ABSTRACT

BOSKO, LINDSEY R. Schur Multipliers of Nilpotent Lie Algebras. (Under the direction of Ernest L. Stitzinger.)

The multiplier of a group was first discovered by I. Schur in the early twentieth century. It can be defined using the second cohomology group, a free presentation, or a central extension. We examine the Schur multiplier for Lie algebras which are nilpotent. We compute multipliers for a particular type of nilpotent Lie algebra, categorizing them in regard to a certain invariant. Our computations indicate the existence of a bound on the dimension of the multiplier in terms of the dimension of the algebra. We prove this conjecture is a theorem. The analogue of this result is also shown to hold for a certain type of $p-$group. We use results from Berkovich [5], Ellis [8], and Zhou [21] to prove the result. Additionally, we develop another bound for the dimension of the multiplier in terms of its class and number of generators. We compare this bound to a known result in [12]. There are many results concerning the Schur multiplier of a group being trivial. See [13], [18], [6], and [19]. We examine sufficient conditions for making the Schur multiplier of a Lie algebra nontrivial proving an elegant theorem involving the dimension of the Lie algebra.

Schur Multipliers of Nilpotent Lie Algebras

by
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DEDICATION

To all that love learning, logic, and challenges.

BIOGRAPHY

Lindsey's love of mathematics bloomed at an early age. Her mom cites a visit to the doctor's office where Lindsey occupied herself by counting ceiling tiles in the waiting room. '24 tiles,' she proudly concluded. Attending Heyer Elementary in Waukesha, WI she earned the title of 'Math Whiz' from her second grade teacher. Lindsey graduated from Elkhorn Area High School in 2002. While attending Elizabethtown College, she began to follow a path toward teaching and earned a degree in mathematics secondary education in 2006. Inspired by college professors and friends, Lindsey felt a strong pull to pursue higher levels of mathematics by attending graduate school. At North Carolina State University, Lindsey studied algebra under the direction of Dr. Ernie Stitzinger. Her dissertation was successfully defended on May 3, 2011. She will begin work at West Liberty University in the Fall of 2011.

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TABLE OF CONTENTS

List of Tables . . . . **vi**

List of Figures . . . . **vii**

Chapter 1 Introduction **1**

Chapter 2 Preliminaries **6**

Chapter 3 Lie Algebras of Maximal Class and Small t(L) **11**

3.1 Algorithm 11
3.2 Computations 12
3.2.1 t(L)=0 12
3.2.2 t(L)=1 12
3.2.3 t(L)=2 12
3.2.4 t(L)=3 12
3.2.5 t(L)=4 14
3.2.6 t(L)=5 & t(L)=6 14
3.2.7 t(L)=7 14
3.2.8 t(L)=8, 9 & 10 15
3.2.9 t(L)=11 & L/Z(L) $\cong$ L(7,5,1,7) 16
3.2.10 t(L)=11 & L/Z(L) $\cong$ L′(7,5,1,7) 19
3.2.11 t(L)=14, 15, & 16 22
3.3 Remarks 22

Chapter 4 Bounding the Dimension of L given t(L) **24**

Chapter 5 Other Bounds on the Dimension of the Schur Multiplier **29**

5.0.1 Nontrivial M(L) 30
5.0.2 An Upper Bound for dim M(L) 31

References **35**

Appendix **37**

Appendix A The Dimensions of Lower Central Factors of Free Lie Algebras 38

LIST OF TABLES

Table 3.1 Maximal Class Lie Algebras 23

Table 4.1 Maximal Class Groups 27

LIST OF FIGURES

Figure 1.1 Projective Representation on G, Lifting to a Representation on C 2

Figure A.1 Commuting Diagram of a Free Lie Algebra 40

Chapter 1

Introduction

Definition 1. *A* Lie Algebra*, L, is a vector space over a field, $\mathcal{K}$, with a product* $[\ ,\ ] : L \times L \to L$ *called bracket such that*

(i) $[\ ,\ ]$ *is $\mathcal{K}$-bilinear*

(ii) $[x, x] = 0$ *for all* $x \in L$

(iii) $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$ *for all* $x, y, z \in L$*. This particular derivation is known as the Jacobi identity.*

Example 2. *To clarify the definition, several examples of Lie algebras are given below.*

(i) The set of $n \times n$ matrices in $\mathcal{K}$, $M(n, \mathcal{K})$, with a commutation bracket: $[A, B] = AB - BA$.

(ii) $\mathbb{R}^3$ *with unit vector basis* $\{x, y, z\}$ *and* $[v, w] = v \times w$ *where* $\times$ *is the usual cross product of vectors.*

(iii) $L = span_{\mathcal{K}}\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z\}$ *with* $[x_i, y_j] = \delta_{ij} z$ *and all other products are zero. This is the Heisenberg Lie algebra.*

There are many parallels that exist between different algebraic structures. We will examine groups and Lie algebras for new similarities and differences regarding their Schur multipliers. We begin by describing several equivalent definitions of the Schur multiplier in some historical context. We develop this by first reverting back to the theory of groups where the Schur multiplier was first defined and studied.

Definition 3. *A* representation *of a group, G, on a vector space, V, is a group homomorphism from G into the general linear group on V, $GL(V)$.*

Representations are very useful for their ability to describe abstract groups (or other algebraic structures) in terms of linear transformations. In a representation, the group elements are studied as matrices with the group operation becoming matrix multiplication. Representations can give some level of concreteness to abstract objects. In the early twentieth century, Issai Schur began studying a related topic, projective representations of groups.

Definition 4. *A* projective representation *of a group, G, on a vector space, V, is a group homomorphism from G into the projective general linear group, $PGL(V) = GL(V)/Z(V)$. $Z(V)$ is the center of V, which is just the set of scalar matrices.*

Schur discovered that projective representations are related to regular representations in the following way. For a finite group, G, there exists a group, C, such that each irreducible projective representation of G can be lifted to an ordinary representation of C over the same vector space. Pictorially,

$$\begin{tikzcd} C \arrow[r, "\Omega"] \arrow[d, "\omega"] & GL(V) \arrow[d, "\phi"] \\ G \arrow[r, "\pi"] & PGL(V) \end{tikzcd}$$

Figure 1.1: Projective Representation on G, Lifting to a Representation on C

In Figure 1, π is a projective representation of the group G that can be lifted to an ordinary representation, Ω of the group C. Using modern terminology, Schur found this lifting to involve cohomology and, in particular, categorizing 2-cocylces modulo 2-coboundaries. With this, Schur gave the first definition for his multiplier.

Definition 5. *For a group, G, its Schur multiplier, $M(G)$, can be defined as the second cohomology group $H^2(G, \mathbb{C}^*)$ where the modular multiplication acts identically: $g \cdot c = c$ for $g \in G,\ c \in \mathbb{C}^*$ and $\mathbb{C}^*$ represents the nonzero complex numbers.*

Before giving an equivalent definition known as Hopf's formula, we define a few necessary terms.

Definition 6. *Given a group, G, its* commutator subgroup, *G^2, is the group generated by all commutators: $[g, h] = ghg^{-1}h^{-1}$ where $g, h \in G$*

Definition 7. *A* free presentation *of a group G is a short exact sequence of groups*

$$1 \to R \to F \to G \to 1$$

where F is a free group on the generators of G and R is a normal subgroup of F generated by the relations.

Definition 8 (Schur 1907, Hopf 1941)**.** *If $1 \to R \to F \to G \to 1$ is a free presentation of G, then $C \cong F/S$ for a normal subgroup $S \subseteq [F, R]$. In addition,*

$$M(G) \cong F^2 \cap R/[F, R]$$

where $[F, R]$ is the group generated by all elements of the form $frf^{-1}r^{-1}$ for $f \in F$ and $r \in R$.

We have one remaining equivalent definition for the Schur multiplier developed from central extensions.

Definition 9. *We call a group G^* a* central extension *of a group G if there exists an onto homomorphism $\Theta : G^* \to G$ such that $\ker \Theta \subseteq Z(G^*)$. If $\ker \Theta \subseteq Z(G^*) \cap (G^*)^2$ then G^* is called a* cover *of G.*

Definition 10. *Given a finite group, G, a* defining pair *is two groups (C, M) such that:*

$$(i) G \cong C/M$$
$$(ii) M \subseteq Z(C) \cap C^2.$$

In this definition, C is a cover and the Schur multiplier is isomorphic to the M paired with the C of maximum order. There may be more than one cover of maximum order, but if M_1 and M_2 are each paired with C_1 and C_2 of maximum order, respectively, then $M_1 \cong M_2 \cong M(G)$. Furthermore, C_1 and C_2 are not necessarily isomorphic, but have been proven to be *isoclinic* meaning $C_1/Z(C_1) \cong C_2/Z(C_2)$ and $C_1^2 \cong C_2^2$.

Now we provide three analogous definitions for the Schur multiplier of a Lie algebra, the first of which uses central extensions [16].

Definition 11. *A pair of Lie algebras (C, M) is a* defining pair *for Lie algebra L if*

$$(i)\ L \cong C/M \text{ and}$$
$$(ii)\ M \subseteq Z(C) \cap C^2.$$

If $\dim L = n$ then $\dim C \leq \frac{1}{2}n(n+1)$ and there is a C whose dimension is maximal. This is called a *cover* of L and the M paired with a maximal C is defined to be the *Schur multiplier*.

Unlike the group theory definition, covers of a Lie algebra are isomorphic [2]. We note that the Schur multiplier of a Lie algebra is central. Thus, computing the dimension of the Schur multiplier will completely categorize it as an abelian Lie algebra. The Schur multiplier can also be defined in terms of free Lie algebras [3].

Theorem 12. *Let L be a finite-dimensional Lie algebra and let $L \cong F/R$ where F is a free Lie algebra with ideal R. Then*

$$M(L) \cong \frac{F^2 \cap R}{[F, R]}.$$

Lastly, we can define the Schur multiplier as a second cohomology group [3].

Theorem 13. *Let L be a finite dimensional Lie algebra over a field $\mathcal{K}$. Consider $\mathcal{K}$ as a trivial $L-$module. Then,*

$$M(L) = H^2(L, \mathcal{K})$$

where $l \cdot k = 0$ for $l \in L,\ k \in \mathcal{K}$.

K. Moneyhun computed covers and Schur multipliers for several specific types of Lie algebras including the upper triangular matrices, triangular matrices, and Heisenberg algebras. In [15], L. Levy computed multipliers for the central series of strictly upper triangular matrices. P. Hardy considered an inverse of this problem. In [11] and [12] he searched for nilpotent Lie algebras whose central extension was as large as possible in regard to a certain invariant. Following Hardy's lead, we also study nilpotent Lie algebras and the same invariant.

Definition 14. *Given a Lie algebra, L, its lower central series is*

$$L = L^1 \supset L^2 \supset L^3 \supset \cdots \supset L^t \supset L^{t+1} \supset \cdots$$

where $L^i = [L^{i-1}, L]$. L is nilpotent *if the series terminates, meaning $L^t = 0$ for some t and all higher powers of L are also 0.*

Example 15. *An example of a nilpotent Lie algebra is the linear span of the set of strictly upper triangular $n \times n$ matrices. Recall that the bracket product is commutation. So, if $L = span_{\mathcal{K}}\{E_{ij} | i < j \leq n\}$ then $L^2 = span_{\mathcal{K}}\{E_{ij} | i + 1 < j \leq n\}$ and $L^n = span_{\mathcal{K}}\{E_{ij} | i + (n-1) < j \leq n\} = \{0\}$ where E_{ij} represents the elementary matrix with a 1 in the i^{th} row and j^{th} column and zeros elsewhere.*

Definition 16. *A nilpotent Lie algebra, L, has* class *c if c is the smallest integer for which $L^c \neq 0$ and $L^{c+1} = 0$.*

Example 17. *If L is the Lie algebra of strictly upper triangular matrices of size $n \times n$ of Example 15 then L has class $n-1$ since $L^{n-1} = span_{\mathcal{K}}\{E_{1,n}\}$ and $L^n = 0$.*

The group theory analogue of nilpotent Lie algebras are $p-$groups or, more generally, nilpotent groups.

Definition 18. *A $p-$group is any group of order p^n where p is prime and n is a nonnegative integer.*

We return now to Hardy who studied Schur multipliers of nilpotent Lie algebras in regard to an invariant. This invariant, t, was first developed in the study of Schur multipliers of $p-$groups. Since there is an upper bound on the order of the cover, there is also an upper bound on the order of the multiplier. Namely, for a group G of order p^n, $|M(G)| \leq p^{\frac{1}{2}n(n-1)}$ ([10]).

Definition 19. *For a group G of order p^n we define $t(G)$ as the nonnegative integer satisfying the equation*

$$|M(G)| = p^{\frac{1}{2}n(n-1)-t(G)}.$$

So, $t(G)$ is a measure of how far the order of the multiplier is from being maximal. We have a natural analogue of this in Lie algebras. Again, both the orders of the cover and multiplier are bounded above. In particular, $\dim M(L) \leq \frac{1}{2}n(n-1)$.

Definition 20. *For a Lie algebra L of dimension n, $t(L)$ is the nonnegative integer satisfying the equation*

$$\dim M(L) = \frac{1}{2}n(n-1) - t(L).$$

Remark 21. *It is shown in [16] that $\dim M(L) = \frac{1}{2}n(n-1)$ if and only if L is abelian. In this case, $t(L) = 0$.*

In this paper, we study a subset of nilpotent Lie algebras in regard to the invariant, $t(L)$ and arrive at an inequality involving the dimension of L and $t(L)$. Using the order of the group and $t(G)$, the analogue of this result is shown to hold in groups. This work can be found in Chapter 4 after we directly compute Lie algebras and their Schur multipliers for various $t(L)$ in Chapter 3. Furthermore, we obtain a new bound on the dimension of the Schur multiplier for nilpotent Lie algebras and compare it to an existing known bound. We also examine sufficient conditions for the the Schur multiplier of any nilpotent Lie algebra to be nontrivial. This result proves to be more elegant than theorems examining the conditions wherein the Schur multiplier of a group in nontrivial. These results can be found in Chapter 5.

Chapter 2

Preliminaries

Hardy categorized nilpotent Lie algebras and their Schur multipliers for $t(L) \leq 8$. What Hardy discovered was that for bigger values of t, one uncovered families of Lie algebras. It is still open for someone to examine larger values of t to find all Lie algebras and their corresponding Schur multipliers which satisfy $t(L) = \frac{1}{2}n(n-1) - \dim M(L)$ for a given $t > 8$. Rather than continue with Hardy's categorization of all nilpotent Lie algebras for larger values of t, we take a cue from Hall and Blackburn. They found that by putting restrictions on $p-$groups one could uncover more information. Therefore, we will take Hardy's approach on a subset of nilpotent Lie algebra which meet a certain requirement.

Definition 22. *If* $\dim L/L^2 = 2$ *and* $\dim L^i/L^{i+1} = 1$ *for* $i = 2, 3, \ldots, c$ *then we say* L *has* maximal class*. An equivalent definition is* $c = n - 1$ *where* $n = \dim L$*.*

Example 23. *The Lie algebra studied in Example 15 is non-abelian of maximal class only if* $n = 3$*.*

Maximal class Lie algebras were first studied by Vergne in the 1960's under the alias of filiform Lie algebras [17]. She defined and classified low dimension cases. Hall and Blackburn studied the group theory counterpart called maximal class $p-$groups. Again, their intention was to uncover information about $p-$groups by enforcing restrictions on the group. Before venturing forth, we include a few known results based on the central series of a nilpotent Lie algebra, which will be the focus of Chapter 3.

Definition 24. *The* lower central series *of a nilpotent Lie algebra,* L *is*

$$L = L^1 \supset L^2 \supset \cdots \supset L^c \supset L^{c+1} = 0$$

where $L^i = [L, L^{i-1}]$*.*

Maximal class Lie algebras have the longest possible lower central series of any nilpotent Lie algebras. Since the minimum number of elements required to generate a nilpotent Lie algebra is the dimension of L/L^2, Lie algebras of maximal class are two generated. Additionally, there is another important central series for a nilpotent Lie algebra.

Definition 25. *The* upper central series *of nilpotent L is*

$$0 = Z_0(L) \subset Z_1(L) \subset Z_2(L) \subset \ldots \subset Z_c(L) = L$$

where $Z_i(L)$ is the ideal in L such that $Z_i(L)/Z_{i-1}(L) = Z\left(L/Z_{i-1}(L)\right)$. Thus, $Z_1(L) = Z(L)$.

Maximal class Lie algebras have the property that $Z_i(L) = L^{c-i+1}$ for $0 \leq i \leq c$ where c is the class of L. In particular, we will use $Z(L) = L^c$, which implies that $\dim Z(L) = 1$ and $L/Z(L)$ is also a maximal class Lie algebra. With these definitions, we now develop several theorems needed to begin computing Schur multipliers of maximal class Lie algebras.

Theorem 26. *For any finite dimensional Lie algebra, L, over a field, $\mathcal{K}$,*

$$\dim L^2(\dim L^2 + 1) \leq 2t(L).$$

Proof. By Corollary 3.5 of [20],

$$\dim H^2(L, \mathcal{K}) \leq \dim H^2(L/L^2, \mathcal{K}) + (\dim L^2)(\dim L/L^2 - 1)$$

where $H^2(L, \mathcal{K})$ represents the second cohomology group of L over field $\mathcal{K}$. With Theorem 13,

$$\dim M(L) \leq \dim M(L/L^2) + (\dim L^2)(\dim(L/L^2) - 1).$$

Now, we use the invariant $t(L)$ and substitute in $\dim L = n,\ \dim L^2 = s$, and $\dim L/L^2 = n - s$ to obtain

$$\begin{aligned}
&\frac{1}{2}n(n-1) - t(L) \leq \frac{1}{2}(n-s)(n-s-1) - t(L/L^2) + s(n-s-1) \\
\Rightarrow &\frac{1}{2}n(n-1) - t(L) \leq \frac{1}{2}(n+s)(n-s-1) - t(L/L^2) \\
\Rightarrow &t(L) \geq \frac{1}{2}s(s+1) + t(L/L^2).
\end{aligned}$$

Since L/L^2 is abelian, $t(L/L^2) = 0$. Thus, $s^2 + s \leq 2t(L)$. $\square$

The next few results will be used throughout the following section. The group theory analogue of these can be found in [14] on pages 57-58. We provide the proofs of the Lie algebra versions for completeness.

Theorem 27. *Let L be a finite dimensional Lie algebra with ideal B and set $A = L/B$. Then there exists a finite dimensional Lie algebra G with an ideal M such that*

$$\begin{aligned} &(i)\ L^2 \cap B \cong G/M \\ &(ii)\ M \cong M(L) \\ &(iii)\ M(A) \text{ is a homomorphic image of } G. \end{aligned}$$

Proof. Let $0 \to R \to F \to L \to 0$ be a free presentation of L and suppose $B = S/R$ for some ideal S in F. Then $A = L/B \cong \frac{F/R}{S/R} \cong F/S$. Now, set $M = (F^2 \cap R)/[F, R]$ and $G = F^2 \cap S/[F, R]$. Then

$$\begin{aligned} L^2 \cap B &\cong (F/R)^2 \cap S/R \cong ((F^2 + R) \cap S)/R = ((F^2 \cap S) + R)/R \\ &\cong F^2 \cap S/(R \cap (F^2 \cap S)) = F^2 \cap S/F^2 \cap R \cong \frac{F^2 \cap S/[F, R]}{F^2 \cap R/[F, R]} \\ &= G/M. \end{aligned}$$

Thus, (i) has been proven. By definition, $M(L) \cong M$ and (ii) has been proven. Also by definition,

$$M(A) \cong F^2 \cap S/[F, S] \cong \frac{F^2 \cap S/[F, R]}{[F, S]/[F, R]} \cong \frac{G}{[F, S]/[F, R]}.$$

Therefore, $M(A)$ is the image of G under some homomorphism, whose kernel is $[F, S]/[F, R]$. □

This leads to the following result.

Corollary 28. *Let L be a finite dimensional Lie algebra, B be any ideal of L, and $A = L/B$. Then* $\dim M(A) \leq \dim M(L) + \dim(L^2 \cap B)$.

Proof. From Theorem 27, $\dim G = \dim(L^2 \cap B) + \dim M(L)$ and $\dim M(A) \leq \dim G$. Thus, $\dim M(A) \leq \dim M(L) + \dim(L^2 \cap B)$. □

Lemma 29. *Let L be a finite dimensional Lie algebra with free presentation*

$$0 \to R \to F \to L \to 0.$$

Let B be an ideal of L contained in $Z(L)$ with $B = S/R$. Define $A = L/B \cong F/S$. Then $[F, S]/([F, R] + S^2)$ is a homomorphic image of $A/A^2 \otimes B/B^2$.

Proof. Define

$$\lambda : A/A^2 \times B/B^2 \to [F,S]/([F,R]+S^2) \text{ by}$$
$$\lambda\big(f+(F^2+S), x+R\big) = [f,x] + ([F,R]+S^2) \text{ where } f \in F,\ x \in S.$$

Note that $A/A^2 \times B/B^2 = \frac{F/S}{(F^2+S)/S} \times S/R \cong F/(F^2+S) \times S/R$. We claim λ is well-defined. Let $f' \equiv f \pmod{F^2+S}$ and $x' \equiv x \pmod{R}$. This implies that $f' = f+g+s$ and $x' = x+r$ for some $g \in F^2, s \in S, r \in R$. Now,

$$[f',x'] = [f+g+s, x+r] = [f,x] + [f,r] + [g,x] + [g,r] + [s,x] + [s,r].$$

From here, notice that $[f,r] \in R$ and $[s,x],[s,r] \in S^2$ since R is an ideal in S, which is an ideal in F. Also,

$$[g,x] \in [F^2,S] = [[F,F],S] \subseteq [F,[S,F]] + [[S,F],F] \subseteq [F,R]$$

since $0 = [B,L] = [S/R, F/R]$ which implies $[S,F] \subseteq R$. Thus, $[f',x'] \equiv [f,x] \pmod{[F,R]+S^2}$ and λ is well-defined. So, there exists a unique homomorphism $\lambda^* : A/A^2 \otimes B/B^2 \to [F,S]/([F,R]+S^2)$. Since the linear span of the image of λ is $[F,S]/([F,R]+S^2)$, λ^* is onto. $\square$

Theorem 30. *Let L be a finite dimensional Lie algebra and $B \subseteq Z(L)$ an ideal. Let $A = L/B$. Then*

$$\dim M(L) + \dim L^2 \cap B \le \dim M(A) + \dim M(B) + \dim(A/A^2 \otimes B/B^2).$$

Proof. As before, begin with a free presentation of $L:\ 0 \to R \to F \to L \to 0$. Let B be a central ideal of L. Then $B = S/R$ for some ideal S of F. Then $A = L/B \cong F/S$. By Lemma 29 and its proof, $[F,S] \subseteq R$. By Theorem 27 and its proof,

$$\dim M(L) + \dim L^2 \cap B = \dim M(A) + \dim [F,S]/[F,R].$$

Since $\frac{[F,S]/[F,R]}{([F,R]+S^2)/[F,R]} \cong [F,S]/([F,R]+S^2)$,

$$\dim M(L) + \dim L^2 \cap B = \dim M(A) + \dim([F,R]+S^2)/[F,R] + \dim [F,S]/([F,R]+S^2).$$

But, $\left([F,R]+S^2\right)/[F,R] \cong S^2/\left([F,R] \cap S^2\right) \cong \frac{S^2/[S,R]}{(S^2 \cap [F,R])/[S,R]}$. Now, $S^2 \subseteq [F,S] \subseteq R \Rightarrow S^2/[S,R] = S^2 \cap R/[S,R] \cong M(B)$ since $0 \to R \to S \to B \to 0$ is a free presentation of B.

Thus,

$$\begin{aligned}
\dim M(L) + \dim L^2 \cap B &= \dim M(A) + \dim([F,R] + S^2)/[F,R] \\
&\quad + \dim[F,S]/([F,R] + S^2) \\
&= \dim M(A) + \dim M(B) - \dim \left(S^2 \cap [F,R]\right)/[S,R] \\
&\quad + \dim[F,S]/([F,R] + S^2) \\
&\leq \dim M(A) + \dim M(B) + \dim A/A^2 \otimes B/B^2.
\end{aligned}$$

□

This concludes the theorems which parallel group theory results found in [14]. As a consequence of Theorem 30 we have a new result.

Corollary 31. *If L is an $n-$dimensional nilpotent Lie algebra of maximal class, then*

$$t(L/Z(L)) + \dim L^2 \leq t(L).$$

Proof. Use Theorem 30 with $B = Z(L)$. Then,

$$\begin{aligned}
\dim M(L) + \dim L^2 \cap Z(L) &\leq \dim M(L/Z(L)) + \dim M(Z(L)) \\
&\quad + \dim \frac{L/Z(L)}{(L/Z(L))^2} \otimes \frac{Z(L)}{(Z(L))^2}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \dim M(L) + 1 \leq \dim M(L/Z(L)) + 0 + [n-1-(n-1-2)](1) \\
&\Rightarrow \dim M(L) + 1 \leq \dim M(L/Z(L)) + 2 \\
&\Rightarrow \frac{1}{2}n(n-1) - t(L) \leq \frac{1}{2}(n-1)(n-2) - t(L/Z(L)) + 1 \\
&\Rightarrow t(L/(Z(L)) + n - 2 \leq t(L) \\
&\Rightarrow t(L/(Z(L)) + \dim L^2 \leq t(L).
\end{aligned}$$

□

We will use Corollary 31 throughout the next chapter in which we begin computing Lie algebras of maximal class and small $t(L)$. In addition, we determine their Schur multipliers.

Chapter 3

Lie Algebras of Maximal Class and Small t(L)

We begin by computing Lie algebras of maximal class with $t(L) = 0$. This implies $\frac{1}{2}n(n-1) = \dim M(L)$ where n is the dimension of L. We continue the computations by increasing the value of $t(L)$ up to $t(L) = 16$. Recall, for an $n-$dimensional Lie algebra of maximal class, $\dim L^2 = n-2$. Using this fact and an algorithm for each $t(L) = \frac{1}{2}n(n-1) - \dim M(L)$, we compose all of our results in Table 3.1 following this section. We employ notation developed by Hardy through most of the computations.

3.1 Algorithm

For a given value of $t(L)$, the algorithm will proceed by first examining the result of Theorem 26 which states

$$\dim L^2 \left(\dim L^2 + 1\right) \leq t(L).$$

Since $\dim L^2 = n-2$, we can determine an upper bound for $\dim L = n$. Appealing to Corollary 31, which states that a maximal class Lie algebra must satisfy the equation

$$t(L/Z(L)) + \dim L^2 \leq t(L),$$

the possibilities for $t(L/Z(L))$ can be known. Since $L/Z(L)$ is a maximal class Lie algebra of dimension $n-1$, its categorization must be known (i.e. it will exist in our previous work). We can now determine L, a one-dimensional central extension of $L/Z(L)$.

Knowing the structure of L, we can determine its cover, C, and Schur multiplier, $M(L)$. By definition, $L \cong C/M(L)$ where $M(L) \subseteq Z(C) \cap C^2$. We look at those central extensions of L that are as large as possible. We examine all possible dependencies by employing change of variables and the Jacobi identity. The remaining central elements that also appear in the derived algebra of C are a basis for $M(L)$. With L and $M(L)$ known, we can also compute C. We show the details of this algorithm most thoroughly in the case $t(L) = 3$. For larger values of $t(L)$, the formalities are dropped, but provide the important details.

3.2 Computations

3.2.1 t(L)=0

Suppose $t(L) = 0$, then L is abelian. Since L is of maximal class, L is two-dimensional and denoted as $A(1)$. In particular, $L = span x, y$ with $[x, y] = 0$ and $M(L)$ is one-dimensional abelian.

3.2.2 t(L)=1

Suppose $t(L) = 1$. Then by Corollary 31, $\dim L^2 \leq 1$. If $\dim L^2 = 0$, then L is the two-dimensional abelian Lie algebra and $t(L) = 0$. Thus, $\dim L^2 = 1$ and $n = 3$ which gives us $L = \langle x, y, z\rangle$ and $t(L/Z(L)) = 0$. From above, $L/Z(L) \cong A(1)$ and since L is of maximal class, $\dim Z(L) = 1$. Therefore L has the following multiplication: $[x, y] = z$ where $Z(L) = \langle z\rangle$. We see that L is the three-dimensional Heisenberg Lie algebra, which we denote as $H(1)$. Then $M(L)$ is spanned by $\{s_1, s_2, s_3\}$ where

$$[x, y] = z + s_1 \qquad [x, z] = s_2 \qquad [y, z] = s_3.$$

We can relabel using $z' = z + s_1$. Since the Jacobi identity does not yield any dependencies, $M(L) = \langle s_2, s_3\rangle$ and $t(L) = \frac{1}{2}n(n-1) - 2 = 1$.

3.2.3 t(L)=2

Suppose $t(L) = 2$. Using Theorem 26, $\dim L^2(\dim L^2 + 1) \leq 4$. Thus, $\dim L^2 \leq 1$ and $n \leq 3$. Hence, $L \cong A(1)$ with $t(L) = 0$ or $L \cong H(1)$ with $t(L) = 1$. Therefore, there are no maximal class Lie algebras for which $t(L) = 2$.

3.2.4 t(L)=3

Suppose $t(L) = 3$. Then $\dim L^2(\dim L^2 + 1) \leq 6 \Rightarrow \dim L^2 \leq 2$. From above, $\dim L^2 \neq 0, 1$. If $\dim L^2 = 2$ then $n = 4$ and $t(L/Z(L)) \leq 1$ by Corollary 31. We know $t(L/Z(L)) = 0$ leads to $L = H(1)$ and $t(L) = 1$. Thus, we examine $t(L/Z(L)) = 1$ where $L/Z(L) \cong H(1)$ and

$Z(L) = \langle r \rangle$ for some $r \in L$. L has a basis of $\{x, y, z, r\}$ and nonzero multiplication:

$$[x, y] = z + \alpha_1 r,\ [x, z] = \alpha_2 r,\ [y, z] = \alpha_3 r$$

where α_2 and α_3 are not both zero. Without loss of generality, we can assume that $\alpha_2 \neq 0$ and then let $z' = z + \alpha_1 r$, $y' = y - \frac{\alpha_3}{\alpha_2} x$, $r' = \alpha_2 r$. This change of basis yields the following non-zero multiplication for $L : [x, y'] = z'$ and $[x, z'] = r'$. Now, for brevity, we eliminate the primes, leaving them to be understood.

Next, we will compute $M(L)$ using the fact that $L \cong C/M(L)$ and $M(L) \subseteq C^2 \cap Z(C)$ where C is the cover for L. Letting $\pi \in Hom(C, L)$ with $\ker \pi = M(L)$ we define a linear mapping μ to be a section for π. That is, $\mu : L \to C$ such that $\pi \circ \mu$ is the identity mapping on L. For $x, y \in L$ we have

$$\pi \left(\mu[x, y] - [\mu(x), \mu(y)]\right) = \pi \left(\mu[x, y]\right) - [\pi(\mu(x)), \pi(\mu(y))] = 0.$$

Thus, $\mu[x, y] - [\mu(x), \mu(y)] \in \ker \pi = M(L)$. To compute a basis for $M(L)$, we perform multiplications in $C = M(L) \oplus \mu(L)$. For the above $L = \langle x, y, z, r \rangle$ with aforementioned relations, the multiplication for C is given below. We use the simplified notation of x to represent $\mu(x)$ for all $x \in L$, leaving the action of μ to be understood.

$$\begin{array}{lll} [x, y] = z + s_1 & [x, z] = r + s_2 & [x, r] = s_3 \\ [y, z] = s_4 & [y, r] = s_5 & [z, r] = s_6 \end{array}$$

where $\{s_1, s_2, s_3, s_4, s_5, s_6\}$ generate $M(L)$ and $\{s_1, s_2, s_3, s_4, s_5, s_6, x, y, z, r\}$ generate C. Using the Jacobi identity in C we see that

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \Rightarrow s_5 = 0$$
$$[x, [y, r]] = [[x, y], r] + [y, [x, r]] \Rightarrow s_6 = 0$$

with all other Jacobi identities providing no new information. Using another change of basis in which we define $z' = z + s_1$ and $r' = r + s_2$ we conclude that $M(L) = \langle s_3, s_4 \rangle$. Thus, $t(L) = \frac{1}{2}(4)(3) - 2 = 4$. We denote this Lie algebra as $L(3, 4, 1, 4)$ where each of the four numbers represents, in order, the fact that this Lie algebra emerged from examining the $t(L) = 3$ case, the dimension of L, the dimension of $Z(L)$, and the value of $t(L)$. This notation has been adopted from [12]. Since $t(L) = 4$, there are no Lie algebras of maximal class for which $t(L) = 3$.

3.2.5 t(L)=4

Since we have examined the cases $t(L) = 0, 1, 2,$ and 3 now suppose $t(L) = 4$. Then $\dim L^2(\dim L^2 + 1) \leq 8$ and, thus, $\dim L^2 \leq 2$. We have already considered the cases where $\dim L^2 = 0$ and 1. Now, if $\dim L^2 = 2$, we can employ Corollary 31 to conclude that $t(L/Z(L)) \leq 2$. The cases in which $t(L/Z(L)) = 0$ and 1 will lead to $L \cong H(1)$ and $L(3,4,1,4)$, respectively. Since there are no maximal class Lie algebras for which $t(L) = 2$, $t(L/Z(L)) \neq 2$. Thus, there is only one Lie algebra of maximal class for which $t(L) = 4$, $L(3,4,1,4)$.

3.2.6 t(L)=5 & t(L)=6

We consider the cases of $t(L) = 5$ and 6 together. For a maximal class Lie algebra, L, we know that $L/Z(L)$ must also be of maximal class. Thus, if L is a new maximal class Lie algebra, $L/Z(L)$ is isomorphic to a maximal class Lie algebra we have already categorized. With Corollary 31 using $L/Z(L) \cong L(3,4,1,4)$, we conclude $t(L)$ is at least 7. Therefore, there are no Lie algebras of maximal class for which $t(L) = 5$ or 6.

3.2.7 t(L)=7

Suppose $t(L) = 7$. From the explanation above, we begin by considering $L/Z(L) \cong L(3,4,1,4)$. Thus, $L \cong \langle x, y, z, r, c \rangle$ with multiplication

$$\begin{array}{lll} [x,y] = z + \alpha_1 c & [x,z] = r + \alpha_2 c & [x,r] = \alpha_3 c \\ [y,z] = \alpha_4 c & [y,r] = \alpha_5 c & [z,r] = \alpha_6 c. \end{array}$$

By a change of basis, we can conclude $\alpha_1 = \alpha_2 = 0$. Using the Jacobi identities we get $\alpha_5 = \alpha_6 = 0$. Since $r \notin Z(L) = \langle c \rangle$, $\alpha_3 \neq 0$. Now we have two cases, either $\alpha_4 = 0$ or $\alpha_4 \neq 0$.

Case 1: $\alpha_4 = 0$

If $\alpha_4 = 0$, then our resulting Lie algebra has multiplication $[x,y] = z$, $[x,z] = r$, $[x,r] = c$ and to compute $M(L)$ we begin with the following multiplication:

$$\begin{array}{lllll} [x,y] = z + s_1 & [x,z] = r + s_2 & [x,r] = c + s_3 & [x,c] = s_4 & [y,z] = s_5 \\ [y,r] = s_6 & [y,c] = s_7 & [z,c] = s_8 & [z,r] = s_9 & [r,c] = s_{10}. \end{array}$$

We relabel to achieve $s_1, s_2, s_3 = 0$. Expanding with the Jacobi identity, we obtain

$$\begin{aligned}
[x,[y,z]] &= [[x,y],z] + [y,[x,z]] \Rightarrow s_6 = 0 \\
[x,[y,r]] &= [[x,y],r] + [y,[x,r]] \Rightarrow s_9 = -s_7 \\
[x,[y,c]] &= [[x,y],c] + [y,[x,c]] \Rightarrow s_8 = 0 \\
[x,[z,c]] &= [[x,z],c] + [z,[x,c]] \Rightarrow s_{10} = 0.
\end{aligned}$$

So $M(L) = \langle s_4, s_5, s_7 \rangle$ and $t(L) = \frac{1}{2}(5)(4) - 3 = 7$. We label this Lie algebra as $L(7,5,1,7)$.

Case 2: $\alpha_4 \neq 0$

If $\alpha_4 \neq 0$, we can relabel to get

$$[x,y] = z,\ [x,z] = r,\ [x,r] = c,\ [y,z] = \alpha c$$

then let $y' = \frac{1}{\alpha}y,\ z' = \frac{1}{\alpha}z,\ r' = \frac{1}{\alpha}r\ c' = \frac{1}{\alpha}c$. The multiplication becomes

$$[x,y'] = z',\ [x,z'] = r',\ [x,r'] = c',\ [y',z'] = c'.$$

Thus, in $M(L)$ we have

$$\begin{array}{lllll}
[x,y] = z + s_1 & [x,z] = r + s_2 & [x,r] = c + s_3 & [x,c] = s_4 & [y,z] = c + s_5 \\
[y,r] = s_6 & [y,c] = s_7 & [z,c] = s_8 & [z,r] = s_9 & [r,c] = s_{10}.
\end{array}$$

Notice that we have relabeled to drop the use of the primes. We relabel again to achieve $s_1 = s_2 = s_3 = 0$. Expanding with the Jacobi identity, we obtain

$$\begin{aligned}
[x,[y,z]] &= [[x,y],z] + [y,[x,z]] \Rightarrow s_4 = s_6 \\
[x,[y,r]] &= [[x,y],r] + [y,[x,r]] \Rightarrow s_9 = -s_7 \\
[x,[y,c]] &= [[x,y],c] + [y,[x,c]] \Rightarrow s_8 = 0 \\
[x,[z,c]] &= [[x,z],c] + [z,[x,c]] \Rightarrow s_{10} = 0
\end{aligned}$$

So $M(L) = \langle s_4, s_5, s_7 \rangle$ and $t(L) = \frac{1}{2}(5)(4) - 3 = 7$. We label this Lie algebra as $L'(7,5,1,7)$.

3.2.8 t(L)=8, 9 & 10

We will handle the cases of $t(L) = 8, 9,$ and 10 together. Since $L/Z(L)$ is a Lie algebra of maximal class, we can conclude the next case in which we potentially derive a new maximal class Lie algebra occurs when $L/Z(L) \cong L(7,5,1,7)$ or $L'(7,5,1,7)$. For either of these cases, $t(L) = 11$ is the first case to consider. We derive this from Corollary 31 with dim $L^2 = 4$. Thus, there are no Lie algebras of maximal class for which $t(L) = 8,\ 9$, or 10.

3.2.9 t(L)=11 & L/Z(L) $\cong$ L(7,5,1,7)

If $t(L) = 11$ then $L/Z(L) \cong L(7,5,1,7)$ or $L'(7,5,1,7)$. If $L/Z(L) \cong L(7,5,1,7)$ then $L = \langle x, y, z, r, c, d\rangle$ with multiplication

$$\begin{array}{lllll} [x,y] = z + \alpha_1 d & [x,z] = r + \alpha_2 d & [x,r] = c + \alpha_3 d & [x,c] = \alpha_4 d & [y,z] = \alpha_5 d \\ [y,r] = \alpha_6 d & [y,c] = \alpha_7 d & [z,r] = \alpha_8 d & [z,c] = \alpha_9 d & [r,c] = \alpha_{10} d. \end{array}$$

Using a change of variable we obtain $\alpha_1, \alpha_2, \alpha_3 = 0$. As before, we use the Jacobi identity to eliminate several variables. Specifically, we obtain that $\alpha_6 = \alpha_9 = \alpha_{10} = 0$ and $\alpha_8 = -\alpha_7$. Next, expand the following Jacobi identities

$$\begin{aligned} [x,[y,z]] &= [[x,y],z] + [y,[x,z]] \Rightarrow \alpha_6 = 0 \\ [x,[y,r]] &= [[x,y],r] + [y,[x,r]] \Rightarrow \alpha_8 = -\alpha_7 \\ [x,[y,r]] &= [[x,y],r] + [y,[x,r]] \Rightarrow \alpha_9 = 0 \\ [x,[z,c]] &= [[x,z],c] + [z,[x,c]] \Rightarrow \alpha_{10} = 0 \end{aligned}$$

Our multiplication is now

$$\begin{array}{llll} [x,y] = z & [x,z] = r & [x,r] = c & [x,c] = \alpha_4 d \\ [y,z] = \alpha_5 d & [y,c] = \alpha_7 d & [z,r] = -\alpha_7 d. & \end{array}$$

Since $c \notin Z(L) = \langle d\rangle$, α_4 and α_7 are not both 0.

Case 1: $\alpha_7 = 0 = \alpha_5$

We first assume $\alpha_7 = 0$. Then $\alpha_4 \neq 0$. If α_5 is also 0 then our multiplication with a change of variable becomes

$$[x,y] = z,\ [x,z] = r,\ [x,r] = c,\ [x,c] = d.$$

We will denote this Lie algebra as $L_1(11)$.

Case 2: $\alpha_7 = 0 \ \&\alpha_5 \neq 0$

If $\alpha_5 \neq 0$ we let $\alpha = \frac{\alpha_4}{\alpha_5}$ then

$$x' = \alpha x,\ y' = \alpha^4 y,\ z' = \alpha^5 z,\ r' = \alpha^6 r,\ c' = \alpha^7 c,\ d' = \alpha^9 \alpha_5 d.$$

So, our multiplication becomes

$$[x', y'] = z' \quad [x', z'] = r' \ [x', r'] = c' \quad \left[x', c'\right] = d' \quad \left[y', z'\right] = d'.$$

We will denote this Lie algebra as $L_2(11)$.

Case 3: $\alpha_7 \neq 0$

If $\alpha_7 \neq 0$ then we define the following variables

$$x' = x - \frac{\alpha_4}{\alpha_7} y,\ y' = y - \frac{\alpha_5}{2\alpha_7} r,\ z' = z - \frac{\alpha_5}{2\alpha_7} c,\ r' = r - \frac{\alpha_4 \alpha_5}{\alpha_7} d,\ d' = \alpha_7 d.$$

So, our nonzero multiplication is

$$[x', y'] = z',\ [x', z'] = r',\ [x', r'] = c,\ [x', c] = 0,\ [y', z'] = 0,\ [y', c] = d',\ [z', r'] = -d'.$$

We will denote this Lie algebra as $L_3(11)$.

$L_1(11)$ **Computations**

Now, we must compute the multipliers for these three algebras, beginning with $L = L_1(11)$. Using the multiplication listed above, its multiplication in the cover will be

$$\begin{array}{lllll} [x, y] = z + s_1 & [x, z] = r + s_2 & [x, r] = c + s_3 & [x, c] = d + s_4 & [x, d] = s_5 \\ [y, z] = s_6 & [y, r] = s_7 & [y, c] = s_8 & [y, d] = s_9 & [z, r] = s_{10} \\ [z, c] = s_{11} & [z, d] = s_{12} & [r, c] = s_{13} & [r, d] = s_{14} & [c, d] = s_{15}. \end{array}$$

Using a change of variable, we obtain $s_1, s_2, s_3, s_4 = 0$. Using Jacobi identities, $s_7 = s_9 = s_{11} = s_{12} = s_{13} = s_{14} = s_{15} = 0$, and $s_{10} = -s_8$. Specifically,

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]] \Rightarrow s_7 = 0$$
$$[x,[y,r]] = [[x,y],r] + [y,[x,r]] \Rightarrow s_{10} = -s_8$$
$$[x,[y,c]] = [[x,y],c] + [y,[x,c]] \Rightarrow s_{11} = -s_9$$
$$[x,[y,d]] = [[x,y],d] + [y,[x,d]] \Rightarrow s_{12} = 0$$
$$[x,[z,r]] = [[x,z],r] + [z,[x,r]] \Rightarrow s_{11} = 0$$
$$[x,[z,c]] = [[x,z],c] + [z,[x,c]] \Rightarrow s_{13} = 0$$
$$[x,[z,d]] = [[x,z],d] + [z,[x,d]] \Rightarrow s_{14} = 0$$
$$[x,[r,d]] = [[x,r],d] + [r,[x,d]] \Rightarrow s_{15} = 0$$

Then $M(L) = \langle s_5, s_6, s_8 \rangle$ and $t(L) = \frac{1}{2}(6)(5) - 3 = 12$. We denote this Lie algebra as $L_1(11) = L(11, 6, 1, 12)$.

$L_2(11)$ **Computations**

Let us now examine $L = L_2(11)$ whose multiplication in C will be

$$\begin{array}{lllll} [x,y] = z + s_1 & [x,z] = r + s_2 & [x,r] = c + s_3 & [x,c] = d + s_4 & [x,d] = s_5 \\ [y,z] = d + s_6 & [y,r] = s_7 & [y,c] = s_8 & [y,d] = s_9 & [z,r] = s_{10} \\ [z,c] = s_{11} & [z,d] = s_{12} & [r,c] = s_{13} & [r,d] = s_{14} & [c,d] = s_{15}. \end{array}$$

A change of variables will mean that $s_1, s_2, s_3, s_4 = 0$. Using the Jacobi identities, we obtain $s_5 = s_7$, $s_{10} = -s_8$, and $s_9 = s_{11} = s_{12} = s_{13} = s_{14} = s_{15} = 0$. Specifically,

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]] \Rightarrow s_5 = s_7$$
$$[x,[y,r]] = [[x,y],r] + [y,[x,r]] \Rightarrow s_{10} = -s_8$$
$$[x,[y,c]] = [[x,y],c] + [y,[x,c]] \Rightarrow s_{11} = -s_9$$
$$[x,[y,d]] = [[x,y],d] + [y,[x,d]] \Rightarrow s_{12} = 0$$
$$[x,[z,r]] = [[x,z],r] + [z,[x,r]] \Rightarrow s_{11} = 0$$
$$[x,[z,c]] = [[x,z],c] + [z,[x,c]] \Rightarrow s_{13} = 0$$
$$[x,[z,d]] = [[x,z],d] + [z,[x,d]] \Rightarrow s_{14} = 0$$
$$[x,[r,d]] = [[x,r],d] + [r,[x,d]] \Rightarrow s_{15} = 0$$

Then $M(L) = \langle s_5, s_6, s_8\rangle$ and $t(L) = \frac{1}{2}(6)(5) - 3 = 12$. We denote this Lie algebra as $L_2(11) = L'(11, 6, 1, 12)$.

$L_3(11)$ **Computations**

Lastly we examine $L = L_3(11)$ whose multiplication in C will be

$$\begin{array}{lllll} [x,y] = z + s_1 & [x,z] = r + s_2 & [x,r] = c + s_3 & [x,c] = s_4 & [x,d] = s_5 \\ [y,z] = s_6 & [y,r] = s_7 & [y,c] = d + s_8 & [y,d] = s_9 & [z,r] = -d + s_{10} \\ [z,c] = s_{11} & [z,d] = s_{12} & [r,c] = s_{13} & [r,d] = s_{14} & [c,d] = s_{15}. \end{array}$$

A change of variables will mean that $s_1, s_2, s_3, s_8 = 0$ Using the Jacobi identities, we obtain $s_5 = s_7 = s_9 = s_{10} = s_{11} = s_{12} = s_{13} = s_{14} = s_{15} = 0$. Specifically,

$$\begin{aligned} [x,[y,z]] &= [[x,y],z] + [y,[x,z]] \Rightarrow s_7 = 0 \\ [x,[y,r]] &= [[x,y],r] + [y,[x,r]] \Rightarrow s_{10} = 0 \\ [x,[y,c]] &= [[x,y],c] + [y,[x,c]] \Rightarrow s_5 = s_{11} \\ [x,[y,d]] &= [[x,y],d] + [y,[x,d]] \Rightarrow s_{12} = 0 \\ [x,[z,r]] &= [[x,z],r] + [z,[x,r]] \Rightarrow -s_5 = s_{11} \\ [x,[z,c]] &= [[x,z],c] + [z,[x,c]] \Rightarrow s_{13} = 0 \\ [x,[z,d]] &= [[x,z],d] + [z,[x,d]] \Rightarrow s_{14} = 0 \\ [x,[r,d]] &= [[x,r],d] + [r,[x,d]] \Rightarrow s_{15} = 0 \\ [y,[z,r]] &= [[y,z],r] + [z,[y,r]] \Rightarrow s_9 = 0 \end{aligned}$$

Then $M(L) = \langle s_4, s_6\rangle$ and $t(L) = \frac{1}{2}(6)(5) - 2 = 13$. We denote this Lie algebra as $L_3(11) = L(11, 6, 1, 13)$.

3.2.10 t(L)=11 & L/Z(L) $\cong$ L$'$(7,5,1,7)

We must also consider if $L/Z(L) \cong L'(7, 5, 1, 7)$ which would give $L = \langle x, y, z, r, c, d\rangle$ with the following multiplication

$$\begin{array}{lllll} [x,y] = z + \alpha_1 d & [x,z] = r + \alpha_2 d & [x,r] = c + \alpha_3 d & [x,c] = \alpha_4 d & [y,z] = c + \alpha_5 d \\ [y,r] = \alpha_6 d & [y,c] = \alpha_7 d & [z,r] = \alpha_8 d & [z,c] = \alpha_9 d & [r,c] = \alpha_{10} d. \end{array}$$

Using a change of variable, $\alpha_1, \alpha_2, \alpha_3 = 0$. Next, expand using the Jacobi identities to obtain $\alpha_4 = \alpha_6,\ \alpha_8 = -\alpha_7$, and $\alpha_9 = \alpha_{10} = 0$. Specifically,

$$\begin{aligned}
[x,[y,z]] &= [[x,y],z] + [y,[x,z]] \Rightarrow \alpha_4 = \alpha_6 \\
[x,[y,r]] &= [[x,y],r] + [y,[x,r]] \Rightarrow \alpha_9 = -\alpha_7 \\
[x,[y,c]] &= [[x,y],c] + [y,[x,c]] \Rightarrow \alpha_8 = 0 \\
[x,[z,c]] &= [[x,z],c] + [z,[x,c]] \Rightarrow \alpha_{10} = 0
\end{aligned}$$

Now our multiplication is

$$\begin{array}{llll}
[x,y] = z & [x,z] = r & [x,r] = c & [x,c] = \alpha_4 d \\
[y,z] = c + \alpha_5 d & [y,r] = \alpha_4 d & [y,c] = \alpha_7 d & [z,r] = -\alpha_7 d.
\end{array}$$

Since $c \notin Z(L)$, α_4 and α_7 are not both 0.

Case 1: $\alpha_7 = 0$

Suppose $\alpha_7 = 0$ and perform the following change of variables

$$x' = x + \frac{\alpha_5}{2\alpha_4} y,\ r' = r + \frac{\alpha_5}{2\alpha_4}(c + \alpha_5 d),\ c' = c + \alpha_5 d,\ d' = \alpha_4 d.$$

Then we have

$$[x',y] = z,\ [x',z] = r',\ [x',r'] = c',\ [x',c'] = d',\ [y,z] = c',\ [y,r'] = d'.$$

We denote this Lie algebra as $L_1'(11)$.

Case 2: $\alpha_7 \neq 0$

Now, suppose $\alpha_7 \neq 0$ and change variables as follows:

$$\begin{array}{lll}
x' = x - \frac{\alpha_4}{\alpha_7} y, & y' = y + \frac{\alpha_4}{\alpha_7} z - \frac{\alpha_5}{2\alpha_7} r, & z' = z + \frac{\alpha_4}{\alpha_7} r - \left(\frac{\alpha_5}{2\alpha_7} + \frac{\alpha_4^2}{\alpha_7^2}\right) c - \frac{\alpha_4^2 \alpha_5}{2\alpha_7^2} d \\
r' = r - \left(\frac{\alpha_4 \alpha_5}{\alpha_7} + \frac{\alpha_4^3}{\alpha_7^2}\right) d, & c' = c - \frac{\alpha_4^2}{\alpha_7} d & d' = \alpha_7 d.
\end{array}$$

Then our multiplication becomes

$$[x',y'] = z',\ [x',z'] = r',\ [x',r'] = c',\ [y',z'] = c',\ [y',c'] = d',\ [z',r'] = -d'$$

and we denote the Lie algebra as $L_2'(11)$.

$L_1'(11)$ **Computations**

To compute the multiplier of $L = L_1'(11)$, we first look at the multiplication in C which is

$$\begin{array}{lllll} [x,y] = z + s_1 & [x,z] = r + s_2 & [x,r] = c + s_3 & [x,c] = d + s_4 & [x,d] = s_5 \\ [y,z] = c + s_6 & [y,r] = d + s_7 & [y,c] = s_8 & [y,d] = s_9 & [z,r] = s_{10} \\ [z,c] = s_{11} & [z,d] = s_{12} & [r,c] = s_{13} & [r,d] = s_{14} & [c,d] = s_{15}. \end{array}$$

Using a change of variable, $s_1 = s_2 = s_3 = s_4 = 0$. Proceed by examining the Jacobi identities to yield $s_7 = s_9 = s_{11} = s_{12} = s_{13} = s_{14} = s_{15} = 0$ and $s_5 = s_{10} + s_8$. Specifically,

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]] \Rightarrow s_7 = 0$$
$$[x,[y,r]] = [[x,y],r] + [y,[x,r]] \Rightarrow s_5 = s_{10} + s_8$$
$$[x,[y,c]] = [[x,y],c] + [y,[x,c]] \Rightarrow s_9 = -s_{11}$$
$$[x,[y,d]] = [[x,y],d] + [y,[x,d]] \Rightarrow s_{12} = 0$$
$$[x,[z,r]] = [[x,z],r] + [z,[x,r]] \Rightarrow s_{11} = 0$$
$$[x,[z,c]] = [[x,z],c] + [z,[x,c]] \Rightarrow s_{12} = -s_{13}$$
$$[x,[z,d]] = [[x,z],d] + [z,[x,d]] \Rightarrow s_{14} = 0$$
$$[x,[r,d]] = [[x,r],d] + [r,[x,d]] \Rightarrow s_{15} = 0$$

Thus, we have $M(L) = \langle s_5, s_6, s_8 \rangle$ and $t(L) = \frac{1}{2}(6)(5) - 3 = 12$. We now denote this Lie algebra as $L''(11, 6, 1, 12)$.

$L_2'(11)$ **Computations**

Lastly, we wish to compute the multiplier of $L = L_2'(11)$ which has multiplication

$$\begin{array}{lllll} [x,y] = z + s_1 & [x,z] = r + s_2 & [x,r] = c + s_3 & [x,c] = s_4 & [x,d] = s_5 \\ [y,z] = c + s_6 & [y,r] = s_7 & [y,c] = d + s_8 & [y,d] = s_9 & [z,r] = -d + s_{10} \\ [z,c] = s_{11} & [z,d] = s_{12} & [r,c] = s_{13} & [r,d] = s_{14} & [c,d] = s_{15}. \end{array}$$

Using a change of variable, $s_1 = s_2 = s_3 = s_8 = 0$. Proceed by examining Jacobi identities to yield $s_4 = s_7$ and $s_5 = s_9 = s_{10} = s_{11} = s_{12} = s_{13} = s_{14} = s_{15} = 0$. Specifically,

$$
\begin{aligned}
[x,[y,z]] = [[x,y],z] + [y,[x,z]] &\Rightarrow s_4 = s_7 \\
[x,[y,r]] = [[x,y],r] + [y,[x,r]] &\Rightarrow s_{10} = -s_8 \\
[x,[y,c]] = [[x,y],c] + [y,[x,c]] &\Rightarrow s_5 = s_{11} \\
[x,[y,d]] = [[x,y],d] + [y,[x,d]] &\Rightarrow s_{12} = 0 \\
[x,[z,r]] = [[x,z],r] + [z,[x,r]] &\Rightarrow -s_5 = s_{11} \\
[x,[z,c]] = [[x,z],c] + [z,[x,c]] &\Rightarrow s_{13} = 0 \\
[x,[z,d]] = [[x,z],d] + [z,[x,d]] &\Rightarrow s_{14} = 0 \\
[x,[r,d]] = [[x,r],d] + [r,[x,d]] &\Rightarrow s_{15} = 0 \\
[y,[z,r]] = [[y,z],r] + [z,[y,r]] &\Rightarrow s_9 = s_{13}
\end{aligned}
$$

Thus, we have $M(L) = \langle s_4, s_6 \rangle$ and $t(L) = \frac{1}{2}(6)(5) - 2 = 13$. We now denote this Lie algebra as $L'(11, 6, 1, 13)$. In conclusion, there are no Lie algebras of maximal class for which $t(L) = 11$, three for which $t(L) = 12$, and two for which $t(L) = 13$.

3.2.11 t(L)=14, 15, & 16

Since $L/Z(L)$ is a Lie algebra of maximal class, we can conclude the next case in which we potentially derive a new maximal class Lie algebra occurs when $L/Z(L)$ is isomorphic to $L(11, 6, 1, 12)$ or $L'(11, 6, 1, 12)$ or $L''(11, 6, 1, 12)$. For this case, $t(L) = 17$ is the first to consider since we need $\dim L^2 = 5$ and $t(L/Z(L)) + \dim L^2 \leq t(L)$ from Corollary 31. Thus, there are no Lie algebras of maximal class for which $t(L) = 14, 15$, or 16.

3.3 Remarks

We can continue to compute Schur multipliers with this method or, more interestingly, we can examine our results as seen in Table 3.1 and conjecture on the relationship between the dimension of L and $t(L)$. This will be the content of the next chapter.

Table 3.1: Maximal Class Lie Algebras

t(L)	Name	Basis	Non-Zero Multiplication
0	A(1)	$\{x, y\}$	
1	H(1)	$\{x, y, z\}$	[x,y]=z
2	none		
3	none		
4	L(3,4,1,4)	$\{x, y, z, r\}$	[x,y]=z, [x,z]=r
5	none		
6	none		
7	L(7,5,1,7)	$\{x, y, z, r, c\}$	[x,y]=z, [x,z]=r, [x,r]=c
7	L′(7,5,1,7)	$\{x, y, z, r, c\}$	[x,y]=z, [x,z]=r, [x,r]=c, [y,z]=c
8	none		
9	none		
10	none		
11	none		
12	L(11,6,1,12)	$\{x, y, z, r, c, d\}$	[x,y]=z, [x,z]=r, [x,r]=c, [x,c]=d
12	L′(11,6,1,12)	$\{x, y, z, r, c, d\}$	[x,y]=z, [x,z]=r, [x,r]=c, [x,c]=d, [y,z]=d
12	L″(11,6,1,12)	$\{x, y, z, r, c, d\}$	[x,y]=z, [x,z]=r, [x,r]=c, [x,c]=d, [y,z]=c, [y,r]=d
13	L(11,6,1,13)	$\{x, y, z, r, c, d\}$	[x,y]=z, [x,z]=r, [x,r]=c, [y,c]=d, [r,z]=d
13	L′(11,6,1,13)	$\{x, y, z, r, c, d\}$	[x,y]=z, [x,z]=r, [x,r]=c, [y,z]=c, [y,c]=d, [r,z]=d
14	none		
15	none		
16	none		

Chapter 4

Bounding the Dimension of L given t(L)

If L is an $n-$dimensional Lie algebra then by Theorem 26, $\frac{1}{2}(n-1)(n-2) \leq t(L)$. By definition we have $t(L) \leq \frac{1}{2}n(n-1)$. Thus, we obtain the compound inequality for all finite-dimensional Lie algebras

$$(n-1)(n-2) \leq 2t(L) \leq n(n-1). \tag{4.1}$$

Eq. 4.1 indicates that a given value for $t(L)$ will result in two possible choices for n. For instance if $t(L) = 3$, then both $n = 3$ and $n = 4$ satisfy Eq. 4.1. From the results collected in Table 3.1, we do not see that to be the case for maximal class Lie algebras. Specifically, there appears to be only one possible dimension of L for any given $t(L)$. Accordingly, we conjecture that the lower bound can be made a strict inequality: $(n-1)(n-2) < 2t(L)$ when $n > 3$. Additionally, this Lie algebra result indicates that a similar conclusion may be made in group theory. This is the content of the following.

Theorem 32. *Let L be a Lie algebra of maximal class with* $\dim L = n > 3$. *Then*

$$(n-2)(n-1) < 2t(L) \leq (n-1)n.$$

Proof. Let $t = t(L)$ and suppose $(n-2)(n-1) = 2t$. Let $B = Z(L)$. Since L is of maximal class, L/B is of maximal class with $\dim(L/B) = n-1$. With Theorem 26, $\frac{1}{2}(n-2)(n-3) \leq t(L/B)$. By Corollary 31, $t(L/B) \leq t-(n-2) = \frac{1}{2}(n-2)(n-3)$. Thus, $t(L/B) = \frac{1}{2}(n-2)(n-3)$. We can continue this process of factoring out by the center to obtain a maximal class Lie algebra, J of maximal class with $\dim J = 4$. This implies $t(J) = 3$. Upon examining Table 3.1, we see no such algebra J exists. Hence, the assumption that $(n-2)(n-1) = 2t$ has been contradicted and the result holds. □

We summarize this for all $t(L)$ by stating

Corollary 33. *For each $t \in \mathbb{Z}^+$, for which there exists a maximal class Lie algebra, L with $t(L) = t$, the dimension of L is unique.*

Having established a result on the relationship between the dimension of a maximal class Lie algebra, L, and $t(L)$ we conjecture that an analogous result holds for a group of maximal class. We must first address the necessary definitions and propositions to build up the machinery to prove this conjecture.

Definition 34. *The* lower central series *for a group, G, is*

$$G = G^1 \supseteq G^2 \supseteq G^3 \supseteq G^4 \supseteq \cdots$$

where $G^i = [G, G^{i-1}]$, the subgroup formed by $ghg^{-1}h^{-1}$ for $g \in G$ and $h \in G^{i-1}$.

Definition 35. *G is* nilpotent *if there exists some positive integer c such that $G^c \neq \{1\}$ and $\{1\} = G^t$ for all $t > c$. Furthermore, G is said to have* class c *if c is the least such integer for which $G^c \neq \{1\}$ and $G^{c+1} = \{1\}$.*

Definition 36. *A group with order p^n has* maximal class *if its class is $n-1$.*

Recall, that if G is a group of order p^n, where p is prime then $|M(G)|$ divides $p^{\frac{1}{2}n(n-1)}$ where $M(G)$ is the Schur multiplier of G. We defined $t(G)$ to be the non-negative integer which makes the following equation true:

$$|M(G)| = p^{\frac{1}{2}n(n-1)-t(G)}.$$

Whereas nilpotent Lie algebras have been classified for $t(L) \leq 8$, only the $p-$groups with $t(G) \leq 3$ have been determined ([8], [21], and [5]) . Since results for groups are more limited, we have not made a conjecture based on them. Rather, our initial prediction is that a Lie algebra result will hold true for groups. Thus, for a maximal class group of order p^n we will derive the analogue of Theorem 32 showing that

$$(n-1)(n-2) < 2t(G) \leq (n-1)n.$$

For the remainder of this chapter we will take G to be of maximal class and of order p^n, $n > 2$ and p is prime unless otherwise noted. It will be of use to note that $G/Z(G)$ is of maximal class with $|Z(G)| = p$. As was the case in maximal class Lie algebras, both G and $G/Z(G)$ are two-generated. Now, we introduce several results, which are proved in [14].

Theorem 37. *(Schur 1907) Let G be a group with $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$, where $n_{i+1}|n_i$ for all $i \in 1, \ldots, k-1$ and $k \geq 2$, and let $\mathbb{Z}_n^{(m)}$ denote the direct product of m copies of $\mathbb{Z}_n$. Then*

$$M(G) \cong \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3}^{(2)} \times \cdots \times \mathbb{Z}_{n_k}^{(k-1)}.$$

Corollary 38. *If $G \cong \underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{n-times}$, $n \geq 2$ and p, prime then $M(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p^{(2)} \times \cdots \times \mathbb{Z}_p^{(n-1)}$. Thus, $|M(G)| = p^{\frac{1}{2}n(n-1)}$.*

Definition 39. *The Frattini subgroup of a group, G is the intersection of all maximal subgroups of G and is denoted as $\Phi(G)$.*

Definition 40. *A group, G, is said to be an* elementary abelian $p-$group *if it is finite abelian and every nontrivial element has order p.*

Lemma 41. *Let G be a finite p-group and let $d(G)$ denote the minimal number of generators of G. Then $G/\Phi(G)$ is elementary abelian of order p^d where $d = d(G)$.*

Theorem 42. *If $|G| = p^n$, $n > 2$, and G is of maximal class then G/G^2 is elementary abelian.*

Proof. Since G is of maximal class, $|G/G^2| = p^2$. For any p-group, $G^2 \subseteq \Phi(G) \subsetneq G$. If G has two distinct maximal subgroups M and N then $|M \cap N| = p^{n-2}$ and

$$p^{n-2} = |G^2| \leq |\Phi(G)| \leq |M \cap N| = p^{n-2}.$$

Thus, $G^2 = \Phi(G)$ since $G^2 \subseteq \Phi(G)$ and $|G^2| = |\Phi(G)|$ and we conclude that G/G^2 is elementary abelian by Lemma 41. Now, suppose G has only one maximal subgroup, M and choose $x \in G$, $x \notin M$. This implies $\langle x \rangle \nsubseteq M$. Since M is a maximal subgroup, $\langle x \rangle = G$ and G is cyclic and, thus, abelian with $G^2 = \{1\}$. Hence, $|G| \leq p^2$, which is a contradiction. Therefore, the result holds. □

Theorem 43. *Let G be a p-group of maximal class with $|G| = p^n$ where $n > 3$ and p is prime. Then*

$$(n-2)(n-1) \leq 2t(G) \leq (n-1)n.$$

Proof. From [9], we know $|M(G)| \leq |M(G/G^2)|\ |G^2|^{d(G/Z(G))-1}$ where $d(G)$ denotes the minimal number of generators of G. So,

$$\begin{aligned}
\frac{1}{2}n(n-1) - t(G) &\leq \log_p|M(G/G^2)| + (d(G/Z(G)) - 1)\log_p|G^2| \\
&= \log_p p + (d(G/Z(G)) - 1)\log_p p^{n-2} \\
&= 1 + (2-1)(n-2) = n-1.
\end{aligned}$$

Thus,

$$\begin{aligned} &\frac{1}{2}n(n-1) - (n-1) \leq t(G) \\ &\Rightarrow \frac{1}{2}(n-1)(n-2) \leq t(G). \end{aligned} \tag{4.2}$$

By definition, $t(G) \leq \frac{1}{2}n(n-1)$. Together with Eq. 4.2, we have the result. □

From Theorem 2.5.5 in [14],

Theorem 44. *For any finite group G with central subgroup B and $A = G/B$,*

$$\left|M(G)\right| \left|G^2 \cap B\right| \textit{ divides } \left|M(A)\right| \left|M(B)\right| \left|A/A^2 \otimes B/B^2\right|.$$

This theorem is used to prove the group theory analogue of Corollary 31.

Corollary 45. *If $|G| = p^n$ and G is of maximal class, then $t(G/Z(G)) + n - 2 \leq t(G)$.*

Proof. Use Theorem 44 with $B = Z(G)$. Then,

$$\begin{aligned} &\left|M(G)\right| \left|G^2 \cap Z(G)\right| \;\; \textit{divides} \;\; \left|M(G/(Z(G))\right| \left|M(Z(G))\right| \left|\frac{G/Z(G)}{(G/Z(G))^2} \otimes \frac{Z(G)}{Z(G)^2}\right| \\ &\quad \Rightarrow p^{\frac{1}{2}n(n-1)-t(G)} \; p \;\; \textit{divides} \;\; p^{\frac{1}{2}(n-1)(n-2)-t(G/Z(G))} \left|G/G^2 \otimes Z(G)\right| \\ &\quad \Rightarrow p^{\frac{1}{2}n(n-1)-t(G)} \; p \;\; \textit{divides} \;\; p^{\frac{1}{2}(n-1)(n-2)-t(G/Z(G))} \; p^2. \end{aligned}$$

By taking logarithms and simplifying, we have $t(G) \geq n - 2 + t(G/Z(G))$. □

To prove the next theorem, we will need the following table, which details all groups of maximal class for which $t(G) \leq 3$ with $|G| = p^n$.

Table 4.1: Maximal Class Groups

t(G)	G	n
0	$\mathbb{Z}_p \times \mathbb{Z}_p$	2
1	E_1	3
2	D_8	3
3	Q_8	3
3	E_2	3

In Table 4.1, D_8 is the dihedral group of order 8, Q_8 is the quaternion group of order 8, E_1 is a p-group of order p^3 with odd prime exponent p, and E_2 is a p-group of order p^3 with odd prime exponent p^2. The exponent of a group is defined to be the the least common multiple of all orders of elements of the group.

Theorem 46. *Let G be a p-group of maximal class with $|G| = p^n$ where $n > 3$ and p is prime. Then*

$$(n-2)(n-1) < 2t(G) \leq (n-1)n.$$

Proof. Let $t = t(G)$ and suppose $(n-2)(n-1) = 2t$. Since G is of maximal class, $|Z(G)| = p$ and $G/Z(G)$ is of maximal class with $|(G/Z(G))| = p^{n-1}$. Hence, by Theorem 43, $\frac{1}{2}(n-2)(n-3) \leq t(G/Z(G))$ and by Corollary 45, $t(G/Z(G)) \leq t(G) - (n-2) = \frac{1}{2}(n-2)(n-3)$. Thus, $t(G/Z(G)) = \frac{1}{2}(n-2)(n-3)$. We can continue this process to obtain a group H of maximal class with $|H| = p^4$. This implies $t(H) = 3$. Examining Table 4.1, we see that no such group of maximal class exists. Thus, $(n-2)(n-1) \neq 2t$ and we conclude that $(n-2)(n-1) < 2t(G) \leq (n-1)n$. $\square$

Finally, we have the following corollary:

Corollary 47. *For each $t \in \mathbb{Z}^+$, for which there exists a $p-$group of maximal class, G with $t(G) = t$, the dimension of G is unique.*

Chapter 5

Other Bounds on the Dimension of the Schur Multiplier

In Chapter 4, we obtained a boundary on the value of $t(L) = \frac{1}{2}n(n-1) - \dim M(L)$ in terms of the dimension of L where L is a Lie algebra of maximal class. We then transfered the result from Lie algebras to group theory. In this chapter, we consider the conditions needed for $\dim M(L) > 0$. This involves searching for sufficient conditions for the Schur multiplier to be nontrivial. We also will address another bound involving the dimension of the multiplier for nilpotent Lie algebras, which depends on the class and minimal number of generators of L.

An important definition follows.

Definition 48. *The Möbius function μ is the function on $\mathbb{N}$ defined as*

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \\ 0 & \text{if } n \text{ is divisible by the square of a prime} \end{cases}$$

This leads to a well-known formula. Suppose that L is generated by n elements. Let F be a free Lie algebra generated by n elements and $L \cong F/R$. Since R is an ideal in F, R is also free. Witt's formula from [1] gives us

$$\dim F^d/F^{d+1} = \frac{1}{d}\sum_{m|d} \mu(m) n^{d/m} \equiv l_n(d) \tag{5.1}$$

where μ is the Möbius function. We provide details of the formula's derivation and a running example in Appendix A.

5.0.1 Nontrivial M(L)

To determine a lower bound for $\dim M(L)$, we note the classic problem of determining when the Schur multiplier of a p-group is necessarily nontrivial. Suppose that X is a minimal generating set for the finite p-group, G. If G has trivial multiplier and G is non-cyclic then there exists $x \in X$ such that $\langle x \rangle \cap \langle X \backslash \{x\} \rangle G^2 \neq 1$ and $|X| < 4$. As a consequence, G cannot have a generating set that consists entirely of elements of order p. This remarkable result is shown by Johnson in [13] with a simple proof provided in [19]. See also [14]. Webb in [18] shows that p-groups of class two have non-trivial multiplier. Recently, B. Eick showed that there are only finitely many $p-$groups with trivial Schur multiplier of a given coclass if p is an odd prime. Coclass of a group whose order is p^n is defined to be $n - c$ where c is its nilpotentcy class. She went on to develop an algorithm to compute these finitely many $p-$groups using GAP ([6]). Note that covers and multipliers can be computed using the GAP program [7]. New results arise when we consider the Lie algebra analogue of the group results cited above. Specifically, we have been able to give an elementary proof of the Lie algebra version of Webb's result.

Let L be a nilpotent Lie algebra generated by $n > 1$ elements. Hence, $\dim L/L^2 = n$. Let F be a free Lie algebra generated by n elements with $L \cong F/R$. Suppose that L has class c. Hence, $F^{c+2} \subsetneq F^{c+1} \subseteq R$ using the result in the last section. Furthermore, F/F^{c+2} is finite dimensional and nilpotent of class $c+1$. Then,

$$n = \dim L/L^2 = \dim \frac{F/R}{(F/R)^2} = \dim \frac{F}{F^2 + R} \leq \dim F/F^2 = n. \tag{5.2}$$

Lemma 49. *If L is nilpotent and $\dim L = n > 1$, then $R \subseteq F^2$ and $M(L) \cong R/[F,R]$.*

Proof. Using Eq. 5.2, $R \subseteq F^2$. Together with Hopf's formula: $M(L) \cong \frac{F^2 \cap R}{[F,R]}$, we obtain the result. □

Theorem 50. *If L is a finite dimensional nilpotent Lie algebra of dimension greater than 1 then $M(L) \neq 0$.*

Proof. Suppose L is a finite dimensional nilpotent Lie algebra and $\dim L = n > 1$. Assume, on the contrary, that $M(L) = 0$. Then by Lemma 49, $[F,R] = R$. Since $R \subseteq F^2$, bracket with F to obtain $[F,R] \subseteq [F,F^2]$. Thus, $R \subseteq F^3$ Inductively, $[F,R] = R \subseteq F^s$ for all $s > 1$. But, $0 = L^{c+1} \cong (F/R)^{c+1}$ implies $F^{c+1} \subseteq R$. With $R \subseteq F^{c+1} \subseteq R$, $F^{c+1} = R$ and $[F, F^{c+1}] = [F,R]$. Thus, $F^{c+2} = R = F^{c+1}$, but $\dim F^s/F^{s+1} \neq 0$ from Witt's formula. With $F^{c+2} \neq F^{c+1}$ we arrive at a contradiction. Thus, $M(L) \neq 0$.

□

In conclusion, we now have conditions for which the Schur multiplier of a Lie algebra is nontrivial. Unlike the main results from Chapter 4, the sufficient conditions we obtained do not directly follow from or translate to information on nontrivial multipliers in group theory.

5.0.2 An Upper Bound for dim M(L)

It is known that if G is a nilpotent group with r generators and class c then the order of the Schur multiplier, $M(G)$, has an upper bound given in terms of r and c [14]. This bound is then shown to be the best possible. We will find a similar bound in the Lie algebra case and compare it to a known bound to determine if it is the best possible.

Begin with a free presentation of $L:\ 0 \to R \to F \to L \to 0$. Let N be an ideal in L and S be an ideal in F such that $(S+R)/R \cong N$. Recall that $M(L) = (F^2 \cap R)/[F,R]$ ([4]). Then $M(L/N) \cong (F^2 \cap (S+R))/[F, S+R]$. We will verify that there is a natural exact sequence

$$0 \to \frac{R \cap [F,S]}{[F,R] \cap [F,S]} \to M(L) \to M(L/N) \to \frac{N \cap L^2}{[N,L]} \to 0. \tag{5.3}$$

We proceed by examining the following map:

$$A/B \xrightarrow{\sigma} C/D$$

where $B \subseteq A \subseteq C$ and $B \subseteq D$ that is defined by $\sigma(x+B) = x+D$. We claim σ is well-defined. $A \hookrightarrow C$ by inclusion. So, $A \to C/D$ is also well-defined. Since, $B \subseteq D$, $B \subseteq \ker\sigma$ and σ is well-defined. Now, we prove Eq. 5.3 is an exact sequence.

Proof. We have $L/N \cong \frac{F/R}{(S+R)/R} \cong F/(S+R)$. Then, from Hopf's formula

$$M(L) \cong (F^2 \cap R)/[F,R]$$
$$M(L/N) \cong \left(F^2 \cap (S+R)\right)/[F, S+R]$$

We can rewrite $(N \cap L^2)/[N,L] \cong \frac{((S+R)/R) \cap \left((F^2+R)/R\right)}{[(S+R)/R, F/R]} \cong \frac{(F^2+R)\cap(S+R)}{[F,S]+R}$. Now, it suffices to show the following sequence is exact

$$0 \to \frac{R \cap [F,S]}{[F,R] \cap [F,S]} \to \frac{F^2 \cap R}{[F,R]} \to \frac{F^2 \cap (S+R)}{[F,S+R]} \to \frac{(F^2+R)\cap(S+R)}{[F,S]+R} \to 0. \tag{5.4}$$

Define $\pi : (R \cap [F,S])/([F,R] \cap [F,S]) \to (F^2 \cap R)/[F,R]$ *by*

$$\pi(x + [F,R] \cap [F,S]) = x + [F,R]\ \forall x \in R \cap [F,S]$$

Note that $R \cap [F,S] \subset R \cap F^2$ and $[F,R] \cap [F,S] \subset [F,R]$. Now, suppose $\pi(x + [F,R] \cap [F,S]) =$

$x + [F,R] = [F,R]$. This implies $x \in [F,R]$. Thus, $x \in [F,S] \cap [F,R]$ and then the kernel of π is $[F,R] \cap [F,S]$, which is the identity of $(R \cap [F,S])/([F,R] \cap [F,S])$. Thus π is injective and well-defined.

$$\textit{Define } \sigma : (F^2 \cap R)/[F,R] \to \left(F^2 \cap (S+R)\right)/[F,S+R] \textit{ by}$$
$$\sigma(y + [F,R]) = y + [F,S+R]$$

We have $F^2 \cap R \subset F^2 \cap (S+R)$ and $[F,R] \subset [F,S+R]$. Then $\sigma\pi(x + [F,R] \cap [F,S]) = \sigma(x + [F,R]) = x + [F,S+R] = [F,S+R]$ for $x \in [F,S] \subset [F,S+R]$. Thus, the $Im(\pi) \subseteq Ker(\sigma)$. Now suppose $\sigma(y + [F,R]) \in [F,S+R]$ for $y \in F^2 \cap R$. Then $y \in [F,S+R]$ and our goal is to show that $y + [F,R] \in Im(\pi)$. It will suffice to show $y + [F,R] \in (R \cap [F,S])/[F,R]$. Already having the fact that $y \in R$, we need only show $y \equiv [f,s] \pmod{[F,R]}$ for some $f \in F,\ s \in S$. $y \in [F,S+R] \Rightarrow y = [f_1, s_1 + r] = [f_1,s_1] + [f_1,r]$ for $f_1 \in F,\ s_1 \in S,\ r \in R$. Thus, $y \equiv [f_1,s_1] \pmod{[F,R]}$ and we have $Ker(\sigma) = Im(\pi)$.

$$\textit{Define } \tau : \left(F^2 \cap (S+R)\right)/[F,S+R] \to \frac{(F^2+R) \cap (S+R)}{[F,S]+R} \textit{ by}$$
$$\tau(z + [F,S+R]) = z + ([F,S]+R)$$

We have $F^2 \cap (S+R) \subset (F^2+R) \cap (S+R)$ and $[F,S+R] \subset [F,S] + [F,R] \subset [F,S]+R$. Then, for $y \in F^2 \cap R$, $\tau\sigma(y + [F,R]) = y + ([F,S]+R) = [F,S]+R$ since $y \in R$. Thus, $Im(\sigma) \subseteq Ker(\tau)$. Next, suppose that for some $z \in F^2 \cap (S+R)$ we have $\tau(z + [F,S+R]) \in [F,S]+R$. Thus, $z \in [F,S]+R$ as well. We need to show that $z + [F,S+R] \in Im(\sigma)$. This occurs only if $z + [F,S+R] \in \left(F^2 \cap R\right)/[F,S+R]$ and it will suffice to show that $z \equiv r \pmod{[F,S+R]}$. From above we have $z = [f,s]+r$ for some $f \in F,\ s \in S,\ r \in R$. Thus, $z - r = [f,s] \Rightarrow z - r \equiv 0 \pmod{[F,S]} \Rightarrow z \equiv r \pmod{[F,S+R]}$. Therefore, $Ker(\tau) = Im(\sigma)$.

Lastly, we show that τ is onto. Let $w + [F,S] + R \in \frac{(F^2+R)\cap(S+R)}{[F,S]+R} \Rightarrow w = f' + r_1 = s + r_2$. So, $w \in (S+R)$. Additionally, $w + [F,S] + R = f' + r + [F,S] + R = f' + [F,S] + R$. Thus, $w \in F^2$ and τ is onto. $\square$

Suppose L has class $c \geq 2$. Let $N = L^c$ and $S = F^c$ in Eq. 5.3. Then $L^c \cong (F^c + R)/R$ and $[F,S] = F^{c+1} \subseteq R$ since $L^{c+1} = 0$. Hence, Eq. 5.3 becomes

$$0 \to \frac{F^{c+1}}{[F,R] \cap F^{c+1}} \xrightarrow{\sigma} M(L) \to M(L/L^c) \to L^c \to 0. \tag{5.5}$$

Theorem 51. *Let L be a nilpotent Lie algebra of class c which is generated by n elements.*

Then

$$\dim M(L) \leq \sum_{j=1}^{c} l_n(j+1) \tag{5.6}$$

where $l_n(d) = \frac{1}{d}\sum_{m|d} \mu(m)n^{d/m}$.

Proof. Induct on c. Let F be free of rank n where $L \cong F/R$. If $c = 1$, then F/R is abelian, $F^2 \subseteq R$ and $M(L) = F^2/[F,R]$. Since $F^3 \subseteq [F,R]$, $\dim M(L) \leq \dim F^2/F^3 = l_n(2)$. Now, suppose that $c > 1$. By induction hypothesis, $\dim M(L/L^c) \leq t := \sum_{j=1}^{c-1} l_n(j+1)$. In Eq. 5.5, let $A = Im(\sigma)$. Then $M(L)/A \cong B \subseteq M(L/L^c)$. Hence, $\dim M(L)/A \leq t$. But $F^{c+1} \subseteq R$ and $F^{c+2} \subseteq [F,R] \cap F^{c+1}$. Thus, A is the homomorphic image of F^{c+1}/F^{c+2}. Therefore $\dim A \leq l_n(c+1)$ by Eq. 5.1 and $\dim M(L) \leq t + l_n(c+1)$ as desired. □

This result is the analogue of a known boundary for $M(G)$ which is proved to be the best possible. We compare Eq. 5.6 to the upper bound for $\dim M(L)$ given in [12] to examine whether it is the best possible bound. The other bound is

Theorem 52. *If L is a Lie algebra of dimension n, then*

$$\dim M(L) \leq \frac{1}{2}n(n-1) - \dim L^2.$$

We now examine the Theorems 52 and 51 applied to different Lie algebras.

Example 53. *Let F be a free Lie algebra on 2 generators and $L = F/F^3$. Then L is a nilpotent Lie algebra of 2 generators and class 2. So, $L \supseteq L^2 \supseteq L^3 = 0$. Then, $\dim L/L^2 = l_2(1) = 2$, $\dim L^2/L^3 = l_2(2) = \frac{1}{2}[\mu(1)2^2 + \mu(2)2] = \frac{1}{2}(4-2) = 1$. Thus, $\dim L = 3$ and by Theorem 52, $\dim M(L) \leq 2$. By Theorem 51,*

$$\begin{aligned}\dim M(L) \leq \sum_{j=1}^{2} l_2(j+1) &= l_2(2) + l_2(3)\\ &= 1 + \frac{1}{3}(\mu(1)2^3 + \mu(3)2)\\ &= 1 + \frac{1}{3}(6) = 3.\end{aligned}$$

Thus, using the result of Theorem 52 proves to be a better bound than the one obtained by Theorem 51.

Example 54. *Let F be a free Lie algebra of 2 generators and $L = F/F^4$. Then L is a nilpotent Lie algebra of 2 generators and class 3. So, $L \supseteq L^2 \supseteq L^3 \supseteq L^4 = 0$. Then, $\dim L/L^2 = 2$,*

$\dim L^2/L^3 = 1$, $\dim L^3/L^4 = \frac{1}{3}[\mu(1)2^3 + \mu(3)2] = 2$. *Thus,* $\dim L = 5$ *and by Theorem 52,* $\dim M(L) \leq 7$. *By Theorem 51,*

$$\begin{aligned}
\dim M(L) \leq \sum_{j=1}^{3} l_2(j+1) &= l_2(2) + l_2(3) + l_2(4) \\
&= 1 + 2 + \frac{1}{4}[\mu(1)2^4 + \mu(2)2^2 + \mu(4)2] \\
&= 3 + \frac{1}{4}(16 - 4) \\
&= 3 + 3 = 6
\end{aligned}$$

Therefore, we see that in this case, Theorem 51 creates a better upper bound for $\dim M(L)$ *than the previously known result in Theorem 52.*

From Examples 53 and 54, we cannot say either boundary is the best possible.

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APPENDIX

Appendix A

The Dimensions of Lower Central Factors of Free Lie Algebras

Recall,

Definition 55. *The Möbius function μ is the function on $\mathbb{N}$ defined as*

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \\ 0 & \text{if } n \text{ is divisible by the square of a prime} \end{cases}$$

Lemma 56. *The Möbius function satisfies* $\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$ *for* $n \in \mathbb{N}$.

We will use the Möbius inversion formula which we list as:

Theorem 57. *Möbius Inversion Formula*
Let h and H be two functions from $\mathbb{N}$ into G, an additive abelian group. Then,

$$H(n) = \sum_{d|n} h(d) \ \forall n \in \mathbb{N} \iff h(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) H(d) = \sum_{d|n} \mu(d) H\left(\frac{n}{d}\right) \ \forall n \in \mathbb{N}.$$

where μ is the Möbius function.

Proof. ($\Rightarrow$) Assume $H(n) = \sum_{d|n} h(d) \ \forall n \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{d|n} \mu\left(\frac{n}{d}\right) H(d) &= \sum_{d|n} \mu(d) H\left(\frac{n}{d}\right) \\
&= \sum_{d|n} \mu(d) \sum_{c|\frac{n}{d}} h(c) \text{ by assumption} \\
&= \sum_{c|n} \sum_{d|\frac{n}{c}} \mu(d)h(c) = \sum_{c|n} h(c) \sum_{d|\frac{n}{c}} \mu(d) = \sum_{n|n} h(n) \text{ by Lemma 56} \\
&= h(n).
\end{aligned}$$

($\Leftarrow$) Assume $h(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) H(d) = \sum_{d|n} \mu(d) H\left(\frac{n}{d}\right)$ $\forall n \in \mathbb{N}$. Then,

$$\begin{aligned}
\sum_{d|n} h(d) &= \sum_{d|n} \left(\sum_{c|d} \mu\left(\frac{d}{c}\right) H(c) \right) \\
&= \sum_{d|n} H(d) \sum_{c|\frac{n}{d}} \mu(c) = \sum_{n|n} H(n) \text{ by Lemma 56} \\
&= H(n).
\end{aligned}$$

$\square$

Now, we wish to use the Möbius inversion formula to create a bound for $M(L)$ where L is a Lie algebra. We begin by counting the dimensions of the lower central factors of free Lie algebras. We follow a familiar path to do this (see [1]). We include the details for clarity and completeness.

Given a finite set X and a field $\mathbb{K}$, we consider all words of the form $x_{i_1}x_{i_2}\ldots x_{i_j}$, $x_i \in X$. These words have the multiplicative operation of juxtaposition. Specifically,

$$(x_{i_1}x_{i_2}\ldots x_{i_j})(x_{l_1}x_{l_2}\ldots x_{l_k}) = x_{i_1}x_{i_2}\ldots x_{i_j}x_{l_1}x_{l_2}\ldots x_{l_k}.$$

Now, we define $A(X)$ to be all $\mathbb{K}$-linear combinations of these words. Thus, $A(X)$ is a free associative algebra that can be decomposed into an infinite direct sum of $A_n(X)$, $n \in \mathbb{Z}_{\geq 0}$ where $A_n(X)$ is the linear span of all words of length n.

Example 58. *Suppose $X = \{x, y\}$. Then,*

$$
\begin{aligned}
A_1(X) &= \langle x, y \rangle \\
A_2(X) &= \langle xx, xy, yx, yy \rangle \\
A_3(X) &= \langle xxx, xxy, xyx, yxx, xyy, yxy, xyy, yyy \rangle \\
&\ \vdots
\end{aligned}
$$

Note, $\dim A_1(X) = 2^1$, $\dim A_2(X) = 2^2$, *and* $\dim A_3(X) = 2^3$. *In general,* $\dim A_n(X) = |X|^n$ *where* $|X| = 2$ *in this example.*

To generalize the example, if $|X| = q$, then $\dim A_n(X) = q^n$. A generating function for $\dim A_n(X) = q^n$ given any set X with order q and $n \in \mathbb{Z}_{\geq 0}$ is

$$1 + qt + q^2t^2 + \ldots + q^nt^n + \cdots = \frac{1}{1 - qt}. \tag{A.1}$$

Our goal will be to develop a second generating function that also counts the dimension of $A_n(X)$, which we will then equate to Eq. A.1.

Next, we form a Lie algebra $[A(X)]$ from $A(X)$ by commutation: $[S, T] = ST - TS$. Also, define $F(X)$ to be the free Lie algebra generated by X. The identity map on X gives rise to a map $\psi : X \to [A(X)]$. By the definition of a free Lie algebra, there exists a unique homomorphism $\phi : F(X) \to [A(X)]$ such that the following diagram commutes:

$$
\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[d] & {[A(X)]} \\
F(X) \arrow[ur, "\phi"'] &
\end{tikzcd}
$$

Figure A.1: Commuting Diagram of a Free Lie Algebra

Define $F_n(X)$ to be the span of all commutators of length $n \in \mathbb{Z}_{\geq 0}$. Then

$$F(X) = F_0(X) \oplus F_1(X) \oplus F_2(X) \oplus \ldots\ .$$

Example 59. *If* $|X| = 2$ *as in Example 58, then*

$$
\begin{aligned}
F_1(X) &= span_{\mathbb{K}}\{x, y\} \\
F_2(X) &= span_{\mathbb{K}}\{[x, y]\} \\
F_3(X) &= span_{\mathbb{K}}\{[x, [x, y]], [y, [x, y]]\} \\
F_4(X) &= span_{\mathbb{K}}\{[x, [x, [x, y]]], [x, [y, [x, y]]], [y, [y, [y, x]]]\} \\
&\vdots
\end{aligned}
$$

$F_4(X)$ is composed of all commutators of length four. We provide details that $F_4(X)$ is spanned by the three elements given. In total, there are eight nonzero commutators in $F_4(X)$. Three are listed above: $[x, [x, [x, y]]]$, $[x, [y, [x, y]]]$, $[y, [y, [y, x]]]$. The other five are linear combinations of these three.

$$
\begin{aligned}
[y, [x, [x, y]]] &= \ \ [x, [y, [x, y]]] \\
[y, [y, [x, y]]] &= -[y, [y, [y, x]]] \\
[x, [x, [y, x]]] &= -[x, [x, [x, y]]] \\
[y, [x, [y, x]]] &= -[x, [y, [x, y]]] \\
[x, [y, [y, x]]] &= -[x, [y, [x, y]]]
\end{aligned}
$$

Now, we ask if there is a formula to determine the dimension of $F_n(X)$ if $|X| = q$. We define $l_q(n) = \dim F_n(X)$. We create an order relation among the commutators formed from X, inductively. To initialize, we arbitrarily order the elements of X : $x_1 < x_2 < \ldots < x_q$. Now, suppose we have ordered all commutators w_i of length less than some k, $k > 1$. Next, we form certain commutators of length k using the already established order relations. Form $[w_i, w_j]$ such that the length of w_i and w_j sum to k where $w_i > w_j$ and if $w_i = [w_s, w_t]$, then $w_j \geq w_t$. Now choose any ordering of the commutators of length k. Once the arbitrary order is established, we need only keep this order when determining commutators whose length is greater than k.

Example 60. *If $X = \{x, y\}$ we arbitrarily order X by $x < y$. Then we have*

$$
\begin{array}{llllllll}
w_1 = & x < w_2 = y & < & w_3 = [y, x] & < & w_4 = [[y, x], x] & < & w_5 = \ [[y, x], y]] \quad < \\
w_6 = & [[[y, x], x], x] & < & w_7 = & & [[[y, x], x], y] & < & w_8 = \ [[[y, x], y], y] \quad < \\
w_9 = & [[[[y, x], x], x], x] & < & w_{10} = & & [w_6, w_2] & < & w_{11} = \ [w_7, w_2] \quad < \\
w_{12} = & [w_8, w_2] & < & w_{13} = & & [w_4, w_3] & < & w_{14} = \ [w_5, w_3] \quad < \ \ldots
\end{array}
$$

$$
\begin{array}{llllll}
w_1 = x & & w_2 = y & & w_3 = [y,x] & xxy = w_4 + w_2 w_1^2 - 2w_3 w_1 \\
xyx & = w_2 w_1^2 - w_3 w_1 & & & yxx & = w_2 w_1^2 \\
xyy & = -2w_3 w_2 + w_2^2 w_1 + w_5 & & & yxy & = -w_3 w_2 + w_2^2 w_1 + w_5 \\
yyx & = w_2^2 w_1 & & & yyy & = w_2^3
\end{array}
$$

We claim $A_3(X)$ is the span of $w_5, w_4, w_3w_1, w_3w_2, w_2^3, w_2^2w_1, w_2w_1^2, w_1^3$. Note that there are eight elements listed above. Since this is the dimension of $A_3(X)$, the elements are a basis if they span $A_3(X)$. Indeed,

$$
\begin{array}{llllll}
xxx & = & w_1^3 & xxy & = & w_4 + w_2 w_1^2 - 2w_3 w_1 \\
xyx & = & w_2 w_1^2 - w_3 w_1 & yxx & = & w_2 w_1^2 \\
xyy & = & -2w_3 w_2 + w_2^2 w_1 + w_5 & yxy & = & -w_3 w_2 + w_2^2 w_1 + w_5 \\
yyx & = & w_2^2 w_1 & yyy & = & w_2^3
\end{array}
$$

In general, it has been proved in [1] that given a finite set X and $n \in \mathbb{N}$, $A_n(X)$ has a basis composed of those elements $w_{i_1} w_{i_2} \ldots w_{i_s}$ in $[A(X)]$ such that $d_{i_1} + d_{i_2} + \ldots + d_{i_s} = n$ where $d_i = \deg w_i$ and $i_1 \geq i_2 \geq \ldots \geq i_s$. Now, if $w_1, w_2, \ldots, w_s$ are all basic monomials with $d_j = \deg w_j$, then $w_1^{e_1} w_2^{e_2} \ldots w_s^{e_s} \in A_n(X) \Leftrightarrow \sum_{i=1}^{s} e_i d_i = n$. Thus, we can now form a second generating function for $\dim A_n(X)$.

$$
\begin{aligned}
&(1 + t^{d_1} + t^{2d_1} + \ldots)(1 + t^{d_2} + t^{2d_2} + \ldots)(1 + t^{d_3} + t^{2d_3} + \ldots)\ldots \\
&\qquad = \left(\frac{1}{1 - t^{d_1}}\right)\left(\frac{1}{1 - t^{d_2}}\right)\left(\frac{1}{1 - t^{d_3}}\right)\ldots = \prod_{j=1}^{\infty} \frac{1}{1 - t^{d_j}}. \qquad \text{(A.2)}
\end{aligned}
$$

Since both equations A.1 and A.2 count the the dimension of $A_n(X)$, they can be equated. Thus,

$$
\prod_{j=1}^{\infty} \frac{1}{1 - t^{d_j}} = \frac{1}{1 - qt}. \qquad \text{(A.3)}
$$

Now, $d_u \neq d_{u+1} = d_{u+2} = d_{u+3} = \ldots = d_{u+v} \neq d_{u+v+1}$ if there exist v monomials of length $m = d_{u+1}$. So $v = \dim F_m(X) = l_q(m)$ where $F_m(X)$ has been previously defined to be the span of all commutators of length m. Thus, the left hand side of Eq. A.3 is equal to $\prod_{m=1}^{\infty} \frac{1}{(1 - t^m)^{l_q(m)}}$. If we now apply logarithms, we have

$$
\log \prod_{m=1}^{\infty} \frac{1}{(1 - t^m)^{l_q(m)}} = \log \frac{1}{1 - qt} \qquad \text{(A.4)}
$$

Using the fact that $-\log(1-x) = \sum_{k=1}^{\infty} \frac{1}{k} x^k$ for $|x| < 1$, the left hand side of equation (A.4) will become

$$\sum_{m=1}^{\infty} l_q(m) \log \frac{1}{1-t^m} = \sum_{m=1}^{\infty} l_q(m) \left(\sum_{v=1}^{\infty} \frac{1}{v} t^{mv} \right) = \sum_{m,v=1}^{\infty} l_q(m) \frac{1}{v} t^{mv}$$

and the right hand side will become $\sum_{n=1}^{\infty} \frac{1}{n}(qt)^n$. Thus,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n}(qt)^n &= \sum_{m,v=1}^{\infty} l_q(m) \frac{1}{v} t^{mv} \\
\Rightarrow \frac{1}{n} q^n &= \sum_{\substack{m,v \in \mathbb{N} \\ mv=n}} \frac{1}{v} l_q(m) \ \textit{by equating powers of } t \\
\Rightarrow q^n &= \sum_{\substack{m,v \in \mathbb{N} \\ mv=n}} m l_q(m) = \sum_{\substack{m \in \mathbb{N} \\ m|n}} m l_q(m) \ \forall n \in \mathbb{N}
\end{aligned}$$

Applying the Möbius inversion formula with $H(n) = q^n$ and $h(n) = nl_q(n)$, we now have

$$\begin{aligned}
nl_q(n) &= \sum_{m|n} \mu\left(\frac{n}{m}\right) q^m = \sum_{m|n} \mu(m) q^{n/m} \\
\Rightarrow l_q(n) &= \frac{1}{n} \sum_{m|n} \mu(m) q^{n/m}.
\end{aligned}$$

In conclusion,

Theorem 61. *Let F be a free Lie algebra of rank q with generating set X, where X contains q elements and $F_n(X)$ is the linear span of products of elements from X of length n. Then $F_n(X)$ is a grading for F and* $\dim F_n(X) = \frac{1}{n} \sum_{m|n} \mu(m) q^{n/m}$ *where μ is the Möbius function.*

Example 62. *Suppose, again, $X = \{x, y\}$. Then $q = |X| = 2$ and according to Theorem 61,*

$$\begin{aligned}
\dim F_1(X) &= \mu(1)2 = 2 \\
\dim F_2(X) &= \frac{1}{2}(\mu(1)2^2 + \mu(2)2) = \frac{1}{2}(4-2) = 1 \\
\dim F_3(X) &= \frac{1}{3}(\mu(1)2^3 + \mu(3)2) = \frac{1}{3}(8-2) = 2 \\
\dim F_4(X) &= \frac{1}{4}(\mu(1)2^4 + \mu(2)2^2 + \mu(4)2) = \frac{1}{4}(16-4) = 3 \\
\dim F_5(X) &= \frac{1}{5}(\mu(1)2^5 + \mu(5)2) = \frac{1}{5}(32-2) = 6
\end{aligned}$$

From explicit computations in Example 59, we see these to be the correct dimensions of $F_i(X)$ for $i = 1, 2, 3, 4$.