ABSTRACT

BANCROFT, ERIN ELIZABETH. Shard Intersections and Cambrian Congruence Classes in Type A. (Under the direction of Nathan Reading.)

This thesis is a contribution to the study of the shard intersection order on permutations and the Tamari lattice. Both objects are closely connected to W-Catalan combinatorics, although we will focus on the case where W is the symmetric group. The shard intersection order is a new lattice structure on a finite Coxeter group W which has the noncrossing partition lattice as a sublattice. When W is the symmetric group, we characterize shard intersections as certain pre-orders and use this characterization to determine properties of the order. In particular, we give an EL-labeling of the order. The Tamari lattice is a Cambrian lattice of type A. We characterize the Tamari congruence classes as certain posets and use this characterization to give tools for navigating within a class and a formula for enumerating the elements of a class. In the process, we define a new quantity called the permutation hook length, which we use to establish several permutation statistics.

Shard Intersections and Cambrian Congruence Classes in Type A

by
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DEDICATION

To Eric- the wonderful, Godly man who shows me he loves me in a hundred different ways every day. I am so proud to be your wife.

BIOGRAPHY

The author was born in Fort Collins, Colorado to Chris and Anne Lukasiewicz. As the daughter of an officer in the US Army, she soon moved from Colorado to the first of many military bases in states including California, Virginia, Missouri, Washington and Maryland and abroad in Germany. Along the way, Erin was joined by two sisters, Jessica and Tracy, a brother, Mark, and two special dogs, Jake and Nikki. She has many fond memories of road trips across the U.S. and Europe with everyone piled into their big brown van. To this day, Erin loves to travel, especially by car.

Erin attended Grove City College, a small Christian school in northwestern Pennsylvania, where she earned a Bachelors of Science in Mathematics along with Secondary Education Certification. Growing up, she had a passion both for teaching and for math and she intended to teach high school mathematics after college. But God had other plans, and after some encouragement from one of her professors at GCC, Erin applied to graduate school to pursue a Ph.D. in mathematics. In the fall of 2007, she started graduate school at North Carolina State University and almost immediately met and fell in love with her perfect help-meet, Eric Bancroft, another mathematics graduate student. They were engaged by February and married during winter break the following December. Outside of doing math, Erin enjoys playing games, baking, trying new foods, cross-stitching, and decorating cakes. Both Eric and Erin received their Doctorate of Philosophy in Mathematics in the summer of 2011, and they will begin their academic careers as assistant professors at Grove City College in the fall.

ACKNOWLEDGEMENTS

I want to begin by thanking my advisor, Dr. Nathan Reading, for sharing so much of his time and knowledge with me. I have grown immensely by working with him. I also want to thank Dr. Reading for introducing me to the beauty of combinatorics, a subject for which I have developed a passion. He has been an inspiring teacher and mentor; one that I hope to emulate as I begin my academic career.

Thank you to my husband for his encouragement, patience, and advice. I could not have made it through without him.

I also want to thank my family for their love and support from afar. Through my parents' example, I have learned the importance of perseverance and doing your best for the glory of God.

Thank you to my undergraduate professor, Dr. Gary Thompson, for encouraging me to pursue a graduate degree. He gave me the confidence I needed to follow in his footsteps.

I also want to thank one of my dearest friends, Marie Tyler, for sticking by me for the last eight years. Our visits and phone conversations have allowed me to keep my sanity.

Thank you to my church family at Redeemer Presbyterian Church in Raleigh. My time spent fellowshipping and woshipping with them has made Raleigh feel like home.

Most importantly, I want to thank God for gifting me with the ability to do mathematics and for guiding me to where I am today. To God alone be the glory.

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The Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{1.1}$$

enumerate various combinatorial objects. For instance

triangulations: decomposition of a convex $(n+2)$-gon into n triangles

parenthesizations: binary parenthesizations of a string of $n+1$ letters

noncrossing partitions of [n]: partitions of $[n]$ such that there do not exist $1 \leq a < b < c < d \leq n$ with a and c together in a block and b and d together in a different block

nonnesting partitions of [n]: partitions of $[n]$ such that if $1 \leq a < b < d < e \leq n$ with a and e together in a block B and b and d together in a different block B', then there exists a $c \in B$ satisfying $b < c < d$

312-avoiding permutations: permutations $x = x_1 x_2 \cdots x_n \in S_n$ such that there does not exist a triple x_i, x_j, x_k satisfying $x_j < x_k < x_i$ for $i < j < k$.

(See Figures 1.1 and 1.2 for examples.) An exercise in [18] asks the reader to show that 66 different combinatorial objects are counted by the Catalan numbers, and Richard Stanley maintains a growing list of additional objects.

$$((ab)c)d, \qquad (a(bc))d, \qquad a((bc)d), \qquad a(b(cd)), \qquad (ab)(cd)$$

Figure 1.1: Parenthesizations for $n = 3$

Examples	Non-Examples
1|24|3|5	14|235
123|45	13|24|5
14|23|5	134|25

(a) Noncrossing Partitions

Examples	Non-Examples
135|24	14|23|5
12|345	15|234
13|24|5	1|25|34

(b) Nonnesting Partitions

Examples	Non-Examples
12345	31245
21435	42153
34251	15234

(c) 312-avoiding Permutations

Figure 1.2: Examples and Non-examples of objects counted by the Catalan number for $n = 5$

In recent years, connections between the Catalan numbers and Coxeter groups have been studied, including a generalization of the Catalan number to any Coxeter group W. A Coxeter group W is a group with presentation given by a set of generators S and relations $(ss')^{m(s,s')} = 1$ for all $s, s' \in S$. Here $m(s,s) = 1$ and $m(s,s') \geq 2$ for $s \neq s'$, and $m(s,s') = m(s',s)$ for all $s, s' \in S$. We will call S the simple generators. W is represented by a Coxeter diagram, which is a graph whose nodes are the simple generators and edges connect pairs s, s' if $m(s,s') \geq 3$. By convention, edges are labeled by the value $m(s,s')$ only if $m(s,s') \geq 4$.

Every finite Coxeter group W can be realized as a group generated by orthogonal reflections. The reflections of W are all of the elements of W which act as orthogonal reflections. The set of reflecting hyperplanes for the reflections of W is called the Coxeter arrangement $\mathcal{A}$ associated to W. The arrangement $\mathcal{A}$ cuts space into simplicial cones, called regions, and the regions are in one-to-one correspondence with the elements of W. The weak order is a natural partial order on the elements of W, and it is also defined in terms of the geometry of Coxeter arrangements.

The classic example of a Coxeter group is the symmetric group $S_n = A_{n-1}$, and it will be the focus of the thesis. The simple generators for S_n are the adjacent transpositions $(i, i+1)$ for $i \in \{1, \ldots, n-1\}$. The Coxeter diagram for S_n is a path with $n-1$ nodes and $n-2$ unlabeled edges. By switching two adjacent entries in a permutation π so as to put them out of numerical order, you obtain a permutation π' which covers π in the weak order on S_n.

The W-Catalan number of a finite Coxeter group W is

$$\mathbf{Cat}(W) = \prod_{i=1}^{n} \frac{h + e_i + 1}{e_i + 1} \tag{1.2}$$

where h and $e_1, \ldots, e_n$ are certain numerical invariants of W. In the case where $W = S_n$, the W-Catalan number is just the Catalan number because $h = n$ and $e_1, \ldots, e_{n-1}$ are equal to $1, \ldots, n-1$ respectively.

As with the classical Catalan numbers, the W-Catalan numbers are combinatorially rich and have appeared in many different fields. They have also been shown to count generalizations of several of the combinatorial objects counted by classical Catalan numbers. For example, given any finite Coxeter group, there is a construction of W-noncrossing partitions that are counted by the W-Catalan number. Another example is triangulations of a convex polygon. In this case, the W-Catalan number enumerates the clusters of almost positive roots, which are the vertices of generalized associahedron. Nonnesting partitions of W are defined to be the set of antichains in the root poset and are again counted by the W-Catalan number. Finally, there is a definition of c-sortable elements of a Coxeter group W which are counted by the W-Catalan number, and when W is the symmetric group, these are just the 312-avoiding permutations. For definitions and details, see [2, Section 1.1], [9, Lecture 5], [15, Sections 2 and 4].

There also exist partial orders on many of these combinatorial objects and their generalizations. For instance, there is a refinement order on noncrossing partitions, as well as a partial order on the generalized W-noncrossing partitions. The nonnesting partitions can be ordered as a distributive lattice of order ideals in the root poset. A partial order on triangulations gives the Tamari lattice, which is isomorphic to the restriction to 312-avoiding permutations of the weak order on S_n. The corresponding generalizations in this last case are the Cambrian lattices.

Although this thesis concentrates on the symmetric group and is thus tied to the "classical" Catalan combinatorics, both objects of study in the thesis came out of an effort to further understand the more general W-Catalan combinatorics and in particular, noncrossing partitions and Cambrian lattices.

In Chapter 2 we begin with the shard intersection order. The shard intersection order is a new lattice structure on a finite Coxeter group W which encodes the geometry of the reflection arrangement and the lattice theory of the weak order. It was shown in [16] that the noncrossing partition lattice is a sublattice of the shard intersection order. In the case where W is the symmetric group, we characterize shard intersections as certain pre-orders which we call permutation pre-orders. We then use this combinatorial characterization to determine properties of the shard intersection order. In particular, in Section 2.4 of Chapter 2, we give an EL-labeling. We conclude the chapter by describing the set of permutation pre-orders which correspond to the c-sortable elements of S_n. These are closely related to noncrossing partitions.

In Chapter 3 we consider the Tamari lattice, which is one of the Cambrian lattices of type A. A Cambrian lattice can be understood in two ways: as a quotient of the weak order on a Coxeter group W with respect to a certain lattice congruence or as the weak order on c-sortable elements of W. In this chapter we describe each Tamari congruence class as a poset on $\{1, \ldots, n\}$, called a C-poset, and characterize which C-posets occur. We also give several tools for navigating within a congruence class and a formula to enumerate the elements of a class. We conclude by using a new quantity called the permutation hook length to define several permutation statistics.

Chapter 2

The Shard Intersection Order on Permutations

2.1 Introduction

Shards were introduced in [13] as a way to understand lattice congruences of the weak order on a finite Coxeter group. They are defined in terms of the geometry of the associated simplicial hyperplane arrangement. The collection Ψ of arbitrary intersections of shards, studied in [16], forms a lattice under reverse containment. This lattice is called the *shard intersection order*. Surprisingly, Ψ was found to be in bijection with the elements of the finite Coxeter group W, and thus the shard intersection order defines a new lattice structure on W. This lattice is graded and contains the W-noncrossing partition lattice $\mathrm{NC}(W)$ as a sublattice. Indeed, for any Coxeter element c, the subposet induced by c-sortable elements [15] is a sublattice isomorphic to $\mathrm{NC}(W)$. A formula for calculating the Möbius numbers of lower intervals was given in [16], but overall the structure of the shard intersection order is not yet well-understood.

In this chapter we consider the most classical Coxeter group, the symmetric group, whose associated hyperplane arrangement is the braid arrangement. In Section 2.2, we give necessary background information on hyperplane arrangements and the general construction of shards.

We then specifically describe shards and shard intersections in the symmetric group. Throughout the chapter, no prior knowledge of Coxeter groups will be assumed. Following a suggestion from Aguiar [1], in Section 2.3 we characterize shard intersections of type A by realizing them combinatorially as certain pre-orders, which we call *permutation pre-orders*. In Section 2.4, we realize the shard intersection order as an order on the permutation pre-orders and use this realization to determine properties of the order, including an EL-labeling. Finally, in Section 2.5 we characterize *noncrossing pre-orders*, which correspond to c-sortable permutations.

2.2 Shards and Shard Intersections

In this section we begin by defining a central hyperplane arrangement and the construction of shards within it. We then focus on the symmetric group, starting with necessary background and concluding with an explicit description of the correspondence between shard intersections and permutations. A *linear hyperplane* in a real vector space V is a codimension-1 linear subspace of V. A *central hyperplane arrangement* $\mathcal{A}$ in V is a finite collection of linear hyperplanes. The *regions* of $\mathcal{A}$ are the closures of the connected components of $V \setminus (\bigcup \mathcal{A})$. Each region is a closed convex polyhedral cone whose dimension equals $\dim(V)$.

Fix a *base region* B in the set of regions. We define a partial order on the set of regions called the *poset of regions*. In the poset of regions, Q is below R if and only if the set of hyperplanes separating Q from B is contained in the set of hyperplanes separating R from B. The unique minimal element in the poset is B and the unique maximal element is $-B$, the region antipodal to B. A region R covers Q if and only if R and Q share a facet-defining hyperplane which separates R from B but does not separate Q from B. (More information on the poset of regions can be found in [4, 8].)

A region is *simplicial* if the normal vectors to its facet-defining hyperplanes form a linearly independent set. A central hyperplane arrangement is *simplicial* if each of its regions is simplicial. A *rank-two subarrangement* $\mathcal{A}'$ of $\mathcal{A}$ is a hyperplane arrangement consisting of all of the hyperplanes of $\mathcal{A}$ which contain some subspace of codimension-2, provided $|\mathcal{A}'| \geq 2$. In this subarrangement there exists a unique region B' containing B, and the two facet-defining hyperplanes of B' are called the *basic hyperplanes* of A'.

Shards are defined by "cutting" the hyperplanes in $\mathcal{A}$. For each nonbasic hyperplane H in a rank-two subarrangement $\mathcal{A}'$, cut H into connected components by removing the subspace $\cap\mathcal{A}'$ from H. A hyperplane H is cut in every rank-two subarrangement in which it is non-basic, so it may be cut many times. Cutting every hyperplane H in this way, we obtain a set of connected components, the closures of which are the *shards* of $\mathcal{A}$. The set of *intersections of shards* of an arrangement $\mathcal{A}$ is denoted $\Psi(\mathcal{A}, B)$ or simply Ψ when $\mathcal{A}$ and the choice of B are clear. The empty intersection of shards is the entire space V.

A shard Σ is a *lower shard* of a region R if it contains a facet-defining hyperplane of R which separates R from a region covered by R. One of the primary results regarding shard intersections is a bijection [16, Proposition 4.7] between regions of a simplicial hyperplane arrangement and intersections of shards. The bijection sends a region R to the intersection of the lower shards of R. The shard intersections form a lattice under reverse containment, which induces, via the bijection, a partial order on the regions. Called the *shard intersection order*, this partial order is different from the poset of regions and will be discussed further in Section 2.4.

Now that we have considered the construction of shards in a general setting, let us turn to the symmetric group S_n. We begin with some background on S_n and its Coxeter arrangement. Throughout this thesis, permutations $\pi \in S_n$ will be written in one-line notation as $\pi = \pi_1\pi_2\cdots\pi_n$ with $\pi_i = \pi(i)$. An *inversion* of π is a pair (π_i, π_j) such that $i < j$ and $\pi_i > \pi_j$. A *descent* of π is a pair $\pi_i\pi_{i+1}$ such that $\pi_i > \pi_{i+1}$. A *descending run* of π is a maximal descending sequence $\pi_i\pi_{i+1}\cdots\pi_s$. In this context, maximal implies that either $i = 1$ or $\pi_i > \pi_{i-1}$ and either $s = n$ or $\pi_s < \pi_{s+1}$. For example, the permutation $\pi = 1642735$ has descending runs $1, 642, 73$ and 5.

The *braid arrangement*, also known as the *Coxeter arrangement* $\mathcal{A}(S_n)$ associated to the symmetric group, is the simplicial arrangement consisting of the hyperplanes $H_{ij} = \{\vec{x} \in \mathbb{R}^n : x_i = x_j\}$ for $1 \leq i < j \leq n$. The permutations in S_n are in bijection with the regions of $\mathcal{A}(S_n)$

as follows: the permutation $\pi \in S_n$ corresponds to the region $R_\pi = \{\vec{x} \in \mathbb{R}^n : x_{\pi_1} \leq x_{\pi_2} \leq \cdots \leq x_{\pi_n}\}$. We choose the base region to be the one corresponding to $\pi = 123 \cdots n$.

There are 2^{j-i-1} shards in each hyperplane H_{ij}. Each shard is obtained by choosing $\epsilon_k \in \{\pm 1\}$ for each k with $i < k < j$, and defining Σ to be the cone $\{\vec{x} : x_i = x_j$ and $\epsilon_k x_i \leq \epsilon_k x_k$ for $i < k < j\}$. An intersection of shards can be represented similarly, by taking the union of the equalities and inequalities defining the shards being intersected.

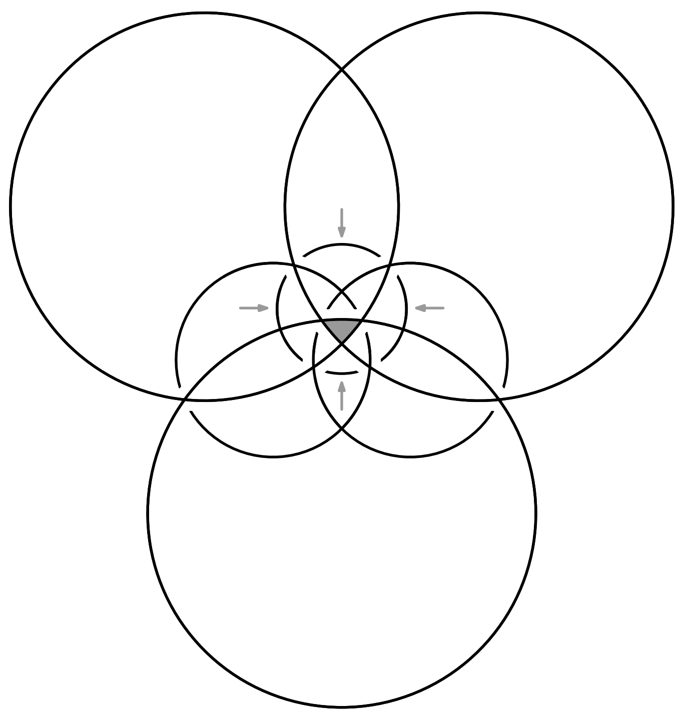

Figure 2.1: Shards in the Coxeter arrangement $\mathcal{A}(S_4)$

Example 2.1. The Coxeter arrangement $\mathcal{A}(S_4)$ consists of six hyperplanes through the origin in $\mathbb{R}^3$. These planes, intersected with the unit sphere in $\mathbb{R}^3$, define an arrangement of six great circles on the sphere. A stereographic projection yields an arrangement of six circles in the plane. This arrangement of circles is shown in Figure 2.1. The three largest circles are the hyperplanes H_{12} on the top left, H_{23} on the bottom, and H_{34} on the top right. The two medium circles are the hyperplanes H_{13} on the left and H_{24} on the right. The smallest circle is the hyperplane H_{14}. Regions of $\mathcal{A}(S_4)$ appear as curve-sided triangles. The base region, corresponding to 1234, is shaded in gray. The shards are closed 2-dimensional cones, so they appear as full circles or as circular arcs in the figure. To clarify the picture, where shards intersect, certain shards are offset slightly from the intersection to indicate that they do not continue through the intersection. The four shards contained in the hyperplane H_{14} are marked by arrows in the figure. The top shard in H_{14} is defined by $\{\vec{x} : x_1 = x_4,\ x_1 \leq x_2,\ -x_1 \leq -x_3\}$. The left shard in H_{14} is defined by $\{\vec{x} : x_1 = x_4,\ x_1 \leq x_2,\ x_1 \leq x_3\}$. The bottom shard in H_{14} is defined by $\{\vec{x} : x_1 = x_4,\ -x_1 \leq -x_2,\ x_1 \leq x_3\}$. The right shard in H_{14} is defined by $\{\vec{x} : x_1 = x_4,\ -x_1 \leq -x_2,\ -x_1 \leq -x_3\}$.

Given a permutation π and the corresponding region R, the descents of π correspond to the hyperplanes containing the lower shards of R. For example, 43 and 31 are descents in $\pi = 4312$ and the hyperplanes containing the shards below R are H_{13} and H_{34}. The cone formed by the intersection of the lower shards of R satisfies $x_i = x_j$ for each descent ji of π. Now we need to determine which shards from these hyperplanes are the lower shards of R. The shard below R contained in H_{ij} is the shard on the same side as R of each hyperplane cutting H_{ij}. Thus, for each k with $i < k < j$, the cone satisfies $x_i \geq x_k$ if and only if (k, i) is an inversion of π, and the cone satisfies $x_i \leq x_k$ if and only if (k, i) is not an inversion of π. Continuing Example 2.1, the shard below R contained in H_{13} will satisfy $x_1 \leq x_2$, since $(2, 1)$ is not an inversion in $\pi = 4312$. The following proposition summarizes this explicit description for the shard intersection associated to a given permutation.

Proposition 2.2. *For a permutation π, the corresponding shard intersection is the cone consisting of points $(x_1, \ldots, x_n)$ satisfying the following conditions for each descent ji of π.*

1. $x_i \equiv x_j$
2. $x_i \geq x_k$ *if and only if* $i < k < j$ *and* (k, i) *is an inversion of* π

3. $x_i \leq x_k$ if and only if $i < k < j$ and (k,i) is not an inversion of π.

We conclude this section with a lemma relating inversions of a permutation to intersections between descending runs. The notation $[a,b]$ stands for the set $\{c \in \mathbb{Z} : a \leq c \leq b\}$. We will say that two descending runs $D = d_1 d_2 \cdots d_m$ and $E = e_1 e_2 \cdots e_n$ *overlap* if $[d_m, d_1] \cap [e_n, e_1] \neq \emptyset$. This lemma will be used in Section 2.3.

Lemma 2.3. *Suppose (i,j) is an inversion in π such that i and j are in distinct descending runs $i_1 i_2 \cdots i_s$ and $j_1 j_2 \cdots j_t$ respectively. If $[i_s, i_1] \cap [j_t, j_1] = \emptyset$, then there exists a chain of descending runs $D_1, D_2, \ldots, D_n$ between i_s and j_1 such that $[i_s, i_1] \cap D_1 \neq \emptyset, D_n \cap [j_t, j_1] \neq \emptyset$, and D_k and D_{k+1} overlap for $1 \leq k \leq n-1$.*

Proof. Let $\pi = \cdots i_1 i_2 \cdots i_s \cdots j_1 j_2 \cdots j_t \cdots$ as in the statement of the lemma. Since $[i_s, i_1] \cap [j_t, j_1] = \emptyset$ and $i > j$, we know that $i_x > j_y$ for all $x \in [s]$ and $y \in [t]$. We claim there must be a descending run $d_1 d_2 \cdots d_p$ between i_s and j_1 such that $d_1 > i_s$ and $d_p < i_s$. We know $i_s > j_1$ and i_s and j_1 are both part of different descending runs. Directly to the right of i_s is the descending run $f_1 \cdots f_r$ with $f_1 > i_s$ since it is a descending run distinct from $i_1 \cdots i_s$. Suppose $f_1 \cdots f_r$ does not satisfy the claim. Then $f_z > i_s$ for all $z \in [r]$. Directly to the right of f_r is the descending run $g_1 \cdots g_p$ with $g_1 > f_r$ since it is a distinct descending run. This implies $g_1 > i_s$. Suppose $g_1 \cdots g_l$ does not satisfy the claim. Then $g_z > f_r > i_s$ for all $z \in [l]$. We can continue considering adjacent descending runs until we reach the descending run $h_1 \cdots h_w$ which is directly to the left of j_1. Again $h_1 > i_s$ and if $h_1 \cdots h_w$ does not satisfy the claim, then $h_z > i_s$ for all $z \in [w]$. From this we obtain $h_z > i_s > j_1$ for all $z \in [w]$ and in particular $h_w > j_1$. This implies that $h_1 \cdots h_w$ and $j_1 \cdots j_t$ are not separate descending runs, which is a contradiction. Therefore, at least one descending run $d_1 \ldots d_p$ between i_s and j_1 must satisfy the claim. Thus $[i_s, i_1] \cap [d_p, d_1] \neq \emptyset$. Let $d_1 \cdots d_p$ be D_1. Also, since $d_1 > i_s$, we know that $d_1 > j_1$. Then, by induction on the number of descending runs between i_s and j_1, there exists a chain of descending runs $D_1, D_2, \ldots, D_n$ between i_s and j_1 such that $[i_s, i_1] \cap D_1 \neq \emptyset, D_n \cap [j_t, j_1] \neq \emptyset$, and D_k and D_{k+1} overlap for $1 \leq k \leq n-1$. ∎

2.3 Permutation Pre-orders

This section begins with background and notation concerning pre-orders, leading to an injection from shard intersections to pre-orders. Composing with the bijection between shard intersections and permutations yields an injection μ from permutations to pre-orders. We characterize the image of μ and we finish the section with the description of the inverse of μ.

A pre-order is a reflexive, transitive, binary relation. A pre-order P on $[n]$ defines an equivalence relation on $[n]$ by setting $i \equiv j$ if and only if $i \preceq j \preceq i$. We will call the classes of this equivalence relation *blocks*. We simultaneously think of P as a pre-order on $[n]$ and as a partial order on its blocks. Notationally, when considering a specific pre-order P, a block will be labeled as $B_{[i,j]}$ where i is the minimal element in the block and j is the maximal element in the block. For example, a singleton block containing the element $l \in [n]$ will be labeled as $B_{[l,l]}$. $B(v)$ is defined to be the block in P containing the element $v \in [n]$. We will say blocks $B_{[i,j]}$ and $B_{[k,l]}$ *overlap* if the intervals $[i, j]$ and $[k, l]$ have a non-empty intersection. A block $B_{[i,j]}$ is covered by a block $B_{[k,l]}$ if $B_{[i,j]} \prec B_{[k,l]}$ and there does not exist a block $B_{[s,t]}$ such that $B_{[i,j]} \prec B_{[s,t]} \prec B_{[k,l]}$. To indicate that block $B_{[k,l]}$ covers block $B_{[i,j]}$ we will write $B_{[i,j]} \prec\!\!\cdot\, B_{[k,l]}$. An element i is covered by an element j in P if $i \prec j$ and for any s such that $i \preceq s \preceq j$, either $s \equiv i$ or $s \equiv j$.

We now describe an injection from the set Ψ of shard intersections in $\mathcal{A}(S_n)$ to the set $\mathcal{P}$ of pre-orders on $[n]$. The pre-order $P \in \mathcal{P}$ corresponding to a shard intersection $\Gamma \in \Psi$ is found by only considering the indices in the equalities and inequalities defining Γ. Thus for $i, j \in [n]$, $i \preceq j$ in P if and only if the inequality $x_i \leq x_j$ holds in Γ. In particular, if $x_i = x_j$ in Γ then $i \preceq j \preceq i$ in P.

Example 2.4. The shard intersection $\Gamma = \{\vec{x} : x_1 = x_4,\ x_6 = x_7,\ x_1 \leq x_2,\ -x_1 \leq -x_3\}$ in $\mathbb{R}^7$ corresponds to the pre-order in Figure 2.2 which has blocks $B_{[3,3]} = \{3\}$, $B_{[1,4]} = \{1,4\}$, $B_{[2,2]} = \{2\}$, $B_{[5,5]} = \{5\}$ and $B_{[6,7]} = \{6,7\}$. Γ is the intersection of the shards $\{\vec{x} : x_1 = x_4,\ x_1 \leq x_2,\ -x_1 \leq -x_3\}$ and $\{\vec{x} : x_6 = x_7\}$ in S_7.

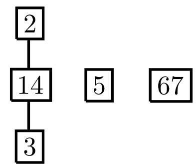

Figure 2.2: The pre-order representing $\Gamma = \{\vec{x} \in \mathbb{R}^7 : x_1 = x_4,\ x_6 = x_7,\ x_1 \leq x_2,\ -x_1 \leq -x_3\}$

Let $\mu : S_n \to \mathcal{P}$ be the map that takes a permutation $\pi \in S_n$ to a pre-order $P \in \mathcal{P}$ as follows: each descending run $\pi_m \pi_{m+1} \cdots \pi_{m+t}$ in π is a block $B_{[\pi_{m+t}, \pi_m]} = \{\pi_m, \pi_{m+1}, \ldots, \pi_{m+t}\}$ in P. For distinct blocks $B_{[\pi_{i+j}, \pi_i]}$ and $B_{[\pi_{k+l}, \pi_k]}$ which overlap, $B_{[\pi_{k+l}, \pi_k]} \succeq B_{[\pi_{i+j}, \pi_i]}$ in P if and only if $\pi_k \cdots \pi_{k+l}$ occurs to the right of $\pi_i \cdots \pi_{i+j}$ in π, i.e. $(i + j) < k$. The transitive closure of these relations defines the pre-order P.

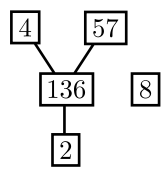

Figure 2.3: $\mu(26314758)$

Example 2.5. For $\pi = 26314758$ the descending runs form the following blocks: $B_{[2,2]} = \{2\}, B_{[1,6]} = \{1, 3, 6\}, B_{[4,4]} = \{4\}, B_{[5,7]} = \{5, 7\}, B_{[8,8]} = \{8\}$. The pre-order $\mu(\pi)$ is shown in Figure 2.3.

Proposition 2.6. *μ is the composition of the bijection between permutations and shard intersections with the injection from shard intersections to pre-orders.*

Proof. All that needs to be shown is that the relations between elements established by μ correspond to the relations given in Proposition 2.2. Suppose ji is a descent in π. Then i and j

are in the same descending run in π and in $\mu(\pi)$ they will be in the same block, implying that $i \equiv j$. Suppose k is an element of π with $i < k < j$. In $\mu(\pi)$ the block $B(k)$ must be related to the block $B(i)$ since they overlap. If (k, i) is an inversion in π, then we have $B(i) \succeq B(k)$ implying $i \succeq k$. If (k, i) is not an inversion, then we have $B(i) \preceq B(k)$ implying $i \preceq k$. This corresponds to the relations given in Proposition 2.2. $\square$

Define a *permutation pre-order* as a pre-order P on $[n]$ satisfying the following two conditions:

(P1) if any two blocks in P overlap, they must be comparable in P

(P2) all covering relationships in P must be between blocks that overlap.

Let Ω be the set of permutation pre-orders.

As noted previously, a single shard $\Sigma \in \Psi$ can be represented as $\Sigma = \{\vec{x} : x_i = x_j \text{ and } \epsilon_k x_i \leq \epsilon_k x_k \text{ for } i < k < j\}$. Thus Σ corresponds to a pre-order P with one block $B_{[i,j]}$ of size two and $n-2$ singleton blocks: $B_{[v,v]}$ for $v \in [n] \setminus \{i, j\}$. Each block $B_{[k,k]}$ for $i < k < j$ will be comparable to $B_{[i,j]}$ in P and all other blocks will be incomparable to it. All covering relationships in P involve overlapping blocks. Therefore $P \in \Omega$. Define $\Omega^{\Sigma} \subset \Omega$ to be the set of permutation pre-orders corresponding to single shards.

Proposition 2.7. *μ is a bijection from permutations in S_n to pre-orders in Ω.*

Proof. By Proposition 2.6, μ is a bijection onto its image. All that remains to be shown is that Ω is the image of μ. It is clear from the definition of μ that the image of μ is contained in Ω.

Let $\omega \in \Omega$. It is sufficient to find a set of pre-orders in Ω^{Σ} such that the transitive closure of the unions of these pre-orders is ω. For each $B_{[i,k]}$ in ω consider each $s \in B_{[i,k]}$ with $i < s$. Define ω_{is} to be the pre-order on $[n]$ with one two element block $B_{[i,s]}$ and the remaining elements in singleton blocks $B_{[v,v]}$ for $v \in [n] \setminus \{i, s\}$ with the following order relations:

$$(1)\ \text{for } v \in [i,s] : \begin{cases} \text{if } i \preceq v \text{ in } \omega, \text{then } i \preceq v \text{ in } \omega_{is} \\ \text{if } i \succeq v \text{ in } \omega, \text{then } i \succeq v \text{ in } \omega_{is} \\ \text{if } i \equiv v \text{ in } \omega, \text{then } i \preceq v \text{ in } \omega_{is} \end{cases}$$

((P1) rules out the possibility that i and v are incomparable.)

(2) for $v \notin [i,s]$: v is incomparable in ω_{is} to x for all $x \in [n] \setminus \{v\}$.

Clearly, $\omega_{is} \in \Omega^{\Sigma}$. Taking the transitive closure of unions of ω_{is} for each pair i,s we obtain all the blocks of ω. (P2) guarantees that all of the covering relationships of ω are obtained as well. Thus, taking the transitive closure of unions of the set of pre-orders ω_{is} (for each pair i,s described above) gives ω. $\square$

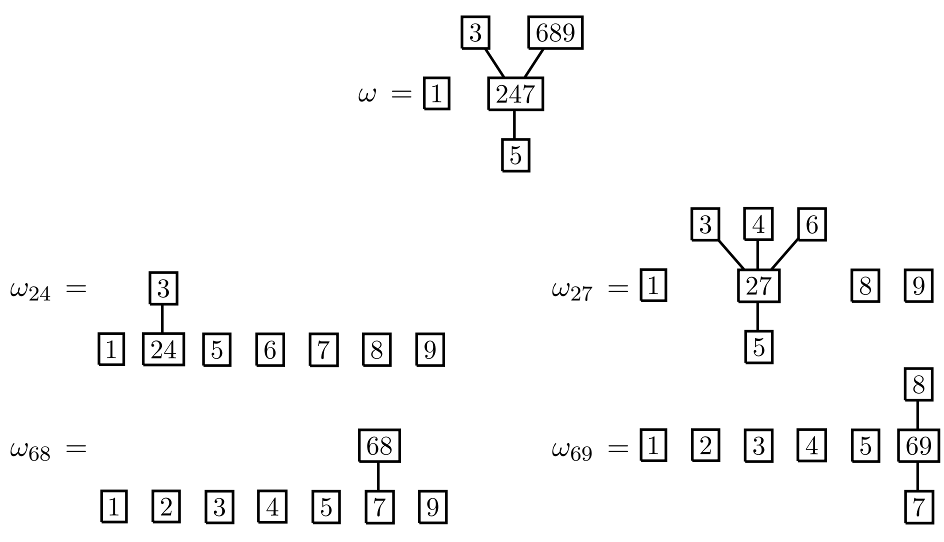

Figure 2.4: Illustration of the proof of Proposition 2.7

We now describe the inverse map to μ. Define $\lambda : \Omega \to S_n$ as follows: given $\omega \in \Omega$, we define a permutation $\pi = \lambda(\omega)$ such that each block in ω is a descending run in π. For any blocks $B_1, B_2 \in \omega$ if $B_1 \prec B_2$ in ω, then we place the descending run containing the elements of B_2 to the right of the descending run containing the elements of B_1 in π. Let $B_{[i,j]}$ and $B_{[k,l]}$ be blocks that are incomparable in ω. By (P1), the two blocks do not overlap, so without loss of generality, $j < k$. We place the descending run containing the elements of $B_{[k,l]}$ to the right of the descending run containing the elements of $B_{[i,j]}$ in π.

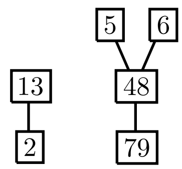

Figure 2.5: The permutation pre-order ω

Example 2.8. Suppose ω is the permutation pre-order shown in Figure 2.5. Then $\lambda(\omega) =$ 231978456.

Next we give a lemma establishing that λ is indeed a permutation and conclude the section with the proof that $\lambda = \mu^{-1}$.

Lemma 2.9. *For any permutation pre-order ω, the image $\lambda(\omega)$ is a permutation.*

Proof. The definition of $\lambda(\omega)$ defines a directed graph on the blocks of ω such that any two blocks are connected by a single directed edge. We will show that the digraph is acyclic, meaning that we have defined a total order on the blocks of ω and $\lambda(\omega)$ is a permutation. It is enough to show that any set of three blocks from ω is acyclically ordered by λ. Suppose $B_{[i,j]}, B_{[k,l]}, B_{[s,t]} \in \omega$. There are four cases to consider.

Case 1: If $B_{[i,j]}, B_{[k,l]}$, and $B_{[s,t]}$ are all incomparable in ω, then $[i,j], [k,l]$ and $[s,t]$ are pairwise disjoint, and λ orders them numerically based on the positions of $[i,j], [k,l]$ and $[s,t]$ in $\mathbb{R}$. Thus, they will be ordered acyclically.

Case 2: Suppose exactly two blocks are comparable. Without loss of generality let $B_{[i,j]} \prec B_{[k,l]}$

and let $B_{[s,t]}$ be incomparable to both $B_{[i,j]}$ and $B_{[k,l]}$. We claim that either $t < i$ and $t < k$ or $s > j$ and $s > l$. Since $B_{[s,t]}$ is incomparable to both $B_{[i,j]}$ and $B_{[k,l]}$ in ω, we know that $[s,t] \cap [i,j] = \emptyset$ and $[s,t] \cap [k,l] = \emptyset$. There are two subcases to consider: either $[i,j] \cap [k,l] \neq \emptyset$ or $[i,j] \cap [k,l] = \emptyset$.

Case 2a: If $[i,j] \cap [k,l] \neq \emptyset$, then it follows that $[s,t] \cap [\min(i,k), \max(j,l)] = \emptyset$ and the claim is true.

Case 2b: If $[i,j] \cap [k,l] = \emptyset$, then by (P2) there exists a chain of covers $B_{[i,j]} \prec\!\cdot\, B_1 \prec\!\cdot\, B_2 \prec\!\cdot\, \cdots \prec\!\cdot\, B_n \prec\!\cdot\, B_{[k,l]}$ in ω such that $B_{[i,j]}$ and B_1 overlap, B_n and $B_{[k,l]}$ overlap, and B_x and B_{x+1} overlap for $1 \leq x \leq n-1$. Since $B_{[s,t]}$ is incomparable to both $B_{[i,j]}$ and $B_{[k,l]}$, it must be incomparable to B_x for $x \in [n]$. Therefore B_x and $B_{[s,t]}$ do not overlap for $x \in [n]$. Again it follows that $[s,t] \cap [\min(i,k), \max(j,l)] = \emptyset$ and the claim is true.

Since the claim is true, the definition of λ implies that $B_{[s,t]}$ is to the left or to the right of both $B_{[i,j]}$ and $B_{[k,l]}$ in $\lambda(\omega)$. Thus the blocks will be ordered acyclically by λ.

Case 3: Suppose two pairs of blocks are comparable and one pair is incomparable. Without loss of generality, either $B_{[s,t]} \prec B_{[i,j]}$ and $B_{[s,t]} \prec B_{[k,l]}$ or $B_{[i,j]} \prec B_{[s,t]}$ and $B_{[k,l]} \prec B_{[s,t]}$, with $B_{[i,j]}$ and $B_{[k,l]}$ incomparable in both instances. In the first instance, by the definition of λ, $B_{[s,t]}$ is to the left of both $B_{[i,j]}$ and $B_{[k,l]}$ in λ. Since $B_{[i,j]}$ and $B_{[k,l]}$ are incomparable in ω, we know that $[i,j] \cap [k,l] = \emptyset$, so the blocks $B_{[i,j]}$ and $B_{[k,l]}$ are ordered numerically in λ based on the positions of $[i,j]$ and $[k,l]$ in $\mathbb{R}$. Thus the three blocks will be ordered acyclically by λ. The second instance follows similarly, and again the three blocks will be ordered acyclically by λ.

Case 4: Finally, if $B_{[i,j]}, B_{[k,l]}$ and $B_{[s,t]}$ are all comparable in ω, then by the transitivity of ω they must be ordered acyclically.

Therefore, $\lambda(\omega)$ defines a total order on the blocks of ω, which corresponds to a unique permutation π. $\square$

Proposition 2.10. $\lambda = \mu^{-1}$

Proof. Since we know that μ is a bijection and we proved in Lemma 2.9 that λ is well-defined, it is enough to show that $\lambda(\mu(\pi)) = \pi$ to complete the proof. This is equivalent to showing that the relative positions of any two descending runs in π remain the same in $\lambda(\mu(\pi))$. Descending runs in π are preserved in $\lambda(\mu(\pi))$ because μ converts the descending runs to blocks and λ converts each block back to the original descending run. (The descending runs do not increase in length since covering pairs $B_{[i,j]}$ and $B_{[k,l]}$ have $[i,j] \cap [k,l] \neq \emptyset$ and λ places incomparable blocks in increasing order.) Let $i_1 i_2 \cdots i_s$ and $j_1 j_2 \cdots j_t$ be descending runs in π with $\pi = \cdots i_1 i_2 \cdots i_s \cdots j_1 j_2 \cdots j_t \cdots$. Each of the descending runs in π will become blocks in $\mu(\pi)$, so $B_{[i_s,i_1]}$ and $B_{[j_t,j_1]}$ are blocks in $\mu(\pi)$. There are two cases to consider.

Case 1: Suppose $[i_s, i_1] \cap [j_t, j_1] \neq \emptyset$. Then $B_{[i_s,i_1]} \prec B_{[j_t,j_1]}$ in $\mu(\pi)$ and λ will place the descending run $i_1 i_2 \cdots i_s$ to the left of the descending run $j_1 j_2 \cdots j_t$ in $\lambda(\mu(\pi))$.

Case 2: Suppose $[i_s, i_1] \cap [j_t, j_1] = \emptyset$. Then we have two subcases to consider: either $i_x > j_y$ for all $x \in [s]$ and $y \in [t]$ or $i_x < j_y$ for all $x \in [s]$ and $y \in [t]$.

Case 2a: Suppose $i_x > j_y$ for all $x \in [s]$ and $y \in [t]$. Then by Lemma 2.3, $B_{[i_s,i_1]} \prec B_{[j_t,j_1]}$ in $\mu(\pi)$ and λ will place the descending run $i_1 i_2 \cdots i_s$ to the left of the descending run $j_1 j_2 \cdots j_t$ in $\lambda(\mu(\pi))$.

Case 2b: Suppose $i_x < j_y$ for all $x \in [s]$ and $y \in [t]$. For the purpose of contradiction, suppose that the descending run $i_1 i_2 \cdots i_s$ is to the right of the descending run $j_1 j_2 \cdots j_t$ in $\lambda(\mu(\pi))$. Since $i_x < j_y$ for $x \in [s]$ and $y \in [t]$, the descending runs were not ordered based on their positions in $\mathbb{R}$ implying that $B_{[j_t,j_1]} \prec B_{[i_s,i_1]}$ in $\mu(\pi)$. Since $[i_s, i_1] \cap [j_t, j_1] = \emptyset$, we know that $B_{[j_t,j_1]}$ is not covered by $B_{[i_s,i_1]}$. Thus we have a chain of covers $B_{[j_t,j_1]} \prec\!\!\cdot\; B_1 \prec\!\!\cdot\; B_2 \prec\!\!\cdot\; \cdots \prec\!\!\cdot\; B_n \prec\!\!\cdot\; B_{[i_s,i_1]}$ in $\mu(\pi)$ where $n \geq 1$. Since $B_{[j_t,j_1]} \prec\!\!\cdot\; B_1$ in $\mu(\pi)$, the descending run $j_1 \cdots j_t$ must be to the left of the descending run containing the elements of B_1 in π. Similarly, the descending run containing the elements of B_x must be to the left of the descending run containing the elements of B_{x+1} in π for $1 \leq x \leq n-1$ and the descending run containing the elements of B_n must be to the left of the descending run $i_1 \cdots i_s$ in π. Thus $j_1 j_2 \cdots j_t$ is to the left of $i_1 i_2 \cdots i_s$ in π, which is a contradiction. Therefore the descending run $i_1 i_2 \cdots i_s$ is to the left of the descending run $j_1 j_2 \cdots j_t$ in $\lambda(\mu(\pi))$. $\square$

2.4 The Shard Intersection Order on S_n

In this section we describe the shard intersection order on S_n in terms of an order on the permutation pre-orders. We then define a labeling for the order and spend the majority of the section proving various lemmas concerning cover relations and the labeling. The section, and the chapter, culminate in the proof that the given labeling is an EL-labeling of the shard intersection order on S_n.

The *shard intersection order* is the set $\Psi(\mathcal{A}, B)$ partially ordered by reverse containment. It was shown in [16] to be a graded, atomic and coatomic lattice and is denoted as $(\Psi(\mathcal{A}, B), \supseteq)$. We will continue considering the case where $W = S_n$. Thus $(\Psi(\mathcal{A}(S_n), B), \supseteq)$ can be realized as the permutation pre-orders partially ordered by containment of relations, which we will denote as $(\Omega, \leq_S)$. (See Figure 2.6 for the shard intersection order on S_4.)

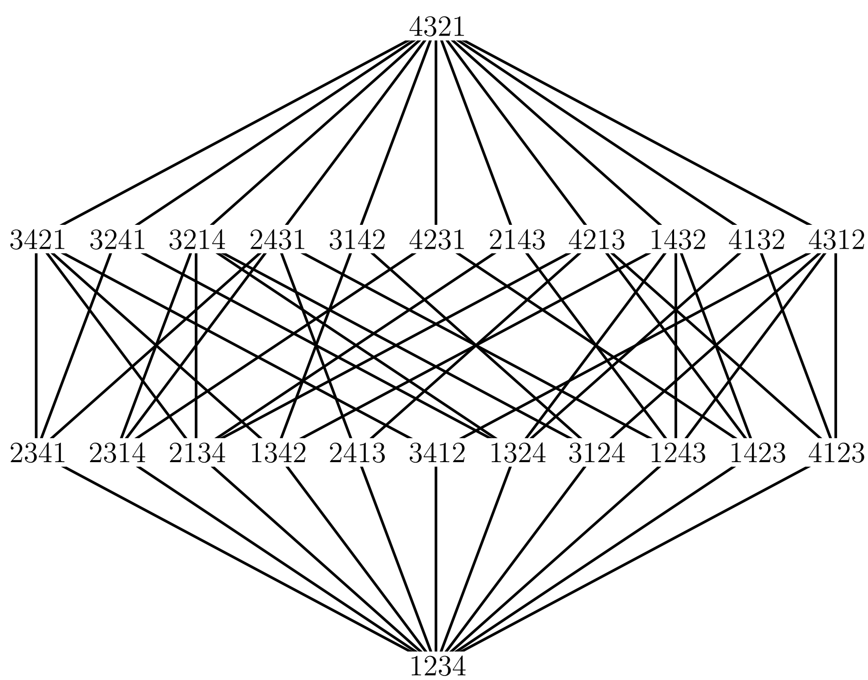

Figure 2.6: The shard intersection order on S_4

To avoid confusion, we remind the reader that the three order relations appearing in the chapter are the total order $\leq$ on real numbers, the partial order $\preceq$ on blocks in a given permutation pre-order and the order relation $\leq_S$ on permutation pre-orders in $(\Omega, \leq_S)$. The minimal element $\hat{0}$ is the empty intersection, which is the permutation pre-order with each number in a singleton block and no order relations among the blocks. The maximal element $\hat{1}$ is the permutation pre-order with all numbers together in one block. In [16] it was shown that the rank of $\Gamma \in \Psi$ is the codimension of Γ. From this, one easily deduces the following result for permutation pre-orders. The notation $|\omega|$ stands for the number of blocks in ω.

Proposition 2.11. *The rank of* $\omega \in \Omega$ *in the graded lattice* $(\Omega, \leq_S)$ *is* $n - |\omega|$.

To go up by a cover from a shard intersection Γ in $(\Psi(\mathcal{A}, B), \supseteq)$ we intersect Γ with one additional shard Σ, chosen so that $\dim(\Sigma \cap \Gamma) = \dim(\Gamma) - 1$. Thus to go up by a cover from ω in $(\Omega, \leq_S)$, any two blocks which are unrelated or related by a cover can be combined. All relations from ω are maintained in the permutation pre-order that covers it, and relations involving either of the combined blocks from ω become an order relation involving the new block in the cover. If by combining two unrelated blocks in ω an interval is formed which now intersects the intervals of other unrelated blocks, then the newly overlapping blocks may be greater than or less than the combined block in the cover.

Example 2.12. Figure 2.7 shows a permutation pre-order labeled ω and the eight permutation pre-orders labeled *(A)* through *(H)* which cover it. Notice that if we combine blocks $B_{[1,3]}$ and $B_{[5,5]}$ in ω, blocks $B_{[1,5]}$ and $B_{[4,4]}$ overlap in the permutation pre-order which covers ω. By (P2) this means that $B_{[1,5]}$ and $B_{[4,4]}$ must be related. Since $B_{[4,4]}$ is unrelated to blocks $B_{[1,3]}$ and $B_{[5,5]}$ in ω there is no restriction on the direction of this relation, so $B_{[4,4]}$ can be either greater than or less than $B_{[1,5]}$. Thus we obtain the two permutation pre-orders *(D)* and *(E)*. Each of these pre-orders arise by intersecting ω with a permutation pre-order corresponding to a single shard in Ω^{Σ} with 3 and 5 together in one block.

In ω, any two blocks can be combined to form a permutation pre-order which covers ω. On

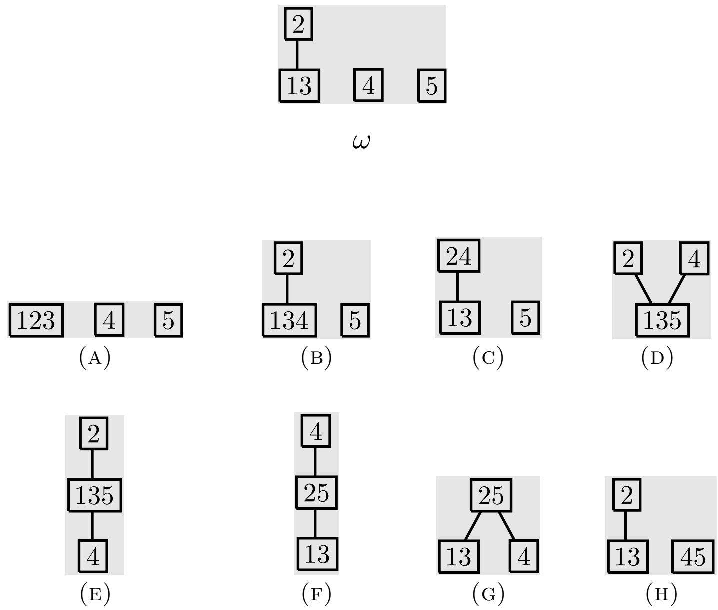

Figure 2.7: Covers of a permutation pre-order $\omega \in (\Omega, \leq_S)$

the other hand, in the permutation pre-order labeled *(F)*, blocks $B_{[1,3]}$ and $B_{[4,4]}$ cannot be combined to form a cover since this would force all numbers to be in one block and the rank would go up by two instead of one.

The main result of this chapter is an edge-lexicographic or EL-labeling [3, 5] of $(\Omega, \leq_S)$. A sequence $a_1a_2\cdots a_n$ is *lexicographically smaller* than a sequence $b_1b_2\cdots b_n$ if there exists a $j \in [n]$ such that $a_i = b_i$ for all $i < j$ but $a_j < b_j$. Define a labeling of the poset to be an *EL-labeling* if the edges of the poset are labeled with positive integers such that the following two conditions hold.

(EL1) Each interval (ω, ω') in the poset has a unique maximal chain $\omega = \omega_0 \lessdot_S \omega_1 \lessdot_S \cdots \lessdot_S \omega_k = \omega'$ whose edge label sequence, $\sigma(\omega_0, \omega_1)\sigma(\omega_1, \omega_2)\cdots\sigma(\omega_{k-1}, \omega_k)$ is weakly increasing in value, and

(EL2) this weakly increasing edge label sequence is lexicographically smaller than the label sequences of all other maximal chains from ω to ω'.

Given a permutation pre-order ω, number the descending runs in $\lambda(\omega)$ from 1 to $|\omega|$, proceeding from left to right. Define this number to be the *placement* of the descending run $i_1 \cdots i_n$ in $\lambda(\omega)$ or the *placement* of its corresponding block $B_{[i_n,i_1]}$ in ω. We will denote the placement of a block B in ω as $pl_\omega(B)$. The total order on the blocks of ω defined by the placements is equivalent to the total order defined by λ. Also, if $B_{[i,j]} \prec B_{[k,l]}$ in ω, then $pl_\omega(B_{[i,j]}) < pl_\omega(B_{[k,l]})$ since the descending run corresponding to $B_{[i,j]}$ is to the left of the descending run corresponding to $B_{[k,l]}$ in $\lambda(\omega)$.

For $(\Omega, \leq_S)$ we will describe a labeling σ. Let $\sigma(\omega_i, \omega_{i+1})$ = the maximum of the placements of the two blocks from ω_i that were combined to form ω_{i+1}. We will prove the following theorem.

Theorem 2.13. *σ is an EL-labeling.*

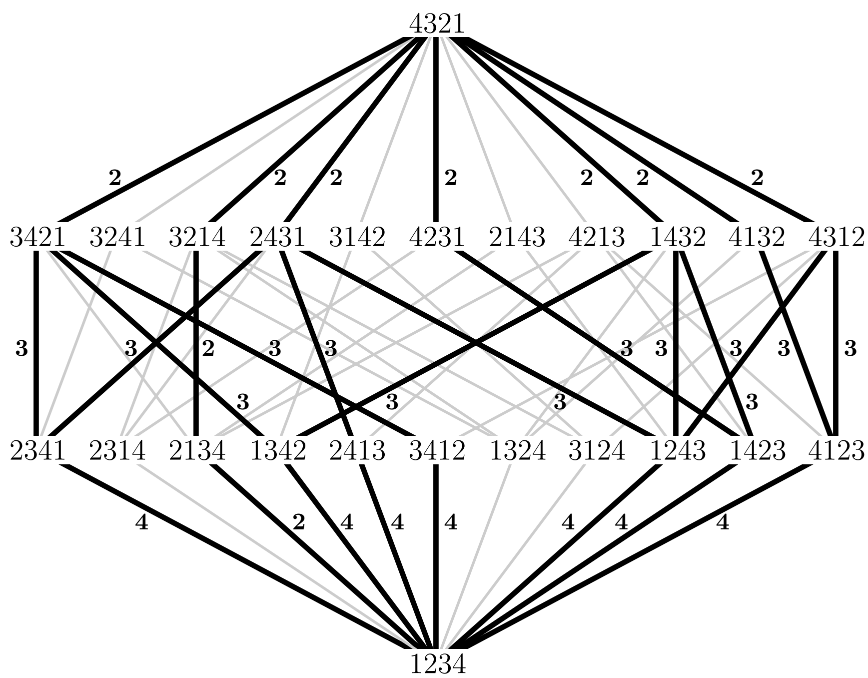

Figure 2.8: Labeled maximal chains in the shard intersection order

Example 2.14. The unique increasing maximal chain and the thirteen strictly decreasing maximal chains in the shard intersection order on S_4 with respect to the labeling σ are shown in Figure 2.8.

The proof of Theorem 2.13 requires a proposition and several lemmas concerning relations in $(\Omega, \leq_S)$. Given a permutation pre-order ω and the interval (ω, ω') in $(\Omega, \leq_S)$, we will denote by $T(\omega, \omega')$ the set of pairs of blocks in ω that could be combined to form a permutation pre-order in (ω, ω') which covers ω. A pair of blocks in ω will be called *combinable* if either one block covers the other or the pair is incomparable in ω.

Lemma 2.15. *Suppose ω and ω' are permutation pre-orders with $\omega <_S \omega'$. Let B_1 and B_2 be combinable blocks in ω such that they are contained in the same block in ω'. Then there exists a permutation pre-order ω'' with $\omega \lessdot_S \omega'' \leq_S \omega'$ such that B_1 and B_2 are in the same block in ω''.*

Proof. Label the elements of B_1 as $a_1, a_2, \ldots, a_m$ with $a_i < a_{i+1}$ for $1 \leq i \leq m-1$ and the elements of B_2 as $b_1, b_2, \ldots, b_n$ with $b_j < b_{j+1}$ for $1 \leq j \leq n-1$. Without loss of generality suppose that $a_1 < b_1$. Since blocks B_1 and B_2 are in the same block in ω' we know that $a_i \equiv b_j$ in ω' for $i \in [m], j \in [n]$. Define ω^* to be the permutation pre-order on $[n]$ with one two element block $B_{[a_1,b_1]}$ and the remaining elements in singleton blocks $B_{[v,v]}$ for $v \in [n] \setminus \{a_1, b_1\}$ with the following order relations:

$$(1) \text{ for } v \in [a_1, b_1]: \begin{cases} \text{if } a_1 \preceq v \text{ in } \omega', \text{then } a_1 \preceq v \text{ in } \omega^* \\ \text{if } a_1 \succeq v \text{ in } \omega', \text{then } a_1 \succeq v \text{ in } \omega^* \\ \text{if } a_1 \equiv v \text{ in } \omega', \text{then } a_1 \preceq v \text{ in } \omega^* \end{cases}$$

((P1) rules out the possibility that a_1 and v are incomparable.)

(2) for $v \notin [a_1, b_1]$: v is incomparable in ω^* to x for all $x \in [n] \setminus \{v\}$.

Clearly, $\omega^* \in \Omega^{\Sigma}$. Taking the transitive closure of the union of ω with ω^* gives the permutation pre-order ω''. $\square$

Proposition 2.16. *Let ω and ω' be permutation pre-orders with $\omega \lessdot_S \omega'$. Suppose B_1 and B_2 are the blocks in ω that are combined to form a block B in ω', with $pl_\omega(B_1) = a < c = pl_\omega(B_2)$. Let B^* be in ω.*

1. *If $a < pl_\omega(B^*) < c$, then B^* is comparable to B in ω'.*

2. *If B^* is incomparable to B_1 and B_2 in ω and comparable to B in ω', then $a < pl_\omega(B^*) < c$.*

3. *If $pl_\omega(B^*) < a$, then $pl_{\omega'}(B^*) = pl_\omega(B^*)$. If $a \leq pl_\omega(B^*) \leq c$, then $a \leq pl_{\omega'}(B^*) \leq c - 1$. If $pl_\omega(B^*) > c$, then $pl_{\omega'}(B^*) = pl_\omega(B^*) - 1$.*

4. *Let B' be a block in ω' such that $pl_{\omega'}(B') = b < c$. If blocks B and B' are combinable in ω', then B' is combinable with B_1 in ω.*

5. *If $pl_\omega(B^*) = b < c$, then B', the block containing B^* in ω', has $pl_{\omega'}(B') < c$.*

Proof. Let $B_1 = B_{[i,j]}$ and $B_2 = B_{[k,l]}$. Then $B = B_{[s,t]}$ for $s = \min(i, k)$ and $t = \max(j, l)$.

Proof of (1): There are two cases to consider. Suppose, in the first case, that $B_{[i,j]}$ and $B_{[k,l]}$ form a cover in ω. Arguing as in Case 2a of Lemma 2.9, any block which is incomparable to both $B_{[i,j]}$ and $B_{[k,l]}$ must be to the left of both $B_{[i,j]}$ and $B_{[k,l]}$ or to the right of both $B_{[i,j]}$ and $B_{[k,l]}$ in $\lambda(\omega)$ and thus does not have a placement between a and c. Therefore, all blocks B^* with $a < pl_\omega(B^*) < c$ must be comparable to either $B_{[i,j]}$ or $B_{[k,l]}$ in ω. Since all relations from ω must also be in ω', the blocks B^* must be comparable to $B_{[s,t]}$ in ω'.

In the second case we suppose that $B_{[i,j]}$ and $B_{[k,l]}$ are incomparable in ω. Again, any block B^* with $a < pl_\omega(B^*) < c$ which is comparable to either $B_{[i,j]}$ or $B_{[k,l]}$ in ω must be comparable to $B_{[s,t]}$ in ω'. Thus, all that is left to consider is a block $B_{[p,q]}$ with $a < pl_\omega(B_{[p,q]}) < c$ which is incomparable to both $B_{[i,j]}$ and $B_{[k,l]}$ in ω. Since the three blocks $B_{[i,j]}, B_{[k,l]}$ and $B_{[p,q]}$ are incomparable in ω, we know that $[i, j], [k, l]$ and $[p, q]$ are pairwise disjoint and the three blocks are ordered in $\lambda(\omega)$ based on the positions of $[i, j], [k, l]$ and $[p, q]$ in $\mathbb{R}$. Given that the placements of the three blocks correspond to their ordering in $\lambda(\omega)$ and $pl_\omega(B_{[i,j]}) < pl_\omega(B_{[p,q]}) < pl_\omega(B_{[k,l]})$, we can conclude that $i \leq j < p \leq q < k \leq l$. Combining $B_{[i,j]}$ and $B_{[k,l]}$ forms the block

$B_{[i,l]} = B_{[s,t]}$, and we know that $[p,q] \subset [i,l] = [s,t]$. Therefore by (P1), $B_{[p,q]}$ must be related to $B_{[s,t]}$ in ω'.

Proof of (2): Suppose for the purpose of contradiction, a block $B_{[f,g]}$ with $pl_\omega(B_{[f,g]}) < a$ is incomparable to both $B_{[i,j]}$ and $B_{[k,l]}$ in ω and is comparable to $B_{[s,t]}$ in ω'. Since λ orders incomparable blocks based on their intervals, $g < i$ and $g < k$. Thus, $g < \min(i,k)$ and $[f,g] \cap [s,t] = \emptyset$. This implies that $B_{[f,g]}$ and $B_{[s,t]}$ do not cover one another in ω' and therefore must be related by a chain of covers: $B_{[f,g]} \prec\!\cdot\, B_{[x_1,x_2]} \prec\!\cdot\, B_{[x_3,x_4]} \prec\!\cdot \cdots \prec\!\cdot\, B_{[x_{d-1},x_d]} \prec\!\cdot\, B_{[s,t]}$. By (P2), we have $[f,g] \cap [x_1,x_2] \neq \emptyset, [x_1,x_2] \cap [x_3,x_4] \neq \emptyset, \ldots, [x_{d-1},x_d] \cap [s,t] \neq \emptyset$. Thus, some interval $[x_m, x_{m+1}]$ for $1 \leq m \leq d-1$ must overlap either $[i,j]$ or $[k,l]$. The block $B_{[x_m,x_{m+1}]}$ corresponding to this interval must be less than either $B_{[i,j]}$ or $B_{[k,l]}$ in ω (whichever it overlaps) because if it were greater than either of them it would be greater than $B_{[s,t]}$ in ω' and thus would not be part of the chain of covers. Since all blocks except for $B_{[i,j]}$ and $B_{[k,l]}$ are the same in both ω and ω', we have that $B_{[f,g]}$ is related to either $B_{[i,j]}$ or $B_{[k,l]}$ by a chain of covers in ω. This contradicts our assumption that $B_{[f,g]}$ is incomparable to both $B_{[i,j]}$ and $B_{[k,l]}$ in ω and thus $pl_\omega(B_{[f,g]}) \not< a$. Similarly, if $pl_\omega(B_{[f,g]}) > c$ we see that $f > \max(j,l)$ and there must exist a chain of covers between $B_{[s,t]}$ and $B_{[f,g]}$ such that one block in the chain has an interval that overlaps either $[i,j]$ or $[k,l]$. Reaching the same contradiction we conclude that $pl_\omega(B_{[f,g]}) \not> c$ and thus $a < pl_\omega(B_{[f,g]}) < c$.

Proof of (3): Let $B_{[u,v]}$ be a block in ω with $pl_\omega(B_{[u,v]}) = x < a$. We first show that $pl_{\omega'}(B_{[u,v]}) < pl_{\omega'}(B_{[s,t]})$. Suppose that $B_{[u,v]}$ is incomparable to both $B_{[i,j]}$ and $B_{[k,l]}$. Since $x < a$ and $x < c$, $v < \min(i,k) = s$ and the descending run corresponding to $B_{[s,t]}$ must be to the right of the descending run corresponding to $B_{[u,v]}$ in $\lambda(\omega')$. Now suppose that $B_{[u,v]}$ is comparable to $B_{[i,j]}$ or $B_{[k,l]}$ in ω. Then either $B_{[u,v]} \prec B_{[i,j]}$ or $B_{[u,v]} \prec B_{[k,l]}$, since $x < a$ and $x < c$. All relations from ω must also hold in ω' so $B_{[u,v]} \prec B_{[s,t]}$ in ω'. Thus the descending run corresponding to the new block $B_{[s,t]}$ will always be to the right of the descending run corresponding to $B_{[u,v]}$ in $\lambda(\omega')$.

We next show that $pl_\omega(B_{[u,v]}) = pl_{\omega'}(B_{[u,v]})$. Let $B_{[f,g]}$ be any block in ω other than

$B_{[i,j]}, B_{[k,l]}$ and $B_{[u,v]}$. If $B_{[u,v]}$ is incomparable to $B_{[f,g]}$ in both ω and ω' then the descending runs corresponding to $B_{[u,v]}$ and $B_{[f,g]}$ must be ordered the same in both $\lambda(\omega)$ and $\lambda(\omega')$, since λ orders them based on the positions of $[u,v]$ and $[f,g]$ in $\mathbb{R}$. If $B_{[u,v]}$ is comparable to $B_{[f,g]}$ in ω, then again the descending runs corresponding to each of them must be ordered the same in both $\lambda(\omega)$ and $\lambda(\omega')$, since relations from ω are preserved in ω'. If $B_{[u,v]}$ is incomparable to $B_{[f,g]}$ in ω and comparable to it in ω', then we claim that $v < f$ and $B_{[u,v]} \prec B_{[f,g]}$ in ω'. Two blocks which are comparable in ω' and incomparable in ω can only occur when both blocks are comparable through the new block $B_{[s,t]}$ in ω'. Since we already showed that $B_{[u,v]}$ must be less than $B_{[s,t]}$ in ω' if they are comparable, we must have $B_{[u,v]} \prec B_{[s,t]} \prec B_{[f,g]}$ in ω'. In order for the relation $B_{[s,t]} \prec B_{[f,g]}$ to exist in ω', the descending run corresponding to $B_{[f,g]}$ must be to the right of the descending run corresponding to $B_{[i,j]}$ in $\lambda(\omega)$. Since $B_{[u,v]}$ and $B_{[f,g]}$ are incomparable in ω, this implies that $v < f$. Thus the claim is true. Therefore, all descending runs to the right of the descending run corresponding to $B_{[u,v]}$ in $\lambda(\omega)$ will remain to the right in $\lambda(\omega')$ and all descending runs to the left in $\lambda(\omega)$ will remain to the left in $\lambda(\omega')$. Thus $B_{[u,v]}$ will retain the placement x in ω', i.e. $pl_\omega(B_{[u,v]}) = pl_{\omega'}(B_{[u,v]})$.

Now let $B_{[u,v]}$ be a block in ω with $pl_\omega(B_{[u,v]}) > c$. By similar arguments, all descending runs to the right of the descending run corresponding to $B_{[u,v]}$ in $\lambda(\omega)$ will remain to the right in $\lambda(\omega')$ and all descending runs to the left in $\lambda(\omega)$ will remain to the left in $\lambda(\omega')$. Since the descending runs corresponding to $B_{[i,j]}$ and $B_{[k,l]}$ have been combined to form a single descending run, there is one less descending run to the left of the descending run corresponding to $B_{[u,v]}$. Thus $pl_{\omega'}(B_{[u,v]}) = pl_\omega(B_{[u,v]}) - 1$. We can now also conclude that if $a \leq pl_\omega(B^*) \leq c$, then $a \leq pl_{\omega'}(B^*) \leq c - 1$.

Proof of (4): Suppose, for the purpose of contradiction, B_1 and B' are not combinable in ω. Then B_1 and B' are comparable by a chain of covers in ω, i.e. either $B_1 \prec\!\!\cdot\, A_1 \prec\!\!\cdot\, A_2 \prec\!\!\cdot\, \cdots \prec\!\!\cdot\, A_k \prec\!\!\cdot\, B'$ or $B' \prec\!\!\cdot\, A_1 \prec\!\!\cdot\, A_2 \prec\!\!\cdot\, \cdots \prec\!\!\cdot\, A_k \prec\!\!\cdot\, B_1$ for $k \geq 1$. Since B_1 and B_2 are combinable in ω they must either be incomparable or related by a cover in ω. If B_1 and B_2 are incomparable in ω then we would have either $B \prec\!\!\cdot\, A_1 \prec\!\!\cdot\, A_2 \prec\!\!\cdot\, \cdots \prec\!\!\cdot\, A_k \prec\!\!\cdot\, B'$ or $B' \prec\!\!\cdot\, A_1 \prec\!\!\cdot\, A_2 \prec\!\!\cdot\, \cdots \prec\!\!\cdot\, B$ in ω'.

Thus B and B' would not be combinable in ω', which is a contradiction. Therefore B_1 and B_2 are related by a cover in ω. If B_1 and B_2 are related by a cover then $k = 1$ and $B_2 = A_1$ in the chain of covers since B and B' are combinable in ω'. Thus we have either $B_1 \prec\!\cdot\, B_2 \prec\!\cdot\, B'$ or $B' \prec\!\cdot\, B_2 \prec\!\cdot\, B_1$ in ω. $B' \prec\!\cdot\, B_2 \prec\!\cdot\, B_1$ is not possible because $pl_\omega(B_2) > pl_\omega(B_1)$. $B_1 \prec\!\cdot\, B_2 \prec\!\cdot\, B'$ implies by part 3 of this Proposition that $pl_{\omega'}(B') \geq c$, which is a contradiction. Therefore B_1 and B' are combinable in ω'.

Proof of (5): There are two cases to consider. In the first case suppose that B^* is B_1 (it cannot be B_2 due to its placement). Then $B' = B$. By part 3 of this Proposition, $a = b \leq pl_{\omega'}(B') \leq c-1$. In the second case suppose that B^* is neither B_1 nor B_2. Then $B' = B^*$. If $b < a$, then by part 3 of this Proposition, $pl_{\omega'}(B') = b$. If $a < b < c$, then again by part 3 of this Proposition, $pl_{\omega'}(B') \leq c - 1$. □

Lemma 2.17. *Let B and B' be blocks in $\omega \in (\Omega, \leq_S)$ with $pl_\omega(B) = a$ and $pl_\omega(B') = a + 1$. Then B and B' are combinable in ω.*

Proof. If B and B' are incomparable in ω then they are combinable and we are done. If B and B' are comparable in ω suppose that one block does not cover the other. Then there exists a block C such that $B \prec C \prec B'$ in ω. This implies that $pl_\omega(B) < pl_\omega(C) < pl_\omega(B')$, which is a contradiction. Thus B and B' are related by a cover and are therefore combinable. □

Lemma 2.18. *Given ω and an interval $(\omega, \omega') \in (\Omega, \leq_S)$, there exists a unique pair of blocks B_1 and B_2 whose larger placement is minimal among all pairs in $T(\omega, \omega')$. Furthermore, there exists a unique cover of ω in which B_1 and B_2 are combined.*

Proof. Suppose for the purpose of contradiction there are two pairs of blocks with minimal larger placement in ω. Thus we have placements a, b and c such that $a < b < c$ and the blocks with placements a and c and the blocks with placements b and c are combinable in ω. For convenience we will denote the blocks by their respective placements. If blocks a and b are incomparable in ω or block b covers block a in ω, then they are combinable in ω. Thus the pair (a, b) is in $T(\omega, \omega')$, contradicting the minimality of c. Block a cannot be greater than block b

in ω due to their placements, so the only case left to consider is when block b is greater than block a but does not cover it. In this case there exists a chain of covers from a to b such that all of the blocks in the chain have placements less than b, i.e. $a \prec a_1 \prec a_2 \prec \cdots \prec a_k \prec b$ where $a_j < b$ for all $j \in [k]$. Therefore (a, a_1) is in $T(\omega, \omega')$, contradicting the minimality of c.

We have established that there exists a unique pair of blocks B_1 and B_2 whose larger placement c is minimal among all pairs in $T(\omega, \omega')$. We know by Lemma 2.15 that there exists at least one cover of ω in which B_1 and B_2 are combined. Suppose, again for the purpose of contradiction, there is more than one cover that can be obtained by combining B_1 and B_2 in ω. The only way for this to occur is if a block is incomparable to both B_1 and B_2 in ω and comparable to the combined block in the covers. Thus we have the following situation. B_1 and B_2 are combined to form block B in ω_1 and ω_2 for $\omega \lessdot_S \omega_1$ and $\omega \lessdot_S \omega_2$. Block B' is incomparable to both B_1 and B_2 in ω. In ω_1, $B \prec B'$ and in ω_2, $B \succ B'$ implying that B' and B are in the same block in ω'. This means that B' is also in the same block as B_1 and B_2 in ω'. By Proposition 2.16 part 2 we know that $pl_\omega(B') = b$ such that $a < b < c$. Since B' and B_1 are incomparable in ω they are a pair in $T(\omega, \omega')$, which contradicts the minimality of c. Therefore, there exists a unique cover of ω in which B_1 and B_2 are combined. $\square$

We now have the necessary tools to prove Theorem 2.13.

Proof. We need to establish that σ satisfies conditions (EL1) and (EL2). Let (ω, ω') be an interval in $(\Omega, \leq_S)$. Denote by ζ the maximal chain $\omega = \omega_0 \lessdot_S \omega_1 \lessdot_S \cdots \lessdot_S \omega_k = \omega'$ where each ω_{i+1} is obtained by combining a pair of blocks from $T(\omega_i, \omega')$ such that the larger of the two placements is minimal among all pairs. By Lemma 2.18, this maximal chain is unique and thus does not share its label sequence with any other chain.

Now we will show that the edge label sequence for ζ is weakly increasing. Suppose, for the purpose of contradiction, there is a descent in the edge label sequence, i.e. $\sigma(\omega_j, \omega_{j+1}) = c$ and $\sigma(\omega_{j+1}, \omega_{j+2}) = b$ for $c > b$. Let B and B' with $pl_{\omega_{j+1}}(B) = a < b = pl_{\omega_{j+1}}(B')$ be the blocks in ω_{j+1} that were combined to form ω_{j+2}. We claim that the pair of blocks combined in ω_j did not have minimal larger placement.

Proof of claim: There are three cases to consider. In the first case we suppose that B and B' are the same blocks in ω_j and ω_{j+1}. Since blocks B and B' are not the blocks that were combined in ω_j to form ω_{j+1} and they both have placements less than c in ω_{j+1}, Proposition 2.16 part 3 tells us that $pl_{\omega_j}(B) < c$ and $pl_{\omega_j}(B') < c$. The only way in which B and B' are not combinable in ω_j but are combinable in ω_{j+1} is if $B \prec\!\!\cdot\, B^* \prec\!\!\cdot\, B'$ in ω_j and either B and B^* or B^* and B' are combined to form ω_{j+1}. Since we are assuming that B and B' are the same blocks in ω_j and ω_{j+1}, this cannot occur. This implies that B and B' are combinable in ω_j. Thus B and B' are a combinable pair of blocks in ω_j with larger placement less than c. In the second case, suppose that B is the new block in ω_{j+1} formed by combining blocks B_1 and B_2 in ω_j where $pl_{\omega_j}(B_1) = c_1 < c = pl_{\omega_j}(B_2)$. Then by Proposition 2.16 part 4, B' and B_1 are a combinable pair in ω_j with larger placement less than c. In the third case, suppose that B' is the new block in ω_{j+1} formed by combining blocks B'_1 and B'_2 where $pl_{\omega_j}(B'_1) = c_2 < c = pl_{\omega_j}(B'_2)$. Then again by Proposition 2.16 part 4, B and B'_1 are a combinable pair in ω_j with larger placement less than c. In each case, we obtain a pair of blocks in ω_j such that the larger placement of the pair is less than c. Therefore, the claim is true and we have reached a contradiction. Thus, there cannot be a descent in the edge label sequence for ζ, meaning ζ is weakly increasing.

All that remains to be shown for (EL1) is that ζ is the only weakly increasing maximal chain. As established above, ζ does not share its label sequence with any other chain. Suppose we follow some other maximal chain χ from ω to ω'. At some step these chains must differ, meaning that in ζ we combine two blocks from ω_j to form ω_{j+1} such that the larger placement of the pair of blocks in ω_j is minimal among pairs in $T(\omega_j, \omega')$, whereas in χ we combine two blocks from ω_j to form ω^*_{j+1} such that the larger placement of the pair of blocks in ω_j is not minimal among pairs in $T(\omega_j, \omega')$. Thus we have the following situation. Let B_1 and B_2 be blocks of ω_j with $pl_{\omega_j}(B_1) = a < c = pl_{\omega_j}(B_2)$ that were combined to form ω_{j+1}. Let B^*_1 and B^*_2 be blocks of ω_j with $pl_{\omega_j}(B^*_1) = b < d = pl_{\omega_j}(B^*_2)$, for $d > c$, that were combined to form ω^*_{j+1}. Since B_1 and B_2 are a pair in $T(\omega_j, \omega')$, they must be in the same block in ω'. Thus, at some later step in χ, a block containing B_1 must be combined with a block containing B_2. By Proposition 2.16 part 5, if we continue to pick pairs of blocks that cause the label sequence of χ to increase, the placements of the blocks containing B_1 and B_2 in ω_k^* for $k > j$ will always be smaller than the edge label leading to ω_k^* assigned by σ (because in ω_{j+1}^* B_1 and B_2 have edge labels less than d). Eventually we are forced to combine the blocks containing B_1 and B_2 (or a pair with smaller larger placement) creating a descent in the edge label sequence for χ. Thus, all other maximal chains have at least one descent. Therefore, ζ satisfies (EL1).

Since we chose the smallest possible label at each step in ζ, its label sequence is lexicographically smaller than the label sequences of all other maximal chains. Therefore, ζ satisfies (EL2). $\square$

Since $(\Omega, \leq_S)$ is a graded poset, Theorem 2.13 implies that it is EL-shellable and hence shellable [3, Theorem 2.3]. Thus the Möbius number of an interval in $(\Omega, \leq_S)$, $\mu(\omega, \omega')$, is equal to $(-1)^{|\omega|-|\omega'|}$ times the number of strictly decreasing maximal chains from ω to ω'. To count the strictly decreasing maximal chains from ω to ω', we need to determine the number of ways in which the combinable block with the highest placement can be combined with blocks with lower placements at each step in the chain. At each step in a maximal chain there may be multiple ways to go up by a cover and obtain the maximum possible label on that edge. For this reason, counting strictly decreasing maximal chains is not completely straightforward, and at this time we do not know how to count strictly decreasing maximal chains for general intervals. Interestingly, the Möbius number of the entire lattice has a simple description [16]. It is the number of indecomposable permutations, or equivalently, the number of permutations with no global descents. See Sequence A003319 of [17] for details and references.

2.5 Noncrossing Pre-orders

In this final section of Chapter 2, we consider a subset of S_n, the c-sortable permutations, and describe the corresponding subset of the permutation pre-orders.

A Coxeter group W is generated by a set S of *simple generators.* For $W = S_n$ the simple generators are $S = \{s_1, s_2, \ldots, s_{n-1}\}$ where $s_i = (i, i+1)$ for $1 \leq i \leq n-1$. A *Coxeter element* c is the product of the simple generators in any order. For a chosen Coxeter element $c \in S_n$, we can define a *barring* on the numbers $\{2, 3, \ldots, n-1\}$ as follows:

1. if s_{i-1} is before s_i in c then i is lower-barred, and we denote this by $\underline{i}$
2. if s_{i-1} is after s_i in c then i is upper-barred, and we denote this by $\overline{i}$.

A Coxeter element in S_n can be written as an n-cycle using the barring by placing 1 at the 12 o'clock position on a circle followed clockwise by the lower-barred numbers in ascending numerical order, the number n, and then the upper-barred numbers in descending numerical order.

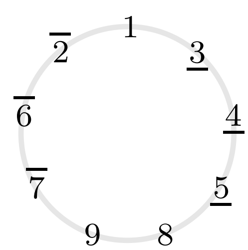

Figure 2.9: Cycle corresponding to $c = s_2s_1s_3s_7s_6s_4s_5s_8$

Example 2.19. For $c = s_2s_1s_3s_7s_6s_4s_5s_8 \in S_9$ the lower-barred numbers are $\underline{3}, \underline{4}, \underline{5}, \underline{8}$ and the upper-barred numbers are $\overline{2}, \overline{6}, \overline{7}$. This gives us the cycle shown in Figure 2.9.

For a permutation $\pi \in S_n$ and $x \in S_m$ we will say that π *contains the pattern* x if there are integers $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that for all $1 \leq j < k \leq m$ we have $x_j < x_k$ if and only if $\pi_{i_j} < \pi_{i_k}$. Otherwise we will say that π *avoids the pattern* x. In this section we will be concerned with patterns involving barred numbers, specifically the patterns $\overline{2}31$ and $31\underline{2}$. For $\pi \in S_n$ with chosen Coxeter element c, we say that π *contains the pattern* $\overline{2}31$ if it contains an instance of the pattern 231 in which the number representing the 2 in π is upper-barred. Similarly, we say π *contains the pattern* $31\underline{2}$ if it contains an instance of the pattern 312 in which the number representing the 2 in π is lower-barred. If these conditions do not hold, we again say that π *avoids the given pattern.*

Example 2.20. As a continuation of Example 2.19, suppose $c = s_2s_1s_3s_7s_6s_4s_5s_8 \in S_9$. Then $\pi = 163425897$ contains four instances of the pattern $31\underline{2}$: $63\underline{4}$, $63\underline{5}$, $64\underline{5}$, and $62\underline{5}$, and avoids the pattern $\overline{2}31$.

There is a general definition of c-sortable elements of a Coxeter group which can be found in [15]. For this chapter, the following characterization [15, Lemmas 4.1 and 4.8] of c-sortable elements in S_n is sufficient. Given a Coxeter element c, a permutation $\pi \in S_n$ is *c-sortable* if and only if π avoids the patterns $\overline{2}31$ and $31\underline{2}$.

A *c-noncrossing partition* is a partition of $[n]$ into blocks such that when the blocks are drawn on the cycle c, the convex hulls of the blocks do not overlap.

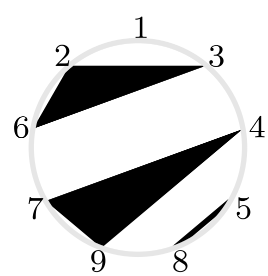

Figure 2.10: c-noncrossing partition

Example 2.21. Continuing Example 2.19, a c-noncrossing partition of the cycle c is shown in Figure 2.10.

Define a *noncrossing pre-order* to be a pre-order $\omega \in \Omega$ on $[n]$ with respect to a Coxeter element c with the following two conditions:

1. blocks in ω form a c-noncrossing partition of the cycle c, and

2. suppose blocks $B_{[i_s,i_1]}$ and $B_{[j_t,j_1]}$ intersect in ω, so that, without loss of generality, there exists a $j_x \in B_{[j_t,j_1]}$ for $x \in [t]$ such that $i_s < j_x < i_1$. If j_x is upper-barred with respect to c then $B_{[i_s,i_1]} \prec B_{[j_t,j_1]}$ in ω and if j_x is lower-barred with respect to c then $B_{[i_s,i_1]} \succ B_{[j_t,j_1]}$ in ω.

Let $\Omega_c^{NC} = \{\omega \in \Omega : \omega \text{ is a noncrossing pre-order}\}$, and let Ω^c be the image of the c-sortable permutations under μ.

Proposition 2.22. $\Omega^c = \Omega_c^{NC}$

Proof. For any c-noncrossing partition there is at most one partial order on the blocks which defines a noncrossing pre-order. In [15, Theorem 6.1] it was shown that there exists a bijection between c-noncrossing partitions and c-sortable permutations. Thus $|\Omega^c| \geq |\Omega_c^{NC}|$, and it is enough to show that $\Omega^c \subseteq \Omega_c^{NC}$. Let $\omega \in \Omega^c$ and let $\lambda(\omega) = \pi$. Suppose $i_1 \cdots i_s$ and $j_1 \cdots j_t$ are distinct descending runs in π corresponding respectively to blocks $B_{[i_s,i_1]}$ and $B_{[j_t,j_1]}$ in ω such that $B_{[i_s,i_1]}$ and $B_{[j_t,j_1]}$ overlap. Then without loss of generality there exists a $j_x \in B_{[j_t,j_1]}$ such that $i_s < j_x < i_1$ for $x \in [s]$. Suppose j_x is upper-barred with respect to c. Then if $B_{[i_s,i_1]} \succ B_{[j_t,j_1]}$ in ω, $\pi = \cdots j_1 \cdots j_t \cdots i_1 \cdots i_s \cdots$ and $j_x i_1 i_s$ forms a $\overline{2}31$ pattern, which is a contradiction of π being c-sortable. Thus $B_{[i_s,i_1]} \prec B_{[j_t,j_1]}$ in ω. Suppose j_x is lower-barred with respect to c. Then if $B_{[i_s,i_1]} \prec B_{[j_t,j_1]}$, $\pi = \cdots i_1 \cdots i_s \cdots j_1 \cdots j_t \cdots$ and $i_1 i_s j_x$ forms a $31\underline{2}$ pattern which is a contradiction of π being c-sortable. Thus $B_{[i_s,i_1]} \succ B_{[j_t,j_1]}$ in ω. $\square$

The choice $c = s_1 s_2 \cdots s_{n-1}$ is the case where every number is lower-barred and the choice $c = s_{n-1} s_{n-2} \cdots s_1$ is the case where every number is upper-barred. In these specific cases, if two distinct blocks $B_{[i,j]}$ and $B_{[k,l]}$ overlap in ω, then either $[i,j] \subset [k,l]$ or $[k,l] \subset [i,j]$. Thus we have the following corollaries.

Corollary 2.23. *Let $c = s_1 s_2 \cdots s_{n-1}$ and let $\omega \in \Omega_c^{NC}$. Then $B_{[i,j]} \preceq B_{[k,l]}$ in ω if and only if $[i,j] \subset [k,l]$.*

Corollary 2.24. *Let $c = s_{n-1}s_{n-2}\cdots s_1$ and let $\omega \in \Omega_c^{NC}$. Then $B_{[i,j]} \preceq B_{[k,l]}$ in ω if and only if $[k,l] \subset [i,j]$.*

Chapter 3

Tamari Congruence Classes

3.1 Introduction

Cambrian lattices were introduced in [14] as quotients of the weak order on a Coxeter group W with respect to certain lattice congruences called Cambrian congruences. Combinatorial realizations of the Cambrian lattices in type A in terms of triangulations and permutations were also given. In particular, it was shown that the Tamari lattice is one of the Cambrian lattices of type A. We will use the term Tamari congruence to name the Cambrian congruences whose associated Cambrian lattice is the Tamari lattice. In this chapter, our purpose is to better understand the classes of the Tamari congruence. Section 3.2 begins with necessary background on the weak order on S_n. We then define the Tamari congruence classes as the fibers of a map from permutations to triangulations and realize each class combinatorially as a partial order on $\{1, \ldots, n\}$, which we call a C-poset. In Section 3.3 we give several characterizations of which posets occur as C-posets. Section 3.4 gives multiple tools for navigating within Tamari congruence classes and between a Tamari congruence class and its C-poset. We conclude the chapter in Section 3.5 by describing the enumeration of the elements in a Tamari congruence class, given any element in the class, and by using a new quantity called the permutation hook length to define some permutation statistics.

3.2 Tamari Congruence Classes

We will begin this section by discussing the weak order on the symmetric group. We then define Tamari congruence classes and discuss how to realize them combinatorially as partial orders on $\{1, 2, \ldots, n\}$. We finish with two propositions that present useful properties of any Tamari congruence class.

The *inversion set* of a permutation x, denoted by $I(x)$, is the set of all inversions (see Chapter 2, Section 2.2) of x. The *(right) weak order* on S_n is a partial order which sets the permutations $x \leq y$ if and only if $I(x) \subseteq I(y)$. For the remainder of this chapter, "the weak order" will be taken to mean "the weak order on S_n." To go up by a cover in the weak order, a pair of adjacent entries in x are transposed so that they are out of numerical order. The weak order is a lattice with $x = 12\cdots n$ as the minimal element and $x = n(n-1)\cdots 1$ as the maximal element. If we take $\mathcal{A}(S_n)$ to be the Coxeter arrangement as described in Chapter 2, then the poset of regions is the weak order on S_n.

General Cambrian congruences are certain families of lattice congruences on the weak order on a finite Coxeter group W. The quotient of the weak order modulo a Cambrian congruence is a Cambrian lattice. In every case where W is the symmetric group, the *Cambrian congruence classes* can be defined as the fibers of a map from permutations to triangulations. We will be considering the simplest case of this map, the fibers of which give the *Tamari congruence*. In the language of [14], we are considering the case where W is the symmetric group and every edge in the Coxeter diagram is oriented $s_{i+1} \to s_i$, or equivalently, every symbol is upper-barred (see Chapter 2, Section 2.5). In this case, the Cambrian lattice is the Tamari lattice.

Let Q be a convex polygon with vertices $0, 1, \ldots, n+1$, having 0 and $n+1$ on a horizontal line and all other numbered vertices positioned above them in numerical order. The map η from permutations to triangulations works as follows. Start with the path along the bottom of Q. Moving from left to right in the permutation π, at each step add the vertex π_i to the path as illustrated in Figure 3.1. The union of all of the paths defines a triangulation.

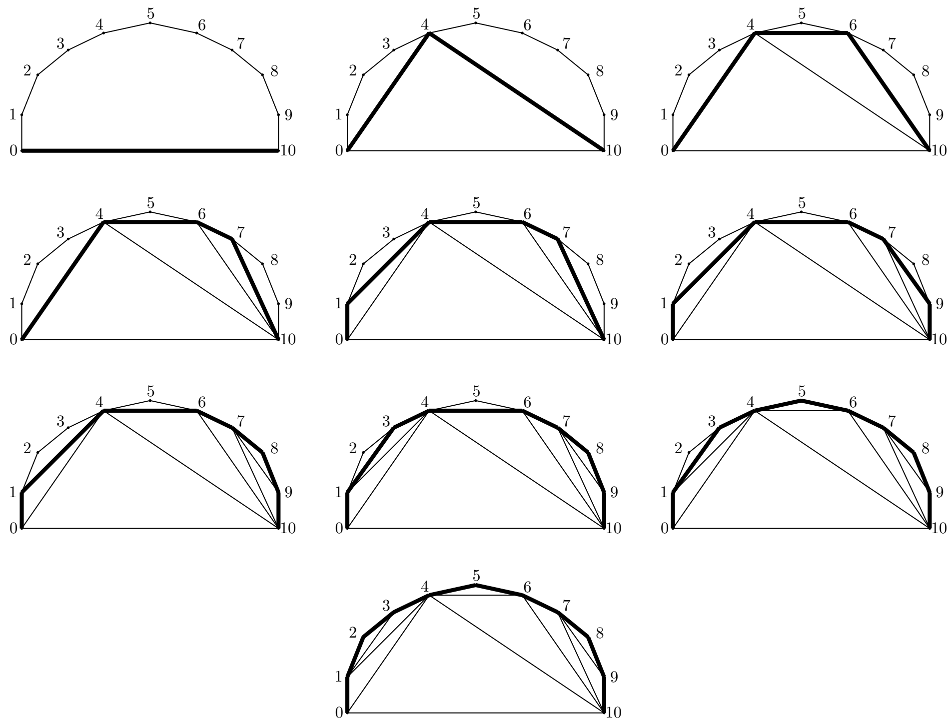

Figure 3.1: $\eta(467198352)$

Example 3.1. Figure 3.1 shows the step-by-step construction (reading from left to right and top to bottom) of $\eta(467198352)$. The permutations $467918352, 461798352$, and 416798352 (among others) also map to this triangulation.

Each time a vertex is added to the path, a single triangle is added to the triangulation. Thus we can label a triangle by the vertex added to create it. Since no edges added to the triangulation in the map η are vertical, if two triangles are adjacent (i.e. they share an edge), then one can be considered "above" the other in the usual sense. This leads to a partial order on $\{1, \ldots, n\}$ called the *above/below poset*. If triangles i and j are adjacent and i is above j in the triangulation, then $j \lessdot i$ in the above/below poset. If triangles i and j are adjacent and i is below j in the triangulation, then $i \lessdot j$ in the above/below poset. The above/below poset is the transitive closure of these cover relations. The map taking triangulations $\eta(x)$ to their above/below posets is a bijection.

Let τ be the map taking x to the above/below poset of $\eta(x)$. Thus the set of fibers of $\tau(x)$ is equal to the set of fibers of $\eta(x)$. Up to an antisymmetry of the weak order (reversing the order of entries in x) and a symmetry of the plane, this is the same map studied by A. Björner and M. Wachs in [7]. The map τ is defined recursively by $\tau(\text{empty permutation}) = \text{empty tree}$, and for $n > 0$ and $x \in S_n$, $\tau(x)$ is the binary tree whose root is x_1, left subtree is $\tau(x^-)$ and right subtree is $\tau(x^+)$, where x^- and x^+ are the subwords of x consisting of all letters less than x_1 and all letters greater than x_1, respectively. Here, a *binary tree* is taken to be a tree rooted at the bottom and growing upwards in which each node has either no children, a left child, a right child, or both a left and right child. A *subtree* of a tree T is defined to be the tree formed by a node of T with all of its descendants.

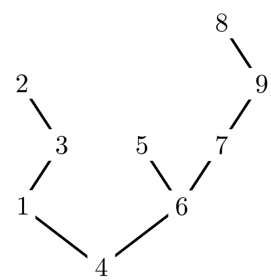

Figure 3.2: $\tau(467198352)$

Example 3.2. Continuing Example 3.1, Figure 3.2 shows $\tau(467198352)$, the above/below poset for the triangulation given in Figure 3.1.

The following proposition is essentially [7, Proposition 9.10]. For the generalization to other Cambrian congruences, see [14]. (See Chapter 2, Section 2.5 for the definition of pattern avoidance.)

Proposition 3.3. *For any triangulation T, the fiber $\eta^{-1}(T)$ is a non-empty interval $[x, x']$ in the weak order. An element x is the bottom element of its class if and only if it avoids the pattern* 231 *and x' is the top element of its class if and only if it avoids the pattern* 213.

Let $\pi^{\uparrow}x$ denote the top element of the Tamari class containing the permutation x, and likewise $\pi_{\downarrow}x$ the bottom element of the Tamari class containing x. Define the C-poset for a Tamari class containing an element x as $\tau(x)$, the above/below poset for $\eta(x)$.

Inside the proof of [6, Proposition 5.1], Björner and Wachs show that $\tau^{-1}(T) = \mathscr{L}(T)$ where $\mathscr{L}(T)$ is the set of linear extensions of the tree T produced by τ. Thus the elements of the Tamari class of x (the fiber $\eta^{-1}(\eta(x))$) are the linear extensions of the C-poset. In the geometric setting of Chapter 2, the linear extensions of the C-poset are the permutations corresponding to regions contained in the cone defined by the C-poset. The facets of this cone are defined by the ascents of $\pi^{\uparrow}x$ and the descents of $\pi_{\downarrow}x$. This implies that the covers in the C-poset are $i \lessdot j$ such that ij is an ascent in $\pi^{\uparrow}x$ and $j \lessdot i$ such that ji is a descent in $\pi_{\downarrow}x$.

We conclude this section with a proposition that describes how to move within a Tamari class, which will be used in Section 3.4. Suppose the entries x_i, x_j and x_{j+1} form a 213 pattern in the permutation x. Switching the entries x_j and x_{j+1} is called a $213 \to 231$-*move* and gives the permutation y, which covers x in the weak order. The following is a special case of [14, Proposition 5.3].

Proposition 3.4. *The permutation x is covered by y in the weak order and $\eta(x) = \eta(y)$ if and only if y is obtained from x by a $213 \to 231$-move.*

3.3 C-poset Characterizations

In this section we will give several equivalent characterizations of the C-posets. In order to do this, we will need the idea of *poset pattern avoidance.* We say a poset P *avoids the pattern* 213 or P *avoids* 213 if there does not exist $1 \leq a < b < c \leq n$ with $b \prec a \prec c$ in P. Similarly, we say P *avoids* 231 if there does not exist $1 \leq a < b < c \leq n$ with $b \prec c \prec a$ in P. We say a poset P *avoids the vee pattern* 213 or P *avoids* $\vee 213$ if there does not exist $1 \leq a < b < c \leq n$ with $a \precdot b$ and $a \precdot c$ in P. We say a poset P *avoids the wedge pattern* 213 or P *avoids* $\wedge 213$ if there does not exist $1 \leq a < b < c \leq n$ with $b \precdot a$ and $c \precdot a$ in P. One can similarly define the notion of posets avoiding $\vee 231, \wedge 231$, and $\wedge 123$.

Theorem 3.5. *Given a poset P on $[n]$ the following are equivalent*

(i) *P is a C-poset.*

(ii) *P is a rooted planar binary tree where at each node, the left subtree contains only numerically smaller descendants and the right subtree contains only numerically larger descendants.*

(iii) *P is a rooted tree that avoids $213, 231, \vee 213$, and $\vee 231$.*

(iv) *P is a connected poset avoiding $213, 231, \vee 213, \vee 231, \wedge 123, \wedge 231$, and $\wedge 213$.*

Proof. Björner and Wachs proved in [7, Proposition 9.10] that the map τ is surjective onto the trees described in (ii). Thus, (i) and (ii) are equivalent. Now we will show that (iii) and (ii) are equivalent.

Let P be a poset as described in (iii) and let x be a node in P. Since P avoids 213 and 231, a subtree at x cannot contain both smaller and larger descendants. Suppose P has a node x with two subtrees T_a and T_b that each contain only descendants that are smaller than x. Let a and b be the elements covering x in subtress T_a and T_b respectively. Then axb or bxa forms a $\vee 231$ pattern. Thus each node in P can have at most one subtree with smaller descendants. Similarly, since P avoids $\vee 213$, each node in P can have at most one subtree with larger descendants. We make the convention to draw P such that the left subtree at each node only contains smaller descendants and the right subtree at each node only contains larger descendants. Thus P is a rooted planar binary tree where at each node, the left subtree contains only smaller descendants and the right subtree contains only larger descendants. Therefore, all posets in the set described in (iii) are contained in the set of posets described in (ii).

Let P^* be a poset as described in (ii). Since the left subtree at each node of P^* contains only smaller descendants and the right subtree at each node of P^* contain only larger descendants, P must avoid $213, 231, \vee 213$, and $\vee 231$. Thus, all posets in the set described in (ii) are contained in the set of posets described in (iii).

The conditions that P avoids $\wedge 123, \wedge 231$, and $\wedge 213$ and that P is connected are equivalent to the condition that P is rooted, thus (iii) and (iv) are equivalent. $\square$

3.4 Algorithms

This section gives several algorithms which are used to navigate within a Tamari class. We will begin with algorithms that can be used on any element of the class. Then, using the pattern avoidance characteristics of the bottom and top elements of the class, we will give more streamlined algorithms that can be used on $\pi^{\uparrow}x$ or $\pi_{\downarrow}x$.

Define a *left-noncrossing descending run* or *ln-descending run* of a permutation x as a maximal descending sequence $x_j x_{j+1} \cdots x_s$ of adjacent entries such that there does not exist an x_i with $x_j > x_i > x_s$ for $i < j$. In this context, maximal implies that either $j = 1$ or $x_j > x_{j-1}$ or there exists an x_i such that $x_{j-1} > x_i > x_j$ for $i < j-1$ and either $s = n$ or $x_s < x_{s+1}$ or there exists an x_i such that $x_s > x_i > x_{s+1}$ for $i < j$. Define a *left-noncrossing ascending run* or *ln-ascending run* of x as a maximal ascending sequence $x_k x_{k+1} \cdots x_t$ such that there does not exist an x_f with $x_k < x_f < x_t$ for $f < k$. In this context, maximal implies that either $k = 1$ or $x_k < x_{k-1}$ or there exists and x_f such that $x_{k-1} < x_f < x_k$ for $f < k-1$ and either $t = n$ or $x_t > x_{t+1}$ or there exists an x_f such that $x_t < x_f < x_{t+1}$ for $f < k$. This first algorithm takes any element x of a Tamari class to $\pi^{\uparrow}x$.

Algorithm 3.6. (x to $\pi^{\uparrow}x$)

- Step 1: Label the ln-descending runs D_1 through D_r, with D_1 the first on the left and D_r the last on the right.

- Step 2: Starting with D_2, label the leftmost element of D_2 as x_k. Let x_i be the largest element to the left of x_k that is still smaller than x_k. As a group, move all of the elements of D_2 leftward until either x_i is reached or an element larger than x_k is reached. Call the new permutation $x^{(2)}$. (If x_i does not exist, i.e. $x_p > x_k$ for all $p < k$, then proceed to Step 3.)

- Step 3: Repeat step 2 in $x^{(2)}$ with D_3.
- Step 4: Continue the process with right successive ln-descending runs (i.e. repeat step 2 in $x^{(i)}$ with D_{i+1} for successive i's) until you create $x^{(r)}$. The permutation $x^{(r)}$ is the top element of the class, $x^{(r)} = \pi^{\uparrow}x$.

Example 3.7. Let $x = 418376529$. Then Algorithm 3.6 proceeds as follows:

$$x = x^{(1)} = \underset{D_1}{\mathbf{41}}\ \ \underset{D_2}{\underline{8}}\ \underset{D_3}{\underline{3}}\ \underset{D_4}{\underline{765}}\ \underset{D_5}{\underline{2}}\ \underset{D_6}{\underline{9}}$$

$$x^{(2)} = 4\,\mathbf{8}\,1\ \ \underline{3}\,\underline{765}\,\underline{2}\,\underline{9}$$

$$x^{(3)} = 481\,\mathbf{3}\ \ \underline{765}\,\underline{2}\,\underline{9}$$

$$x^{(4)} = 48\,\mathbf{765}\,13\ \ \underline{2}\,\underline{9}$$

$$x^{(5)} = 4876513\,\mathbf{2}\ \ \underline{9}$$

$$x^{(6)} = 48\,\mathbf{9}\,765132$$

In each step above, the ln-descending run that was moved is in bold and the ln-descending runs remaining to be moved are left underlined.

This next algorithm takes any element x of a Tamari class to $\pi_{\downarrow}x$.

Algorithm 3.8. (x to $\pi_{\downarrow}x$)

- Step 1: Label the ln-ascending runs A_1 through A_s, with A_1 the first on the left and A_s the last on the right.
- Step 2: Starting with A_2, label the leftmost element of A_2 as x_k. Let x_i be the smallest element to the left of x_k that is still larger than x_k. As a group, move all of the elements

of A_2 leftward until either x_i is reached or an element smaller than x_k is reached. Call the new permutation $x^{(2)}$. (If x_i does not exist, i.e. $x_p < x_k$ for all $p < k$, then proceed to Step 3.)

- Step 3: Repeat step 2 in $x^{(2)}$ with A_3.
- Step 4: Continue the process with right successive ln-runs (i.e. repeat step 2 in $x^{(i)}$ with A_{i+1} for successive i's) until you create $x^{(s)}$. The permutation $x^{(s)}$ is the bottom element of the class, $x^{(s)} = \pi_{\downarrow} x$.

Example 3.9. Let $x = 452679318$. Then Algorithm 3.8 proceeds as follows:

$$x = x^{(1)} = \underset{A_1}{\mathbf{45}}\ \underset{A_2}{\underline{2}}\ \underset{A_3}{\underline{679}}\ \underset{A_4}{\underline{3}}\ \underset{A_5}{\underline{1}}\ \underset{A_6}{\underline{8}}$$

$$x^{(2)} = 4\,\mathbf{2}\,5\ \ \underline{679}\,\underline{3}\,\underline{1}\,\underline{8}$$

$$x^{(3)} = 425\,\mathbf{679}\ \ \underline{3}\,\underline{1}\,\underline{8}$$

$$x^{(4)} = 42\,\mathbf{3}\,5679\ \ \underline{1}\,\underline{8}$$

$$x^{(5)} = 42\,\mathbf{1}\,35679\ \ \underline{8}$$

$$x^{(6)} = 42135679\,\mathbf{8}$$

Now we will prove that Algorithms 3.6 and 3.8 produce the desired output. The following Lemma will be used in the proof.

Lemma 3.10. *If a permutation $x \in S_n$ contains a 213 pattern, then it contains an instance of the pattern where the "1" and the "3" are adjacent.*

Proof. Suppose $x \in S_n$ contains a 213 pattern. Specifically let $1 \leq a < b < c \leq n$ be entries in x such that bac forms the pattern 213. Also, choose (a, b, c) among such triples so as to minimize the number of entries occurring between c and a. Assume that a and c are not adjacent, thus there exists at least an element t between a and c. If $t > b$ then bat forms a 213 pattern. If $t < b$, then btc forms a 213. Thus the minimality of the number of entries between c and a is contradicted in either case, so a and c must be adjacent in x. □

Proposition 3.11. *Algorithm 3.6 constructs the top element $\pi^{\uparrow}x$ of the Tamari class of x. Algorithm 3.8 constructs the bottom element $\pi_{\downarrow}x$ of the Tamari class of x.*

Proof. To prove that Algorithm 3.6 constructs the top element of the Tamari class of x, by Propositions 3.3 and 3.4 it is sufficient to show that $x^{(r)}$ is related to x by a series of $213 \to 231$-moves and that it avoids the pattern 213. In step i of Algorithm 3.6, ln-descending run D_i will be moved in the permutation $x^{(i-1)}$ to create the permutation $x^{(i)}$. Let $D_i = d_1 d_2 \cdots d_t$, let a be the largest entry to the left of d_1 in x that is still smaller than d_1, let h be the entry that stops the leftward movement of the elements of D_i, let $b_1 b_2 \cdots b_r$ denote the sequence of entries between a and h, and let $c_1 c_2 \cdots c_s$ denote the sequence of entries between h and d_1. Note that a may be equal to h, so $x^{(i-1)}$ either looks like

$$x^{(i-1)} = \cdots a\, b_1 b_2 \cdots b_r\, h\, c_1 c_2 \cdots c_s\, d_1 \cdots d_t \cdots \tag{3.1}$$

or looks like

$$x^{(i-1)} = \cdots a\, c_1 c_2 \cdots c_s\, d_1 \cdots d_t \cdots \tag{3.2}$$

Claim 1: We first claim that $c_j < d_k$ for all $j \in [s]$ and $k \in [t]$.

Proof of Claim 1: Since $d_1 \cdots d_t$ moves past $c_1 \cdots c_s$ to create $x^{(i)}$, we know that $c_j < d_1$ for all $j \in [s]$. Suppose $c_x > d_y$ for some $x \in [s]$ and $y \in \{2, \ldots, t\}$. Then $d_1 > c_x > d_t$. In the steps of Algorithm 3.6 leading up to the formation of $x^{(i-1)}$, the entries to the left of d_1 in x are rearranged and the entries to the right of and including d_1 in x are not moved. Thus, c_x is to the left of d_1 in x, which implies that D_i is not an ln-descending run. This contradiction proves the claim.

By Claim 1, the leftward movement of each ln-descending run in Algorithm 3.6 can be completed in a series of $213 \to 231$-moves, by moving each entry of a ln-descending run separately, one move at a time. Specifically, when D_i is moved, the $213 \to 231$-moves are $ac_jd_k \to ad_kc_j$ for $j \in [s]$ and $k \in [t]$.

Claim 2: Let $x^{(i)}|_{D_i}$ be the permutation created at step i of Algorithm 3.6, truncated to include only ln-descending runs D_1 through D_i (e.g. $x^{(1)}|_{D_1} = D_1$ and $x^{(r)}|_{D_r} = x^{(r)}$). We claim that all ascents in $x^{(i)}|_{D_i}$ must be of the form ef, where f is the first entry of its ln-descending run in x and e is the largest entry left of f in x which is smaller than f.

Proof of Claim 2: The claim is vacuously true for $x^{(1)}|_{D_1}$ since it has no ascents. We will proceed by induction on i. Consider the permutation $x^{(i-1)}$ given in (3.1) and (3.2). The truncated permutation $x^{(i)}|_{D_i}$ is formed when Algorithm 3.6 moves D_i to the right of entry a or h of $x^{(i-1)}|_{D_{i-1}}$. By the induction hypothesis, all ascents in $x^{(i)}|_{D_i}$ that are also in $x^{(i-1)}|_{D_{i-1}}$ satisfy the claim. By Claim 1, d_tc_1 cannot be an ascent. Thus, the only way to form a new ascent in $x^{(i)}|_{D_i}$ is if $a = h$ as in 3.2, and the ascent will be ad_1, which satisfies the claim. Therefore, $x^{(i)}|_{D_i}$ satisfies the claim and by induction on i, Claim 2 is true.

Suppose, for the purpose of contradiction, that $x^{(r)}$ contains a 213 pattern. Specifically, let $1 \le a < b < c \le n$ be entries in $x^{(r)}$ such that bac forms the pattern 213. Also, choose (a, b, c) among such triples so that a and c are adjacent (which is possible due to Lemma 3.10). By Claim 1, b is to the left of c in x. Thus, the entries ac in $x^{(r)}$ form an ascent not of the form given in Claim 2, since a is not the largest number to the left of c in x that is still smaller than c. This contradiction shows that $x^{(r)}$ must not have a 213 pattern and $x^{(r)} = \pi^{\uparrow}x$.

Thus Algorithm 3.6 is verified. To prove that Algorithm 3.8 constructs the bottom element of the Tamari class, we would need to show that $x^{(s)}$ is related to x by a series of $231 \to 213$-moves and that it avoids the pattern 231. The arguments follow similarly to the previous case. □

In the cases where we know either the bottom or top element of a Tamari class and we want to find either $\pi^{\uparrow}x$ or $\pi_{\downarrow}x$ for that class, we can use the pattern avoidance conditions given in

Proposition 3.3 to simplify the algorithms. Since the bottom element of a Tamari class is 231-avoiding, all ln-descending runs are descending runs (and vice versa) and there are no elements x_j to stop the leftward progression of the descending runs. Thus Algorithm 3.6 simplifies as follows.

Algorithm 3.12. ($x = \pi_{\downarrow}x$ to $\pi^{\uparrow}x$)

- Step 1: Label the descending runs D_1 through D_r, with D_1 the first on the left and D_r the last on the right.
- Step 2: Starting with D_2, label the leftmost element of D_2 as x_i. As a group, move all of the elements of D_2 leftward until you reach the largest element to the left of x_i that is still smaller than x_i (i.e. move all of D_2 directly to the right of the element x_j where $x_i > x_j$ such that $j < i$ and there is no element x_k with $k < i$ such that $x_j < x_k < x_i$). Call the new permutation $x^{(2)}$.
- Step 3: Repeat step 2 in $x^{(2)}$ with D_3.
- Step 4: Continue the process with right successive descending runs (i.e. repeat step 2 in $x^{(i)}$ with D_{i+1} for successive i's) until you create $x^{(r)}$. The permutation $x^{(r)}$ is the top element of the congruence class, $x^{(r)} = \pi^{\uparrow}x$.

Example 3.13. Let $x = 219534768$. Then Algorithm 3.12 proceeds as follows:

$$x = x^{(1)} = \underset{D_1}{\mathbf{21}}\ \underset{D_2}{\underline{953}}\ \underset{D_3}{\underline{4}}\ \underset{D_4}{\underline{76}}\ \underset{D_5}{\underline{8}}$$

$$x^{(2)} = 2\,\mathbf{953}\,1\ \ \underline{4}\,\underline{76}\,\underline{8}$$

$$x^{(3)} = 2953\,\mathbf{4}\,1\ \ \underline{76}\,\underline{8}$$

$$x^{(4)} = 295\,\mathbf{76}\,341\ \ \underline{8}$$

$$x^{(5)} = 2957\,\mathbf{8}\,6341$$

Similarly, Algorithm 3.8 simplifies when $x = \pi^{\uparrow}x$.

Algorithm 3.14. ($x = \pi^{\uparrow}x$ to $\pi_{\downarrow}x$)

- Step 1: Label the ascending runs A_1 through A_s, with A_1 the first on the left and A_s the last on the right.
- Step 2: Starting with A_2, label the leftmost element of A_2 as x_i. As a group, move all of the elements of A_2 leftward until you reach the smallest element to the left of x_i that is still larger than x_i (i.e. move all of A_2 directly to the right of the element x_j where $x_i < x_j$ such that $j < i$ and there is no element x_k with $k < i$ such that $x_j < x_k < x_i$). Call the new permutation $x^{(2)}$.
- Step 3: Repeat step 2 in $x^{(2)}$ with A_3.
- Step 4: Continue the process with right successive ascending runs (i.e. repeat step 2 in $x^{(i)}$ with A_{i+1} for successive i's) until you create $x^{(s)}$. The permutation $x^{(s)}$ is the bottom element of the congruence class, $x^{(s)} = \pi_{\downarrow}x$.

Example 3.15. Let $x = 967824531$. Then Algorithm 3.14 proceeds as follows:

$$x = x^{(1)} = \underset{A_1}{\mathbf{9}}\ \underset{A_2}{\underline{678}}\,\underset{A_3}{\underline{245}}\ \underset{A_4}{\underline{3}}\ \underset{A_5}{\underline{1}}$$

$$x^{(2)} = 9\,\mathbf{678}\ \ \underline{245}\,\underline{3}\,\underline{1}$$

$$x^{(3)} = 96\,\mathbf{245}\,78\ \ \underline{3}\,\underline{1}$$

$$x^{(4)} = 9624\,\mathbf{3}\,578\ \ \underline{1}$$

$$x^{(5)} = 962\,\mathbf{1}\,43578$$

If instead of finding $\pi^{\uparrow}x$ or $\pi_{\downarrow}x$, we want to create the C-poset from a bottom or top element of a Tamari class, we can adjust Algorithms 3.12 and 3.14 to obtain the following.

Algorithm 3.16. ($x = \pi_{\downarrow}x$ to the C-poset)

- Step 1: Underline the descending runs of x.
- Step 2: Starting on the left, write the first descending run as a chain with the first element of the descending run as the bottom element of the chain and the proceeding elements placed as left descendants.
- Step 3: Moving rightward in x, place the next descending run as a chain with the first element of the descending run as the right child of the biggest number from the previous descending runs that is still smaller than it, and the proceeding elements placed as its left descendants.
- Step 4: Repeat step 3 until all descending runs in x have been placed.

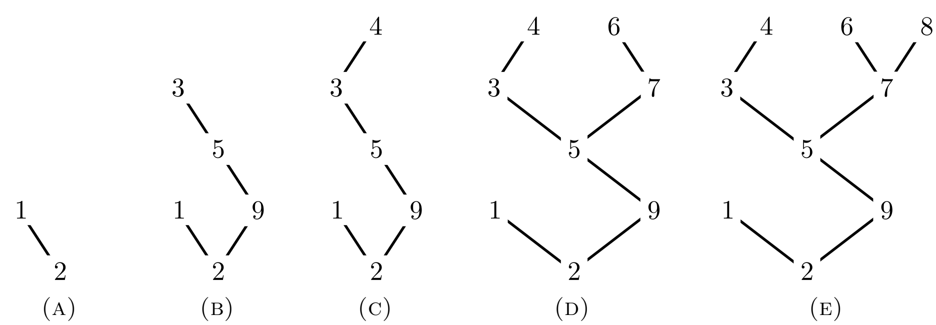

Figure 3.3: Algorithm 3.12 acting on $x = 219534768$

Example 3.17. Consider the permutation $x = 219534768$ as in Example 3.13. Acting on x by Algorithm 3.12 we obtain the series of trees given in Figure 3.3. The final tree, shown in Figure 3.3 (E), is the C-poset for the Tamari class containing x.

Algorithm 3.18. ($x = \pi^{\uparrow}x$ to the C-poset)

- Step 1: Underline the ascending runs of x.

- Step 2: Starting on the left, write the first ascending run as a chain with the first element of the ascending run as the bottom element of the chain and the proceeding elements placed as right descendants.

- Step 3: Moving rightward in x, place the next ascending run as a chain with the first element of the ascending run as the left child of the smallest number in the poset so far, that is still larger than it, and the proceeding elements placed as its right descendants.

- Step 4: Repeat step 3 until all ascending runs in x have been placed.

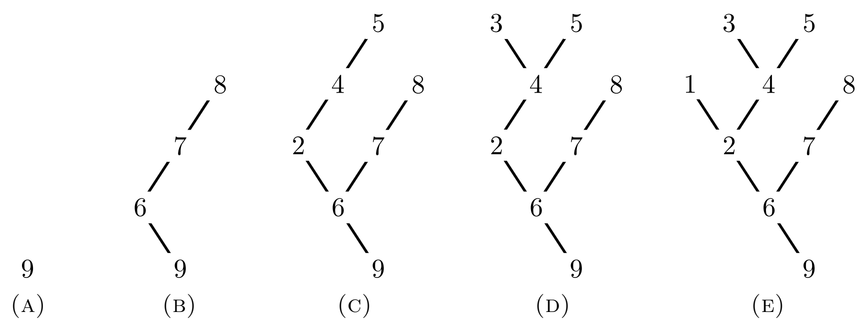

Figure 3.4: Algorithm 3.14 acting on $x = 967824531$

Example 3.19. Figure 3.4 shows how Algorithm 3.14 acts on the permutation $x = 967824531$ from Example 3.15.

3.5 Sizes of Tamari Classes

In this final section, we describe how to enumerate the elements in the Tamari class of a given element x using a new quantity we call the *permutation hook length*. We then employ this new quantity to define some permutation statistics. Recall that an inversion of x is a pair (i, j) such that $i > j$ and i is before j in x. We will continue to denote the set of inversions of x as $I(x)$.

For a given permutation x and for $j \in [n]$, let

$$\alpha_j(x) = \min\left(\{i : (i,j) \in I(x)\} \cup \{n+1\}\right). \tag{3.3}$$

That is, $\alpha_j(x)$ is the smallest entry left of j in the permutation x that is greater than j, or $\alpha_j = n+1$ if no such entry exists. Also, let

$$\beta_j(x) = \max\left(\{i : (i,j) \notin I(x)\} \cup \{0\}\right). \tag{3.4}$$

That is, $\beta_j(x)$ is the largest entry left of j in the permutation x that is less than j, or $\beta_j = 0$ if no such entry exists. Then we define the *permutation hook length of j in x* as

$$h_j(x) = \alpha_j(x) - \beta_j(x) - 1. \tag{3.5}$$

These definitions allow us to determine the number of elements in the Tamari class of x, denoted as $|\mathscr{T}(x)|$.

Theorem 3.20. *The number of elements in the Tamari class of a permutation x is*

$$|\mathscr{T}(x)| = \frac{n!}{\prod_{j\in[n]} h_j(x)} \tag{3.6}$$

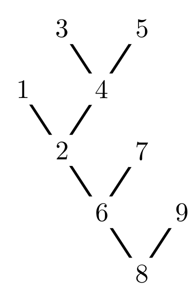

Figure 3.5: The C-poset for the Tamari class containing $x = 869724315$

Example 3.21. Let $x = 869724315$. Then $h_1(x) = 9, h_2(x) = 7, h_5(x) = 5, h_6(x) = 3$ and $h_i(x) = 1$ for all other values of j. Thus $|\mathscr{T}(x)| = \dfrac{9!}{9 \cdot 7 \cdot 1 \cdot 1 \cdot 1 \cdot 5 \cdot 3 \cdot 1 \cdot 1 \cdot 1}$, meaning there are 384 elements in the Tamari class containing x. The C-poset corresponding to this class is given in Figure 3.5.

Proof. As we stated in Section 3.2, the elements of the Tamari class are the linear extensions of the C-poset corresponding to that class. Thus counting the elements of the Tamari class of x is the same as counting the number of linear extensions of the tree $\tau(x)$. The number of linear extensions of a binary tree T labeled by $[n]$ is given by Knuth's hook length formula [11]

$$\mathscr{L}(T) = \frac{n!}{\prod_{j \in [n]} h_j} \tag{3.7}$$

where h_j is the *hook length* at node j (the size of the subtree rooted at j). All that remains to be shown then, is that the permutation hook length of j in x equals the hook length of j in $\tau(x)$.

Define the sets $\mathcal{L}_j(x)$ and $\mathcal{G}_j(x)$ as follows. $\mathcal{L}_j(x)$ is the set of numbers to the right of j which are less than j and do not act as a "1" in a 231 pattern with j acting as the "3." $\mathcal{G}_i(x)$ is the set of numbers to the right of j which are greater than j and do not act as "3" in a 213 pattern with j acting as the "1." Then $\alpha_j - j - 1 = |\mathcal{G}_j|$ and $j - \beta_j - 1 = |\mathcal{L}_j|$. Using these relations, we can rewrite equation (3.5) as

$$h_j(x) = |\mathcal{G}_j(x)| + |\mathcal{L}_j(x)| + 1. \tag{3.8}$$

It is immediate from the definition of τ that $\{j\} \cup \mathcal{L}_j(x) \cup \mathcal{G}_j(x)$ is the set of elements in the subtree rooted at j. □

We now consider some permutation statistics that can be defined using permutation hook lengths. There are several known identities involving hook lengths of nodes of binary trees T.

These take the form

$$\sum_T \prod_j f(h_j) = g(n) \tag{3.9}$$

where the sum is over all binary trees with n nodes numbered from 1 to n and the product is over $j \in [n]$. For instance, since the Catalan number counts binary trees T,

$$\sum_T 1 = \frac{1}{n+1}\binom{2n}{n}. \tag{3.10}$$

There are also less obvious identities, which appear as [12, Corollary 7.5], [10, Equation 4] and [10, Equation 5]:

$$\sum_T \prod_j \left(1 + \frac{1}{h_j}\right) = \frac{2^n(n+1)^{n-1}}{n!} \tag{3.11}$$

$$\sum_T \prod_j \frac{1}{h_j 2^{h_j-1}} = \frac{1}{n!} \tag{3.12}$$

$$\sum_T \prod_j \frac{1}{(2h_j+1)2^{2h_j-1}} = \frac{1}{(2n+1)!}. \tag{3.13}$$

Using Theorem 3.20, we can rewrite any identity of the form (3.9) as a sum over permutations x as follows

$$\sum_x \prod_j f(h_j(x)) \cdot h_j(x) = n! \cdot g(n). \tag{3.14}$$

For example, rewriting identities (3.10), (3.11), (3.12) and (3.13) respectively produces the following identities, where each sum is over S_n:

$$\sum_x \prod_j h_j(x) = \frac{2n!}{(n+1)!} \tag{3.15}$$

$$\sum_x \prod_j (h_j(x)+1) = 2^n(n+1)^{n-1} \tag{3.16}$$

$$\sum_x \prod_j \frac{1}{2^{h_j(x)-1}} = 1 \tag{3.17}$$

$$\sum_x \prod_j \frac{h_j(x)}{(2h_j(x)+1)2^{2h_j(x)-1}} = \frac{n!}{(2n+1)!}. \tag{3.18}$$

As a future goal, it would be interesting to find simple bijective proofs for (3.15), (3.16), (3.17), and (3.18). We would also like to use permutation hook lengths to find new identities for sums over binary trees. As an example of how the process might work, using the identity

$$\sum_\pi 1 = n! \tag{3.19}$$

and converting to a sum over binary trees, we obtain the known identity

$$\sum_T \prod_j \frac{1}{h_j} = 1. \tag{3.20}$$

See, for example [10, Equation 2]. In order for this process to work, we would need to use tree-dependent statistics in the sense of [6, Section 5].

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