ABSTRACT

ABERNATHY, ZACHARY JOHN. Nonlinear Dynamic Equations subject to Global and Periodic Boundary Conditions. (Under the direction of Jesús Rodríguez.)

In this manuscript, we study nonlinear dynamic equations subject to global and periodic boundary conditions. We first analyze nonlinear difference equations of the form

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \psi(x(t)) = G(x(t))$$

subject to the global boundary conditions

$$\begin{cases} \alpha x(a) + \beta \Delta x(a) + \eta_1(x) = \phi_1(x) \\ \\ \gamma x(b+1) + \delta \Delta x(b+1) + \eta_2(x) = \phi_2(x). \end{cases}$$

By using properties of the nonlinearities which occur in both the dynamic equation and in the boundary conditions, we are able to provide sufficient conditions for the existence of solutions.

We then study nonlinear differential equations of the form

$$(p(t)x'(t))' + q(t)x(t) + \psi(x(t)) = G(x(t))$$

subject to

$$\begin{cases} \alpha x(0) + \beta x'(0) + \eta_1(x) = \phi_1(x) \\ \\ \gamma x(1) + \delta x'(1) + \eta_2(x) = \phi_2(x). \end{cases}$$

Again, we establish sufficient conditions for the solvability of these equations by analyzing the relationship between the nonlinearities and a related linear Sturm-Liouville problem.

We conclude by finding periodic solutions to a system of nonlinear difference equations of the form

$$\Delta x(t) = f(\epsilon, t, x(t)).$$

The solution space of the corresponding linear homogeneous equation is n-dimensional, and accordingly we use a projection scheme and fixed point argument to establish the existence of solutions.

Nonlinear Dynamic Equations subject to Global and Periodic Boundary Conditions

by
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A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh, North Carolina

2011

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BIOGRAPHY

Zachary John Abernathy was born on August 20, 1984 in Winston-Salem, North Carolina and grew up in the neighboring town of Lewisville. His parents are Amy and Ronnie Abernathy, who at the time of this writing have been happily married for 28 years. He attended West Forsyth High School where he spent his time participating in Math ACE tournaments and running on the varsity cross country team, and graduated as the salutatorian in 2002. He then went on to Wake Forest University as a Nancy Susan Reynolds Scholar and graduated Summa Cum Laude with honors in mathematics in 2006 with a Bachelor of Science degree in both mathematics and physics.

It was during his junior year at Wake that Zach met his future wife, Kristen, who was then obtaining her Master of Arts degree in mathematics. They both decided to go to North Carolina State University to pursue a Ph.D. in mathematics, and they were married on June 28th, 2008.

While at North Carolina State University, Zach has taught a wide variety of courses, including the full three-semester calculus sequence and both ordinary and partial differential equations. He received his Master of Science degree in mathematics in 2008 and was given the Outstanding Teaching Assistant Award in 2008 and 2009. In 2010, he graduated from the Certificate of Accomplishment in Teaching program and became a Preparing the Professoriate Fellow. He also became a "Thank a Teacher" recipient in 2011.

ACKNOWLEDGEMENTS

I wish to thank Dr. Jesús Rodríguez for his continued guidance and support throughout the entirety of the five years that I've known him. I truly appreciate the amount of time and effort you have invested in the development of this work, and your confidence in my abilities has always helped me navigate the rollercoaster ride that is mathematical research. I have always enjoyed our discussions about the joys and challenges found in family life and academia, and I hope to emulate our relationship with my own research students in the future.

My graduate school experience has been enriched by the wonderful people I have met in the mathematics department at NC State. I would like to thank Miriam Ansley, Seyma Bennett-Shabbir, Di Buckland, Carolyn Gunton, Nicole Newkirk Dahlke, Denise Seabrooks, and Charlene Wallace for their advice, humor, and help. You have brightened my day many a time. I would also like to thank Dr. Robert Martin, Dr. Jesús Rodríguez, Dr. Stephen Schecter, and Dr. James Selgrade for taking the time out of their schedules to be on my thesis advisory committee. Another big thank-you goes to Dr. Alina Chertock, Dr. John Griggs, Dr. Barbi Honeycutt, and Dr. Jesús Rodríguez for preparing letters of recommendation for the job application process.

I would also like to thank my fellow graduate students for their companionship throughout my years at NC State, especially Ralph Abbey, Catherine Buell, Jeb Collins, Rebecca Kalhorn, Nathaniel Schwartz, Kristen Stagg, Ryan Therkelsen, and Tom Wears. Whether it has been discussing teaching techniques, playing/talking basketball, or supporting each other through the job search process, your friendship is much appreciated.

My success in graduate school is due in large part to the wonderful math faculty I worked with during my undergraduate studies at Wake Forest University. I want to thank Dr. John Baxley and Dr. Kenneth Berenhaut for introducing me to the process of mathematical research and encouraging my development as a mathematician. A special thank-you goes to Dr. Stephen Robinson. You have continued to be an amazing mentor as both a teacher and scholar, and I appreciate all of the opportunities you have given me to facilitate my professional development.

I would finally like to thank my family for all of their love and support. Tony and Shelly, you have become as close as family to us and your friendship has made our lives in Raleigh so much more fulfilled. Pat and Lenny, thank you so much for your unselfishness and guidance. You have always been ready to help us at the drop of a hat. Mom and Dad, you have taught me so much about patience, hard work, and the importance of family. Your example motivates me more than you know, and you always help me remember to laugh.

To my son, Justin, thank you for bringing a smile to my face every day. Watching you grow up always puts things in perspective. And to my wife and best friend, Kristen, your love and kindness has brought me so much joy. Going on this journey with you has been one of best experiences of my life.

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Chapter 1

Introduction

This paper is devoted to the study of various general classes of discrete and continuous nonlinear boundary value problems. In Chapters 2 and 3, we consider nonlinear Sturm-Liouville problems subject to global nonlinear boundary conditions. We use the Global Inverse Function Theorem together with either Brouwer's or Schauder's fixed point theorem to provide sufficient conditions for the existence of solutions. In Chapter 4, we establish the existence of periodic solutions to a system of nonlinear difference equations using the Lyapunov-Schmidt procedure along with Brouwer's fixed point theorem.

We begin in Chapter 2 by studying nonlinear difference equations of the form

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \psi(x(t)) = G(x(t))$$

subject to the global boundary conditions

$$\begin{cases} \alpha x(a) + \beta\Delta x(a) + \eta_1(x) = \phi_1(x) \\ \\ \gamma x(b+1) + \delta\Delta x(b+1) + \eta_2(x) = \phi_2(x). \end{cases}$$

We provide sufficient conditions for the existence of solutions based on properties of the nonlinearities and the eigenvalues of an associated linear Sturm-Liouville problem.

In Chapter 3, we consider a continuous analog of the problem in Chapter 2. Namely, we study the nonlinear differential equation

$$(p(t)x'(t))' + q(t)x(t) + \psi(x(t)) = G(x(t))$$

subject to general non-local boundary conditions of the form

$$\begin{cases} \alpha x(0) + \beta x'(0) + \eta_1(x) = \phi_1(x) \\ \\ \gamma x(1) + \delta x'(1) + \eta_2(x) = \phi_2(x). \end{cases}$$

We are again able to establish sufficient conditions for the existence of solutions using properties of the nonlinearities and their relationship with the eigenvalues of an associated linear Sturm-Liouville problem. However, due to the continuous nature of the problem, our function spaces become infinite-dimensional and the resulting analysis is much more delicate.

The purpose of Chapter 4 is to search for periodic solutions to a system of nonlinear difference equations of the form

$$\Delta x(t) = f(\epsilon, t, x(t)).$$

The corresponding linear homogeneous system has an n-dimensional kernel, i.e. the system is at full resonance. We provide sufficient conditions for the existence of periodic solutions based on asymptotic properties of the nonlinearity f when $\epsilon = 0$. By allowing for higher-dimensional solution spaces of the associated linear problem as well as for more general asymptotic behavior of the nonlinear function f, our results complement previous work in the study of periodic discrete dynamical systems.

As evidenced above, the reader should note that a similar approach is used in Chapters 2 and 3, with Chapter 4 using a distinct methodology. However, each chapter is self-contained and may be fully understood without any prerequisite knowledge of the other chapters.

Chapter 2

Nonlinear Discrete Sturm-Liouville Problems with Global Boundary Conditions

2.1 Introduction

We study the existence of solutions of the discrete boundary value problem

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \psi(x(t)) = G(x(t)) \tag{2.1}$$

subject to the global boundary conditions

$$\begin{cases} \alpha x(a) + \beta\Delta x(a) + \eta_1(x) = \phi_1(x) \\ \\ \gamma x(b+1) + \delta\Delta x(b+1) + \eta_2(x) = \phi_2(x). \end{cases} \tag{2.2}$$

Throughout our discussion, X will denote the set of real-valued functions defined on the integers $[a, b+2]$ and Y will denote the set of real-valued functions defined on the integers $[a+1, b+1]$. We shall assume G, ϕ_1, and ϕ_2 are continuous, where $G : X \to Y$, $\phi_1 : X \to \mathbb{R}$, and $\phi_2 : X \to \mathbb{R}$. The functions η_1 and η_2 are continuously Fréchet differentiable from X into $\mathbb{R}$. We also assume $\psi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $p(t)$ is defined and positive on $[a, b+1]$, $q(t)$ is defined on $[a+1, b+1]$, and the boundary conditions (2.2) are such that $\alpha^2+\beta^2 \neq 0, \gamma^2+\delta^2 \neq 0, \alpha \neq \beta, \gamma \neq \delta$. In order to study the boundary value problem (2.1)-(2.2), we will first need to consider the related boundary value problem

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \psi(x(t)) = h(t) \tag{2.3}$$

subject to

$$\begin{cases} \alpha x(a) + \beta \Delta x(a) + \eta_1(x) = v_1 \\ \\ \gamma x(b+1) + \delta \Delta x(b+1) + \eta_2(x) = v_2 \end{cases} \tag{2.4}$$

where $h(t)$ is defined on $[a+1, b+1]$ and $v_1, v_2 \in \mathbb{R}$. With a non-resonance type assumption on ψ, together with bounds on the Fréchet derivatives of η_1, η_2, we will use the Global Inverse Function Theorem to show the existence of a unique solution to (2.3)-(2.4). If, in addition, G, ϕ_1, and ϕ_2 satisfy appropriate growth hypotheses, we will use the above-mentioned result in conjunction with the Brouwer Fixed Point Theorem to obtain the existence of at least one solution to (2.1)-(2.2).

The results presented in this chapter complement previous work in the field of non-linear boundary value problems. Readers interested in differential equations subject to global constraints may consult [3, 7, 22, 25, 28, 31]. Discrete systems subject to a variety of boundary conditions appear in [1, 10, 23, 24, 26, 29, 30]. Related references in the study of periodic behavior for discrete dynamical systems are [9, 11].

We shall rewrite the boundary value problem (2.1)-(2.2) as an operator equation of the form

$$\mathcal{L}x - \Psi x = \mathcal{G}x.$$

The study of equations of this form has been frequent in the literature, where $\mathcal{L}$ is a linear differential expression, Ψ is a continuously Fréchet differentiable operator, and $\mathcal{G}$ is an operator with bounded range. Indeed, Dolph [8] studied analagous Hammerstein integral equations, while the case when $\mathcal{L}$ is an ordinary differential operator has been studied extensively in papers such as Leach [19], Lazer and Leach [17], Lazer and Sanchez [18], and Brown [4]. Brown and Lin [5] were able to allow for the case when $\mathcal{G}$ was unbounded but was subject to a sub-linear growth requirement. Landesman and Lazer [16] considered the case when $\mathcal{L}$ is a self adjoint partial differential operator.

Our formulation of the operator equation $\mathcal{L}x - \Psi x = \mathcal{G}x$ incorporates both the dynamics and the boundary conditions. The non-local boundary conditions (2.2) and (2.4) are significantly more general than those that have previously appeared in the above-mentioned analogous results for differential equations.

2.2 Preliminaries

Let $N = b - a + 1$, let X be the set of real-valued functions x defined on $[a, b+2]$, and let Y be the set of real-valued functions y defined on $[a+1, b+1]$.

X and Y are finite-dimensional Hilbert spaces with respect to the discrete inner product

$$< x_1, x_2 >= \sum_t x_1(t)x_2(t),$$

where the sum is taken over all integer t values in the above intervals of definition. We shall use the inner product norm

$$\|x\| = \sqrt{< x, x >}$$

on each set. When we compute the norm of a bounded operator, we will use the operator norm and denote it $|||\cdot|||$. We will also use $|\cdot|$ to denote the Euclidean norm on $\mathbb{R}^n$ for any positive integer n. Finally, we shall use the norm $\left\| \begin{bmatrix} h \\ v \end{bmatrix} \right\| = \max\{\|h\|, |v|\}$ on the set $Y \times \mathbb{R}^2$.

We define $\mathcal{A} : X \to Y$ and $\mathcal{B} : X \to \mathbb{R}^2$ by

$$\mathcal{A}x(t) = \Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) \text{ for } t \in [a+1, b+1],$$

$$\mathcal{B}(x) = \begin{bmatrix} \alpha x(a) + \beta \Delta x(a) \\ \gamma x(b+1) + \delta \Delta x(b+1) \end{bmatrix},$$

and we let $\mathcal{L} : X \to Y \times \mathbb{R}^2$ be given by

$$\mathcal{L}(x) = \begin{bmatrix} \mathcal{A}(x) \\ \mathcal{B}(x) \end{bmatrix}.$$

We define $\Psi : X \to Y \times \mathbb{R}^2$ by

$$\Psi(x) = \begin{bmatrix} -\omega(x) \\ -\eta(x) \end{bmatrix},$$

where $\omega : X \to Y$ is given by $\omega(x)(t) = \psi(x(t))$ for $t \in [a+1, b+1]$, and $\psi \in C^1(\mathbb{R}, \mathbb{R})$. The map $\eta : X \to \mathbb{R}^2$ is given by $\eta(x) = \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \end{bmatrix}$, where η_1 and η_2 are continuously Fréchet differentiable from $X \to \mathbb{R}$.

Let $\mathcal{G} : X \to Y \times \mathbb{R}^2$ be given by

$$\mathcal{G}(x) = \begin{bmatrix} G(x) \\ \phi(x) \end{bmatrix},$$

where G is a continuous function from $X \to Y$ and $\phi(x) = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix}$, where ϕ_1 and ϕ_2 are continuous functions from $X \to \mathbb{R}$.

Note that the problem (2.1)-(2.2) is equivalent to solving

$$\mathcal{L}x - \Psi(x) = \mathcal{G}x$$

and the problem (2.3)-(2.4) is equivalent to solving

$$\mathcal{L}x - \Psi(x) = \begin{bmatrix} h \\ v \end{bmatrix},$$

where $h \in Y,\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$.

We shall first consider the related linear Sturm-Liouville problem

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \lambda x(t) = 0 \tag{2.5}$$

together with the homogeneous boundary conditions

$$\begin{cases} \alpha x(a) + \beta \Delta x(a) = 0 \\ \gamma x(b+1) + \delta \Delta x(b+1) = 0 \end{cases} \tag{2.6}$$

which may be equivalently expressed as

$$\mathcal{A}x + \lambda x = 0, \quad \mathcal{B}x = 0.$$

The following proposition is taken from Kelley, Peterson [14].

Proposition 2.2.1. *Suppose that* $\alpha^2 + \beta^2 \neq 0, \gamma^2 + \delta^2 \neq 0, \alpha \neq \beta, \gamma \neq \delta$. *The Sturm-Liouville problem* (5)-(6) *has* N *real, simple eigenvalues* $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ *and* N *corresponding linearly independent eigenfunctions* $x_i(t), 1 \leq i \leq N$. *If* λ_n, λ_m *are distinct eigenvalues, then* x_n, x_m *are orthogonal, i.e.* $< x_n, x_m >= 0$.

Remark 2.2.2. *The eigenfunctions may be chosen to be real and normalized, i.e.* $< x_k, x_k > = 1, \ \ 1 \leq k \leq N$. *We shall make this convention throughout the remainder of the chapter.*

Remark 2.2.3. *If* h *is an arbitrary real-valued function defined on* $[a+1, b+1]$, *we may* *express the Fourier series of* h *with respect to the eigenfunctions of the Sturm-Liouville problem* (5)-(6) *as*

$$h(t) = \sum_{k=1}^{N} < h, x_k > x_k(t).$$

2.3 The $\mathcal{L}x - \Psi(x) = [h, v]^T$ Case

Recall that solving the problem (2.3)-(2.4) is equivalent to solving

$$\mathcal{L}x - \Psi(x) = \begin{bmatrix} h \\ v \end{bmatrix}.$$

Since X is finite-dimensional with dimension $N+2$, $\mathcal{L}$ is in fact a bounded, linear operator, and hence is continuously Fréchet differentiable, with:

$$\mathcal{L}'(x)(u)(t) = \mathcal{L}u(t) \quad \text{for all} \;\; u, x \in X, t \in [a+1, b+1].$$

Ψ is also continuously Fréchet differentiable, with:

$$\Psi'(x)(u)(t) = \begin{bmatrix} -\psi'(x(t))u(t) \\ -\eta'(x)u \end{bmatrix} \quad \text{for all} \;\; u, x \in X, t \in [a+1, b+1].$$

To establish the existence of a solution to (2.3)-(2.4), we shall use the following version of the Global Inverse Function Theorem, proved in Chow, Hale [7]:

Theorem 2.3.1. *Suppose* X, Y *are Banach spaces,* $\Phi \in C^1(X, Y)$*, and for each* $x \in X$*,*

$\Phi'(x)$ is a bijection from X onto Y. If there is a constant K such that $|||[\Phi'(x)]^{-1}||| \leq K$ for all $x \in X$, then Φ is a homeomorphism of X onto Y.

Throughout the remainder of the chapter, we will assume that the following two conditions hold:

(**H.1**) There exists an interval $[c, d]$ which does not contain any of the eigenvalues λ_i of the Sturm-Liouville problem (5)-(6) and such that $c \leq \psi'(s) \leq d, \quad s \in \mathbb{R}$.

(**H.2**) There exists a constant ζ_0 such that the non-linear boundary operator η satisfies

$$\sup\{|||\eta'(x)||| : x \in X\} = \zeta_0 < \infty.$$

There are three cases in which condition (H.1) may arise. First, we may have

$$\lambda_{m-1} < c \leq \psi'(s) \leq d < \lambda_m, \quad s \in \mathbb{R},$$

where λ_{m-1}, λ_m are consecutive eigenvalues of (2.5)-(2.6). This is the usual form of condition (H.1) considered in the continuous analogs of equations (2.1) and (2.3). However, we also have two other possibilities for satisfying (H.1):

$$c \leq \psi'(s) \leq d < \lambda_1$$

and, due to the finite number of eigenvalues in the discrete case,

$$\lambda_N < c \leq \psi'(s) \leq d.$$

We let

$$\mu = \frac{d+c}{2}, \quad \Gamma = \frac{d-c}{2},$$

and define $\mathcal{L}_\mu = \begin{bmatrix} \mathcal{A} + \mu I \\ \mathcal{B} \end{bmatrix}$.

Let $\{w_1, w_2\}$ form a basis for the two-dimensional solution space of the second-order difference equation

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \mu x(t) = 0,$$

or equivalently, $(\mathcal{A} + \mu I)x = 0$. We assume without loss of generality that w_1, w_2 are taken such that $\|w_1\|^2 + \|w_2\|^2 \leq 1$. We shall denote $w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$.

We define the 2×2 matrix $B = [\ \mathcal{B}(w_1) \mid \mathcal{B}(w_2)\ ]$.

Recall that the eigenvalues and corresponding eigenfunctions of the linear Sturm-Liouville problem (2.5)-(2.6) are denoted by $\{\lambda_k, x_k\}$ for $1 \leq k \leq N$. In our discussion of the solvability of the boundary value problem (2.3)-(2.4), we will use the following properties of the operator $\mathcal{L}_\mu$.

Lemma 2.3.2. *$\mathcal{L}_\mu$ is a bijection from X onto $Y \times \mathbb{R}^2$. For any* $\begin{bmatrix} h \\ v \end{bmatrix} \in Y \times \mathbb{R}^2$,

$$\mathcal{L}_\mu^{-1} \begin{bmatrix} h \\ v \end{bmatrix} = \sum_{k=1}^{N} \frac{< h, x_k >}{\mu - \lambda_k} x_k + w(t)^T B^{-1} v.$$

Moreover,

$$|||\mathcal{L}_\mu^{-1}||| \leq \rho + |||B^{-1}|||,$$

where $\rho = \sup\{|\mu - \lambda_k|^{-1}, k \in [1, N]\}$.

Proof. First note that $\mu \neq \lambda_i$ implies that $\mathcal{L}_\mu$ is one-to-one. Given any $\begin{bmatrix} h \\ v \end{bmatrix} \in Y \times \mathbb{R}^2$, we search for a solution to $\mathcal{L}_\mu(x) = \begin{bmatrix} h \\ v \end{bmatrix}$ of the form

$$x(t) = \sum_{k=1}^{N} \alpha_k x_k(t) + c_1 w_1(t) + c_2 w_2(t), \tag{2.7}$$

where $\{x_k\}$ are the eigenfunctions of the Sturm-Liouville problem (2.5)-(2.6), i.e. $\mathcal{A}x_k + \lambda_k x_k = 0$, $\mathcal{B}x_k = 0$.

Since $(\mathcal{A} + \mu I)(c_1 w_1 + c_2 w_2) = 0$ and $\mathcal{B}\left(\sum_{k=1}^{N} \alpha_k x_k\right) = 0$, we see that $\mathcal{L}_\mu(x) = \begin{bmatrix} h \\ v \end{bmatrix}$ if and only if

$$(\mathcal{A} + \mu I) \sum_{k=1}^{N} \alpha_k x_k = h \tag{2.8}$$

and

$$\mathcal{B}(c_1 w_1 + c_2 w_2) = v. \tag{2.9}$$

Given $h \in Y$, we can use the Fourier series expansion in terms of the eigenfunctions $\{x_k\}$

to solve for α_k in (2.8) as follows:

$$(\mathcal{A}+\mu I)\sum_{k=1}^{N}\alpha_k x_k = h$$

$$(\mathcal{A}+\mu I)\sum_{k=1}^{N}\alpha_k x_k = \sum_{k=1}^{N} < h, x_k > x_k$$

$$\sum_{k=1}^{N}(-\lambda_k+\mu)\alpha_k x_k = \sum_{k=1}^{N} < h, x_k > x_k.$$

Hence, $\alpha_k = \frac{<h,x_k>}{\mu-\lambda_k}$.

Next, note that (2.9) may be rewritten

$$c_1\mathcal{B}(w_1) + c_2\mathcal{B}(w_2) = v$$

or

$$B\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = v,$$

where the 2×2 matrix $B = [\ \mathcal{B}(w_1) \mid \mathcal{B}(w_2)\ ]$. The matrix B is invertible, due to the fact that the boundary value problem $(\mathcal{A}+\mu I)x = 0, \mathcal{B}x = 0$ has only the trivial solution. Hence, we have found

$$c_1 w_1(t) + c_2 w_2(t) = w(t)^T B^{-1} v.$$

Therefore, for any $\begin{bmatrix} h \\ v \end{bmatrix} \in Y \times \mathbb{R}^2$,

$$\mathcal{L}_\mu^{-1} \begin{bmatrix} h \\ v \end{bmatrix} = \sum_{k=1}^{N} \frac{< h, x_k >}{\mu - \lambda_k} x_k + w(t)^T B^{-1} v.$$

To estimate the norm of $\mathcal{L}_\mu^{-1}$, we observe that since the eigenfunctions $\{x_k\}$ form an orthonormal set,

$$\| \sum_{k=1}^{N} \frac{< h, x_k >}{\mu - \lambda_k} x_k \| \leq \left(\sup_{k \in [1,N]} \frac{1}{|\mu - \lambda_k|} \right) \|h\|,$$

and therefore, recalling that $\|w_1\|^2 + \|w_2\|^2 \leq 1$, we obtain

$$|||\mathcal{L}_\mu^{-1}||| \leq \rho + |||B^{-1}|||.$$

$\square$

The value of ρ will depend on which of the three cases of condition (H.1) occurs. If we have

$$\lambda_{m-1} < c \leq \psi'(s) \leq d < \lambda_m, \quad s \in \mathbb{R},$$

then

$$\rho = \sup_{k\in[1,N]} \frac{1}{|\mu - \lambda_k|} = \max\{\frac{1}{\mu - \lambda_{m-1}}, \frac{1}{\lambda_m - \mu}\}.$$

Similarly, if $c \le \psi'(s) \le d < \lambda_1$, then

$$\rho = \sup_{k\in[1,N]} \frac{1}{|\mu - \lambda_k|} = \frac{1}{\lambda_1 - \mu},$$

and if $\lambda_N < c \le \psi'(s) \le d$, then

$$\rho = \sup_{k\in[1,N]} \frac{1}{|\mu - \lambda_k|} = \frac{1}{\mu - \lambda_N}.$$

It is an immediate consequence of the definitions of ρ, Γ that $\rho\Gamma < 1$. This fact is of fundamental importance in the results that follow.

We now present sufficient conditions for the solvability of the boundary value problem (2.3)-(2.4).

Theorem 2.3.3. *Suppose conditions (H.1) and (H.2) hold. If*

$$\rho\Gamma + |||B^{-1}|||\zeta_0 < 1,$$

then for each h defined on $[a+1, b+1]$ and each $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$*, the boundary value problem* (2.3)-(2.4) *has a unique solution.*

Proof. The boundary value problem (2.3)-(2.4) is equivalent to the operator equation

$$\mathcal{L}x - \Psi(x) = \begin{bmatrix} h \\ v \end{bmatrix}.$$

Since $\mathcal{L}$ and Ψ are both Fréchet differentiable, so is $\mathcal{L} - \Psi$.

We shall first establish the fact that for each $x \in X$, $\mathcal{L} - \Psi'(x)$ is a bijection from X onto $Y \times \mathbb{R}^2$.

We define the operator $\Psi_\mu : X \to Y \times \mathbb{R}^2$ by

$$\Psi_\mu(x) = \begin{bmatrix} -\omega(x) + \mu I \\ -\eta(x) \end{bmatrix},$$

where $\omega : X \to Y$ is given by $\omega(x)(t) = \psi(x(t))$ for $t \in [a+1, b+1]$.

Consider the equation

$$[\mathcal{L} - \Psi'(x)]u = \begin{bmatrix} h \\ v \end{bmatrix}, \quad u \in X, \begin{bmatrix} h \\ v \end{bmatrix} \in Y \times \mathbb{R}^2.$$

Note that the equation $[\mathcal{L} - \Psi'(x)]u = \begin{bmatrix} h \\ v \end{bmatrix}$ is equivalent to

$$\mathcal{L}_\mu u - \Psi'_\mu(x)u = \begin{bmatrix} h \\ v \end{bmatrix}$$

$$u - \mathcal{L}_\mu^{-1}\Psi'_\mu(x)u = \mathcal{L}_\mu^{-1}\begin{bmatrix} h \\ v \end{bmatrix}$$

$$[I - \mathcal{L}_\mu^{-1}\Psi'_\mu(x)]u = \mathcal{L}_\mu^{-1}\begin{bmatrix} h \\ v \end{bmatrix}.$$

Our next goal is to estimate the norm of $\mathcal{L}_\mu^{-1}\ \Psi'_\mu(x)$.

It is clear that

$$\Psi'_\mu(x)(u) = \begin{bmatrix} -\omega'(x)u + \mu u \\ -\eta'(x)u \end{bmatrix},$$

where $\omega'(x)(u)(t) = -\psi'(x(t))u(t)$.

Using conditions (H.1) and (H.2), we obtain

$$|-\psi'(x(t)) + \mu| \leq \Gamma \text{ for all } x \in X \text{ and } t \in [a+1, b+1],$$

and

$$\sup\{|||\eta'(x)||| : x \in X\} = \zeta_0.$$

We now appeal to the formula for $\mathcal{L}_\mu^{-1}$ in Lemma 2.3.2 to obtain

$$|||\mathcal{L}_\mu^{-1}\Psi'_\mu(x)||| \leq \rho\Gamma + |||B^{-1}|||\zeta_0.$$

Recall, for any bounded operator A with $|||A||| < 1$, we have that $I - A$ has a bounded inverse and $|||(I-A)^{-1}||| \leq (1 - |||A|||)^{-1}$.

Thus, by choosing ζ_0 small enough so that $\rho\Gamma + |||B^{-1}|||\zeta_0 < 1$, the operator $I - \mathcal{L}_\mu^{-1}\Psi'_\mu(x)$

has a bounded inverse which implies $(\mathcal{L} - \Psi'(x))^{-1}$ exists for each $x \in X$, and

$$(\mathcal{L} - \Psi'(x))^{-1} = [I - \mathcal{L}_\mu^{-1}\Psi'_\mu(x)]^{-1}\mathcal{L}_\mu^{-1}$$

$$|||(\mathcal{L} - \Psi'(x))^{-1}||| \leq \left(\frac{1}{1 - (\rho\Gamma + |||B^{-1}|||\zeta_0)}\right)(\rho + |||B^{-1}|||)$$

$$= \frac{\rho + |||B^{-1}|||}{1 - \rho\Gamma - |||B^{-1}|||\zeta_0} = K,$$

a bound independent of x. Hence, $\mathcal{L} - \Psi$ satisfies the hypotheses of Theorem 2.3.1, which implies there is a unique solution to the boundary value problem (2.3)-(2.4) for each $h(t)$ defined on $[a+1, b+1]$ and each $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$. This completes the proof. $\square$

We mention as a corollary to Theorem 2.3.3 the case when the non-linear boundary operators η_1 and η_2 are identically zero.

Corollary 2.3.4. *Suppose condition (H.1) holds. Then the boundary value problem*

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \psi(x(t)) = h(t)$$

subject to

$$\begin{cases} \alpha x(a) + \beta \Delta x(a) = v_1 \\ \\ \gamma x(b+1) + \delta \Delta x(b+1) = v_2 \end{cases}$$

has a unique solution for each h *defined on* $[a+1, b+1]$ *and each* $\begin{bmatrix} v_1 \\ \\ v_2 \end{bmatrix} \in \mathbb{R}^2$.

Proof. Using the above notation in the proof of Theorem 2.3.3, we have that $\zeta_0 = 0$, which together with the fact that $\rho\Gamma < 1$, implies that $(\mathcal{L} - \Psi'(x))^{-1}$ exists for each $x \in X$, and

$$(\mathcal{L} - \Psi'(x))^{-1} = [I - \mathcal{L}_\mu^{-1}\Psi'_\mu(x)]^{-1}\mathcal{L}_\mu^{-1}$$

$$|||(\mathcal{L} - \Psi'(x))^{-1}||| \le \left(\frac{1}{1-\rho\Gamma}\right)\left(\rho + |||B^{-1}|||\right)$$

$$= \frac{\rho + |||B^{-1}|||}{1-\rho\Gamma} = K',$$

a bound independent of x. Hence, $\mathcal{L} - \Psi$ again satisfies the hypotheses of Theorem 2.3.1, which implies the result. □

Remark 2.3.5. *We note that the above estimate of the norm of* $(\mathcal{L} - \Psi'(x))^{-1}$ *in Theorem* 2.3.3 *provides some insight into the relationship between the distribution of the eigenvalues to the linear Sturm-Liouville problem* (2.5)-(2.6)*, the permissible range of the derivative of the nonlinearity* ψ*, and the size of* $|||\eta'(x)|||$*. As we allow* $\psi'(s)$ *to lie closer to consecutive* *eigenvalues of the linear problem (by letting $\rho\Gamma$ approach 1), we must be more restrictive with $\eta'(x)$ and choose ζ_0 small enough to ensure $\rho\Gamma + |||B^{-1}|||\zeta_0 < 1$. On the other hand, if we are more restrictive with ψ' by keeping the interval $[c, d]$ far away from any eigenvalues of* (2.5)-(2.6)*, we can loosen the condition on $\eta'(x)$ and allow ζ_0 to be larger. We also observe that when $d < \lambda_1$ or $c > \lambda_N$, all we further require of ψ' is for it to be bounded, which in the latter case would not typically be possible in a continuous analog with an infinite number of eigenvalues.*

2.4 The $\mathcal{L}x - \Psi(x) = \mathcal{G}x$ Case

We now study the existence of solutions to the boundary value problem (2.1)-(2.2), or equivalently,

$$\mathcal{L}x - \Psi(x) = \mathcal{G}x,$$

where $\mathcal{G} : X \to Y \times \mathbb{R}^2$ is continuous.

Continuing with our supposition of conditions (H.1), (H.2), we first mention the following important property of the operator $(\mathcal{L} - \Psi)^{-1}$. This property was observed in the continuous analog of equations (2.1) and (2.3) by Brown [4]. We are able to supply a different proof using the Fréchet differentiability of $\mathcal{L}$.

Proposition 2.4.1. $(\mathcal{L} - \Psi)^{-1} : Y \times \mathbb{R}^2 \to X$ *is continuously Fréchet differentiable and Lipschitz continuous with Lipschitz constant* $K = \frac{\rho + |||B^{-1}|||}{1 - \rho\Gamma - |||B^{-1}|||\zeta_0}$.

Proof. For each $x \in X$, $\begin{bmatrix} y \\ v \end{bmatrix} \in Y \times \mathbb{R}^2$ such that $(\mathcal{L} - \Psi)(x) = \begin{bmatrix} y \\ v \end{bmatrix}$, we have that

$(\mathcal{L}-\Psi'(x))^{-1}$ exists by Lemma 2.3.2 and is the derivative of $(\mathcal{L}-\Psi)^{-1}\begin{bmatrix} y \\ v \end{bmatrix}$. Moreover, since $(\mathcal{L}-\Psi'(x))^{-1}$ has a uniform bound of K, it follows that $(\mathcal{L}-\Psi)^{-1}$ is Lipschitz continuous with constant K by the Mean Value Theorem for Fréchet derivatives. $\square$

In the following result, we provide sufficient conditions for the solvability of the boundary value problem (2.1)-(2.2), which is equivalent to

$$\mathcal{L}x - \Psi(x) = \mathcal{G}x,$$

where $\mathcal{G}(x) = \begin{bmatrix} G(x) \\ \phi(x) \end{bmatrix}$.

In [5], Brown and Lin establish the existence of solutions to a continuous analog of equation (2.1) subject to linear homogeneous boundary conditions. It should be observed that the operator G represents a nonlinearity in the dynamic equation, and the nonlinear boundary operator ϕ allows for the possibility of more general global boundary conditions.

Let $C = \|(\mathcal{L}-\Psi)^{-1}(0)\|$.

Theorem 2.4.2. *Suppose ψ, η satisfy conditions (H.1), (H.2), $\rho\Gamma + |||B^{-1}|||\zeta_0 < 1$, and $\mathcal{G} : X \to Y \times \mathbb{R}^2$ is continuous. If there exists a constant $M > 0$ such that for $\|x\| \le M$, $\|\mathcal{G}(x)\| \le \frac{M-C}{K}$, then there exists at least one solution of the boundary value problem (2.1)-(2.2).*

Proof. As noted earlier, solving (2.1)-(2.2) is equivalent to solving

$$\mathcal{L}x - \Psi(x) = \mathcal{G}(x)$$

$$x = (\mathcal{L} - \Psi)^{-1}\mathcal{G}(x).$$

Let $H = (\mathcal{L} - \Psi)^{-1} \circ \mathcal{G}$, and define

$$\mathcal{B}_M = \{x \in X : \|x\| \leq M\}.$$

Then $H(\mathcal{B}_M) \subseteq \mathcal{B}_M$, since for $\|x\| \leq M$,

$$\begin{aligned}
\|H(x)\| &= \|(\mathcal{L} - \Psi)^{-1}\mathcal{G}(x)\| \\
&\leq K\|\mathcal{G}(x)\| + \|(\mathcal{L} - \Psi)^{-1}(0)\| \\
&= K\|\mathcal{G}(x)\| + C \\
&\leq K\left(\frac{M - C}{K}\right) + C = M.
\end{aligned}$$

Since H is a continuous function, the Brouwer Fixed Point Theorem guarantees existence of at least one fixed point of H in $\mathcal{B}_M$. □

Remark 2.4.3. *With the appearance of the Lipschitz constant K for $(\mathcal{L} - \Psi)^{-1}$ in the bound for $\|\mathcal{G}(x)\|$ on the ball $\mathcal{B}_M$ above, we again note the connection between the distribution of eigenvalues of the linear problem* (2.5)-(2.6) *and the permissible size of the nonlinearity $\mathcal{G}$. The further apart the eigenvalues of* (2.5)-(2.6)*, the smaller the value of K can be made, and hence the size of $\|\mathcal{G}(x)\|$ is allowed to be larger.*

As with the corollary to Theorem 2.3.3, we state the previous result in the case when the non-linear boundary operators η_1 and η_2 are identically zero.

Corollary 2.4.4. *Suppose condition (H.1) holds. If there exists a constant $M > 0$ such that for $\|x\| \leq M$, $\|\mathcal{G}(x)\| \leq \frac{M-C}{K}$, then the boundary value problem*

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \psi(x(t)) = G(x(t))$$

subject to

$$\begin{cases} \alpha x(a) + \beta \Delta x(a) = \phi_1(x) \\ \\ \gamma x(b+1) + \delta \Delta x(b+1) = \phi_2(x) \end{cases}$$

has at least one solution.

We mention as another corollary to Theorem 2.4.2 the special case when $\mathcal{G}$ obeys a sub-linear growth condition. The proof of this corollary is immediate.

Corollary 2.4.5. *Suppose (H.1) and (H.2) hold, $\rho\Gamma + \||B^{-1}\||\zeta_0 < 1$, and there is an $0 \leq \epsilon < 1$ such that $\|\mathcal{G}(x)\| \leq b_1 + b_2\|x\|^\epsilon$. Then there exists at least one solution of the boundary value problem* (2.1)-(2.2).

We next note that when G is a Nemytskii-type operator, i.e. if there exists a $g : [a+1, b+1] \times \mathbb{R} \to \mathbb{R}$ such that $G(x)(t) = g(t, x(t))$, we are able to state a condition on g, ϕ to satisfy the growth hypothesis on $\mathcal{G}$ in Theorem 2.4.2, and thus establish the existence of a solution to (2.1)-(2.2) in this case.

Corollary 2.4.6. *Suppose (H.1) and (H.2) hold, $\rho\Gamma + |||B^{-1}|||\zeta_0 < 1$. Assume $g(t,\cdot)$ is continuous for each $t \in [a+1, b+1]$ and there exists an $M > 0$ such that $|\phi(x)| \le \frac{M-C}{K}$ for all $\|x\| \le M$ and*

$$|g(t,s)| \le \frac{M-C}{K\sqrt{N}}$$

for all $|s| \le M$ and $t \in [a+1, b+1]$. Then $\mathcal{G} : X \to Y \times \mathbb{R}^2$ defined by $\mathcal{G}(x) = \begin{bmatrix} g(\cdot, x(\cdot)) \\ \phi(x) \end{bmatrix}$ satisfies the hypotheses of Theorem 2.4.2.

Proof. Choose $x \in X$ such that $\|x\| \le M$, where M is defined as above. Then $|x(t)| \le M$ for each $t \in [a+1, b+1]$, which implies $|g(t, x(t))| \le \frac{M-C}{K\sqrt{N}}$ for each $t \in [a+1, b+1]$. It follows that

$$\|\mathcal{G}(x)\| = \max\{\left(\sum_{t=a+1}^{b+1} [g(t,x(t))]^2\right)^{1/2}, |\phi(x)|\}$$

$$\le \max\{\left(\sum_{t=a+1}^{b+1} \left(\frac{M-C}{K\sqrt{N}}\right)^2\right)^{1/2}, \frac{M-C}{K}\} = \frac{M-C}{K},$$

hence $\mathcal{G}$ satisfies the hypotheses of Theorem 2.4.2. □

Remark 2.4.7. *In a similar fashion, we mention that when G is the equivalent of an integral operator in the discrete time scale, i.e. there exists a $g : [a+1, b+1] \times \mathbb{R} \to \mathbb{R}$ such that $G(x)(t) = \sum_{k=a+1}^{t} g(k, x(k))$, we can again give conditions on g, ϕ to ensure that $\mathcal{G}$ satisfies the hypotheses of Theorem* 2.4.2. *Indeed, suppose $g(t,\cdot)$ is continuous for each $t \in [a+1, b+1]$ and there exists an $M > 0$ such that $|\phi(x)| \le \frac{M-C}{K}$ for all $\|x\| \le M$ and*

$$|g(t,s)| \leq \frac{M-C}{KN^{3/2}}$$

for all $|s| \leq M$ *and for each* $t \in [a+1, b+1]$*. Then* $\mathcal{G} : X \to Y \times \mathbb{R}^2$ *defined by* $\mathcal{G}(x) = \begin{bmatrix} \sum_{k=a+1}^{\cdot} g(k, x(k)) \\ \phi(x) \end{bmatrix}$ *satisfies the hypotheses of Theorem* 2.4.2 *by a straightforward calculation of* $\|\mathcal{G}(x)\|$ *as in the previous corollary.*

Chapter 3

On the Solvability of Sturm-Liouville Problems with Non-Local Boundary Conditions

3.1 Introduction

We consider the solvability of the boundary value problem

$$(p(t)x'(t))' + q(t)x(t) + \psi(x(t)) = G(x(t)) \tag{3.1}$$

subject to the global boundary conditions

$$\begin{cases} \alpha x(0) + \beta x'(0) + \eta_1(x) = \phi_1(x) \\ \\ \gamma x(1) + \delta x'(1) + \eta_2(x) = \phi_2(x). \end{cases} \tag{3.2}$$

We assume $\psi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $p(t) > 0$ and $q(t)$ is real on $[0,1]$, p, p', q are continuous on $(0,1)$, and the boundary conditions (3.2) are such that $\alpha^2 + \beta^2 \neq 0, \gamma^2 + \delta^2 \neq 0$. $G, \eta_1, \eta_2, \phi_1$, and ϕ_2 shall be nonlinear operators defined on a function space. To study the solvability of the boundary value problem (3.1)-(3.2), we will begin by considering the related boundary value problem

$$(p(t)x'(t))' + q(t)x(t) + \psi(x(t)) = h(t) \tag{3.3}$$

subject to

$$\begin{cases} \alpha x(0) + \beta x'(0) + \eta_1(x) = v_1 \\ \\ \gamma x(1) + \delta x'(1) + \eta_2(x) = v_2 \end{cases} \tag{3.4}$$

where h is square-integrable on $[0,1]$ and $v_1, v_2 \in \mathbb{R}$. The solvability of (3.3)-(3.4) will be established using the Global Inverse Function Theorem. In order to do so, we will first ensure that a related linearized boundary value problem is non-resonant. Having done so, we will use the Schauder Fixed Point Theorem to prove the existence of solutions to (3.1)-(3.2).

The results obtained in this chapter depend in a crucial way on the relationship between the eigenvalues of a linear Sturm-Liouville problem and the rate of growth of nonlinearities present in both the differential equation and the boundary conditions. In this respect, our work complements that of Dolph [8], Landesman and Lazer [16], Lazer and Leach [17], Lazer and Sanchez [18], Leach [19], Brown [4], Brown and Lin [5], and Rodríguez and Abernathy [27]. Relevant references for those interested in nonlocal boundary value problems in differential equations are $[3, 7, 22, 25, 28, 31]$. For results concerning discrete systems under a variety of boundary conditions, the interested reader may consult [1, 9, 10, 11, 23, 24, 26, 29, 30].

3.2 Preliminaries

Our goal shall be to restate the above problems (3.1)-(3.2) and (3.3)-(3.4) as operator equations defined on a function space. To this end, let $L^2 = L^2[0,1]$ be the set of real-valued square-integrable functions defined on the interval $[0,1]$. L^2 is a Hilbert space with respect to the inner product

$$< x_1, x_2 >= \int_0^1 x_1(t)x_2(t)dt.$$

This inner product induces the usual inner product norm

$$\|x\| = \sqrt{< x, x >}$$

on this space. The operator norm will be used to compute the norm of any bounded operator, and we denote it $|||\cdot|||$. The Euclidean norm on $\mathbb{R}^n$ for any positive integer n will be denoted by $|\cdot|$. Finally, the norm used on the set $L^2 \times \mathbb{R}^2$ will be $\| \begin{bmatrix} h \\ v \end{bmatrix} \| = \max\{\|h\|, |v|\}$.

We define $\mathcal{L} : D(\mathcal{L}) \to L^2 \times \mathbb{R}^2$ by

$$\mathcal{L}(x) = \begin{bmatrix} \mathcal{A}(x) \\ \mathcal{B}(x) \end{bmatrix},$$

where $D(\mathcal{L}) \subset L^2$ is defined to be

$D(\mathcal{L}) = \{x \in L^2 : x, x' \text{ are absolutely continuous and } x'' \in L^2\}$, and $\mathcal{A} : D(\mathcal{L}) \to L^2$ and $\mathcal{B} : D(\mathcal{L}) \to \mathbb{R}^2$ are given by

$$\mathcal{A}x(t) = (p(t)x'(t))' + q(t)x(t) \text{ for } t \in [0,1],$$

$$\mathcal{B}(x) = \begin{bmatrix} \alpha x(0) + \beta x'(0) \\ \gamma x(1) + \delta x'(1) \end{bmatrix}.$$

We define $\Psi : D(\mathcal{L}) \to L^2 \times \mathbb{R}^2$ by

$$\Psi(x) = \begin{bmatrix} -\psi \circ x \\ -\eta(x) \end{bmatrix},$$

where $\psi \in C^1(\mathbb{R}, \mathbb{R})$. We assume η_1, η_2 are continuously Fréchet differentiable functions from $D(\mathcal{L})$ into $\mathbb{R}$, and the map $\eta : D(\mathcal{L}) \to \mathbb{R}^2$ is given by $\eta(x) = \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \end{bmatrix}$.

Similarly, we define $\mathcal{G} : D(\mathcal{L}) \to L^2 \times \mathbb{R}^2$ to be

$$\mathcal{G}(x) = \begin{bmatrix} G(x) \\ \phi(x) \end{bmatrix},$$

where we assume G is a continuous function from $D(\mathcal{L}) \to L^2$ and $\phi(x) = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix}$, where ϕ_1, ϕ_2: $D(\mathcal{L}) \to \mathbb{R}$ are continuous.

We may now conclude that the problem (3.1)-(3.2) is equivalent to solving

$$\mathcal{L}x - \Psi(x) = \mathcal{G}x$$

and the problem (3.3)-(3.4) is equivalent to solving

$$\mathcal{L}x - \Psi(x) = \begin{bmatrix} h \\ v \end{bmatrix},$$

where $h \in L^2,\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$.

In order to study either of these nonlinear problems, we will first analyze the linear Sturm-Liouville problem

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = 0 \tag{3.5}$$

subject to

$$\begin{cases} \alpha x(0) + \beta x'(0) = 0 \\ \gamma x(1) + \delta x'(1) = 0 \end{cases} \tag{3.6}$$

which, using our above notation, may be rewritten as

$$\mathcal{A}x + \lambda x = 0, \quad \mathcal{B}x = 0.$$

Suppose that $\alpha^2 + \beta^2 \neq 0, \gamma^2 + \delta^2 \neq 0$. It is well-known (see, for instance, Kelley and Peterson [15]) that the Sturm-Liouville problem (3.5)-(3.6) has infinitely many real, simple eigenvalues $\lambda_1 < \lambda_2 < \cdots$ and corresponding linearly independent eigenfunctions $x_1, x_2, \cdots$. If λ_n, λ_m are distinct eigenvalues, then x_n, x_m are orthogonal, i.e. $< x_n, x_m >= 0$. Without loss of generality, we make the convention that the eigenfunctions are chosen to be real and normalized, i.e.

$$< x_k, x_k > = 1, \quad k \in \mathbb{N}.$$

Furthermore, if h is an arbitrary real-valued square-integrable function defined on $[0, 1]$, then h has a generalized Fourier series with respect to the eigenfunctions of the Sturm-Liouville problem (3.5)-(3.6) of the form

$$\sum_{k=1}^{\infty} < h, x_k > x_k(t).$$

3.3 The $\mathcal{L}x - \Psi(x) = [h, v]^T$ Case

With the linear Sturm-Liouville problem understood, we may now begin studying the solvability of the problem (3.3)-(3.4), which may be expressed

$$\mathcal{L}x - \Psi(x) = \begin{bmatrix} h \\ v \end{bmatrix}.$$

It is straightforward to show that $\mathcal{L}$ is a closed operator with domain $D(\mathcal{L})$ dense in L^2.

It follows that $D(\mathcal{L})$ is a Banach space with respect to the graph norm

$$|||x|||_{gr} = \|x\|_{L^2} + \|\mathcal{L}x\|_{L^2\times\mathbb{R}^2} = \|x\|_{L^2} + \max\{\|\mathcal{A}x\|_{L^2}, |\mathcal{B}x|\}.$$

With this norm, $\mathcal{L}$ is continuous by the closed graph theorem, from which it follows that $\mathcal{L}$ is continuously Fréchet differentiable, and:

$$\mathcal{L}'(x)(u)(t) = \mathcal{L}u(t) \quad \text{for all } \; u, x \in D(\mathcal{L}), t \in [0,1].$$

We will assume that η is continuously Fréchet differentiable from $D(\mathcal{L})$ into $\mathbb{R}^2$. Using the fact that $\psi \in C^1(\mathbb{R},\mathbb{R})$, it is straightforward to verify that Ψ is also continuously Fréchet differentiable, and:

$$\Psi'(x)(u)(t) = \begin{bmatrix} -\psi'(x(t))u(t) \\ -\eta'(x)u \end{bmatrix} \quad \text{for all } \; u, x \in D(\mathcal{L}), t \in [0,1].$$

Noting the differentiability of the operators $\mathcal{L}$ and Ψ, our principal tool in proving the solvability of (3.3)-(3.4) will be the following Global Inverse Function Theorem (Chow, Hale [7]):

Theorem 3.3.1. *Suppose X, Y are Banach spaces, $\Phi \in C^1(X,Y)$, and for each $x \in X$, $\Phi'(x)$ is a bijection from X onto Y. If there is a constant K such that $|||[\Phi'(x)]^{-1}||| \leq K$ for all $x \in X$, then Φ is a homeomorphism of X onto Y.*

We will assume the following two critically important conditions for the remainder of the chapter:

(**H.1**) For all $s \in \mathbb{R}$, $c \leq \psi'(s) \leq d$, where the interval $[c,d]$ does not contain any of the eigenvalues λ_i of the Sturm-Liouville problem (3.5)-(3.6).

(**H.2**) The Fréchet derivative of the non-linear boundary operator η is bounded; that is, there exists a constant ζ_0 for which η satisfies

$$\sup\{|||\eta'(x)||| : x \in D(\mathcal{L})\} = \zeta_0 < \infty.$$

Condition (H.1) may occur in two different cases. The first is when the derivative of ψ lies in a compact interval in between consecutive eigenvalues λ_{m-1}, λ_m of (3.5)-(3.6):

$$\lambda_{m-1} < c \leq \psi'(s) \leq d < \lambda_m, \quad s \in \mathbb{R}.$$

However, it is also possible for the interval $[c, d]$ to lie to the left of the first eigenvalue:

$$c \leq \psi'(s) \leq d < \lambda_1.$$

Next, let

$$\mu = \frac{d+c}{2}, \qquad \Gamma_0 = \frac{d-c}{2},$$

and define $\mathcal{L}_\mu = \begin{bmatrix} \mathcal{A} + \mu I \\ \mathcal{B} \end{bmatrix}$.

Now consider the equation $(\mathcal{A} + \mu I)x = 0$, which corresponds to the second-order differential equation

$$(p(t)x'(t))' + q(t)x(t) + \mu x(t) = 0.$$

We choose a basis $\{w_1, w_2\}$ for the two-dimensional solution space of this differential equation, and we assume without loss of generality that the basis is chosen so that $\|w_1\|^2 + \|w_2\|^2 \leq 1$.

We let $w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$, and define the 2×2 matrix $B = [\ \mathcal{B}(w_1) \mid \mathcal{B}(w_2)\ ]$.

Recall that $\{\lambda_k, x_k\}$ for $k \in \mathbb{N}$ are the eigenvalues and corresponding eigenfunctions of the linear Sturm-Liouville problem (3.5)-(3.6). The next lemma provides some useful properties of the operator $\mathcal{L}_\mu$ that will facilitate the use of Theorem 3.3.1 in solving the nonlinear boundary value problem (3.3)-(3.4).

Lemma 3.3.2. *$\mathcal{L}_\mu$ is a bijection from $D(\mathcal{L})$ onto $L^2 \times \mathbb{R}^2$, and for any* $\begin{bmatrix} h \\ v \end{bmatrix} \in L^2 \times \mathbb{R}^2$,

$$\mathcal{L}_\mu^{-1} \begin{bmatrix} h \\ v \end{bmatrix} = \sum_{k=1}^{\infty} \frac{< h, x_k >}{\mu - \lambda_k} x_k + w(\cdot)^T B^{-1} v.$$

Furthermore,

$$|||\mathcal{L}_\mu^{-1}||| \leq A_0 + B_0,$$

where $A_0 = \sup\limits_{k\in\mathbb{N}} \frac{1}{|\mu-\lambda_k|} + \sup\limits_{k\in\mathbb{N}} \left| \frac{\lambda_k}{\mu-\lambda_k} \right|$ *and* $B_0 = |||B^{-1}||| + \max\{|\mu||||B^{-1}|||, 1\}$.

Proof. It is clear that $\mathcal{L}_\mu$ is one-to-one since $\mu \neq \lambda_i$. To show $\mathcal{L}_\mu$ is onto, given any $\begin{bmatrix} h \\ v \end{bmatrix} \in L^2 \times \mathbb{R}^2$, we consider the equation $\mathcal{L}_\mu(x) = \begin{bmatrix} h \\ v \end{bmatrix}$ and attempt a solution of the form

$$x(t) = \sum_{k=1}^{\infty} \alpha_k x_k(t) + c_1 w_1(t) + c_2 w_2(t), \tag{3.7}$$

where the eigenfunctions $\{x_k\}$ satisfy $\mathcal{A}x_k + \lambda_k x_k = 0$, $\mathcal{B}x_k = 0$.

Upon applying $\mathcal{L}_\mu$ to the form above, we note that $(\mathcal{A} + \mu I)(c_1w_1 + c_2w_2) = 0$. The following norm will prove convenient in some estimates to follow:

$$|||u|||_m = \|u\|_\infty + \|u'\|_\infty + \|u''\|_{L^2}.$$

Now consider the following graph norm estimate:

$$\begin{aligned}
|||u|||_{gr} &= \|u\|_{L^2} + \max\{\|\mathcal{A}u\|_{L^2}, |\mathcal{B}u|\} \\
&\le \|u\|_{L^2} + c_1\|\mathcal{A}u\|_{L^2} + c_2|\mathcal{B}u| \\
&\le c_3\|u\|_{L^2} + c_4\|u'\|_{L^2} + c_5\|u''\|_{L^2} + c_6\|u\|_\infty + c_7\|u'\|_\infty \\
&\le C_1\left(\|u\|_\infty + \|u'\|_\infty + \|u''\|_{L^2}\right) \\
&= C_1|||u|||_m,
\end{aligned}$$

where $\|\cdot\|_\infty$ represents the sup norm. But the Open Mapping Theorem then implies that there exists a constant C_2 such that

$$|||u|||_m \le C_2|||u|||_{gr}.$$

Hence $\|u\|_\infty \le |||u|||_m \le C_2|||u|||_{gr}$ and we conclude that convergence with respect to the graph norm implies uniform convergence. It is now clear that $\mathcal{B}\left(\sum_{k=1}^\infty \alpha_k x_k\right) = \sum_{k=1}^\infty \alpha_k \mathcal{B}x_k = 0$, and thus solving $\mathcal{L}_\mu(x) = \begin{bmatrix} h \\ v \end{bmatrix}$ is equivalent to solving

$$(\mathcal{A} + \mu I)\sum_{k=1}^\infty \alpha_k x_k = h \tag{3.8}$$

and

$$\mathcal{B}(c_1 w_1 + c_2 w_2) = v. \tag{3.9}$$

Given $h \in L^2$, we can solve for the coefficients α_k in (3.8) by expanding h as a Fourier series in terms of the eigenfunctions $\{x_k\}$:

$$(\mathcal{A} + \mu I) \sum_{k=1}^{\infty} \alpha_k x_k = h$$

$$(\mathcal{A} + \mu I) \sum_{k=1}^{\infty} \alpha_k x_k = \sum_{k=1}^{\infty} < h, x_k > x_k$$

$$\sum_{k=1}^{\infty} (-\lambda_k + \mu) \alpha_k x_k = \sum_{k=1}^{\infty} < h, x_k > x_k.$$

Thus, $\alpha_k = \frac{<h,x_k>}{\mu - \lambda_k}$.

Next, we restate (3.9) as

$$c_1 \mathcal{B}(w_1) + c_2 \mathcal{B}(w_2) = v$$

or

$$B \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = v,$$

where we recall the 2×2 matrix $B = [\ \mathcal{B}(w_1) \mid \mathcal{B}(w_2)\ ]$. Since the boundary value problem $(\mathcal{A} + \mu I)x = 0, \mathcal{B}x = 0$ has only the trivial solution, B must be invertible. It follows that

$$c_1 w_1(t) + c_2 w_2(t) = w(t)^T B^{-1} v.$$

Hence, for any $\begin{bmatrix} h \\ v \end{bmatrix} \in L^2 \times \mathbb{R}^2$,

$$\mathcal{L}_\mu^{-1} \begin{bmatrix} h \\ v \end{bmatrix} = \sum_{k=1}^{\infty} \frac{< h, x_k >}{\mu - \lambda_k} x_k + w(\cdot)^T B^{-1} v.$$

Our second goal is to estimate the operator norm of $\mathcal{L}_\mu^{-1}$. Using the orthonormality of the eigenfunctions $\{x_k\}$, we have

$$\| \sum_{k=1}^{\infty} \frac{< h, x_k >}{\mu - \lambda_k} x_k \| \leq \left(\sup_k \frac{1}{|\mu - \lambda_k|} \right) \|h\|.$$

Recalling that $D(\mathcal{L})$ is equipped with the graph norm, we calculate

$$\||\mathcal{L}_\mu^{-1} \begin{bmatrix} h \\ v \end{bmatrix} \||_{gr} \leq \||\sum_{k=1}^{\infty} \frac{< h, x_k >}{\mu - \lambda_k} x_k \||_{gr} + \||w(\cdot)^T B^{-1} v\||_{gr}$$

$$\leq \| \sum_{k=1}^{\infty} \frac{< h, x_k >}{\mu - \lambda_k} x_k \| + \| \mathcal{L} \left(\sum_{k=1}^{\infty} \frac{< h, x_k >}{\mu - \lambda_k} x_k \right) \|_{L^2 \times \mathbb{R}^2} + \|w(\cdot)^T B^{-1} v\| + \| \mathcal{L} \left(w(\cdot)^T B^{-1} v \right) \|_{L^2 \times \mathbb{R}^2}$$

$$\leq \left(\sup_k \frac{1}{|\mu - \lambda_k|} \right) \|h\| + \max\{ \| \mathcal{A} \left(\sum_{k=1}^{\infty} \frac{< h, x_k >}{\mu - \lambda_k} x_k \right) \|, | \mathcal{B} \left(\sum_{k=1}^{\infty} \frac{< h, x_k >}{\mu - \lambda_k} x_k \right) | \} + \|w(\cdot)^T B^{-1} v\|$$

$$+\max\{\|\mathcal{A}\left(w(\cdot)^T B^{-1} v\right)\|, |\mathcal{B}\left(w(\cdot)^T B^{-1} v\right)|\}$$

$$\leq \left(\sup_k \frac{1}{|\mu-\lambda_k|}\right)\|h\| + \|\sum_{k=1}^{\infty} \frac{\lambda_k < h, x_k >}{\mu-\lambda_k} x_k\| + \|w(\cdot)^T B^{-1} v\| + \max\{|\mu|\|w(\cdot)^T B^{-1} v\|, |v|\}$$

$$\leq \left(\sup_k \frac{1}{|\mu-\lambda_k|}\right)\|h\| + \left(\sup_k \left|\frac{\lambda_k}{\mu-\lambda_k}\right|\right)\|h\| + \|w(\cdot)^T B^{-1} v\| + \max\{|\mu|\|w(\cdot)^T B^{-1} v\|, |v|\},$$

where we have again used $(\mathcal{A}+\mu I)(c_1 w_1 + c_2 w_2) = 0$ and $\mathcal{B}\left(\sum_{k=1}^{\infty} \alpha_k x_k\right) = 0$.

Recalling that $\|w_1\|^2 + \|w_2\|^2 \leq 1$, we then have

$$|||\mathcal{L}_\mu^{-1}\begin{bmatrix} h \\ v \end{bmatrix}|||_{gr} \leq \left(\sup_k \frac{1}{|\mu-\lambda_k|} + \sup_k \left|\frac{\lambda_k}{\mu-\lambda_k}\right|\right)\|h\| + \left(|||B^{-1}||| + \max\{|\mu||||B^{-1}|||, 1\}\right)|v|,$$

hence

$$|||\mathcal{L}_\mu^{-1}||| \leq A_0 + B_0.$$

□

We may simplify the expression for A_0 based on which case of condition (H.1) occurs. In the first case, where

$$\lambda_{m-1} < c \le \psi'(s) \le d < \lambda_m, \quad s \in \mathbb{R},$$

then

$$A_0 = \sup_k \frac{1}{|\mu - \lambda_k|} + \sup_k \left| \frac{\lambda_k}{\mu - \lambda_k} \right| = \max\{\frac{1+\lambda_{m-1}}{\mu - \lambda_{m-1}}, \frac{1+\lambda_m}{\lambda_m - \mu}\}.$$

Otherwise, if $c \le \psi'(s) \le d < \lambda_1$, we have

$$A_0 = \sup_k \frac{1}{|\mu - \lambda_k|} + \sup_k \left| \frac{\lambda_k}{\mu - \lambda_k} \right| = \frac{1+\lambda_1}{\lambda_1 - \mu}.$$

In our first main result, the next theorem shall provide sufficient conditions for the existence of solutions to the nonlinear boundary value problem (3.3)-(3.4).

Theorem 3.3.3. *If ψ, η satisfy conditions (H.1) and (H.2), and if*

$$A_0\Gamma_0 + B_0\zeta_0 < 1,$$

then the boundary value problem (3.3)-(3.4) *has a unique solution for each square-integrable function h defined on $[0, 1]$ and each* $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$.

Proof. Since the boundary value problem (3.3)-(3.4) is equivalent to

$$\mathcal{L}x - \Psi(x) = \begin{bmatrix} h \\ v \end{bmatrix},$$

our goal is to show that the operator $\mathcal{L} - \Psi$ satisfies the hypotheses of Theorem 3.3.1. First, note that $\mathcal{L} - \Psi \in C^1$ since $\mathcal{L}$ and Ψ are each Fréchet differentiable on $D(\mathcal{L})$.

Define the operator $\Psi_\mu : D(\mathcal{L}) \to L^2 \times \mathbb{R}^2$ by

$$\Psi_\mu(x) = \begin{bmatrix} -\psi \circ x + \mu I \\ -\eta(x) \end{bmatrix}.$$

The Fréchet derivative of Ψ_μ is then given by

$$\Psi'_\mu(x)(u) = \begin{bmatrix} -(\psi' \circ x)u + \mu u \\ -\eta'(x)u \end{bmatrix}.$$

In order to show that for each $x \in D(\mathcal{L})$, $\mathcal{L} - \Psi'(x)$ is a bijection from $D(\mathcal{L})$ onto $L^2 \times \mathbb{R}^2$, we consider the equation

$$[\mathcal{L} - \Psi'(x)]u = \begin{bmatrix} h \\ v \end{bmatrix}, \quad u \in D(\mathcal{L}), \begin{bmatrix} h \\ v \end{bmatrix} \in L^2 \times \mathbb{R}^2.$$

Using the above definitions of $\mathcal{L}_\mu$ and Ψ_μ, this equation becomes

$$\mathcal{L}_\mu u - \Psi'_\mu(x)u = \begin{bmatrix} h \\ v \end{bmatrix}$$

$$u - \mathcal{L}_\mu^{-1}\Psi'_\mu(x)u = \mathcal{L}_\mu^{-1} \begin{bmatrix} h \\ v \end{bmatrix}$$

$$[I - \mathcal{L}_\mu^{-1}\Psi'_\mu(x)]u = \mathcal{L}_\mu^{-1}\begin{bmatrix} h \\ v \end{bmatrix}.$$

To proceed, we use conditions (H.1) and (H.2) and obtain

$$|-\psi'(x(t)) + \mu| \le \Gamma_0 \text{ for all } x \in D(\mathcal{L}) \text{ and } t \in [0,1],$$

and

$$\sup\{|||\eta'(x)||| : x \in D(\mathcal{L})\} = \zeta_0.$$

Using these estimates together with the formula for $\mathcal{L}_\mu^{-1}$ derived in Lemma 3.3.2, we observe that

$$|||\mathcal{L}_\mu^{-1}\Psi'_\mu(x)||| \le A_0\Gamma_0 + B_0\zeta_0.$$

For any bounded operator A with $|||A||| < 1$, it is well known that $I - A$ has a bounded inverse and $|||(I-A)^{-1}||| \le (1 - |||A|||)^{-1}$.

Hence, to ensure the operator $I - \mathcal{L}_\mu^{-1}\Psi'_\mu(x)$ has a bounded inverse, we choose Γ_0 and ζ_0 small enough so that $A_0\Gamma_0 + B_0\zeta_0 < 1$. It follows that $(\mathcal{L} - \Psi'(x))^{-1}$ exists for each $x \in D(\mathcal{L})$, and

$$(\mathcal{L} - \Psi'(x))^{-1} = [I - \mathcal{L}_\mu^{-1}\Psi_\mu'(x)]^{-1}\mathcal{L}_\mu^{-1}$$

$$|||(\mathcal{L} - \Psi'(x))^{-1}||| \leq \left(\frac{1}{1 - (A_0\Gamma_0 + B_0\zeta_0)}\right)(A_0 + B_0)$$

$$= \frac{A_0 + B_0}{1 - A_0\Gamma_0 - B_0\zeta_0} = K,$$

a bound independent of x. Thus $\mathcal{L} - \Psi$ satisfies the hypotheses of Theorem 3.3.1, and we conclude the boundary value problem (3.3)-(3.4) has a unique solution for each square-integrable h defined on $[0, 1]$ and each $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$. □

Remark 3.3.4. *We call attention to the generality of the nonlinear boundary operators* η_1 *and* η_2 *that appear in the boundary value problem* (3.3)-(3.4). *As an important special case, these operators may allow for nonlinear multipoint boundary value problems, if we take for example*

$$\eta_1(x) = \sum_{i=1}^{n} f_i(x(t_i)),$$

$$\eta_2(x) = \sum_{j=1}^{m} g_j(x(t_j)),$$

where each f_i, g_j *is a* C^1 *function and* $t_i, t_j \in [0, 1]$.

Remark 3.3.5. *We also note that the above estimate of the norm of* $(\mathcal{L} - \Psi'(x))^{-1}$ *in Theorem* 3.3.3 *illustrates the interplay between the distribution of the eigenvalues to the* *linear Sturm-Liouville problem* (3.5)-(3.6)*, the range of the derivative of the nonlinearity* ψ*, and the allowable size of* $|||\eta'(x)|||$*. If* $\psi'(s)$ *lies close to consecutive eigenvalues of the linear problem or in between eigenvalues of large magnitude (causing* $A_0\Gamma_0$ *to approach* 1*), we must choose* ζ_0 *smaller to ensure* $A_0\Gamma_0 + B_0\zeta_0 < 1$*, forcing a tighter bound for* $\eta'(x)$*. On the other hand, if the length of the interval* $[c, d]$ *is decreased (thus being more restrictive with* ψ'*) and* $[c, d]$ *is placed between eigenvalues of smaller magnitude, we have more freedom to choose a larger value for* ζ_0*. Of course, if* $\sup \psi'(s) < \lambda_1$*, the only further requirement is for* ψ' *to be bounded.*

We conclude by stating a corollary to Theorem 3.3.3 which analyzes the case in which the nonlinear boundary operators η_1 and η_2 are identically zero.

Corollary 3.3.6. *Suppose condition (H.1) holds. If* $A_0\Gamma_0 < 1$*, then the nonhomogeneous boundary value problem*

$$(p(t)x'(t))' + q(t)x(t) + \psi(x(t)) = h(t)$$

subject to

$$\begin{cases} \alpha x(0) + \beta x'(0) = v_1 \\ \\ \gamma x(1) + \delta x'(1) = v_2 \end{cases}$$

has a unique solution for each square-integrable h *on* $[0, 1]$ *and each* $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$.

Proof. Since $\zeta_0 = 0$, the assumption that $A_0\Gamma_0 < 1$ immediately implies that $\mathcal{L} - \Psi'(x)$ is invertible for each $x \in D(\mathcal{L})$ and

$$(\mathcal{L} - \Psi'(x))^{-1} = [I - \mathcal{L}_\mu^{-1}\Psi'_\mu(x)]^{-1}\mathcal{L}_\mu^{-1}$$

$$|||(\mathcal{L} - \Psi'(x))^{-1}||| \leq \left(\frac{1}{1 - A_0\Gamma_0}\right)(A_0 + B_0)$$

$$= \frac{A_0 + B_0}{1 - A_0\Gamma_0} = K',$$

a bound independent of x. Hence, Theorem 3.3.1 implies that $\mathcal{L} - \Psi$ is again a homeomorphism of $D(\mathcal{L})$ onto $L^2 \times \mathbb{R}^2$, and the result follows. □

3.4 The $\mathcal{L}x - \Psi(x) = \mathcal{G}x$ Case

Our next goal is to establish the solvability of the boundary value problem (3.1)-(3.2), which we recall may be rewritten as

$$\mathcal{L}x - \Psi(x) = \mathcal{G}x,$$

where $\mathcal{G} : D(\mathcal{L}) \to L^2 \times \mathbb{R}^2$ is continuous.

Continuing with the assumptions made in Theorem 3.3.3, we now observe the operator $(\mathcal{L} - \Psi)^{-1}$ is Lipschitz.

Remark 3.4.1. *The map $(\mathcal{L} - \Psi)^{-1} : L^2 \times \mathbb{R}^2 \to D(\mathcal{L})$ is Lipschitz continuous with Lipschitz constant $K = \frac{A_0+B_0}{1-A_0\Gamma_0-B_0\zeta_0}$. In the proof of Theorem 3.3.3, we saw that for* *each $x \in D(\mathcal{L})$, $(\mathcal{L} - \Psi'(x))^{-1}$ exists and $|||(\mathcal{L} - \Psi'(x))^{-1}||| \le K$. If $\begin{bmatrix} y \\ v \end{bmatrix} \in L^2 \times \mathbb{R}^2$ is such that $(\mathcal{L} - \Psi)(x) = \begin{bmatrix} y \\ v \end{bmatrix}$, then $(\mathcal{L} - \Psi'(x))^{-1} = [(\mathcal{L} - \Psi)^{-1}]' \begin{bmatrix} y \\ v \end{bmatrix}$. It is now a consequence of the Mean Value Theorem for Fréchet derivatives that $(\mathcal{L} - \Psi)^{-1}$ is Lipschitz continuous with constant K.*

We would like to mention that the essential elements of the above observation are due to Brown [4].

Using this Lipschitz property, we are now able to establish sufficient conditions for the existence of at least one solution to the boundary value problem (3.1)-(3.2), or equivalently

$$\mathcal{L}x - \Psi(x) = \mathcal{G}x,$$

where $\mathcal{G}(x) = \begin{bmatrix} G(x) \\ \phi(x) \end{bmatrix}$.

We note that Brown and Lin [5] obtain an existence result for a boundary value problem similar to equation (3.1) but subject to linear homogeneous boundary conditions. The following result holds for more general nonlocal boundary conditions due to the presence of the nonlinear boundary operator ϕ.

Theorem 3.4.2. *Suppose the hypotheses of Theorem* 3.3.3 *are satisfied, and $\mathcal{G} : D(\mathcal{L}) \to L^2 \times \mathbb{R}^2$ is continuous. If there exists a constant $M > 0$ such that $\mathcal{G}$ satisfies $\|\mathcal{G}(x)\| \le$*

$K^{-1}\left(M - |||(\mathcal{L}-\Psi)^{-1}(0)|||_{gr}\right)$ for all $|||x|||_{gr} \le M$, then there exists at least one solution of the boundary value problem (3.1)-(3.2)*.*

Proof. Since $\mathcal{L}-\Psi$ is invertible, (3.1)-(3.2) may be rewritten

$$x = (\mathcal{L}-\Psi)^{-1}\mathcal{G}(x).$$

Denote $H = (\mathcal{L}-\Psi)^{-1} \circ \mathcal{G}$, and let

$$\mathcal{B}_M = \{x \in D(\mathcal{L}) : |||x|||_{gr} \le M\}.$$

It follows that

$$\begin{aligned}
|||H(x)|||_{gr} &= |||(\mathcal{L}-\Psi)^{-1}\mathcal{G}(x)|||_{gr} \\
&\le K\|\mathcal{G}(x)\| + |||(\mathcal{L}-\Psi)^{-1}(0)|||_{gr} \\
&\le K\left(K^{-1}\left(M - |||(\mathcal{L}-\Psi)^{-1}(0)|||_{gr}\right)\right) + |||(\mathcal{L}-\Psi)^{-1}(0)|||_{gr} = M
\end{aligned}$$

for all $|||x|||_{gr} \le M$, hence $H(\mathcal{B}_M) \subseteq \mathcal{B}_M$. Recall our earlier estimates for the graph norm in the proof of Lemma 3.3.2, namely $|||u|||_{gr} \le C_1|||u|||_m$ and $|||u|||_m \le C_2|||u|||_{gr}$. We may make a similar estimate for the Sobolev norm $|||u|||_S = \|u\|_{L^2} + \|u'\|_{L^2} + \|u''\|_{L^2}$:

$$\begin{aligned}
|||u|||_S &= \|u\|_{L^2} + \|u'\|_{L^2} + \|u''\|_{L^2} \\
&\le \|u\|_\infty + \|u'\|_\infty + \|u''\|_{L^2} \\
&= |||u|||_m.
\end{aligned}$$

But another application of the Open Mapping Theorem then implies that there exists a constant C_3 such that

$$|||u|||_m \leq C_3|||u|||_S.$$

We conclude that the graph norm $|||u|||_{gr}$ is equivalent to the Sobolev norm $|||u|||_S$, and a Sobolev embedding theorem then implies that $D(\mathcal{L})$ has a compact embedding into $C^1[0,1]$ (see Brown and Lin [5]). Hence $\mathcal{G}$ is compact and continuous. H is then a completely continuous function, and thus there exists at least one fixed point of H in $\mathcal{B}_M$ by the Schauder Fixed Point Theorem. This fixed point corresponds to a solution of the boundary value problem (3.1)-(3.2). □

Remark 3.4.3. *The Lipschitz constant K for $(\mathcal{L}-\Psi)^{-1}$ appears in the bound for $\|\mathcal{G}(x)\|$ on the ball $\mathcal{B}_M$ above, again illustrating how the distribution of eigenvalues of the linear problem* (3.5)-(3.6) *affects the allowable size of the nonlinearity $\mathcal{G}$. For example, if the eigenvalues of* (3.5)-(3.6) *are far apart relative to the size of the interval $[c,d]$, the value of K can be made smaller, permitting the size of $\|\mathcal{G}(x)\|$ to be larger.*

We again provide a corollary concerning the case when the nonlinear boundary operators η_1 and η_2 vanish.

Corollary 3.4.4. *Suppose ψ satisfies condition (H.1) and $A_0\Gamma_0 < 1$. If there exists a constant $M > 0$ such that $\|\mathcal{G}(x)\| \leq K^{-1}\left(M - |||(\mathcal{L}-\Psi)^{-1}(0)|||_{gr}\right)$ for all $|||x|||_{gr} \leq M$, then the boundary value problem*

$$(p(t)x'(t))' + q(t)x(t) + \psi(x(t)) = G(x(t))$$

subject to

$$\begin{cases} \alpha x(0) + \beta x'(0) = \phi_1(x) \\ \\ \gamma x(1) + \delta x'(1) = \phi_2(x) \end{cases}$$

has at least one solution.

If we consider the special case where $\mathcal{G}$ obeys a sub-linear growth condition, we immediately obtain the following corollary.

Corollary 3.4.5. *Suppose the hypotheses of Theorem* 3.3.3 *are satisfied, and there exists an* $0 \le \epsilon < 1$ *such that* $\|\mathcal{G}(x)\| \le b_1 + b_2 |||x|||_{gr}^{\epsilon}$. *Then the boundary value problem* (3.1)-(3.2) *has at least one solution.*

As another corollary, let us consider the case where G is a Nemytskii-type operator, i.e. there exists a $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ such that $G(x)(t) = g(t, x(t))$. Let $\overline{M} = K^{-1}\left(M - |||(\mathcal{L} - \Psi)^{-1}(0)|||_{gr}\right)$.

Corollary 3.4.6. *Suppose the hypotheses of Theorem* 3.3.3 *are satisfied. If* $g(t, \cdot)$ *is continuous for each* $t \in [0,1]$ *and there exists an* $M > 0$ *such that* $|\phi(x)| \le \overline{M}$ *for all* $|||x|||_{gr} \le M$ *and* $|g(t,s)| \le \overline{M}$ *for all* $|s| \le MC_2$ *and* $t \in [0,1]$, *then the mapping* $\mathcal{G} : D(\mathcal{L}) \to L^2 \times \mathbb{R}^2$ *given by* $\mathcal{G}(x) = \begin{bmatrix} g(\cdot, x(\cdot)) \\ \phi(x) \end{bmatrix}$ *satisfies the hypotheses of Theorem* 3.4.2.

Proof. Consider any $x \in D(\mathcal{L})$ such that $|||x|||_{gr} \le M$, with M defined as above. Then

$|x(t)| \leq \|x\|_\infty \leq C_2 |||x|||_{gr} \leq MC_2$ for each $t \in [0,1]$, and $|g(t, x(t))| \leq \overline{M}$ for each $t \in [0,1]$. Hence

$$\|\mathcal{G}(x)\| = \max\{\left(\int_0^1 |g(t,x(t))|^2 \, dt\right)^{1/2}, |\phi(x)|\}$$
$$\leq \max\{\left(\int_0^1 \left(\overline{M}\right)^2 \, dt\right)^{1/2}, \overline{M}\} = \overline{M}.$$

□

Remark 3.4.7. *We conclude by stating another important special case of Theorem* 3.4.2 *when G is an integral operator, i.e. there exists a $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ such that $G(x)(t) = \int_0^t g(k, x(k)) \, dk$. Suppose $g(t, \cdot)$ is continuous for each $t \in [0,1]$ and there exists an $M > 0$ such that $|\phi(x)| \leq \overline{M}$ for all $|||x|||_{gr} \leq M$ and $|g(t,s)| \leq \overline{M}$ for all $|s| \leq MC_2$ and for each $t \in [0,1]$. Let $\mathcal{G}(x) = \begin{bmatrix} \int_0^{\cdot} g(k, x(k)) \, dk \\ \phi(x) \end{bmatrix}$ be a mapping from $D(\mathcal{L}) \to L^2 \times \mathbb{R}^2$. Then an immediate calculation of $\|\mathcal{G}(x)\|$ as in Corollary* 3.4.6 *implies that $\mathcal{G}$ again satisfies the hypotheses of Theorem* 3.4.2.

Chapter 4

Existence of Periodic Solutions to Nonlinear Difference Equations at Full Resonance

4.1 Introduction

In this chapter, we study the solvability of nonlinear discrete systems of the form

$$\Delta x(t) = f(\epsilon, t, x(t)). \tag{4.1}$$

In particular, we are interested in finding N-periodic solutions of the above system, where we assume $f(\epsilon, t, x) = f(\epsilon, t+N, x)$. Here, $f = (f_1, f_2, \ldots, f_n)^T$ where $f_i : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ for $i = 1, 2, \ldots, n$. Note that the solution space of the corresponding linear homogeneous system is n-dimensional, i.e. the system is at full resonance. Our approach in providing sufficient conditions for the existence of periodic solutions to (4.1) depends significantly on this resonance along with asymptotic properties of the nonlinear function $f(0,t,x)$.

Since the solution space of the associated linear homogeneous equation is non-trivial, we will use a projection scheme (Lyapunov-Schmidt procedure) together with the Brouwer fixed point theorem to analyze the nonlinear problem (4.1). A similar approach has often been used in the study of both continuous and discrete dynamical systems (see, for instance, [1,2,6,7,9-11,13,14,20-22,25,26,28-32]). Our results complement previous work in the study of periodic discrete dynamical systems. We allow for higher-dimensional solution spaces of the associated linear problem as well as for more general asymptotic behavior of the nonlinear function f.

4.2 Preliminaries

For each natural number $N \geq 2$, let X_N be the set of all sequences x from $\{0,1,2,3,...\}$ into $\mathbb{R}^n$ that are N-periodic; that is, $x(l+N) = x(l)$ for every $l \in \{0,1,2,3,...\}$. For $x \in X_N$, let $\|x\|_\infty = \sup\{|x(l)| : l = 0,1,2,3,...\}$.

We define $L : X_N \to X_N$ by

$$Lx(t) = \Delta x(t) = x(t+1) - x(t) \text{ for } t = 0,1,2,3,...,$$

and define $F_\epsilon : \mathbb{R} \times X_N \to X_N$ by

$$F_\epsilon(x)(t) = \begin{bmatrix} f_1(\epsilon,t,x(t)) \\ f_2(\epsilon,t,x(t)) \\ \vdots \\ f_n(\epsilon,t,x(t)) \end{bmatrix} \text{ for } t = 0,1,2,3,....$$

We assume each f_i is a continuous map from $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}$ for $i = 1, 2, \ldots, n$. It follows that F_ϵ is continuous. It is assumed that for some $m \in \mathbb{R}$, $|f_i(0, t, x)| \leq m$ for $i = 1, 2, \ldots, n$. Hence, for all $x \in X_N$, $\|F_0(x)\|_\infty \leq m$.

Our problem, finding periodic solutions to the system

$$\Delta x(t) = f(\epsilon, t, x(t))$$

is equivalent to solving

$$Lx = F_\epsilon(x).$$

Since L is not invertible, we cannot apply the Brouwer Fixed Point Theorem directly. We shall decompose X_N using the methods described in [9]. We find projections, P, of X_N onto $ker(L)$, and E, of X_N onto $Im(L)$, so that we may write $X_N = ker(L) \oplus Im(I - P)$ and $X_N = Im(L) \oplus Im(I - E)$. The projections we use are those devised by Rodríguez [23].

Let

$$(I - E)x(t) = \frac{1}{N} \sum_{l=0}^{N-1} x(l) \text{ for } t = 0, 1, 2, 3, \ldots.$$

Let

$$Px(t) = \frac{1}{N} \sum_{l=0}^{N-1} x(l) \text{ for } t = 0, 1, 2, 3, \ldots.$$

Remark 4.2.1. *If $\tilde{L}$ is the restriction of L to $Im(I - P)$ then $Im(\tilde{L}) = Im(L)$. $\tilde{L}$, viewed as a map from $Im(I - P)$ into $Im(L)$ is invertible. We denote $(\tilde{L})^{-1}$ by M. One may then verify*

i. M is bounded and linear

ii. $MLx = (I - P)x$ for all $x \in D(L)$

iii. $LMh = h$ *for all* $h \in Im(L)$

iv. $EL = L$ *and* $(I - E)L = 0$

v. $PM = 0$

Proposition 4.2.2. $Lx = F_\epsilon x$ *is equivalent to*

$$\begin{cases} x = Px + MEF_\epsilon(x) \\ (I - E)F_\epsilon(Px + MEF_\epsilon(x)) = 0 \end{cases} \tag{4.2}$$

Proof: We have $Lx = F_\epsilon x$ if and only if

$$\begin{cases} E(Lx - F_\epsilon x) = 0 \\ (I - E)(Lx - F_\epsilon x) = 0 \end{cases}$$

if and only if

$$\begin{cases} Lx = EF_\epsilon(x) \\ (I - E)F_\epsilon(x) = 0 \end{cases}$$

if and only if

$$\begin{cases} (I - P)x = MEF_\epsilon(x) \\ (I - E)F_\epsilon(x) = 0 \end{cases}$$

if and only if

$$\begin{cases} x = Px + MEF_\epsilon(x) \\ (I - E)F_\epsilon(Px + MEF_\epsilon(x)) = 0. \end{cases}$$

4.3 Main Results

Since $ker(L)$=span$\{e_1, e_2, \ldots, e_n\}$, where e_i is the ith standard unit basis vector, we may rewrite (4.2) of Proposition 4.2.2 as the equivalent system of $n+1$ equations

$$\begin{cases} x = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n + MEF_\epsilon(x) \\ 0 = \displaystyle\sum_{l=0}^{N-1} f_i(\epsilon, l, \alpha_1 + [MEF_\epsilon(x)]_1(l), \ldots, \alpha_n + [MEF_\epsilon(x)]_n(l)),\ i = 1, \ldots, n. \end{cases}$$

The proof of Theorem 4.3.1 relies on techniques that appear in [2,6,7,12,21,22,25,28].

Theorem 4.3.1. *Suppose that*

(i) $f_i : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *for* $i = 1, 2, \ldots, n$ *is continuous, and for some* $m \in \mathbb{R}$, $|f_i(0, t, x)| \leq m$ *for* $i = 1, 2, \ldots, n$.

(ii) For each $i = 1, 2, \ldots, n$, *there exist constants* $L_i, P_i, N_i > 0$ *such that for all* $x_i > L_i$, $f_i(0, t, x_1, \ldots, x_i, \ldots, x_n) \geq P_i$ *and* $f_i(0, t, x_1, \ldots, -x_i, \ldots, x_n) \leq -N_i$ *for all* $t = 0, 1, 2, \ldots$ *and all* $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in \mathbb{R}$.

Then, there exists an $\epsilon_0 > 0$ *such that for* $\epsilon \in [0, \epsilon_0]$, *there exists at least one periodic solution of*

$$\Delta x(t) = f(\epsilon, t, x(t)).$$

Proof: We define mappings

$$H_1 : \mathbb{R} \times X_N \times \mathbb{R}^n \to X_N,$$

$$H_{i+1} : \mathbb{R} \times X_N \times \mathbb{R}^n \to \mathbb{R},$$

and

$$H : \mathbb{R} \times X_N \times \mathbb{R}^n \to X_N \times \mathbb{R}^n$$

by

$$H_1(\epsilon, x, \alpha_1, \ldots, \alpha_n) = \alpha_1 e_1 + \ldots + \alpha_n e_n + MEF_\epsilon(x),$$

and for $i = 1, \ldots, n$,

$$H_{i+1}(\epsilon, x, \alpha_1, \ldots, \alpha_n) = \alpha_i - \sum_{l=0}^{N-1} f_i(\epsilon, l, \alpha_1 + [MEF_\epsilon(x)]_1(l), \ldots, \alpha_n + [MEF_\epsilon(x)]_n(l)).$$

H is then given by

$$H(\epsilon, x, \alpha_1, \ldots, \alpha_n) = (H_1(\epsilon, x, \alpha_1, \ldots, \alpha_n), \ldots, H_{n+1}(\epsilon, x, \alpha_1, \ldots, \alpha_n)).$$

We shall first analyze the case when $\epsilon = 0$. Note that for $i = 1, 2, \ldots, n$, $|[MEF_0(x)]_i(l)| \leq \|ME\| m$ for every $l \in \{0, 1, 2, ...\}$ and every $x \in X_N$.

Consider $H_{i+1}(0, x, \alpha_1, \ldots, \alpha_n)$ for each $i = 1, 2, \ldots, n$. If α_i is sufficiently large, we may ensure

$$f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \geq P_i > 0$$

and

$$f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, -\alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \leq -N_i < 0$$

for every $l \in \{0, 1, 2, ...\}$ and every $x \in X_N$. Therefore there is some $\gamma_i > Nm + 1 > 0$ such that for all $x \in X_N$ and for all $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n \in \mathbb{R}$,

$H_{i+1}(0, x, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) < \alpha_i$ and $H_{i+1}(0, x, \alpha_1, \ldots, -\alpha_i, \ldots, \alpha_n) > -\alpha_i$ for $\alpha_i > \gamma_i$. We let $\delta_i = \gamma_i + Nm + 1$.

Now if $\alpha_i \in [\gamma_i, \delta_i]$, then for all $x \in X_N$ and $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n \in \mathbb{R}$ we have $H_{i+1}(0, x, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$

$$= \alpha_i - \sum_{l=0}^{N-1} f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))$$

$$\geq \alpha_i - \sum_{l=0}^{N-1} |f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))|$$

$$\geq \alpha_i - Nm$$

$$> \alpha_i - \gamma_i$$

$$\geq 0$$

and

$$H_{i+1}(0, x, \alpha_1, \ldots, -\alpha_i, \ldots, \alpha_n)$$

$$\begin{aligned}
&= -\alpha_i - \sum_{l=0}^{N-1} f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, -\alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \\
&\leq -\alpha_i + \sum_{l=0}^{N-1} |f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, -\alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))| \\
&\leq -\alpha_i + Nm \\
&< -\alpha_i + \gamma_i \\
&\leq 0.
\end{aligned}$$

Thus for all $x \in X_N$, $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n \in \mathbb{R}$, and $\alpha_i \in [\gamma_i, \delta_i]$,

$$H_{i+1}(0, x, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_n), H_{i+1}(0, x, \alpha_1, \ldots, -\alpha_i, \ldots, \alpha_n) \in [-\alpha_i, \alpha_i] \subseteq [-\delta_i, \delta_i].$$

Furthermore, if $0 \leq \alpha_i < \gamma_i$, for all $x \in X_N$ and

$\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n \in \mathbb{R}$,

$$|H_{i+1}(0, x, \alpha_1, \ldots, \pm\alpha_i, \ldots, \alpha_n)|$$

$$\begin{aligned}
&\leq |\pm \alpha_i| \\
&\quad + \sum_{l=0}^{N-1} |f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \pm\alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))| \\
&\leq \gamma_i + Nm \\
&< \delta_i.
\end{aligned}$$

We have shown that for $\epsilon = 0$, H_{i+1} maps $X_N \times [-\delta_i, \delta_i] \times \mathbb{R}^{n-1}$ into $[-\delta_i, \delta_i]$ for each

$i = 1, 2, \ldots, n$.

Define $\mathcal{B} = \{(x, \alpha_1, \ldots, \alpha_n) \in X_N \times \mathbb{R}^n : \|x\|_\infty \leq \delta_1 + \ldots + \delta_n + \|ME\|m + 1$ and $|\alpha_i| \leq \delta_i$ for each $i = 1, 2, \ldots, n\}$, and note that $\mathcal{B}$ is a non-empty, closed, bounded, convex set.

From the above results, it follows that for $(x, \alpha_1, \ldots, \alpha_n) \in \mathcal{B}$, $H(0, x, \alpha_1, \ldots, \alpha_n) \in \mathcal{B}$. For if $(x, \alpha_1, \ldots, \alpha_n) \in \mathcal{B}$, then $H_{i+1}(0, x, \alpha_1, \ldots, \alpha_n) \in [-\delta_i, \delta_i]$, while

$$\begin{aligned}\|H_1(0, x, \alpha_1, \ldots, \alpha_n)\|_\infty &\leq |\alpha_1|\,\|e_1\|_\infty + \ldots + |\alpha_n|\,\|e_n\|_\infty + \|MEF_0(x)\|_\infty \\ &\leq \delta_1 + \ldots + \delta_n + \|ME\|m \\ &< \delta_1 + \ldots + \delta_n + \|ME\|m + 1.\end{aligned}$$

Since H is a continuous function, the Brouwer Fixed Point Theorem guarantees existence of at least one fixed point of H in $\mathcal{B}$.

Now consider the case when $\epsilon > 0$. We will show that there exists $\epsilon_0 \in \mathbb{R}$ such that for each $\epsilon \leq \epsilon_0$, $H(\epsilon, x, \alpha_1, \ldots, \alpha_n) \in \mathcal{B}$ whenever $(x, \alpha_1, \ldots, \alpha_n) \in \mathcal{B}$.

Choose ϵ small enough so that

$$\Big| f_i(\epsilon, l, \alpha_1 + [MEF_\epsilon(x)]_1(l), \ldots, \pm\alpha_i + [MEF_\epsilon(x)]_i(l), \ldots, \alpha_n + [MEF_\epsilon(x)]_n(l)) -$$

$$f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \pm\alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \Big|$$

$$< \min\left\{\frac{P_i}{2}, \frac{N_i}{2}, \frac{1}{N}\right\}$$

for all $(x, \alpha_1, \ldots, \alpha_n) \in \mathcal{B}$.

Note that we may now assume ϵ lies in some compact interval of $\mathbb{R}$, from which it follows that for all $x \in \mathcal{B}_x = \{\|x\|_\infty \leq \delta_1 + \ldots + \delta_n + \|ME\|m + 1\}$, $\|F_\epsilon(x) - F_0(x)\|_\infty$ can be made arbitrarily small for sufficiently small ϵ. For our purposes, choose ϵ small enough so that for all $x \in \mathcal{B}_x$,

$$\begin{aligned}
\|MEF_\epsilon(x)\|_\infty &\leq \|ME\|(\|F_\epsilon(x) - F_0(x)\|_\infty + \|F_0(x)\|_\infty) \\
&\leq \|ME\|\left(\frac{1}{\|ME\|} + m\right) \\
&= 1 + \|ME\|m.
\end{aligned}$$

For each ϵ satisfying the above properties, it now follows that $H(\epsilon, x, \alpha_1, \ldots, \alpha_n) \in \mathcal{B}$ whenever $(x, \alpha_1, \ldots, \alpha_n) \in \mathcal{B}$. First observe that for all $(x, \alpha_1, \ldots, \alpha_n) \in \mathcal{B}$,

$$\begin{aligned}
\|H_1(\epsilon, x, \alpha_1, \ldots, \alpha_n)\|_\infty &\leq |\alpha_1|\,\|e_1\|_\infty + \ldots + |\alpha_n|\,\|e_n\|_\infty + \|MEF_\epsilon(x)\|_\infty \\
&\leq \delta_1 + \ldots + \delta_n + \|ME\|m + 1.
\end{aligned}$$

Next, for all $\alpha_i \in [\gamma_i, \delta_i]$,

$$
\begin{aligned}
&H_{i+1}(\epsilon, x, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \\
&= \alpha_i - \sum_{l=0}^{N-1} \Big(f_i(\epsilon, l, \alpha_1 + [MEF_\epsilon(x)]_1(l), \ldots, \alpha_i + [MEF_\epsilon(x)]_i(l), \ldots, \alpha_n + [MEF_\epsilon(x)]_n(l)) \\
&\qquad - f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \Big) \\
&\qquad - \sum_{l=0}^{N-1} f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \\
&\\
&\le \alpha_i - \sum_{l=0}^{N-1} \Big(f_i(\epsilon, l, \alpha_1 + [MEF_\epsilon(x)]_1(l), \ldots, \alpha_i + [MEF_\epsilon(x)]_i(l), \ldots, \alpha_n + [MEF_\epsilon(x)]_n(l)) \\
&\qquad - f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \Big) \\
&\qquad - NP_i \\
&< \alpha_i - \tfrac{NP_i}{2} \\
&< \alpha_i,
\end{aligned}
$$

while a similar calculation shows that $H_{i+1}(\epsilon, x, \alpha_1, \ldots, -\alpha_i, \ldots, \alpha_n) > -\alpha_i$ for all $\alpha_i \in [\gamma_i, \delta_i]$.

Also, for all $\alpha_i \in [\gamma_i, \delta_i]$,

$$
\begin{aligned}
&H_{i+1}(\epsilon, x, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \\
&= \alpha_i - \sum_{l=0}^{N-1} \Big(f_i(\epsilon, l, \alpha_1 + [MEF_\epsilon(x)]_1(l), \ldots, \alpha_i + [MEF_\epsilon(x)]_i(l), \ldots, \alpha_n + [MEF_\epsilon(x)]_n(l)) \\
&\quad - f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \Big) \\
&\quad - \sum_{l=0}^{N-1} f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))
\end{aligned}
$$

$$
\begin{aligned}
&\geq \alpha_i - \sum_{l=0}^{N-1} \Big| f_i(\epsilon, l, \alpha_1 + [MEF_\epsilon(x)]_1(l), \ldots, \alpha_i + [MEF_\epsilon(x)]_i(l), \ldots, \alpha_n + [MEF_\epsilon(x)]_n(l)) \\
&\quad - f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \Big| \\
&\quad - \sum_{l=0}^{N-1} \Big| f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l) \Big|
\end{aligned}
$$

$$
\begin{aligned}
&\geq \alpha_i - \sum_{l=0}^{N-1} \Big| f_i(\epsilon, l, \alpha_1 + [MEF_\epsilon(x)]_1(l), \ldots, \alpha_i + [MEF_\epsilon(x)]_i(l), \ldots, \alpha_n + [MEF_\epsilon(x)]_n(l)) \\
&\quad - f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \Big| \\
&\quad - Nm \\
&> \alpha_i - Nm - 1 \\
&> \alpha_i - \gamma_i \\
&\geq 0,
\end{aligned}
$$

while a similar calculation shows that $H_{i+1}(\epsilon, x, \alpha_1, \ldots, -\alpha_i, \ldots, \alpha_n) \leq 0$ for all $\alpha_i \in [\gamma_i, \delta_i]$.

Finally, for all $\alpha_i \in [0, \gamma_i]$,

$$
\begin{aligned}
&|H_{i+1}(\epsilon, x, \alpha_1, \dots, \pm\alpha_i, \dots, \alpha_n)| \\
&\le |\pm\alpha_i| + \sum_{l=0}^{N-1} \Big| f_i(\epsilon, l, \alpha_1 + [MEF_\epsilon(x)]_1(l), \dots, \pm\alpha_i + [MEF_\epsilon(x)]_i(l), \dots, \alpha_n + [MEF_\epsilon(x)]_n(l)) \\
&\qquad - f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \dots, \pm\alpha_i + [MEF_0(x)]_i(l), \dots, \alpha_n + [MEF_0(x)]_n(l)) \Big| \\
&\qquad + \sum_{l=0}^{N-1} \Big| f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \dots, \pm\alpha_i + [MEF_0(x)]_i(l), \dots, \alpha_n + [MEF_0(x)]_n(l)) \Big| \\
&\le \gamma_i + \sum_{l=0}^{N-1} \Big| f_i(\epsilon, l, \alpha_1 + [MEF_\epsilon(x)]_1(l), \dots, \pm\alpha_i + [MEF_\epsilon(x)]_i(l), \dots, \alpha_n + [MEF_\epsilon(x)]_n(l)) \\
&\qquad - f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \dots, \pm\alpha_i + [MEF_0(x)]_i(l), \dots, \alpha_n + [MEF_0(x)]_n(l)) \Big| \\
&\qquad + Nm \\
&< \gamma_i + Nm + 1 \\
&= \delta_i.
\end{aligned}
$$

Hence for each ϵ sufficiently small, $H(\epsilon, x, \alpha_1, \dots, \alpha_n) \in \mathcal{B}$ whenever $(x, \alpha_1, \dots, \alpha_n) \in \mathcal{B}$. Since H is a continuous function, the Brouwer Fixed Point Theorem guarantees existence of at least one fixed point of H in $\mathcal{B}$. This fixed point is a periodic solution of

$$\Delta x(t) = f(\epsilon, t, x(t)).$$

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