

ABSTRACT

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(Under the direction of Dr. Ernest Stitzinger.)

Giovanni Frattini introduced his subgroup, now called the Frattini subgroup, in the nineteenth century. It has inspired investigations since it appeared. Besides being studied in its own right, it has been generalized in group theory, transformed into what is the Jacobson radical in ring theory and copied in Lie and other algebras. In the present work, we make further contributions to the theory of the Frattini subalgebra and introduce Lie algebra generalizations, which have group theory counterparts.

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Generalizations and Analogs of the Frattini Subalgebra

by
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Chapter 1

Introduction

A major subject of study in algebra is the intersection of all maximal “things”. In groups, Giovanni Frattini investigated the intersection of all maximal subgroups, now called the Frattini subgroup. The Frattini subgroup, denoted $Frat(G)$, for a group G has been continually investigated since Frattini published his results in a paper in 1885. In the first half of the 20th century various authors investigated this concept in group theory. Two of the main contributors to this work were Wolfgang Gaschütz and Bertram Huppert.

In the 1960’s Frattini subgroup properties were considered in Lie algebras. E.I. Marshall began investigating the intersection of all maximal subalgebras in Lie algebras in [20]. In Lie algebras, the intersection of all maximal subalgebras is called the Frattini subalgebra, denoted $F(L)$ for a Lie algebra L . Unlike the Frattini subgroup, which is always normal, the Frattini subalgebra is not always an ideal. Thus, we denote the Frattini ideal, $\phi(L)$, to be the largest ideal of L contained in $F(L)$. Several of the main contributors to this work are Donald Barnes, Ernest Stitzinger, and David Towers.

At the same time that the Frattini subgroup was considered in Lie algebras, generalizations of the Frattini subgroup were introduced in group theory. A central theme in this paper is finding correlations in Lie algebras to known facts in group theory. More specifically, we have combined these two ideas and investigated generalizations and analogs of the Frattini subalgebra in Lie algebras.

Chapter 2

Preliminaries

To begin, all Lie algebras that are investigated in this paper will be finite dimensional. We will first introduce definitions and notation that will be used throughout this paper. Using the same notation as David Towers in [33], the Frattini subalgebra, $F(L)$, is the intersection of all maximal subalgebras in a Lie algebra L . As this is not always an ideal, we refer to the Frattini ideal, $\phi(L)$, as the largest ideal of L contained in $F(L)$. The only known example where the Frattini subalgebra is not an ideal is as follows: For the 3-dimensional Lie algebra $L = \langle x, y, z \rangle$ over the field of 2 elements with multiplication $[x, y] = z, [y, z] = x, [z, x] = y$, $F(L) = \langle x + y + z \rangle$, which is not an ideal. In this example, $\phi(L) = 0$.

Throughout this paper we will have the occasion to use three very important series for L , namely

- (i) the derived series for L is the chain of subalgebras

$$L = L^{(1)} \supseteq L^{(2)} \supseteq L^{(3)} \supseteq \dots \supseteq L^{(c)} \supseteq L^{(c+1)} \supseteq \dots$$

where $L^{(m)} = [L^{(m-1)}, L^{(m-1)}]$. L is solvable if the series terminates.

(ii) the lower central series for L is the chain of ideals

$$L = L^1 \supset L^2 \supset L^3 \supset \cdots \supset L^c \supset L^{c+1} \supset \cdots$$

where $L^i = [L^{i-1}, L]$. L is nilpotent if the series terminates. L is nilpotent of length s if $L^{s+1} = 0$.

(iii) the upper central series for L is the chain of ideals

$$Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_c \subseteq Z_{c+1} \subseteq \cdots$$

where $Z_0 = \{0\}$, $Z_1 = Z(L)$, and Z_i/Z_{i-1} is the center of L/Z_{i-1} . The hypercenter of L , denoted $Z^*(L)$, is the terminal member of this series.

In [7], Beidleman and Seo generalized some of the fundamental properties of the Frattini subgroup of a finite group. They defined a generalized Frattini subgroup to be any proper normal subgroup H of a group G such that $G = HN_G(P)$ implies $G = N_G(P)$ for any Sylow p -subgroup P of a normal subgroup K of G . They showed that the generalized Frattini subgroup satisfies the property: If G is a finite group with A and B normal subgroups in G such that $B \subset \text{Frat}(G)$ and A/B nilpotent, then A is nilpotent. In Chapter 3, using the correspondence between the Frattini subalgebra and the Frattini subgroup, we show the Lie algebra analogs to these group theory properties. As we do not have Sylow p -subgroups in Lie algebras, we must introduce the use of Cartan subalgebras to play this role. A subalgebra C of a Lie algebra L is called a Cartan subalgebra of L if C is nilpotent and $N_L(C) = C$. Following this definition, a proper ideal H in L is

generalized Frattini in L if for each ideal K in L and each Cartan subalgebra C of K , whenever $L = H + N_L(C)$, then $L = N_L(C)$. We are able to show that a generalized Frattini subalgebra also satisfies the equivalent property in Lie algebras, often referred to as Barnes' Theorem. We also show relationships of the nilpotent radical, the unique nilpotent ideal which contains all nilpotent ideals of L , denoted $Nil(L)$, and the radical, the maximal solvable ideal containing all solvable ideals of a Lie algebra L , denoted $Rad(L)$, to generalized Frattini subalgebras.

The second section of Chapter 3 deals with two ideals that are closely related to the Frattini ideal. $R(L)$ is the intersection of all maximal subalgebras of L which are also ideals of L , putting $R(L) = L$ if no such maximal subalgebras exists. The subalgebra $T(L)$ is the intersection of all maximal subalgebras of L which are not ideals of L , again putting $T(L) = L$ if no such maximal subalgebras exist, and $\tau(L)$ will be the largest ideal of L contained in $T(L)$. The first four lemmas are results of Towers in [34]. Towers shows relationships between these new ideals and the Frattini ideal of L . We use these results to show that $\tau(L)$ is generalized Frattini in L .

In Chapter 4, we continue to look at the theory of generalized Frattini subalgebras based off the work done by Beidleman in [8]. Similar to Beidleman, we break this section up into three sections. The first section involves the property $\mathcal{M}(L)$. We define a proper ideal of a K Lie algebra L to satisfy property $\mathcal{M}(L)$ if and only if it satisfies the three properties $\phi(L/K) = 0$, L/K contains a unique minimal ideal, and $\dim(L/K) > 1$. The second section we look at the core of a maximal subalgebra that is not an ideal in a solvable Lie algebra. For a maximal subalgebra M that is not an ideal, we call the core of M , denoted $core(M)$, the largest ideal of L that is contained in M . The last section

we look at generalized Frattini in A -Lie algebras. An A -Lie algebra is a Lie algebra in which all nilpotent subalgebras are abelian.

Chapter 5 contains Lie algebra analogs to the group results of Kappe and Kirtland in [16]. We go back to the ideal $R(L)$ that we briefly investigated in Chapter 3 and add to this a new ideal that is also closely related to the Frattini ideal. The ideal, $nFrat(L)$, is the intersection of all maximal ideals of L . We find characterizations of $nFrat(L)$ and $R(L)$ by non-generators, find to what extent our analogs are nilpotent, and find containment relations and a characterization of nilpotency.

Moore and Rodrigues investigated the Frattini extension in groups in their paper [22]. They find several conditions for when an extension is a Frattini extension as well as look at results when an extension is a Frattini extension. For two Lie algebras N and H , an extension of N by H is a Lie algebra L that has an ideal $K \cong N$ and $L/K \cong H$. We denote this extension by (L, ϵ) . We call an extension (L, ϵ) a Frattini extension if the kernel of ϵ is contained in the Frattini ideal of L . In Chapter 6, we find similar results to that of Moore and Rodrigues for Frattini extensions in Lie algebras.

There is still work being done on the Frattini subgroup itself. A group G is called elementary if the Frattini subgroup of each subgroup of G is the identity. Following this definition, a Lie algebra L is elementary if the Frattini ideal of each subalgebra of L is 0. Elementary Lie algebras have been investigated by several authors. A group (Lie Algebra) is minimal non-elementary if it is not elementary but each of the proper subgroups (subalgebras) is elementary. Kirtland has shown in [17] that a finite group is minimal non-elementary if and only if G is either cyclic of order p^2 , p any prime, or

a non-abelian p -group of order p^3 , p an odd prime. Hence all minimal non-elementary finite groups are p -groups. The analogous concept in Lie algebras admits non-nilpotent examples. We find all such finite dimensional Lie algebras with nilpotent derived algebra over an algebraically closed field in Chapter 7.

Chapter 3

Generalized Frattini

3.1 General Properties

The purpose of this chapter is to generalize some of the fundamental properties of the Frattini subalgebra of a finite dimensional Lie algebra. Barnes shows that the Frattini ideal satisfies the following property. We shall call it

Barnes' Theorem. *Let A and B be ideals of L such that $B \subseteq A \cap F(L)$ then if A/B is nilpotent then A is nilpotent.*

We want to investigate other ideals that have this property. A generalized Frattini ideal is such an ideal that satisfies Barnes' Theorem. Similar to in the group theory case, we are able to show that every proper ideal of a nilpotent Lie algebra L is generalized Frattini in L .

Theorem 1. *Let H be generalized Frattini in a Lie algebra L . Then*

1. *H is nilpotent,*
2. *An ideal of L contained in H is generalized Frattini,*

3. $H + \phi(L)$ is generalized Frattini,
4. $H + Z(L)$ is generalized Frattini, whenever it is a proper subalgebra.

Proof:

1. Let C be a Cartan subalgebra of H and H be generalized Frattini. Then $H + N_L(C) = L$ by Barnes [5]. Since H is generalized Frattini $N_L(C) = L$. Hence $N_H(C) = H$. Since C a Cartan subalgebra of H , $N_H(C) = C$. Thus $H = C$. Since C is nilpotent, H is nilpotent.
2. Let N be an ideal of L such that N is contained in H . Let K be an ideal of L with C , Cartan subalgebra of K , such that $N + N_L(C) = L$. But $L = H + N_L(C)$, hence $L = N_L(C)$. So N is generalized Frattini.
3. Let $\phi(L)$ be the Frattini ideal. Suppose that K an ideal of L with Cartan subalgebra, C such that $H + \phi(L) + N_L(C) = L$. Suppose M is a maximal ideal with $H + N_L(C) \subseteq M$. But $\phi(L) \subseteq M$ which contradicts $H + \phi(L) + N_L(C) = L$. So no such M exists and $H + N_L(C) = L$. Since H is generalized Frattini, this implies $N_L(C) = L$. Thus $H + \phi(L)$ is generalized Frattini.
4. Consider $H + Z(L) + N_L(C) = L$. Since $Z(L)$ is contained in every normalizer, $Z(L) \subseteq N_L(C)$. So $H + Z(L) + N_L(C) = H + N_L(C) = L$. Since H is generalized Frattini, $N_L(C) = L$. Thus $H + Z(L)$ is generalized Frattini. \square

As a consequence of Theorem 1 we obtain the following.

Corollary 1. *The Frattini ideal, $\phi(L)$ and the center, $Z(L)$ are generalized Frattini in L .*

Proof: The proof is the same as in Theorem 1 parts (3) and (4) and take $H = 0$. \square

Theorem 2. *Let H be generalized Frattini in L . If K is an ideal of L and K/H nilpotent, then K is nilpotent.*

Proof: Let C be Cartan in K with $H \subseteq K$. Then $C + H/H$ is Cartan in K/H . By definition $N_{K/H}(C+H/H) = C+H/H$. Since K/H is nilpotent, $K/H \subseteq N_{K/H}(C+H/H)$ and $N_{K/H}(C + H/H) \neq C + H/H$ unless $K/H = C + H/H$. So $K/H = C + H/H$. Hence $K = C + H$. By Barnes, $L = N_L(C) + K = N_L(C) + C + H = N_L(C) + H$. Since H is generalized Frattini, $L = N_L(C)$ and $K = N_K(C) = C$ is nilpotent. \square

Corollary 2. *Let H be generalized Frattini in L . Then L is nilpotent if and only if L/H is nilpotent.*

Corollary 3. *A Lie algebra L is nilpotent if and only if L' is generalized Frattini.*

Proof: Suppose L' is generalized Frattini. Then L/L' is nilpotent. By Theorem 2, L is nilpotent.

Conversely, suppose that L is nilpotent. Then for any ideal H of L and $L' \subseteq H$, H/L' nilpotent implies that H is nilpotent. Thus L' is generalized Frattini. \square

Theorem 3. *Let H be generalized Frattini in L . If K is an ideal in L such that $K^\omega \subseteq H$ then $K^\omega = 0$ and K is nilpotent.*

Proof: Since $K^\omega \subseteq H$ then K/H is nilpotent. Thus K is nilpotent by Theorem 2, and hence $K^\omega = 0$. \square

Corollary 4. *A proper ideal K of L is nilpotent if and only if its commutator subalgebra K' is generalized Frattini in L .*

Proof: If K' is generalized Frattini then K/K' is nilpotent. Thus K is nilpotent.

Conversely, suppose K is nilpotent. Then $K' = \phi(K) \subseteq \phi(L)$. If J is an ideal of L such that J/K' is nilpotent, then $(J + \phi(L))/\phi(L)$ is nilpotent. Thus J is nilpotent, and hence K' is generalized Frattini. \square

The next theorem is an equivalent definition for generalized Frattini in place of using Cartan subalgebras.

Theorem 4. *Let H be an ideal in L . H is generalized Frattini in L if and only if for all ideals J in L such that H is contained in J and J/H nilpotent then J is nilpotent.*

Proof: Let H be generalized Frattini in L . By Theorem 2 if $H \subseteq J$ and J/H nilpotent, then J is nilpotent and the condition holds.

Conversely, suppose the condition holds. Let K be an ideal of L and C a Cartan subalgebra of K such that $L = N_L(C) + H$. Then $H + C/H$ is Cartan in $K + H/H$ and $C + H/H$ is an ideal in L/H since $L = N_L(C) + H$ and $[N_L(C) + H, C + H] \subseteq C + H$. Therefore, $H + C/H$ is an ideal in $K + H/H$. But $N_{K/H}(C + H/H) = C + H/H = K + H/H$. Therefore, $K + H/H$ is nilpotent, and so $K + H$ is nilpotent. Thus K is nilpotent and $C = K$. Hence $N_L(C) = N_L(K) = L$. Therefore, H is generalized Frattini. \square

Consequently, we get the following.

Corollary 5. *All proper ideals in nilpotent Lie algebras are generalized Frattini.*

We now give an example to illustrate the use of our new definition for generalized Frattini.

Example 1. Let $L = \langle x, y, z \rangle$ and bracket structure $[z, x] = x$, $[z, y] = y$, and $[x, y] = 0$. Then the ideals $H = \langle x \rangle$ and $K = \langle y \rangle$ are generalized Frattini in L .

Consider the ideal $J = \langle x, y \rangle$. Then $J^1 = [J, J] = 0$ so J is nilpotent and $J/H = \langle y \rangle$ is nilpotent. Thus by Proposition 4, or our new definition, $H = \langle x \rangle$ is generalized Frattini. Similarly, for $K = \langle y \rangle$.

Remark 1. The sum of generalized Frattini subalgebras of L may not be generalized Frattini.

Example 2. Let $L = \langle x, y, z \rangle$ and bracket structure $[z, x] = x$, $[z, y] = y$, and $[x, y] = 0$. Then $H = \langle x \rangle$ and $K = \langle y \rangle$ are generalized Frattini but $H + K = \langle x, y \rangle$ is not.

Theorem 5. *Let H be generalized Frattini in L and let K be an ideal of L containing H . Then K/H is generalized Frattini in L/H if and only if K is generalized Frattini in L .*

Proof: Suppose that K is generalized Frattini in L . Let J/H be an ideal in L/H such that $\frac{J/H}{K/H}$ is nilpotent. $\frac{J/H}{K/H} \cong J/K$ so J/K is nilpotent. Since K is generalized Frattini this implies J is nilpotent. But then J/H is nilpotent. Thus by Proposition 4, K/H is generalized Frattini in L/H .

Conversely, suppose that K/H is generalized Frattini in L/H . Let J be an ideal in L such that J/K is nilpotent. Since $\frac{J/H}{K/H} \cong J/K$, $\frac{J/H}{K/H}$ is nilpotent. So by Proposition 4, J/H is nilpotent. So then J is nilpotent since H is generalized Frattini. Again by Proposition 4, K is generalized Frattini in L . \square

Recall that the nilpotent radical, $Nil(L)$, is the unique nilpotent ideal which contains all nilpotent ideals of L and that the radical, $Rad(L)$, is the maximal solvable ideal

containing all solvable ideals of a Lie algebra L . The next theorems involve the nilpotent radical and the radical as well as provide a necessary and sufficient condition for the nilpotent radical to be generalized Frattini in L .

Theorem 6. *If $Nil(L)$ is generalized Frattini, then every solvable ideal of L is nilpotent.*

Proof: Let H be a solvable ideal of L and let k be the smallest positive integer such that $H^{(k+1)} = 0$. So $H^{(k)}$ is abelian. Thus $H^{(k)}$ is nilpotent and so contained in $Nil(L)$. Since $H^{(k)}$ is contained in $Nil(L)$, $H^{(k)}$ is generalized Frattini by Theorem 1. Since $H^{(k-1)}/H^{(k)}$ is abelian, by Theorem 2, $H^{(k-1)}$ is nilpotent. Hence $Nil(L)$ contains $H^{(k-1)}$. Continuing in this way we get $H^{(1)}$ is contained in $Nil(L)$. By Theorem 1, $H^{(1)}$ is generalized Frattini. So then $H/H^{(1)}$ is nilpotent and $H^{(1)}$ is generalized Frattini and by Theorem 2, this implies H is nilpotent. \square

Corollary 6. *If $Nil(L)$ is generalized Frattini in L then L can not be solvable.*

Proof: If L is solvable, then by Theorem 6, L is nilpotent and $L = Nil(L)$. Then $Nil(L)$ is not a proper ideal so it can not be generalized Frattini. \square

Example 3. Let $L = \langle x, y, z \rangle$ and bracket structure $[z, x] = x, [z, y] = y$, and $[x, y] = 0$. L is solvable and $Nil(L) = \langle x, y \rangle$ as we have already seen is not generalized Frattini.

Theorem 7. *If H is generalized Frattini in L , then $Nil(L/H) = Nil(L)/H$.*

Proof: Since H is generalized Frattini in L , H is nilpotent so $H \subseteq Nil(L)$. Then $Nil(L)/H$ is nilpotent, so $Nil(L)/H \subseteq Nil(L/H)$. On the other hand, if $B/H =$

$Nil(L/H)$, then B is nilpotent by Theorem 2. Then $B \subseteq Nil(L)$ and $B/H \subseteq Nil(L)/H$. Hence the result holds. \square

Example 4. Let $L = \langle x, y, z \rangle$ and bracket structure $[z, x] = x, [z, y] = y$, and $[x, y] = 0$. Let $H = \langle x, y \rangle$. Then the left-hand side gives $Nil(L/H) = z + H = L$ and the right-hand side gives $Nil(L)/H = 0$.

Theorem 8. *Let L be non-nilpotent. If $Rad(L) = Nil(L)$, then $Nil(L)$ is generalized Frattini in L .*

Proof: Let N be an ideal of L containing $Nil(L)$ such that $N/Nil(L)$ is nilpotent. Then $N/Nil(L)$ is solvable and N is solvable. Hence $N = Nil(L) = Rad(L)$. Then N is nilpotent. Therefore, $Nil(L)$ is generalized Frattini in L by Propostion 4. \square

Theorem 9. *Let L be non-nilpotent. $Nil(L)$ is generalized Frattini in L if and only if $Nil(L) = Rad(L)$.*

Proof: If $Nil(L) = Rad(L)$ then by Theorem 8, $Nil(L)$ is generalized Frattini. If $Nil(L)$ is generalized Frattini in L , then $Rad(L)$ is nilpotent by Theorem 6. Hence $Rad(L) = Nil(L)$. \square

Example 5. Let $L = gl(2, F)$. Then $Z(L) = \{\alpha I | \alpha \in F\} = Nil(L) = Rad(L)$. Thus $Nil(L)$ is generalized Frattini.

If $x \in L$, then let

$$L_0 = \{y \in L | y(ad^n x) = 0 \text{ for some } n\}$$

$$L_1 = \{y \in L | \text{for each } n, \exists z_n \in L \ni z_n(ad^n x) = y\}$$

L_0 is the Fitting null component of L and x acts nilpotently on L_0 . L_0 is also a subalgebra of L . L_1 is the Fitting one component of L and x acts on L non-singularly. The following give rise to our next theorem which is also an equivalent definition for generalized Frattini.

Theorem 10. *Let H be an ideal of L . H is generalized Frattini in L if and only if for each ideal K in L and each Cartan subalgebra C of K such that $L = H + L_0(C)$ then $L_0(C) = L$.*

Proof: Assume H satisfies the conditions and let K be an ideal in L such that K/H is nilpotent. Let C be a Cartan subalgebra of K . Then $L_1(C) \subseteq H$ so $L = K + L_0(C) = H + C + L_0(C) = H + L_0(C)$. Hence $L = L_0(C)$ and $K = K_0(C) = C$. Therefore, K is nilpotent and H is generalized Frattini in L by Proposition 4.

Conversely, let H be generalized Frattini in L . Let K be an ideal in L and C be a Cartan subalgebra in K such that $L = L_0(C) + H$. Then $C + H/H$ is a Cartan subalgebra in $K + H/H$. Now C acts nilpotently on L/H , so also on $K + H/H$. Thus $C + H/H$ acts nilpotently on $K + H/H$. Therefore, $C + H/H = K + H/H$ and $K + H/H$ is nilpotent. Hence $K + H$ is nilpotent by Proposition 4, and K is nilpotent and $C = K$. Hence $L_0(C) = L$ and the condition holds. \square

From this equivalent definition it is easy to see the following result.

Corollary 7. *The hypercenter $Z^*(L)$ is generalized Frattini in L .*

Proof: By definition, $Z^*(L)$ is contained in $L_0(C)$ for any nilpotent subalgebra C of L . Hence the result follows from Theorem 10. \square

3.2 $R(L)$ and $\tau(L)$

In this section we look at generalized Frattini in regards to two ideals that are closely related to the Frattini ideal. The first four lemmas are from Towers in [34]. Towers uses the notation $\sigma(L)$ for $R(L)$. We will discuss $R(L)$ in further detail in Chapter 5.

Recall that $R(L)$ is the intersection of all maximal subalgebras of L which are also ideals of L , putting $R(L) = L$ if no such maximal subalgebras exists. The subalgebra $T(L)$ is the intersection of all maximal subalgebras of L which are not ideals of L , again putting $T(L) = L$ if no such maximal subalgebras exist, and $\tau(L)$ will be the largest ideal of L contained in $T(L)$. We look for properties that lend these subalgebras to be generalized Frattini in L .

Lemma 1. *(i) $F(L) = R(L) \cap T(L)$; (ii) $\phi(L) = R(L) \cap \tau(L)$.*

Lemma 2. $L^2 \subseteq R(L)$.

Lemma 3. *If $\phi(L) = 0$, then $\tau(L) = Z(L) = Z^*(L)$*

Lemma 4. $Z^*(L) \cap L^2 \subseteq \phi(L)$

Theorem 11. *$\tau(L)$ is generalized Frattini in L .*

Proof: Theorem 3 gives $\tau(L)/\phi(L) = Z(L/\phi(L))$. Since $\phi(L)$ is generalized Frattini in L and $Z(L/\phi(L))$ is generalized Frattini in $L/\phi(L)$, $\tau(L)$ is generalized Frattini in L

by Theorem 5. □

Theorem 12. *A maximal generalized Frattini ideal H of a non-nilpotent Lie algebra L contains $\tau(L)$.*

Proof: Let H be maximal generalized Frattini in L . Then $H + \phi(L) = H$ by Theorem 1. Also $H + \tau(L)/\phi(L) = H/\phi(L) + \tau(L)/\phi(L) = H/\phi(L) + Z(L/\phi(L))$ is generalized Frattini in $L/\phi(L)$ by Theorem 1. Hence $H + \tau(L)$ is generalized Frattini in L . Since H is maximal, $\tau(L) \subseteq H$. □

Theorem 13. *A maximal generalized Frattini ideal H of a non-nilpotent Lie algebra L contains $Z^*(L)$.*

Proof: $(Z^*(L) + \phi(L))/\phi(L) \subseteq Z^*(L/\phi(L)) = \tau(L)/\phi(L)$. Hence $Z^*(L) \subseteq \tau(L) \subseteq H$. □

Chapter 4

More Generalized Frattini

4.1 Special Type of Generalized Frattini

In this section we will continue to look at the theory of generalized Frattini in a Lie algebra. We will specifically find results of ideals that satisfy the property $\mathcal{M}(L)$. If a proper ideal K of L satisfies property $\mathcal{M}(L)$, then we denote this fact by $K \in \mathcal{M}(L)$. Similar to Beidleman in the group theory case in [8], if L is a solvable Lie algebra and $K \in \mathcal{M}(L)$, then K is generalized Frattini in L if and only if K is properly contained in $Nil(L)$.

We recall that a non-trivial ideal H of L is called a minimal ideal of L if it contains no proper non-trivial ideals of L . A proper ideal of a K Lie algebra L is said to satisfy property $\mathcal{M}(L)$ if and only if $\phi(L/K) = 0$, L/K contains a unique minimal ideal, and $\dim(L/K) > 1$.

Lemma 5. *Let K be a proper ideal of L such that K is in $\mathcal{M}(L)$. Then K contains*

$\phi(L)$ and L/K is non-nilpotent. In particular, L is non-nilpotent

Proof: Since $\phi(L/K) = 0$, it follows that K contains $\phi(L)$. Suppose that L/K is nilpotent and let A/K be the unique minimal ideal of L/K . Since L/K is nilpotent, A/K is abelian. By [29] (Theorem 3), it follows that $L/K = \text{Nil}(L/K) = A/K$, and therefore L/K is abelian. Hence, $\dim(L/K) = 1$, which is impossible. Therefore, L/K is non-nilpotent. \square

Theorem 14. *Let L be a solvable Lie algebra and let K be in $M(L)$. Then K is generalized Frattini if and only if K is a proper subalgebra of $\text{Nil}(L)$.*

Proof: Let A/K denote the unique minimal ideal of L/K . Since L is solvable, $\text{Nil}(L/K) = A/K$. Suppose that K is generalized Frattini in L . By Theorem 7, $\text{Nil}(L)/K = A/K$. Hence, $\text{Nil}(L) = A$, and so K is a proper subalgebra of $\text{Nil}(L)$.

Conversely, let K be a proper subalgebra of $\text{Nil}(L)$. Then $\text{Nil}(L)/K = A/K = \text{Nil}(L/K)$. Let H be an ideal of L such that H contains K and H/K is nilpotent. Then $H/K \subseteq \text{Nil}(L/K) = \text{Nil}(L)/K$, so $H \subseteq \text{Nil}(L)$. By Theorem 4, K is generalized Frattini in L . \square

Theorem 15. *Let L be a solvable Lie algebra and let K be in $\mathcal{M}(L)$. Let A/K be the unique minimal ideal of L/K . Then K is generalized Frattini if and only if $A = \text{Nil}(L)$.*

Proof: Suppose K is generalized Frattini. Then $\text{Nil}(L/K) = \text{Nil}(L)/K$ by Theorem 7. Since L solvable, $\text{Nil}(L/K) = A/K$. Then $A/K = \text{Nil}(L/K) = \text{Nil}(L)/K$ implies $A = \text{Nil}(L)$.

Conversely, suppose $A = \text{Nil}(L)$. Since L is solvable, $\text{Nil}(L/K) = A/K$. Thus $\text{Nil}(L/K) = A/K = \text{Nil}(L)/K$. But then K is a proper subalgebra of $\text{Nil}(L)$. Thus K

is generalized Frattini by Theorem 14. □

Corollary 8. *Let L be a solvable Lie algebra and let K be in $\mathcal{M}(L)$. If K is generalized Frattini, then K is a maximal generalized Frattini subalgebra of L .*

Proof: Let H be generalized Frattini such that $K \subseteq H$. By Theorem 1, H is nilpotent, so $H \subseteq \text{Nil}(L)$. Now let A/K be the unique minimal ideal in L/K . By Theorem 15, $A = \text{Nil}(L)$. Therefore, $H = K$ or $H = \text{Nil}(L)$. Suppose that $H = \text{Nil}(L)$. Then by Theorem 6, every solvable ideal of L is nilpotent. However, L is solvable, so nilpotent. This contradicts that L must be non-nilpotent by Lemma 5. Thus K is maximal. □

4.2 Core of a Solvable Lie Algebra

In this section we will look at the core of a maximal subalgebra that is not an ideal. The core of a maximal subalgebra M is the largest ideal of L that is contained in M . We will denote the core of M to be $\text{core}(M)$. The results in this section resemble the results in the group theory case with little differences.

Theorem 16. *Let L be a solvable Lie algebra and let M be a maximal subalgebra that is not an ideal. Let $K = \text{core}(M)$. Then K is in $\mathcal{M}(L)$.*

Proof: Consider L/K and M/K and assume $K = 0$. Let A be a minimal ideal in L , $A \not\subseteq M$. So $\dim(M + A) > \dim M$ this implies $M + A = L$.

Claim: $B = A \cap M = 0$

Proof of claim: Suppose $B \neq 0$. Since $B \subseteq M$, $[B, A] \subseteq [A \cap M, A] = 0$ and

$[B, M] \subseteq [A \cap M, M] \subseteq A \cap M$. This implies $[B, A + M] \subseteq 0 + A \cap M = B$ so B is an ideal in L . Thus $B = 0$. Therefore, $L = M \oplus A$.

Suppose that C is another minimal ideal. Then $\dim(A \oplus C) > \dim A$ implies $(A \oplus C) \cap M \neq 0$. For the same reason as above, $(A \oplus C) \cap M$ is an ideal in L contained in M . But then $(A \oplus C) \cap M = 0$, which is a contradiction. So there is no C , and A must be unique. Thus condition 2 of the definition holds. Now $\dim L = \dim A + \dim M > 1$, so condition 3 of the definition holds. Lastly, $\phi(L)$ is an ideal of L . Since L is solvable and $\phi(L) \subseteq M$, this implies $\phi(L) = 0$. Thus condition 1 of the definition holds and K is in $\mathcal{M}(L)$. \square

Theorem 17. *Let M be a maximal subalgebra of a solvable Lie algebra, L , that is not an ideal. Let $K = \text{core}(M)$. Then K is generalized Frattini if and only if K is a proper subalgebra of $\text{Nil}(L)$.*

Proof: By Theorem 16, $K = \text{core}(M) \in \mathcal{M}(L)$. Thus by Theorem 14, K is generalized Frattini if and only if $K \subsetneq \text{Nil}(L)$. \square

Corollary 9. *Let L be a solvable Lie algebra and let K be the core of a maximal subalgebra of L that is not an ideal. If K and L' are nilpotent, then K is generalized Frattini.*

Proof: Since L/K is non-abelian $L' \not\subseteq K$. Hence $K \subsetneq L' + K \subseteq \text{Nil}(L)$. Thus by Theorem 17, K is generalized Frattini. \square

Corollary 10. *Let L be a solvable Lie algebra and let M be a nilpotent maximal subalgebra in L that is not an ideal and whose core in L is K . If L' is nilpotent, then K is generalized Frattini.*

Proof: Since M is nilpotent then K is nilpotent. Thus $L' \not\subseteq K$. Hence $K \subsetneq L' + K \subseteq \text{Nil}(L)$. Thus by Theorem 17 K is generalized Frattini. \square

Theorem 18. *Let L be a supersolvable Lie algebra and let $K \in \mathcal{M}(L)$. If K is nilpotent, then K is generalized Frattini.*

Proof: Let L be supersolvable. Then L' is nilpotent. Since $K \in \mathcal{M}(L)$ and K is nilpotent, $L' \not\subseteq K$. Thus $K \subsetneq \text{Nil}(L)$, so by Theorem 14, K is generalized Frattini. \square

The assumption in Theorem 18 that L must be supersolvable cannot be omitted.

Example 6. Assume L is over a field F with $\text{Char}(F) = p$. Let E and F be elements of L whose effect on a basis $\{x_1, \dots, x_p\}$ for V is given by
 $Ex_i = x_{i+1} \pmod{p}$, $Fx_i = ix_i$

$$\begin{aligned}
[E, F]x_i &= EFx_i - FEx_i \\
&= iEx_i - Fx_{i+1} \\
&= ix_{i+1} - (i+1)x_{i+1} \\
&= -x_{i+1} \\
[F, E]x_i &= FEx_i - EFx_i \\
&= (i+1)x_{i+1} - ix_{i+1} \\
&= x_{i+1} \\
[F, E]x_p &= x_1 = Ex_p
\end{aligned}$$

Note: $[A + w, B + u] = [A, B] + Au - Bw$.

Let $L = [F, E] + V$. Let $H = \langle F \rangle + V$, which is self normalizing and maximal. Let

$K = \text{core}(H) = V$. Then $(E + K)/K$ is nilpotent. But $\langle E \rangle + K$ is not nilpotent since $E(x_1 + \cdots + x_p) = x_1 + \cdots + x_p$.

Corollary 11. *Let L be supersolvable and $K = \text{core}(M)$, where M is a nilpotent maximal subalgebra of L that is not an ideal. Then K is generalized Frattini.*

Proof: Let L be supersolvable. Then L' is nilpotent. Thus by Corollary 10, K is generalized Frattini. \square

Different from the group theory case, the next several theorems we must require that L be solvable. In the group theory case, since all subgroups of G are nilpotent when G is non-nilpotent, G is solvable.

Theorem 19. *Let L be a solvable, non-nilpotent Lie algebra all of whose proper subalgebras are nilpotent. Then $\phi(L)$ is the unique maximal generalized Frattini subalgebra of L .*

Proof: Let L be solvable and let K be a maximal generalized Frattini subalgebra of L . From Theorem 1, $\phi(L)$ is contained in K . Suppose that $\phi(L)$ is properly contained in K . Then there exists a maximal subalgebra M such that $L = K + M$. Then $L/K = (K + M)/K$ is nilpotent and thus L is nilpotent. This contradicts L being non-nilpotent. Therefore, $\phi(L) = K$. \square

Corollary 12. *Let L be a solvable, non-nilpotent Lie algebra all of whose proper subalgebras are nilpotent. If K is the core of a maximal subalgebra in L that is not an ideal, then $K = F(L) = \phi(L)$.*

Proof: Since L is solvable and L' is nilpotent, by Corollary 10, K is generalized Frattini. Since K contains $F(L)$, by Theorem 19, $K = F(L) = \phi(L)$. \square

Theorem 20. *Let L be a solvable, non-nilpotent Lie algebra such that every proper subalgebra of $L/\phi(L)$ is nilpotent. Then*

1. $L/\phi(L)$ is non-nilpotent;
2. $\phi(L)$ is the unique maximal generalized Frattini ideal of L ;
3. Every proper ideal of L is nilpotent.

Proof:

1. This was proved by D. W. Barnes using cohomology. I will offer a different proof.
Let $x \in L$ and L_0 be the Fitting null component of ad_x acting on L . If $L/\phi(L)$ is nilpotent, then $L_0 + \phi(L) = L$. This implies there exists a subalgebra that supplements $\phi(L)$, which is a contradiction. Thus ad_x is a nilpotent transformation for $x \in L$. This implies L is nilpotent by Engel's Theorem. Which contradicts L being non-nilpotent. Thus $L/\phi(L)$ is non-nilpotent.
2. Let K be a maximal generalized Frattini subalgebra of L . By Theorem 1, $\phi(L) \subseteq K$. Hence $K/\phi(L)$ is generalized Frattini of $L/\phi(L)$ by Theorem 5. By Theorem 19, $K = \phi(L)$. Therefore, $\phi(L)$ is unique maximal generalized Frattini.
3. Let H be an ideal of L . Then $(H + \phi(L))/\phi(L)$ is nilpotent. Since $\phi(L)$ is generalized Frattini, by Theorem 2, $H + \phi(L)$ is nilpotent. Hence, H is nilpotent. \square

4.3 Generalized Frattini in an A -Lie Algebra

This section will look at generalized Frattini subalgebras of A -Lie algebras. We use the fact that in an A -Lie algebra all nilpotent subalgebras are abelian. The results are analogous to those of Beidleman in the group theory case.

Lemma 6. *Let L be a finite dimensional solvable Lie algebra. Then $C_L(\text{Nil}(L)) \subseteq \text{Nil}(L)$.*

Proof: Since L is solvable, $L \supseteq L^{(1)} \cdots L^{(k)} = 0$. Let $D = C_L(\text{Nil}(L))$ and $D^{(i)}$ be the last term of the derived series of D not in $\text{Nil}(L)$. (i.e. $D^{(i)} \not\subseteq \text{Nil}(L)$, but $D^{(i+1)} \subseteq \text{Nil}(L)$.) Then $[D^{(i)} + \text{Nil}(L), D^{(i)}] \subseteq \text{Nil}(L)$. So $D^{(i)} + \text{Nil}(L)$ is nilpotent and larger than $\text{Nil}(L)$, which is a contradiction.

Theorem 21. *Let L be a solvable Lie algebra and let $K \in \mathcal{M}(L)$. Let H be a subalgebra of L which contains K and $\text{Nil}(H)$ is abelian. If K is generalized Frattini in H , then K is generalized Frattini in L .*

Proof: Let K be generalized Frattini in H . Then K is a nilpotent ideal of both H and L , hence $K \subseteq \text{Nil}(H)$ and $K \subseteq \text{Nil}(L)$. If L is nilpotent, then the result is clear, so assume that L is not nilpotent. If H is nilpotent, then $K \neq H = \text{Nil}(H)$ by the definition of generalized Frattini. If H is not nilpotent, then $K \neq \text{Nil}(H)$ using Theorem 9, since $\text{Nil}(H) \neq \text{Rad}(H)$. In either case, $K \subsetneq \text{Nil}(H)$. Suppose that $K = \text{Nil}(L)$. Then $\text{Nil}(L) = K \subsetneq \text{Nil}(H)$. Since $\text{Nil}(H)$ is abelian, $[\text{Nil}(L), \text{Nil}(H)] = 0$ and $\text{Nil}(H) \subseteq C_L(\text{Nil}(L)) \subseteq \text{Nil}(L) = K \subseteq \text{Nil}(H)$. Hence $K = \text{Nil}(H)$, which is a contradiction. Therefore, $K \subsetneq \text{Nil}(L)$. By Theorem 14, K is generalized Frattini in L . \square

Corollary 13. *Let L be solvable. Let H be an abelian maximal subalgebra that is not an ideal. Then $K = \text{core}(H)$ is generalized Frattini of L .*

Proof: K is generalized Frattini in H by Corollary 5. So $K \in M(L)$ by Theorem 17. Therefore, the result follows from Theorem 21. \square

Corollary 14. *Let L be solvable and let H be an abelian maximal subalgebra that is not an ideal. Then H does not contain $\text{Nil}(L)$.*

Proof: Suppose $\text{Nil}(L) \subseteq H$. Since $K = \text{core}(H)$ is the largest ideal of L contained in H , $\text{Nil}(L) \subseteq K$. Because of Theorem 21 and Theorem 1, $\text{Nil}(L)$ is generalized Frattini in L . This contradicts Theorem 6. Thus H does not contain $\text{Nil}(L)$. \square

Corollary 15. *Let L be a solvable Lie algebra and let $K \in \mathcal{M}(L)$. Let H be an A -Lie algebra of L containing K . If K is generalized Frattini in H , then K is generalized Frattini in L .*

Proof: Note: H an A -Lie algebra implies all nilpotent subalgebras of H are abelian. Suppose K is generalized Frattini in H and H an A -Lie algebra. Then $\text{Nil}(H)$ is abelian. Hence by Theorem 21, K is generalized Frattini in L . \square

Theorem 22. *Let L be an A -Lie algebra of nilpotent length two and let $K \in \mathcal{M}(L)$. Then K is generalized Frattini in L if and only if K is abelian.*

Proof: Suppose K is generalized Frattini in L . By Theorem 1, K is nilpotent and hence abelian as L is an A -Lie algebra.

Now suppose K is abelian. Let $T = \bigcap L^k$, the intersection of the lower central series of L . Since L has nilpotent length 2, then T is nilpotent. Thus $T \subseteq Nil(L)$. Suppose $K = Nil(L)$. Then $T \subseteq K$, so L/K is nilpotent. This contradicts Lemma 5, and so $K \subsetneq Nil(L)$. Therefore, K is generalized Frattini in L by Theorem 14. \square

Chapter 5

Analogs of the Frattini Subalgebra

5.1 Characterization by Non-generators

The present chapter contains Lie algebra analogs to the results in [16]. We investigate properties of $nFrat(L)$ and $R(L)$. We find characterizations of the concepts by non-generators, find to what extent our concepts are nilpotent, find containments relations and a characterization of nilpotency. All Lie algebras considered here are finite dimensional over a field, F . Recall that $nFrat(L)$ is the intersection of all maximal ideals of L and $R(L)$ is the intersection of all maximal subalgebras which are also ideals of L , putting $R(L) = L$ if no such maximal subalgebras exist.

The following is an example where $nFrat(L)$ is different from $R(L)$.

Example 7. Let $L = gl(n, F)$. If $\text{char}(F) = 0$, then there are two maximal ideals L^2 and $Z(L)$ and $L^2 \cap Z(L) = 0$. Then $nFrat(L) = \{0\}$ and $R(L) = L^2$.

For a Lie algebra L we define the following: (1) $\mathcal{M} = \{M | M \text{ maximal subalgebra of } L\}$,

(2) $\mathcal{N} = \{N | N \neq L, N \text{ maximal ideal of } L\}$, and (3) $\mathcal{R} = \{R | R \text{ in } \mathcal{M} \cap \mathcal{N}\} = \text{all maximal subalgebras that are ideals.}$

Lemma 7. *Let N be an ideal of L .*

1. $N \in \mathcal{N}$ if and only if L/N is simple.
2. $N \in \mathcal{R}$ if and only if $\dim(L/N) = 1$.

It is true in group theory that $\text{Frat}(G)$ is the set of non-generators. It is also known that this concept carries over for $n\text{Frat}(G)$ and $R(G)$, [16]. We are going to study the idea of non-generators for these concepts in Lie algebras. It is widely recognized that the Frattini subalgebra is equal to the set of non-generators. We will provide this proof as it fits with the rest.

We define a subset S of a Lie algebra L to be normal in L if $ad_x(S) \subseteq S$ for all $x \in L$. An element $x \in L$ is called a normal non-generator if $L = \langle x, T \rangle$ for a normal subset T in L implies $L = \langle T \rangle$.

Proposition 1. *$F(L)$ equals the set of non-generators.*

Proof: Let $x \in F(L)$. Let $L = \langle H, x \rangle$. If $H \neq L$, then $H \subseteq M$, where M is a maximal subalgebra. Also, $x \in F(L) \subseteq M$. Hence $\langle H, x \rangle = M$, which is a contradiction. So $H = L$ and x is a non-generator.

Now suppose $x \notin F(L)$. Let M be a maximal subalgebra such that $x \notin M$. Then $M \subset \langle x, M \rangle \subseteq L$, so $\langle x, M \rangle = L$. But $M \neq L$, so x is a generator for L . \square

Theorem 23. *$R(L)$ equals the set of all normal non-generators of L .*

Proof: Suppose $x \notin R(L)$. Then $x \notin M$, a maximal subalgebra that is an ideal. M is a normal subset of L and $L = \langle x, M \rangle$ but $L \neq M$. Hence x is a normal generator.

Now suppose $x \in R(L)$ and $L = \langle x, S \rangle$ for a normal subset S of L . If $\langle S \rangle \neq L$, then $\dim(L) = \dim(S) + 1$, so $\langle S \rangle \in \mathcal{R}$. But $x \in R(L) \subseteq \langle S \rangle$ so $\langle S \rangle = L$, which is a contradiction. \square

If we let X be a subset of L . Then $X^L = \langle [x, l_1, \dots, l_k] \rangle$ where $x \in X$ and $l_i \in L$ and $k = 0, 1, \dots$. We call an element $x \in L$ an n -nongenerator of L if for every subset X of L , $L = X^L$ whenever $L = \langle x, X \rangle^L$.

Lemma 8. For any element $g \in L$ any subset X of L , $\langle g, X \rangle^L = \langle g^L, X^L \rangle = g^L + X^L$.

Proof: Both g^L and X^L are contained in $\langle g, X \rangle^L$, so $\langle g^L, X^L \rangle \subseteq \langle g, X \rangle^L$ and $g^L + X^L \subseteq \langle g, X \rangle^L$. Since, $\langle g, X \rangle \subseteq \langle g^L, X^L \rangle$ this implies $\langle g, X \rangle^L \subseteq \langle g^L, X^L \rangle$. Also, $g^L \subseteq g^L + X^L$ and $X^L \subseteq g^L + X^L$, thus $\langle g^L, X^L \rangle \subseteq g^L + X^L$.

Alternate Proof: Let g and h be in L and X a subset of L with $x \in X$.

$$x^L = \sum a_i [x, h_{i_1}, h_{i_2}, \dots, h_{i_k}]$$

$$g^L = \sum b_i [g, h_{i_1}, h_{i_2}, \dots, h_{i_k}]$$

$$\begin{aligned} \langle g, x \rangle^L &= [\alpha g + \beta x, h_1, \dots, h_n] \\ &= [\alpha g, h_1, \dots, h_n] + [\beta x, h_1, \dots, h_n] \\ &\in g^L + X^L \text{ and } \in \langle g^L, X^L \rangle \end{aligned}$$

So $\langle g, X \rangle^L \subseteq g^L + X^L$ and $\langle g, X \rangle^L \subseteq \langle g^L, X^L \rangle$. Since $g \in \langle h, X \rangle^L$ then $g^L \in \langle g, X \rangle^L$ and similarly for X and X^L . So $g^L + X^L \subseteq \langle g, X \rangle^L$. Since $g^L + X^L$ is a vector space, $\langle g^L, X^L \rangle \subseteq g^L + X^L$. Thus we get

$$\langle g, X \rangle^L \subseteq \langle g^L, X^L \rangle \subseteq g^L + X^L \subseteq \langle g, X \rangle^L$$

and the theorem holds. \square

Theorem 24. *For a Lie algebra L , $nFrat(L)$ is the set of n -nongenerators of L .*

Proof: Let $T = \{x | x \text{ is a } n\text{-nongenerator of } L\}$. Since L is finite dimensional, there exists maximal ideals so $nFrat(L) \neq L$. Suppose $x \in T$ and $x \notin nFrat(L)$. There exists $N \in \mathcal{N}$ such that $x \notin N$. Now either $x^L + N = N$ or $x^L + N = L$. If $x^L + N = N$, then $x \in N$. But $x \notin N$ so $x^L + N \neq N$. Thus $x^L + N = L$. This implies $\langle x, N \rangle^L = L$, so $N = N^L = L$ since x is an n -nongenerator. But $N \neq L$, and so this contradiction establishes $x \in N$ for all $N \in \mathcal{N}$ and $x \in nFrat(L)$. Thus $T \subseteq nFrat(L)$.

Conversely, let $x \in nFrat(L)$ and suppose x is not an n -nongenerator. Thus there exists $S \subseteq L$ such that $L = \langle x, S \rangle^L$, but $L \neq S^L$. Hence S^L is a proper ideal of L and $x \notin S^L$. By lemma 8, $L = \langle x, S \rangle^L = x^L + S^L$. Let M be maximal with respect to the properties for S^L : $x \notin M, M \triangleleft L, S^L \subseteq M, L = x^L + M$.

We claim $M \in \mathcal{N}$. If not, there exists N such that $M \subsetneq N \subsetneq L, N \trianglelefteq L$. Then $L = x^L + M = x^L + N$. If $x \notin N$, then N can replace M in the condition above the claim, which is a contradiction. Thus $x \in N$ and $x^L \subseteq N$. Hence $L = N$, which is a contradiction. Hence $M \in \mathcal{N}$, but $x \notin M$, so $x \notin nFrat(L)$, which is a contradiction. Hence whenever $L = \langle x, S \rangle^L$ implies $L = S^L$ and x is an n -nongenerator. Thus $nFrat(L) \subseteq T$.

\square

5.2 Nilpotency and Containments

In this section, we collect information about the Frattini subalgebra and its generalizations which are inspired by the results in group theory. We begin by reviewing known properties of the Frattini subalgebra or Frattini ideal.

The following theorem is proven in [29].

Theorem 25. *Let L be a Lie algebra and N an ideal of L .*

1. $F(L) + N/N \subseteq F(L/N)$.
2. If $N \subseteq F(L)$, then $F(L)/N = F(L/N)$.

We have similar results for $nFrat(L)$.

Theorem 26. *Let L be a Lie algebra and N an ideal of L . Then*

1. $(nFrat(L) + N)/N \subseteq nFrat(L/N)$;
2. If $N \subseteq nFrat(L)$, then $nFrat(L)/N = nFrat(L/N)$.

Proof:

1. For each M with $M/N \in \mathcal{N}(L/N)$, we have $M \in \mathcal{N}(L)$. Thus $nFrat(L) \subseteq \bigcap_{M/N \in \mathcal{N}(L/N)} M$ and $(nFrat(L) + N)/N \subseteq nFrat(L/N)$.

2. Since $N \subseteq nFrat(L)$, $N \subseteq M$ for all $M \in \mathcal{N}(L)$. Also, $M/N \in \mathcal{N}(L/N)$ if and only if $M \in \mathcal{N}(L)$. Therefore, $nFrat(L)/N = (\bigcap_{M \in \mathcal{N}(L)} M)/N = \bigcap_{M \in \mathcal{N}(L)} M/N = nFrat(L/N)$. \square

In group theory, if G is finite then $Frat(G)$ nilpotent. In Lie algebras, it is known that the Frattini ideal is nilpotent.

Theorem 27. *Let L be a Lie algebra, then $\phi(L)$ is nilpotent.*

Proof: Let $x \in \phi(L)$. Then $L_1(x) \subseteq \phi(L)$ since $\phi(L) \trianglelefteq L$. Thus $L_0(x) + \phi(L) = L$ and $L_0(x) + F(L) = L$. Thus $L_0(x) = L$ and x is nilpotent. Hence $\phi(L)$ is nilpotent. \square

As in group theory, it is not always true that $nFrat(L)$ or $R(L)$ are nilpotent.

Example 8. Let $L = gl(n, F)$. If $\text{char}(F) = 0$, then there are two maximal ideals L' and $Z(L)$. Then $L' \cap Z(L) = 0$. However, if $\text{char}(F) = p$, where $p \neq 2$ and $p \mid n$, then $Z(L) \subseteq L' = sl(n, F)$. Thus the only maximal ideal is $sl(n, F)$, and so $nFrat(L) = sl(n, F)$. Therefore, $nFrat(L)$ is not nilpotent. In this example, $R(L) = nFrat(L) = sl(n, F)$. Thus $R(L)$ is also not nilpotent.

It is known in group theory that if N is a normal subgroup in G then $Frat(N) \subseteq Frat(G)$. This result does not hold in complete generality for Lie algebras. However, it does hold in the following case.

Theorem 28. *Let N be an ideal of L over a field F , then if $\text{char}(F) = 0$ or if L is solvable, then $F(N) \subseteq F(L)$.*

Proof: If $\text{char}(F) = 0$, then $F(N)$ is an ideal in L by [33]. If L is solvable, then $F(N)$ is an ideal in L by [5]. Thus $F(N) \subseteq F(L)$. See Theorem 29. \square

If L is over a field of characteristic p it is possible for $F(N) \not\subseteq F(L)$ when N is an ideal in L . The following is an example based off an similar example given by Jacobson in [14].

Example 9. Let A be a vector space over a field of characteristic p and let $\{x_0, x_1, \dots, x_{p-1}\}$ be a basis for A . Define linear transformations R and S of A by

$$R(x_i) = x_{i+1}, \quad i < p-1$$

$$R(x_{p-1}) = x_0$$

$$S(x_i) = ix_{i-1}, \quad i(\bmod p) > 0$$

$$S(x_0) = 0$$

Then $[R, S] = I$ where I is the identity linear transformation. Let B be the three dimensional Lie algebra with basis $\{R, S, I\}$. B is a Heisenberg Lie algebra. Let L be the semi direct sum of A and B with multiplication given by $[b, a] = b(a)$ for $b \in B$, $a \in A$. Let $K = A + \langle S \rangle$. K is nilpotent since S is a nilpotent linear transformation. Hence $F(K) = K^2$ and K is not an ideal in L . Also $F(K) = K^2 = \langle x_0, x_1, \dots, x_{p-2} \rangle$ is not an ideal in L . Now let $H = A + \langle S, I \rangle$. H is not nilpotent since $[I, x_0] = x_0$. H is solvable and K is an ideal in H . $F(K) = K^2$ is also an ideal in H . Hence $F(K) \subseteq F(H)$. Also $F(H) \subsetneq H^2$ and $\dim H^2 = \dim K^2 + 1$. Thus $F(K) = K^2 \subseteq F(H) \subsetneq H^2$ yields that $F(K) = F(H)$. Therefore, $F(H)$ is not an ideal in L even though H is ideal in L . Furthermore, $F(L) = 0$ as we now will show. B acts irreducibly on A , hence B is maximal in L . Thus $F(L) \subseteq B$. Since L is solvable, $F(L)$ is an ideal of L and therefore,

$[F(L), A] \subseteq F(L) \cap A \subseteq B \cap A = 0$. Since $F(L)$ consists of linear transformations of A , $F(L) = 0$. Hence $F(H) \subsetneq F(L)$ and $F(L) = 0$ does not imply that $F(H) = 0$ even though H is an ideal in L .

Under certain conditions in Lie algebras, we get that $F(N) \subseteq F(L)$. This property also carries over to the Frattini ideal, $nFrat(L)$, and $R(L)$.

Theorem 29. *If $N \subseteq L$ and $F(N)$ is an ideal of L , then $F(N) \subseteq F(L)$.*

Proof: Suppose not. Then there exists a maximal subalgebra M in L such that $F(N) \not\subseteq M$. Then $F(N) + M = L$. So $(F(N) + M) \cap N = F(N) + (N \cap M) = N$. Thus $N \cap M = N$ which implies that $F(N) \subset N \subseteq M$ which is a contradiction. Therefore, $F(N) \subseteq F(L)$. \square

Theorem 30. *If $N \subseteq L$ and $\phi(N)$ is an ideal of L , then $\phi(N) \subseteq \phi(L)$.*

Proof: Suppose not. $\phi(N) + \phi(L)$ is an ideal in L . So if $\phi(N) + \phi(L) \subseteq F(L)$, then $\phi(N) \subseteq \phi(L)$, which is a contradiction. So there exists a maximal subalgebra M of L such that $\phi(N) \not\subseteq M$. Then $\phi(N) + M = L$. This implies that $\phi(N) + (N \cap M) = N$ as above. This implies that $N \cap M = N$. Thus $N \subseteq M$, and hence $\phi(N) \subseteq M$, which is a contradiction. Therefore, $\phi(N) \subseteq \phi(L)$. \square

Theorem 31. *If $N \subseteq L$ and $nFrat(N)$ is an ideal of L , then $nFrat(N) \subseteq nFrat(L)$.*

Proof: Suppose not. Then there exists a maximal ideal M of L such that $nFrat(N) \not\subseteq M$. Then $nFrat(N) + M = L$ and $nFrat(N) + (M \cap N) = N$, which implies $M \cap N = N$.

Thus $nFrat(N) \subseteq N \subseteq M$, a contradiction. Therefore, $nFrat(N) \subseteq nFrat(L)$. \square

Theorem 32. *If $N \subseteq L$ and $R(N)$ is an ideal of L , then $R(N) \subseteq R(L)$.*

Proof: Suppose not. Then there exists a maximal ideal M of L such that $R(N) \not\subseteq M$. Then $R(N) + M = L$. Then $R(N) + (M \cap N) = N$, which implies $M \cap N = N$. Thus $R(N) \subseteq N \subseteq M$, which is a contradiction. Therefore, $R(N) \subseteq R(L)$. \square

Here we consider what happens when L is the direct sum of ideals. Like in group theory, we get equality for the Frattini subalgebra. This equality carries over for the Frattini ideal, $nFrat(L)$, and $R(L)$ as well. The proofs of the first two are shown by Towers in [33].

Lemma 9. *If $L = L_1 \oplus \dots \oplus L_n$, then $F(L) = F(L_1) \oplus \dots \oplus F(L_n)$.*

Theorem 33. *If $L = L_1 \oplus \dots \oplus L_n$, then $\phi(L) = \phi(L_1) \oplus \dots \oplus \phi(L_n)$.*

Theorem 34. *If $L = L_1 \oplus \dots \oplus L_n$, then $nFrat(L) = nFrat(L_1) \oplus \dots \oplus nFrat(L_n)$.*

Proof: Since $L = L_1 \oplus \dots \oplus L_n$, $nFrat(L_i) \subsetneq L_i$ for each i . Let $M_j \in \mathcal{N}(L_j)$. Then we have $L_1 \oplus \dots \oplus L_{j-1} \oplus L_{j+1} \oplus \dots \oplus L_n \oplus M_j \in \mathcal{N}(L)$. Thus, $nFrat(L) \subseteq L_1 \oplus \dots \oplus L_{j-1} \oplus L_{j+1} \oplus \dots \oplus L_n \oplus nFrat(M_j)$. Therefore, $nFrat(L) \subseteq nFrat(L_1) \oplus \dots \oplus nFrat(L_n)$.

Now consider L_j . Since L_j is an ideal of L , $nFrat(L_j) \subseteq nFrat(L)$ by Theorem 31. Thus, $nFrat(L_1) \oplus \dots \oplus nFrat(L_n) \subseteq nFrat(L)$. \square

Theorem 35. *If $L = L_1 \oplus \dots \oplus L_n$, then $R(L) = R(L_1) \oplus \dots \oplus R(L_n)$.*

Proof: The proof is similar to that of $nFrat(L)$. □

5.3 Characterizations of Nilpotency

In this section, relations between $\phi(L)$, $nFrat(L)$, and $R(L)$ are investigated. We also find a characterization of nilpotency in terms of the equality of these ideals.

Lemma 10. *In any Lie algebra L , $\phi(L) \subseteq R(L)$ and $nFrat(L) \subseteq R(L)$.*

Theorem 36. $\phi(L) \subseteq nFrat(L)$.

Proof: For each $N \in \mathcal{N}$, $\phi(L) + N \trianglelefteq L$. Thus $\phi(L) + N = N$ or $\phi(L) + N = L$. If $\phi(L) + N = L$ then $N = L$ as $\phi(L)$ cannot be supplemented, which is a contradiction. Thus $\phi(L) + N = N$, so $\phi(L) \subseteq N$ for all $N \in \mathcal{N}$. Therefore, $\phi(L) \subseteq nFrat(L)$. □

Corollary 16. *For a Lie algebra L , $\phi(L) \subseteq nFrat(L) \subseteq R(L)$.*

The following is an example of Corollary 16.

Example 10. Let L be the nonabelian two dimensional Lie algebra, $L = \text{span}\{x, y\}$ with $[x, y] = y$. Then y is the only maximal ideal of L . So $\phi(L) = 0$ and $nFrat(L) = R(L) = y$. Thus $\phi(L) \subset nFrat(L) \subseteq R(L)$.

Lemma 11. *If L is a solvable Lie algebra, then $R(L) = nFrat(L)$.*

Proof: Let L be a solvable Lie algebra. Let N be a maximal ideal of L . Then $\dim(L/N) = 1$ for any $N \in \mathcal{N}$. This is true if and only if $N \in \mathcal{R}$ by Lemma 7. But

then the set of all maximal ideals is equal to the set of all maximal subalgebras that are ideals, $\mathcal{N} = \mathcal{R}$. So $nFrat(L) = R(L)$. \square

The converse of Lemma 11 is not true.

Example 11. Let $L = gl(n, F)$. If $\text{char}(F) = p$, where $p \neq 2$ and $p \mid n$, then we have seen $nFrat(L) = R(L) = sl(n, F)$. However, L is not solvable.

Lemma 11 is not true if L is not solvable. The following is an example of a non-solvable Lie algebra with $R(L) \neq nFrat(L)$.

Example 12. Let $L = sl(2, F)$. L is not solvable. Since the only maximal ideal is $\{0\}$, then $nFrat(L) = \{0\}$. However, L contains no maximal subalgebras that are ideals, so $\mathcal{R} = \emptyset$ which implies $R(L) = sl(2, F)$. Thus $R(L) \neq nFrat(L)$.

Theorem 37. *Let L be a Lie algebra. Then L is nilpotent if and only if $\phi(L) = nFrat(L) = R(L)$.*

Proof: If L is nilpotent then L is solvable. Hence $\phi(L) = nFrat(L) = R(L)$.

So suppose $\phi(L) = nFrat(L) = R(L)$. Let M be a maximal ideal of L such that $M = \bigoplus M_i$. Consider the Lie algebra homomorphism $\Pi : L \longrightarrow \bigoplus (L/M_i)$, where $\Pi(x) = (x + M_1, x + M_2, \dots, x + M_n)$ for $x \in L$. So each L/M_i is a 1-dimensional subalgebra of L and hence an abelian subalgebra. The $\text{Ker}\Pi = \bigcap M_i = R(L)$. But then $L/\text{Ker}\Pi$ is abelian since it is the direct sum of abelian subalgebras. Thus $L/\text{Ker}\Pi = L/R(L)$, which equals $L/\phi(L)$, is nilpotent, and so L is nilpotent. \square

Chapter 6

Frattini Extensions

In this chapter we will establish some properties of the Frattini extension. Recall if N and H are two Lie algebras, then an extension of the algebra N by the algebra H is a Lie algebra L having an ideal $K \cong N$ and $L/K \cong H$. Equivalently an extension can be defined in terms of short exact sequences of Lie algebras and homomorphisms as follows: Let ϕ and ψ be the isomorphisms described above. Consider $N \xrightarrow{\phi} K \xrightarrow{i} L$ and $L \xrightarrow{\pi} L/K \xrightarrow{\psi} H$, where i is the inclusion map and π is the natural homomorphism. Let $\alpha = i \circ \phi$ and $\epsilon = \psi \circ \pi$. Then we have the following short exact sequence $\{0\} \longrightarrow N \xrightarrow{\alpha} L \xrightarrow{\epsilon} H \longrightarrow \{0\}$, which we say it represents the extension (L, ϵ) . We say an extension (L, ϵ) is a Frattini extension if the Kernel of ϵ is contained in the Frattini ideal, $\phi(L)$, of L .

We will use L -modules throughout this chapter. A vector space V over a field F is an L -module if there is an operation $L \times V \rightarrow V$, $(x, v) \mapsto x \cdot v$, such that the following axioms hold for all $x, y \in L, u, v \in V$ and $a, b \in F$: (i) $x \cdot (au + bv) = a(x \cdot u) + b(x \cdot v)$, (ii) $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$, and (iii) $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$. If V is an L -module, a subspace U of V is a submodule if $x \cdot u \in U$ for all $x \in L$ and $u \in U$.

Example 13. Any Lie algebra L is an L -module under the adjoint action (i.e. $x \cdot y = ad_x(y) = [x, y]$ for all $x, y \in L$).

Remark 2. It is clear from the definition of submodule that any submodule of L under the adjoint action is an ideal of L .

Suppose that we have the given extension $\{0\} \longrightarrow N \xrightarrow{\alpha} L \xrightarrow{\epsilon} H \longrightarrow \{0\}$ with N abelian. H acts on N by $h \cdot n = [l, n]$ where $\epsilon(l) = h$. The action is independent of the l used since N is abelian. This product amounts to the adjoint action of L on N and hence the H -submodules of N are the L -submodules of N , which are the ideals of L contained in N .

Throughout this section we will be using the relationship between being an H -submodule of N and an ideal of L contained in N .

Lemma 12. *Let (L, ϵ) be an extension by H in which $Ker(\epsilon)$ is an abelian, finite dimensional H -module. If J is a maximal subalgebra of L which does not contain $Ker(\epsilon)$, then (J, α) is an extension with $\alpha = \epsilon|_J$ and $Ker(\alpha)$ is a maximal H -submodule of $Ker(\epsilon)$.*

Proof: Let $\{0\} \longrightarrow N \longrightarrow L \xrightarrow{\epsilon} H \longrightarrow \{0\}$ be the given extension where $N = Ker(\epsilon)$. We need to show that:

1. (J, α) is an extension of $J \cap N$ by H , 2. $Ker(\alpha) = J \cap N$ is an ideal of L , and 3. $Ker(\alpha)$ is a maximal H -submodule of $Ker(\epsilon)$.

1. Since $N \not\subseteq J$ and J a maximal subalgebra of L , $J \subset J+N \subseteq L$. Since J is maximal, this implies that $J+N = L$. Since (L, ϵ) is an extension we have $(L/N) \cong H$, so $J+N/N = L/N \cong H$. But $J+N/N \cong J/J \cap N \cong H$. Thus J is an extension of $J \cap N$ by H .

2. Recall that N is abelian and $L = J + N$. Then

$$\begin{aligned} [J \cap N, L] &= [J \cap N, N] + [J \cap N, J] \\ &= 0 + J \cap N. \end{aligned}$$

So $J \cap N$ is an ideal of L .

3. We may assume that $N \cap J = 0$.

Claim: N is minimal ideal of L .

Proof of Claim: Let K be an ideal of L such that $0 \subset K \subsetneq N$. Then

$$\dim J < \dim J + \dim K = \dim(J + K) + \dim(J \cap K) = \dim(J + K).$$

So $J \subsetneq J + K$.

$$\dim L = \dim(J + N) = \dim J + \dim N - \dim(J \cap N)$$

$$= \dim J + \dim N > \dim J + \dim K = \dim(J + K)$$

This contradicts that J is maximal. So there is no such K . Thus N is minimal.

Therefore, $N \cap J$ is maximal in N . □

An L -module V is said to be irreducible if $V \neq \{0\}$ and V has no proper submodules (i.e. $\{0\}$ and V are the only submodules of V .) As in the group theory case in [22], if $\text{Ker}(\epsilon)$ is an irreducible H -module, we can use Lemma 12, to prove the following result.

Theorem 38. *Let (L, ϵ) be an extension by H where the $\text{Ker}(\epsilon)$ is abelian. Suppose that $\text{Ker}(\epsilon)$ is a non-trivial irreducible H -module. Then (L, ϵ) is a Frattini extension if and only if it is non-split.*

Proof: Suppose that

$$\{0\} \longrightarrow N \longrightarrow L \xrightarrow{\epsilon} H \longrightarrow \{0\} \quad (6.1)$$

is a non-split extension. We need to show that $N = \text{Ker}(\epsilon)$ is a subalgebra of $\phi(L)$. If $N \not\subseteq \phi(L)$, then there exists J a maximal subalgebra of L such that $N \not\subseteq J$, and so by Lemma 12 we have the (J, α) is an extension, with $\alpha = \epsilon|_J$. Moreover, $\text{Ker}(\alpha)$ is a maximal H -submodule of $\text{Ker}(\epsilon)$ and since $\text{Ker}(\epsilon)$ is irreducible as an H -module we have that $\text{Ker}(\alpha) = N \cap J = \{0\}$ and therefore α is a monomorphism. But α is an epimorphism, since (J, α) is an extension. Hence α is an isomorphism. Thus (6.1) splits, which is a contradiction.

Conversely, suppose that (L, ϵ) is a Frattini extension. We need to show that (6.1) is non-split. Suppose that (6.1) splits. Then there exists a homomorphism say ψ from H into L such that $\epsilon\psi = I_H$, and so ψ is a monomorphism and therefore $H \cong \text{Im}(\psi)$. Hence $\psi : H \rightarrow \text{Im}(\psi)$ is an isomorphism. Also, we have that $\text{Im}(\psi) \subseteq L$ and $\text{Ker}(\epsilon) \subseteq L$. Since the extension splits, $L = \text{Ker}(\epsilon) + \text{Im}(\psi)$ and $\text{Ker}(\epsilon) \cap \text{Im}(\psi) = \{0\}$. Since $\text{Ker}(\epsilon) \not\subseteq \text{Im}(\psi)$ then $\text{Im}(\psi)$ is not a maximal subalgebra of L . Thus there exists J a maximal subalgebra of L such that $\text{Im}(\psi) \subset J \subset L$. But maximality of J implies that $\text{Ker}(\epsilon) \subseteq \phi(L) \subseteq J$. Now, $\text{Ker}(\epsilon) \subseteq J$ and $\text{Im}(\psi) \subset J$ implies that $L = \text{Ker}(\epsilon) + \text{Im}(\psi) \subseteq J$, which is a contradiction. \square

Theorem 39. *If (L, ϵ) is a Frattini extension by H , then $(\phi(L), \epsilon)$ is an extension of*

$\text{Ker}(\epsilon)$ by $\phi(H)$.

Proof: Since (L, ϵ) is an extension, we have $L/\text{Ker}(\epsilon) \cong H$, and so $\phi(L/\text{Ker}(\epsilon)) \cong \phi(H)$. Since $\text{Ker}(\epsilon) \trianglelefteq \phi(L)$, we have that $\phi(L/\text{Ker}(\epsilon)) = \phi(L)/\text{Ker}(\epsilon)$. \square

A Lie algebra L is perfect if $L = L'$. This brings us to our next theorem.

Theorem 40. *Let L be a finite dimensional Lie algebra and let (L, ϵ) be a Frattini extension of N by H with H a perfect Lie algebra. Then L is perfect.*

Proof: Suppose that L is not perfect. Let $N = \text{Ker}(\epsilon)$. Since (L, ϵ) is a Frattini extension by H we have that $L/\text{Ker}(\epsilon) = L/N \cong H$. Since H is perfect we get that $(L/N)' \cong H' = H$ which implies $L' + N/N \cong (L/N)' \cong L/N$. Since L is finite dimensional $L' + N = L$. Since $N \subseteq \phi(L)$, N has no proper supplement, so $L' = L$. \square

Theorem 41. *If $\epsilon : J \longrightarrow M$ is an epimorphism of a finite dimensional Lie algebra and L is minimal among the subalgebras of J , with $\epsilon(L) = M$, then $(L, \epsilon|_L)$ is a Frattini extension.*

Proof: Let K be a maximal subalgebra of L . Suppose that $\text{Ker}(\epsilon|_L) \not\subseteq K$ then $K \subseteq K + \text{Ker}(\epsilon|_L) \subseteq L$. Maximality of K implies that $K + \text{Ker}(\epsilon|_L) = L$. Also we have $\text{Ker}(\epsilon) \triangleleft J$, $\text{Ker}(\epsilon|_L) \subseteq \text{Ker}(\epsilon)$, and $K \subset J$, thus

$$K \subseteq L = K + \text{Ker}(\epsilon|_L) \subseteq K + \text{Ker}(\epsilon) \subseteq J. \quad (6.2)$$

Let $W = K + \text{Ker}(\epsilon)$. Then

$$\begin{aligned}
\epsilon(W) &= \{\epsilon(k+x) | k \in K \text{ and } x \in \text{Ker}(\epsilon)\} \\
&= \{\epsilon(k) + \epsilon(x) | k \in K \text{ and } x \in \text{Ker}(\epsilon)\} \\
&= \{\epsilon(k) | k \in K\} \\
&= \epsilon(K)
\end{aligned}$$

By (6.2), we have $\epsilon(L) \subseteq \epsilon(W) \subseteq \epsilon(J)$, which is $M \subseteq \epsilon(W) \subseteq M$. Thus $\epsilon(W) = M$ and hence $\epsilon(K) = M$. Since K is a subalgebra of L and $\epsilon(K) = M$, minimality of L yields a contradiction. Thus $\text{Ker}(\epsilon|_L)$ is contained in every maximal ideal of L and hence in $\phi(L)$. Therefore, $(L, \epsilon|_L)$ is a Frattini extension. \square

Proposition 2. *If (L, ϵ) is a Frattini extension by H , and $\beta : H \longrightarrow L$ is a homomorphism such that $\alpha\beta$ is surjective, then β is surjective.*

Proof: Let $K = \text{Im}(\beta) = \beta(H)$. Since $\alpha\beta$ is surjective, we have that $\alpha(L) = H = (\alpha\beta)(H) = \alpha(\beta(H)) = \alpha(K)$. So for any $l \in L$, there exists $k \in K$ such that $\alpha(l) = \alpha(k)$. So $\alpha(l) = \alpha(k)$ implies that $\alpha(l) - \alpha(k) = \alpha(l - k) = \{0\}$. This implies that $l - k \in \text{Ker}(\alpha)$. So $l \in \text{Ker}(\alpha) + K$ for all $l \in L$. Thus $L \subseteq \text{Ker}(\alpha) + K$. Now, $K \subseteq L$ and $\text{Ker}(\alpha) \trianglelefteq L$ implies that $\text{Ker}(\alpha) + K \subseteq L$. Hence $L = \text{Ker}(\alpha) + K$. Since $\text{Ker}(\alpha) \trianglelefteq \phi(L)$ and as in the proof of Theorem 40, there exists no ideal, M , of L such that $M + \text{Ker}(\alpha) = L$. Therefore, $K = L$ and β is surjective. \square

Theorem 42. *Composites of Frattini extensions are Frattini extensions.*

Proof: Consider $\{0\} \longrightarrow \text{Ker}(\alpha) \longrightarrow L \xrightarrow{\alpha} H \longrightarrow \{0\}$ with α an epimorphism and

$Ker(\alpha) \leq \phi(L)$. Let $\{0\} \longrightarrow Ker(\beta) \longrightarrow H \xrightarrow{\beta} M \longrightarrow \{0\}$ with β an epimorphism and $Ker(\beta) \leq \phi(H)$. We need to show that $\beta\alpha$ is an epimorphism and that $Ker(\beta\alpha) \leq \phi(L)$.

1. $\beta\alpha$ is an epimorphism:

Since $h \in H$ and α is an epimorphism, then there exists $l \in L$ such that $\alpha(l) = h$.

If $m \in M$, then there exists $h \in H$ such that $\beta(h) = m$ since β is an epimorphism.

Thus $(\beta\alpha)(l) = \beta(\alpha(l)) = \beta(h) = m$. Therefore, $\beta\alpha$ is an epimorphism.

2. $Ker(\beta\alpha) \subseteq \phi(L)$:

Since $\alpha(L) = H$ we have $\alpha(\phi(L)) = \phi(\alpha(L)) = \phi(H)$. Thus $\phi(L) = \alpha^{-1}(\phi(H))$, the inverse image of $\phi(H)$. But

$$\begin{aligned} Ker(\beta\alpha) &= \{l | (\beta\alpha)(l) = 0, l \in L\} \\ &= \{l | \beta(\alpha(l)) = 0, l \in L\} \\ &= \{l | \alpha(l) \in Ker(\beta), l \in L\} \\ &= \{l | l \in \alpha^{-1}(Ker(\beta))\} \end{aligned}$$

Since $Ker(\beta) \subseteq \phi(H)$ we get $\alpha^{-1}Ker(\beta) \subseteq \alpha^{-1}(\phi(H)) = \phi(L)$. Hence $Ker(\beta\alpha) \subseteq \phi(L)$. □

Lemma 13. *Let N be a minimal ideal of L . If N has a complement in L , then $N \cap \phi(L) = \{0\}$.*

Proof: If N is complemented in L then $L = N + H$ and $N \cap H = \{0\}$ for some $H \subsetneq L$. Since $N \triangleleft L$ and $\phi(L) \triangleleft L$ we have $N \cap \phi(L) \triangleleft L$. Now minimality of N implies that

$N \cap \phi(L) = \{0\}$ or N . If $N \cap \phi(L) = N$, then $N \subseteq \phi(L)$, and hence $N + H \subseteq \phi(L) + H$. So $L = \phi(L) + H$. But this implies $H = L$, which is a contradiction. Thus $N \cap \phi(L) = \{0\}$. \square

Theorem 43. *Let L be an extension of N by H , where N is abelian. Then L is a minimal Frattini extension of N if and only if N is a minimal non-complemented ideal of L .*

Proof: Let N be a minimal non-complemented ideal of L . If $N \not\subseteq \phi(L)$, then there exists a maximal subalgebra M of L such that $N \not\subseteq M$. Now $N \leq L$ and $M \subseteq L$ implies that $M \subset N + M \subseteq L$. Maximality of M implies that $N + M = L$. Since $N \cap M \triangleleft M$ and $N \cap M \triangleleft N$, we have that $N \cap M \triangleleft N + M = L$. Minimality of N in L implies that $N \cap M = \{0\}$, and hence M is a complement of N . This is a contradiction. So $N \subseteq \phi(L)$, and therefore, L is a minimal Frattini extension.

Conversely, if N is a complemented minimal ideal of L then by Lemma 13, $N \cap \phi(L) = \{0\}$, and hence N is not a subalgebra of $\phi(L)$. Therefore, L is not a minimal Frattini extension. \square

Chapter 7

Minimal Non-Elementary Lie Algebras

In this chapter we will classify minimal non-elementary Lie algebras. Minimal non-elementary finite groups must be nilpotent. The Lie algebra analog admits non-nilpotent examples. The Lie algebras considered here are solvable. Hence the Frattini subalgebra coincides with the Frattini ideal. Let $F(L)$ denote the Frattini subalgebra of L . For solvable Lie algebras over a field of characteristic 0, the derived algebra is nilpotent, hence the theorem finds all minimal non-elementary Lie algebras in this case when the field is algebraically closed. For characteristic p , if L is nilpotent of length 2, then the final term, L^w , in the lower central series is nilpotent, hence abelian. Then L is the semi-direct sum of L^w and a Cartan subalgebra C of L [[3], Theorem 8]. Then C is abelian and $L^2 = L^w$. Hence the theorem applies to any Lie algebra of nilpotent length 2 over an algebraically closed field.

Lemma 14. *Let L be a minimal non-elementary finite dimensional solvable Lie algebra*

with L^2 nilpotent.

1. If L is nilpotent, then L is Heisenberg.
2. If L is not nilpotent and C is a Cartan subalgebra, then L is the semi-direct sum $L = C + L^2$.

Proof:

1. Suppose that L is nilpotent. Then $F(H) = H^2 = 0$ for every proper subalgebra H of L and $F(L) = L^2$. If $\dim(L/L^2) > 2$, then every pair of elements of L are in a proper subalgebra of L and hence commute. Thus L is abelian, a contradiction. Hence two elements, x and y , generate L . Since $\langle x, L^2 \rangle$ and $\langle y, L^2 \rangle$ are proper subalgebras of L , they are abelian and $L^2 \subseteq Z(L)$. Hence $L = \langle x, y, z = [x, y] \rangle$ and L is Heisenberg.
2. Suppose that L is not nilpotent. Let C be a Cartan subalgebra of L . As on page 57 of [14], L decomposes as a vector space direct sum of C and C_1 where $[C, C_1] = C_1$, the Fitting decomposition of L with respect to C . Both C and L^2 are nilpotent proper subalgebras of L , hence they are abelian. Since $C_1 \subseteq L^2$, it follows that C_1 is an ideal of L . If $C_1 \neq L^2$, then $C^2 \cong (L/C_1)^2 \cong L^2/C_1 \neq 0$, a contradiction. Hence $C_1 = L^2$ and $L = C + L^2$ is a semi-direct sum. \square

Theorem 44. *Let L be a finite dimensional Lie algebra over an algebraically closed field K and suppose that L^2 is nilpotent. Then L is minimal non-elementary if and only if L has a basis x, y, z with multiplication $[x, y] = \alpha y + z$, $[x, z] = \alpha z$ and $[y, z] = 0$, where $\alpha \in K$.*

Proof: Assume L minimal non-elementary. If L is nilpotent, then it has the desired presentation by part (i) of Lemma. If L is not nilpotent, then by part (ii) of the Lemma, L is the semi-direct sum $L = C + L^2$, where C is a Cartan subalgebra of L . Suppose that $\dim L/L^2 > 1$ and let $x \in C$. Then $H = \langle x \rangle + L^2$ is a proper subalgebra of L and $F(H) = 0$. In particular, H is not nilpotent. If it were nilpotent then H would be abelian. Hence x is in the center of L and $\langle x \rangle$ is a direct summand of L , which is a contradiction. If D is a Cartan subalgebra of H , then $H = D + H^2$ is a semi-direct sum using the same argument as in part (ii) of Lemma. Since L^2 is abelian, the Fitting null component of $\text{ad}(x)$ acting on H is nilpotent, hence it is a Cartan subalgebra of H . Hence we can let D be the Fitting null component. Since $F(H) = 0$, Proposition 1 of [28] yields that D , hence $\text{ad}(x)$, acts completely reducibly on H since $x \in D$, $H = D + H^2$ and D is abelian. Since K is algebraically closed, $\text{ad}(x)$ acts diagonally on H and on the ideals L^2 and $F(L)$ that are contained in H . These results hold for all $x \in C$ and, since C is abelian, the $\text{ad}(x)$ are simultaneously diagonalizable on L^2 and on $F(L)$. Let $\{x_1, \dots, x_n, y_1, \dots, y_t\}$ be a basis of these common eigenvectors where the $x_i \in \phi(L)$ and $y_i \in L^2$, but $y_i \notin \phi(L)$. If $M = \langle x_2, \dots, x_n, y_1, \dots, y_t \rangle + C$, then M is a maximal subalgebra of L . This is a contradiction since $x_1 \notin M$, but $x_1 \in F(L)$. Hence $\dim L/L^2 = 1$. Thus we can let $L = \langle x \rangle + L^2$.

We claim that L^2 is not the direct sum of two non-zero ideals of L . Suppose that M and N show this statement to be false. Let $A = \langle x \rangle + N$ and $B = \langle x \rangle + M$. Both $F(A)$ and $F(B)$ are 0. Consider the set consisting of each maximal subalgebra of A added to M . The intersection of the elements of this set is M , since $F(A) = 0$ and L is the semi-direct sum of A and M . The Fitting null component E of $\text{ad}(x)$ acting on A is nilpotent, since N is abelian. Hence E is a Cartan subalgebra of A . Then $A = E + A^2$ is a semi-direct sum, E is abelian and the action of E on A^2 is the action of $\text{ad}(x)$ on

A^2 . By Proposition 1 of [28], $\text{ad}(x)$ is diagonalizable on A^2 and also on A since E is abelian. Similarly, $\text{ad}(x)$ acts diagonally on B and hence on L^2 . By Proposition 1 of [28], $F(L) = 0$ since $\langle x \rangle$ is a Cartan subalgebra of L . This contradiction establishes that $\text{ad}(x)$ has only one Jordan block when acting on L^2 . Hence there exists a basis $\{x_1, \dots, x_n\}$ of L^2 such that $[x, x_i] = \alpha x_i + x_{i+1}$ for $i = 1, \dots, n-1$ and $[x, x_n] = \alpha x_n$, where $0 \neq \alpha \in K$.

If $n > 2$, then $B = \langle x, x_2, \dots, x_n \rangle$ has $F(B) = 0$ and $\text{ad}(x)$ acts diagonally on B^2 by Proposition 1 of [28] which contradicts the multiplication just given for L . Thus $n = 2$ and the result holds. The converse is clear. \square

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