ABSTRACT

WEARS, THOMAS HANNAN. Signature Varieties of Polynomial Functions. (Under the direction of Dr. Irina Kogan.)

In this thesis we study the equivalence of polynomial functions in m variables over the field of complex numbers under the equivalence relation generated by the action of an algebraic subgroup $\mathcal{G}$ of $GL_m(\mathbb{C})$. The study is carried out using a combination of techniques from differential and algebraic geometry. We introduce the notion of a $\mathcal{G}$-signature correspondence which is a map from the space of polynomials in m variables to the space of algebraic subvarieties of some algebraic variety over $\mathbb{C}$. For an almost complete $\mathcal{G}$-signature correspondence, two generic polynomial functions are equivalent under the action of $\mathcal{G}$ if and only if their corresponding algebraic subvarieties under the $\mathcal{G}$-signature correspondence are equal. While the notion of a $\mathcal{G}$-signature correspondence is completely constructive, its implementation relies on elimination algorithms, which are known to have high computational complexity. The advantage of using a $\mathcal{G}$-signature correspondence to address the $\mathcal{G}$-equivalence of polynomials in comparison with other methods lies in its universality: the same construction applies to polynomials in any number of variables of any degree.

Signature Varieties of Polynomial Functions

by
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DEDICATION

To my mother, Katherine Hannan Wears, and my wife, Shelby Wears.

Every man needs at least one good woman in his life. Some men are fortunate enough

to have two.

BIOGRAPHY

The author was born[1] in the 1980's in a small town located on the shores of the mighty St. Lawrence River[2]. With one brother and two sisters, the author's childhood was filled with a variety of mischief and misadventures that could be found in any healthy (or unhealthy) American family of the same time period. By graduating from O.C.C.S in the spring of 1993 the author began to show the promise that once prompted his mother to declare that he would "either end up in prison or become a saint."[3] Around the same time, the author also had another transformational event in his life; he took his future wife, Shelby Rae Spriggs, on their first date to the movies.[4] Despite the fact that Ms. Spriggs' opinion of (and interest in) the author would change on a number of occasions over the next 10 years, the author was persistent in his pursuit and in late 2005 the author finally convinced Ms. Spriggs to marry him. In late 2006 the author and Ms. Spriggs were married in the cathedral located in the small river town of the author's birth and in April of 2010 they welcomed a son, Owen Thomas Wears, to the world.

After graduating from O.C.C.S, the author's love for mathematics began to show during his middle school years. During this time period it was not uncommon to find the author working on his math homework during Ms. Winchester's art class. After earning his high school diploma from O.F.A around the turn of the century, the author enrolled at Duquesne University, where he would graduate with a B.S. in mathematics.

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[1] Anyone that is reading this with the hope of trying to determine the author's date of birth is going to be disappointed.

[2] Despite rumors to the contrary the small town of the author's birth was on the United States side of the St. Lawrence River.

[3] It is likely that the prediction made by the author's mother will not end up being correct.

[4] *Indian Summer.*

Upon graduating from Duquesne University, the author immediately enrolled in graduate school at North Carolina State University to pursue his Ph.D. in mathematics. The author plans to continue studying mathematics and hopes to share his love of mathematics and life with anyone interested.

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There is an almost mystical, unexplainable togetherness that unites the world as human beings and the far reaching impact and influence that our actions have on others is something that we will never truly be able to understand, explain or comprehend. As such, I would first like to acknowledge several people whom I have never formally met, and yet, without them, none of this would have been possible. They will likely never know that they have been acknowledged for their help in the completion of this thesis, and you will almost certainly never read their biography on your Kindle.[5] They are real people with real problems and real treasures who handle them with grace and dignity, with no hope or aspirations for fame or recognition, and yet they inspire others in ways that they will never know. So, I would like to thank the families of Jenna Watkins and Matthew Higgins.

And now for the more formal. . . I would like to express my most sincere thanks and gratitude to my advisor Dr. Kogan. I am forever grateful for the opportunities and support that you have provided me with. In addition, I would also like to thank Dr. Hoon Hong, Dr. Nathaniel Reading, Dr. Ernest Stizinger and Dr. Loek Helminck. You have all had a tremendous impact on my life (both mathematical and otherwise) in ways that I will never be able to fully describe in words and in ways that I am sure you are completely unaware. I am, and will be, forever thankful.

I would like to express my most sincere thanks and gratitude to Dr. John Griggs for helping me evolve and develop as a teacher of mathematics. When confronted with problems, he has, with grace and dignity, nurtured and helped innumerable graduate

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[5]Or Nook, etc...

students deal with difficulties in the classroom. Your work and contributions to the NCSU Department of Mathematics do not go unnoticed. Additionally, I would also like to thank the staff of the NCSU Department of Mathematics. In particular, I would like to thank Di Buckland, Julia Reynolds, Carolyn Gunton, Denise Seabrooks, Seyma Bennett-Shabbir, Charlene Wallace, Frankie Stephenson, Janice Gaddy, Nicole Dahlke, and Brenda Currin. You, too, are regularly called upon when people are in need; you have always delivered.

I would also like to take this opportunity to thank a few people from my past who profoundly impacted my life at an early age and helped make possible any success that I may achieve. Thank you to Bill Merna, Mark Brady, Joe Harrison, Lori Rafferty, Shelly Amo, Bill Cavanaugh, David Valois, and Dr. Eric Rawdon. Real heroes and role models can live in small towns and make big impacts.

Finally, I would like to thank my friends and family. To characterize you all as friends and family is a gross understatement. At various times you have played the roles of support staff, therapist, counselor, advisor, baby sitter [6], etc . . . You have offered motivational words and support when I have needed it most and you have helped keep me grounded and in check when appropriate. You have been willing to listen and offer advice when needed. You have helped me through some difficult times and you have made the good times more enjoyable. You have laughed at bad jokes and you have failed to laugh at good jokes, but you have at least usually laughed (even if it was at me). You have listened to me retell the same stories to the point where I am probably the only one being entertained. And through all of this, what startles me most is that you have never

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[6]For Owen, Shelby. Not for me. Fine, sometimes for me.

asked for anything in return. It leaves me (almost) speechless. I sit around sometimes and think of how fortunate I am to have all of you in my life and I wonder what exactly it is that I *bring* to the table. Before any of you start to sit around and wonder the same thing, I have now decided that I will at least bring my own chair.

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Introduction and Statement of the Problem

1.1 Introduction

The aim of this thesis is to study the action of an algebraic subgroup $\mathcal{G} \subset GL_m(\mathbb{C})$ on the set of polynomials in m variables, $\mathcal{P}_m$, induced by a linear change of variables. The primary problem is to decide when two polynomials are equivalent under the action of $\mathcal{G}$. This problem has its roots in classical invariant theory and dates back to the mid 19th century and the likes of Cayley [4], Clebsch [5], Gordan [12], Hilbert [14] [15], and Sylvester [29] [30].

In the 19th century, the primary focus of study centered around the invariants and covariants of a homogeneous polynomial f in m variables, and, more generally, invariants of tensor components under a linear change of basis. In classical terminology, an invariant ϕ is a function that depends only on the coeffecients of f and remains unchanged

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under the action of $\mathcal{G}$, whereas a covariant also depends on the indeterminate variables of the polynomial. During this time period, the main approach was computational and the desire was to prove that the ring of invariants for a system of homogeneous polynomials was finitely generated as a $\mathbb{C}$ algebra. However, this approach changed drastically in the 1890's when Hilbert proved his now celebrated 'Hilbert Basis Theorem.' Rejected at the time for being nonconstructive, Hilbert proved that any finite system of homogeneous polynomials admitted a 'Hilbert Basis'[1] for its covariants. Hilbert's history altering result changed the course of the development of mathematics, as the computational and constructive approaches fell by the wayside and almost completely out of favor by the 1920's, and in their place rose subjects and topics such as commutative algebra [1], geometric invariant theory [20], algebraic geometry [6], and representation theory [11]. And yet, despite the ominous tone in the description of the field, classical invariant theory has recently undergone something of a modern revival [7], [28] due in large part to advances in computer science, symbolic computation and computer algebra systems.

Our approach to studying the action of $\mathcal{G}$ on $\mathcal{P}_m$ blends techniques from differential and algebraic geometry and finds its inspiration in the recent works of Fels, Hubert, Kogan, Moreno Maza, and Olver, [2], [9], [10], [18], [19], [17], [23]. In [23] Olver made the novel observation that for a given $f \in \mathcal{P}_m$, one can view the graph of f, $\Gamma(f)$, as a hypersurface in $\mathbb{C}^{m+1}$ and the study of a linear change of variables can be recast as an equivalence problem for submanifolds of $\mathbb{C}^{m+1}$. In this formulation, the goal is to generate a fundamental[2] set of differential invariants that can be used to parameterize a signature set of f, whereby one is (hopefully) able to determine that polynomials f and g

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[1]Meaning that any other covariant can be written as polynomial function of a finite number generators.
[2]Meaning that any other differential invariant can be expressed locally as a function of the fundamental set.

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are equivalent if and only if they have the same signature set. At approximately the same time, Fels and Olver were developing their equivariant method of moving frames [9], [10] which extended the applicability of moving frame techniques in addressing equivalence problems to a wide variety of situations, including that of classical invariant theory and the equivalence of polynomials under a linear change of variables. In short, the equivariant method of moving frames provides a pseudo-algorithmic approach to generating a fundamental set of differential invariants and differential invariant operators for the action of a Lie group $\mathcal{G}$ on a manifold M.

Signatures parametrized by differential invariants have been used in [2], [18], [19], [23] to great effect to address the equivalence and symmetry of homogeneous polynomials in two variables and homogeneous polynomials in three variables of degree three. These works, however, underscore the challenges one would face in applying these methods to homogeneous polynomials in four or more variables. In the case of polynomials in two variables, the equivariant method of moving frames leads to a 'nice' fundamental set of rational differential invariants [23], [18], whereas in the case of polynomials in three variables one obtains a set of local algebraic invariants [18], which is decidedly more difficult to use in practice. To our knowledge, the computation of differential invariants for polynomials in more than three variables via the moving frame construction was never performed. In [3], the authors define signatures for polynomials in two and three variables based on rational differential invariants obtained by other methods, but the methods do not have a straightforward generalization to polynomials in more than three variables.

In Chapter 2 of the thesis, we review the basics of the equivariant moving frame construction following for the most part the presentation of [10]. We prove that signatures

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of polynomials parametrized by normal differential invariants characterize equivalence classes of polynomials on an open (in the metric topology) subset of the set of polynomials. The proof is an adaptation to the case of polynomial functions of established proofs of more difficult equivalence theorems for generic smooth submanifolds. We illustrate these results in the case of polynomials in two and three variables.

In Chapter 3 of the thesis, we propose a new signature construction that completely circumvents explicit computation of invariants. We define a map from the set of polynomials in m variables to a set of subvarieties of an affine space of sufficiently large dimension. The signature variety of a polynomial can be computed directly from the equations of the group action via elimination algorithms [6]. This construction is completely algorithmic and can be applied to polynomials in any number of variables, although the high complexity of the elimination algorithms may prevent explicit computations of signature varieties. We prove that signature varieties characterize equivalence classes of polynomials on an open (in the Zariski topology) subset of the set of polynomials. The construction is illustrated by examples.

1.2 Statement of the Problem

Let $\mathcal{P}_m$ denote the set of polynomials in m variables. The set of polynomials in m variables of degree less than or equal to d will be denoted by $\mathcal{P}_m^d$ and the set of homogeneous polynomials in m variables of degree d will be denoted by $\mathcal{HP}_m^d$. Let $GL_m(\mathbb{C})$ denote the general linear group and let $\mathcal{G} \subset GL_m(\mathbb{C})$ denote an algebraic subgroup. Elements of $\mathcal{G}$ will be represented by $m \times m$ matrices over $\mathbb{C}$. The standard action of $\mathcal{G}$ on $\mathbb{C}^m$ induces

a linear action of $\mathcal{G}$ on $\mathcal{P}_m$ defined by

$$\Lambda \star f(\mathbf{x}) = f(\Lambda^{-1} \cdot \mathbf{x}) \qquad \forall \Lambda \in \mathcal{G}, \ \mathbf{x} \in \mathbb{C}^m \tag{1.1}$$

Remark 1.2.1. *Both $\mathcal{P}_m^d$ and $\mathcal{HP}_m^d$ are closed under the action of $\mathcal{G}$ on $\mathcal{P}_m$ and thus the action of $\mathcal{G}$ on $\mathcal{P}_m$ can be restricted to either $\mathcal{P}_m^d$ or $\mathcal{HP}_m^d$*

Definition 1.2.1 (Equivalence). *Let $f, g \in \mathcal{P}_m$. We say that f and g are $\mathcal{G}$-equivalent if there exists $\Lambda \in \mathcal{G}$ such that $\Lambda \star f = g$ and we write $f \cong_\mathcal{G} g$ or $f \cong g$ if the group $\mathcal{G}$ is understood.*

Example 1.2.2. *Let $m = 2$ and let points of $\mathbb{C}^2$ be denoted by $\mathbf{x} = (x, y)$. Let $f = x^3 + y^3$, $g = 9x^3 + 15x^2 y + 9xy^2 + 2y^3 \in \mathcal{P}_m$ and let $\Lambda = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \in GL_2(\mathbb{C})$. Then $\Lambda \star f = g$. We merely calculate*

$$\begin{aligned} f\left(\Lambda^{-1} \cdot \mathbf{x}\right) &= (2x + y)^3 + (x + y)^3 \\ &= 9x^3 + 15x^2 y + 9xy^2 + 2y^3 \\ &= g(\mathbf{x}) \end{aligned}$$

Remark 1.2.2. *Observe that $\det(\Lambda) = 1$ and we can also regard f and g as being $SL_2(\mathbb{C})$-equivalent.*

Definition 1.2.3 ($\mathcal{G}$-Signature Correspondence and $\mathcal{G}$-Signature Variety). *A $\mathcal{G}$-signature correspondence for the action of $\mathcal{G}$ on $\mathcal{P}_m$ is a map from $\mathcal{P}_m$ to the set of algebraic subvarieties of some algebraic variety $\mathcal{C}$ over $\mathbb{C}$, such that if the image of $f \in \mathcal{P}_m$ under*

this correspondence is denoted by $\mathcal{S}_f$, then

$$f \cong_{\mathcal{G}} g \Rightarrow \mathcal{S}_f = \mathcal{S}_g.$$

The image of $f \in \mathcal{P}_m$ under a $\mathcal{G}$-signature correspondence is said to be the $\mathcal{G}$-signature variety of f determined by the given $\mathcal{G}$-signature correspondence.

Definition 1.2.4 (Complete and Almost Complete Signature Correspondence). *We say that the $\mathcal{G}$-signature correspondence is* complete *if for all f and $g \in \mathcal{P}_m$,*

$$f \cong_{\mathcal{G}} g \iff \mathcal{S}_f = \mathcal{S}_g. \tag{1.2}$$

A $\mathcal{G}$-signature correspondence is almost complete *if there exists d_0 such that for all $d > d_0$ there exists a Zariski open subset $\mathcal{T}_m^d \subset \mathcal{P}_m^d$ such that (1.2) holds for all f and $g \in \mathcal{T}_m^d$*

Remark 1.2.3. *Given a subset $\mathcal{F} \subset \mathcal{P}_m$ which is closed under the action of $\mathcal{G}$ (e.g., $\mathcal{P}_m^d$ or $\mathcal{HP}_m^d$), then we can speak of a $\mathcal{G}$-signature correspondence for the action of $\mathcal{G}$ on $\mathcal{F}$.*

Using a blend of techniques from differential and algebraic geometry, we present a constructive definition of an almost complete $GL_m(\mathbb{C})$-signature correspondence for $\mathcal{P}_m$ with m arbitrary, and for $d \geq 3$ we give a constructive definition of an almost complete $GL_m(\mathbb{C})$-signature correspondence for $\mathcal{HP}_3^d$.

Chapter 2

Moving Frames and Differential Invariant Signatures

We begin this chapter by first introducing the material for the nth order jet space of analytic functions $f : \mathbb{C}^m \to \mathbb{C}$, which will be denoted by $J^n(\mathbb{C}^m, \mathbb{C})$. All future constructions, including those of a more algebraic nature, will take place on a jet space of appropriate order. We briefly discuss the prolonged action of a linear algebraic subgroup $\mathcal{G}$ of $GL_m(\mathbb{C})$ on $J^n(\mathbb{C}^m, \mathbb{C})$ and then review some of the fundamental geometry associated with $J^n(\mathbb{C}^m, \mathbb{C})$ and the prolonged action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$, including the total derivative operators and (horizontal) total differential operators, both of which will play a role in the method of equivariant moving frames. We follow this with a brief review of local and global differential invariants for the prolonged action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$, which will prove useful both in the context of and independently of the equivariant moving frame method. After a basic discussion of the relevant geometry of $J^n(\mathbb{C}^m, \mathbb{C})$, we provide a brief review and summary of the relevant material from the method of equivariant moving frame maps. Finally, we provide an analysis of the actions of $SL_2(\mathbb{C})$ on $\mathcal{P}_2$ and

$\mathcal{HP}_2^d$ as well as a brief analysis of the action of $GL_3(\mathbb{C})$ on $\mathcal{HP}_3^3$ and $\mathcal{P}_3$ using techniques based on the equivariant method of moving frames and global differential invariants. The chapter will conclude with a brief discussion of the problems that one faces when trying to extend the use of either differential invariants or the method of equivariant moving frames beyond the cases analyzed in this chapter.

2.1 Jet Spaces and Prolongation

2.1.1 Jet Space and Jets of Functions

Let $x = (x^1, \ldots, x^m)$ denote the standard coordinate functions on $\mathbb{C}^m$. Let $GL_m(\mathbb{C})$ denote the group of $m \times m$ matrices over $\mathbb{C}$ with nonzero determinant. Let $\lambda = (\lambda_j^i), 1 \leq i, j \leq m$ denote coordinates on $\mathbb{C}^{m^2}$, which will be used as parameters for the group $GL_m(\mathbb{C})$. For $\Lambda = (\lambda_j^i) \in GL_m(\mathbb{C})$, we will use the standing convention that λ_j^i denotes the entry of the matrix Λ in the ith row and j^{th} column and $\hat{\lambda}_j^i$ will denote the entry of the matrix Λ^{-t} that resides in the ith column and the jth row, where Λ^{-t} denotes the transpose of Λ^{-1}. We will view $GL_m(\mathbb{C})$ as an algebraic subvariety of $\mathbb{C}^{m^2+1}$, where we take coordinates on $\mathbb{C}^{m^2+1}$ to be (λ_j^i, s). The ideal defining $GL_m(\mathbb{C}) \subset \mathbb{C}^{m^2+1}$ is $\left(\det(\lambda_j^i)s - 1\right) \subset \mathbb{C}[\lambda_j^i, s]$, where $\mathbb{C}[\lambda_j^i, s]$ denotes the polynomial ring over $\mathbb{C}$ in the m^2+1 variables (λ_j^i, s). Any algebraic subgroup $\mathcal{G} \subset GL_m(\mathbb{C})$ can be realized as a subvariety of $GL_m(\mathbb{C})$ defined by a radical ideal $G \subset \mathbb{C}[\lambda_j^i, s]$, where $\left(\det(\lambda_j^i)s - 1\right) \subset G$. There are natural (left) actions of $\mathcal{G}$ on $\mathbb{C}^m$ and on the set of functions $f : \mathbb{C}^m \to \mathbb{C}$. The action of $\mathcal{G}$ on $\mathbb{C}^m$ is given by standard matrix multiplication. For $\Lambda = (\lambda_j^i) \in \mathcal{G}$ and

$x = \left(x^1, \ldots, x^m\right) \in \mathbb{C}^m$, the action is given by[1]

$$\Lambda \cdot x = X = \left(X^1, \ldots, X^m\right) = \left(\lambda_i^1 x^i, \ldots, \lambda_i^m x^i\right), \qquad 1 \le i \le m. \tag{2.1}$$

where $X = (X^1, \ldots X^m)$ is a second set of coordinates on $\mathbb{C}^m$ which we view as being the target coordinates for the action map. The action of $\mathcal{G}$ on $\mathbb{C}^m$ is then extended to an action on functions $f : \mathbb{C}^m \to \mathbb{C}$ as follows. For $\Lambda \in \mathcal{G}$ and $f : \mathbb{C}^m \to \mathbb{C}$, the action of Λ on f is denoted by $\Lambda \star f$, where $\Lambda \star f : \mathbb{C}^m \to \mathbb{C}$ is defined by either of the following (equivalent) formulations:

$$\left(\Lambda \star f\right)(\mathbf{x}) = f\left(\Lambda^{-1} \cdot \mathbf{x}\right), \tag{2.2}$$

or

$$\left(\Lambda \star f\right)\left(\Lambda \cdot \mathbf{x}\right) = f\left(\mathbf{x}\right). \tag{2.3}$$

Let $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ denote the nth order jet space of functions from $\mathbb{C}^m \to \mathbb{C}$. Coordinates on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ are given by

$$z^{(n)} = (x, u, u_K) = \left(x^1, \ldots, x^m, u, u_1, u_2, \ldots, u_K, \ldots\right),$$

where K ranges over (symmetric) partial derivative multi-indices[2] with $|K| \le n$.

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[1]The summation convention will be used throughout.

[2]We will adopt the convention that an order n partial derivative multi-index depending on m independent variables is an n-tuple $K = (i_1, i_2, \ldots, i_n)$ where $1 \le i_j \le m$. Any two such tuples will be regarded as being equal if they differ only by a permutation of the entries. When listing out coordinates on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ we will always assume that the indices of a fixed order have been ordered lexicographically with $1 < 2 < \ldots < m$. When there is no chance for confusion, we will drop both the parentheses and the commas from our multi-indices.

Remark 2.1.1. *Note that $J^n(\mathbb{C}^m, \mathbb{C})$ is thus canonically identified with $\mathbb{C}^{N(n,m)}$, where*

$$N(n,m) = m + \binom{m+n}{n}$$

.

Points in $\mathbb{C}^m$ will be denoted by $\mathbf{x}$ or $\mathbf{X}$ and points in $J^n(\mathbb{C}^m, \mathbb{C})$ will be denoted by $\mathbf{z}^{(n)}$ or $\mathbf{Z}^{(n)}$.[3]

Notation 2.1.1. *For a smooth function $f : \mathbb{C}^m \to \mathbb{C}$ and a partial derivative multi-index $K = (k_1, k_2, \ldots k_l)$, we will denote the partial derivative of f with respect to the multi-index K by*

$$f_K = f_{k_1 k_2 \cdots k_l} = \frac{\partial^{|K|} f}{\partial x^{k_l} \cdots \partial x^{k_2} \partial x^{k_1}}. \tag{2.4}$$

Definition 2.1.1 (Jet of a Function). *The nth jet of a smooth function $f : \mathbb{C}^m \to \mathbb{C}$ is the function $\mathrm{j}_n \, f : \mathbb{C}^m \to J^n(\mathbb{C}^m, \mathbb{C})$ defined by*

$$\mathrm{j}_n \, f(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x}), f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_K(\mathbf{x}), \ldots), \ \forall \mathbf{x} \in \mathbb{C}^m, 1 \leq |K| \leq n. \tag{2.5}$$

Example 2.1.2. *Let $m = n = 2$ and set $x^1 = x$ and $x^2 = y$. Let $f = x^3 + 5x^2 y + 9y^3$. Then $\mathrm{j}_2 \, f : \mathbb{C}^2 \to J^2(\mathbb{C}^2, \mathbb{C})$ is*

$$\mathrm{j}_2 \, f(x, y) = \left(x, \, y, \, x^3 + 5x^2 y + 9y^3, \, 3x^2 + 10xy, \, 5x^2 + 27y^2, \, 6x + 10y, \, 10x, \, 54y \right).$$

Remark 2.1.2. *Note that for any smooth function $f : \mathbb{C}^m \to \mathbb{C}$, $\bigcup_{\mathbf{x} \in \mathbb{C}^m} \mathrm{j}_0 \, f(\mathbf{x}) \subset \mathbb{C}^m \times \mathbb{C}$ is the graph of f and that $\bigcup_{\mathbf{x} \in \mathbb{C}^m} \mathrm{j}_n \, f(\mathbf{x}) \subset J^n(\mathbb{C}^m, \mathbb{C})$ is an m-dimensional submanifold.*

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[3]We use capital letters for points that are being viewed as 'target points' for psychological reasons. We regard the points as residing in the same space.

2.1.2 Prolongation of the $\mathcal{G}$ Action

The action of $\mathcal{G}$ on $\mathbb{C}^m$ can be prolonged (see for instance [22]) to an action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$ satisfying

$$\Lambda \cdot (j_n \ f(\mathbf{x})) = j_n \ (\Lambda \star f)(\Lambda \cdot \mathbf{x}), \tag{2.6}$$

for all $\Lambda \in \mathcal{G}$ and for all $f : \mathbb{C}^m \to \mathbb{C}$. We will denote the prolonged action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$ by $\alpha : \mathcal{G} \times J^n(\mathbb{C}^m, \mathbb{C}) \to J^n(\mathbb{C}^m, \mathbb{C})$. We also introduce a second set of coordinates $Z^{(n)} = (X^i, U, U_K)$, $|K| \le n$, for the target coordinates of the action map α. The action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$ is given explicitly in coordinates by $\alpha\left(\lambda_j^i, z^{(n)}\right) = Z^{(n)} = (X^i, U, U_K)$, where

$$X^j = \lambda_i^j x^i, \tag{2.7}$$

$$U = u \tag{2.8}$$

$$U_j = \hat{\lambda}_j^i u_i \tag{2.9}$$

$$U_{j_1 j_2} = \hat{\lambda}_{j_1}^{i_1} \hat{\lambda}_{j_2}^{i_2} u_{i_1 i_2} \tag{2.10}$$

$$\vdots$$

$$U_{j_1 j_2 \cdots j_n} = \hat{\lambda}_{j_1}^{i_1} \hat{\lambda}_{j_2}^{i_2} \cdots \hat{\lambda}_{j_n}^{i_n} u_{i_1 i_2 \cdots i_n}, \tag{2.11}$$

$1 \le i, j, \ldots \le m$, $|K| \le n$.

Remark 2.1.3. *The components of the action map $\alpha : \mathcal{G} \times J^n(\mathbb{C}^m, \mathbb{C}) \to J^n(\mathbb{C}^m, \mathbb{C})$ are given by globally defined polynomial functions in λ_j^i and s.*

Notation 2.1.2. *The orbit of a point $\mathbf{z}^{(n)} \in J^n(\mathbb{C}^m, \mathbb{C})$ under the action of $\mathcal{G}$ will be denoted by $\mathcal{O}_{\mathbf{z}^{(n)}}$.*

Example 2.1.3. *Let $m = n = 2$. For $\Lambda = \left(\lambda_j^i\right) \in \mathcal{G}$ and $\mathbf{z}^{(2)} = \left(x^1, x^2, u, u_1, u_2, u_{11}, u_{12}, u_{22}\right) \in$*

$J^2\left(\mathbb{C}^2,\mathbb{C}\right)$, *the action of* $\mathcal{G}$ *on* $J^2\left(\mathbb{C}^2,\mathbb{C}\right)$ *is given by*

$$\Lambda \cdot z^{(2)} = Z^{(2)} = \left(X^1, X^2, U, U_1, U_2, U_{11}, U_{12}, U_{22}\right), \qquad (2.12)$$

where

$$X^i = \lambda_j^i x^j$$
$$U = u,$$
$$U_i = \hat{\lambda}_i^j u_j,$$
$$U_{ij} = \hat{\lambda}_i^k \hat{\lambda}_j^l u_{kl},$$

with $1 \leq i, j, k, l \leq 2$.

2.1.3 Basic Geometry of the Prolonged $\mathcal{G}$ Action

We will briefly review some of the fundamental geometry and structure associated with $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ and with the prolonged action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$. We refer the reader to [21] or [22] for a detailed discussion of these topics.

Associated to the jet spaces $J^k\left(\mathbb{C}^m, \mathbb{C}\right)$ and $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ (assuming that $k > n$), there are natural projection $\pi_n^k : J^k\left(\mathbb{C}^m, \mathbb{C}\right) \to J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ which are equivariant with respect to the actions of $\mathcal{G}$ on $J^k\left(\mathbb{C}^m, \mathbb{C}\right)$ and $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$. That is, for all $\Lambda \in \mathcal{G}$ and for all $\mathbf{z}^{(k)} \in J^k\left(\mathbb{C}^m, \mathbb{C}\right)$ we have

$$\pi_n^k\left(\Lambda \cdot \mathbf{z}^{(k)}\right) = \Lambda \cdot \pi_n^k\left(\mathbf{z}^{(k)}\right). \qquad (2.13)$$

Let $\mathbf{z}^{(k)} \in J^k\left(\mathbb{C}^m, \mathbb{C}\right)$ and $\mathbf{z}^{(n)} \in J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ be points such that $\pi_n^k\left(\mathbf{z}^{(k)}\right) = \mathbf{z}^{(n)}$. Then the

orbit $\mathcal{O}_{\mathbf{z}^{(k)}}$ for the action of $\mathcal{G}$ on $J^k\left(\mathbb{C}^m, \mathbb{C}\right)$ passing through the point $\mathbf{z}^{(k)}$ projects onto the orbit $\mathcal{O}_{\mathbf{z}^{(n)}}$ for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ passing through the point $\mathbf{z}^{(n)}$. That is, $\pi_n^k\left(\mathcal{O}_{\mathbf{z}^{(k)}}\right) = \mathcal{O}_{\mathbf{z}^{(n)}}$ whenever $\pi_n^k\left(\mathbf{z}^{(k)}\right) = \mathbf{z}^{(n)}$, or more generally, whenever $\pi_n^k\left(\mathbf{z}^{(k)}\right)$ and $\mathbf{z}^{(n)}$ are equivalent under the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$. Thus, the dimension of the maximal orbits for the action of $\mathcal{G}$ on $J^k\left(\mathbb{C}^m, \mathbb{C}\right)$ is nondecreasing as a function of k. Furthermore, since the dimension of the orbits is bounded by the dimension of the group, there is a minimal $0 < n_0 < \infty$ such that the maximal orbit dimension is first obtained on $J^{n_0}\left(\mathbb{C}^m, \mathbb{C}\right)$.

Notation 2.1.3. *We will let s_n denote the maximum dimension of the orbits for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ and we will let $\max\left(s_n\right) = \max\{s_n\}_{n=1}^{\infty}$.*

For a fixed m, we will denote by $J^{\infty}\left(\mathbb{C}^m, \mathbb{C}\right)$ the inverse limit formed by the finite jet spaces $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ and their projection maps $\pi_n^k : J^k\left(\mathbb{C}^m, \mathbb{C}\right) \to J^n\left(\mathbb{C}^m, \mathbb{C}\right)$, where we are assuming $k > n$.

Definition 2.1.4 (Stable Orbit Dimension). *The* stable orbit dimension *for the action of $\mathcal{G}$ on $J^{\infty}\left(\mathbb{C}^m, \mathbb{C}\right)$ is defined to be $\max\left(s_n\right)$.*

Definition 2.1.5 (Order of Stabilization). *The* order of stabilization *for the action of $\mathcal{G}$ on $J^{\infty}\left(\mathbb{C}^m, \mathbb{C}\right)$ is the minimal n_0 such that the stable orbit dimension is obtained.*

A result due to Ovsiannikov [25] shows that the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ is locally free [4] on a dense open subset of $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ for n sufficiently large. We will state the theorem for a general manifold M and a general Lie group $\mathcal{G}$.

---

[4]The dimension of the orbits is equal to the dimension of the group.

Theorem 2.1.6. *The action of a Lie group $\mathcal{G}$ on M is locally effective on subsets if and only if the prolonged action of $\mathcal{G}$ on $J^k(M, p)$ is locally free on a dense open subset for k sufficiently large.*

Remark 2.1.4. *It will be proven later (see Theorem 3.7.1) that the prolonged action of an algebraic subgroup $\mathcal{G} \subset GL_m(\mathbb{C})$ on $J^n(\mathbb{C}^m, \mathbb{C})$ is locally free for $n \geq m$. This bound is not sharp, however (see Remark 3.7.2).*

Definition 2.1.7 (Regular Jet). *A point $\mathbf{z}^{(n)} \in J^n(\mathbb{C}^m, \mathbb{C})$ is said to be a regular jet if the dimension of the orbit passing through $\mathbf{z}^{(n)}$ is of maximal dimension.*

The standard total derivative operators on $J^\infty(\mathbb{C}^m, \mathbb{C})$ are denoted by D_i, $1 \leq i \leq m$, where D_i denotes the total derivative with respect to x^i. The total derivative operators are given in local coordinates $z^{(n)} = (x, u, u_K)$ by the formula

$$D_i = \frac{\partial}{\partial x^i} + u_{K,i} \frac{\partial}{\partial u_K}, \qquad 1 \leq i \leq m, \tag{2.14}$$

where there is summation over all partial derivative multi-indices K. When applying the total derivative operators to a differential function $\phi : J^n(\mathbb{C}^m, \mathbb{C}) \to \mathbb{C}$, then the summation in (2.14) can be truncated to a finite sum. Application of a total derivative operator D_i to a differential function ϕ results in a differential function $D_i \phi : J^{n+1}(\mathbb{C}^m, \mathbb{C}) \to \mathbb{C}$.

Notation 2.1.4. *Let $f : \mathbb{C}^m \to \mathbb{C}$ and let $\phi : J^n(\mathbb{C}^m, \mathbb{C}) \to \mathbb{C}$ be a function. The restriction of ϕ to the n-jet of f will be denoted by either $\phi[f]$ or $\phi[\mathrm{j}_n\, f]$ if the order of the jet space on which the function is being evaluated needs additional emphasis.*

Remark 2.1.5. *For a function $\phi : J^n(\mathbb{C}^m, \mathbb{C}) \to \mathbb{C}$ and a function $f : \mathbb{C}^m \to \mathbb{C}$, $\phi[f]$*

can be viewed as a function from $\mathbb{C}^m \to \mathbb{C}$, defined by

$$\phi[f](\mathbf{x}) = \phi\left(\mathrm{j}_n \ f(\mathbf{x})\right), \qquad \forall \mathbf{x} \in \mathbb{C}^m.$$

The total derivative operators capture implicit differentiation in the following sense. If $\phi : J^n(\mathbb{C}^m, \mathbb{C})$ is a differential function and $f : \mathbb{C}^m \to \mathbb{C}$, then

$$(D_i \phi)\,[\mathrm{j}_{n+1} \ f] = \frac{\partial}{\partial x^i}(\phi[\mathrm{j}_n \ f]).$$

The total derivative operators form a basis for the m-dimensional vector space of total differential operators. Any total differential operator on $J^\infty(\mathbb{C}^m, \mathbb{C})$ is thus uniquely expressible as

$$\mathcal{D} = Q^i(x, u, u_K) D_i \qquad 1 \le i \le m, \tag{2.15}$$

and satisfies

$$(\mathcal{D}\phi)\,[\mathrm{j}_{n+1} \ f] = Q^i[\mathrm{j}_n \ f]\frac{\partial}{\partial x^i}(\phi[\mathrm{j}_n \ f]),$$

where ϕ is any differential function and $f : \mathbb{C}^m \to \mathbb{C}$.

Remark 2.1.6. *The prolonged action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$ (2.7)-(2.11) can be defined using the total derivative operators. For $\Lambda \in \mathcal{G}$ define the total differential operator*

$$\mathcal{E}_i = \hat{\lambda}_i^j D_j, \qquad 1 \le i, j \le m. \tag{2.16}$$

Then for any partial derivative multi-index $K = (k_1, \ldots, k_j)$, (2.7)-(2.11) are equivalent to

$$\mathcal{E}_K u = U_K, \tag{2.17}$$

where $\mathcal{E}_K u = \mathcal{E}_{k_j} \cdots \mathcal{E}_{k_1} u$.

2.1.4   Differential Invariants

We will now briefly introduce the notion of local and global differential invariants for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$.

Definition 2.1.8 (Differential Invariants). *Let $U \subset J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ be an open set. A function $\phi : U \to \mathbb{C}$ is said to be a* local differential invariant *if for all $\mathbf{z}^{(n)} \in U$ there exists an open neighborhood $V \subset \mathcal{G}$ about the identity matrix such that for all $\Lambda \in V$,*

$$\phi\left(\Lambda \cdot \mathbf{z}^{(n)}\right) = \phi\left(\mathbf{z}^{(n)}\right).$$

If $\phi\left(\Lambda \cdot \mathbf{z}^{(n)}\right) = \phi\left(\mathbf{z}^{(n)}\right)$ for all $\mathbf{z}^{(n)} \in U$ and for all $\Lambda \in \mathcal{G}$ such that $\Lambda \cdot \mathbf{z}^{(n)} \in U$, then we say that ϕ is a global differential invariant *on U.*

Remark 2.1.7. *In the definition of a local differential invariant, note that the open neighborhood $V \subset \mathcal{G}$ about the identity may depend on the point $\mathbf{z}^{(n)} \in U$.*

Remark 2.1.8. *A global differential invariant for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ can be viewed as a function that is constant on the $\mathcal{G}$-orbits.*

Example 2.1.9. *Let $\mathcal{G} = SL_2\left(\mathbb{C}\right)$. The function $\phi : J^2\left(\mathbb{C}^2, \mathbb{C}\right) \to \mathbb{C}$ defined by $\phi\left(\mathbf{z}^{(2)}\right) = u_{11}u_{22} - u_{12}^2$ is a global differential invariant for the action of $\mathcal{G}$ on $J^2\left(\mathbb{C}^2, \mathbb{C}\right)$. Let $\mathbf{z}^{(2)} = \left(x^1, x^2, u, u_1, u_2, u_{11}, u_{12}, u_{22}\right) \in J^2\left(\mathbb{C}^2, \mathbb{C}\right)$ and let $\Lambda = \left(\lambda_j^i\right) \in SL_2\left(\mathbb{C}\right)$. We merely*

note that

$$\phi \left(\Lambda \cdot \mathbf{z}^{(2)} \right) = U_{11} U_{22} - U_{12}^2$$

$$= \left(\hat{\lambda}_1^i \hat{\lambda}_1^j u_{ij} \right) \left(\hat{\lambda}_2^k \hat{\lambda}_2^l u_{kl} \right) - \left(\hat{\lambda}_1^r \hat{\lambda}_2^s u_{rs} \right)^2$$

$$= \left(\hat{\lambda}_1^1 \hat{\lambda}_2^2 - \hat{\lambda}_1^2 \hat{\lambda}_2^1 \right)^2 \left(u_{11} u_{22} - u_{12}^2 \right)$$

$$= u_{11} u_{22} - u_{12}^2$$

$$= \phi \left(\mathbf{z}^{(2)} \right),$$

as $\left(\hat{\lambda}_1^1 \hat{\lambda}_2^2 - \hat{\lambda}_1^2 \hat{\lambda}_2^1 \right) = \det \Lambda^{-t} = 1.$

Remark 2.1.9. *If ϕ is a differential invariant for the action of $\mathcal{G}$ on $J^n \left(\mathbb{C}^m, \mathbb{C} \right)$ and $\Lambda \in \mathcal{G}$, then $\Lambda \star \phi[f] = \phi[\Lambda \star f]$ under the action of $\mathcal{G}$ on functions from $\mathbb{C}^m \to \mathbb{C}$.*

Example 2.1.10. *Let $\mathcal{G} = SL_2 \left(\mathbb{C} \right)$, $n = 2$, and set $x^1 = x, x^2 = y$. Let $\phi : J^2 \left(\mathbb{C}^2, \mathbb{C} \right) \to \mathbb{C}$ be the differential invariant $\phi \left(\mathbf{z}^{(2)} \right) = u_{11} u_{22} - u_{12}^2$ as given in Example 2.1.9. Let f, $g \in \mathcal{P}_2^3$ and $\Lambda \in SL_2 \left(\mathbb{C} \right)$ be given by*

$$f = x^3 + y^3, \qquad g = 9x^3 + 15x^2 y + 9xy^2 + 2y^3, \qquad \Lambda = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Then $\phi[f] = 36xy$ and $\phi[g] = (54\,x + 30\,y)\,(18\,x + 12\,y) - (30\,x + 18\,y)^2$. A straightfor-

ward calculation along the lines of that in Example 1.2.2 shows

$$\phi[g](X, Y) = (54\ (x - y) + 30\ (-x + 2y))\ (18\ (x - y) + 12\ (-x + 2y))$$

$$- (30\ (x - y) + 18\ (-x + 2y))^2$$

$$= 36xy$$

$$= \phi[f](x, y).$$

Thus, by (2.3), $\Lambda \star \phi[f] = \phi[g] = \phi[\Lambda \star f]$.

2.2 Group Equivariant Moving Frames

In this section we introduce the notion of a local moving map for the action of $\mathcal{G}$ on $J^\infty(\mathbb{C}^m, \mathbb{C})$. The notion of a moving frame map has a wider range of applicability and can be used to address the local equivalence problem for submanifolds of a manifold M under the action of a Lie group G in far greater generality than presented here (see [10]). However, we aim to be self-contained and we would like to present proofs for the algebraically minded reader that are as accessible as possible. In particular, we avoid (in as much as is possible) any reliance on equivalence of coframes, contact invariant coframes, contact distributions, etc . . . The proof offered here of Theorem 2.2.13 will not work in the general smooth setting, although it will carry over to the analytic setting under some additional hypothesis on transversality.

2.2.1 Definitions and Properties of Moving Frames

Definition 2.2.1 (Local Moving Frame Map)**.** *Let $\mathbf{z}^{(n)} \in J^n(\mathbb{C}^m, \mathbb{C})$. A local moving frame map for the action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$ about $\mathbf{z}^{(n)}$ consists of*

1. *an open neighborhood $\mathcal{W}^{(n)}$ about $\mathbf{z}^{(n)}$ and*

2. *a smooth map $\rho : \mathcal{W}^{(n)} \subset J^n\left(\mathbb{C}^m, \mathbb{C}\right) \to \mathcal{G}$ that is locally equivariant[5]:*

   For all $\mathbf{z}^{(n)} \in \mathcal{W}^{(n)}$, there exists an open neighborhood $\mathcal{U} \subset \mathcal{G}$ about the identity element such that for all $\Lambda \in \mathcal{U}$, $\rho\left(\Lambda \cdot \mathbf{z}^{(n)}\right) = \rho\left(\mathbf{z}^{(n)}\right) \cdot \Lambda^{-1}$.

Remark 2.2.1. *When the point $\mathbf{z}^{(n)} \in J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ in Definition 2.2.1 is not important, then we will say that $\rho : \mathcal{W}^{(n)} \subset J^n\left(\mathbb{C}^m, \mathbb{C}\right) \to \mathcal{G}$ is a local moving frame map for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$.*

We recall a proposition that can be found in [10] and [17] that gives necessary and sufficient conditions for the existence of a local moving frame map.

Proposition 2.2.2. *A local moving frame map for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ exists about $\mathbf{z}^{(n)} \in J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ if and only if $\mathcal{G}$ acts locally freely at $\mathbf{z}^{(n)}$. [6]*

As a result, a local moving frame for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ exists about $\mathbf{z}^{(n)} \in J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ if and only if $\mathbf{z}^{(n)}$ is a regular jet and n is $\geq$ order of stabilization for the action of $\mathcal{G}$ on $J^\infty\left(\mathbb{C}^m, \mathbb{C}\right)$.

Remark 2.2.2. *If $k > n$ and $\rho : \mathcal{W}^{(n)} \subset J^n\left(\mathbb{C}^m, \mathbb{C}\right) \to \mathcal{G}$ is a local moving frame map for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$, then ρ can also be used to define a local moving frame map for the action of $\mathcal{G}$ on $J^k\left(\mathbb{C}^m, \mathbb{C}\right)$. Let $\mathcal{W}^{(k)} = \{\mathbf{z}^{(k)} \in J^k\left(\mathbb{C}^m, \mathbb{C}\right) \,\big|\, \pi_n^k\left(\mathbf{z}^{(k)}\right) \in \mathcal{W}\} = \left(\pi_n^k\right)^{-1}\left(\mathcal{W}^{(n)}\right)$ and define $\rho^{(k)} : \mathcal{W}^{(k)} \subset J^k\left(\mathbb{C}^m, \mathbb{C}\right) \to \mathcal{G}$ by*

$$\rho^{(k)}\left(\mathbf{z}^{(k)}\right) = \rho\left(\pi_n^k\left(\mathbf{z}^{(k)}\right)\right).$$

Local equivariance follows immediately from the equivariance of the projection maps.

---

[5]We assume that the left action of $\mathcal{G}$ on itself is given by right inverse translation.

[6]Continuity implies that the action will be locally free in a neighborhood of $\mathbf{z}^{(n)}$.

Definition 2.2.3 (Order of a Moving Frame). *Let $\rho : \mathcal{W}^{(k)} \subset J^k(\mathbb{C}^m, \mathbb{C}) \to \mathcal{G}$ be a local moving frame map for the action of $\mathcal{G}$ on $J^k(\mathbb{C}^m, \mathbb{C})$. The* order *of the moving frame map ρ is the minimum order of the jet space on which it is defined.*

Remark 2.2.3. *As a result of Remark 2.2.2, a local moving frame map of order n defines a local moving frame map on all higher order jet spaces. As such, we will denote a local moving frame map for the action of $\mathcal{G}$ on $J^\infty(\mathbb{C}^m, \mathbb{C})$ by $\rho : \mathcal{W}^{(\infty)} \subset J^\infty(\mathbb{C}^m, \mathbb{C}) \to \mathcal{G}$ and the order of the moving frame map will be mentioned when needed.*

Definition 2.2.4 (Invariantized Position Map). *Let $\rho : \mathcal{W}^{(\infty)} \subset J^\infty(\mathbb{C}^m, \mathbb{C}) \to \mathcal{G}$ be an order n local moving frame map for the action of $\mathcal{G}$ on $J^\infty(\mathbb{C}^m, \mathbb{C})$. For $k \geq n$, the* invariantized position map of order k *associated to the moving frame map ρ is the map $\iota^{(k)} : \mathcal{W}^{(k)} \subset J^k(\mathbb{C}^m, \mathbb{C}) \to J^k(\mathbb{C}^m, \mathbb{C})$ defined by*

$$\iota^{(k)}\left(\mathbf{z}^{(k)}\right) = \rho\left(\mathbf{z}^{(k)}\right) \cdot \mathbf{z}^{(k)}. \tag{2.18}$$

For a moving frame map $\rho : \mathcal{W}^{(\infty)} \subset J^\infty(\mathbb{C}^m, \mathbb{C}) \to \mathcal{G}$, the associated invariantized position map identifies a (local) normal form for the representatives of the $\mathcal{G}$-orbits for the action of $\mathcal{G}$ on $J^k(\mathbb{C}^m, \mathbb{C})$. This is reflected in the following proposition.

Proposition 2.2.5. *Let $\rho : \mathcal{W}^{(\infty)} \subset J^\infty(\mathbb{C}^m, \mathbb{C}) \to \mathcal{G}$ be an order n local moving frame map for the action of $\mathcal{G}$ on $J^\infty(\mathbb{C}^m, \mathbb{C})$ and let $\mathbf{z}^{(k)} \in \mathcal{W}^{(k)}$ $(k \geq n)$. Then, for all $\Lambda \in \mathcal{G}$ sufficiently close to the identity,*

$$\iota^{(k)}\left(\mathbf{z}^{(k)}\right) = \iota^{(k)}\left(\Lambda \cdot \mathbf{z}^{(k)}\right).$$

Proof. The proof follows immediately from the definitions and, in particular, from the

local equivariance of the moving frame map ρ. Namely,

$$
\begin{aligned}
\iota^{(k)}\big(\Lambda \cdot \mathbf{z}^{(k)}\big) &= \rho\left(\Lambda \cdot \mathbf{z}^{(k)}\right) \cdot \left(\Lambda \cdot \mathbf{z}^{(k)}\right) \\
&= \rho(\mathbf{z}^{(k)})\Lambda^{-1} \cdot \left(\Lambda \cdot \mathbf{z}^{(k)}\right) \qquad \text{(Local equivariance of } \rho) \\
&= \rho(\mathbf{z}^{(k)}) \cdot \mathbf{z}^{(k)} \\
&= \iota^{(k)}(\mathbf{z}^{(k)})
\end{aligned}
$$

$\square$

2.2.2 Invariantization and Recurrence Formula

We will now briefly outline the process of local invariantization associated with a local moving frame map $\rho : \mathcal{W}^{(\infty)} \subset J^\infty\left(\mathbb{C}^m, \mathbb{C}\right) \to \mathcal{G}$.

Definition 2.2.6 (Invariantization). *Let $\rho : \mathcal{W}^{(\infty)} \subset J^\infty\left(\mathbb{C}^m, \mathbb{C}\right) \to \mathcal{G}$ be a local moving frame map of order n for the action of $\mathcal{G}$ on $J^\infty\left(\mathbb{C}^m, \mathbb{C}\right)$ with the associated invariantized position map $\iota : \mathcal{W}^{(\infty)} \subset J^\infty\left(\mathbb{C}^m, \mathbb{C}\right) \to J^\infty\left(\mathbb{C}^m, \mathbb{C}\right)$. Let $\phi : \mathcal{W}^{(\infty)} \subset J^\infty\left(\mathbb{C}^m, \mathbb{C}\right) \to \mathbb{C}$ be a smooth differential function of finite order p. The* local invariantizion *of ϕ corresponding to the moving frame ρ is defined to be the differential function of order $\max\{n, p\}$ defined by $\iota^*\left(\phi\right)$.*

Proposition 2.2.7. *The local invariantization of a differential function ϕ as in Definition 2.2.6 is a local differential invariant.*

Proof. Let $\mathbf{z}^{(k)} \in \mathcal{W}^{(k)}$ and assume $k \geq n$. Then, for all $\Lambda \in \mathcal{G}$ sufficiently close to the

identify,

$$\iota^{(k)*}(\phi)\left(\Lambda \cdot \mathbf{z}^{(k)}\right) = \phi\left(\iota^{(k)}\left(\Lambda \cdot \mathbf{z}^{(k)}\right)\right)$$

$$= \phi\left(\iota^{(k)}\left(\mathbf{z}^{(k)}\right)\right) \qquad (\text{ By Proposition 2.2.5})$$

$$= \iota^{(k)*}(\phi)\left(\mathbf{z}^{(k)}\right).$$

Thus, $\iota^{(k)*}(\phi)$ is a local differential invariant. $\qquad\square$

The process of invariantizing differential functions on $J^{\infty}(\mathbb{C}^m, \mathbb{C})$ through the use of a local moving frame map $\rho : \mathcal{W}^{(\infty)} \subset J^{\infty}(\mathbb{C}^m, \mathbb{C}) \to \mathcal{G}$ and its corresponding (local) invariantized position map offers the following geometric interpretation. Given a function $\phi : \mathcal{W}^{(\infty)} \subset J^{\infty}(\mathbb{C}^m, \mathbb{C}) \to \mathbb{C}$, the corresponding local invariant $\iota^*(\phi)$ is obtained by evaluating ϕ on the local normal forms (as determined by ρ) of the orbits and spreading the values along the orbits. Furthermore, if $\phi : \mathcal{W}^{(\infty)} \subset J^{\infty}(\mathbb{C}^m, \mathbb{C}) \to \mathbb{C}$ is a local differential invariant, then $\iota^*(\phi) = \phi$.

A key part of the invariantization process is based on the invarintization of the coordinate functions $z^{(n)} = (x, u, u_K)$ on $J^n(\mathbb{C}^m, \mathbb{C})$ and the construction of a local frame of horizontal invariant differential operators.

Definition 2.2.8 (Normalized Invariants). *Let $\rho : \mathcal{W}^{(\infty)} \subset J^{\infty}(\mathbb{C}^m, \mathbb{C}) \to \mathcal{G}$ be a local moving frame map for the action of $\mathcal{G}$ on $J^{\infty}(\mathbb{C}^m, \mathbb{C})$. The local normalized invariants associated to ρ are the local invariants obtained by invariantizing the local coordinate functions $z^{(n)} = (x, u, u_K)$.*

Notation 2.2.1. *The local normalized invariants corresponding to a local moving frame*

map $\rho : \mathcal{W}^{(\infty)} \subset J^\infty \left(\mathbb{C}^m, \mathbb{C}\right) \to \mathcal{G}$ will be denoted by

$$I^i = \iota^* \left(x^i\right), \qquad\qquad\qquad 1 \le i \le m$$

$$I = \iota^* \left(u\right) = u,$$

$$I_K = \iota^* \left(u_K\right), \qquad\qquad\qquad 1 \le |K|.$$

Remark 2.2.4. *The local normalized invariants associated to a local moving frame map ρ of order k provide a fundamental set of local differential invariants for the action of $\mathcal{G}$ on $J^k \left(\mathbb{C}^m, \mathbb{C}\right)$ in the sense that* all *other differential invariants of order k can be expressed locally as functions of $\left(I^i, I, I_K\right)$.*

The next step in the invariantization process is the construction of a local $\mathcal{G}$-invariant, frame[7] of total differential operators. We refer the reader to section 10 of [10] for details.

Proposition 2.2.9 (Invariant Differential Operators). *Let $\mathcal{E}_i = \hat{\lambda}_i^j D_j$, $1 \le i, j \le m$, denote the prolonged implicit differential operators from (2.16) and let $\rho : \mathcal{W}^{(\infty)} \subset J^\infty \left(\mathbb{C}^m, \mathbb{C}\right) \to \mathcal{G}$ be a local moving frame of order n. The total differential operators*

$$\mathcal{D}_i = \hat{\lambda}_i^j \left(\rho \left(\mathbf{z}^{(n)}\right)\right) D_j, \qquad 1 \le i, j \le m, \tag{2.19}$$

where $\hat{\lambda}_i^j \left(\rho \left(\mathbf{z}^{(n)}\right)\right)$ denotes the $\hat{\lambda}_i^j$ coordinate function of the moving frame map ρ, form a local $\mathcal{G}$-invariant frame of total differential operators.

Remark 2.2.5. *Informally, given a moving frame map $\rho : \mathcal{W}^{(\infty)} \subset J^\infty \left(\mathbb{C}^m, \mathbb{C}\right) \to \mathcal{G}$, obtaining the local invariant differential operators amounts to substitution of the component functions from the moving frame map ρ into the group parameters appearing in the prolonged implicit differential operators (2.16).*

---

[7]Whereby 'frame' in this context we mean m-linearly independent total differential operators.

The invariant differential operators $\mathcal{D}_i$ associated to a local moving frame map, ρ, map local differential invariants to local differential invariants. Namely, if $\phi : \mathcal{W}^{(k)} \subset J^k(\mathbb{C}^m, \mathbb{C}) \to \mathbb{C}$ is a local differential invariant of order k, then applying $\mathcal{D}_i$ to ϕ results in a local differential invariant of order $\max\{n, k+1\}$. In particular, we can apply the invariant differential operators $\mathcal{D}_i$, $1 \le i \le m$, to the local normalized differential invariants I_K of order $|K|$. The fundamental observation due to Fels and Over ([10]) that relates $\mathcal{D}_i I_K$ with the local normalized invariant $I_{K,i}$ and lies at the heart of the applicability of the equivariant moving frame method to equivalence problems is given below.

Lemma 2.2.10 (Recurrence Formula). *Let $\rho : \mathcal{W}^{(\infty)} \subset J^\infty(\mathbb{C}^m, \mathbb{C}) \to \mathcal{G}$ be local moving frame map of order n for the action of $\mathcal{G}$ on $J^\infty(\mathbb{C}^m, \mathbb{C})$. For any normalized local differential invariant I^j or I_K,*

$$\mathcal{D}_i I^j = \delta_i^j + M_i^j, \qquad 1 \le i, j \le m,$$

$$\mathcal{D}_i I_K = I_{K,i} + M_{K,i},$$

where δ_i^j denotes the usual Kronecker-delta, and M_i^j and $M_{K,i}$ are local differential invariants of order $n+1$ and $\max\{|K|, n+1\}$, respectively.

Remark 2.2.6. *While the normaliziation process (i.e. invariantizing the local coordinate functions) and invariant differentiation on $J^\infty(\mathbb{C}^m, \mathbb{C})$ do not commute (i.e. $\mathcal{D}_i I_K \neq I_{K,i}$), the recurrence formula puts a bound on the order of the differential invariant $M_{K,i}$ which serves as the correction term accounting for the difference between $\mathcal{D}_i I_K$ and $I_{K,i}$. The importance of the recurrence formula is that when I_K is a normalized local differential invariant of order $|K| \ge n+1$, the correction term $M_{K,i} = \mathcal{D}_i I_K - I_{K,i}$ is a differential*

invariant of order at most $|K|$ as opposed to being a differential invariant of order $|K|+1$ as one would naively expect.

2.2.3 Signatures Parametrized by Local Normalized Invariants

In this subsection, given a local moving frame map $\rho : \mathcal{W}^{(\infty)} \subset J^\infty\left(\mathbb{C}^m, \mathbb{C}\right) \to \mathcal{G}$ for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^\infty, \mathbb{C}\right)$, then for $f \in \mathcal{P}_m$ we introduce the signature map of f associated to ρ and the corresponding signature set of f. We will then show how the signature sets can be used to address the issue of determining when $f, g \in \mathcal{P}_m$ are $\mathcal{G}$-equivalent.

Definition 2.2.11 (Signature Map - Moving Frame)**.** *Let $\rho : \mathcal{W}^{(\infty)} \subset J^\infty\left(\mathbb{C}^m, \mathbb{C}\right) \to \mathcal{G}$ be a local moving frame map of order n. For $k \geq n$ and $f \in \mathcal{P}_m$, let $U = \left\{ \mathbf{x} \in \mathbb{C}^m \,\middle|\, \mathrm{j}_k\, f\left(\mathbf{x}\right) \in \mathcal{W}^{(k)} \right\}$. The kth order signature map of f associated to the local moving frame ρ is the map*

$$S_\rho^k[\,f\,] : U \to J^k\left(\mathbb{C}^m, \mathbb{C}\right), \tag{2.20}$$

defined by

$$S_\rho^k[\,f\,]\left(\mathbf{x}\right) = \left(I^i[f]\left(\mathbf{x}\right), I[f]\left(\mathbf{x}\right), I_K[f]\left(\mathbf{x}\right)\right), \quad \mathbf{x} \in \mathbb{C}^m, \ \ 1 \leq i \leq m, \ \ 1 \leq |K| \leq k.$$
$$\tag{2.21}$$

Remark 2.2.7. *Note that $S_\rho^k[\,f\,]$ is nothing other than the map obtained by restricting the order k invariantized position map assoicated with the moving frame map ρ to the k-jet of f.*

Definition 2.2.12 (Signature Set - Moving Frame)**.** *Let $\rho : \mathcal{W}^{(\infty)} \subset J^\infty\left(\mathbb{C}^m, \mathbb{C}\right) \to \mathcal{G}$*

be a local moving frame map of order n and let $k \geq n$. The kth order signature set of f associated to the local moving frame map ρ is the image of $S_\rho^k[f]$. The kth order signature set of f associated to the local moving frame map ρ will be denoted by $ImS_\rho^k[f]$.

Remark 2.2.8. *We have the geometric interpretation that $S_\rho^k[f](\mathbf{x})$ is the projection of $j_k \, f(\mathbf{x})$ along the group orbit to the local normal form determined by the moving frame map ρ.*

Theorem 2.2.13. *Let $\rho : \mathcal{W}^{(\infty)} \subset J^\infty(\mathbb{C}^m, \mathbb{C}) \to \mathcal{G}$ be a local moving frame map of order n for the action of $\mathcal{G}$ on $J^\infty(\mathbb{C}^m, \mathbb{C})$. Let f and $g \in \mathcal{P}_m$. If there exist open sets $U, V \subset \mathbb{C}^m$ and an integer $t \geq n$ such that*

1. *The rank of $\left.S_\rho^t[f]\right|_U$ is constant on U,*

2. *$\left.rank\ S_\rho^t[f]\right|_U = \left.rank\ S_\rho^{t+1}[f]\right|_U$, and*

3. *$S_\rho^{t+1}[f](U) = S_\rho^{t+1}[g](V) \neq \emptyset$*

then f and g are equivalent under the action of $\mathcal{G}$.

The proof of the theorem follows from a series of lemmas.

Lemma 2.2.14. *Let $\rho : \mathcal{W}^{(\infty)} \subset J^\infty(\mathbb{C}^m, \mathbb{C}) \to \mathcal{G}$ be a local moving frame map of order n and let $f \in \mathcal{P}_m$. If there exists an open set $U \subset \mathbb{C}^m$ and an integer $t \geq n$, such that*

1. *The rank of $S_\rho^t[f]$ is constant on U, and*

2. *$\left.rank\ S_\rho^t[f]\right|_U = \left.rank\ S_\rho^{t+1}[f]\right|_U.$*

Then,

1. $\forall\, k \geq t$, $rank\, S_\rho^k[f]\big|_U = rank\, S_\rho^t[f]\big|_U$, and

2. $\forall\, \mathbf{x} \in U$, $\exists$ an open neighborhood $U_{\mathbf{x}} \subset U$ about $\mathbf{x}$ and $r \left(= rank\, S_\rho^t[f]\big|_U\right)$ local

   normalized differential invariants $I_{K_1}, I_{K_2}, \ldots, I_{K_r}$ of order $\leq t$ such that all other

   normalized invariants I_K can be expressed as functions of $I_{K_1}, I_{K_2}, \ldots, I_{K_r}$ when

   evaluated on the jet of f.

Remark 2.2.9. *Note that the local normalized differential invariants* $I_{K_1}, \ldots, I_{K_r}$ *ap-*

pearing in part 2 of the conclusion of the above lemma may include local normalized

differential invariants I^j. *We use* $I_{K_1}, \ldots I_{K_r}$ *for notational simplicity.*

Proof. Let $r = rank\, S_\rho^t[f]\big|_U = rank\, S_\rho^{t+1}[f]\big|_U$ and let $\mathbf{x} \in U \subset \mathbb{C}^m$ be arbitrary. We

will show that the rank of $S_\rho^{t+2}[f]\big|_U$ is also r. Since the rank of $S_\rho^t[f]\big|_U$ is constant on

U and it is equal to the rank of $S_\rho^{t+1}[f]\big|_U$, then in a neighborhood of $\mathbf{x}$, we can choose

r normalized local differential invariants $I_{K_1}, \ldots, I_{K_r}$ of order $\leq t$ such that any other

differential invariant of order $\leq t+1$ can be expressed as a function of $I_{K_1}, \ldots, I_{K_r}$ when

evaluated on the jet of f. In particular, for each normalized local invariant

$$I^i, \quad 1 \leq i \leq m$$

$$I_K, \quad 0 \leq |K| \leq t+1,$$

there exists a corresponding smooth function of r variables,

$$F^i\left(y^1, \ldots, y^r\right), \quad 1 \leq i \leq m$$

$$F_K\left(y^1, \ldots, y^r\right), \quad 0 \leq |K| \leq t+1,$$

such that in a neighborhood $\mathbf{x}$ we have

$$I^i[f] = F^i\left(I_{K_1}, \ldots, I_{K_r}\right)[f], \qquad\qquad 1 \leq i \leq m \text{, and} \qquad (2.22)$$

$$I_K[f] = F_K\left(I_{K_1}, \ldots, I_{K_r}\right)[f], \qquad\qquad 0 \leq |K| \leq t+1. \qquad (2.23)$$

In addition, by the recurrence formula (Lemma 2.2.10) for the invariant differential operators $\mathcal{D}_i$, $1 \leq i \leq m$, we have

$$\mathcal{D}_i I^j = \delta_i^j + M_i^j \qquad\qquad 1 \leq j \leq m, \qquad (2.24)$$

$$\mathcal{D}_i I_K = I_{K,i} + M_{K,i} \qquad\qquad 0 \leq |K|, \qquad (2.25)$$

where I^j, I_K represent the local normalized differential invariants and the correction terms (M_i^j and $M_{K,i}$) are local differential invariants of order $\leq \max\{n+1, |K|\}$.

Taking I^j or I_K to be any local normalized invariant of order $\leq t$ and noting that the right hand sides of (2.24) and (2.25) will be a differential invariant of order at most $t+1$, then for $1 \leq i \leq m$, we have (in a neighborhood of $\mathbf{x}$)

$$\mathcal{D}_i I^j[f] = H_i^j\left(I_{K_1}, \ldots, I_{K_r}\right)[f] \qquad\qquad 1 \leq j \leq m, \qquad (2.26)$$

$$\mathcal{D}_i I_K[f] = H_{K,i}\left(I_{K_1}, \ldots, I_{K_r}\right)[f], \qquad\qquad (0 \leq |K| \leq t) \qquad (2.27)$$

where $H_i^j\left(y^1, \ldots, y^r\right)$ and $H_{K,i}\left(y^1, \ldots y^r\right)$ are smooth functions of r variables.

If I_K is now taken to be a normalized differential invariant of order $t+1$, then we can

express the local normalized differential invariants of order $t + 2$ as

$$I_{K,i} = \mathcal{D}_i I_K - M_{K,i}, \qquad (|K| = t + 1). \tag{2.28}$$

In a neighborhood of $\mathbf{x}$, for $|K| = t + 1$, we thus have,

$$I_{K,i}[f] = (\mathcal{D}_i I_K)[f] - M_{K,i}[f] \qquad \qquad \text{(By (2.28))}$$

$$\tag{2.29}$$

$$= (\mathcal{D}_i F_K (I_{K_1}, \ldots, I_{K_r}))[f] - M_{K,i}[f] \qquad \text{(By (2.23))}$$

$$\tag{2.30}$$

$$= \sum_{j=1}^{r} \left(\frac{\partial F_K}{\partial y^j} (I_{K_1}, \ldots, I_{K_r}) \mathcal{D}_i I_{K_j} \right)[f] - M_{K,i}[f] \qquad \text{(Chain Rule)}$$

$$\tag{2.31}$$

$$= \sum_{j=1}^{r} \left(\frac{\partial F_K}{\partial y^j} (I_{K_1}, \ldots, I_{K_r}) H_{K_j,i} (I_{K_1}, \ldots, I_{K_r}) \right)[f] - M_{K,i}[f] \quad \text{(By (2.27))}.$$

$$\tag{2.32}$$

Since $M_{K,i}$ is a differential invariant of order at most $t + 1$, then $I_{K,i}[f]$ can be expressed as function of $I_{K_1}[f], \ldots, I_{K_r}[f]$ in a neighborhood of $\mathbf{x}$. Thus, in a neighborhood of $\mathbf{x}$ all differential invariants of order $t + 2$ can be expressed as functions of $I_{K_1}, \ldots, I_{K_r}$ when evaluated on the jet of f. Since $\mathbf{x} \in U \subset \mathbb{C}^m$ was arbitrary, the rank of $S_\rho^{t+2}[f]$ is constant of r on U. The result follows. $\qquad \square$

Lemma 2.2.15. *Let $f, g \in \mathcal{P}_m$ satisfying the conditions of Theorem 2.2.13. Then $\forall k \geq n$ (where n is the order of the moving frame map ρ), there exists $\mathbf{x}_0 \in U$ and $\mathbf{x}_1 \in V$ such that $S_\rho^k[f](\mathbf{x}_0) = S_\rho^k[g](\mathbf{x}_1)$.*

Proof. If $k \leq t + 1$, then note that $S_\rho^{t+1}[f](U) = S_\rho^{t+1}[g](V)$ implies $S_\rho^k[f](U) = S_\rho^k[g](V)$, and thus, there exists $\mathbf{x}_0 \in U$ and $\mathbf{x}_1 \in V$ such that $S_\rho^k[f](\mathbf{x}_0) = S_\rho^k[g](\mathbf{x}_1)$.

We now assume that $k > t + 1$. The assumption that $S_\rho^{t+1}[f](U) = S_\rho^{t+1}[g](V) \neq \emptyset$ implies that $\exists \mathbf{x}_0 \in U$ and $\exists \mathbf{x}_1 \in V$ such that $S_\rho^t[f](\mathbf{x}_0) = S_\rho^t[g](\mathbf{x}_1)$. By hypothesis, both f and g satisfy the conditions of Lemma 2.2.14. Thus, there exists an open neighborhood $U_{\mathbf{x}_0} \subset U$ about $\mathbf{x}_0$ and r normalized differential invariants $I_{K_1}^f, \ldots, I_{K_r}^f$ of order $\leq t$ such that all other normalized differential invariants can be written as functions of $I_{K_1}^f, \ldots, I_{K_r}^f$ when restricted to the jet of f. In particular, for the normalized differential invariants I_K of order $t + 1$ we have

$$I_K[f] = H_K^f(I_{K_1}^f, \ldots I_{K_r}^f)[f], \qquad |K| = t + 1, \tag{2.33}$$

where the $H_K^f(y^1, \ldots, y^r)$ are smooth functions of r variables. Likewise, for g, there exists an open neighborhood $V_{\mathbf{x}_1} \subset V$ about $\mathbf{x}_1$ and r normalized differential invariants $I_{K_1}^g, \ldots, I_{K_r}^g$ of order $\leq t$ such that all other normalized differential invariants can be written as functions of $I_{K_1}^g, \ldots, I_{K_r}^g$ when restricted to the jet of g. For the local normalized differential invariants I_K of order $t + 1$ we have

$$I_K[g] = H_K^g(I_{K_1}^g, \ldots I_{K_r}^g)[g] \qquad |K| = t + 1, \tag{2.34}$$

where each $H_K^g(y^1, \ldots, y^r)$ is a smooth function of r variables. The assumption that $S_\rho^{t+1}[f](U) = S_\rho^{t+1}[g](V)$ implies that we can choose the *same* normalized differential invariants $I_{K_1}, \ldots I_{K_r}$ that we will express all other normalized differential invariants in

terms of. That is,

$$I_{K_i}^f = I_{K_i}^g \qquad 1 \le i \le r. \tag{2.35}$$

On account of the assumption that $S_\rho^{t+1}[\,f\,](U) = S_\rho^{t+1}[\,g\,](V)$, then we can further assume that for the normalized differential invariants I_K of order $t+1$, the functions H_K^f and H_K^g in (2.33) and (2.34) are also the same. The result now follows from the calculation (2.29) - (2.32) appearing in the proof of Lemma 2.2.14. All higher order normalized differential invariants I_K, $|K| \ge t+1$, will (locally) have the same functional relationships in terms of $I_{K_1}, \ldots, I_{K_r}$ when restricted to the jets of f and g respectively. Since

$$I_{K_i}[f](\mathbf{x}_0) = I_{K_i}[g](\mathbf{x}_1), \qquad 1 \le i \le r, \tag{2.36}$$

we conclude that for any normalized differential invariant I_K, we have

$$I_K[f](\mathbf{x}_0) = I_K[g](\mathbf{x}_1). \tag{2.37}$$

Thus, for all $k \ge n$, there exists $\mathbf{x}_0 \in U$ and $\mathbf{x}_1 \in V$ such that $S_\rho^k[\,f\,](\mathbf{x}_0) = S_\rho^k[\,g\,](\mathbf{x}_1)$. $\qquad \square$

Lemma 2.2.16. *Let $\rho : \mathcal{W}^{(\infty)} \subset J^\infty\,(\mathbb{C}^m, \mathbb{C}) \to \mathcal{G}$ be a local moving frame map of order n and let $f, g \in \mathcal{P}_m^d$. If the $\max\{d, n\}^{th}$ order moving frame signature sets of f and g have a point in common then f and g are $\mathcal{G}$-equivalent.*

Proof. For simplicity, we will assume $d \ge n$. If $d < n$, the same argument holds by considering the nth order signatures and utilizing the properties of the projection maps $\pi_d^n : J^n\,(\mathbb{C}^m, \mathbb{C}) \to J^d\,(\mathbb{C}^m, \mathbb{C})$.

By the hypothesis, there exists $\mathbf{x}_0, \mathbf{x}_1 \in \mathbb{C}^m$ such that $S_\rho^d[\,f\,]\,(\mathbf{x}_0) = S_\rho^d[\,g\,]\,(\mathbf{x}_1)$. By

definition of the signature map corresponding to the moving frame map ρ, we have

$$\rho\left(\mathrm{j}_d\ f\left(\mathbf{x}_0\right)\right)\cdot\mathrm{j}_d\ f\left(\mathbf{x}_0\right)=\rho\left(\mathrm{j}_d\ g\left(\mathbf{x}_1\right)\right)\cdot\mathrm{j}_d\ g\left(\mathbf{x}_1\right), \tag{2.38}$$

Set $\Lambda=\rho\left(\mathrm{j}_d\ g\left(\mathbf{x}_1\right)\right)^{-1}\rho\left(\mathrm{j}_d\ f\left(\mathbf{x}_1\right)\right)\in\mathcal{G}$. Then

$$\Lambda\cdot\mathrm{j}_d\ f\left(\mathbf{x}_0\right)=\mathrm{j}_d\ g\left(\mathbf{x}_1\right),$$

and by condition (2.6) we have

$$\Lambda\cdot\mathrm{j}_d\ f\left(\mathbf{x}_0\right)=\mathrm{j}_d\left(\Lambda\star f\right)\left(A\cdot\mathbf{x}_0\right)=\mathrm{j}_d\ g\left(\mathbf{x}_1\right)$$

Thus, the d jets of $\Lambda\star f$ and g are identical at $\mathbf{x}_1\in\mathbb{C}^m$. Since a polynomial of degree d is completely determined by its d-jet at a point, this implies $\Lambda\star f=g$. $\qquad\square$

The proof of Theorem 2.2.13 follows immediately. If f and $g\in\mathcal{P}_m^d$ satisfy the conditions of Theorem 2.2.13 and $d\geq n$, then their signatures of order d have a point in common. If $d<n$, then we consider $ImS_\rho^n[\,f\,]$ and $ImS_\rho^n[\,g\,]$ instead.

2.3 Constructing Moving Frame Maps

We will now briefly discuss how one obtains a moving frame map in practice. For simplicity, we will work on $J^n\left(\mathbb{C}^m,\mathbb{C}\right)$, where n is $\geq n_0$, the order of stabilization for the action of $\mathcal{G}$ on $J^\infty\left(\mathbb{C}^m,\mathbb{C}\right)$. In accordance with Proposition 2.2.2, this ensures that there exists a local moving frame map in the neighborhood of any regular jet $\mathbf{z}^{(n)}$. As Proposition 2.2.5 shows, any local moving frame map $\rho:\mathcal{W}^{(n)}\subset J^n\left(\mathbb{C}^m,\mathbb{C}\right)\to\mathcal{G}$ defined in a neighborhood

of a regular jet $\mathbf{z}^{(n)}$ determines a local normal form for the orbits. To *construct* a local moving frame map in a neighborhood of a regular jet $\mathbf{z}^{(n)}$, one *begins* by first specifying the local normal forms for the orbits and *then* one defines $\rho : \mathcal{W}^{(n)} \subset J^n(\mathbb{C}^m, \mathbb{C}) \to \mathcal{G}$ by the condition that $\rho(\mathbf{y}^{(n)}) \in \mathcal{G}$ is the/a group element which will bring $\mathbf{y}^{(n)} \in \mathcal{W}^{(n)}$ to its specified local normal form.

Before illustrating how this construction is formally carried out, we first recall the definition of the infinitesimal generators for the action of a finite dimensional Lie group $\mathcal{G}$ on a smooth manifold M. See [22] for additional details. For simplicity of notation, for the definition we will use $\alpha : \mathcal{G} \times M \to M$ to denote the action of $\mathcal{G}$ on M.

Notation 2.3.1. *For a manifold M and a point $\mathbf{m} \in M$, we will denote the tangent space to M at $\mathbf{m}$ by $T_{\mathbf{m}}M$.*

Notation 2.3.2. *For a vector field V on a manifold M, we will use either $V(\mathbf{m})$ or $V|_{\mathbf{m}}$ to denote the value of V at $\mathbf{m}$.*

Definition 2.3.1. *Let $\mathcal{G}$ be an r-dimensional Lie group acting smoothly on a smooth manifold M. Let $V_1, \ldots, V_r$ form a basis for $\mathfrak{g} = T_e\mathcal{G}$, where e denotes the identity element of $\mathcal{G}$, and let $\exp(tV_i)$ denote the one-parameter subgroup generated by V_i. The infinitesimal generators for the action of $\mathcal{G}$ on M are the vector fields $\hat{V}_i$ on M defined by*

$$\hat{V}_i(\mathbf{m}) = \frac{d}{dt}\left(\alpha\left(\exp(tV_i), \mathbf{m}\right)\right)\Big|_{t=0}, \qquad \forall \mathbf{m} \in M, 1 \leq i \leq r.$$

Remark 2.3.1. *Note that for all $\mathbf{m} \in M$, the tangent space to the $\mathcal{G}$-orbit at $\mathbf{m}$ is spanned by $\hat{V}_i(\mathbf{m})$, $1 \leq i \leq r = \dim(\mathcal{G})$. Specifically, if $\mathbf{m} \in M$ is fixed, then $\alpha(g, \mathbf{m}) : \mathcal{G} \to M$*

and the pushforward $\alpha_*(e, \mathbf{m}) : T_e\mathcal{G} \to T_\mathbf{m}M$ maps the tangent space at $e \in \mathcal{G}$ to the tangent space at $\mathbf{m} \in M$. The image of any basis for $T_e\mathcal{G}$ will thus span the tangent space to the $\mathcal{G}$-orbit passing through $\mathbf{m}$.

Notation 2.3.3. *When referencing the infinitesimal generators for a given $\mathcal{G}$-action, we will omit the 'hat' from the notation.*

We will now introduce the notion of a cross-section as it is applied to the construction of local moving frame maps. A cross-section will be used to specify the local normal forms for the $\mathcal{G}$-orbits of the action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$. We use the definition of a cross-section from [17] (see pp. 8, Definition 1.5) to the action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$.

Definition 2.3.2 (Cross-section). *An embedded submanifold $\mathcal{C}^{(n)} \subset J^n(\mathbb{C}^m, \mathbb{C})$ is a local cross-section to the orbits if there is an open set $\mathcal{U} \subset J^n(\mathbb{C}^m, \mathbb{C})$ such that*

1. *$\forall \mathbf{z}^{(n)} \in \mathcal{U}$, $\mathcal{C}^{(n)}$ intersects $\mathcal{O}^0_{\mathbf{z}^{(n)}} \cap \mathcal{U}$ at a unique point, where $\mathcal{O}^0$ is the connected component of $\mathcal{O}_{\mathbf{z}^{(n)}} \cap \mathcal{U}$ containing $\mathbf{z}^{(n)}$,*

2. *$\forall \mathbf{z}^{(n)} \in \mathcal{C}^{(n)} \cap \mathcal{U}$, $\mathcal{O}^0_{\mathbf{z}^{(n)}}$ and $\mathcal{C}^{(n)}$ are transversal and of complementary dimensions.*

Remark 2.3.2. *Note that the second condition in the above definition amounts to requiring that $T_{\mathbf{z}^{(n)}}J^n(\mathbb{C}^m, \mathbb{C}) = T_{\mathbf{z}^{(n)}}\mathcal{C}^{(n)} \oplus T_{\mathbf{z}^{(n)}}\mathcal{O}_{\mathbf{z}^{(n)}}$.*

When the cross-section $\mathcal{C}^{(n)} \subset J^n(\mathbb{C}^m, \mathbb{C})$ is defined locally as the common zero set of $r = \dim(\mathcal{G})$ functions $F^i : J^n(\mathbb{C}^m, \mathbb{C}) \to \mathbb{C}$, $1 \leq i \leq r$, then the transversality conditions can be checked as follows. Let $F : J^n(\mathbb{C}^m, \mathbb{C}) \to \mathbb{C}^r$ be the function defined by

$$F(z^{(n)}) = \left(F^1(z^{(n)}), \ldots, F^r(z^{(n)})\right). \tag{2.39}$$

Let V_i, $1 \leq i \leq r$, denote the infinitesimal generators for the action of $\mathcal{G}$ on $\mathbb{C}^m$ and denote the corresponding infinitesimal generators for the action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$ [8] by $\mathrm{pr}^{(n)}(V_i)$, $1 \leq i \leq r$. The tangent space to the cross-section $\mathcal{C}^{(n)}$ at a point $\mathbf{z}^{(n)} \in \mathcal{C}^{(n)}$ is the kernel of the pushforward $F_* : T_{\mathbf{z}^{(n)}} J^n(\mathbb{C}^m, \mathbb{C}) \to \mathbb{C}^r$. Since the infinitesimal generators $\mathrm{pr}^{(n)}(V_i)|_{\mathbf{z}^{(n)}}$ span $T_{\mathbf{z}^{(n)}} \mathcal{O}_{\mathbf{z}^{(n)}}$, then $\mathcal{C}^{(n)}$ will fail to be transversal to $\mathcal{O}_{\mathbf{z}^{(n)}}$ at $\mathbf{z}^{(n)}$ if and only if the kernel of F_* contains a nontrivial element of the span of the infinitesimal generators $\mathrm{pr}^{(n)}(V_i)|_{\mathbf{z}^{(n)}}$, $1 \leq i \leq r$. This is equivalent to the condition that the $r \times r$ matrix $L(i, j) = \left(\mathrm{pr}^{(n)}(V_i)(F^j) \right)$ have nonzero determinant at the point $\mathbf{z}^{(n)}$.

Now, assuming that $\mathcal{C}^{(n)} \subset J^n(\mathbb{C}^m, \mathbb{C})$ is a local cross-section to the $\mathcal{G}$ orbits defined locally by the zero-set of a submersion $F : J^n(\mathbb{C}^m, \mathbb{C}) \to \mathbb{C}^r$, then one uses $\mathcal{C}^{(n)}$ to define a local normal form for the orbits. The local moving frame map associated to the local normal forms determined by $\mathcal{C}^{(n)}$ is defined by the condition that $\rho(\mathbf{z}^{(n)}) \in \mathcal{G}$ satisfy $\rho(\mathbf{z}^{(n)}) \cdot \mathbf{z}^{(n)} \in \mathcal{C}^{(n)}$. To find ρ, one must solve the *the normalization equations*

$$F^i\left(Z^{(n)}\right) = F^i\left(\left(\alpha\left(\lambda_j^i, z^{(n)}\right)\right)\right) = 0, \qquad 1 \leq i \leq r, \tag{2.40}$$

for the group parameters λ_j^i in terms of the the jet variables $z^{(n)}$. While the Implicit Function Theorem guarantees that one is able to solve for the group parameters λ_j^i as a function of the jet variables $z^{(n)}$, the resulting equations are often nonlinear and difficult to deal with directly. Furthermore, matters are complicated by the fact that an orbit may intersect the given cross-section more than once, which means that one will often be forced to deal with multi-valued functions, branch-cuts, and other like matters.

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[8]Note that the infinitesimal generators for the action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$ can be obtained directly from the action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$ or by prolonging the infinitesimal generators V_i for the action of $\mathcal{G}$ on $\mathbb{C}^m$ to the appropriate order. See ([22], Chapter 4) for details.

2.4 Examples

We will first present a full and thorough investigation of the actions of $SL_2(\mathbb{C})$ on $J^\infty(\mathbb{C}^2, \mathbb{C})$ and $\mathcal{P}_2$. The simplicity of this case results from the fact that the cross-section introduced below is a global cross-section on a dense open subset of $J^1(\mathbb{C}^2, \mathbb{C})$. Similar results are available in [2], [18], [23] , and [3]. In [2], [18], and [23] the equivalence of homogenous polynomials in two variables is considered under the action of $GL_2(\mathbb{C})$. Their results are obtained by working with the dehomogenized version of homogeneous polynomials in two variables and viewing the graph as a curve in the complex projective plane subject to the appropriate $GL_2(\mathbb{C})$ transformations. In [2], [18], and [23], the authors also used moving frame methods to make a number of interesting observations pertaining to the symmetery group of a binary form under the action of $GL_2(\mathbb{C})$. The results in [3] are obtained in a fashion similar to the results described below. However, the authors bypass the notion of a moving frame and are able to proceed directly to the invariant differential operators.

The analysis will lead us naturally to the notion of a differential invariant signature map and a differential invariant signature correspondence. We pursue these notions for the action of $SL_2(\mathbb{C})$ on $\mathcal{HP}_2$, the set of homogeneous polynomials in two variables. We show that one can determine the $SL_2(\mathbb{C})$-equiavalence of homogeneous polynomials in $m = 2$ variables using a differential invariant signature correspondence determined by three polynomial differential invariants. Using this as motivation, we then consider the case of a differential invariant signature correspondence with no knowledge of results obtained via moving frame analysis and we apply this to the action of $GL_3(\mathbb{C})$ on $\mathcal{HP}_3^3$, the set of homogeneous polynomials of degree three in three variables. This is followed

by illustrating that while one can construct (and solve for!) an appropriate moving frame map for the action of $GL_3(\mathbb{C})$ on $J^3\left(\mathbb{C}^3,\mathbb{C}\right)$ and the action of $GL_3(\mathbb{C})$ on $\mathcal{P}_3$, there are inherent difficulties that one faces in using such an approach.

2.4.1 Moving Frames Applied to $\mathcal{P}_2$

As before, when considering the action of $SL_2(\mathbb{C})$ on $J^\infty\left(\mathbb{C}^2,\mathbb{C}\right)$ and $\mathcal{P}_2$, we will set $x^1 = x$ and $x^2 = y$ throughout the section. The action $\alpha : SL_2(\mathbb{C}) \times J^2\left(\mathbb{C}^2,\mathbb{C}\right) \to J^2\left(\mathbb{C}^2,\mathbb{C}\right)$ of $SL_2(\mathbb{C})$ on $J^2\left(\mathbb{C}^2,\mathbb{C}\right)$ is recorded below in coordinates for reference. Letting $\Lambda = (\lambda_j^i) \in SL_2(\mathbb{C})$ and $\mathbf{z}^{(2)} = (x, y, u, u_1, u_2, u_{11}, u_{12}, u_{22})$, the action is given by

$$X = \lambda_1^1 x + \lambda_2^1 y \tag{2.41}$$

$$Y = \lambda_1^2 x + \lambda_2^2 y \tag{2.42}$$

$$U = u \tag{2.43}$$

$$U_1 = \hat{\lambda}_1^i u_i = \hat{\lambda}_1^1 u_1 + \hat{\lambda}_1^2 u_2 \tag{2.44}$$

$$U_2 = \hat{\lambda}_2^i u_i = \hat{\lambda}_2^1 u_1 + \hat{\lambda}_2^2 u_2 \tag{2.45}$$

$$U_{11} = \hat{\lambda}_1^i \hat{\lambda}_1^j u_{ij} = (\hat{\lambda}_1^1)^2 u_{11} + 2\hat{\lambda}_1^1 \hat{\lambda}_1^2 u_{12} + (\hat{\lambda}_1^2)^2 u_{22} \tag{2.46}$$

$$U_{12} = \hat{\lambda}_1^i \hat{\lambda}_2^j u_{ij} = \hat{\lambda}_1^1 \hat{\lambda}_2^1 u_{11} + \left(\hat{\lambda}_1^1 \hat{\lambda}_2^2 + \hat{\lambda}_1^2 \hat{\lambda}_2^1\right) u_{12} + \hat{\lambda}_1^2 \hat{\lambda}_2^2 u_{22} \tag{2.47}$$

$$U_{22} = \hat{\lambda}_1^i \hat{\lambda}_1^j u_{ij} = (\hat{\lambda}_1^1)^2 u_{11} + 2\hat{\lambda}_1^1 \hat{\lambda}_1^2 u_{12} + (\hat{\lambda}_1^2)^2 u_{22}, \tag{2.48}$$

where $1 \le i, j \le 2$.

For $SL_2(\mathbb{C})$ we also note that the following conditions hold on the group parameters λ_j^i and $\hat{\lambda}_j^i$:

$$\lambda_1^1 = \hat{\lambda}_2^2, \quad -\lambda_2^1 = \hat{\lambda}_2^1, \quad -\lambda_1^2 = \hat{\lambda}_1^2, \quad \lambda_1^1 = \hat{\lambda}_2^2 \tag{2.49}$$

For $\Lambda \in SL_2(\mathbb{C})$, the prolonged implicit differential operators (2.16) corresponding to the action of $SL_2(\mathbb{C})$ on $J^\infty(\mathbb{C}^2, \mathbb{C})$ are

$$\mathcal{E}_1 = \lambda_2^2 D_1 - \lambda_1^2 D_2, \tag{2.50}$$

and

$$\mathcal{E}_2 = -\lambda_2^1 D_1 + \lambda_1^1 D_2, \tag{2.51}$$

where

$$D_1 = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + u_{11} \frac{\partial}{\partial u_1} + u_{12} \frac{\partial}{\partial u_2} + \dots,$$
$$D_2 = \frac{\partial}{\partial y} + u_2 \frac{\partial}{\partial u} + u_{12} \frac{\partial}{\partial u_1} + u_{22} \frac{\partial}{\partial u_2} + \dots$$

are the total derivative operators on $J^\infty(\mathbb{C}^2, \mathbb{C})$. Note that prolonged transformation formulas for $SL_2(\mathbb{C})$ on $J^\infty(\mathbb{C}^2, \mathbb{C})$ can be obtained by successively applying (2.50) and (2.51) to u. Specifically, for any partial derivative multi-index K we have

$$\mathcal{E}_K u = U_K. \tag{2.52}$$

In addition, the infinitesimal generators for the action of $SL_2(\mathbb{C})$ on $\mathbb{C}^2$ are

$$V_1 = x \frac{\partial}{\partial y}, \qquad V_2 = y \frac{\partial}{\partial x}, \qquad V_3 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \tag{2.53}$$

and the infinitesimal generators for the action of $SL_2(\mathbb{C})$ on $J^1(\mathbb{C}^2, \mathbb{C})$ are

$$\mathrm{pr}^{(1)}(V_1) = x\frac{\partial}{\partial y} - u_2\frac{\partial}{\partial u_1},$$

$$\mathrm{pr}^{(1)}(V_2) = y\frac{\partial}{\partial x} - u_1\frac{\partial}{\partial u_2},$$

$$\mathrm{pr}^{(1)}(V_3) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}.$$

The prolonged infinitesimal generators are generically linearly independent on $J^1\left(\mathbb{C}^2,\mathbb{C}\right)$, meaning that the action of $SL_2\left(\mathbb{C}\right)$ on $J^1\left(\mathbb{C}^2,\mathbb{C}\right)$ is locally free. The stabilization order for the action of $SL_2\left(\mathbb{C}\right)$ on $J^\infty\left(\mathbb{C}^2,\mathbb{C}\right)$ is thus $n_0 = 1$.

Proposition 2.4.1. *The submanifold $\mathcal{C}^{(1)} \subset J^1\left(\mathbb{C}^2,\mathbb{C}\right)$, where*

$$\mathcal{C}^{(1)} = \left\{ \mathbf{z}^{(1)} \in J^1\left(\mathbb{C}^2,\mathbb{C}\right) \,\middle|\, x = 0, y = 1, u_1 = 0, u_2 \neq 0 \right\},$$

is cross-section to the $SL_2\left(\mathbb{C}\right)$ orbits on $J^1\left(\mathbb{C}^2,\mathbb{C}\right)$. Furthermore, $\mathcal{C}^{(1)}$ intersects each orbit at most once.

Proof. $\mathcal{C}^{(1)}$ is a subset of the zero set of the function $F : J^1\left(\mathbb{C}^2,\mathbb{C}\right) \to \mathbb{C}^3$ defined by $F(\mathbf{z}^{(1)}) = (x, y-1, u_1)$. Denote the component functions of F by $F^1 = x$, $F^2 = y-1$, and $F^3 = u_1$ and let $L(i,j)$ be the 3×3 matrix $L(i,j) = \left(\mathrm{pr}^{(1)}(V_i)(F^j)\right)$. The determinant of $L(i,j)$ at a point $\mathbf{z}^{(1)} \in \mathcal{C}^{(1)}$ is

$$\det(L(i,j))(\mathbf{z}^{(1)}) = u_2(\mathbf{z}^{(1)}),$$

which is nonzero on $\mathcal{C}^{(1)}$. Thus, $\mathcal{C}^{(1)}$ intersects the orbits transversally.

To show that $\mathcal{C}^{(1)}$ intersects each orbit at most once is a straightforward calculation. Let $\mathbf{z}^{(1)} \in \mathcal{C}^{(1)}$ be a point on $\mathcal{C}^{(1)}$ and let $\Lambda = (\lambda_j^i) \in SL_2(\mathbb{C})$ be arbitrary. Any another point on the orbit of $\mathbf{z}^{(1)}$ that also belongs to $\mathcal{C}^{(1)}$ must satisfy the equations determined by $\Lambda \cdot \mathbf{z}^{(1)} \in \mathcal{C}^1$ for some $\Lambda \in SL_2(\mathbb{C})$. Letting $\Lambda = (\lambda_j^i) \in SL_2(\mathbb{C})$ be arbitrary, then combining (2.41), (2.44), (2.49) with the fact that $\mathbf{z}^{(1)} \in \mathcal{C}^{(1)}$, the requirement that $\Lambda \cdot \mathbf{z}^{(1)} \in \mathcal{C}^{(1)}$ leads to the following system of equations:

$$X = \lambda_2^1 = 0$$

$$Y = \lambda_2^2 = 1$$

$$U_1 = -\lambda_1^2 u_1 = 0$$

$$\det(\Lambda) = \lambda_1^1 \lambda_2^2 - \lambda_2^1 \lambda_1^2 = 1.$$

We immediately conclude that $\Lambda = (\lambda_j^i) = Id$ and thus, the orbit through $\mathbf{z}^{(1)}$ intersects the coordinate cross-section exactly once. $\qquad\square$

The moving frame map $\rho : J^1\left(\mathbb{C}^2, \mathbb{C}\right) \to SL_2(\mathbb{C})$ corresponding to the cross-section $\mathcal{C}^{(1)}$ is then obtained as follows. For $\mathbf{z}^{(1)} \in J^1\left(\mathbb{C}^2, \mathbb{C}\right)$, $\rho(\mathbf{z}^{(1)}) \in SL_2(\mathbb{C})$ is the unique group element that brings $\mathbf{z}^{(1)}$ to the cross-section $\mathcal{C}^{(1)}$. Letting $\mathbf{z}^{(1)} \in J^1\left(\mathbb{C}^2, \mathbb{C}\right)$ be fixed but arbitrary and letting $\rho(\mathbf{z}^{(1)}) = (\lambda_j^i)$ leads to the normalization equations determined by the condition $\rho(\mathbf{z}^{(1)}) \cdot \mathbf{z}^{(1)} \in \mathcal{C}^{(1)}$. The normalization equations corresponding to the cross-section $\mathcal{C}^{(1)}$ are thus

$$X = \lambda_1^1 x + \lambda_2^1 y = 0 \tag{2.54}$$

$$Y = \lambda_1^2 x + \lambda_2^2 y = 1 \tag{2.55}$$

$$U_1 = \lambda_2^2 u_1 - \lambda_1^2 u_2 = 0 \tag{2.56}$$

$$\det(\lambda_j^i) = \lambda_1^1 \lambda_2^2 - \lambda_2^1 \lambda_1^2 = 1 \tag{2.57}$$

Solving the normalization equations (2.54) - (2.57) for the group parameters λ_j^i yields the moving frame map $\rho : J^1\left(\mathbb{C}^2, \mathbb{C}\right) \to SL_2\left(\mathbb{C}\right)$.

Proposition 2.4.2. *The moving frame map* $\rho : \mathcal{W}^{(1)} \subset J^1\left(\mathbb{C}^2, \mathbb{C}\right) \to SL_2\left(\mathbb{C}\right)$ *is given by*

$$\rho(\mathbf{z}^{(1)}) = \begin{pmatrix} y & -x \\ \frac{u_1}{x\,u_1 + y\,u_2} & \frac{u_2}{x\,u_1 + y\,u_2} \end{pmatrix}, \tag{2.58}$$

where the domain of definition is $\mathcal{W}^{(1)} = \left\{ \mathbf{z}^{(1)} \in J^1\left(\mathbb{C}^2, \mathbb{C}\right) \,\middle|\, xu_1 + yu_2 \neq 0 \right\}$.

Remark 2.4.1. *If* $f \neq 0 \in \mathcal{HP}_2^d$, *then on account of* f *being a solution to Euler's Partial Differential Equation*

$$xu_1 + yu_2 = du,$$

the domain of $S_\rho^1[f]$ *will be nonempty. In particular,* $S_\rho^1[f]$ *is defined on all of* $\mathbb{C}^2/$ Var (f), *where* Var (F) *denotes the variety of* f. *Further, the only* $f \in \mathcal{P}_m$ *for which the domain of the moving frame signature map* $S_\rho^1[f]$ *is empty are solutions of the* $SL_2\left(\mathbb{C}\right)$ *invariant partial differential equation* $xu_1 + yu_2 = 0$.

Associated to the moving frame map $\rho : J^n\left(\mathbb{C}^2, \mathbb{C}\right) \to SL_2\left(\mathbb{C}\right)$ and the the cross-section $\mathcal{C}^{(n)}$, the corresponding invariantization of functions (Definition 2.2.6) is given

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by

$$\iota^* \left(\phi\right) \left(\mathbf{z}^{(n)}\right) = \phi \left(\rho \left(\mathbf{z}^{(n)}\right) \cdot \mathbf{z}^{(n)}\right), \tag{2.59}$$

where $\phi : J^n \left(\mathbb{C}^2, \mathbb{C}\right) \to \mathbb{C}$. The normalized invariants (Definition 2.2.8) on $J^n \left(\mathbb{C}^2, \mathbb{C}\right)$ provides a fundamental set of invariants for the action of $SL_2 \left(\mathbb{C}\right)$ on $J^n \left(\mathbb{C}^2, \mathbb{C}\right)$. The normalized invariants on $J^2 \left(\mathbb{C}^2, \mathbb{C}\right)$ corresponding to the moving frame map ρ are recorded below.

$$I^1 = \iota^*(x)(\mathbf{z}^{(n)}) = x(\rho(\mathbf{z}^{(n)}) \cdot \mathbf{z}^{(n)}) \qquad = 0 \tag{2.60}$$

$$I^2 = \iota^*(y)(\mathbf{z}^{(n)}) = y(\rho(\mathbf{z}^{(n)}) \cdot \mathbf{z}^{(n)}) \qquad = 1 \tag{2.61}$$

$$I = \iota^*(u)(\mathbf{z}^{(n)}) = u(\rho(\mathbf{z}^{(n)}) \cdot \mathbf{z}^{(n)}) \qquad = u \tag{2.62}$$

$$I_1 = \iota^*(u_1)(\mathbf{z}^{(n)}) = u_1(\rho(\mathbf{z}^{(n)}) \cdot \mathbf{z}^{(n)}) \qquad = 0 \tag{2.63}$$

$$I_2 = \iota^*(u_2)(\mathbf{z}^{(n)}) = u_2(\rho(\mathbf{z}^{(n)}) \cdot \mathbf{z}^{(n)}) \qquad = xu_1 + yu_2 \tag{2.64}$$

$$I_{11} = \iota^* \left(u_{11}\right) \left(\mathbf{z}^n\right) = u_{11} \left(\rho \left(\mathbf{z}^{(n)}\right) \cdot \mathbf{z}^{(n)}\right) \quad = \frac{(u_2)^2 u_{11} - 2u_1 u_2 + u_{22} (u_1)^2}{(xu_1 + yu_2)^2} \tag{2.65}$$

$$I_{12} = \iota^* \left(u_{12}\right) \left(\mathbf{z}^n\right) = u_{12} \left(\rho \left(\mathbf{z}^{(n)}\right) \cdot \mathbf{z}^{(n)}\right) \quad = \frac{xu_2 u_{1,1} + (u_2 y - u_1 x) u_{1,2} - u_1 y u_{2,2}}{xu_1 + yu_2} \tag{2.66}$$

$$I_{22} = \iota^* \left(u_{22}\right) \left(\mathbf{z}^n\right) = u_{22} \left(\rho \left(\mathbf{z}^{(n)}\right) \cdot \mathbf{z}^{(n)}\right) \quad = x^2 u_{11} + 2xyu_{12} + y^2 u_{22} \tag{2.67}$$

The invariant differential operators (Lemma 2.2.9) on $J^\infty \left(\mathbb{C}^2, \mathbb{C}\right)$ corresponding to the moving frame map ρ are obtained by substituting the coordinate formula for the moving frame map (2.58) for the corresponding group parameters in (2.50) and (2.51). The invariant differential operators are thus

$$\mathcal{D}_1 = \frac{u_2}{xu_1 + yu_2} D_1 - \frac{u_1}{xu_1 + yu_2} D_2, \tag{2.68}$$

and

$$\mathcal{D}_2 = xD_1 + yD_2. \tag{2.69}$$

The results outlined in the general case all carry over directly. Additionally, due to the fact that $\mathcal{C}^{(1)}$ is a global cross-section on $\mathcal{W}^{(1)}$, there are a number of results that can be slightly strengthened. First, all results become "if and only if," and second, the invariantization of a differential function results in a global differential invariant. The second fact will lead us to introducing the notion of a differential invariant signature map and a differential invariant signature set which is independent of both the notion of a moving frame and a cross-section.

Furthermore, due to the fact that the dimension of a moving frame signature set of a polynomial in $m = 2$ variables is at most two, then we are able to conclude that the $SL_2(\mathbb{C})$ equivalence of polynomials in $m = 2$ variables is completely decided by the moving frame signature sets of order three. Specifically, we have, the following.

Theorem 2.4.3. *Let $f, g \in \mathcal{P}_2$. Then f and g are $SL_2(\mathbb{C})$ equivalent if and only if $ImS_\rho^3[f] = ImS_\rho^3[g]$.*

2.4.2 Differential Invariant Signature Correspondence

As previously mentioned, the normalized invariants on $J^3(\mathbb{C}^2, \mathbb{C})$ are global differential invariants for the action of $SL_2(\mathbb{C})$ on $J^3(\mathbb{C}^2, \mathbb{C})$. The list of normalized differential invariants on $J^3(\mathbb{C}^2, \mathbb{C})$ is

$$\left(I^1, I^2, u, I_1, I_2, I_{11}, I_{12}, I_{22}, I_{111}, I_{112}, I_{122}, I_{222}\right), \tag{2.70}$$

where $(I^1, I^2, u, I_1, I_2, I_{11}, I_{12}, I_{22})$ have been recorded above in (2.60) - (2.67). We will show that if we fix the degree d and restrict our attention to $f, g \in \mathcal{HP}_2^d$, then the above result (Theorem 2.4.3) can be sharpened to show that that the list of differential invariants $\Psi = (u, I_{11}, I_{111})$, can be used to determine necessary and sufficient conditions for f and g to be $SL_2(\mathbb{C})$ equivalent.

We will record I_{111} for the sake of completeness.

$$I_{111} = \frac{(u_2)^3 u_{111} - 3u_1(u_2)^2 u_{112} + 3(u_1)^2 u_2 u_{122} - (u_1)^3 u_{222}}{(xu_1 + yu_2)^3}$$

The result will follow immediately from Theorem 2.4.3 and the properties of homogenous functions. In particular, we will rely on the fact that a homogenous function in two variables of degree d satisfies the partial differential equation $xu_1 + yu_2 = d\,u$.

These notions will prove useful in the analysis of the action of $GL_3(\mathbb{C})$ on $\mathcal{HP}_3^3$ in 2.4.3, and as such, we will state two definitions and prove a proposition in general before continuing our analysis of the action of $SL_2(\mathbb{C})$ on $\mathcal{HP}_2^d$.

Definition 2.4.4 (Differential Invariant Signature Map and Set). *Let $\mathcal{G}$ be an algebraic subgroup of $GL_m(\mathbb{C})$ and let $\Phi = (\phi_1, \phi_2, \ldots, \phi_r)$ be a list of global differential invariants for the prolonged action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$ and let $f \in \mathcal{P}_m$ The Φ- differential invariant signature map of f is the map $S_\Phi[f] : \mathbb{C}^m \to \mathbb{C}^r$ defined by*

$$S_\Phi[f](\mathbf{x}) = (\phi_1[f](\mathbf{x}), \ldots, \phi_r[f](\mathbf{x})). \tag{2.71}$$

The Φ-differential invariant signature set of f is the image of $S^{(\Phi)}[f]$ and will be denoted by $ImS_\Phi[f]$.

As an immediate consequence of the jet space transformation laws (condtion (2.6)),

we have the following proposition.

Proposition 2.4.5. *Let $\Phi = (\phi_1, \phi_2, \ldots, \phi_r)$ be a list of global differential invariants for the prolonged action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$ and let $f, g \in \mathcal{P}_m$. If f and g are $\mathcal{G}$ equivalent, then $ImS_\Phi[f] = ImS_\Phi[g]$.*

Proof. Let $\Lambda \in \mathcal{G}$ such that $\Lambda \star f = g$ and let $\mathbf{x} \in \mathbb{C}^m$ be such that $\Lambda \cdot \mathbf{x}$ belongs to the domain of definition of $S_\Phi[g]$. Then,

$$
\begin{aligned}
S_\Phi[g](A \cdot \mathbf{x}) &= S_\Phi[(\Lambda \star f)](\Lambda \cdot \mathbf{x}) \\
&= (\phi_1[(\Lambda \star f)](\Lambda \cdot \mathbf{x}), \ldots, \phi_r[(\Lambda \star f)](\Lambda \cdot \mathbf{x})) \\
&= (\phi_1[f](\mathbf{x}), \ldots, \phi_r[f](\mathbf{x})) \qquad\qquad \text{(Invariance of } \phi_i\text{)} \\
&= S_\Phi[f](\mathbf{x})
\end{aligned}
$$

$\square$

Definition 2.4.6 (Differential Invariant Signature Correspondence). *A list of differential invariants for the action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$ determines a complete differential invariant signature correspondence for the action of $\mathcal{G}$ on $\mathcal{P}_m$ if $\forall f, g \in \mathcal{P}_m$*

$$ f \cong_{\mathcal{G}} g \iff ImS_\Phi[f] = ImS_\Phi[g]. \tag{2.72} $$

Example 2.4.7. *Let Φ be the list of differential invariants in (2.70). Then, Φ determines a $SL_2(\mathbb{C})$ differential invariant signature correspondence for the action of $SL_2(\mathbb{C})$ on $\mathcal{P}_2$.*

Returning to our regularly scheduled programming and our discussion of the action of $SL_2(\mathbb{C})$ on $\mathcal{HP}_2^d$, we have the following.

Proposition 2.4.8. *Let $f, g \in \mathcal{HP}_2^d$, and let Ψ be the list of differential invariants $\Psi = (u, I_{11}, I_{111})$. Then f and g are equivalent if and only if $ImS_\Psi[f] = ImS_\Psi[g]$.*

Proof. ($\Longrightarrow$) By Proposition 2.4.5, if f and g are equivalent then $ImS_\Psi[f] = ImS_\Psi[g]$. ($\Longleftarrow$) Let Φ be the list of differential invariants given in (2.70). Note that as a result of Theorem 2.4.3, if $ImS_\Phi[f] = ImS_\Phi[g]$, then f and g are equivalent. We will show that if $ImS_\Psi[f] = ImS_\Psi[g]$, then $ImS_\Phi[f] = ImS_\Phi[g]$. This amounts to showing that all of the differential invariants in Φ can be expressed as (the same) functions of (u, I_{11}, I_{111}) when restricted to $j_3 \, f$ and $j_3 \, g$. Let $\mathbf{x} \in \mathbb{C}^2 / \mathrm{Var}\,(f)$. Then, by definition,

$$I_{11}[f](\mathbf{x}) = u_{11} \circ \left(\rho^{(3)} \left(j_3 \, f(\mathbf{x}) \right) \cdot j_3 \, f(\mathbf{x}) \right),$$

and

$$I_{111}[f](\mathbf{x}) = u_{111} \circ \left(\rho^{(3)} \left(j_3 \, f(\mathbf{x}) \right) \cdot j_3 \, f(\mathbf{x}) \right). \tag{2.73}$$

Let $A = \rho^{(3)} \left(j_3 \, f(\mathbf{x}) \right) \in SL_2\,(\mathbb{C})$ and set $A \cdot f = \bar{f}$. Then

$$\rho^{(3)} \left(j_3 \, f(\mathbf{x}) \right) \cdot j_3 \, f(\mathbf{x}) = j_3 \, \bar{f}(0, 1).$$

By homogeneity of f (and thus homogeneity of $\bar{f}$), all derivatives of $\bar{f}$ at $\mathbf{x}_0 = (0, 1)$ involving partial derivatives of order less than or equal to 3 that involve partial differentiation with respect to y can be completely solved for in terms of $\bar{f}(0, 1)$, $\bar{f}_1(0, 1)$,

$\bar{f}_{11}(0,1)$, and $\bar{f}_{111}(0,1)$. The general formula (up to partials of order three) are given by

$$\bar{f}_2 = \frac{d\,\bar{f} - x\bar{f}_1}{y}$$

$$\bar{f}_{12} = \frac{(d-1)\bar{\bar{f}}_1 - x\bar{f}_{11}}{y}$$

$$\bar{f}_{22} = \frac{d(d-1)\bar{f} - 2(d-1)x\bar{f}_1 + x^2\bar{f}_{11}}{y^2}$$

$$\bar{f}_{112} = \frac{(d-2)\bar{f}_{11} - x\bar{f}_{111}}{y}$$

$$\bar{f}_{122} = \frac{(d-1)(d-2)\bar{f}_1 - 2(d-2)x\bar{f}_{11} + x^2\bar{f}_{111}}{y^2}$$

$$\bar{f}_{222} = \frac{d(d-1)(d-2)\bar{f} - 3(d-1)(d-2)x\bar{f}_1 + 3(d-2)x^2\bar{f}_{11} - x^3\bar{f}_{111}}{y^3}$$

This gives

$$I_2[f](\mathbf{x}_0) = d\bar{f}(0,1) = d\,f(\mathbf{x}_0)$$

$$I_{12}[f](\mathbf{x}_0) = (d-1)\bar{f}_1(0,1) = (d-1)I_1(\mathbf{x}_0)$$

$$I_{22}[f](\mathbf{x}_0) = d(d-1)\bar{f}(0,1) = d(d-1)f(\mathbf{x}_0)$$

$$I_{112}[f](\mathbf{x}_0) = (d-2)\bar{f}_{11}(0,1) = (d-2)I_{11}[f](\mathbf{x}_0)$$

$$I_{122}[f](\mathbf{x}_0) = (d-1)(d-2)\bar{f}_1(0,1) = (d-1)(d-2)I_1[f](\mathbf{x}_0)$$

$$I_{222}[f](\mathbf{x}_0) = d(d-1)(d-2)\bar{f}(0,1) = d(d-1)(d-2)f(\mathbf{x}_0).$$

Since $\mathbf{x}_0 \in \mathbb{C}^2/\mathrm{Var}(f)$ is arbitrary, we conclude that

$$I_2[f] = d\,u$$

$$I_{12}[f] = (d-1)\,I_1[f]$$

$$I_{22}[f] = d(d-1)\,u$$

$$I_{112}[f] = (d-2)\,I_{11}[f]$$

$$I_{122}[f] = (d-1)(d-2)\,I_1[f]$$

$$I_{222}[f] = d(d-1)(d-2)\,u.$$

The same argument also holds for g. Thus, $ImS_\Psi[f] = ImS_\Psi[g] \iff ImS_\Phi[f] = ImS_\Phi[g] \iff f$ and g are $SL_2(\mathbb{C})$ equivalent.

$\square$

Finally, we note that for all d, we can obtain a list of polynomial differential invariants that determine a differential invariant signature correspondence for the action of $\mathcal{G}$ on $\mathcal{HP}_2^d$.

Corollary 2.4.9. *Let f and $g \in \mathcal{HP}_2^d$. Let $\Sigma = (\sigma_1, \sigma_2, \sigma_3)$ where*

$$\sigma_1 = u,$$

$$\sigma_2 = (u_2)^2 u_{11} - 2u_1 u_2 u_{12} + (u_1)^2 u_{22}$$

$$\sigma_3 = (u_2)^3 u_{111} - 3u_1(u_2)^2 u_{112} + 3(u_1)^2 u_2 u_{122} - (u_1)^3 u_{222}.$$

Then, $ImS_\Sigma[f] = ImS_\Sigma[g] \iff f$ and g are $SL_2(\mathbb{C})$ equivalent.

Proof. Let $\mathbb{H}^3 = \left\{ \mathbf{y} \in \mathbb{C}^3 \middle| y^1 \neq 0 \right\}$ and let Ψ be the list of differential invariants from

Proposition 2.4.8. Let $h \in \mathcal{HP}_2^d$. Define $\Theta : \mathbb{H}^3 \to \mathbb{H}^3$ by

$$\Theta(\mathbf{y}) = \left(y^1, y^2 \left(d\, y^1 \right)^2, y^3 (d\, y^1)^3 \right).$$

Θ defines a diffeomorphism of $\mathbb{H}^3$ which carries $ImS_\Psi[\,f\,]$ into $ImS_\Sigma[\,f\,]$. Since $h \in \mathcal{HP}_2^d$ is arbitrary, the result follows immediately. $\qquad\square$

As an application of the previous considerations, we will use the list of differential invariants Σ to compute the Σ-signatures of the monomials $x^{19-i}y^i$, $1 \leq i \leq 19$. Note that for an arbirtary $f \in \mathcal{HP}_2^{19}$ to be $SL_2\,(\mathbb{C})$ equivalent to a monomial, f must satisfy one of the conditions in 2.1. The choice of $d = 19$ is completely arbitrary and similar results for arbitrary d are easily obtained. For a given monomial $x^{19-i}y^i$, $\Sigma[x^{19-i}y^i] : \mathbb{C}^2 \to \mathbb{C}^3$, and we will use coordinates $\left(s^1, s^2, s^3 \right)$ on the image space $\mathbb{C}^3$. Using a straight forward elimination algorithm, we give the ideal in the polynomial ring $\mathbb{C}[s^1, s^2, s^3]$, which corresponds to the Zariski closure if the image of $\Sigma[x^{19-i}y^i]$. The ideals of the Zariski closure of the Σ-signature sets were computed in Maple using a straight forward elimination algorithm.

Table 2.1: Implicit Formulas For Σ-signatures of monomials $x^{19-i}y^i$, $1 \le i \le 19$

x^{19}, y^{19}	$\left(s^2, s^3\right)$
$x^{18}y$, xy^{18}	$\left(578\left(s^2\right)^3 + 171(s^3)^2 s^1\right)$
$x^{17}y^2$, $x^2 y^{17}$	$\left(450\left(s^2\right)^3 + 323(s^3)^2 s^1\right)$
$x^{16}y^3$, $x^3 y^{16}$	$\left(169\left(s^2\right)^3 + 228(s^3)^2 s^1\right)$
$x^{15}y^4$, $x^4 y^{15}$	$\left(121\left(s^2\right)^3 + 285(s^3)^2 s^1\right)$
$x^{14}y^5$, $x^5 y^{14}$	$\left(162\left(s^2\right)^3 + 665(s^3)^2 s^1\right)$
$x^{13}y^6$, $x^6 y^{13}$	$\left(98\left(s^2\right)^3 + 741(s^3)^2 s^1\right)$
$x^{12}y^7$, $x^7 y^{12}$	$\left(25\left(s^2\right)^3 + 399(s^3)^2 s^1\right)$
$x^{11}y^8$, $x^8 y^{11}$	$\left(9\left(s^2\right)^3 + 418(s^3)^2 s^1\right)$
$x^{10}y^9$, $x^9 y^{10}$	$\left(2\left(s^2\right)^3 + 855(s^3)^2 s^1\right)$

2.4.3 Ternary Cubics and Differential Invariant Signature Correspondences

We will pursue the notions of a differential invariant signature map and a differential invariant signature correspondence from Definition 2.4.4 and Definition 2.4.6. In particular, we focus on $\mathcal{HP}_3^3$, homogeneous polynomials of degree three in three variables, or as they are referred to in the classical literature, ternary cubics. We will take $\mathcal{G} = GL_3(\mathbb{C})$ throughout the section and we will make use of the known classification of non-degenerate ternary cubics and an elementary elimination algorithm similar to that used in the computations of the Σ-signatures for monomials in $\mathcal{HP}_2^{19}$ in the previous subsection. The approach presented here is similar to that found in [19], although we bypass the notion of the moving frame entirely.

Definition 2.4.10. *A homogeneous polynomial $f \in \mathcal{P}_m^d$ is said to be* degenerate *if there exists $\Lambda \in GL_m(\mathbb{C})$ such that $\Lambda \star f$ is a polynomial in* less than m *variables.*

When $m \leq 4$, there is a classical result due to Hesse[9] which characterizes degenerate homogenesous polynomials (see [23]). Recall that the Hessian of a function f in m variables is given by $H[f] = \det(f_{ij})$, $1 \leq i, j \leq m$.

Theorem 2.4.11. *Let $f \in \mathcal{P}_m^d$ with $m \leq 4$. Then f is degenerate if and only if $Hess(f)$ is identically zero.*

We will make use of Hesse's Theorem in order to provide a differential invariant signature correspondence for $ND = \left\{ f \in \mathcal{P}_3^3 \middle| f \text{ is non-degenerate} \right\}$. Observe that $ND \subset \mathcal{HP}_3^3$ is an invariant subset under the action of $GL_3(\mathbb{C})$.

---

[9]Hesse originally believed the result to be true for all m.

First, we recall the classification of $\mathcal{G}$-equivalence classes of $\mathcal{HP}_3^3$ as presented in [19].

Theorem 2.4.12. *Let $f \in \mathcal{HP}_3^3$ be irreducible.*

1. *If $f(x, y, z)$ defines a nonsingular projective variety then $f(x, y, z)$ is $GL_3(\mathbb{C})$-equivalent to one of the following:*

   (a) *a cubic in a one-parameter family: $x^3 + axz^2 + z^3 - y^2 z$, where $a \neq 0$, $a^3 \neq -\dfrac{27}{4}$,*

   (b) *$x^3 + xz^2 - y^2 z$, or*

   (c) *$x^3 + z^3 - y^2 z$*

2. *If $f(x, y, z)$ defines a singular projective variety then it is equivalent to one of the following:*

   (a) *$x^3 - y^2 z$, or*

   (b) *$x^2(x + z) - y^2 z$.*

Remark 2.4.2. *If $f \in \mathcal{HP}_3^3$ is equivalent to $x^3 + axz^2 + z^3 - y^2 z$ where*

1. *$a = 0$, then f is irreducible, defines a singular projective variety and is equivalent to $x^2(x + z) - y^2 z$.*

2. *$a = -\dfrac{27}{4}$, then f is a reducible cubic and is equivalent to $z(x^2 + y^2 + z^2)$ (See below).*

Theorem 2.4.13. *Let $f \in \mathcal{HP}_3^3$ be reducible.*

1. *If f is a product of quadratic and linear factors then it is equivalent to either*

   (a) *$z(x^2 + yz)$, or*

   (b) *$z(x^2 + y^2 + z^2)$.*

2. If f is a product of three linear factors and

    (a) the three factors are linearly independent, then f is equivalent to xyz,

    (b) the three factors are linearly dependent, but any pair of them is linearly independent, then f is equivalent to $xy(x+y)$,

    (c) exactly two of the factors are the same, then f is equivalent to x^2y, or

    (d) all three factors are the same, then f is equivalent to x^3.

Remark 2.4.3. *The $GL_3(\mathbb{C})$-equivalence classes of degenerate ternary cubics are represented by the canonical forms 2.(b), (c), and (d) appearing in Theorem 2.4.13.*

We will use a list of differential invariants $\Sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, where three of the differential invariants in the list are generated from classical tensor algebra or classical invariant theory methods (see Chapter 9 of [27] or Chapter 12 of [13] for further details).

Let ϵ^{ijk} , $1 \le i, j, k \le 3$, be the symbol [10] which takes the value ± 1 when (i, j, k) is a permutation of $(1, 2, 3)$, with the $\pm$ depending on the sign of the permutation, and zero otherwise. We will view ϵ^{ijk} as a relative tensor and generate differential invariants through tensor contraction.

Example 2.4.14. *Let $1 \le i_l, j_l, k_l \le 3$. Then*

$$\epsilon^{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} u_{i_1 j_1} u_{i_2 j_2} u_{i_3 j_3} \tag{2.74}$$

is $6H$, where $H = \det(u_{ij})$ is the Hessian of a function in three variables.

---

[10] In classical tensor algebra, the ϵ^{ijk} form the components of a relative contravariant tensor. In the physics literature this is often referred to as the Levi-Civita symbol. Further, this generalizes to $m \ge 3$.

Proposition 2.4.15. *Let $H = \det(u_{ij})$, $1 \le i, j \le 3$ and let $\sigma_2, \sigma_3, \sigma_4$ be given by*

$$\sigma_2 = \frac{\epsilon^{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} \epsilon^{k_1 k_2 k_3} \epsilon^{l_1 l_2 l_3} u_{i_1 j_1 k_1} u_{l_1 i_2 j_2} u_{k_2 l_2 i_3} u_{j_3 k_3 l_3}}{H^2}$$

$$\sigma_3 = \frac{\epsilon^{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} H_{i_1} H_{j_1} u_{i_2 j_2} u_{i_3 j_3}}{H^3}$$

$$\sigma_4 = \frac{\epsilon^{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} \epsilon^{k_1 k_2 k_3} \epsilon^{l_1 l_2 l_3} H_{i_1} H_{j_1} H_{k_1} u_{l_1 i_2} u_{j_2 k_2} u_{l_2 i_3} u_{j_3 k_3 l_3}}{H^5},$$

$1 \le i_s, j_s, k_s, l_s \le 3$. Then $\sigma_2, \sigma_3, \sigma_4$ are global differential invariants for the action of $GL_3(\mathbb{C})$ on $J^3(\mathbb{C}^3, \mathbb{C})$.

Theorem 2.4.16. *Let $ND = \left\{ f \in \mathcal{P}_3^3 \middle| f \text{ is non-degenerate} \right\}$. Let H, σ_2, σ_3, and σ_4 be given as above and let $\Sigma = (u, \sigma_2, \sigma_3, \sigma_4)$. Then Σ provides a complete differential invariant signature correspondence for ND.*

Proof. For any $f \in ND$, $S_\Sigma[f] : \mathbb{C}^3 / \operatorname{Var}(H[f]) \to \mathbb{C}^4$, where we will take coordinates (s_1, s_2, s_3, s_4) on $\mathbb{C}^4$. The ideals corresponding to the Zariski closures of the Σ-signature sets for the canonical forms of the equivalence classes of non-degenerate ternary cubics are all distinct. This completes the result. □

Remark 2.4.4. *The differential invariants and elimination algorithms implemented in this section were carried out in Maple. A maple file containing the code and the implicit forms of the signature sets of non-degenerate ternary cubics is (and will be) maintained at the author's website.* [11]

---

[11] At the time of publication the maple file is available at
`http://www.longwood.edu/staff/wearsth/thesiscode.html`

2.4.4 Moving Frames Applied to $\mathcal{P}_3$

We will now briefly outline the construction of moving frame map for the action of $GL_3(\mathbb{C})$ on $J^\infty(\mathbb{C}^3, \mathbb{C})$. The main purpose of this will be to illuminate some of the difficulties that one encounters when trying to carry out a direct implementation of the equivariant moving frame method to address the $GL_3(\mathbb{C})$-equivalence of polynomials in three variables. In particular, we will see that while one can solve the normalization equations corresponding to a particular cross-section, the resulting moving frame map itself will be of little practical use. The cross-section introduced below below will be generalized in Section 3.7 to the action of $GL_m(\mathbb{C})$ on $J^\infty(\mathbb{C}^m, \mathbb{C})$ and it will be used again in Section 3.7 to study the $GL_3(\mathbb{C})$-equivalence of homogeneous polynomials in three variables.

The action of $GL_3(\mathbb{C})$ on $J^3(\mathbb{C}^3, \mathbb{C})$ is given by equations (2.7) - (2.11), where $m = n = 3$. The infinitesimal generators for the action of $GL_3(\mathbb{C})$ on $\mathbb{C}^3$ are given by

$$
\begin{aligned}
V_1 &= x^1 \frac{\partial}{\partial x^1}, & V_2 &= x^1 \frac{\partial}{\partial x^2}, & V_3 &= x^1 \frac{\partial}{\partial x^3} \\
V_4 &= y \frac{\partial}{\partial x^1}, & V_5 &= x^2 \frac{\partial}{\partial x^2}, & V_6 &= x^2 \frac{\partial}{\partial x^3} \\
V_7 &= x^3 \frac{\partial}{\partial x^1}, & V_8 &= x^3 \frac{\partial}{\partial x^2}, & V_9 &= x^3 \frac{\partial}{\partial x^3},
\end{aligned}
$$

and we will denote the corresponding prolongations of the V_i to $J^3(\mathbb{C}^3, \mathbb{C})$ will be denoted by $\mathrm{pr}^{(3)}(V_i)$, $1 \le i \le 9$. We will not record the explicit coordinate form of the (ultimately unnecessary) prolonged infinitesimal generators.

Proposition 2.4.17. *The submanifold $\mathcal{C}^{(3)} \subset J^3(\mathbb{C}^3, \mathbb{C})$, where*

$$
\mathcal{C}^{(3)} = \left\{ \mathbf{z}^{(3)} \in J^3(\mathbb{C}^3, \mathbb{C}) \, \middle| \, x^i = u_i = u_{ii} = 0, x^3 = u_{iii} = 1, u_3 u_{12} \ne 0 \right\},
$$

where $i = 1, 2$, is a local cross-section for the action of $GL_3(\mathbb{C})$ on $J^3(\mathbb{C}^3, \mathbb{C})$.

Proof. Let $F : J^3(\mathbb{C}^3, \mathbb{C}) \to \mathbb{C}^9$ be defined by

$$F(\mathbf{z}^{(3)}) = \left(x^1, x^2, x^3 - 1, u_1, u_2, u_{11}, u_{22}, u_{111} - 1, u_{222} - 1 \right),$$

and denote the component functions of F by F^i, $1 \le i \le 9$. The determinant of the 9×9 matrix $L(i, j) = \left(\mathrm{pr}^{(3)}(V_i)(F^j) \right)$ at a point $\mathbf{z}^{(3)} \in \mathcal{C}^{(3)}$ is

$$\det(L(i, j)) \left(\mathbf{z}^{(3)} \right) = 36(u_3 u_{12})^2,$$

which is nonzero on account of $\mathbf{z}^{(3)} \in \mathcal{C}^{(3)}$, ensuring that $\mathcal{C}^{(3)}$ intersects each $GL_3(\mathbb{C})$-orbit transversally. $\qquad\square$

The resulting normalization equations for the moving frame map are thus

$$X^i = \lambda_j^i x^j = 0 \tag{2.75}$$

$$X^3 = \lambda_j^3 x^j = 0 \tag{2.76}$$

$$U_i = \hat{\lambda}_i^j u_j = 0 \tag{2.77}$$

$$U_{ii} = \hat{\lambda}_i^j \hat{\lambda}_i^k u_{jk} = 0 \tag{2.78}$$

$$U_{iii} = \hat{\lambda}_i^j \hat{\lambda}_i^k \hat{\lambda}_i^l u_{jkl} = 1, \tag{2.79}$$

$1 \le i \le 2$, $1 \le j, k, l \le 3$. One can solve the normalization equations for the group parameters λ_j^i in terms if the jet coordinates (z, u, u_K), and the resulting inverse transpose $\Lambda^{-t} = \left(\hat{\lambda}_j^i \right)$ is given in coordinates by

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$$\Lambda^{-t} = \begin{pmatrix} -\dfrac{S\,u_3}{\sqrt[3]{A}} & -\dfrac{P\,u_3}{\sqrt[3]{A}} & \dfrac{S\,u_1+P\,u_2}{\sqrt[3]{A}} \\[12pt] -\dfrac{R\,u_3}{\sqrt[3]{B}} & -\dfrac{S\,u_3}{\sqrt[3]{B}} & \dfrac{R\,u_1+S\,u_2}{\sqrt[3]{B}} \\[12pt] x^1 & x^2 & x^3 \end{pmatrix}, \tag{2.80}$$

where

$$P = u_{1,1}\,u_3{}^2 - 2\,u_{1,3}\,u_1\,u_3 + u_{3,3}\,u_1{}^2,$$

$$Q = u_{1,2}\,u_3{}^2 - u_{1,3}\,u_2\,u_3 - u_{2,3}\,u_1\,u_3 + u_{3,3}\,u_1\,u_2,$$

$$R = u_{2,2}\,u_3{}^2 - 2\,u_{2,3}\,u_2\,u_3 + u_{3,3}\,u_2{}^2,$$

$$S = -Q \pm \sqrt{Q^2 - PR},$$

$$A = \left(3\,u_{2,2,3}\,u_3{}^2\,u_2 - 3\,u_{2,3,3}\,u_3\,u_2{}^2 + u_{3,3,3}\,u_2{}^3 - u_{2,2,2}\,u_3{}^3\right)P^3$$

$$+ \left(-6\,u_{2,3,3}\,u_3\,u_1\,u_2 + 3\,u_{2,2,3}\,u_3{}^2\,u_1 - 3\,u_{1,2,2}\,u_3{}^3 + 3\,u_{3,3,3}\,u_1\,u_2{}^2 + 6\,u_{1,2,3}\,u_3{}^2\,u_2 - 3\,u_{1,3,3}\,u_3\,u_2{}^2\right)SP^2$$

$$+ \left(-3\,u_{2,3,3}\,u_3\,u_1{}^2 + 3\,u_{1,1,3}\,u_3{}^2\,u_2 - 6\,u_{1,3,3}\,u_3\,u_1\,u_2 + 3\,u_{3,3,3}\,u_1{}^2\,u_2 - 3\,u_{1,1,2}\,u_3{}^3 + 6\,u_{1,2,3}\,u_3{}^2\,u_1\right)S^2P$$

$$+ \left(-3\,u_{1,3,3}\,u_3\,u_1{}^2 + u_{3,3,3}\,u_1{}^3 - u_{1,1,1}\,u_3{}^3 + 3\,u_{1,1,3}\,u_3{}^2\,u_1\right)S^3,$$

$$B = \left(-3\,u_{1,3,3}\,u_3\,u_1{}^2 + u_{3,3,3}\,u_1{}^3 - u_{1,1,1}\,u_3{}^3 + 3\,u_{1,1,3}\,u_3{}^2\,u_1\right)R^3$$

$$+ \left(-3\,u_{2,3,3}\,u_3\,u_1{}^2 + 3\,u_{1,1,3}\,u_3{}^2\,u_2 - 6\,u_{1,3,3}\,u_3\,u_1\,u_2 + 3\,u_{3,3,3}\,u_1{}^2\,u_2 - 3\,u_{1,1,2}\,u_3{}^3 + 6\,u_{1,2,3}\,u_3{}^2\,u_1\right)SR^2$$

$$+ \left(-6\,u_{2,3,3}\,u_3\,u_1\,u_2 + 3\,u_{2,2,3}\,u_3{}^2\,u_1 - 3\,u_{1,2,2}\,u_3{}^3 + 3\,u_{3,3,3}\,u_1\,u_2{}^2 + 6\,u_{1,2,3}\,u_3{}^2\,u_2 - 3\,u_{1,3,3}\,u_3\,u_2{}^2\right)S^2R$$

$$+ \left(3\,u_{2,2,3}\,u_3{}^2\,u_2 - 3\,u_{2,3,3}\,u_3\,u_2{}^2 + u_{3,3,3}\,u_2{}^3 - u_{2,2,2}\,u_3{}^3\right)S^3,$$

and we use the $\pm$ in the definition of S to indicate that we can choose either branch of the square root function. Complications arise because the solutions to the normalization equations (2.75) - (2.79) are not unique and resulting moving frame map is multi-valued. Indeed, generic $GL_3(\mathbb{C})$-orbits intersects the cross-section $\mathcal{C}^{(3)}$ in 18 places. Thus, even

after choosing branches of the square and cubic root functions appearing in (2.80) and obtaining a moving frame map ρ, there is no reason to assume $GL_3(\mathbb{C})$ equivalent points of $J^3(\mathbb{C}^3, \mathbb{C})$ will map to the same normal form of the cross-section $\mathcal{C}^{(3)}$. Thus, if one restricts the resulting moving frame map to the jets of polynomial functions, it is not immediately obvious that equivalent polynomials will have identical moving frame signature sets. Further, it is not clear (without using a limiting argument) that the resulting moving frame map is even defined on the cross-section $\mathcal{C}^{(3)}$. Lastly, due to the presence of the algebraic functions in (2.80), one is not able to use methods of algebraic geometry to determine when the moving frame signature sets parametrized by the local normalized invariants of two polynomial functions are the same.

2.5   Discussion

Before proceeding to the notion of $\mathcal{G}$-signature varieties, $\mathcal{G}$-signature ideals, and $\mathcal{G}$-signature correspondences, a few remarks are in order. First, the remarkable success and efficiency of using the method of equivariant moving frames, differential invariant signature maps, and differential invariant signature correspondences in order to address the question of when two polynomials in $m = 2$ variables are $\mathcal{G}$-equivalent is completely misleading to what happens when one tries to apply either method to a more general case. Any practical use of the method of equivariant moving frames results in solving a system of nonlinear equations and the resulting differential invariants will be local and be given by algebraic functions, which prevents the use of Groebner basis techniques for addressing the question of when the corresponding signature sets are equal. Further complications also arise which will be addressed in the introduction to the next chapter.

The case of $\mathcal{G} = SL_2(\mathbb{C})$ or $\mathcal{G} = GL_2(\mathbb{C})$ presents an overly simplified, albeit pretty, picture of the applicability of *both* the equivariant method of moving frames and the notion of a differential invariant signature map and correspondence. Pursuing these notions with the case $m = 3$ variables begins to reveal some of the major hurdles that one will be forced to overcome. For example, the method used to prove that the list of differential invariants given by Σ in Section 2.4.3 provides a differential invariant signature correspondence on the set of non-degenerate ternary cubics essentially amounts to knowing the classification of ternary cubics and then being able to educatedly pick the proper differential invariants. We remark, however, that despite the benign appearance of the differential invariants appearing in Proposition 2.4.15, they are creatures of a rather monstrous sort. For example, the numerator of $\sigma_4 = \dfrac{\epsilon^{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} \epsilon^{k_1 k_2 k_3} \epsilon^{l_1 l_2 l_3} H_{i_1} H_{j_1} H_{k_1} u_{l_1 i_2} u_{j_2 k_2} u_{l_2 i_3} u_{j_3 k_3 l_3}}{H^5}$ is (before simplification) an expression with approximately 1.7×10^6 terms in the jet coordinates. After simplification and upon restriction to $f \in \mathcal{HP}_3^d$ the same numerator will result in a polynomial of degree $13d - 30$. Furthermore, attempting to generalize this approach to $m > 3$ variables causes more problems. For example, if one attempts to address the issue of equivalence of polynomials in $m = 4$ variables using the methods found in [13] and [27] to generate differential invariants, then one can do no better than using differential invariants which involve, before simplification, approximately 2×10^8 terms in the jet variables. For these reasons, we propose the notion of $\mathcal{G}$-signature varieties and $\mathcal{G}$-signature correspondences to combine the local power of the method of equivariant moving frames while retaining the effectiveness and global power of algebraic varieties.

Chapter 3

$\mathcal{G}$-Signature Varieties and

$\mathcal{G}$-Signature Correspondences

Despite the efficiency of the equivariant moving frame method in settling equivalence issues for homogeneous polynomials in $m = 2$ variables, a direct application of moving frames to address the equivalence of homogeneous polynomials in $m = 3$ variables faces rather severe difficulties. First, as previously mentioned, one typically finds a moving frame map by introducing a cross-section and solving a system of nonlinear equations. The introduction of a cross-section is arbitrary and one usually aims to introduce a cross-section that simplifies the corresponding normalization equations. However, the introduced cross-section does not (in general) have to be reflective of the geometry of the problem, but only satisfy the appropriate transversality conditions. Second, the moving frame construction is inherently local and it is often not clear where the results pertaining to equivalence are applicable. An 'instructive example' can be found in [24] (Section 6, pp. 16-19) where the author uses the equivariant method of moving frames to address the equivalence of space curves under the action of the Euclidean group. It is well-known

(see [8]) that under suitable hypothesis on smoothness and nondegeneracy, a space curve is determined up to congruence by its curvature, κ, and torsion, τ, functions. However, in [24], after the introduction of a non-traditional local cross-section, the author obtains similar results which are valid only on the class of space curves satisfying $\kappa\tau > 1$. Thus a solution of the equivalence problem based on the equivariant moving frame method is only valid on a certain open subset whose relative "size" is not known a priori. The advantage of the algebraic signature construction presented here is that it provides us with a solution of the equivalence problem on a Zariski open subset of the set of polynomials of sufficiently high degree, and is therefore valid almost everywhere.

In this chapter, we will show that any ideal $C^{(n)}$ belonging to the ring of polynomial functions in the jet variables $Z^{(n)} = (X^i, U, U_K)$ gives rise to a $\mathcal{G}$-signature correspondence between $\mathcal{P}_m$ and algebraic subvarieties of $\mathcal{C}^{(n)} \subset J^n(\mathbb{C}^m, \mathbb{C})$, where $\mathcal{C}^{(n)} = \mathrm{Var}\left(C^{(n)}\right)$ denotes the variety of the ideal $C^{(n)}$. For $f \in \mathcal{P}_m$, the restriction of the action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$ to $\mathrm{j}_n f$ naturally leads to the notions of a $\mathcal{G}$-signature ideal, a $\mathcal{C}^{(n)}$-signature set, and a $\mathcal{G}$-signature variety. We will also show that by fixing the degree d, then we can (trivially) obtain a complete signature correspondence for $\mathcal{P}_m^d$. We will then introduce the notion of a cross-section ideal for the the action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$. To each cross-section ideal, there will be a corresponding cross-section variety. The notion of a cross-section ideal was introduced in [17] where the authors studied rational actions of an algebraic group G on an affine space $\mathbb{K}^n$. We present a slightly altered definition of a cross-section ideal in order to put the transversality conditions at the forefront. After introducing a cross-section ideal, the corresponding cross-section variety will play an analogous role to that of a cross-section in the moving frame constructions presented above. In particular, we will use the cross-section variety to determine local normal forms for the action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$. However, using the algebraic constructions, should the cross-section

variety intersect the $\mathcal{G}$ orbits more than one time, we do not have to worry about deciding which local normal form we will project a point $\mathbf{z}^{(n)} \in J^n(\mathbb{C}^m, \mathbb{C})$ to. Instead, we will project to all local normal forms at once. Combining these constructions with the local techniques from the equivariant moving frame method, then for all $d \geq m$, we are able to produce an almost complete signature correspondence for $\mathcal{P}_m^d$ at a finite order of the jet space that is independent of the degree d. For the basics of the algebraic geometry and elimination theory that are used, we refer the reader to Chapters 2, 3 and 4 of [6].

3.1 Preliminaries

Notation 3.1.1. *We will denote the ring of polynomial functions on $J^n(\mathbb{C}^m, \mathbb{C})$ in the source coordinates by $\mathbb{C}[x, u, u_K] = \mathbb{C}[z^{(n)}]$ and in the target coordinates by $\mathbb{C}[X, U, U_K] = \mathbb{C}[Z^{(n)}]$. The variety of an ideal C contained in polynomial ring will be denoted by either $\mathrm{Var}(C)$ or $\mathcal{C}$. The radical of an ideal C will be denoted by $\sqrt{C}$.*

We will briefly recall the statements and conventions from Section 1.2 and Section 2.1. The group $GL_m(\mathbb{C})$ will be realized as the variety of the ideal $\left(\det\left(\lambda_j^i\right) s - 1\right) \subset \mathbb{C}[\lambda_j^i, s]$, $1 \leq i, j \leq m$. Any algebraic subgroup $\mathcal{G}$ of $GL_m(\mathbb{C})$ can be realized as a subvariety of $GL_m(\mathbb{C})$ defined by a radical ideal $G \subset \mathbb{C}[\lambda_j^i, s]$, where $\left(\det(\lambda_j^i)s - 1\right) \subset G$. Before proceeding, we recall Definition 3.1.1 and Definition 3.1.2.

Definition 3.1.1 ($\mathcal{G}$-Signature Correspondence and $\mathcal{G}$-Signature Variety)**.** *A $\mathcal{G}$-signature correspondence for the action of $\mathcal{G}$ on $\mathcal{P}_m$ is a map from $\mathcal{P}_m$ to the set of algebraic subvarieties of some algebraic variety $\mathcal{C}$ over $\mathbb{C}$, such that if the image of $f \in \mathcal{P}_m$ under*

this correspondence is denoted by $\mathcal{S}_f$, then

$$f \cong_{\mathcal{G}} g \Rightarrow \mathcal{S}_f = \mathcal{S}_g.$$

The image of $f \in \mathcal{P}_m$ under a $\mathcal{G}$-signature correspondence is said to be the $\mathcal{G}$-signature variety of f determined by the given $\mathcal{G}$-signature correspondence.

Definition 3.1.2 (Complete and Almost Complete Signature Correspondence). *We say that the $\mathcal{G}$-signature correspondence is* complete *if for all f and $g \in \mathcal{P}_m$,*

$$f \cong_{\mathcal{G}} g \iff \mathcal{S}_f = \mathcal{S}_g. \tag{3.1}$$

A $\mathcal{G}$-signature correspondence is almost complete *if there exists d_0 such that for all $d > d_0$ there exists a Zariski open subset $\mathcal{T}_m^d \subset \mathcal{P}_m^d$ such that (3.1) holds for all f and $g \in \mathcal{T}_m^d$*

Remark 3.1.1. *On account of $\mathcal{P}_m^d$ being closed under the action of $\mathcal{G}$, then one can also speak of an almost complete $\mathcal{G}$-signature correspondence for the action of $\mathcal{G}$ on $\mathcal{P}_m^d$ in the obvious manner.*

3.2   The Action Ideals

Definition 3.2.1 (Action Ideal). *Let $\mathcal{G}$ be an algebraic subgroup of $GL_m(\mathbb{C})$ defined by the ideal $G = (\gamma_1, \ldots, \gamma_t) \subset \mathbb{C}[\lambda_j^i, s]$. The* action ideal *for the action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$*

is the ideal contained in $\mathbb{C}[\lambda_j^i, s, z^{(n)}, Z^{(n)}]$ *generated by the equations*

$$\gamma_l = 0$$

$$X^i - \lambda_j^i x^j = 0$$

$$U - u = 0$$

$$U_j - \hat{\lambda}_j^i u_i = 0$$

$$\vdots$$

$$U_{j_1 \cdots j_k} - \hat{\lambda}_{j_1}^{i_1} \cdots \hat{\lambda}_{j_k}^{i_k} u_{i_1 \cdots i_k} = 0$$

$$\vdots$$

$$U_{j_1 \cdots j_n} - \hat{\lambda}_{j_1}^{i_1} \cdots \hat{\lambda}_{j_n}^{i_n} u_{i_1 \cdots i_n} = 0$$

where $1 \leq l \leq t$, $1 \leq i_s, j_s, \ldots \leq m$, *and* $1 \leq |K| \leq n$.

Remark 3.2.1. *Note that the* $\hat{\lambda}_j^i$ *are expressible as polynomials in* λ_j^i *and* s, *and satisfy the relations* $\lambda_j^i \hat{\lambda}_k^j = \delta_k^i$ *and* $\hat{\lambda}_j^i \lambda_k^j = \delta_k^i$.

Remark 3.2.2. *Going forward, we will assume that* $\mathcal{G}$ *is a fixed algebraic subgroup of* $GL_m(\mathbb{C})$ *given as the variety of a radical, unmixed dimensional ideal* $G = (\gamma_1, \ldots, \gamma_t) \subset \mathbb{C}[\lambda_j^i, s]$.

Notation 3.2.1. *The action ideal for the action of* $\mathcal{G}$ *on* $J^n(\mathbb{C}^m, \mathbb{C})$ *will be denoted by*

$$A^{(n)} = \left(G + \left(Z^{(n)} - \alpha \left(\lambda_j^i, z^{(n)} \right) \right) \right). \tag{3.2}$$

Definition 3.2.2 (Restricted Action Ideal of f). *Let* $f \in \mathcal{P}_m$. *The restriction of the* $\mathcal{G}$

action on $J^n\left(\mathbb{C}^m,\mathbb{C}\right)$ to the n-jet of f is given by the restricted action ideal

$$A^{(n)}[f] = \left(G + \left(Z^{(n)} - \alpha\left(\lambda_j^i, \mathrm{j}_n\, f\left(x\right)\right)\right)\right) \subset \mathbb{C}\left[\lambda_j^i, s, x, Z^{(n)}\right]. \tag{3.3}$$

3.3 $C^{(n)}$-Projection Ideals and $\mathcal{G}$- Signature Varieties

We will now let $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be any ideal and we will introduce $C^{(n)}$ into the action ideal (3.2) and into the restricted action ideal (3.3). Geometrically, we aim to capture the notion of projecting along the $\mathcal{G}$ orbits of $J^n\left(\mathbb{C}^m,\mathbb{C}\right)$ to $\mathrm{Var}\left(C^{(n)}\right) = \mathcal{C}^{(n)}$, the variety of the ideal $C^{(n)}$.

Notation 3.3.1. *When needed, we will denote an ideal $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ by the ominous notation $C^{(n)}_{Z^{(n)}}$ to place additional emphasis on the fact that we are viewing $C^{(n)}$ as in ideal in the polynomial ring of the target coordinates of $J^n\left(\mathbb{C}^m,\mathbb{C}\right)$.*

Notation 3.3.2. *We will denote the variety of an ideal $C^{(n)}, S^{(n)}, etc... \subset \mathbb{C}[Z^{(n)}]$ by the corresponding calligraphic letter $\mathcal{C}^{(n)}$, $\mathcal{S}^{(n)}$, etc . . . For all other ideals (e.g. $A^{(n)}[f]$), we will explicitly write $\mathrm{Var}\left(\right)$ (e.g. $\mathrm{Var}\left(A^{(n)}[f]\right)$).*

Notation 3.3.3. *We will denote sets of points that do not form a variety by $\check{\mathcal{C}}^{(n)}$, $\check{\mathcal{S}}^{(n)}$, etc . . . Generally, the corresponding calligraphic letter $\mathcal{C}^{(n)}$, $\mathcal{S}^{(n)}$, etc . . . will denote the variety corresponding to the algebraic closure.*

Definition 3.3.1 ($C^{(n)}$-Projection Ideals)**.** *Let $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be any ideal. The ideal*

$$\left(A^{(n)} + C^{(n)}\right) = \left(G + \left(Z^{(n)} - \alpha\left(\lambda_j^i, z^{(n)}\right)\right) + C^{(n)}_{Z^{(n)}}\right) \subset \mathbb{C}[\lambda_j^i, s, z^{(n)}, Z^{(n)}] \tag{3.4}$$

is said to be the $C^{(n)}$-projection ideal for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m,\mathbb{C}\right)$.

For $f \in \mathcal{P}_m$, the ideal

$$\left(A^{(n)}[f] + C^{(n)}\right) = \left(G + \left(Z^{(n)} - \alpha\left(\lambda_j^i, \mathrm{j}_n\ f(x)\right)\right) + C_{Z^{(n)}}^{(n)}\right) \subset \mathbb{C}[\lambda_j^i, s, x, Z^{(n)}] \quad (3.5)$$

is the $C^{(n)}$- projection ideal for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ restricted to f.

Remark 3.3.1. *For $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ and $f \in \mathcal{P}_m$, the varieties of the $C^{(n)}$-projection ideal and the $C^{(n)}$-projection ideal restricted to f are, respectively,*

$$\mathrm{Var}\left(A^{(n)} + C^{(n)}\right) = \left\{\left(\Lambda, \mathbf{z}^{(n)}, \mathbf{Z}^{(n)}\right) \in \mathcal{G} \times J^n\left(\mathbb{C}^m, \mathbb{C}\right) \times J^n\left(\mathbb{C}^m, \mathbb{C}\right) \left| \Lambda \cdot \mathbf{z}^{(n)} = \mathbf{Z}^{(n)} \in \mathcal{C}^{(n)}\right.\right\},$$

$$(3.6)$$

and

$$\mathrm{Var}\left(A^{(n)}[f] + C^{(n)}\right) = \left\{\left(\Lambda, \mathbf{x}, \mathbf{Z}^{(n)}\right) \in \mathcal{G} \times \mathbb{C}^m \times J^n\left(\mathbb{C}^m, \mathbb{C}\right) \left| \Lambda \cdot \left(\mathrm{j}_n\ f\left(\mathbf{x}\right)\right) = \mathbf{Z}^{(n)} \in \mathcal{C}^{(n)}\right.\right\}.$$

$$(3.7)$$

Proposition 3.3.2. *Let $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be any ideal and let $f, g \in \mathcal{P}_m$. If f and g are $\mathcal{G}$-equivalent, then $\mathrm{Var}\left(A^{(n)}[f] + C^{(n)}\right)$ and $\mathrm{Var}\left(A^{(n)}[g] + C^{(n)}\right)$ are isomorphic.*

Proof. Let $\Lambda \in \mathcal{G}$ such that $\Lambda \star f = g$. We will use the group element $\Lambda \in \mathcal{G}$ defining the equivalence between f and g to define an isomorphism between $\mathrm{Var}\left(A^{(n)}[f] + C^{(n)}\right)$ and $\mathrm{Var}\left(A^{(n)}[g] + C^{(n)}\right)$. Letting $\Lambda_0 \in \mathcal{G}$ be arbitrary, the jet space transformation laws

imply

$$\Lambda_0 \Lambda^{-1} \cdot (j_n \ g \left(\Lambda \cdot \mathbf{x} \right)) =$$

$$= \Lambda_0 \Lambda^{-1} \cdot (j_n \ \left(\Lambda \star f \right) \left(\Lambda \cdot \mathbf{x} \right)) \qquad \text{(By Hypothesis)}$$

$$= \Lambda_0 \Lambda^{-1} \cdot (\Lambda \cdot (j_n \ f \left(\mathbf{x} \right))) \qquad \text{(By (2.6))}$$

$$= \Lambda_0 \cdot (j_n \ f \left(\mathbf{x} \right)) .$$

Setting $\Lambda_0 \cdot (j_n \ f \left(\mathbf{x} \right)) = \mathbf{Z}^{(n)}$, then by (3.7) we conclude that $\left(\Lambda_0, \mathbf{x}, \mathbf{Z}^{(n)} \right) \in \text{Var} \left(A^{(n)}[f] + C^{(n)} \right)$ if and only if $\left(\Lambda_0 \Lambda^{-1}, \Lambda \cdot \mathbf{x}, \mathbf{Z}^{(n)} \right) \in \text{Var} \left(A^{(n)}[g] + C^{(n)} \right)$. Thus, the (an) isomorphism from $\text{Var} \left(A^{(n)}[f] + C^{(n)} \right)$ to $\text{Var} \left(A^{(n)}[g] + C^{(n)} \right)$ is given by $\left(\Lambda_0, \mathbf{x}, \mathbf{Z}^{(n)} \right) \mapsto \left(\Lambda_0 \Lambda^{-1}, \Lambda \cdot \mathbf{x}, \mathbf{Z}^{(n)} \right) .$

$\square$

Definition 3.3.3 ($\mathcal{C}^{(n)}$-Signature Set). *Let $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be an ideal and let $f \in \mathcal{P}_m$. The $\mathcal{C}^{(n)}$-signature set of f for the action of $\mathcal{G}$ on $J^n \left(\mathbb{C}^m, \mathbb{C} \right)$ is*

$$\breve{\mathcal{S}}_{\mathcal{C}}^{(n)}[f] = \left\{ \mathbf{Z}^{(n)} \in \mathcal{C}^{(n)} \Big| \exists \Lambda \in \mathcal{G}, \mathbf{x} \in \mathbb{C}^m \ s.t. \ \Lambda \cdot (j_n \ f \left(\mathbf{x} \right)) = \mathbf{Z}^{(n)} \right\},$$

Remark 3.3.2. *Note that the $\mathcal{C}^{(n)}$-signature set of f is the projection of $\text{Var} \left(A^{(n)}[f] + C^{(n)} \right)$ onto $J^n \left(\mathbb{C}^m, \mathbb{C} \right)$.*

Proposition 3.3.4. *Let $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be an ideal and let $f, g \in \mathcal{P}_m$. If f and g are $\mathcal{G}$-equivalent, then $\breve{\mathcal{S}}_{\mathcal{C}}^{(n)}[f] = \breve{\mathcal{S}}_{\mathcal{C}}^{(n)}[g]$.*

Proof. The result follows immediately from the isomorphism between the varieties $\text{Var} \left(A^{(n)}[f] + C^{(n)} \right)$ and $\text{Var} \left(A^{(n)}[g] + C^{(n)} \right)$ established in Proposition 3.3.2. Let $\Lambda \in \mathcal{G}$ such that $\Lambda \star f = g$ and note that the isomorphism which maps $\left(\Lambda_0, \mathbf{x}, \mathbf{Z}^{(n)} \right) \in \text{Var} \left(A^{(n)}[f] + C^{(n)} \right)$ to $\left(\Lambda_0 \Lambda^{-1}, \Lambda \cdot \mathbf{x}, \mathbf{Z}^{(n)} \right) \in \text{Var} \left(A^{(n)}[g] + C^{(n)} \right)$ is the identity on the

$J^n(\mathbb{C}^m, \mathbb{C})$ component. This proves the claim. □

Combining the above considerations with standard elimination theory motivates the following.

Definition 3.3.5 ($\mathcal{G}$-Signature Ideal). *Let $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be any ideal and let $f \in \mathcal{P}_m$. The $\mathcal{G}$-signature ideal of f associated to the ideal $C^{(n)}$ is the elimination ideal*

$$S_C^{(n)}[f] = (A^{(n)}[f] + C^{(n)}) \cap \mathbb{C}[Z^{(n)}]. \tag{3.8}$$

Definition 3.3.6 ($\mathcal{G}$-Signature Variety). *Let $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be any ideal and let $f \in \mathcal{P}_m$. The $\mathcal{G}$-signature variety of f associated to the ideal $C^{(n)}$ is the variety of $S_C^{(n)}[f]$. The signature variety of f will be denoted $\mathcal{S}_C^{(n)}[f]$.*

Remark 3.3.3. *Note that for any ideal $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ and for any $f \in \mathcal{P}_m$, the $\mathcal{G}$-signature variety, $\mathcal{S}_C^{(n)}[f]$, is the algebraic closure of the $C^{(n)}$-signature set, $\breve{\mathcal{S}}_C^{(n)}[f]$. This is a consequence of the closure theorem in elimination theory ([6], pp. 125).*

As an immediate corollary, we have the following.

Corollary 3.3.7. *Let $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be an ideal and let $f, g \in \mathcal{P}_m$. If f and g are $\mathcal{G}$-equivalent, then the $\mathcal{G}$-signature varieties $\mathcal{S}_C^{(n)}[f]$ and $\mathcal{S}_C^{(n)}[g]$ are equal.*

Proof. Proposition 3.3.4 implies that the $C^{(n)}$-signature sets are equal, i.e.,

$$\breve{\mathcal{S}}_C^{(n)}[f] = \breve{\mathcal{S}}_C^{(n)}[g].$$

Their Zariski closures $\mathcal{S}_C^{(n)}[f]$ and $\mathcal{S}_C^{(n)}[g]$ are also then equal. □

Corollary 3.3.8. *If $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ is any ideal and $f, g \in \mathcal{P}_m$ are $\mathcal{G}$-equivalent, then $\sqrt{S_C^{(n)}[f]} = \sqrt{S_C^{(n)}[g]}$.*

Remark 3.3.4. *Corollary 3.3.7 shows that any ideal $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ determines a $\mathcal{G}$-signature correspondence.*

We will now prove a series of relatively simple lemmas which help to illuminate the geometrical significance of the $\mathcal{G}$-signature variety $\mathcal{S}_{\mathcal{C}}^{(n)}[f]$ for $f \in \mathcal{P}_m$. We will also continue to make use of the $\mathcal{C}^{(n)}$-signature set of f, $\check{\mathcal{S}}_{\mathcal{C}}^{(n)}[f]$.

Lemma 3.3.9. *Let $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be an ideal and let $f, g \in \mathcal{P}_m$. If $\check{\mathcal{S}}_{\mathcal{C}}^{(n)}[f] \cap \check{\mathcal{S}}_{\mathcal{C}}^{(n)}[g]$ is nonempty, then $\exists \mathbf{y} \in \mathbb{C}^m$ and $\Lambda_0, \Lambda_1 \in \mathcal{G}$ such that $\mathrm{j}_n (\Lambda_0 \star f) (\mathbf{y}) = \mathrm{j}_n (\Lambda_1 \star g) (\mathbf{y})$.*

Proof. Let $\mathcal{C}^{(n)} \subset J^n (\mathbb{C}^m, \mathbb{C})$ be the variety of $C^{(n)}$ and let $\mathbf{Z}^{(n)} \in \check{\mathcal{S}}_{\mathcal{C}}^{(n)}[f] \cap \check{\mathcal{S}}_{\mathcal{C}}^{(n)}[g] \subset \mathcal{C}^{(n)}$ be a common point on the $\mathcal{C}^{(n)}$-signature sets of f and g. Let $\pi^n (\mathbf{Z}^{(n)}) = \mathbf{y} \in \mathbb{C}^m$ denote the projection of $\mathbf{Z}^{(n)}$ onto $\mathbb{C}^m$. By definition of the $\mathcal{C}^{(n)}$-signature sets of f and g, there exists $\mathbf{x}_0, \mathbf{x}_1 \in \mathbb{C}^m$ and $\Lambda_0, \Lambda_1, \in \mathcal{G}$ such that

$$\Lambda_0 \cdot (\mathrm{j}_n \, f (\mathbf{x}_0)) = \Lambda_1 \cdot (\mathrm{j}_n \, g (\mathbf{x}_1)) = \mathbf{Z}^{(n)} \in \mathcal{C}^{(n)}. \tag{3.9}$$

Condition (3.9) implies that

$$\Lambda_0 \cdot \mathbf{x}_0 = \Lambda_1 \cdot \mathbf{x}_1 = \mathbf{y} \in \mathbb{C}^m, \tag{3.10}$$

and condition (2.6) implies

$$\mathrm{j}_n (\Lambda_0 \star f) (\Lambda_0 \cdot \mathbf{x}_0) = \mathrm{j}_n (\Lambda_1 \star g) (\Lambda_1 \cdot \mathbf{x}_1). \tag{3.11}$$

Thus,

$$\mathrm{j}_n (\Lambda_0 \star f) (\mathbf{y}) = \mathrm{j}_n (\Lambda_1 \star g) (\mathbf{y}). \tag{3.12}$$

$\square$

Definition 3.3.10 ($\mathcal{G}$-Regular)**.** *Let $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be an ideal and let $f \in \mathcal{P}_m$. We say that f is $\mathcal{G}$-regular with respect to the ideal $C^{(n)}$ if the $\mathcal{G}$-signature variety $\mathcal{S}_\mathcal{C}^{(n)}[f]$ is nonempty.*

Remark 3.3.5. *For any ideal $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ and any $f \in \mathcal{P}_m$ note that the $\mathcal{G}$-signature variety $\mathcal{S}_\mathcal{C}^{(n)}[f] \neq \emptyset \iff$ the $C^{(n)}$-signature set $\check{\mathcal{S}}_\mathcal{C}^{(n)}[f] \neq \emptyset$.*

We will now bound the degree d of our polynomials under consideration and show that, as one might expect, the $\mathcal{G}$-signature varieties corresponding to any ideal $C^{(d)} \subset \mathbb{C}[Z^{(d)}]$ completely determine when two $\mathcal{G}$-regular polynomials $f, g \in \mathcal{P}_m^d$ are $\mathcal{G}$-equivalent.

Lemma 3.3.11. *Let $C^{(d)} \subset \mathbb{C}[Z^{(d)}]$ be an ideal and let $f, g \in \mathcal{P}_m^d$ be $\mathcal{G}$-regular with respect to $C^{(d)}$. Then f and g are $\mathcal{G}$-equivalent if and only if $\check{\mathcal{S}}_\mathcal{C}^{(d)}[f] \cap \check{\mathcal{S}}_\mathcal{C}^{(d)}[g]$ is nonempty.*

Proof. By Proposition 3.3.4, if f and g are equivalent then $\check{\mathcal{S}}_\mathcal{C}^{(d)}[f] = \check{\mathcal{S}}_\mathcal{C}^{(d)}[g]$. Since f and g are assumed to be $\mathcal{G}$-regular, we know that $\check{\mathcal{S}}_\mathcal{C}^{(d)}[f]$ and $\check{\mathcal{S}}_\mathcal{C}^{(d)}[g]$ are nonempty and thus, $\check{\mathcal{S}}_\mathcal{C}^{(d)}[f] \cap \check{\mathcal{S}}_\mathcal{C}^{(d)}[g] \neq \emptyset$.

Now, we will assume that the $C^{(d)}$-signature sets $\check{\mathcal{S}}_\mathcal{C}^{(d)}[f]$ and $\check{\mathcal{S}}_\mathcal{C}^{(d)}[g]$ have a point in common. Lemma 3.3.9 implies that there exists $\Lambda_0, \Lambda_1 \in \mathcal{G}$ and $\mathbf{y} \in \mathbb{C}^m$ such that

$$\mathrm{j}_d \left(\Lambda_0 \star f \right) (\mathbf{y}) = \mathrm{j}_d \left(\Lambda_1 \star g \right) (\mathbf{y}). \tag{3.13}$$

Since $\Lambda_0 \star f, \Lambda_1 \star g \in \mathcal{P}_m^d$ and their d-jets at $\mathbf{y} \in \mathbb{C}^m$ are equal, then $\Lambda_0 \star f = \Lambda_1 \star g$. Setting $\Lambda = (\Lambda_1)^{-1} \Lambda_0$, we thus have $\Lambda \star f = g$, and we conclude that f and g are equivalent. $\square$

Corollary 3.3.12. *Let $C^{(d)} \subset \mathbb{C}[Z^{(d)}]$ be an ideal and let $f, g \in \mathcal{P}_m^d$ be $\mathcal{G}$-regular with respect to $C^{(d)}$. Then f and g are $\mathcal{G}$-equivalent if and only if $\mathcal{S}_\mathcal{C}^{(d)}[f] = \mathcal{S}_\mathcal{C}^{(d)}[g]$.*

Proof. The result follows immediately from Proposition 3.3.4 and the fact that the $\mathcal{G}$-signature varieties $\mathcal{S}_{\mathcal{C}}^{(d)}[f]$ and $\mathcal{S}_{\mathcal{C}}^{(d)}[g]$ are the Zariski closures of the $\mathcal{C}^{(d)}$-signature sets $\breve{\mathcal{S}}_{\mathcal{C}}^{(d)}[f]$ and $\breve{\mathcal{S}}_{\mathcal{C}}^{(d)}[g]$, respectively. $\qquad\square$

We will now show that we obtain (trivially) a complete $\mathcal{G}$-signature correspondence on $\mathcal{P}_m^d$ by considering the ideal $C^{(d)} = (0) \subset \mathbb{C}[Z^{(d)}]$.

Proposition 3.3.13. *Let $C^{(d)} \subset \mathbb{C}[Z^{(d)}]$ be the zero ideal. Then $\forall f, g \in \mathcal{P}_m^d$, f and g are $\mathcal{G}$-equivalent if and only if $\mathcal{S}_{\mathcal{C}}^{(d)}[f] = \mathcal{S}_{\mathcal{C}}^{(d)}[g]$.*

Proof. The variety of $C^{(d)}$ is $J^d(\mathbb{C}^m, \mathbb{C})$. Therefore, every $h \in \mathcal{P}_m$ is $\mathcal{G}$-regular with respect to $C^{(d)}$. The result now follows immediately from Corollary 3.3.12. $\qquad\square$

Remark 3.3.6. *Observe that when one lets $C^{(d)}$ be the zero ideal, then for $f \in \mathcal{P}_m$, $\breve{\mathcal{S}}_{\mathcal{C}}^{(d)}[f]$ can be viewed as the orbit of the submanifold determined by $\mathrm{j}_d\, f \subset J^d(\mathbb{C}^m, \mathbb{C})$. $\breve{\mathcal{S}}_{\mathcal{C}}^{(d)}[f]$ will then be, generically, $(\dim(\mathcal{G}) + m)$- dimensional provided that d is greater than or equal to the or the order of stabilization for the action of $\mathcal{G}$ on $J^\infty(\mathbb{C}^m, \mathbb{C})$. At the alternative end of the spectrum, if one lets $C^{(d)} = \mathbb{C}[Z^{(d)}]$, then the corresponding variety $\mathcal{C}^{(d)}$ is the empty set, and as a result, for all $f \in \mathcal{P}_m$, $\breve{\mathcal{S}}_{\mathcal{C}}^{(d)}[f]$ and $\mathcal{S}_{\mathcal{C}}^{(d)}[f]$ are also empty and there are no polynomials which are $\mathcal{G}$-regular with respect to $\mathbb{C}[Z^{(d)}]$.*

3.4 Cross-Section Ideals and Cross-Section Varieties

We will now introduce the notion of a cross-section ideal which will specialize the algebraic constructions of the previous sections and make them amenable to the local techniques of the moving frame while still maintaing their global effectiveness. The definition of a cross-section ideal presented here has been slightly altered from that in [16] and [17] in

order to put the condition that the corresponding cross-section variety be transversal to the $\mathcal{G}$ orbits at the forefront. Compare with Proposition 3.2 in [16].

Definition 3.4.1 (Cross-Section Ideal)**.** *Let s_n denote the dimension of the generic orbits for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m,\mathbb{C}\right)$ and let $\mathrm{pr}^{(n)}(V_i)$, $1 \leq i \leq r = \dim(\mathcal{G})$, denote the infinitesimal generators for the action for $\mathcal{G}$ on $J^n\left(\mathbb{C}^m,\mathbb{C}\right)$. Further, let $C^{(n)} \subset \mathbb{C}\left[Z^{(n)}\right]$ be a prime ideal of codimension s_n which is given by a set of generators $(F_1, F_2, \ldots, F_t)$ with corresponding variety $\mathcal{C}^{(n)}$.*

$C^{(n)}$ is said to be a cross-section ideal *for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m,\mathbb{C}\right)$ if the generic rank of the $r \times t$ matrix $L(i,j) = \left(\mathrm{pr}^{(n)}(V_i)(F_j)\right)\Big|_{\mathcal{C}^{(n)}}$ is s_n.*

Definition 3.4.2 (Cross-Section Variety)**.** *Let $C^{(n)} \subset \mathbb{C}\left[Z^{(n)}\right]$ be a cross-section ideal for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m,\mathbb{C}\right)$. The* cross-section variety *defined by the ideal $C^{(n)}$ is the variety $\mathrm{Var}\left(C^{(n)}\right) = \mathcal{C}^{(n)} \subset J^n\left(\mathbb{C}^m,\mathbb{C}\right)$.*

Definition 3.4.3 (Transversal Point, Transversally Regular)**.** *Let $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be a cross-section ideal for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m,\mathbb{C}\right)$ of order n and let $\mathcal{C}^{(n)} \subset J^n\left(\mathbb{C}^m,\mathbb{C}\right)$ be the corresponding cross-section variety.*

1. *A nonsingular point $\mathbf{Z}^{(n)} \in \mathcal{C}^{(n)}$ is said to be a* transversal point *for the cross-section ideal $C^{(n)}$ if the cross-section $\mathcal{C}^{(n)}$ is transversal to the $\mathcal{G}$- orbit $\mathcal{O}_{\mathbf{Z}^{(n)}}$ at $\mathbf{Z}^{(n)}$.*

2. *Let $f \in \mathcal{P}_m$. We say that f is* transversally regular *with respect to the cross-section ideal $C^{(n)}$ if the $\mathcal{C}^{(n)}$-signature set of f, $\check{\mathcal{S}}_{\mathcal{C}}^{(n)}[f]$, contains a transversal point.*

Remark 3.4.1. *A non-singular point $\mathbf{Z}^{(n)} \in \mathcal{C}^{(n)}$ is a transversal point for the cross-section ideal $C^{(n)}$ if and only if $T_{\mathbf{Z}^{(n)}}\mathcal{O}_{\mathbf{Z}^{(n)}} \oplus T_{\mathbf{Z}^{(n)}}\mathcal{C}^{(n)} = T_{\mathbf{Z}^{(n)}}J^n\left(\mathbb{C}^m,\mathbb{C}\right)$.*

Remark 3.4.2. *Let $\mathrm{pr}^{(n)}(V_i)$ denote the infinitesimal generators for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m,\mathbb{C}\right)$ and assume that $C^{(n)}$ is a cross-section ideal given by a set of generators*

$F_1, \ldots, F_t \in C^{(n)}[Z^{(n)}]$. Then a nonsingular point $\mathbf{Z}^{(n)} \in \mathcal{C}^{(n)}$ is a transversal point if and only if the rank of the $r \times t$ matrix $L(i,j) = \left(\mathrm{pr}^{(n)}(V_i)F_j\right)$ at $\mathbf{Z}^{(n)}$ is equal to s_n. Furthermore, the set of all points $\mathbf{Z}^{(n)} \in \mathcal{C}^{(n)}$ which fail to be transversal points lie in a proper subvariety $\mathcal{W}^{(n)} \subset \mathcal{C}^{(n)}$ whose defining ideal $W^{(n)}$ can be easily described. Let $P^{(n)}$ denote the ideal generated by the set of all $s_n \times s_n$ minors of the matrix[1] $L(i,j)(Z^{(n)})$ (i.e., $P^{(n)}$ is the ideal defining the condition that the rank of $L(i,j)(Z^{(n)}) < s_n$), then

$$W^{(n)} = C^{(n)} + P^{(n)} \subset \mathbb{C}[Z^{(n)}]. \tag{3.14}$$

As a consequence of the above remark, we immediately have the following.

Corollary 3.4.4. *A prime ideal $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ whose codimension is equal to the dimension of the generic $\mathcal{G}$-orbits defines a cross-section ideal for the action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$ if and only if $\sqrt{W^{(n)}} \neq C^{(n)}$, where $W^{(n)}$ is defined as in (3.14).*

We will now prove two propositions which serve to show that for a given cross-section ideal $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$, most $\mathcal{G}$-orbits $\mathcal{O}_{\mathbf{z}^{(n)}}$ will intersect the cross-section variety $\mathcal{C}^{(n)}$ transversally.

Proposition 3.4.5. *Let $C^{(n)}$ be a cross-section ideal for the action of $\mathcal{G}$ on $J^n(\mathbb{C}^m, \mathbb{C})$ with corresponding cross-section variety $\mathcal{C}^{(n)}$ and let $\breve{\Pi}^{(n)} = \left\{ \mathbf{z}^{(n)} \in J^n(\mathbb{C}^m, \mathbb{C}) \,\middle|\, \mathcal{O}_{\mathbf{z}^{(n)}} \cap \mathcal{C}^{(n)} = \emptyset \right\}$. Then $\breve{\Pi}^{(n)} \subset \mathcal{Y}^{(n)} \subset J^n(\mathbb{C}^m, \mathbb{C})$, where $\mathcal{Y}^{(n)}$ is a proper subvariety of $J^n(\mathbb{C}^m, \mathbb{C})$.*

Proof. Let $\mathcal{W}^{(n)} \subset \mathcal{C}^{(n)}$ be the subvariety of $\mathcal{C}^{(n)}$ where the cross-section variety $\mathcal{C}^{(n)}$ fails to intersect the $\mathcal{G}$-obrbits transversally (Remark 3.4.2). Then by the definition of the cross-section ideal $C^{(n)}$, we have $\dim(\mathcal{C}^{(n)}/\mathcal{W}^{(n)}) = \dim(J^n(\mathbb{C}^m, \mathbb{C})) - s_n$, where s_n

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[1]$L(i,j)$ is being viewed as a map from $J^n(\mathbb{C}^m, \mathbb{C}) \to M_{r \times s}$, the set of all $r \times s$ matrices.

denotes the dimension of the generic orbits for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m,\mathbb{C}\right)$. Define

$$\breve{\mathcal{U}} = \left\{\mathbf{z}^{(n)} \in J^n\left(\mathbb{C}^m,\mathbb{C}\right) \middle| \exists \Lambda \in \mathcal{G} \text{ and } \mathbf{Z}^{(n)} \in \left(\mathcal{C}^{(n)}/\mathcal{W}^{(n)}\right) \; s.t. \; \Lambda \cdot \mathbf{z}^{(n)} = \mathbf{Z}^{(n)}\right\},$$

and

$$\breve{\mathcal{V}} = \left\{\mathbf{z}^{(n)} \in J^n\left(\mathbb{C}^m,\mathbb{C}\right) \middle| \exists \Lambda \in \mathcal{G} \text{ and } \mathbf{Z}^{(n)} \in \mathcal{C}^{(n)} \; s.t. \; \Lambda \cdot z^{(n)} = \mathbf{Z}^{(n)}\right\}.$$

and observe that $\breve{\mathcal{U}} \subset \breve{\mathcal{V}}$. Now consider the elimination ideal $V = \left(A^{(n)} + C^{(n)}\right) \cap \mathbb{C}[z^{(n)}]$, where $\left(A^{(n)} + C^{(n)}\right)$ is then $C^{(n)}$-projection ideal (Definition 3.3.1). Note that $\text{Var}\left(V\right)$ is the algebraic closure of $\breve{\mathcal{V}}$ and thus $\breve{\mathcal{U}} \subset \text{Var}\left(V\right) \subset J^n\left(\mathbb{C}^m,\mathbb{C}\right)$.

On account of $C^{(n)}$ defining a cross-section ideal then $\breve{\mathcal{U}}$ contains a metric topology open set and the topological dimension of $\breve{\mathcal{U}}$ satisfies

$$\dim(\breve{\mathcal{U}}) = \dim\left(\mathcal{C}^{(n)}/\mathcal{W}^{(n)}\right) + s_n = \dim\left(J^n\left(\mathbb{C}^m,\mathbb{C}\right)\right).$$

Since $J^n\left(\mathbb{C}^m,\mathbb{C}\right)$ is an irreducible variety and on account of the fact that $\breve{\mathcal{U}} \subset \text{Var}\left(V\right)$ then we conclude that $\text{Var}\left(V\right) = J^n\left(\mathbb{C}^m,\mathbb{C}\right)$. By elimination theory, there is a proper subvariety $\mathcal{Y}^{(n)} \subset J^n\left(\mathbb{C}^m,\mathbb{C}\right)$ such that $\left(J^n\left(\mathbb{C}^m,\mathbb{C}\right)/\left(\mathcal{Y}^{(n)}\right)\right) \subset \breve{\mathcal{V}}$, or equivalently, $\left(J^n\left(\mathbb{C}^m,\mathbb{C}\right)/\breve{\mathcal{V}}\right) \subset \mathcal{Y}^{(n)}$. Now, merely note that $\breve{\Pi}^{(n)} = J^n\left(\mathbb{C}^m,\mathbb{C}\right)/\breve{\mathcal{V}}$. $\qquad\square$

Proposition 3.4.6. *Let $C^{(n)}$ be a cross-section ideal for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m,\mathbb{C}\right)$ and let $\mathcal{W}^{(n)}$ denote the subvariety of $\mathcal{C}^{(n)}$ containing the non-transversal points of the cross-section variety (see Remark 3.4.2). Define*

$$\breve{\Upsilon}^{(n)} = \left\{\mathbf{z}^{(n)} \in J^n\left(\mathbb{C}^m,\mathbb{C}\right) \middle| \exists \Lambda \in \mathcal{G} \; \text{ and } \mathbf{Z}^{(n)} \in \mathcal{W}^{(n)} \; s.t. \; \Lambda \cdot \mathbf{z}^{(n)} = \mathbf{Z}^{(n)}\right\}.$$

Then $\check{\Upsilon}^{(n)}$ *lies in a proper subvariety* $\mathcal{Q}^{(n)}$ *of* $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$.

Proof. Let $W^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be the ideal from Remark 3.4.2 that defines the variety $\mathcal{W}^{(n)}$ that contains the non-transversal points for the cross-section ideal $C^{(n)}$. Then consider the ideal $\left(A^{(n)} + W^{(n)}\right) \subset \mathbb{C}[\lambda_j^i, s, z^{(n)}, Z^{(n)}]$, where $A^{(n)}$ is the action ideal for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$. The variety of the elimination ideal

$$Q = \left(A^{(n)} + W^{(n)}\right) \cap \mathbb{C}[z^{(n)}] \tag{3.15}$$

is the Zariski closure of $\check{\Upsilon}^{(n)}$. We will denote the variety of Q by $\mathrm{Var}\left(Q\right) = \mathcal{Q}^{(n)}$. Note that $\dim \check{\Upsilon}^{(n)} \leq \dim(\mathcal{W}^{(n)}) + s_n < \dim\left(J^n\left(\mathbb{C}^m, \mathbb{C}\right)\right)$, where s_n denotes the dimension of the generic orbits for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$. Thus $\dim\left(\mathcal{Q}^{(n)}\right) \leq \dim\left(\mathcal{W}^{(n)}\right)$. Therefore, $\check{\Upsilon}^{(n)}$ lies in a proper subvariety of $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$. $\qquad\square$

Corollary 3.4.7. *Let* $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ *be a cross-section ideal for the action of* $\mathcal{G}$ *on* $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$. *The set of all* $\mathbf{z}^{(n)} \in J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ *such that the* $\mathcal{G}$-*orbit* $\mathcal{O}_{\mathbf{z}^{(n)}}$ *fails to intersect* $\mathcal{C}^{(n)}$ *at a transversal point belongs to a proper subvariety of* $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$.

Proof. Let $\mathcal{Y}^{(n)} \subset J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ denote the variety from Proposition 3.4.5 and let $\mathcal{Q}^{(n)} \subset J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ denote the variety appearing in Proposition 3.4.6. Then we consider the variety $\mathcal{Q}^{(n)} \cup \mathcal{Y}^{(n)} \subset J^n\left(\mathbb{C}^m, \mathbb{C}\right)$. Since both $\mathcal{Q}^{(n)}$ and $\mathcal{Y}^{(n)}$ are proper subvarieties of $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ and $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ is irreducible, it follows that

$$\dim\left(\mathcal{Q}^{(n)} \cup \mathcal{Y}^{(n)}\right) = \max\left\{\dim(\mathcal{Q}^{(n)}), \dim(\mathcal{Y}^{(n)})\right\}.$$

Furthermore, $\mathcal{Q}^{(n)} \cup \mathcal{Y}^{(n)}$ contains the set of all $\mathbf{z}^{(n)} \in J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ such that either

1. $\mathcal{O}_{\mathbf{z}^{(n)}} \cap \mathcal{C}^{(n)} = \emptyset$, or

2. $\mathcal{O}_{\mathbf{z}^{(n)}} \cap \mathcal{C}^{(n)}$ contains a non-transversal point.

The conclusion follows. □

Theorem 3.4.8. *Let $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be a cross-section ideal for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ with corresponding cross-section variety $\mathcal{C}^{(n)}$. Let $\mathcal{R}^{(n)} \subset J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ be a subvariety that contains the set of all $\mathbf{z}^{(n)} \in J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ whose $\mathcal{G}$-orbits $\mathcal{O}_{\mathbf{z}^{(n)}}$ do not contain a point that intersects $\mathcal{C}^{(n)}$ transversally and let $R^{(n)} \subset \mathbb{C}[z^{(n)}]$ be an ideal defining $\mathcal{R}^{(n)}$. Define*

$$\check{\mathcal{B}}_m^d = \left\{ f \in \mathcal{P}_m^d \,\middle|\, \mathrm{j}_n\, f\left(\mathbf{x}\right) \in \mathcal{R}, \forall \mathbf{x} \in \mathbb{C}^m \right\}.$$

Then for all $d \geq n$, the set $\check{\mathcal{B}}_m^d$ lies in a proper subvariety of $\mathcal{P}_m^d$.

Remark 3.4.3. *Since $\mathcal{P}_m^d$ is an irreducible variety, any proper subvariety is necessarily of lower dimension.*

Proof. Let $f \in \mathcal{P}_m^d$ $(d \geq n)$ be an arbitrary polynomial with undetermined coefficients $c = (c_K)$, where we will assume that the standard basis of monomials spanning $\mathcal{P}_m^d$ has been ordered in accordance with our ordering on partial derivative multi-indices. The coefficients $c = (c_K)$ thus range over all partial derivative multi-indices with $|K| \leq d$. Let $Q \subset \mathbb{C}[x, c, z^{(n)}]$ be the ideal

$$Q = \left(\left(z^{(n)} - \mathrm{j}_n\, f(x)\right) + R^{(n)} \right) \subset \mathbb{C}[x, c, z^{(n)}],$$

and let $Y^{(n)}$ be the elimination ideal $Y^{(n)} = Q \cap \mathbb{C}[x, c]$. The variety of $Y^{(n)}$, $\mathrm{Var}\left(Y^{(n)}\right)$, is the Zariski closure of the set

$$\check{\mathcal{Y}}^{(n)} = \left\{ (\mathbf{x}, f) \in \mathbb{C}^m \times \mathcal{P}_m^d \,\middle|\, \mathrm{j}_n\, f(\mathbf{x}) \in \mathcal{R}^{(n)} \right\}.$$

Let $\phi_1(x, c), \ldots, \phi_t(x, c) \in \mathbb{C}[x, c]$ be a set of generators for the ideal $Y^{(n)}$ and rewrite each $\phi_i(x, c)$ as $\tilde{\phi}_i \in \mathbb{C}[c][x]$. Now, let $B_m^d \subset \mathbb{C}[c]$ be the ideal generated by the set the coefficient functions of the $\tilde{\phi}_i$. For any $f \in \mathcal{P}_m^d$, we have $(\mathbf{x}, f) \in \breve{\mathcal{Y}}^{(n)} \ \forall \mathbf{x} \in \mathbb{C}^m$ if and only if the coefficients of f, $c_f = (c_K)_f$, are a zero of the ideal B_m^d. This implies that $\breve{\mathcal{B}}_m^d \subset \mathrm{Var}\left(B_m^d\right)$.

In order to show (for all $d \geq n$) that $\mathrm{Var}\left(B_m^d\right)$ is a proper subvariety of $\mathcal{P}_m^d$, we will first show that $\breve{\mathcal{Y}}^{(n)} = \mathrm{Var}\left(Y^{(n)}\right)$. Since we are working over $\mathbb{C}$, elimination theory implies that $\breve{\mathcal{Y}}^{(n)}$ is dense in $\mathrm{Var}\left(Y^{(n)}\right)$ in the metric topology of $\mathbb{C}^m \times \mathcal{P}_m^d$. Therefore, for all $(\mathbf{x}, f) \in \mathrm{Var}\left(Y^{(n)}\right)$, there exists a sequence $(\mathbf{x}_i, f_i) \in \breve{\mathcal{Y}}^{(n)}$, $i = 1, \ldots, \infty$, that converges to $(\mathbf{x}, f)$ in the metric topology of $\mathbb{C}^m \times \mathcal{P}_m^d$. Thus, $\lim_{i \to \infty} \mathrm{j}_n f_i(\mathbf{x}_i) = \mathrm{j}_n f(\mathbf{x})$ in the metric topology of $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$. Furthermore, note that for all $i = 1, \ldots, \infty$, $\mathrm{j}_n f_i(\mathbf{x}_i) \in \mathcal{R}^{(n)}$, and that $\mathcal{R}^{(n)}$ is closed in both the Zariski and metric topologies of $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$. Thus, $\lim_{i \to \infty} \mathrm{j}_n f_i(\mathbf{x}_i) = \mathrm{j}_n f(\mathbf{x}) \in \mathcal{R}^{(n)}$ and we conclude that $\breve{\mathcal{Y}}^{(n)} = \mathrm{Var}\left(Y^{(n)}\right)$.

We will now show that for $d \geq n$, $\mathrm{Var}\left(B_m^d\right)$ is a proper subvariety of $\mathcal{P}_m^d$. We will do so by constructing a polynomial $f \in \mathcal{P}_m^d$ that does not belong to the variety $\mathrm{Var}\left(B_m^d\right)$. Let $\mathbf{z}^{(n)} \in J^n\left(\mathbb{C}^m, \mathbb{C}\right)/\mathcal{R}^{(n)}$ and take $\pi^n(\mathbf{z}^{(n)}) = \mathbf{x} \in \mathbb{C}^m$. Since the n-jet of a polynomial f of degree n at a point $\mathbf{x} \in \mathbb{C}^m$ uniquely determines the polynomial f, then we can take f to be the polynomial of degree n determined by $\mathrm{j}_n f(\mathbf{x}) = \mathbf{z}^{(n)}$. Thus $(\mathbf{x}, f)$ does not belong to $\breve{\mathcal{Y}}^{(n)} = \mathrm{Var}\left(Y^{(n)}\right)$, Thus, for $d \geq n$, $\mathrm{Var}\left(B_m^d\right) \subset \mathcal{P}_m^d$ is a proper subvariety. On account of $\mathcal{P}_m^d$ being irreducible, we conclude that $\mathrm{Var}\left(B\right)_n^d$ has a lower dimension. $\square$

3.5 Stabilization of $\mathcal{G}$-Signature Varieties

Our primary interest for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ lies with cross-section ideals which are of codimension $r = \dim(\mathcal{G})$. Recall that a cross-section ideal $C^{(n)}$ can be of codimension $r = \dim(\mathcal{G})$ if and only if the defining polynomials for $C^{(n)}$ are explicitly dependent on the jet variables of order n_0 (where n_0 is the order of stabilization for the action of $\mathcal{G}$ on $J^\infty\left(\mathbb{C}^m, \mathbb{C}\right)$) or higher. Furthermore, if $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ is a cross-section ideal for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ with $n \geq n_0$, then for all $k \geq n$, the extension ideal of $C^{(n)}$ in $\mathbb{C}[Z^{(k)}]$ also serves to define a cross-section ideal for the action of $\mathcal{G}$ on $J^k\left(\mathbb{C}^m, \mathbb{C}\right)$. We will now set notation to account for the fact that a cross-section ideal of codimension $r = \dim(\mathcal{G})$ gives rise to a sequence of cross-section ideals

Notation 3.5.1. *If $n \geq n_0$ and $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ is a cross-section ideal for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$, then for $k \geq n$, we will denote the extension ideal of $C^{(n)}$ in $\mathbb{C}[Z^{(k)}]$ by $\tilde{C}^{(k)} \subset \mathbb{C}[Z^{(k)}]$. We will denote the cross-section variety corresponding to the ideal $\tilde{C}^{(k)}$ by $\tilde{\mathcal{C}}^{(k)}$ and for $f \in \mathcal{P}_m$, we will denote the $\tilde{\mathcal{C}}^{(k)}$-signature set of f and the $\mathcal{G}$-signature variety associated to $\tilde{C}^{(k)}$ by the usual $\breve{\mathcal{S}}_\mathcal{C}^{(k)}[f]$ and $\mathcal{S}_\mathcal{C}^{(k)}[f]$, respectively.*

We will now record a simple proposition that will be used in the theorem that follows.

Proposition 3.5.1. *Let n_0 denote the stabilization order for the action of $\mathcal{G}$ on $J^\infty\left(\mathbb{C}^m, \mathbb{C}\right)$ and let $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be a cross-section ideal for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ with $n \geq n_0$. Then, for all $k \geq n$, if $\mathbf{Z}^{(k)} \in \tilde{\mathcal{C}}^{(k)}$ and $\pi_n^k\left(\mathbf{Z}^{(k)}\right) \in \mathcal{C}^{(n)}$ is a transversal point for the cross-section ideal $\mathcal{C}^{(n)}$, then $\mathbf{Z}^{(k)}$ is also a transversal point for the cross-section ideal $\tilde{C}^{(k)}$.*

Theorem 3.5.2. *Let $C^{(n)} \subset \mathbb{C}[Z^{(n)}]$ be a cross-section ideal for the action of $\mathcal{G}$ on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ $(n \geq n_0)$ with corresponding cross-section $\mathcal{C}^{(n)} \subset J^n\left(\mathbb{C}^m, \mathbb{C}\right)$, and let $f, g \in$*

$\mathcal{P}_m$.

If f and g are transversally regular with respect to the cross-section ideal $C^{(n)}$ and there exists $k \geq n$ such that

1. *$\mathcal{S}_{\mathcal{C}}^{(k+1)}[f] = \mathcal{S}_{\mathcal{C}}^{(k+1)}[g]$, and*

2. *$\dim(\mathcal{S}_{\mathcal{C}}^{(k)}[f]) = \dim(\mathcal{S}_{\mathcal{C}}^{(k+1)}[f])$,*

then f and g are equivalent.

Proof. The second assumption implies that for all t with $n \leq t \leq k+1$, then we also have $\mathcal{S}_{\mathcal{C}}^{(t)}[f] = \mathcal{S}_{\mathcal{C}}^{(t)}[g]$. Let $\mathcal{V}^{(k)} = \mathcal{S}_{\mathcal{C}}^{(k)}[f] = \mathcal{S}_{\mathcal{C}}^{(k)}[g]$ and let $\mathcal{V}^{(k+1)} = \mathcal{S}_{\mathcal{C}}^{(k+1)}[f] = \mathcal{S}_{\mathcal{C}}^{(k+1)}[g]$. By hypothesis (and also on account of the fact that the $\mathcal{G}$-signature varieties are the Zariski closures of the $\mathcal{C}^{(n)}$-signature sets), there exists $\mathbf{Z}^{(n)} \in \breve{\mathcal{S}}_{\mathcal{C}}^{(n)}[f] \cap \breve{\mathcal{S}}_{\mathcal{C}}^{(n)}[g]$ such that $\mathbf{Z}^{(n)}$ is a transversal point with respect to the cross-section ideal $C^{(n)}$. Furthermore, there exists $\mathbf{Z}^{(k)} \in \breve{\mathcal{S}}_{\mathcal{C}}^{(k)}[f] \cap \breve{\mathcal{S}}_{\mathcal{C}}^{(k)}[g] \subset \tilde{\mathcal{C}}^{(k)}$ and $\mathbf{Z}^{(k+1)} \in \breve{\mathcal{S}}_{\mathcal{C}}^{(k+1)}[f] \cap \breve{\mathcal{S}}_{\mathcal{C}}^{(k+1)}[g] \subset \tilde{\mathcal{C}}^{(k+1)}$ such that

1. $\pi_k^{k+1}(\mathbf{Z}^{(k+1)}) = \mathbf{Z}^{(k)}$,

2. $\pi_n^k(\mathbf{Z}^{(k)}) = \mathbf{Z}^{(n)}$,

3. $\mathbf{Z}^{(k+1)}$ lies in an irreducible component of $\mathcal{V}^{(k+1)}$ of maximal dimension, and

4. $\mathbf{Z}^{(k)}$ also lies in an irreducible component of $\mathcal{V}^{(k)}$ of maximal dimension.

On account of Proposition 3.5.1, $\mathbf{Z}^{(k)}$ and $\mathbf{Z}^{(k+1)}$ are transversal points with respect to the cross-section ideals $\tilde{C}^{(k)}$ and $\tilde{C}^{(k+1)}$, respectively.

By definition of $\breve{\mathcal{S}}_{\mathcal{C}}^{(k+1)}[f]$ and $\breve{\mathcal{S}}_{\mathcal{C}}^{(k+1)}[g]$, there exists $\Lambda_0, \Lambda_1 \in \mathcal{G}$ and $\mathbf{x}_0, \mathbf{x}_1 \in \mathbb{C}^m$ such that

$$\Lambda_0 \cdot \mathrm{j}_{k+1} \, f(\mathbf{x}_0) = \Lambda_1 \cdot \mathrm{j}_{k+1} \, g(\mathbf{x}_1) = \mathbf{Z}^{(k+1)} \in \tilde{\mathcal{C}}^{(k+1)}. \tag{3.16}$$

We now replace f and g with the equivalent polynomials $\Lambda_0 \star f = \bar{f}$ and $\Lambda_1 \star g = \bar{g}$ and we will denote the m-dimensional submanifolds of $J^k\left(\mathbb{C}^m, \mathbb{C}\right)$ and $J^{k+1}\left(\mathbb{C}^m, \mathbb{C}\right)$ determined by the jets $\mathrm{j}_k \ \bar{f}$ and $\mathrm{j}_{k+1} \ \bar{f}$ by $M_{\bar{f}}^{(k)}$ and $M_{\bar{f}}^{(k+1)}$, respectively. We can further assume that $\mathbf{Z}^{(k)}$ and $\mathbf{Z}^{(k+1)}$ are points such that

1. $m - \dim\left(T_{\mathbf{Z}^{(k)}} M_{\bar{f}}^{(k)} \cap T_{\mathbf{Z}^{(k)}} \mathcal{O}_{\mathbf{Z}^{(k)}}\right)$ is equal to the dimension of $\mathcal{S}_{\mathcal{C}}^{(k)}[\bar{f}]$ at $Z^{(k)}$, and

2. $m - \dim\left(T_{\mathbf{Z}^{(k+1)}} M_{\bar{f}}^{(k+1)} \cap T_{\mathbf{Z}^{(k+1)}} \mathcal{O}_{\mathbf{Z}^{(k+1)}}\right)$ is equal to the dimension of $\mathcal{S}_{\mathcal{C}}^{(k+1)}[\bar{f}]$ at $Z^{(k+1)}$,

and likewise for $\bar{g}$. This ensures that the jets of $\bar{f}$ and $\bar{g}$ are as fully transverse to the orbits as possible at $\mathbf{Z}^{(k)}$ and $\mathbf{Z}^{(k+1)}$.

Due to Corollary 3.3.7, the $\mathcal{G}$-signature varieties $\mathcal{S}_{\mathcal{C}}^{(k)}[\bar{f}]$, $\mathcal{S}_{\mathcal{C}}^{(k+1)}[\bar{f}]$, $\mathcal{S}_{\mathcal{C}}^{(k)}[\bar{g}]$ and $\mathcal{S}_{\mathcal{C}}^{(k+1)}[\bar{g}]$ satisfy the same conditions on their signatures as those of f and g, and the hypothesis of the theorem still apply. Since $\mathbf{Z}^{(n)}$ is a transversal point for the cross-section ideal $C^{(n)}$, there exists a local moving frame map (of order n) in a neighborhood of $\mathbf{Z}^{(n)}$ which corresponds to the local cross-section determined by the cross-section variety $\mathcal{C}^{(n)}$. Furthermore, $\forall t \geq n$, the moving frame signature sets of $\bar{f}$ and $\bar{g}$ order t will correspond (locally) with the $\tilde{\mathcal{C}}^{(t)}$-signature sets of $\bar{f}$ and $\bar{g}$, *and* the $\tilde{\mathcal{C}}^{(t)}$-signature sets will be locally parameterized by the signature maps of $\bar{f}$ and $\bar{g}$ associated to the local moving frame map. Theorem 2.2.13 implies $\bar{f}$ and $\bar{g}$ are equivalent. Thus, f and g are equivalent. $\quad\square$

3.6 Example

We will now illustrate the previous constructions with a simple example. We will take $\mathcal{G} = SL_2\left(\mathbb{C}\right)$ and we will consider the actions of $\mathcal{G}$ on $J^1\left(\mathbb{C}^2, \mathbb{C}\right)$, $J^2\left(\mathbb{C}^2, \mathbb{C}\right)$ and on $\mathcal{P}_2$.

We will realize $SL_2(\mathbb{C})$ as the variety of the ideal $(\lambda_1^1\lambda_2^2 - \lambda_2^1\lambda_1^2 - 1) \subset \mathbb{C}[\lambda_j^i]$, where we have eliminated the variable s appearing in the realization of $GL_2(\mathbb{C})$ as an affine variety. Recall from Chapter 2.4.1 that the action of $SL_2(\mathbb{C})$ is locally free on $J^1(\mathbb{C}^2, \mathbb{C})$. We set $x^1 = x$ and $x^2 = y$ and the set the index range for the section to be $1 \leq i, j \leq 2$. The action ideal of Definition 3.2.1 for the action of $SL_2(\mathbb{C})$ on $J^1(\mathbb{C}^2, \mathbb{C})$ is the ideal $A^{(1)} \subset \mathbb{C}[\lambda_j^i, x, y, u, u_1, u_2, X, Y, U, U_1, U_2]$ generated by the equations

$$\lambda_1^1\lambda_2^2 - \lambda_2^1\lambda_1^2 - 1 = 0 \tag{3.17}$$

$$X - \lambda_1^1 x - \lambda_2^1 y = 0 \tag{3.18}$$

$$Y - \lambda_1^2 x - \lambda_2^2 y = 0 \tag{3.19}$$

$$U - u = 0 \tag{3.20}$$

$$U_1 - \lambda_2^2 u_1 + \lambda_1^2 u_2 = 0 \tag{3.21}$$

$$U_2 + \lambda_2^1 u_1 - \lambda_1^1 u_2 = 0, \tag{3.22}$$

where we have used the fact that $\hat{\lambda}_1^1 = \lambda_2^2$, $\hat{\lambda}_2^2 = \lambda_1^1$, $\hat{\lambda}_1^2 = -\lambda_1^2$ and $\hat{\lambda}_2^1 = -\lambda_2^1$ (see equation (2.49)).

Remark 3.6.1. *Due to the simplicity of the relationships between λ_j^i and $\hat{\lambda}_j^i$, we will insert them into the transformation laws without mention.*

The action ideal $A^{(2)}$ for the action of $SL_2(\mathbb{C})$ on $J^2(\mathbb{C}^2, \mathbb{C})$ is obtained by adjoining the equations

$$U_{11} - \left(\lambda_2^2\right)^2 u_{11} + 2\lambda_2^2\lambda_1^2 u_{12} - \left(\lambda_1^2\right)^2 u_{22} = 0 \tag{3.23}$$

$$U_{12} + \lambda_2^2\lambda_2^1 u_{11} - \left(\lambda_2^2\lambda_1^1 + \lambda_1^2\lambda_2^1\right) u_{12} + \lambda_1^2\lambda_1^1 u_{22} = 0 \tag{3.24}$$

$$U_{22} - \left(\lambda_2^1\right)^2 u_{11} + 2\lambda_2^1\lambda_1^1 u_{12} - \left(\lambda_1^1\right)^2 u_{22} = 0 \tag{3.25}$$

to the equations (3.17)-(3.22) generating $A^{(1)}$.

The ideal

$$C^{(1)} = \left(X, U^1, Y - 1 \right) \subset \mathbb{C}[X, Y, U, U_1, U_2] \tag{3.26}$$

defines the variety

$$\mathcal{C}^{(1)} = \left\{ \mathbf{Z}^{(1)} \in J^1 \left(\mathbb{C}^2, \mathbb{C} \right) \middle| X = U_1 = 0, Y = 1 \right\} \subset J^1 \left(\mathbb{C}^2, \mathbb{C} \right),$$

and gives rise to the $C^{(1)}$-projection ideal (Definition 3.3.1) for the action of $SL_2\left(\mathbb{C}\right)$ on $J^1\left(\mathbb{C}^2, \mathbb{C}\right)$,

$$\left(A^{(1)} + C^{(1)}_{Z^{(1)}} \right) \subset \mathbb{C}[\lambda^i_j, z^{(1)}, Z^{(1)}].$$

The generators for $\left(A^{(1)} + C^{(1)}_{Z^{(1)}} \right)$ are obtained by adjoining the equations

$$X = 0, \quad Y - 1 = 0, \quad U_1 = 0, \tag{3.27}$$

to the equations (3.17) - (3.22) generating the action ideal $A^{(1)}$. Similarly, taking $\tilde{C}^{(2)}$ to be the extension ideal of $C^{(1)}$ in $\mathbb{C}[Z^{(2)}]$, then the generators for the $\tilde{C}^{(2)}$-projection ideal for the action of $SL_2\left(\mathbb{C}\right)$ on $J^2\left(\mathbb{C}^2, \mathbb{C}\right)$ by adjoining the equations (3.23) - (3.25) to the generators of the $C^{(1)}$-projection ideal.

We will now let $f = x^3 + y^3 \in \mathcal{P}_2$ and we will carry out the construction of the restricted action ideal from Definition 3.2.2 and the restricted $C^{(1)}$-projection ideal from Definition 3.3.1. The two jet of f is

$$j_2\, f\left(x, y\right) = \left(x, y, x^3 + y^3, 3x^2, 3y^2, 6x, 0, 6y \right),$$

and the restricted action ideals $A^{(1)}[f]$ and $A^{(2)}[f]$ can be viewed as nothing more than substitution of the components of $j_2\, f$ into the $J^2\left(\mathbb{C}^2,\mathbb{C}\right)$ source variables $z^{(2)} = (x, y, u, u_1, u_2, u_{11}, u_{12}, u_{22})$ in the equations $(3.17) - (3.25)$ generating the action ideals $A^{(1)}$ and $A^{(2)}$. The restricted action ideal $A^{(1)}[f] \subset \mathbb{C}[\lambda_j^i, x, y, Z^{(1)}]$ is thus generated by the equations

$$\lambda_1^1 \lambda_2^2 - \lambda_2^1 \lambda_1^2 - 1 = 0, \tag{3.28}$$

$$X - \lambda_1^1 x - \lambda_2^1 y = 0, \tag{3.29}$$

$$Y - \lambda_1^2 x - \lambda_2^2 y = 0, \tag{3.30}$$

$$U - u = 0, \tag{3.31}$$

$$U_1 - \lambda_2^2 \, 3x^2 + \lambda_1^2 \, 3y^2 = 0, \tag{3.32}$$

$$U_2 + \lambda_2^1 \, 3x^2 - \lambda_1^1 \, 3y^2 = 0, \tag{3.33}$$

and the equations generating the restricted action ideal $A^{(2)}[f] \subset \mathbb{C}[\lambda_j^i, x, y, Z^{(2)}]$ are obtained by adjoining the equations

$$U_{11} - \left(\lambda_2^2\right)^2 6x - \left(\lambda_1^2\right)^2 6y = 0, \tag{3.34}$$

$$U_{12} + \lambda_2^2 \lambda_2^1 6x + \lambda_1^2 \lambda_1^1 6y = 0, \tag{3.35}$$

$$U_{22} - \left(\lambda_2^1\right)^2 6x - \left(\lambda_1^1\right)^2 6y = 0 \tag{3.36}$$

to the generators (3.28) - (3.33) of $A^{(1)}[f]$.

The equations generating the $C^{(1)}$ and $\tilde{C}^{(2)}$-projection ideals restricted to f, $\left(A^{(1)}[f] + C^{(1)}\right)$ and $\left(A^{(2)}[f] + \tilde{C}^{(2)}\right)$, are obtained by adjoining the equations (3.27) generating C to the

generators (3.28) - (3.33) of $A^{(1)}[f]$, as well as to the generators (3.28) - (3.36) of $A^{(2)}[f]$ respectively.

We will now use the $SL_2(\mathbb{C})$-signature varieties to show how the above constructions can be applied towards the problem of deciding when two elements of $\mathcal{P}_2$ are $SL_2(\mathbb{C})$-equivalent. We will continue using $f = x^3 + y^3 \in \mathcal{P}_2$ and the cross-section ideal $C^{(1)} = (X, Y - 1, U_1) \subset \mathbb{C}[Z^{(1)}]$. Let $g = 9x^3 + 15x^2 + 9xy^2 + 2y^3 \in \mathcal{P}_2$ be given as in Example 2.1.10 and let $h = x^3 \in \mathcal{P}_2$. Observe that f and g are $SL_2(\mathbb{C})$-equivalent, but f and h are not $SL_2(\mathbb{C})$-equivalent.

The one jet of g is

$$\mathrm{j}_1\, g = \left(x, y, 9\,x^3 + 15\,x^2 y + 9\,xy^2 + 2\,y^3, 27\,x^2 + 30\,xy + 9\,y^2, 15\,x^2 + 18\,xy + 6\,y^2\right),$$

and we will denote the two jet of g by

$$\mathrm{j}_2\, g = \left(\mathrm{j}_1\, g, 54x + 30y, 30x + 18y, 18x + 12y\right).$$

The two jet of h is $\mathrm{j}_2\, h = \left(x, y, x^3, 3\,x^2, 0, 6\,x, 0, 0\right).$

The equations generating $\left(A^{(1)}[g] + C^{(1)}\right) \subset \mathbb{C}[\lambda_j^i, x, y, Z^{(1)}]$ are

$$\lambda_1^1 \lambda_2^2 - \lambda_2^1 \lambda_1^2 - 1 = 0$$

$$X - \lambda_1^1 x - \lambda_2^1 y = 0$$

$$Y - \lambda_1^2 x - \lambda_2^2 y = 0$$

$$U - (g(x,y)) = 0$$

$$U_1 - \lambda_2^2 g_1(x,y) + \lambda_1^2 g_2(x,y) = 0$$

$$U_2 + \lambda_2^1 g_1(x,y) - \lambda_1^1 g_2(x,y) = 0$$

$$X = 0$$

$$Y = 1$$

$$U_1 = 0$$

where $g(x,y), g_1(x,y), g_2(x,y)$ denote the corresponding components of $j_1\, g$, and the generators of $\left(A^{(2)}[g] + \tilde{C}^{(2)}\right) \subset \mathbb{C}[\lambda_j^i, x, y, Z^{(2)}]$ are obtained by adjoining

$$U_{11} - \left(\lambda_2^2\right)^2 g_{11}(x,y) + 2\lambda_2^2 \lambda_1^2 g_{12}(x,y) - \left(\lambda_1^2\right)^2 g_{22}(x,y) = 0$$

$$U_{12} + \lambda_2^2 \lambda_2^1 g_{11}(x,y) - \left(\lambda_2^2 \lambda_1^1 + \lambda_1^2 \lambda_2^1\right) g_{12}(x,y) + \lambda_1^2 \lambda_1^1 g_{22}(x,y) = 0$$

$$U_{22} - \left(\lambda_2^1\right)^2 g_{11}(x,y) + 2\lambda_2^1 \lambda_1^1 g_{12}(x,y) - \left(\lambda_1^1\right)^2 g_{22}(x,y) = 0$$

to the equations generating $\left(A^{(1)}[g] + C^{(1)}\right)$.

Similarly, for $h = x^3$, the generators of $\left(A^{(2)}[h] + \tilde{C}^{(2)} \right) \subset \mathbb{C}[\lambda_j^i, x, y, Z^{(2)}]$ are

$$\lambda_1^1 \lambda_2^2 - \lambda_2^1 \lambda_1^2 - 1 = 0$$

$$X - \lambda_1^1 x - \lambda_2^1 y = 0,$$

$$Y - \lambda_1^2 x - \lambda_2^2 y = 0,$$

$$U - x^3 = 0,$$

$$U_1 - \lambda_2^2 3x^2 = 0,$$

$$U_2 + \lambda_2^1 3x^2 = 0,$$

$$X = 0,$$

$$Y = 1,$$

$$U_1 = 0,$$

$$U_{11} - \left(\lambda_2^2 \right)^2 6x = 0,$$

$$U_{12} + \lambda_2^2 \lambda_2^1 6x = 0,$$

$$U_{22} - \left(\lambda_2^1 \right)^2 6x = 0,$$

where the generators for $\left(A^{(1)}[h] + C^{(1)} \right) \subset \mathbb{C}[\lambda_j^i, x, y, Z^{(1)}]$ are obtained in the obvious manner.

The $SL_2\,(\mathbb{C})$-signature ideals (Definition 3.3.5) $S_C^{(1)}[f], S_C^{(1)}[g], S_C^{(1)}[h] \subset \mathbb{C}[Z^{(1)}]$ are generated by the equations

$$X = 0, \quad Y - 1 = 0, \quad U_1 = 0, \quad U_2 - 3U = 0$$

which implies that the $SL_2\,(\mathbb{C})$-signature varieties $\mathcal{S}_C^{(1)}[f], \mathcal{S}_C^{(1)}[g]$ and $\mathcal{S}_C^{(1)}[h]$ are all equal. At order two, however, we observe the following. The $SL_2\,(\mathbb{C})$-signature ideals $S_C^{(2)}[f], S_C^{(2)}[g] \subset$

$\mathbb{C}[Z^{(2)}]$ are generated by the equations

$$X = 0, \quad Y - 1 = 0, \quad U_1 = 0, \quad U_2 - 3U = 0, \quad U_{12} = 0, \quad U_{22} - 6U = 0$$

As expected, in accordance with Corollary 3.3.7, this implies the $SL_2(\mathbb{C})$-signature varieties are equal: $\mathcal{S}_\mathcal{C}^{(2)}[f] = \mathcal{S}_\mathcal{C}^{(2)}[g]$. On the other hand, the $SL_2(\mathbb{C})$-signature ideal $S_C^{(2)}[h]$ is generated by the equations

$$X = 0, \quad Y - 1 = 0, \quad U_1 = 0, \quad U_{12} = 0,$$

$$(U_{11})^2 = 0, \quad U_{11}U = 0, \quad U_2 - 3U = 0, \quad U_{22} - 6U = 0,$$

and we can see immediately that

$$\mathcal{S}_\mathcal{C}^{(2)}[f] = \mathcal{S}_\mathcal{C}^{(2)}[g] \neq \mathcal{S}_\mathcal{C}^{(2)}[h]. \tag{3.37}$$

Thus, the $SL_2(\mathbb{C})$-signature varieties $\mathcal{S}_\mathcal{C}^{(2)}[f]$, $\mathcal{S}_\mathcal{C}^{(2)}[g]$ and $\mathcal{S}_\mathcal{C}^{(2)}[h]$ serve to distinguish the $SL_2(\mathbb{C})$-equivalent polynomials f and g from the inequivalent polynomial h.

3.7 $GL_m(\mathbb{C})$-Signature Varieties

In this section we will take $\mathcal{G} = GL_m(\mathbb{C})$ and we will apply the previous constructions and results to the action of $GL_m(\mathbb{C})$ on $J^\infty(\mathbb{C}^m, \mathbb{C})$ and the corresponding action of $GL_m(\mathbb{C})$ on $\mathcal{P}_m$. We will introduce a specific cross-section ideal $C^{(m)} \subset \mathbb{C}[Z^{(m)}]$ for the action of $GL_m(\mathbb{C})$ on $J^m(\mathbb{C}^m, \mathbb{C})$, which for all $d \geq m$, gives rise to an almost complete signature correspondence for the action of $GL_m(\mathbb{C})$ on $\mathcal{P}_m^d$. After carrying this out in general, we will specialize to the case $m = 3$ and the actions of $GL_3(\mathbb{C})$ on $J^\infty(\mathbb{C}^3, \mathbb{C})$

and $\mathcal{HP}_3$, the set of homogeneous polynomials in three variables. We will show that the cross-section ideal $C^{(m)}$ introduced in the case of m variables applied to $m = 3$ produces, for all $d \geq 3$, an almost complete signature correspondence for the action of $GL_m(\mathbb{C})$ on $\mathcal{HP}_3^d$.

3.7.1 $GL_m(\mathbb{C})$-Signatures Applied to $\mathcal{P}_m$

Proposition 3.7.1. *The prolonged action of $GL_m(\mathbb{C})$ is locally free on a dense open subset of $J^m(\mathbb{C}^m, \mathbb{C})$.*

Proof. The m^2-infinitesimal generators for the action of $GL_m(\mathbb{C})$ on $\mathbb{C}^m$ are given in local coordinates by $V_j^i = x^i \frac{\partial}{\partial x^j}$, $1 \leq i, j \leq m$. We will denote the corresponding prolonged infinitesimal generators for the action of $GL_m(\mathbb{C})$ on $J^m(\mathbb{C}^m, \mathbb{C})$ by $\mathrm{pr}^{(m)}(V_j^i)$. We will order the vector fields $\mathrm{pr}^{(m)}(V_j^i)$ by ordering the index pairs (i, j) lexicographically. Using the established ordering of the coordinate functions $z^{(m)} = (x, u, u_K)$ on $J^m(\mathbb{C}^m, \mathbb{C})$, we define an $m^2 \times \left(m + \binom{2m}{m}\right)$ matrix $L(s, t)$ as follows. The entry of L in the sth row and the tth column of the matrix L is defined to be the sth infinitesimal generator $\mathrm{pr}^{(m)}(V_j^i)_s$ applied to the the tth coordinate function $z_t^{(m)}$. Thus, the rows of L correspond to the coefficient functions of the prolonged infinitesimal generators with respect to the (ordered) basis of vector fields $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, \frac{\partial}{\partial u_K}\right)$ on $J^m(\mathbb{C}^m, \mathbb{C})$. As previously remarked (see Remark 2.3.1) the rank of the matrix $L(s, t)$ at a point $\mathbf{z}^{(m)} \in J^m(\mathbb{C}^m, \mathbb{C})$ denotes the dimension of $T_{\mathbf{z}^{(m)}} \mathcal{O}_{\mathbf{z}^{(m)}}$, the tangent space to the orbit $\mathcal{O}_{\mathbf{z}^{(m)}}$ at $\mathbf{z}^{(m)}$.

We will show that there exists an $m \times m$ submatrix $\hat{L}$ of L and a point $\mathbf{z}^{(m)} \in J^m(\mathbb{C}^m, \mathbb{C})$ for which $\det\left(\hat{L}(\mathbf{z}^{(m)})\right) \neq 0$, combining this with the fact that maximal rank is a Zariski open condition (see Remark 3.4.2), we will conclude that the dimension of

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the generic orbits for the action of $GL_m(\mathbb{C})$ on $J^m(\mathbb{C}^m, \mathbb{C})$ is m.

Define the $m \times m$ submatrix $\hat{L}$ of L to be the matrix whose columns are indexed by the variables $x^1, \ldots, x^m, u_{[i^k]}$, where $1 \le i \le (m-1), 1 \le k \le m$, and $[i^k]$ denotes the partial derviative multi-index of length k that represents k partial derivatives all taken with respect to the ith variable. A direct computation shows that

$$\text{pr}^{(m)}(V_j^i)_s(x^k) = \delta_j^k x^i, \qquad 1 \le s \le m^2, \ 1 \le k \le m, \tag{3.38}$$

and

$$\text{pr}^{(m)}(V_j^i)_s(u_{[p^k]}) = \delta_j^p k u_{i,[j^{k-1}]}, \qquad 1 \le s \le m^2, \ 1 \le k \le m, \tag{3.39}$$

where δ_j^k and δ_j^p denote the usual Kronecker-delta. We now define a subvariety $\mathcal{C}^{(m)} \subset J^m(\mathbb{C}^m, \mathbb{C})$ by the equations

$$x^i = 0 \qquad\qquad\qquad 1 \le i \le m-1$$

$$x^m = 1$$

$$u_{[i^k]} = 0 \qquad\qquad\qquad 1 \le i, k \le m-1$$

$$u_{[i^m]} = 1 \qquad\qquad\qquad 1 \le i \le m-1,$$

and for $\mathbf{z}^{(m)} \in \mathcal{C}^{(m)}$, a direct computation shows that

$$\det(\hat{L})\Big|_{\mathcal{C}^{(m)}} = (m! \, u_m)^{m-1} \prod_{i=1}^{m-1} \sum_{\sigma(k_1, \ldots, \hat{k}_i, \ldots, k_{m-1})} sign(\sigma) u_{\sigma(k_1),i} u_{\sigma(k_2),[i^2]} \cdots u_{\sigma(k_{m-2}),[i^{m-1}]}, \tag{3.40}$$

where $\left(k_1, \ldots, \hat{k}_i, \ldots, k_{m-1} \right)$ denotes the partial derivative multi-index of length $(m-2)$

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obtained by omitting the ith index. Thus, if $\mathbf{z}^{(m)} \in \mathcal{C}^{(m)}$ and (3.40) is non-zero, then the dimension of the orbit $\mathcal{O}_{\mathbf{z}^{(m)}}$ is m^2 and the action of $\mathcal{G}$ is locally free on $J^m\left(\mathbb{C}^m, \mathbb{C}\right)$. $\square$

Remark 3.7.1. *Note that if $\mathcal{G}$ is taken to be an algebraic subgroup of $GL_m\left(\mathbb{C}\right)$, then Proposition 3.7.1 shows that the prolonged action of $GL_m(\mathbb{C})$ is also locally free on $J^m\left(\mathbb{C}^m, \mathbb{C}\right)$.*

Remark 3.7.2. *The above bound is not sharp. The action of $GL_m(\mathbb{C})$ can be, and almost certainly is, locally free on $J^n\left(\mathbb{C}^m, \mathbb{C}\right)$ with $n < m$. For example, the action of $GL_3(\mathbb{C})$ is locally free on $J^2\left(\mathbb{C}^3, \mathbb{C}\right)$. This can be seen by noting that the generic rank of the 9×13 matrix $L(s,t)$ occurring in the proof of Theorem 3.7.1 that corresponds to the action of $GL_3(\mathbb{C})$ on $J^2\left(\mathbb{C}^3, \mathbb{C}\right)$ is $9 = \dim(GL_3\left(\mathbb{C}\right))$.*

The subvariety $\mathcal{C}^{(m)} \subset J^m\left(\mathbb{C}^m, \mathbb{C}\right)$ defined in the course of the proof of Proposition 3.7.1 will play a special role in what follows. $\mathcal{C}^{(m)}$ is defined as the variety of the prime ideal $C^{(m)} \subset \mathbb{C}[Z^{(m)}]$ generated by the m^2 *linear* equations

$$x^i = 0 \qquad\qquad 1 \leq i \leq m-1 \qquad\qquad (3.41)$$

$$x^m = 1 \qquad\qquad\qquad\qquad (3.42)$$

$$u_{[i^k]} = 0 \qquad\qquad 1 \leq i, k \leq m-1 \qquad\qquad (3.43)$$

$$u_{[i^m]} = 1 \qquad\qquad 1 \leq i \leq m-1, \qquad\qquad (3.44)$$

and is thus of codimension m^2 in $J^m\left(\mathbb{C}^m, \mathbb{C}\right)$. Furthermore, the determinant of the $m^2 \times m^2$ matrix $\hat{L}(s,t)\big|_{\mathcal{C}^{(m)}}$ is precisely the check (see Definition 3.4.1) required to ensure that $\mathcal{C}^{(m)}$ is generically transversal to the $GL_m(\mathbb{C})$-orbits. Thus, equations (3.41) - (3.44) define a cross-section ideal $C^{(m)}$ for the action of $GL_m(\mathbb{C})$ on $J^m\left(\mathbb{C}^m, \mathbb{C}\right)$ of codimension $m^2 = \dim\left(GL_m(\mathbb{C})\right)$.

As a consequence of our prior results, we now have that for all pairs of positive integers (m, d), there exists an almost complete $GL_m(\mathbb{C})$-signature correspondence on $\mathcal{P}_m^d$ at the prolongation order $\min\{m + 1, d\}$.

Theorem 3.7.2. *If $d \leq m$, then by Proposition 3.3.13, the ideal $C^{(d)} = (0) \subset \mathbb{C}[Z^{(d)}]$ determines a complete $GL_m(\mathbb{C})$-signature correspondence for the action of $GL_m(\mathbb{C})$ on $\mathcal{P}_m^d$.*

Remark 3.7.3. *If $d < m$, then no polynomials belonging to $\mathcal{P}_m^d$ are transversally regular with respect to the cross-section ideal $C^{(m)}$ defined by equations (3.41)-(3.44). This is due to the fact that $u_{[i^m]} = 1$ on the corresponding cross-section variety $\mathcal{C}^{(m)}$.*

Theorem 3.7.3. *If $d > m$, there exists a cross-section ideal $C^{(m)} \subset \mathbb{C}[Z^{(m)}]$ for the action of $GL_m(\mathbb{C})$ on $J^m(\mathbb{C}^m, \mathbb{C})$ that determines an almost complete $GL_m(\mathbb{C})$-signature correspondence for the action of $GL_m(\mathbb{C})$ on $\mathcal{P}_m^d$.*

Proof. We will take $C^{(m)} \subset \mathbb{C}[Z^{(m)}]$ to be the ideal generated by the equations (3.41) - (3.44). It has just been shown that $C^{(m)}$ is a cross-section ideal for the action of $GL_m(\mathbb{C})$ on $J^m(\mathbb{C}^m, \mathbb{C})$, and by Theorem 3.4.8, the set of polynomials of degree d that are transversally regular with respect to the cross-section ideal $C^{(m)}$ is Zariski open in $\mathcal{P}_m^d$. Denote by $\mathcal{T}_m^d$ the set of all $f \in \mathcal{P}_m^d$ that are transversally regular with respect to the cross-section ideal $C^{(m)}$. The $GL_m(\mathbb{C})$-signature varieties of all $f \in \mathcal{T}_m^d$ will have their dimensions repeat by order at least $2m$ (i.e. $\dim(\mathcal{S}_\mathcal{C}^{(k)}[f]) = \dim(\mathcal{S}_\mathcal{C}^{(k+1)}[f])$ with $m \leq k \leq 2m - 1$). Furthermore, generic $f \in \mathcal{T}_m^d$ will satisfy $\mathcal{S}_\mathcal{C}^{(m)}[f] = \mathcal{S}_\mathcal{C}^{(m+1)}[f]$. Thus, the cross-section ideal $C^{(m)}$ (and its appropriate extension ideals) determines an almost complete signature correspondence for the action of $GL_m(\mathbb{C})$ on $\mathcal{P}_m^d$ at the $(m + 1)^{th}$ order of prolongation. $\square$

Remark 3.7.4. *The statement of the above theorem is equivalent to the statement that the cross-section ideal $C^{(m)} \subset \mathbb{C}[Z^{(m)}]$ determines an almost complete $GL_m(\mathbb{C})$-signature correspondence for the action of $GL_m(\mathbb{C})$ on $\mathcal{P}_m$.*

3.7.2 $GL_3(\mathbb{C})$-Signature Varieties Applied to $\mathcal{HP}_3$

We will carry out a thorough analysis of the actions of $GL_3(\mathbb{C})$ on $J^\infty (\mathbb{C}^3, \mathbb{C})$ and $\mathcal{HP}_3$. The results of this analysis will show that the cross-section ideal $C^{(3)} \subset \mathbb{C}[Z^{(3)}]$ introduced in the previous subsection will provide an almost complete signature correspondence for the action of $GL_3(\mathbb{C})$ on $\mathcal{HP}_3^d$, where $d \geq 3$.

We consider the action of $GL_3(\mathbb{C})$ on $J^\infty (\mathbb{C}^3, \mathbb{C})$ and we will set $x^1 = x, x^2 = y, x^3 = z$ in the source coordinates and $X^1 = X$, $X^2 = Y$, $X^3 = Z$ in the target coordinates. We will use the cross-section ideal for the action of $GL_3 (\mathbb{C})$ on $J^3 (\mathbb{C}^3, \mathbb{C})$ defined in the previous subsection by the equations (3.41) - (3.44). The cross-section ideal $C^{(3)} \subset \mathbb{C}[Z^{(3)}]$ for the action of $GL_3(\mathbb{C})$ on $J^3 (\mathbb{C}^3, \mathbb{C})$ is thus generated by the equations

$$X = 0, \ Y = 0, \ Z - 1 = 0, \ U_1 = 0, \ U_2 = 0, \ U_{11} = 0, \ U_{22} = 0, \ U_{111} - 1 = 0, \ U_{222} - 1 = 0.$$

The corresponding cross-section variety is

$$\mathcal{C}^{(3)} = \left\{ \mathbf{Z}^{(3)} \in J^3 \left(\mathbb{C}^3, \mathbb{C} \right) \, \middle| \, X = Y = U_1 = U_2 = U_{11} = U_{22} = 0 \, , \, Z = U_{111} = 1 = U_{222} \right\},$$

$$(3.45)$$

and for $n \geq 3$, the cross-section variety is

$$\tilde{\mathcal{C}}^{(n)} = \left\{ \mathbf{Z}^{(n)} \in J^n \left(\mathbb{C}^3, \mathbb{C} \right) \, \middle| \, \pi_3^n(\mathbf{Z}^{(3)}) \in \mathcal{C}^{(3)} \right\}. \qquad (3.46)$$

On account of Proposition 3.7.1, the action of $GL_3(\mathbb{C})$ is locally free on $J^3\left(\mathbb{C}^3, \mathbb{C}\right)$, and the transversality conditions for the cross-section ideal C are given by (3.40). With $m = 3$, the transversality conditions are thusly,

$$36\left(U_3 U_{12}\right)^2 \neq 0. \tag{3.47}$$

Remark 3.7.5. *For the remainder of the section, we will refer to $C^{(3)} \subset \mathbb{C}[Z^{(3)}]$ as the cross-section ideal for the the action of $GL_3(\mathbb{C})$ on $J^3\left(\mathbb{C}^3, \mathbb{C}\right)$ and we will say that $U_{12}U_3 \neq 0$ are the transversality conditions of the cross-section ideal $C^{(3)}$.*

We now aim to show that for all positive integers $d \geq 3$ there exists a Zariski open subset $\mathcal{T}_3^d \subset \mathcal{HP}_3^d$, such that all $f \in \mathcal{T}_3^d$ are transversally regular with respect to the cross-section ideal $C^{(3)}$. As a consequence, we will see that for all $f, g \in \mathcal{T}_3^d$,

$$f \cong g \iff \mathcal{S}_{\mathcal{C}}^{(6)}[f] = \mathcal{S}_{\mathcal{C}}^{(6)}[g].$$

Since the cross-section ideal $C^{(3)} \subset \mathbb{C}[Z^{(3)}]$ and the corresponding cross-section variety $\mathcal{C}^{(3)}$ are defined as a common level set of the coordinate functions $X, Y, Z, U_1, U_2, U_{11}, U_{22}$, U_{111}, U_{222}, then we can seek to bring a point $\mathbf{z}^{(3)} \in J^3\left(\mathbb{C}^3, \mathbb{C}\right)$ to $\mathcal{C}^{(3)}$ in a succession of steps. In doing so, we will prove the following theorem.

Theorem 3.7.4. *If $f \neq 0 \in \mathcal{HP}_3^d$ with $d \geq 3$, then*

1. *$\exists \Lambda \in GL_3(\mathbb{C})$ and $\mathbf{x} \neq 0 \in \mathbb{C}^3$ such that $\Lambda \cdot (\mathrm{j}_2\ f(\mathbf{x})) = \mathrm{j}_2\ (\Lambda \star f)(\Lambda \cdot \mathbf{x})$ satisfies*

$$X = Y = U_1 = U_2 = U_{11} = U_{22} = 0, \quad Z = 1, \quad U_3 \neq 0$$

2. *$\exists \Lambda \in GL_3(\mathbb{C})$ such that $\Lambda \star f$ has the following properties:*

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(a) The coefficient of $z^d \neq 0$.

(b) The coefficients of $x^1 z^{d-1}, y^1 z^{d-1}, x^2 z^{d-2}, y^2 z^{d-2}$ are all zero.

We will carry this proof out in a series of lemmas, but first we introduce some terminology.

Definition 3.7.5 (Satisfies the t-jet conditions). *For $f \in \mathcal{P}_3$ and $0 \leq t \leq 3$, we will say that f satisfies the t-jet conditions of $\mathcal{C}^{(3)}$ at $\mathbf{x} \in \mathbb{C}^3$ if there exists $\Lambda \in GL_3(\mathbb{C})$ such that $\Lambda \cdot \mathrm{j}_t \, f(\mathbf{x}) \in \pi_t^3 \left(\mathcal{C}^{(3)} \right).$[2]*

Example 3.7.6. *Since the action of $GL_3(\mathbb{C})$ on $\mathbb{C}^3 - \mathbf{0}$ is transitive, then $\forall f \in \mathcal{P}_3$ and $\forall \mathbf{x} \neq \mathbf{0} \in \mathbb{C}^3, f \in \mathcal{P}_3$ satisfies the zero-jet conditions $(X = 0, Y = 0, Z = 1)$ of $\mathcal{C}^{(3)}$ at $\mathbf{x}$.*

Remark 3.7.6. *Observe that if $f \in \mathcal{P}_3$ satisfies the k-jet conditions of $\mathcal{C}^{(3)}$ at $\mathbf{x} \in \mathbb{C}^3$, then (assuming $t \leq k$), f necessarily satisfies the t-jet conditions of $\mathcal{C}^{(3)}$ at $\mathbf{x}$.*

Warning 3.7.1. *In the lemmas that follow, we apply a sequence of transformations to an arbitrary function $f \in \mathcal{HP}_3$ and its jets. However, after each transformation, we continue to denote the function by f.*

Lemma 3.7.7. *If $f \neq 0 \in \mathcal{HP}_3$, then $\exists \mathbf{x} \in \mathbb{C}^3$ such that f satisfies the zero-jet condition of $\mathcal{C}^{(3)}$ at $\mathbf{x}$. Furthermore, we can take $\Lambda \in GL_3(\mathbb{C})$ such that $U_3 \left(\Lambda \cdot \left(\mathrm{j}_1 \, f(\mathbf{x}) \right) \right) = U_3 \left(\mathrm{j}_1 \left(\Lambda \star f \right) (0, 0, 1) \right) \neq 0.$*

Proof. Note that there exists $\mathbf{x} = (x, y, z) \in \mathbb{C}^3$ such that $z \neq 0$ and $\mathbf{x} \notin \mathrm{Var} \, (f)$ and define $\Lambda^{-t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & z \end{pmatrix}$. The jet space transformation laws imply $X = 0, Y = 0, Z = 1$,

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[2]Equivalently, we can bring $\mathrm{j}_t \, f(\mathbf{x})$ to $\pi_t^3(\mathcal{C}^{(3)})$.

and $U_3 = xu_1 + yu_2 + zu_3$. Since f is assumed to be homogeneous (and a solution of Euler's PDE: $xu_1 + yu_2 + zu_3 = du$, where d is the degree of f), we conclude that $U_3 \neq 0$ on account of $\mathbf{x} \notin \mathrm{Var}\,(f)$. $\qquad\square$

Remark 3.7.7. *As a result of Lemma 3.7.7, then going forward, we will assume that* $f \neq 0 \in \mathcal{HP}_3$ *and satisfies* $f_3(0,0,1) \neq 0$.

Lemma 3.7.8. *If* $f \neq 0 \in \mathcal{HP}_3$, *then* f *satisfies the one-jet condition of* $\mathcal{C}^{(3)}$ *at* $\mathbf{x} = (0,0,1)$.

Proof. The matrices preserving the zero-jet condition (and the assumed transversality condition $U_3 \neq 0$) of $\mathcal{C}^{(3)}$ under the action of $GL_3(\mathbb{C})$ on $J^1\,(\mathbb{C}^3, \mathbb{C})$ are of the form

$$\Lambda^{-t} = \begin{pmatrix} \hat{\lambda}_1^1 & \hat{\lambda}_1^2 & \hat{\lambda}_1^3 \\ \hat{\lambda}_2^1 & \hat{\lambda}_2^2 & \hat{\lambda}_2^3 \\ 0 & 0 & 1 \end{pmatrix}, \text{ where } \det\left(\Lambda^{-t}\right) \neq 0. \text{ For } f \in \mathcal{HP}_3, \text{ taking } \Lambda^{-t} = \begin{pmatrix} 1 & 0 & \frac{-f_1}{f_3} \\ 0 & 1 & \frac{-f_2}{f_3} \\ 0 & 0 & 1 \end{pmatrix},$$

where the components are to be evaluated at $\mathbf{x} = (0,0,1)$, and applying the jet space transformations shows that f satisfies the 1-jet condtions of $\mathcal{C}^{(3)}$ at $\mathbf{x} = (0,0,1)$. $\qquad\square$

Remark 3.7.8. *As a result of Lemma 3.7.2, going forward, we will now assume that* $f \neq 0 \in \mathcal{HP}_3$ *satisfies the 1-jet conditions of* $\mathcal{C}^{(3)}$ *at* $\mathbf{x} = (0,0,1)$ *as well as the transversality condition* $f_3\,(0,0,1) \neq 0$.

Lemma 3.7.9. *If* $f \neq 0 \in \mathcal{HP}_3$, *then* f *satisfies the two-jet conditions of* $\mathcal{C}^{(3)}$ *at* $\mathbf{x} = (0,0,1)$.

Proof. The matrices preserving the one-jet condition of $\mathcal{C}^{(3)}$ (with the assumed transversality condition $U_3 \neq 0$) under the action of $GL_3(\mathbb{C})$ on $J^\infty\,(\mathbb{C}^3, \mathbb{C})$ are of the form

$$\Lambda^{-t} = \begin{pmatrix} \hat{\lambda}_1^1 & \hat{\lambda}_1^2 & 0 \\ \hat{\lambda}_2^1 & \hat{\lambda}_2^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ where } \det\left(\Lambda^{-t}\right) \neq 0. \text{ If } f_{11}(0,0,1) = 0 \text{ and/or } f_{22}(0,0,1) = 0, \text{ then}$$

we can take the first and/or second rows of Λ^{-t} to be $(1,0,0)$ and/or $(0,1,0)$ respectively. If we assume that $f_{11}(0,0,1) \neq 0$ and $f_{22}(0,0,1) \neq 0$, then we can take

$$\Lambda^{-t} = \begin{pmatrix} \frac{f_{12}+\sqrt{(f_{12})^2-f_{11}f_{22}}}{f_{11}} & 1 & 0 \\ 1 & \frac{f_{12}+\sqrt{(f_{12})^2-f_{11}f_{22}}}{f_{22}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.48}$$

where the components of Λ^{-t} are to be evaluated at $\mathbf{x} = (0,0,1)$. Applying the jet space transformation laws shows that f satisfies the zero-jet, one-jet, and two-jet conditions of $\mathcal{C}^{(3)}$ at $\mathbf{x} = (0,0,1)$. $\qquad\square$

Remark 3.7.9. *In (3.48), we can take either of the two roots of $\sqrt{(f_{12})^2 - f_{11}f_{22}}$.*

Remark 3.7.10. *As a result of the three previous lemmas, we can now assume that $f \neq 0 \in \mathcal{HP}_3$ can be given by its jet at $\mathbf{x} = (0,0,1)$ and that $j_2\, f(0,0,1)$ satisfies*

$$f_1 = f_2 = f_{11} = f_{22} = 0 \quad and \quad f_3 \neq 0. \tag{3.49}$$

However, the above analysis does not lead us to any conclusions on f_{12} and whether or not f can be assumed to satisfy the transversality condition $U_{12} \neq 0$ of the cross-section $\mathcal{C}^{(3)}$.

At this point, assuming we are at a transversal point, we merely note that the group elements $\Lambda \in GL_3(\mathbb{C})$ preserving the two-jet conditions of $\mathcal{C}^{(3)}$ are of the form

$$\Lambda^{-t} = \begin{pmatrix} \hat{\lambda}_1^1 & 0 & 0 \\ 0 & \hat{\lambda}_2^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or } \Lambda^{-t} = \begin{pmatrix} 0 & \hat{\lambda}_1^2 & 0 \\ \hat{\lambda}_2^1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.50}$$

Note that this immediately implies that if f satisfies the two-jet conditions of $\mathcal{C}^{(3)}$ and the transversality conditions of $\mathcal{C}^{(3)}$ at $\mathbf{x} = (0,0,1)$, then f satisfies the three jet conditions of $\mathcal{C}^{(3)}$ at $\mathbf{x} = (0,0,1)$ if and only if $f_{111}(0,0,1) \neq 0$ and $f_{222}(0,0,1) \neq 0$. As a result, in order to be transversally regular with respect to the cross-section ideal C, then the degree of f must be greater than or equal to three.

For an arbitrary $f \in \mathcal{HP}_3^{d \geq 3}$ we will say that f is in *a quasi-normal form* if the coefficients of f with respect to the standard basis $\langle x^i y^j z^k \rangle$ of $\mathcal{HP}_3^d$ satisfy the conditions of part 2 in Theorem 3.7.4. Observe that if $f \in \mathcal{HP}_3^d$ ($d \geq 3$) and f is in a quasi-normal form such that the coefficients of $x^3 z^{d-3}, y^3 z^{d-3}$, and xyz^{d-2} are all non-zero, then f is transversally regular with respect to the cross-section ideal C. Furthermore, if f is in a quasi-normal form such that some of (or all of) the coefficients of $x^3 z^{d-3}, y^3 z^{d-3}$ and xyz^{d-2} are zero, then f may still be transversally regular with respect to the cross-section ideal C!

As an immediate consequence, we have the following.

Corollary 3.7.10. *For all $d \geq 3$, the set of all $f \in \mathcal{HP}_3^d$ which are transversally regular with respect to the cross-section ideal C is Zariski open.*

Proof. Let $V = \langle x^i y^j z^k \rangle_{0 \leq i,j,k \leq d, i+j+k=d}$ denote the standard basis for $\mathcal{HP}_3^d$, which we will identify with $\mathbb{C}^{\binom{2+d}{d}}$. Denote standard coordinates on $\mathbb{C}^{\binom{2+d}{d}}$ by (t^i) and assume that V has been ordered with respect to our convention on partial-derivative multi-indices. Let $W = \langle x^1 z^{d-1}, y^1 z^{d-1}, x^2 z^{d-2}, y^2 z^{d-2} \rangle$ and note that the quasi-normal forms are contained in the set spanned by $V - W$. We will assume that $V - W$ inherits the order on its basis from that of the ordering on V, and under our identification of V with $\mathbb{C}^{\binom{2+d}{d}}$, we will denote the coordinates on $\mathbb{C}^{\binom{2+d}{d}}$ corresponding to $V - W$ by $(t^i)_{V-W}$. An element

h contained in the set spanned by $V - W$ can be expressed as

$$h = \sum_{V-W} F^{(i,j,k)} x^i y^j z^k.$$

Then the set of all $h \in \mathcal{HP}_3^d$ which fail to be transversally regular with respect to the cross-section ideal C belong to the variety of the ideal $\left(F^{(1,1,d-2)}, F^{(3,0,d-3)}, F^{(0,3,d-3)}, F^{(0,0,d)} \right) \subset$ $\mathbb{C}[(t^i)_{V-W}]$, where $\mathbb{C}[(t^i)_{V-W}]$ denotes the polynomial ring in the $\binom{2+d}{d} - 4$ variables corresponding to the ordered basis $V - W$. $\qquad\square$

Corollary 3.7.11. *For all $d \geq 3$, the cross-section ideal $C^{(3)} \subset \mathbb{C}[Z^{(3)}]$ for the action of $GL_3(\mathbb{C})$ on $J^3(\mathbb{C}^3, \mathbb{C})$ determines an almost complete signature correspondence for the action of $GL_3(\mathbb{C})$ on $\mathcal{HP}_3^d$.*

Remark 3.7.11. *Alternatively, the cross-section ideal $C^{(3)} \subset \mathbb{C}[Z^{(3)}]$ determines an almost complete signature correspondence for the action of $GL_3(\mathbb{C})$ on $\mathcal{HP}_3$.*

Corollary 3.7.12. *If $f, g \in \mathcal{HP}_3^d$ with f and g transversally regular with respect to the cross-section ideal $C^{(3)}$, then f and g are equivalent if and only if $\mathcal{S}_C^{(6)}[f] = \mathcal{S}_C^{(6)}[g]$.*

Proof. The proof follows from a simple dimension count. We merely note that for all $g \in \mathcal{HP}_3$ that are transversally regular with respect to the cross-section ideal $C^{(3)}$, the conditions of Theorem 3.5.2 are satisfied no later than $k = 5$, and generically, the conditions of Theorem 3.5.2 will be satisfied when $k = 3$. $\qquad\square$

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