switchings, the shooting method is reported to converge fast and to generate very accurate solutions. Direct methods, on the other hand, consider time optimal problems as genuine nonlinear programming problems. They are used in several variants, which frequently involve reparametrization of the controls as the unknowns. The works [12] and [14] fall in the first category, whereas [16] and [11] fall in the latter. Much of the literature for time optimal control problem up to the late sixties is covered in [7]. Many recent results have been documented in [15].

In our method, the control function in the necessary optimality system is of the bang-bang form (for proof, refer [6]) and hence non-differentiable. To overcome this difficulty, a regularized problem is formulated and the semi-smooth Newton method is applied to solve the associated optimality system. Shooting method is used to implement the semi-smooth Newton algorithm efficiently. The associated L^2 minimum norm problem is solved to give a good initial estimate of the shooting parameters, and continuation method based on sensitivity analysis is applied to improve this guess.

1.1.3 L^∞-minimum effort control problem

This section of control problems also finds wide applications, one example being minimizing the current range in an electrical circuit.

The L^∞ minimum effort problem addresses the question about how small the control effort can be made in order to drive the system from a given state to a target state within a fixed time T. The problem can be stated as

$$\begin{aligned} &\min \sum_{j=1}^{m} \frac{1}{2}|\gamma_j|^2 \\ \text{subject to } &\frac{dx(t)}{dt} = f(u(t), x(t)),\ t \in [0, T], \end{aligned} \tag{1.4}$$

with initial state constraint $x(0) = x_0$, target state constraint $\Phi(x(T)) = 0$ and control constraint $|u_j(t)| \leq \gamma_j$, where the control vector $u = \{u_j\}$ for $j = 1, ..., m$.

The minimum effort control problem was introduced in [16] and solved in [17] by reducing it to n-dimensional parameter optimization problem. In [1], it was attempted to solve this problem by reducing it to another parameter optimization problem. An important application of this problem, the slewing of flexible spacecraft, is also described and solved numerically.

In our method, similar to the time optimal case, a regularized problem is formulated and semi-smooth Newton method is used to solve the optimality system.

The solution to the three control problems stated above is described and tested for two dimensional linear systems in the later chapters.

1.2 Thesis Organization

This thesis is organized as follows. In Chapter 2, we start with sketching the solution to the control problem (P). Then, time optimal problem is discussed in detail and a regularized problem is formulated, which we attempt to solve.

In Chapter 3, we introduce the semi-smooth Newton method used in solving the time optimal control problem and describe the procedure to implement it. Also, we describe the solution to the minimum norm problem which is solved to initialize the semi-smooth Newton approach for the time optimal problem.

In Chapter 4, we write our proposed solution step by step in the form of an algorithm.

In Chapter 5, we discuss the numerics, that is, the results obtained by computationally solving different cases of the control problem.

Chapter 6 contains some concluding remarks and indicates some directions for future work.

Chapter 2

Necessary Optimality System

The critical point in solving the control problem (P) is to derive the optimality conditions for x, u and τ. In order to derive the necessary optimality system, we introduce the adjoint variable $p(t) \in \Re^n$, which can be interpreted as the Lagrange multiplier associated with the dynamic constraint (1.2). The Hamiltonian associated with (P) is given by

$$\mathcal{H}(t, x, u, p) = f^0(t, u, x) + p^T f(t, u, x). \tag{2.1}$$

Let (x^*, u^*) be the optimal pair for problem (P) for the case where the terminal time T is fixed and (x^*, u^*, τ^*) be the optimal triplet for the case where $T = \tau$ is a free variable. Let p^* be the corresponding adjoint state. The first order necessary optimality conditions are

$$\frac{dx^*(t)}{dt} = \frac{\partial \mathcal{H}}{\partial p}(t, x^*, u^*, p^*);\ x^*(0) = x_0 \tag{2.2}$$

$$-\frac{dp^*(t)}{dt} = \frac{\partial \mathcal{H}}{\partial x}(t, x^*, u^*, p^*), \tag{2.3}$$

$$u^*(t) = argmin_{u \in U} \mathcal{H}(t, x^*(t), u, p^*(t)), \tag{2.4}$$

$$\Phi(x^*(T)) = 0, \tag{2.5}$$

and transversality condition

$$\mathcal{H}(t, x^*, u^*, p^*) = 0 \, \forall \, t \in [0, T]. \tag{2.6}$$

Another approach to solve the control problem (P) is based on the Hamilton-Jacobi Bellman equation given as

$$\frac{\partial V}{\partial t}(x, t) + min_{u \in U}(\nabla V(x, t) \cdot f(t, u, x) + f^0(t, u, x)) = 0, \tag{2.7}$$

i.e., the optimal feedback law is given by

$$u^*(t) = argmin_{u \in U}(\nabla V(x^*(t), t) \cdot f(t, u, x^*(t)) + f^0(t, u, x^*(t))), \tag{2.8}$$

subject to the terminal condition

$$V(x, T) = h(x). \tag{2.9}$$

It can be shown that $V(x, t)$ is the optimal value function for the cost incurred from starting in state x at time t and controlling the system from then until time T.

2.1 The time optimal control problem and its regularization

We here first describe how to obtain the necessary optimality system for the linear time optimal problem and then regularize it (refer [9]) to obtain the smoother optimality system. The same can be applied to a bilinear system also. Consider the time-optimal control problem for the linear multi-input system.

$$min_{\tau \geq 0} \int_0^\tau dt$$

subject to

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t),$$
$$x(0) = x_0,\ x(\tau) = x_f,$$
$$|u(t)| \leq \gamma, \tag{2.10}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $x_0 \in \Re^n$ and $x_f \in \Re^n$ are known. It is assumed that the system is controllable, that is, A and B satisfy the Kalman-Rank condition for controllability given as

$$\text{rank}\,[B, AB, A^2B, ..., A^{n-1}B] = n \tag{2.11}$$

So, (P) has a solution with the optimal time denoted by τ^*, associated state trajectory x^* and control u^*.

The Hamiltonian for the linear control problem is given by

$$\mathcal{H}(x, u, p) = 1 + p^T(Ax + Bu). \tag{2.12}$$

The first order optimality system for (P) can be obtained as

$$x' = Ax + Bu,\ x(0) = x_0, \tag{2.13}$$

$$-p' = A^T p, \tag{2.14}$$

$$u(t) = -sign(B^T p(t)), \tag{2.15}$$

$$x(\tau) = x_f, \tag{2.16}$$

$$1 + p(\tau)^T(Ax(\tau) + Bu(\tau)) = 0, \tag{2.17}$$

where

$$sign(s) = \begin{cases} [-\gamma, \gamma] & if\ s = 0 \\ \gamma \frac{s}{|s|} & otherwise. \end{cases} \tag{2.18}$$

We note that if $B^T p(t)$ is zero in an interval, then $u(t)$ is undetermined in that interval. So, the formula (2.15) may not necessarily determine $u(t)$ for the complete interval $[0, \tau]$. The optimal control given by (2.15) is in the bang-bang form. We use the Newton method to solve the system (2.13)-(2.17) for x, u, τ, p and μ. But (2.15) is non-differentiable. To overcome this difficulty, we introduce the regularized problem (P_ϵ), based on the penalty approach given in [10] as follows.

$$min_{\tau \geq 0} \int_0^\tau (1 + \frac{\varepsilon}{2}|u(s)|^2) ds$$

subject to

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t), \\ x(0) = x_0,\ x(\tau) &= x_f, \\ |u(t)| &\leq \gamma. \end{aligned} \tag{2.19}$$

The cost functional converges to the minimum time τ as $\varepsilon \to 0^+$. Hamiltonian for (P_ε) is given by

$$\mathcal{H}(x, u, p) = 1 + \frac{\varepsilon}{2}|u|^2 + p^T(Ax + uB). \tag{2.20}$$

We again minimize this Hamiltonian function with respect to the control variable to obtain

$$u(t) = -sign_\varepsilon(B^T p(t)), \tag{2.21}$$

where

$$sign_\varepsilon(s) = \begin{cases} \gamma \frac{s}{|s|} & if\ |s| \geq 1 \\ 0 & if\ |s| \leq 1-\varepsilon \\ \frac{\gamma}{\varepsilon}\frac{s}{|s|}(|s| - (1-\varepsilon)) & otherwise. \end{cases} \tag{2.22}$$

We see that $sign_\varepsilon(s)$ is a uniquely defined single valued function, so now $u(t)$ can be determined for the complete interval $[0, \tau]$. Since $sign_\varepsilon(s)$ is a piecewise continuously differentiable function, $sign_\varepsilon(s)$ belongs to the category of semi-smooth functions.

Hence, the necessary optimality system for (P_ε) is obtained as

$$\begin{cases} x' = Ax + Bu,\ x(0) = x_0, \\ -p' = A^T p \\ u(t) = -sign_\varepsilon(B^T p(t)) \\ x(\tau) = x_f \\ 1 + \frac{\varepsilon}{2}|u(\tau)|^2 + p(\tau)^T(Ax(\tau) + Bu(\tau)) = 0, \end{cases} \tag{2.23}$$

where $'$ implies derivative with respect to t; $t \in [0, \tau]$.

Since $p(t)$ is continuous, the optimal control as defined for the regularized problem is also continuous.

2.2 Coordinate transformation for time

Since τ appears repeatedly as an implicit variable in (2.23), we use the coordinate transformation $\hat{t} = t/\tau$ in (P_ε). The constraint equations for (P_ε) thus reduce to

$$\frac{dx(\hat{t})}{d\hat{t}} = \tau\, f(\hat{t}, u(\hat{t}), x(\hat{t})),\ \hat{t} \in [0,1],$$
$$x(0) = x_0,\ x(1) = x_f, \tag{2.24}$$

and the cost functional becomes

$$\tau \int_0^1 f^0(\hat{t}, u(\hat{t}), x(\hat{t}))d\hat{t}. \tag{2.25}$$

So, now τ appears as an explicit variable. It can be shown that the necessary optimality system for the transformed problem $(\overline{P})$ is given by the following set of equations

$$x' = \tau(Ax + Bu),\ x(0) = x_0, \tag{2.26}$$

$$-p' = \tau A^T p,\ p(1) = \mu \tag{2.27}$$

$$u(\hat{t}) = -sign_\varepsilon(B^T p(\hat{t})) \tag{2.28}$$

$$x(1) = x_f \tag{2.29}$$

$$1 + \frac{\varepsilon}{2}|u(1)|^2 + \mu^T(Ax(1) + Bu(1)) = 0, \tag{2.30}$$

where $'$ implies derivative with respect to $\hat{t}$, $\hat{t} \in [0, 1]$, and τ appears explicitly.

The necessary optimality system hence obtained is a two point boundary value problem in x, u, τ, p and μ, which is solved using iterative methods.

2.3 Discretization of control dynamics and the running cost

We here discretize the transformed time optimal problem $(\overline{P})$. To discretize x and u in time, we divide the time interval $[0, 1]$ into n parts with equal step size dt. The state can be approximated by applying the Crank-Nicholson scheme for the dynamic constraint as

$$\frac{x_{k+1} - x_k}{dt} = \tau(A\,\frac{x_{k+1} + x_k}{2} + Bu_k); x_0 = x(0), \tag{2.31}$$

and running cost can be approximated by the mid point rule as

$$\tau \sum_{k=1}^{n} f^0(u_k, \frac{x_k + x_{k+1}}{2})dt + h(x_n), \tag{2.32}$$

where u_k is computed at the middle point of each time interval $[t_k, t_{k+1}]$ and $k = 0, 1.....n$. So, our objective is to find $u_k \in U_{ad}$ that minimizes (2.32) subject to (2.31) and the target constraint $\Phi(x_n) = 0$.

It can be shown that the equations corresponding to (2.27) - (2.30) in the optimality system for the discretized problem are obtained as

$$-\frac{p_{k+1} - p_k}{dt} = \tau A^T \frac{p_{k+1} + p_k}{2}; p_n = \mu, \tag{2.33}$$

$$u_k = -sign_\varepsilon(B^T \frac{p_{k+1} + p_k}{2}), \tag{2.34}$$

$$x_n = x_f, \tag{2.35}$$

$$1 + \frac{\varepsilon}{2}|u_n|^2 + \mu^T(Ax_n + Bu_n) = 0. \tag{2.36}$$

We will use the semismooth Newton method to solve the system (2.31), (2.33) - (2.36)for x_k, u_k, τ, p_k and μ, for $k = 1, 2, ..., n$, as described in the next chapter.

Chapter 3

Semi-smooth Newton algorithm and Sensitivity analysis

In this chapter, we use the semi-smooth Newton method to solve the necessary optimality system for the regularized time optimal control problem P_ε. An introduction to the semi-smooth Newton method is given in Appendix A.

3.1 Semi-smooth Newton Algorithm

In this section, we apply the semi-smooth Newton method to the regularized time optimal problem. Let us define the function

$$F : D_F \subset X \to L^2(0,1;\Re^n) \times U_{ad} \times \Re \times L^2(0,1;\Re^n) \times \Re^n,$$

where

$$D_F = W^{1,2}(0,1) \times L^2(0,1) \times \Re \times W^{1,2}(0,1) \times \Re^n, \tag{3.1}$$

and

$$F(x,u,\tau,p,\mu) = \begin{pmatrix} x' - \tau Ax - \tau Bu \\ -p' - \tau A^T p \\ u + sign_\varepsilon(B^T p) \\ x(1) - x_f \\ 1 + \frac{\varepsilon}{2}|u(1)|^2 + \mu^T(Ax(1) + Bu(1)) \end{pmatrix}. \tag{3.2}$$

That is, $F(x,u,\tau,p,\mu) = 0$ is the necessary optimality system for $(\overline{P})$. To solve it, we use a generalized derivative for $sign_\varepsilon(s)$, which is given by

$$G_\varepsilon(s) := \begin{cases} \gamma/\varepsilon & if\ (1-\varepsilon) < |s| < 1 \\ 0 & otherwise \end{cases}. \tag{3.3}$$

The semi-smooth Newton iteration step is given by

$$DF(x,u,\tau,p,\mu)(\delta x, \delta u, \delta\tau, \delta p, \delta\mu) = -F(x,u,\tau,p,\mu), \tag{3.4}$$

where DF is the generalized Jacobian. (3.4) is equivalent to

$$\begin{cases} \frac{d}{dt}\delta x - \tau A\delta x - \tau B\delta u - \delta\tau(Ax + Bu) = -F_1,\ \delta x(0) = \delta x_0 \\ \frac{d}{dt}\delta p - \tau A^T \delta p - \delta\tau A^T p = -F_2,\ \delta p(1) = \delta\mu \\ \delta u + G_\varepsilon(B^T p)B^T \delta p = -F_3 \\ \delta x(1) = -F_4 \\ \mu^T(A\delta x(1) + B\delta u(1)) + (Ax(1) + Bu(1))^T \delta\mu + \varepsilon u(1)^T \delta u(1) = -F_5 \end{cases}, \tag{3.5}$$

where $F = (F_1; F_2; F_3; F_4; F_5)$ are given by (3.5). (3.5) is a semi-smooth Newton step, but we reduce this step to finite rank update by the following deductions.

Given τ and μ, $p(t)$ is determined by solving the adjoint equation (2.33). Then we evaluate the control $u(t)$ by the formula (2.34), and then $x(t)$ is determined by the state equation (2.31).

In summary, equation $F(x, u, \tau, p, \mu) = 0$ can be reduced to $\varphi(\tau, \mu) = 0$, where

$$\varphi(\tau, \mu) = \begin{pmatrix} x(1) - x_f \\ 1 + \frac{\varepsilon}{2}|u(1)|^2 + \mu^T(Ax(1) + Bu(1)) \end{pmatrix}. \tag{3.6}$$

Again, $x(t)$, $u(t)$ and $p(t)$ are functions of τ and μ. The Newton step for solving the reduced system $\varphi(\tau, \mu) = 0$ is given

$$D\varphi(\tau, \mu)(\delta\tau, \delta\mu) = -\varphi(\tau, \mu), \tag{3.7}$$

which can be written as

$$\begin{pmatrix} \delta x(1) = -\varphi_1 \\ \mu^T(A\delta x(1) + B\delta u(1)) + (Ax(1) + Bu(1))^T\delta\mu + \varepsilon u(1)^T\delta u(1) = -\varphi_2 \end{pmatrix} \tag{3.8}$$

where $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ are given by (3.8), and δx and δu are evaluated in terms of $\delta\tau$ and $\delta\mu$ using (3.5).

Similarly, we consider the discretized system (2.31), (2.33) - (2.36), i.e., we use x_k, u_k and p_k determined from the system of equations (2.31), (2.33) - (2.36), instead of $x(t)$, $u(t)$ and $p(t)$. For the discretized system, (3.7) uses δx_k, δu_k and δp_k, determined by

$$\frac{\delta x_{k+1} - \delta x_k}{dt} = \tau A\frac{\delta x_{k+1} + \delta x_k}{2} + (A\frac{x_{k+1} + x_k}{2} + Bu_k)\delta\tau + \tau B\delta u_k + f_k,\ \delta x_0 = \delta x(0) = 0, \tag{3.9}$$

$$\delta u_k = -G_\epsilon\,(B^T\,\frac{p_{k+1} + p_k}{2})\,B^T\,\frac{\delta p_{k+1} + \delta p_k}{2}\ +\ h_k, \tag{3.10}$$

$$-\frac{\delta p_{k+1} - \delta p_k}{dt} = \tau A^T\,\frac{\delta p_{k+1} + \delta p_k}{2} + A^T\,\frac{p_{k+1} + p_k}{2}\,\delta\tau + g_k,\ \delta p_n = \delta\mu, \tag{3.11}$$

where f_k, g_k and h_k correspond to F_1, F_2 and F_3 in (3.5) for the continuous system.

We describe how to get a good initial guess for μ and τ in the next section.

3.2 Initialization

In order to have convergence for the semi-smooth Newton method, we should have a good initial guess of μ in the algorithm. For a fixed time horizon T, we can find the exact adjoint $\hat{p}(1) = \hat{\mu}$ for the L^2 minimum norm problem. Using the approach described in Chapter 2, the necessary optimality system for the L^2 minimum norm problem is obtained as

$$\begin{aligned} \frac{dx}{dt} &= T(Ax + Bu)\, x(0) = x_0, \\ -\frac{dp}{dt} &= TA^T p,\ p(1) = \hat{\mu}, \\ \hat{u} &= -B^T p, \\ x(1) &= x_f. \end{aligned} \tag{3.12}$$

Thus, $\hat{\mu}$ is given by

$$\exp(At)x_0 + \mathcal{L}\mathcal{L}^* \hat{\mu} = 0, \tag{3.13}$$

where

$$\mathcal{L}^* \hat{\mu} = B^T \exp(A^T t) \hat{\mu}, \tag{3.14}$$

and

$$\hat{\mu} = -(\mathcal{L}\mathcal{L}^*)^{-1} \exp(At) x_0, \tag{3.15}$$

with

$$\mathcal{L}\mathcal{L}^* = \int_0^1 \exp(A^T t) B B^T \exp(At) dt \in \Re^{n \times n}. \tag{3.16}$$

Note that $\mathcal{L}\mathcal{L}^*$ is a symmetric Then, using (2.30), we scale the vector $\hat{\mu}$ by a factor of

$$\beta = \frac{1}{|\hat{\mu}^T (Ax(1) + B\hat{u}(1))|}, \tag{3.17}$$

such that

$$\mu = \beta \hat{\mu}. \tag{3.18}$$

Hence, we are able to have a good initial guess of μ for the L^1 minimum norm problem, time optimal problem and minimum effort problems. For the time optimal control problem, a good initialization of τ can be found by chosing a value of $\tau = T$ such that $sup\ |\hat{u}(t)|_{t\in[0,1]}$ is close to γ.

3.3 Continuation method using sensitivity analysis

In this section, we discuss the continuation method for solving $\varphi(\mu, v) = 0$ for the L^1 norm minimization, where v represents one or more of the varying parameters, x_0, x_f, τ and γ.

First, given v, we solve $\varphi(\mu, v) = 0$ as explained above. Then, by implicit function theory, $\delta\mu$ satisfies

$$\frac{\partial\varphi}{\partial\mu}\delta\mu + \frac{\partial\varphi}{\partial v}\delta v = 0, \tag{3.19}$$

that is,

$$J\delta\mu + \frac{\partial\varphi}{\partial v}\delta v = 0, \tag{3.20}$$

where J is the Jacobian of $\varphi(\mu, v)$ at the solution (μ, v) of $\varphi(\mu, v) = 0$ at given parameter set v. In (3.20), $\partial\varphi/\partial v$ is computed by using the variational system (3.5). Then we use the update $\mu^+ = \mu + \delta\mu$ as an initial guess for the problem $\varphi(\mu, v + \delta v) = 0$.

Using a similar procedure, we can evaluate the update for τ for the time optimal case using

$$J \begin{pmatrix} \delta\tau \\ \delta\mu \end{pmatrix} + \frac{\partial\varphi}{\partial v}\delta v = 0. \tag{3.21}$$

Assuming J is invertible, the update for (τ, μ) is given by

$$\begin{aligned} \tau^+ &= \tau + \delta\tau, \\ \mu^+ &= \mu + \delta\mu. \end{aligned} \tag{3.22}$$

In summary, we first obtain a good initial guess for the unknown parameters, and then using the variational system, we obtain the Jacobian for the discretized system. The Jacobian obtained is then used for the sensitivity analysis so as to get a new initial guess when the known parameters are varied. With this initial condition, the semi-smooth Newton method converges provided $|\delta v|$ is sufficiently small.

Chapter 4

Algorithm and implementation

In Chapter 2, we obtained the discretized system (2.31), (2.33) - (2.36) for the regularized time optimal problem P_ε. In this Chapter, we give the algorithm to solve this system using the procedure described in the previous chapter, as follows.

1. Set $k = 0$ and select an initial guess of $\tau^{(0)}$ and $\mu^{(0)}$;

2. Compute the adjoint variable p_k backwards from $k = n$ to $k = 1$ using (2.33), and compute u_k simultaneously using (2.34);

3. Compute the state x_k forwards in time from $k = 1$ to $k = n$ using (2.31) and record the terminal state $x(1)$;

4. Solve (3.7) for the update

$$\delta z^{(k)} = (\delta\tau^{(k)}, \delta\mu^{(k)}), \tag{4.1}$$

   using (3.20);

5. Determine the damping factor $\alpha^{(k)} \in (0; 1]$ numerically so as to minimize

$$||F(z^{(k)} + \alpha^{(k)}\delta z^{(k)}||_2; \tag{4.2}$$

6. Update

$$\tau^{(k+1)} = \tau^{(k)} + \alpha^{(k)}\delta\tau^{(k)}$$
$$\mu^{(k+1)} = \mu^{(k)} + \alpha^{(k)}\delta\mu^{(k)}; \tag{4.3}$$

7. Stop if $||\varphi(\tau^{(k)}, \mu^{(k)})|| \leq \delta$ for very small $\delta > 0$, or set $k = k + 1$ and return to step 2.

In our test examples, step 5 is substituted by setting α as 0.1, 0.25 or 1 for different cases. The implementation of this algorithm in Matlab is explained in Appendix B.

Chapter 5

Numerical Results

In this chapter, we present numerical results for two dimensional test problems. We describe our numerical method and result for L^1 minimum norm problem, i.e., where the terminal time T and γ are fixed, time optimal problem (refer Section 1.1.2), i.e., where γ is fixed and τ is to be determined, and then the L^∞ minimum effort problem (refer Section 1.1.3), i.e., where terminal time T is fixed and γ is to be determined. We begin by solving the L^2 minimum norm problem (refer Section 1.1.1) to obtain an initial guess for the L^1 minimum norm problem, time optimal problem and the minimum effort problem (refer Section 3.2). We use the continuation method (refer Section 3.3) to solve the family of control problems for different values of terminal time, γ, x_0 and x_f.

5.1 L^2 minimum norm problem

We solve the problem

$$min \int_0^{12} \frac{1}{2}|u(t)|^2 dt$$

subject to

$$\frac{dx(t)}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t);\ t \in [0, 12],$$

$$x(0) = \begin{pmatrix} -2 \\ 0 \end{pmatrix},\ x(12) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5.1}$$

to obtain an initial guess for the time optimal and minimum effort problems.

The optimal control obtained and the phase diagram are shown in Fig. 5.1.

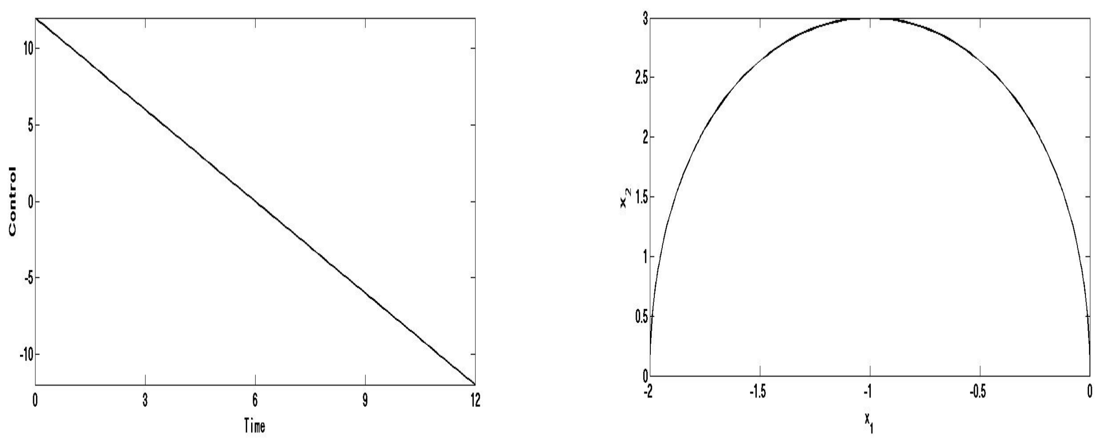


Figure 5.1: Control and phase diagram for the L^2 minimum norm problem with $A = [0, 1; 0, 0]$, $B = [0; 1]$, $x_0 = [-2; 0]$, $x_f = [0; 0]$, $T = 12$

5.2 L^1 minimum norm problem

We solve the problem

$$min \int_0^{4.5} |u(t)|dt$$

subject to

$$\frac{dx(t)}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t);\ t \in [0, 4.5],$$
$$x(0) = \begin{pmatrix} -2 \\ 0 \end{pmatrix},\ x(12) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$|u(t)| \leq 1. \tag{5.2}$$

We start with an initial guess $\mu = \begin{pmatrix} -0.1823 \\ 1.5000 \end{pmatrix}$, and update μ by $\alpha = 0.25$ upto 30 iterations and by full Newton step after that. The optimal control obtained and the state trajectory are shown in Fig. 5.2.

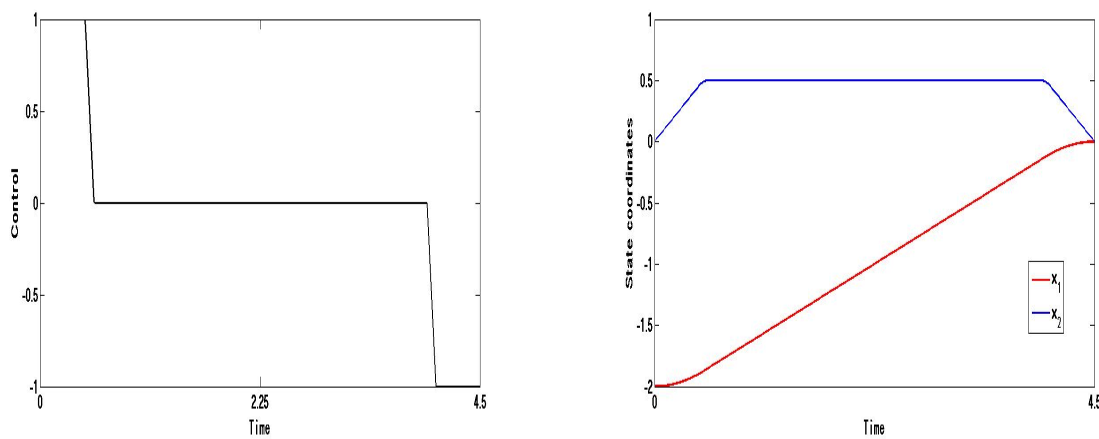


Figure 5.2: Control diagram and state trajectory for the L^1 minimum norm problem with $A = [0, 1; 0, 0]$, $B = [0; 1]$, $x_0 = [-2; 0]$, $x_f = [0; 0]$, $T = 4.5$, $gamma = 1$

5.3 Time optimal problem

We here present our solution to three cases.

Case 1

Here, we solve

$$min \int_0^\tau dt$$

subject to

$$\frac{dx(t)}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t);\ t \in [0, \tau],$$
$$x(0) = \begin{pmatrix} -2 \\ 0 \end{pmatrix},\ x(\tau) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$|u(t)| \leq 0.55, \tag{5.3}$$

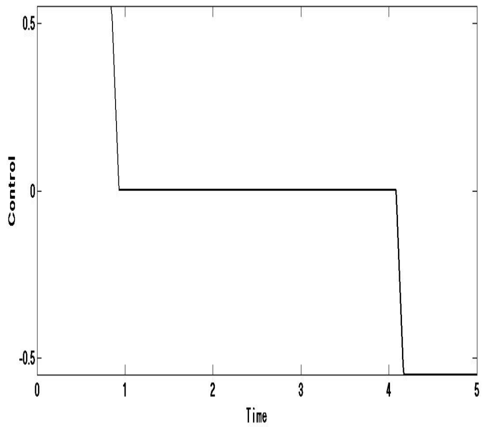
.

We start with an initial guess $\tau = 12$ and $\mu = \begin{pmatrix} -0.1823 \\ 1.5000 \end{pmatrix}$, and update τ using full Newton step. μ is updated by $\alpha = 0.25$ upto 30 iterations and by full Newton step after that. The final value of $\tau = 5.00$ is hence obtained. The optimal control obtained and the phase diagram are shown in Fig. 5.3.

Case 2

Here, we solve

$$min \int_0^\tau dt$$



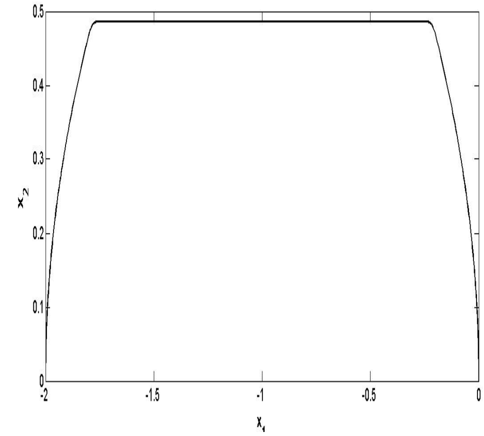


Figure 5.3: Control and phase diagram for the time optimal problem with $A = [0, 1; 0, 0]$, $B = [0; 1]$, $x_0 = [-2; 0]$, $x_f = [0; 0]$, $\gamma = 0.55$

subject to

$$\frac{dx(t)}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t);\ t \in [0, \tau],$$

$$x(0) = x_0,\ x(\tau) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$|u(t)| \leq \gamma, \tag{5.4}$$

.

We here obtain the optimal τ for different values of x_0 and γ. We start with an initial guess for $\mu = \begin{pmatrix} 9.1801 \\ 4.5918 \end{pmatrix}$. We then update τ and μ both by $\alpha = .25$. The results for this case are given in Table 5.1 and in Figs. 5.4 and 5.5.

Table 5.1: Results for the time optimal control problem with $A = [0, 1; -1, 0]$, $B = [0; 1]$, $x_f = [0; 0]$

x_0	γ	Initial guess for τ	τ obtained	Number of switches
[-2;0]	.5	5	5.92	4
[-2;.15]	.5	5.92	5.84	3
[-2;.4]	.55	5.84	5.82	3

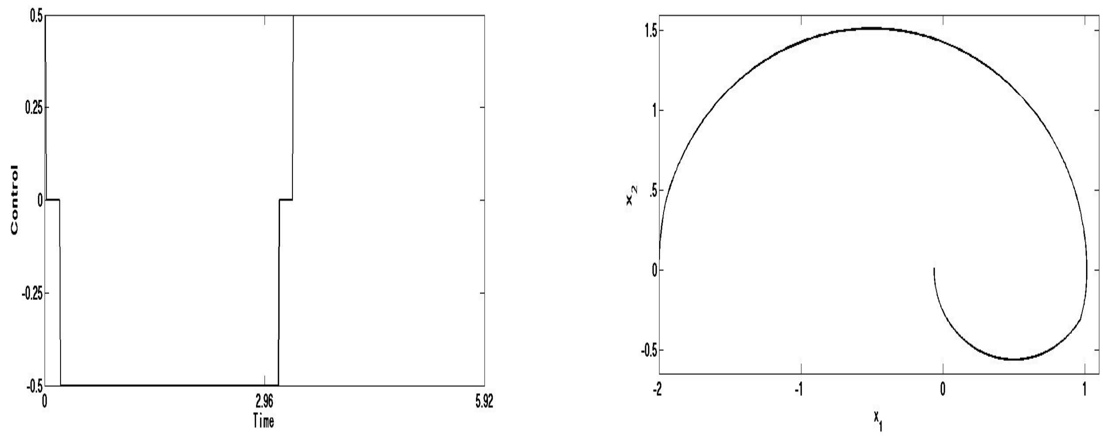


Figure 5.4: Control and phase diagram for the time optimal problem with $A = [0, 1; -1, 0]$, $B = [0; 1]$, $x_0 = [-2; 0]$, $x_f = [0; 0]$, $\gamma = 0.5$.

Case 3

Here, we solve

$$min \int_0^\tau dt$$

subject to

$$\frac{dx(t)}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t);\ t \in [0, \tau],$$

$$x(0) = \begin{pmatrix} -2 \\ 0 \end{pmatrix},\ x(\tau) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$|u(t)| \leq 10, \tag{5.5}$$

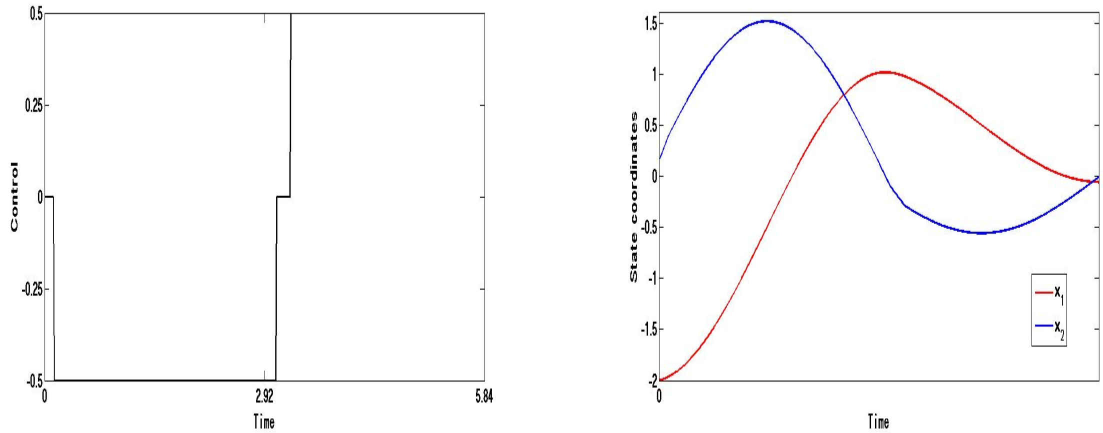


Figure 5.5: Control diagram and state trajectory for the time optimal problem with $A = [0, 1; -1, 0]$, $B = [0; 1]$, $x_0 = [-2; .15]$, $x_f = [0; 0]$, $\gamma = 0.5$

.

We start with an initial guess $\tau = 1$ and $\mu = \begin{pmatrix} -3.0069 \\ 1.1000 \end{pmatrix}$. We then update τ using $\alpha = 0.25$, and μ using full Newton step. The final value of $\tau = 0.99$ is hence obtained. The optimal control obtained and the phase diagram are shown in Fig. 5.6.

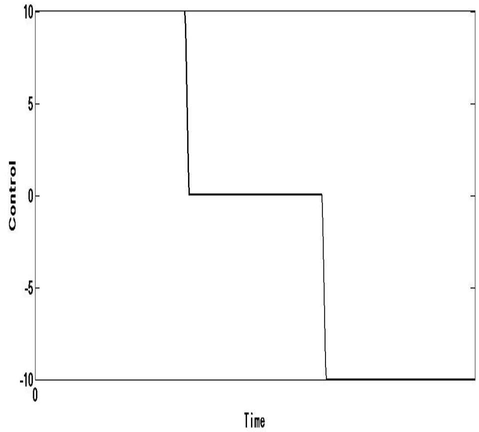
Variant of time optimal control problem

Here, we solve the problem

$$min\,\tau \int_0^1 (1 + |u(t)|)dt$$

subject to

$$\frac{dx(t)}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t),$$
$$x(0) = \begin{pmatrix} -2 \\ 0 \end{pmatrix},\; x(\tau) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$|u(t)| \leq 5. \tag{5.6}$$



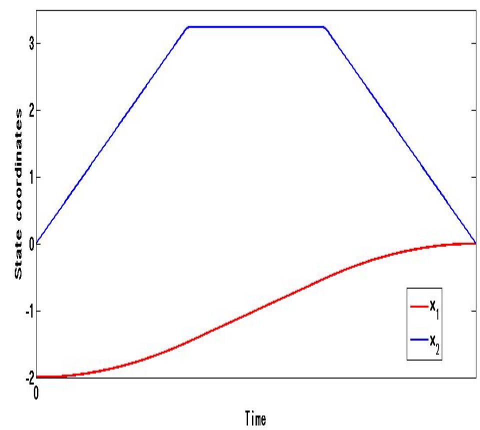


Figure 5.6: Control and state trajectory for the time optimal problem with $A = [0, 1; 1, 0]$, $B = [0; 1]$, $x_0 = [-2; 0]$, $x_f = [0; 0]$, $\gamma = 10$

We start with an initial guess $\tau = 1$ and $\mu = \begin{pmatrix} -3.0069 \\ 1.1000 \end{pmatrix}$, and update μ using full Newton step. τ is updated by $\alpha = 0.25$. The final value of $\tau = 1.49$ is hence obtained. The optimal control obtained and the phase diagram are shown in Fig. 5.7.

5.4 L^∞ minimum effort problem

Here, we solve

$$min\ \frac{1}{2}\gamma^2 dt$$

subject to

$$\frac{dx(t)}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t);\ t \in [0, T],$$
$$x(0) = \begin{pmatrix} -2 \\ 0 \end{pmatrix},\ x(T) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$|u(t)| \leq \gamma, \tag{5.7}$$

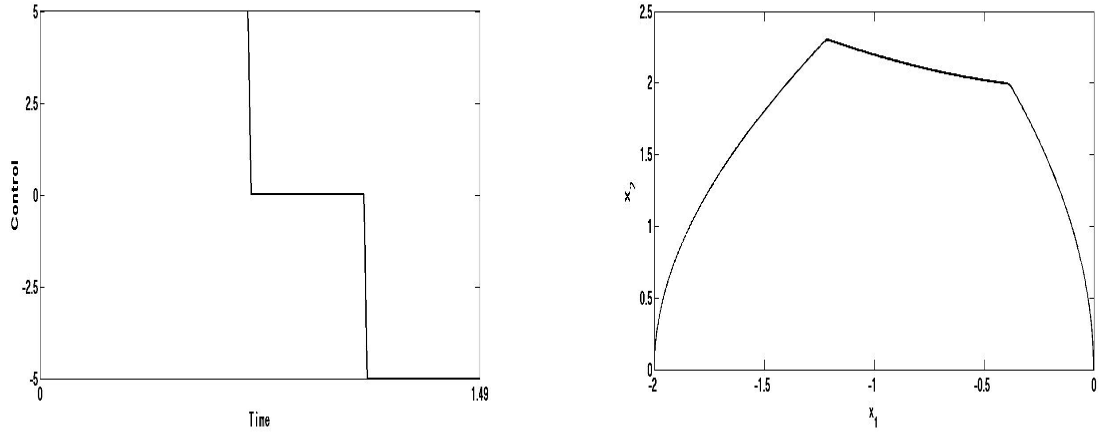


Figure 5.7: Control and phase diagram for a variant of the time optimal problem with $A = [0,1;1,0]$, $B = [0;1]$, $x_0 = [-2;0]$, $x_f = [0;0]$, $\gamma = 5$

Table 5.2: Results for the L^∞ minimum effort problem with $A = [0,1;0,0]$, $B = [0;1]$, $x_0 = [-2;0]$, $x_f = [0;0]$

T	Initial guess for γ	γ obtained
11	.5	.11
8.2	.5	.15

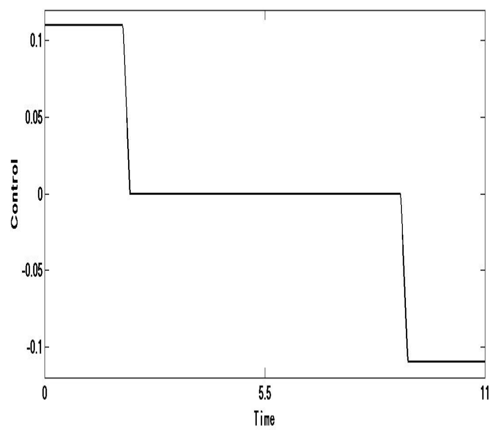
.

We here obtain the optimal value of γ two different values of T. We start with an initial guess $\mu = \begin{pmatrix} -0.1823 \\ 1.5000 \end{pmatrix}$. We then update μ using $\alpha = 0.25$ upto 30 iterations, and then convert to full Newton step. γ is updated using full Newton step. Two switches are obtained in the optimal control for each case. The results for this case are given in Table 5.2 and in Figs. 5.8 and 5.9.

Variant of L^∞ minimum effort control problem

Here, we solve

$$min\ \frac{1}{2}\gamma^2 + \int_0^{2.25} |u(t)|dt$$



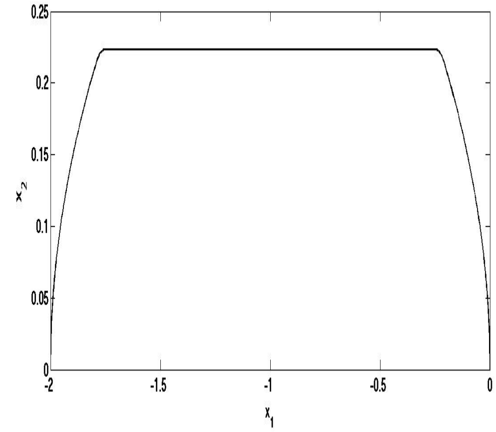


Figure 5.8: Control and phase diagram for the L^∞ minimum effort problem with $A = [0, 1; 0, 0]$, $B = [0; 1]$, $x_0 = [-2; 0]$, $x_f = [0; 0]$, $T = 11$

subject to

$$\frac{dx(t)}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t),$$
$$x(0) = \begin{pmatrix} -2 \\ 0 \end{pmatrix},\ x(2.25) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$|u(t)| \leq \gamma. \tag{5.8}$$

We start with an initial guess $\gamma = 10$ and $\mu = \begin{pmatrix} -3.0069 \\ 1.1000 \end{pmatrix}$, and update μ using full Newton step. γ is updated by $\alpha = 0.25$. The final value of $\gamma = 2.88$ is hence obtained. The optimal control obtained and the state trajectory are shown in Fig. 5.10.

In summary, we need to have a good initial guess μ for a specifically chosen triplet τ, γ and x_0. This method of continuation based on the sensitivity calculation is efficient in solving all cases that we have tested.

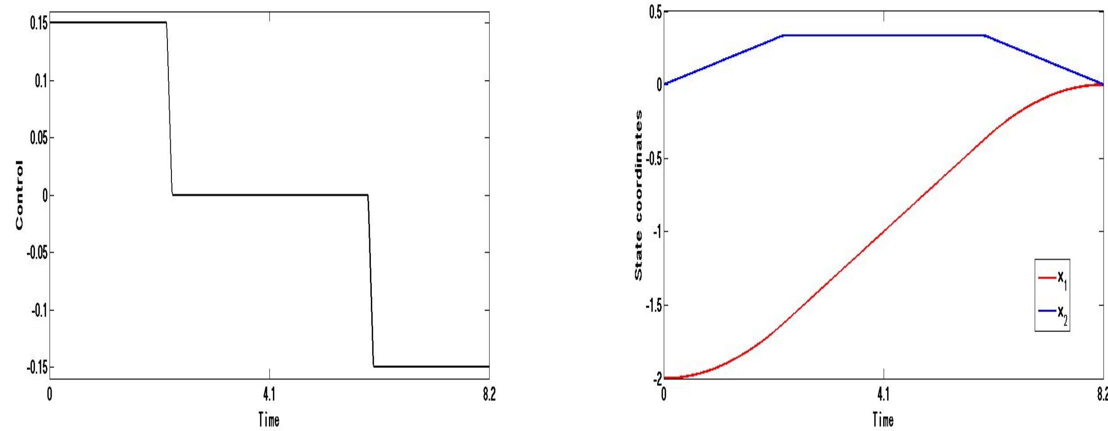


Figure 5.9: Control and state trajectory for the L^∞ minimum effort problem with $A = [0, 1; 0, 0]$, $B = [0; 1]$, $x_0 = [-2; 0]$, $x_f = [0; 0]$, $T = 8.2$

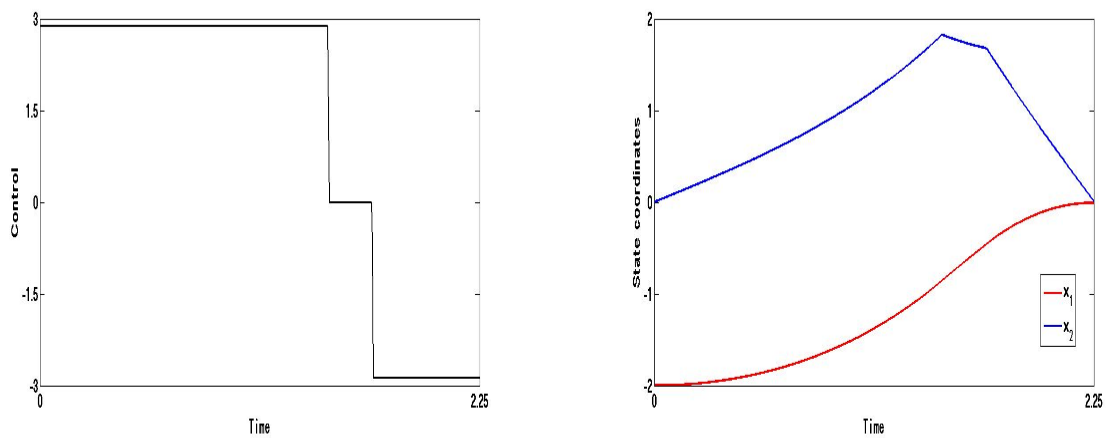


Figure 5.10: Control diagram and state trajectory for a variant of L^∞ minimum effort control problem with $A = [0, 1; 0, 0]$, $B = [0; 1]$, $x_0 = [-2; 0]$, $x_f = [0; 0]$, $T = 2.25$

Chapter 6

Conclusions and Future Work

6.1 Conclusions

In this thesis, we study the optimal control problem for linear systems, which arise in several areas, for example, quantum systems, electric circuits, fuel regulators, tumor control therapy, spacecraft control and equitable resource allocation. Specifically, we investigate and solve L^1 and L^2 minimum norm problems, time optimal problem, and L^∞ minimum effort problems for linear two dimensional time-invariant systems.

Control theory has been an active research area since the 1950s. In Chapter 1, we introduce the problems of interest and motivation to solve them. We also discuss the solution to the minimum norm problems here using the lagrange multiplier theory and the control grammian. In Chapter 2, we sketch a solution to the general problem described in Chapter 1, and then apply it to the time optimal problem and two mixed problems to obtain the necessary optimality system. The necessary optimality system obtained for the time optimal, minimum effort problems and the mixed problems are non-smooth, unlike that for the minimum norm problems. This results in bang-bang (non-differentiable) control for these problems, so we describe the regularization formulation to overcome this difficulty, and hence obtain a semi-smooth optimality system. In Chapter 3, we describe how to solve this system using semi-smooth Newton method.

Two critical aspects of the semi-smooth algorithm are the initialization of the unknown parameters and obtaining solutions efficiently for varying known parameters. Initialization based on solutions of the L^2 minimum norm problem and continuation of solution based on sensitivity analysis are then described. In Chapter 4, we state the algorithm based on the solution described in Chapters 2 and 3. Although the solution has been described only for the time optimal problem, it has been extended to solve the minimum effort problem also. It can also be extended to bilinear systems and higher dimensional systems. The key aspects in effectively applying the stated algorithm are, good initail guess for the unknown parameters, and the damping factor α.

Our method is easy to implement and does not require accurate information on the switching structure in advance, in contrast to the existing shooting method. Numerical results for a lot of test cases are presented in Chapter 5 to demonstrate the effectiveness of our method.

6.2 Future Work

Some directions for future work in this area are stated below.

- We have applied the semi-smooth Newton method to solve time optimal and minimum effort control problems iteratively. An important issue is how to adjust the damping factor adaptively to have global convergence of the numerical method. We propose to apply the Armijo-rule or Levenberg-Marquardt Method to get a good estimate for the damping factor.

- Although we have tested our method only for linear time-invariant systems, our method could be extended to bilinear time-invariant systems and other systems governed by partial differential equations.

- We have tested the effectiveness of our method for two dimensional systems. Our method could also be extended to higher dimensional systems.

- We have stated some areas where the control problems studied are of significance. Our method could be applied to any of these areas in particular.

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APPENDICES

Appendix A

Introduction to Semi-smooth Newton method

Consider the nonlinear equation $F(y) = 0$ in a Banach space X The generalized Newton update is given by

$$y^{k+1} = y^k - V_k^{-1} F(y^k), \tag{A.1}$$

where V_k is a generalized Jacobian of F at y^k. In the finite dimensional space for a locally Lipschitz continuous function F, let D_F denote the set of points at which F is differentiable. For $x \in X = R^n$ we define $\partial_B F(x)$ as

$$\partial_B F(x) = \left\{ J : J = \lim_{x_i \to x,\, x_i \in D_F} \nabla F(x_i) \right\}, \tag{A.2}$$

where D_F is dense by Rademacher's theorem, which states that every locally Lipschitz continuous function in the finite dimensional space is differentiable almost everywhere. Thus, we take $V_k \in \partial_B F(y^k)$.

In infinite dimensional spaces, notions of generalized derivatives for functions which are not C^1, cannot rely on Rademacher's theorem. Here, instead, we shall mainly utilize a concept of generalized derivative that is sufficient to guarantee superlinear convergence of Newton's method. This notion of differentiability is called Newton derivative and will be defined below. Let $X,\ Z$ be real Banach spaces and let $D \subset X$ be an open set.

Definition

(1) *$F\colon D \subset X \to Z$ is called Newton differentiable at x, if there exists an open neighborhood $N(x) \subset D$ and mappings $G\colon N(y) \to \mathcal{L}(X, Z)$ such that*

$$\lim_{|h| \to 0} \frac{|F(y+h) - F(y) - G(y+h)h|_Z}{|h|_X} = 0. \tag{A}$$

The family $\{G(x) : x \in N(x)\}$ is called a N-derivative of F at x.

(2) F is called semi-smooth at y, if it is Newton differentiable at y and

$$\lim_{t \to 0^+} G(y + t\,h)h \quad \textit{exists unifomly in } |h| = 1.$$

Semi-smoothness was originally introduced for scalar-valued functions. Convex functions and real-valued C^1 functions are examples for such semi-smooth functions in the finite dimensional space.

For example, if $F(y)(s) = \psi(y(s))$, point-wise, then $G(y)(s) \in \psi_B(y(s))$ is an N derivative in $L^p(\Omega) \to L^q(\Omega)$ under appropriate conditions. We use often $\psi(s) = |s|$ and $\psi(s) = max(0, s)$ for our model examples.

Suppose $F(y^*) = 0$, Then, $y^* = y^k + (y^* - y^k)$ and

$$|y^{k+1} - y^*| = |V_k^{-1}(F(y^*) - F(y^k) - V_k(y^* - y^k)| \leq |V_k^{-1}|o(|y^k - y^*|)$$

Thus, the semi-smooth Newton method is q-superlinear convergent provided the Jacobian sequence V_k is uniformly invertible as $y^k \to y^*$. That is, if one can select a sequence of quasi-Jacobian V_k that consistent, i.e., $|V_k - G(y^*)|$ as $y^k \to y^*$ and invertible, (A.1) is still q-superlinear convergent.

More details on the Semi-smooth Newton method can be found in [10].

Appendix B

Matlab code for time optimal problem

Here, we describe how we implemented the algorithmic procedure given in Chapter 4 for the time optimal problem in Matlab.

We set the values of A, B, x_0, x_f, γ, α and n, so that $dt = 1/n$. Set initial guess for μ and τ. Specify the dimension of the system as $m = 2$.

Stopping criterion (7) in Chapter 4 is written as

$$up = ones(3,1);\ j = 0;\ while\, max(|up(:)|) > 10^{-4}$$

Let

$$\begin{aligned} M &= \tau * A * dt; \\ S &= (speye(m) - 0.5 * M^T)^{-1}(speye(m) + 0.5 * M^T); \end{aligned} \tag{B.1}$$

(2.33) can be written as

$$p_k = S\, p_{k+1}.$$

So, p_k is evaluated below.

$$
\begin{aligned}
for\; k = n : -1 : 1;\; ptmp = p;\; p = S * p; \\
pp(:, k) = p;\; pmiddle(:, k) = (ptmp + p)/2;
\end{aligned}
\tag{B.2}
$$

The residue term g_k appearing in (3.11) is computed as

$$
g(:, k) = ((ptmp - p) + M^T * pmiddle(:, k))/dt;\; end;
$$

We now compute the control u_k and its derivative.

$$
for\; k = 1 : n;\; v = -B^T * pmiddle(:, k);
$$

So, from (2.34), we obtain $u^{'} = G_\epsilon(v)$, which is evaluated below.

$$
\begin{aligned}
if((|v| > (1 - \epsilon))\&\&(|v| < 1)),\; ud(k) = \gamma/\epsilon; \\
else\; ud(k) = 0;\; end;
\end{aligned}
\tag{B.3}
$$

$u = sign_\epsilon(v)$ is computed below.

$$
\begin{aligned}
if(|v| \leq (1 - \epsilon)),\; u = 0; \\
elseif(|v| >= 1),\; u = (v/|v|) * \gamma; \\
else\; u = (v/|v|) * (\gamma/\epsilon) * (|v| - (1 - \epsilon)); end; \\
uu(k) = u;
\end{aligned}
\tag{B.4}
$$

The state x_k is computed iteratively using (2.31)

$$xtmp = x;$$
$$x = (speye(m) - 0.5 * M)^{-1}((speye(m) + 0.5 * M) * x + dt * \tau * u * B);$$
$$xx(:, k) = x; xmiddle(:, k) = (xtmp + x)/2; \qquad \text{(B.5)}$$

The residue term f_k appearing in (3.9) is computed as

$$f(:, k) = ((x - xtmp) - M * xmiddle(:, k))/dt - \tau * u * B;\ end;$$

(3.11) can be written as

$$\delta p_k = S * \delta p_{k+1} + (I - M^T/2)^{-1}(A * pmiddle(:, k) + g(:, k))dt.$$

Since $\delta p_n = \delta\mu$, δp_k is evaluated in an interative manner below.

$$M1 = speye(m);\ M7 = zeros(m, 1);\ M8 = zeros(m, 1);$$
$$for\ k = n : -1 : 1;\ M1 = S * M1; MM1(:, k) = M1(:);$$
$$tmp7 = (speye(m) - 0.5 * M^T)^{-1} * A^T * pmiddle(:, k) * dt;$$
$$M7 = tmp7 + S * M7;\ MM7(:, k) = M7(:);$$
$$tmp8 = (speye(m) - 0.5 * M^T)^{-1} * g(:, k) * dt;$$
$$M8 = tmp8 + S * M8;\ MM8(:, k) = M8(:);\ end; \qquad \text{(B.6)}$$

So,

$$\delta p_k = MM1(:, k)\,\delta\mu + MM7(:, k)\,\delta\tau + MM8(:, k). \qquad \text{(B.7)}$$

(3.9) can be written as

$$\delta x_{k+1} = S * \delta x_k + (I - M/2)^{-1}((A * xmiddle(:, k) + B * u_k)\delta\tau + \tau B\delta u_k + f(:, k))dt,$$

where δu_k is computed from (3.10) using (B.7), below.

$$M2 = zeros(m,1);\ M3 = zeros(m,1);$$
$$tmp2 = zeros(m,1);\ tmp3 = zeros(m,1);$$
$$M4 = zeros(m,m);\ M5 = zeros(m,1);\ M6 = zeros(m,1); \tag{B.8}$$

$$for\ k = 1:n;\ tmp = S^T;$$
$$tmp4 = -(speye(m) - 0.5 * M)^{-1} * ud(k) * \tau * B * B^T * reshape(MM1(:,k),m,m) * dt;$$
$$tmp5 = -(speye(m) - 0.5 * M)^{-1} * ud(k) * \tau * B * B^T * MM7(:,k) * dt;$$
$$tmp6 = -(speye(m) - 0.5 * M)^{-1} * ud(k) * \tau * B * B^T * MM8(:,k) * dt;$$
$$M4 = tmp4 + tmp * M4;\ M5 = tmp5 + tmp * M5;\ M6 = tmp6 + tmp * M6; \tag{B.9}$$

So,

$$\delta u_n = M4\,\delta\mu + M5\,\delta\tau + M6.$$

Now, δx_k is computed.

$$tmp2 = (speye(m) - 0.5 * M)^{-1} * (A * xmiddle(:,k) + uu(k) * B) * dt;$$
$$tmp3 = (speye(m) - 0.5 * M)^{-1} * f(:,k) * dt;$$
$$M2 = tmp2 + tmp1 * M2;\ M3 = tmp3 + tmp * M3;\ end; \tag{B.10}$$

So,

$$\delta x_n = \delta u_n + M2\,\delta\tau + M3.$$

Let the Jacobian be

$$NN1 = \begin{pmatrix} \frac{\partial \varphi_1}{\partial \mu} & \frac{\partial \varphi_1}{\partial \tau} \\ \frac{\partial \varphi_2}{\partial \mu} & \frac{\partial \varphi_2}{\partial \tau} \end{pmatrix}.$$

Now,

$$\frac{\partial \varphi_1}{\partial \mu} = \frac{\partial x_n}{\partial \mu},$$

which we have already computed as $M4$. Similarly,

$$\frac{\partial \varphi_1}{\partial \tau} = \frac{\partial x_n}{\partial \tau},$$

computed above as $M2 + M5$.

Now, from (3.8),

$$\frac{\partial \varphi_2}{\partial \mu} = (A * x_n + u_n * B)^T \tau + \mu^T (A\tau \frac{\partial x_n}{\partial \mu} + B \frac{\partial u_n}{\partial \mu}),$$

where

$$\frac{\partial u_n}{\partial \mu} = -ud(n) * B_T.$$

So, it can be written as $T1 + T2 * M4$, where

$$T1 = ((A * x_n + u_n * B)^T - ud(n) * \mu^T * B * B^T) * \tau;\ T2 = \mu^T * A * \tau;$$

Now, from (3.8),

$$\frac{\partial \varphi_2}{\partial \tau} = \mu^T \tau (A \frac{\partial x_n}{\partial \tau} + B \frac{\partial u_n}{\partial \tau}) + \mu^T (Ax_n + Bu_n),$$

where

$$\frac{\partial u_n}{\partial \mu} = -ud(n) * B_T.$$

Hence,

$$NN1 = \begin{pmatrix} M4 & M2 + M5 \\ T1 + T2 * M4 & T2 * (M2 + M5) + \mu^T * (A * x_n + u_n * B) \end{pmatrix};$$

Here, φ is given by

$$NN2 = \begin{pmatrix} M3 + M6 + x_n - x_f \\ T2 * (M3 + M6) + 1 + \mu^T * (A * x_n + u_n * B) * \tau \end{pmatrix};$$

Hence, the update

$$\delta z = \begin{pmatrix} \delta\mu \\ \delta\tau \end{pmatrix}$$

is computed below and μ and τ updated before the next iteration.

$$\begin{aligned} up &= -NN1^{-1} NN2; \\ \mu = \mu + \alpha * up(1:2);\ \tau &= \tau + \alpha * up(3); \\ j &= j + 1;\ end; \end{aligned} \tag{B.11}$$