ABSTRACT

DUNBAR, JONATHAN D. The Affine Lie Algebra $\hat{sl}_n(\mathbb{C})$ and its $\mathbf{Z}$-algebra Representation. (Under the direction of Dr. Kailash Misra and Dr. Naihuan Jing.)

At the end of the 1960s, Victor Kac and Robert V. Moody independently discovered the infinite dimensional analogs of finite semisimple Lie algebras which we now refer to as Kac-Moody Lie algebras. These algebras have provided many avenues of interesting research, especially in the case of the *affine* Kac-Moody Lie algebras, which have led to important developments in vertex operators and other areas of mathematics and physics.

James Lepowsky and Robert L. Wilson introduced, in 1981, $\mathbf{Z}$-algebras associated with any integrable module of an affine Lie algebra. $\mathbf{Z}$-algebras are generated by Z-operators which centralize the action of the Heisenberg subalgebra and hence act on the vacuum space of the module. Their work was especially significant for providing Lie theoretic proofs of Roger-Ramanujan identities. In 1986, Minoru Wakimoto published his family of vertex operator realizations of $\hat{sl}(2)$. This family offered many realizations of $\hat{sl}(2)$ at arbitrary level, via creation and annihilation operators acting on an infinite-dimensional Fock space of Laurent polynomials. Subsequently, Boris Feigin and Edward Frenkel generalized Wakimoto's realizations for the affine Lie algebra $\hat{sl}(n)$. In particular, they gave explicit formulas for the action of the simple root vectors. Later, they extended these results to all affine Lie algebras.

In Chapter 2, we revisit Wakimoto's family of representations for $\hat{sl}(2)$. Through this explicit realization, we construct the associated Lepowsky-Wilson $\mathbf{Z}$-algebra. Chapter 3 extends Wakimoto's representation to $\hat{sl}(n)$, following the work of Feigin and Frenkel. Next, we give a general Wakimoto-style formula for the action of all positive root vectors, which is followed by a brief remark on the construction of the $\mathbf{Z}$-algebras associated with $\hat{sl}(n)$. We then give the explicit Wakimoto realization for $\hat{sl}(3)$ in Chapter 4 and offer the defining relations for its associated $\mathbf{Z}$-algebra. In the last chapter, we give a new realization, at the critical level, of the $\mathbf{Z}$-algebra associated with the Wakimoto modules of $\hat{sl}(2)$ acting on a Clifford-type algebra and calculate the character of the associated vacuum space.

The Affine Lie Algebra $\hat{sl}_n(\mathbb{C})$ and its $\mathbf{Z}$-algebra Representation

by
Jonathan D. Dunbar

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APPROVED BY:

Dr. Ernie Stitzinger	Dr. Tom Lada
Dr. Naihuan Jing Co-chair of Advisory Committee	Dr. Kailash Misra Co-chair of Advisory Committee

DEDICATION

To Lindsey

BIOGRAPHY

Jonathan Daniel Dunbar was born in Durham, North Carolina on June 23, 1983 to Amy and Daniel Dunbar. He moved with his family to Pittsford, New York in 1999 where he graduated from Pittsford-Mendon High School on the day of his eighteenth birthday. He then attended East Carolina University, initially with the intention of taking a major in Anthropology and a minor in Mathematics. Soon discovering a love for the abstract, he inverted his major and minor subjects and graduated with a B.A. in Mathematics in May of 2004. Staying at East Carolina University, he earned a M.A. in May of 2006 for his thesis, *Representations of the Heisenberg Group*, under the direction of Dr. Gail Ratcliff. That August, he began attending North Carolina State University, the same school that both his parents had attended some 25 years earlier. There he successfully defended his dissertation on December 1, 2010 and earned a Doctorate of Philosophy in Mathematics in May of 2011, under the direction of Dr. Naihuan Jing and Dr. Kailash Misra.

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Chapter 1

Introduction

Over the last 150 years, finite dimensional semisimple Lie algebras have gone from brand-new objects to well-understood structures. The work of Wilhelm Killing and Élie Cartan classified all complex simple Lie algebras into the so-called classical types, denoted A_n, B_n, C_n, and D_n, and the five exceptional types, denoted E_6, E_7, E_8, F_4, G_2. Later, Claude Chevalley would extend this classification to algebras over fields of characteristic 0 (cf. [3]). Through the works of Chevally and Jean-Pierre Serre, we learn that we can realize these algebras in terms of generators and relations. (cf. [4], [10] and [13], et.al.)

Infinite dimensional analogs of these algebras were discovered independently in 1967-1968 by Victor Kac, in [14] and [15], and Robert Moody, in [22] and [23]. Among the Kac-Moody Lie algebras, we are particularly interested in those of affine type. Given a Lie algebra $\mathfrak{g}$ of classic type, the affine Kac-Moody Lie algebra $\hat{\mathfrak{g}}$ of type 1, can be constructed by extending the associated loop-algebra for $\mathfrak{g}$ by a one-dimensional center and a degree derivation d. That is, $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}\left[t, t^{-1}\right] \oplus \mathbb{C}C \oplus \mathbb{C}d$. For elements of $\hat{\mathfrak{g}}$, we write $x \otimes t^m = x(m)$ and have the following bracket relations:

$$\begin{gathered}
[x_1(m), x_2(n)] = [x_1, x_2](m+n) + m\mathrm{Tr}(x_1 x_2) C \delta_{m+n,0} \\
[C, x] = 0,\ \forall\ x \in \hat{\mathfrak{g}} \qquad [d, x(m)] = mx(m).
\end{gathered}$$

When considering representations of affine Lie algebras, we note that the central element C will act as a scalar on the module. This scalar is the *level* of the representation (cf. [21]). This level is the very same as the concept of levels of highest weight modules. That is, as we write the highest weight as the linear combination of the fundamental weights, the level of the module equals the sum of the coefficients ([10], et. al.). The level is said to be *critical* when it is equal to the negative of the dual Coexeter number, $-h$ (cf. [18]).

The representation theory of affine Kac-Moody Lie algebras has become an important topic in mathematics and physics since Kac and Moody discovered the algebras. In particular, this theory has resulted in the rise in importance of the subject of *vertex operators*, involving an arrangement of creation and annihilation operators. Vertex operators were discovered as physicists delved into subjects like conformal field theory and string theory (cf. [13], [18]). Algebras of vertex operators were important in the famous moonshine module of the Monster simple group (See [9]). Several mathematicians have since constructed vertex operator realizations of affine Kac-Moody Lie algebras. Of particular interest is the affine Kac-Moody Lie algebras $\hat{sl}(n)$. In [24], [20] and others, we have explicit realizations of $\hat{sl}(n)$ which utilize vertex operators. For more information on vertex operators and examples of constructions using them, consult [16], [13], [18], [20], [21], [19], and notably [24]. The realization of $\hat{sl}(2)$ by Minoru Wakimoto found in [24] is of special significance, and we will revisit it in Chapter 2.

In 1981, James Lepowsky and Robert Wilson introduced the concept of a **Z**-algebra associated with an integrable module to better understand the structure of its vacuum space, which is the subspace of vectors that are annihilated by the positive part of the Heisenberg subalgebra. The underlying property of the **Z**-algebra is that it centralizes the action of the entire Heisenberg subalgebra, and thus acts on the vacuum space. In [21], Lepowsky and Wilson described the basis of the vacuum space of integrable highest-weight modules as monomials of **Z**-operators acting on the module's highest weight vector, in the principle picture. By calculating character formulas of the vacuum space, they then were able to give Lie theoretic proofs of Rogers-Ramanujan identities. In Chapter 2 we reintroduce the **Z**-algebra associated with a Wakimoto module following the work of Lepowsky and Wilson in [21] and Lepowsky and Mirko Primc in [19].

In the chapters that follow, we shall first recall the definition of the affine special linear Lie algebra $\hat{sl}(2)$ and Wakimoto's family of representations. Using this explicit realization of $\hat{sl}(2)$, we then introduce the method of constructing the associated Lepowsky-Wilson **Z**-algebra. In Chapter 3, we show the method for extending Wakimoto's representation to $\hat{sl}(n)$ for arbitrary n, which was first done by Igor Feigin and Edward Frenkel in [5] for the simple root vectors. They later showed in [6] that this construction can be extended to all affine Lie algebras. We offer formulae for the Wakimoto-style operator realization for all positive root vectors, and briefly comment on the general method for constructing **Z**-algebras associated with these modules for $\hat{sl}(n)$. In Chapter 4 we discuss in detail the Wakimoto realization for $\hat{sl}(3)$ and also present the defining relations for its associated **Z**-algebra in the form of the "multiplication tables" in Tables 4.1 and 4.2. In Chapter 5, we present an explicit realization of the **Z**-algebra associated with the affine Lie algebra $\hat{sl}(2)$ at the critical level -2 acting on a Clifford type algebra. Finally, we use this explicit realization to calculate the character of the associated vacuum space.

Chapter 2

$\hat{sl}(n)$ and its Wakimoto Representations

2.1 The Special Linear Lie Algebra

Here we will define some basic ideas and constructions which are necessary for comprehension of this work. Typically, these definitions will not be specifically referenced in later chapters, but they are fundamentally important to understanding this dissertation.

Notationally, we let E_{ij} refer to the elementary matrix with a 1 as the i^{th} row and j^{th} column entry and zeroes everywhere else. Also, recall that the *trace* of an $n \times n$ matrix is the sum of the elements along its main diagonal. Define the *commutator bracket* of matrices A and B to be $[A, B] := AB - BA$, where AB implies the typical matrix product.

Let $\mathfrak{g} = sl_n(\mathbb{C})$ denote the set of $n \times n$ complex trace zero matrices. Since we will always take our base field to be the complex numbers, we may refer to this set equivalently as $\mathfrak{g}$, sl_n or $sl(n)$.

Under the commutator bracket defined above, $sl_n(\mathbb{C})$ is a Lie algebra, because it satisfies these properties: (Let $x, y, z \in sl_n$.)

1. $[\ ,\ ] : sl_n \times sl_n \mapsto sl_n$ is a bilinear map.
2. $[x, x] = 0$
3. The map also satisfies the *Jacobi Identity*, $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$

Remark 2.1.1. At this time we also introduce the *anti-commutator* $\{A, B\} := AB + BA$. This will become an important definition in later chapters.

Let $\mathfrak{h} = \text{span}\,[H_i = E_{ii} - E_{i+1,i+1} | 1 \leq i \leq n-1]$ be the subalgebra of sl_n consisting of trace 0 diagonal matrices. We call $\mathfrak{h}$ the *Cartan subalgebra* of $\mathfrak{g}$. For $x \in sl(n)$, ad_x denotes the linear transformation given by $ad_x(y) = [x, y]$. Then ad_h acts semisimply on $\mathfrak{g}$ for all $h \in \mathfrak{h}$.

For $i = 1, 2, 3, \ldots, n$, define the linear functionals $\epsilon_i \in \mathfrak{h}^*$, such that

$$\epsilon_i \begin{bmatrix} c_1 & 0 & \ldots & 0 \\ 0 & c_2 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & c_n \end{bmatrix} = c_i.$$

Then, we take the following notation for some important linear functionals:

$$\alpha_i := \epsilon_i - \epsilon_{i+1}, \ 1 \leq i \leq n-1 \tag{2.1.1}$$

$$\Lambda_i := \epsilon_1 + \epsilon_2 + \cdots + \epsilon_i, \ 1 \leq i \leq n-1. \tag{2.1.2}$$

Definition 2.1.2. For $\alpha \in \mathfrak{h}^*$, define the root space $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} | \, [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$.

The linear functional $0 \neq \alpha \in \mathfrak{h}^*$ is a *root* if $\mathfrak{g}_\alpha \neq \{0\}$. Note $\mathfrak{g}_{\alpha_i} = span\{X_{i,i+1} := E_{i,i+1}\}$ and $\mathfrak{g}_{-\alpha_i} = span\{Y_{i,i+1} := E_{i+1,i}\}$. The set $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\}$ is called the set of *simple roots*. Since $\epsilon_i - \epsilon_j = \alpha_i + \ldots + \alpha_{j-1}$ for $i < j$, then let $\Phi^+ = \{\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1} | 1 \leq i < j \leq n\}$ denote the set of positive roots, and let $\Phi^- = \{-(\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}) \, | 1 \leq i < j \leq n\}$ denote the set of negative roots. $\Phi = \Phi^+ \cup \Phi^-$ is the set of all roots. For $i < j$, $\mathfrak{g}_{\epsilon_i - \epsilon_j} = span\{X_{ij} := E_{ij}\}$. The Lie algebra sl_n has the *root space decomposition* $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^\pm = \bigoplus_{\pm\alpha \in \Delta^+} \mathfrak{g}_\alpha$.

For sl_n, we define the inner product of two roots in Δ. The definition then extends to all of $\mathfrak{h}^*$.

$$(\alpha_i, \alpha_j) := \alpha_i(H_j) = \begin{cases} 2, & i = j \\ -1, & i = j \pm 1 \\ 0, & \text{otherwise} \end{cases} \tag{2.1.3}$$

Definition 2.1.3. A set of *generators* of a Lie algebra $\mathfrak{g}$ is a set of elements $S \subset \mathfrak{g}$ such that all elements in $\mathfrak{g}$ can be written as the linear combination of elements, or brackets of elements in S. If the set $S = \{s_1, s_2, \ldots, s_n\}$ generates the Lie algebra $\mathfrak{g}$, then we write $\mathfrak{g} = \langle s_1, s_2, \ldots, s_n \rangle$.

Letting $E_i = X_{i,i+1} := E_{i,i+1}$, $F_i = Y_{i,i+1} := E_{i+1,i}$ and $H_i := E_{ii} - E_{i+1,i+1}$, then $\mathfrak{g} = sl_n = \langle E_i, F_i, H_i | 1 \leq i \leq n-1 \rangle$.

Definition 2.1.4. A *Lie algebra homomorphism* is a linear map $f : \mathfrak{g}_1 \longmapsto \mathfrak{g}_2$ between Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, with respective brackets $[\ ,\ ]_1$ and $[\ ,\ ]_2$, such that for any $g, \ g' \in \mathfrak{g}_1$, the relation $f([g, g']_1) = [f(g), f(g')]_2$ holds.

When a homomorphism is both injective and surjective, we call it an *isomorphism*. An isomorphism between a Lie algebra and itself is an *automorphism*.

Definition 2.1.5. For any vector space V, $gl(V) = \{\text{linear maps } \tau : V \mapsto V\}$ is a Lie algebra under the commutator bracket. A *representation* of $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism $\pi : \mathfrak{g} \longmapsto gl(V)$. We call V the *representation space.*

Given such a representation, we say that V is a $\mathfrak{g}$-module via the action $g \cdot v = \pi(g)(v)\ \forall\ g \in \mathfrak{g},\ v \in V$. When defining a representation π of $\mathfrak{g}$, it is sufficient to define how the map π acts on its generators.

Definition 2.1.6. Suppose that the vector space V has the decomposition $V = \bigoplus_{m \in \mathbb{Z}} V(m)$, such that $\dim(V(m)) < \infty$ for all $m \in \mathbb{Z}$. Then we say that V has a $\mathbb{Z}$*-grading.* The *character* of a representation is the $\mathbb{Z}$-graded q-dimension of the module V, denoted by $ch(V)$, and defined

$$ch(V) = \sum_{m \in \mathbb{Z}} \dim(V(m)) q^m.$$

2.2 The Infinite Dimensional Affine Lie Algebra $\hat{sl}(n)$

In 1968, Victor Kac and Robert V. Moody independently defined infinite dimensional analogs of semisimple Lie algebras. The finite dimensional Lie algebra $\mathfrak{g} = sl(n)$ has such an analog called the *affine* special linear Lie algebra and denoted $\hat{\mathfrak{g}} = \hat{sl}(n)$. We define it as

$$\hat{\mathfrak{g}} = \hat{sl}_n = sl_n \otimes \mathbb{C}\left[t, t^{-1}\right] \oplus \mathbb{C}C \oplus \mathbb{C}d,$$

where C is a *central element*, meaning that it commutes with all elements of $\hat{\mathfrak{g}}$. The element $d = 1 \otimes t\frac{\partial}{\partial t}$ is the *degree derivation.* For $x \in \hat{\mathfrak{g}}$ where $[d, x] = mx$, we say that m is the degree of x. Elements in $sl_n \otimes \mathbb{C}\left[t, t^{-1}\right]$ are denoted $x(m) := x \otimes t^m$ and brackets for $\hat{\mathfrak{g}}$ are defined

$$[x_1(m), x_2(n)] = [x_1, x_2](m+n) + mC\mathrm{Tr}(x_1 x_2)\,\delta_{m+n,0}, \tag{2.2.1}$$

$$[C, x(m)] = 0, \qquad\qquad [d, x(m)] = m(m),$$

where x, x_1 and $x_2 \in \mathfrak{g}$, $[x_1, x_2]$ is the usual commutator bracket for $\mathfrak{g}$, $\mathrm{Tr}(x_1 x_2)$ is the trace of the matrix product $x_1 x_2$, and $\delta_{m+n,0}$ is the Kroenecker delta function. (cf. [4], [13])

Definition 2.2.1. Let a matrix A have the following properties:

- $a_{ii} = 2,\ \forall\ i = 1, 2, \ldots, n-1$
- $a_{ij} \leq 0,\ i \neq j$
- $a_{ij} = 0 \iff a_{ji} = 0$

Any matrix that satisfies these properties is called a *generalized Cartan matrix* (denoted GCM). A GCM $A = (a_{ij})$ is symmetrizable if there is a nonsingular diagonal matrix D such that DA is symmetric. A positive-definite GCM is called a *Cartan matrix*.

Definition 2.2.2. Given a GCM A, we can define a Lie algebra L over the field $\mathbb{C}$, generated by $\{e_i, f_i, h_i | 1 \leq i \leq n-1\}$ (called the *Chevalley generators*), with relations (called Serre relations):

(1)	$[h_i, h_j] = 0,\ \forall\ i, j = 1, 2, \ldots, n-1$	(4)	$[e_i, f_j] = \delta_{ij} h_i$
(2)	$[h_j, f_i] = -a_{ij} f_i$	(5)	$(ad_{e_i})^{1-a_{ij}}\, e_j = 0,\ \forall\ i \neq j$
(3)	$[h_j, e_i] = a_{ij} e_i$	(6)	$(ad_{f_i})^{1-a_{ij}}\, f_j = 0,\ \forall\ i \neq j$

These algebras are called *Kac-Moody Lie algebras*. (See [13].)

Example 2.2.3. Define the $(n-1) \times (n-1)$ matrix $A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$.

The Kac-Moody Lie algebra L, defined by generators $\{e_i, f_i, h_i |\ 1 \leq i \leq n-1\}$, Cartan matrix A, and the Serre relations in Definition 2.2.2, is the finite dimensional simple Lie algebra $\mathfrak{g} = sl_n$.

The affine Lie algebra $\hat{\mathfrak{g}} = \hat{sl}_n$ can also be defined by the Chevalley generators $\{e_i, f_i, h_i |\ 0 \leq i \leq n-1\}$ and the Serre relations. The GCM associated with $\hat{\mathfrak{g}}$ is

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, \text{ for } n = 2 \text{ and } A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \ddots & & 0 \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ 0 & & \ddots & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}, \text{ for } n > 2. \quad (2.2.2)$$

2.3 Wakimoto's Representation of $\hat{sl}(2)$ at All Levels

In 1986, Minoru Wakimoto published the construction of his family of representations of $\hat{sl}_2(\mathbb{C})$ (see [24]). This family defined, for each pair $\mu, \nu \in \mathbb{C}$, a representation $\pi_{\mu,\nu}$ on the space of polynomials $V = \mathbb{C}\,[x_i,\ y_j]_{\substack{i \in \mathbb{Z} \\ j \geq 0}}$. His definition requires the use of the following differential operators

$$a(i) = \begin{cases} x_i & \text{if } i \geq 0 \\ \frac{-\partial}{\partial x_i} & \text{if } i < 0 \end{cases} \qquad a^*(i) = \begin{cases} \frac{\partial}{\partial x_i} & \text{if } i \geq 0 \\ x_i & \text{if } i < 0 \end{cases} \qquad b(i) = \begin{cases} iy_i & \text{if } i > 0 \\ 0 & \text{if } i = 0 \\ \frac{-\partial}{\partial y_{-i}} & \text{if } i < 0 \end{cases} .$$

Notice that our operators take two forms: multiplication by a variable and differentiation by a variable. The former we shall refer to as a *creation* operator. The later shall be known as an *annihilation* operator.

It will be necessary to employ a different type of product here.

Definition 2.3.1. We define the (bosonic) *normally ordered product* of operators as follows:

$$:a(i)a^*(j):= \begin{cases} a(i)a^*(j) & \text{if } i > 0 \\ \frac{1}{2}\left(a(0)a^*(j) + a^*(j)a(0)\right) & \text{if } i = 0 \\ a^*(j)a(i) & \text{if } i < 0 \end{cases} .$$

(Note that this is a slightly modified version from the one given in [24].)

The effect of this type of product is that it moves annihilation operators to the right of creation operators, so that they act first. This makes the operators well-defined as they act on the representation space. Utilizing this normal ordering, we define the operator $E(m) = \sum_{j \in \mathbb{Z}} :a(j+m)a^*(j):$.

Given these operators, we can consider their various commutator brackets as they act on V. Once we define the representation we will need these relations and others to prove the bracket relations for its action on $\hat{sl}_2$. Here are some examples of the calculated relations:

- $[a(i), a(j)] = 0 = [a^*(i), a^*(j)]$
- $[a^*(i), a(j)] = \delta_{ij}$
- $[:a(i)a^*(j):, a(k)] = a(i)\delta_{jk}$,
  $[:a(i)a^*(j):, a^*(k)] = -a^*(j)\delta_{ik}$
- $[E(m), E(n)] = m\delta_{m,-n} = [b(m), b(n)]$

Use the following naming system for these matrices in sl_2: $X = E_{12}$, $H = E_{11} - E_{22}$, and $Y = E_{21}$. Then, we can define the family of Wakimoto representations of $\hat{sl}_2$ after choosing the following as its Chevalley generators

$$\begin{array}{ll} e_0 = X(1), & e_1 = Y(0), \\ f_0 = Y(-1), & f_1 = X(0), \\ h_0 = H(0) + C, & h_1 = -H(0) \end{array} .$$

Definition 2.3.2. The representation $\pi_{\mu,\nu} : \hat{sl}_2 \longrightarrow GL(V)$ is defined, for all $\mu, \nu \in \mathbb{C}$ as follows:

$$\begin{aligned} \pi_{\mu,\nu}(X(n)) &= a(-n) \\ \pi_{\mu,\nu}(Y(n)) &= \left(\mu - 1 - \left(\frac{\nu^2}{2} + 1\right) n\right) a^*(n) - \sum_{j \in \mathbb{Z}} :E(j)a^*(j+n): + \nu \sum_{j \in \mathbb{Z}} b(j)a^*(j+n) \\ \pi_{\mu,\nu}(H(n)) &= 2E(-n) - \nu b(-n) + (1-\mu)\delta_{n,0} \\ \pi_{\mu,\nu}(C) &= -\left(2 + \frac{\nu^2}{2}\right) \\ \pi_{\mu,\nu}(d) &= -\sum_{j \in \mathbb{Z}} j :a(j)a^*(j): + \nu \sum_{j=1}^{\infty} b(j)b(-j). \end{aligned}$$

The way in which C acts on the representation space is the called the *level* of the representation. Notice that the level depends the value of ν, and can itself take any value in $\mathbb{C}$. Observe that in the case of $\nu = 0$ the representation given above greatly simplifies to:

$$\begin{aligned} \pi_{\mu,\nu}(X(n)) &= a(-n) \\ \pi_{\mu,\nu}(Y(n)) &= (\mu - 1 - n)\, a^*(n) - \sum_{j \in \mathbb{Z}} :E(j)a^*(j+n): \\ \pi_{\mu,\nu}(H(n)) &= 2E(-n) + (1-\mu)\,\delta_{n,0} \\ \pi_{\mu,\nu}(C) &= -2 \\ \pi_{\mu,\nu}(d) &= -\sum_{j \in \mathbb{Z}} j :a(j)a^*(j): . \end{aligned}$$

The level in this case is -2,which is the negative dual Coxeter number for $sl(2)$. When the level of the representation equals the negative dual Coxeter number, we call that the *critical level.*

In general, verifying the bracket relations for $\hat{sl}_2$ would be difficult since there are so many elements. So, to do the calculations all at once we employ generating functions, as in [5], where we take the elements to be coefficients in a formal power series.

For $x \in \hat{sl}_2$, define the series $x(z) = \sum_{n \in \mathbb{Z}} x(n)z^{-n}$. We call the homogeneous components $x(n)$ the *nodes* of $x(z)$.

Clearly, to define generating functions for elements of the algebra, we must also construct them for our operators in a way that matches that definition. Fortunately, we can define them so that the definition is consistent, and also so that the power series of the operators interact with one another in the expected way.

$$\text{Define } a(z) = \sum_{n\in\mathbb{Z}} a(n)z^n,\ a^*(z) = \sum_{n\in\mathbb{Z}} a^*(n)z^{-n},\ b(z) = \sum_{n\in\mathbb{Z}} b(n)z^n$$
$$\text{and } E(z) = \sum_{m\in\mathbb{Z}} E(m)z^m = {:}a(z)a^*(z){:}\,.$$

Observe that the product $x_1(z)x_2(z)$ does not make sense because, for instance, we have an infinite set of coefficients for the constant term. In fact, this is true for all terms. One way to make sense of this is through the normal ordering of a product of power series.

Definition 2.3.3. Define the *normal ordered product* of two formal power series by

$$:x_1(z)x_2(w): = \sum_{m,n\in\mathbb{Z}} :x_1(m)x_2(n):\, z^m w^n.$$

Note that $:x_1(z)x_2(w): = :x_2(w)x_1(z):$.

The other way we make sense of the product of formal power series is by taking products of series with different variables, as in $x_1(z)x_2(w)$. Here we have little trouble determining the coefficient of the $z^m w^n$ term, namely $(x_1(m)x_2(n))$. When calculating commutator brackets of our generating functions, we will only consider those in different variables.

The convenience of switching to formal power series is that they provide commutator relations analogous to those in [24]. To calculate such brackets, we require the employment of contractions. Physicists call these *operator product expansions*, or *OPE*s.

Definition 2.3.4. Given formal power series $x_1(z)$, and $x_2(w)$ with normal ordering $:x_1(z)x_2(w):$, then their *contraction* is denoted and defined to be
$\underbrace{x_1(z)x_2(w)} = x_1(z)x_2(w) - :x_1(z)x_2(w):$.

Physicists' notation utilizes the tilde to denote the contraction of formal series. That is, $x_1(z)x_2(w) \sim \underbrace{x_1(z)x_2(w)}$. This emphasizes the property that $x_1(z)x_2(w)$ behaves like its contraction in commutator brackets.

We can observe this here, and thus see the usefulness of contractions.

$$\begin{aligned}
[x_1(z), x_2(w)] &= x_1(z)x_2(w) - x_2(w)x_1(z) \\
&= \left(\underbrace{x_1(z)x_2(w)} + :x_1(z)x_2(w):\right) - \left(\underbrace{x_2(w)x_1(z)} + :x_2(w)x_1(z):\right) \\
&= \underbrace{x_1(z)x_2(w)} - \underbrace{x_2(w)x_1(z)}
\end{aligned}$$

Example 2.3.5. Here we calculate an example contraction, using Definition 2.3.1 and Definition 2.3.3.

$$\begin{aligned}\underbrace{a(z)a^*(w)} &= a(z)a^*(w) - :a(w)a^*(z): = \sum_{m,n\in\mathbb{Z}} a(m)a^*(n)z^m w^{-n} - \sum_{m,n\in\mathbb{Z}} :a(m)a^*(n): z^m w^{-n} \\ &= \sum_{m<0,\ n\in\mathbb{Z}} \left[a(m), a^*(n)\right] z^m w^{-n} + \frac{1}{2}\sum_{n\in\mathbb{Z}} \left[a(0), a^*(n)\right] w^{-n} \\ &= \sum_{m<0,\ n\in\mathbb{Z}} -\delta_{mn} z^m w^{-n} + \frac{1}{2}\sum_{n\in\mathbb{Z}} -\delta_{0n} w^{-n} = -\sum_{m<0} z^m w^{-m} - \frac{1}{2} \\ &= -\sum_{m>0}\left(\frac{w}{z}\right)^m - \frac{1}{2} = -\left(\frac{w}{z-w}\right) - \frac{1}{2} = -\left(\frac{z+w}{2(z-w)}\right)\end{aligned}$$

Similarly, we can find $\underbrace{a^*(w)a(z)} = \dfrac{z+w}{2(w-z)}$.
Hence, $[a^*(w), a(z)] = \dfrac{z+w}{2(w-z)} + \dfrac{z+w}{2(z-w)} = \displaystyle\sum_{n\in\mathbb{Z}}\left(\frac{w}{z}\right)^n$.

We will also encounter products of two or more normally ordered products, for example products that include $E(z) = :a(z)a^*(z):$, we need to have a method for understanding these products.

Theorem 2.3.6. *(Wick's Formula)*
Let $A_1(z), \ldots, A_m(z), B_1(w), \ldots, B_n(w)$ be formal power series of creation and annihilation operators with normal ordering and contractions as defined in Definition 2.3.3 and Definition 2.3.4, respectively. Then,

$$\begin{aligned}&:A_1(z)\cdots A_m(z)::B_1(w)\cdots B_n(w): \\ &\quad= \sum_{s=0}^{min\{m,n\}} \sum_{\substack{i_1<\cdots<i_s \\ j_1\neq\cdots\neq j_s}} \left(\underbrace{A_{i_1}(z)B_{j_1}(w)}\cdots\underbrace{A_{i_s}(z)B_{j_s}(w)}\right)\cdot \\ &\qquad\cdot(:A_1(z)\cdots A_{i_1-1}(z)A_{i_1+1}(z)\cdots A_m(z)\cdot B_1(w)\cdots B_{j_1-1}(w)B_{j_1+1}(w)\cdots B_n(w):)\end{aligned}$$

That is, the product of two normally ordered products is the sum of all possible products of normal orderings and non-zero contractions where, in each term, each of $A_1(z), \ldots, A_m(z)$, $B_1(w), \ldots, B_n(w)$ is present in either the normally ordered product or a contraction.

Example 2.3.7. Perhaps, this is best understood through an example calculation:

$$\begin{aligned} E(z)E(w) &= :a(z)a^*(z)::a(w)a^*(w): \\ &= :a(z)a^*(z)a(w)a^*(w): + :a(z)a^*(w): \underbrace{a^*(z)a(w)} \\ &\quad + :a^*(z)a(w): \underbrace{a(z)a^*(w)} + \underbrace{a^*(z)a(w)}\,\underbrace{a(z)a^*(w)} \\ &= :a(z)a^*(z)a(w)a^*(w): + :a(z)a^*(w): \left(\frac{z+w}{2(z-w)}\right) \\ &\quad - :a^*(z)a(w): \left(\frac{z+w}{2(z-w)}\right) - \left(\frac{z+w}{2(z-w)}\right)^2 \end{aligned}$$

Also for any generating function $x(z)$, we vacuously define $:x(z): = x(z)$. With these products fully understood, we can now calculate a variety of bracket relations for the formal power series realizations of our Wakimoto operators.

- $[a(z), a(w)] = 0 = [a^*(z), a^*(w)]$
- $[a^*(z), a(w)] = \delta\left(\frac{w}{z}\right)$, (as seen in example 2.3.5)
- $[E(z), a(w)] = a(w)\delta\left(\frac{w}{z}\right), \qquad [E(z), a^*(w)] = -a^*(w)\delta\left(\frac{w}{z}\right)$
- $[E(z), E(w)] = -w\partial_w\delta\left(\frac{w}{z}\right) = [b(z), b(w)]$
- $[:E(z)a^*(z):, a(w)] = 2E(w)\delta\left(\frac{w}{z}\right) - w\partial_w\delta\left(\frac{w}{z}\right)$
- $[:E(z)a^*(z):, a^*(w)] = -a^*(w)a^*(w)\delta\left(\frac{w}{z}\right)$
- $[:E(z)a^*(z):, E(w)] = :E(w)a^*(w):\delta\left(\frac{w}{z}\right) - a^*(w)w\partial_w\delta\left(\frac{w}{z}\right)$

Here, $\delta\left(\frac{w}{z}\right) = \sum_{n\in\mathbb{Z}}\left(\frac{w}{z}\right)^n$ is known as the *Dirac delta function* (See [13], [16], [5]).

The delta function has the following critical properties:

$$f(z,w)\delta\left(\frac{w}{z}\right) = f(z,z)\delta\left(\frac{w}{z}\right) = f(w,w)\delta\left(\frac{w}{z}\right) \tag{2.3.1}$$

$$f(z,w)\partial_w\delta\left(\frac{w}{z}\right) = \partial_w\left[f(z,w)\delta\left(\frac{w}{z}\right)\right] - \partial_w\left[f(z,w)\right]\delta\left(\frac{w}{z}\right) \tag{2.3.2}$$

$$f(z,w)\partial_w\delta\left(\frac{w}{z}\right) = f(z,z)\partial_w\delta\left(\frac{w}{z}\right) - \partial_w\left[f(z,w)\right]\delta\left(\frac{w}{z}\right) \tag{2.3.3}$$

$$z\partial_z\delta\left(\frac{w}{z}\right) = -w\partial_w\delta\left(\frac{w}{z}\right) \tag{2.3.4}$$

Formulas (2.3.2) and (2.3.3) are equivalent, following from (2.3.1).

Remarks 2.3.8.

- $\delta\left(\frac{w}{z}\right) = \left(\frac{z+w}{2}\right)\left(\frac{1}{z-w} + \frac{1}{w-z}\right)$
- $w\partial_w \delta\left(\frac{w}{z}\right) = \left(\frac{z+w}{2}\right)^2\left(\frac{1}{(z-w)^2} - \frac{1}{(w-z)^2}\right)$

In the first remark, we view the term $\frac{1}{z-w}$ as the sum of the geometric series $w^{-1}\sum_{m>0}\left(\frac{w}{z}\right)^m$ where $|w| < |z|$. Similarly, $\frac{1}{w-z}$ is a geometric series sum where $|z| < |w|$. In [16], we see that $\partial_w^{(j)}\delta(z-w) = \dfrac{1}{(z-w)^{j+1}} - \dfrac{1}{(w-z)^{j+1}} = \sum_{m\in\mathbb{Z}}\begin{pmatrix} m \\ j \end{pmatrix} z^{-m-1}w^{m-j}$, where $\partial_w^{(j)} = \frac{1}{j!}\partial_w$ and $\delta(z-w) = z^{-1}\sum_{m\in\mathbb{Z}}\left(\frac{w}{z}\right)^m$. We know that $\delta(z-w)$ and $\delta\left(\frac{w}{z}\right)$ share properties (2.3.1), (2.3.2), and (2.3.4).

It is a worthwhile exercise to verify the formulas in remark 2.3.8. Observe

$$\left(\frac{z+w}{2}\right)\left(\frac{1}{z-w} + \frac{1}{w-z}\right) = \left(\frac{z+w}{2}\right)\delta\left(z-w\right) = z\delta\left(z-w\right) = \delta\left(\frac{w}{z}\right), \text{ and}$$

$$\begin{aligned}
&\left(\frac{z+w}{2}\right)^2\left(\frac{1}{(z-w)^2} - \frac{1}{(w-z)^2}\right) \\
&\quad = \left(\frac{z+w}{2}\right)^2\left(\partial_w\delta(z-w)\right) = \partial_w\left[\left(\frac{z+w}{2}\right)^2\delta(z-w)\right] - \partial_w\left[\left(\frac{z+w}{2}\right)^2\right]\delta(z-w) \\
&\quad = \partial_w\left[w^2\delta(z-w)\right] - \partial_w\left[\left(\frac{z+w}{2}\right)^2\right]\delta(z-w) = \partial_w\left[\sum_{m\in\mathbb{Z}} z^{-m-1}w^{m+2}\right] - w\delta(z-w) \\
&\quad = \sum_{m\in\mathbb{Z}}(m+2)z^{-m-1}w^{m+1} - \sum_{n\in\mathbb{Z}}\left(\frac{w}{z}\right)^n = \sum_{m\in\mathbb{Z}} m\left(\frac{w}{z}\right)^m = w\partial_w\delta\left(\frac{w}{z}\right).
\end{aligned}$$

Now, for the sake of simpler arguments, we define the series of generators

$$X(z) = \sum_{m\in\mathbb{Z}} X(m)z^{-m},\ Y(z) = \sum_{m\in\mathbb{Z}} Y(m)z^{-m},\ H(z) = \sum_{m\in\mathbb{Z}} H(m)z^{-m}.$$

Then, following (2.2.1), the commutator bracket relations of $\hat{sl}_2$ as generating functions are as follows:

$$[H(z), X(w)] = \sum_{m,n\in\mathbb{Z}} [H(m), X(n)]\, z^{-m}w^{-n} = \sum_{m,n\in\mathbb{Z}} 2X(m+n)z^{-m}w^{-n} \tag{2.3.5}$$
$$= 2\sum_{m,n\in\mathbb{Z}} X(n)w^{-n}\left(\frac{w}{z}\right)^m = 2X(z)\delta\left(\frac{w}{z}\right)$$

$$[H(z), Y(w)] = \sum_{m,n\in\mathbb{Z}} [H(m), Y(n)]\, z^{-m}w^{-n} = -2Y(z)\delta\left(\frac{w}{z}\right) \tag{2.3.6}$$

$$[X(z), Y(w)] = \sum_{m,n\in\mathbb{Z}} [X(m), Y(n)]\, z^{-m}w^{-n} \tag{2.3.7}$$
$$= \sum_{m,n\in\mathbb{Z}} \left(H(m+n) + mC\delta_{m,-n}\right) z^{-m}w^{-n}$$
$$= H(w)\delta\left(\frac{w}{z}\right) + Cw\partial_w\delta\left(\frac{w}{z}\right)$$

$$[H(z), H(w)] = \sum_{m,n\in\mathbb{Z}} [H(m), H(n)]\, z^{-m}w^{-n} = 2Cw\partial_w\delta\left(\frac{w}{z}\right) \tag{2.3.8}$$

Using formal power series, we define the Wakimoto representation in this way:

$$\pi_{\mu,\nu}(X(z)) = a(z)$$
$$\pi_{\mu,\nu}(Y(z)) = (\mu - 1)\,\chi a^*(z) + \left(\frac{\nu^2}{2} + 1\right) z\partial_z a^*(z) - :E(z)a^*(z): + \nu b(z)a^*(z)$$
$$\pi_{\mu,\nu}(H(z)) = 2E(z) - \nu b(z) + (1-\mu)\chi, \text{ where } \chi(H(n)) = \delta_{n,0}$$

In general, we prefer to leave the map $\pi_{\mu,\nu}$ as assumed but unwritten.
Bracket calculations can now be performed, for general levels, with the generating functions. To confirm that we have a representation, we verify the relations.

$$[H(z), X(w)] = [2E(z) - \nu b(z) + (1-\mu)\,\chi, a(w)] = 2\,[E(z), a(w)] \tag{2.3.9}$$
$$= 2a(w)\delta\left(\frac{w}{z}\right) = 2X(w)\delta\left(\frac{w}{z}\right)$$

Thus, we have calculated $[H(m), X(n)]$ for all $m, n \in \mathbb{Z}$, and confirmed (2.3.5).

$$
\begin{aligned}
[H(z), Y(w)] &= -2\,[E(z), : E(w)a^*(w) :] + 2(\mu-1)\chi\,[E(z), a^*(w)] + 2\nu b(w)\,[E(z), a^*(w)] \\
&\quad + 2\left(\frac{\nu^2}{2}+1\right) w\partial_w\,[E(z), a^*(w)] - \nu^2\,[b(z), b(w)]\,a^*(w) \\
&= 2 : E(w)a^*(w) : \delta\left(\frac{w}{z}\right) - 2a^*(w)w\partial_w\delta\left(\frac{w}{z}\right) + 2(1-\mu)\chi a^*(w)\delta\left(\frac{w}{z}\right) \\
&\quad - 2\nu b(w)a^*(w)\delta\left(\frac{w}{z}\right) - (\nu^2+2)\left(w\partial_w(a^*(w))\delta\left(\frac{w}{z}\right) - a^*(w)w\partial_w\delta\left(\frac{w}{z}\right)\right) \\
&\quad - \nu^2 a^*(z)w\partial_w\delta\left(\frac{w}{z}\right) \qquad (2.3.10) \\
&= -2Y(w)\delta\left(\frac{w}{z}\right)
\end{aligned}
$$

$$
\begin{aligned}
[X(z), Y(w)] &= (\mu-1)\,\chi\,[a(z), a^*(w)] + \left(\frac{\nu^2}{2}+1\right)[a(z), w\partial_z a^*(w)] \qquad (2.3.11) \\
&\quad - [a(z), : E(w)a^*(w) :] + \nu b(w)\,[a(z), a^*(w)] \\
&= (1-\mu)\chi\delta\left(\frac{w}{z}\right) - \left(-2E(w)\delta\left(\frac{w}{z}\right) + w\partial_w\delta\left(\frac{w}{z}\right)\right) \\
&\quad - \nu b(w)\delta\left(\frac{w}{z}\right) - \left(1+\frac{\nu^2}{2}\right) w\partial_w\delta\left(\frac{w}{z}\right) \\
&= (2E(w) - \nu b(w) + (1-\mu)\chi)\,\delta\left(\frac{w}{z}\right) - \left(2+\frac{\nu^2}{2}\right) w\partial_w\delta\left(\frac{w}{z}\right) \\
&= H(w)\delta\left(\frac{w}{z}\right) + Cw\delta_w\delta\left(\frac{w}{z}\right)
\end{aligned}
$$

$$
[H(z), H(w)] = 4\,[E(z), E(w)] + \nu^2\,[b(z), b(w)] = 2\left(2+\frac{\nu^2}{2}\right)\left(-w\partial_w\delta\left(\frac{w}{z}\right)\right) = 2Cw\partial_w\delta\left(\frac{w}{z}\right) \qquad (2.3.12)
$$

So, we have verified that the formal power series of the representation is consistent with the known relations for $\hat{sl}(2)$.

2.4 The Algebra $\mathcal{Z}$

As introduced in [21] and following the construction in [19] we construct an algebra $\mathcal{Z}$ corresponding to the affine Lie algebra $\hat{\mathfrak{g}}$. Let $\mathcal{Z}(n)$ denote the **Z**-algebra associated with $\hat{sl}(n)$. Letting Φ be the set of roots for $\hat{\mathfrak{g}}$, then for all $\alpha \in \Phi$ and all $m \in \mathbb{Z}$, we shall define operators $\mathbf{Z}(\alpha, m) \in \mathcal{Z}(n)$ which act on the vector space V.

For the Lie algebra $\mathfrak{g}$, we have the root space decomposition, as in Definition 2.1.2, with Cartan subalgebra $\mathfrak{h}$. Then we now define the Heisenberg subalgebra as in [19].

Definition 2.4.1. For affine Lie algebra $\hat{\mathfrak{g}}$, we define the *Heisenberg subalgebra* as

$$\hat{\mathfrak{h}} = \bigoplus_{n \in \mathbb{Z}_{\neq 0}} (\mathfrak{h} \otimes t^n) \oplus \mathbb{C}C \oplus \mathbb{C}d.$$

Also, decompose $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_0 \oplus \hat{\mathfrak{h}}_-$ where we define $\hat{\mathfrak{h}}_\pm = \bigoplus\limits_{\pm n > 0} (\mathfrak{h} \otimes t^n)$ and $\hat{\mathfrak{h}}_0 = \mathfrak{h} \otimes 1 \oplus \mathbb{C}C \oplus \mathbb{C}d$.

Suppose we have a representation of $\hat{\mathfrak{g}}$ acting on a space V. Then as $\mathcal{Z}$ also acts on V, it is a necessary property of $\mathcal{Z}$ that its action commutes with the action of $\hat{\mathfrak{h}}$. Now, under the action of $\hat{\mathfrak{h}}$, we decompose V into the tensor product of an irreducible module of the Heisenberg subalgebra and the vacuum space of V.

Definition 2.4.2. The vacuum space, denoted by Ω_V, is the subspace of V annihilated by $\hat{\mathfrak{h}}_+$. That is,

$$\Omega_V = \left\{ v \in V | \hat{\mathfrak{h}}_+ \cdot v = 0 \right\}.$$

Remark 2.4.3. As $\mathcal{Z}$ acts on V, it generates Ω_V. Let v_0 be a highest weight vector, then $\Omega_V = \{ \mathbf{Z}(\beta_1, m_1) \cdots \mathbf{Z}(\beta_1, m_1) \cdot v_0 | k \geq 0,\ \beta_i \in \Phi \}$. As noted in [19], realizing Ω_V allows us to further construct a realization of the Lie algebra $\hat{\mathfrak{g}}$ operating on V.

We now shall go about the construction of $\mathcal{Z}$.

Allow us to now denote the function $\overline{H}(z) = -\int \frac{H(z)}{Cz} dz = \sum\limits_{m \neq 0} \frac{H(n)}{nC} z^{-n}$. We will more often utilize its decomposition $\overline{H}(z) = \overline{H}_+(z) + \overline{H}_-(z)$, where $\overline{H}_\pm(z) = \sum\limits_{\mp n > 0} \frac{H(n)}{nC} z^{-n}$.

Now, let $e^{\overline{H}(z)}$ be the formal exponential of $\overline{H}(z)$. We will need to be able to work with this exponential operator, and so before we fully introduce $\mathcal{Z}$, we discuss $e^{\overline{H}(z)}$.

Theorem 2.4.4. *(by Campbell-Baker-Hausdorff)*
Given formal power series $x_1 = x_1(z)$ and $x_1 = x_1(z)$ where $[x_1, [x_1, x_2]] = 0 = [x_2, [x_1, x_2]]$, then

$$e^{x_1} e^{x_2} = e^{x_2} e^{x_1} e^{[x_1, x_2]}.$$

Equivalently, $[e^{x_1}, e^{x_2}] = e^{x_2} e^{x_1} \left(e^{[x_1, x_2]} - 1 \right)$.

Theorem 2.4.4 allows us to determine the following identities:

$$e^{\overline{H}_-(z)}e^{\overline{H}_+(w)} = e^{\overline{H}_+(w)}e^{\overline{H}_-(z)}\left(1-\frac{w}{z}\right)^{\frac{2}{C}} \tag{2.4.1}$$

$$e^{-\overline{H}_-(z)}e^{-\overline{H}_+(w)} = e^{-\overline{H}_+(w)}e^{-\overline{H}_-(z)}\left(1-\frac{w}{z}\right)^{\frac{2}{C}} \tag{2.4.2}$$

$$e^{\overline{H}_-(z)}e^{-\overline{H}_+(w)} = e^{-\overline{H}_+(w)}e^{\overline{H}_-(z)}\left(1-\frac{w}{z}\right)^{\frac{-2}{C}} \tag{2.4.3}$$

$$e^{-\overline{H}_-(z)}e^{\overline{H}_+(w)} = e^{\overline{H}_+(w)}e^{-\overline{H}_-(z)}\left(1-\frac{w}{z}\right)^{\frac{-2}{C}} \tag{2.4.4}$$

These identities follow directly from Theorem 2.4.4 and noting that $\sum\limits_{n>0}\frac{1}{n}\left(\frac{w}{z}\right)^n$ is the series expansion of $-\ln\left(1-\frac{w}{z}\right)$, where $|w| < |z|$. We observe may observe this in the following example calculation:

Example 2.4.5. As mentioned above, first note that Theorem 2.4.4 implies that

$$e^{\overline{H}_-(z)}e^{\overline{H}_+(w)} = e^{\overline{H}_+(w)}e^{\overline{H}_-(z)}e^{\left[\overline{H}_-(z),\overline{H}_+(w)\right]}.$$

Then the identities follow directly from this calculation:

$$\begin{aligned}\left[\overline{H}_-(z),\overline{H}_+(w)\right] &= \sum_{\substack{m>0\\n<0}}\frac{[H(m),H(n)]}{mnC^2}z^{-m}w^{-n} = \sum_{\substack{m>0\\n<0}}\frac{2mC\delta_{m,-n}}{mnC^2}z^{-m}w^{-n}\\ &= \sum_{m>0}\frac{2\delta_{m,-n}}{nC}z^{-m}w^{-n} = \frac{-2}{C}\sum_{m>0}\frac{1}{m}\left(\frac{w}{z}\right)^m\\ &= \frac{2}{C}\ln\left(1-\frac{w}{z}\right) = \ln\left(1-\frac{w}{z}\right)^{\frac{2}{C}}.\end{aligned}$$

Remark 2.4.6. Another effect of Theorem 2.4.4, is that $[x_1, e^{x_2}] = e^{x_2}[x_1, x_2]$.

This fact is clear once we note that $x_1 = \partial_s\left(e^{sx_1}\right)_{s=0}$ and then apply the Campbell-Baker-Hausdorff rule. Observe:

$$\begin{aligned}[x_1, e^{x_2}] &= \left[\partial_s\left(e^{sx_1}\right)_{s=0}, e^{x_2}\right] = \partial_s\left(\left[e^{sx_1}, e^{x_2}\right]\right)_{s=0} = \partial_s\left[e^{x_2}e^{sx_1}\left(e^{[sx_1,x_2]}-1\right)\right]_{s=0}\\ &= \left[e^{x_2}x_1e^{sx_1}\left(e^{[sx_1,x_2]}-1\right) + e^{x_2}e^{sx_1}[x_1,x_2]e^{s[x_1,x_2]}\right]_{s=0} = e^{x_2}[x_1,x_2]\end{aligned}$$

In truth, remark 2.4.6 will be especially useful once we begin computing the bracket relations $\mathcal{Z}(2)$, but first we define its elements.

Definition 2.4.7. For each root, $\pm\alpha$ of $\hat{sl}_2$, we define $\mathbf{Z}$-algebra operators $\mathbf{Z}^{\pm}(m) = \mathbf{Z}(\pm\alpha, m)$, by using the normally ordered product to define the generating functions $\mathbf{Z}^{\pm}(z)$. Let

$$\mathbf{Z}^{+}(z) = \mathbf{Z}(\alpha, z) =: e^{\overline{H}(z)} X(z) := e^{\overline{H}_{+}(z)} X(z) e^{\overline{H}_{-}(z)},$$

$$\mathbf{Z}^{-}(z) = \mathbf{Z}(-\alpha, z) =: e^{-\overline{H}(z)} Y(z) := e^{-\overline{H}_{+}(z)} Y(z) e^{-\overline{H}_{-}(z)}.$$

Note that the normal ordering seen here is consistent with the normal ordering from Definition 2.3.1, because the annihilation operators are moved to the right.

Now, as previously mentioned, $\mathcal{Z}$ should commute with the Heisenberg subalgebra of $\hat{g} = \hat{sl}_2$. That is, $\left[H(n), \mathbf{Z}^{\pm}(z)\right]$ for all $n \neq 0$. We will take the time to verify that now for $\mathbf{Z}^{+}(z)$, but the argument for $\mathbf{Z}^{-}(z)$ is similarly made.

By first realizing that $\left[H(n), \overline{H}_{\pm}(z)\right] = \begin{cases} -2z^n & \pm n > 0 \\ 0, & \pm n < 0 \end{cases}$, we can see that $\left[H(n), \left[H(n), \overline{H}_{\pm}(z)\right]\right] = 0 = \left[\overline{H}_{\pm}(n), \left[H(n), \overline{H}_{\pm}(z)\right]\right]$. Thus, Theorem 2.4.6 applies in this situation.

$$\begin{aligned}
&\left[H(n), \mathbf{Z}^{+}(z)\right] = \left[H(n), : e^{\overline{H}(z)} X(z) :\right] = \left[H(n), e^{\overline{H}_{+}(z)} X(z) e^{\overline{H}_{-}(z)}\right] \\
&= \left[H(n), e^{\overline{H}_{+}(z)}\right] X(z) e^{\overline{H}_{-}(z)} + e^{\overline{H}_{+}(z)} \left[H(n), X(z)\right] e^{\overline{H}_{-}(z)} + e^{\overline{H}_{+}(z)} X(z) \left[H(n), e^{\overline{H}_{-}(z)}\right] \\
&= e^{\overline{H}_{+}(z)} \left[H(n), \overline{H}_{+}(z)\right] X(z) e^{\overline{H}_{-}(z)} + e^{\overline{H}_{+}(z)} \left[H(n), X(z)\right] e^{\overline{H}_{-}(z)} \\
&\qquad + e^{\overline{H}_{+}(z)} X(z) e^{\overline{H}_{-}(z)} \left[H(n), \overline{H}_{-}(z)\right] \\
&= e^{\overline{H}_{+}(z)} \left(\sum_{m<0} \frac{2nC\delta_{n,-m}}{mC} X(z) + \sum_{m \in \mathbb{Z}} 2X(m+n) z^{-m} + \sum_{m>0} \frac{2nC\delta_{n,-m} z^{-m}}{mC} X(z) \right) e^{\overline{H}_{-}(z)} \\
&= e^{\overline{H}_{+}(z)} \left(\sum_{m \neq 0} -2\delta_{n,-m} z^{-m} \right) X(z) e^{\overline{H}_{-}(z)} + (2z^n) e^{\overline{H}_{+}(z)} X(z) e^{\overline{H}_{-}(z)} \\
&= 2z^n \mathbf{Z}^{+}(z) - 2z^n \mathbf{Z}^{+}(z) = 0
\end{aligned}$$

Remark 2.4.8. The necessary effect of the above calculation is that

$$\left[H(n), \mathbf{Z}^{\pm}(z)\right] = 0 \Longrightarrow \left[e^{\overline{H}_{\pm}(z)}, \mathbf{Z}^{\pm}(z)\right] = 0.$$

The previous remark allows us to calculate the product of two generating functions in $\mathcal{Z}$.

Example 2.4.9.

$$\begin{aligned}
\mathbf{Z}^+(z)\mathbf{Z}^+(w) &= \mathbf{Z}^+(z)\left(e^{\overline{H}_+(w)}X(w)e^{\overline{H}_-(w)}\right) = e^{\overline{H}_+(w)}\mathbf{Z}^+(z)X(w)e^{\overline{H}_-(w)} \\
&= e^{\overline{H}_+(w)}e^{\overline{H}_+(z)}X(z)e^{\overline{H}_-(z)}\left(e^{-\overline{H}_+(w)}e^{\overline{H}_+(w)}\right)X(w)e^{\overline{H}_-(w)} \\
&= e^{\overline{H}_+(w)}e^{\overline{H}_+(z)}X(z)e^{-\overline{H}_+(w)}e^{\overline{H}_+(w)}X(w)e^{\overline{H}_-(w)}e^{\overline{H}_-(z)}\left(1-\frac{w}{z}\right)^{\frac{-2}{C}} \\
&= e^{\overline{H}_+(z)+\overline{H}_+(w)}X(z)X(w)e^{\overline{H}_-(z)+\overline{H}_-(w)}\left(1-\frac{w}{z}\right)^{\frac{-2}{C}}
\end{aligned}$$

It is clear, then, that $\mathbf{Z}^+(z)\mathbf{Z}^+(w) = e^{\overline{H}_+(z)+\overline{H}_+(w)}X(w)X(z)e^{\overline{H}_-(z)+\overline{H}_-(w)}\left(1-\frac{z}{w}\right)^{\frac{-2}{C}}$. Notice that the presence of $\left(1-\frac{w}{z}\right)^{\frac{-2}{C}}$ and $\left(1-\frac{z}{w}\right)^{\frac{-2}{C}}$ make it extremely difficult — if not impossible — to calculate the commutator bracket of these two generating functions. We therefore introduce the following definition.

Definition 2.4.10. Given roots $\phi_1,\ \phi_2$ of the affine Lie algebra $\hat{\mathfrak{g}}$ and their associated generating functions $\mathbf{Z}(\phi_1, z),\ \mathbf{Z}(\phi_2, w)$ in $\mathcal{Z}$, then we define the *generalized commutator bracket* as

$$[[\mathbf{Z}(\phi_1,z),\mathbf{Z}(\phi_2,w)]] = \mathbf{Z}(\phi_1,z)\mathbf{Z}(\phi_2,w)\left(1-\frac{w}{z}\right)^{\frac{(\phi_1,\phi_2)}{C}} - \mathbf{Z}(\phi_2,w)\mathbf{Z}(\phi_1,z)\left(1-\frac{z}{w}\right)^{\frac{(\phi_1,\phi_2)}{C}}$$

where $(\phi_1\phi_2)$ is their inner-product as given by (2.2.2).

This definition does not restrict us to the case of $\mathcal{Z}(2)$, though for the time being we want to solely consider that case, noting that then $\phi_1, \phi_2 \in \{\pm\alpha\}$. Furthermore, Definition 2.4.10 makes no restrictions to the level of the representation, but since the definition depends on C it is apparent that relations will differ between levels.

Theorem 2.4.11. *For the case of* $\hat{sl}_2$, $\left[\left[\boldsymbol{Z}^\pm(z), \boldsymbol{Z}^\pm(w)\right]\right] = 0$ *and*

$$\left[\left[\boldsymbol{Z}^+(z), \boldsymbol{Z}^-(w)\right]\right] = H(0)\delta\left(\frac{w}{z}\right) + Cw\partial_w\delta\left(\frac{w}{z}\right).$$

Proof. Recall example 2.4.9. Then

$$\left[\left[\mathbf{Z}^+(z),\mathbf{Z}^+(w)\right]\right] = e^{\overline{H}_+(z)+\overline{H}_+(w)}\left[X(z),X(w)\right]e^{\overline{H}_-(z)+\overline{H}_-(w)} = 0.$$

Similarly, we are able to easily compute

$$\left[\left[\mathbf{Z}^-(z),\mathbf{Z}^-(w)\right]\right] = e^{-\overline{H}_+(z)-\overline{H}_+(w)}\left[Y(z),Y(w)\right]e^{-\overline{H}_-(z)-\overline{H}_-(w)} = 0.$$

The case of $\left[\left[\mathbf{Z}^+(z),\mathbf{Z}^-(w)\right]\right]$ is notably more interesting in computation and result. First consider the products.

$$
\begin{aligned}
\mathbf{Z}^+(z)\mathbf{Z}^-(w) &= \mathbf{Z}^+(z)e^{-\overline{H}_+(w)}Y(w)e^{-\overline{H}_-(w)} = e^{-\overline{H}_+(w)}\mathbf{Z}^+(z)Y(w)e^{-\overline{H}_-(w)} \\
&= e^{-\overline{H}_+(w)}e^{\overline{H}_+(z)}X(z)e^{\overline{H}_-(z)}Y(w)e^{-\overline{H}_-(w)} \\
&= e^{-\overline{H}_+(w)}e^{\overline{H}_+(z)}X(z)e^{\overline{H}_-(z)}\left(e^{\overline{H}_+(w)}e^{-\overline{H}_+(w)}\right)Y(w)e^{-\overline{H}_-(w)} \\
&= e^{-\overline{H}_+(w)}e^{\overline{H}_+(z)}X(z)e^{\overline{H}_-(z)}e^{\overline{H}_+(w)}\mathbf{Z}^-(w) \\
&= e^{-\overline{H}_+(w)}e^{\overline{H}_+(z)}X(z)e^{\overline{H}_+(w)}e^{\overline{H}_-(z)}\left(1-\frac{w}{z}\right)^{\frac{2}{C}}\mathbf{Z}^-(w) \\
&= e^{-\overline{H}_+(w)}e^{\overline{H}_+(z)}X(z)e^{\overline{H}_+(w)}\mathbf{Z}^-(w)e^{\overline{H}_-(z)}\left(1-\frac{w}{z}\right)^{\frac{2}{C}} \\
&= e^{-\overline{H}_+(w)}e^{\overline{H}_+(z)}X(z)e^{\overline{H}_+(w)}e^{-\overline{H}_+(w)}Y(w)e^{-\overline{H}_-(w)}e^{\overline{H}_-(z)}\left(1-\frac{w}{z}\right)^{\frac{2}{C}} \\
&= e^{\overline{H}_+(z)-\overline{H}_+(w)}X(z)Y(w)e^{\overline{H}_-(z)-\overline{H}_-(w)}\left(1-\frac{w}{z}\right)^{\frac{2}{C}}.
\end{aligned}
$$

Similarly, $\mathbf{Z}^-(w)\mathbf{Z}^+(z) = e^{\overline{H}_+(z)-\overline{H}_+(w)}Y(w)X(z)e^{\overline{H}_-(z)-\overline{H}_-(w)}\left(1-\frac{z}{w}\right)^{\frac{2}{C}}$.

Here we call upon (2.3.11) and (2.3.2) to complete the proof.

$$
\begin{aligned}
\left[\left[\mathbf{Z}^+(z),\mathbf{Z}^-(w)\right]\right] &= e^{\overline{H}_+(z)-\overline{H}_+(w)}\left[X(z),Y(w)\right]e^{\overline{H}_-(z)-\overline{H}_-(w)} \\
&= e^{\overline{H}_+(z)-\overline{H}_+(w)}\left(H(w)\delta\left(\frac{w}{z}\right)+Cw\partial_w\delta\left(\frac{w}{z}\right)\right)e^{\overline{H}_-(z)-\overline{H}_-(w)} \\
&= H(w)\delta\left(\frac{w}{z}\right)+Cw\left(\partial_w\delta\left(\frac{w}{z}\right)-\left(\sum_{m\neq 0}\frac{H(m)}{C}w^{-m-1}\right)\delta\left(\frac{w}{z}\right)\right) \\
&= H(0)\delta\left(\frac{w}{z}\right)+Cw\partial_w\delta\left(\frac{w}{z}\right)
\end{aligned}
$$

This completes the proof of our relations for $\mathcal{Z}(2)$ at any level. $\square$

Chapter 3

Z-algebras for Higher Level $\hat{sl}(n)$

3.1 Extending the Wakimoto Representation to $\hat{sl}(n)$

We wish to expand on the concepts of Wakimoto in [24], Lepowsky, Wilson and Primc in [19, 21] by extending Wakimoto's work to $\hat{sl}(n)$, following [5], and developing relations for the Lepowsky-Wilson **Z**-algebra at general level.

Note that formulas realizing the Wakimoto representation of $\hat{sl}(n)$ have been found by Boris Feigin and Edward Frenkel in [5] for the elements associated with the simple roots. We know that elements of $\hat{\mathfrak{g}}$ associated with the non-simple roots of $\mathfrak{g}$ can be obtained by taking brackets of simple root vectors. Our goal, however, is to be able to write such formula that gives the operator realization for all elements in $\hat{\mathfrak{g}}$. We provide such a formula for all positive root vectors.

A quick glance at Definition 2.3.2, and one can clearly see the relationship between the upper-triangular element $X \in sl(2)$ and the operator a. In the same turn, one will notice a similar tie between the lower-triangular element $Y \in sl(2)$ and the operator a^*. As we extend his definition to $sl(n)$, we aim to maintain the symmetry of Wakimoto's representation, and indeed, the symmetry of $sl(n)$ itself. Keep in mind that for each n, the number of upper-triangular elements (and symmetrically, lower-triangular elements) in $sl(n)$ is $\frac{n(n-1)}{2} = |\Phi^+|$. This should imply to us that if we wish to define a representation that maps each upper-triangular element to an operator that is independent from the others, then we will need to define $\frac{n(n-1)}{2}$ operators, and consequently the vector space of our representation will require the same number of families of variables.

Because of our dependence on the root system of $\mathfrak{g}$, it will be important to have a convention for denoting a root and its corresponding element.

Definition 3.1.1.

- For $E_{ij} \in sl_n$ with $1 \leq i < j \leq n$, we shall denote the corresponding elements in $\hat{sl}_n$ by $X_{ij}(m) = X_{\alpha_{ij}}(m) := E_{ij} \otimes t^m$, where $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$, the positive root associated with E_{ij}.

- For $-\alpha_{ij} = -\alpha_i - \alpha_{i+1} - \cdots - \alpha_{j-1}$, associate the element $Y_{ij}(m) = Y_{-\alpha_{ij}}(m) := E_{ji} \otimes t^m$.

- We continue to use the convention $H_i(m) := (E_{ii} - E_{i+1,i+1}) \otimes t^m$, for $1 \leq i < n$.

Our representations will act on the Fock space $V = \mathbb{C}[x_{ij,m}, y_{ij,p} | \ 1 \leq i < j \leq n, m \in \mathbb{Z},\ p \in \mathbb{N}]$.

We define the operators

$$a_{ij}(m) = \begin{cases} x_{ij,m} & \text{if } m \geq 0 \\ \frac{-\partial}{\partial x_{ij,m}} & \text{if } m < 0 \end{cases} \quad a^*_{ij}(m) = \begin{cases} \frac{\partial}{\partial x_{ij,m}} & \text{if } m < 0 \\ x_{ij,m} & \text{if } m \geq 0 \end{cases} \quad b_{ij}(m) = \begin{cases} m y_{ij,m} & \text{if } m > 0 \\ 0 & \text{if } m = 0 \\ \frac{-\partial}{\partial y_{ij,-m}} & \text{if } m < 0 \end{cases}$$

The normally ordered product of operators $a_{ij}(m)$ and $a^*_{kl}(n)$ is as before:

$$:a_{ij}(m)a^*_{kl}(n):= \begin{cases} a_{ij}(m)a^*_{kl}(n) & m > 0 \\ \frac{1}{2}\left(a_{ij}(0)a^*_{kl}(n) + a^*_{kl}(n)a_{ij}(0)\right) & m = 0 \\ a^*_{kl}(n)a_{ij}(m) & m < 0. \end{cases}$$

Then let $\mathcal{E}_{ij}(m) = \sum\limits_{k \in \mathbb{Z}} :a_{ij}(k+m)a^*_{ij}(k):$. Notice the change in notation from the $\hat{sl}_2$ case. This is to alleviate confusion between the operator $\mathcal{E}_{ij}(z)$ and our notation of the matrix E_{ij}.

Now, as Wakimoto did in [24], we wish to address the various commutation relations of these operators before formally developing and writing out the representation of $\hat{sl}(n)$. In this work, though, the realization will take place among the formal power series realizations of these operators and, correspondingly, the loop algebra elements of $\hat{sl}(n)$. In order to do this, however, we must understand the basic commutation relations of the operators themselves.

Observe that if $i \neq k$ and $j \neq l$, then the operators $a^*_{ij}(m)$ and $a_{kl}(n)$ will commute. If, however, we have both $i = k$ and $j = l$ then the commutator bracket will behave just as in the $\hat{sl}(2)$ case.

$$\left[a^*_{ij}(m), a_{kl}(n)\right] = \delta_{mn}\delta_{ik}\delta_{jl}$$

In fact, this can be said about all of the basic commutation relations.

$$[\mathcal{E}_{ij}(m), a_{kl}(n)] = a_{ij}(m+n)\delta_{ik}\delta_{jl}, \tag{3.1.1}$$

$$[\mathcal{E}_{ij}(m), a^*_{kl}(n)] = -a^*_{ij}(n-m)\delta_{ik}\delta_{jl}, \text{ and} \tag{3.1.2}$$

$$[\mathcal{E}_{ij}(m), \mathcal{E}_{kl}(n)] = m\delta_{m,-n}\delta_{ik}\delta_{jl} = [b_{ij}(m), b_{kl}(n)]. \tag{3.1.3}$$

Now, we introduce the formal power series realizations for each operator.

$$
\begin{aligned}
a_{ij}(z) &= \sum_{n\in\mathbb{Z}} a_{ij}(n)z^n, \qquad a^*_{ij}(z) = \sum_{n\in\mathbb{Z}} a^*_{ij}(n)z^{-n} \\
b_{ij}(z) &= \sum_{n\in\mathbb{Z}} b_{ij}(n)z^n, \qquad \mathcal{E}_{ij}(z) =: a_{ij}(z)a^*_{ij}(z) := \sum_{n\in\mathbb{Z}} \mathcal{E}_{ij}(n)z^n
\end{aligned}
\tag{3.1.4}
$$

As before, we start with the simplest commutator, which we see behaves as expected.

$$[a^*_{ij}(z), a_{kl}(w)] = \delta\left(\frac{w}{z}\right)\delta_{ik}\delta_{jl}$$

This follows from the computation of the contractions below. We calculate others, as well.

$$
\begin{aligned}
\underbrace{a^*_{ij}(z)a_{kl}(w)} &= \left(\frac{z+w}{2(z-w)}\right)\delta_{ik}\delta_{jl}, & \underbrace{a_{ij}(z)a^*_{kl}(w)} &= -\left(\frac{z+w}{2(z-w)}\right)\delta_{ik}\delta_{jl}, \\
\underbrace{a^*_{ij}(z)\mathcal{E}_{kl}(w)} &= a^*_{kl}(z)\left(\frac{z+w}{2(z-w)}\right)\delta_{ik}\delta_{jl}, & \underbrace{\mathcal{E}_{kl}(z)a^*_{ij}(w)} &= -a^*_{kl}(w)\left(\frac{z+w}{2(z-w)}\right)\delta_{ik}\delta_{jl}
\end{aligned}
$$

Now we are able to calculate a variety of commutators.

$$[a_{ij}(z), a_{kl}(w)] = 0 = [a^*_{ij}(z), a^*_{kl}(w)] \qquad [b_{ij}(z), a_{kl}(w)] = 0 = [b_{ij}(z), a^*_{kl}(w)]$$

$$[b_{ij}(z), b_{kl}(w)] = z\partial_z\delta\left(\frac{w}{z}\right)\delta_{ik}\delta_{jl} = -w\partial_w\delta\left(\frac{w}{z}\right)\delta_{ik}\delta_{jl} = [\mathcal{E}_{ij}(m), \mathcal{E}_{kl}(m)]$$

Above, we have the basic commutators from which we are able to calculate the brackets for our representation. However, notice that in the case that the indices are not equivalent, then the operators and their respective generating functions commute. Thus, the above brackets amount to $\frac{n^2-n}{2}$ copies of the relations from $\hat{sl}(2)$. This parallels the work in [24]. Below we show our method of calculating brackets with these different varieties of operators.

Example 3.1.2. Consider the formal power series of operators $a^*_{ij}(z)$, $E_{ij}(z)$, $a^*_{ij}(w)$ and $a_{kl}(w)$ with $i \neq k$ and $j \neq l$. Then suppose a calculation requires us to compute the bracket $\left[:\mathcal{E}_{ij}(z)a^*_{ij}(z):\,,:a_{kl}(w)a^*_{ij}(w):\right]$. Since $a^*_{ij}(z)$ and $a_{kl}(w)$ commute when $i \neq k$ and $j \neq l$, then we know that $:a_{kl}(w)a^*_{ij}(w):= a_{kl}(w)a^*_{ij}(w)$. Thus,

$$
\begin{aligned}
\left[:\mathcal{E}_{ij}(z)a^*_{ij}(z):, :a_{kl}(w)a^*_{ij}(w):\right] &= a_{kl}(w)\left[:\mathcal{E}_{ij}(z)a^*_{ij}(z):, a^*_{ij}(w)\right] \\
&= a_{kl}(w)\left(-a^*_{ij}(z)a^*_{ij}(z)\delta\left(\frac{w}{z}\right)\right) \\
&= -\left(:a_{kl}(w)a^*_{ij}(z)a^*_{ij}(z): + 2a^*_{ij}(z)\,\underbrace{a_{kl}(w)a^*_{ij}(z)}\right)\delta\left(\frac{w}{z}\right) \\
&= -:a_{kl}(w)a^*_{ij}(w)a^*_{ij}(w):\delta\left(\frac{w}{z}\right)
\end{aligned}
$$

The second to last step requires us to utilize the fact that
$x_1x_2x_3 =: x_1x_2x_3 : + x_1 \underbrace{x_2x_3} + x_2 \underbrace{x_1x_3} + x_2 \underbrace{x_1x_2}$, which is a generalization of Definition 2.3.4.

Remark 3.1.3. Note that a generalized Wakimoto representation has already been determined (see [5] and [1]), which gives specific formula for writing the operator forms of the elements associated with the simple roots of $\hat{sl}(2)$. For completeness, we provide these formulas below. It is worthwhile to note that the definitions for H_i and C are already generalized.

$$
\begin{aligned}
X_{i,i+1}(z) &= a_{i,i+1}(z) + \sum_{k=i+2}^{n} :a_{ik}(z)a^*_{i+1,k}(z): \\
Y_{i,i+1}(z) &= -\sum_{k=1}^{i} :\mathcal{E}_{k,i+1}(z)a^*_{i,i+1}(z): + \sum_{k=1}^{i-1} :\mathcal{E}_{k,i}(z)a^*_{i,i+1}(z): + \sum_{k=i+2}^{n} :a_{i+1,k}(z)a^*_{ik}(z): \\
&\quad - \sum_{k=1}^{i-1} :a_{ki}(z)a^*_{k,i+1}(z): + \left(\frac{n\nu^2}{4}+1\right) z\partial_z a^*_{i,i+1}(z) + (\mu-1)\, a^*_{i,i+1}(z) \\
&\quad + \left(\nu b_{i,i+1}(z) - \frac{\nu}{2}\sum_{k=1}^{i-1} b_{ki}(z) + \frac{\nu}{2}\sum_{k=1}^{i-1} b_{ki}(z) - \frac{\nu}{2}\sum_{k=i+2}^{n} b_{i+1,k}(z) + \frac{\nu}{2}\sum_{k=i+2}^{n} b_{ik}(z)\right) a^*_{i,i+1}(z) \\
H_i(z) &= 2\mathcal{E}_{i,i+1}(z) - \sum_{k=1}^{i-1}\mathcal{E}_{ki}(z) + \sum_{k=1}^{i-1}\mathcal{E}_{k,i+1}(z) - \sum_{k=i+2}^{n}\mathcal{E}_{i+1,k}(z) + \sum_{k=i+2}^{n}\mathcal{E}_{ik}(z) \\
&\quad -\nu b_{i,i+1}(z) + \frac{\nu}{2}\sum_{k=1}^{i-1} b_{ki}(z) - \frac{\nu}{2}\sum_{k=1}^{i-1} b_{k,i+1}(z) + \frac{\nu}{2}\sum_{k=i+2}^{n} b_{i+1,k}(z) - \frac{\nu}{2}\sum_{k=i+2}^{n} b_{ik}(z) \\
C &= -n\left(\frac{\nu^2}{4}+1\right)
\end{aligned}
$$

We have been able to generalize this construction for the elements associated with the positive roots. That is, we have a formula for the realization of $X_{ij}(z)$ for all $1 \le i < j \le n$, using the power series of operators described in (3.1.4). Then, recall that for general basis matrices $E_{ij},\ E_{kl} \in sl_n$, we know $[E_{ij}, E_{kl}] = E_{il}\delta_{jk} - E_{kj}\delta_{il}$. So, for upper-triangular elements X_{ij} and X_{kl}, we should expect that $[X_{ij}(z), X_{kl}(w)] = (X_{il}(w)\delta_{jk} - X_{kj}(w)\delta_{il})\,\delta\left(\frac{w}{z}\right)$.

Theorem 3.1.4. $X_{\alpha_{ij}}(z) = X_{ij}(z) = a_{ij}(z) + \sum_{k=j+1}^{n} :a_{ik}(z)a^*_{jk}(z):$.

Proof. First note that, $:a_{ik}(z)a^*_{jk}(z) := a_{ik}(z)a^*_{jk}(z)$, except when $i = j$, since the operators themselves will commute. However, $i < j$ as subscripts for $X_{ij}(z)$. With that in mind, we calculate

$$
\begin{aligned}
[X_{ij}(z), X_{kl}(w)] &= \left[a_{ij}(z) + \sum_{r=j+1}^{n} :a_{ir}(z)a^*_{jr}(z):,\ a_{kl}(w) + \sum_{s=l+1}^{n} :a_{ks}(w)a^*_{ls}(w):\right] \\
&= \left[a_{ij}(z) + \sum_{r=j+1}^{n} a_{ir}(z)a^*_{jr}(z),\ a_{kl}(w) + \sum_{s=l+1}^{n} a_{ks}(w)a^*_{ls}(w)\right] \\
&= \sum_{r=j+1}^{n} a_{ir}(z)\left[a^*_{jr}(z), a_{kl}(w)\right] + \sum_{s=l+1}^{n} a_{ks}(w)\left[a_{ij}(z), a^*_{ls}(w)\right] \\
&\qquad + \sum_{\substack{r=j+1\\ s=l+1}}^{n} \left[a_{ir}(z)a^*_{jr}(z), a_{ks}(w)a^*_{ls}(w)\right] \\
&= \sum_{r=j+1}^{n} a_{ir}(w)\delta_{jk}\delta_{rl}\delta\left(\frac{w}{z}\right) - \sum_{s=l+1}^{n} a_{ks}(w)\delta_{il}\delta_{js}\delta\left(\frac{w}{z}\right) \\
&\qquad + \sum_{\substack{r=j+1\\ s=l+1}}^{n} \left(a_{ir}(w)a^*_{ls}(w)\delta_{jk}\delta_{rs}\delta\left(\frac{w}{z}\right) - a_{ks}(w)a^*_{jr}(w)\delta_{il}\delta_{rs}\delta\left(\frac{w}{z}\right)\right) \\
&= \delta_{jk}\left(a_{il}(w) + \sum_{r=j+1}^{n} a_{ir}(w)a^*_{lr}(w)\right)\delta\left(\frac{w}{z}\right) \\
&\qquad -\delta_{il}\left(a_{kj}(w) + \sum_{s=j+1}^{n} a_{ks}(w)a^*_{js}(w)\right)\delta\left(\frac{w}{z}\right) \\
&= \left(X_{il}(w)\delta_{jk} - X_{kj}(w)\delta_{il}\right)\delta\left(\frac{w}{z}\right).
\end{aligned}
$$

□

3.2 Constructing the algebra $\mathcal{Z}$ Associated with $\hat{sl}(n)$

Given remark 3.1.3, we can set about with the task of constructing $\mathcal{Z}(n)$, the **Z**-algebra associated with $\hat{sl}(n)$. Recall that, notationally, we define the root $\alpha_{ij} = \alpha_i + \cdots + \alpha_{j-1}$. As in the case of $\mathcal{Z}(2)$, we will construct a generating function $\mathbf{Z}(\beta, z)$ for each root $\beta \in \Phi$.

Definition 3.2.1. For the Lie algebra $\hat{sl}_n$, and the root α_{ij} in Φ, of $sl(n)$, define $\mathbf{Z}^{\pm}_{ij}(m) = \mathbf{Z}(\pm\alpha_{ij}, m) \in \mathcal{Z}(n)$ by defining the generating function $\mathbf{Z}^{\pm}_{ij}(z) = \sum_{m\in\mathbb{Z}} \mathbf{Z}(\pm\alpha_{ij}, m)z^{-m}$ where

$$
\begin{aligned}
\mathbf{Z}^{+}_{ij}(z) &= :e^{\left(\overline{H}_i(z)+\cdots+\overline{H}_i(z)\right)}X_{ij}(z): \\
\mathbf{Z}^{-}_{ij}(z) &= :e^{\left(-\overline{H}_i(z)-\cdots-\overline{H}_i(z)\right)}Y_{ij}(z):
\end{aligned}
$$

Then, we can determine relations for $\mathcal{Z}(n)$ using the same generalized commutator bracket from Definition 2.4.10 as in the $\hat{sl}(2)$ case, and then calculate them with our knowledge of Theorem 2.4.4 and the bracket relations of the algebra. Recall that Definition 2.4.10 utilizes the inner-product of the roots of $sl(n)$. We will continue to define these inner-products using the generalized Cartan matrix from Section 2.2.

In Section 4.2 we will use this construction to determine all the relations for $\mathcal{Z}(3)$.

Chapter 4

An Explicit Look at the Construction for $\hat{sl}(3)$ at All Levels

4.1 The Wakimoto Representation for $\hat{sl}(3)$

We now take the opportunity to show the explicit realization at all levels for the affine Lie algebra $\hat{\mathfrak{g}} = \hat{sl}_3$. Recall that for $x_1, x_2 \in sl_3$, then the commutator bracket for $\hat{\mathfrak{g}}$ is defined by $[x_1(m), x_2(n)] = [x_1, x_2]\,(m+n) + mC\mathrm{Tr}(x_1 x_2)\delta_{m,-n}$.

We choose generators for $\hat{\mathfrak{g}}$ in this manner:

$$X_{12} := X_{\alpha_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_{23} := X_{\alpha_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_{13} := X_{\alpha_1+\alpha_2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Y_{12} := Y_{-\alpha_1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y_{23} := Y_{-\alpha_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Y_{13} := Y_{-\alpha_1-\alpha_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$H_1 := H_{\alpha_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad H_2 := H_{\alpha_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note that by this notation, $H_{\alpha_1+\alpha_2} = H_1 + H_2$.

Our representation space is the Fock space $V = \mathbb{C}\,[x_{ij,m}, y_{ij,p} | 1 \le i < j \le 3, \forall m \in \mathbb{Z}, p \in \mathbb{N}]$.

Then, as in Section 3, for $1 \le i < j \le 3$, we have operators

$$a_{ij}(m) = \begin{cases} x_{ij,m} & \text{if } m \ge 0 \\ \frac{-\partial}{\partial x_{ij,m}} & \text{if } m < 0 \end{cases} \quad a^*_{ij}(m) = \begin{cases} \frac{\partial}{\partial x_{ij,m}} & \text{if } m < 0 \\ x_{ij,m} & \text{if } m \ge 0 \end{cases} \quad b_{ij}(m) = \begin{cases} m y_{ij,m} & \text{if } m > 0 \\ 0 & \text{if } m = 0 \\ \frac{-\partial}{\partial y_{ij,-m}} & \text{if } m < 0 \end{cases}$$

and $\mathcal{E}_{ij}(m) = \sum_{k\in\mathbb{Z}} :a_{ij}(k+m)a^*_{ij}(k):$.

Definition 4.1.1. Using remark 3.1.3 and Theorem 3.1.4, we define the family of representations, $\{\pi_{\mu,\nu}\}$, such that $\pi_{\mu,\nu} : \hat{\mathfrak{g}} \longrightarrow gl(V)$ for all $\mu, \nu \in \mathbb{C}$, where $\pi_{\mu,\nu}$ maps the generators of $\hat{\mathfrak{g}}$ in the following way:

$$\begin{aligned}
X_{12}(z) &\mapsto a_{12}(z) + a_{13}(z)a^*_{23}(z), \qquad X_{23}(z) \mapsto a_{23}(z), \qquad X_{13}(z) \mapsto a_{13}(z),\\
Y_{12}(z) &\mapsto (\mu-1)a^*_{12}(z) + \left(\frac{3\nu^2}{4}+1\right) z\partial_z a^*_{12}(z) - :\mathcal{E}_{12}(z)a^*_{12}(z): + :a_{23}(z)a^*_{13}(z):\\
&\qquad + \left(\nu b_{12}(z) - \frac{\nu}{2}b_{23}(z) + \frac{\nu}{2}b_{13}(z)\right)a^*_{12}(z)\\
Y_{23}(z) &\mapsto (\mu-1)a^*_{23}(z) + \left(\frac{3\nu^2}{4}+2\right) z\partial_z a^*_{23}(z) - :\mathcal{E}_{23}(z)a^*_{23}(z): - :\mathcal{E}_{13}(z)a^*_{23}(z):\\
&\qquad + :\mathcal{E}_{12}(z)a^*_{23}(z): - :a_{12}(z)a^*_{13}(z): + \left(\frac{-\nu}{2}b_{12}(z) + \nu b_{23}(z) + \frac{\nu}{2}b_{13}(z)\right)a^*_{23}(z)\\
Y_{13}(z) &\mapsto 2(\mu-1)a^*_{13}(z) + \left(\frac{3\nu^2}{4}+2\right) z\partial_z a^*_{13}(z) - :\mathcal{E}_{13}(z)a^*_{13}(z): - :\mathcal{E}_{12}(z)a^*_{13}(z):\\
&\qquad - :\mathcal{E}_{23}(z)a^*_{13}(z): + :\mathcal{E}_{12}(z)a^*_{12}(z): a^*_{23}(z) + \left(\frac{\nu}{2}b_{12}(z) + \frac{\nu}{2}b_{23}(z) + \nu b_{13}(z)\right)a^*_{13}(z)\\
&\quad - \left(\frac{3\nu^2}{4}+1\right) z\partial_z\left(a^*_{12}(z)a^*_{23}(z)\right) + (1-\mu-\nu b_{12}(z) + \frac{\nu}{2}b_{23}(z) - \frac{\nu}{2}b_{13}(z))a^*_{12}(z)a^*_{13}(z)\\
H_1(z) &\mapsto 2\left(\mathcal{E}_{12}(z) - \frac{\nu}{2}b_{12}(z)\right) - \left(\mathcal{E}_{23}(z) - \frac{\nu}{2}b_{23}(z)\right) + \left(\mathcal{E}_{13}(z) - \frac{\nu}{2}b_{13}(z)\right) + (1-\mu)\chi\\
H_2(z) &\mapsto -\left(\mathcal{E}_{12}(z) - \frac{\nu}{2}b_{12}(z)\right) + 2\left(\mathcal{E}_{23}(z) - \frac{\nu}{2}b_{23}(z)\right) + \left(\mathcal{E}_{13}(z) - \frac{\nu}{2}b_{13}(z)\right) + (1-\mu)\chi\\
C &\mapsto -3\left(\frac{\nu^2}{4}+1\right)
\end{aligned}$$

As before, we suppress the notation of the map $\pi_{\mu,\nu}$ and simply directly associate the element in $\hat{\mathfrak{g}}$ and its realization as an operator on V.

We verify some, but not all relations here. All calculations are determined using relations from Section 3.

Example 4.1.2.

$$\begin{aligned}[X_{12}(z), X_{23}(z)] &= [a_{12}(z) + a_{13}(z)a^*_{23}(z), a_{23}(w)] = a_{13}(z)\left[a^*_{23}(z), a_{23}(w)\right] \\ &= a_{13}(z)\delta\left(\frac{w}{z}\right) = X_{13}(w)\delta\left(\frac{w}{z}\right)\end{aligned}$$

Example 4.1.3.

$$\begin{aligned}[H_1(z), X_{12}(w)] &= 2\left[\mathcal{E}_{12}(z), a_{12}(w)\right] + \left[\mathcal{E}_{13}(z), a_{13}(w)\right]a^*_{23}(w) - a_{13}(w)\left[\mathcal{E}_{23}(z), a^*_{23}(w)\right] \\ &= 2a_{12}(z)\delta\left(\frac{w}{z}\right) + \left(a_{13}(z)\delta\left(\frac{w}{z}\right)\right)a^*_{23}(w) - a_{13}(w)\left(-a^*_{23}(z)\delta\left(\frac{w}{z}\right)\right) \\ &= 2\left(a_{12}(w) + a_{13}(w)a^*_{23}(w)\right)\delta\left(\frac{w}{z}\right) = 2X_{12}(w)\delta\left(\frac{w}{z}\right) \\ [H_1(z), X_{23}(w)] &= -\left[\mathcal{E}_{23}(z), a_{23}(w)\right] = -a_{23}(z)\delta\left(\frac{w}{z}\right) = -X_{23}(w)\delta\left(\frac{w}{z}\right) \\ [H_1(z), X_{13}(w)] &= \left[\mathcal{E}_{13}(z), a_{13}(w)\right] = a_{13}(z)\delta\left(\frac{w}{z}\right) = X_{13}(w)\delta\left(\frac{w}{z}\right)\end{aligned}$$

Example 4.1.4.

$$\begin{aligned}[Y_{12}(z), X_{23}(w)] &= 0 \\ [Y_{12}(z), X_{13}(w)] &= [:a_{23}(z)a^*_{13}(z):, a_{13}(w)] = a_{23}(z)\delta\left(\frac{w}{z}\right) = X_{23}(w)\delta\left(\frac{w}{z}\right) \\ [Y_{12}(z), X_{12}(w)] &= \left[\left((\mu-1) + \nu b_{12}(z) - \frac{\nu}{2}b_{23}(z) + \frac{\nu}{2}b_{13}(z)\right)a^*_{12}(z) - :\mathcal{E}_{12}(z)a^*_{12}(z):, a_{12}(w)\right] \\ &\quad + \left[\left(\frac{3\nu^2}{4}+1\right)z\partial_z a^*_{12}(z), a_{12}(w)\right] + \left[a_{13}(z)a^*_{23}(z), :a_{23}(w)a^*_{13}(w):\right] \\ &= \left((\mu-1) + \nu b_{12}(z) - \frac{\nu}{2}b_{23}(z) + \frac{\nu}{2}b_{13}(z)\right)\delta\left(\frac{w}{z}\right) - \mathcal{E}_{12}(w)\delta\left(\frac{w}{z}\right) \\ &\quad - :a_{12}(w)a^*_{12}(w):\delta\left(\frac{w}{z}\right) + \left(\frac{3\nu^2}{4}+1\right)z\partial_z\delta\left(\frac{w}{z}\right) - w\partial_w\delta\left(\frac{w}{z}\right) \\ &\quad + \left(a_{13}(w)a^*_{13}(w) - a_{23}(w)a^*_{23}(w)\right)\delta\left(\frac{w}{z}\right) \\ &= \left((\mu-1) + \nu b_{12}(z) - \frac{\nu}{2}b_{23}(z) + \frac{\nu}{2}b_{13}(z)\right)\delta\left(\frac{w}{z}\right) - 2\mathcal{E}_{12}(w)\delta\left(\frac{w}{z}\right) \\ &\quad - \left(\frac{3\nu^2}{4}+2\right)w\partial_w\delta\left(\frac{w}{z}\right) + :a_{13}(w)a^*_{13}(w): - :a_{23}(w)a^*_{23}(w): - w\partial_w\delta\left(\frac{w}{z}\right) \\ &= \left((\mu-1) + \nu b_{12}(z) - \frac{\nu}{2}b_{23}(z) + \frac{\nu}{2}b_{13}(z)\right)\delta\left(\frac{w}{z}\right) - 2\mathcal{E}_{12}(w)\delta\left(\frac{w}{z}\right) \\ &\quad - \mathcal{E}_{23}(w)\delta\left(\frac{w}{z}\right) + \mathcal{E}_{13}(w)\delta\left(\frac{w}{z}\right) - 3\left(\frac{\nu^2}{4}+1\right)w\partial_w\delta\left(\frac{w}{z}\right) \\ &= -H_1(w)\delta\left(\frac{w}{z}\right) + Cw\partial_w\delta\left(\frac{w}{z}\right)\end{aligned}$$

4.2 Explicit Defining Relations for the Z-algebra Corresponding to $\hat{sl}(3)$

Define $\overline{H}_i(z) := -\int \frac{H_i(z)}{Cz}dz = \sum\limits_{m\neq 0} \frac{H_i(n)}{nC} z^{-n}$. Then we decompose $\overline{H}_i(z) = \overline{H}_{i+}(z) + \overline{H}_{i-}(z)$, where $\overline{H}_{i\pm}(z) = \sum\limits_{\mp n>0} \frac{H_i(n)}{nC} z^{-n}$.

Let $e^{\overline{H}_i(z)}$ again be the formal exponential of $\overline{H}_i(z)$, and note the following identities:

$$e^{\pm\overline{H}_{i-}(z)}e^{\pm\overline{H}_{i+}(w)} = e^{\pm\overline{H}_{i+}(w)}e^{\pm\overline{H}_{i-}(z)}\left(1-\frac{w}{z}\right)^{\frac{2}{C}}, \ \forall\ i=1,2 \tag{4.2.1}$$

$$e^{\overline{H}_{1-}(z)}e^{\overline{H}_{2+}(w)} = e^{\overline{H}_{2+}(w)}e^{\overline{H}_{1-}(z)}\left(1-\frac{w}{z}\right)^{\frac{-1}{C}} \tag{4.2.2}$$

$$e^{\overline{H}_{1-}+\overline{H}_{2-}(z)}e^{\pm\overline{H}_{2+}(w)\pm\overline{H}_{2+}(w)} = e^{\pm\overline{H}_{2+}(w)\pm\overline{H}_{2+}(w)}e^{\overline{H}_{1-}+\overline{H}_{2-}(z)}\left(1-\frac{w}{z}\right)^{\frac{\pm 2}{C}} \tag{4.2.3}$$

Then, we define the **Z**-algebra $\mathcal{Z}(3)$ by defining the following generating functions:

$$\mathbf{Z}^+_{12}(z) = \mathbf{Z}(\alpha_1, z) =: e^{\overline{H}_1(z)}X_{12}(z) : \qquad \mathbf{Z}^+_{23}(z) = \mathbf{Z}(\alpha_2, z) =: e^{\overline{H}_2(z)}X_{23}(z) :$$

$$\mathbf{Z}^+_{13}(z) = \mathbf{Z}(\alpha_1+\alpha_2, z) =: e^{\overline{H}_1(z)+\overline{H}_2(z)}X_{13}(z) :$$

$$\mathbf{Z}^-_{12}(z) = \mathbf{Z}(-\alpha_1, z) =: e^{-\overline{H}_1(z)}Y_{12}(z) : \qquad \mathbf{Z}^-_{23}(z) = \mathbf{Z}(\alpha_2, z) =: e^{-\overline{H}_2(z)}Y_{23}(z) :$$

$$\mathbf{Z}^+_{13}(z) = \mathbf{Z}(-\alpha_1-\alpha_2, z) =: e^{-\overline{H}_1(z)-\overline{H}_2(z)}Y_{13}(z) :$$

Recall, from Definition 2.4.10 our generalized commutator bracket

$$[[\mathbf{Z}(\phi_1,z),\mathbf{Z}(\phi_2,w)]] = \mathbf{Z}(\phi_1,z)\mathbf{Z}(\phi_2,w)\left(1-\frac{w}{z}\right)^{\frac{(\phi_1,\phi_2)}{C}} - \mathbf{Z}(\phi_2,w)\mathbf{Z}(\phi_1,z)\left(1-\frac{z}{w}\right)^{\frac{(\phi_1,\phi_2)}{C}}$$

where the bilinear inner product (ϕ_1,ϕ_2) of roots $\alpha_1,\ \alpha_2,\ \alpha_1+\alpha_2$ can be defined by the Cartan matrix for $sl(3)$, $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

Example 4.2.1. Consider some products in $\mathcal{Z}(3)$. Observe that relations among elements in subsets $\left\{\mathbf{Z}^+_{ij}(z), \mathbf{Z}^-_{ij}(w)\right\}$, for $1 \le i < j \le 3$, behave just as in the $\mathcal{Z}(2)$ case seen in Section 2.1.

$$\begin{aligned}
\mathbf{Z}^+_{12}(z)\mathbf{Z}^+_{13}(w) &= e^{\overline{H}_{1+}(z)}X_{12}(z)e^{\overline{H}_{1-}(z)}e^{\overline{H}_{1+}(w)+\overline{H}_{2+}(w)}X_{13}(w)e^{\overline{H}_{1-}(w)+\overline{H}_{2-}(w)} \\
&= e^{\overline{H}_{1+}(w)+\overline{H}_{2+}(w)}e^{\overline{H}_{1+}(z)}X_{12}(z)e^{\overline{H}_{1-}(z)}X_{13}(w)e^{\overline{H}_{1-}(w)+\overline{H}_{2-}(w)} \\
&= e^{\overline{H}_{1+}(z)+\overline{H}_{1+}(w)+\overline{H}_{2+}(w)}X_{12}(z)X_{13}(w)e^{\overline{H}_{1-}(w)+\overline{H}_{2-}(w)}\left(1-\frac{w}{z}\right)^{\frac{1}{C}}
\end{aligned}$$

Thus, we use the product of these elements to compute their generalized commutator bracket relation.

$$\left[\left[\mathbf{Z}_{12}^{+}(z),\mathbf{Z}_{13}^{+}(w)\right]\right]=e^{\overline{H}_{1+}(z)+\overline{H}_{1+}(w)+\overline{H}_{2+}(w)}\left[X_{12}(z),X_{13}(w)\right]e^{\overline{H}_{1-}(z)\overline{H}_{1-}(w)+\overline{H}_{2-}(w)}=0$$

We see that this bracket is simply 0, but as in the sl_2 case, there are non-trivial relations.

Example 4.2.2.

$$\begin{aligned}\left[\left[\mathbf{Z}_{12}^{+}(z),\mathbf{Z}_{23}^{+}(z)\right]\right]&=e^{\overline{H}_{1+}(z)+\overline{H}_{2+}(w)}\left[X_{12}(z),X_{23}(w)\right]e^{\overline{H}_{1-}(z)+\overline{H}_{2-}(w)}\\&=e^{\overline{H}_{1+}(z)+\overline{H}_{2+}(w)}\left(X_{13}(w)\delta\left(\frac{w}{z}\right)\right)e^{\overline{H}_{1-}(z)+\overline{H}_{2-}(w)}\\&=\mathbf{Z}_{13}^{+}(w)\delta\left(\frac{w}{z}\right)\end{aligned}$$

$$\begin{aligned}\left[\left[\mathbf{Z}_{12}^{+}(z),\mathbf{Z}_{13}^{-}(z)\right]\right]&=e^{\overline{H}_{1+}(z)-\overline{H}_{1+}(w)-\overline{H}_{2+}(w)}\left[X_{12}(z),Y_{13}(w)\right]e^{\overline{H}_{1-}(z)-\overline{H}_{1-}(w)-\overline{H}_{2-}(w)}\\&=e^{\overline{H}_{1+}(z)-\overline{H}_{1+}(w)-\overline{H}_{2+}(w)}\left(-Y_{23}(w)\delta\left(\frac{w}{z}\right)\right)e^{\overline{H}_{1-}(z)-\overline{H}_{1-}(w)-\overline{H}_{2-}(w)}\\&=-\mathbf{Z}_{23}^{-}(w)\delta\left(\frac{w}{z}\right)\end{aligned}$$

Example 4.2.3. For space and simplicity's sake, in this example we let $\overline{H}_{3\pm}(z):=\overline{H}_{1\pm}(z)+\overline{H}_{2\pm}(z)$, and $H_3(n):=H_1(n)+H_2(n)$ where needed.

$$\begin{aligned}\left[\left[\mathbf{Z}_{13}^{+}(z),\mathbf{Z}_{13}^{-}(w)\right]\right]&=e^{\overline{H}_{3+}(z)-\overline{H}_{3+}(w)}\left[X_{13}(z),Y_{13}(w)\right]e^{\overline{H}_{3-}(z)-\overline{H}_{3-}(w)}\\&=e^{\overline{H}_{3+}(z)-\overline{H}_{3+}(w)}\left(\left(H_1(w)+H_2(w)\right)\delta\left(\frac{w}{z}\right)+Cw\partial_w\delta\left(\frac{w}{z}\right)\right)e^{\overline{H}_{3-}(z)-\overline{H}_{3-}(w)}\\&=\left(H_1(w)+H_2(w)\right)\delta\left(\frac{w}{z}\right)+Cw\partial_w\delta\left(\frac{w}{z}\right)-Cw\partial_w\left(e^{\overline{H}_3(z)-\overline{H}_3(w)}\right)\delta\left(\frac{w}{z}\right)\\&=\left(H_1(w)+H_2(w)\right)\delta\left(\frac{w}{z}\right)+Cw\partial_w\delta\left(\frac{w}{z}\right)-\left(\sum_{n\neq 0}\left(H_3(n)\right)w^{-n}\right)\delta\left(\frac{w}{z}\right)\\&=\left(H_1(0)+H_2(0)\right)\delta\left(\frac{w}{z}\right)+Cw\partial_w\delta\left(\frac{w}{z}\right)\end{aligned}$$

From the above examples, we can get some sense of the structure of the algebra $\mathcal{Z}(3)$. To view the full picture of $\mathcal{Z}(3)$, however, we require the "multiplication tables" for the general commutator bracket (See page 31). Plainly, we can see parallels to $\mathcal{Z}(2)$. With $\mathcal{Z}(3)$, though, we have several more non-trivial brackets that come in two varieties of product; those resulting in another element from $\mathcal{Z}(3)$ and those resulting the zero nodes of the associated diagonal elements.

Bracket "Multiplication Tables" for $\mathcal{Z}(3)$

Table 4.1: Brackets of positive root elements of $\mathcal{Z}(3)$

$[[\cdot,\cdot]]$	$\mathbf{Z}_{12}^{+}(w)$	$\mathbf{Z}_{23}^{+}(w)$	$\mathbf{Z}_{13}^{+}(w)$
$\mathbf{Z}_{12}^{+}(z)$	0	$\mathbf{Z}_{13}^{+}(w)\delta\left(\frac{w}{z}\right)$	0
$\mathbf{Z}_{23}^{+}(z)$	$-\mathbf{Z}_{13}^{+}(w)\delta\left(\frac{w}{z}\right)$	0	0
$\mathbf{Z}_{13}^{+}(z)$	0	0	0
$\mathbf{Z}_{12}^{-}(z)$	$-H_1(0)\delta\left(\frac{w}{z}\right)$ $-Cw\partial_w\delta\left(\frac{w}{z}\right)$	0	$\mathbf{Z}_{23}^{-}(w)\delta\left(\frac{w}{z}\right)$
$\mathbf{Z}_{23}^{-}(z)$	0	$-H_2(0)\delta\left(\frac{w}{z}\right)$ $-Cw\partial_w\delta\left(\frac{w}{z}\right)$	$-\mathbf{Z}_{12}^{+}(w)\delta\left(\frac{w}{z}\right)$
$\mathbf{Z}_{13}^{-}(z)$	$\mathbf{Z}_{23}^{-}(w)\delta\left(\frac{w}{z}\right)$	$-\mathbf{Z}_{12}^{-}(w)\delta\left(\frac{w}{z}\right)$	$-\left(H_1(0)+H_2(0)\right)\delta\left(\frac{w}{z}\right)$ $-Cw\partial_w\delta\left(\frac{w}{z}\right)$

Table 4.2: Brackets of negative root elements of $\mathcal{Z}(3)$

$[[\cdot,\cdot]]$	$\mathbf{Z}_{12}^{-}(w)$	$\mathbf{Z}_{23}^{-}(w)$	$\mathbf{Z}_{13}^{-}(w)$
$\mathbf{Z}_{12}^{+}(z)$	$H_1(0)\delta\left(\frac{w}{z}\right)$ $+Cw\partial_w\delta\left(\frac{w}{z}\right)$	0	$-\mathbf{Z}_{23}^{-}(w)\delta\left(\frac{w}{z}\right)$
$\mathbf{Z}_{23}^{+}(z)$	0	$H_2(0)\delta\left(\frac{w}{z}\right)$ $+Cw\partial_w\delta\left(\frac{w}{z}\right)$	$\mathbf{Z}_{12}^{-}(w)\delta\left(\frac{w}{z}\right)$
$\mathbf{Z}_{13}^{+}(z)$	$-\mathbf{Z}_{23}^{-}(w)\delta\left(\frac{w}{z}\right)$	$\mathbf{Z}_{12}^{-}(w)\delta\left(\frac{w}{z}\right)$	$\left(H_1(0)+H_2(0)\right)\delta\left(\frac{w}{z}\right)$ $+Cw\partial_w\delta\left(\frac{w}{z}\right)$
$\mathbf{Z}_{12}^{-}(z)$	0	$-\mathbf{Z}_{13}^{-}(w)\delta\left(\frac{w}{z}\right)$	0
$\mathbf{Z}_{23}^{-}(z)$	$\mathbf{Z}_{13}^{-}(w)\delta\left(\frac{w}{z}\right)$	0	0
$\mathbf{Z}_{13}^{-}(z)$	0	0	0

Chapter 5

A Realization of $\hat{sl}(2)$ at the Critical Level and the associated Lepowsky-Wilson Z-algebra

5.1 A New Realization of $\hat{sl}(2)$ at the Critical Level

We now introduce a new representation of $\hat{sl}_2(\mathbb{C})$, following the work of Jing in [11]. We will then see that this realization provides us with a representation of the associated **Z**-algebra $\mathcal{Z}(2)$. From these constructions we shall be able to calculate the character of the module, and hence we can also see the character of its vacuum space, as generated by our realization of $\mathcal{Z}(2)$.

To define our representation space we require an understanding of the following spaces:

- $S(\mathfrak{h})$ is the space of symmetric functions on $\mathfrak{h} = \langle H(m) | m \in \mathbb{Z}_{\neq 0} \rangle$. Later, we will also require the subspace $S(\mathfrak{h}^-)$, where $\mathfrak{h}^- = \langle H(-m) | m > 0 \rangle$. The reason for this is that, by the action introduced in [24], $H(m) \cdot 1 = 0$ for all $m > 0$. Once we have defined our representation for $\hat{sl}(2)$, we will wish to only consider the subspace of non-zero action.

- $\mathbb{C}[\mathbb{Z}\frac{\alpha}{2}]$ is the group algebra, defined $\forall\ \beta, \lambda \in \mathbb{Z}\frac{\alpha}{2}$ by these relations:

$$e^{\beta} \cdot e^{\lambda} = e^{\beta+\lambda}$$
$$z^{\beta} \cdot e^{\lambda} = z^{(\beta,\lambda)} e^{\lambda}$$

  where (β, λ) is the inner-product as defined in Section 2.2. For our purposes, we may simply recall that $(\alpha, \alpha) = 2$ and then extend linearly.

- Let $V = \mathbb{Z} + \frac{1}{2}$ as a vector space. Notationally, we write $\underline{m} \in V$ to distinguish the vector $\underline{m}$ from its numerical value m. Then, define $\Lambda(V)$ to be the space of semi-infinite wedge products of these component vectors (See [13]). Elements $v \in \Lambda(V)$ are of the form $v = \underline{n_0} \wedge \underline{n_{-1}} \wedge \underline{n_{-2}} \wedge \cdots$, where the component vectors are ordered by $n_0 > n_{-1} > n_{-2} > \cdots \in \mathbb{Z} + \frac{1}{2}$. These wedge products are called *semi-infinite* because for each $v \in \Lambda(v)$ there is a certain $t \in \mathbb{Z}$ such that $n_{-r} = n_{-(r+1)} - 1$ for all $r > t$. This means that each v has a "tail" where the component vectors numerically descend by 1 with each consecutive vector as we read the component vectors from left to right.

For a vector $v \in \Lambda(V)$, let the set of its component vectors, $\{\underline{n_0},\ \underline{n_{-1}},\ \underline{n_{-2}}, \dots\} \subset \mathbb{Z} + \frac{1}{2}$, be called the *support* of v. We will denote this set by *supp*(v). Also, we define the space $\Lambda(V)_{\frac{1}{2}} = \left\{v \in \Lambda(V) | \frac{1}{2} \in supp(v)\right\}$.

For $k \in \mathbb{Z} + \frac{1}{2}$, denote $|k\rangle = \underline{k} \wedge \underline{k-1} \wedge \underline{k-2} \wedge \underline{k-3} \wedge \dots$.

The space $\hat{V} = S(\mathfrak{h}^-) \otimes \Lambda(V)_{\frac{1}{2}} \otimes \mathbb{C}\left[\mathbb{Z}\frac{\alpha}{2}\right]$ will be our representation space.

Definition 5.1.1. Define the set of operators $\mathcal{A} = \left\{A(m), A^*(m) | \forall m, n \in \mathbb{Z} + \frac{1}{2}\right\}$ acting on the space $\Lambda(V)$ in the following way:

$$A(m) \cdot v = \left(m - \frac{1}{2}\right) \underline{-m} \wedge v, \qquad A^*(m) \cdot v = \left(m - \frac{1}{2}\right) \frac{\partial}{\partial \underline{m}} (v).$$

Notice that the operators in $\mathcal{A}$ have the following relations as they act on $\Lambda(V)$:

$$\{A(m), A(n)\} = 0 = \{A^*(m), A^*(n)\} \tag{5.1.1}$$

$$\{A(m), A^*(n)\} = -\left(m^2 - \frac{1}{4}\right) \delta_{m+n,0} \tag{5.1.2}$$

using the *anti-commutator* from Definition 2.1.1.

Definition 5.1.2. Define the *fermionic* normal ordering of objects $x_1, x_2 \in \mathcal{A}$ by

$$:x_1(m)x_2(n): = \begin{cases} x_1(m)x_2(n), & m < 0 \\ -x_2(n)x_1(m), & m > 0 \end{cases}$$

As throughout this work, calculations are performed using generating functions. In this instance

$$x(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} x(n) z^{-n-\frac{1}{2}},\ \forall\ x = A, A^*. \tag{5.1.3}$$

We notice certain important characteristics of the operators in $\mathcal{A}$ when we look at the normal ordering of the generating functions. Again, consider $x_1, x_2 \in \{A, A^*\}$.

$$:x_1(z)x_2(w): = \sum_{\substack{n\in\mathbb{Z}\\ m>0}} x_1(m)x_2(n)z^{-m-\frac{1}{2}}w^{-n-\frac{1}{2}} - \sum_{\substack{n\in\mathbb{Z}\\ m<0}} x_2(n)x_1(m)z^{-m-\frac{1}{2}}w^{-n-\frac{1}{2}} \tag{5.1.4}$$

$$\begin{aligned}\text{Thus, } :x_1(z)x_2(w): + :x_2(w)x_1(z): \quad &= -\sum_{\substack{n\in\mathbb{Z}\\ m<0}} \{x_1(m), x_2(n)\}\, z^{-m-\frac{1}{2}}w^{-n-\frac{1}{2}} \\ &\quad + \sum_{\substack{m\in\mathbb{Z}\\ n>0}} \{x_1(m), x_2(n)\}\, z^{-m-\frac{1}{2}}w^{-n-\frac{1}{2}} = 0.\end{aligned}$$

So, we see that both $:A(z)A^*(w): = \ -:A^*(w)A(z):$ and $:A(z)A(w): = \ -:A(w)A(z):$. We may therefore conclude that $:A(w)A(w): = 0$ and thus

$$:A(z)A(w):\delta\left(\frac{w}{z}\right) = 0. \tag{5.1.5}$$

Likewise, we may argue $:A^*(z)A^*(w):\delta\left(\frac{w}{z}\right) = 0$.

Despite the change from bosonic to fermionic normally ordered products, we continue using Definition 2.3.4 to define contractions. Since $A(m)$ (resp. $A^*(m)$) anti-commutes with $A(n)$ (resp. $A^*(n)$), for all $m, n \in \mathbb{Z} + \frac{1}{2}$, then contractions $\underbrace{A(z)A(w)} = \underbrace{A^*(z)A^*(w)} = 0$. So, we will only calculate the contractions for the opposing operators.

$$\begin{aligned}\underbrace{A(z)A^*(w)} &= A(z)A^*(w) - :A(z)A^*(w): \\ &= \frac{1}{2}\sum_{n\in\mathbb{Z}}\{A(0), A^*(n)\}w^{-n-\frac{1}{2}} + \sum_{\substack{m>0\\ n\in\mathbb{Z}+\frac{1}{2}}}\{A(m), A^*(n)\}z^{-m-\frac{1}{2}}w^{-n-\frac{1}{2}} \\ &= \sum_{\substack{m>0\\ n\in\mathbb{Z}+\frac{1}{2}}} -\left(m^2 - \frac{1}{4}\right) z^{-m-\frac{1}{2}}w^{m-\frac{1}{2}} = \frac{-2zw}{(z-w)^3}\end{aligned} \tag{5.1.6}$$

Similarly, we calculate $\underbrace{A^*(w)A(z)} = \frac{-2zw}{(w-z)^3}$.

Now, define the series $E^{\pm}(z) \in S(\mathfrak{h})$, which we decompose as $E^{\pm}(z) = E_+^{\pm}(z)E_-^{\pm}(z)$, where

$$E_+^{\pm}(z) = \exp\left(\mp\sum_{n>0}\frac{H(-n)}{2n}z^n\right) \qquad E_-^{\pm}(z) = \exp\left(\pm\sum_{n>0}\frac{H(n)}{2n}z^{-n}\right).$$

Then, as operators, we have the following relations:

$$
\begin{aligned}
E^{+}_{\pm}(z)E^{-}_{\pm}(z) &= 1\\
E^{\pm}_{-}(z)E^{\pm}_{+}(w) &= E^{\pm}_{+}(w)E^{\pm}_{-}(z)\left(1-\frac{w}{z}\right)^{-1}\\
E^{+}_{-}(z)E^{-}_{+}(w) &= E^{-}_{+}(w)E^{+}_{-}(z)\left(1-\frac{w}{z}\right)\\
E^{+}_{\pm}(z)E^{+}_{\pm}(w) &= E^{+}_{\pm}(w)E^{+}_{\pm}(z)\\
E^{-}_{\pm}(z)E^{+}_{\pm}(w) &= E^{+}_{\pm}(w)E^{-}_{\pm}(z)\\
E^{-}_{\pm}(z)E^{-}_{\pm}(w) &= E^{-}_{\pm}(w)E^{-}_{\pm}(z)\\
\partial_z\left(E^{\pm}_{+}(z)E^{\pm}_{-}(z)\right) &= E^{\pm}_{+}(z)E^{\pm}_{-}(z)\left(\sum_{n\neq 0}\frac{H(n)}{2}z^{-n-1}\right)
\end{aligned}
$$

Theorem 5.1.3. *The map $\psi : \hat{sl}_2 \longrightarrow gl(\hat{V})$ defined below, is a representation of $\hat{sl}_2$, acting on the space $\hat{V} = S(\mathfrak{h}^-)\otimes\Lambda(V)_{\frac{1}{2}}\otimes\mathbb{C}\left[\mathbb{Z}\frac{\alpha}{2}\right]$.*

$$
\begin{aligned}
X(z) &\longmapsto E^{+}_{+}(z)E^{+}_{-}(z)\otimes A(z)e^{\alpha}z^{-\frac{\alpha}{2}}\\
Y(z) &\longmapsto E^{-}_{+}(z)E^{-}_{-}(z)\otimes A^{*}(z)e^{-\alpha}z^{\frac{\alpha}{2}}\\
H(z) &\longmapsto H(z)\otimes 1\\
C &\longmapsto -2\\
d &\longmapsto 1\otimes z\frac{\partial}{\partial_z}+z\frac{\partial}{\partial z}\otimes 1
\end{aligned}
$$

By custom, suppress the map ψ and simply associate the left-hand and right-hand sides in the above definition. Furthermore, when appropriate, we will suppress the tensors.

Proof. To show that ψ defines a representation, we must verify the bracket relations for $\hat{sl}(2)$.

$$
\begin{aligned}
X(z)X(w) &= E^{+}_{+}(z)E^{+}_{-}(z)E^{+}_{+}(w)E^{+}_{-}(w)\otimes A(z)e^{\alpha}z^{-\frac{\alpha}{2}}A(w)e^{\alpha}w^{-\frac{\alpha}{2}}\\
&= E^{+}_{+}(z)E^{+}_{+}(w)E^{+}_{-}(z)E^{+}_{-}(w)\left(1-\frac{w}{z}\right)^{-1}\otimes A(z)A(w)e^{2\alpha}z^{-\frac{\alpha}{2}-1}w^{-\frac{\alpha}{2}}\\
&= E^{+}_{+}(z)E^{+}_{+}(w)E^{+}_{-}(z)E^{+}_{-}(w)\otimes\left(:A(z)A(w):+\underbrace{A(z)A(w)}\right)(z-w)^{-1}e^{2\alpha}(zw)^{-\frac{\alpha}{2}}\\
&= E^{+}_{+}(z)E^{+}_{+}(w)E^{+}_{-}(z)E^{+}_{-}(w)\otimes :A(z)A(w):(z-w)^{-1}e^{2\alpha}(zw)^{-\frac{\alpha}{2}}
\end{aligned}
$$

Hence, $X(w)X(z) = E_+^+(z)E_+^+(w)E_-^+(z)E_-^+(w) \otimes : A(w)A(z) : (w-z)^{-1}\, e^{2\alpha}(zw)^{-\frac{\alpha}{2}}$
$= E_+^+(z)E_+^+(w)E_-^+(z)E_-^+(w) \otimes (- : A(z)A(w) :)\,(w-z)^{-1}\, e^{2\alpha}(zw)^{-\frac{\alpha}{2}}$

So then, $[X(z), X(w)] = X(z)X(w) - X(w)X(z)$
$= E_+^+(z)E_+^+(w)E_-^+(z)E_-^+(w) \otimes : A(z)A(w) : \left(\frac{1}{z-w} + \frac{1}{w-z}\right) e^{2\alpha}(zw)^{-\frac{\alpha}{2}}$
$= E_+^+(z)E_+^+(w)E_-^+(z)E_-^+(w) \otimes : A(z)A(w) : \delta\left(\frac{w}{z}\right) e^{2\alpha}(zw)^{-\frac{\alpha}{2}}$
$= 0$, from (5.1.5)

Similarly, $[Y(z), Y(w)] = 0$, as expected.

Now, to verify $[X(z), Y(w)] = H(w)\delta\left(\frac{w}{z}\right) + Cw\partial_w\delta\left(\frac{w}{z}\right)$, recalling that C acts as the level of the representation, we require (5.1.6).

$$\begin{aligned}
X(z)Y(w) &= E_+^+(z)E_-^+(z)E_+^-(w)E_-^-(w) \otimes A(z)e^{\alpha}z^{-\frac{\alpha}{2}}A^*(w)e^{-\alpha}w^{\frac{\alpha}{2}} \\
&= E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w)\left(1 - \frac{w}{z}\right) \\
&\qquad \otimes \left(: A(z)A^*(w) : + \underbrace{A(z)A^*(w)}\right) e^{\alpha-\alpha}z^{-\frac{\alpha}{2}}w^{\frac{\alpha}{2}} \\
&= E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w) \otimes : A(z)A^*(w) : (z-w)\, z^{-\frac{\alpha}{2}}w^{\frac{\alpha}{2}} \\
&\qquad + E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w) \otimes \left(\frac{-2zw}{(z-w)^3}\right)(z-w)\, z^{-\frac{\alpha}{2}}w^{\frac{\alpha}{2}} \\
&= E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w) \otimes : A(z)A^*(w) : (z-w)\, z^{-\frac{\alpha}{2}}w^{\frac{\alpha}{2}} \\
&\qquad + E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w) \otimes \left(\frac{-2zw}{(z-w)^2}\right)\left(\frac{w}{z}\right)^{\frac{\alpha}{2}}, \text{ and}
\end{aligned}$$

$$
\begin{aligned}
Y(w)X(z) &= E_+^-(w)E_-^-(w)E_+^+(z)E_-^+(z) \otimes A^*(w)e^{-\alpha}w^{\frac{\alpha}{2}}A(z)e^{\alpha}z^{-\frac{\alpha}{2}} \\
&= E_+^-(w)E_+^+(z)E_-^-(w)E_-^+(z)\left(1-\frac{z}{w}\right) \otimes A^*(w)A(z)e^{-\alpha+\alpha}z^{-\frac{\alpha}{2}}w^{\frac{\alpha}{2}+1} \\
&= E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w) \otimes :A^*(w)A(z):(w-z)\,z^{-\frac{\alpha}{2}}w^{\frac{\alpha}{2}} \\
&\quad + E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w) \otimes (w-z)\left(\frac{-2zw}{(w-z)^3}\right)z^{-\frac{\alpha}{2}}w^{\frac{\alpha}{2}} \\
&= E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w) \otimes :A(z)A^*(w):(z-w)\,z^{-\frac{\alpha}{2}}w^{\frac{\alpha}{2}} \\
&\quad + E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w) \otimes \left(\frac{-2zw}{(w-z)^2}\right)\left(\frac{w}{z}\right)^{\frac{\alpha}{2}}
\end{aligned}
$$

So then the bracket relation follows directly.

$$
\begin{aligned}
[X(z),Y(w)] &= X(z)Y(w) - Y(w)X(z) \\
&= E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w) \otimes (:A(z)A^*(w): - :A(z)A^*(w):)\,(z-w)\,z^{-\frac{\alpha}{2}}w^{\frac{\alpha}{2}} \\
&\quad + E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w) \otimes \left(\frac{-2zw}{(z-w)^2} - \frac{-2zw}{(w-z)^2}\right)\left(\frac{w}{z}\right)^{\frac{\alpha}{2}} \\
&= -2E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w)\left(\frac{w}{z}\right)^{\frac{\alpha}{2}}\left[w\partial_w\delta\left(\frac{w}{z}\right)\right] \\
&= -2w\partial_w\left[E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w)\left(\frac{w}{z}\right)^{\frac{\alpha}{2}}\delta\left(\frac{w}{z}\right)\right] \\
&\quad + 2w\partial_w\left[E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w)\right]\left(\frac{w}{z}\right)^{\frac{\alpha}{2}}\delta\left(\frac{w}{z}\right) \\
&\quad + 2wE_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w)\partial_w\left[\left(\frac{w}{z}\right)^{\frac{\alpha}{2}}\right]\delta\left(\frac{w}{z}\right) \\
&= -2w\partial_w\delta\left(\frac{w}{z}\right) + 2w\left(E_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w)\left(\sum_{n\neq 0}\frac{H(n)}{2}w^{-n-1}\right)\delta\left(\frac{w}{z}\right)\right) \\
&\quad + 2wE_+^+(z)E_+^-(w)E_-^+(z)E_-^-(w)\left[\frac{\alpha}{2}w^{\frac{\alpha}{2}-1}z^{\frac{-\alpha}{2}}\right]\delta\left(\frac{w}{z}\right) \\
&= -2w\partial_w\delta\left(\frac{w}{z}\right) + \sum_{n\neq 0}H(n)w^{-n}\delta\left(\frac{w}{z}\right) + \alpha\delta\left(\frac{w}{z}\right) \\
&= \sum_{n\neq 0}H(n)w^{-n}\delta\left(\frac{w}{z}\right) + H(0)\delta\left(\frac{w}{z}\right) - 2w\partial_w\delta\left(\frac{w}{z}\right) \\
&= H(w)\delta\left(\frac{w}{z}\right) - 2w\partial_w\delta\left(\frac{w}{z}\right) \qquad (5.1.7)
\end{aligned}
$$

To verify $[H(z), X(w)] = 2X(w)\delta\left(\frac{w}{z}\right)$, we require the following calculations:

$$\left[H(z), E_+^+(w)\right] = \sum_{m\in\mathbb{Z}} \left[H(m), e^{-\sum\limits_{n>0} \frac{H(-n)}{2n} w^n}\right] z^{-m} = E_+^+(w) \sum_{\substack{m\in\mathbb{Z} \\ n>0}} \frac{-\left[H(m), H(-n)\right]}{2n} z^{-m} w^n$$

$$= E_+^+(w) \sum_{\substack{m\in\mathbb{Z} \\ n>0}} \frac{-(mC\delta_{mn})}{2n} z^{-m} w^n = E_+^+(w) \sum_{n>0} 2\left(\frac{w}{z}\right)^n$$

Similarly, $\left[H(z), E_-^+(w)\right] = E_-^+(w) \sum_{n<0} 2\left(\frac{w}{z}\right)^n$ and $[H(0), e^\alpha] = (\alpha, \alpha)e^\alpha = 2e^\alpha$

$$\begin{aligned}\text{Thus, } [H(z), X(w)] &= \left[H(0), E_+^+(w)\right] E_-^+(w) \otimes A(w)e^\alpha w^{-\frac{\alpha}{2}} \\ &+ E_+^+(w) \left[H(0), E_-^+(w)\right] \otimes A(w)e^\alpha w^{-\frac{\alpha}{2}} \\ &+ E_+^+(w) E_-^+(w) \otimes A(w) \left[H(0), e^\alpha\right] w^{-\frac{\alpha}{2}} \\ &= 2E_+^+(w)E_-^+(w) \otimes A(w)e^\alpha w^{-\frac{\alpha}{2}} \delta\left(\frac{w}{z}\right) = 2X(w)\delta\left(\frac{w}{z}\right)\end{aligned}$$

The above argument translates easily to the case of $[H(z), Y(w)] = 2Y(w)\delta\left(\frac{w}{z}\right)$, as well. So, we see that we, in fact, have a representation of $\hat{sl}_2$. Indeed, by comparing (5.1.7) with the known bracket relation for $[X(z), Y(w)]$, we see that $C = -2$. Hence, the level of our representation is -2, the critical level. □

Take our Chevalley generators for $\hat{sl}_2$ as follows:

$$\begin{array}{ll} e_0 = Y(1) & e_1 = X(0) \\ f_0 = X(-1) & f_1 = Y(0) \\ h_0 = -H(0) + C & h_1 = H(0) \end{array}$$

Definition 5.1.4. A *weight vector* is a vector v in the representation space, with *weight* $\lambda \in \mathfrak{h}^*$, such that $h \cdot v = \lambda(h)v$, for all $h \in \mathfrak{h}$. (In this case, $h = h_0,\ h_1$.)

A weight vector that is annihilated by the positive root elements of the Lie algebra is called a *highest weight vector*. (In our case, a vector v such that $e_0 \cdot v = e_1 \cdot v = 0$.) Such a weight λ is called a *highest weight*.

Theorem 5.1.5. *The vector* $\left|\frac{1}{2}\right\rangle = 1 \otimes \left|\frac{1}{2}\right\rangle \in \hat{V}$ *is a highest weight vector.*

Proof.

$$\begin{aligned}
\text{Observe, } X(z)\cdot\left|\frac{1}{2}\right\rangle &= E_+^+(z)\cdot 1\otimes \sum_{n\in\mathbb{Z}+\frac{1}{2}} A(n)\cdot\left|\frac{1}{2}\right\rangle z^{-n-\frac{1}{2}}\\
&= \exp\left(-\sum_{m>0}\frac{H(-m)}{2m}z^m\right)\cdot 1\otimes \sum_{n\in\mathbb{Z}+\frac{1}{2}}\left(A(n)\cdot\left|\frac{1}{2}\right\rangle\right) z^{-n-\frac{1}{2}}\\
&= \exp\left(-\sum_{m>0}\frac{H(-m)}{2m}z^m\right)\cdot 1\otimes \sum_{n\in\mathbb{Z}+\frac{1}{2}}\left(n-\frac{1}{2}\right)\underline{-n}\wedge\left|\frac{1}{2}\right\rangle z^{-n-\frac{1}{2}}\\
&= \exp\left(-\sum_{m>0}\frac{H(-m)}{2m}z^m\right)\cdot 1\otimes \sum_{n<-\frac{1}{2}}\left(n-\frac{1}{2}\right)\underline{-n}\wedge\left|\frac{1}{2}\right\rangle z^{-n-\frac{1}{2}}
\end{aligned}$$

So, $X(m)\cdot\left|\frac{1}{2}\right\rangle = 0,\ \forall\ m\geq 0$, and $X(m)\cdot\left|\frac{1}{2}\right\rangle \neq 0,\ \forall\ m<0$.

$$\begin{aligned}
Y(z)\cdot\left|\frac{1}{2}\right\rangle &= E_+^-(z)\cdot 1\otimes \sum_{n\in\mathbb{Z}+\frac{1}{2}} A^*(n)\cdot\left|\frac{1}{2}\right\rangle z^{-n-\frac{1}{2}}\\
&= \exp\left(\sum_{m>0}\frac{H(-m)}{2m}z^m\right)\cdot 1\otimes \sum_{n\in\mathbb{Z}+\frac{1}{2}}\left(A^*(n)\cdot\left|\frac{1}{2}\right\rangle\right) z^{-n-\frac{1}{2}}\\
&= \exp\left(\sum_{m>0}\frac{H(-m)}{2m}z^m\right)\cdot 1\otimes \sum_{n\in\mathbb{Z}+\frac{1}{2}}\left(n-\frac{1}{2}\right)\frac{\partial}{\partial\underline{n}}\left(\left|\frac{1}{2}\right\rangle\right) z^{-n-\frac{1}{2}}\\
&= \exp\left(\sum_{m>0}\frac{H(-m)}{2m}z^m\right)\cdot 1\otimes \sum_{n\leq-\frac{1}{2}}\left(n-\frac{1}{2}\right)\frac{\partial}{\partial\underline{n}}\left(\left|\frac{1}{2}\right\rangle\right) z^{-n-\frac{1}{2}}
\end{aligned}$$

So, $Y(m)\cdot\left|\frac{1}{2}\right\rangle = 0,\ \forall\ m>0$ and $Y(m)\cdot\left|\frac{1}{2}\right\rangle \neq 0,\ \forall\ m\leq 0$.

Thus, $e_0\cdot\left|\frac{1}{2}\right\rangle = Y(1)\cdot\left|\frac{1}{2}\right\rangle = 0$ and $e_1\cdot\left|\frac{1}{2}\right\rangle = X(0)\cdot\left|\frac{1}{2}\right\rangle = 0$. This is sufficient to show that $\left|\frac{1}{2}\right\rangle$ is a highest weight vector. □

Furthermore, we can observe that $[X(0),Y(0)]\cdot\left|\frac{1}{2}\right\rangle = H(0)\cdot\left|\frac{1}{2}\right\rangle = 0$. Hence,

$$h_0\cdot\left|\frac{1}{2}\right\rangle = -2\left|\frac{1}{2}\right\rangle,\qquad h_1\cdot\left|\frac{1}{2}\right\rangle = 0\qquad\text{and}\qquad d\cdot\left|\frac{1}{2}\right\rangle = 0.$$

Therefore, our highest weight is $\lambda = -2\Lambda_0$, where Λ_0 is the fundamental weight as described in Definition 2.1.2.

Now we determine the character of $\hat{V} = S(\mathfrak{h}^-)\otimes\Lambda(V)_{\frac{1}{2}}\otimes\mathbb{C}\left[\mathbb{Z}\frac{\alpha}{2}\right]$, with the degree determined by the degree operator d as defined in Definition 5.1.3. The action of $A(m)$ on $\left|\frac{1}{2}\right\rangle$ allows us to form the wedge product of any $\underline{m}$ with $\left|\frac{1}{2}\right\rangle$ where $m \geq \frac{3}{2}$. We see this clearly when we let the generating function for $A(z)$ act on the vector.

$$\begin{aligned} A(z) \cdot \left|\frac{1}{2}\right\rangle &= \sum_{m \in \mathbb{Z}+\frac{1}{2}} A(m) \cdot \left|\frac{1}{2}\right\rangle z^{-m-\frac{1}{2}} = \sum_{m \in \mathbb{Z}+\frac{1}{2}} \left(m - \frac{1}{2}\right) \underline{-m} \wedge \left|\frac{1}{2}\right\rangle z^{-m-\frac{1}{2}} \\ &= \sum_{m \leq \frac{-3}{2}} \left(m - \frac{1}{2}\right) \underline{-m} \wedge \left|\frac{1}{2}\right\rangle z^{-m-\frac{1}{2}} \end{aligned} \tag{5.1.8}$$

We can define the action of d on the operator $A(m)$ by $d \cdot A(m) = \left(m - \frac{1}{2}\right) A(m)$. Then, since $d \cdot \left|\frac{1}{2}\right\rangle = 0$, then we see that

$$d \cdot \left(A(m_l) \cdots A(m_1) \cdot \left|\frac{1}{2}\right\rangle \right) = \left(\sum_{i=1}^{l} m_i - \frac{1}{2} \right) A(m_l) \cdots A(m_1) \cdot \left|\frac{1}{2}\right\rangle \tag{5.1.9}$$

Notice that the degree of the operators are integers greater than or equal to 1, when $m \leq \frac{-3}{2}$. So, from $A(z)$, the contribution to the character is $\prod_{m>0} (1 + q^m)$.

Similarly consider the action of $A^*(m)$ on $\left|\frac{1}{2}\right\rangle$.

$$\begin{aligned} A^*(z) \cdot \left|\frac{1}{2}\right\rangle &= \sum_{m \in \mathbb{Z}+\frac{1}{2}} A^*(m) \cdot \left|\frac{1}{2}\right\rangle z^{-m-\frac{1}{2}} = \sum_{m \in \mathbb{Z}+\frac{1}{2}} \left(m - \frac{1}{2}\right) \frac{\partial}{\partial \underline{m}} \left(\left|\frac{1}{2}\right\rangle \right) z^{-m-\frac{1}{2}} \\ &= \sum_{m \leq \frac{-1}{2}} \left(m - \frac{1}{2}\right) \frac{\partial}{\partial \underline{m}} \left(\left|\frac{1}{2}\right\rangle \right) z^{-m-\frac{1}{2}} \end{aligned} \tag{5.1.10}$$

The only powers of z with non-zero coefficients occur when $m \leq \frac{-1}{2}$, which corresponds to degrees greater than or equal to 0. Then, we may define $d \cdot A^*(n) = -n - \frac{1}{2}$. And so, similar to (5.1.9), we see that

$$d \cdot \left(A^*(n_k) \cdots A^*(n_1) \cdot \left|\frac{1}{2}\right\rangle \right) = \left(\sum_{i=1}^{k} -n_i - \frac{1}{2} \right) A^*(n_k) \cdots A^*(n_1) \cdot \left|\frac{1}{2}\right\rangle \tag{5.1.11}$$

The contribution here to the character is $\prod_{m \geq 0} (1 + q^m) = 2 \prod_{m>0} (1 + q^m)$.

Thus, the complete contribution to the character of $\hat{V}$ from the infinite wedge product space $\Lambda(V)_{\frac{1}{2}}$ is taken by the product of (5.1.8) and (5.1.10) to obtain $2 \prod_{m>0} (1 + q^m)^2$.

Let us consider this another way. An arbitrary vector in the vacuum space is of the form $v = \underline{j_k} \wedge \cdots \wedge \underline{j_1} \wedge \underline{1/2} \wedge \cdots \wedge -\widehat{\underline{i_1}} \wedge \cdots \wedge -\widehat{\underline{i_l}} \wedge \cdots$, where $j \in \mathbb{Z}_{\geq 0} + \dfrac{1}{2}$, $i \in \mathbb{Z}_{>0} + \dfrac{1}{2}$ and $\hat{\underline{i}}$ implies the removal of $\underline{i}$. We may view $\Lambda(V)$ as $\Lambda(V^+ \oplus V^-)$ where $V^+ = \mathbb{Z}_{\geq 0} + \dfrac{1}{2}$ and $V^- = \mathbb{Z}_{<0} + \dfrac{1}{2}$. It is understood that $\Lambda(V^+ \oplus V^-) = \Lambda(V^+) \otimes \Lambda(V^-)$

Keep in mind the following

- We assume that $j_k > \cdots > j_1$ and $i_l > \cdots > i_1$.
- It is not necessarily the case that $j_r = j_{r-1} + 1$ or $i_r = i_{r-1} + 1$ for any r.
- A component vector $\underline{j_r}$ (respectively, the removal of $\underline{i_s}$) has degree $j_r - \frac{1}{2}$ (respectively, $i_s - \frac{1}{2}$).

$$\text{So, } \deg(v) = \sum_{r=1}^{k} \left(j_r - \frac{1}{2} \right) + \sum_{s=1}^{l} \left(i_s - \frac{1}{2} \right).$$

Now, $j_r - \frac{1}{2} > 0$ and $i_s - \frac{1}{2} \geq 0$ are integers. So, to count the dimension of the degree m portion of $\Lambda\left(V\right)_{\frac{1}{2}}$, we need to count the number of ways of expressing an integer $m = (n_1 + n_2 + \cdots + n_k) + (p_1 + p_2 + \cdots + p_\ell)$ such that $n_1 < n_2 < \cdots < n_k$ and $p_1 < p_2 < \cdots < p_\ell$, where $n_i \in \left\{ j_r - \frac{1}{2} \middle| \; j_r \in \mathbb{Z}_{>0} + \frac{1}{2} \right\}$ and $p_i \in \{ i_s - \frac{1}{2} | \; i_s \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$. Then, we know that $\prod_{n>0} (1 + q^n)$ counts the partitions of the number $n = (n_1 + n_2 + \cdots + n_k)$ from distinct parts greater than 0, and $\prod_{p \geq 0} (1 + q^p)$ counts the partitions of $p = (p_1 + p_2 + \cdots + p_\ell)$ from distinct parts greater than or equal to 0.

Hence,

$$ch\left(\Lambda(V^+ \oplus V^-)\right) = ch\left(\Lambda(V^+)\right) \cdot ch\left(\Lambda(V^-)\right) = \prod_{n>0} (1 + q^n) \prod_{p \geq 0} (1 + q^p) = 2 \prod_{m \geq 0} (1 + q^m)^2.$$

Importantly, we see that the calculation of the character of $\Lambda(V)_{\frac{1}{2}}$ by knowing the degree of an arbitrary vector $v \in \Lambda(V)_{\frac{1}{2}}$, is consistent with the degree calculations based on the actions of operators A and A^*.

Now, the characters of the subspaces $S(\mathfrak{h}^-)$ and $\mathbb{C}\left[\mathbb{Z}\frac{\alpha}{2}\right]$ are well-known.

$$ch\left(S(\mathfrak{h}^-)\right) = \prod_{n>0} \left(\frac{1}{1 - q^n} \right) \quad \text{and} \quad ch\left(\mathbb{C}\left[\mathbb{Z}\frac{\alpha}{2}\right]\right) = \sum_{p \in \mathbb{Z}} q^{\frac{p^2}{2}} \quad \text{(cf. [13])}$$

Combining all of this information, we see that the character of $\hat{V}$ is

$$
\begin{aligned}
ch(\hat{V}) &= ch\left(S\left(\mathfrak{h}^{-}\right)\right) \cdot ch\left(\Lambda(V)_{\frac{1}{2}}\right) \cdot ch\left(\mathbb{C}\left[\mathbb{Z}\frac{\alpha}{2}\right]\right) \\
&= \frac{2 \prod\limits_{m>0} (1+q^m)^2 \sum\limits_{p\in\mathbb{Z}} q^{\frac{p^2}{2}}}{\prod\limits_{n>0} (1-q^n)} \qquad (5.1.12)
\end{aligned}
$$

5.2 A Representation of the Associated Z-algebra

We define the **Z**-operator algebra for $\hat{sl}(2)$ as before:

Definition 5.2.1. For each root $\pm\alpha$ of $sl_2(\mathbb{C})$, we define the functions

$$\mathbf{Z}^{\pm}(z) = \mathbf{Z}(\pm\alpha, z) = \sum_{m\in\mathbb{Z}} \mathbf{Z}^{\pm}(m) z^{-m}.$$

As introduced in Section 2.4, we have algebra $\mathcal{Z}(2)$ of these operators $\mathbf{Z}^{+}(m)$ and $\mathbf{Z}^{-}(m)$, $\forall\ m \in \mathbb{Z}$, defined by the generating functions

$$
\begin{aligned}
\mathbf{Z}^{+}(z) &= {:}E^{-}(z)X(z){:} = E^{-}_{+}(z)X(z)E^{-}_{-}(z), \\
\mathbf{Z}^{-}(z) &= {:}E^{+}(z)Y(z){:} = E^{+}_{+}(z)Y(z)E^{+}_{-}(z).
\end{aligned}
$$

Additionally, for the critical level, we determined in Theorem 2.4.11 defining relations for the **Z**-algebra associated with $\hat{sl}_2(\mathbb{C})$. Namely,

$$\left[\left[\mathbf{Z}^{\pm}(z), \mathbf{Z}^{\pm}(w)\right]\right] = 0, \qquad (5.2.1)$$

$$\left[\left[\mathbf{Z}^{+}(z), \mathbf{Z}^{-}(w)\right]\right] = H(0)\delta\left(\frac{w}{z}\right) - 2w\partial_w \delta\left(\frac{w}{z}\right). \qquad (5.2.2)$$

Theorem 5.2.2. *We define a representation of the* $\boldsymbol{Z}$*-algebra on the space* $\Omega_{\hat{V}} = \Lambda(V) \otimes \mathbb{C}\left[\mathbb{Z}\frac{\alpha}{2}\right]$ *via the following map (supressing tensors):*

$$\boldsymbol{Z}^{+}(z) \mapsto A(z)e^{\alpha}z^{\frac{-\alpha}{2}} \qquad \textit{and} \qquad \boldsymbol{Z}^{-}(z) \mapsto A^{*}(z)e^{-\alpha}z^{\frac{\alpha}{2}}$$

Proof. We will first show that (5.2.1) holds for $\mathbf{Z}^+$. The argument for $\mathbf{Z}^-$ is similarly made.

$$\begin{aligned}\mathbf{Z}^+(z)\mathbf{Z}^+(w) &\mapsto A(z)e^{\alpha}z^{\frac{-\alpha}{2}}A(w)e^{\alpha}w^{\frac{-\alpha}{2}} = A(z)A(w)e^{2\alpha}z^{-1}(zw)^{\frac{-\alpha}{2}} \\ &= \left(:A(z)A(w): + \underbrace{A(z)A(w)}\right)e^{2\alpha}z^{-1}(zw)^{\frac{-\alpha}{2}} \\ &= :A(z)A(w): e^{2\alpha}z^{-1}(zw)^{\frac{-\alpha}{2}}\end{aligned}$$

$$\begin{aligned}\big[\![\mathbf{Z}^+(z), \mathbf{Z}^+(w)]\!\big] &= \mathbf{Z}^+(z), \mathbf{Z}^+(w)\left(1-\frac{w}{z}\right)^{-1} - \mathbf{Z}^+(w), \mathbf{Z}^+(z)\left(1-\frac{z}{w}\right)^{-1} \\ &\mapsto :A(z)A(w): e^{2\alpha}z^{-1}(zw)^{\frac{-\alpha}{2}}\left(1-\frac{w}{z}\right)^{-1} \\ &\quad - :A(w)A(z): e^{2\alpha}w^{-1}(zw)^{\frac{-\alpha}{2}}\left(1-\frac{z}{w}\right)^{-1} \\ &= :A(z)A(w): e^{2\alpha}(zw)^{\frac{-\alpha}{2}}\left(\frac{1}{z-w}+\frac{1}{w-z}\right) \\ &= :A(z)A(w): e^{2\alpha}(zw)^{\frac{-\alpha}{2}}w^{-1}\delta\left(\frac{w}{z}\right) = 0, \text{ as needed, using (5.1.5).}\end{aligned}$$

Similarly, we show (5.2.2) using contractions in (5.1.6).

$$\begin{aligned}\mathbf{Z}^+(z)\mathbf{Z}^-(w) &\mapsto A(z)e^{\alpha}z^{\frac{-\alpha}{2}}A^*(w)e^{-\alpha}w^{\frac{\alpha}{2}} = A(z)A^*(w)\left(\frac{w}{z}\right)^{\frac{\alpha}{2}}z \\ &= \left(:A(z)A^*(w): + \underbrace{A(z)A^*(w)}\right)\left(\frac{w}{z}\right)^{\frac{\alpha}{2}} \\ &= \left(:A(z)A^*(w): - \frac{2zw}{(z-w)^3}\right)z\left(\frac{w}{z}\right)^{\frac{\alpha}{2}} \\ \text{Similarly, } \mathbf{Z}^-(w)\mathbf{Z}^+(z) &\mapsto \left(- :A(z)A^*(w): - \frac{2zw^2}{(w-z)^3}\right)\left(\frac{w}{z}\right)^{\frac{\alpha}{2}}\end{aligned}$$

$$\text{Hence, } \big[\![\mathbf{Z}^+(z), \mathbf{Z}^-(w)]\!\big] = H(0)\delta\left(\frac{w}{z}\right) - 2w\partial_w\delta\left(\frac{w}{z}\right).$$

This result is clear once we notice that $-2w\partial_w\left[\delta\left(\frac{w}{z}\right)\right]\left(\frac{w}{z}\right)^{\frac{\alpha}{2}} = H(0)\delta\left(\frac{w}{z}\right) - 2w\partial_w\delta\left(\frac{w}{z}\right)$, using properties of the derivative of the delta function, and also the fact that $H(0) = \alpha$. Then, because relations (5.2.1) and (5.2.2) hold and $C = -2$, we do in fact have a **Z**-algebra representation for $\hat{sl}_2(\mathbb{C})$ at the critical level. $\square$

We know from Section 2.4 (and [21]) that $\mathcal{Z}$ acts strictly on $\Omega_{\hat{V}}$, the vacuum subspace of

$\hat{V}$. Hence, considering the action of $\mathcal{Z}(2)$,

$$\Omega_{\hat{V}} = \Lambda(V)_{\frac{1}{2}} \otimes \mathbb{C}\left[\mathbb{Z}\frac{\alpha}{2}\right].$$

Thus, we conclude that the character of the vacuum space follows from (5.1.12), giving

$$ch(\Omega_{\hat{V}}) = 2\prod_{m>0}(1+q^m)^2 \sum_{p\in\mathbb{Z}} q^{\frac{p^2}{2}}.$$

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