ABSTRACT

ALLOCCA, MICHAEL P. L_∞ Algebra Representation Theory. (Under the direction of Dr. Thomas Lada).

L_∞ algebras are natural generalizations of Lie algebras from a homotopy theoretical point of view. This concept was originally motivated by a problem in mathematical physics, both as a supporting role in deformation theory and more recently in closed field string theory. Many elementary properties and classical theorems of Lie algebras have been proven to hold true in the homotopy context. Specifically, representation theory of Lie algebras is a subject of current research. Lada and Markl proved the existence of a homotopy theoretic version of Lie algebra representations in the form of L_∞ algebra representations and constructed a one-to-one correspondence between these representations and the homotopy version of Lie modules, L_∞ modules [9]. This dissertation further explores L_∞ modules, highly motivated by classical Lie algebra representation theory.

L_∞ Algebra Representation Theory

by
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BIOGRAPHY

Michael Allocca was born in 1982 in New York City. Following a memorable upbringing in the city that never sleeps, he attended Fairfield University in Fairfield, CT, where he double majored in Computer Science and Mathematics and earned his Bachelor of Science degree in 2004. After spending a year outside of full time academia, he then migrated to Raleigh, NC to attend graduate school at North Carolina State University, where he received his Master's degree in 2007. As a graduate student, he published in an international journal and won two teaching awards. He is an avid New York Mets fan, and extends his support of jaded sports franchises to the New York Jets and NC State athletics. He has accepted a full-time faculty position at the University of Scranton, and looks forward to a long and rewarding academic career.

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Chapter 1

Introduction

One of the most interesting phenomena in topology is the loss of certain algebraic structures on topological spaces that is observed when continuously deforming one space into another. Often times several relations will no longer hold as equalities, but rather *up to homotopy.* A homotopy can be viewed as a continuous deformation of one map into another. For example, associativity of multiplication is among the simplest of algebraic structures. If we consider multiplication to be a binary map, m_2, on a topological space, the usual associativity relation states the following:

$$m_2 \circ (m_2 \times id) = m_2 \circ (id \times m_2)$$

When examining the loss of algebraic structures from a topological point of view, it is interesting to investigate what happens if we instead have the following:

$$m_2 \circ (m_2 \times id) \simeq m_2 \circ (id \times m_2)$$

where $\simeq$ means 'homotopic'. More specifically, if X is a space endowed with an associative multiplication and Y is homotopy equivalent to X, then the multiplication inherited by Y from X need not be associative but rather homotopy associative in the above sense. In fact, Y inherits a collection of compatible higher homotopies $m_n : K_n \times Y^n \to Y$ where K_n is a polytope of dimension $n-2$ for $n \geq 2$ as defined by Stasheff in [12].

As also explored by Stasheff in [12], all of this may be expressed in an analogous setting over a chain complex using chain homotopies. In building a coderivation d on the suspended complex satisfying $d^2 = 0$, homotopy associativity requires similar higher homotopies in the form of multilinear n-ary operations, m_n, that build an algebraic structure known as an A_∞ algebra.

Similarly, the Jacobi identity is a significant algebraic structure that is central to any Lie algebra. If a Lie bracket is expressed as a bilinear map, l_2, then the Jacobi identity states the following:

$$l_2 \circ l_2 = 0$$

where l_2 is extended via skew-symmetry when evaluated on three elements. Following the same type of reasoning in an algebraic setting over a chain complex, the Jacobi identity can then be expressed *up to homotopy* by the following relation:

$$l_2 \circ l_2 \simeq 0$$

Through similar reasoning outlined above, the need arises for higher homotopies in the form of skew-symmetric multilinear n-ary operations, l_n, that form an algebraic structure known as an L_∞ algebra. This concept of a Lie algebra *up to homotopy* was originally motivated by a problem in deformation theory [11], and more recently in closed field string theory [13] [16]. From a purely mathematical viewpoint, A_∞ and L_∞ algebras function as natural generalizations of associative and Lie algebras respectively.

In spite of the robust topological background and applications in theoretical physics, the very nature of L_∞ algebras as generalizations of Lie algebras warrants further study from a purely algebraic point of view. L_∞ algebras were first explored in depth in [10] and further in [9]. Since then, many basic classical Lie algebra ideas have proven to generalize beautifully to the homotopy context. However, due to the intense computations associated with homotopy algebras, many unanswered questions that correspond to even the most basic concepts in classical Lie algebra remain. Specifically, Lie algebra representation theory is a current topic of fruitful research that warrants an investigation of the manner in which it generalizes to the homotopy context. In [9], the generalization of a Lie algebra representation up to homotopy was explored and it was shown that there exists a one-to-one correspondence between these structures and natural generalizations of Lie modules. There has been little further investigation on this subject, which has great potential to facilitate the study of L_∞ algebras in a manner similar to classical Lie algebra representation theory.

This dissertation explores L_∞ algebra representations in the equivalent language of L_∞ modules. We will synthesize and contribute to basic concepts from current literature. We will construct a finite dimensional L_∞ module and a new finite dimensional L_∞ algebra structure. We will also explore structure-preserving maps between L_∞ modules, a concept that is central to classical Lie algebra representation theory and generalizes remarkably up to homotopy.

Chapter 2

Background and Notation

We now establish some of the conventions and definitions that are central to this thesis. We will then proceed with an understanding of the sign computations associated with a graded setting and basic combinatorics that are relevant to the main results of this dissertation.

2.1 Graded Vector Spaces and Koszul Signs

The investigation of algebraic structures up to homotopy is mostly conducted in an analogous chain complex setting. The sequence of abelian groups or modules in this setting will be interpreted as a more general version of a vector space.

Definition 2.1.1. A $\mathbb{Z}$-*graded vector space* over a field $\mathbb{F}$ is a direct sum $V = \bigoplus_{i\in\mathbb{Z}} V_i$ of vector spaces over $\mathbb{F}$. For $n \in \mathbb{Z}$, elements $x \in V_n$ are said to have *degree* n, denoted $|x| = n$.

Remark 2.1.2. For simplicity, we will assume that any vector space we encounter will be defined over a field of characteristic 0.

Remark 2.1.3. Given two $\mathbb{Z}$-graded vector spaces, V and W, their direct sum inherits the same gradation in the following sense:

$$(V \oplus W)_i = V_i \oplus W_i$$

Hence, for an element $x \oplus y \in V \oplus W$, $|x \oplus y| = |x| = |y|$.

As expected, adding vector multiplication to this type of structure will yield a more general version of an algebra.

Definition 2.1.4. A $\mathbb{Z}$-*graded associative algebra* over a field $\mathbb{F}$ is a $\mathbb{Z}$-graded vector space over $\mathbb{F}$, $A = \bigoplus_{i \in \mathbb{Z}} A_i$, equipped with associative multiplication, denoted '$\cdot$', that satisfies the following:

$$A_i \cdot A_j \subseteq A_{i+j}$$

Remark 2.1.5. To avoid ambiguity, from now on we will assume that any graded structure adheres to a $\mathbb{Z}$-grading. It is, however, worth noting that many of the topics that will be addressed in the following chapters have been studied in a $\mathbb{Z}_2$-graded setting.

To generalize a binary map in a graded setting, we also consider an n-ary map on the tensor product of elements.

Definition 2.1.6. Let V be a graded vector space. A map $l_n : V^{\otimes n} \to V_n$ is said to be of *degree* k if $l_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n) \in V_{k+|x_1|+|x_2|+\cdots+|x_n|}$.

Example 2.1.7. Let V be a graded vector space and suppose $x_1 \in V_4$, $x_2 \in V_{-1}$, and $x_3 \in V_0$. Then $|x_1| = 4$, $|x_2| = -1$ and $|x_3| = 0$. If $l_3 : V \otimes V \otimes V \to V$ is of degree 1, then $l_3(x_1 \otimes x_2 \otimes x_3) \in V_{1+4-1+0} = V_4$.

In the context of a graded vector space, objects often adhere to the "Koszul sign" convention [14]. That is, whenever two "things" of degree p and q are permuted, we multiply the result by $(-1)^{pq}$. This applies to the degrees of vector space elements as well as the degrees of maps. Given a permutation σ acting on a string of symbols, we denote the Koszul sign of σ by $\epsilon(\sigma)$.

Commutativity is a rare luxury in a graded setting. However, skew-symmetry is more common. When defining an n-ary map on elements in a graded setting in which elements do not commute, the Koszul sign is often times employed in conjunction with a permutation sign.

Definition 2.1.8. Let V be a graded vector space and $l_n : V^{\otimes} \to V$ be a multilinear map on V. We say l_n is *skew-symmetric* if

$$l_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (-1)^{\sigma}\epsilon(\sigma) l_n\big(\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\big)$$

for all $\sigma \in S_n$, where $(-1)^{\sigma}$ is the permutation sign and $\epsilon(\sigma)$ is the Koszul sign.

Example 2.1.9. Let V be a graded vector space and $x_1, x_2, x_3 \in V$. Let $\sigma = (132)$ be a permutation in S_3 and $l_3 : V^{\otimes} \to V$ be a multilinear skew-symmetric map on V. Then $l_3(x_1, x_2, x_3) = -(-1)^{|x_2||x_3|} l_3(x_1, x_3, x_2)$. Here, -1 is the permutation sign and $\epsilon(\sigma) = (-1)^{|x_2||x_3|}$ is the Koszul sign.

Remark 2.1.10. In a graded associative algebra, multiplication is a binary map that in general is not commutative nor skew-symmetric. We will, however, primarily encounter graded structures that do include skew-symmetric maps for the majority of this dissertation.

2.2 Unshuffles

When n-ary maps are extended to be evaluated on more than n elements, often times a certain pattern is utilized involving a collection of permutations that maintain an ascending order in two groups.

Definition 2.2.1. Let S_n denote the symmetric group of degree n. A permutation $\sigma \in S_n$ is a $(j, n-j)$-*unshuffle* for $0 \leq j \leq n$ if

$$\sigma(1) < \sigma(2) \cdots < \sigma(j) \text{ and } \sigma(j+1) < \sigma(j+2) < \cdots < \sigma(n).$$

Example 2.2.2. $\sigma = (1\ 5\ 3\ 2)(4\ 6) \in S_7$ is a $(4,3)$-unshuffle since

$$\sigma(\{1,2,3,4,5,6,7\}) = \{\underbrace{2}_{\sigma(1)}, \underbrace{3}_{\sigma(2)}, \underbrace{4}_{\sigma(3)}, \underbrace{6}_{\sigma(4)}\}\{\underbrace{1}_{\sigma(5)}, \underbrace{4}_{\sigma(6)}, \underbrace{7}_{\sigma(7)}\}$$

To avoid confusion, we note that $\sigma(i)$ in this definition refers to the element in the ith position rather than the position to which i is sent via σ. For example, σ sends 1 to 5 in the previous example, however $\sigma(1) = 2$ since 2 sits in the first position in the resulting string. This convention is utilized in order to remain consistent with the current literature.

We may further define unshuffles to include permutations that arrange more than two groups of string elements in ascending order.

Definition 2.2.3. A permutation $\sigma \in S_n$ is a $(i_1, i_2, \cdots, i_k)$-*unshuffle* (with $i_1 + i_2 + \cdots + i_k = n$) if

$$\begin{gathered} \sigma(1) < \cdots < \sigma(i_1), \\ \sigma(i_1 + 1) < \cdots < \sigma(i_1 + i_2), \\ \vdots \\ \text{and } \sigma(i_1 + \cdots + i_{k-1} + 1) < \cdots < \sigma(n). \end{gathered}$$

Example 2.2.4. $\sigma = (26475) \in S_7$ is a (2,3,2)-unshuffle since

$$\sigma(\{1,2,3,4,5,6,7\}) = \{\underbrace{1}_{\sigma(1)}, \underbrace{5}_{\sigma(2)}\}\{\underbrace{3}_{\sigma(3)}, \underbrace{6}_{\sigma(4)}, \underbrace{7}_{\sigma(5)}\}\{\underbrace{2}_{\sigma(6)}, \underbrace{4}_{\sigma(7)}\}$$

The same interpretation of $\sigma(i)$ is used for these generalized unshuffles.

In the homotopy setting associated with the main results of this thesis, n-ary maps are often extended on strings of elements greater than n through unshuffles.

Example 2.2.5. Let $l_2 : V \otimes V \to V$ be a bilinear map on a vector space V. Let $x, y, z \in V$. We may extend $l_2 : V \otimes V \otimes V \to V \otimes V$ on three inputs by summing over all $(2-1)$-unshuffles in the following sense:

$$l_2(x \otimes y \otimes z) = l_2(x \otimes y) \otimes z \pm l_2(x \otimes z) \otimes y \pm l_2(y \otimes z) \otimes x$$

The signs associated with the above relation are dependent on whether l_2 is skew-symmetric and the use of the Koszul sign in the given algebraic structure. It is worth noting that the Jacobi identity of a Lie algebra discussed in the introduction is extended via skew-symmetry on three inputs over all $(2-1)$-unshuffles. This topic will be addressd in greater deatil in the next chapter.

Remark 2.2.6. This type of map extension over unshuffles coincides with the extension of a linear map on a graded vector space V to a coderivation on the symmetric coalgebra $\bigwedge^* V$. The extension of a linear map on V to a coderivation on the tensor coalgebra T^*V involves a different construction, however the main results of this dissertation in the context of L_∞ algebras do not utilize it. For a more explicit description of these coderivations, see [7].

In general, a skew-symmetric multilinear map $l_k : V^{\otimes n} \to V^{\otimes(n-k+1)}$ may be extended on $n > k$ elements by the following definition:

$$l_k(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \sum_{\sigma} (-1)^{\sigma} \epsilon(\sigma)(l_k(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}) \otimes v_{\sigma(k+1)} \otimes \cdots \otimes v_{\sigma(n)}$$

where the summation is taken over all $(k, n-k)$-unshuffles.

If the bilinear map l_2 of example 2.2.5 were to be skew-symmetric, it would extend to three inputs as follows:

$$l_2(x \otimes y \otimes z) = l_2(x \otimes y) \otimes z - (-1)^{|y||z|} l_2(x \otimes z) \otimes y + (-1)^{|x|(|y|+|z|)} l_2(y \otimes z) \otimes x$$

Remark 2.2.7. An extension of a skew-symmetric multilinear map itself need not be skew-symmetric. For example, $l_1(x \otimes y \otimes z) \neq -(-1)^{|x||y|} l_1(y \otimes x \otimes z)$ since

$$l_1(x \otimes y \otimes z) = l_1(x) \otimes y \otimes z - (-1)^{|x||y|} l_1(y) \otimes x \otimes z + (-1)^{|z|(|x|+|y|)} l_1(z) \otimes x \otimes y$$

and

$$\begin{aligned}
&-(-1)^{|x||y|} l_1(y \otimes x \otimes z) \\
&= -(-1)^{|x||y|} \Big[l_1(y) \otimes x \otimes z - (-1)^{|x||y|} l_1(x) \otimes y \otimes z + (-1)^{|z|(|x|+|y|)} l_1(z) \otimes y \otimes x \Big] \\
&= l_1(x) \otimes y \otimes z - (-1)^{|x||y|} l_1(y) \otimes x \otimes z - (-1)^{|x||y|+z|(|x|+|y|)} l_1(z) \otimes y \otimes x
\end{aligned}$$

However, when paired with another skew-symmetric multilinear map, the composition will itself be skew-symmetric. That is, if $l_i : V^{\otimes i} \to V$ and $l_j : V^{\otimes j} \to V$ are skew-symmetric and $i + j = n + 1$, then $l_i \circ l_j : V^{\otimes n} \to V$ is skew-symmetric. For example, if $l_3 : V^{\otimes 3} \to V$ is also multilinear and skew-symmetric, then $l_3 l_1(x \otimes y \otimes z) =$

$-(-1)^{|x||y|}l_3l_1(y \otimes x \otimes z)$ since

$$l_3l_1(x \otimes y \otimes z) = l_3(l_1(x) \otimes y \otimes z) - (-1)^{|x||y|}l_3(l_1(y) \otimes x \otimes z) \\ + (-1)^{|z|(|x|+|y|)}l_3(l_1(z) \otimes x \otimes y)$$

and

$$\begin{aligned}
&- (-1)^{|x||y|}l_3l_1(y \otimes x \otimes z) \\
&= -(-1)^{|x||y|}\Big[l_3(l_1(y) \otimes x \otimes z) - (-1)^{|x||y|}l_3(l_1(x) \otimes y \otimes z) \\
&\qquad\qquad + (-1)^{|z|(|x|+|y|)}l_3(l_1(z) \otimes y \otimes x)\Big] \\
&= l_3(l_1(x) \otimes y \otimes z) - (-1)^{|x||y|}l_3(l_1(y) \otimes x \otimes z) - (-1)^{|x||y|+z|(|x|+|y|)}l_3(l_1(z) \otimes y \otimes x) \\
&= l_3(l_1(x) \otimes y \otimes z) - (-1)^{|x||y|}l_3(l_1(y) \otimes x \otimes z) + (-1)^{|z|(|x|+|y|)}l_3(l_1(z) \otimes x \otimes y)
\end{aligned}$$

This will be a significant consideration in the later chapters, as it will eliminate unnecessary computations.

It is also worth noting that if a permutation transposes consecutive elements of odd order in a skew-symmetric setting, then no sign change takes place.

Example 2.2.8. Let V be a graded vector space and $x_1, x_2, x_3 \in V$ with $x_2 \in V_{-1}$ and $x_3 \in V_3$. Let $\sigma = (132)$ be a permutation in S_3 and $l_3 : V^{\otimes} \to V$ be a multilinear skew-symmetric map on V. Then

$$l_3(x_1, x_2, x_3) = -(-1)^{|x_2||x_3|}l_3(x_1, x_3, x_2) = -(-1)^{-3}l_3(x_1, x_3, x_2) = l_3(x_1, x_3, x_2).$$

2.3 Shorthand Notation

For brevity, we will often times adopt a shorter notation for several expressions when convenient:

- At times, we will omit '| |' when describing degrees for the sake of concise computations. For example, in lieu of $(-1)^{|x||y|}$ we will often write $(-1)^{xy}$.

- When expressing maps on elements in $V^{\otimes n}$, we will sometimes omit '$\otimes$' in favor of commas for ease of viewing. For example, $l_4(x_1 \otimes x_2 \otimes x_3 \otimes x_4)$ will be synonymous with $l_4(x_1, x_2, x_3, x_4)$. When the context is clear, we will also adopt this concision for elements in direct sums. For example, if $x_1 \oplus x_2$ is an element of the direct sum $V \oplus W$, we will sometimes write (x_1, x_2) instead.

- In the case of skew-symmetry, we will use $\chi(\sigma)$ to denote the total sign effect of a permutation on an element of a graded vector space. That is, for any unshuffle σ, $\chi(\sigma) = (-1)^{\sigma}\epsilon(\sigma)$ where $(-1)^{\sigma}$ is the permutation sign and $\epsilon(\sigma)$ is the Koszul sign.

Example 2.3.1. Let V be a graded vector space, $x_1, x_2, x_3 \in V$, and $l_2 : V \otimes V \to V$ be a degree 0 skew-symmetric map. Then

$$l_2(l_2(x_1, x_2), x_3) = \underbrace{\underbrace{(-1)}_{(-1)^{\sigma}}\underbrace{(-1)^{(0+x_1+x_2)(x_3)}}_{\epsilon(\sigma)}}_{\chi(\sigma)} l_2(x_3, l_2(x_1, x_2))$$

It is important to observe that even though the degree of l_2 is 0, it is taken into account when permuting string elements with l_2. For Koszul signs, this is true for all n-ary maps, as their degrees must also be included when reordering strings. The permutation sign is present as a result of the skew-symmetry of l_2. These will be important considerations in the later chapters, as we will be working with n-ary skew-symmetric maps of various degrees.

Chapter 3

Representations of Lie Algebras

The study of Lie algebras is not only intrinsically beautiful from a mathematical viewpoint, it is also useful in many areas of theoretical physics. Furthermore, Lie algebra representation theory has grown into a flourishing area of current research that facilitates the study of Lie algebras in a more concrete manner. It is this robust nature that provides the main motivation for the topics that this dissertation addresses. In this chapter, we will review some basic concepts from the study of classical Lie algebras that will be relevant in building the main results of the later chapters. Most of the ideas in this chapter are summed up and proven concisely in [3] and [4].

3.1 Lie Algebras

Generally speaking, Lie algebras are vector spaces equipped with a special type of "vector multiplication" that need not be commutative nor associative. They are closely related to Lie groups and differentiable manifolds, as difficult problems with Lie groups can be reduced to simpler problems in their associated Lie algebras. In the context of this thesis, we focus on the basic properties of Lie algebras for the sake of generalization up to homotopy.

Definition 3.1.1. A *Lie algebra* is a vector space L over a field $\mathbb{F}$ equipped with an operation $[\ ,\ ] : L \times L \to L$ called the bracket that satisfies the following conditions for all $x, y, z \in L$:

1. $[\ ,\ ]$ is bilinear

2. $[x, x] = 0$

3. $[[x, y], z] - [[x, z], y] + [[y, z], x] = 0$ (Jacobi identity)

Property 2 implies skew-symmetry, provided $char(\mathbb{F}) \neq 2$. That is, for all $x, y \in L$, $[x, y] = -[y, x]$. This is an important consideration for L_∞ algebras. Property 3 is equivalent to the expression $l_2 \circ l_2 = 0$ described in previous chapters by interpreting the bracket as l_2 and extending via skew-symmetry on three elements x, y, z and summing over all $(2-1)$-unshuffles.

It is fairly elementary to verify that given any associative algebra, A, its multiplication '$\cdot$' induces a Lie algebra on A through the commutator bracket:

$$[x, y] = x \cdot y - y \cdot x$$

If we consider a finite dimensional vector space, V, then the set of linear transformations on V (denoted $end(V)$) can be given the structure of a more concrete Lie algebra and is easily computed by representing linear transformations as matrices. This is because $end(V)$ is an associative algebra under composition, hence its commutator gives it the structure of a Lie algebra. This also holds true for infinite dimensional vector spaces. Due to its significance, this Lie algebra is given a special name.

Definition 3.1.2. Let V be a vector space and $end(V)$ denote the associative algebra of linear transformations on V. Define a bracket operator on $end(V)$ by $[x, y] = x \circ y - y \circ x$. Under this operator, $end(V)$ forms a Lie algebra called the *general linear algebra*, denoted $gl(V)$.

Central to the study of any algebraic constructs are structure-preserving maps between them. These maps preserve structure, for example, by respecting the addition and/or multiplication of elements. In the case of Lie algebras, a very natural definition of this arises in preserving the bracket.

Definition 3.1.3. Let L and L' be Lie algebras over a field $\mathbb{F}$. A linear transformation $\varphi : L \to L'$ is a *(Lie algebra) homomoprhism* if $\varphi([x, y]) = [\varphi(x), \varphi(y)]\ \forall\ x, y \in L$.

This type of structure-preserving map is especially significant when associating a Lie algebra with the general linear algebra, which gives rise to the basic ideas behind representation theory.

3.2 Representations

Given a vector space V, the general linear algebra $gl(V)$ is very concrete in the sense that the bracket is ultimately computed through matrix multiplication. When a more abstract but finite dimensional Lie algebra is associated with $gl(V)$ through a structure-preserving map, much can be told about it via this concrete nature. In fact, for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, this is always true; by Ado's theorem [4], for some finite dimensional vector space V, L is isomorphic to a subalgebra of $gl(V)$.

This is the general idea behind Lie algebra representation theory. We operate under the philosophy that the study of Lie algebras can be difficult, whereas linear algebra can be much easier, so it is useful to represent Lie algebra problems as linear algebra problems through homomorphisms.

Definition 3.2.1. Let L be a finite dimensional Lie algebra over a field $\mathbb{F}$ and V a vector space over $\mathbb{F}$. A *representation* of L is a homomorphism $\varphi : L \to gl(V)$.

Remark 3.2.2. V and L both need not be finite dimensional.

Example 3.2.3. Let L be a finite dimensional Lie algebra over $\mathbb{F}$ and $x \in L$. Define $ad_x : L \to L$ by $ad_x(y) = [x, y]\ \forall\ y \in L$. Then $ad_x \in end(V)$, hence $ad_x \in gl(V)$. Furthermore, if we define $ad : L \to gl(V)$ by $ad(x) = ad_x$, then ad is a representation called the *adjoint representation*.

The adjoint representation plays a particularly significant role in the study of semisimple Lie algebras, which may be classified by the weights of their adjoint representations.

3.3 Lie Modules

Given a Lie algebra representation, it is often more convenient to express it in a different but equivalent convention.

Definition 3.3.1. Let L be a Lie algebra over a field $\mathbb{F}$ and V a vector space over $\mathbb{F}$. V is an *L-module* if there exists an operation $L \times V \to V$ given by $(x, v) \to x \cdot v$ such that

1. $x \cdot (au + bv) = a(x \cdot u) + b(x \cdot v)$
2. $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$
3. $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$

The use of module language is equivalent to working with representations by defining the homomorphism $\varphi : L \to gl(V)$ by

$$\varphi(x)(v) = x \cdot v. \tag{3.3.1}$$

That is, given a Lie module V, the equation $\varphi(x)(v) = x \cdot v$ defines a representation of L. Conversely, given a representation φ of L, using equation 3.3.1 to define a module action induces an L-module, as all three properties of definition 3.3.1 can easily be verified. Hence there is a one-to-one correspondence between representations of a Lie algebra L and L-modules.

Any structure-preserving map between two L-modules must preserve the module actions. Subsequently, we have the following definition.

Definition 3.3.2. Let L be a Lie algebra over $\mathbb{F}$ and M, M' be L-modules. An *(L-module) homomorphism* is a linear map $\psi : M \to M'$ such that $\psi(x \cdot m) = x \cdot \psi(m)$ $\forall x \in L,\ m \in M$.

The most fundamental example of a Lie module occurs over itself.

Example 3.3.3. Let L be a Lie algebra over $\mathbb{F}$. Then L is an L-module via the action $x \cdot y = [x, y] = ad_x(y)$.

It is fairly simple to verify the module axioms for this example using the Jacobi identity. Furthermore, this simple case where a Lie algebra L is itself an L-module corresponds to the adjoint representation of L.

Chapter 4

L_∞ Algebras

From a purely algebraic viewpoint, an L_∞ algebra is a natural generalization of a Lie algebra. Its construction stems from a similar construction of an A_∞ algebra, which functions as a natural generalization of an associative algebra. In this chapter we summarize many of the key results of [10] and [9], which in conjunction with the previous chapters and recent developments will lay the groundwork for the main results of this dissertation. The Koszul sign convention described earlier will be employed in all graded settings.

4.1 A_∞ Motivation

The study of A_∞ algebras has grown into a fruitful area of research. Most of it can be traced back to Stasheff's work in [12]. It is beyond the scope of this thesis to investigate these structures in detail. We will, however, summarize some of his results for the sake of motivating recent work with L_∞ Algebras.

We first consider a more general version of definition 2.1.4.

Definition 4.1.1. A *differential graded associative algebra* is a graded associative algebra A equipped with a graded differential $\partial : A \to A$ of degree -1 or 1 that satisfies the following:

1. $\partial \circ \partial = 0$

2. $\partial(x_1 \cdot x_2) = \partial(x_1) \cdot x_2 + (-1)^{x_1} x_1 \cdot \partial(x_2)$

for all $x_1, x_2 \in A$.

Equation 1 endows A with the structure of a chain or cochain complex, depending on the degree of ∂. For the sake of uniform terminology, we denote ∂ by m_1 and '$\cdot$' by m_2. Hence m_1 is a degree -1 (or 1 for cochain complexes) map, and m_2 is of degree 0. Furthermore, equation 2 is a graded version of the Leibniz formula.

Since A is a graded associative algebra, its multiplication, m_2, must satisfy the associative relation:

$$m_2 \circ (m_2 \times id) - m_2 \circ (id \times m_2) = 0$$

We motivate further study by considering what happens when this relation holds only up to homotopy:

$$m_2 \circ (m_2 \times id) - m_2 \circ (id \times m_2) \simeq 0$$

In the (co)chain complex setting in which A lies, we then require a (co)chain homotopy $m_3 : A \otimes A \otimes A \to A$ that must satisfy the following equation:

$$m_1 \circ m_3 + m_3 \circ m_1 + m_2 \circ (m_2 \times id) - m_2 \circ (id \times m_2) = 0 \tag{4.1.1}$$

A few remarks are in order. Omitting some details and by 'abuse of notation', the significance of this arises in building a coderivation, d, of degree -1 (or 1 for cochain complexes) on the suspended complex of A satisfying $d^2 = 0$. We will define the suspension of A (denoted $\uparrow A$) as the graded vector space with indices given by $(\uparrow A)_n = A_{n-1}$, and the suspension operator, $\uparrow\colon A \to (\uparrow A)$ (resp: desuspension operator, $\downarrow\colon V \to (\downarrow V)$) in the natural sense. The coderivation can be built by letting $d = m_1' + m_2'$ where $m_k' : (\uparrow A^{\otimes k}) \to \uparrow A$ is given by $m_k' = (-1)^{\frac{k(k-1)}{2}} \uparrow \circ m_k \circ \downarrow^{\otimes k}$ and then extending m_1' and m_2' as coderivations on the tensor coalgebra $T^*(\uparrow A)$ as explained in [8]. Under this construction, the equation $d^2 = 0$ is satisfied when evaluated on any two variables. However, when evaluated on three variables, this equation is only satisfied if associativity holds. In the case of homotopy associativity,

m_3 is necessary in the construction of the differential in the sense that $d^2(x_1, x_2, x_3) = 0$ if $d = m_1' + m_2' + m_3'$. It is also worth noting that since the maps m_1' and m_2' are extended as coderivations on the tensor coalgebra when evaluated on higher numbers of elements, the maps m_1 and m_2 exhibit the same type of extension when evaluated on higher numbers of elements in A.

When evaluated on an element $(x_1, x_2, x_3) \in A \otimes A \otimes A$ and extending m_1 and m_2 on three inputs as described above, equation 4.1.1 is then equivalent to

$$\begin{aligned} &m_1(m_3(x_1, x_2, x_3)) + m_3(m_1(x_1), x_2, x_3) + (-1)^{|x_1|} m_3(x_1, m_1(x_2), x_3) \\ &+ (-1)^{|x_1||x_2|} m_3(x_1, x_2, m_1(x_3)) + m_2(m_2(x_1, x_2), x_3) - m_2(x_1, m_2(x_2, x_3)) = 0 \end{aligned}$$

Since m_1 is of degree 1 or -1 depending on the setting as a cochain or chain complex, m_3 must be of degree -1 or 1 respectively.

Continuing in this fashion, we may consider what happens when homotopies are themselves homotopic. That is, a similar problem arises on the suspended complex when evaluating d^2 on four inputs. To address this, we construct the first higher homotopy, $m_4 : A^{\otimes 4} \to A$ of degree 2 (or -2 for cochain complexes) and its analog m_4' on the suspended complex. Continuing in this manner, the need for further higher homotopies, $m_5, m_6, m_7 \ldots$ arise and the relations that they satisfy are ultimately encoded in an A_∞ (strong homotopy associative) structure as follows [12].

Definition 4.1.2. Let V be a graded vector space. An A_∞ *algebra* structure on V is a collection of multilinear maps $m_k : V^{\otimes k} \to V$ of degree $k-2$ that satisfy the identity

$$\sum_{\lambda=0}^{n-1} \sum_{k=1}^{n-\lambda} \alpha \, m_{n-k+1}(x_1 \otimes \cdots \otimes x_\lambda \otimes m_k(x_{\lambda+1} \otimes \cdots \otimes x_{\lambda+k}) \otimes x_{\lambda+k+1} \otimes \cdots \otimes x_n) = 0$$

where $\alpha = (-1)^{k+\lambda+k\lambda+kn+k(|x_1|+\cdots+|x_\lambda|)}$, for all $n \geq 1$.

This utilizes the chain complex convention. One may alternatively utilize the cochain complex convention by requiring each map m_k to have degree $2-k$. For the remainder of this thesis we will employ the chain complex convention.

Remark 4.1.3. An A_∞ structure on V is equivalent to the existence of a degree -1 coderivation $d := \sum_{k=1}^{\infty} m'_k$ where m'_k was previously defined.

Remark 4.1.4. For $n = 3$, definition 4.1.2 reduces to the familiar expression

$$\begin{aligned} &m_1(m_3(x_1, x_2, x_3)) + m_3(m_1(x_1), x_2, x_3) + (-1)^{|x_1|} m_3(x_1, m_1(x_2), x_3) \\ &+ (-1)^{|x_1||x_2|} m_3(x_1, x_2, m_1(x_3)) + m_2(m_2(x_1, x_2), x_3) - m_2(x_1, m_2(x_2, x_3)) = 0 \end{aligned}$$

If $V = \bigoplus_{i \in \mathbb{Z}} V_i$ with $V_i = 0$ if $i \neq 0$, then for degree reasons this reduces to

$$m_2(m_2(x_1, x_2), x_3) - m_2(x_1, m_2(x_2, x_3)) = 0$$

which is precisely the elementary relation that multiplication must satisfy in an associative algebra. Hence, it is perfectly reasonable to view an A_∞ algebra as a natural generalization of an associative algebra.

The mathematical reasoning exhibited here motivates a similar investigation in the context of Lie algebras.

4.2 Graded Lie Algebras

Just as graded vector spaces may be equipped with an associative multiplication that respects the vector space's gradation, we may also endow them with a graded bracket structure.

Definition 4.2.1. A *graded Lie algebra* is a graded vector space L with a graded Lie bracket $[\ ,\ ] : L_p \otimes L_q \to L_{p+q}$ that satisfies the following:

1. $[x, y] = -(-1)^{xy}[y, x]$
2. $[[x, y], z] - (-1)^{yz}[[x, z], y] + (-1)^{x(y+z)}[[y, z], x] = 0$

for all $x, y, z \in L$.

Property (1) illustrates graded skew-symmetry, utilizing the Koszul sign in addition to the permutation sign. Property (2) is the graded analog of the Jacobi identity.

We may once again extend this type of structure to form a chain complex by equipping it with a graded differential that satisfies the Leibniz formula.

Definition 4.2.2. A *differential graded Lie algebra* is a graded Lie algebra L equipped with a graded differential $\partial : L \to L$ of degree -1 that satisfies the following:

1. $\partial \circ \partial = 0$

2. $\partial[x_1, x_2] = [\partial(x_1), x_2] + (-1)^{x_1}[x_1, \partial(x_2)]$

for all $x_1, x_2 \in L$.

As expected, a differential graded associative algebra will induce a differential graded Lie algebra through the graded commutator:

$$[x_1, x_2] = x_1 \cdot x_2 - (-1)^{x_1 x_2} x_2 \cdot x_1.$$

One can easily verify that property 2 of definition 4.2.2 is satisfied under this bracket.

4.3 Higher Homotopies

Following a similar line of reasoning in building differential graded associative algebras, we denote ∂ by l_1 and the graded bracket by l_2, and express the graded Jacobi identity in terms of maps as follows:

$$l_2 \circ l_2 = 0$$

where l_2 is extended via skew-symmetry over $(2-1)$-unshuffles when evaluated on an element $(x_1, x_2, x_3) \in L \otimes L \otimes L$.

Subsequently, we investigate what happens if the above relation holds only up to homotopy:

$$l_2 \circ l_2 \simeq 0$$

In the chain complex setting in which L lies, we then require a chain homotopy $l_3 : L \otimes L \otimes L \to L$ that must satisfy the following relation:

$$l_1 \circ l_3 + l_3 \circ l_1 + l_2 \circ l_2 = 0 \tag{4.3.1}$$

We may repeat the same type of investigation of the associative case here on the suspended complex $\uparrow L$. That is, if the Jacobi identity only holds up to homotopy, then the homotopy l_3 is necessary in constructing a degree -1 coderivation satisfying the equation $d^2 = 0$ when evaluated on three inputs on the suspended complex. The Lie analog to the associative setting involves the symmetric coalgebra $\bigwedge^*(\uparrow L)$ in lieu of the tensor coalgebra. Hence coderivations in this case are extended using unshuffles, as explained in remark 2.2.6.

When evaluated on an element $(x_1, x_2, x_3) \in L \otimes L \otimes L$ and extending l_1 and l_2 on three inputs, equation 4.3.1 is equivalent to

$$\begin{aligned}
&l_1(l_3(x_1, x_2, x_3)) + l_3(l_1(x_1), x_2, x_3) - (-1)^{x_1 x_2} l_3(l_1(x_2), x_1, x_3) \\
&+ (-1)^{x_3(x_1+x_2)} l_3(l_1(x_3), x_1, x_2) + l_2(l_2(x_1, x_2), x_3) \\
&- (-1)^{x_2 x_3} l_2(l_2(x_1, x_3), x_2)) + (-1)^{x_1(x_2+x_3)} l_2(l_2(x_2, x_3), x_1) = 0
\end{aligned}$$

Continuing in the same fashion that was exhibited in the associative case, we may explore what happens when homotopies are homotopic by introducing higher homotopies $l_4, l_5, l_6, \cdots$. This was originally exposed by Lada and Stasheff in [10] and further by Lada and Markl in [9]. This data is encoded concisely in an L_∞ (strong homotopy Lie) structure as follows.

Definition 4.3.1. Let V be a graded vector space. An L_∞ *algebra* structure on V is a collection of multilinear maps $\{l_k : V^{\otimes k} \to V\}$ of degree $k-2$ which are skew-symmetric in the sense that

$$l_k(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)}) = \chi(\sigma) l_k(x_1, x_2, \dots, x_k)$$

for all $\sigma \in S_k,\ x_i \in V$, with $\chi(\sigma) = (-1)^\sigma \epsilon(\sigma)$, and are also required to satisfy the generalized form of the Jacobi identity:

$$\sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(v_{\sigma(1)}, \dots, v_{\sigma(i)}), v_{\sigma(i+1)}, \dots, v_{\sigma(n)}) = 0$$

where the inner summation is taken over all $(i, n-i)$-unshuffles, $i \geq 1$, $n \geq 1$.

Remark 4.3.2. For $n = 3$, this reduces to the familiar expression

$$\begin{aligned} &l_1(l_3(x_1, x_2, x_3)) + l_3(l_1(x_1), x_2, x_3) - (-1)^{x_1 x_2} l_3(l_1(x_2), x_1, x_3) \\ &+ (-1)^{x_3(x_1+x_2)} l_3(l_1(x_3), x_1, x_2) + l_2(l_2(x_1, x_2), x_3) \\ &- (-1)^{x_2 x_3} l_2(l_2(x_1, x_3), x_2)) + (-1)^{x_1(x_2+x_3)} l_2(l_2(x_2, x_3), x_1) = 0 \end{aligned}$$

If $V = \bigoplus\limits_{i \in \mathbb{Z}} V_i$ with $V_i = 0$ if $i \neq 0$, then for degree reasons this yields

$$l_2(l_2(x_1, x_2), x_3) - l_2(l_2(x_1, x_3), x_2)) + l_2(l_2(x_2, x_3), x_1) = 0$$

which is precisely the classic Jacobi identity of a Lie algebra when interpreting l_2 as a Lie bracket. Hence, an L_∞ algebra serves as a natural generalization of a Lie algebra from a homotopy theoretical point of view.

4.4 A Classical Relationship

Among the most elegant characteristics of A_∞ and L_∞ algebras is the manner in which they behave as algebraic generalizations of their classical associative and Lie algebra counterparts. As explored in an earlier chapter, one of the most fundamental results in the study of classical Lie algebras is the relationship between associative and Lie algebras through commutators, which also holds between differential graded associative and differential graded Lie algebras. Remarkably, this relationship also holds in the homotopy context. That is, given an A_∞ algebra, commutators of its structure maps will induce an L_∞ structure.

Theorem 4.4.1 (Lada, Markl [9]). *Let V be a graded vector space and $\{m_k : V^{\otimes k} \to V\}$ define an A_∞ structure on V. For $n \geq 1$, define $l_n : V^{\otimes n} \to V$ by*

$$l_n(x_1, x_2, \cdots, x_n) = \sum_{\sigma \in S_n} \chi(\sigma) m_n(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)})$$

Then $\{l_k : V^{\otimes k} \to V\}$ defines an L_∞ structure on V.

Remark 4.4.2. For $n = 2$ in a "one nonzero term" graded vector space ($V = \bigoplus_{i\in\mathbb{Z}} V_i$ with $V_i = 0$ if $i \neq 0$), this is

$$l_2(x_1, x_2) = m_2(x_1, x_2) - (-1)^0 m_2(x_2, x_1)$$

which is precisely the classical relationship between an associative and Lie algebra with l_2 representing the bracket and m_2 representing multiplication.

4.5 A Finite Dimensional Example

Until recently, nontrivial finite dimensional examples of L_∞ algebras have been somewhat elusive. Trivial examples are plentiful, as any Lie algebra may be canonically embedded in an L_∞ structure by associating its underlying vector space V with a simple graded vector space consisting of V in degree 0 and trival vector spaces in degree $\neq 0$, and setting all homotopies and the graded differential equal to the zero map. In [2], Daily constructed one of the first "interesting" examples of a finite dimensional L_∞ algebra that consists of a Lie algebra together with a non-Lie action on another vector space.

Theorem 4.5.1 (Daily, Lada [2])**.** *Let L be the graded vector space given by $L = \bigoplus_{i\in\mathbb{Z}} L_i$ where L_0 has basis $< v_1, v_2 >$, L_{-1} has basis $< w >$, and $L_i = 0$ for $i \neq 0, -1$ with skew-symmetric multilinear maps $l_n : L^{\otimes n} \to L$ defined by the following:*

$$\begin{aligned} l_1(v_1) = l_1(v_2) &= w \\ l_2(v_1 \otimes v_2) &= v_1 \\ l_2(v_1 \otimes w) &= w \\ For\ n \geq 3 :\ l_n(v_2 \otimes w^{\otimes n-1}) &= C_n w \end{aligned}$$

where $C_3 = 1$ and $C_n = (-1)^{n-1}(n-3)C_{n-1}$ and $l_n = 0$ when evaluated on any element of $L^{\otimes n}$ that is not listed above. Then (L, l_n) is an L_∞ algebra.

Remark 4.5.2. When convenient, the recursive definition of C_n may be recognized more explicitly by

$$C_n = (-1)^{\frac{(n-2)(n-3)}{2}}(n-3)!.$$

Remark 4.5.3. In this example, V_0 is a strict Lie algebra under the bracket $[v_1, v_2] = l_2(v_1 \otimes v_2)$.

Remarkably, this small structure turns out to be surprisingly rich. For example, recently in [5] Kadeishvili and Lada showed that this structure yields an example of an open-closed homotopy algebra (OCHA), as defined by Kajiura and Stasheff [6]. Furthermore, it has been an ongoing subject of interest to expand homotopy algebra structures over this same graded vector space. Recently, an A_∞ structure has been defined over it as follows.

Theorem 4.5.4 (Allocca, Lada [1])**.** *Let V denote the graded vector space given by $V = \bigoplus_{i\in\mathbb{Z}} V_n$ where V_0 has basis $< v_1, v_2 >$, V_{-1} has basis $< w >$, and $V_n = 0$ for $n \neq 0, 1$. Define a structure on V by the following multilinear maps $m_n : V^{\otimes n} \to V$:*

$$m_1(v_1) = m_1(v_2) = w$$

$$For\ n \geq 2: \quad m_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes(n-2)-k}) = (-1)^k s_n v_1,\ 0 \leq k \leq n-2$$

$$m_n(v_1 \otimes w^{\otimes(n-2)} \otimes v_2) = s_{n+1} v_1$$

$$m_n(v_1 \otimes w^{\otimes(n-1)}) = s_{n+1} w$$

where $s_n = (-1)^{\frac{(n+1)(n+2)}{2}}$, and $m_n = 0$ when evaluated on any element of $V^{\otimes n}$ that is not listed above. Then (V, m_n) is an A_∞ algebra.

Remark 4.5.5. Although this A_∞ structure is defined over the same graded vector space described in the L_∞ example in [2], its commutators do not induce it. This creates an interesting area of future research, as the commutators will induce a different unknown finite dimensional L_∞ algebra example. Furthermore, the question of which A_∞ structure induces the example in [2] remains open.

4.6 L_∞ Modules and Representations

In light of the nature of L_∞ algebras as generalizations of Lie algebras, we proceed with significant motivation to generalize the most elementary properties of classical Lie algebras to the homotopy context. We now explore the basics of classical Lie algebra representation theory on this level.

To begin, we consider Lie modules as a language equivalent to Lie algebra representations and aim to generalize the idea of a Lie module up to homotopy. As previously explored, in classical Lie theory the most fundamental example of an L-module occurs over itself. That is, a Lie algebra L is an L-module under the adjoint action: $x \cdot y := [x, y]$. Hence, in order to generalize the concept of a Lie module, we require that an L_∞ algebra be an L_∞ module over itself via a generalization of the adjoint action. Under this reasonable assumption, a natural definition arises, as given by Lada and Markl in [9]:

Definition 4.6.1. Let (L, l_k) be an L_∞ algebra and M a differential graded vector space with graded differential k_1. A *(left) L-module* on M is a collection of skew-symmetric multilinear maps $\{k_n : L^{\otimes n-1} \otimes M \to M | 1 \le n < \infty\}$ of degree $n-2$ such that the following identity holds:

$$\sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} k_j(k_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)}), \xi_{\sigma(i+1)}, \dots, \xi_{\sigma(n)}) = 0 \qquad (4.6.1)$$

where σ ranges over all $(i, n-i)$-unshuffles, $i \ge 1$, with $n \ge 1$, $\xi_1, \cdots, \xi_{n-1} \in L$, and $\xi_n \in M$.

This relation follows the same pattern of the generalized Jacobi identity, motivated by the need to generalize the adjoint representation. A few observations are appropriate here:

- By definition of an unshuffle, either $\xi_{\sigma(i)} = \xi_n$ or $\xi_{\sigma(n)} = \xi_n$.

- Since we have $k_n : L^{\otimes n-1} \otimes M \to M$, we must utilize the skew-symmetry of k_n

in the case where $\xi_{\sigma(i)} = \xi_n$ as follows:

$$k_j(\underbrace{k_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)})}_{\in M}, \xi_{\sigma(i+1)} \dots, \xi_{\sigma(n)}) = \alpha k_j(\xi_{\sigma(i+1)} \dots \xi_{\sigma(n)}, \underbrace{k_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(i)})}_{\in M})$$

With $\alpha = (-1)^{j-1}(-1)^{(i+\sum_{k=1}^{i}|\xi_{\sigma(k)}|)(\sum_{k=i+1}^{n}|\xi_{\sigma(k)}|)}$.

- If $\xi_1, \cdots, \xi_n \in L$, then we define $k_n(\xi_1, \cdots, \xi_n) = l_n(\xi_1, \cdots, \xi_n)$.

If $n = 3$ and $L = \bigoplus_{i \in \mathbb{Z}} L_i$ with $L_i = 0$ if $i \neq 0$, and M is defined similarly, equation 4.6.1 reduces to the following:

$$\begin{aligned} k_3(k_1(\xi_1), \xi_2, \xi_3) - (-1)^{\xi_1\xi_2} k_3(k_1(\xi_2), \xi_1, \xi_3) + (-1)^{\xi_3(\xi_1+\xi_2)} k_3(k_1(\xi_3), \xi_1, \xi_2) \\ +k_2(k_2(\xi_1, \xi_2), \xi_3) - (-1)^{\xi_2\xi_3} k_2(k_2(\xi_1, \xi_3), \xi_2) + (-1)^{\xi_1(\xi_2+\xi_3)} k_2(k_2(\xi_2, \xi_3), \xi_1) \\ +k_1(k_3(\xi_1, \xi_2, \xi_3)) = 0 \end{aligned}$$

For degree reasons and using $k_2 = l_2$ where appropriate, this simplifies to:

$$\begin{aligned} & k_2(k_2(\xi_1, \xi_2), \xi_3) - k_2(k_2(\xi_1, \xi_3), \xi_2) + k_2(k_2(\xi_2, \xi_3), \xi_1) = 0 \\ \implies & k_2(l_2(\xi_1, \xi_2), \xi_3) + (-1)^0 k_2(\xi_2, k_2(\xi_1, \xi_3)) - (-1)^0 k_2(\xi_1, k_2(\xi_2, \xi_3)) = 0 \\ \implies & k_2(l_2(\xi_1, \xi_2), \xi_3) = k_2(\xi_1, k_2(\xi_2, \xi_3)) - k_2(\xi_2, k_2(\xi_1, \xi_3)) \end{aligned}$$

Interpreting k_2 as a module action '$\cdot$' in the classical Lie case, this is precisely property 3 of a Lie module (definition 3.3.1).

We now shift our attention to a homotopy analog of Lie algebra representations, which should be equivalent to L_∞ module language. Given an L_∞ module M, its differential k_1 induces a graded analog to the associative algebra of endomorphisms. Let $End(M)$ denote the graded vector space of linear maps from M to M. Under composition, together with the graded differential k_1, $End(M)$ forms a differential graded associative algebra. Furthermore, $End(M)$ induces a differential graded Lie algebra through commutators, denoted $End(M)_L$, which can be viewed as a homotopy analog of $gl(V)$. Hence in order to generalize a Lie algebra representation, one requires a structure-preserving map (homomorphism) between an L_∞ algebra and a differential graded Lie algebra. Lada and Markl defined this in [9].

Definition 4.6.2. Let (L, l_i) be an L_∞ algebra and $(A, \delta, [\ ,\ ])$ a differential graded Lie algebra. A *weak L_∞ map (homomorphism)* from L to A is a collection $\{f_n : L^{\otimes n} \to A\}$ of skew-symmetric multilinear maps of degree $n-1$ such that

$$\delta f_n(\xi_1, \cdots, \xi_n) + \sum_{j+k=n+1} \sum_{\sigma} \chi(\sigma)(-1)^{k(j-1)+1} f_j(l_k(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(k)}), \xi_{\sigma(k+1)}, \cdots, \xi_{\sigma(n)})$$
$$+ \sum_{s+t=n} \sum_{\tau} \chi(\tau)(-1)^{s-1}(-1)^{(t-1)(\sum_{p=1}^{s} \xi_{\tau(p)})} \left[f_s(\xi_{\tau(1)}, \cdots, \xi_{\tau(s)}), f_t(\xi_{\tau(s+1)}, \cdots, \xi_{\tau(n)}) \right]$$
$$= 0$$

Where σ runs through all $(k, n-k)$-unshuffles and τ runs through all $(s, n-s)$-unshuffles such that $\tau(1) < \tau(s+1)$, and $[\ ,\ ]$ denotes the graded bracket on A, and $\xi_1, \cdots, \xi_n \in L$.

One can easily verify that this reduces to the classical definition of a Lie algebra homomorphism if both L and A are regular Lie algebras.

We may then associate this with the natural analog of a classical Lie algebra representation (definition 3.2.1) as follows.

Definition 4.6.3. Let (L, l_n) be an L_∞ algebra and M a differential graded vector space. A *representation* of L on M is a weak L_∞ map $L \to End(M)_L$.

The equivalence of modules and representations in the homotopy context is summed up as concisely as it is in the classical Lie case.

Theorem 4.6.4 (Lada, Markl [9])**.** *Let (L, l_n) be an L_∞ algebra and M a differential graded vector space. Then there exists a one-to-one correspondence between L-module structures and weak L_∞ maps $L \to End(M)_L$.*

For the entirety of our investigation of L_∞ algebra representation theory, we will utilize L_∞ module language and proceed under the assumption that all results may also be described in terms of equivalent representations.

4.7 L_∞ Homomorphisms

In generalizing Lie algebra representations to the context of L_∞ structures, one only requires a structure-preserving map between an L_∞ algebra and a differential graded Lie algebra. Any differential graded Lie algebra can be canonically embedded in an L_∞ structure, however it is natural to wonder what a map between two structures that are strictly L_∞ looks like. In [15], Frégier, Markl and Yau outlined this in terms of maps. When evaluated on a collection of elements, we have the following equivalent definition.

Definition 4.7.1. Let (L, l_i) and (L', l'_i) be L_∞ algebras. An L_∞ *homomorphism* from L to L' is a collection $\{f_n : L^{\otimes n} \to L'\}$ of skew-symmetric multilinear maps of degree $n-1$ such that

$$\sum_{j+k=n+1} \sum_{\sigma} \chi(\sigma)(-1)^{k(j-1)+1} f_j(l_k(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(k)}), \xi_{\sigma(k+1)}, \cdots, \xi_{\sigma(n)}) +$$
$$\sum_{\substack{1\le t\le n \\ i_1+\cdots+i_t=n \\ i_r\ge 1}} \sum_{\tau} \alpha\, l'_t\big(f_{i_1}(\xi_{\tau(1)}, \cdots \xi_{\tau(i_1)}), f_{i_2}(\xi_{\tau(i_1+1)}, \cdots, \xi_{\tau(i_1+i_2)}), \cdots$$
$$\cdots, f_{i_t}(\xi_{\tau(i_1+\cdots+i_{t-1}+1)}, \cdots, \xi_{\tau(i_t)})\big) = 0$$

Where $\xi_1, \cdots, \xi_n \in L$, and σ runs through all $(k, n-k)$-unshuffles and τ runs through all $(i_1, \cdots, i_t)$-unshuffles satisfying $\tau(i_1+\cdots+i_{l-1}+1) < \tau(i_1+\cdots+i_l+1)$ if $i_l = i_{l+1}$, and $\alpha = \chi(\tau)(-1)^{\frac{t(t-1)}{2}+\sum_{k=1}^{t-1} i_k(t-k)}\nu$ with ν representing the Koszul sign that results from evaluating $(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_t})$ on $(\xi_{\tau(1)} \otimes \xi_{\tau(2)} \otimes \cdots \otimes \xi_{\tau(n)})$.

Remark 4.7.2. Explicitly,

$$\nu = (-1)^{(i_t-1)(\sum_{k=1}^{n-i_t} \xi_{\tau(k)})+(i_{t-1}-1)(\sum_{k=1}^{n-(i_t+i_{t-1})} \xi_{\tau(k)})+\cdots+(i_2-1)(\sum_{k=1}^{n-(i_t+i_{t-1}+\cdots+i_2)} \xi_{\tau(k)})}$$

Remark 4.7.3. If $L' = A$, $\delta = l'_1$, and $[\ ,\ ] = l'_2$, then this agrees with definition 4.6.2.

Remark 4.7.4. As expected, for $n = 3$ and $L = \bigoplus_{i\in\mathbb{Z}} L_i$ with $L_i = 0$ and L' defined similarly, this reduces to

$$f_1(l_2(x_1, x_2)) - l'_2(f_1(x_1), f_1(x_2)) = 0$$

which is precisely a Lie algebra homomorphism (definition 3.1.3):

$$\varphi([x_1, x_2]) = [\varphi(x_1), \varphi(x_2)].$$

L_∞ homomorphisms will play a significant role in constructing homomoprhisms between L_∞ modules.

Chapter 5

A Finite Dimensional L_∞ Module

As illustrated by the importance of the finite dimensional example of an L_∞ algebra in [2], concrete examples of homotopy algebra structures are often times elusive. Given the significance of Lie algebra representation theory and the established introductory results in the homotopy context, we aim to further expand on its theory by first building a concrete example of a finite dimensional L_∞ module.

5.1 Groundwork

We focus on the example given in [2] over which we will build a module. Equivalently, we will be constructing a representation of this L_∞ algebra. So let $L = \bigoplus_{i\in\mathbb{Z}} L_i$ where L_0 has basis $< v_1, v_2 >$, L_{-1} has basis $< w >$, and $L_i = 0$ for $i \neq 0, -1$ with skew-symmetric multilinear maps $l_n : L^{\otimes n} \to L$ defined by the following:

$$l_1(v_1) = l_1(v_2) = w$$

$$l_2(v_1 \otimes v_2) = v_1$$

$$l_2(v_1 \otimes w) = w$$

$$For\ n \geq 3 :\ l_n(v_2 \otimes w^{\otimes n-1}) = C_n w$$

where $C_3 = 1$ and $C_n = (-1)^{n-1}(n-3)C_{n-1}$ and $l_n = 0$ when evaluated on any element of $L^{\otimes n}$ that is not listed above.

With this we must associate a graded vector space $M = \bigoplus_{i \in \mathbb{Z}} M_i$ and equip it with a collection of skew-symmetric multilinear maps $k_n : L^{\otimes n-1} \otimes M \to M$ of degree $n-2$ such that the following relation holds for all $n \geq 1$ (equation 4.6.1):

$$\sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} k_j(k_i(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(i)}), \xi_{\sigma(i+1)}, \ldots, \xi_{\sigma(n)}) = 0$$

where σ ranges over all $(i, n-i)$-unshuffles, $i \geq 1$, with $n \geq 1$, $\xi_1, \cdots, \xi_{n-1} \in L$, and $\xi_n \in M$.

For the sake of minimizing computation and mirroring the construction of L, we build M to be finite dimensional with two nonzero vector spaces that also reside in degrees 0 and -1. For further simplification, we construct both of them to be one dimensional. So let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ where M_0 is a one dimensional vector space with basis $< m >$, M_{-1} is a one dimensional vector space with basis $< u >$ and $M_i = 0$ for $i \neq 0, -1$. We require all module actions k_n to work in tandem with equation 4.6.1. Our strategy is to use linear algebra in conjunction with "educated guesses" to define $k_1, k_2, \cdots$.

Remark 5.1.1. Since all maps k_n must be multilinear and will be extended via skew-symmetry, it suffices to define them solely on various numbers of basis elements.

5.2 The Differential and Module Action

We first define the graded differential, $k_1 : M \to M$, which can only act nontrivially on two elements: m and u. Since k_1 must be of degree $1 - 2 = -1$, there are two possibilities:

$$k_1(m) = cu \text{ for some constant } c$$
$$k_1(u) = 0$$

For simplicity, let $c = 1$. So the graded differential is defined by $k_1(m) = u$ and is defined to be 0 when evaluated on any other element.

We now shift our focus to the construction of $k_2 : L \otimes M \to M$, which must be of degree $2-2=0$. We define k_2 by how it acts on elements $(x,m) \in L \otimes M$ where x is any one of the three basis elements in L and m is any one of the two basis elements in M. For degree reasons, some combinations of basis elements may be eliminated. For example, the degree of $k_2(w,u)$ is $0-1-1=-2$, hence $k_2(w,u)=0$. Therefore there are five general ways to describe the manner in which k_2 acts nontrivially on basis elements:

$$\begin{aligned}
k_2(v_1,m) &= a_1 m\\
k_2(v_2,m) &= a_2 m\\
k_2(w,m) &= b_1 u\\
k_2(v_1,u) &= b_2 u\\
k_2(v_2,u) &= b_3 u
\end{aligned}$$

for some constants a_i, b_j. The choices for these constants are entirely dependent on whether they satisfy equation 4.6.1 for $n=2$ when evaluated on basis elements:

$$k_1k_2(x,q) - k_2(l_1(x),q) - (-1)^x k_2(x,k_1(q)) = 0 \tag{5.2.1}$$

for $x \in \{v_1, v_2, w\}$, $q \in \{m, u\}$.

Since $deg(k_1k_2(x,q)) = -1+0+|x|+|q| = deg(k_2k_1(x,q))$ and every element in equation 5.2.1 must be located in index 0 or -1 in order for the equation to be nontrivial, it suffices to evaluate equation 5.2.1 on elements (x,q) such that $|x| = |q| = 0$. For example, if $x=w$ and $q=m$, then for degree reasons equation 5.2.1 reduces to $0=0$.

Hence, there are only two pairs of basis elements to check: (v_1,m) and (v_2,m). Equation 5.2.1 then dictates the following:

$$\begin{aligned}
k_1k_2(v_1,m) - k_2(l_1(v_1),m) - (-1)^{v_1} k_2(v_1,k_1(m)) = 0\\
k_1k_2(v_2,m) - k_2(l_1(v_2),m) - (-1)^{v_2} k_2(v_2,k_1(m)) = 0
\end{aligned}$$

Hence,

$$a_1u - b_1u - b_2u = 0$$
$$a_2u - b_1u - b_3u = 0$$

or equivalently

$$a_1 - b_1 - b_2 = 0$$
$$a_2 - b_1 - b_3 = 0$$

A small degree of guesswork is involved here, as there are infinitely many solutions to this system of equations. Our strategy is to choose constants (preferrably natural numbers) that simplify k_2 enough to facilitate the construction of the homotopies through a reasonable level of computation. One such "simple" solution is to let $a_1 = b_2 = 1$ and $b_1 = a_2 = a_3 = 0$. Hence we may proceed with the following definition for k_2 that satisfies equation 5.2.1:

$$k_2(v_1, m) = m$$
$$k_2(v_1, u) = u$$

with $k_2 = 0$ when evaluated on any other element. We also extend k_2 via skew-symmetry.

5.3 First Homotopy

We follow similar reasoning to construct the first homotopy, $k_3 : L \otimes L \otimes M \to M$. We define k_3 by how it acts on any number of basis elements and will extend this definition via skew-symmetry. Since k_3 is of degree $3 - 2 = 1$ and $deg(k_3(x, y, q)) = 1 + |x| + |y| + |q|$, we have the following possibilities for nonzero actions of k_3 on basis elements:

$$k_3(v_1, v_1, u) = a_1 m$$
$$k_3(v_1, v_2, u) = a_2 m$$
$$k_3(v_2, v_2, u) = a_3 m$$
$$k_3(v_1, w, m) = b_1 m$$
$$k_3(v_2, w, m) = b_2 m$$
$$k_3(v_1, w, u) = c_1 u$$
$$k_3(v_2, w, u) = c_2 u$$
$$k_3(w, w, m) = c_3 u$$

for some some constants a_i, b_j, c_k. The nonzero actions of k_3 on basis elements must work in conjunction with the previously defined maps k_1 and k_2 to satisfy equation 4.6.1 for $n = 3$:

$$\begin{gathered} k_3(l_1(x), y, q) - (-1)^{xy} k_3(l_1(y), x, q) + (-1)^{x+y} k_3(x, y, k_1(q)) \\ + k_2(l_2(x, y), q) + (-1)^{xy} k_2(y, k_2(x, q)) - k_2(x, k_2(y, q)) + k_1 k_3(x, y, q) = 0 \end{gathered} \tag{5.3.1}$$

for $x, y \in \{v_1, v_2, w\}$, $q \in \{m, u\}$.

Since $deg(k_i k_j(x, y, q)) = |x| + |y| + |q|$ for all $i + j = 4$ and each element in equation 5.3.1 must reside in degree 0 or -1, it suffices to examine this equation when evaluated only on basis elements whose degrees add up to 0 or -1. Hence, any nonzero actions of k_3 on basis elements must adhere to the following equations:

$$\begin{aligned}
k_3(l_1(v_1),v_1,u)-(-1)^{v_1v_1}k_3(l_1(v_1),v_1,u)+(-1)^{v_1+v_1}k_3(v_1,v_1,k_1(u))\\
+k_2(l_2(v_1,v_1),u)+(-1)^{v_1v_1}k_2(v_1,k_2(v_1,u))-k_2(v_1,k_2(v_1,u))+k_1k_3(v_1,v_1,u)=0\\
k_3(l_1(v_1),v_2,u)-(-1)^{v_1v_2}k_3(l_1(v_2),v_1,u)+(-1)^{v_1+v_2}k_3(v_1,v_2,k_1(u))\\
+k_2(l_2(v_1,v_2),u)+(-1)^{v_1v_2}k_2(v_2,k_2(v_1,u))-k_2(v_1,k_2(v_2,u))+k_1k_3(v_1,v_2,u)=0\\
k_3(l_1(v_2),v_2,u)-(-1)^{v_2v_2}k_3(l_1(v_2),v_2,u)+(-1)^{v_2+v_2}k_3(v_2,v_2,k_1(u))\\
+k_2(l_2(v_2,v_2),u)+(-1)^{v_2v_2}k_2(v_2,k_2(v_2,u))-k_2(v_2,k_2(v_2,u))+k_1k_3(v_2,v_2,u)=0\\
k_3(l_1(v_1),w,m)-(-1)^{v_1w}k_3(l_1(w),v_1,m)+(-1)^{v_1+w}k_3(v_1,w,k_1(m))\\
+k_2(l_2(v_1,w),m)+(-1)^{v_1w}k_2(w,k_2(v_1,m))-k_2(v_1,k_2(w,m))+k_1k_3(v_1,w,m)=0\\
k_3(l_1(v_2),w,m)-(-1)^{v_2w}k_3(l_1(w),v_2,m)+(-1)^{v_2+w}k_3(v_2,w,k_1(m))\\
+k_2(l_2(v_2,w),m)+(-1)^{v_2w}k_2(w,k_2(v_2,m))-k_2(v_2,k_2(w,m))+k_1k_3(v_2,w,m)=0\\
k_3(l_1(v_1),v_1,m)-(-1)^{v_1v_1}k_3(l_1(v_1),v_1,m)+(-1)^{v_1+v_1}k_3(v_1,v_1,k_1(m))\\
+k_2(l_2(v_1,v_1),m)+(-1)^{v_1v_1}k_2(v_1,k_2(v_1,m))-k_2(v_1,k_2(v_1,m))+k_1k_3(v_1,v_1,m)=0\\
k_3(l_1(v_1),v_2,m)-(-1)^{v_1v_2}k_3(l_1(v_2),v_1,m)+(-1)^{v_1+v_2}k_3(v_1,v_2,k_1(m))\\
+k_2(l_2(v_1,v_2),m)+(-1)^{v_1v_2}k_2(v_2,k_2(v_1,m))-k_2(v_1,k_2(v_2,m))+k_1k_3(v_1,v_2,m)=0\\
k_3(l_1(v_2),v_2,m)-(-1)^{v_2v_2}k_3(l_1(v_2),v_2,m)+(-1)^{v_2+v_2}k_3(v_2,v_2,k_1(m))\\
+k_2(l_2(v_2,v_2),m)+(-1)^{v_2v_2}k_2(v_2,k_2(v_2,m))-k_2(v_2,k_2(v_2,m))+k_1k_3(v_2,v_2,m)=0
\end{aligned}$$

Hence,

$$
\begin{aligned}
u - u + a_1 u &= 0 \\
-c_2 u + c_1 u + u + a_2 u &= 0 \\
a_3 u &= 0 \\
c_3 u - c_1 u + b_1 u &= 0 \\
c_3 u - c_2 u + b_2 u &= 0 \\
a_1 m &= 0 \\
-b_2 m + b_1 m + a_2 m + m &= 0 \\
a_3 m &= 0
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
a_1 &= 0 \\
a_2 + c_1 - c_2 &= -1 \\
a_3 &= 0 \\
b_1 - c_1 + c_3 &= 0 \\
b_2 - c_2 + c_3 &= 0 \\
a_1 &= 0 \\
a_2 + b_1 - b_2 &= -1 \\
a_3 &= 0
\end{aligned}
$$

As a matrix equation, we have:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

or as an equivalent row-reduced augmented matrix:

$$\left(\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \end{array}\right) \longrightarrow \begin{array}{c} \begin{array}{cccccccc} a_1 & a_2 & a_3 & b_1 & b_2 & c_1 & c_2 & c_3 \end{array} \\ \left(\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Due to the infinite amount of solutions to this system, we must once again employ clever guesswork that will permit the formation of patterns for higher homotopies. One such solution would let $b_2 = c_2 = 1$ and all other constants be equal to 0. Through this solution, k_3 is defined as follows:

$$k_3(v_2, w, m) = m$$
$$k_3(v_2, w, u) = u$$

with $k_3 = 0$ when evaluated on any other element. We also extend k_3 via skew-symmetry.

There is an important observation to be made regarding the previous computations. The equations $a_1 = 0$ and $a_3 = 0$ all stemmed from evaluating equation 5.3.1

on basis elements containing multiple instances of v_1 or v_2. The following remark addresses this, and will facilitate the computations associated with building the higher homotopies.

Remark 5.3.1. For degree reasons, maps on basis elements that contain two or more copies of v_1 or v_2 must be zero since they will be extended via skew-symmetry. For example, in building k_3 we could immediately assume $k_3(v_1, v_1, u) = 0$ since in order for k_3 to be skew-symmetric, the following must be true:

$$\begin{aligned}
&k_3(v_1, v_1, u) = -(-1)^{v_1 v_1} k_3(v_1, v_1, u) \\
\Rightarrow\ &k_3(v_1, v_1, u) = -k_3(v_1, v_1, u) \\
\Rightarrow\ &k_3(v_1, v_1, u) = 0
\end{aligned}$$

A similar observation may be made for $k_3(v_2, v_2, u)$. This will apply to all higher homotopies, k_n. Furthermore, skew-symmetry will eliminate the need to compute equation 4.6.1 for $n \geq 3$ when evaluated on any basis elements that contain two or more copies of v_1 or v_2 since this property extends to all maps $k_i k_j$, $i + j = n + 1$ (per remark 2.2.7). By properties of the Koszul sign, this does not, however, apply to multiple instances of elements in degree -1.

5.4 Higher Homotopies

With no clear pattern emerging in the construction of k_1, k_2, and k_3, it is necessary to make similar computations in building the first higher homotopy, $k_4 : L \otimes L \otimes L \to M$, which must be of degree $4 - 2 = 2$. For degree reasons similar to those exhibited in the previous sections, k_4 may act nontrivially on a collection of basis elements whose degrees sum up to either -2 or -3. Furthermore, remark 5.3.1 eliminates any collection that contains more than one instance of v_1 or v_2. Hence the only candidates for nontrivial actions of k_4 on basis elements are the following:

$$k_4(v_1, v_2, w, u) = a_1 m$$
$$k_4(v_1, w, w, m) = a_2 m$$
$$k_4(v_2, w, w, m) = a_3 m$$
$$k_4(v_1, w, w, u) = b_1 u$$
$$k_4(v_2, w, w, u) = b_2 u$$
$$k_4(w, w, w, m) = b_3 u$$

for some some constants a_i, b_j, c_k. The nonzero actions of k_4 on basis elements must work in conjunction with the previously defined maps k_1, k_2, and k_3 to satisfy equation 4.6.1 for $n = 4$:

$$\begin{aligned}
-k_4\big(l_1(x), y, z, q\big) + (-1)^{xy} k_4\big(l_1(y), x, z, q\big) - (-1)^{z(x+y)} k_4\big(l_1(z), x, y, q\big) \\
-(-1)^{x+y+z} k_4\big(x, y, z, k_1(q)\big) + k_3\big(l_2(x, y), z, q\big) - (-1)^{yz} k_3\big(l_2(x, z), y, q\big) \\
+(-1)^{x(y+z)} k_3\big(y, z, k_2(x, q)\big) + (-1)^{x(y+z)} k_3\big(l_2(y, z), x, q\big) \\
-(-1)^{yz} k_3\big(x, z, k_2(y, q)\big) + k_3\big(x, y, k_2(z, q)\big) - k_2\big(l_3(x, y, z), q\big) \\
-(-1)^{z(x+y+1)} k_2\big(z, k_3(x, y, q)\big) - (-1)^{x} k_2\big(x, k_3(y, z, q)\big) + k_1 k_4(x, y, z, q) = 0 \quad (5.4.1)
\end{aligned}$$

for $x, y, z \in \{v_1, v_2, w\}$, $q \in \{m, u\}$.

Since $deg(k_i k_j(x, y, z, q)) = 1 + |x| + |y| + |z| + |q|$ for all $i + j = 5$ and each element in equation 5.4.1 must reside in index 0 or -1, it suffices to examine this equation when evaluated only on collections of basis elements such that $|x| + |y| + |z| + |q|$ equals -1 or -2. Furthermore, the reasoning exhibited in remark 5.3.1 eliminates the need to compute this on any collection of basis elements that contain more than one instance of v_1 or v_2. Hence, any nonzero actions of k_4 on basis elements must adhere to the following equations:

$$
\begin{aligned}
&-k_4\big(l_1(v_1),v_2,w,u\big)+(-1)^{v_1v_2}k_4\big(l_1(v_2),v_1,w,u\big)-(-1)^{w(v_1+v_2)}k_4\big(l_1(w),v_1,v_2,u\big)\\
&-(-1)^{v_1+v_2+w}k_4\big(v_1,v_2,w,k_1(u)\big)+k_3\big(l_2(v_1,v_2),w,u\big)-(-1)^{v_2w}k_3\big(l_2(v_1,w),v_2,u\big)\\
&+(-1)^{v_1(v_2+w)}k_3\big(v_2,w,k_2(v_1,u)\big)+(-1)^{v_1(v_2+w)}k_3\big(l_2(v_2,w),v_1,u\big)\\
&-(-1)^{v_2w}k_3\big(v_1,w,k_2(v_2,u)\big)+k_3\big(v_1,v_2,k_2(w,u)\big)-k_2\big(l_3(v_1,v_2,w),u\big)\\
&-(-1)^{w(v_1+v_2+1)}k_2\big(w,k_3(v_1,v_2,u)\big)-(-1)^{v_1}k_2\big(v_1,k_3(v_2,w,u)\big)+k_1k_4(v_1,v_2,w,u)=0\\
&-k_4\big(l_1(v_1),w,w,m\big)+(-1)^{v_1w}k_4\big(l_1(w),v_1,w,m\big)-(-1)^{w(v_1+w)}k_4\big(l_1(w),v_1,w,m\big)\\
&-(-1)^{v_1+w+w}k_4\big(v_1,w,w,k_1(m)\big)+k_3\big(l_2(v_1,w),w,m\big)-(-1)^{ww}k_3\big(l_2(v_1,w),w,m\big)\\
&+(-1)^{v_1(w+w)}k_3\big(w,w,k_2(v_1,m)\big)+(-1)^{v_1(w+w)}k_3\big(l_2(w,w),v_1,m\big)\\
&-(-1)^{ww}k_3\big(v_1,w,k_2(w,m)\big)+k_3\big(v_1,w,k_2(w,m)\big)-k_2\big(l_3(v_1,w,w),m\big)\\
&-(-1)^{w(v_1+w+1)}k_2\big(w,k_3(v_1,w,m)\big)-(-1)^{v_1}k_2\big(v_1,k_3(w,w,m)\big)+k_1k_4(v_1,w,w,m)=0\\
&-k_4\big(l_1(v_2),w,w,m\big)+(-1)^{v_2w}k_4\big(l_1(w),v_2,w,m\big)-(-1)^{w(v_2+w)}k_4\big(l_1(w),v_2,w,m\big)\\
&-(-1)^{v_2+w+w}k_4\big(v_2,w,w,k_1(m)\big)+k_3\big(l_2(v_2,w),w,m\big)-(-1)^{ww}k_3\big(l_2(v_2,w),w,m\big)\\
&+(-1)^{v_2(w+w)}k_3\big(w,w,k_2(v_2,m)\big)+(-1)^{v_2(w+w)}k_3\big(l_2(w,w),v_2,m\big)\\
&-(-1)^{ww}k_3\big(v_2,w,k_2(w,m)\big)+k_3\big(v_2,w,k_2(w,m)\big)-k_2\big(l_3(v_2,w,w),m\big)\\
&-(-1)^{w(v_2+w+1)}k_2\big(w,k_3(v_2,w,m)\big)-(-1)^{v_2}k_2\big(v_2,k_3(w,w,m)\big)+k_1k_4(v_2,w,w,m)=0\\
&-k_4\big(l_1(v_1),v_2,w,m\big)+(-1)^{v_1v_2}k_4\big(l_1(v_2),v_1,w,m\big)-(-1)^{w(v_1+v_2)}k_4\big(l_1(w),v_1,v_2,m\big)\\
&-(-1)^{v_1+v_2+w}k_4\big(v_1,v_2,w,k_1(m)\big)+k_3\big(l_2(v_1,v_2),w,m\big)-(-1)^{v_2w}k_3\big(l_2(v_1,w),v_2,m\big)\\
&+(-1)^{v_1(v_2+w)}k_3\big(v_2,w,k_2(v_1,m)\big)+(-1)^{v_1(v_2+w)}k_3\big(l_2(v_2,w),v_1,m\big)\\
&-(-1)^{v_2w}k_3\big(v_1,w,k_2(v_2,m)\big)+k_3\big(v_1,v_2,k_2(w,m)\big)-k_2\big(l_3(v_1,v_2,w),m\big)\\
&-(-1)^{w(v_1+v_2+1)}k_2\big(w,k_3(v_1,v_2,m)\big)-(-1)^{v_1}k_2\big(v_1,k_3(v_2,w,m)\big)+k_1k_4(v_1,v_2,w,m)=0
\end{aligned}
$$

Hence,

$$\begin{aligned} b_2u - b_1u + u + u - u + a_1u &= 0 \\ -b_3u - b_1u + a_2u &= 0 \\ -b_3u - b_2u + a_3 &= 0 \\ a_3m - a_2m + a_1m + m + m - m &= 0 \end{aligned}$$

or equivalently

$$\begin{aligned} &a_1 - b_1 + b_2 = -1 \\ &a_2 - b_1 - b_3 = 0 \\ &a_3 - b_2 - b_3 = 0 \\ &a_1 - a_2 + a_3 = -1 \end{aligned}$$

which may be expressed as a row-reduced augmented matrix:

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & -1 \end{array}\right) \to \begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \end{array} \\ \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Among the infinitely many solutions to this system, if we let $a_3 = b_2 = -1$ and other constants be equal to 0, then we obtain a potential pattern for the next higher homotopies. Hence, we define k_4 as follows:

$$\begin{aligned} k_4(v_2, w, w, m) &= -m \\ k_4(v_2, w, w, u) &= -u \end{aligned}$$

with $k_4 = 0$ when evaluated on any other element and k_4 is extended via skew-symmetry.

The potential pattern with these choices of constants mirrors that of the constants in the higher order structure maps of the L_∞ algebra example in [2]. That is, in this example the homotopies adhere to the following sign pattern:

$$C_3 = 1$$
$$C_4 = -1$$
$$\vdots$$
$$C_n = (-1)^{n-1}(n-3)C_{n-1}$$

By repeating the arguments in the past few sections for $n = 5, 6, \cdots$, we find that this pattern does continue for $k_5, k_6, \cdots$. For brevity, we omit these computations and form a finite dimensional L_∞ module under the assumption that this pattern continues.

Theorem 5.4.1. *Let L denote the L_∞ structure in [2] given by $L = \bigoplus_{i\in\mathbb{Z}} L_i$ where L_0 has basis $< v_1, v_2 >$, L_{-1} has basis $< w >$, and $L_i = 0$ for $i \neq 0, -1$ with skew-symmetric multilinear maps $l_n : L^{\otimes n} \to L$ defined by the following:*

$$l_1(v_1) = l_1(v_2) = w$$
$$l_2(v_1 \otimes v_2) = v_1$$
$$l_2(v_1 \otimes w) = w$$
$$For\ n \geq 3:\ \ l_n(v_2 \otimes w^{\otimes n-1}) = C_n w$$

where $C_3 = 1$ and $C_n = (-1)^{n-1}(n-3)C_{n-1}$ and $l_n = 0$ when evaluated on any element of $L^{\otimes n}$ that is not listed above. Now let M denote the graded vector space given by $M = \bigoplus_{i\in\mathbb{Z}} M_i$ where M_0 is a one dimensional vector space with basis $< m >$, M_{-1} is a one dimensional vector space with basis $< u >$ and $M_i = 0$ for $i \neq 0, -1$. Define a structure on M by the following multilinear maps $k_n : L^{\otimes n-1} \otimes M \to M$:

$$k_1(m) = u$$
$$k_2(v_1 \otimes m) = m$$
$$k_2(v_1 \otimes u) = u$$
$$For\ n \geq 3:\ \ k_n(v_2 \otimes w^{\otimes n-2} \otimes m) = C_n m$$
$$k_n(v_2 \otimes w^{\otimes n-2} \otimes u) = C_n u$$

Extend these maps to be skew-symmetric and define $k_n = 0$ when evaluated on any element of $L^{\otimes n-1} \otimes M$ that is not listed above. Then (M, k_n) is an L-module.

Remark 5.4.2. This utilizes the chain complex convention, whereas the equivalent cochain complex convention is assumed in [2]. Hence $|v_1| = |v_2| = |m| = 0$, $|w| = |u| = -1$ and l_n and k_n are of degree $n-2$. They may be adapted to the cochain complex convention by requiring $|w| = |u| = 1$ and defining each map k_n to be of degree $2-n$.

Proof. We aim to prove the following:

$$\sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} k_j(k_i(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(i)}), \xi_{\sigma(i+1)}, \ldots, \xi_{\sigma(n)}) = 0$$

where σ ranges over all $(i, n-i)$-unshuffles, $i \geq 1$, with $n \geq 1$, $\xi_1, \cdots, \xi_{n-1} \in L$, and $\xi_n \in M$.

In shorthand notation, this is equivalent to showing that

$$\sum_{s=1}^{n} (-1)^{s(n-s)} k_{n-s+1} k_s(\xi_1, \xi_2, \cdots, \xi_n) = 0 \tag{5.4.2}$$

where it is understood that k_s will be extended on $n > s$ elements over $(s, n-s)$-unshuffles.

Since k_n is multilinear and skew-symmetric, it suffices to show this relation holds when evaluated only on basis elements and in any string order. Each element in equation 5.4.2 also has degree $(n-s-1-2) + (s-2) + |\xi_1| + |\xi_2| + \cdots + |\xi_n| = n - 3 + |\xi_1| + |\xi_2| + \cdots + |\xi_n|$, which must equal 0 or -1 in order for the elements to be nonzero. So $(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) \in L^{\otimes n-1} \otimes M$ must contain either 2 or 3 elements in degree 0. If this tensor product contains v_1 or v_2 twice, then equation 5.4.2 holds trivially since $|v_1| = |v_2| = 0$ and k_n is skew-symmetric (as explored in remark 5.3.1). For example,

$$\begin{aligned} k_{n-s+1} k_s(v_1 \otimes v_1 \otimes w \otimes^{n-3} \otimes u) &= -(-1)^{v_1 v_1} k_{n-s+1} k_s(v_1 \otimes v_1 \otimes w \otimes^{n-3} \otimes u) \\ &= -k_{n-s+1} k_s(v_1 \otimes v_1 \otimes w \otimes^{n-3} \otimes u) \end{aligned}$$

by permuting the first two elements. So $k_{n-s+1}k_s(v_1 \otimes v_1 \otimes w \otimes^{n-3} \otimes u) = 0$. Hence it suffices to prove that equation 5.4.2 holds on the following string choices for $(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n)$:

$$(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes u) \tag{5.4.3}$$

$$(v_1 \otimes w^{\otimes n-2} \otimes m) \tag{5.4.4}$$

$$(v_2 \otimes w^{\otimes n-2} \otimes m) \tag{5.4.5}$$

$$(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes m) \tag{5.4.6}$$

For string 5.4.3, in regards to the summands of equation 5.4.2 we observe the following:

$$\begin{aligned}
k_1k_n(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes u)) &= k_1(0) = 0 \\
k_2k_{n-1}(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes u) &= (-1)^{n-1}k_2\big(k_{n-1}(v_2 \otimes w^{\otimes n-2} \otimes u) \otimes v_1\big) \\
&= (-1)^n C_{n-1} k_2(v_1 \otimes u) \\
&= (-1)^n C_{n-1} u
\end{aligned}$$

$k_r k_{n-r+1}(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes u) = 0$ for $3 \leq r \leq n-2$ since k_r and k_{n-r+1} are only nonzero when they are evaluated on a tensor product containing v_2.

$$\begin{aligned}
k_{n-1}k_2(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes u) &= -(n-3)k_{n-1}\big(l_2(v_1 \otimes w) \otimes v_2 \otimes w^{\otimes n-4} \otimes u\big) \\
&\quad - k_{n-1}\big(k_2(v_1 \otimes u) \otimes v_2 \otimes w^{\otimes n-3}\big) \\
&= (n-3)k_{n-1}(v_2 \otimes w^{\otimes n-3} \otimes u) \\
&\quad + k_{n-1}(v_2 \otimes w^{\otimes n-3} \otimes u) \\
&= (n-2)k_{n-1}(v_2 \otimes w^{\otimes n-3} \otimes u) \\
&= (n-2)C_{n-1}u \\
k_n k_1(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes u) &= k_n\big(k_1(v_1) \otimes v_2 \otimes w^{\otimes n-3} \otimes u\big) \\
&= k_n(w \otimes v_2 \otimes w^{\otimes n-3} \otimes u) \\
&= -k_n(v_2 \otimes w^{\otimes n-2} \otimes u) \\
&= -C_n u \\
&= -(-1)^{n-1}(n-3)C_{n-1}u \\
&= (-1)^n(n-3)C_{n-1}u
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{s=1}^{n}(-1)^{s(n-s)}k_{n-s+1}k_s(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes u) &= (-1)^{1(n-1)}(-1)^n(n-3)C_{n-1}u \\
&\quad + (-1)^{2(n-2)}(n-2)C_{n-1}u \\
&\quad + (-1)^{(n-1)(n-(n-1))}(-1)^n C_{n-1}u \\
&= -(n-3)C_{n-1}u \\
&\quad + (n-2)C_{n-1}u \\
&\quad - C_{n-1}u \\
&= 0
\end{aligned}$$

The case where equation 5.4.2 is evaluated on string 5.4.4 is trivial by definition of k_n and l_n.

Regarding string 5.4.5, we observe the following:

$$k_1 k_n(v_2 \otimes w^{\otimes n-2} \otimes m) = k_1(C_n m) = C_n u$$

$k_r k_{n-r+1}(v_2 \otimes w^{\otimes n-2} \otimes m) = 0$ for $2 \leq r \leq n-1$ for similar reasons encountered above.

$$\begin{aligned} k_n k_1(v_2 \otimes w^{\otimes n-2} \otimes m) &= (-1)^{n-1} k_n\big(k_1(m) \otimes v_2 \otimes w^{\otimes n-2}\big) \\ &= -(-1)^{n-1} k_n(v_2 \otimes w^{\otimes n-2} \otimes u) \\ &= (-1)^n C_n u \end{aligned}$$

Hence

$$\begin{aligned} \sum_{s=1}^{n} (-1)^{s(n-s)} k_{n-s+1} k_s(v_2 \otimes w^{\otimes n-2} \otimes m) &= (-1)^{1(n-1)}(-1)^n C_n u + (-1)^{n(n-n)} C_n u \\ &= -C_n u + C_n u \\ &= 0 \end{aligned}$$

For string 5.4.6, we observe the following:

$$\begin{aligned} k_1 k_n(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes m) &= k_1(0) = 0 \\ k_2 k_{n-1}(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes m) &= (-1)^{n-1} k_2\big(k_{n-1}(v_2 \otimes w^{\otimes n-2} \otimes m) \otimes v_1\big) \\ &= (-1)^n C_{n-1} k_2(v_1 \otimes m) \\ &= (-1)^n C_{n-1} m \end{aligned}$$

$k_r k_{n-r+1}(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes m) = 0$ for $3 \leq r \leq n-2$ for similar reasons encountered above.

$$
\begin{aligned}
k_{n-1}k_2(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes m) &= -(n-3)k_{n-1}\big(l_2(v_1 \otimes w) \otimes v_2 \otimes w^{\otimes n-4} \otimes m\big) \\
&\quad + (-1)^{n-2}k_{n-1}\big(k_2(v_1 \otimes m) \otimes v_2 \otimes w^{\otimes n-3}\big) \\
&= -(n-3)k_{n-1}(w \otimes v_2 \otimes w^{\otimes n-4} \otimes m) \\
&\quad + (-1)^{n-2}k_{n-1}(m \otimes v_2 \otimes w^{\otimes n-3}) \\
&= (n-3)k_{n-1}(v_2 \otimes w^{\otimes n-3} \otimes m) \\
&\quad + (-1)^{(n-2)+(n-2)}k_{n-1}(v_2 \otimes w^{\otimes n-3} \otimes m) \\
&= (n-3)C_{n-1}m \\
&\quad + C_{n-1}m \\
&= (n-2)C_{n-1}m \\
k_nk_1(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes m) &= k_n\big(k_1(v_1) \otimes v_2 \otimes w^{\otimes n-3} \otimes m\big) \\
&= -k_n(v_2 \otimes w^{\otimes n-2} \otimes m) \\
&= -C_nm \\
&= -(-1)^{n-1}(n-3)C_{n-1}m \\
&= (-1)^n(n-3)C_{n-1}m
\end{aligned}
$$

Hence

$$
\begin{aligned}
\sum_{s=1}^{n}(-1)^{s(n-s)}k_{n-s+1}k_s(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes m) &= (-1)^{1(n-1)}(-1)^n(n-3)C_{n-1}m \\
&\quad + (-1)^{2(n-2)}(n-2)C_{n-1}m \\
&\quad + (-1)^{(n-1)(n-(n-1))}(-1)^nC_{n-1}m \\
&= -(n-3)C_{n-1}m \\
&\quad + (n-2)C_{n-1}m \\
&\quad - C_{n-1}m \\
&= 0
\end{aligned}
$$

In all 4 cases, equation 5.4.2 holds. Hence M is an L-module. $\square$

With the creation of a concrete example of an L_∞ module, we hope that interesting interpretations of it will arise in various topics in homotopy algebra. Furthermore, a concrete L_∞ module will induce another interesting L_∞ algebra structure in a manner that generalizes the relationship between a classical Lie module and a canonical Lie algebra. This will be addressed in the next chapter.

Chapter 6

A Canonical L_∞ Algebra

Given a Lie algebra L and an L-module M, a rather simple vector space can be formed via direct sum, $L \oplus M$. This vector space may also be endowed with an elementary Lie structure. In this chapter we review these results in both the classical setting and the homotopy analog. Subsequently, a new finite dimensional L_∞ structure will be defined using the results of the previous chapter.

6.1 Lie Structures on $L \oplus M$

One of the most fundamental results in the study of classical Lie algebras, as given concisely in [4], is that given a Lie algebra L and an $L-$module M, the vector space $L \oplus M$ forms a Lie algebra via the bracket

$$\big[(x_1, m_1), (x_2, m_2)\big] = \Big([x_1, x_2], x_1 \cdot m_2 - x_2 \cdot m_1\Big)$$

where '$\cdot$' denotes the module action in M. The bilinearity and skew-symmetry of the Lie bracket in L permits an easy verification of properties 1 and 2 of a Lie algebra for this bracket. Similarly, it is fairly elementary to verify that this bracket satisfies the Jacobi identity. Hence, any Lie module will induce a new Lie algebra on the direct sum.

It is not surprising that a homotopy theoretic version of the classical Lie algebra $L \oplus M$ exists. Given an L_∞ structure, L, and an L-module, M, we may construct a new graded vector space $L \oplus M$ that can be endowed with its own L_∞ structure as follows.

Theorem 6.1.1 (Lada, [8]). *Let (L, l_k) be an L_∞ algebra and (M, k_n) be an L-module. Then the graded vector space $L \oplus M$ inherits a canonical L_∞ structure under the collection of maps $\{j_n : (L \oplus M)^{\otimes n} \to L \oplus M\}$ defined by*

$$j_n\big((x_1, m_1) \otimes \cdots \otimes (x_n, m_n)\big)$$
$$= \Big(l_n(x_1 \otimes \cdots \otimes x_n), \sum_{i=1}^{n} (-1)^{n-i}(-1)^{m_i \sum_{k=i+1}^{n} x_k} k_n(x_1 \otimes \cdots \otimes \hat{x}_i \otimes \cdots \otimes x_n \otimes m_i)\Big)$$

where $\hat{x}_i$ means omit x_i.

Remark 6.1.2. By definition,

$$j_2\big((x_1, m_1), (x_2, m_2)\big) = \Big(l_2(x_1, x_2), k_2(x_1, m_2) - (-1)^{x_2 m_1} k_2(x_2, m_1)\Big)$$

which reduces to the familiar lie bracket when L is a strict Lie algebra under the bracket l_2 and M is a an L-module under the action k_2:

$$\big[(x_1, m_1), (x_2, m_2)\big] = \Big([x_1, x_2], x_1 \cdot m_2 - (-1)^0 x_2 \cdot m_1\Big)$$

Hence, any L_∞ module will induce a different L_∞ algebra on the direct sum of graded vector spaces.

6.2 Induced L_∞ Structure

Given the newly constructed concrete example of an L_∞ module in theorem 5.4.1, it is natural to investigate the type of L_∞ algebra structure it induces. Let L and M denote the L_∞ algebra and L-module in theorem 5.4.1. That is, $L = \bigoplus_{i \in \mathbb{Z}} L_i$ where $L_i = 0$ if $i \neq 0, -1$ and

$$L_0 =< v_1, v_2 >$$
$$L_{-1} =< w >$$

and $M = \bigoplus_{i\in\mathbb{Z}} M_i$ where $M_i = 0$ if $i \neq 0, -1$ and

$$M_0 =< m >$$
$$M_{-1} =< u >$$

As a graded vector space, elements $(x, m) \in L \oplus M$ must satisfy $|x| = |m|$. Hence $L \oplus M = \bigoplus_{i\in\mathbb{Z}} (L \oplus M)_i$ where $(L \oplus M)_i = 0$ if $i \neq 0, -1$ and

$$(L \oplus M)_0 =< (v_1, m), (v_2, m) >$$
$$(L \oplus M)_{-1} =< (w, u) >$$

Using the definition given in theorem 6.1.1, we may explicitly define the structure maps $\{j_n\}$ on $L \oplus M$. As a result of multilinearity and skew-symmetry, it suffices to define these maps by how they act on any number of basis elements.

Since $deg(j_1) = -1$, $j_1(w, u) = 0$. Furthermore,

$$j_1(v_1, m) = (l_1(v_1), k_1(m)) = (w, u)$$
$$j_1(v_2, m) = (l_1(v_2), k_1(m)) = (w, u)$$

Hence j_1 is very simply defined by

$$j_1(v_1, m) = j_1(v_2, m) = (w, u)$$

Examining all combinations of basis elements, we find j_2 to be defined as follows:

$$j_2\big((v_1, m), (v_1, m)\big) = \Big(l_2(v_1, v_1), k_2(v_1, m) - (-1)^{v_1 m} k_2(v_1, m)\Big) = (0, 0)$$
$$j_2\big((v_2, m), (v_2, m)\big) = \Big(l_2(v_2, v_2), k_2(v_2, m) - (-1)^{v_2 m} k_2(v_2, m)\Big) = (0, 0)$$
$$j_2\big((v_1, m), (v_2, m)\big) = \Big(l_2(v_1, v_2), k_2(v_1, m) - (-1)^{v_2 m} k_2(v_2, m)\Big) = (v_1, m)$$
$$j_2\big((v_1, m), (w, u)\big) = \Big(l_2(v_1, w), k_2(v_1, u) - (-1)^{wm} k_2(w, m)\Big) = (w, u)$$
$$j_2\big((v_2, m), (w, u)\big) = \Big(l_2(v_2, w), k_2(v_2, u) - (-1)^{wm} k_2(w, m)\Big) = (0, 0)$$
$$j_2\big((w, u), (w, u)\big) = (0, 0) \text{ since } deg(j_2) = 0$$

Hence

$$j_2\big((v_1,m),(v_2,m)\big) = (v_1,m)$$
$$j_2\big((v_1,m),(w,u)\big) = (w,u)$$

with $j_2 = 0$ when evaluated on any other element.

It is apparent that the graded differential j_1 and bracket j_2 are acting in precisely the same manner as their counterparts in L. The homotopies, however, do not. Let $n \geq 3$ and consider $j_n : (L \oplus M)^{\otimes n} \to L \oplus M$. By definition of j_n in theorem 6.1.1, in conjunction with the definitions of l_n and k_n, the only nonzero action of j_n on basis elements occurs on $(v_2, m) \otimes (w,u)^{\otimes n-1}$ since both l_n and k_n require v_2 as an input when $n \geq 3$.

$$\begin{aligned} j_n\big((v_2,m)\otimes(w,u)^{\otimes n-1}\big) &= \Big(l_n(v_2 \otimes w^{\otimes n-1}), \\ &\underbrace{k_n(v_2 \otimes w^{\otimes n-2} \otimes u) + k_n(v_2 \otimes w^{\otimes n-2} \otimes u) + \cdots + k_n(v_2 \otimes w^{\otimes n-2} \otimes u)}_{(n-1)k_n(v_2 \otimes w^{\otimes n-2} \otimes u)} + 0\Big) \\ &= (C_n w, (n-1)C_n u) \\ &= C_n(w,(n-1)u) \end{aligned}$$

These structure maps form a new L_∞ algebra as follows.

Theorem 6.2.1. *Let $L \oplus M = \bigoplus_{i\in\mathbb{Z}} (L\oplus M)_i$ where $(L \oplus M)_0$ is two dimensional with basis $< (v_1,m),(v_2,m) >$ and $(L\oplus M)_{-1}$ is one dimensional with basis $< (w,u) >$ and $(L \oplus M)_i = 0$ for $i \neq 0,-1$. Define a structure on $(L \oplus M)$ by the following multilinear maps $\{j_n : (L\oplus M)^{\otimes n} \to L \oplus M\}$:*

$$j_1(v_1,m) = j_1(v_2,m) = (w,u)$$
$$j_2\big((v_1,m)\otimes(v_2,m)\big) = (v_1,m)$$
$$j_2\big((v_1,m)\otimes(w,u)\big) = (w,u)$$
$$j_n\big((v_2,m)\otimes(w,u)^{\otimes n-1}\big) = C_n(w,(n-1)u)$$

where $C_3 = 1$, $C_n = (-1)^{n-1}(n-3)C_{n-1}$, and $j_n = 0$ when evaluated on any element of $(L\oplus M)^{\otimes n}$ that is not listed above. Then $(L \oplus M, j_n)$ is an L_∞ algebra.

The proof is an immediate consequence of the previous computations and theorem 6.1.1.

It is worth noting that this is another example of an L_∞ structure that is a strict Lie algebra in degree 0. Finding an example that is not strictly Lie in degree 0 remains an interesting open question.

Chapter 7

Homomorphisms of L_∞ Modules

Central to Lie algebra representation theory are structure-preserving maps (homomorphisms) between Lie modules. We aim here to investigate the homotopy theoretic version of these maps in the form of L_∞ module homomorphisms. In order to construct such homomorphisms, we must first analyze classical properties that they will mirror. Our strategy is to accomplish this through the relationship among a Lie algebra L, an L-module M, and the direct sum vector space $L \oplus M$.

7.1 Relationship Between Homomorphisms of Lie Algebras and Modules

As explored in the previous chapter, a Lie algebra L and an L-module M form a canonical Lie algebra on $L \oplus M$. Furthermore, if we define another L-module, M', another canonical Lie algebra is formed on $L \oplus M'$. The following theorems illustrate a basic relationship between Lie module homomorphisms $\psi : M \to M'$ (in the sense of definition 3.3.2) and Lie algebra homomorphisms $\varphi : L \oplus M \to L \oplus M'$ (in the sense of definition 3.1.3).

Theorem 7.1.1. *Suppose L is a Lie algebra and M, M' are L-modules. Let $\psi : M \to M'$ be an L-module homomorphism. Define $\varphi : L \oplus M \to L \oplus M'$ by $\varphi(x, m) =$*

$(x, \psi(m))$. *Then* φ *is a Lie algebra homomorphism.*

Proof. Let $(x_1, m_1), (x_2, m_2) \in L \oplus M'$. Then

$$\begin{aligned}
\varphi([(x_1, m_1), (x_2, m_2)]) &= \varphi([x_1, x_2], x_1 \cdot m_2 - x_2 \cdot m_1) \\
&= ([x_1, x_2], \psi(x_1 \cdot m_2 - x_2 \cdot m_1)) \\
&= ([x_1, x_2], \psi(x_1 \cdot m_2) - \psi(x_2 \cdot m_1)) \\
&= ([x_1, x_2], x_1 \cdot \psi(m_2) - x_2 \cdot \psi(m_1)) \\
&= [(x_1, \psi(m_1)), (x_2, \psi(m_2))] \\
&= [\varphi(x_1, m_1), \varphi(x_2, m_2)]
\end{aligned}$$

□

Theorem 7.1.2. *Suppose* L *is a Lie algebra and* M, M' *are* L*-modules. Let* $\varphi : L \oplus M \to L \oplus M'$ *be a Lie algebra homomorphism such that* $\varphi = id_L \times \psi$ *where* $\psi : M \to M'$. *Then* ψ *is a Lie module homomorphism.*

Proof. Define $\iota : M \to L \oplus M$ by $\iota(m) = 0 \oplus m = (0, m)$. We note that $\psi = \pi_2 \circ \varphi \circ \iota$ where π_2 is the projection map since

$$(\pi_2 \circ \varphi \circ \iota)(m) = \pi_2(\varphi(0, m)) = \pi_2(0, \psi(m)) = \psi(m) \ \forall m \in M$$

Hence

$$\begin{aligned}
\psi(x \cdot m) &= (\pi_2 \circ \varphi \circ \iota)(x \cdot m) \\
&= (\pi_2 \circ \varphi)(\iota(x \cdot m)) \\
&= \pi_2(\varphi(0, x \cdot m)) \\
&= \pi_2(\varphi([(x, 0), (0, m)])) \\
&= \pi_2([\varphi(x, 0), \varphi(0, m)]) \\
&= \pi_2([(x, 0), (0, \psi(m)]) \\
&= \pi_2(0, x \cdot \psi(m) - 0) \\
&= x \cdot \psi(m) \ \forall x \in L, \ m \in M
\end{aligned}$$

□

These elementary results provide significant motivation for building a structure-preserving map between L_∞ modules in the homotopy context. Any candidate for such a map should mirror the behavior exhibited here in the classical case. That is, homomorphisms between L_∞ structures are well-understood and definition 4.7.1 may be applied to the two canonical L_∞ algebras $L \oplus M$ and $L \oplus M'$. Any candidate for an L_∞ module homomorphism should be similarly related to this L_∞ algebra homomorphism.

7.2 Homotopy Context

Given the fact that the homotopy theoretic versions of Lie algebras L, Lie modules M, canonical Lie algebras $L \oplus M$, and homomorphisms among the algebras are well-defined, we may proceed to build a homomorphism between L_∞ modules by mirroring the classical relationships.

Unless noted otherwise, (L, l_n) will denote an L_∞ structure. (M, k_n) and (M', k'_n) will denote L_∞ modules with $(L \oplus M, j_n)$ and $(L \oplus M', j'_n)$ their canonical L_∞ algebras respectively.

We will refer to an L_∞ algebra homomorphism $F : L \oplus M \to L \oplus M'$ by structure maps $\{f_n : (L \oplus M)^{\otimes n} \to L \oplus M'\}$, and our candidate for an L_∞ module homomorphism $H : M \to M'$ by structure maps $\{h_n : L^{\otimes n-1} \otimes M \to M'\}$.

With many types of maps being utilized simultaneously, it is beneficial to remember the following degrees:

- deg $l_n = n - 2$.
- deg $k_n = n - 2$.
- deg $f_n = n - 1$.
- deg $h_n = n - 1$.

We start with the relationships between Lie algebra homomorphisms and L-module homomorphisms given in theorems 7.1.1 and 7.1.2:

$$\varphi(x, m) = (x, \psi(m))$$

In this notation, we may view φ and ψ as degree 0 homomorphisms, say $\varphi = f_1$ and $\psi = h_1$. Hence

$$f_1(x, m) = (x, h_1(m)).$$

The compatibility between h_1 and f_1 provides significant motivation for the construction of higher homotopies in this setting. In general, f_n and h_n should be compatible.

7.3 First Homotopy

We now explore the generalized setting where (L, l_n) is an L_∞ algebra, (M, k_n) and (M', k'_n) are L-modules, and $(L \oplus M, j_n)$ and $(L \oplus M', j'_n)$ are the induced L_∞ algebras.

h_1 being a Lie module homomorphism entails commutativity of the following diagram:

$$\begin{tikzcd} L \otimes M \arrow[r, "id \otimes h_1"] \arrow[d, "k_2"] & L \otimes M' \arrow[d, "k'_2"] \\ M \arrow[r, "h_1"] & M' \end{tikzcd}$$

If the above diagram only commutes up to homotopy, we introduce our first homotopy, h_2 as follows:

$$\begin{tikzcd} L \otimes M \arrow[r, "id \otimes h_1"] \arrow[d, "k_2"] \arrow[dr, dashed, "h_2"] & L \otimes M' \arrow[d, "k'_2"] \\ M \arrow[r, "h_1"] & M' \end{tikzcd}$$

So if k_1 and k'_1 are graded differentials on M and M' respectively, then $k'_1 h_2 + h_2 k_1 = h_1 k_2 - k'_2(id \otimes h_1)$ (by definition of chain homotopy). Evaluated on an element

$(x, m) \in L \oplus M$, this means :

$$k_1' h_2(x,m) + h_2(l_1(x),m) + (-1)^x h_2(x,k_1(m)) - h_1 k_2(x,m) + k_2'(x,h_1(m)) = 0 \tag{7.3.1}$$

We now shift our attention to $f_2 : (L \oplus M) \otimes (L \oplus M) \to L \oplus M'$. Motivation from the classical Lie setting dictates that f_2 should be compatible with h_2 in a manner that permits the L_∞ algebra homomorphism relation (definition 4.7.1) to hold on 2 inputs, $(x_1, m_1),\ (x_2, m_2) \in L \oplus M$. That is, the following equation should hold true:

$$\begin{aligned} j_1' f_2((x_1,m_1),(x_2,m_2)) + j_2'(f_1(x_1,m_1), f_1(x_2,m_2)) + f_2(j_1(x_1,m_1),(x_2,m_2)) \\ -(-1)^{(x_1,m_1)(x_2,m_2)} f_2(j_1(x_2,m_2),(x_1,m_1)) - f_1 j_2((x_1,m_1),(x_2,m_2)) = 0 \end{aligned} \tag{7.3.2}$$

We also observe that f_2 is of degree $2-1=1$. This limitation encourages a simple definition for f_2, as each coordinate must share this degree. The first coordinate of $f_2\big((x_1,m_1),(x_2,m_2)\big)$, located in L, must combine x_1 and x_2 using a degree 1 map. The second coordinate, located in M', ought to combine x_1, x_2, m_1, and m_2 using a degree 1 map as well, creating an expected location for h_2. A natural choice for f_2 that generalizes the properties observed in the classical Lie setting is:

$$f_2\big((x_1,m_1),(x_2,m_2)\big) := \big(\ 0\ ,\ h_2(x_1,m_2) - (-1)^{x_2 m_1} h_2(x_2,m_1)\ \big) \tag{7.3.3}$$

We also note that the second coordinate here adheres to the same pattern as the second coordinate of the j_n maps in the L_∞ structure $L \oplus M$ (theorem 6.1.1), using h_n in lieu of k_n.

Combining equations 7.3.2 and 7.3.3, we find:

$$\begin{aligned} &\Big(\quad 0 \quad , k_1' h_2(x_1,m_2) - (-1)^{x_2 m_1} k_1' h_2(x_2,m_1)\Big) \\ &+\Big(l_2(x_1,x_2), k_2'(x_1,h_1(m_2) - (-1)^{x_2 h_1(m_1)} k_2'(x_2,h_1(m_1))\Big) \\ &+\Big(\quad 0 \quad , h_2(l_1(x_1),m_2) - (-1)^{x_2 k_1(m_1)} h_2(x_2,k_1(m_1))\Big) \\ -(-1)^{(x_1,m_1)(x_2,m_2)} &\Big(\quad 0 \quad , h_2(l_1(x_2),m_1) - (-1)^{x_1 k_1(m_2)} h_2(x_1,k_1(m_2))\Big) \\ &-\Big(l_2(x_1,x_2), h_1 k_2(x_1,m_2) - (-1)^{x_2 m_1} h_1 k_2(x_2,m_1)\Big) \\ &= \Big(\ 0\ ,\ 0\ \Big) \end{aligned}$$

In the first coordinate, we have $0 + l_2(x_1, x_2) + 0 - 0 - l_2(x_1, x_2) = 0$.

In the second coordinate, we group together terms containing (x_1, m_2) and (x_2, m_1) respectively, and find:

$$\begin{aligned} k_1' h_2(x_1, m_2) + k_2'(x_1, h_1(m_2)) + k_2(l_1(x_1), m_2) + \\ (-1)^{x_1 m_2 + x_1 m_2 + x_1} h_2(x_1, k_1(m_2)) - h_1 k_2(x_1, m_2) \\ -(-1)^{x_2 m_1} \Big[k_1' h_2(x_2, m_1) + k_2'(x_2, h_1(m_1)) + k_2(l_1(x_2), m_1) \\ + (-1)^{x_2} h_2(x_2, k_1(m_1)) - h_1 k_2(x_2, m_1) \Big] \\ = 0 \end{aligned}$$

which holds true by applying equation 7.3.1 twice. Hence, equation 7.3.2 holds true, implying that the relation observed in 7.3.1 is a natural generalization of the behavior observed in the classical Lie case up to the first homotopy. As we investigate higher homotopies, this behavior will validate any relations we derive.

7.4 Key Reasoning for Higher Homotopies

Before attempting to determine the relation that the first higher homotopy, h_3, must satisfy, we consider what would happen if we were not aware of the relation in equation 7.3.1 that h_2 must satisfy.

Our reasoning in determining equation 7.3.3 would not change. We would then require a relation with h_2 that permits equation 7.3.2 to hold true. Once again, the first coordinate here would simply sum up to 0. The second coordinate would simplify to

$$\begin{aligned} k_1' h_2(x_1, m_2) + k_2'(x_1, h_1(m_2)) + k_2(l_1(x_1), m_2) + \\ (-1)^{x_1 m_2 + x_1 m_2 + x_1} h_2(x_1, k_1(m_2)) - h_1 k_2(x_1, m_2) \\ -(-1)^{x_2 m_1} \Big[k_1' h_2(x_2, m_1) + k_2'(x_2, h_1(m_1)) + k_2(l_1(x_2), m_1) \\ + (-1)^{x_2} h_2(x_2, k_1(m_1)) - h_1 k_2(x_2, m_1) \Big] \end{aligned}$$

We would then require this expression to be equal to 0. We have two groupings of the same relation on two different inputs ((x_1, m_2) and (x_2, m_1)). If this expression equals 0, then equation 7.3.2 holds. This would build the h_2 relation from the ground up without the knowledge that stemmed from a simple chain homotopy definition (a luxury we will not have for h_3), and is the type of logic we will employ in investigating the next higher homotopies.

7.5 h_3 Relation

We now execute the same type of strategy in constructing the first homotopy; h_3 ought to be compatible with f_3 in a manner such that the f_3 relation (L_∞ algebra homomorphism on 3 inputs) holds:

$$
\begin{aligned}
&j'_1 f_3\big((x_1,m_1),(x_2,m_2),(x_3,m_3)\big)\\
+(-1)^{(x_1,m_1)}&j'_2\big(f_1(x_1,m_1), f_2((x_2,m_2),(x_3,m_3)\big)\\
+(-1)^{(x_3,m_3)[(x_1,m_1)+(x_2,m_2)+1]}&j'_2\big(f_1(x_3,m_3), f_2((x_1,m_1),(x_2,m_2))\big)\\
-(-1)^{(x_2,m_2)[(x_1,m_1)+1]}&j'_2\big(f_1(x_2,m_2), f_2((x_1,m_1),(x_3,m_3))\big)\\
+&j'_3\big(f_1(x_1,m_1), f_1(x_2,m_2), f_1(x_3,m_3)\big)\\
-&f_3\big(j_1(x_1,m_1),(x_2,m_2),(x_3,m_3)\big)\\
+(-1)^{(x_1,m_1)(x_2,m_2)}&f_3\big(j_1(x_2,m_2),(x_1,m_1),(x_3,m_3)\big)\\
-(-1)^{(x_3,m_3)((x_1,m_1)+(x_2,m_2))}&f_3\big(j_1(x_3,m_3),(x_1,m_1),(x_2,m_2)\big)\\
-&f_2\big(j_2((x_1,m_1),(x_2,m_2)),(x_3,m_3)\big)\\
+(-1)^{(x_2,m_2)(x_3,m_3)}&f_2\big(j_2((x_1,m_1),(x_3,m_3)),(x_2,m_2)\big)\\
-(-1)^{(x_1,m_1)((x_2,m_2)+(x_3,m_3))}&f_2\big(j_2((x_2,m_2),(x_3,m_3)),(x_1,m_1)\big)\\
-&f_1 j_3\big((x_1,m_1),(x_2,m_2),(x_3,m_3)\big)\\
=&\ 0
\end{aligned}
\tag{7.5.1}
$$

We do not know what relation h_3 is governed by , but we know that the construction of f_3 should involve h_3. Once again, we observe that f_3 is of degree $3-1=2$. Following similar reasoning in the construction of f_2, a natural choice for f_3 would be:

$$f_3\big((x_1,m_1),(x_2,m_2),(x_3,m_3)\big) = \Big(\ 0\ ,\ h_3(x_1,x_2,m_3) - (-1)^{x_3m_2}h_3(x_1,x_3,m_2) \\ + (-1)^{m_1(x_2+x_3)}h_3(x_2,x_3,m_1)\Big) \qquad (7.5.2)$$

Combining equation 7.5.2 with the left hand side of equation 7.5.1, we get:

$$\begin{aligned}
& j_1'\Big(\ 0\ , h_3(x_1,x_2,m_3) - (-1)^{m_2x_3}h_3(x_1,x_3,m_2)) \\
& \quad + (-1)^{m_1(x_2+x_3)}h_3(x_2,x_3,m_1)\Big) \\
& +(-1)^{(x_1,m_1)}j_2'\Big((x_1,h_1(m_1)),\ (\ 0\ , h_2(x_2,m_3) - (-1)^{x_3m_2}h_2(x_3,m_2))\Big) \\
& +(-1)^{(x_3,m_3)[(x_1,m_1)+(x_2,m_2)+1]}j_2'\Big((x_3,h_1(m_3)),\ (\ 0\ , h_2(x_1,m_2) - (-1)^{x_2m_1}h_2(x_2,m_1))\Big) \\
& -(-1)^{(x_2,m_2)[(x_1,m_1)+1]}j_2'\Big((x_2,h_1(m_2)),\ (\ 0\ , h_2(x_1,m_3) - (-1)^{x_3m_1}h_2(x_3,m_1))\Big) \\
& +j_3'\Big((x_1,h_1(m_1)),(x_2,h_1(m_2)),(x_3,h_1(m_3))\Big) \\
& -f_3\Big((l_1(x_1),k_1(m_1)),(x_2,m_2),(x_3,m_3)\Big) \\
& +(-1)^{(x_1,m_1)(x_2,m_2)}f_3\Big((l_1(x_2),k_1(m_2)),(x_1,m_1),(x_3,m_3)\Big) \\
& -(-1)^{(x_3,m_3)[(x_1,m_1)+(x_2,m_2)]}f_3\Big((l_1(x_3),k_1(m_3)),(x_1,m_1),(x_2,m_2)\Big) \\
& -f_2\Big(\big(l_2(x_1,x_2),k_2(x_1,m_2) - (-1)^{x_2m_1}k_2(x_2,m_1)\big),\big(x_3,m_3\big)\Big) \\
& +(-1)^{(x_2,m_2)(x_3,m_3)}f_2\Big(\big(l_2(x_1,x_3),k_2(x_1,m_3) - (-1)^{x_3m_1}k_2(x_3,m_1)\big),\big(x_2,m_2\big)\Big) \\
& -(-1)^{(x_1,m_1)[(x_2,m_2)+(x_3,m_3)]}f_2\Big(\big(l_2(x_2,x_3),k_2(x_2,m_3) - (-1)^{x_3m_2}k_2(x_3,m_2)\big),\big(x_1,m_1\big)\Big) \\
& -f_1\Big(l_3(x_1,x_2,x_3),k_3(x_1,x_2,m_3) - (-1)^{x_3m_2}k_3(x_1,x_3,m_2) \\
& \quad + (-1)^{m_1(x_2+x_3)}k_3(x_2,x_3,m_1)\Big)
\end{aligned}$$

This expression simplifies to

$$
\begin{aligned}
&\Big(\ 0\ ,\ k_1' h_3(x_1,x_2,m_3) - (-1)^{m_2x_3} k_1' h_3(x_1,x_3,m_2) \\
&\qquad + (-1)^{m_1(x_2+x_3)} k_1' h_3(x_2,x_3,m_1)\ \Big) \\
+(-1)^{(x_1,m_1)} &\Big(\ 0\ ,\ k_2'\big(x_1, h_2(x_2,m_3) - (-1)^{x_3m_2} h_2(x_3,m_2)\big) - 0\ \Big) \\
+(-1)^{(x_3,m_3)[(x_1,m_1)+(x_2,m_2)+1]} &\Big(\ 0\ ,\ k_2'\big(x_3, h_2(x_1,m_2) - (-1)^{x_2m_1} h_2(x_2,m_1)\big) - 0\ \Big) \\
-(-1)^{(x_2,m_2)[(x_1,m_1)+1]} &\Big(\ 0\ ,\ k_2'\big(x_2, h_2(x_1,m_3) - (-1)^{x_3m_1} h_2(x_3,m_1)\big) - 0\ \Big) \\
+&\Big(\ l_3(x_1,x_2,x_3)\ ,\ k_3'\big(x_1,x_2,h_1(m_3)\big) \\
&\qquad - (-1)^{x_3m_2} k_3'\big(x_1,x_3,h_1(m_2)\big) \\
&\qquad + (-1)^{m_1(x_2+x_3)} k_3'\big(x_2,x_3,h_1(m_1)\big)\ \Big) \\
-&\Big(\ 0\ ,\ h_3\big(l_1(x_1),x_2,m_3\big) - (-1)^{m_2x_3} h_3\big(l_1(x_1),x_3,m_2\big) \\
&\quad + (-1)^{(m_1+1)(x_2+x_3)} h_3\big(x_2,x_3,k_1(m_1)\big)\ \Big) \\
+(-1)^{(x_1,m_1)(x_2,m_2)} &\Big(\ 0\ ,\ h_3\big(l_1(x_2),x_1,m_3\big) - (-1)^{m_1x_3} h_3\big(l_1(x_2),x_3,m_1\big) + \\
&\qquad (-1)^{(m_2+1)(x_1+x_3)} h_3\big(x_1,x_3,k_1(m_2)\big)\ \Big) \\
-(-1)^{(x_3,m_3)[(x_1,m_1)+(x_2,m_2)]} &\Big(\ 0\ ,\ h_3\big(l_1(x_3),x_1,m_2\big) - (-1)^{m_1x_2} h_3\big(l_1(x_3),x_2,m_1\big) + \\
&\qquad (-1)^{(m_3+1)(x_1+x_2)} h_3\big(x_1,x_2,k_1(m_3)\big)\ \Big) \\
-&\Big(\ 0\ ,\ h_2\big(l_2(x_1,x_2),m_3\big) - (-1)^{x_3(x_1+m_2)} h_2\big(x_3, k_2(x_1,m_2) \\
&\qquad - (-1)^{x_2m_1} k_2(x_2,m_1)\big)\ \Big) \\
+(-1)^{(x_2,m_2)(x_3,m_3)} &\Big(\ 0\ ,\ h_2\big(l_2(x_1,x_3),m_2\big) - (-1)^{x_2(x_1+m_3)} h_2\big(x_2, k_2(x_1,m_3) \\
&\qquad - (-1)^{x_3m_1} k_2(x_3,m_1)\big)\ \Big) \\
-(-1)^{(x_1,m_1)[(x_2,m_2)+(x_3,m_3)]} &\Big(\ 0\ ,\ h_2\big(l_2(x_2,x_3),m_1\big) - (-1)^{x_1(x_2+m_3)} h_2\big(x_1, k_2(x_2,m_3) \\
&\qquad - (-1)^{x_3m_2} k_2(x_3,m_2)\big)\ \Big) \\
-&\Big(\ l_3(x_1,x_2,x_3)\ ,\ h_1k_3(x_1,x_2,m_3) \\
&\qquad - (-1)^{x_3m_2} h_1k_3(x_1,x_3,m_2) \\
&\qquad + (-1)^{m_1(x_2+x_3)} h_1k_3(x_2,x_3,m_1)\ \Big)
\end{aligned}
\tag{7.5.3}
$$

Remark 7.5.1. $|(x_i, m_j)| = |x_i| = |m_j|$

Remark 7.5.2. $|x_i| + |m_j| = |x_j| + |m_i|$

To satisfy equation 7.5.1, expression 7.5.3 must equal $(0, 0)$. It is clear that the first coordinate sums up to 0. Setting the second coordinate equal to 0 should reveal what relation h_3 is governed by in a manner similar to the way the h_2 relation appeared twice in the previous section.

In the previous 2-input case, we were able to group all terms involving (x_1, m_2) and all terms involving (x_2, m_1) and found two copies of the same relation, which was coincidentally the h_2 relation. We would expect to encounter the same type of behavior on strings of 3 inputs. There are multiple expressions that repeat with inputs (x_1, x_2, m_3), (x_1, x_3, m_2), or (x_2, x_3, m_1), up to some sign. We thus place all elements of the second coordinate of expression 7.5.3 into one of three groups:

$$\begin{aligned}
k_1' h_3(x_1, x_2, m_3) & \qquad\qquad (x_1, x_2, m_3 \text{ terms}) \\
+k_3'\big(x_1, x_2, h_1(m_3)\big) & \\
-h_3\big(l_1(x_1), x_2, m_3\big) & \\
+(-1)^{x_1 x_2} h_3\big(l_1(x_2), x_1, m_3\big) & \\
-(-1)^{x_1 + x_2} h_3\big(x_1, x_2, k_1(m_3)\big) & \\
-h_2\big(l_2(x_1, x_2), m_3\big) & \\
-h_1 k_3(x_1, x_2, m_3) & \\
+(-1)^{x_1} k_2'\big(x_1, h_2(x_2, m_3)\big) & \\
-(-1)^{x_2(x_1+1)} k_2'\big(x_2, h_2(x_1, m_3)\big) & \\
-(-1)^{x_1 x_2} h_2\big(x_2, k_2(x_1, m_3)\big) & \\
+h_2\big(x_1, k_2(x_2, m_3)\big) &
\end{aligned}$$

$$
\begin{aligned}
&-(-1)^{x_3m_2}k_1'h_3(x_1,x_3,m_2) && (x_1,x_3,m_2 \text{ terms})\\
&-(-1)^{x_3m_2}k_3'\big(x_1,x_3,h_1(m_2)\big)\\
&+(-1)^{x_3m_2}h_3\big(l_1(x_1),x_3,m_2\big)\\
&-(-1)^{x_3m_2+x_1x_3}h_3\big(l_1(x_3),x_1,m_2\big)\\
&+(-1)^{x_3m_2+x_1+x_3}h_3\big(x_1,x_3,k_1(m_2)\big)\\
&+(-1)^{x_3m_2}h_2\big(l_2(x_1,x_3),m_2\big)\\
&+(-1)^{x_3m_2}h_1k_3(x_1,x_3,m_2)\\
&-(-1)^{x_3m_2+x_1}k_2'\big(x_1,h_2(x_3,m_2)\big)\\
&+(-1)^{x_3m_2+x_3(x_1+1)}k_2'\big(x_3,h_2(x_1,m_2)\big)\\
&+(-1)^{x_3m_2+x_1x_3}h_2\big(x_3,k_2(x_1,m_2)\big)\\
&-(-1)^{x_3m_2}h_2\big(x_1,k_2(x_3,m_2)\big)
\end{aligned}
$$

$$
\begin{aligned}
&(-1)^{m_1(x_2+x_3)}k_1'h_3(x_2,x_3,m_1) && (x_2,x_3,m_1 \text{ terms})\\
&+(-1)^{m_1(x_2+x_3)}k_3'\big(x_2,x_3,h_1(m_1)\big)\\
&-(-1)^{m_1(x_2+x_3)}h_3\big(l_1(x_2),x_3,m_1\big)\\
&+(-1)^{m_1(x_2+x_3)+x_2x_3}h_3\big(l_1(x_3),x_2,m_1\big)\\
&-(-1)^{m_1(x_2+x_3)+x_2+x_3}h_3\big(x_2,x_3,k_1(m_1)\big)\\
&-(-1)^{m_1(x_2+x_3)}h_2\big(l_2(x_2,x_3),m_1\big)\\
&-(-1)^{m_1(x_2+x_3)}h_1k_3(x_2,x_3,m_1)\\
&+(-1)^{m_1(x_2+x_3)+x_2}k_2'\big(x_2,h_2(x_3,m_1)\big)\\
&-(-1)^{m_1(x_2+x_3)+x_3(x_2+1)}k_2'\big(x_3,h_2(x_2,m_1)\big)\\
&-(-1)^{m_1(x_2+x_3)+x_2x_3}h_2\big(x_3,k_2(x_2,m_1)\big)\\
&+(-1)^{m_1(x_2+x_3)}h_2\big(x_2,k_2(x_3,m_1)\big)
\end{aligned}
$$

As expected, all three groups follow the same pattern on different strings of inputs. This is more apparent after factoring out a common sign of $-(-1)^{x_2x_3}$ in the second grouping, and $(-1)^{m_1(x_2+x_3)}$ in the third. Requiring all three groups to sum up to zero yields the h_3 relation as follows:

$$\begin{aligned}k'_1h_3(x,y,m) + k'_3\big(x,y,h_1(m)\big) - h_3\big(l_1(x),y,m\big) + (-1)^{xy}h_3\big(l_1(y),x,m\big)\\ -(-1)^{x+y}h_3\big(x,y,k_1(m)\big) - h_2\big(l_2(x,y),m\big) - h_1k_3(x,y,m) + (-1)^{x}k'_2\big(x,h_2(y,m)\big)\\ -(-1)^{y(x+1)}k'_2\big(y,h_2(x,m)\big) - (-1)^{xy}h_2\big(y,k_2(x,m)\big) + h_2\big(x,k_2(y,m)\big)\\ =0\end{aligned}$$

$\forall x, y \in L, m \in M$

7.6 Higher Homotopies

After examining the previous relations on 2 and 3 inputs respectively, a rather simple pattern evolves, up to some signs. It would appear that all terms involving h_jk_i contain an extra sign of $(-1)^{i(j-1)+1}$ stemming from the L_∞ algebra homomorphism definition (4.7.1). Also considering the pattern that emerges from each k'_rh_s term, we have a likely natural description of an L_∞ module homomorphism:

Definition 7.6.1. Let (L, l_i) be an L_∞ algebra and $(M, k_i), (M', k'_i)$ be two L-modules. An L_∞ *module homomorphism* from M to M' is a collection $\{h_n : L^{\otimes(n-1)} \otimes M \to M'\}$ of skew-symmetric multilinear maps of degree $n-1$ such that

$$\begin{aligned}\sum_{i+j=n+1}\sum_{\sigma}\chi(\sigma)(-1)^{i(j-1)+1}h_j(k_i(\xi_{\sigma(1)},\cdots,\xi_{\sigma(i)}),\xi_{\sigma(i+1)},\cdots,\xi_{\sigma(n)})+\\ \sum_{r+s=n+1}\sum_{\tau}\chi(\tau)(-1)^{(s-1)(\sum_{t=1}^{n-s}x_{\tau(t)})}k'_r\big(x_{\tau(1)},\cdots,x_{\tau(n-s)},h_s(x_{\tau(n-s+1)},\cdots,x_{\tau(n-1)},m)\big)\\ =0\end{aligned}$$

Where σ runs through all $(i, n-i)$-unshuffles and τ runs through all $(n-s, s-1)$-unshuffles .

A few remarks are in order.

Remark 7.6.2. We define $k_i = l_i$ when evaluated strictly on elements in L. This is precisely the same requirement in the definition (4.6.1) of an L_∞ module.

Remark 7.6.3. Since we have $h_n : L^{\otimes(n-1)} \otimes M \to M'$, we must utilize skew-symmetry whenever the 'm' element is not in the n^{th} position. For example, $h_3(x, m, y) = (-1)^{ym} h_3(x, y, m)$.

Following this definition, the h_4 relation is constructed as follows:

$$
\begin{aligned}
k_1' h_4(x,y,z,m) \qquad & - h_1 k_4(x,y,z,m) \\
+k_2'(x, h_3(y,z,m)) \qquad & + h_2(l_3(x,y,z), m) \\
-(-1)^{xy} k_2'(y, h_3(x,z,m)) \qquad & - (-1)^{zm} h_2(k_3(x,y,m), z) \\
+(-1)^{z(x+y)} k_2'(z, h_3(x,y,m)) \qquad & + (-1)^{y(z+m)} h_2(k_3(x,z,m), y) \\
+k_3'(x,y,h_2(z,m)) \qquad & - (-1)^{x(y+z+m)} h_2(k_3(y,z,m), x) \\
-(-1)^{yz+x+z} k_3'(x,z,h_2(y,m)) \qquad & - h_3(l_2(x,y), z, m) \\
+(-1)^{(x+1)(y+z)} k_3'(y,z,h_2(x,m)) \qquad & + (-1)^{yz} h_3(l_2(x,z), y, m) \\
+k_4'(x,y,z,h_1(m)) \qquad & - (-1)^{m(y+z)} h_3(k_2(x,m), y, z) \\
& + (-1)^{xy+m(x+z)} h_3(k_2(y,m), x, z) \\
& - (-1)^{x(y+z)} h_3(l_2(y,z), x, m) \\
& - (-1)^{(x+y)(z+m)} h_3(k_2(z,m), x, y) \\
& + h_4(l_1(x), y, z, m) \\
& - (-1)^{xy} h_4(l_1(y), x, z, m) \\
& + (-1)^{z(x+y)} h_4(l_1(z), x, y, m) \\
& - (-1)^{m(x+y+z)} h_4(k_1(m), x, y, z) \\
& = 0
\end{aligned}
$$

We consider remark 7.6.3 here to view this as in a more familiar form similar to equation 7.3.1:

$$
\begin{aligned}
k_1'h_4(x,y,z,m) \qquad & - h_1k_4(x,y,z,m) \\
+k_2'(x,h_3(y,z,m)) \qquad & + h_2(l_3(x,y,z),m) \\
-(-1)^{xy}k_2'(y,h_3(x,z,m)) \qquad & - (-1)^{z(x+y)+z}h_2(z,k_3(x,y,m)) \\
+(-1)^{z(x+y)}k_2'(z,h_3(x,y,m)) \qquad & + (-1)^{xy+y}h_2(y,k_3(x,z,m)) \\
+k_3'(x,y,h_2(z,m)) \qquad & - (-1)^{x}h_2(x,k_3(y,z,m)) \\
-(-1)^{yz+x+z}k_3'(x,z,h_2(y,m)) \qquad & - h_3(l_2(x,y),z,m) \\
+(-1)^{(x+1)(y+z)}k_3'(y,z,h_2(x,m)) \qquad & + (-1)^{yz}h_3(l_2(x,z),y,m) \\
+k_4'(x,y,z,h_1(m)) \qquad & - (-1)^{x(y+z)}h_3(y,z,k_2(x,m)) \\
& + (-1)^{yz}h_3(x,z,k_2(y,m)) \\
& - (-1)^{x(y+z)}h_3(l_2(y,z),x,m) \\
& - h_3(x,y,k_2(z,m)) \\
& + h_4(l_1(x),y,z,m) \\
& - (-1)^{xy}h_4(l_1(y),x,z,m) \\
& + (-1)^{z(x+y)}h_4(l_1(z),x,y,m) \\
& - h_4(x,y,z,k_1(m)) \\
& = 0
\end{aligned}
$$

The $h_5, h_6, \cdots$ relations follow similar patterns.

Given the relationship between Lie module and Lie algebra homomorphisms observed in the classical Lie case and lower homotopy contexts, this definition for L_∞ module homomorphisms is valid if the same relationship holds for all higher homotopies. That is, all higher homotopies should adhere to the following.

Theorem 7.6.4. *Let (L, l_i) be an L_∞ algebra and $(M, k_i), (M', k_i')$ be two L-modules with $(L \oplus M, j_i)$, $(L \oplus M', j_i')$ their canonical L_∞ structures. Let $H = \{h_n : L^{\otimes(n-1)} \otimes M \to M'\}$ be an L-module homomorphism. Let $F = \{f_n : (L \oplus M)^{\otimes n} \to L \oplus M'\}$ be* *defined by*

$$f_1(x,m) = (x, h_1(m))$$

$$f_n\big((x_1,m_1),\cdots,(x_n,m_n)\big) = \Big(0, \sum_{i=1}^{n}(-1)^{n-i}(-1)^{m_i\sum_{k=i+1}^{n}x_k}h_n(x_1\otimes\cdots \\ \cdots\otimes\hat{x}_i\otimes\cdots\otimes x_n, m_i)\Big)$$

for $n\geq 2$, where $\hat{x}_i$ means omit x_i. Then F is an L_∞ algebra homomorphism.

Proof. We aim to prove that F is an L_∞ algebra homomorphism by showing that the following relation holds:

$$\sum_{r+s=n+1}\sum_{\sigma}\chi(\sigma)(-1)^{s(r-1)+1}f_r\Big(j_s\big((x_{\sigma(1)},m_{\sigma(1)}),\cdots,(x_{\sigma(s)},m_{\sigma(s)})\big) \\ ,(x_{\sigma(s+1)},m_{\sigma(s+1)}),\cdots,(x_{\sigma(n)},m_{\sigma(n)})\Big) + \\ \sum_{\substack{1\leq t\leq n\\ i_1+\cdots+i_t=n\\ i_1\leq\cdots\leq i_t}}\sum_{\tau}\lambda\, j'_t\Big(f_{i_1}\big((x_{\tau(1)},m_{\tau(1)}),\cdots,(x_{\tau(i_1)},m_{\tau(i_1)})\big), f_{i_2}\big((x_{\tau(i_1+1)},m_{\tau(i_1+1)}),\cdots \\ \cdots(x_{\tau(i_1+i_2)},m_{\tau(i_1+i_2)})\big),\cdots,f_{i_t}\big((x_{\tau(i_1+\cdots+i_{t-1}+1)},m_{\tau(i_1+\cdots+i_{t-1}+1)}),\cdots,(x_{\tau(i_t)},m_{\tau(i_t)})\big)\Big) \\ = 0$$

Where σ runs through all $(s,n-s)$-unshuffles and τ runs through all $(i_1,\cdots,i_t)$-unshuffles satisfying $\tau(i_1+\cdots+i_{l-1}+1) < \tau(i_1+\cdots+i_l+1)$ if $i_l = i_{l+1}$, and $\lambda = \chi(\tau)(-1)^{\frac{t(t-1)}{2}+\sum_{k=1}^{t-1}i_k(t-k)}\nu$ with ν representing the Koszul sign that results from evaluating $(f_{i_1}\otimes f_{i_2}\otimes\cdots\otimes f_{i_t})$ on $(\xi_{\tau(1)}\otimes\xi_{\tau(2)}\otimes\cdots\otimes\xi_{\tau(n)})$.

A few observations are in order:

- Each element in the above relation lies in $L\oplus M'$. The first coordinate fairly simply sums up to 0, as the only nonzero first coordinate element in the first double sum is $-l_n(x_1,\cdots,x_n)$ and the only nonzero first coordinate element in the second double sum is $l_n(x_1,\cdots,x_n)$. The second coordinate ought to have n copies of definition 7.6.1 as previously observed for $n=2$ and $n=3$. This is where the difficulty of the proof lies.

- In the second coordinate, the $f_r j_s$ terms produce $h_i k_j$ terms, and the $j'_t(f_{i_1} \otimes \cdots \otimes f_{i_t})$ terms produce $k'_i h_j$ terms.

First consider the $f_r j_s$ terms:

$$\sum_{r+s=n+1} \sum_{\sigma} \chi(\sigma)(-1)^{s(r-1)+1} f_r\Big(j_s\big((x_{\sigma(1)}, m_{\sigma(1)}), \cdots, (x_{\sigma(s)}, m_{\sigma(s)})\big)$$

$$, (x_{\sigma(s+1)}, m_{\sigma(s+1)}), \cdots, (x_{\sigma(n)}, m_{\sigma(n)})\Big)$$

$$= \sum_{r+s=n+1} \sum_{\sigma} \chi(\sigma)(-1)^{s(r-1)+1} f_r\Big(\big(l_s(x_{\sigma(1)}, \cdots, x_{\sigma(s)}),$$

$$\sum_{i=1}^{s}(-1)^{s-i}(-1)^{m_{\sigma(i)} \sum_{k=i+1}^{s} x_{\sigma(k)}} k_s(x_{\sigma(1)}, \cdots, \hat{x_{\sigma(i)}}, \cdots, x_{\sigma(s)}, m_{\sigma(i)})\big),$$

$$(x_{\sigma(s+1)}, m_{\sigma(s+1)}), \cdots, (x_{\sigma(n)}, m_{\sigma(n)})\Big)$$

Where $\hat{x}_i$ means omit x_i. If $r = 1$, this is

$$- f_r\Big(l_n(x_1, \cdots, x_n), \sum_{i=1}^{n}(-1)^{n-i}(-1)^{m_i \sum_{k=i+1}^{n} x_i} k_n(x_1, \cdots, \hat{x}_i, \cdots, x_n, m_i)\Big)$$

$$= -\Big(l_n(x_1, \cdots, x_n), \sum_{i=1}^{n}(-1)^{n-i}(-1)^{m_i \sum_{k=i+1}^{n} x_i} h_1 k_n(x_1, \cdots, \hat{x}_i, \cdots, x_n, m_i)\Big)$$

If $r \neq 1$, this is

$$\Big(0, \sum_{j=2}^{r}(-1)^{n-j}(-1)^{m_{\sigma(s+j-1)} \sum_{k=s+j}^{n} x_{\sigma(k)}} h_r\big(l_s(x_{\sigma(1)}, \cdots, x_{\sigma(s)}), x_{\sigma(s+1)}, \cdots$$

$$\cdots x_{\widehat{\sigma(s+j-1)}}, \cdots, x_{\sigma(n)}, m_{\sigma(s+j-1)}\big)$$

$$+ (-1)^{n-1}\ \alpha\ h_r\big(x_{\sigma(s+1)}, \cdots, x_{\sigma(n)},$$

$$\sum_{i=1}^{s}(-1)^{s-i}(-1)^{m_{\sigma(i)} \sum_{k=i+1}^{s} x_{\sigma(k)}} k_s\big(x_{\sigma(1)}, \cdots, \hat{x_{\sigma(i)}}, \cdots, x_{\sigma(s)}, m_{\sigma(i)})\big)\big)\Big)$$

where $\alpha = (-1)^{\left(s-2+x_{\sigma(1)}+\cdots+x_{\sigma(n-1)}+m_{\sigma(n)}\right)\left(x_{\sigma(s+1)}+\cdots+x_{\sigma(n)}\right)}$

Remark 7.6.5. By properties of degrees in $L \oplus M$,

$$\begin{aligned}\alpha =&(-1)^{\left(s-2+x_{\sigma(1)}+\cdots+x_{\sigma(n-2)}+x_{\sigma(n)}+m_{\sigma(n-1)}\right)\left(x_{\sigma(s+1)}+\cdots+x_{\sigma(n)}\right)}\\ =&(-1)^{\left(s-2+x_{\sigma(1)}+\cdots+x_{\sigma(n-3)}+x_{\sigma(n-1)+x_{\sigma(n)})}+m_{\sigma(n-2)}\right)\left(x_{\sigma(s+1)}+\cdots+x_{\sigma(n)}\right)}\\ &etc.\end{aligned}$$

By graded skew-symmetry, this becomes:

$$\begin{aligned}\Big(0,\sum_{j=2}^{r}(-1)^{n-j}(-1)^{m_{\sigma(s+j-1)}\sum_{k=s+j}^{n}x_{\sigma(k)}}h_r\big(l_s(x_{\sigma(1)},\cdots,x_{\sigma(s)}),x_{\sigma(s+1)},\cdots\\ \cdots x_{\sigma(\hat{s+j}-1)},\cdots,x_{\sigma(n)},m_{\sigma(s+j-1)}\big)\\ +(-1)^{n-1}h_r\big(\sum_{i=1}^{s}(-1)^{s-i}(-1)^{m_{\sigma(i)}\sum_{k=i+1}^{s}x_{\sigma(k)}}k_s(x_{\sigma(1)},\cdots,x_{\hat{\sigma(i)}},\cdots,x_{\sigma(s)},m_{\sigma(i)}),\\ x_{\sigma(s+1)},\cdots,x_{\sigma(n)}\big)\Big)\end{aligned}$$

Given that each of the above elements in the second coordinate omit one term, we find n different groups of elements in a similar fashion to what was exhibited when $n = 2,3$. That is, we can divide the above collection into groups with strings of the following terms:

$$\begin{aligned}(x_1,x_2,\cdots,x_{n-1},m_n)&\text{ terms}\\ (x_1,x_2,\cdots,x_{n-2},x_n,m_{n-1})&\text{ terms}\\ &\vdots\\ (x_2,x_3,\cdots,x_n,m_1)&\text{ terms}\end{aligned}$$

Each of these groups ought to have a Koszul sign in common. For example, all $(x_2,x_3,\cdots,x_n,m_1)$ terms should have a common factor of $(-1)^{n-1+m_1(x_2+\cdots+x_n)}$. In general, all $(x_1,x_2,\cdots,x_{\alpha-1},x_{\alpha+1},\cdots x_n,m_\alpha)$ terms should have a common factor of $(-1)^{n-\alpha+m_\alpha\sum_{i=\alpha+1}^{n}x_i}$.

Another significant observation is that each specific unshuffle provides precisely one of each different type of $(x_1,x_2,\cdots,x_{\alpha-1},x_{\alpha+1},\cdots x_n,m_\alpha)$ terms. That is, for each $1 \leq \alpha \leq n$ there is a 1-1 correspondence between $\sigma \in unsh(s,n-s)$ and

$(x_1, x_2, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots x_n, m_\alpha)$ terms, where $unsh(s, n-s)$ is the set of all $(s, n-s)$-unshuffles.

Now let σ be a fixed $(s, n-s)$-unshuffle. Let $r + s = n + 1$ and $1 \leq \alpha \leq n$ be fixed. Consider the following expression:

$$\begin{aligned}\chi(\sigma)(-1)^{s(r-1)+1} f_r\Big(j_s\big((x_{\sigma(1)}, m_{\sigma(1)}), \cdots, (x_{\sigma(s)}, m_{\sigma(s)})\big),\\ (x_{\sigma(s+1)}, m_{\sigma(s+1)}), \cdots, (x_{\sigma(n)}, m_{\sigma(n)})\Big)\end{aligned} \tag{7.6.1}$$

As a result of the 1-1 correspondence, there is precisely one term of the form $(x_1, x_2, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots x_n, m_\alpha)$ in the second coordinate of the above expression. It is formed by the omission of (x_α, m_α) through the definition of f_r or j_s. This yields 2 cases that depend on where σ places (x_α, m_α):

Case 1: σ places (x_α, m_α) outside j_s. Then $\alpha = \sigma(p)$ for some $s + 1 \leq p \leq n$. Therefore expression 7.6.1 becomes

$$\begin{aligned}&\chi(\sigma)(-1)^{s(r-1)+1} f_r\Big(j_s\big((x_{\sigma(1)}, m_{\sigma(1)}), \cdots, (x_{\sigma(s)}, m_{\sigma(s)})\big), (x_{\sigma(s+1)}, m_{\sigma(s+1)}), \cdots\\ &\qquad \cdots, (x_{\sigma(p-1)}, m_{\sigma(p-1)}), (x_\alpha, m_\alpha), (x_{\sigma(p+1)}, m_{\sigma(p+1)}), \cdots\\ &\qquad \cdots (x_{\sigma(n)}, m_{\sigma(n)})\Big)\end{aligned}$$

$$\begin{aligned}=&\chi(\sigma')(-1)^{s(r-1)+1}(-1)^{n-\alpha+m_\alpha \sum_{i=\alpha+1}^{n} x_i} f_r\Big(j_s\big((x_{\sigma(1)}, m_{\sigma(1)}), \cdots, (x_{\sigma(s)}, m_{\sigma(s)})\big),\\ &\qquad (x_{\sigma(s+1)}, m_{\sigma(s+1)}), \cdots, (x_{\sigma(p-1)}, m_{\sigma(p-1)}),\\ &\qquad (x_{\sigma(p+1)}, m_{\sigma(p+1)}), \cdots (x_{\sigma(n)}, m_{\sigma(n)}), (x_\alpha, m_\alpha)\Big)\end{aligned}$$

Where σ' is the $(s, n-s)$-unshuffle such that

$$\begin{aligned}&\sigma'\big((x_1, x_2, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots x_n, m_\alpha)\big)\\ =&(x_{\sigma(1)}, \cdots, x_{\sigma(s)}, x_{\sigma(s+1)}, \cdots, x_{\sigma(p-1)}, x_{\sigma(p+1)}, \cdots x_{\sigma(n)}, m_\alpha)\end{aligned}$$

Taking the r^{th} term of the second coordinate of f_r, we have:

$$\chi(\sigma')(-1)^{s(r-1)+1}(-1)^{n-\alpha+m_\alpha \sum_{i=\alpha+1}^{n} x_i} h_r\Big(l_s(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(s)}), x_{\sigma(s+1)}, \cdots x_{\sigma(n)}, m_\alpha\Big)$$

which is the $(x_1, x_2, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots x_n, m_\alpha)$ term corresponding to σ.

Case 2: σ places (x_α, m_α) inside j_s. Then $\alpha = \sigma(p)$ for some $1 \leq p \leq s$. Then expression 7.6.1 becomes

$$\chi(\sigma)(-1)^{s(r-1)+1} f_r\Big(j_s\big((x_{\sigma(1)}, m_{\sigma(1)}), \cdots, (x_{\sigma(p-1)}, m_{\sigma(p-1)}), (x_\alpha, m_\alpha), (x_{\sigma(p+1)}, m_{\sigma(p+1)}), \cdots, (x_{\sigma(s)}, m_{\sigma(s)})\big), (x_{\sigma(s+1)}, m_{\sigma(s+1)}), \cdots, (x_{\sigma(n)}, m_{\sigma(n)})\Big)$$

$$= \chi(\sigma)(-1)^{s(r-1)+1}(-1)^{s-p+m_\alpha(x_{\sigma(p+1)}+\cdots+x_{\sigma(s)})} f_r\Big(j_s\big((x_{\sigma(1)}, m_{\sigma(1)}), \cdots, (x_{\sigma(p-1)}, m_{\sigma(p-1)}), (x_{\sigma(p+1)}, m_{\sigma(p+1)}), \cdots, (x_{\sigma(s)}, m_{\sigma(s)}), (x_\alpha, m_\alpha)\big), (x_{\sigma(s+1)}, m_{\sigma(s+1)}), \cdots, (x_{\sigma(n)}, m_{\sigma(n)})\Big)$$

Taking the s^{th} term of the second coordinate of j_s, we have:

$$\chi(\sigma)(-1)^{s(r-1)+1}(-1)^{s-p+m_\alpha(x_{\sigma(p+1)}+\cdots+x_{\sigma(s)})} f_r\Big(\big(l_s(x_{\sigma(1)}, \cdots, x_{\sigma(p-1)}, x_{\sigma(p+1)}, \cdots \cdots x_{\sigma(s)}, x_\alpha), k_s(x_{\sigma(1)}, \cdots, x_{\sigma(p-1)}, x_{\sigma(p+1)}, \cdots, x_{\sigma(s)}, m_\alpha)\big), (x_{\sigma(s+1)}, m_{\sigma(s+1)}), \cdots, (x_{\sigma(n)}, m_{\sigma(n)})\Big)$$

$$= \beta f_r\Big((x_{\sigma(s+1)}, m_{\sigma(s+1)}), \cdots, (x_{\sigma(n)}, m_{\sigma(n)}), \big(l_s(x_{\sigma(1)}, \cdots, x_{\sigma(p-1)}, x_{\sigma(p+1)}, \cdots, x_{\sigma(s)}, x_\alpha), k_s(x_{\sigma(1)}, \cdots, x_{\sigma(p-1)}, x_{\sigma(p+1)}, \cdots, x_{\sigma(s)}, m_\alpha)\big)\Big)$$

Where

$$\begin{aligned}\beta =& \chi(\sigma)(-1)^{s(r-1)+1}(-1)^{s-p+m_\alpha(x_{\sigma(p+1)}+\cdots+x_{\sigma(s)})} \\ &(-1)^{n-s+(s-2+x_{\sigma(1)}+\cdots+x_{\sigma(p-1)}+x_{\sigma(p+1)}+\cdots+x_{\sigma(s)}+m_\alpha)(x_{\sigma(s+1)}+\cdots+x_{\sigma(n)})} \\ =& \chi(\sigma)(-1)^{s(r-1)+1}(-1)^{n-p+m_\alpha(x_{\sigma(p+1)}+\cdots+x_{\sigma(s)})} \\ &(-1)^{(s-2+x_{\sigma(1)}+\cdots+x_{\sigma(p-1)}+x_{\sigma(p+1)}+\cdots+x_{\sigma(s)}+m_\alpha)(x_{\sigma(s+1)}+\cdots+x_{\sigma(n)})}\end{aligned}$$

Taking the r^{th} term of the second coordinate of f_r, we have:

$$\beta h_r\Big(x_{\sigma(s+1)},\cdots,x_{\sigma(n)},k_s(x_{\sigma(1)},\cdots,x_{\sigma(p-1)},x_{\sigma(p+1)},\cdots,x_{\sigma(s)},m_\alpha)\Big) \tag{7.6.2}$$

Now since σ is an unshuffle, $\chi(\sigma)$ does not contain any sign of the form $(-1)^{m_\alpha x_\gamma}$ with $\gamma > \alpha$. Hence

$$\begin{aligned}\beta =&\chi(\sigma)(-1)^{s(r-1)+1}(-1)^{(s-2+x_{\sigma(1)}+\cdots+x_{\sigma(p-1)}+x_{\sigma(p+1)}+\cdots+x_{\sigma(s)})(x_{\sigma(s+1)}+\cdots+x_{\sigma(n)})}\\ &(-1)^{n-p+m_\alpha\sum_{i=\alpha+1}^n x_i}\end{aligned}$$

Expression 7.6.2 then becomes

$$\begin{aligned}&\beta(-1)^{(s-2+x_{\sigma(1)}+\cdots+x_{\sigma(p-1)}+x_{\sigma(p+1)}+\cdots+x_{\sigma(s)}+m_\alpha)(x_{\sigma(s+1)}+\cdots+x_{\sigma(n)})}\\ &\qquad h_r\Big(k_s(x_{\sigma(1)},\cdots,x_{\sigma(p-1)},x_{\sigma(p+1)},\cdots,x_{\sigma(s)},m_\alpha),x_{\sigma(s+1)},\cdots,x_{\sigma(n)}\Big)\\ =&\ \chi(\sigma)(-1)^{s(r-1)+1}(-1)^{m_\alpha(x_{\sigma(s+1)}+\cdots+x_{\sigma(n)})}(-1)^{n-p+m_\alpha\sum_{i=\alpha+1}^n x_i}\\ &\qquad h_r\Big(k_s(x_{\sigma(1)},\cdots,x_{\sigma(p-1)},x_{\sigma(p+1)},\cdots,x_{\sigma(s)},m_\alpha),x_{\sigma(s+1)},\cdots,x_{\sigma(n)}\Big)\\ =&\ \chi(\sigma')(-1)^{s(r-1)+1}(-1)^{n-\alpha+m_\alpha\sum_{i=\alpha+1}^n x_i}\\ &\qquad h_r\Big(k_s(x_{\sigma(1)},\cdots,x_{\sigma(p-1)},x_{\sigma(p+1)},\cdots,x_{\sigma(s)},m_\alpha),x_{\sigma(s+1)},\cdots,x_{\sigma(n)}\Big)\end{aligned}$$

Where σ' is the $(s,n-s)$-unshuffle such that

$$\begin{aligned}&\sigma'\big((x_1,x_2,\cdots,x_{\alpha-1},x_{\alpha+1},\cdots x_n,m_\alpha)\big)\\ =&(x_{\sigma(1)},\cdots,x_{\sigma(s)},x_{\sigma(s+1)},\cdots,x_{\sigma(p-1)},x_{\sigma(p+1)},\cdots x_{\sigma(n)},m_\alpha)\end{aligned}$$

and since $\alpha \geq p$.

In both cases, we have a common sign that factors out. Since we also sum over all $(s,n-s)$-unshuffles, the collection of $(x_1,x_2,\cdots,x_{\alpha-1},x_{\alpha+1},\cdots x_n,m_\alpha)$ terms in the second coordinate of the first double sum of the L_∞ algebra homomorphism relation simplifies to the following:

$$(-1)^{n-\alpha+m_\alpha\sum_{i=\alpha+1}^n x_i}\sum_{i+j=n+1}\sum_\sigma \chi(\sigma)(-1)^{i(j-1)+1}h_j(k_i(\xi_{\sigma(1)},\cdots,\xi_{\sigma(i)}),\xi_{\sigma(i+1)},\cdots,\xi_{\sigma(n)})$$

which contains the first double sum of our L_∞ module homomorphism definition.

Now consider the $j'_t(f_{i_1} \otimes \cdots \otimes f_{i_t})$ terms:

$$\sum_{\substack{1 \le t \le n \\ i_1 + \cdots + i_t = n \\ i_1 \le \cdots \le i_t}} \sum_{\tau} \lambda \, j'_t\Big(f_{i_1}\big((x_{\tau(1)}, m_{\tau(1)}), \cdots, (x_{\tau(i_1)}, m_{\tau(i_1)})\big), f_{i_2}\big((x_{\tau(i_1+1)}, m_{\tau(i_1+1)}), \cdots$$

$$\cdots (x_{\tau(i_1+i_2)}, m_{\tau(i_1+i_2)})\big), \cdots, f_{i_t}\big((x_{\tau(i_1+\cdots+i_{t-1}+1)}, m_{\tau(i_1+\cdots+i_{t-1}+1)}), \cdots, (x_{\tau(i_t)}, m_{\tau(i_t)})\big)\Big)$$

with $\lambda = \chi(\tau)(-1)^{\frac{t(t-1)}{2} + \sum_{k=1}^{t-1} i_k(t-k)} \nu$.

We note again that the only nonzero element in the first coordinate here is $l_n(x_1, \cdots, x_n)$, and it occurs when $t = n$. Hence it suffices to examine only the second coordinate. We now aim to show that this double sum in the second coordinate induces a collection of groups of elements equal to the second double sum of our L_∞ module homomorphism definition corresponding to $(x_1, x_2, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots x_n, m_\alpha)$ terms, up to some common sign. That will give us n copies of the L_∞ module homomorphism definition, summing up to 0 as needed.

We follow similar reasoning here to track down $x_1, x_2, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots x_n, m_\alpha$ terms.

We make the following observations:

- By definition of j'_t and f_i, unless $i_1 = i_2 = \cdots = i_{\beta-1} = i_{\beta+1} = \cdots = i_t = 1$ for some $1 \le \beta \le t$, $j'_t(f_{i_1} \otimes \cdots \otimes f_{i_t}) = 0$.
- $\beta = t$ since $i_1 \le i_2 \le \cdots \le i_t$.
- $j'_t(f_{i_1} \otimes \cdots \otimes f_{i_t})$ will not produce a $(x_1, x_2, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots x_n, m_\alpha)$ term unless τ places (x_α, m_α) inside f_{i_t} (by definition of j'_t).

Subsequently,

$$\nu = (i_t - 1)\Big(\sum_{k=1}^{t-1} x_{\tau(k)}\Big)$$

and since $\frac{t(t-1)}{2} = 1 + 2 + \cdots + t - 1$,

$$(-1)^{\frac{t(t-1)}{2} + \sum_{k=1}^{t-1} i_k(t-k)} = (-1)^{1+2+\cdots+t-1(t-1)+(t-2)+\cdots(t-(t-1))} = 1$$

Therefore $\lambda = \chi(\tau)(-1)^{(i_t-1)(\sum_{k=1}^{t-1} x_{\tau(k)})}$.

We now fix $1 \leq t \leq n$ and τ, and assume τ places m_α in f_{i_t} and $\alpha = \tau(p)$ for some $t \leq p \leq n$.

Consider the following:

$$\begin{aligned} \lambda\, j_t'\Big(& f_{i_1}\big((x_{\tau(1)}, m_{\tau(1)}), \cdots (x_{\tau(i_1)}, m_{\tau(i_1)})\big), f_{i_2}\big((x_{\tau(i_1+1)}, m_{\tau(i_1+1)}), \cdots \\ & \cdots, (x_{\tau(i_1+i_2)}, m_{\tau(i_1+i_2)}\big), \cdots \\ & \cdots, f_{i_t}\big((x_{\tau(i_1+\cdots+i_{t-1}+1)}, m_{\tau(i_1+\cdots+i_{t-1}+1)}), \cdots, (x_{\tau(i_t)}, m_{\tau(i_t)})\big)\Big) \end{aligned}$$

Since $i_1 = i_2 = \cdots = i_{t-1}$, this is

$$\begin{aligned} & \lambda\, j_t'\Big(f_1\big((x_{\tau(1)}, m_{\tau(1)})\big), f_1\big((x_{\tau(2)}, m_{\tau(2)})\big), \cdots, f_1\big((x_{\tau(t-1)}, m_{\tau(t-1)})\big), \\ & \quad f_{i_t}\big((x_{\tau(t)}, m_{\tau(t)}), \cdots, (x_{\tau(p-1)}, m_{\tau(p-1)}), (x_\alpha, m_\alpha), \\ & \qquad (x_{\tau(p+1)}, m_{\tau(p+1)}), \cdots, (x_{\tau(n)}, m_{\tau(n)})\big)\Big) \\ & = \lambda\, j_t'\Big((x_{\tau(1)}, h_1(m_{\tau(1)})), (x_{\tau(2)}, h_1(m_{\tau(2)})), \cdots, (x_{\tau(t-1)}, h_1(m_{\tau(t-1)})), \\ & \quad f_{i_t}\big((x_{\tau(t)}, m_{\tau(t)}), \cdots, (x_{\tau(p-1)}, m_{\tau(p-1)}), (x_\alpha, m_\alpha), \\ & \qquad (x_{\tau(p+1)}, m_{\tau(p+1)}), \cdots, (x_{\tau(n)}, m_{\tau(n)})\big)\Big) \\ & = \lambda\delta\, j_t'\Big((x_{\tau(1)}, h_1(m_{\tau(1)})), (x_{\tau(2)}, h_1(m_{\tau(2)})), \cdots, (x_{\tau(t-1)}, h_1(m_{\tau(t-1)})), \\ & \quad f_{i_t}\big((x_{\tau(t)}, m_{\tau(t)}), \cdots, (x_{\tau(p-1)}, m_{\tau(p-1)}), (x_{\tau(p+1)}, m_{\tau(p+1)}), \cdots \\ & \qquad \cdots, (x_{\tau(n)}, m_{\tau(n)}), (x_\alpha, m_\alpha)\big)\Big) \end{aligned} \tag{7.6.3}$$

where $\delta = (-1)^{n-p+m_\alpha(x_{\tau(p+1)}+\cdots+x_{\tau(n)})}$.

Now

$$\begin{aligned} \lambda\delta &= \chi(\tau)(-1)^{(i_t-1)(\sum_{k=1}^{t-1} x_{\tau(k)})}(-1)^{n-p+m_\alpha(x_{\tau(p+1)}+\cdots+x_{\tau(n)})} \\ &= \chi(\tau')(-1)^{(i_t-1)(\sum_{k=1}^{t-1} x_{\tau(k)})}(-1)^{n-\alpha+m_\alpha \sum_{i=\alpha+1}^{n} x_i} \end{aligned}$$

Where τ' is the unshuffle such that

$$\tau'\big((x_1, x_2, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots x_n, m_\alpha)\big) = (x_{\tau(1)}, \cdots, x_{\tau(p-1)}, x_{\tau(p+1)}, \cdots x_{\tau(n)}, m_\alpha)$$

Taking the i_t^{th} element of the second coordinate of f_{i_t} and the t^{th} element of the second coordinate of j_t' in expression 7.6.3, we have

$$\chi(\tau')(-1)^{(i_t-1)(\sum_{k=1}^{t-1} x_{\tau(k)})}(-1)^{n-\alpha+m_\alpha \sum_{i=\alpha+1}^{n} x_i}$$
$$k_t'\big(x_{\tau(1)}, \cdots, x_{\tau(t-1)}, h_{i_t}(x_{\tau(t)}, \cdots, x_{\tau(p-1)}, x_{\tau(p+1)}, \cdots x_{\tau(n)}, m_\alpha)\big)$$

Now since τ is a $(i_1, i_2, \cdots, i_t)$-unshuffle with $i_1 = i_2 = \cdots = i_{t-1} = 1$ and the property that $\tau(i_1 + \cdots + i_{l-1} + 1) < \tau(i_1 + \cdots + i_l + 1)$ if $i_l = i_{l+1}$ and τ places (x_α, m_α) in f_{i_t}, there is a 1-1 correspondence between these unshuffles and $(n - i_t, i_t - 1)$-unshuffles. Since we sum over all of these unshuffles, the collection of $(x_1, x_2, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots x_n, m_\alpha)$ terms simplifies to the following:

$$(-1)^{n-\alpha+m_\alpha \sum_{i=\alpha+1}^{n} x_i} \sum_{r+s=n+1} \sum_{\tau} \chi(\tau)(-1)^{(s-1)(x_{\tau(1)}+\cdots+x_{\tau(n-s)})}$$
$$k_r'\big(x_{\tau(1)}, \cdots, x_{\tau(n-s)}, h_s(x_{\tau(n-s+1)}, \cdots, x_{\tau(n-1)}, m)\big)$$

Where τ runs through $(n - s, s - 1)$-unshuffles, which is the second double sum in definition 7.6.1.

Hence, grouping all $(x_1, x_2, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots x_n, m_\alpha)$ terms together in the second coordinate of the L_∞ algebra homomorphism yields the left hand side of definition 7.6.1 with a common sign of $(-1)^{n-\alpha+m_\alpha \sum_{i=\alpha+1}^{n} x_i}$, allowing the second coordinate to sum up to 0.

Since the first coordinate also sums up to 0, F is an L_∞ algebra homomorphism.

$\square$

This theorem validates definition 7.6.1 of an L_∞ module homomorphism by illustrating that it serves as a generalization of properties exhibited by classical Lie algebras. Any further investigation of L_∞ modules may use this as a structure-preserving map.

Chapter 8

Further Implications

Given the results of [9] and the further exposition provided by this dissertation, the basic tenets of L_∞ algebra representation theory should be well-understood. The main results of this dissertation expand this theory by constructing a concrete L_∞ module and defining the concept of an L_∞ module homomorphism. Despite the computationally intense nature of these results, they serve as the homotopy theoretical version of elementary results that would likely be addressed in the first few days of a course on Lie algebra representation theory. Many basic ideas still remain unexplored. We list a few of these here in the hopes that they are resolved in the near future.

- The concepts of submodules and irreducible modules are central to classical Lie algebra representation theory. How are the homotopy theoretic versions of these concepts defined?

- How can one define the kernel and image of a Lie algebra homomoprhism in the L_∞ context? How does it relate to sub(L_∞) algebras?

- Does an analog of Schur's lemma [3] exist in the homotopy context?

- Is there a homotopy analog to complete reducibility for finite dimensional L_∞ modules? That is, can an L_∞ module be classified as a direct sum of irreducible submodules?

- Is there a homotopy version of the representation theory of $sl(2, \mathbb{F})$?
- Can more concrete examples of L_∞ modules be constructed in order to provide an explicit L_∞ module homomorphism using the one constructed in this dissertation?

We believe that these unexplored ideas will lead to a fruitful expansion of L_∞ algebra representation theory.

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