ABSTRACT

ABERNATHY, KRISTEN KOBYLUS. Existence of Solutions to Nonlinear Boundary Value Problems at Resonance. (Under the direction of Jesús Rodríguez.)

The focus of this paper is the study of nonlinear dynamical time-systems subject to general boundary conditions. We first consider nonlinear differential equations of the form

$$y^{(n)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = f(y(t)) + (Gy)(t); \quad 0 \le t \le 1$$

subject to the boundary conditions

$$\begin{aligned}
&b_{11}y(0) + \cdots + b_{1n}y^{(n-1)}(0) + d_{11}y(1) + \cdots + d_{1n}y^{(n-1)}(1) = 0\\
&b_{21}y(0) + \cdots + b_{2n}y^{(n-1)}(0) + d_{21}y(1) + \cdots + d_{2n}y^{(n-1)}(1) = 0\\
&\quad\vdots\\
&b_{n1}y(0) + \cdots + b_{nn}y^{(n-1)}(0) + d_{n1}y(1) + \cdots + d_{nn}y^{(n-1)}(1) = 0.
\end{aligned}$$

We provide sufficiency conditions for existence of solutions based on the dimension of the solution space of the corresponding linear, homogeneous boundary value problem, the asymptotic behavior of the nonlinear real-valued function f, and the "size" of the nonlinear function G.

Next we consider parameter dependent vector equations of the form

$$\dot{x}_i(t) = a_i(t)x_i(t) + f_i(\epsilon, t, x_1(t), \cdots, x_n(t)), \quad i = 1, 2, \cdots, n,$$

subject to two-point boundary conditions

$$b_i x_i(0) + d_i x_i(1) = 0, \quad i = 1, 2, \cdots, n.$$

We present an argument for existence of solutions for the case when the corresponding linear, homogeneous boundary value problem is at full resonance.

We conclude by analyzing discrete, nonlinear systems of the form

$$y(k+n) + \cdots + a_0(k)y(k) = f(y(k)) + \sum_{l=0}^{J} w(k,l)g(l, y(l), \cdots, y(l+n-1))$$

subject to the multipoint boundary conditions

$$\sum_{j=1}^{n} b_{ij}(0)y(j-1) + \sum_{j=1}^{n} b_{ij}(1)y(j) + \cdots + \sum_{j=1}^{n} b_{ij}(J)y(j+J-1) = 0$$

for $i = 1, 2, \cdots, n$. Again, we formulate sufficiency conditions based on the assumption that the corresponding linear, homogeneous system has a nontrivial solution space. The Lyapunov-Schmidt Procedure plays a crucial role in establishing existence of solutions and we offer a self-contained presentation of the basic ideas of the Lyapunov-Schmidt Procedure.

Existence of Solutions to Nonlinear Boundary Value Problems at Resonance

by
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BIOGRAPHY

Kristen Kobylus Abernathy was born on January 15, 1983 in the northeastern city made popular by the television show "The Office," Scranton, Pennsylvania. She spent most of her childhood in Scranton, moved to Waco, Texas for a couple years, and settled in Salisbury, North Carolina by her teenage years. From early childhood, she wanted to follow in the footsteps of her mother, Pat Kobylus, and become a teacher when she grew up.

In 2001, Kristen graduated from North Rowan High School as salutatorian of her class. More importantly, in that same year, she gave birth to her son, Justin Barrett Kobylus. At the time of this writing, Justin is nine years old and a fourth grader at Sacred Heart Cathedral School in Raleigh, North Carolina. He is an avid basketball player, a skilled pianist, and a scientist in training.

After high school, Kristen attended Catawba College in Salisbury, North Carolina, where she majored in mathematics and minored in biology. In 2004, she received Catawba College's Outstanding Mathematics Student Award and graduated Summa Cum Laude. The following two years she attended Wake Forest University on a graduate assistantship, obtaining her Master of Arts degree in May 2006. Following Wake Forest University, she went on to doctoral work at North Carolina State University.

During Kristen's time at Wake Forest University, she met her future husband, Zach Abernathy. They both graduated in 2006 and were both accepted into the graduate program at North Carolina State University. They were married on June 28, 2008.

While at North Carolina State Univeristy, Kristen has taught a variety of mathematics courses, including the three-semester Calculus sequence, ordinary differential equations, and an introductory proof writing course. In 2008, Kristen won the Maltbie award for excellence in teaching. She was also awarded Outstanding Teaching Assistant for the Mathematics Department and was a "Thank a Teacher" recipient. Kristen participated in the Certificate of Accomplishment in Teaching program and was selected as a Preparing the Professoriate Fellow. At the time of this writing, she is looking forward to a career in academia.

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The journey toward the completion of my doctoral dissertation has been influenced by many people along the way. My mathematics career truly began while at Catawba College, Dr. Jason Hunt, Dr. John Zerger, and Dr. Sharon Sullivan introduced me to the beautiful world of mathematics. I am forever grateful for their support and guidance throughout the years. After my undergraduate work, I can attribute Wake Forest University to refining my love of mathematics. Dr. John Baxley and Dr. Stephen Robinson are amazingly brilliant mathematicians and I am fortunate to have them as mentors and friends.

Of course, nothing in my life would be possible without the love and support of my family. I consider Tony and Shelly Pecorella part of my family and I am so thankful to have them in my life. My extended family is absolutely amazing and I am so lucky to be surrounded by constant love and encouragement. My Mom and Dad Abernathy have given me support and brought so much happiness and laughter into my life.

Words cannot express how thankful I am to my parents. My Mom and Dad raised me with the belief that anything is possible with the Lord. They have shown me unconditional love and encouragement throughout my life and I have them to thank for all my success.

Finally, I am incredibly blessed to have such a wonderful husband and an amazing son. Justin is a daily inspiration and a constant reminder of my many blessings in life. My husband, Zach, is the most brilliant, kind, patient, and encouraging person I know. I am so thankful to have shared this experience with him.

TABLE OF CONTENTS

Chapter 1

Introduction

This paper is devoted to the analysis of nonlinear boundary value problems for both continuous and discrete dynamical systems. In Chapters 2 and 3, we study nonlinear differential equations subject to two-point boundary conditions, while in Chapter 4 we consider nonlinear discrete time-systems subject to nonlocal boundary conditions. In each chapter, we will assume that the solution space to the corresponding linear, homogeneous problem is nontrivial. We establish the existence of solutions using the Lyapunov-Schmidt Procedure in conjunction with Brouwer's and Schauder's Fixed Point theorems.

In Chapter 2, we consider nonlinear differential equations of the form

$$y^{(n)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = f(y(t)) + (Gy)(t); \quad 0 \le t \le 1$$

subject to the general boundary conditions

$$
\begin{aligned}
&b_{11}y(0) + \cdots + b_{1n}y^{(n-1)}(0) + d_{11}y(1) + \cdots + d_{1n}y^{(n-1)}(1) = 0 \\
&b_{21}y(0) + \cdots + b_{2n}y^{(n-1)}(0) + d_{21}y(1) + \cdots + d_{2n}y^{(n-1)}(1) = 0 \\
&\quad \vdots \\
&b_{n1}y(0) + \cdots + b_{nn}y^{(n-1)}(0) + d_{n1}y(1) + \cdots + d_{nn}y^{(n-1)}(1) = 0.
\end{aligned}
$$

In this chapter, we assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that the map $G : (\mathcal{C}([0,1],\mathbb{R}), \|\cdot\|_\infty) \to (\mathcal{C}([0,1],\mathbb{R}), \|\cdot\|_\infty)$ is nonlinear and continuous. We formulate sufficient conditions for the existence of solutions based on the dimension of the solution space of the corresponding linear, homogeneous equation and the "size" of the nonlinear terms.

The focus of Chapter 3 is the study of parameter dependent vector equations of the form

$$\dot{x}_i(t) = a_i(t)x_i(t) + f_i(\epsilon, t, x_1(t), \cdots, x_n(t)), \quad i = 1, 2, \cdots, n,$$

subject to two-point boundary conditions

$$b_i x_i(0) + d_i x_i(1) = 0, \quad i = 1, 2, \cdots, n.$$

We present the case where the solution space of the corresponding linear, homogeneous vector equation is at full resonance. The asymptotic behavior of $f_i(0, t, x_1(t), \cdots, x_n(t))$ and the solution space of the linear, homogeneous boundary value problem play crucial roles in establishing sufficient conditions.

Our goal in Chapter 4 is to provide sufficient conditions for the existence of solutions to discrete, nonlinear systems of the form

$$y(k+n)+\cdots+a_0(k)y(k)=f(y(k))+\sum_{l=0}^{J}w(k,l)g(l,y(l),\cdots,y(l+n-1))$$

subject to the multipoint boundary conditions

$$\sum_{j=1}^{n}b_{ij}(0)y(j-1)+\sum_{j=1}^{n}b_{ij}(1)y(j)+\cdots+\sum_{j=1}^{n}b_{ij}(J)y(j+J-1)=0$$

for $i=1,2,\cdots,n$. The criteria we present depends on the size of the nonlinearities and the set of solutions to the corresponding linear, homogeneous boundary value problems. The results presented in this chapter extend the previous work of D. Etheridge and J. Rodriguez [6], [7] and J. Rodriguez and P. Taylor [19], [20].

It should be noted that the results in each chapter are independent of other chapters. It follows that each of the chapters is presented in a self-contained manner and it is not necessary to read the chapters in any particular order.

Chapter 2

Nonlinear Boundary Value Problems in a Continuous Time-Setting

2.1 Introduction

In this chapter, we consider boundary value problems of the form

$$y^{(n)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = f(y(t)) + (Gy)(t); \quad 0 \le t \le 1 \tag{2.1}$$

subject to

$$\begin{aligned}
&b_{11}y(0) + \cdots + b_{1n}y^{(n-1)}(0) + d_{11}y(1) + \cdots + d_{1n}y^{(n-1)}(1) = 0 \\
&b_{21}y(0) + \cdots + b_{2n}y^{(n-1)}(0) + d_{21}y(1) + \cdots + d_{2n}y^{(n-1)}(1) = 0 \\
&\quad\vdots \\
&b_{n1}y(0) + \cdots + b_{nn}y^{(n-1)}(0) + d_{n1}y(1) + \cdots + d_{nn}y^{(n-1)}(1) = 0.
\end{aligned} \tag{2.2}$$

We assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that the limits $f(\infty)$ and $f(-\infty)$ exist.

The map $G : (\mathcal{C}([0,1],\mathbb{R}), \|\cdot\|_\infty) \to (\mathcal{C}([0,1],\mathbb{R}), \|\cdot\|_\infty)$ is nonlinear and continuous.

We concern ourselves with problems where the corresponding linear, homogeneous boundary value problem

$$y^{(n)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = 0 \tag{2.3}$$

subject to (2.2) has a one dimensional solution space. For such problems, we provide sufficient conditions for the existence of solutions to (2.1)-(2.2). These conditions are based on the limiting behavior of the real valued function f, the properties of the solution space of the linear homogeneous boundary value problem (2.3)-(2.2), and the "size" of the nonlinear map G. It is significant to observe that the results we obtain may be applied to boundary value problems for integro-differential equations of the form

$$y^{(n)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = f(y(t)) + \int_0^1 k(t,s)g(t,y(s))ds; \quad 0 \le t \le 1$$

subject to (2.2) as well as to classical boundary value problems of the form

$$y^{(n)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = f(y(t)) + g(t,y(t)); \quad 0 \le t \le 1$$

subject to (2.2).

Our approach is based on the Lyapunov-Schmidt Procedure (Alternative Method). The results we present here allow us to establish the solvability of boundary value problems that do not fall within the scope of the results previously obtained by Rodríguez and Taylor [21]. Ideas and techniques similar to the ones we use in this chapter have been successfully applied to the study of periodic behavior in discrete and continuous dynamical systems [3], [5], [6], [8], [10], [22] boundary value problems for differential and difference equations [1], [7], [12], [13], [15], [16], [18], [19], [20], [21], and more general systems [2], [23].

2.2 Preliminaries

In order to analyze the boundary value problem (2.1)-(2.2), we formulate it in system form.

The matrix $A(t)$ is defined by

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ -a_n(t) & -a_{n-1}(t) & -a_{n-2}(t) & \cdots & -a_1(t) \end{bmatrix}.$$

The vector

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is given by $x_1 = y$, $x_2 = y'$, $\cdots$, $x_n = y^{(n-1)}$ and the boundary matrices B and D are

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & & \ddots & & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix}, D = \begin{bmatrix} d_{11} & d_{12} & d_{13} & \cdots & d_{1n} \\ d_{21} & d_{22} & d_{23} & \cdots & d_{2n} \\ \vdots & & \ddots & & \vdots \\ d_{n1} & d_{n2} & d_{n3} & \cdots & d_{nn} \end{bmatrix}.$$

It is clear that the boundary value problem (2.1)-(2.2) is equivalent to

$$\dot{x}(t) = A(t)x(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(x_1(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ G(x_1(t)) \end{bmatrix}, 0 \leq t \leq 1 \tag{2.4}$$

subject to

$$Bx(0) + Dx(1) = 0. \tag{2.5}$$

Throughout the chapter we will assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that it has finite limits at ∞ and $-\infty$. We write

$$f(\infty) = \lim_{s\to\infty} f(s)$$

and

$$f(-\infty) = \lim_{s\to-\infty} f(s).$$

For any integer $p \geq 1$ the space $(\mathcal{C}([0,1], \mathbb{R}^p), \|\cdot\|_\infty)$ will denote $\{\phi : [0,1] \to \mathbb{R}^p : \phi \text{ is continuous}\}$. The norm used on this space is the sup norm; this is, $\|\phi\|_\infty = \sup\{|\phi(t)| : 0 \leq t \leq 1\}$ where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^p$.

The map $G : (\mathcal{C}([0,1], \mathbb{R}), \|\cdot\|_\infty) \to (\mathcal{C}([0,1], \mathbb{R}), \|\cdot\|_\infty)$ is continuous and there exists an M such that for any $\phi \in (\mathcal{C}([0,1], \mathbb{R}), \|\cdot\|_\infty)$ $\sup\{|G(\phi(t))| : 0 \leq t \leq 1\} \leq M < \infty$.

So as to be able to use functional analytic ideas we introduce the following notation.

The space $X = \{x \in (\mathcal{C}([0,1], \mathbb{R}^n), \|\cdot\|_\infty) : Bx(0) + Dx(1) = 0\}$. $\mathcal{F} : X \to$

$(\mathcal{C}([0,1],\mathbb{R}^n),\|\cdot\|_\infty)$ is defined by $(\mathcal{F}x)(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(x_1(t)) \end{bmatrix}$ and $\mathcal{G} : X \to (\mathcal{C}([0,1],\mathbb{R}^n),\|\cdot\|_\infty)$ is given by $(\mathcal{G}x)(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ G(x_1(t)) \end{bmatrix}$. It is obvious that $\mathcal{F}$ and $\mathcal{G}$ are continuous maps from X into $(\mathcal{C}([0,1],\mathbb{R}^n),\|\cdot\|_\infty)$ and that $\sup\{\|\mathcal{F}(x)\|_\infty : x \in X\}$ and $\sup\{\|\mathcal{G}(x)\|_\infty : x \in X\}$ are both finite.

We define the operator $L : D(L) \to (\mathcal{C}([0,1],\mathbb{R}^n),\|\cdot\|_\infty)$ by $(Lx)(t) = \dot{x}(t) - A(t)x(t)$ where $D(L)$ consists of the continuously differentiable functions in X. It is evident that the boundary value problem (2.1)-(2.2) is equivalent to

$$Lx = \mathcal{F}(x) + \mathcal{G}(x). \tag{2.6}$$

Since the properties of the solution space of the linear homogeneous boundary value problem (2.3)-(2.2) play a role in the solvability of (2.1)-(2.2), we must first consider the linear problem

$$Lx = 0.$$

Proposition 2.2.1 *$Lx = 0$ if and only if*

$$x(t) = \Gamma(t)v$$

where $\Gamma(t)$ is the principal matrix solution of $\dot{x}(t) = A(t)x(t)$ and $v \in ker(B + D\Gamma(1))$.

Proof:

$$Lx = 0$$

if and only if

$$\dot{x}(t) - A(t)x(t) = 0 \text{ and } Bx(0) + Dx(1) = 0$$

if and only if

$$x(t) = \Gamma(t)C \text{ for some } C \text{ and } B\Gamma(0)C + D\Gamma(1)C = 0$$

if and only if

$$[B + D\Gamma(1)]C = 0$$

if and only if

$$C \in ker(B + D\Gamma(1)).$$

Corollary 2.2.2 *$ker(B + D\Gamma(1))$ and $ker(L)$ have the same dimension.*

It is well documented that solutions of

$$\dot{x}(t) = A(t)x(t) + h(t)$$

are given by the variation of constants formula

$$x(t) = \Gamma(t)x(0) + \Gamma(t)\int_0^t \Gamma^{-1}(s)h(s)ds.$$

Proposition 2.2.3 *$Lx = h$ if and only if x is given by the variation of constants formula* *above, where $x(0)$ must satisfy*

$$[B + D\Gamma(1)]x(0) = -D\Gamma(1)\int_0^1 \Gamma^{-1}(s)h(s)ds.$$

Proof:

$$Lx = h$$

if and only if

$$x(t) = \Gamma(t)x(0) + \Gamma(t)\int_0^t \Gamma^{-1}(s)h(s)ds$$

and

$$Bx(0) + Dx(1) = 0$$

if and only if

$$Bx(0) + D[\Gamma(1)x(0) + \Gamma(1)\int_0^1 \Gamma^{-1}(s)h(s)ds] = 0$$

if and only if

$$[B + D\Gamma(1)]x(0) = -D\Gamma(1)\int_0^1 \Gamma^{-1}(s)h(s)ds.$$

Corollary 2.2.4 *L is a bijection on $D(L)$ if and only if $(B + D\Gamma(1))$ is invertible.*

2.3 The Case of Invertible L

It should be observed that if L is invertible and the nonlinearities f and G are bounded, it is straightforward to establish the existence of solutions of (2.1)-(2.2). In fact, (2.1)-(2.2) is solvable if and only if the operator $L^{-1}(\mathcal{F} + \mathcal{G})$ has a fixed point. The existence of such a fixed point is an immediate consequence of Schauder's Theorem once we observe that $L^{-1}(\mathcal{F} + \mathcal{G})$ is compact.

2.4 The Case of Singular L

Since the existence of solutions is relatively straightforward when L is invertible, the more interesting case is when $ker(L)$ or, equivalently, $ker(B + D\Gamma(1))$ is nontrival. In this chapter, we consider the case when the dimension of $ker(L)$ is one. For the reader's convenience, we offer a self-contained presentation of the basic ideas of the Lyapunov-Schmidt reduction. These ideas have been applied to a large class of problems in differential and difference equations [3], [6], [7], [8], [12], [13], [14], [15], [16], [18], [19], [20], [21]. For an abstract formulation and for a vast number of applications, we refer the reader to [4], [5], [9].

We know that $Lx = 0$ if and only if $x(t) = \Gamma(t)v$, where $v \in ker(B + D\Gamma(1))$. We now wish to examine when $Lx = h$ has a solution. According to Proposition 2.2.3, $h \in Im(L)$ if and only if there is some $x_0 \in \mathbb{R}^n$ such that $[B + D\Gamma(1)]x_0 = -D\Gamma(1) \int_0^1 \Gamma^{-1}(s)h(s)ds$; that is, if and only if $\int_0^1 D\Gamma(1)\Gamma^{-1}(s)h(s)ds \in Im(B + D\Gamma(1))$. Since $Im(B + D\Gamma(1)) = [ker(B + D\Gamma(1))^T]^\perp$, $h \in Im(L)$ if and only if $W^T \int_0^1 D\Gamma(1)\Gamma^{-1}(s)h(s)ds = 0$, where the columns of the n by n matrix W form a basis for $ker(B + D\Gamma(1))^T$.

We define

$$\Psi^T(t) = W^T D\Gamma(1)\Gamma^{-1}(t).$$

By the argument outlined above, $Lx = h$ if and only if $\int_0^1 \Psi^T(t)h(t)dt = 0$.

Since L is not invertible, we can't apply the Schauder Fixed Point Theorem directly. We will use the splittings of X and $(\mathcal{C}([0,1], \mathbb{R}^n), \|\cdot\|_\infty)$ typically used in the Lyapunov-Schmidt procedure.

We find projections, P, of X onto $ker(L)$, and E, of $(\mathcal{C}([0,1], \mathbb{R}^n), \|\cdot\|_\infty)$ onto $Im(L)$, so that we may write $X = ker(L) \oplus Im(I - P)$ and $(\mathcal{C}([0,1], \mathbb{R}^n), \|\cdot\|_\infty) = Im(L) \oplus Im(I - E)$.

Let $\Phi(t) = \Gamma(t)V$ where the vector V forms a basis for $ker(B + D\Gamma(1))$. Let

$$C_1 = \int_0^1 \Phi^T(t)\Phi(t)dt$$

and

$$C_2 = \int_0^1 \Psi^T(t)\Psi(t)dt.$$

Proposition 2.4.1 *C_1 is invertible and C_2 is invertible when $[B : D]$ has full rank.*

Proof: To show C_1 is invertible, assume $C_1 a = 0$ and define $q(t) = \Phi(t)a$. Then $a^T C_1 a = \int_0^1 q^T(t)q(t)dt = 0$ which implies $q(t) = 0$ for all $t \in [0, 1]$. This implies $a = 0$ because $\Phi(t)$ is a nonzero vector.

To show C_2 is invertible, we need to show the columns of $\Psi^T(t)$ are linearly independent. Let $\Psi_j^T(t)$ be the jth column of $\Psi^T(t)$. If $[B : D]$ has full rank, $a \in ker(B^T)$ and $a \in ker(D^T)$ implies $a = 0$. Now,

$$\sum_{i=1}^{n} c_i \Psi_i^T(t) = 0$$

if and only if

$$(c_1, c_2, \cdots, c_n)W^T D = 0$$

if and only if

$$W(c_1, c_2, \cdots, c_n)^T \in ker(D^T).$$

Since $W(c_1, c_2, \cdots, c_n)^T \in ker[B + D\Gamma(1)]^T$, $W(c_1, c_2, \cdots, c_n)^T \in ker(B^T)$ and hence $W(c_1, c_2, \cdots, c_n)^T = (0, 0, \cdots, 0)$. It follows that since W forms a basis for $ker(B + D\Gamma(1))^T$, $(c_1, c_2, \cdots, c_n)^T = (0, 0, \cdots, 0)$.

Let

$$(I-E)x(t) = \Psi(t)C_2^{-1}\int_0^1 \Psi^T(s)x(s)ds.$$

Let

$$Px(t) = \Phi(t)C_1^{-1}\int_0^1 \Phi^T(s)x(s)ds.$$

For the reader's convenience, we have presented a detailed construction of the projections onto the $ker(L)$ and $Im(L)$. For the case of periodic boundary conditions, we refer the reader to D.C. Lewis [9]; for discrete boundary value problems, we suggest Rodríguez [14]. Although the projections we have constructed here are a special case of those that appear in Spealman and Sweet [23] and Rodríguez and Taylor [21], we have chosen present this construction due to the fact that we do not need to appeal to the full generality of the results mentioned previously.

We now use the standard techniques of the Lyapunov-Schmidt method to analyze $Lx = \mathcal{F}(x) + \mathcal{G}(x)$.

Remark 2.4.2 *If $\tilde{L}$ is the restriction of L to $D(L) \cap Im(I-P)$ then $Im(\tilde{L}) = Im(L)$. $\tilde{L}$, viewed as a map from $D(L) \cap Im(I-P)$ into $Im(L)$ is invertible. We denote $(\tilde{L})^{-1}$ by M. From this, it follows that $MLx = (I-P)x$. Later, we will use the obvious fact that M is compact.*

Proposition 2.4.3 *$Lx = \mathcal{F}(x) + \mathcal{G}(x)$ is equivalent to*

$$\begin{cases} x = Px + ME\mathcal{F}(x) + ME\mathcal{G}(x) \\ \textit{and} \\ (I-E)\mathcal{F}(Px + ME(\mathcal{F}(x) + \mathcal{G}(x))) + (I-E)\mathcal{G}(Px + ME(\mathcal{F}(x) + \mathcal{G}(x))) = 0. \end{cases}$$

Proof: We have $Lx = \mathcal{F}(x) + \mathcal{G}(x)$ if and only if

$$\begin{cases} E(Lx - (\mathcal{F}(x) + \mathcal{G}(x))) = 0 \\ \text{and} \\ (I - E)(Lx - (\mathcal{F}(x) + \mathcal{G}(x))) = 0 \end{cases}$$

if and only if

$$\begin{cases} Lx = E(\mathcal{F}(x) + \mathcal{G}(x)) \\ \text{and} \\ (I - E)(\mathcal{F}(x) + \mathcal{G}(x)) = 0 \end{cases}$$

if and only if

$$\begin{cases} (I - P)x = ME(\mathcal{F}(x) + \mathcal{G}(x)) \\ \text{and} \\ (I - E)(\mathcal{F}(x) + \mathcal{G}(x)) = 0 \end{cases}$$

if and only if

$$\begin{cases} x = Px + ME(\mathcal{F}(x) + \mathcal{G}(x)) \\ \text{and} \\ (I - E)(\mathcal{F}(Px + ME(\mathcal{F}(x) + \mathcal{G}(x))) + \mathcal{G}(Px + ME(\mathcal{F}(x) + \mathcal{G}(x)))) = 0. \end{cases}$$

We have limited our presentation of the Lyapunov-Schmidt Procedure to only those aspects necessary for the problem at hand. This approach, as well as its generalization, the Alternative Method, is well documented [2], [3], [4], [5], [9], [12], [13], [15], [18]. For those interested in the study of periodicity, in either discrete or continuous dynamical systems, we suggest [3], [5], [6], [8], [10], [22]. For applications in the field of discrete boundary value problems, the reader may consult [7], [14], [16], [19], [20]. An abstract and very general presentation appears in [2], [5], [9].

2.5 Main Results

The conditions of 2.4.3 may be rewritten as

$$\begin{cases} x = \alpha\Phi(t) + ME\mathcal{F}(x) + ME\mathcal{G}(x) \\ \text{and} \\ 0 = \int_0^1 \Psi_n(t) f(\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt + \int_0^1 \Psi_n(t) G(\alpha\Phi_1(t) \\ +[ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt \end{cases}$$

where $\Phi_i(t)$, $\Psi_i(t)$, and $[ME(\mathcal{F}(x) + \mathcal{G}(x))]_i(t)$ are the ith entries of $\Phi(t)$, $\Psi(t)$, and $\alpha\Phi(t) + ME(\mathcal{F}(x) + \mathcal{G}(x))(t)$, respectively.

We will assume that there are finite numbers, which we designate $f(\infty)$ and $f(-\infty)$, such that

$$\lim_{r\to\infty} f(r) = f(\infty)$$

and

$$\lim_{r\to-\infty} f(r) = f(-\infty).$$

We define J_1 and J_2 as

$$J_1 = f(\infty)\int_{\{t\in[0,1]:\Phi_1(t)>0\}} \Psi_n(t)dt + f(-\infty)\int_{\{t\in[0,1]:\Phi_1(t)<0\}} \Psi_n(t)dt$$

and

$$J_2 = f(-\infty)\int_{\{t\in[0,1]:\Phi_1(t)>0\}} \Psi_n(t)dt + f(\infty)\int_{\{t\in[0,1]:\Phi_1(t)<0\}} \Psi_n(t)dt.$$

Theorem 2.5.1 *Suppose that*

1. *dim*(*ker*$(B + D\Gamma(1))) = 1$ *where* $\Gamma(t)$ *is the principal matrix solution of* $\dot{x}(t) = A(t)x(t)$;

2. $[B : D]$ *has full rank;*

3. $f : \mathbb{R} \to \mathbb{R}$ *is continuous;*

4. $f(\infty)$ *and* $f(-\infty)$ *exist;*

5. $J_1 J_2 < 0$;

6. $G : (\mathcal{C}([0,1], \mathbb{R}), \|\cdot\|_\infty) \to (\mathcal{C}([0,1], \mathbb{R}), \|\cdot\|_\infty)$ *is continuous and*

$$\sup\{\|G(w)\|_\infty : w \in (\mathcal{C}([0,1], \mathbb{R}), \|\cdot\|_\infty)\} \leq \min\{|J_1|, |J_2|\}.$$

Then there exists at least one solution of

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = f(y(t)) + (Gy)(t)$$

that satisfies

$$b_{11}y(0) + \cdots + b_{1n}y^{(n-1)}(0) + d_{11}y(1) + \cdots + d_{1n}y^{(n-1)}(1) = 0$$

$$b_{21}y(0) + \cdots + b_{2n}y^{(n-1)}(0) + d_{21}y(1) + \cdots + d_{2n}y^{(n-1)}(1) = 0$$

$$\vdots$$

$$b_{n1}y(0) + \cdots + b_{nn}y^{(n-1)}(0) + d_{n1}y(1) + \cdots + d_{nn}y^{(n-1)}(1) = 0.$$

Proof: Let $J = \min\{|J_1|, |J_2|\}$. We define mappings

$$H_1 : \mathbb{R} \times (\mathcal{C}([0,1], \mathbb{R}n), \|\cdot\|_\infty) \to (\mathcal{C}([0,1], \mathbb{R}^n), \|\cdot\|_\infty)$$

$$H_2 : \mathbb{R} \times (\mathcal{C}([0,1], \mathbb{R}^n), \|\cdot\|_\infty) \to \mathbb{R}$$

$$H : \mathbb{R} \times (\mathcal{C}([0,1], \mathbb{R}^n), \|\cdot\|_\infty) \to \mathbb{R} \times (\mathcal{C}([0,1], \mathbb{R}^n), \|\cdot\|_\infty)$$

by

$$H_1(\alpha, x) = \alpha\Phi(t) + ME\mathcal{F}(x) + ME\mathcal{G}(x),$$

$$H_2(\alpha, x) = \alpha - (\textstyle\int_0^1 \Psi_n(t) f(\alpha\Phi_1 + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt +$$
$$\textstyle\int_0^1 \Psi_n(t) G(\alpha\Phi_1 + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt),$$

and

$$H(\alpha, x) = (H_1(\alpha, x), H_2(\alpha, x)).$$

Since $\{t : \Phi_1(t) = 0\}$ has Lebesgue measure zero, it follows that

$$\int_0^1 \Psi_n(t) f(\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt =$$

$$\int_{\{t\in[0,1]:\Phi_1(t)>0\}} \Psi_n(t) f(\alpha\Phi_1 + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt +$$

$$\int_{\{t\in[0,1]:\Phi_1(t)<0\}} \Psi_n(t) f(\alpha\Phi_1 + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt.$$

Since $ME(\mathcal{F} + \mathcal{G})$ is bounded, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{\alpha\to\infty} \int_0^1 \Psi_n(t) f(\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt =$$

$$f(\infty)\int_{\{t\in[0,1]:\Phi_1(t)>0\}}\Psi_n(t)dt + f(-\infty)\int_{\{t\in[0,1]:\Phi_1(t)<0\}}\Psi_n(t)dt = J_1.$$

Similarly,

$$\lim_{\alpha\to-\infty}\int_0^1 \Psi_n(t)f(\alpha\Phi_1(t)+[ME(\mathcal{F}(x)+\mathcal{G}(x))]_1(t))dt =$$

$$f(-\infty)\int_{\{t\in[0,1]:\Phi_1(t)>0\}}\Psi_n(t)dt + f(\infty)\int_{\{t\in[0,1]:\Phi_1(t)<0\}}\Psi_n(t)dt = J_2.$$

Without loss of generality, we assume $J_2 < 0 < J_1$.

Assuming that $\Psi_n(t)$ is not identically zero, we can choose our basis for $ker(B + D\Gamma(1))^T$ so that $\|\Psi\|_\infty \leq 1$. Therefore, there is some $\alpha_0 \geq m$ where $m = sup\{|f(t)| : t \in \mathbb{R}\}$ such that for all $\alpha \geq \alpha_0$, $\int_0^1 \Psi_n(t)f(\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt \geq J$ and $\int_0^1 \Psi_n(t)f(-\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt \leq -J$. Since $|G(\alpha\Phi_1 + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))| \leq J$ for all $t \in \mathbb{R}$, for $\alpha \geq \alpha_0$ and $x \in (\mathcal{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$, $H_2(\alpha, x) = \alpha - (\int_0^1 \Psi_n(t)f(\alpha\Phi_1 + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt + \int_0^1 \Psi_n(t)G(\alpha\Phi_1 + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt)$ $\leq \alpha - (J - J) = \alpha$. Similarly, for $\alpha \geq \alpha_0$ and $x \in (\mathcal{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$, $H_2(-\alpha, x) \geq -\alpha$.

Letting $\delta = \alpha_0 + (m + J)$, define $\mathcal{B} = \{(\alpha, x) \in \mathbb{R} \times (\mathcal{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty) : |\alpha| \leq \delta$ and $\|x\|_\infty \leq \delta\|\Phi\|_\infty + \|ME\|(m + J)\}$. Here, $\|ME\|$ denotes the operator norm of the bounded, linear map ME.

Note that $\|ME\mathcal{F}(x)\|_\infty \leq \|ME\|m$ and $\|ME\mathcal{G}(x)\|_\infty \leq \|ME\|J$ for every $x \in (\mathcal{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$.

Now if $\alpha \in [\alpha_0, \delta]$, for all $x \in (\mathcal{C}([0,1], \mathbb{R}^n), \|\cdot\|_\infty)$, we have

$$\begin{aligned}
H_2(\alpha, x) = \alpha - (\int_0^1 \Psi_n(t) f(\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt + \\
\int_0^1 \Psi_n(t) G(\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt) \\
\geq \alpha - (\int_0^1 |\Psi_n(t)||f(\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))|dt + \\
\int_0^1 |\Psi_n(t)||G(\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))|dt) \\
\geq \alpha - (m + J) \\
\geq \alpha - \alpha_0 - J \\
\geq -J \\
\geq -\delta
\end{aligned}$$

and

$$\begin{aligned}
H_2(-\alpha, x) = -\alpha - (\int_0^1 \Psi_n(t) f(\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt + \\
\int_0^1 \Psi_n(t) G(\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))dt) \\
\leq -\alpha + \int_0^1 |\Psi_n(t)||f(\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))|dt + \\
\int_0^1 |\Psi_n(t)||G(\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))|dt) \\
\leq -\alpha + (m + J) \\
\leq -\alpha + \alpha_0 + J \\
\leq J \\
\leq \delta.
\end{aligned}$$

Thus, for all $x \in (\mathcal{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$ and $\alpha \in [\alpha_0, \delta]$, $H_2(\alpha, x)$, $H_2(-\alpha, x) \in [-\alpha, \alpha] \subseteq [-\delta, \delta]$.

Furthermore, if $0 \le \alpha < \alpha_0$, for all $x \in (\mathcal{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$,

$$\begin{aligned}
|H_2(\pm\alpha, x)| \le | \pm \alpha| + \int_0^1 & |\Psi_n(t)||f(\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))|dt + \\
& \int_0^1 |\Psi_n(t)||G(\alpha\Phi_1(t) + [ME(\mathcal{F}(x) + \mathcal{G}(x))]_1(t))|dt \\
& \le \alpha_0 + (m + J) \\
& \le \delta.
\end{aligned}$$

We have shown that H_2 maps $[-\delta, \delta] \times (\mathcal{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$ into $[-\delta, \delta]$ when $\Psi_n(t)$ is not identically zero. However, if $\Psi_n(t)$ is identically zero, $H_2(\alpha, x) = \alpha$ and so H_2 will map $[-\delta, \delta] \times (\mathcal{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$ into $[-\delta, \delta]$. From this it follows that $H(\mathcal{B}) \subseteq \mathcal{B}$. For if $(\alpha, x) \in \mathcal{B}$, then $H_2(\alpha, x) \in [-\delta, \delta]$, while

$$\begin{aligned}
\|H_1(\alpha, x)\|_\infty &\le |\alpha| \|\Phi\|_\infty + \|ME(\mathcal{F}(x) + \mathcal{G}(x))\|_\infty \\
&\le \delta \|\Phi\|_\infty + \|ME\| m + \|ME\| J.
\end{aligned}$$

Since M is compact and E, $\mathcal{F}$, and $\mathcal{G}$ are continuous and map bounded sets to bounded sets, H is completely continuous. So, the completely continuous function H maps the non-empty, closed, bounded, convex set $\mathcal{B}$ into itself. Hence, the Schauder Fixed Point Theorem guarantees existence of at least one fixed point, $\tilde{x}$, of H in $\mathcal{B}$. For each such $\tilde{x}$, $\tilde{y} = \tilde{x}_1$ is a solution of

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = f(y(t)) + (Gy)(t)$$

which satisfies

$$b_{11}y(0) + \cdots + b_{1n}y^{(n-1)}(0) + d_{11}y(1) + \cdots + d_{1n}y^{(n-1)}(1) = 0$$

$$b_{21}y(0) + \cdots + b_{2n}y^{(n-1)}(0) + d_{21}y(1) + \cdots + d_{2n}y^{(n-1)}(1) = 0$$

$$\vdots$$

$$b_{n1}y(0) + \cdots + b_{nn}y^{(n-1)}(0) + d_{n1}y(1) + \cdots + d_{nn}y^{(n-1)}(1) = 0.$$

2.6 Final Remarks

In the case of a classical boundary value problem of the form

$$y^{(n)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = f(y(t)) + g(t, y(t))$$

subject to (2.2), we can ensure solvability whenever $\sup\{|g(u,v)| : (u,v) \in \mathbb{R}^2\} \leq \min\{|J_1|, |J_2|\}$. Similarly, in the case of a integro-differential boundary value problem of the form

$$y^{(n)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = f(y(t)) + \int_0^1 g(t, y(s))ds$$

subject to (2.2), we obtain the existence of solutions if $\sup\{|g(u,v)| : (u,v) \in \mathbb{R}^2\} \leq \min\{|J_1|, |J_2|\}$.

Chapter 3

Boundary Value Problems at Full Resonance

3.1 Introduction

In this chapter, we establish criteria for the existence of solutions to the parameter dependent vector equation

$$\dot{x}_i(t) = a_i(t)x_i(t) + f_i(\epsilon, t, x_1(t), \cdots, x_n(t)), \quad i = 1, 2, \cdots, n, \tag{3.1}$$

subject to the boundary conditions

$$b_i x_i(0) + d_i x_i(1) = 0, \quad i = 1, 2, \cdots, n. \tag{3.2}$$

We focus on the case where the solution space of the corresponding linear, homogeneous vector equation

$$\dot{x}_i(t) - a_i(t)x_i(t) = 0, \quad i = 1, 2, \cdots, n, \tag{3.3}$$

subject to boundary conditions (3.2) is n-dimensional. We will provide sufficient conditions for existence of solutions to (3.1), (3.2). The asymptotic behavior of $f_i(0, t, x_1(t), \cdots, x_n(t))$ and the solution space of the linear, homogeneous boundary value problem (3.3), (3.2) will play crucial roles in establishing sufficient conditions.

Our technique used to establish existence of solutions to (3.1), (3.2) relies on the Lyapunov-Schmidt Procedure. Ideas and techniques similar to the ones used in this chapter were successfully applied to the study of discrete and continuous dynamical systems [2], [3], [5], [6], [7], [8], [10], [12], [13], [15], [16], [17], [18], [19], [20], [21], [22], [23].

3.2 Preliminaries

Before we establish solvability criteria for (3.1),(3.2), we will first analyze the linear, homogeneous boundary value problem (3.3), (3.2). It is easily verified that solutions to (3.3), (3.2) are of the form

$$\phi(t) = \Phi(t)v$$

where $v \in \mathbb{R}^n$ and $\Phi(t)$ is the matrix

$$\Phi(t) = \begin{bmatrix} e^{\int_0^t a_1(s)ds} & 0 & 0 & \cdots & 0 \\ 0 & e^{\int_0^t a_2(s)ds} & 0 & \cdots 0 & \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & e^{\int_0^t a_n(s)ds} \end{bmatrix}.$$

Note that solutions to the nonhomogeneous equation

$$\dot{x}_i(t) = a_i(t)x_i(t) + h_i(t), \quad i = 1, 2, \cdots, n \tag{3.4}$$

have the form

$$x_i(t) = \int_0^t e^{-\int_0^s a_i(r)dr} h_i(s)ds.$$

Thus, the boundary value problem (3.4), (3.2) is solvable when

$$d_i \int_0^1 e^{-\int_0^s a_i(r)dr} h_i(s)ds = 0$$

for each $i = 1, 2, \cdots, n$.

We now wish to analyze the solvability of (3.1), (3.2). In order to do this, we will introduce notation that allows us to proceed using functional analysis tools.

We define $L : D(L) \to \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty)$ by

$$Lx = \dot{x} - Ax$$

where

$$D(L) = \mathcal{C}^1([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \cap X$$

and

$$X = \{x \in \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \mid Bx(0) + Dx(1) = 0\}.$$

Here, we use the notation

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, A = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix},$$

$$B = \begin{bmatrix} b_1 & 0 & 0 & \cdots & 0 \\ 0 & b_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & b_n \end{bmatrix}, \text{ and } D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}.$$

The space $\mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty)$ will denote $\{\phi : [0,1] \to \mathbb{R}^n : \phi \text{ is continuous}\}$ and $\mathcal{C}^1([0,1], \mathbb{R}^n, \|\cdot\|_\infty)$ will denote $\{\phi : [0,1] \to \mathbb{R}^n : \phi \text{ is continuously differentiable}\}$. The norm used on these spaces is the sup norm; that is, $\|\phi\|_\infty = \sup\{|\phi(t)| : 0 \leq t \leq 1\}$ where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$.

We let $F : \mathbb{R} \times \mathbb{R} \times \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \to \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty)$ be given by

$$F(\epsilon, x)(t) = \begin{bmatrix} f_1(\epsilon, t, x(t)) \\ f_2(\epsilon, t, x(t)) \\ \vdots \\ f_n(\epsilon, t, x(t)) \end{bmatrix}.$$

For simplicity, we will write $F(0, x)(t) = F(x)(t)$. We assume f_i for $i = 1, \cdots, n$ is continuous and $\sup\{|f_i(0,u)| : u \in \mathbb{R}^{n+1}\} \leq m$ for some $m \in \mathbb{R}$. Hence, F is continuous and, for all $x \in X$, $\|F(x)\|_\infty \leq m$.

With this notation, the problem (3.1),(3.2) is equivalent to $Lx = F(\epsilon, x)$. We first consider the particular case when $\epsilon = 0$. This is equivalent to the operator equation $Lx = Fx$.

The fact that L is not invertible makes it impossible to establish the solvability of $Lx = Fx$ by a direct use of the Schauder Fixed Point Theorem. Instead, we will analyze this operator equation with a projection scheme usually referred to as the Lyapunov-Schmidt Procedure. For the reader's convenience, we provide all the necessary background. We will exploit the structure of the linear system, discussed above, in the construction of the projections. For an abstract formulation of the methods used below and for a vast number of applications of these methods, we refer the interested reader to [4],[5],[9].

For $h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$, if follows that $Lx = h$ if and only if $d_i \int_0^1 e^{-\int_0^s a_i(r)dr} h_i(s)ds = 0$ for each $i = 1, 2, \cdots, n$. We see that this simplifies to $Lx = h$ if and only if $\int_0^1 \Phi^{-1}(t)h(t)dt = 0$.

The projections we define below are familiar to the Lyapunov-Schmidt Procedure. We will now provide a self-contained presentation of the Lyapunov-Schmidt Procedure for the reader's convenience.

By direct computation, we can verify that the maps $P : X \to X$ defined by

$$Px_i(t) = \frac{e^{\int_0^t a_i(s)ds}}{\int_0^1 e^{2\int_0^t a_i(s)ds} dt} \int_0^1 e^{\int_0^s a_i(l)dl} x_i(s)ds$$

and $E : \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \to \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty)$ defined by

$$Ex_i(t) = x_i(t) - \frac{e^{-\int_0^t a_i(s)ds}}{\int_0^1 e^{-2\int_0^t a_i(s)ds}dt} \int_0^1 e^{-\int_0^s a_i(l)dl} x_i(s)ds$$

are projections and $Im(P) = \ker(L)$ and $Im(E) = Im(L)$. This allows us to write $X = ker(L) \oplus Im(I-P)$ and $\mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) = Im(L) \oplus Im(I-E)$.

Remark 3.2.1 *If $\tilde{L}$ is the restriction of L to $D(L) \cap Im(I-P)$ then $Im(\tilde{L}) = Im(L)$. $\tilde{L}$, viewed as a map from $D(L) \cap Im(I-P)$ into $Im(L)$ is invertible. We denote $(\tilde{L})^{-1}$ by M and note that $MLx = (I-P)x$. Later, we will use the fact that M is compact.*

Proposition 3.2.2 *$Lx = F(x)$ is equivalent to*

$$\begin{cases} x = Px + MEF(x) \\ \quad \textit{and} \\ (I-E)F(Px + MEF(x)) = 0 \end{cases}$$

Proof: Using the fact that E is a projection, we have $Lx = Fx$ if and only if

$$\begin{cases} E(Lx - Fx) = 0 \\ \quad \text{and} \\ (I-E)(Lx - Fx) = 0. \end{cases}$$

Since $(I-E)L = 0$ and $EL = L$, this is equivalent to

$$\begin{cases} Lx = EF(x) \\ \quad \text{and} \\ (I-E)F(x) = 0. \end{cases}$$

Applying M to the first equation, we obtain

$$\begin{cases} (I-P)x = MEF(x) \\ \text{and} \\ (I-E)F(x) = 0. \end{cases}$$

From this, we conclude that $Lx = F(x)$ is equivalent to

$$\begin{cases} x = Px + MEF(x) \\ \text{and} \\ (I-E)F(Px + MEF(x)) = 0. \end{cases}$$

3.3 Main Results

According to Proposition 3.2.2, $Lx = Fx$ if and only if

$$\begin{cases} x = \beta_1\Phi_1(t) + \cdots + \beta_n\Phi_n(t) + MEF(x) \\ 0 = \int_0^1 (\Phi_1(t))^{-1} f_1(0, t, \beta_1\Phi_1(t) + \cdots + \beta_n\Phi_n(t) + MEF(x)(t))dt \\ \vdots \\ 0 = \int_0^1 (\Phi_n(t))^{-1} f_n(0, t, \beta_1\Phi_1(t) + \cdots + \beta_n\Phi_n(t) + MEF(x)(t))dt) \end{cases} \tag{3.5}$$

where $\Phi_i(t) = e^{\int_0^t a_i(s)ds}$.

Lemma 3.3.1 *Suppose that*

i. $b_i + d_i e^{\int_0^1 a_i(s)ds} = 0$ *for all* $i = 1, 2, \cdots, n$;

ii. $f_i : \mathbb{R}^{n+2} \to \mathbb{R}$ *is continuous for all* $i = 1, \cdots, n$;

iii. For each $i = 1, \cdots, n$, there exists $\gamma_i \in \mathbb{R}$ such that

$$f_i(0, t, \beta_1, \cdots, \beta_i, \cdots, \beta_n) f_i(0, t, \beta_1, \cdots, -\beta_i, \cdots, \beta_n) < 0 \text{ whenever } \beta_i \geq \gamma_i.$$

Then (3.5) *has a solution.*

Proof: Assume, without loss of generality, that for $\beta_i \geq \gamma_i$, $(\Phi_i(t))^{-1} f_i(0, t, \beta_1, \cdots, \beta_i, \cdots, \beta_n) > 0$. We define mappings

$$\begin{aligned} &H_1 : \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \times \mathbb{R}^n \to \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \\ &H_{i+1} : \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \times \mathbb{R}^n \to \mathbb{R}, \text{ for } i = 1, \cdots, n, \end{aligned}$$

by

$$\begin{aligned} &H_1(x, \beta_1, \cdots, \beta_n) = \beta_1 \Phi_1(t) + \cdots + \beta_n \Phi_n(t) + MEF(x), \\ &H_{i+1}(x, \beta_1, \cdots, \beta_n) = \beta_i - \textstyle\int_0^1 (\Phi_1(t))^{-1} f_i(0, t, \beta_1 \Phi_1(t) + \cdots + \beta_n \Phi_n(t) + MEF(x)(t)) dt, \end{aligned}$$

and

$$H(x, \beta_1, \cdots, \beta_n) = (H_1(x, \beta_1, \cdots, \beta_n), \cdots, H_{n+1}(x, \beta_1, \cdots, \beta_n)).$$

If β_i is sufficiently large, we have

$$\Phi_i(t)^{-1} f_i(0, t, \beta_1 \Phi_1(t) + \cdots + \beta_i \Phi_i(t) + \cdots + \beta_n \Phi_n(t) + MEF(x)(t)) > 0$$

and

$$\Phi_i(t)^{-1} f_i(0, t, \beta_1 \Phi_1(t) + \cdots - \beta_i \Phi_i(t) + \cdots + \beta_n \Phi_n(t) + MEF(x)(t)) < 0$$

for all $t \in [0,1]$ and every $x \in \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty)$. Therefore there is some $\alpha_i \geq m\|(\Phi_i)^{-1}\|_\infty$ such that for all $\beta_i \geq \alpha_i$, $x \in \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty)$,

$$H_{i+1}(x, \beta_1, \cdots, \beta_i, \cdots, \beta_n) < \beta_i$$

and

$$H_{i+1}(x, \beta_1, \cdots, -\beta_i, \cdots, \beta_n) > -\beta_i.$$

Letting $\delta = \max\{\alpha_i + m\|(\Phi_i)^{-1}\|_\infty\}$, define $\mathcal{B} = \{(x, \beta_1, \cdots, \beta_n) \in \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \times \mathbb{R}^n : \|x\|_\infty \leq \delta(\|\Phi_1\|_\infty + \cdots + \|\Phi_n\|_\infty) + |\|ME\||m, |\beta_i| \leq \delta \text{ for } i = 1, \cdots, n\}$. Here, $|\|ME\||$ denotes the operator norm of the bounded, linear map ME. Since M is compact, we will show that the completely continuous function H maps the non-empty, closed, bounded, convex set $\mathcal{B}$ into itself. Then the Schauder Fixed Point Theorem will guarantee the existence of a fixed point, $(x, \beta_1, \cdots, \beta_n)$, of H in $\mathcal{B}$. This fixed point is a solution of (3.5).

Note that $\|MEF(x)\|_\infty \leq |\|ME\||m$ for every $x \in \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty)$.

Now if $\beta_i \in [\alpha_i, \delta]$, for all $x \in \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty)$, we have

$$\begin{aligned}
&H_{i+1}(x, \beta_1, \cdots, \beta_i, \cdots, \beta_n) \\
&= \beta_i - \textstyle\int_0^1 (\Phi_i(t))^{-1} f_i(0, t, \beta_1\Phi_1(t) + \cdots + \beta_i\Phi_i(t) + \cdots + \beta_n\Phi_n(t) + MEF(x)(t))dt \\
&\geq \beta_i - \textstyle\int_0^1 |(\Phi_i(t))^{-1}||f_i(0, t, \beta_1\Phi_1(t) + \cdots + \beta_i\Phi_i(t) + \cdots + \beta_n\Phi_n(t) + MEF(x)(t))|dt \\
&\geq \beta_i - \|(\Phi_i)^{-1}\|_\infty m \\
&\geq \beta_i - \alpha_i \\
&\geq 0
\end{aligned}$$

and

$$
\begin{aligned}
&H_{i+1}(x, \beta_1, \cdots, -\beta_i, \cdots, \beta_n) \\
&= -\beta_i - \int_0^1 (\Phi_i(t))^{-1} f_i(0, t, \beta_1\Phi_1(t) + \cdots - \beta_i\Phi_i(t) + \cdots + \beta_n\Phi_n(t) + MEF(x)(t))dt \\
&\leq -\beta_i + \int_0^1 |(\Phi_i(t))^{-1}||f_i(0, t, \beta_1\Phi_1(t) + \cdots - \beta_i\Phi_i(t) + \cdots + \beta_n\Phi_n(t) + MEF(x)(t))|dt \\
&\leq -\beta_i + \|(\Phi_i)^{-1}\|_\infty m \\
&\leq -\beta_i + \alpha_i \\
&\leq 0.
\end{aligned}
$$

Thus, for all $x \in \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty)$ and $\beta_i \in [\alpha_i, \delta]$, $H_{i+1}(x, \beta_1, \cdots, \beta_i, \cdots, \beta_n)$, $H_{i+1}(x, \beta_1, \cdots, -\beta_i, \cdots, \beta_n) \in [-\beta_i, \beta_i] \subseteq [-\delta, \delta]$ for $i = 1, \cdots, n$.

Furthermore, if $0 \leq \beta_i < \alpha_i$, for all $x \in \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty)$,

$$
\begin{aligned}
&|H_{i+1}(x, \beta_1, \cdots, \pm\beta_i, \cdots, \beta_n)| \\
&\leq |\pm\beta_i| + \int_0^1 |(\Phi_i(t))^{-1}||f_i(0, t, \beta_1\Phi_1(t) + \cdots \pm \beta_i\Phi_i(t) + \cdots + \beta_n\Phi_n(t) + MEF(x)(t))dt| \\
&\leq \alpha_i + \|(\Phi_i)^{-1}\|_\infty m \\
&\leq \delta
\end{aligned}
$$

for $i = 1, \cdots, n$.

We have shown that H_{i+1} maps $\mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \times [-\delta, \delta] \times \mathbb{R}^{n-1}$ into $[-\delta, \delta]$.

From this it follows that $H(\mathcal{B}) \subseteq \mathcal{B}$. For if $(x, \beta_1, \cdots, \beta_n) \in \mathcal{B}$, then $H_{i+1}(x, \beta_1, \cdots, \beta_n) \in [-\delta, \delta]$ for $i = 1, \cdots, n$, while

$$
\begin{aligned}
&\|H_1(x, \beta_1, \cdots, \beta_n)\|_\infty \\
&\leq |\beta_1|\|\Phi_1\|_\infty + \cdots + |\beta_n|\|\Phi_n\|_\infty + \|MEF(x)\|_\infty \\
&\leq \delta(\|\Phi_1\|_\infty + \cdots + \|\Phi_n\|_\infty) + |\|ME\||m.
\end{aligned}
$$

We now establish existence of solutions of (3.1), (3.2) for values of ϵ different from zero. It is significant to observe that the nonlinearities $f_i(\epsilon, t, x_1(t), \cdots, x_n(t))$ are allowed to be unbounded.

Theorem 3.3.2 *Suppose that*

i. $b_i + d_i e^{\int_0^1 a_i(s)ds} = 0$ *for all* $i = 1, 2, \cdots, n$;

ii. $f_i : \mathbb{R}^{n+2} \to \mathbb{R}$ *is continuous for all* $i = 1, \cdots, n$;

iii. *For each* $i = 1, \cdots, n$, *there exists* $\gamma_i \in \mathbb{R}$ *such that*
$f_i(0, t, \beta_1, \cdots, \beta_i, \cdots, \beta_n) f_i(0, t, \beta_1, \cdots, -\beta_i, \cdots, \beta_n) < 0$ *whenever* $\beta_i \geq \gamma_i$.

Then, there exists an ϵ_0 *such that for* $\epsilon \in [0, \epsilon_0]$, *there is at least one solution of*

$$\dot{x}_i(t) = a_i(t)x_i(t) + f_i(\epsilon, t, x_1(t), \cdots, x_n(t)), \quad i = 1, 2, \cdots, n,$$

that satisfies

$$b_i x_i(0) + d_i x_i(1) = 0, \quad i = 1, 2, \cdots, n.$$

Proof: As above, we define mappings

$$\begin{aligned}
&H_1 : \mathbb{R} \times \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \times \mathbb{R}^n \to \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \\
&H_{i+1} : \mathbb{R} \times \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \times \mathbb{R}^n \to \mathbb{R} \\
&H : \mathbb{R} \times \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \times \mathbb{R}^n \to \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \times \mathbb{R}^n
\end{aligned}$$

by

$$\begin{aligned}
&H_1(\epsilon, x, \beta_1, \cdots, \beta_n) = \beta_1 \Phi_1 + \cdots + \beta_n \Phi_n + MEF(\epsilon, x), \\
&H_{i+1}(\epsilon, x, \beta_1, \cdots, \beta_n) = \beta_i - \textstyle\int_0^1 \Phi_i(t)^{-1} f_i(\epsilon, t, \beta_1\Phi_1(t) + \cdots + \beta_n\Phi_n(t) + MEF(\epsilon, x)(t)),
\end{aligned}$$

and

$$H(\epsilon, x, \beta_1, \cdots, \beta_n) = (H_1(\epsilon, x, \beta_1, \cdots, \beta_n), \cdots, H_{n+1}(\epsilon, x, \beta_1, \cdots, \beta_n)).$$

By the proof of Lemma 3.3.1, redefining $\alpha_i \geq \|(\Phi_i)^{-1}\|_\infty(m+K)$ and $\delta = \max\{\alpha_i + \|\Phi_i^{-1}\|_\infty(m+K) : i = 1, \cdots, n\}$ for some fixed real number K, we can create a nonempty, convex set $\mathcal{B} = \{(x, \beta_1, \cdots, \beta_n) \in \mathcal{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \times \mathbb{R}^n : \|x\|_\infty \leq \delta(\|\Phi_1\|_\infty + \cdots + \|\Phi_n\|_\infty) + \||ME\||(m+K)$ and $|\beta_i| \leq \delta$ for $i = 1, \cdots, n\}$ such that, when $\epsilon = 0$, the following hold true:

1. for all $\beta_i \geq \alpha_i \geq \|(\Phi_i)^{-1}\|_\infty(m+K)$, $H_{i+1}(0, x, \beta_1, \cdots, \beta_i, \cdots, \beta_n) \leq \beta_i - K$ and $H_{i+1}(0, x, \beta_1, \cdots, -\beta_i, \cdots, \beta_n) \geq -\beta_i + K$;
2. for $\beta_i \in [\alpha_i, \delta]$, $H_{i+1}(0, x, \beta_1, \cdots, \beta_i, \cdots, \beta_n) \geq -K$ and $H_{i+1}(0, x, \beta_1, \cdots, -\beta_i, \cdots, \beta_n) \leq K$;
3. for $0 \leq \beta_i < \alpha_i$, $|H_{i+1}(0, x, \beta_1, \cdots, \pm\beta_i, \cdots, \beta_n)| \leq \delta + K$; and
4. $\|H_1(0, x, \beta_1, \cdots, \beta_n)\|_\infty \leq \delta(\|\Phi_1\|_\infty + \cdots + \|\Phi_n\|_\infty) + \||ME\||(m+K)$.

It is evident that

$$\inf_{(x,\beta_1,\cdots,\beta_n)\in\mathcal{B}} dist(H(0, x, \beta_1, \cdots, \beta_n), \partial\mathcal{B}) > 0;$$

that is, when $\epsilon = 0$, there is a positive distance between the boundary of the set $\mathcal{B}$ and the set of $H(0, x, \beta_1, \cdots, \beta_n)$ for $(x, \beta_1, \cdots, \beta_n) \in \mathcal{B}$. Since $\{\beta_1\Phi_1 + \cdots + \beta_n\Phi_n + MEF(x) | (\beta_1, \cdots, \beta_n, x) \in \mathcal{B}\}$ is equicontinuous and uniformly bounded, it is compact by Arzela-Ascoli's Theorem. This implies that if we choose a positive value, $\tilde{\epsilon}$, so that we restrict ϵ to the interval $[0, \tilde{\epsilon}]$, the map $(\epsilon, \beta_1, \cdots, \beta_n, x) \mapsto H(\epsilon, \beta_1, \cdots, \beta_n, x)$ is uniformly continuous on $\mathcal{B}$. From this it follows that there exists ϵ_0 such that if $|\epsilon| \leq \epsilon_0$,

$$H(\epsilon, \beta_1, \cdots, \beta_n, x) \in \mathcal{B}$$

for all $(\beta_1, \cdots, \beta_n, x) \in \mathcal{B}$. The solvability of the parameter dependent vector equation

$$\dot{x}_i(t) = a_i(t)x_i(t) + f_i(\epsilon, t, x_1(t), \cdots, x_n(t)), \quad i = 1, 2, \cdots, n,$$

that satisfies

$$b_i x_i(0) + d_i x_i(1) = 0, \quad i = 1, 2, \cdots, n,$$

is now a consequence of Schauder's Fixed Point Theorem.

Chapter 4

Discrete Nonlinear Multipoint Boundary Value Problems

4.1 Introduction

In this chapter we study nonlinear discrete systems of the form

$$y(k+n)+\cdots+a_0(k)y(k)=f(y(k))+\sum_{l=0}^{J}w(k,l)g(l,y(l),\cdots,y(l+n-1)) \qquad (4.1)$$

subject to the multipoint boundary conditions

$$\sum_{j=1}^{n}b_{ij}(0)y(j-1)+\sum_{j=1}^{n}b_{ij}(1)y(j)+\cdots+\sum_{j=1}^{n}b_{ij}(J)y(j+J-1)=0 \qquad (4.2)$$

for $i=1,2,\cdots,n$. It will be assumed that the maps f and g are continuous, $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}^{n+1}\to\mathbb{R}$. The function w is real valued and defined for each (k,l) in

$\{0, 1, \cdots, J\} \times \{0, 1, \cdots, J\}$. Notice that when

$$w(k,l) = \begin{cases} 1 \text{ if } k = l \\ 0 \text{ if } k \neq l \end{cases},$$

(4.1) becomes the classical difference equation

$$y(k+n) + \cdots + a_0(k)y(k) = f(y(k)) + g(k, y(k), \cdots, y(k+n-1)).$$

Our attention will be focused on the case where the corresponding linear, homogeneous difference equation

$$y(k+n) + a_{n-1}(k)y(k+(n-1)) + \cdots + a_0(k)y(k) = 0 \tag{4.3}$$

subject to the boundary conditions (4.2) has a one-dimensional solution space. We establish existence of solutions to (4.1),(4.2) using the Brouwer Fixed Point Theorem in conjunction with the Lyapunov-Schmidt Procedure. Our results depend on the limiting behavior of the function f, the solution space of the boundary value problem (4.3),(4.2), and on the size of the nonlinear function g.

Approaches similar to the one presented in this chapter have been successfully used in the analysis of nonlinear boundary value problems for both differential and difference equations. For readers interested in the study of periodicity in discrete or continuous dynamical systems, we suggest [3], [5], [6], [8], [10], [22]. Those interested in nonlocal boundary value problems may consult [7], [12], [13], [15], [16], [18], [19], [20], [21]. Abstract general formulations and applications to strongly nonlinear equations appear in [2], [23].

4.2 Preliminaries

In order to study the solvability of (4.1),(4.2), we rewrite this boundary value problem (4.3),(4.2) in system form. The $n \times n$ matrix $A(k)$ is defined by

$$A(k) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ -a_0(k) & -a_1(k) & -a_2(k) & \cdots & -a_{n-1}(k) \end{bmatrix}$$

and we assume $a_0(k) \neq 0$ for all k. The $n \times n$ boundary matrices B_0, B_1, $\cdots$, B_J are given by

$$B_l = [b_{ij}(l)]$$

for $l \in \{0, 1, \cdots, J\}$. The vector valued function x is given by

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}$$

where $x_1(k) = y(k)$, $x_2(k) = y(k+1), \cdots, x_n(k) = y(k+n-1)$.

We define

$$Z = \{h : \{0, 1, 2, ..., J-1\} \to \mathbb{R}^n\}$$

and

$$X = \{x : \{0, 1, 2, ..., J\} \to \mathbb{R}^n : B_0 x(0) + B_1 x(1) + \cdots + B_J x(J) = 0\}.$$

In each of these spaces we will use the supremum norm; that is, for $x \in X$, $\|x\| = \sup\{|x(k)| : k = 0, 1, \cdots, J\}$ and for $h \in Z$, $\|h\| = \sup\{|h(k)| : k = 0, 1, 2, ..., J-1\}$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$.

The operators L, F, and G are maps from X into Z and are given by

$$L(x)(k) = x(k+1) - A(k)x(k),$$

$$F(x)(k) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(x_1(k)) \end{bmatrix},$$

and

$$G(x)(k) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \sum_{l=0}^{J} w(k,l) g(l, x_1(l), \cdots, x_{n-1}(l)) \end{bmatrix}.$$

It is evident that the boundary value problem (4.1), (4.2) is equivalent to

$$Lx = F(x) + G(x). \tag{4.4}$$

We first consider the linear problem

$$x(k+1) = A(k)x(k) + h(k) \tag{4.5}$$

subject to

$$B_0 x(0) + B_1 x(1) + \cdots + B_J x(J) = 0 \tag{4.6}$$

where we assume that for each nonnegative integer k, $a_0(k) \neq 0$ and that both $x(k)$ and $h(k)$ belong to $\mathbb{R}^n$.

In order to avoid redundancy in the statement of the boundary conditions, we suppose throughout the chapter that the $n \times n(J+1)$ augmented matrix

$$[B_0 : B_1 : \cdots : B_J]$$

has rank n. It should be noted that the rank of the augmented matrix $[B_0 : B_1 : \cdots : B_J]$ is n if and only if $\bigcap_{l=0}^{J} \ker(B_l^T) = \{0\}$.

Using the variation of constants formula, we can write solutions of

$$x(k+1) = A(k)x(k) + h(k)$$

as

$$x(k) = \Gamma(k)x(0) + \Gamma(k)\sum_{l=0}^{k-1} \Gamma^{-1}(l+1)h(l),$$

where $\Gamma(k)$ is the fundamental matrix solution of the homogeneous system $x(k+1) = A(k)x(k)$; that is, $\Gamma(k) = A(k-1)A(k-2)\cdots A(0)$ for $k = 1, 2, ...$ and $\Gamma(0) = I_{nxn}$.

Consequently, x solves the boundary value problem (4.5),(4.6) if and only if

$$x(k) = \Gamma(k)x(0) + \Gamma(k)\sum_{l=0}^{k-1} \Gamma^{-1}(l+1)h(l) \tag{4.7}$$

where

$$\begin{aligned}[B_0 + B_1\Gamma(1) + \cdots + B_J\Gamma(J)]x(0) = -[&B_1\Gamma(1)\Gamma^{-1}(1)h(0) + \cdots \\ &+ B_J\Gamma(J)\sum_{l=0}^{J-1} \Gamma^{-1}(l+1)h(l)].\end{aligned} \tag{4.8}$$

This establishes the fact that (4.1),(4.2) is solvable when $B_1\Gamma(1)\Gamma^{-1}(1)h(0) + \cdots + B_J\Gamma(J)\sum_{l=0}^{J-1}\Gamma^{-1}(l+1)h(l) \in Im(B_0 + B_1\Gamma(1) + \cdots + B_J\Gamma(J))$. It follows that if $B_0 + B_1\Gamma(1) + \cdots + B_J\Gamma(J)$ is invertible, then L is a bijection from X onto Z and the formula for L^{-1} is given by

$$L^{-1}(h)(k) = \Gamma(k)(B_0 + B_1\Gamma(1) + \cdots + B_J\Gamma(J))^{-1}(-(B_1\Gamma(1)\Gamma^{-1}(1)h(0) + \cdots$$

$$+B_J\Gamma(J)\sum_{l=0}^{J-1}\Gamma^{-1}(l)h(l))) + \Gamma(k)\sum_{l=0}^{k-1}\Gamma^{-1}(l)h(l).$$

We will concern ourselves with the case when L is not invertible. We refer the reader to Rodriguez and Taylor [20] for results in the case when L is invertible.

4.3 The Case of Singular L

We now wish to consider the case when the kernel of L is one-dimensional. Since L is not invertible, we can not apply Brouwer's Fixed Point Theorem directly. The ideas presented in this section are standard with the Lyanpuov-Schmidt Procedure and are included for the reader's convenience. The projections we construct have previously appeared in the setting of discrete boundary value problems [19], [20]. Similar projections have also been used in differential and difference equations [2], [5], [6], [7], [11], [13], [14], [15], [16], [17], [21]. The techniques that appear below have been applied to a large number of problems in differential and difference equations [3], [12], [18]. For an abstract formulation of the Lyapunov-Schmidt Procedure and a discussion of applications, we refer the interested reader to [4], [5], [9].

Proposition 4.3.1 *$ker(L)$ and $ker(B_0 + B_1\Gamma(1) + \cdots + B_J\Gamma(J))$ have the same dimension.*

Proof: $Lx = 0$ if and only if $x(k) = \Gamma(k)v$ for some $v \in \ker[(B_0+B_1\Gamma(1)+\cdots+B_J\Gamma(J))]$.

4.3.1 Projection onto ker(L)

Let $\phi(k) = \Gamma(k)v$ where the vector v spans $ker(B_0 + B_1\Gamma(1) + \cdots + B_J\Gamma(J))$. Let

$$C_1 = \sum_{l=0}^{J} |\phi(l)|^2.$$

Clearly, $C_1 \neq 0$.

The proof of the following proposition appears in Rodriguez and Taylor [20].

Proposition 4.3.2 *If we define $P : X \to X$ by*

$$Px(k) = \phi(k)C_1^{-1} \sum_{l=0}^{J} \phi^T(l)x(l),$$

then P is a projection onto ker(L).

4.3.2 Projection onto Im(L)

Before we define our projection, we first need to define components that are vital to the construction of a projection onto the image of L. We define $\psi : \{0, 1, \cdots, J-1\} \to \mathbb{R}^n$ by

$$\psi(k) = \sum_{l=k+1}^{J} [B_l\phi(l)\phi^{-1}(k+1)]^T w$$

where w spans $\ker((B_0+B_1\Gamma(1)+\cdots+B_J\Gamma(J))^T)$. In order to simplify future estimates, we choose w such that $\sup\{|\psi(k)| : k = 0, 1, \cdots, J-1\} \leq \frac{1}{J(J+1)}$.

The proof of the following proposition appears in Rodriguez and Taylor [20].

Proposition 4.3.3 *$Lx = h$ if and only if $\sum_{l=0}^{J-1} \psi^T(l)h(l) = 0$.*

With the above notation, we define

$$C_2 = \sum_{l=0}^{J-1} |\psi(l)|^2.$$

Note that, according to a lemma in Rodriguez and Taylor [20], since $\bigcap_{l=0}^{J} \ker(B_l^T) = \{0\}$, ψ is not the zero map.

We now have the tools we need to define a projection onto the image of L.

Proposition 4.3.4 *If we define $E : Z \to Z$ by*

$$Ex(k) = x(k) - \psi(k) C_2^{-1} \sum_{l=0}^{J-1} \psi^T(l) x(l),$$

then E is a projection onto Im(L).

The proofs showing that P is a projection onto the kernel of L and E is a projection onto the image of L can be found in Rodriguez and Taylor [20].

With the projections described above, we may now write $X = ker(L) \oplus Im(I - P)$ and $Z = Im(L) \oplus Im(I - E)$. Note that $L : Im(I - P) \to Im(L)$ is a bijection and thus there exists a bounded and linear map $M : Im(L) \to Im(I - P)$ such that

1. $LMh = h$ for all $h \in Im(L)$;
2. $MLx = (I - P)x$ for all $x \in X$.

We now analyze $Lx = F(x) + G(x)$ using the Lyapunov-Schmidt Procedure.

Proposition 4.3.5 *$Lx = F(x) + G(x)$ is equivalent to*

$$\begin{cases} x = Px + MEF(x) + MEG(x) \\ \qquad\qquad\qquad\qquad \textit{and} \\ (I - E)F(Px + ME(F(x) + G(x))) + (I - E)G(Px + ME(F(x) + G(x))) = 0. \end{cases}$$

Proof: Clearly $Lx = F(x) + G(x)$ if and only if

$$\begin{cases} E(Lx - (F(x) + G(x))) = 0 \\ \qquad\qquad\qquad\qquad \text{and} \\ (I - E)(Lx - (F(x) + G(x))) = 0. \end{cases}$$

Since E is a projection onto the $Im(L)$, the above set of equations is equivalent to

$$\begin{cases} Lx = E(F(x) + G(x)) \\ \qquad\qquad\qquad\qquad \text{and} \\ (I - E)(F(x) + G(x)) = 0. \end{cases}$$

Using the fact that $MLx = (I - P)x$, we conclude that $Lx = F(x) + G(x)$ if and only if

$$\begin{cases} x = Px + ME(F(x) + G(x)) \\ \qquad\qquad\qquad\qquad \text{and} \\ (I - E)(F(Px + ME(F(x) + G(x))) + G(Px + ME(F(x) + G(x)))) = 0. \end{cases}$$

4.4 Main Results

Using Proposition 4.3.5 it is evident that $Lx = F(x) + G(x)$ is equivalent to

$$\begin{cases} x = \alpha\phi + MEF(x) + MEG(x) \\ \qquad\qquad\qquad \text{and} \\ 0 = \sum_{k=0}^{J-1} \psi_n(k) f(\alpha\phi_1(k) + [ME(F(x) + G(x))]_1(k)) + \\ \\ \sum_{k=0}^{J-1} \psi_n(k) (\sum_{l=0}^{J} w(k,l) g(l, \alpha\phi_1(l) + [ME(F(x) + G(x))](l))) \end{cases}$$

where $\phi_i(k)$, $\psi_i(k)$, and $[ME(F(x) + G(x))]_i(k)$ are the ith entries of $\phi(k)$, $\psi(k)$, and $ME(F(x) + G(x))(k)$, respectively.

Throughout our discussion, we will assume that

$$\lim_{r\to\infty} f(r)$$

and

$$\lim_{r\to-\infty} f(r)$$

both exist. We will denote them as follows

$$\lim_{r\to\infty} f(r) = f(\infty)$$

and

$$\lim_{r\to-\infty} f(r) = f(-\infty).$$

We introduce the notation:

$$O_1 = \{k \in \{0,1,...,J\} : \phi_1(k) > 0\},$$
$$O_2 = \{k \in \{0,1,...,J\} : \phi_1(k) = 0\},$$
$$O_3 = \{k \in \{0,1,...,J\} : \phi_1(k) < 0\}.$$

The number of elements in the set O_2 will be denoted by γ and we define $d = m\gamma$, where $m = \sup\{|f(t)| : t \in \mathbb{R}\}$.

The constants K_1 and K_2 will be given by

$$K_1 = f(\infty)\sum_{O_1}\psi_n(k) + f(-\infty)\sum_{O_3}\psi_n(k),$$

$$K_2 = f(-\infty)\sum_{O_1}\psi_n(k) + f(\infty)\sum_{O_3}\psi_n(k).$$

In our next two results, it is of fundamental importance that $K_1K_2 < 0$ and $d \leq \min\{|K_1|, |K_2|\}$. For the reader's convenience and the sake of simplicity, we will assume that $K_2+d < 0 < K_1-d$. The modifications needed for the case where $K_1+d < 0 < K_2-d$ are straightforward.

Lemma 4.4.1 *Suppose f and g are continuous maps and $f(\infty)$ and $f(-\infty)$ exist. If $K_2 + d < 0 < K_1 - d$ and $|w(k,l)g(s))| < \min\{|K_2 + d|, |K_1 - d|\}$ for all $(k,l) \in \{0,1,\cdots,J\} \times \{0,1,\cdots,J\}$ and $s \in \mathbb{R}^{n+1}$, then there exists a real number α_0 such that for all $\alpha \geq \alpha_0$,*

$$\sum_{k=0}^{J-1}\psi_n(k)f(\alpha\phi_1(k) + [ME(F(x)+G(x))]_1(k)) + \sum_{k=0}^{J-1}\psi_n(k)(\sum_{l=0}^{J} w(k,l)g(l,\alpha\phi_1(l)+$$

$$[ME(F(x)+G(x))](l))) \geq 0$$

and

$$\sum_{k=0}^{J-1} \psi_n(k) f(-\alpha\phi_1(k) + [ME(F(x)+G(x))]_1(k)) + \sum_{k=0}^{J-1} \psi_n(k) (\sum_{l=0}^{J} w(k,l) g(l, -\alpha\phi_1(l) +$$

$$[ME(F(x)+G(x))](l))) \leq 0.$$

Proof: Recall that we have chosen our basis for $\ker(B + D\Gamma(J))$ so that $\|\psi\| \leq \frac{1}{J(J+1)}$. Note that

$$\sum_{k=0}^{J-1} \psi_n(k) f(\alpha\phi_1(k) + [ME(F(x)+G(x))]_1(k)) =$$

$$\sum_{O_1} \psi_n(k) f(\alpha\phi_1(k) + [ME(F(x)+G(x))]_1(k)) +$$

$$\sum_{O_2} \psi_n(k) f(\alpha\phi_1(k) + [ME(F(x)+G(x))]_1(k)) +$$

$$\sum_{O_3} \psi_n(k) f(\alpha\phi_1(k) + [ME(F(x)+G(x))]_1(k)).$$

Since $ME(F+G)$ is bounded,

$$\lim_{\alpha\to\infty} \sum_{k=0}^{J-1} \psi_n(k) f(\alpha\phi_1(k) + [ME(F(x)+G(x))]_1(k)) =$$

$$f(\infty) \sum_{O_1} \psi_n(k) + \sum_{O_2} \psi_n(k) f([ME(F(x)+G(x))]_1(k))$$

$$+ f(-\infty) \sum_{O_3} \psi_n(k) \geq K_1 - d.$$

Similarly,

$$\lim_{\alpha\to-\infty} \sum_{k=0}^{J-1} \psi_n(k) f(\alpha\phi_1(k) + [ME(F(x)+G(x))]_1(k)) =$$

$$f(-\infty)\sum_{O_1}\psi_n(k)+\sum_{O_2}\psi_n(k)f([ME(F(x)+G(x))]_1(k))$$

$$+f(\infty)\sum_{O_3}\psi_n(k)\leq K_2+d.$$

Define $K = \sup\{|w(k,l)g(s)|\}$ for all $(k,l) \in \{0,1,\cdots,J\} \times \{0,1,\cdots,J\}$ and $s \in \mathbb{R}^{n+1}$. Since $K_2+d<0<K_1-d$, there is some α_0 such that for all $\alpha \geq \alpha_0$, $\sum_{k=0}^{J-1}\psi_n(k)f(\alpha\phi_1(k)+[ME(F(x)+G(x))]_1(k)) \geq K$ and $\sum_{k=0}^{J-1}\psi_n(k)f(-\alpha\phi_1(k)+[ME(F(x)+G(x))]_1(k)) \leq -K$. Since $|w(k,l)g(l,\alpha\phi_1(l)+[ME(F(x)+G(x))]_1(l))| \leq K$ for all $(k,l) \in \{0,1,2,...\} \times \{0,1,2,...\}$, for $\alpha \geq \alpha_0$ and $x \in X$, $\sum_{k=0}^{J-1}\psi_n(k)f(\alpha\phi_1(k)+[ME(F(x)+G(x))]_1(k))+\sum_{k=0}^{J-1}\psi_n(k)(\sum_{l=0}^{J}w(k,l)g(l,\alpha\phi_1(l)+[ME(F(x)+G(x))]_1(l)))$ $\geq (K-K)=0$. Similarly, for $\alpha \geq \alpha_0$ and $x \in X$, $\sum_{k=0}^{J-1}\psi_n(k)f(-\alpha\phi_1(k)+[ME(F(x)+G(x))]_1(k))+$
$\sum_{k=0}^{J-1}\psi_n(k)(\sum_{l=0}^{J}w(k,l)g(l,\alpha\phi_1(l)+[ME(F(x)+G(x))]_1(l))) \leq (-K+K)=0$.

We will now use this lemma to prove the following theorem:

Theorem 4.4.2 *Suppose that*

1. *$dim(ker(B_0+B_1\Gamma(1)+\cdots+B_J\Gamma(J)))=1$;*
2. *$f:\mathbb{R}\to\mathbb{R}$ is continuous and $f(\infty)$ and $f(-\infty)$ exist;*
3. *$K_2+d<0<K_1-d$;*
4. *$g:\mathbb{R}^{n+1}\to\mathbb{R}$ is continuous, w is real valued and defined for each (k,l) in $\{0,1,\cdots,J\}\times\{0,1,\cdots,J\}$, and $|w(k,l)g(s)|<\min\{|K_2+d|,|K_1-d|\}$ for all $s\in\mathbb{R}^{n+1}$.*

Then there exists at least one solution of

$$y(k+n)+\cdots+a_1(k)y(k+1)+a_0(k)y(k) = f(y(k))+\sum_{l=0}^{J} w(k,l)g(l,y(l),\cdots,y(l+n-1))$$

that satisfies

$$\sum_{j=1}^{n} b_{ij}(0)y(j-1)+\sum_{j=1}^{n} b_{ij}(1)y(j)+\cdots+\sum_{j=1}^{n} b_{ij}(J)y(j+J-1) = 0.$$

Proof: We define mappings

$$H_1 : \mathbb{R}\times X \to X$$
$$H_2 : \mathbb{R}\times X \to \mathbb{R}$$
$$H : \mathbb{R}\times X \to \mathbb{R}\times X$$

by

$$H_1(\alpha,x) = \alpha\phi + MEF(x) + MEG(x),$$
$$H_2(\alpha,x) = \alpha - (\textstyle\sum_{k=0}^{J-1}\psi_n(k)f(\alpha\phi_1(k)+[ME(F(x)+G(x))]_1(k))+$$
$$\textstyle\sum_{k=0}^{J-1}\psi_n(k)(\sum_{l=0}^{J} w(k,l)g(l,\alpha\phi_1(l)+[ME(F(x)+G(x))]_1(l))),$$

and

$$H(\alpha,x) = (H_1(\alpha,x),H_2(\alpha,x)).$$

As in the proof of Lemma 4.4.1, we define $K = \sup\{|w(k,l)g(s)|\}$ for all $(k,l)\in\{0,1,\cdots,J\}\times\{0,1,\cdots,J\}$ and $s\in\mathbb{R}^{n+1}$. Then, by Lemma 4.4.1, there exists $\alpha_0 \geq m$ where $m = \sup\{|f(t)| : t\in\mathbb{R}\}$ such that for all $\alpha\geq\alpha_0$, $H_2(\alpha,x)\leq\alpha$ and $H_2(-\alpha,x)\geq -\alpha$.

Letting $\delta = \alpha_0 + (m+K)$, define $\mathcal{B} = \{(\alpha,x)\in\mathbb{R}\times X : |\alpha|\leq\delta$ and $\|x\|\leq \delta\|\phi\| + |||ME|||(m+K)\}$. Here, we denote by $|||ME|||$ the norm on the space of bounded linear functions.

Note that $\|MEF(x)\| \leq \||ME\||m$ and $\|MEG(x)\| \leq \||ME\||K$ for every $x \in X$.

For the next step in the proof, recall that $\|\psi\| \leq \frac{1}{J(J+1)}$. Now if $\alpha \in [\alpha_0, \delta]$, for all $x \in X$, we have

$$
\begin{aligned}
H_2(\alpha, x) = \alpha - (&\sum_{k=0}^{J-1} \psi_n(k) f(\alpha\phi_1(k) + [ME(F(x)+G(x))]_1(k)) + \\
&\sum_{k=0}^{J-1} \psi_n(k)(\sum_{l=0}^{J} w(k,l) g(l, \alpha\phi_1(l) + [ME(F(x)+G(x))]_1(l)))) \\
&\geq \alpha - (\sum_{k=0}^{J-1} |\psi_n(k)||f(\alpha\phi_1(k) + [ME(F(x)+G(x))]_1(k))| + \\
&\sum_{k=0}^{J-1} |\psi_n(k)|(\sum_{l=0}^{J} |w(k,l)||g(l, \alpha\phi_1(l) + [ME(F(x)+G(x))]_1(l))|)) \\
&\geq \alpha - (m + K) \\
&\geq \alpha - \alpha_0 - K \\
&\geq -K \\
&\geq -\delta
\end{aligned}
$$

and

$$
\begin{aligned}
H_2(-\alpha,x) &= -\alpha - (\sum_{k=0}^{J-1}\psi_n(k)f(\alpha\phi_1(k)+[ME(F(x)+G(x))]_1(k))+ \\
&\sum_{k=0}^{J-1}\psi_n(k)(\sum_{l=0}^{J}w(k,l)g(l,\alpha\phi_1(l)+[ME(F(x)+G(x))]_1(l)))) \\
&\leq -\alpha + \sum_{k=0}^{J-1}|\psi_n(k)||f(\alpha\phi_1(k)+[ME(F(x)+G(x))]_1(k))|+ \\
&\sum_{k=0}^{J-1}|\psi_n(k)|(\sum_{l=0}^{J}|w(k,l)||g(l,\alpha\phi_1(l)+[ME(F(x)+G(x))]_1(l))|) \\
&\leq -\alpha + (m+K) \\
&\leq -\alpha + \alpha_0 + K \\
&\leq K \\
&\leq \delta.
\end{aligned}
$$

Thus, for all $x \in X$ and $\alpha \in [\alpha_0, \delta]$, $H_2(\alpha, x)$, $H_2(-\alpha, x) \in [-\alpha, \alpha] \subseteq [-\delta, \delta]$.

Furthermore, if $0 \leq \alpha < \alpha_0$, for all $x \in X$,

$$
\begin{aligned}
|H_2(\pm\alpha,x)| &\leq |\pm\alpha| + \sum_{k=0}^{J-1}|\psi_n(k)||f(\alpha\phi_1(k)+[ME(F(x)+G(x))]_1(k))|+ \\
&\sum_{k=0}^{J-1}|\psi_n(k)|\left(\sum_{l=0}^{J}|w(k,l)||g(l,\alpha\phi_1(l)+[ME(F(x)+G(x))]_1(l))|\right) \\
&\leq \alpha_0 + (m+K) \\
&\leq \delta.
\end{aligned}
$$

We have shown that H_2 maps $[-\delta,\delta]\times X$ into $[-\delta,\delta]$. From this it follows that $H(\mathcal{B}) \subseteq \mathcal{B}$.

For if $(\alpha, x) \in \mathcal{B}$, then $H_2(\alpha, x) \in [-\delta, \delta]$, while

$$\begin{aligned}\|H_1(\alpha, x)\| &\leq |\alpha|\|\phi\| + \|ME(F(x) + G(x))\| \\ &\leq \delta\|\phi\| + \||ME\||m + \|ME\|K.\end{aligned}$$

Hence, the continuous function H maps the non-empty, closed, bounded, convex set $\mathcal{B}$ into itself. Therefore, the Brouwer Fixed Point Theorem guarantees existence of at least one fixed point, $\tilde{x}$, of H in $\mathcal{B}$. For each such $\tilde{x}$, $\tilde{y} = \tilde{x}_1$ is a solution of

$$y(k+n) + \cdots + a_1(k)y(k+1) + a_0(k)y(k) = f(y(k)) + \sum_{l=0}^{J} w(k,l)g(l, y(l), \cdots, y(l+n-1))$$

which satisfies

$$\sum_{j=1}^{n} b_{ij}(0)y(j-1) + \sum_{j=1}^{n} b_{ij}(1)y(j) + \cdots + \sum_{j=1}^{n} b_{ij}(J)y(j+J-1) = 0.$$

4.5 Unbounded Perturbation

In this section, we consider the case where the perturbation of f is allowed to be unbounded, but controlled by a small parameter ϵ. More precisely, we consider dynamic equations of the form

$$y(k+n) + \cdots + a_0(k)y(k) = f(y(k)) + \epsilon \sum_{l=0}^{J} w(k,l)g(l, y(l), \cdots, y(l+n-1))$$

subject to boundary conditions

$$\sum_{j=1}^{n} b_{ij}(0)y(j-1) + \sum_{j=1}^{n} b_{ij}(1)y(j) + \cdots + \sum_{j=1}^{n} b_{ij}(J)y(j+J-1) = 0.$$

Theorem 4.5.1 *Suppose that*

1. $\dim(\ker(B_0 + B_1\Gamma(1) + \cdots + B_J\Gamma(J))) = 1$;

2. $f : \mathbb{R} \to \mathbb{R}$ *is continuous and* $f(\infty)$ *and* $f(-\infty)$ *exist;*

3. $K_2 + d < 0 < K_1 - d$;

4. $g : \mathbb{R}^{n+1} \to \mathbb{R}$ *is continuous and* w *is real valued and defined for each* $(k, l) \in \{0, 1, \cdots, J\} \times \{0, 1, \cdots, J\}$.

Then, there exists an ϵ_0 *such that for* $\epsilon \in [0, \epsilon_0]$, *there exists at least one solution of*

$$y(k+n)+\cdots+a_1(k)y(k+1)+a_0(k)y(k) = f(y(k))+\epsilon\sum_{l=0}^{J} w(k,l)g(l, y(l), \cdots, y(l+n-1))$$

that satisfies

$$\sum_{j=1}^{n} b_{ij}(0)y(j-1) + \sum_{j=1}^{n} b_{ij}(1)y(j) + \cdots + \sum_{j=1}^{n} b_{ij}(J)y(j+J-1) = 0.$$

Proof: Let $G : \mathbb{R} \times X \to Z$ be defined by

$$G(\epsilon, x)(k) = \begin{bmatrix} 0 \\ \vdots \\ \epsilon \sum_{l=0}^{J} w(k,l)g(l, x_1(l), \cdots, x_{n-1}(l)) \end{bmatrix}.$$

As above, we define mappings

$$H_1 : \mathbb{R} \times \mathbb{R} \times X \to X$$
$$H_2 : \mathbb{R} \times \mathbb{R} \times X \to \mathbb{R}$$
$$H : \mathbb{R} \times \mathbb{R} \times X \to \mathbb{R} \times X$$

by

$$H_1(\epsilon, \alpha, x) = \alpha\phi + MEF(x) + MEG(\epsilon, x),$$

$$H_2(\epsilon, \alpha, x) = \alpha - (\textstyle\sum_{k=0}^{J-1} \psi_n(k) f(\alpha\phi_1(k) + [ME(F(x) + G(\epsilon, x))]_1(k)) +$$

$$\epsilon \textstyle\sum_{k=0}^{J-1} \psi_n(k) (\sum_{l=0}^{J} w(k,l) g(l, \alpha\phi_1(l) + [ME(F(x) + G(x))]_1(l)))),$$

and

$$H(\epsilon, \alpha, x) = (H_1(\epsilon, \alpha, x), H_2(\epsilon, \alpha, x)).$$

We choose a value $K > 0$ so that $K < \min\{|K_2 + d|, |K_1 - d|\}$. By the proof of Theorem 4.4.2, again defining α_0 and $\delta = \alpha_0 + (m + K)$ as above, we can create a nonempty, convex set $\mathcal{B} = \{(\alpha, x) \in \mathbb{R} \times X : |\alpha| \le \delta \text{ and } \|x\| \le \delta\|\phi\| + \||ME\||(m + K)\}$ such that, when $\epsilon = 0$, the following hold true:

1. for all $\alpha \ge \alpha_0 \ge m$, $H_2(0, \alpha, x) \le \alpha - K$ and $H_2(0, -\alpha, x) \ge -\alpha + K$;
2. for $\alpha \in [\alpha_0, \delta]$, $H_2(0, \alpha, x) \ge 0$ and $H_2(0, -\alpha, x) \le 0$;
3. for $0 \le \alpha < \alpha_0$, $|H_2(0, \pm\alpha, x)| \le \delta + K$; and
4. $\|H_1(0, \alpha, x)\| \le \delta\|\phi\| + \||ME\||m$.

It is evident that

$$\inf_{(\alpha,x)\in\mathcal{B}} dist(H(0, \alpha, x), \partial\mathcal{B}) > 0;$$

that is, when $\epsilon = 0$, there is a positive distance between the boundary of the set $\mathcal{B}$ and the set of $H(0, \alpha, x)$ for $(\alpha, x) \in \mathcal{B}$. Since X, Z are finite dimensional, it is obvious that $\mathcal{B}$ is compact. Therefore, if we choose a positive value, $\tilde{\epsilon}$, so that we restrict ϵ to the interval $[0, \tilde{\epsilon}]$, the map $(\epsilon, \alpha, x) \mapsto H(\epsilon, \alpha, x)$ is uniformly continuous on $\mathcal{B}$. From this it follows that there exists ϵ_0 such that if $|\epsilon| \leq \epsilon_0$,

$$H(\epsilon, \alpha, x) \in \mathcal{B}$$

for all $(\alpha, x) \in \mathcal{B}$. The solvability of the difference equation

$$y(k+n)+\cdots+a_1(k)y(k+1)+a_0(k)y(k) = f(y(k))+\epsilon\sum_{l=0}^{J} w(k,l)g(l, y(l), \cdots, y(l+n-1))$$

that satisfies

$$\sum_{j=1}^{n} b_{ij}(0)y(j-1) + \sum_{j=1}^{n} b_{ij}(1)y(j) + \cdots + \sum_{j=1}^{n} b_{ij}(J)y(j+J-1) = 0$$

is now a consequence of Brouwer's Fixed Point Theorem.

4.6 Example

The example, which we now consider, is a generalization of the example found in Rodriguez and Taylor [20]. The theory which we have developed in this chapter allows us to consider more general nonlinearities in the dynamic equation. We consider the difference equation

$$y(k+2) + 3y(k+1) + 2y(k) = f(y(k)) + \sum_{l=0}^{J} w(k,l)g(l, y(l), y(l+1)) \tag{4.9}$$

subject to boundary conditions

$$y(0) + (2 - 2^{\frac{3-J}{2}})y\left(\frac{J-1}{2}\right) + (1 - 2^{\frac{3-J}{2}})y\left(\frac{J+1}{2}\right) = 0,$$
$$y(J) + y(J+1) = 0, \tag{4.10}$$

where J and $\frac{J-1}{2}$ are odd integers and J is larger than 1.

In system form, (4.9),(4.10) becomes

$$x(k+1) = A(k)x(k) + F(x(k)) + G(x(k))$$
$$B_0 x(0) + B_{\frac{J-1}{2}} x\left(\frac{J-1}{2}\right) + B_J x(J) = 0 \tag{4.11}$$

where $x_1(k) = y(k)$, $x_2(k) = y(k+1)$,

$$A(k) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \text{ for all k,}$$

$$B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_{\frac{J-1}{2}} = \begin{bmatrix} 2 - 2^{\frac{3-J}{2}} & 1 - 2^{\frac{3-J}{2}} \\ 0 & 0 \end{bmatrix}, \quad B_J = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix},$$

$$F(x(k)) = \begin{bmatrix} 0 \\ f(x_1(k)) \end{bmatrix}, \text{ and } G(x(k)) = \begin{bmatrix} 0 \\ \sum_{l=0}^{J} w(k,l)g(l,x(l)) \end{bmatrix}.$$

Since $A(k)$ is constant, $\Gamma(k) = A^k$ for $k = 0, 1, 2, \cdots, J$. The linear structure of this problem is the same as that in Rodriguez and Taylor [20]. Here, we utilize the calculations that appear in this previous paper. Since J is odd,

$$\Gamma(J) = A^J = \begin{bmatrix} 2^J - 2 & 2^J - 1 \\ 2 - 2^{J+1} & 1 - 2^{J+1} \end{bmatrix}$$

and

$$B_0 + B_{\frac{J-1}{2}} A^{\frac{J-1}{2}} + B_J A^J = \begin{bmatrix} 1 & 1 \\ 2^J & 2^J \end{bmatrix},$$

which gives

$$\ker(B_0 + B_{\frac{J-1}{2}} A^{\frac{J-1}{2}} + B_J A^J) = \text{ span } \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

From this, we conclude that

$$\phi(k) = A^k \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} (-1)^k \\ (-1)^{k+1} \end{bmatrix}$$

for $k = 0, 1, 2, \cdots, J$. It follows that $O_1 = \{k \in \{0, 1, ..., J\} : \phi_1(k) > 0\} = \{0, 2, 4, \cdots, J-1\}$, $O_2 = \{k \in \{0, 1, ..., J\} : \phi_1(k) = 0\} = \emptyset$, and $O_3 = \{k \in \{0, 1, ..., J\} : \phi_1(k) < 0\} = \{1, 3, 5, \cdots, J\}$.

It is easy to verify that $\bigcap_{i=0}^{J} \ker(B_i^T) = \{0\}$. In [20], we see that

$$\ker((B_0 + B_{\frac{J-1}{2}} A^{\frac{J-1}{2}} + B_J A^J)^T) = \text{ span } \left\{ \begin{bmatrix} 2^J \\ -1 \end{bmatrix} \right\}.$$

This gives

$$\psi(k)=\begin{cases}\begin{pmatrix}(-1)^{k+1}2^J(2-2^{-(k+1)})\\(-1)^{k+1}2^J(1-2^{-(k+1)})\end{pmatrix} & \text{for } k=0,1,\cdots,\frac{J-3}{2},\\ \begin{pmatrix}(-1)^{k+1}2^{J-(k+1)}\\(-1)^{k+1}2^{J-(k+1)}\end{pmatrix} & \text{for } k=\frac{J-1}{2},\cdots,J-1.\end{cases},$$

$$K_1=-f(\infty)\left(2^{J-2}(J+1)+\left(\frac{-2^{J+1}+2^{\frac{J+3}{2}}-1}{3}\right)\right)+$$
$$f(-\infty)\left(2^{J-2}(J-3)+\left(\frac{2^{\frac{J+3}{2}}-2^J-2}{3}\right)\right),$$

and

$$K_2=-f(-\infty)\left(2^{J-2}(J+1)+\left(\frac{-2^{J+1}+2^{\frac{J+3}{2}}-1}{3}\right)\right)+$$
$$f(\infty)\left(2^{J-2}(J-3)+\left(\frac{2^{\frac{J+3}{2}}-2^J-2}{3}\right)\right).$$

It can easily be shown that if $J\geq 5$ and $f(\infty)<\frac{19}{6}f(-\infty)$, we are guaranteed that $K_2<0<K_1$. Then, according to Theorem 4.4.2, if $|w(k,l)g(s)|<\min\{|K_2|,|K_1|\}$ for all $s\in\mathbb{R}^3$, (4.9),(4.10) will have a solution. A simple example would be the case where:

$$f(k)=\begin{cases}\frac{30}{\pi}\tan^{-1}(k) & \text{for } k\geq 0,\\ \frac{-100}{\pi}\tan^{-1}(k) & \text{for } k<0.\end{cases}$$

It can be verified that for $J\geq 5$, $K_2\leq-\frac{4300}{3}$ and $K_1\geq 25$. From this, it follows that the boundary value problem (4.9),(4.10) will have a solution whenever $|w(k,l)g(s)|<25$ for all $(k,l)\in\{0,1,2,...,J\}\times\{0,1,2,...,J\}$ and $s\in\mathbb{R}^3$.

REFERENCES

[1] R.P. Agarwal, S. Djebali, T. Moussaoui, and O.G. Mustafa, On the asymptotic integration of nonlinear differential equations, *Journal of Computational and Applied Mathematics* **2** (2007), pp. 352-376.

[2] S. Bancroft, J.K. Hale and D. Sweet, Alternative problems for nonlinear functional equations, *Journal of Differential Equations* **4** (1968), pp. 40-56.

[3] L. Cesari, Functional analysis and periodic solutions of nonlinear differential equations, *Contributions to Differential Equations* **1** (1963), pp. 149-187.

[4] L. Cesari, Functional analysis and Galerkin's method, *Michigan Mathematical Journal* **11** (1964), pp. 385-414.

[5] S. Chow and J.K. Hale, Methods of Bifurcation Theory, Springer, Berlin (1982).

[6] Debra L. Etheridge and Jesús Rodríguez, Periodic solutions of nonlinear discrete-time systems, *Applicable Analysis* **62** (1996), pp. 119-137.

[7] Debra L. Etheridge and Jesús Rodríguez, Scalar discrete nonlinear two-point boundary value problems, *Journal of Difference Equations and Applications* **4** (1998), pp. 127-144.

[8] A. Halanay, Solutions periodiques et presque-periodiques des systemes d'equations aux differences finies, *Arch Rational Mech. Anal.* **12** (1963), pp. 134-149.

[9] J.K. Hale, Applications of alternative problems, *Lecture Notes* **vol. 71-1**, Brown University, Providence, RI (1971).

[10] J.K. Hale, Ordinary Differential Equations, Robert E. Kreiger Publishing Company, Malabar, Florida (1980).

[11] D.C. Lewis, On the role of first integrals in the pertubation of periodic solutions, *The Annals of Mathematics, Second Series* **63** (1956), pp. 535-548.

[12] Jesús Rodríguez, An alternative method for boundary value problems with large nonlinearities, *Journal of Differential Equations* **43** (1982), pp. 157-167.

[13] Jesús Rodríguez, Nonlinear differential equations under Stieltjes boundary conditions, *Nonlinear Analysis* **7** (1983), pp. 107-116.

[14] Jesús Rodríguez, On resonant discrete boundary value problems, *Applicable Analysis*, **19** (1985), pp. 265-274.

[15] Jesús Rodríguez, Galerkin's method for ordinary differential equations subject to generalized nonlinear boundary conditions, *Journal of Differential Equations* **97** (1992), pp. 112-126.

[16] Jesús Rodríguez, Nonlinear discrete Sturm-Liouville problems, *Journal of Mathematical Analysis and Applications* **308** (2005), pp. 380-391.

[17] Jesús Rodríguez and Kristen Abernathy, On the solvability of nonlinear boundary value problems, *Differential Equations & Applications* **2** (2010), pp. 487-501.

[18] Jesús Rodríguez and D. Sweet, Projection methods for nonlinear boundary value problems, *Journal of Differential Equations* **58** (1985), pp. 282-293.

[19] Jesús Rodríguez and Padraic Taylor, Weakly nonlinear discrete multipoint boundary value problems, *Journal of Mathematical Analysis and Applications* **329** (2007), pp. 77-91.

[20] Jesús Rodríguez and Padraic Taylor, Scalar discrete nonlinear multipoint boundary value problems, *Journal of Mathematical Analysis and Applications* **330** (2007), pp. 876-890.

[21] Jesús Rodríguez and Padraic Taylor, Multipoint boundary value problems for nonlinear ordinary differential equations, *Nonlinear Analysis* **68** (2008), pp. 3465-3474

[22] N. Rouche and J. Mawhin, Ordinary Differential Equations, Pitman, London (1980).

[23] Wayne Spealman and Daniel Sweet, The alternative method for solutions in the kernel of a bounded linear functional, *Journal of Differential Equations* **37** (1980), pp. 297-302.