ABSTRACT

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Fully integrable PDEs, those with an infinite number of conservation laws, are of particular interest to the modeling community as exact solutions can often be found. There are several methods currently used to study integrable systems. Constructing integrable hierarchies using the language of vertex algebras provides a simple formulation of the structure of the hierarchy. It also allows us to easily find desirable solutions, such as soliton solutions to the KP hierarchy. For classical hierarchies, such as the KP and Toda hierarchies, this construction relies on the boson-fermion correspondence.

The vertex algebra formed by two charged free bosons does not have a fermionic Fock space. To investigate the charged free boson integrable hierarchy, Friedan-Martinec-Shenker bosonization is used in place of the boson-fermion correspondence. The charged free bosons were studied by Kac and Radul; while the hierarchy was mentioned it was not studied in any detail. We thoroughly investigate this hierarchy and several of its reductions. This gives a generalization of the vertex algebra methods currently used to study integrable hierarchies. We also find several interesting PDEs, including the Euler beam equation which was known to be integrable but had not been studied using an algebraic approach.

The Charged Free Boson Integrable Hierarchy

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BIOGRAPHY

The student writing this thesis has enjoyed her work in vertex algebras and would recommend the field to other graduate students. She also does not like writing biographies.

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TABLE OF CONTENTS

Chapter 1 Introduction **1**

Chapter 2 Lie Algebra **5**
- 2.0.1 Introduction 5
- 2.1 Lie Algebra 5
- 2.2 Schur polynomials 10
- 2.3 gl_∞ 11
- 2.4 Heisenberg Algebra 12
- 2.5 Virasoro Algebra 13

Chapter 3 Vertex Algebra **14**
- 3.0.1 Introduction 14
- 3.1 Vertex algebras 14
- 3.2 Lattice Vertex Algebras 16
- 3.3 Boson-fermion correspondence 18
- 3.4 Constructing the KP hierarchy 21
- 3.5 Lax Equations 25
- 3.6 Solitons and the Virasoro Algebra 27
- 3.7 Twisted modules 28

Chapter 4 Results **31**
- 4.0.1 introduction 31
- 4.1 Charged free bosons 31
- 4.2 Casimir 33
- 4.3 Boson-Boson Correspondence 35
- 4.4 Constructing Hirota Equations 37
- 4.5 Soliton-like Solutions 40
- 4.6 Twisted reduction 41
- 4.7 $\alpha = 0$ reduction 45
- 4.8 $\beta = 0$ reduction 48
- 4.9 Virasoro solutions with $\beta = 0$ 51
- 4.10 Future Research: Lax Equations 52

Chapter 1

Introduction

Since the dawn of modern mathematics partial differential equations have been essential to modeling. However, it is often difficult or impossible to find exact solutions, or even in many cases reasonable approximate solutions, to these equations. Fully integrable PDEs, those with an infinite number of conservation laws, are the exception to this rule. There are several methods currently used to study integrable systems, each with its own advantages.

The KP equation describes the flow of shallow water in a 2D channel. Historically and even today flow is often restricted to a 1D channel by eliminating the y variable. This gives the KdV equation for shallow water. The KdV equation has been used to model an incredible number of physical situations. It is used not only in water waves, but also everything from glaciers to air flow near the wing of an airplane.

Of course, the KdV equation was studied using numerical methods long before it was known to be integrable. However, fluids problems are usually difficult to solve numerically as the surface of the fluid reacts to the movement of the fluid. The surface is one of the boundaries needed to solve the boundary problem so this creates obvious complications. The KdV equation is produced by turning this extremely difficult boundary value problem into an initial value problem. This eliminates the complication of the free boundary, but the partial differential equation, the KdV equation, is more complex than the original PDE used for the boundary value problem. The KdV equation has been solved numerically, however the solutions are given in terms of elliptic functions and are often too complicated to be of any use. The Kyoto group's groundbreaking series of papers developed the idea of using vertex algebra structures to find desirable solutions to the KdV and KP equations.

We begin with a review of the Lie algebra material necessary for the reader to understand this thesis. Although we introduce all necessary notation, we highly recommend the reader refer to a text on Lie algebras if they are unfamiliar with the subject. We then quickly discuss Schur polynomials, which will be used frequently in computations.

We review gl_∞, the algebra of $\mathbb{Z} \times \mathbb{Z}$ matrices with finitely many nonzero diagonals, in some detail as it will later provide solutions to our PDEs. The Virasoro and Heisenberg algebras are also defined. Representations of these algebras on the space of polynomials and derivatives are discussed, as this is how algebraic equations are translated into PDEs.

We then begin a fairly thorough review of vertex algebras focusing on lattice vertex algebras. The standard examples are computed. With this, we give the classical vertex algebra construction of the KP hierarchy. An alternate, although equivalent construction will be given during the research portion of the the thesis as a reduction of the charged free boson hierarchy.

To construct the KP hierarchy we first define the fermionic and bosonic Fock spaces. The bosonic Fock space, the polynomials in infinitely many variables, is also used later to construct the charged free boson hierarchy. The Casimir element of gl_∞ gives PDEs after some amount of expansion. The Kyto group, after some cleaver simplifications, found the smallest PDE contained in the hierarchy was the KP equation, which we re-derive for the reader.

The Lax formulation of the hierarchy is then derived from the Hirota equations by reproducing van Morebeke's computations. The Lax formulation of a hierarchy is often useful to study physical properties of the PDEs including their conservation laws.

We discuss soliton solutions for the KP hierarchy. These are perhaps the most desirable solutions for the modeling community as they are standing waves. Solitons describe physical phenomena such as tsunamis and light traveling through a fiber optic cable. They are also found at the leading edge of shock waves in dispersive systems.

The research section begins with a review of the charged free boson algebra and the boson-boson correspondence as our notation differs from that of previous authors. We then find PDEs in the hierarchy and some solutions.

We look at several reductions, made both in the algebra and by setting variables equal to zero in the equations. Using twisted modules eliminates indices from solutions and allows us to make reductions without guesswork. This is a new perspective on how to make reductions even in the case of known hierarchies; however, when used on known hierarchies it gives known reductions. For example, one can produce the BKP hierarchy from the KP hierarchy using this method. We explore other reductions produced by simply setting variables equal to zero. These reductions give interesting PDEs, such as Euler's beam equation, and additional solutions. We also discuss some of the difficulties of making these reductions.

Our section of open problems is devoted to attempts at finding a Lax pair formulation of the hierarchy. We show this problem is significantly more difficult for the charged free boson hierarchy than for the KP hierarchy. We successfully find wave equations and make significant progress towards finding Lax operators.

Chapter 2

Lie Algebra

2.0.1 Introduction

This chapter will review the notions from Lie algebra needed to formulate the charged free boson hierarchy. We also review the Lie algebras commonly used in this thesis, including the Heisenberg algebra and gl_∞.

2.1 Lie Algebra

We begin with a review Lie algebras and Lie algebra notation used in this thesis. For a thorough introduction we refer the reader to Humphreys' text [H].

All Lie algebras mentioned in this thesis are over the complex numbers unless stated otherwise. A (complex) Lie algebra is a vector space L over $\mathbb{C}$ with a bilinear bracket operation $[,] : L \times L \to L$ that satisfies skew symmetry and the Jacobi identity. For completeness we list these here.

Jacobi identity:

$[a,[b,c]] = [[a,b],c] + [b,[a,c]]$;

skew symmetry:

$[a,a] = 0$.

Perhaps the most classical example of a Lie algebra is gl_n, the $n \times n$ matrices, with bracket structure $[a,b] = ab - ba$. Note this example makes clear why the words "bracket" and "commutator" are both used for $[a,b]$.

The standard basis for gl_n is the set of elementary matrices $E_{i,j}$. Here $E_{i,j}$ is the matrix with a 1 in the i,jth position and zeros elsewhere. The bracket of two basis elements is given by

$$[E_{i,j}, E_{k,l}] = E_{i,j}E_{k,l} - E_{k,l}E_{i,j} = \delta_{j,k}E_{i,l} - \delta_{i,l}E_{k,j}$$

.

A subspace of a Lie algebra is called a (Lie) subalgebra if it is closed under taking brackets. A Lie algebra representation is a linear map to the space of matrices which preserves the bracket structure. An equivalent notion is that of a module, which is a vector space with an action of a Lie algebra satisfying $[a,b]v = a(bv) - b(av)$ for a,b in the Lie algebra and v in the module.

One Lie subalgebra of gl_n is sl_n, the $n \times n$ matrices with trace zero. We check this is closed under taking brackets.
Let $a, b \in sl_n$ then $tr([a,b]) = tr(ab - ba) = tr(ab) - tr(ba) = 0$.

An example of a module, and hence a representation, of gl_n is its action on the space $\mathbb{C}^n$. Let e_i be the vector with a 1 in the ith component and zeros elsewhere. Then

$$[E_{i,j}, E_{k,l}]e_n = \delta_{j,k}E_{i,l}e_n - \delta_{i,l}E_{k,j}e_n = \delta_{j,k}\delta_{l,n}e_i - \delta_{i,l}\delta_{j,n}e_k,$$

which equals

$$E_{i,j}(E_{k,l}e_n) - E_{k,l}(E_{i,j}e_n) = E_{i,j}\delta_{l,n}e_k - E_{k,l}\delta_{j,n}e_i = \delta_{j,k}\delta_{l,n}e_i - \delta_{i,l}\delta_{j,n}e_k.$$

To introduce a particularly useful type of module called a highest weight module we will need a few definitions from representation theory. The standard example we will here is sl_2 with basis $e = E_{1,2}, f = E_{2,1}, h = E_{1,1} - E_{2,2}$ and commutators
$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$

A Cartan subalgebra H of a Lie algebra is defined to be a nilpotent subalgebra with the property that $[x,y] \in H$ for all $x \in H$ implies $y \in H$.
For our example of sl_2 the Cartan subalgebra is just $\mathbb{C}h$.

We define a weight space V_λ with weight $\lambda \in H^*$ for a module V to be $V_\lambda = \{v \in V | hv = \lambda(h)v \ \forall h \in H\}$.

For sl_2 as an sl_2-module under the adjoint action we have
$V_2 = \mathbb{C}e \qquad V_{-2} = \mathbb{C}f \qquad V_0 = \mathbb{C}h.$

We can divide gl_n into three parts $\{E_{i,j} | i < j\}$, $\{E_{i,i}\}$, and $\{E_{i,j} | i > j\}$. These are the upper triangular, diagonal, and lower triangular matrices respectively and this process is called a triangular decomposition. This works in a more general setting, for example in sl_2, e is upper triangular, h is diagonal, and f is lower triangular.

With this definition we can now define a highest weight and a highest weight vector. A highest vector is an element v_0 of V such that the upper triangular part of the Lie algebra acts on v_0 as zero. The highest weight is the weight of v_0.

For example, sl_2 as an sl_2-module has a highest weight vector e since $[e, e] = 0$. The highest weight is 2, the weight of e.

A representation is called irreducible if it does not have a nontrivial proper subrepresentation.

The functionals ω_i that act on the Cartan subalgebra of gl_n or sl_n by $\omega_i(E_{i,i}) = 1$, $\omega_i(E_{j,j}) = 0$ for $j \neq i$ are known as fundamental weights.

Any irreducible gl_n-module is a highest weight module with highest weight $c_1\omega_1 +$

$c_2\omega_2 + \ldots$ where the c_i are non-negative integers, and it is uniquely determined by its highest weight. Conversely, every $c_1\omega_1 + c_2\omega_2 + \ldots$ where the c_i are non-negative integers is the highest weight of an irreducible gl_n-module.

A module is called completely reducible if it can be written as a direct sum of irreducible modules. Weyl's theorem states any finite dimensional representation of any finite dimensional semi-simple Lie algebra is completely reducible. However, this is not true for infinite dimensional Lie algebras.

A Lie algebra is called reductive if it is completely reducible as a module over itself under the adjoint action. One example of a reductive Lie algebra is gl_n.

We will need GL_n, the Lie group of invertible $n \times n$ matrices. The only fact we will use from Lie group theory is that exponentation of elements of gl_n gives elements of GL_n.

We will frequently let an n-dimensional Lie algebra act on the space of complex polynomials in n variables.

For example, below is a very nice representation of gl_n on the space of polynomials in$x_1, x_2, x_3, \ldots, x_n$. Note this vector space will be important later.

Let $E_{i,j} = x_i \partial_{x_j}$ for $i, j \geq 1$. Then

$$[E_{i,j}, E_{k,l}] = [x_i \partial_{x_j}, x_k \partial_{x_l}] = \delta_{j,k} x_i \partial_{x_l} - \delta_{i,l} x_k \partial_{x_j}.$$

This gives us a natural action on the space of polynomials. For example, $E_{1,3} \cdot x_2 x_3 = x_1 \partial_{x_3} \cdot x_2 x_3 = x_1 x_2$ This representation is very similar to the one used to construct the 2-KP hierarchy.

2.2 Schur polynomials

In this thesis we will often need infinite dimensional algebras. We want to extend $\mathbb{C}[x_1, x_2, \ldots, x_n]$ to $\mathbb{C}[x_1, x_2, \ldots]$, the polynomials in infinitely many variables.

Normally the degree of x_n is defined to be 1, however in $\mathbb{C}[x_1, x_2, \ldots]$ this would produce infinitely many basis vectors of each degree. If the degree of x_n is instead defined to be n there are only finitely many basis vectors of each degree. For example, the basis vectors of degree 2 are x_1^2 and x_2.

We will use a basis of the algebra $\mathbb{C}[x_1, x_2, \ldots]$ known as the Schur polynomials. Note that our definition is a change of basis from the combinatorial definition. The generating function for the elementary Schur polynomials is

$$\sum_{n \geq 0} S_n(x) z^n = exp(\sum_{k \geq 1} x_k z^k). \tag{2.1}$$

For example, $S_0 = 1$, $S_1 = x_1$, $S_2 = \frac{x_1^2}{2} + x_2$, $S_3 = \frac{x_1^3}{3!} + x_2 x_1 + x_3$ and

$$S_n(x) = \sum_{n_1+2n_2+\cdots+rn_r=n} \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_r^{n_r}}{n_r!}.$$

2.3 gl_∞

The remainder of this chapter follows [KR2] and [K1].

The Lie algebra we will use most is gl_∞ which is the collection of $\mathbb{Z} \times \mathbb{Z}$ matrices with only finitely many nonzero diagonals. This condition allows multiplication to be well defined.

Note most texts introduce gl_∞ as matrices with finitely many nonzero elements and then arrive at the algebra with finitely many nonzero diagonals by adding a central extension. This first algebra is too small to be of use - for example it does not contain the identity matrix - so we will start with the larger algebra.

A standard basis for gl_∞ is $E_{i,j}$, $i, j \in \mathbb{Z}$ with central element c. Commutators are given by the formula

$$[E_{i,j}, E_{k,l}] = \delta_{j,k} E_{i,l} - \delta_{i,l} E_{k,j} + \delta_{i,l\leq 0}\delta_{k,j>0} c$$

.

One of the most commonly used representations is to let gl_∞ act on $\mathbb{C}^\infty$, the space of vectors indexed by the integers with finitely many nonzero entries. Once again this condition allows multiplication by a matrix to be well defined. Note $\mathbb{C}^\infty$ has basis e_i $i \in \mathbb{Z}$. As with the finite dimensional case $E_{i,j}e_k = \delta_{j,k}e_i$ and we let $ce_k = 0$ for all k.

2.4 Heisenberg Algebra

The Heisenberg algebra has basis $\{a_n, c | n \in \mathbb{Z}\}$ where c is central. Commutations relations are

$$[a_m, a_n] = m\delta_{m,-n}c.$$

We can find a representation of the Heisenberg algebra on the space of polynomials $\mathbb{C}[x_1, x_2, \dots]$:

$a_n \to \partial_{x_n}, \quad a_{-n} \to nx_n, \quad a_0 \to k, \quad c \to c, \quad n > 0.$

Also the representation of the Heisenberg algebra in gl_∞ used to construct the KP hierarchy is given by $a_n = \sum_{i \in \mathbb{Z}} E_{i,i+n}$. These are commonly referred to as the shift matrices as they map e_k to e_{k-n}.

2.5 Virasoro Algebra

The Virasoro algebra has a basis that consists of elements L_n $n \in \mathbb{Z}$ and a central element c. Commutation relations are given by

$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c.$

We can write the Virasoro algebra in terms of the Heisenberg algebra

$L_i = \frac{1}{2}\sum_{n\in\mathbb{Z}} : a_{-n}a_{n+i} :.$

We can also find L_i in gl_∞

$$L_i = \sum_{j\in\mathbb{Z}} jE_{j,j+i}.$$

Chapter 3

Vertex Algebra

3.0.1 Introduction

In this chapter we begin with a review of vertex algebras and, in particular, lattice vertex algebras. We then discuss two formulations of the KP hierarchy, the Hirota and Lax equations, in detail. We review some of the more useful solutions of the KP equation that can be produced using vertex operators. We end by introducing twisted modules for lattice vertex algebras which will be needed in chapter 4.

3.1 Vertex algebras

This section follows [K1]. A vertex algebra is constructed over a vector space V. Elements of V are called states. The vacuum vector $\mathbb{1}$ will act as an "identity" for the vertex algebra. A field or vertex operator is a series $a(z) = Y(a,z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ where $a_n \in End(V)$ and $a_n(v) = 0$ for n sufficiently large.

For the purpose of this construction, one can think of z as a formal variable and a_n as linear operators. In this case we will call $a(z)$ the generating function of the algebra. For example, once again consider the Heisenberg algebra, and write the generating series for the algebra as $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$. This is a vertex operator with commutation relation $[a(z), a(w)] = \partial_w \delta(z-w)$. We can obtain $[a_i, a_j]$ by finding the coefficient of $z^{-i-1}w^{-j-1}$.

Here we define $\delta(z-w) = \frac{1}{z-w} + \frac{1}{w-z} = \sum_{n \in \mathbb{Z}} z^n w^{-n-1}$.

We will now present two equivalent definitions of a vertex algebra. We will need these definitions at various times.

Definition 3.1.1. *A vertex algebra consists of states, the vacuum vector, and a linear map called the state-field correspondence, $a \to Y(a,z)$, with the axioms below.*
Translation covariance: $[T, Y(a,z)] = \partial_z Y(a,z)$, where T is defined by $Tv = v_{-2}\mathbb{1}$.
Vacuum: $Y(\mathbb{1}, z) = I$ where I is the identity operator on V and $Y(a,z)\mathbb{1}|_{z=0} = a$.
Locality: $(z-w)^N [Y(a,z), Y(b,w)] = 0$ for N sufficiently large.

The first two axioms ensure an identity and a derivative. Normally multiplication, and hence the commutator, of two formal Laurent series is not necessarily well defined. Axiom 3 ensures we can bracket vertex operators.

Definition 3.1.2. *A vertex algebra consists of states, the vacuum vector, and a linear map called the state-field correspondence, $a \to Y(a,z)$, with the axioms below.*
Vacuum: $Y(\mathbb{1}, z) = I$ where I is the identity operator on V and $a_{-1}\mathbb{1} = a$.

Borcherds identity:

$$\sum_{j\geq 0}\binom{m}{j}(a_{n+j}b)_{m+k-j}c = \sum_{j\geq 0}(-1)^j\binom{n}{j}a_{m+n-j}b_{k+j}c - \sum_{j\geq 0}(-1)^{j+n}\binom{n}{j}b_{n+k-j}a_{m+j}c \tag{3.1}$$

The Borcherds identity is not as intuitive as Definition 1, but it will be extremely useful later on.

Also useful is a special case of Borcherds formula called the commutator formula.

$$[Y(a,z),Y(b,w)] = \sum_{n\geq 0} Y(a_nb,w)\frac{\partial_w^n}{n!}\delta(z-w). \tag{3.2}$$

We also include the commutator formula written in terms of modes

$$[a_m,b_n] = \sum_{j\geq 0}\binom{m}{j}(a_jb)_{m+n-j}, \tag{3.3}$$

and the -1st product identity

$$(a_{-1}b)_n = \sum_{j<0} a_jb_{n-j-1} + \sum_{j\geq 0} b_{n-j-1}a_j. \tag{3.4}$$

3.2 Lattice Vertex Algebras

This section follows [K1]. We will give the general definition of a lattice vertex algebra. The reader that finds this unclear can think of the lattice relations producing the commutation relations.

An integral lattice L is a free abelian group L with a bilinear form $L \times L \to \mathbb{Z}$.

A lattice vertex algebra with lattice $L = \gamma_1\mathbb{Z} \times \cdots \times \gamma_n\mathbb{Z}$ is a vertex algebra with elements of the form $(\gamma_1)_{i_1} \dots (\gamma_1)_{i_j} \dots (\gamma_n)_{k_1} \dots (\gamma_n)_{k_l} e^{\gamma}$ for $\gamma \in L$.

For an example we will look at the algebra with lattice $\alpha\mathbb{Z}$ with $(\alpha, \alpha) = 1$ which will be used in the construction of the KP hierarchy next section.
It is generated by $\alpha(z) = \sum_{j \in \mathbb{Z}} \alpha_j z^{-j-1}$ and
$Y(e^{\pm\alpha}, z) = e^{\alpha} : exp(\pm \int a(z)) := z^{\pm\alpha_0} e^{\alpha} exp(\pm \sum_{j>0} \alpha_{-j} \frac{z^j}{j}) exp(\pm \sum_{j>0} \alpha_j \frac{z^{-j}}{-j})$.
Here $e^{\pm\alpha}$ are elements of the group algebra $\mathbb{C}[L]$ with multiplication $e^{\alpha}e^{\beta} = e^{\alpha+\beta}$

To compute commutators we will need the following formulas:

$$[\alpha_n, e^{\gamma}_m] = (\alpha, \gamma) e^{\gamma}_{m+n}, \tag{3.5}$$

and

$$e^{\gamma_1}_n e^{\gamma_2} =: S_{-(\gamma_1,\gamma_2)-n-1}(\gamma_1, \frac{\partial\gamma_1}{2!}, \frac{\partial^2\gamma_1}{3!} ...) e^{\gamma_1+\gamma_2} : \tag{3.6}$$

and the commutator formula, equation 3.3.
Here we define the normally ordered product, $: a(z)b(w) := a(z)_+ b(w) + b(w) a(z)_-$, where $a(z)_+$ is the part of $a(z)$ with nonnegative powers of z and $a(z)_-$ is the part with negative powers of z.

As an exercise we now compute

$$[Y(e^{\alpha}, z), Y(e^{\alpha}, w)] = \sum_{n/ge0} Y(e^{\alpha}_n e^{\alpha}, w) \frac{\partial_w^n}{n!} \delta(z-w).$$

Now $e^{\alpha}_n e^{\alpha} =: S_{-n-2}(\alpha, \frac{\partial \alpha}{2!}, \frac{\partial^2 \alpha}{3!} ...)e^{2\alpha}$: but $S_n = 0$ for $n < 0$ which gives

$$[Y(e^{\alpha}, z), Y(e^{\alpha}, w)] = 0.$$

Similarly,

$$[Y(e^{\alpha}, z), Y(e^{-\alpha}, w)] = \sum_{n/ge0} Y(e^{\alpha}_n e^{-\alpha}, w) \frac{\partial_w^n}{n!} \delta(z-w).$$

Here $e^{\alpha}_n e^{-\alpha} =: S_{-n}(\gamma_1, \frac{\partial \alpha}{2!}, \frac{\partial^2 \alpha}{3!} ...)e^0$: so we get a contribution of 1 when $n = 0$, which gives

$$[Y(e^{\alpha}, z), Y(e^{-\alpha}, w)] = \delta(z-w).$$

3.3 Boson-fermion correspondence

This section follows [KR2]. Before beginning construction of the KP hierarchy, we have to describe the space that gl_{∞} will be acting on, which is called the Fock space. In previous sections the algebra of polynomials in $\mathbb{N}$ variables was mentioned. This is one description of the Fock space called the bosonic Fock space. This section will also introduce the fermionic description. The isomorphism between the two spaces is the famous boson-fermion correspondence. Physicists think of bosons as light waves, this is because squaring a polynomial just gives another polynomial and having photons of the same frequency in the same place just gives a laser beam. Fermions have the property of squaring to zero; equivalently, two pieces of matter can not be in the same place at the same time.

For continuity we define here the bosonic Fock space $B = \mathbb{C}[x_1, x_2, \dots]$. We have two gradings, or measurements of degree. One is given by the usual definition of the degree of a polynomial. The other, which will often be more useful, is to declare the degree of x_n to be n.

To familiarize the reader with the bosonic Fock space we examine the action of the Heisenberg algebra.

Recall from section 2.4 the Heisenberg algebra has an irreducible representation

$a_n \to \partial_{x_n}, \quad a_{-n} \to nx_n, \quad a_0 \to k, \quad c \to c, \quad n > 0.$

Notice if we apply sums and products of various a_{-n} to the polynomial 1 we can create any polynomial. If we apply the appropriate a_ns to any element of the Fock space it will become 1, and $a_n 1 = 0$ for $n > 0$. Due to this a_{-n} are referred to as creation operators and a_n are annihilation operators. The polynomial 1 is referred to as the vacuum. This has the physical interpretation of creating and annihilating photons from the vacuum of space.

The fermionic Fock space does not have as intuitive a definition as the bosonic Fock space. We begin by introducing the wedge product, commonly used in topology.

Define the wedge product $a \wedge b$ to be the tensor product with the added relation $a \wedge a = 0$. The fermionic Fock space is then

$$F = \bigoplus_{n \in \mathbb{Z}} F_n$$

where F_n consists of vectors of the form

$$f = v_{i_n} \wedge v_{i_{n-1}} \wedge \dots$$

with the conditions $i_n > i_{n-1} > \dots$ and $i_j = j + n$ for j sufficiently less than zero.

The degree of f is defined to be $deg(f) = \sum_{j \leq n} i_j - j - n$.

The vector

$$f_n = v_n \wedge v_{n-1} \wedge \dots$$

is the vector of degree zero in F_n and will play the role of the vacuum vector.

We will now once again look at the action of the Heisenberg algebra. To do this we define two operators.

Let $\Psi_k(f) = \Psi_k(v_{i_n} \wedge v_{i_{n-1}} \wedge \dots) = v_k \wedge v_{i_n} \wedge v_{i_{n-1}} \wedge \dots$ where $\Psi_k(f)$ is then reordered so the indices are once again decreasing, picking up a sign in the process. Notice if v_k was already a component of f then $\Psi_k(f) = 0$ by definition.

Let $\Psi_k^*(f) = \Psi_k^*(v_{i_n} \wedge v_{i_{n-1}} \wedge \dots)$ first permute v_k to the front of f, picking up a sign in the process, then remove v_k. If f does not contain v_k we define $\Psi_k^*(f) = 0$.

The Heisenberg operators act by

$$a_n = \sum_{k>0} \Psi_k \Psi^*_{k+n} + \sum_{k\leq 0} \Psi^*_{k+n} \Psi_k.$$

Here, as with the bosonic Fock space, the annihilation operators take the vacuum to zero and the creation operators applied to the vacuum generate the Fock space. These representation are isomorphic and the isomorphism gives the famed boson-fermion correspondence.

We also define the generating series $\Psi(z) = \sum_{k\in\mathbb{Z}} \Psi_k z^{k-1}$ and $\Psi^*(z) = \sum_{k\in\mathbb{Z}} \Psi^*_k z^{-k}$ for use in the next section.

3.4 Constructing the KP hierarchy

This section follows [KR2]. To construct an integrable system using vertex operators one essentially picks an operator in a clever way and that operator is mapped to a space of polynomials and derivatives giving PDEs. Solutions to the hierarchy are then any operator that commutes with the one producing PDEs. This intuition suggests to choose the Casimir of gl_∞ to give PDEs as it commutes with all of GL_∞ producing a large number of solutions. For the reader familiar with Lie algebras what we call a Casimir operator does not always coincide with the classical definition, but will still commute with the action of gl_∞.

We reviewed the fermionic Fock space in the previous section. To construct the KP hierarchy we now need to find the Casimir element of gl_∞. We will denote the fermionic Casimir operator by Ω_F:

$$\Omega_F = Res_z \Psi(z) \otimes \Psi^*(z) = \sum_{k \in \mathbb{Z}} \Psi_k \otimes \Psi_k^*.$$

The KP hierarchy is the collection of equations found by expanding $\Omega_F(\tau \otimes \tau) = 0$. Solutions, τ, to this equation are called tau functions. However, this is not the usual family of differential equations one thinks of as the KP hierarchy. To construct these PDEs we need to use the boson-fermion correspondence. We will denote the Casimir operator after applying the boson-fermion correspondence as Ω_B.

The boson-fermion correspondence maps $\Psi(z) \to Y(e^\alpha, z)$, $\Psi^*(z) \to Y(e^{-\alpha}, z)$ where

$$Y(e^{\pm\alpha}, z) = z^{\pm\alpha_0} e^{\pm\alpha} exp(\pm \sum_{j>0} \alpha_{-j} \frac{z^j}{j}) exp(\pm \sum_{j>0} \alpha_j \frac{z^{-j}}{-j})$$

as mentioned in section 3.2 and $\alpha(z)$ is the Heisenberg algebra from section 2.4.

This gives

$$\Omega_B = Res_z z^{\alpha_0} e^\alpha exp(\sum_{j>0} \alpha_{-j} \frac{z^j}{j}) exp(\sum_{j>0} \alpha_j \frac{z^{-j}}{-j}) \otimes z^{-\alpha_0} e^{-\alpha} exp(-\sum_{j>0} \alpha_{-j} \frac{z^j}{j}) exp(-\sum_{j>0} \alpha_j \frac{z^{-j}}{-j}).$$

From this formula we can find the KP and KdV equations. A similar calculation will be used in later sections to produce PDEs in the charged free boson hierarchy and its reductions.

We begin by using that $\alpha(z)$ is the Heisenberg algebra to construct a representation on polynomial algebra. Specifically, we map $\alpha_{-n} \to nx_n$, $\alpha_n \to \partial_{x_n}$ for $n > 0$.

This gives

$$\Omega_B = Res_z z^{\alpha_0} e^{\alpha} exp(\sum_{j>0} x_j z^j) exp(\sum_{j>0} \partial_{x_j} \frac{z^{-j}}{-j}) \otimes z^{-\alpha_0} e^{-\alpha} exp(-\sum_{j>0} x_j z^j) exp(-\sum_{j>0} \partial_{x_j} \frac{z^{-j}}{-j}).$$

When acting on $\tau \in \mathbb{C}[x_1, x_2, \ldots]$, α_0 acts trivially, thus in the above formula we can remove the prefactors $z^{\pm\alpha_0} e^{\pm\alpha}$.

We denote the first tensor factor by primes and the second by double primes. Then

$$\Omega_B(\tau \otimes \tau) = Res_z exp\big(\sum_{j>0} x'_j z^j\big) exp\big(\sum_{j>0} \partial_{x'_j} \frac{z^{-j}}{-j}\big)$$

$$exp(-\sum_{j>0} x''_j z^j) exp(-\sum_{j>0} \partial_{x''_j} \frac{z^{-j}}{-j})(\tau(x') \otimes \tau(x''))$$

Since the two tensor factors commute, we get

$$\Omega_B = Res_z exp\big(\sum_{j>0} (x'_j - x''_j) z^j\big) exp\big(\sum_{j>0} (\partial_{x'_j} - \partial_{x''_j}) \frac{z^{-j}}{-j}\big).$$

We introduce the following change of variables:

$x' \to x - y, \quad x'' \to x - y$, and obtain

$$Res_z(exp(\sum_{n>0} -2y_n z^n)exp(\sum_{n>0} \partial_{y_n} \frac{z^{-n}}{n})(\tau(x-y)\tau(x+y)) = 0$$

Recalling formula 2.1 we can expand the exponentials using Schur polynomials:

$$Res_z\big(\sum_{n>0} S_n(-2y)z^n\big)\big(\sum_{n>0} S_n(\bar{\partial_y})\frac{z^{-n}}{n}\big)(\tau(x-y)\tau(x+y)) = 0.$$

where $\bar{\partial_x} = (\partial_{x_1}, \frac{1}{2}\partial_{x_2}, \frac{1}{3}\partial_{x_3} \dots)$. Taking the residue gives

$$\sum_{n \geq 0} S_n(-2y)S_{n+1}(\bar{\partial_y})(\tau(x-y)\tau(x+y)) = 0.$$

We rewrite this as

$$\sum_{n \geq 0} S_n(-2y)S_{n+1}(\bar{\partial_u})(\tau(x-y-u)\tau(x+y+u))|_{u=0} = 0$$

and use Taylor's formula to get

$$\sum_{n \geq 0} S_n(-2y)S_{n+1}(\bar{\partial_u})exp(\sum_{m>0} y_m \partial_{u_m})(\tau(x-u)\tau(x+u))|_{u=0} = 0. \qquad (3.7)$$

Notice here that we have eliminated all y derivatives. Because of this we can now use y as an indexing variable.

The KP equation is the coefficient of y_3:

$$\partial_{u_3}\partial_{u_1} - 2S_4(\bar{\partial_u})(\tau(x-u)\tau(x+u))|_{u=0} = 0$$

after the change of variables

$$f(x_1, x_2, x_3) = \frac{\partial^2}{\partial x_1^2} log(\tau(x_1, x_2, x_3)).$$

3.5 Lax Equations

Instead of the Hirota equations it is sometimes advantageous to study what are known as Lax equations. For example, it is easier to derive conservation laws from Lax equations and Lax equations from conservation laws. When the Lax equations for the KP hierarchy were first formulated it was not known how they related to the Hirota equations. Since then several authors, including [M] who we follow for this section, have thoroughly studied the connections between the Lax and Hirota equations.

We start by computing wave functions by solving $Y(e^{\alpha}, z)\tau = \Psi\tau$ and $Y(e^{-\alpha}, z)\tau = \Psi^*\tau$. Expanding we get:

$$exp\big(\sum_{n>0} x_n z^n\big) exp\big(-\sum_{n>0} \partial_{x_n} \frac{z^n}{n}\big)\tau(x) = \Psi(x,z)\tau(x),$$

$$exp\big(-\sum_{n>0} x_n z^n\big) exp\big(\sum_{n>0} \partial_{x_n} \frac{z^n}{n}\big)\tau(x) = \Psi^*(x,z)\tau(x).$$

We use Taylor's formula and divide by τ to get

$$\Psi(x,z) = exp\big(\sum_{n>0} x_n z^n\big)\frac{\tau(x-[z^{-1}])}{\tau(x)},$$

$$\Psi^*(x,z) = exp\big(-\sum_{n>0} x_n z^n\big)\frac{\tau(x+[z^{-1}])}{\tau(x)},$$

where $\tau(x-[z^{-1}]) = \tau(x_1 - z^{-1}, x_2 - 1/2z^{-2}, x_3 - 1/3z^{-3}\dots)$.

Our dressing operators are now the wave functions excluding the exponential after replacing z with ∂_{x_1}.

$$S = \frac{\tau(x-[\partial_{x_1}])}{\tau(x)}, \qquad S^* = \frac{\tau(x+[\partial_{x_1}])}{\tau(x)}.$$

The operators $L = S\partial_{x_1}S^{-1}$, and $L^* = S^*(\partial_{x_1})(S^*)^{-1}$ satisfy Lax equations

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L] \qquad n > 0$$

. Here L_+ is the portion of L with only non-negative powers of ∂_{x_1}.

Note $L = \partial_{x_1} + \sum_{n>0} f_n(x)\partial_{x_1}^{-n}$ is called a pseudo-differential operator. Multiplication is given by Leibnitz's rule

$$f(x)\partial_{x_1}^n \cdot g(x)\partial_{x_1}^m = \sum_{k\geq 0} \binom{n}{k} f(x)^{(k)} g(x)\partial_{x_1}^{n+m-k}$$

which coincides with the usual definition when n and m are positive.

3.6 Solitons and the Virasoro Algebra

This section follows [K1]. In shallow water wave theory, a soliton looks like a hump of water that moves along the surface without changing shape or speed. The KP equation has solutions of this form. Remarkably, these solutions have very nice mathematical properties that allow for explicit construction using vertex algebras. For an algebraist soliton solutions are a wider class of functions that can be found by exponentiating gl_∞.

This construction starts by finding a generating series $\Gamma(z,w)$ for gl_∞:

$$\Gamma(z,w) = \sum_{i,j\in\mathbb{Z}} E_{i,j} z^{i-1} w^{-j} = exp(\sum_{i\geq 0}(z^i - w^i)x_i) exp(-\sum_{i\geq 0}(z^{-i} - w^{-i})\frac{\partial_{x_i}}{i})$$

We can see this by expanding $\Gamma(z,w) =: Y(e^\alpha, z)Y(e^{-\alpha}, w)$ where $Y(e^{\pm\alpha}, z)$ are the lattice operators from section 3.2.

An n soliton solution is of the form $\tau = exp(\sum_{i=1}^{n} c_i \Gamma(z_i, w_i))1$. This is τ function because 1 is a τ function and the Casimir commutes with gl_∞.

For example, a one soliton solution to the KP hierarchy is $\tau = exp(c\Gamma(z,w))1$.

Since $\Gamma(z,w)^2 = 0$, we have $\tau(x) = 1 + c\,\Gamma(z,w)1$. Expanding gives

$$\tau = 1 + c\,exp(\sum_{i\geq 0}(z^i - w^i)x_i)exp(-\sum_{i\geq 0}(z^{-i} - w^{-i})\frac{\partial_{x_i}}{i})1$$

Evaluation on 1 kills the partials giving $\tau = 1 + c\,exp(\sum_{i\geq 0}(z^i - w^i)x_i)1$.
If we set all but x_1, x_2, x_3 to zero we get that

$$\tau = 1 + c\,exp(x_1(z-w) + x_2(z^2 - w^2) + x_3(z^3 - w^3))1$$

is a 1 soliton solution to the KP equation. Applying the usual change of variables will give the formula seen in fluid mechanics texts.

Solutions can be obtained from the Virasoro algebra in a similar fashion. Virasoro is given by the generating series:

$$: Y(e^{\alpha}, z)\partial_z Y(e^{-\alpha}, z) := \frac{1}{2} : \alpha(z)\alpha(z) :$$

3.7 Twisted modules

In this section we will review twisted modules of a lattice vertex algebra following the work of [BK].

Let V_Q be a lattice vertex algebra with lattice Q. Let σ be an automorphism of Q of order N. We give the definition of a σ-twisted module.

Definition 3.7.1. *A σ-twisted module M of a vertex algebra V is equipped with a linear* *map from* V *to* N*-twisted fields,* $a \to Y^{TW}(a,z) = \sum_{n \in \frac{1}{N}\mathbb{Z}} a_n^{TW} z^{-n-1}$, $a_n^{TW} \in End(M)$, *with the axioms below.*

Vacuum: $Y^{TW}(|0\rangle, z) = I$ *where* I *is the identity in the module,*

Automorphism: $Y^{TW}(\sigma a, z) = Y^{TW}(a, e^{2\pi i} z)$,

Borcherds identity:

$$\sum_{j\geq 0} \binom{m}{j} (a_{n+j}^{TW} b^{TW})_{m+k-j} c^{TW} = \sum_{j\geq 0} (-1)^j \binom{n}{j} a_{m+n-j}^{TW} (b_{k+j}^{TW} c^{TW})$$

$$- \sum_{j\geq 0} (-1)^{j+n} \binom{n}{j} b_{n+k-j}^{TW} (a_{m+j}^{TW} c^{TW}) \tag{3.8}$$

for $a, b \in V$, $c \in M$.

We specifically want twisted modules over a lattice vertex algebra. Thus we will now expand [BK]'s formula for a twisted lattice vertex algebra module:

$$Y^{TW}(e^{\gamma}, z) = z^{b_\gamma} U_\gamma^{TW} E_\gamma^{TW}(z) \tag{3.9}$$

We now need to describe b, U, and E:

$$b_\gamma = \frac{|\gamma_0|^2 - |\gamma|^2}{2} \tag{3.10}$$

U^{TW} is defined by the relation

$$U_{\gamma_1}^{TW} U_{\gamma_2}^{TW} = \epsilon(\gamma_1, \gamma_2) B_{\gamma_1, \gamma_2}^{-1} U_{\gamma_1+\gamma_2}^{TW} \tag{3.11}$$

where

$$\epsilon(\gamma_1,\gamma_2)\epsilon(\gamma_2,\gamma_1) = (-1)^{(\gamma_1,\gamma_2)+|\gamma_1|^2|\gamma_1|^2} \tag{3.12}$$

and for a lattice automorphism σ of order N

$$B_{\gamma_1,\gamma_2} = N^{-(\gamma_1,\gamma_2)} \prod_{k=1}^{N-1} (1 - e^{\frac{2\pi i k}{N}})^{(\sigma^k \gamma_1,\gamma_2)}. \tag{3.13}$$

Also

$$E_\gamma^{TW}(z) =: exp(\int \gamma^{TW}(z)) : \tag{3.14}$$

where $\gamma^{TW}(z) = \sum_{n \in \frac{1}{N}\mathbb{Z}} \gamma_n^{TW} z^{-n-1}$.

Here $\gamma^{TW}(z)$ are the twisted analog of the Heisenberg currents, and act on the Fock space as an irreducible highest weight module with highest weight 1.

Chapter 4

Results

4.0.1 introduction

In this chapter we begin with a review of the charged free bosons. We then construct the Hirota equations for the charged free boson hierarchy and some explicit equations and solutions are found. The majority of the chapter discusses reductions of the hierarchy. At the end of the chapter we discuss progress towards an important open problem.

4.1 Charged free bosons

This section follows [W].

Consider the charged free bosons

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n}, \qquad a^*(z) = \sum_{n\in\mathbb{Z}} a^*_n z^{-n-1}$$

where commutation relations are given by

$$[a_n, a_m^*] = \delta_{m,-n}, \qquad [a_n, a_m] = [a_n^*, a_m^*] = 0.$$

We can also write the commutators in terms of the operators as follows:

$$[a(z), a^*(w)] = \delta(z-w) = \sum_{n \in \mathbb{Z}} z^n w^{-n-1}.$$

Lemma 4.1.1. *A representation of gl_∞ is given by*

$$E_{n,m} = a_n^* a_{-m} + \delta_{n,m \leq 0}.$$

We can check this by commuting the commutators:

$$[a_n^* a_m, a_i^* a_j] = a_n^* a_m a_i^* a_j - a_i^* a_j a_n^* a_m$$

$$= a_n^* a_i^* a_m a_j + \delta_{m,-i} a_n^* a_j - a_i^* a_j a_n^* a_m = \delta_{m,-i} a_n^* a_j - \delta_{n,-j} a_i^* a_m.$$

The above relation can also be written in terms of generating functions as the coefficient of $z^{-n-1} w^{-n-1}$ in $: a^*(z) a(w) :$ is $: a_n^* a_m := E_{n,-m}$.

We say that $: a^*(z) a(w) :$ generates gl_∞.

4.2 Casimir

At this point it is possible to compute the Casimir element of gl_∞, which was first done by Kac and Van de Leur [KV], and find one formulation of the charged free boson hierarchy. These are simple equations and can be solved without techniques of integrable systems. However, as with the KP hierarchy, we are able to easily compute solutions to these equations. The image of these solutions under the boson-boson correspondence become interesting. We compute the hierarchy here and discuss solutions more thoroughly in a later section.

The Casimir element Ω is

$$\Omega = Res_z a^*(z) \otimes a(z) = \sum_{n\in\mathbb{Z}} a_n^* \otimes a_{-n}.$$

The hierachy is given by $\Omega(\tau \otimes \tau) = 0$ where solutions to the PDEs are call τ functions.

Proposition 4.2.1. *The Casimir commutes with the diagonal action of gl_∞.*

Proof. We check:

$$[a_n^* \otimes a_{-n}, 1 \otimes a_i^* a_j + a_i^* a_j \otimes 1]$$

$$= a_n^* \otimes a_{-n} a_i^* a_j + a_n^* a_i^* a_j \otimes a_{-n} - a_n^* \otimes a_i^* a_j a_{-n} - a_i^* a_j a_n^* \otimes a_{-n}$$

$$= \delta_{n,i} a_n^* \otimes a_j - \delta_{n,-j} a_i^* \otimes a_{-n} = 0.$$

□

To produce PDEs a representation of the charged free bosons on the space of polynomials is needed. The generating series $a^*(z)$ and $a(z)$ give polynomials in

$\mathbb{C}[x_1, x_2, x_3 \ldots ; y_0, y_1, y_2 \ldots]$ using the below isomorphism:

$a_n = y_{-n}$ for $n \leq 0$, $a_n = \partial_{x_n}$ for $n > 0$,

$a^*_n = x_{-n}$ for $n < 0$, $a^*_n = -\partial_{y_n}$ for $n \geq 0$.

Note that this is an irreducible highest weight representation with highest weight vector 1 satisfying $a_n 1 = 0$ for $n > 0$ and $a^*_n 1 = 0$ for $n \geq 0$.

Applying this isomorphism to $\Omega(\tau \otimes \tau) = 0$ gives

$(\sum_{n>0} x_n \otimes \partial_{x_n} - \sum_{n\geq 0} \partial_{y_n} \otimes y_n)(\tau \otimes \tau) = 0.$

Solutions to this equation are of the form $\tau = exp(\sum_{m>0,n\geq 0} c_{n,m} x_m y_n)$.

We check this directly. We have:

$$\partial_{x_k} \tau = (\sum_{n>0} c_{n,k} y_n) exp(\sum_{m>0,n\geq 0} c_{n,m} x_m y_n),$$

$$\partial_{y_k} \tau = (\sum_{m>0} c_{k,m} x_m) exp(\sum_{m>0,n\geq 0} c_{n,m} x_m y_n);$$

therefore

$$(\sum_{n>0} x_n \otimes \partial_{x_n} - \sum_{n>0} \partial_{y_n} \otimes y_n)(\tau \otimes \tau)$$

$$= \sum_{n>0} x_n exp(\sum_{m,k>0} c_{k,m} x_m y_k) \otimes (\sum_{k>0} c_{k,n} y_k) exp(\sum_{m>0,k\geq 0} c_{k,m} x_m y_k)$$

$$-(\sum_{m>0} c_{n,m} x_m) exp(\sum_{m,k>0} c_{k,m} x_m y_k) \otimes y_n exp(\sum_{m,k>0} c_{k,m} x_m y_k) = 0.$$

4.3 Boson-Boson Correspondence

In [A] and [W] the charged free boson algebra was embedded into a lattice vertex algebra using the boson-boson correspondence.

Consider the lattice $L = \mathbb{Z}\alpha + \mathbb{Z}\beta$ with the following relations:
$|\alpha|^2 = 1$, $|\beta|^2 = -1$, and $\langle \alpha, \beta \rangle = 0$.

The boson-boson correspondence is given by:
$a^*(z) = Y(e^{-\alpha-\beta}, z)$,
$a(z) =: \alpha(z)Y(e^{\alpha+\beta}, z) := Y(\alpha_{-1}e^{\alpha+\beta}, z)$.

Recall from sections 3.1 and 3.2
$[Y(a,z), Y(b,w)] = \sum_{n/ge0} Y(a_n b, w)\frac{\partial_w^n}{n!}\delta(z-w)$,
$[\alpha_n, e_m^{\pm(\alpha+\beta)}] = \pm e_{m+n}^{\pm(\alpha+\beta)}$,
and $e_n^{\gamma_1} e^{\gamma_2} =: S_{-n-1}(\gamma_1, \frac{\partial\gamma_1}{2!}, \frac{\partial^2\gamma_1}{3!}...)e^{\gamma_1+\gamma_2}$ for $\gamma_1, \gamma_2 = \pm(\alpha+\beta)$

We can now check the commutation relations. We have

$$[Y(e^{-\alpha-\beta}, z), Y(e^{-\alpha-\beta}, w)] = \sum_{n/ge0} Y(e_n^{-\alpha-\beta}e^{-\alpha-\beta}, w)\partial_w^n\delta(z-w).$$

Notice $e_n^{-\alpha-\beta}e^{-\alpha-\beta} = 0$ for $n \geq 0$ since Schur polynomials with negative indices are zero by definition. Thus $[Y(e^{-\alpha-\beta}, z), Y(e^{-\alpha-\beta}, w)] = 0$.

Similarly

$$[Y(\alpha_{-1}e^{\alpha+\beta}, z), Y(\alpha_{-1}e^{\alpha+\beta}, w)] = \sum_{n/ge0} Y((\alpha_{-1}e^{\alpha+\beta})_n \alpha_{-1}e^{\alpha+\beta}, w)\partial_w^n \delta(z-w).$$

To compute $(\alpha_{-1}e^{\alpha+\beta})_n \alpha_{-1}e^{\alpha+\beta}$ we will need the -1st product identity given in section 3.1.

Applying the -1st product identity,

$$(\alpha_{-1}e^{\alpha+\beta})_n \alpha_{-1}e^{\alpha+\beta} = \sum_{j<0} \alpha_j e^{\alpha+\beta}_{n-j-1} \alpha_{-1}e^{\alpha+\beta} + \sum_{j\geq 0} e^{\alpha+\beta}_{n-j-1} \alpha_j \alpha_{-1} e^{\alpha+\beta}.$$

After a lengthy computation using the above formulas one finds

$$[Y(\alpha_{-1}e^{\alpha+\beta}, z), Y(\alpha_{-1}e^{\alpha+\beta}, w)] = 0.$$

Finally

$$[Y(e^{-\alpha-\beta}, z), Y(\alpha_{-1}e^{\alpha+\beta}, w)] = \sum_{n/ge0} Y(e_n^{-\alpha-\beta} \alpha_{-1}e^{\alpha+\beta}, w)\partial_w^n \delta(z-w).$$

Now

$$e_n^{-\alpha-\beta} \alpha_{-1} e^{\alpha+\beta} = \alpha_{-1} e_n^{-\alpha-\beta} e^{\alpha+\beta} - e_{n-1}^{-\alpha-\beta} e^{\alpha+\beta} = -1.$$

Thus

$$[Y(e^{-\alpha-\beta}, z), Y(\alpha_{-1}e^{\alpha+\beta}, w)] = -\delta(z-w).$$

Under the boson boson correspondence gl_∞ is now generated by

$: Y(e^{-\alpha-\beta}, z) : \alpha(w) Y(e^{\alpha+\beta}, w) ::$

4.4 Constructing Hirota Equations

We now construct Hirota equations for the charged free boson integrable hierarchy. We begin by computing the Casimir element Ω of gl_∞ which generates PDEs under the boson-boson correspondence. Solutions to $\Omega(\tau \otimes \tau) = 0$ are called τ functions.

The Casimir element, originally computed by Kac and Van de Leur [KV], is $\Omega = Res_z a^*(z) \otimes a(z)$. The boson-boson correspondence maps this to $\Omega = Res_z Y(e^{-\alpha-\beta}, z) \otimes : \alpha(z) Y(e^{\alpha+\beta}, z) :$.

To expand $\Omega(\tau \otimes \tau) = 0$ we need the following representation of the lattice vertex algebra in the space of polynomials and derivatives in $\mathbb{C}[e^{\alpha+\beta}, e^{-\alpha-\beta}, x_1, x_2, x_3 \ldots ; y_1, y_2 \ldots]$: for $n > 0$

$\alpha_{-n} = n x_n, \quad \alpha_n = \partial_{x_n},$

$\beta_{-n} = n y_n, \quad \beta_n = -\partial_{y_n}.$

This gives

$$\Omega = Res_z e^{-\alpha-\beta} exp(-\sum_{n>0} (x_n + y_n) z^n) exp(\sum_{n>0} (\partial_{x_n} - \partial_{y_n}) \frac{z^n}{n}) \otimes$$

$$: (\sum_{k \geq 0} k x_k z^{k-1} + \sum_{k>0} \partial_{x_k} z^{-k-1}) e^{\alpha+\beta} exp(\sum_{n>0} (x_n + y_n) z^n) exp(-\sum_{n>0} (\partial_{x_n} - \partial_{y_n}) \frac{z^n}{n}) : \ .$$

Denoting the first tensor factor by primes and the second by double primes gives:

$$\Omega = Res_z e^{-\alpha'-\beta'} e^{\alpha''+\beta''} exp(\sum_{n>0}(x''_n + y''_n - x'_n - y'_n) z^n)$$

$$(\sum_{k\geq 0} k x''_k z^{k-1} + \sum_{k>0} \partial_{x''_k} z^{-k-1}) exp(\sum_{n>0}(\partial_{x'_n} - \partial_{y'_n} - \partial_{x''_n} + \partial_{y''_n}) \frac{z^n}{n}).$$

We now make the following change of variables:

$x'' \to w + t$, $x' \to w - t$, $y'' \to x + y$, $y' \to x - y$, $e^{\alpha+\beta} \to q$.

This gives:

$$\Omega = Res_z q'^{-1} q'' exp(\sum_{n>0}(2y_n + 2t_n) z^n)$$

$$(\sum_{k\geq 0} k(w_k + t_k) z^{k-1} + \sum_{k>0}(\partial_{w_k} + \partial_{t_k}) z^{-k-1}) exp(\sum_{n>0}(\partial_{y_n} - \partial_{t_n}) \frac{z^n}{n}).$$

Expanding in terms of Schur polynomials and taking the residue:

$$\Omega = q'^{-1} q'' \sum_{i,j\geq 0} S_i(2y + 2t)$$

$$\big((j-i)(w_{j-i} + t_{j-i}) + \partial_{w_{i-j}} + \partial_{t_{i-j}} + \delta_{i,j}(w_0 + t_0)\big) S_j(\bar{\partial}_y - \bar{\partial}_t)$$

We also expand using Taylor's formula finally arrive at:

Theorem 4.4.1. *We have* $q'^{-1}q'' \sum_{i,j\geq 0} S_i(2y+2t)$
$\big((j-i)(w_{j-i}+t_{j-i}) + \partial_{w_{i-j}} + \partial_{\mu_{i-j}} + \delta_{i,j}(w_0+t_0)\big)$
$S_j(\bar{\partial_\lambda} - \bar{\partial_\mu}) exp(\sum_{k\geq 1} y_k \partial_{\lambda_k}) exp(\sum_{l\geq 1} t_l \partial_{\mu_l}) \tau(x-\lambda, w-\mu; q')\tau(x+\lambda, w+\mu; q'')|_{\lambda=\mu=0} = 0$

where $\bar{\partial_x} = (\partial_{x_1}, \frac{1}{2}\partial_{x_2}, \frac{1}{3}\partial_{x_3} \dots)$ *and any variable with a negative index is 0.*

The charged free boson hierarchy is most easily compared to the 2-KP hierarchy. As with the 2-KP hierarchy τ functions will have two indices due to the q's and individual PDEs can be found by looking at coefficients of y and t. However, neither hierarchy is a reduction of the other. Also, each coefficient of $y_p t_q$ will split into one equation plus a second one multiplied by w_0.

The coefficient of $y_p t_q w_0$ is actually quite nice so we include the general formula:

$$(4S_{p+q}(\bar{\partial_\lambda} - \bar{\partial_\mu}) + 2S_q(\bar{\partial_\lambda} - \bar{\partial_\mu})\partial_{\lambda_p} + 2S_p(\bar{\partial_\lambda} - \bar{\partial_\mu})\partial_{\mu_q} + \partial_{\mu_q}\partial_{\lambda_p})$$

$$\tau(x-\lambda, w-\mu; q')\tau(x+\lambda, w+\mu; q'')|_{\lambda=\mu=0} = 0.$$

This gives us the following PDEs:

$p=1, q=1, f = \partial_{x_1} log(\tau(x_1, w_1)), \quad (4\partial_{x_1} - 3\partial_{w_1})f = 0,$

$p=2, q=1, f = log(\tau(x_1, w_1, x_2, w_2)), \quad (4\partial_{x_1,x_2} - 2\partial_{x_1,w_2} - 2\partial_{w_1,x_2} + 2\partial_{w_1,w_2})f = 0,$

$p=1, q=2, f = \partial_{x_2} log(\tau(x_1, w_1, x_2, w_2)), \quad (3\partial_{x_1} - 2\partial_{w_1})f = 0,$

4.5 Soliton-like Solutions

Computing soliton solutions for the KP hierarchy is easy due to the fact that fermions are square free. For the charged free boson hierarchy it is non-trivial to explicitly compute elements of GL_∞ by exponentiation. Note that our representation of gl_∞ is a Lie algebra representation, multiplication in the vertex algebra often does not correspond to multiplication in gl_∞.

Formally, 1-soliton solutions are derived by computing $exp(c\,X)$ where c is a constant and $X \in gl_\infty$. In terms of vertex operators we use the generating series Γ for gl_∞ computed in section 4.3:

$$\Gamma(z,w) =: Y(e^{-\alpha-\beta}, z) : \alpha(w) Y(e^{\alpha+\beta}, w) :: \ .$$

In general, this exponential can only be formally expanded as a series. To find explicit formulas we set $z = w$. This gives shift operators, which function identically to those used in computations with fermions [KR2].

$\Gamma(z,z)$ was originally computed by Wang [W], note that his notation is quite different.

Lemma 4.5.1. $\Gamma(z,z) = -\beta(z)$.

Proof. We compute

$$\Gamma(z,z) =: Y(e^{-\alpha-\beta}, z) : \alpha(z) Y(e^{\alpha+\beta}, z) ::$$

$$=: Y(e^{-\alpha-\beta}, z)Y(\alpha_{-1}e^{\alpha+\beta}, z):$$

$$= Y(e^{-\alpha-\beta}_{-1}\alpha_{-1}e^{\alpha+\beta}, z):\ .$$

Now

$$e^{-\alpha-\beta}_{-1}\alpha_{-1}e^{\alpha+\beta} = \alpha_{-1}e^{-\alpha-\beta}_{-1}e^{\alpha+\beta} + e^{-\alpha-\beta}_{-2}e^{\alpha+\beta}$$

$$= \alpha_{-1} + S_1(-\alpha-\beta) = -\beta.$$

□

Thus $exp(c\Gamma(z,z)) = exp(-c\beta(z))$. Since modes of $-\beta(z)$ are shift operators, by definition the corresponding matrices $-\beta_n \to B_n$ where $B_n = \sum_{k\in\mathbb{Z}} E_{k,k+n}$ have the property $B_iB_j = B_{i+j}$. We can now explicitly compute $exp(c\Gamma(z,z)) = e^{cB_0}exp(c\sum_{n\neq 0} B_1^n)$.

Through computation one also finds $\lim_{w\to z}\Gamma(z,w)$ is well defined and equals $\Gamma(z,z)$. Thus in a neighborhood around $z = w$ explicit formulas for soliton-like solutions can be approximated.

4.6 Twisted reduction

One of the advantages of using a lattice vertex algebra is the ease of finding reductions for a hierarchy. The methods we use here can also be applied to the N-KP hierarchies to find their natural reductions quickly and without any guesswork. This is clear from our method of construction.

We find reductions by twisting the root lattice. As with 2-KP, the automorphism

$x \to -x$, $x = \alpha, \beta$ gives τ functions that do not have indices by eliminating the zero modes.

Using general formulas from [BK] we calculate the twisted $:\alpha(z)Y(e^{\alpha+\beta}, z):$ and $Y(e^{-\alpha-\beta}, z)$. Denote twisted operators with TW.

Recall from section 3.7 that $Y^{TW}(e^{\gamma}, z) = z^{b_\gamma} U_\gamma^{TW} E_\gamma^{TW}(z)$. We now find b, U, and E for $\gamma = \pm(\alpha + \beta)$.

From section 3.7, $b_\gamma = \frac{|\gamma_0|^2 - |\gamma|^2}{2}$ but $|\pm(\alpha + \beta)|^2 = 0$ so $b_{\pm(\alpha+\beta)} = 0$.

Also from section 3.7, $U_{\gamma_1}^{TW} U_{\gamma_2}^{TW} = \epsilon(\gamma_1, \gamma_2) B_{\gamma_1,\gamma_2}^{-1} U_{\gamma_1+\gamma_2}^{TW}$ where $\epsilon(\gamma_1, \gamma_2)\epsilon(\gamma_2, \gamma_1) = (-1)^{(\gamma_1,\gamma_2)+|\gamma_1|^2|\gamma_1|^2}$ and for a lattice automorphism σ of order 2 we have
$B_{\gamma_1,\gamma_2} = 2^{-(\gamma_1,\gamma_2)}(1 - e^{\pi i})^{(\sigma\gamma_1,\gamma_2)}$.

Thus $\epsilon(\alpha + \beta, -\alpha - \beta) = 1$.

and $B_{\alpha+\beta,-\alpha-\beta} = 2^{-(\alpha+\beta,-\alpha-\beta)}(1 - e^{\pi i})^{(-\alpha-\beta,-\alpha-\beta)} = 1$.

This gives $U_{\alpha+\beta}^{TW} U_{-\alpha-\beta}^{TW} = \epsilon(\alpha + \beta, -\alpha - \beta) B_{\alpha+\beta,-\alpha-\beta}^{-1} U_0^{TW} = 1$.

We will pick $U_{\alpha+\beta}^{TW} = U_{-\alpha-\beta}^{TW} = 1$.

Now this just leaves us to find $E_\gamma^{TW}(z)$. Once again we refer the reader back to section 3.7 giving $E_\gamma^{TW}(z) =: exp(\int \gamma^{TW}(z)) :$. Fortunately $\gamma^{TW}(z)$ for an automorphism of order 2 is easy to calculate:

$$\gamma^{TW}(z) = \sum_{n\in\frac{\mathbb{Z}}{2}} \gamma_n^{TW} z^{-n-1} \text{ where } \gamma_n^{TW} = \gamma_n \text{ for } n \in \mathbb{Z} \text{ and 0 otherwise.}$$

This gives us
$Y^{TW}(e^{-\alpha-\beta}, z) = exp(-\sum_{n\in\mathbb{Z}+1/2} (\alpha_{-n} + \beta_{-n})\frac{z^n}{n})$ and
$Y^{TW}(e^{\alpha+\beta}, z) = exp(\sum_{n\in\mathbb{Z}+1/2} (\alpha_{-n} + \beta_{-n})\frac{z^n}{n})$.

We now have to calculate $a_{TW}(z)$. To do this we use the following nth product formula from[BK] with $n = -1$:
$Y^{TW}(\alpha, z)_{-1} Y^{TW}(e^{\alpha+\beta}, z) = \sum_{m\geq 0} \binom{-1/2}{m} z^{-m} Y^{TW}(\alpha_{m-1} e^{\alpha+\beta}, z)$
and recall from section 3.2 that $\alpha_{m-1}e^{\alpha+\beta} = 0$ for $m > 1$ and $\alpha_0 e^{\alpha+\beta} = e^{\alpha+\beta}$.

Proposition 4.6.1. *We obtain* $a^*_{TW}(z) = exp(-\sum_{n\in\mathbb{Z}+1/2} (\alpha_{-n} + \beta_{-n})\frac{z^n}{n})$,
$a_{TW}(z) =: (\sum_{n\in\mathbb{Z}+1/2} \alpha_n z^{-n-1}) exp(\sum_{n\in\mathbb{Z}+1/2} (\alpha_{-n} + \beta_{-n})\frac{z^n}{n}) :$
$-\frac{1}{2z} exp(\sum_{n\in\mathbb{Z}+1/2} (\alpha_{-n} + \beta_{-n})\frac{z^n}{n})$.

Once again define $\Omega_{TW} = Res_z a^*_{TW}(z) \otimes a_{TW}(z)$. However, due to the correction in the normally ordered product this is now no longer a highest weight module. Thus a second correction term is needed and the correct definiton is $\Omega_{TW} = a^*_{TW}(z) \otimes a_{TW}(z) + \frac{1}{2}$.

Denoting the first tensor factor by primes and the second by double primes gives:

$$\Omega_{TW} = Res_z exp(\sum_{n=1/2}^{\infty}(x''_n+y''_n-x'_n-y'_n)z^n)(\sum_{k=1/2}^{\infty}kx''_k z^{k-1}+\sum_{k=1/2}^{\infty}\partial_{x''_k}z^{-k-1})exp(\sum_{n=1/2}^{\infty}(\partial_{x'_n}-\partial_{y'_n}-\partial_{x''_n}+\partial_{y''_n})\frac{z^n}{n})$$
$$-\tfrac{1}{2z}exp(\sum_{n=1/2}^{\infty}(x''_n+y''_n-x'_n-y'_n)z^n)exp(\sum_{n=1/2}^{\infty}(\partial_{x'_n}-\partial_{y'_n}-\partial_{x''_n}+\partial_{y''_n})\frac{z^n}{n})+\frac{1}{2}$$

We now make the following change of variables:

$x'' \to w+t$, $x' \to w-t$, $y'' \to x+y$, $y' \to x-y$

This gives:

$$\Omega_{TW} = Res_z exp(\sum_{n=1/2}^{\infty}(2y_n+2t_n)z^n)(\sum_{k=1/2}^{\infty}k(w_k+t_k)z^{k-1}+\sum_{k=1/2}^{\infty}(\partial_{w_k}+\partial_{t_k})z^{-k-1})exp(\sum_{n=1/2}^{\infty}(\partial_{y_n}-\partial_{t_n})\frac{z^n}{n})-\frac{1}{2z}exp(\sum_{n=1/2}^{\infty}(2y_n+2t_n)z^n)exp(\sum_{n=1/2}^{\infty}(\partial_{y_n}-\partial_{t_n})\frac{z^n}{n})+\frac{1}{2}$$

Theorem 4.6.2. *We have:* $\sum_{i,j\geq 0} S_i(2y+2t)\left((j-i)(w_{j-i}+t_{j-i})+\partial_{w_{i-j}}+\partial_{\mu_{i-j}}\right)$
$S_j(\bar{\partial_\lambda}-\bar{\partial_\mu})exp(\sum_{k\geq 1}y_k\partial_{\lambda_k})exp(\sum_{l\geq 1}t_l\partial_{\mu_l})\tau(w-\lambda,x-\mu)\tau(w+\lambda,x+\mu)|_{\lambda=\mu=0}$

$$-\left(\sum_{i\geq 0}S_i(2y+2t)S_i(\bar{\partial_\lambda}-\bar{\partial_\mu})-1\right)exp(\sum_{k\geq 1}y_k\partial_{\lambda_k})exp(\sum_{l\geq 1}t_l\partial_{\mu_l})\tau(x-\lambda,w-\mu)\tau(x+\lambda,w+\mu)|_{\lambda=\mu=0}=0$$

This gives us the following PDEs:

$p=1, q=1, f=log(\tau(x_1,w_1)), \quad (16\partial_{x_1,w_1}+3\partial_{x_1,x_1}+\partial_{w_1,w_1})f=0,$

$p=3, q=1, f=\partial_{x_1}log(\tau(x_1,x_3)), \quad 11\partial_{w_1,w_1,w_1}f+66(\partial_{x_1}f)^2-32\partial_{w_3}f=0,$

$p=1, q=3, f=\partial_{x_1}log(\tau(x_1,x_3)), \quad 13\partial_{w_1,w_1,w_1}f+78(\partial_{x_1}f)^2+32\partial_{w_3}f=0.$

4.7 $\alpha = 0$ reduction

We show that setting $\alpha = 0$ gives the KP hierarchy. However, as we see below, setting $\alpha = 0$ in the algebra and in the equations is not equivalent. While the Hirota equations behave the same regardless of where we make the reduction, setting variables equal to zero in the generating series for gl_∞ does not give another copy of gl_∞ using fewer variables. This means that most solutions of the charged free boson integrable hierarchy with some variables set equal to zero can not be found by setting setting variables equal to zero in the generating series for gl_∞.

Define the following operators:

$$a_B^*(z) = Y(e^{-\beta}, z), \qquad a_B(z) = Y(e^{\beta}, z), \qquad \Omega_B = Res_z a_B^*(z) \otimes a_B(z).$$

We write

$$Y(e^{\pm\beta}, z) = z^{\pm\beta_0} e^{\pm\beta} exp(\sum_{n>0} \pm\beta_{-n} \frac{z^n}{n}) exp(\sum_{n<0} \pm\beta_n \frac{z^{-n}}{-n})$$

when it is convenient.

We will call Ω_B our Casimir operator.

First we show that Ω_B does not commute with $Y(e^{-\beta}, z)Y(e^{\beta}, w)$, which should be the generating series for gl_∞. Note we left off the normally ordered product as $Y(e^{-\beta}, z)$ and $Y(e^{\beta}, w)$ commute. We show this using the commutator formula from section 3.1 and the formulas from section 3.2.

$$[Y(e^{-\beta},z),Y(e^{\beta},w)] = \sum_{n>0} Y(e_n^{-\beta}e^{\beta},w)\frac{\partial_w^n}{n!}\delta(z-w)$$

But this is zero since $e_n^{-\beta}e^{\beta} =: S_{-n-2}(-\beta, \frac{\partial -\beta}{2!}, \frac{\partial^2 -\beta}{3!}...) : e^0$ and Schur polynomials with negative indices are zero by definition.

We also need the following commutators, note the computation of $[Y(e^{\beta},z),Y(e^{\beta},w)]$ is similar :

$$[Y(e^{\beta},z),Y(e^{\beta},w)] = \sum_{n>0} Y(e_n^{\beta}e^{\beta},w)\frac{\partial_w^n}{n!}\delta(z-w)$$

where $e_n^{\beta}e^{\beta} =: S_{-n}(\beta, \frac{\partial\beta}{2!}, \frac{\partial^2\beta}{3!}...) : e^{2\beta}$.

Here we get a contribution from $n=0$ giving

$$[Y(e^{\pm\beta},z),Y(e^{\pm\beta},w)] = Y(e^{\pm 2\beta},w)\delta(z-w).$$

Unlike the previous cases discussed in this thesis, $Y(e^{-\beta},z)$ and $Y(e^{\beta},w)$ do not close a subalgebra.

Now we compute

$$[Res_x Y(e^{-\beta},x)\otimes Y(e^{\beta},x), 1\otimes Y(e^{-\beta},z)Y(e^{\beta},w) + Y(e^{-\beta},z)Y(e^{\beta},w)\otimes 1]$$

$$= Res_x Y(e^{-\beta},x)\otimes[Y(e^{\beta},x),Y(e^{-\beta},z)Y(e^{\beta},w)] + [Y(e^{-\beta},x),Y(e^{-\beta},z)Y(e^{\beta},w)]\otimes Y(e^{\beta},x)$$

$$= Res_x Y(e^{-\beta},x)\otimes Y(e^{-\beta},z)Y(e^{2\beta},w)\delta(x-w) - Y(e^{-2\beta},z)Y(e^{\beta},w)\delta(x-z)\otimes Y(e^{\beta},x)$$

$$= Y(e^{-\beta},w)\otimes Y(e^{-\beta},z)Y(e^{2\beta},w) - Y(e^{-2\beta},z)Y(e^{\beta},w)\otimes Y(e^{\beta},z).$$

Here we changed the sign since the computations involved odd operators. Notice the Casimir only commutes with $Y(e^{-\beta}, z)Y(e^{\beta}, w)$ when $z = w$. This happens for other reductions.

We expand $\Omega_B(\tau \otimes \tau)$ to produce the KP hierarchy:

$$\Omega_B(\tau \otimes \tau) = Res_z e^{-\beta} z^{-\beta_0} exp(-\sum_{n>0} \beta_{-n} \frac{z^n}{n}) exp(\sum_{n<0} \beta_n \frac{z^{-n}}{n})$$

$$\otimes z^{\beta_0} e^{\beta} exp(\sum_{n>0} \beta_{-n} \frac{z^n}{n}) exp(-\sum_{n<0} \beta_n \frac{z^{-n}}{n})(\tau \otimes \tau).$$

We use the change of variables from section 4.4 to obtain

$$\Omega_B = Res_z z^{-x_0} e^{-\beta} exp(-\sum_{n>0} x_n z^n) exp(-\sum_{n>0} \partial_{x_n} \frac{z^{-n}}{n}) \otimes z^{x_0} e^{\beta} exp(\sum_{n>0} x_n z^n) exp(\sum_{n>0} \partial_{x_n} \frac{z^{-n}}{n}).$$

Denote the first tensor factor by primes and the second by double primes:

$$\Omega = Res_z e^{-\beta'} e^{\beta''} exp(\sum_{n>0} (x''_n - x'_n) z^n) exp(\sum_{n>0} (-\partial_{x'_n} + \partial_{x''_n}) \frac{z^n}{n}).$$

We now make the change of variables:
$x'' \to x + y$, $x" \to x - y$, $e^{\beta} \to q$ and get

$$\Omega_B(\tau \otimes \tau) = Res_z q'^{-1} q'' exp(\sum_{n>0} (2y_n) z^n) exp(\sum_{n>0} (\partial_{y_n}) \frac{z^n}{n})(\tau(x - y) \otimes \tau(x + y)).$$

Expanding in terms of Schur polynomials gives

$$Res_z q'^{-1} q'' (\sum_{n>0} S_n(2y) z^n)(\sum_{n>0} S_n(\bar{\partial}_y) z^{-n})(\tau(x - y) \otimes \tau(x + y)).$$

We then take the residue and expand using Taylor's formula:

$$q'^{-1}q''(\sum_{n>0} S_n(2y)S_{n+1}(\bar{\partial_\mu})exp(\sum_{l\geq 1} t_l\partial_{\mu_l})\tau(x-\mu)\tau(x+\mu)|_{\mu=0} = 0.$$

This may look slightly different from the KP hierarchy, but setting $t = iy$ and assuming τ is independent of q gives the usual formula.

4.8 $\beta = 0$ reduction

We can also make a reduction by setting $\beta = 0$. This can be done in a straightforward manner using the equations but we must be careful with the solutions. Define the following operators:

$$a_A^*(z) = Y(e^{-\alpha}, z),$$
$$a_A(z) =: \alpha(z)Y(e^{\alpha}, z) : .$$

The Casimir is the expected formula, $\Omega_A = Res_z a_A^*(z) \otimes a_A(z)$ and the expected generating series for gl_∞ is $: a_A^*(z)a_A(w) :$. As in the last section, Ω_A commutes with $: a_A^*(z)a_A(w) :$ only when $z = w$. This computation is similar to the computation in the last section.

Performing the usual algebra to expand the Casimir we arrive at:

$$\Omega_A = Res_z e^{-\alpha}z^{-\alpha_0}exp(-\sum_{n>0}(x_n)z^n)exp(\sum_{n>0}(\partial_{x_n})\frac{z^n}{n})\otimes$$

$$: (\sum_{k\geq 0} kx_k z^{k-1} + \sum_{k>0} \partial_{x_k} z^{-k-1}) e^{\alpha} z^{\alpha_0} exp(\sum_{n>0}(x_n)z^n) exp(-\sum_{n>0}(\partial_{x_n})\frac{z^n}{n}) :$$

Denoting the first tensor factor by primes and the second by double primes gives:

$$\Omega_A = Res_z z^{-\alpha_0'+\alpha_0''} e^{-\alpha''+\alpha''} exp(\sum_{n>0}(x_n''-x_n')z^n)$$

$$(\sum_{k\geq 0} kx_k'' z^{k-1} + \sum_{k>0} \partial_{x_k''} z^{-k-1}) exp(\sum_{n>0}(\partial_{x_n'}-\partial_{x_n''})\frac{z^n}{n}).$$

We now make the following change of variables:

$x'' \to w+t$, $x' \to w-t$, $e^{\alpha} \to q$.

This gives:

$$\Omega_A = Res_z q'^{-1} q'' exp(\sum_{n>0}(2t_n)z^n)$$

$$(\sum_{k\geq 0} k(w_k+t_k)z^{k-1} + \sum_{k>0}(\partial_{w_k}+\partial_{t_k})z^{-k-1}) exp(\sum_{n>0}(-\partial_{t_n})\frac{z^n}{n}).$$

Expanding in terms of Schur polynomials and taking the residue, we have

$$\Omega_A = q'^{-1}q'' \sum_{i,j\geq 0} S_i(2t)\big((j-i)(w_{j-i}+t_{j-i}) + \partial_{w_{i-j}} + \partial_{t_{i-j}} + \delta_{i,j}(w_0+t_0)\big)S_j(-\bar{\partial}_t).$$

Theorem 4.8.1. *We have:*

$$q'^{-1}q'' \sum_{i,j\geq 0} S_i(2t)\left((j-i)(w_{j-i}+t_{j-i}) + \partial_{w_{i-j}} + \partial_{\mu_{i-j}} + \delta_{i,j}(w_0+t_0)\right)$$

$$S_j(-\bar{\partial}_\mu) exp(\sum_{l\geq 1} t_l \partial_{\mu_l}) \tau(x-\mu;q')\tau(x+\mu;q'')|_{\mu=0} = 0.$$

By stripping off coefficients of t_p, assuming τ only depends on the first N variables for some N, and using the change of variables $f = log(\tau)$ we can find the following PDEs:

$p = 1,\ \tau(t_0, t_1, t_2),$

$f_x + yf_{x,y}$=0,

$f_{xxxx} + 6f_{xx}^2 - 3f_{yy} = 0,$

$p = 1,\ \tau(t_0, t_1, t_2, t_3),$

$4f_x + 4yf_{xy} - zf_{xxxx} - 6zf_{xx}^2 + 3zf_{yy} + 4zf_{xz} = 0,$

$4f_{yz} - 6f_{xy}f_{xx} - f_{xxxy} = 0,$

$p = 2,\ \tau(t_0, t_1, t_2),$

$6f_z + 6f_{xyy} + 12f_xf_{yy} + zf_{yyyy} + zf_{yy}^2 - 3zf_{zz} = 0.$

The equation $(\star)$ $f_{xxxx} + 6f_{xx}^2 - 3f_{yy} = 0$ is particularly nice so we will thoroughly investigate its solutions. We will take y to be the time variable and x position. This choice gives that f_{yy} is acceleration.

First let $f \to \epsilon f$. Plugging in to $(\star)$ gives $\epsilon f_{xxxx} + 6\epsilon^2 f_{xx}^2 - 3\epsilon f_{yy} = 0$ Factoring out an ϵ and letting $\epsilon \to 0$ removes the nonlinearity. The resulting equation $f_{xxxx} - 3f_{yy} = 0$ is Euler's beam equation.

For completeness, the dispersion relation can be calculated by linearizing and computing the Forier transform. To linearize, rewrite $(\star)$ as $f_{xxxx} + 6af_{xx} - 3f_{yy} = 0$ where $a = f_{xx}$. Computing the Forier transform and examining the exponent, or equivalently replacing $\partial_y \to -i\omega$ and $\partial_x \to ik$, gives $3\omega^2 = -6ak^2 + k^4$.

To force solutions to approach 0 for large time, we need ω to be real. This happens when $6a < k^2$.

From the dispersion relation it is easy to calculate the group velocity $\omega'(k)$ of traveling wave solutions. We get $\omega'(k) = \frac{-2ak+2/3k^3}{(-2ak^2+1/3k^4)^{(1/2)}}$ provided that $6a < k^2$.

To look for traveling wave solutions, assume $f(x, y) = g(x - cy)$. Plugging into $(\star)$, we get $3c^2g'' = g^{(4)} + 6(g'')^2$. Letting $h = g''$ gives $h'' = 3c^2h - 6h^2$. This gives solutions which are elliptic functions.

We also notice that if $f(x, y)$ is a solution to $(\star)$, so is $f(cx, c^2y)$. This motivates us to look for similarity solutions $f(x, y) = g(\frac{x}{\sqrt{y}})$. Once again plugging into $(\star)$ and denoting $\frac{x}{\sqrt{y}} = w$ gives:
$\frac{g^{(4)}}{y^2} + \frac{g''^2}{y^2} = \frac{x^2g''}{4y^3} + \frac{3xg'}{4y^{5/2}}$. We can write this as an equation for $g' = h$ giving $4h^{(3)} + 24(h')^2 = 3w^2h' + 9wh$.

4.9 Virasoro solutions with $\beta = 0$

Virasoro with $\beta = 0$ is by definition $: a_A^*(z)\partial_z a_A(z) :$. Expanding in terms of lattice operators gives $: \alpha(z)\alpha(z) : + \partial_z\alpha(z)$.

We further expand in terms of modes to get:

$$\sum_{m,n>0} nmx_nx_mz^{n+m-2} + 2\sum_{m,n>0} nx_n\partial_{x_m}z^{n-m-2} + \sum_{m,n>0} \partial_{x_n}\partial_{x_m}z^{-n-m-2}$$
$$+2\sum_{n>0} nx_nx_0z^{n-2} + 2\sum_{n>0} x_0\partial_{x_n}z^{-n-2} + x_0^2z^{-2} + 2\sum_{n>1} n(n-1)x_nz^{n-2}$$
$$+2\sum_{n>0}(-n-1)\partial_{x_n}z^{-n-2} - x_0z^{-2}$$

Taking the coefficient of z^{-k-2} gives

$$L_k = \sum_{m>0}(-k-m)mx_{-k-m}x_m + 2\sum_{m>0}(m-k)x_{m-k}\partial_{x_m} + \sum_{m>0}\partial_{x_{k-m}}\partial_{x_m}$$

$$-2kx_{-k}x_0 + 2x_0\partial_{x_k} + (x_0^2 - x_0)\delta_{k,0} + 2k(k+1)x_{-k} - 2(k+1)\partial_{x_k}$$

where variables with negative indices are zero.

We now exponentiate $c\,L_k$ and apply to 1 which gives the following solutions:

$L_0 \to \tau(x_0) = exp(c\,x_0^2 - c\,x_0)$,

$L_{-1} \to \tau(x_0, x_1, x_2) = exp(4c\,x_0x_1 - 2c\,x_0x_2)$,

$L_{-2} \to \tau(x_0, x_1, x_2, x_3, x_4) = exp(-3c\,x_3^2 - 16c\,x_4x_0 - 6c\,x_3x_1 - 8c\,x_4 + c\,x_1^2 + 4c\,x_0x_2 + 2c\,x_2)$.

4.10 Future Research: Lax Equations

To derive and solve Lax equations for the charged free boson hierarchy we will reverse engineer them from the Hirota equations and then prove we have found the correct operators. Note we caution the reader to observe that while this section appears to be complete we have no proof that the operator S is invertible. This is actually quite a deep question and may lead to new algebra results. The partial results are worthwhile as we have formulated wave equations. These do not depend on S^{-1} so this part of the section is complete.

Since Hirota equations are derived from Lax equations using S^{-1} and we have Hirota equations, it is reasonable to assume some analog of S^{-1} exists. However, ∂_z^{-1} does not exist in the charged free boson vertex algebra. We suspect that this question can be answered algebraically by restricting the space on which S operates. A similar question has also been partially answered geometrically for the super KP hierarchy.

We start by computing wave functions by solving $a(z)\tau = \Psi\tau$ and $a^*(z)\tau = \Psi^*\tau$. Expanding we get:

$$\partial_{z_1} z_1^{x_0} z^{y_0} exp(\sum_{n>0}(x_n z_1^n + y_n z^n) exp(-\sum_{n>0}(\partial_{x_n}\frac{z_1^n}{n} - \partial_{y_n}\frac{z^n}{n})\tau(x,y)|_{z_1=z} = \Psi\tau(x,y),$$

$$z^{-x_0-y_0} exp(-\sum_{n>0}(x_n + y_n)z^n) exp(\sum_{n>0}(\partial_{x_n} - \partial_{y_n})\frac{z^n}{n})\tau(x,y) = \Psi^*\tau(x,y).$$

We use Taylor's formula and divide by τ and make the change of variables $y \to iy$ to obtain the following result.

Theorem 4.10.1. *We have*

$$\Psi = \partial_{z_1} exp(\sum_{n>0}(x_n z_1^n + y_n z^n)\frac{\tau(x - [z_1^{-1}], y - [z^{-1}])}{\tau(x,y)}|_{z_1=z},$$

$$\Psi^* = exp(-\sum_{n>0}(x_n + y_n)z^n)\frac{\tau(x + [z^{-1}], y + [z^{-1}])}{\tau(x,y)},$$

where $\tau(x - [z^{-1}]) = \tau(x_1 - z^{-1}, x_2 - 1/2z^{-2}, x_3 - 1/3z^{-3}...)$.

Our dressing operators are now the wave functions excluding the exponential:

$$S = \partial_{z_1}\frac{\tau(x - [z_1^{-1}], y - [z^{-1}])}{\tau(x,y)}|_{z_1=z},$$

$$S^* = \frac{\tau(x + [z^{-1}], y + [z^{-1}])}{\tau(x,y)}.$$

We will now attempt to derive Lax equations for $L = S(\partial_{x_1} + \partial_{y_1})S^{-1}$, and $L^* =$

$S^*(\partial_{x_1} + \partial_{y_1})(S^*)^{-1}$.

First we note that ∂ is a linear operator so we split L and L^* into x and y parts. The y equations are identical to the KP Lax equations. The x equations require more proof.

Let

$$\Psi_A = \partial_z exp(\sum_{n>0} x_n z^n)\frac{\tau(x - [z^{-1}])}{\tau(x)},$$

$$\Psi_A^* = exp(-\sum_{n>0} x_n z^n)\frac{\tau(x + [z^{-1}])}{\tau(x)},$$

$$S_A = \partial_z \frac{\tau(x - [z^{-1}])}{\tau(x)}|_{z=\partial_{x_1}},$$

$$S_A^* = \frac{\tau(x + [\partial_{x_1}^{-1}])}{\tau(x)},$$

$$L_A = S_A \partial_{x_1} S_A^{-1},$$

$$L_A^* = S_A^* \partial_{x_1} (S_A^*)^{-1}.$$

Also define $\bar{S} = \frac{\tau(x - [\partial_{x_1}^{-1}])}{\tau(x)}$.

We first show

Lemma 4.10.2.

$$Res_z(\sum_{i\geq 0} a_i(\partial_{x_1})z^{-i}\tau(x))(\sum_{n\geq 0} n x_n z^{n-1})(\sum_{j\geq 0} b_j(\partial_{y_1})z^{-j}\tau(y))exp((x_1 - y_1)z)$$

is well defined and zero only if $a_i b_j = 0$ for $x_1 > y_1 > 0$.

Proof. We begin by expanding the exponential and moving any terms independent of z

outside of the residue:

$$\sum_{i,j,n,k\geq 0} a_i(\partial_{x_1})z^{-i}\tau(x)(nx_nz^{n-1})b_j(\partial_{y_1})z^{-j}\tau(y)\frac{(x_1-y_1)^k}{k!}Res_z z^{-i-j+n-1+k}.$$

Taking the residue gives

$$\sum_{i,j,n\geq 0} a_i(\partial_{x_1})z^{-i}\tau(x)(nx_nz^{n-1})b_j(\partial_{y_1})z^{-j}\tau(y)\frac{(x_1-y_1)^{i+j-n}}{(i+j-n)!}$$

which is clearly well defined. $\square$

We now compute $(\frac{\partial}{\partial x_n} - (L_A)^n_+)$ acting on Ψ_A where $(L_A)^n_+$ is the portion of L_A with only positive powers of ∂_{x_1}:

$$(\frac{\partial}{\partial x_n} - (L_A)^n_+)\Psi_A = (\frac{\partial}{\partial x_n} - (L_A)^n_+)S_A exp(\sum_{i>0} x_i z^i)$$

$$= (\frac{\partial S_A}{\partial x_n} + z^n S_A - (L_A)^n_+ S_A) exp(\sum_{i>0} x_i z^i) = (\frac{\partial S_A}{\partial x_n} + S_A\frac{\partial}{\partial x_1^n} - (L_A)^n_+ S_A) exp(\sum_{i>0} x_i z^i)$$

$$= (\frac{\partial S_A}{\partial x_n} + L_A^n S_A - (L_A)^n_+ S_A) exp(\sum_{i>0} x_i z^i) = (\frac{\partial S_A}{\partial x_n} + (L_A)^n_- S_A) exp(\sum_{i>0} x_i z^i).$$

Now consider $Res_z((\frac{\partial}{\partial x_n} - (L_A)^n_+)\Psi_A(x))\Psi^*_A(y)$. This is zero due to the Hirota equations. We now show that implies $(\frac{\partial}{\partial x_n} - (L_A)^n_+)S_A = 0$.

Start by setting $y_i = x_i, i > 1$ giving

$$Res_z((\frac{\partial}{\partial x_n} - (L_A)^n_+)S_A e^{x_1 z})S^*_A(y_1, x)e^{-y_1 z} = 0.$$

Rearranging gives

$$Res_z(\frac{\partial S_A}{\partial x_n} + (L_A)^n_- S_A)e^{x_1 z} S_A^* e^{-y_1 z} = 0.$$

By lemma 4.10.1 this is well defined and $(\frac{\partial S_A}{\partial x_n} + (L_A)^n_- S_A)S_A^* = 0$.

All that is left is to show S_A^* is invertible, but this is clear from the vertex algebra structure. Thus if we find where S_A^{-1} is defined, we will have a Lax formulation of the charged free boson hierarchy.

[A] Adamovic, Drazen. "Representations of the Vertex Algebra $W_{1+\infty}$ with a Negative Integer Central Charge"
Communications in Algebra 29.7 (2001).

[BK] B. Bakalov and V. Kac "Twisted modules over lattice vertex algebras"
Lie theory and its applications in physics V pp. 3-26 (2004).

[H] Humphreys, James. "Introduction to Lie Algebras and Representation Theory"
Springer (1973).

[K1] Kac, Victor. "Vertex Algebras for Beginners"
American Mathematical Society (1997).

[KR1] V. Kac and A. Radul "Representation theory of the vertex algebra $W_{1+\infty}$"
Transf. Groups, Vol. 1 pp. 4170 (1996).

[KR2] V. Kac and A. Raina "Bombay Lectures On Highest Weight Representations of Infinite Dimensional Lie Algebras"
World Scientific Pub Co Inc (1988).

[KV1] V. Kac and J. Van De Leur. "The n-Component KP Hierarchy and Representation Theory"
Springer Series in Nonlinear Dynamics pp. 302 - 343 (1993).

[M] P. van Morebeke. Lectures on Integrable Systems - Integrable Foundations of String Theory
World Scientific (1994).

[W] Wang, Weiqiang. "$W_{1+\infty}$ Algebra, W_3 Algebra, and Friedan-Martinec-Shenker bosonization"

Commun.Math.Phys. 195 pp. 95-111 (1998).