This dissertation deals with Stochastic Target Problem. The target problem is a new class of stochastic control problem. In finance, it is closely related to option pricing and hedging problem in a high-frequency trading market. We study the minimal initial capital needed in order to super-replicate a given contingent claim at a specified maturity time.

A Levy process is introduced to describe the dynamics of the state process and two different ways to solve the target problem are given in this work. In the first method we use dynamic programming to derive an integro-differential equation to characterize the solution. In the second method, we make perturbations on the optimal control to get a maximum principle which consists of a system of forward-backward stochastic differential equations.

A classical way to model the stock price process is using a linear stochastic differential equation with an assumption that the investor is "small", which means his/her trading strategy and financial status cannot affect the stock price. Therefore, in a classical model of stock price process, the coefficient of the process is independent of the wealth process and the trading strategy of the investor. However, in recent years, some people have started discussing the behavior of "large investor" models in which the stock price can be affected by the investor’s financial status and trading strategy. Furthermore, the dynamics of stock prices becomes nonlinear.

In the first two chapters, we follow Bensoussan’s work ([5]) and summarize necessary conditions for deterministic and stochastic control problems. In the third chapter we introduce the Levy process in our control model. A set of necessary conditions for the new model is derived adapting the techniques from the first two chapters. A stochastic target problem with jump diffusion is presented in Chapter 4 along with and two solving strategies. Finally, in Chapters 5 and 6 we present an example including parameter estimation, algorithm. Techniques for parameter estimation can be found in Hanson ([20], [21], [22]). The numerical work is based on the dynamic programming approach.
Stochastic Target Problem With Jump Diffusion

by
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DEDICATION

To my parents.
BIOGRAPHY

I was born in the United States in 1980 and grew up in Taiwan. I received my bachelor’s degree in mathematics from Chung Yuan University, Taiwan in 2002, and my master’s degree in Mathematics National Tsing Hua University, Taiwan in 2004. I came back to the US to pursue my PhD degree studies in Operations Research at North Carolina State University in 2007. In 2008, in my second year at North Carolina State University, I started my doctoral research under the direction of Dr. Negash Medhin in stochastic control. I was particularly interested in control methodologies in the analysis of option pricing problem and model development in finance.
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TABLE OF CONTENTS

List of Figures .................................................. vii

Chapter 1 Deterministic control .................................. 1
  1.1 Introduction .................................................. 1
  1.2 Deterministic control problem ............................... 1
  1.3 Necessary conditions for deterministic problem ............. 2
  1.4 Conclusion ................................................... 12

Chapter 2 Stochastic control ..................................... 13
  2.1 Introduction .................................................. 13
  2.2 Stochastic control problem ................................... 13
  2.3 Necessary conditions for stochastic problem ................. 14
  2.4 Conclusion ................................................... 31

Chapter 3 Stochastic control with jump diffusion ............. 32
  3.1 Introduction .................................................. 32
  3.2 Jump diffusion in high-frequency trading market ............ 32
  3.3 Stochastic control problem with jump diffusion ............. 40
  3.4 Necessary conditions for jump diffusion problem .......... 42
  3.5 Relation to dynamic programming ............................ 64
  3.6 Sufficient condition for jump diffusion problem ............ 73
  3.7 Conclusion ................................................... 76

Chapter 4 Target problem ......................................... 77
  4.1 Introduction .................................................. 77
  4.2 An example: superreplication problem in finance .......... 78
  4.3 1-dimensional stochastic target problem with jump diffusion 82
  4.4 When the optimal trajectory hits the boundary of the target at maturity time $T$ 84
  4.5 When the optimal trajectory does not hit the boundary of the target at maturity time $T$ 93
  4.6 n-dimensional stochastic target problem with jump diffusion 96
  4.7 Conclusion ................................................... 98

Chapter 5 Application in finance ................................. 99
  5.1 Introduction .................................................. 99
  5.2 Stock price process ........................................... 100
  5.3 The model ..................................................... 106
  5.4 Model parameter estimation ................................... 109
  5.5 Conclusion ................................................... 113
LIST OF FIGURES

Figure 4.1 The penalty function $g$ ........................................... 84
Figure 4.2 The graph of function $\phi_e$ ........................................ 86

Figure 5.1 S&P stock index log-return ................................. 110
Figure 5.2 Sample histogram of S&P500 log-return .............. 111
Figure 5.3 Theoretical histogram of S&P500 log-return .......... 114
Figure 5.4 Error plot ......................................................... 115

Figure 6.1 Cubic Spline Basic functions ............................... 121
Figure 6.2 Linear Spline Basic functions ............................... 122
Figure 6.3 Value function $\Phi(t, y)$ of (P2) by cubic spline family ........ 138
Figure 6.4 Optimal control by cubic spline family ............... 139
Figure 6.5 Value function $\Phi(t, y)$ of (P2) by linear spline family .... 140
Figure 6.6 Optimal control by linear spline family .......... 141
Chapter 1

Deterministic control

1.1 Introduction

In this chapter, we use perturbation method to derive maximum principle for a deterministic continuous-time control problem. Maximum principle is a set of necessary conditions characterizing optimal control and trajectory. We follow Bensoussan [5] in deriving maximum principle.

1.2 Deterministic control problem

We suppose that

\[ f(t, x, u) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} \]
\[ b(t, x, u) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \]
\[ g(x) : \mathbb{R}^n \to \mathbb{R} \]

are continuous differentiable functions, and satisfy the usual conditions that we need to guarantee the existence of solution of the following deterministic control problem.

\[
\begin{align*}
\text{minimize} & \quad J(u(.)) = \int_0^T f(t, x(t), u(t))dt + g(x(T)) \\
\text{subject to} & \quad dx = b(t, x(t), u(t))dt \\
& \quad x(0) = x_0 \\
& \quad u(t) \in K
\end{align*}
\]
The system of differential equation in (P1) is called the state equation. The vector $x$ which belongs to $\mathbb{R}^n$ is called the state variable, and the function $x(t)$ in $H^1(0,T,\mathbb{R}^n)$ which satisfies the state equation will be called the trajectory corresponding to $u(t)$, and $x_0$ is a given vector in $\mathbb{R}^n$. On the other hand, the vector $u$ is called the control variable, and the non-empty, closed, convex set $K$ is the control constraint. We suppose that the state equation has one and only one solution when $u(t)$ is given. Denote the optimal control-solution pair for (P1) is $(u^*(t), x^*(t))$.

A pair of functions $(x(t), u(t))$ is called an admissible pair if $u(t)$ is an admissible control corresponding to trajectory $x(t)$. We suppose the optimal solution of problem (1) always exists, and denote the optimal solution pair by $(x^*(t), u^*(t))$. That is, we can always find $(x^*(t), u^*(t))$, such that

$$J(u^*(\cdot)) \leq J(u(\cdot))$$

for all admissible pairs $(x(t), u(t))$. Our objective is to look for a set of necessary conditions to characterize $u^*(t)$ and $x^*(t)$ in the following sections.

### 1.3 Necessary conditions for deterministic problem

For simplicity, we will sometimes ignore the arguments and use $x^*$ or $x$ to denote the function $x^*(t)$ or $x(t)$. We will also use $u^*$ or $u$ to denote the function $u^*(t)$ or $u(t)$.

We begin with the necessary conditions. First of all, we make perturbations on the optimal control $u^*$. Let

$$u_\theta = u^* + \theta v$$

where $v = v(t)$ is an admissible control. According to our assumptions in Section 2, we can find one and only one $x_\theta$ which is the trajectory corresponding to $u_\theta$. That is

$$dx_\theta = b(t,x_\theta(t),u_\theta(t))dt$$

$$x_\theta(0) = x_0$$

In order to discuss the change in $J$ corresponding to the perturbation in $u^*$, we need to discuss the change in the dynamics first. By mean value theorem, we have the following
\[
x_{\theta}(t) - x^*(t) = x_0 + \int_0^t b(s, x_{\theta}(s), u_{\theta}(s))ds
\]
\[
-x_0 - \int_0^t b(s, x^*(s), u^*(s))ds
\]
\[
= \int_0^t b(s, x_{\theta}, u_{\theta}) - b(s, x^*, u^*)ds
\]
\[
= \int_0^t \int_0^1 \{b_x(s, x^* + \lambda(x_{\theta} - x^*), u^* + \lambda(u_{\theta} - u^*)) (x_{\theta} - x^*)
\]
\[
+ b_u(s, x^* + \lambda(x_{\theta} - x^*), u^* + \lambda(u_{\theta} - u^*)) v\} d\lambda ds
\]

Then, we have
\[
\frac{x_{\theta}(t) - x^*(t)}{\theta} = \int_0^t \int_0^1 \{b_x(s, x^* + \lambda(x_{\theta} - x^*), u^* + \lambda(u_{\theta} - u^*)) \frac{x_{\theta} - x^*}{\theta}
\]
\[
+ b_u(s, x^* + \lambda(x_{\theta} - x^*), u^* + \lambda(u_{\theta} - u^*)) v\} d\lambda ds
\]

where \(b_x\) and \(b_u\) denote the partial derivative of \(b\) with respect to \(x\) and \(u\), etc..

The following Lemma is used to determine the limit of \(\frac{x_{\theta}(t) - x^*(t)}{\theta}\) as \(\theta\) tends to zero.

**Lemma 1.3.1.** We define a function \(Z(t)\) in \(H^1(0, T, \mathbb{R}^n)\) by the system of differential equation

\[
dZ = \{b_x(t, x^*(t), u^*(t)) Z(t) + b_u(t, x^*(t), u^*(t)) v(t)\} dt
\]
\[
Z(0) = 0
\]

Then, \(\frac{x_{\theta}(t) - x^*(t)}{\theta} \rightarrow Z(t)\) in \(L^2\)

**Proof:**
Consider the following functions

**Lemma 1.3.2.**

\[
\begin{align*}
    x_\theta(t) &= x_0 + \int_0^t b(s, x_\theta(s), u_\theta(s)) \, ds \\
    x^*(t) &= x_0 + \int_0^t b(s, x^*(s), u^*(s)) \, ds \\
    Z(t) &= \int_0^t \left\{ b_x(s, x^*(s), u^*(s)) Z(s) + b_u(s, x^*(s), u^*(s)) v(s) \right\} ds
\end{align*}
\]

and let \( y_\theta = \frac{x_\theta - x^*}{\theta} - Z \)

Then, we have

\[
\begin{align*}
y_\theta &= \int_0^t \frac{b(s, x_\theta(s), u_\theta(s)) - b(s, x^*(s), u^*)}{\theta} \, ds \\
&\quad - b_x(s, x^*, u^*) Z - b_u(s, x^*, u^*) v \, ds \\
&= \int_0^t \int_0^1 \left\{ b_x(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) y_\theta \\
+ [b_x(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - b_x(s, x^*, u^*)] Z \\
+ [b_u(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - b_u(s, x^*, u^*)] v \right\} d\lambda \, ds
\end{align*}
\]

Since the functions \( Z \) and \( v \) belong to \( L^2 \), we can take sufficiently large number \( M \), such that both of their 2-norms are less than or equal to \( M \). Also assume that \( |b_x| \leq M \). By Holder’s inequality, we have
\[ |y_\theta|^2 \leq \left( \int_0^t \int_0^1 \{|b_x(s,x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*))| \, |y_\theta| + |b_x(s,x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - b_x(s,x^*, u^*)| \, |Z| \right.
+ |b_u(s,x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - b_u(s,x^*, u^*)| \, |v| \right) \, d\lambda \, ds \right)^2
\]
\[ \leq 2 \left( \int_0^t \int_0^1 |b_x(s,x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*))| \, |y_\theta| \, d\lambda \, ds \right)^2
+ 2 \left( \int_0^t \int_0^1 |b_x(s,x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - b_x(s,x^*, u^*)| \, |Z| \, d\lambda \, ds \right)^2
+ 2 \left( \int_0^t \int_0^1 |b_u(s,x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - b_u(s,x^*, u^*)| \, |v| \, d\lambda \, ds \right)^2
\]
\[ \leq 2M^2 T^2 \int_0^t |y_\theta|^2 \, ds
+ 2M^2 \int_0^t \int_0^1 |b_x(s,x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - b_x(s,x^*, u^*)|^2 \, d\lambda \, ds
+ 2M^2 \int_0^t \int_0^1 |b_u(s,x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - b_u(s,x^*, u^*)|^2 \, d\lambda \, ds
\]
\[ \leq 2M^2 T^2 \int_0^T |y_\theta|^2 \, ds + 2M^2 \vartheta(\theta)
\]

where
\[
\vartheta(\theta) = \int_0^T \int_0^1 |b_x(s,x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - b_x(s,x^*, u^*)|^2 \, d\lambda \, ds
+ \int_0^T \int_0^1 |b_u(s,x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - b_u(s,x^*, u^*)|^2 \, d\lambda \, ds
\]

and clearly \(\vartheta(\theta)\) tends to zero when \(\theta\) tends to zero. By Gronwall’s inequality, we have
\[ |y_\theta|^2 \leq 2M^2 \vartheta(\theta) \, e^{2M^2 T^2 t} \quad \forall t \in [0,T] \]

Therefore
\[ \int_0^T |y_\theta|^2 dt \leq \int_0^T 2M^2 \vartheta(\theta) \, e^{2M^2 T^2 t} dt \]
the right hand side tends to zero when \( \theta \) tends to zero. So, \( y_\theta \) tends to zero in \( L^2 \). That is,

\[
\frac{x_\theta - x^*}{\theta} \rightarrow Z \text{ in } L^2
\]

The proof is done. \( \square \)

Now, we are ready to consider the change in \( J \) as a result of the perturbation in the optimal control \( u^* \).

\[
J(u_\theta) - J(u^*) = g(x_\theta(T)) - g(x^*(T)) + \int_0^T f(t, x_\theta(t), u_\theta(t)) - f(t, x^*(t), u^*(t))dt
\]

\[
= \int_0^1 g_x(x^*(T) + \lambda(x_\theta(T) - x^*(T))) \cdot (x_\theta(T) - x^*(T)) d\lambda
\]

\[
+ \int_0^T \int_0^1 f_x(t, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) \cdot (x_\theta - x^*) d\lambda dt
\]

\[
+ \int_0^T \int_0^1 f_u(t, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) \cdot (u_\theta - u^*) d\lambda dt
\]

Then we divide both sides by \( \theta \)

\[
\frac{J(u_\theta) - J(u^*)}{\theta} = \int_0^1 g_x(x^*(T) + \lambda(x_\theta(T) - x^*(T))) \cdot \frac{x_\theta(T) - x^*(T)}{\theta} d\lambda
\]

\[
+ \int_0^T \int_0^1 f_x(t, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) \cdot \frac{x_\theta(t) - x^*(t)}{\theta} d\lambda dt
\]

\[
+ \int_0^T \int_0^1 f_u(t, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) \cdot v(t) d\lambda dt
\]

Letting \( \theta \) tend to 0, by last Lemma, we can have the Gateaux derivative of \( J \) along the direction
Proposition 1.3.3. The Gateaux derivative of $J$ along the direction $v$ is

$$dJ(u^*; v) = g_x(x^*(T)) \cdot Z(T) + \int_0^T f_x \cdot Z + f_u \cdot v \, dt$$

Our next step is to define dual variables for the costate problem. Consider the equation

$$d\zeta(t) = \{b_x(t, x^*(t), u^*(t))\zeta(t) + \Phi(t)\} \, dt$$

$$\zeta(0) = 0$$

where $\Phi$ is a function in $L^2(0,T;\mathbb{R}^n)$. And we suppose that for each $\Phi$, we can find one and only one $\zeta$ in $H^2(0,T;\mathbb{R}^n)$. Then, for each $\Phi$, we can consider the quantity

$$\int_0^T f_x \cdot \zeta \, dt + g_x(x^*(T)) \cdot \zeta(T)$$

This map is linear and bounded. By Riesz representation theorem, we can find one and only one function $p(t)$ in $L^2(0,T;\mathbb{R}^n)$, such that for all $\Phi$ in $L^2(0,T;\mathbb{R}^n)$, we have

$$\int_0^T p(t) \cdot \Phi(t) \, dt = \int_0^T f_x(t) \cdot \zeta(t) \, dt + g_x(x^*(T)) \cdot \zeta(T)$$

The $p(t)$ will be called the dual variable or adjoint variable. On the other hand, we define the Hamiltonian

$$H(t, x, u, p) = f(t, x, u) + b(t, x, u) \cdot p$$

Using the notation of Hamiltonian with $\Phi = b_u v$ and $\zeta = Z$, we conclude

$$dJ(u^*; v) = \int_0^T H_u(t, x^*(t), u^*(t), p(t)) \cdot v(t) \, dt$$
Proposition 1.3.4. For (P1), we have the maximum principle

\[ H_u(t, x^*(t), u^*(t), p(t)) \cdot (v - u^*(t)) \geq 0 \]

for all \( v \in K \) and almost everywhere \( t \) in \([0, T]\).

Proof:

Since \( (u^*, x^*) \) is the optimal pair of problem (1), we know the Gateaux derivative of \( J(u(.)) \) at \( u^*(.) \) along the direction \( v(.) - u^*(.) \) is non-negative. That is,

\[ \int_0^T H_u(t, x^*(t), u^*(t), p(t)) \cdot (v(t) - u^*(t)) \, dt = dJ(u^*(.); v(.) - u^*(.)) \geq 0 \]

Let \( v \) be any fixed vector in \( K \), \( t \) be a Lebesgue point in \((0, T)\), and \( \varepsilon > 0 \), such that \((t, t + \varepsilon) \in [0, T]\). Then, we consider the function

\[ \overline{v}(s) = \begin{cases} 
  v, & s \in (t, t + \varepsilon) \\
  u^*(s), & \text{otherwise}
\end{cases} \]

Clearly the function always belongs to \( K \). On the other hand, since \( u^* \) belongs to \( L^2 \), we have

\[ \int_0^T |\overline{v} \cdot \overline{v}| \, dt = \int_t^{t+\varepsilon} |v \cdot v| \, dt + \int_{[0, T] \setminus (t, t+\varepsilon)} |u^* \cdot u^*| \, dt 
\leq |v \cdot v|\varepsilon + |u^*|^2 < \infty \]

So, \( \overline{v}(.) \) belongs to \( L^2 \) and it is an admissible control. Then, we have

\[ \int_t^{t+\varepsilon} H_u(t, x^*(t), u^*(t), p(t)) \cdot (v - u^*(t)) \, dt = \int_0^T H_u(t, x^*(t), u^*(t), p(t)) \cdot (\overline{v}(t) - u^*(t)) \, dt \geq 0 \]

Letting \( \varepsilon \) tends to 0

\[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon} H_u(t, x^*(t), u^*(t), p(t)) \cdot (v - u^*(t)) \, dt \longrightarrow H_u(t, x^*(t), u^*(t), p(t)) \cdot (v - u^*(t)) \quad \text{a.e.} \]

So, we conclude that

\[ H_u(t, x^*(t), u^*(t), p(t)) \cdot (v - u^*(t)) \geq 0 \]
for a.e $t \in [0, T]$, and all $v \in K$

The proof is done. □

The next Proposition is another way to characterize the adjoint variable.

**Proposition 1.3.5.** $p(t)$ is the adjoint variable we defined by Riesz representation theorem, if and only if it satisfies the costate equation

$$
\begin{align*}
dp &= -H_x(t, x^*, u^*, p)dt = -\{f_x(t, x^*, u^*) + b_x(t, x^*, u^*)^*p\}dt \\
p(T) &= g_x(x^*(T))
\end{align*}
$$

Where $b^*_x$ is the transpose of $b_x$.

**Proof:**

Firstly, we suppose $p(t)$ is the solution of the costate equation and proof that it satisfies the Riesz representation theorem. According to Integration by Parts, we have

$$
\begin{align*}
d(p(t) \cdot \zeta(t)) &= \zeta(t) \cdot dp(t) + p(t) \cdot d\zeta(t) \\
&= \zeta(t) \cdot \{-f_x(t, x^*, u^*)dt + b_x(t, x^*, u^*)^*p(t)dt\} \\
&\quad + p(t) \cdot \{b_x(t, x^*, u^*)\zeta(t)dt + \Phi(t)dt\} \\
&= -f_x(t, x^*, u^*) \cdot \zeta(t)dt + p(t) \cdot \Phi(t)dt
\end{align*}
$$

Since $\zeta(0) = 0$ and $p(T) = g_x(x^*(T))$, we have

$$
\begin{align*}
g_x(x^*(T)) \cdot \zeta(T) &= p(T) \cdot \zeta(T) \\
&= p(0) \cdot \zeta(0) + \int_0^T -f_x(t, x^*, u^*) \cdot \zeta(t)dt + p(t) \cdot \Phi(t)dt \\
&= \int_0^T -f_x(t, x^*, u^*) \cdot \zeta(t) + p(t) \cdot \Phi(t)dt
\end{align*}
$$
Then,
\[
\int_0^T p(t) \cdot \Phi(t) dt = \int_0^T f_x(t, x^*, u^*) \cdot \zeta(t) dt + g_x(x^*(T)) \cdot \zeta(T)
\]

Therefore \( p(t) \) is the adjoint variable we defined by Riesz representation theorem.

Secondly, let’s suppose \( p(t) \) is given by the Riesz representation theorem, and prove that it must satisfies the costate equation. Consider the matrix equation
\[
d\Psi(t) = b_x(t, x^*, u^*) \Psi(t) dt
\]
\[
\Psi(0) = I
\]

where \( I \) is the identity matrix. Then \( \Psi(t) \) is the fundamental solution of the equation corresponding \( \zeta(t) \). It can be shown that the fundamental equation has exactly one solution, that is, the \( \Psi(t) \) exists and unique [4][5]. Since \( \zeta(0) = 0 \) we have
\[
\zeta(t) = \Psi(t)\zeta(0) + \Psi(t) \int_0^t \Psi^{-1}(s)\Phi(s)ds
\]
\[
= \Psi(t) \int_0^t \Psi^{-1}(s)\Phi(s)ds
\]

**NOTATION:** Here and Chapter 2 and 3, we use \(-*\) as a notation of putting inverse and transpose operations on a matrix. More precisely, we define
\[
A^{-*} \equiv (A^{-1})^*.
\]
According to integration by parts, we have

\[
\int_0^T p(t) \cdot \Phi(t) dt = \int_0^T f_x(t, x^*, u^*) \cdot \zeta(t) dt + g_x(x^*(T)) \cdot \zeta(T)
\]

\[
= \int_0^T f_x(t, x^*) \cdot \{ \Psi(t) \int_0^t \psi^{-1}(s) \Phi(s) ds \} dt + g_x(x^*(T)) \cdot \{ \Psi(T) \int_0^T \psi^{-1}(s) \Phi(s) ds \}
\]

\[
= \int_0^T \int_0^t f_x(t, x^*, u^*) \cdot \{ \Psi(t) \psi^{-1}(s) \Phi(s) ds \} + g_x(x^*(T)) \cdot \{ \Psi(T) \psi^{-1}(s) \Phi(s) ds \}
\]

\[
= \int_0^T \Phi(t) \cdot \psi^{-1}(t) \{ \int_t^T \psi(x^*, u^*) f_x(s, x^*, u^*) ds + \psi(T) g_x(x^*(T)) \} dt
\]

\[
= \int_0^T \Phi(t) \cdot \psi^{-1}(t) \{ \int_0^T \psi(s) f_x(s, x^*, u^*) ds + \psi(T) g_x(x^*(T)) \} dt
\]

\[
= \int_0^T \psi^{-1}(t) \{ \int_0^t \psi(s) g_x(s, x^*, u^*) ds + \psi(T) g_x(x^*(T)) \} dt
\]

for all \( \Phi(t) \in L^2 \). Then,

\[
p(t) = \psi^{-1}(t) \{ \int_0^t \psi(s) f_x(s, x^*, u^*) ds + \int_0^T \psi(s) f_x(s, x^*, u^*) ds + \psi(T) g_x(x^*(T)) \}
\]

By letting \( t = T \), we obtain the boundary condition

\[
p(T) = \psi^{-1}(T) \{ \int_0^T \psi(s) g_x(s, x^*, u^*) ds + \int_0^T \psi(s) g_x(s, x^*, u^*) ds + \psi(T) g_x(x^*(T)) \}
\]

\[
= g_x(x^*(T))
\]

Since \( \frac{d\psi(t)}{dt} = b_x \psi \), we have \( \frac{d\psi(t)}{dt} = \psi^* b_x^* \). From above,

\[
\psi^* p(t) = - \int_0^t \psi(s) f_x(s, x^*, u^*) ds + \int_0^T \psi(s) f_x(s, x^*, u^*) ds + \psi(T) g_x(x^*(T))
\]
Therefore
\[ \frac{d\Psi(t)^*}{dt}p(t) + \Psi(t)^* \frac{dp(t)}{dt} = -\Psi(t)^* f_x(t, x^*, u^*) \]

\[ \frac{dp(t)}{dt} = -f_x(t, x^*, u^*) - b_x(t, x^*, u^*)^* p = -H_x(t, x^*, u^*, p) \]

The proof is done. □

**Proposition 1.3.6. Necessary Conditions for (P1)**

Let \((u^*, x^*)\) be the optimal pair for (P1), then we can find corresponding \(p(t)\) belongs to \(L^2(0, T; \mathbb{R}^n)\), such that

\[
\begin{align*}
    dx^* &= H_p(t, x^*, u^*, p)dt = b(t, x^*, u^*) \\
x(0) &= x_0 \\
\frac{dp}{dt} &= -H_x(t, x^*, u^*, p)dt = -\{f_x(t, x^*, u^*) + b_x(t, x^*, u^*)^* p\} dt \\
p(T) &= g_x(x^*(T))
\end{align*}
\]

\[ H_u(t, x^*(t), u^*(t), p(t)) \cdot (v - u^*(t)) \geq 0 \quad \forall v \in K \]

### 1.4 Conclusion

We described some main basic results in a deterministic control problem in this chapter. In the next chapter, we will add a drift term to a control problem and use similar approach as in this chapter to get necessary conditions for a stochastic control problem.
Chapter 2

Stochastic control

2.1 Introduction

In this chapter, we use perturbation method to derive maximum principle of a stochastic control problem. Maximum principle is a set of necessary conditions characterizing optimal control and trajectory. We follow Bensoussan [5] in deriving maximum principle.

2.2 Stochastic control problem

We suppose that

\[ f(t, x, u) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} \]
\[ b(t, x, u) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \]
\[ \sigma(t, x, u) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^{n \times n} \]
\[ g(x) : \mathbb{R}^n \to \mathbb{R} \]

are continuous differentiable functions, and satisfy the usual conditions that we need to guarantee the existence of solution of the following stochastic control problem.
\[
\begin{aligned}
\text{minimize} & \quad J(u(.)) = E\left[\int_0^T f(t,x(t),u(t))dt + g(x(T))\right] \\
\text{subject to} & \quad dx = b(t,x(t),u(t))dt + \sigma(t,x(t),u(t))dB_t \\
& \quad x(0) = x_0
\end{aligned}
\]

(P1)

The stochastic differential equation is defined on the probability space \((\Omega,F,P)\). \(B_t\) is an \(n\)-dimensional Brownian motion and the filtration \(F_t\) which generated by \(B_t\) satisfies the usual conditions. \(x(t) = x_t = x(t,\omega)\) and \(u(t) = u_t = u(t,\omega)\), for \(\omega \in \Omega\). The stochastic differential equation in (P1) will be called the state equation. The vector \(x\) which belongs to \(\mathbb{R}^n\) will be called the state variable, and the function \(x(t)\) which satisfies the state equation will be called the trajectory corresponding to \(u(t)\), and \(x_0\) is a given vector in \(\mathbb{R}^n\). On the other hand, the vector \(u\) is called the control variable, and the non-empty, closed, convex set \(K\) is the control constraint. We suppose that the state equation has one and only one solution when \(u(t)\) is given. All the state variable and controls in this thesis are predictable process, and it can be shown that their perturbations will be also predictable[4]. Denote the optimal control-solution pair for (P1) by \((u^*(t),x^*(t))\). The state equation can also be written as the form

\[
\begin{aligned}
dx &= b(t,x(t),u(t))dt + \sum_{j=1}^n \sigma^j(t,x(t),u(t))dB_j \\
x(0) &= x_0
\end{aligned}
\]

where \(\sigma^j\) is the \(j\)th column of \(\sigma\) and \(B_j\) is the \(j\)th element of \(B_t\).

\section*{2.3 Necessary conditions for stochastic problem}

For simplicity, we will sometimes ignore the arguments and use \(x^*\) or \(x\) to denote the function \(x^*(t)\) or \(x(t)\). We will also use \(u^*\) or \(u\) to denote the function \(u^*(t)\) or \(u(t)\).

We begin with the necessary conditions. First of all, we make perturbations on the optimal control \(u^*\). Let

\[
u_\theta = u^* + \theta v
\]
where $v = v(t)$ is an admissible control. According to our assumptions in Section 2, we can find one and only one $x_\theta$ which is the trajectory corresponding to $u_\theta$. That is

$$dx_\theta = b(t, x_\theta(t), u_\theta(t))dt + \sigma(t, x_\theta(t), u_\theta(t))dB_t$$

$$x_\theta(0) = x_0$$

In order to discuss the change in $J$ corresponding to the perturbation in $u^*$, we need to discuss the change in the dynamics first. By mean value theorem, we have the following

$$x_\theta(t) - x^*(t) = x_0 + \int_0^t b(s, x_\theta(s), u_\theta(s))ds + \int_0^t \sigma(s, x_\theta(s), u_\theta(s))dB_s$$

$$-x_0 - \int_0^t b(s, x^*(s), u^*(s))ds + \int_0^t \sigma(s, x^*(s), u^*(s))dB_s$$

$$= \int_0^t b(s, x_\theta(s), u_\theta(s)) - b(s, x^*(s), u^*(s))ds$$

$$+ \int_0^t \sigma(s, x_\theta(s), u_\theta(s)) - \sigma(s, x^*(s), u^*(s))dB_s$$

$$= \int_0^t \int_0^1 \left\{ b_x(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*))\right\} dx_\theta - x^*$$

$$+ \sum_{j=1}^n \int_0^t \int_0^1 \left\{ \sigma^j_x(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*))(x_\theta - x^*)$$

$$+ \sigma^j_u(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*))(u_\theta - u^*) \right\} d\lambda dB_j$$

Then, we have
\[
\frac{x_\theta(t) - x^*(t)}{\theta} = \int_0^t \int_0^1 \left\{ b_x(s, x^* + \lambda (x_\theta - x^*), u^* + \lambda (u_\theta - u^*)) \frac{x_\theta - x^*}{\theta} \\
+ b_u(s, x^* + \lambda (x_\theta - x^*), u^* + \lambda (u_\theta - u^*)) v \right\} d\lambda \, ds \\
+ \sum_{j=1}^n \int_0^t \int_0^1 \left\{ \sigma^j_x(s, x^* + \lambda (x_\theta - x^*), u^* + \lambda (u_\theta - u^*)) \frac{x_\theta - x^*}{\theta} \\
+ \sigma^j_u(s, x^* + \lambda (x_\theta - x^*), u^* + \lambda (u_\theta - u^*)) v \right\} d\lambda \, dB_j
\]

where \( b_x, b_u, \sigma_x, \) and \( \sigma_u \) represent the partial derivative of \( b \) and \( \sigma \) with respect to \( x \) and \( u \), etc.

The following Lemma is used to determine the limit of \( \frac{x_\theta(t) - x^*(t)}{\theta} \) as \( \theta \) tends to zero.

**Lemma 2.3.1.** We define a function \( Z(t) = Z(t, \omega) \) by the following stochastic differential equation

\[
dZ = \left\{ b_x(t, x^*(t), u^*(t)) Z(t) \right\} dt \\
+ \sum_{j=1}^n \left\{ \sigma^j_x(t, x^*(t), u^*(t)) Z(t) \right\} dB_j \]

\( Z(0) = 0 \)

Then, \( \frac{x_\theta(t) - x^*(t)}{\theta} \rightarrow Z(t) \) in \( L^2 \)

**Proof:**

Consider the following functions
\[ x_\theta(t) = x_0 + \int_0^t b(s, x_\theta(s), u_\theta(s)) \, ds + \sum_{j=1}^n \int_0^t \sigma_j(s, x_\theta(s), u_\theta(s)) \, dB_j \]
\[ x^*(t) = x_0 + \int_0^t b(s, x^*(s), u^*(s)) \, ds + \sum_{j=1}^n \int_0^t \sigma_j(s, x^*(s), u^*(s)) \, dB_j \]
\[ Z(t) = \int_0^t \{ b_x(s, x^*(s), u^*(s))Z(s) + b_u(s, x^*(s), u^*(s))v(s) \} \, ds \]
\[ + \sum_{j=1}^n \int_0^t \{ \sigma_{j,x}^2(s, x^*(s), u^*(s))Z(s) + \sigma_{j,u}^2(s, x^*(s), u^*(s))v(s) \} \, dB_j \]

and let \( y_\theta = \frac{x_\theta - x^*}{\theta} - Z \)

Then, we have

\[ y_\theta = \int_0^t \left\{ \frac{b(s, x_\theta(s), u_\theta(s)) - b(s, x^*(s), u^*(s))}{\theta} \right. \]
\[ - b_x(s, x^*(s), u^*(s))Z(s) - b_u(s, x^*(s), u^*(s))v(s) \} \, ds \]
\[ + \sum_{j=1}^n \int_0^t \left\{ \frac{\sigma_j(s, x_\theta(s), u_\theta(s)) - \sigma_j(s, x^*(s), u^*(s))}{\theta} \right. \]
\[ - \sigma_{j,x}^2(s, x^*(s), u^*(s))Z(s) - \sigma_{j,u}^2(s, x^*(s), u^*(s))v(s) \} \, dB_j \]
\[ = \int_0^t \frac{b(s, x_\theta, u_\theta) - b(s, x^*, u^*)}{\theta} \]
\[ - b_x(s, x^*, u^*)Z - b_u(s, x^*, u^*)v \, ds \]
\[ = \int_0^t \int_0^1 \left\{ b_x(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*))y_\theta \right. \]
\[ + [b_x(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - b_x(s, x^*, u^*)]Z \]
\[ + [b_u(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - b_u(s, x^*, u^*)]v \} \, d\lambda \, ds \]
\[ + \sum_{j=1}^n \int_0^t \left\{ \sigma_{j,x}^2(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*))y_\theta \right. \]
\[ + [\sigma_{j,x}^2(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - \sigma_{j,x}^2(s, x^*, u^*)]Z \]
\[ + [\sigma_{j,u}^2(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - \sigma_{j,u}^2(s, x^*, u^*)]v \} \, d\lambda \, dB_j \]

Since the functions \( Z \) and \( v \) belong to \( L^2 \), we can take sufficiently large number \( M \), such that both of their 2-norms are less than or equal to \( M \). Also assume that \( |b_x| \leq M \) and \( |\sigma_{j,x}^2| \leq M \).
By Holder’s inequality, we have

\[
E[|y_\theta|^2] \leq E[(\int_0^t \int_0^1 \{ |b_x(s, x^* + \lambda(x\theta - x^*), u^* + \lambda(u\theta - u^*))| |y_\theta| \\
+ |b_x(s, x^* + \lambda(x\theta - x^*), u^* + \lambda(u\theta - u^*)) - b_x(s, x^*, u^*)| |Z| \\
+ |b_u(s, x^* + \lambda(x\theta - x^*), u^* + \lambda(u\theta - u^*)) - b_u(s, x^*, u^*)| |v| \} d\lambda ds \\
+ \sum_{j=1}^n \int_0^t \int_0^1 \{ |\sigma^j_x(s, x^* + \lambda(x\theta - x^*), u^* + \lambda(u\theta - u^*))| |y_\theta| \\
+ |\sigma^j_x(s, x^* + \lambda(x\theta - x^*), u^* + \lambda(u\theta - u^*)) - \sigma^j_x(s, x^*, u^*)| |Z| \\
+ |\sigma^j_u(s, x^* + \lambda(x\theta - x^*), u^* + \lambda(u\theta - u^*)) - \sigma^j_u(s, x^*, u^*)| |v| \} d\lambda dB_j)^2] \\
\leq 2 E[(\int_0^t \int_0^1 |b_x(s, x^* + \lambda(x\theta - x^*), u^* + \lambda(u\theta - u^*))| |y_\theta| d\lambda ds)^2] \\
+ 2 E[(\int_0^t \int_0^1 |b_x(s, x^* + \lambda(x\theta - x^*), u^* + \lambda(u\theta - u^*)) - b_x(s, x^*, u^*)| |Z| d\lambda ds)^2] \\
+ 2 E[(\int_0^t \int_0^1 |b_u(s, x^* + \lambda(x\theta - x^*), u^* + \lambda(u\theta - u^*)) - b_u(s, x^*, u^*)| |v| d\lambda ds)^2] \\
+ 2 \sum_{j=1}^n E[(\int_0^t \int_0^1 |\sigma^j_x(s, x^* + \lambda(x\theta - x^*), u^* + \lambda(u\theta - u^*))| |y_\theta| d\lambda dB_j)^2] \\
+ 2 \sum_{j=1}^n E[(\int_0^t \int_0^1 |\sigma^j_x(s, x^* + \lambda(x\theta - x^*), u^* + \lambda(u\theta - u^*)) - \sigma^j_x(s, x^*, u^*)| |Z| d\lambda dB_j)^2] \\
+ 2 \sum_{j=1}^n E[(\int_0^t \int_0^1 |\sigma^j_u(s, x^* + \lambda(x\theta - x^*), u^* + \lambda(u\theta - u^*))| |v| d\lambda dB_j)^2] \\
\leq (2 M^2 T^2 + 2 n M^2) \int_0^T |y_\theta|^2 ds + 2 M^2 \hat{\vartheta}(\theta)
\]

where
\[
\vartheta(\theta) = \int_0^T \int_0^1 E[|b_x(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - b_x(s, x^*, u^*)|^2] \, d\lambda \, ds \\
+ \int_0^T \int_0^1 E[|b_u(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - b_u(s, x^*, u^*)|^2] \, d\lambda \, ds \\
+ \sum_{j=1}^n \int_0^T \int_0^1 E[|\sigma_x^j(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - \sigma_x^j(s, x^*, u^*)|^2] \, d\lambda \, ds \\
+ \sum_{j=1}^n \int_0^T \int_0^1 E[|\sigma_u^j(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) - \sigma_u^j(s, x^*, u^*)|^2] \, d\lambda \, ds
\]

and clearly \( \vartheta(\theta) \) tends to zero when \( \theta \) tends to zero. By Gronwall’s inequality, we have

\[
|y_\theta|^2 \leq 2M^2 \vartheta(\theta) \, e^{(2M^2T^2 + 2nM^2)t} \quad \forall t \in [0, T]
\]

Therefore

\[
\int_0^T |y_\theta|^2 \, dt \leq \int_0^T 2M^2 \vartheta(\theta) \, e^{(2M^2T^2 + 2nM^2)t} \, dt
\]

the right hand side tends to zero when \( \theta \) tends to zero. So, \( y_\theta \) tends to zero in \( L^2 \). That is,

\[
\frac{x_\theta - x^*}{\theta} \longrightarrow Z \quad \text{in} \quad L^2
\]

The proof is done. \( \square \)

Now, we are ready to consider the change in \( J \) as a result of the perturbation in the optimal control \( u^* \).
\[ J(u_\theta) - J(u^*) = E[ g(x_\theta(T)) - g(x^*(T)) \] 
\[ + \int_0^T f(t, x_\theta(t), u_\theta(t)) - f(t, x^*(t), u^*(t)) dt ] \]
\[ = E[ \int_0^1 g_x(x^*(T) + \lambda(x_\theta(T) - x^*(T)) \cdot (x_\theta(T) - x^*(T)) d\lambda \] 
\[ + \int_0^T \int_0^1 f_x(t, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) \cdot (x_\theta - x^*) d\lambda dt \] 
\[ + \int_0^T \int_0^1 f_u(t, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) \cdot (u_\theta - u^*) d\lambda dt ] \]

Then we divide both sides by \( \theta \)

\[ \frac{J(u_\theta) - J(u^*)}{\theta} = E[ \int_0^1 g_x(x^*(T) + \lambda(x_\theta(T) - x^*(T)) \cdot \frac{x_\theta(T) - x^*(T)}{\theta} d\lambda \] 
\[ + \int_0^T \int_0^1 f_x(t, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) \cdot \frac{x_\theta(t) - x^*(t)}{\theta} d\lambda dt \] 
\[ + \int_0^T \int_0^1 f_u(t, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) \cdot v(t) d\lambda dt ] \]

Letting \( \theta \) tend to 0, we obtain the Gateaux derivative of \( J \) along the direction \( v \).

**Proposition 2.3.2.** The Gateaux derivative of \( J \) along the direction \( v \) is

\[ dJ(u^*; v) = E[ g_z(x^*(T)) \cdot Z(T) + \int_0^T f_z \cdot Z + f_u \cdot v dt ] \]

Our next step is to define dual variables and equation of the control problem. Consider the equation
\[ d\zeta(t) = \{b_x(t, x^*(t), u^*(t))\zeta(t) + \Phi(t)\}dt + \sum_{j=1}^{n}\{\sigma^j_x(t, x^*(t), u^*(t))\zeta(t) + \Psi^j(t)\}dB_j, \]

\[ \zeta(0) = 0 \]

where \(\Phi\) and \(\Psi^j\) are functions in \(L^2_F(0, T; \mathbb{R}^n)\), for all \(j\). And we suppose that for fixed \(\Phi\) and \(\Psi^j\), we can find one and only one \(\zeta\) satisfying the above stochastic differential equation. Then, for fixed \(\Phi\) and \(\Psi^j\), we can consider the quantity

\[ E\left[ \int_0^T f_x \cdot \zeta dt + g_x(x^*(T)) \cdot \zeta(T) \right] \]

This map is linear and bounded. By Riesz representation theorem, we can find functions \(p(t)\) and \(q^j(t)\) in \(L^2_F(0, T; \mathbb{R}^n)\), such that for all \(\Phi\) and \(\Psi^j\) in \(L^2_F(0, T; \mathbb{R}^n)\), we have

\[ E[\int_0^T p(t) \cdot \Phi(t) dt + \sum_{j=1}^{n}\int_0^T q^j(t) \cdot \Psi^j(t) dt] = E[\int_0^T f_x(t) \cdot \zeta(t) dt + g_x(x^*(T)) \cdot \zeta(T)] \]

The \(p(t)\) and \(q^j(t)\) will be called the dual variables or adjoint variables. On the other hand, we define the Hamiltonian

\[ H(t, x, u, p, q) = f(t, x, u) + b(t, x, u) \cdot p + \sum_{j=1}^{n}\sigma^j(t, x, u) \cdot q^j \]

where \(q = [q^1, q^2, q^3, ..., q^n]\). We conclude

\[ dJ(u^*; v) = E[\int_0^T H_u(t, x^*(t), u^*(t), p(t), q(t)) \cdot v(t) dt] \]

**Proposition 2.3.3.** For (P1), we have

\[ H_u(t, x^*(t), u^*(t), p(t), q(t)) \cdot (v - u^*(t)) \geq 0 \]

for all \(v \in K\) and almost everywhere \(t\) in \([0, T]\) and \(w\) in \(\Omega\). 

21
Proof:

Since \((u^*, x^*)\) is the optimal pair for (P1), we know the Gateaux derivative of \(J(u(.))\) at \(u^*(.\) along the direction \(v(.) - u^*(.\) is non-negative. That is,

\[
E\left[ \int_0^T H_u(t, x^*(t), u^*(t), p(t), q(t)) \cdot (v(t) - u^*(t)) \, dt \right] = \, dJ(u^*(.); v(.) - u^*(.)) \geq 0
\]

Letting \(v\) be any fixed vector in \(K\) and the set \(A = \{(t, \omega)|H_u(t, x^*(t), u^*(t), p(t), q(t)) \cdot (v - u^*(t)) < 0\}\). We want to show that the measure of set \(A\) is zero. That is, \(m(A)=0\). Define the function

\[
\bar{v}(t, \omega) = \begin{cases} v, & (t, \omega) \in A \\ u^*(t, \omega), & \text{otherwise} \end{cases}
\]

Fixed a Lebesgue point \(t\) in \((0, T)\) and let \(A_t = \{w \in \Omega|H_u(t, x^*(t), u^*(t), p(t), q(t)) \cdot (v - u^*(t)) < 0\}\). Then the function becomes

\[
\bar{v}(t, \omega) = \begin{cases} v, & w \in A_t \\ u^*(t, \omega), & \text{otherwise} \end{cases}
\]

We can see \(\bar{v}(t, \omega)\) is measurable with respect to \(F_t\) in above, and we know that \(A_t\) is in \(F_t\). Moreover, since

\[
A = \bigcup_{t \in [0, T]} A_t
\]

We know the function \(\bar{v}(t, \omega)\) is adapted with respect to the filtration \(\{F_t\}\). On the other hand, since \(u^*\) belongs to \(L^2\), we have

\[
|\bar{v}| \leq |v \cdot v| T + |u| < \infty
\]

So, \(\bar{v}(.\) belongs to \(L^2\) and it is an admissible control. Then, we have
\[
E[\int_0^T H_u(t, x^*(t), u^*(t), p(t), q(t)) \cdot (v - u^*(t)) \, dt]
= \int_\Omega \int_A H_u(t, x^*(t), u^*(t), p(t), q(t)) \cdot (\bar{v}(t) - u^*(t)) \, dt \, dP \geq 0
\]

and
\[
H_u(t, x^*(t), u^*(t), p(t), q(t)) \cdot (\bar{v}(t) - u^*(t)) < 0 \quad \text{on } A
\]

So, we conclude that \( m(A) = 0 \). That is,
\[
H_u(t, x^*(t), u^*(t), p(t), q(t)) \cdot (v - u^*(t)) \geq 0
\]

for a.e \( t \in [0, T] \) and \( \omega \in \Omega \), and all \( v \in K \).

The proof is done. \( \square \)

The next Proposition is another way to characterize the adjoint variables.

**Proposition 2.3.4.** \( p(t) \) and \( q^j(t) \) are the adjoint variables we defined by Riesz representation theorem, if and only if they satisfy the costate equation

\[
dp = -H_x(t, x^*, u^*, p, q)dt + q dB_t
= -\{f_x(t, x^*, u^*) + b_x(t, x^*, u^*)^* \cdot p + \sum_{j=1}^n \sigma^j_x(t, x^*, u^*)^* \cdot q^j \} dt + \sum_{j=1}^n q^j dB_j
\]

\[
p(T) = g_x(x^*(T))
\]

**Proof:**

Firstly, we suppose \( p(t) \) and \( q^j(t) \) are the solutions of the costate equation and proof that they
satisfy the Riesz representation theorem. According to integration by parts, we have

\[
d(p(t) \cdot \zeta(t)) = \zeta(t) \cdot dp(t) + p(t) \cdot d\zeta(t) + \sum_{j=1}^{n} q^j(t) \cdot \{\sigma^j_x(t, x^*, u^*) \zeta(t) + \Psi^j(t)\} dt
\]

\[
= \zeta \cdot \{-[f_x(t, x^*, u^*) + b_x(t, x^*, u^*)^*] p + \sum_{j=1}^{n} \sigma^j_x(t, x^*, u^*)^* q^j \} dt + \sum_{j=1}^{n} q^j dB_j
\]

\[
+ p \cdot \{[b_x(t, x^*, u^*)] \zeta + \Phi\} dt + \sum_{j=1}^{n} [\sigma^j_x(t, x^*, u^*)] \zeta + \Psi^j dB_j
\]

\[
+ \sum_{j=1}^{n} q^j \cdot [\sigma^j_x(t, x^*, u^*) \zeta + \Psi^j] dt
\]

\[
= \{p \cdot \Phi - \zeta \cdot f_x(t, x^*, u^*) + \sum_{j=1}^{n} q^j \cdot \Psi^j\} dt + \sum_{j=1}^{n} \{\zeta \cdot q^j + p \cdot (\sigma^j_x \zeta + \Psi^j)\} dB_j
\]

Then,

\[
E[p(T) \cdot \zeta(T)] = p(0) \cdot \zeta(0) + E\left[\int_0^T p \cdot \Phi dt - \int_0^T \zeta \cdot f_x(t, x^*, u^*) dt \right.
\]

\[
+ \sum_{j=1}^{n} \int_0^T q^j \cdot \Psi^j dt\]

Since \(\zeta(0) = 0\) and \(p(T) = g_x(x^*(T))\), we have

\[
E[\int_0^T p(t) \cdot \Phi(t) dt + \sum_{j=1}^{n} \int_0^T q^j(t) \cdot \Psi^j(t) dt] = E[\int_0^T \zeta(t) \cdot f_x(t) dt + g_x(x^*(T)) \cdot \zeta(T)]
\]

which is the Riesz representation.

Secondly, let’s suppose \(p(t)\) and \(q^j(t)\) are given by the Riesz representation theorem, and prove
that they must satisfy the costate equation. Consider the matrix equation

\[
\begin{align*}
    dM(t) &= b_x(t, x^*, u^*) M(t) \, dt + \sum_{j=1}^{n} \sigma_x^j(t, x^*, u^*) M(t) dB_j \\
    M(0) &= I
\end{align*}
\]

where I is the identity matrix. Then \( M(t) \) is the fundamental solution of the equation corresponding \( \zeta(t) \). It can be shown that the fundamental equation has exactly one solution, that is, the \( M(t) \) exists and unique [1][4][5].

In order to construct the inverse of \( M \), we consider another matrix equation

\[
\begin{align*}
    dW(t) &= \alpha(t, \omega) \, dt + \sum_{j=1}^{n} \beta(t, \omega) dB_j \\
    W(0) &= I
\end{align*}
\]

We need to find appropriate \( \alpha(t, \omega) \) and \( \beta(t, \omega) \) such that \( M(t)W(t)=I \), for all \( t \) in \([0,T]\). By Ito’s formula

\[
\begin{align*}
    d(W(t)M(t)) &= dW(t)M(t) + W(t)dM(t) + \sum_{j=1}^{n} \beta(t, \omega)M(t) dB_j + \sum_{j=1}^{n} \sigma_x^j(t, x^*, u^*) M(t) dt \\
    &= \alpha(t, \omega)M(t) dt + \sum_{j=1}^{n} \beta(t, \omega)M(t) dB_j + W(t)b_x(t, x^*, u^*) M(t) dt \\
    &\quad + \sum_{j=1}^{n} W(t)\sigma_x^j(t, x^*, u^*) M(t) dB_j + \sum_{j=1}^{n} \beta(t, \omega)\sigma_x^j(t, x^*, u^*) M(t) dt \\
    &= \{ \alpha(t, \omega) M(t) + W(t)b_x(t, x^*, u^*) M(t) + \sum_{j=1}^{n} \beta(t, \omega)\sigma_x^j(t, x^*, u^*) M(t) \} dt \\
    &\quad + \sum_{j=1}^{n} \{ \beta(t, \omega) M(t) + W(t)\sigma_x^j(t, x^*, u^*) M(t) \} dB_j
\end{align*}
\]
We choose
\[
\alpha(t, \omega) = -W(t)b_x(t, x^*, u^*) + \sum_{j=1}^n W(t)\sigma^j_x(t, x^*, u^*)\sigma^j_x(t, x^*, u^*)
\]
\[
\beta^j(t, \omega) = -W(t)\sigma^j_x(t, x^*, u^*)
\]

Then,
\[
d(W(t)M(t)) = 0
\]

So we have \(M(t)W(t)=I\), for all \(t\) in \([0, T]\). Since both \(M(t)\) and \(W(t)\) are square matrices, it is also sufficient to say \(W(t)M(t)=I\), for all \(t\) in \([0, T]\). Let \(\Psi_j = 0\) in the equation of \(\zeta(t)\). We have
\[
d\zeta(t) = \{b_x(t, x^*(t), u^*(t))\zeta(t) + \Phi(t)\}dt + \sum_{j=1}^n \sigma^j_x(t, x^*(t), u^*(t))\zeta(t)dB_j
\]

Then,
\[
d(W(t)\zeta(t)) = W(t)d\zeta(t) + dW(t)\zeta(t) - \sum_{j=1}^n W(t)\sigma^j_x(t, x^*, u^*)\sigma^j_x(t, x^*, u^*)\zeta(t)dt
\]
\[
= W\{(b_x + \Phi)dt + \sum_{j=1}^n \sigma^j_x dB_j\} + \{(Wb_x + \sum_{j=1}^n W\sigma^j_x\sigma^j_x)dt - \sum_{j=1}^n W\sigma^j_x dB_j\}\zeta
\]
\[
- \sum_{j=1}^n W\sigma^j_x\sigma^j_x\zeta dt
\]
\[
= W(t)\Phi(t)dt
\]

which implies
\[
W(t)\zeta(t) = \int_0^t W(s)\Phi(s)ds
\]

and
\[
\zeta(t) = M(t)\int_0^t W(s)\Phi(s)ds
\]
According to Ito’s formula, we have

\[
E\left[ \int_0^T p(t) \cdot \Phi(t) dt \right] = E\left[ \int_0^T f_x(t, x^*, u^*) \cdot \zeta(t) dt + g_x(x^*(T)) \cdot \zeta(T) \right]
\]

\[
= E\left[ \int_0^T \{ M(t) \int_0^t W(s) \Phi(s) ds \} dt \right] + g_x(x^*(T)) \cdot \{ M(T) \int_0^T W(s) \Phi(s) ds \}
\]

\[
= E\left[ \int_0^T \{ M(t) W(t) \Phi(t) \} dt \right] + g_x(x^*(T)) \cdot \{ M(T) W(T) \Phi(T) \}
\]

\[
= E\left[ \Phi(t) \cdot W(t)^* \left\{ \int_t^T M(s)^* f_x(s, x^*, u^*) ds + M(T)^* g_x(x^*(T)) \right\} dt \right]
\]

\[
= E\left[ \Phi(t) \cdot W(t)^* \left\{ - \int_0^t M(s)^* f_x(s, x^*, u^*) ds + \int_t^T M(s)^* f_x(s, x^*, u^*) ds\right\}
\]

\[
+ M(s)^* f_x(s, x^*, u^*) ds + M(T)^* g_x(x^*(T)) \right\} dt \right]
\]

This is true for all \( \Phi(t) \in L^2_F \). We obtain the explicit form of \( p(t) \)

\[
p(t) = W(t)^* \left\{ - \int_0^t M(s)^* f_x(s, x^*, u^*) ds \right\}
\]

\[
+ E\left[ \int_0^T M(s)^* f_x(s, x^*, u^*) ds + M(T)^* g_x(x^*(T)) \right| F_t \}
\]

By letting \( t=T \), it gives the terminal condition

\[
p(T) = g_x(x^*(T))
\]

Notice that the process \( E\left[ \int_0^T M(s)^* f_x(s, x^*, u^*) ds + M(T)^* g_x(x^*(T)) \right| F_t \} \) is a martingale. By
martingale representation theorem, we have

\[
E \left[ \int_0^T M(s)^* f_x(s, x^*, u^*) ds + M(T)^* g_x(x^*(T)) \mid F_t \right]
\]

\[
= E \left[ \int_0^T M(s)^* f_x(s, x^*, u^*) ds + M(T)^* g_x(x^*(T)) \right] + \sum_{j=1}^n \int_0^t G_j(s, \omega) dB_j
\]

where \( G_j \in L^2_{F_t}(0, T; \mathbb{R}^n) \) is uniquely defined. Then,

\[
p(t) = W(t)^* \left\{ - \int_0^t M(s)^* f_x(s, x^*, u^*) ds + \sum_{j=1}^n \int_0^t G_j(s, \omega) dB_j 
+ E \left[ \int_0^T M(s)^* f_x(s, x^*, u^*) ds + M(T)^* g_x(x^*(T)) \right] \right\}
\]

Now, it is time to simplify \( p(t) \) to the form we want. We have

\[
p(t) = W(t)^* \eta(t)
\]

where

\[
d\eta(t) = -M(t)^* f_x(t, x^*, u^*) dt + \sum_{j=1}^n G_j(t, \omega) dB_j
\]

\[
\eta(0) = E \left[ \int_0^T M(s)^* f_x(s, x^*, u^*) ds + M(T)^* g_x(x^*(T)) \right]
\]

and

\[
dW(t)^* = \alpha(t, \omega)^* dt + \sum_{j=1}^n \beta_j(t, \omega)^* dB_j
\]

\[
W(0)^* = I
\]
\[ \alpha(t, \omega)^* = -b_x(t, x^*, u^*)^* W(t)^* + \sum_{j=1}^{n} \sigma^j_x(t, x^*, u^*)^* \sigma^j_x(t, x^*, u^*)^* W(t)^* \]
\[ \beta^j(t, \omega)^* = -\sigma^j_x(t, x^*, u^*)^* W(t)^* \]

By Ito's formula, we have

\[
\begin{align*}
dp(t) &= d(W(t)^* \eta(t)) \\
&= dW(t)^* \eta(t) + W(t)^* d\eta(t) - \sum_{j=1}^{n} \sigma^j_x(t, x^*, u^*)^* W(t)^* G_j(t, \omega) dt \\
&= \{(-b_x W^* + \sum_{j=1}^{n} \sigma^j_x^* \sigma^j_x W^*) dt - \sum_{j=1}^{n} \sigma^j_x^* W^* dB_j \} \eta \\
&\quad + W^* \{-M^* f_x dt + \sum_{j=1}^{n} G_j dB_j \} - \sum_{j=1}^{n} \sigma^j_x^* W^* G_j dt \\
&= \{-b_x^* (W^* \eta) + \sum_{j=1}^{n} \sigma^j_x^* \sigma^j_x (W^* \eta) - \sum_{j=1}^{n} \sigma^j_x^* W^* G_j - f_x \} dt \\
&\quad + \sum_{j=1}^{n} \{-\sigma^j_x^* (W^* \eta) + W^* G_j \} dB_j \\
&= \{-b_x^* p + \sum_{j=1}^{n} \sigma^j_x^* \sigma^j_x p - \sum_{j=1}^{n} \sigma^j_x^* W^* G_j - f_x \} dt \\
&\quad + \sum_{j=1}^{n} \{-\sigma^j_x^* p + W^* G_j \} dB_j
\end{align*}
\]

take

\[ q^j(t, \omega) = -\sigma^j_x(t, x^*, u^*)^* p(t) + W(t)^* G_j(t, \omega) \]
then
\[ dp(t) = \{-b_x^* p - \sum_{j=1}^n \sigma_j^* q^j - f_x\} dt + \sum_{j=1}^n q^j dB_j \]
\[ = -H_x(t, x^*(t), u^*(t), p(t), q(t)) dt + \sum_{j=1}^n q_j(t, \omega) dB_j \]

where
\[ H(t, x, u, p, q) = f(t, x, u) + b(t, x, u) \cdot p + \sum_{j=1}^n \sigma_j(t, x, u) \cdot q^j \]

The proof is done. □

**Proposition 2.3.5. Necessary Conditions for (P1)**

Let \((u^*, x^*)\) be the optimal pair for (P1), then we can find corresponding \(p(t)\) and \(q^j(t)\) belonging to \(L^2_F(0, T; \mathbb{R}^n)\), such that

\[ dx^* = H_p(t, x^*, u^*, p, q) dt + \sum_{j=1}^n \sigma_j^j(t, x^*, u^*) dB_j \]
\[ = b(t, x^*, u^*) + \sum_{j=1}^n \sigma_j^j(t, x^*, u^*) dB_j \]
\[ x^*(0) = x_0 \]

\[ dp = -H_x(t, x^*, u^*, p, q) dt + \sum_{j=1}^n q^j(t, x^*, u^*) dB_j \]
\[ = -\{f_x(t, x^*, u^*) + b_x(t, x^*, u^*)^* p + \sum_{j=1}^n \sigma_j^j(t, x^*, u^*)^* q^j\} dt \]
\[ + \sum_{j=1}^n q_j^j(t, x^*, u^*) dB_j \]

\[ p(T) = g_x(x^*(T)) \]
\[ H_u(t, x^*(t), u^*(t), p(t), q(t)) \cdot (v - u^*(t)) \geq 0 \]

30
2.4 Conclusion

We described some main basic results in stochastic control problem in this chapter. In the next chapter, we will add a jump term to a control problem and use similar approach as in this chapter to get necessary conditions for a stochastic control problem.
Chapter 3

Stochastic control with jump diffusion

3.1 Introduction

In this chapter, we briefly introduce the Levy process and the setting of stochastic control problem with jump diffusion in Section 2 and 3, respectively. In Section 4, we use perturbation method to derive the maximum principle and a set of necessary conditions. In a functional analysis approach, we introduce the costate variables by Riesz representation theorem and connect them with a system of backward stochastic differential equations which is called the costate equation. Hamilton-Jacobi-Bellman (HJB) equation is introduced and a connection with maximum principle is made in Section 5. A sufficient condition is presented in Section 6. The ideas of chapter are based previous ones.

3.2 Jump diffusion in high-frequency trading market

Let $(\Omega, F, P)$ be a probability space with a filtration $\{F_t\}_{t \geq 0}$. A stochastic process $\{\eta_t\}_{t \geq 0}$ is called a Levy process if it satisfies the following conditions

1. $\{\eta_t\}_{t \geq 0}$ is adapted to $\{F_t\}_{t \geq 0}$.
2. $\{\eta_t\}_{t \geq 0}$ has stationary and independent increment.
3. $\{\eta_t\}_{t \geq 0}$ has cadlag (or some people use the notation RCLL) paths which means that the trajectories are right continuous with left limits.
4. $\eta_0 = 0$. 

32
Since the Levy process \( \{\eta_t\}_{t \geq 0} \) is cadlag, we can consider the jump of \( \{\eta_t\}_{t \geq 0} \) at time \( t \), which is defined by

\[
\triangle \eta_t = \eta_t - \eta_{t^-}
\]

In order to count the number of jumps occurred before or at some time \( t \), we define

\[
N(t, U) = N(t, U, \omega) = \sum_{0 < s \leq t} \chi_U(\triangle \eta_s)
\]

for each Borel set \( U \) in \( \mathbb{R} \), supposing 0 is not in \( \overline{U} \), the closure of \( U \). Where \( \chi_U \) is the characteristic function of set \( U \), such that

\[
\chi_U(x) = \begin{cases} 
1, & x \in U \\
0, & \text{otherwise}
\end{cases}
\]

In other words, \( N(t, U) \) is the number of jumps of size \( \triangle \eta_s \in U \) which occurs before or at time \( t \). \( N(t, U) \) is called the Poisson random measure of \( \{\eta_t\}_{t \geq 0} \), and it becomes a measure on Borel sets if we fixed \( w \in \Omega \) and time \( t \). In particular, if we fixed \( t = 1 \), define

\[
v(U) = E[N(1, U)]
\]

for all Borel set \( U \). Then, \( v \) is a measure on Borel sets, and \( v \) is called the Levy measure of \( \{\eta_t\}_{t \geq 0} \). One can proof that there is an one-to-one and onto map between Levy measure and Levy process. People should notice that \( v \) is only \( \sigma \)-finite, not finite, which means it is possible that

\[
\int_{\mathbb{R}_0} \min\{1, |z|\} v(dz) = \infty
\]

this is the case the trajectory of Levy Processes may have many small jumps in a short time period, a situation that high-frequency trading market happens. In contrast with Poisson processes which can only have finite number of jumps in a short time period and it is the low-frequency case. Levy processes can simulate the financial market we need today more precisely, and this is the reason we choose Levy process as our jump dynamics. If we fixed a Borel set \( U \), then the process \( \{N(t, U)\}_{t \geq 0} \) is a Poisson process with intensity \( v(U) \).

We can consider the map

\[
(a, b] \times U \rightarrow N(b, U) - N(a, U)
\]

it is the number of jumps occurred in the time period \( (a, b] \) with size in \( U \) and this map is
indeed a measure. In the following context, we will use the notation

\[ N(dt, dz) \]

to represent the differential form of this measure. And use

\[ \overline{N}(dt, dz) = N(dt, dz) - v(dz)dt \]

to represent the compensated Poisson random measure.

Levy process is very general. Actually, Brownian motions, Poisson processes, compound Poisson processes are all special cases of Levy processes. We have the following Ito-Levy decomposition theorem. The proof can be found in [1][2][3].

**Theorem 3.2.1.** Let \( \{ \eta_t \}_{t \geq 0} \) be a Levy process. Then, \( \{ \eta_t \}_{t \geq 0} \) has the decomposition

\[
\eta_t = wt + \sigma B_t + \int_{|z| < 1} z \overline{N}(t, dz) + \int_{|z| \geq 1} z N(t, dz)
\]

for some constant \( w \) and \( \sigma \) belongs to \( \mathbb{R} \).

If we assume that

\[ E[\eta_t^2] < \infty \]

then we have

\[ \int_{\mathbb{R}_0} z^2 v(dz) < \infty \]

so, we can represent \( \eta_t \) by

\[ \eta_t = wt + \sigma B_t + \int_{\mathbb{R}_0} z \overline{N}(t, dz) \]

To make sense of the stochastic differential equation with jump diffusion, we must give meaning to the integral

\[ \int_0^T \int_{\mathbb{R} \setminus \{0\}} r(t, z, \omega) N(dt, dz) \]

and we need tools to evaluate such integral. Since \( N(dt, dz) \) is a square integrable martingale, it can be handled in a similar way with what we did on Brownian motion in the case of Ito’s integral. And we can also derive the corresponding Ito’s formula and integration by parts for jump diffusion.
If a function $r(t, z, \omega)$ such that

$$E\left[ \int_0^T \int_{\mathbb{R}\setminus\{0\}} |r(t, z, \omega)|^2 v(dz) dt \right] < \infty$$

then,

$$\int_0^T \int_{\mathbb{R}\setminus\{0\}} r(t, z, \omega) N(dt, dz)$$

is well-defined in a similar way with Ito’s integral. We list some properties of such integral here [1][2][3]

(1) mean equals to zero

$$E\left[ \int_0^T \int_{\mathbb{R}\setminus\{0\}} r(t, z, \omega) N(dt, dz) \right] = 0$$

(2) isometry property

$$E\left[ \left( \int_0^T \int_{\mathbb{R}\setminus\{0\}} r(t, z, \omega) N(dt, dz) \right)^2 \right] = E\left[ \int_0^T \int_{\mathbb{R}\setminus\{0\}} |r(t, z, \omega)|^2 v(dz) dt \right]$$

(3)

$$E\left[ \int_0^T \int_{\mathbb{R}\setminus\{0\}} r(t, z, \omega) N(dt, dz) \right] = E\left[ \int_0^T \int_{\mathbb{R}\setminus\{0\}} r(t, z, \omega) v(dz) dt \right]$$

(4) Let

$$M_t = \int_0^t \int_{\mathbb{R}\setminus\{0\}} r(t, z, \omega) N(dt, dz)$$

then $M_t$ belongs to $L^2(\Omega)$ and the process $\{M_t\}_{t \geq 0}$ is a continuous martingale.

We are ready to consider the stochastic differential equation with jump diffusion which sometimes is called an Ito-Levy process in this form
\[ dX_t = b(t,\omega)dt + \sigma(t,\omega)dB_t + \int_{\mathbb{R}\setminus\{0\}} r(t, z, \omega)\mathcal{N}(dt, dz) \]

or it can be written in this form

\[ X_t = X_0 + \int_0^t b(s,\omega)ds + \int_0^t \sigma(s,\omega)dB_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} r(s, z, \omega)\mathcal{N}(ds, dz) \]

\( X_t \) is well-defined provided these functions \( b(t,\omega) \), \( \sigma(t,\omega) \), and \( r(t, z, \omega) \) are predictable processes such that

\[ \int_0^t \left[ |b(s,\omega)| + |\sigma(s,\omega)|^2 + \int_{\mathbb{R}\setminus\{0\}} |r(s, z, \omega)|^2 v(dz) \right] ds < \infty \]

Now, we introduce without proof the famous Ito’s formula for Levy process. The complete proof can be found in [2] and [3].

**Proposition 3.2.2. 1-dimensional Ito’s formula**

Let \( X_t \) be an one dimensional Ito-Levy process and \( g(t,x) \) is a function in \( C^2 \), then the process \( g(t, X_t) \) is also an Ito-Levy process with the differential form

\[
\begin{align*}
    dg(t, X_t) &= g_t(t, X_t)dt \\
    &\quad + g_x(t, X_t)b(t,\omega)dt + g_x(t, X_t)\sigma(t,\omega)dB_t + \frac{1}{2}g_{xx}(t, X_t)\sigma(t,\omega)^2dt \\
    &\quad + \int_{\mathbb{R}\setminus\{0\}} [g(t, X_t + r(t, z)) - g(t, X_t) - g_x(t, X_t)r(t, z)]v(dz)dt \\
    &\quad + \int_{\mathbb{R}\setminus\{0\}} [g(t, X_{t-} + r(t, z)) - g(t, X_{t-})]\mathcal{N}(dt, dz).
\end{align*}
\]

We can extend the above one dimensional problems to multi-dimensional problems. Consider the n-dimensional Ito-Levy process with differential form

\[ dX_t = b(t,\omega)dt + \sum_{j=1}^m \sigma^j(t,\omega)dB_j + \sum_{j=1}^m \int_{\mathbb{R}\setminus\{0\}} r^j(t, z, \omega)\mathcal{N}_j(dt, dz_j) \]
where $X_t = X(t) = [X_1(t), X_2(t), ..., X_m(t)]^*$, $\sigma(t, \omega)$ and $r(t, \omega)$ are $n$ by $m$ matrix with columns $\sigma^j(t, \omega)$ and $r^j(t, \omega)$, respectively. The $B = [B_1, ..., B_m]^*$ is a Brownian motion in $\mathbb{R}^m$, which means the 1-dimensional processes $\{B_j\}$ are independent, 1-dimensional Brownian motions. The $\overline{N}(dt, dz) = [\overline{N}_1(dt, dz_1), \overline{N}_2(dt, dz_2), ..., \overline{N}_m(dt, dz_m)]^*$ is a Poisson random measure in $\mathbb{R}^m$, which means the 1-dimensional processes $\overline{N}_j(dt, dz_j)$ are independent, 1-dimensional Poisson random measure.

$X_t$ can be written in this form

\[ X_t = X_0 + \int_0^t b(s, \omega)ds + \sum_{j=1}^m \int_0^t \sigma^j(s, \omega)dB_j \]

\[ + \sum_{j=1}^m \int_0^t \int_{\mathbb{R}\setminus\{0\}} r^j(s, z, \omega)\overline{N}_j(ds, dz_j) \]

and then we extend the Ito’s formula for Levy process to multi-dimensional case with. The proof can be found in [2] and [3].

**Proposition 3.2.3. multi-dimensional Ito’s formula**

Let $X_t$ be an $n$-dimensional Ito-Levy process and $g(t, x)$ is a real-valued function in $C^2$, then the process $g(t, X_t)$ is also an $n$-dimensional Ito-Levy process with the differential form

\[ dg(t, X_t) = g_t(t, X_t)dt \]

\[ + \sum_{i=1}^n g_{x_i}(t, X_t)b_i(t, \omega)dt + \sum_{i=1}^n \sum_{j=1}^m g_{x_i}(t, X_t)\sigma_{ij}(t, \omega)dB_j \]

\[ + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m g_{x_ix_j}(t, X_t)(\sigma(t, \omega)\sigma(t, \omega)^*)_ij dt \]

\[ + \sum_{j=1}^m \int_{\mathbb{R}\setminus\{0\}} [g(t, X_t + r^j(t, z)) - g(t, X_t) - \sum_{i=1}^n g_{x_i}(t, X_t)r(t, z)_{ij}]v_j(dz_j)dt \]

\[ + \sum_{j=1}^m \int_{\mathbb{R}\setminus\{0\}} [g(t, X_t- + r^j(t, z)) - g(t, X_t-)]\overline{N}_j(dt, dz_j). \]
By the Ito’s formula, we can also extend the integration by parts for Ito-Levy Process.

**Proposition 3.2.4. 1-dimensional integration by parts**

*Suppose we have two 1-dimensional Ito-Levy processes* \( X_t \) *and* \( Y_t \) *with differential forms*

\[
\begin{align*}
    dX_t &= b(t, \omega)dt + \sigma(t, \omega)dB_t + \int_{\mathbb{R}\setminus\{0\}} r(t, z, \omega)N(dt, dz) \\
    dY_t &= a(t, \omega)dt + \phi(t, \omega)dB_t + \int_{\mathbb{R}\setminus\{0\}} \gamma(t, z, \omega)N(dt, dz)
\end{align*}
\]

*then*

\[
    d(X_tY_t) = X_t dY_t + Y_t dX_t + \sigma(t, \omega)\phi(t, \omega)dt \\
    + \int_{\mathbb{R}\setminus\{0\}} r(t, z, \omega)\gamma(t, z, \omega)N(dt, dz)
\]

**Proof:**

Consider the process

\[
    d \begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} b(t, \omega) \\ a(t, \omega) \end{bmatrix} dt + \begin{bmatrix} \sigma(t, \omega) \\ \phi(t, \omega) \end{bmatrix} dB_t \\
    + \int_{\mathbb{R}\setminus\{0\}} \begin{bmatrix} r(t, z, \omega) \\ \gamma(t, z, \omega) \end{bmatrix} N(dt, dz)
\]

By using the multi-dimensional Ito’s formula with \( n = 2, m = 1 \), and \( g(x, y) = xy \), we have
\[ d(X_tY_t) = Y_t(b \, dt + \sigma \, dB_t) + X_t(a \, dt + \phi \, dB_t) + \sigma \phi \, dt \]
\[ + \int_{\mathbb{R}\setminus\{0\}} \{(X_t + \gamma)(Y_t + r) - X_tY_t - Y_t - X_t\gamma\} \, (dz) \, dt \]
\[ + \int_{\mathbb{R}\setminus\{0\}} \{(X_t + \gamma)(Y_t + r) - X_tY_t - Y_t - X_t\gamma\} \, \mathcal{N}(dt, dz) \]
\[ = Y_t(b \, dt + \sigma \, dB_t) + X_t(a \, dt + \phi \, dB_t) + \sigma \phi \, dt \]
\[ + \int_{\mathbb{R}\setminus\{0\}} \{(X_t + \gamma)(Y_t + r) - X_tY_t - Y_t - X_t\gamma\} \, (dz) \, dt \]
\[ + \int_{\mathbb{R}\setminus\{0\}} \{(X_t + \gamma)(Y_t + r) - X_tY_t - Y_t - X_t\gamma\} \, \mathcal{N}(dt, dz) \]
\[ = Y_t(b \, dt + \sigma \, dB_t) + \int_{\mathbb{R}\setminus\{0\}} r \, \mathcal{N}(dt, dz) \]
\[ + X_t(a \, dt + \phi \, dB_t) + \int_{\mathbb{R}\setminus\{0\}} \gamma \, \mathcal{N}(dt, dz) \]
\[ + \int_{\mathbb{R}\setminus\{0\}} \{(X_t + \gamma)(Y_t + r) - X_tY_t - Y_t - X_t\gamma\} \, (dz) \, dt \]
\[ + \int_{\mathbb{R}\setminus\{0\}} \{(X_t + \gamma)(Y_t + r) - X_tY_t - Y_t - X_t\gamma\} \, \mathcal{N}(dt, dz) \]
\[ = Y_tX_t + X_tY_t + \sigma \phi \, dt \]
\[ + \int_{\mathbb{R}\setminus\{0\}} \{(X_t + \gamma)(Y_t + r) - X_tY_t - Y_t - X_t\gamma\} \, N(dt, dz) \]
\[ = X_tY_t + X_tX_t + \sigma(t, \omega) \phi(t, \omega) \, dt + \int_{\mathbb{R}\setminus\{0\}} r(t, z, \omega) \gamma(t, z, \omega) \, N(dt, dz) \]

The proof is done. $\square$

Following the same arguments with $g(x, y) = x \cdot y$, we can derive the integration by parts for multi-dimensional Ito-Levy process. The complete proof can be found in [2][3].

**Proposition 3.2.5.** multi-dimensional integration by parts

Suppose we have two $n$-dimensional Ito-Levy processes $X_t$ and $Y_t$ with differential forms

\[ dX_t = b(t, \omega) \, dt + \sigma(t, \omega) \, dB_t + \int_{\mathbb{R}\setminus\{0\}} r(t, z, \omega) \, \mathcal{N}(dt, dz) \]
\[ dY_t = a(t,\omega)dt + \phi(t,\omega)dB_t + \int_{\mathbb{R}\setminus\{0\}} \gamma(t, z, \omega)N(dt, dz) \]

where \(a(t,\omega)\) and \(b(t,\omega)\) are \(n\) by \(1\) vector, \(\sigma(t,\omega), \phi(t,\omega), r(t, z, \omega)\) and \(\gamma(t, z, \omega)\) are \(n\) by \(n\) matrices. \(B_t\) is a Brownian motion in \(\mathbb{R}^n\) and \(N(dt, dz)\) is a compensated Poisson random measure in \(\mathbb{R}^n\). Then the real-valued process \(X_t \cdot Y_t\) is still an Ito-Levy process with dynamics

\[ d(X_t \cdot Y_t) = X_t \cdot dY_t + Y_t \cdot dX_t + \sum_{j=1}^{n} \sigma_j(t,\omega) \cdot \phi_j(t,\omega)dt \]

\[ + \sum_{j=1}^{n} \int_{\mathbb{R}\setminus\{0\}} r_j(t, z, \omega) \cdot \gamma_j(t, z, \omega)N(dt, dz) \]

One should notice that the jump term in the integration by parts formula is measured by the Poisson random measure, although it is measured by the compensated Poisson random measure in the Ito’s formula.

### 3.3 Stochastic control problem with jump diffusion

In this section, we are going to derive maximum principle for a control problem in a functional analysis approach. Conventionally, people like to use lower-case letters to represent functions in the field of functional analysis, so we will use the notation \(x(t,\omega)\) for a stochastic process instead of using \(X_t(\omega)\) only in this section.

We suppose that

\[ f(t, x, u) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \]

\[ b(t, x, u) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \]

\[ \sigma(t, x, u) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n} \]

\[ \gamma(t, x, u) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n} \]

\[ g(x) : \mathbb{R}^n \rightarrow \mathbb{R} \]
are continuous differentiable functions, and satisfy the usual conditions that we need to promise the existence and uniqueness of solution of the following stochastic control problem.

\[
\text{minimize} \quad J(u(\cdot)) = E\left[ \int_0^T f(t, x(t), u(t)) dt + g(x(T)) \right] \quad (P1)
\]

subject to

\[
\begin{align*}
\text{subject to} \quad dx &= b(t, x(t), u(t)) dt + \sigma(t, x(t), u(t)) dB_t \\
&\quad + \int_{(\mathbb{R}_0)^n} \gamma(t, z, x(t-), u(t-)) N(dt, dz) \\
x(0) &= x_0
\end{align*}
\]

The set \( \mathbb{R}_0 \) means \( \mathbb{R} \setminus \{0\} \). The stochastic differential equation is defined on the complete probability space \((\Omega, F, P)\). \( B_t \) is an \( n \) dimensional Brownian motion, \( N(dt, dz) \) is the Poisson random measure and the filtration \( \{F_t\} \) which is generated by \( B_t \) and \( N(dt, dz) \) satifies the usual conditions. Notice that \( F_t \) is the smallest \( \sigma \)-algebra which contains \( B_s \) and \( N(ds, dz) \), for all \( s \leq t \) and all \( z \) in \( \mathbb{R}_0 \). The last integral is the Levy jump term which is often used to simulate the high-frequency trading market and we always suppose that the Levy process belongs to \( L_2 \), that is \( E[\eta_t^2] < \infty \). \( x(t) = x_t = x(t, \omega) \) and \( u(t) = u_t = u(t, \omega) \), for \( \omega \in \Omega \). The stochastic differential equation in \( (P1) \) will be called the state equation. The vector \( x \) which belongs to \( \mathbb{R}^n \) will be called the state variable, and the function \( x(t) \) which satisfies the state equation will be called the trajectory corresponding to \( u(t) \), and \( x_0 \) is a given vector in \( \mathbb{R}^n \). On the other hand, the vector \( u \) is called the control variable, and the non-empty, closed, convex set \( K \) is the control constraint. We suppose that the state equation has one and only one solution when \( u(t) \) is given. All the state variable and controls in this thesis are predictable process, and it can be shown that their perturbations will be also predictable[4]. Denote the optimal solution pair for \( (P1) \) is \((u^*(t), x^*(t))\). The state equation can also be written as the form

\[
\begin{align*}
\text{dx} &= b(t, x(t), u(t)) dt + \sum_{j=1}^{n} \sigma^j(t, x(t), u(t)) dB_j \\
&\quad + \sum_{j=1}^{n} \int_{\mathbb{R}_0} \gamma^j(t, z, x(t-), u(t-)) N_j(dt, dz) \\
x(0) &= x_0
\end{align*}
\]

where \( \sigma^j \) is the \( j \)th column of \( \sigma \), \( \gamma^j \) is the \( j \)th column of \( \gamma \), \( B_j \) is the \( j \)th element of \( B_t \), and \( N_j(dt, dz) \) is the \( j \)th element of \( N(dt, dz) \).
3.4 Necessary conditions for jump diffusion problem

For simplicity, we will sometimes ignore the arguments and use $x^*$ or $x$ to denote the function $x^*(t)$ or $x(t)$. We will also use $u^*$ or $u$ to denote the function $u^*(t)$ or $u(t)$.

We begin with the necessary conditions. First of all, we make perturbations on the optimal control $u^*$. Let

$$u_\theta = u^* + \theta v$$

where $v = v(t)$ is an admissible control. According to our assumptions in Section 3, we can find one and only one $x_\theta$ which is the trajectory corresponding to $u_\theta$. That is

$$dx_\theta = b(t, x_\theta(t), u_\theta(t))dt + \sigma(t, x_\theta(t), u_\theta(t))dB_t$$
$$+ \int_{(\mathbb{R}_0)^n} \gamma(t, z, x_\theta(t^-), u_\theta(t^-))N(dt, dz)$$
$$x_\theta(0) = x_0$$

In order to discuss the change in $J$ corresponding to the perturbation in $u^*$, we need to discuss the change in the dynamics first. By the mean value theorem, we have the following
\[ x_\theta(t) - x^*(t) = x_0 + \int_0^t b(s, x_\theta(s), u_\theta(s))ds + \int_0^t \sigma(s, x_\theta(s), u_\theta(s))dB_s \\
+ \int_0^t \int_{(\mathbb{R})^n} \gamma(s, z, x_\theta(s-), u_\theta(s-))\mathcal{N}(ds, dz) \\
- x_0 - \int_0^t b(s, x^*(s), u^*(s))ds - \int_0^t \sigma(s, x^*(s), u^*(s))dB_s \\
- \int_0^t \int_{(\mathbb{R})^n} \gamma(s, z, x^*(s-), u^*(s-))\mathcal{N}(ds, dz) \\
= \int_0^t b(s, x_\theta(s), u_\theta(s)) - b(s, x^*(s), u^*(s))ds \\
+ \int_0^t \sigma(s, x_\theta(s), u_\theta(s)) - \sigma(s, x^*(s), u^*(s))dB_s \\
+ \int_0^t \int_{(\mathbb{R})^n} \gamma(s, z, x_\theta(s-), u_\theta(s-)) - \gamma(s, z, x^*(s-), u^*(s-))\mathcal{N}(ds, dz) \\
= \int_0^t \int_0^1 \{b_\lambda(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*))(x_\theta - x^*) \\
+ b_\lambda(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*))(u_\theta - u^*)\} d\lambda ds \\
+ \sum_{j=1}^n \int_0^t \int_0^1 \{\sigma^j_\lambda(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*))(x_\theta - x^*) \\
+ \sigma^j_\lambda(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*))(u_\theta - u^*)\} d\lambda dB_j \\
+ \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0^n} \int_0^1 \{\gamma^j_\lambda(s, z, x^*(s-)+\lambda(x_\theta(s-)-x^*(s-)), \\
\gamma^j_\lambda(s, z, x^*(s-)+\lambda(x_\theta(s-)-x^*(s-))), u^*(s-)+\lambda(u_\theta(s-)-u^*(s-)))(x_\theta(s-)-x^*(s-)) \\
+ \gamma^j_\lambda(s, z, x^*(s-)+\lambda(x_\theta(s-)-x^*(s-)), u^*(s-)+\lambda(u_\theta(s-)-u^*(s-)))(u_\theta(s-)-u^*(s-))\} d\lambda \mathcal{N}_j(ds, dz_j) \]

Then, we have
Then, \[ \frac{x_\theta(t) - x^*(t)}{\theta} = \int_0^t \int_0^1 \{b_x(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) \} \frac{x_\theta(s) - x^*(s)}{\theta} + b_u(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*))v \} \, d\lambda \, ds \\
+ \sum_{j=1}^n \int_0^t \int_0^1 \{\sigma^j_x(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) \} \frac{x_\theta(s) - x^*(s)}{\theta} \\
+ \sigma^j_u(s, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*))v \} \, d\lambda \, dB_j \\
+ \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \{\gamma^j_\theta(s, z, x^*(s-)) + \lambda(x_\theta(s-)) - x^*(s-)) \} \frac{x_\theta(s-)) - x^*(s-))}{\theta} \\
+ \gamma^j_u(s, z, x^*(s-)) + \lambda(x_\theta(s-)) - x^*(s-)) , \\
\sum_{j=1}^n \{\gamma^j_\theta(s, z, x^*(s-)) + \lambda(x_\theta(s-)) - x^*(s-)) \} \, d\lambda \, \mathcal{N}_j(ds, dz_j) \]

where \( b_x, b_u, \sigma_x, \sigma_u, \gamma_x, \) and \( \gamma_u \) represent the partial derivative of \( b, \sigma, \) and \( \gamma \) with respect to \( x \) and \( u, \) etc..

The following Lemma is used to determine the limit of \( \frac{x_\theta(t) - x^*(t)}{\theta} \) as \( \theta \) tends to zero.

**Lemma 3.4.1.** We define a function \( Z(t) = Z(t, \omega) \) by the following stochastic differential equation

\[
dZ = \{b_x(t, x^*(t), u^*(t))Z(t) + b_u(t, x^*(t), u^*(t))v(t)\}dt \\
+ \sum_{j=1}^n \{\sigma^j_x(t, x^*(t), u^*(t))Z(t) + \sigma^j_u(t, x^*(t), u^*(t))v(t)\}dB_j \\
+ \sum_{j=1}^n \int_{\mathbb{R}_0} \{\gamma^j_\theta(t, z, x^*(t-)) + \gamma^j_u(t, z, x^*(t-)) \} \mathcal{N}_j(dt, dz_j) \]

\( Z(0) = 0 \)

Then, \( \frac{x_\theta(t) - x^*(t)}{\theta} \rightarrow Z(t) \) in \( L^2 \)
Proof:

Consider the following functions

\[
x_\theta(t) = x_0 + \int_0^t b(s, x_\theta(s), u_\theta(s)) \, ds + \sum_{j=1}^n \int_0^t \sigma^j(s, x_\theta(s), u_\theta(s)) \, dB_j + \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \gamma^j(s, z, x_\theta(s), u_\theta(s)) \mathcal{N}_j(ds, dz)
\]

\[
x^*(t) = x_0 + \int_0^t b(s, x^*(s), u^*(s)) \, ds + \sum_{j=1}^n \int_0^t \sigma^j(s, x^*(s), u^*(s)) \, dB_j + \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \gamma^j(s, z, x^*(s), u^*(s)) \mathcal{N}_j(ds, dz)
\]

\[
Z(t) = \int_0^t b_x(s, x^*(s), u^*(s))Z(s) + b_u(s, x^*(s), u^*)v(s)ds + \sum_{j=1}^n \int_0^t \sigma^j_x(s, x^*(s), u^*(s))Z(s) + \sigma^j_u(s, x^*(s), u^*)v(s) \, dB_j + \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \{\gamma^j_x(s, z, x^*(s), u^*(s))Z(s) - \gamma^j_u(s, z, x^*(s), u^*(s))v(s)\} \mathcal{N}_j(ds, dz)
\]

and let \( y_\theta = \frac{x_\theta - x^*}{\theta} - Z \)

Then, we have
\[
\begin{align*}
\ dy_\theta &= \left\{ \frac{b(t, x_\theta(t), u_\theta(t)) - b(t, x^*(t), u^*(t))}{\theta} \right. \\
& \quad - b_x(t, x^*(t), u^*(t))Z(t) - b_u(t, x^*(t), u^*(t))v(t) \right\} \ dt \\
& + \sum_{j=1}^{n} \left\{ \frac{\sigma^j(t, x_\theta(t), u_\theta(t)) - \sigma^j(t, x^*(t), u^*(t))}{\theta} \right. \\
& \quad - \sigma^j_x(t, x^*(t), u^*(t))Z(t) - \sigma^j_u(t, x^*(t), u^*(t))v(t) \right\} dB_j \\
& + \sum_{j=1}^{n} \int_{\mathbb{R}_0} \left\{ \frac{\gamma^j(t, z, x_\theta(t-), u_\theta(t-)) - \gamma^j(t, z, x^*(t-), u^*(t-))}{\theta} \right. \\
& \quad - \gamma^j_x(t, z, x^*(t-), u^*(t-))Z(t-)-\gamma^j_u(t, z, x^*(t-), u^*(t-))v(t) \right\} \mathcal{N}_j(dt, dz_j) \\
& = \{ A_1(t)y_\theta(t) + A_2(t) \} dt + \sum_{j=1}^{n} \{ A_{3j}(t)y_\theta(t) + A_{4j}(t) \} dB_j \\
& + \left. \int_{\mathbb{R}_0} \sum_{j=1}^{n} \{ A_{5j}(t-, z)y_\theta(t-) + A_{6j}(t-, z) \} \mathcal{N}_j(dt, dz_j) \right|
\end{align*}
\]

where
\[
\begin{align*}
A_1(t) &= \int_{0}^{1} \{ A_1(t) - b_x(t, x^*(t), u^*(t)) \} Z(t) \\
A_2(t) &= \left\{ A_1(t) - b_x(t, x^*(t), u^*(t)) \right\} \left\{ A_1(t) - b_x(t, x^*(t), u^*(t)) \right\} \\
A_{3j}(t) &= \int_{0}^{1} \{ A_{3j}(t) - \sigma^j_x(t, x^*(t), u^*(t)) \} \left\{ A_{3j}(t) - \sigma^j_x(t, x^*(t), u^*(t)) \right\} v(t) dt \\
A_{4j}(t) &= \left\{ A_{3j}(t) - \sigma^j_x(t, x^*(t), u^*(t)) \right\} \left\{ A_{3j}(t) - \sigma^j_x(t, x^*(t), u^*(t)) \right\} \\
A_{5j}(t, z) &= \int_{0}^{1} \{ A_{5j}(t, z) - \gamma^j_x(t, z, x^*(t), u^*(t)) \} \left\{ A_{5j}(t, z) - \gamma^j_x(t, z, x^*(t), u^*(t)) \right\} v(t) dt \\
A_{6j}(t, z) &= \left\{ A_{5j}(t, z) - \gamma^j_x(t, z, x^*(t), u^*(t)) \right\} \left\{ A_{5j}(t, z) - \gamma^j_x(t, z, x^*(t), u^*(t)) \right\} \\
\end{align*}
\]
By Ito’s formula, we have

\[
d(y_\theta^2(t)) = 2y_\theta(t) \cdot \{[A_1(t)y_\theta(t) + A_2(t)]dt + \sum_{j=1}^{n}[A_{3j}(t)y_\theta(t) + A_{4j}(t)]dB_j + \int_{R_0} \sum_{j=1}^{n}[A_{5j}(t^-)y_\theta(t) + A_{6j}(t^-, z)]N_j(dt, dz) + n\sum_{j=1}^{n}\left[A_{3j}(t)y_\theta(t) + A_{4j}(t)\right]dt + \int_{R_0} \sum_{j=1}^{n}[A_{5j}(t, z)y_\theta(t) + A_{6j}(t, z)]N_j(dt, dz)\}
\]

Therefore, we obtain

\[
E[|y_\theta(t)|^2] = E\left[\int_0^t \left\{2y_\theta(s) \cdot (A_1(s)y_\theta(s) + A_2(s)) + \sum_{j=1}^{n}|A_{3j}(s)y_\theta(s) + A_{4j}(s)|^2\right\}ds\right] + E\left[\int_0^t \int_{R_0} \sum_{j=1}^{n}|A_{5j}(s, z)y_\theta(t) + A_{6j}(s, z)|^2v_j(dz)ds\right] \\
\leq E\left[\int_0^T \left\{2|y_\theta(t)| \cdot (|A_1(t)||y_\theta(t)| + |A_2(t)|) + \sum_{j=1}^{n}(|A_{3j}(t)||y_\theta(t)| + |A_{4j}(t)|)^2\right\}dt\right] + E\left[\int_0^T \int_{R_0} \sum_{j=1}^{n}(|A_{5j}(t, z)||y_\theta(t)| + |A_{6j}(t, z)|)^2v_j(dz)dt\right]
\]

Assume that all the partial derivatives of \(b\), \(\sigma^j\), and \(\gamma^j\) are bounded. Then, we can find a constant \(M\), such that

\[
E[|y_\theta(t)|^2] \leq M\int_0^T E[|y_\theta(t)|^2]dt + o(\theta)
\]

By Gronwall’s inequality, we have \(y_\theta \rightarrow 0\) in \(L^2\). That is,

\[
\frac{x_\theta - x^*}{\theta} \rightarrow Z \quad \text{in} \quad L^2
\]

The proof is done. □
Now, we are ready to consider the change in $J$ as a result of the perturbation in the optimal control $u^*$. 

\[
J(u_\theta) - J(u^*) \\
= E[ g(x_\theta(T)) - g(x^*(T)) \\
+ \int_0^T f(t, x_\theta(t), u_\theta(t)) - f(t, x^*(t), u^*(t)) dt ] \\
= E[ \int_0^1 g_x(x^*(T) + \lambda(x_\theta(T) - x^*(T))) \cdot (x_\theta(T) - x^*(T)) \cdot \lambda d\lambda \\
+ \int_0^T \int_0^1 f_x(t, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) \cdot (x_\theta - x^*) \cdot \lambda d\lambda dt \\
+ \int_0^T \int_0^1 f_u(t, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) \cdot (u_\theta - u^*) \cdot \lambda d\lambda dt ]
\]

Then we divide both sides by $\theta$

\[
\frac{J(u_\theta) - J(u^*)}{\theta} \\
= E[ \int_0^1 g_x(x^*(T) + \lambda(x_\theta(T) - x^*(T))) \cdot \frac{x_\theta(T) - x^*(T)}{\theta} \lambda d\lambda \\
+ \int_0^T \int_0^1 f_x(t, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) \cdot \frac{x_\theta(t) - x^*(t)}{\theta} \lambda d\lambda dt \\
+ \int_0^T \int_0^1 f_u(t, x^* + \lambda(x_\theta - x^*), u^* + \lambda(u_\theta - u^*)) \cdot (u_\theta - u^*) \cdot \lambda(t) d\lambda dt ]
\]

Letting $\theta$ tend to 0, we obtain the Gateaux derivative of $J$ along the direction $v$.

**Proposition 3.4.2.** The Gateaux derivative of $J$ along the direction $v$ is

\[
dJ(u^*; v) = E[g_x(x^*(T)) \cdot Z(T) + \int_0^T f_x \cdot Z + f_u \cdot v dt]
\]

Our next step is to define dual variables and equation of the control problem. Consider the equation
\[ d\zeta(t) = \{b_x(t, x^*(t), u^*(t))\zeta(t) + \Phi(t)\}dt + \sum_{j=1}^{n} \{\sigma^j_x(t, x^*(t), u^*(t))\zeta(t) + \Psi^j(t)\}dB_j \]
\[ + \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{\gamma^j_x(t, z, x^*(t), u^*(t))\zeta(t) + \Theta^j(t, z)\}N_j(dt, dz_j) \]
\[ \zeta(0) = 0 \]

where \( \Phi \) and \( \Psi^j \) are functions in \( L^2_F([0, T]; \mathbb{R}^n) \), and \( \Theta^j \) are functions in \( L^2_F([0, T] \times (\mathbb{R}_0)^n; \mathbb{R}^n) \), for all \( j \). And we suppose that for fixed \( \Phi, \Psi^j, \) and \( \Theta^j \), we can find one and only one \( \zeta \) satisfies the above stochastic differential equation. Then, for fixed \( \Phi, \Psi^j, \) and \( \Theta^j \) we can consider the quantity
\[
E[\int_0^T f_x \cdot \zeta(t) dt + g_x(x^*(T)) \cdot \zeta(T)]
\]

This map is linear and bounded. By Riesz representation theorem, we can find functions \( p(t) \) and \( q^j(t) \) in \( L^2_F([0, T]; \mathbb{R}^n) \), and \( r^j(t, z) \) in \( L^2_F([0, T] \times (\mathbb{R}_0)^n; \mathbb{R}^n) \), such that for all \( \Phi \) and \( \Psi^j \) in \( L^2_F([0, T]; \mathbb{R}^n) \), and \( \Theta^j \) in \( L^2_F([0, T] \times (\mathbb{R}_0)^n; \mathbb{R}^n) \), we have
\[
E[\int_0^T p(t) \cdot \Phi(t) dt + \sum_{j=1}^{n} \int_0^T q^j(t) \cdot \Psi^j(t) dt + \sum_{j=1}^{n} \int_{\mathbb{R}_0} r^j(t, z) \cdot \Theta^j(t, z) v_j(dz_j) dt]
\]
\[ = E[\int_0^T f_x(t) \cdot \zeta(t) dt + g_x(x^*(T)) \cdot \zeta(T)] \]

The \( p(t) \), \( q^j(t) \) and \( r^j(t, z) \) will be called the dual variables or adjoint variables. On the other hand, we define the Hamiltonian
\[
H(t, x, u, p, q, r) = f(t, x, u) + b(t, x, u) \cdot p + \sum_{j=1}^{n} \sigma^j(t, x, u) \cdot q^j + \sum_{j=1}^{n} \int_{\mathbb{R}_0} \gamma^j(t, z, x, u) \cdot r^j v_j(dz_j)
\]
or it can be written in this form

\[
H(t, x, u, p, q, r) = f + p^* b + \text{trace}(q^* \sigma) \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \gamma^j(t, z, x, u) \cdot r^j v_j(dz_j)
\]

where \( q = [q_1, q_2, q_3, ..., q^n] \), \( r = [r_1, r_2, r_3, ..., r^n] \). We conclude

\[
dJ(u^*; v) = E[\int_0^T f_x \cdot Z + g_x(x^*(T)) \cdot Z(T) + f_u \cdot v \, dt]
\]  
\[
= E[\int_0^T \{ p \cdot (b_u v) + \sum_{j=1}^{n} q^j \cdot (\sigma^j_u v) \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} r^j \cdot (\gamma^j_u v) v_j(dz_j) + f_u \cdot v \} \, dt]
\]  
\[
= E[\int_0^T v^* \{ b^*_u p + \sum_{j=1}^{n} \sigma^j_u^* q^j \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \gamma^j_u^* r^j v_j(dz_j) + f_u \} \, dt]
\]  
\[
= E[\int_0^T v(t)^* H_u(t, x^*(t), u^*(t), p(t), q(t), r(t, z)) \, dt]
\]  
\[
= E[\int_0^T H_u(t, x^*(t), u^*(t), p(t), q(t), r(t, z)) \cdot v(t) \, dt]
\]

**Proposition 3.4.3.** For (P1), we have the maximum principle

\[
H_u(t, x^*(t), u^*(t), p(t), q(t), r(t, z)) \cdot (v - u^*(t)) \geq 0
\]

for all \( v \in K \) and almost everywhere \( t \) in \([0, T] \) and \( w \) in \( \Omega \).

**Proof:**
Since \((u^*,x^*)\) is the optimal pair for \((P1)\), we know the Gateaux derivative of \(J(u(\cdot))\) at \(u^*(\cdot)\) along the direction \(v(\cdot) - u^*(\cdot)\) is non-negative. That is,

\[
E\left[ \int_0^T H_u(t, x^*(t), u^*(t), p(t), q(t), r(t, z)) \cdot (v(t) - u^*(t)) \, dt \right] = dJ(u^*(\cdot); v(\cdot) - u^*(\cdot)) \geq 0
\]

Let \(v\) be any fixed vector in \(K\) and the set \(A = \{(t, \omega)|H_u(t, x^*(t), u^*(t), p(t), q(t), r(t, z)) \cdot (v - u^*(t)) < 0\}\). We want to show that the measure of set \(A\) is zero. That is, \(m(A)=0\). Define the function

\[
\bar{v}(t, \omega) = \begin{cases} v, & (t, \omega) \in A \\ u^*(t, \omega), & \text{otherwise} \end{cases}
\]

Fix a Lebesgue point \(t\) in \([0,T]\) and let \(A_t = \{w \in \Omega|H_u(t, x^*(t), u^*(t), p(t), q(t)) \cdot (v - u^*(t)) < 0\}\). Then the function becomes

\[
\bar{v}(t, \omega) = \begin{cases} v, & w \in A_t \\ u^*(t, \omega), & \text{otherwise} \end{cases}
\]

Clearly, we can see \(\bar{v}(t, \omega)\) is measurable with respect to \(F_t\) in the second form, and we know that \(A_t\) is in \(F_t\). Moreover, since

\[
A = \bigcup_{t \in [0,T]} A_t
\]

We know the function \(\bar{v}(t, \omega)\) is adapted with respect to the filtration \(\{F_t\}\). On the other hand, by the definition of \(|\bar{v}|\), we have

\[
|\bar{v}| < \infty
\]

So, \(\bar{v}(\cdot)\) belongs to \(L^2\) and it is an admissible control. Then, we have
\[
E \left[ \int_0^T H_u(t, x^*(t), u^*(t), p(t), q(t), r(t, z)) \cdot (v - u^*(t)) \, dt \right]
\]
\[
= \int_\Omega \int_A H_u(t, x^*(t), u^*(t), p(t), q(t), r(t, z)) \cdot (\bar{\nu}(t) - u^*(t)) \, dtdP
\]
\[\geq 0\]

and
\[
H_u(t, x^*(t), u^*(t), p(t), q(t), r(t, z)) \cdot (\bar{\nu}(t) - u^*(t)) < 0 \quad \text{on} \quad A
\]

So, we conclude that \( m(A) = 0 \). That is,
\[
H_u(t, x^*(t), u^*(t), p(t), q(t), r(t, z)) \cdot (v - u^*(t)) \geq 0
\]

for a.e \( t \in [0, T] \) and \( w \in \Omega \), and all \( v \in K \)

The proof is done. \( \square \)

The next Proposition is another way to characterize the adjoint variables.

**Proposition 3.4.4.** \( p(t), q^j(t), \) and \( r^j(t, z) \) are the adjoint variables we defined by Riesz representation theorem, if and only if they satisfy the costate equation

\[
dp = -H_x(t, x^*, u^*, p, q, r) dt + q \, dB_t + \int_{(\mathbb{R}_0)^n} r \, N(dt, dz)
\]
\[
= -\left\{ f_x(t, x, u) + b_x(t, x, u)^* p + \sum_{j=1}^{n} \sigma^j_x(t, x, u)^* q^j \right\} dt \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \gamma^j_x(t, z, x, u)^* \, r^j v_j(dz_j) dt \\
+ \sum_{j=1}^{n} q^j dB_j + \sum_{j=1}^{n} \int_{\mathbb{R}_0} r^j N_j(dt, dz_j)
\]
\[
p(T) = g_x(x^*(T))
\]
Proof:

Firstly, we suppose \( p(t), q^j(t), \) and \( r^j(t, z) \) are the solutions of the costate equation and prove that they satisfy the Riesz representation theorem. According to integration by parts, we have

\[
d(p(t) \cdot \zeta(t)) = dp \cdot \zeta + p \cdot d\zeta + \sum_{j=1}^{n} q^j \cdot \{ \sigma^j \zeta + \Psi^j \} dt
\]

\[
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} r^j \cdot \{ \gamma^j \zeta + \Theta^j \} N_j(dt, dz_j)
\]

\[
= \{-[f_x + b^*_x p + \sum_{j=1}^{n} \sigma^j x^j + \sum_{j=1}^{n} \int_{\mathbb{R}_0} \gamma^j x^j v_j (dz_j)] dt
\]

\[
+ \sum_{j=1}^{n} q^j dB_j + \sum_{j=1}^{n} \int_{\mathbb{R}_0} r^j N_j(dt, dz_j) \} \cdot \zeta
\]

\[
+p \cdot \{ [b_x \zeta + \Phi] dt + \sum_{j=1}^{n} [\sigma^j x^j + \Psi^j] dB_j
\]

\[
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} [\gamma^j \zeta + \Theta^j] N_j(dt, dz_j)
\]

\[
+ \sum_{j=1}^{n} q^j \cdot \{ \sigma^j \zeta + \Psi^j \} dt + \sum_{j=1}^{n} \int_{\mathbb{R}_0} r^j \cdot \{ \gamma^j \zeta + \Theta^j \} N_j(dt, dz_j)
\]

Then,

\[
E[p(T) \cdot \zeta(T)] = p(0) \cdot \zeta(0) + E[\int_{0}^{T} p \cdot \Phi dt - \int_{0}^{T} f_x \cdot \zeta dt
\]

\[
+ \sum_{j=1}^{n} \int_{0}^{T} q^j \cdot \Psi^j dt + \sum_{j=1}^{n} \int_{\mathbb{R}_0} r^j \cdot \Theta^j v_j (dz_j) dt]
\]

Since \( \zeta(0) = 0 \) and \( p(T) = g_x(x^*(T)) \), we have
\[ E \int_0^T p \cdot \Phi \, dt + \sum_{j=1}^n \int_0^T q^j \cdot \Psi^j \, dt + \sum_{j=1}^n \int_{\mathbb{R}_0} r^j \cdot \Theta^j v_j(dz_j) \, dt \]

\[ = E \left[ \int_0^T f_x \cdot \zeta \, dt + g_x(x^*(T)) \cdot \zeta(T) \right] \]

which is the Riesz representation.

Secondly, let’s suppose \( p(t), q^j(t), \) and \( r^j(t, z) \) are given by the Riesz representation theorem, and prove that they must satisfy the costate equation. Consider the matrix equation

\[
\begin{align*}
\frac{dM(t)}{dt} &= b_x(t, x^*, u^*) M(t) \, dt + \sum_{j=1}^n \sigma^j_x(t, x^*, u^*) M(t) \, dB_j \\
&\quad + \sum_{j=1}^n \int_{\mathbb{R}_0} \gamma^j_x(t, x^*, u^*) M(t) N_j(dt, dz_j) \\
M(0) &= I
\end{align*}
\]

where I is the identity matrix. The above equation is called the fundamental equation of the equation corresponding \( \zeta(t) \) and \( M(t) \) is called the fundamental solution. It can be shown that the fundamental equation has exactly one solution, that is, the \( M(t) \) exists and unique [1][4][5].

In order to construct the inverse of \( M \), we consider another matrix equation

\[
\begin{align*}
\frac{dW(t)}{dt} &= \alpha(t, \omega) \, dt + \sum_{j=1}^n \beta^j(t, \omega) dB_j + \sum_{j=1}^n \int_{\mathbb{R}_0} \phi^j(t, z, \omega) N_j(dt, dz_j) \\
W(0) &= I
\end{align*}
\]

We need to find appropriate \( \alpha(t, \omega), \beta^j(t, \omega), \) and \( \phi^j(t, z, \omega) \) such that \( M(t) W(t) = I \), for all \( t \) in \( [0,T] \). By Ito’s formula.
\[ d(W(t)M(t)) = dW(t)M(t) + W(t)dM(t) + \sum_{j=1}^{n} \beta^j(t, \omega) \sigma^j_x(t, x^*, u^*) M(t) dt \]

\[ + \sum_{j=1}^{n} \int_{\mathbb{R}_0} \phi^j(t, z) \gamma^j_x(t, x^*, u^*) M(t) N_j(dt, dz_j) \]

\[ = \alpha(t, \omega) M(t) dt + \sum_{j=1}^{n} \beta^j(t, \omega) M(t) dB_j \]

\[ + \sum_{j=1}^{n} \int_{\mathbb{R}_0} \phi^j(t, z) M(t) \overline{N}_j(dt, dz_j) + W(t)b_x(t, x^*, u^*) M(t) dt \]

\[ + \sum_{j=1}^{n} W(t) \sigma^j_x(t, x^*, u^*) M(t) dB_j + \sum_{j=1}^{n} \int_{\mathbb{R}_0} W(t) \gamma^j_x(t, x^*, u^*) M(t) \overline{N}_j(dt, dz_j) \]

\[ + \sum_{j=1}^{n} \beta^j(t, \omega) \sigma^j_x(t, x^*, u^*) M(t) dt + \sum_{j=1}^{n} \int_{\mathbb{R}_0} \phi^j(t, z) \gamma^j_x(t, x^*, u^*) M(t) N_j(dt, dz_j) \]

\[ = \{ \alpha(t, \omega) M(t) + W(t)b_x(t, x^*, u^*) M(t) + \sum_{j=1}^{n} \beta^j(t, \omega) \sigma^j_x(t, x^*, u^*) M(t) \} dt \]

\[ + \sum_{j=1}^{n} \{ \beta^j(t, \omega) M(t) + W(t) \sigma^j_x(t, x^*, u^*) M(t) \} dB_j \]

\[ + \sum_{j=1}^{n} \{ \int_{\mathbb{R}_0} \phi^j(t, z) M(t) + W(t) \gamma^j_x(t, x^*, u^*) M(t) \} \overline{N}_j(dt, dz_j) \]

\[ + \sum_{j=1}^{n} \int_{\mathbb{R}_0} \phi^j(t, z) \gamma^j_x(t, x^*, u^*) M(t) N_j(dt, dz_j) \]
\[ d(W(t)M(t)) = \{ \alpha(t, \omega)M(t) + W(t)b_x(t, x^*, u^*)M(t) + \sum_{j=1}^{n} \beta^j(t, \omega)\sigma^j(t, x^*, u^*)M(t) \\
- \sum_{j=1}^{n} \int_{\mathbb{R}_0} \phi^j(t, z)M(t) + W(t)\gamma^j(t, x^*, u^*)M(t)v_j(dz_j) \}dt \\
+ \sum_{j=1}^{n} \{ \beta^j(t, \omega)M(t) + W(t)\sigma^j(t, x^*, u^*)M(t) \}dB_j \\
+ \sum_{j=1}^{n} \{ \int_{\mathbb{R}_0} \phi^j(t, x^*, u^*)M(t) \\
+ W(t)\gamma^j(t, x^*, u^*)M(t) + \phi^j(t, z)\gamma^j(t, x^*, u^*)M(t) \}N_j(dt, dz_j) \]

We choose

\[ \alpha(t, \omega) = -W(t)b_x(t, x^*, u^*) + \sum_{j=1}^{n} W(t)\sigma^j(t, x^*, u^*)\sigma^j(t, x^*, u^*) \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{ -W(t)\gamma^j(t, x^*, u^*)(I + \gamma^j(t, x^*, u^*))^{-1} + W(t)\gamma^j(t, x^*, u^*) \}v_j(dz_j) \]

\[ \beta^j(t, \omega) = -W(t)\sigma^j(t, x^*, u^*) \]

\[ \phi^j(t, z) = -W(t)\gamma^j(t, x^*, u^*)(I + \gamma^j(t, x^*, u^*))^{-1} \]

Then,

\[ d(W(t)M(t)) = 0 \]

Thus, we have \( M(t)W(t)=I \), for all \( t \) in \([0, T]\). Let \( \Psi^j = 0 \) and \( \Theta^j = 0 \) in the equation of \( \zeta(t) \).

We have

\[ d\zeta(t) = \{ b_x(t, x^*(t), u^*(t))\zeta(t) + \Phi(t) \}dt + \sum_{j=1}^{n} \sigma^j(t, x^*(t), u^*(t))\zeta(t)dB_j \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \gamma^j(t, x^*, u^*)\zeta(t)N_j(dt, dz_j) \]
Then,

\[
d(W(t)\zeta(t)) = W(t)d\zeta(t) + dW(t)\zeta(t) - \sum_{j=1}^{n} W(t)\sigma_{2x}^j(t, x^*, u^*)\sigma_{2x}^j(t, x^*, u^*)\zeta(t)dt
\]

\[
- \sum_{j=1}^{n} \int_{\mathbb{R}_0} W(t)\gamma_{2x}(t, x^*, u^*)(I + \gamma_{2x}(t, x^*, u^*))^{-1}\gamma_{2x}(t, x^*, u^*)\zeta(t)N_j(dt, dz_j)
\]

\[
= W\{(b_x\zeta + \Phi)dt + \sum_{j=1}^{n} \sigma_{2x}^j dB_j + \sum_{j=1}^{n} \int_{\mathbb{R}_0} \gamma_{2x}^j N_j(dt, dz_j)\}
\]

\[
+ \{-W b_x + \sum_{j=1}^{n} W\sigma_{2x}^j + \sum_{j=1}^{n} \int_{\mathbb{R}_0} -W\gamma_{2x}(I + \gamma_{2x})^{-1} + W\gamma_{2x} v_j(dz_j)\}dt
\]

\[
- \sum_{j=1}^{n} W\sigma_{2x}^j dB_j - \sum_{j=1}^{n} \int_{\mathbb{R}_0} W\gamma_{2x}(I + \gamma_{2x})^{-1}N_j(dt, dz_j)\zeta
\]

\[
- \sum_{j=1}^{n} W\sigma_{2x}^j dB_j - \sum_{j=1}^{n} \int_{\mathbb{R}_0} W\gamma_{2x}(I + \gamma_{2x})^{-1}N_j(dt, dz_j)\zeta
\]

\[
= W\Phi dt + \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{W\gamma_{2x}^j \zeta - W\gamma_{2x}^j (I + \gamma_{2x})^{-1}\zeta\} (N_j(dt, dz_j) - v_j(dz_j)dt)
\]

\[
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{-W\gamma_{2x}(I + \gamma_{2x})^{-1} + W\gamma_{2x}\} \zeta v_j(dz_j)dt
\]

\[
- \sum_{j=1}^{n} \int_{\mathbb{R}_0} W\gamma_{2x}(I + \gamma_{2x})^{-1}N_j(dt, dz_j)
\]

\[
= W(t)\Phi(t)dt
\]

which implies

\[
W(t)\zeta(t) = \int_{0}^{t} W(s)\Phi(s)ds
\]

and

\[
\zeta(t) = M(t) \int_{0}^{t} W(s)\Phi(s)ds
\]
According to Ito’s formula, we have

\[
E\left[\int_0^T p(t) \cdot \Phi(t) dt\right] = E\left[\int_0^T f_x(t, x^*, u^*) \cdot \zeta(t) dt + g_x(x^*(T)) \cdot \zeta(T)\right]
\]

\[
= E\left[\int_0^T f_x(t, x^*, u^*) \cdot \{M(t) \int_0^t W(s) \Phi(s) ds\} dtight. \\
\vdots
\]

This is true for all \(\Phi(t) \in L^2_{\mathbb{F}}\). We obtain the explicit form of \(p(t)\)

\[
p(t) = W(t)^* \{ - \int_0^t M(s)^* f_x(s, x^*, u^*) ds \\
+ E\left[\int_0^T M(s)^* f_x(s, x^*, u^*) ds + M(T)^* g_x(x^*(T)) | F_t\right] \}
\]

By letting \(t=T\), it gives the terminal condition

\[
p(T) = g_x(x^*(T))
\]

Notice that the process \(E[\int_0^T M(s)^* f_x(s, x^*, u^*) ds + M(T)^* g_x(x^*(T)) | F_t]\) is a martingale. By
martingale representation theorem, we have

\[
E\left[\int_0^T M(s)^* f_x(s, x^*, u^*) ds + M(T)^* g_x(x^*(T)) \mid F_t\right] = E\left[\int_0^T M(s)^* f_x(s, x^*, u^*) ds + M(T)^* g_x(x^*(T)) \right] + \sum_{j=1}^n \int_0^t G_j(s, \omega) dB_j
\]

where \(G_j \in L^2_F(0, T; \mathbb{R}^n)\) is uniquely defined. Then,

\[
p(t) = W(t)^* \left\{- \int_0^t M(s)^* f_x(s, x^*, u^*) ds + \sum_{j=1}^n \int_0^t G_j(s, \omega) dB_j + E\left[\int_0^T M(s)^* f_x(s, x^*, u^*) ds + M(T)^* g_x(x^*(T))\right]\right\}
\]

Now, it is time to simplify \(p(t)\) to the form we want. We have

\[
p(t) = W(t)^* \eta(t)
\]

where

\[
dx(t) = -M(t)^* f_x(t, x^*, u^*) dt + \sum_{j=1}^n G_j(t, \omega) dB_j
\]

\[
\eta(0) = E\left[\int_0^T M(s)^* f_x(s, x^*, u^*) ds + M(T)^* g_x(x^*(T))\right]
\]

and

\[
dW(t)^* = \alpha(t, \omega)^* dt + \sum_{j=1}^n \beta_j(t, \omega)^* dB_j + \sum_{j=1}^n \int_{\mathbb{R}_0} \phi_j(t, z, \omega)^* N_j(dt, dz)
\]

\[
W(0)^* = I
\]
\[ \alpha^* = -b^*_x W^* + \sum_{j=1}^{n} \sigma_x^j \sigma_x^* W^* + \sum_{j=1}^{n} \int_{R_0} \{-(I + \gamma_j^*)^{-1} \gamma_j^* W^* + \gamma_j^* W^*) \} v_j(dz_j) \]

\[ \beta^j = -\sigma_x^j W^* \]

\[ \phi^j = -(I + \gamma_j^*)^{-1} \gamma_j^* W^* \]

By Ito’s formula, we have
\[
dp(t) = d(W(t)^*\eta(t)) \\
= dW(t)^*\eta(t) + W(t)^*d\eta(t) - \sum_{j=1}^{n} \sigma_j^j(t, x^*, u^*) W(t)^*G_j(t, \omega) dt \\
= \{[-b_x^*W^* + \sum_{j=1}^{n} \sigma_x^j \sigma_x^j W^* + \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{(I + \gamma_x^j)^{-*} \gamma_x^j W^* + \gamma_x^j W^*\}v_j(dz_j)]dt \\
- \sum_{j=1}^{n} \sigma_x^j W^* dB_j - \sum_{j=1}^{n} \int_{\mathbb{R}_0} (I + \gamma_x^j)^{-*} \gamma_x^j W^* \mathcal{N}_j(dt, dz_j)\} \eta \\
W^*\{M^* f_x dt + \sum_{j=1}^{n} G_j dB_j\} - \sum_{j=1}^{n} \sigma_x^j W^* G_j dt \\
= \{-b_x^*W^*\eta + \sum_{j=1}^{n} \sigma_x^j \sigma_x^j W^*\eta - W^* M^* f_x - \sum_{j=1}^{n} \sigma_x^j W^* G_j \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{(I + \gamma_x^j)^{-*} \gamma_x^j W^*\eta + \gamma_x^j W^*\eta\}v_j(dz_j) \} dt \\
+ \sum_{j=1}^{n} \{-\sigma_x^j W^* \eta + W^* G_j\} dB_j \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{(I + \gamma_x^j)^{-*} \gamma_x^j W^*\eta \mathcal{N}_j(dt, dz_j)\} \\
= \{-b_x^*p + \sum_{j=1}^{n} \sigma_x^j \sigma_x^j p - f_x - \sum_{j=1}^{n} \sigma_x^j W^* G_j \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{(I + \gamma_x^j)^{-*} \gamma_x^j p + \gamma_x^j p\}v_j(dz_j) \} dt \\
+ \sum_{j=1}^{n} \{-\sigma_x^j p + W^* G_j\} dB_j \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{(I + \gamma_x^j)^{-*} \gamma_x^j p \mathcal{N}_j(dt, dz_j)\} \\
\]

set

\[
q_j = -\sigma_x^j p + W^* G_j \]

\[
r_j = -(I + \gamma_x^j)^{-*} \gamma_x^j p
\]
then the derivative of Hamilton along $x$ becomes

$$H_x(t, x^*, u^*, p, q, r) = f_x + b_x^* p + \sum_{j=1}^{n} \sigma_x^{j*} q_j + \sum_{j=1}^{n} \int_{\mathbb{R}_0} \gamma_x^{j*} r_j v_j(dz_j)$$

$$= f_x + b_x^* p + \sum_{j=1}^{n} \sigma_x^{j*} \{-\sigma_x^{j*} p + W^* G_j\}$$

$$+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \gamma_x^{j*} \{- (I + \gamma_x^j)^{-*} \sigma_x^{j*} p\} v_j(dz_j)$$

$$= b_x^* p - \sum_{j=1}^{n} \sigma_x^{j*} \sigma_x^{j*} p + f_x + \sum_{j=1}^{n} \sigma_x^{j*} W^* G_j$$

$$- \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{ \gamma_x^{j*} (I + \gamma_x^j)^{-*} \gamma_x^{j*} p \} v_j(dz_j)$$

$$= b_x^* p - \sum_{j=1}^{n} \sigma_x^{j*} \sigma_x^{j*} p + f_x + \sum_{j=1}^{n} \sigma_x^{j*} W^* G_j$$

$$- \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{ [\gamma_x^{j*} (I + \gamma_x^j)^{-*} + (I + \gamma_x^j)^{-*} - (I + \gamma_x^j)^{-*}] \gamma_x^{j*} p \} v_j(dz_j)$$

$$= b_x^* p - \sum_{j=1}^{n} \sigma_x^{j*} \sigma_x^{j*} p + f_x + \sum_{j=1}^{n} \sigma_x^{j*} W^* G_j$$

$$- \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{ [I - (I + \gamma_x^j)^{-*}] \gamma_x^{j*} p \} v_j(dz_j)$$

$$= b_x^* p - \sum_{j=1}^{n} \sigma_x^{j*} \sigma_x^{j*} p + f_x + \sum_{j=1}^{n} \sigma_x^{j*} W^* G_j$$

$$- \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{ [- (I + \gamma_x^j)^{-*} \gamma_x^{j*} p + \gamma_x^{j*} p] \} v_j(dz_j)$$

Therefore

$$dp(t) = -H_x(t, x^*(t), u^*(t), p(t), q(t), r(t, z)) dt + \sum_{j=1}^{n} q_j(t, \omega) dB_j$$

$$+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} r_j(t, z) N_j(dt, dz_j)$$

The proof is done. □
Proposition 3.4.5. Necessary Conditions for (P1)

Let \((u^*, x^*)\) be an optimal pair for (P1), then we can find corresponding \(p(t), q^j(t), \) and \(r^j(t, z), \) such that

\[
dx^* = H_p(t, x^*, u^*, p, q, r)dt + \sum_{j=1}^n \sigma^j(t, x^*, u^*)dB_j \\
+ \sum_{j=1}^n \int_{\mathbb{R}_0} \gamma^j(t, x^*, u^*)\mathbb{N}_j(dt, dz_j) \\
= b(t, x^*, u^*)dt + \sum_{j=1}^n \sigma^j(t, x^*, u^*)dB_j \\
+ \sum_{j=1}^n \int_{\mathbb{R}_0} \gamma^j(t, x^*, u^*)\mathbb{N}_j(dt, dz_j) \\
x^*(0) = x_0
\]

\[
dp = -H_x(t, x^*, u^*, p, q, r)dt \\
+ \sum_{j=1}^n q^j(t)dB_j + \sum_{j=1}^n \int_{\mathbb{R}_0} r^j(t, z)\mathbb{N}_j(dt, dz_j) \\
= -\{f_x(t, x^*, u^*) + b_x(t, x^*, u^*)p + \sum_{j=1}^n \sigma^j_x(t, x^*, u^*)q^j \\
+ \sum_{j=1}^n \int_{\mathbb{R}_0} \gamma^j_x(t, x^*, u^*)r^j v_j(dz_j)\}dt \\
+ \sum_{j=1}^n q^jdB_j + \sum_{j=1}^n \int_{\mathbb{R}_0} r^j\mathbb{N}_j(dt, dz_j) \\
p(T) = g_x(x^*(T))
\]

\[
H_u(t, x^*(t), u^*(t), p(t), q(t), r(t)) \cdot (v - u^*(t)) \geq 0 \quad \forall t, v
\]
3.5 Relation to dynamic programming

For the convenience of reader, we restate (P1) here

\[
\begin{align*}
\text{minimize} \quad J(u) &= E\left[ \int_0^T f(t, X(t), u(t)) dt + g(X(T)) \right] \quad (P1) \\
\text{subject to} \quad dX(t) &= b(t, X(t), u(t)) dt + \sigma(t, X(t), u(t)) dB_t \\
&\quad + \int_{(\mathbb{R}_0)^n} \gamma(t, z, X(t^-), u(t^-)) \mathcal{N}(dt, dz) \\
X(0) &= x_0
\end{align*}
\]

In previous section, we spent a lot of effort to derive maximum principle for (P1). However, maximum principle is not the only famous necessary condition of control problem. Another approach to get a set of necessary condition for (P1) is by dynamic programming. In this section, we will introduce the well-known Hamilton-Jacobi-Bellman (HJB) equation and relate it to the maximum principle (see [4] for more information). To put (P1) into a suitable Markovian framework, we let \(X^{s,x}(t)\) to be the solution of

\[
\begin{align*}
dX(t) &= b(t, X(t), u(t)) dt + \sigma(t, X(t), u(t)) dB_t \\
&\quad + \int_{(\mathbb{R}_0)^n} \gamma(t, z, X(t^-), u(t^-)) \mathcal{N}(dt, dz) \\
X(s) &= x
\end{align*}
\]

For \(t \geq s\) and admissible control \(u\), let

\[
J^{s,x}(u) = E\left[ \int_s^T f(t, X^{s,x}(t), u(t)) dt + g(X^{s,x}(T)) \right]
\]

Then we can define the value function \(\Phi(s, x)\) for (P1) by
\( \Phi(s, x) = \min_u J^{s,x}(u) \)

over admissible control \( u \). Suppose an optimal control for \( (P1) \) can be written in the feedback form \( u^* = u^*(t, X^*(t)) \). In other words, the corresponding optimal state process \( X^*(t) \) satisfies the state equation when \( u = u^*(t, X^*(t)) \). Suppose the value function \( \Phi \) in \( C^3 \) and under some suitable conditions \([2][3]\), then we have the following HJB equation for \( (P1) \)

\[
\max_{u \in K} F(t, x, u) = F(t, x, u^*(t, x)) = 0 \\
\Phi(T, x) = g(x)
\]

for all \( x \) and \( t \geq 0 \) where

\[
F(t, x, u) = -f(t, x, u) - \Phi_t(t, x) - \sum_{i=1}^{n} b_i(t, x, u) \Phi_{x_i}(t, x) \\
- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\sigma \sigma^*)_{ij}(t, x, u) \Phi_{x_i x_j}(t, x) \\
- \sum_{j=1}^{n} \int_{\mathbb{R}_0} \left\{ \Phi(t, x + \gamma^j(t, z, x, u)) - \Phi(t, x) \right\} v_j(dz_j) \\
- \sum_{i=1}^{n} \gamma_{ij}(t, z, x, u) \Phi_{x_i}(t, x) v_j(dz_j)
\]

The derivation of HJB equation requires more knowledge in stochastic calculus and can be found in \([2]\) and \([3]\). We will put the emphasis on making the connection between HJB equation and the maximum principle. Suppose \( p, q, \) and \( r \) are the costate process associated with the maximum principle which satisfy Proposition 3.4.5. We want to show that
\[ p(t) = \Phi_x(t, X^*(t)) \quad (3.1) \]
\[ q_{ij}(t) = \sum_{k=1}^{n} \sigma_{kj}(t, X^*(t), u^*(t)) \Phi_{x_i x_k}(t, X^*(t)) \quad (3.2) \]
\[ r_{ij}(t, z) = \Phi_{x_i}(t, X^*(t) + \gamma^j(t, z, X^*(t), u^*(t)) - \Phi_{x_i}(t, X^*(t)) \quad (3.3) \]

Our purpose is to show that the right hand side of (1)(2)(3) satisfy the costate equation, and by uniqueness assumption, we can conclude that they are the costate processes which we defined by maximum principle.

Firstly, we fix a component \( x_h \) and consider the process \( \Phi_{x_h}(t, X^*(t)) \). We hope to show that the costate process at \( h \)th component \( p_h(t) = \Phi_{x_h}(t, X^*(t)) \). Therefore, we need the dynamics of \( \Phi_{x_h}(t, X^*(t)) \). By Ito’s formula, we have

\[
d(\Phi_{x_h}(t, X^*(t))) = \Phi_{x_h t}(t, X^*(t)) \, dt + \sum_{i=1}^{n} \Phi_{x_h x_i}(t, X^*(t)) b_i(t, X^*(t), u^*(t, X^*(t))) \, dt \\
+ \sum_{i=1}^{n} \sum_{j=1}^{n} \Phi_{x_h x_i x_j}(t, X^*(t)) \sigma_{ij}(t, X^*(t), u^*(t, X^*(t))) dB_j \\
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \Phi_{x_h x_i x_j}(t, X^*(t)) (\sigma \sigma^*)_{ij}(t, X^*(t), u^*(t, X^*(t))) \, dt \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{ \Phi_{x_h}(t, X^*(t)) + \gamma^j(t, z, X^*(t), u^*(t, X^*(t))) \} v_j(dz_j) \, dt \\
- \Phi_{x_h t}(t, X^*(t)) - \sum_{i=1}^{n} \Phi_{x_h x_i}(t, X^*(t)) \gamma_{ij}(t, z, X^*(t), u^*(t, X^*(t))) \} v_j(dz_j) \, dt \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{ \Phi_{x_h}(t, X^*(t) -) + \gamma^j(t, z, X^*(t-), u^*(t, X^*(t-))) \} \overline{N}_j(dt, dz_j) \quad (4) \]

Secondly, in order to simplify (4), we evaluate \( F_{x_h}(t, X^*(t)) \) and obtain
\[
F_{x_h}(t, x, u^*(t, x)) = F_u(t, x, u^*(t, x)) \cdot u^*_{x_h}(t, x)
\]
\[
- f_{x_h}(t, x, u^*(t, x)) - \Phi_{t \cdot x_h}(t, x)
\]
\[
- \sum_{i=1}^{n} b_{i \cdot x_h}(t, x, u^*(t, x)) \Phi_{x_i}(t, x) - \sum_{i=1}^{n} b_i(t, x, u^*(t, x)) \Phi_{x_i \cdot x_h}(t, x)
\]
\[
- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \partial(\sigma \sigma^*)_{ij}(t, x, u^*(t, x)) \Phi_{x_i \cdot x_j}(t, x)
\]
\[
- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\sigma \sigma^*)_{ij}(t, x, u^*(t, x)) \Phi_{x_i \cdot x_j \cdot x_h}(t, x)
\]
\[
- \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{ \sum_{i=1}^{n} \Phi_{x_i}(t, x + \gamma^j(t, z, x, u^*(t, x)))(\delta_{ih} + \frac{\partial \gamma_{ij}}{\partial x_h}(t, z, x, u^*(t, x)))
\]
\[
- \Phi_{x_h}(t, x) - \sum_{i=1}^{n} \frac{\partial \gamma_{ij}}{\partial x_h}(t, z, x, u^*(t, x)) \Phi_{x_i}(t, x)
\]
\[
- \sum_{i=1}^{n} \gamma_{ij}(t, z, x, u^*(t, x)) \Phi_{x_i \cdot x_h}(t, x) \} v_j(dz_j)
\]

where \(\delta_{ih}\) equals to 1 if \(i = h\), otherwise zero. Since \(F(t, x, u^*(t, x)) = 0\), we have \(F_{x_h}(t, x, u^*(t, x)) = 0\). By using the fact that \(F_u(t, x, u^*(t, x)) = 0\) and let \(x = X^*(t)\) in the above expression, we have the following equation
0 = \begin{align*}
&-f_{x_h}(t, X^*(t), u^*(t, X^*(t))) - \Phi_{t,x_h}(t, X^*(t)) \\
&- \sum_{i=1}^{n} b_i(t, X^*(t), u^*(t, X^*(t))) \Phi_{x_i}(t, X^*(t)) - \sum_{i=1}^{n} b_i(t, X^*(t), u^*(t, X^*(t))) \Phi_{x_i x_h}(t, X^*(t)) \\
&- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial (\sigma^*)_{ij}}{\partial x_h}(t, X^*(t), u^*(t, X^*(t))) \Phi_{x_i x_j x_h}(t, X^*(t)) \\
&- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\sigma^*)_{ij}(t, X^*(t), u^*(t, X^*(t))) \Phi_{x_i x_j x_h}(t, X^*(t)) \\
&- \sum_{j=1}^{n} \int_{\mathbb{R}_0} \left\{ \sum_{i=1}^{n} \Phi_{x_i}(t, X^*(t) + \gamma^j(t, z, X^*(t), u^*(t, X^*(t)))) (\delta_{i h} + \frac{\partial \gamma_{ij}}{\partial x_h}(t, z, X^*(t), u^*(t, X^*(t)))) \right\} \\
&- \Phi_{x_h}(t, X^*(t)) - \sum_{i=1}^{n} \frac{\partial \gamma_{ij}}{\partial x_h}(t, z, X^*(t), u^*(t, X^*(t))) \Phi_{x_i}(t, X^*(t)) \\
&- \sum_{i=1}^{n} \gamma_{ij}(t, z, X^*(t), u^*(t, X^*(t))) \Phi_{x_i x_h}(t, X^*(t)) \right\} v_j(dz_j)
\end{align*}

Rearranging the equation, we obtain
\[ \Phi_{txh}(t, X^*(t)) = -f_{xh}(t, X^*(t), u^*(t, X^*(t))) \]
\[- \sum_{i=1}^{n} b_{i,xh}(t, X^*(t), u^*(t, X^*(t))) \Phi_{xi}(t, X^*(t)) \]
\[- \sum_{i=1}^{n} b_{i}(t, X^*(t), u^*(t, X^*(t))) \Phi_{xi;th}(t, X^*(t)) \]
\[- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial (\sigma \sigma^*)_{ij}}{\partial x_h}(t, X^*(t), u^*(t, X^*(t))) \Phi_{xi;jh}(t, X^*(t)) \]
\[- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\sigma \sigma^*)_{ij}(t, X^*(t), u^*(t, X^*(t))) \Phi_{xi;jh}(t, X^*(t)) \]
\[- \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{ \sum_{i=1}^{n} \Phi_{xi}(t, X^*(t) + \gamma^j(t, z, X^*(t), u^*(t, X^*(t)))) \]
\[- (\delta_{ih} + \frac{\partial \gamma_{ij}}{\partial x_h}(t, z, X^*(t), u^*(t, X^*(t)))) \]
\[- \Phi_{xh}(t, X^*(t)) - \sum_{i=1}^{n} \frac{\partial \gamma_{ij}}{\partial x_h}(t, z, X^*(t), u^*(t, X^*(t))) \Phi_{xi}(t, X^*(t)) \]
\[- \sum_{i=1}^{n} \gamma_{ij}(t, z, X^*(t), u^*(t, X^*(t))) \Phi_{xi;h}(t, X^*(t)) \} v_j(dz_j) \]

Substituting \( \Phi_{txh}(t, X^*(t)) \) into (4), we get the following
\[
\begin{align*}
\text{d } \Phi_{x_h} &= \left[ -f_{x_h} - \sum_{i=1}^{n} b_i x_h \Phi_{x_i} - \sum_{i=1}^{n} b_i \Phi_{x_i x_h} \\
- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial(\sigma^*)_{ij}}{\partial x_h} \Phi_{x_i x_j} - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\sigma^*)_{ij} \Phi_{x_i x_j} x_h \\
- \sum_{j=1}^{n} \int_{\mathbb{R}_0} \left\{ \sum_{i=1}^{n} \Phi_{x_i}(t, X^* + \gamma^j)(\delta_{ih} + \frac{\partial \gamma_{ij}}{\partial x_h}) \\
- \Phi_{x_h} - \sum_{i=1}^{n} \frac{\partial \gamma_{ij}}{\partial x_h} \Phi_{x_i} - \sum_{i=1}^{n} \gamma_{ij} \Phi_{x_i} x_h \right\} v_j(dz_j) \right] \text{ dt} \\
+ \sum_{i=1}^{n} \Phi_{x_h x_i} b_i \text{ dt} + \sum_{i=1}^{n} \sum_{j=1}^{n} \Phi_{x_h x_i} \sigma_{ij} dB_j \\
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \Phi_{x_h x_i x_j} (\sigma^*)_{ij} \text{ dt} \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \left\{ \Phi_{x_h}(t, X^* + \gamma^j) - \Phi_{x_h} - \sum_{i=1}^{n} \Phi_{x_h x_i} \gamma_{ij} \right\} v_j(dz_j) \text{ dt} \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \left\{ \Phi_{x_h}(t, X^* + \gamma^j) - \Phi_{x_h} \right\} \mathcal{N}_j(dt, dz_j)
\end{align*}
\]

(5)

The \( \delta_{ih} \) in (5) can be erased after rearranging. Then,
\[
d\Phi_{x_h} = \left[ -f_{x_h} - \sum_{i=1}^{n} b_{i} x_{i} \Phi_{x_{i}} - \sum_{i=1}^{n} b_{i} \Phi_{x_{i}x_{h}} \right. \\
- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \left( \sigma \sigma^{*} \right)_{ij}}{\partial x_{h}} \Phi_{x_{i}x_{j}} - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \sigma \sigma^{*} \right)_{ij} \Phi_{x_{i}x_{j}x_{h}} \\
- \sum_{j=1}^{n} \int_{\mathbb{R}_0} \left\{ \sum_{i=1}^{n} \left[ \Phi_{x_{i}}(t, X^{*} + \gamma_{j}) - \Phi_{x_{i}} \right] \frac{\partial \gamma_{ij}}{\partial x_{h}} \right\} v_{j}(dz_{j}) dt \\
+ \Phi_{x_{h}}(t, X^{*} + \gamma_{j}) - \Phi_{x_{h}} - \sum_{i=1}^{n} \gamma_{ij} \Phi_{x_{i}x_{h}} v_{j}(dz_{j}) dt \\
+ \sum_{i=1}^{n} \Phi_{x_{h}x_{i}} b_{i} dt + \sum_{i=1}^{n} \sum_{j=1}^{n} \Phi_{x_{h}x_{i}} \sigma_{ij} dB_{j} \\
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \Phi_{x_{h}x_{i}x_{j}} \left( \sigma \sigma^{*} \right)_{ij} dt \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \left\{ \Phi_{x_{h}}(t, X^{*} + \gamma_{j}) - \Phi_{x_{h}} - \sum_{i=1}^{n} \Phi_{x_{h}x_{i}} \gamma_{ij} \right\} v_{j}(dz_{j}) dt \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \left\{ \Phi_{x_{h}}(t, X^{*} + \gamma_{j}) - \Phi_{x_{h}} \right\} \mathbb{N}_{j}(dt, dz_{j}) \right)
\]

Several terms can be cancelled out and (6) can be simplified as

\[
d\Phi_{x_h} = \left[ -f_{x_h} - \sum_{i=1}^{n} b_{i} x_{i} \Phi_{x_{i}} - \sum_{i=1}^{n} b_{i} \Phi_{x_{i}x_{h}} \right. \\
- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \left( \sigma \sigma^{*} \right)_{ij}}{\partial x_{h}} \Phi_{x_{i}x_{j}} \\
- \sum_{j=1}^{n} \int_{\mathbb{R}_0} \left\{ \Phi_{x_{i}}(t, X^{*} + \gamma_{j}) - \Phi_{x_{i}} \right\} \frac{\partial \gamma_{ij}}{\partial x_{h}} v_{j}(dz_{j}) \} dt \\
+ \sum_{i=1}^{n} \sum_{j=1}^{n} \Phi_{x_{h}x_{i}} \sigma_{ij} dB_{j} \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \left\{ \Phi_{x_{h}}(t, X^{*} + \gamma_{j}) - \Phi_{x_{h}} \right\} \mathbb{N}_{j}(dt, dz_{j}) \}
\]

71
Since

\[
\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial (\sigma \sigma^*)}{\partial x_h} \Phi_{xixj} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_h} (\sum_{k=1}^{n} \sigma_{ik} \sigma_{jk}) \Phi_{xixj}
\]

\[
\quad = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (\frac{\partial \sigma_{ik}}{\partial x_h} \sigma_{jk} + \frac{\partial \sigma_{jk}}{\partial x_h} \sigma_{ik}) \Phi_{xixj}
\]

\[
\quad = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial \sigma_{ij}}{\partial x_h} (\sum_{k=1}^{n} \Phi_{xixk} \sigma_{kj})
\]

We can rewrite (7) as

\[
d \Phi_{x_h} = - \{ f_{x_h} + \sum_{i=1}^{n} b_i x_i \Phi_{x_i} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \sigma_{ij}}{\partial x_h} (\sum_{k=1}^{n} \Phi_{xixk} \sigma_{kj}) \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \sum_{i=1}^{n} [\Phi_{x_i}(t, X^* + \gamma^j) - \Phi_{x_i}] \frac{\partial \gamma_{ij}}{\partial x_h} \sigma_{ij} (d\zeta_j)] \} dt \\
+ \sum_{i=1}^{n} \sum_{j=1}^{n} \Phi_{xixi} \sigma_{ij} dB_j \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \{ \Phi_{x_h}(t, X^* + \gamma^j) - \Phi_{x_h} \} \mathbb{N}_j(dt,d\zeta_j)
\]

(7)

Compare (7) to the hth component form of costate equation in Proposition 3.4.5

\[
d p_h = - \{ f_{x_h} + \sum_{i=1}^{n} b_i x_ip_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial q_{ij}}{\partial x_h} q_{ij} \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \sum_{i=1}^{n} r_{ij} \frac{\partial q_{ij}}{\partial x_h} v_j(d\zeta_j)] \} dt + \sum_{j=1}^{n} q_{hj} dB_j \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} r_{hj} \mathbb{N}_j(dt,d\zeta_j)
\]

Since we suppose the costate processes are unique, we have
\[ p(t) = \Phi_x(t, X^*(t)) \]
\[ q_{ij}(t) = \sum_{k=1}^{n} \sigma_{kj}(t, X^*(t), u^*(t))\Phi_{x_i x_k}(t, X^*(t)) \]
\[ r_{ij}(t, z) = \Phi_{x_i}(t, X^*(t) + \gamma^j(t, z, X^*(t), u^*(t)) - \Phi_{x_i}(t, X^*(t)) \]

The HJB equation and its connection with maximum principle is well-known for deterministic and diffusion cases. The details of HJB equation for deterministic control can be found in [7], and for stochastic diffusion control can be found in [2][3].

### 3.6 Sufficient condition for jump diffusion problem

We have derived the necessary condition from maximum principle for (P1) in Section 4. Here we are interested in the sufficient condition. Actually, with appropriate convexity assumption, the maximum principle will become sufficient. For the convenience of the reader, we restate (P1) here.

\[
\begin{align*}
\minimize & \quad J(u) = E[\int_0^T f(t, X(t), u(t))dt + g(X(T))] \quad (P1) \\
\text{subject to} & \quad dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB_t \\
& \quad + \int_{(\mathbb{R}_0)^n} \gamma(t, z, X(t^-), u(t^-))\mathcal{N}(dt, dz) \\
X(0) & = x_0
\end{align*}
\]

As we mentioned in Section 3, both the state process \( X(t) \) and control \( u(t) \) are predictable and every processes in our setting are in \( L^2 \). We can have the following sufficient condition for (P1).

**Proposition 3.6.1.** Under the same setting in Section 3, suppose we have \( (\hat{X}(t), \hat{u}(t)) \) be an admissible pairs and adaptive processes \( \hat{p}(t), \hat{q}(t), \hat{r}(t, z) \) which satisfy the costate equation in Proposition 3.4.5. Furthermore, we assume

- \( H_u(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t)) \cdot (v - \hat{u}(t)) \geq 0 \quad \forall t \in [0, T], \ \forall v \in K \)
• The map $(x,v) \mapsto H(t,x,v,\hat{p}(t),\hat{q}(t),\hat{r}(t))$ is convex $\forall t \in [0,T]$

• The function $g$ from $\mathbb{R}^n$ to $\mathbb{R}$ is convex

Then, we can conclude that $(\hat{X}(t), \hat{u}(t))$ is the optimal pair and $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z))$ is the corresponding costate process.

**Proof:**

For any given admissible pair $(X(t), u(t))$, we need to show that $J(\hat{u}) \leq J(u)$. In other words, we need the quantity

$$J(\hat{u}) - J(u) = E\left[\int_0^T \{ f(t, \hat{X}(t), \hat{u}(t)) - f(t, X(t), u(t)) \} dt + g(\hat{X}(T)) - g(X(T)) \right]$$

less than or equal to zero.

By using the notation of Hamiltonian, the first part of (8) can be written as

$$E[\int_0^T \{ f(t, \hat{X}(t), \hat{u}(t)) - f(t, X(t), u(t)) \} dt]$$

$$= E[\int_0^T \{ H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t)) - H(t, X(t), u(t), \hat{p}(t), \hat{q}(t), \hat{r}(t))$$

$$- [b(t, \hat{X}(t), \hat{u}(t)) - b(t, X(t), u(t))] \cdot \hat{p}(t) - \sum_{j=1}^n [\sigma^j(t, \hat{X}(t), \hat{u}(t)) - \sigma^j(t, X(t), u(t))] \cdot \hat{q}^j(t)$$

$$- \sum_{j=1}^n \int_{\mathbb{R}_0} [\gamma^j(t, z, \hat{X}(t), \hat{u}(t)) - \gamma^j(t, z, X(t), u(t))] \cdot \hat{r}^j(t,z) v^j(dz) \} dt]$$

(9)

Before starting to evaluate the second part of (8), we find the $d(\hat{p}(t) \cdot (\hat{X}(t) - X(t)))$ by the integration by parts
\[
d(\dot{p}(t) \cdot (\dot{X}(t) - X(t))) = \dot{p}(t) \cdot d(\dot{X}(t) - X(t)) + (\dot{X}(t) - X(t)) \cdot \dot{p}(t) + \sum_{j=1}^{n} [\sigma^j(t, \dot{X}(t), \dot{u}(t)) - \sigma^j(t, X(t), u(t))] \cdot \dot{q}^j(t) \, dt \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} [\gamma^j(t, z, \dot{X}(t), \dot{u}(t)) - \gamma^j(t, z, X(t), u(t))] \cdot \dot{r}^j(t, z) \mathcal{N}_j(dt, dz_j) \\
= \dot{p}(t) \cdot \{ [b(t, \dot{X}, \dot{u}) - b(t, X, u)] dt + \sum_{j=1}^{n} [\sigma^j(t, \dot{X}, \dot{u}) - \sigma^j(t, X, u)] db_j \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} [\gamma^j(t, z, \dot{X}, \dot{u}) - \gamma^j(t, z, X, u)] \mathcal{N}_j(dt, dz_j) \}
+ (\dot{X}(t) - X(t)) \cdot \{-H(t, \dot{X}, \dot{u}, \dot{p}, \dot{q}, \dot{r}) + \\
\sum_{j=1}^{n} q^j(t) db_j + \sum_{j=1}^{n} \int_{\mathbb{R}_0} \dot{r}^j(t, z) \mathcal{N}_j(dt, dz_j) \}
+ \sum_{j=1}^{n} [\sigma^j(t, \dot{X}(t), \dot{u}(t)) - \sigma^j(t, X(t), u(t))] \cdot \dot{q}^j(t) \, dt \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} [\gamma^j(t, z, \dot{X}(t), \dot{u}(t)) - \gamma^j(t, z, X(t), u(t))] \cdot \dot{r}^j(t, z) \mathcal{N}_j(dt, dz_j) \\
(10)
\]

Since \( g \) is convex, \( \dot{X}(0) - X(0) = 0 \), and (10), we can evaluate the second part of (8) as

\[
E[g(\dot{X}(T)) - g(X(T))] \\
\leq E[g(x(\dot{X}(T)) \cdot (\dot{X}(T) - X(T))] \\
= E[\dot{p}(T) \cdot (\dot{X}(T) - X(T))] \\
= E[\int_{0}^{T} \dot{p}(t) \cdot [b(t, \dot{X}(t), \dot{u}(t)) - b(t, X(t), u(t))] dt \\
- \int_{0}^{T} (\dot{X}(t) - X(t)) \cdot H_x(t, \dot{X}, \dot{u}, \dot{p}, \dot{q}, \dot{r}) dt \\
+ \sum_{j=1}^{n} \int_{0}^{T} [\sigma^j(t, \dot{X}(t), \dot{u}(t)) - \sigma^j(t, X(t), u(t))] \cdot \dot{q}^j(t) \, dt \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} \gamma^j(t, z, \dot{X}(t), \dot{u}(t)) - \gamma^j(t, z, X(t), u(t))] \cdot \dot{r}^j(t, z) \mathcal{N}_j(dz_j) \, dt \\
(11)
\]
Combine (3) and (11), we obtain the following inequality

\[
J(\hat{u}) - J(u) \leq E\left[\int_0^T \{ H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t)) - H(t, X(t), u(t), \hat{p}(t), \hat{q}(t), \hat{r}(t)) \right.
\]
\[
- (\hat{X}(t) - X(t)) \cdot H_x(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t)) \left.\} \ dt \right]
\]

Since the map \((x, v) \mapsto H(t, x, v, \hat{p}(t), \hat{q}(t), \hat{r}(t))\) is convex, we have

\[
H(t, \hat{X}, \hat{u}, \hat{p}, \hat{q}, \hat{r}) - H(t, X, u, \hat{p}, \hat{q}, \hat{r}) \leq H_x(t, \hat{X}, \hat{u}, \hat{p}, \hat{q}, \hat{r}) \cdot (\hat{X}(t) - X(t))
\]
\[
+ H_u(t, \hat{X}, \hat{u}, \hat{p}, \hat{q}, \hat{r}) \cdot (\hat{u}(t) - u(t))
\]

Since \(H_u(t, \hat{X}, \hat{u}, \hat{p}, \hat{q}, \hat{r}) \cdot (\hat{u}(t) - u(t)) \leq 0\), we have

\[
H(t, \hat{X}, \hat{u}, \hat{p}, \hat{q}, \hat{r}) - H(t, X, u, \hat{p}, \hat{q}, \hat{r}) - H_x(t, \hat{X}, \hat{u}, \hat{p}, \hat{q}, \hat{r}) \cdot (\hat{X}(t) - X(t))
\]
\[
\leq H_u(t, \hat{X}, \hat{u}, \hat{p}, \hat{q}, \hat{r}) \cdot (\hat{u}(t) - u(t))
\]
\[
\leq 0
\]

Therefore, \(J(\hat{u}) - J(u) \leq 0\).

The proof is done. \(\square\)

The sufficient condition is also true for deterministic control [7] and stochastic diffusion control with appropriate modification [1][5]. People can find more details in Oksendal’s paper [4].

3.7 Conclusion

We described some fundamental results of the stochastic control problem with jump diffusion in this chapter. We are going to use these results to solve the target problem in the following chapters.
Chapter 4

Target problem

4.1 Introduction

In a complete financial market, every contingent claim can be attained by an admissible portfolio and starting with a large enough initial capital. We can find the fair price and replicating portfolio by Black-Scholes theorem. The fair price is the expectation of discounted value of the claim, under the equivalent martingale measure. Thus, the price can be found by Monte-Carlo simulation, PDE method or binomial tree approach [8][14]. One way to find the replicating portfolio is using a generalized version of Clark-Ocone theorem from the Malliavin calculus [3].

On the other hand, in an incomplete market, a typical example of which is that a market in which claims can depend on stocks which are not available for investment, or even in the case that we cannot find an equivalent martingale measure of the market, we can use stochastic control approach to deal with the option pricing and hedging problems.

A classical way to model the stock price process is using a linear stochastic differential equation with an assumption that the investor is "small", which means his/her trading strategy and financial status cannot affect the stock price. Therefore, in a classical model of stock price process, the coefficient of the process is independent of the wealth process and the trading strategy of the investor. However, in resent years, some people start discussing the behaviour of "large investor" models, in which the stock price can be affected by the investor's financial status and trading strategy. Furthermore, the dynamic of stock prices becomes nonlinear.

In this chapter, we define a new class of stochastic control problem, named optimal target problem, which is particularly useful in modelling the behaviour of options in an incomplete high-frequency trading financial market. Furthermore, this is a large investor model with non-
linear stock price processes.

A classical way to deal with the optimal target problem with convex analysis and martingale approach can be found in Cvitanic and Karatzas 1993 [11]. To solve option pricing problems with constrained portfolios which make the market incomplete, they constructed a series of auxiliary complete markets with unconstrained portfolios and use the expectation of discounted value of the claim, under specific equivalent martingale measure, to find the price of the option in each auxiliary complete market, and then take supremum of these prices. The hedging strategy can be characterized by the wealth process, which is a backward stochastic differential equation. One drawback of this approach is that it cannot deal with the more general case of target problem - large investor model with nonlinear stock price process [15].

In the large investor model, we can also construct an equivalent martingale measure for the stock price process just as what people did in a small investor model. However, since the wealth and strategy lie in the coefficients of stock price process, the measure will be both wealth and strategy dependent. On the other hand, there are two related stochastic differential equations in this model: the stock price process which is a forward stochastic differential equation and the wealth process which is a backward stochastic differential equation. Thus, when we derive a system of forward-backward stochastic differential equations, we cannot simply set out target constraint be an equality as what people did in small investor case [10][12]. In Cvitanic and Karatzas 1996 [13], they constructed some comparison theorems to overcome this obstacle. But the model will become more complicated, if we decide to add the high-frequency trading feature and unpredictable Levy jump in it. We will face this problem in this chapter.

4.2 An example: superreplication problem in finance

We consider a financial market which consists of
1. a bank account with constant price process. $B_t = 1$ for all $t$ in $[0, T]$
2. a risky asset $X_t$ with positive price process described by a stochastic differential equation.

To model the risky asset $X_t$, since it is a positive process, we define

$$X_t = e^{\overline{X}_t}, \quad X_0 = x, \quad x > 0$$

where $x$ is the given initial stock price at time 0, and $\overline{X}_t$ satisfies the following stochastic differential equation
\[
\begin{align*}
  dX_t &= \mu(t, X_t, Y_t, u_t)dt + \sigma(t, X_t, Y_t, u_t)dB_t + \int_{\mathbb{R}_0} \gamma(t, z, X_{t-}, Y_{t-}, u_{t-})N(dt, dz) \\
  X_0 &= \ln(x)
\end{align*}
\]

by Ito’s formula, we can easily find the differential form of the risky price process \( X_t \)

\[
\begin{align*}
  dX_t &= \{\mu(t, \ln(X_t), \ln(Y_t), u_t) + \frac{1}{2}\sigma^2(t, \ln(X_t), \ln(Y_t), u_t)\}X_t dt + \sigma(t, \ln(X_t), \ln(Y_t), u_t)X_t dB_t \\
  &\quad + \int_{\mathbb{R}_0} \left\{ e^{\gamma(t, \ln(X_t), \ln(Y_t), u_t)} - 1 - \gamma(t, \ln(X_t), \ln(Y_t), u_t) \right\}X_t v(dz)dt \\
  &\quad + \int_{\mathbb{R}_0} \left\{ e^{\gamma(t, \ln(X_{t-}), \ln(Y_{t-}), u_{t-})} - 1 \right\}X_{t-} N(dt, dz)
\end{align*}
\]

where \( Y_t = e^{Y_t} \). Notice that the Levy jump term in \( X_t \) characterizes the high-frequency trading feature in financial market since it allows infinitely many jumps in a short time. The market can be incomplete or even without equivalent martingale measure since we will always work under the practical measure in the following context. As usual, the assumption that the interest rate of bank account is zero can be dropped by adding appropriate discounting factor.

The filtration \( \{F_t\}_{t \geq 0} \) is generated by \( B_t \) and \( N(t, dz) \), that is, \( F_t \) is the information we have before and at time \( t \). The quantity \( u_t \) is a \( \{F_t\}_{t \geq 0} \) adapted process which represents the fraction of total wealth invested in the risky asset \( X_t \). So, this is a large investor model in which our trading strategy and personal wealth can affect the market price \( X_t \).

Suppose \( Y_t \) is our wealth process with initial capital \( y > 0 \) and \( u_t \) is the proportion of wealth which invested in the risky asset. Then, it is intuitively correct that the change of the wealth in a small time increment \( [t, t + \Delta) \) can be approximated by

\[
Y_{t+\Delta} - Y_t = \frac{Y_t u_t}{X_t} (X_{t+\Delta} - X_t) + \frac{Y_t (1 - u_t)}{B_t} (B_{t+\Delta} - B_t)
\]

\[
= \frac{Y_t u_t}{X_t} (X_{t+\Delta} - X_t)
\]

Thus, the wealth process satisfied the stochastic differential equation
\[ dY_t = \frac{Y_t u_t}{X_t} dX_t \]

\[ Y_0 = y \]

Suppose we do not allow borrowing and short selling, that is, we assume the market is self-financing, then the portfolio

\[ 0 \leq u_t \leq 1 \]

applying Ito’s formula, we can have the differential form of \( Y_t \) and \( Y_t \)

\[ dY_t = \{ \mu(t, \ln(X_t), \ln(Y_t), u_t) + \frac{1}{2} \sigma^2(t, \ln(X_t), \ln(Y_t), u_t) \} Y_t u_t dt + \sigma(t, \ln(X_t), \ln(Y_t), u_t) Y_t u_t dB_t \]

\[ + \int_{R_0} \{ e^{\gamma(t, \ln(X_t), \ln(Y_t), u_t)} - 1 - \gamma(t, \ln(X_t), \ln(Y_t), u_t) \} Y_t u_t v(dz) dt \]

\[ + \int_{R_0} \{ e^{\gamma(t, \ln(X_t), \ln(Y_t), u_t)} - 1 \} Y_t u_t N(dt, dz) \]

and

\[ d\bar{Y}_t = \{ \mu(t, \bar{X}_t, \bar{Y}_t, u_t) + \frac{1}{2} \sigma^2(t, \bar{X}_t, \bar{Y}_t, u_t) u_t - \frac{1}{2} \sigma^2(t, \bar{X}_t, \bar{Y}_t, u_t^2) \} dt \]

\[ + \sigma(t, \bar{X}_t, \bar{Y}_t, u_t) u_t dB_t + \int_{R_0} \{ e^{\gamma(t, \bar{X}_t, \bar{Y}_t, u_t)} - 1 - \gamma(t, \bar{X}_t, \bar{Y}_t, u_t) \} u_t v(dz) dt \]

\[ + \int_{R_0} \ln(1 + (e^{\gamma(t, \bar{X}_t, \bar{Y}_t, u_t)} - 1) u_t) - (e^{\gamma(t, \bar{X}_t, \bar{Y}_t, u_t)} - 1) u_t v(dz) dt \]

\[ + \int_{R_0} \ln(1 + (e^{\gamma(t, \bar{X}_t, \bar{Y}_t, u_t)} - 1) u_t) N(dt, dz) \]

where \( Y_0 = y \) and \( \bar{Y}_t = \ln(Y_t) \). The pair \((X_t, Y_t)\) is the stock-wealth process, and the pair \((\bar{X}_t, \bar{Y}_t)\) is called the log stock-wealth process. Suppose we are the seller of an European option with contingent claim \( f(X_T) \) at maturity time \( T \), we need to ask ourselves what is the minimal price we should charge for this option at the initial time and what is the hedging strategy for us to promise that at least we will not lose our money for sure. We formulate this problem as following
\[
\min y \\
\text{subject to } Y_T \geq f(X_T)
\]

or it is equivalent to consider the log problem

\[
\min y \\
\text{subject to } Y_T \geq f(X_T)
\]

where the initial stock price \( x > 0 \) is given and the inequality constraint is the TARGET we want to achieve at maturity time \( T \) to promise that we will not lose our money at the final time \( T \). Thus, this is called an optimal stochastic target problem in mathematics or superreplication problem in finance.

Every option pricing problem has its dual problem, which is another target problem or superreplication problem. The dual problem is constructed from the buyer’s view point. If the market is complete, the two problems have no duality gap. Their optimal objective values are the same. But in our discussion, we don’t make this completeness assumption.

In a more general case, we consider more than one risky assets and make the constrained set of portfolios a non-empty, closed, and convex set, that is, we have dynamics \( X^i_t, i \) from 1 to \( n \), and the controls

\[
u_t \in K
\]

where \( K \) is a non-empty, closed, and convex set.

It is a reasonable assumption for portfolio selection. For example
1. if the market is unconstrained, we take \( K = \mathbb{R}^n \).
2. if short selling is not allowed in this market, we take \( K = [0, \infty)^n \).
3. if borrowing is not allowed in this market, we take \( K = \{ u \in \mathbb{R}^n | \sum_{i=1}^n u_i \leq 1 \} \).
4. if the market is incomplete, we may take \( K = \{ u \in \mathbb{R}^n | u_i = 0 \ for \ i = m, m + 1...n \} \) for some \( m \).
5. if the market is incomplete and with prohibition of borrowing, we may take \( K = \{ u \in \mathbb{R}^n | \sum_{i=1}^{m-1} u_i \leq 1 \ and \ u_i = 0 \ for \ i = m, m + 1...n \} \) for some \( m \).

Let’s define the optimal target problem with Levy jump precisely in the next section.
4.3 1-dimensional stochastic target problem with jump diffusion

In this section, we suppose both the stock prices process $X_t$ and the wealth process $Y_t$ are 1-dimensional processes and consider the following problem

$$\begin{align*}
\text{minimize} & \quad y \\
\text{subject to} & \\
\quad dX_t &= b_1(t, X_t, Y_t, u_t)dt + \sum_{j=1}^{2} \sigma_{1j}(t, X_t, Y_t, u_t)dB_j \\
& \quad + \sum_{j=1}^{2} \int_{\mathbb{R}_0} \gamma_{1j}(t, z, X_{t-}, Y_{t-}, u_{t-})N_j(dt, dz_j) \\
\quad dY_t &= b_2(t, X_t, Y_t, u_t)dt + \sum_{j=1}^{2} \sigma_{2j}(t, X_t, Y_t, u_t)dB_j \\
& \quad + \sum_{j=1}^{2} \int_{\mathbb{R}_0} \gamma_{2j}(t, z, X_{t-}, Y_{t-}, u_{t-})N_j(dt, dz_j) \\
Y_T &\geq f(X_T) \quad \text{a.s.} \\
X_0 &= x, \quad Y_0 = y > 0, \quad u_t \in K
\end{align*}$$

where $x$ is a given positive number in $\mathbb{R}$ and $f$ is a real-valued function, and $K$ is a non-empty, closed, convex set. $(P1)$ is called a 1-dimensional target problem. $Y_T \geq f(X_T)$ a.s. is the target we need to achieve at the maturity time $T$. Suppose the coefficient functions $b, \sigma, \gamma$ satisfied the usual conditions such that $(P1)$ has one and only one solution. Notice that the control in a target problem is a pair $(u_t, y)$. Thus, in order to get necessary conditions of $(P1)$, we need to make perturbation on both $u$ and $y$.

We denote the unique optimal solution pair by $(u^*_t, y^*)$ and the corresponding stock and wealth processes by $X^*_t$ and $Y^*_t$. In the following sections, we are going to characterize this optimal pair and find necessary conditions for the target problem. In order to do so, we divide the problem into two cases:
1. The optimal trajectory hits the boundary of the target at maturity time $T$, that is
\[ Y_T^* = f(X_T^*) \quad a.s. \]

2. The optimal trajectory does not hit the boundary of the target at maturity time $T$, that is
\[ Y_T^* > f(X_T^*) \quad a.s. \]

In the first case, we will introduce an auxiliary problem and use the maximal principle which we got in Chapter 3 to derive a set of necessary condition for $(u_t^*, y^*)$. In the second case, we will use dynamic programming approach to derive the Hamilton-Jacobi-Bellman (HJB) equation for jump processes which is a second order partial differential equation with suitable boundary condition as our necessary condition. Another approach is characterizing the value function as a discontinuous viscosity solution of a modified HJB equation which can be found in Mete Soner and Nizar Touzi [9]. But their approach works under the continuity assumption of stock price and wealth processes.

Notice that the target constraint $Y_T \geq f(X_T) \ a.s.$ is equivalent to
\[ E[g(f(X_T) - Y_T)] = 0 \]
where $g(x) = \max(x, 0) \ \forall x \in \mathbb{R}$. $g$ is the penalty function we are going to use in the following sections. Please see Figure 1 for function $g$. 
4.4 When the optimal trajectory hits the boundary of the target at maturity time $T$

Suppose $(u_t^*, y_t^*)$ is the optimal pair of (P1) for some given initial stock price $x$, and $X_t^*$, $Y_t^*$ are the corresponding stock price and wealth processes. In this section, we suppose the optimal trajectory hits the boundary of the target at maturity time $T$, that is

$$Y_T^* = f(X_T)^* \text{ a.s.}$$

Firstly, we consider the simplest case, let $f$ be the identity function, that is, $f(x) = x$ for all $x \in \mathbb{R}$. In order to make perturbation on $u_t$, we construct the auxiliary problem
minimize \[ E[g(X_T - Y_T)] \] \hspace{1cm} (P2) 

subject to

\[
\begin{align*}
    dX_t &= b_1(t, X_t, Y_t, u_t)dt + \sum_{j=1}^{2} \sigma_1(t, X_t, Y_t, u_t)dB_j \\
    & \quad + \sum_{j=1}^{2} \int_{\mathbb{R}_0} \gamma_1(t, z, X_{t-}, Y_{t-}, u_{t-})N_j(dt, dz) \\
    dY_t &= b_2(t, X_t, Y_t, u_t)dt + \sum_{j=1}^{2} \sigma_2(t, X_t, Y_t, u_t)dB_j \\
    & \quad + \sum_{j=1}^{2} \int_{\mathbb{R}_0} \gamma_2(t, z, X_{t-}, Y_{t-}, u_{t-})N_j(dt, dz) \\
    X_0 &= x, \quad Y_0 = y > 0, \quad u_t \in K
\end{align*}
\]

where \( K \) is a non-empty, closed, convex set, and \( g \) is the penalty function. \( x \) and \( y \) are two positive given numbers in \( \mathbb{R} \).

Suppose we fix the initial value \( x \) in (P2) the same as the initial price of the stock in (P1). If \( y \) is a feasible initial wealth of (P1), that is, there is a control \( u_t \in K \), such that \( Y_T \geq X_T \) a.s., then \( E[g(X_T - Y_T)] = 0 \). In other words, if \( y \) is a feasible initial wealth of (P1) for the given initial stock price \( x \), then the optimal value (P2) is zero under the same initial stock price and optimal control. Thus, once we get necessary conditions for (P2), we can characterize the feasibility of \( y \) in (P1).

In order to get necessary conditions of the auxiliary problem, we need to build a sequence of control problems and discuss their limit behaviours. We introduce a new function \( \phi(x) \) in this section as follows

\[
\phi(x) = \begin{cases} 
\frac{1}{c}e^{\frac{-1}{1-x^2}}, & |x| < 1 \\
0, & \text{otherwise}
\end{cases}
\]

where \( c = \int_{-1}^{1} e^{\frac{-1}{1-x^2}} dx \) is a finite number which makes \( \int_{-\infty}^{\infty} \phi(x)dx = 1 \). For any \( \varepsilon > 0 \), we define

\[
\phi_{\varepsilon}(x) = \frac{1}{\varepsilon} \phi\left(\frac{x}{\varepsilon}\right)
\]
Figure 4.2: The graph of function $\phi_\epsilon$

From Figure 2, we can see that the function $\phi_\epsilon$ tends to the delta distribution when $\epsilon$ tends to zero. We list some properties of these functions in the following lemma.

**Lemma 4.4.1.**

(a) $\int_{-\infty}^{\infty} \phi_\epsilon(x)dx = \int_{-\infty}^{\infty} \phi(x)dx = 1 \quad \forall \epsilon > 0$

(b) $\lim_{\epsilon \to 0} (\phi_\epsilon * g)(x) = g(x) \quad \forall x \in \mathbb{R}$

(c) Let $(\phi_\epsilon * g)'$ be the distributional derivative of $\phi_\epsilon * g$, then

$$\lim_{\epsilon \to 0} (\phi_\epsilon * g)'(x) = \begin{cases} 
0, & x < 0 \\
\frac{1}{2}, & x = 0 \\
1, & x > 0
\end{cases}$$
where * means the convolution operator, that is,

\[(\phi_\varepsilon * g)(x) = \int_{-\infty}^{\infty} \phi_\varepsilon(x - y)g(y)dy\quad \forall x\]

**Proof:**

(a)

\[
\int_{-\infty}^{\infty} \phi_\varepsilon(x)dx = \int_{-\infty}^{\infty} \frac{1}{\varepsilon}\phi\left(\frac{x}{\varepsilon}\right)dx
\]

\[= \int_{-\infty}^{\infty} \frac{1}{\varepsilon}\phi(y)\varepsilon dy\quad (y = \frac{x}{\varepsilon})\]

\[= \int_{-\infty}^{\infty} \phi(y)dy\]

\[= 1\]

(b) Notice that \(g\) is continuous and let \(\delta\) be the delta distribution

\[
\lim_{\varepsilon \to 0} (\phi_\varepsilon * g)(x) = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \phi_\varepsilon(x - y)g(y)dy
\]

\[= \int_{-\infty}^{\infty} \delta(x - y)g(y)dy\]

\[= g(x)\quad \forall x\]

(c) Notice that the distributional derivative of \(g\) is the Heaviside function, that is

\[
g'(x) = \begin{cases} 
  x, & x \geq 0 \\
  0, & x < 0
\end{cases}
\]
for $x < 0$ and $\varepsilon$ small enough

$$(\phi_\varepsilon * g)'(x) = (\phi_\varepsilon * g')(x)$$
$$= \int_{-\infty}^{\infty} \phi_\varepsilon(x - y)g'(y)dy \quad \forall x$$
$$= \int_{0}^{\infty} \phi_\varepsilon(x - y)dy$$
$$= \int_{-\infty}^{x} \phi_\varepsilon(z)dz \quad (z = x - y)$$
$$= 0$$

Thus,
$$\lim_{\varepsilon \to 0} (\phi_\varepsilon * g)'(x) = 0 \quad \forall x < 0$$

for $x > 0$ and $\varepsilon$ small enough

$$(\phi_\varepsilon * g)'(x) = \int_{-\infty}^{x} \phi_\varepsilon(z)dz$$
$$= 1$$

Thus,
$$\lim_{\varepsilon \to 0} (\phi_\varepsilon * g)'(x) = 1 \quad \forall x > 0$$

for $x = 0$ and $\varepsilon > 0$

$$(\phi_\varepsilon * g)'(x) = (\phi_\varepsilon * g')(0)$$
$$= \int_{-\infty}^{\infty} \phi_\varepsilon(0 - y)g'(y)dy$$
$$= \int_{0}^{\infty} \phi_\varepsilon(-y)dy$$
$$= \int_{-\infty}^{0} \phi_\varepsilon(y)dy$$
$$= \frac{1}{2}$$

Thus,
$$\lim_{\varepsilon \to 0} (\phi_\varepsilon * g)'(0) = \frac{1}{2}$$
The proof is done. □

Now, for each $\epsilon > 0$, we can build a $\epsilon - problem$

\[
\begin{align*}
\text{minimize} & \quad E[(\phi_\epsilon * g)(X_T - Y_T)] \\
\text{subject to} & \quad \\
X_t & = b_1(t, X_t, Y_t, u_t)dt + \sum_{j=1}^{2} \sigma_{1j}(t, X_t, Y_t, u_t)dB_j \\
& \quad + \sum_{j=1}^{2} \int_{\mathbb{R}_0} \gamma_{1j}(t, z, X_{t-}, Y_{t-}, u_{t-})N_j(dt, dz) \\
Y_t & = b_2(t, X_t, Y_t, u_t)dt + \sum_{j=1}^{2} \sigma_{2j}(t, X_t, Y_t, u_t)dB_j \\
& \quad + \sum_{j=1}^{2} \int_{\mathbb{R}_0} \gamma_{2j}(t, z, X_{t-}, Y_{t-}, u_{t-})N_j(dt, dz) \\
X_0 & = x, \quad Y_0 = y, \quad u_t \in K
\end{align*}
\]

define the Hamiltonian

\[
H : [0, T] \times \mathbb{R} \times \mathbb{R} \times K \times \mathbb{R}^{2 \times 2} \times \mathbb{R}^{2 \times 2} \longrightarrow \mathbb{R}
\]

by

\[
H(t, x, y, u, p, q, r) = p^Tb(t, x, y, u) + tr(q^T\sigma)(t, x, y, u) + \int_{(\mathbb{R}_0)^2} tr(r^T\gamma)(t, z, x, y, u)v(dz)
\]

According to the result we got in Chapter 3, we have a set of necessary conditions of the $\epsilon - problem$ as follows
\[\begin{align*}
  d \begin{bmatrix} X_t \\ Y_t \end{bmatrix} &= H_p(t, X_t, Y_t, u_t, p_t, q_t, r(t, z)) dt + \sigma(t, X_t, Y_t, u_t) dB_t \\
  &\quad + \int_{(\mathbb{R}_0)^2} \gamma(t, z, X_{t-}, Y_{t-}, u_{t-}) \mathcal{N}(dt, dz) \\
  \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} &= \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}\]

\[\begin{align*}
  d \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} &= -H_{(x,y)}(t, X_t, Y_t, u_t, p_t, q_t, r(t, z)) dt + q(t) dB_t + \int_{(\mathbb{R}_0)^2} r(t, z) \mathcal{N}(dt, dz) \\
  \begin{bmatrix} p_1(T) \\ p_2(T) \end{bmatrix} &= \begin{bmatrix} (\phi_{\varepsilon} * g)'(X_T - Y_T) \\ -(\phi_{\varepsilon} * g)'(X_T - Y_T) \end{bmatrix}
\end{align*}\]

\[H(t, X_t, Y_t, u_t, p_t, q_t, r(t, z))(v - u_t) \geq 0 \quad \forall v \in K\]

where

\[p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \quad q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}, \quad r = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}, \quad \gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}\]

\(p, q, r\) are costate processes of the \(\varepsilon - \) problem. Since \(\phi_{\varepsilon} * g\) tends to \(g\) as \(\varepsilon\) tends to zero, we can see the both the state and costate processes of \(\varepsilon - \) problems tends to theses of the auxiliary problem (P2). And, we have a set of necessary conditions for (P2) as follows.

\[\begin{align*}
  d \begin{bmatrix} X_t \\ Y_t \end{bmatrix} &= H_p(t, X_t, Y_t, u_t, p_t, q_t, r(t, z)) dt + \sigma(t, X_t, Y_t, u_t) dB_t \\
  &\quad + \int_{(\mathbb{R}_0)^2} \gamma(t, z, X_{t-}, Y_{t-}, u_{t-}) \mathcal{N}(dt, dz) \\
  \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} &= \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}\]
\[ d \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = -H(x,y)(t, X_t, Y_t, u_t, p_t, q_t, r(t, z))dt + q(t, X_t, Y_t, u_t)dB_t \\
+ \int_{(\mathbb{R}_0)^2} r(t, z, X_{t-}, Y_{t-}, u_{t-})\mathcal{N}(dt, dz) \]

\[ \begin{bmatrix} p_1(T) \\ p_2(T) \end{bmatrix} = \begin{bmatrix} \lim_{\varepsilon \to 0} (\phi_{\varepsilon} * g)'(X_T - Y_T) \\ - \lim_{\varepsilon \to 0} (\phi_{\varepsilon} * g)'(X_T - Y_T) \end{bmatrix} \]

\[ H(t, X_t, Y_t, u_t, p_t, q_t, r(t, z))(v - u_t) \geq 0 \quad \forall v \in K \]

Since we suppose the optimal trajectory does hit the boundary of the target at maturity time \( T \), that is \( Y_T^* = X_T^* \), by Lemma 4.1, we know the boundary condition of costate processes is that

\[ p(T) = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \]

Thus, we can characterize the feasibility of (P1) by the following proposition

**Proposition 4.4.2.** For \( y \in \mathbb{R} \), if there are \( u_t, X_t, \) and \( Y_t \) satisfying (P1) with identity function \( f \) and \( Y_T = X_T \) a.s., then, we can find processes \( p, q, r \) such that,

\[ d \begin{bmatrix} X_t \\ Y_t \end{bmatrix} = H_p(t, X_t, Y_t, u_t, p_t, q_t, r(t, z))dt + \sigma(t, X_t, Y_t, u_t)dB_t \\
+ \int_{(\mathbb{R}_0)^2} \gamma(t, z, X_{t-}, Y_{t-}, u_{t-})\mathcal{N}(dt, dz) \quad (C1) \]

\[ \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad (C2) \]
\[
\begin{align*}
    d \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} &= -H_{(x,y)}(t, X_t, Y_t, u_t, p_t, q_t, r(t, z))dt + q(t, X_t, Y_t, u_t)dB_t \\
    &\quad + \int_{(\mathbb{R}_0)^2} r(t, z, X_{t-}, Y_{t-}, u_{t-})\mathcal{N}(dt, dz) \quad (C3)
\end{align*}
\]

\[
    p(T) = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \quad (C4)
\]

\[
    H(t, X_t, Y_t, u_t, p_t, q_t, r(t, z))(v - u_t) \geq 0 \quad \forall v \in K \quad (C5)
\]

Furthermore, we can conclude the following proposition

**Proposition 4.4.3.** (P1) with constraint \( Y_T = X_T \) a.s. and identity function \( f \) is equivalent to

\[
\begin{align*}
    \text{minimize} \quad y & \quad \text{(P3)} \\
    \text{subject to} & \\
    (C1), (C2), (C3), (C4), (C5)
\end{align*}
\]

If the optimal trajectory does not hit the boundary of the target at time \( T \), the boundary condition of costate equation becomes

\[
p(T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Let’s consider the more general case with non-identity function \( f \). For any smooth function \( f \), let \( \overline{X}_t = f(X_t) \). The dynamics of \( \overline{X}_t \) can be found by Ito’s formula.

the objective function of (P2) becomes

\[
E[g(f(X_T) - Y_T)] = E[g(\overline{X}_T - Y_T)]
\]
and the objective function of the $\varepsilon$-problem becomes

$$E[(\phi_\varepsilon \ast g)(f(X_T) - Y_T)] = E[(\phi_\varepsilon \ast g)(\bar{X}_T - Y_T)]$$

follow the same procedure above, we can derive the feasibility condition of (P1). Furthermore, (P1) is equivalent to the problem

$$\begin{align*}
\text{minimize} & \quad y \\
\text{subject to} & \quad (C1), (C2), (C3), (C4), (C5) \text{ with } \bar{X}_t \text{ instead of } X_t
\end{align*}$$

4.5 When the optimal trajectory does not hit the boundary of the target at maturity time $T$

Suppose $(u^*_t, y^*)$ is the optimal pair of the 1-dimensional target problem for some given initial stock price $x$, and $X^*_t, Y^*_t$ are the corresponding stock price and wealth processes. In this section, we suppose the optimal trajectory does not hit the boundary of the target at maturity time $T$, that is

$$Y^*_T > f(X^*_T) \text{ a.s.}$$

Firstly, we consider the simplest case, let $f$ be the identity function. That is, $f(x) = x$ for all $x \in \mathbb{R}$. We construct the auxiliary problem
\[
\begin{align*}
\text{minimize} & \quad E[g(X_T - Y_T)] \quad (P4) \\
\text{subject to} & \quad \\
& dX_s = b_1(s, X_s, Y_s, u_s)ds + \sum_{j=1}^{2} \sigma_{1j}(s, X_s, Y_s, u_s)dB_j \\
& \quad + \sum_{j=1}^{2} \int_{\mathbb{R}_0} \gamma_{1j}(s, z, X_{s-}, Y_{s-}, u_{s-})N_j(ds, dz_j) \quad s > t \\
& dY_s = b_2(s, X_s, Y_s, u_s)ds + \sum_{j=1}^{2} \sigma_{2j}(s, X_s, Y_s, u_s)dB_j \\
& \quad + \sum_{j=1}^{2} \int_{\mathbb{R}_0} \gamma_{2j}(s, z, X_{s-}, Y_{s-}, u_{s-})N_j(ds, dz_j) \quad s > t \\
& X_t = x, \quad Y_t = y, \quad u_s \in K
\end{align*}
\]

where \( t \) is given in \([0, T]\), and \( u_s \) is a Markov control. That is, the stochastic process \( u_s \) can be written in this form

\[
u_s = u(s, \omega) = u(s, X_s(\omega), Y_s(\omega)) \quad s \in [t, T]
\]

It can be shown that under some mild conditions, it is sufficient to consider Markov controls (see \([2]\), e.g., [Oksendal, Theorem 11.2.3]) for a general control problem. \( K \) is a non-empty, closed, convex set, and \( g \) is the penalty function. \( x \) and \( y \) are two positive given numbers in \( \mathbb{R} \). If we fixed the starting time \( t \in [0, T] \), and the starting point of \( X_t = x \) and \( Y_t = y \). Define the function

\[
\Phi(t, x, y) = \min E[g(X_T - Y_T)]
\]

where \( u_t \) is a Markov control in \( K \). \( \Phi \) is called the value function of \((P4)\). We suppose \((P4)\) has one and only one solution for all given \( t \), \( x \), and \( y \), then the value function \( \Phi \) is well-defined. Furthermore, we suppose \( \Phi \) is continuous up to its second derivative in both \( x \) and \( y \), and its first derivative in \( t \). Thus, people can use dynamic programming to derive the HJB equation of \((P4)\) as follows
\[ \Phi_t(t, x, y) = \max_{v \in K} \left\{ -b_1(t, x, y, v)\Phi_x(t, x, y) - b_2(t, x, y, v)\Phi_y(t, x, y) - \frac{1}{2} \text{tr} \left( \nabla_x y \Phi(t, x, y) (\sigma \sigma^T)(t, x, y, v) \right) \right. \\
- \sum_{j=1}^{2} \int_{\mathbb{R}_0} \Phi(t, x + \gamma_1(t, z_j, x, y, v), y + \gamma_2(t, z_j, x, y, v)) - \Phi(t, x, y) \\
- \Phi_x(t, x, y)\gamma_1(t, z_j, x, y, v) - \Phi_y(t, x, y)\gamma_2(t, z_j, x, y, v))v_j(dz_j) \right\} \]

for all \( t \in [0, T] \), and \( x, y \in \mathbb{R} \). With the boundary condition

\[ \Phi(T, x, y) = g(x - y) \quad \forall x, y \]

And the above optimization problem has solution along the optimal control. That is

\[ 0 = \Phi_t(t, x, y) + b_1(t, x, y, u(t, x, y))\Phi_x(t, x, y) + b_2(t, x, y, u(t, x, y))\Phi_y(t, x, y) \\
+ \frac{1}{2} \text{tr} \left( \nabla_x y \Phi(t, x, y) (\sigma \sigma^T)(t, x, y, u(t, x, y)) \right) \]

\[ + \sum_{j=1}^{2} \int_{\mathbb{R}_0} \Phi(t, x + \gamma_1(t, z_j, x, y, u(t, x, y)), y + \gamma_2(t, z_j, x, y, u(t, x, y))) - \Phi(t, x, y) \\
- \Phi_x(t, x, y)\gamma_1(t, z_j, x, y, u(t, x, y)) - \Phi_y(t, x, y)\gamma_2(t, z_j, x, y, u(t, x, y)))v_j(dz_j) \]

for all \( t \in [0, T] \), and \( x, y \in \mathbb{R} \)

We can use the information from the value function of (P4) to tell the feasibility of each \( y \) in the 1-dimensional target problem (P1). We fix the initial stock price \( x \). Since the condition \( g(X_T - Y_T) = 0 \) a.s. is equivalent to \( Y_T \geq X_T \) a.s., we conclude that the (P1) is equivalent to the problem

\[ \min \ y \quad \text{(P5)} \]

subject to

\[ \Phi(0, x, y) = 0 \]

The \( \Phi \) can be characterized by the HJB equation with boundary condition \( \Phi(T, x, y) = g(x - y) \).
(P5) is numerically solvable.

When we want to consider the more general function $f(x)$, we can use Ito’s formula to modify the discussion we have made above. For any smooth function $f$, let $\overline{X}_t = f(X_t)$, by Ito’s formula, we can find its dynamics and the objective function of (P4) becomes

$$E[g(f(X_T) - Y_T)] = E[g(\overline{X}_T - Y_T)]$$

Follow the same procedure above, we can derive the HJB equation for the auxiliary problem to characterize its value function $\Phi$. Furthermore, (P1) is equivalent to the problem

$$\min \ y$$
$$subject \ to$$
$$\Phi(0, x, y) = 0$$

4.6 n-dimensional stochastic target problem with jump diffusion

When we have more than one risky asset, we need to adjust our model to encode the interdependence between different security prices. And it can be achieved by taking the driven Brownian motion and Poisson random measure to be correlated. People can show that, it is equivalently to consider independent Brownian motions and Poisson random measure and drive the asset prices by linear combinations of these.

In this section, we suppose the wealth process $Y_t = Y(t) = Y(t, \omega)$ is 1-dimensional process but the stock prices process $X_t = X(t) = X(t, \omega)$ is $n$-dimensional. $B_t = [B_j]_{1 \leq j \leq n}$ is an $n$-dimensional Brownian motion and $N(dt, dz) = [N_j(dt, dz)]_{1 \leq j \leq n}$ is an $n$-dimensional Poisson random measure. Both of them are column vectors with independent component and independent to each other. Consider the following problem
\begin{align*}
\text{minimize} & \quad y & \quad \text{(P6)} \\
\text{subject to} & \\
\quad dX_i(t) & = b_i(t, X(t), Y(t), u(t))dt + \sum_{j=1}^{n+1} \sigma_{ij}(t, X(t), Y(t), u(t))dB_j \\
& \quad + \sum_{j=1}^{n+1} \int_{\mathbb{R}_0} \gamma_{ij}(t, z, X(t-), Y(t-), u(t-))N_j(dt, dz_j) \quad \forall 1 \leq i \leq n \\
\quad dY(t) & = b_{n+1}(t, X(t), Y(t), u(t))dt + \sum_{j=1}^{n+1} \sigma_{n+1,j}(t, X(t), Y(t), u(t))dB_j \\
& \quad + \sum_{j=1}^{n+1} \int_{\mathbb{R}_0} \gamma_{n+1,j}(t, z, X(t-), Y(t-), u(t-))N_j(dt, dz_j) \\
Y(T) & \geq f(X(T)) \quad \text{a.s.} \\
X_i(0) & = x_i > 0, \quad \forall 1 \leq i \leq n \quad Y(0) = y > 0, \quad u(t) \in K
\end{align*}

We can rewrite (P6) in the matrix form

\begin{align*}
\text{minimize} & \quad y & \quad \text{(P6)} \\
\text{subject to} & \\
\quad d \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} & = b(t, X(t), Y(t), u(t))dt + \sigma(t, X(t), Y(t), u(t))dB_t \\
& \quad + \int_{(\mathbb{R}_0)^{n+1}} \gamma(t, z, X(t-), Y(t-), u(t-))N(dt, dz) \\
Y(T) & \geq f(X(T)) \quad \text{a.s.} \\
X_i(0) & = x_i > 0, \quad \forall 1 \leq i \leq n \quad Y(0) = y > 0, \quad u(t) \in K
\end{align*}

where $x$ is a given positive vector in $\mathbb{R}^n$ and $f$ is a real-valued function. $K$ is a non-empty, closed, convex set. $b = [b_i]_{1 \leq i \leq n}$ is a column vector, $\sigma = [\sigma_{ij}]$ and $\gamma = [\gamma_{ij}]$ are $n$ by $n$ matrix. (P6) is an $n$-dimensional target problem. $Y(T) \geq f(X(T))$ a.s. is the target we need to achieve.
at the maturity time $T$.

Suppose the coefficient functions $b, \sigma, \gamma$ satisfied the usual conditions such that (P6) has one and only one solution, we denote the unique optimal solution pair by $(u^*(t), y^*)$ and the corresponding stock and wealth processes by $X^*(T)$ and $Y^*(T)$. By Ito’s formula, we can find the differential form of the 1-dimensional process $\bar{X}(t) = f(X(t))$

$$
d\bar{X}(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \left( b_i dt + \sum_{j=1}^{n} \sigma_{i,j} dB_j \right) + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma \sigma^T)_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dt \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} f(X(t) + \gamma_j) - f(X(t)) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(t))\gamma_{i,j} v_j(dz_j) dt \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}_0} f(X(t) + \gamma_j) - f(X(t)) N_j(dt,dz_j)
$$

where $\gamma_j$ is the $j$th column of matrix $\gamma$. Thus, the $n$-dimensional target problem has the form

$$
\min \quad y \\
subject \ to \\
Y(T) \geq \bar{X}(T) \quad a.s.
$$

which is a 1-dimensional target problem and can be reformulated into (P3) or (P5) depending on the behaviour of optimal trajectory at maturity time $T$.

### 4.7 Conclusion

In this section, we gave the target problem two different formulations. If the optimal trajectory hits the boundary of the target at maturity time $T$, we use the tools we developed in Chapter 3 to get (P3). On the other hand, if the optimal trajectory does not hit the boundary of the target at maturity time $T$, we use dynamic programming to get another formulation (P5). In the next chapter, we are going to solve them numerically.
Chapter 5

Application in finance

5.1 Introduction

The most popular option pricing model presented by Black-Scholes in 1973 uses a pure diffusion process to describe the uncertainty of stock price. However, there is one assumption that the stock price is always continuous and this rules out the important feature of large fluctuation in the market, such as crashes or rallies. In order to characterize this feature, people include jumps in the financial market models. The jump term should present uncertainties in both time and response. For simulation purpose, we suppose the Levy measure associated with the jump term is finite, so the Levy process can be represented by a compound Poisson process. In other words, we consider the special case that there are only finitely many jumps in each finite interval. Moreover, we suppose the size of jumps are independent and identically normal distributions. The justification for jumps to be modelled by Poisson process can be found in [24].

The jump term cannot only allow discontinuity in the dynamics but also present some other quantitative features in the financial market which cannot be described by pure diffusion processes. In the real market, what investors really care about is the log-return process, because people are interested in the relative change in the stock price instead of the price itself. There are two important features of the log-return distribution in financial market.

1. The skewness coefficient is negative.
2. The kurtosis coefficient is greater than three.

The definitions of these coefficients are in (4) and (5). The skewness coefficient is a statistical quantity used to describe the tendency of a distribution. The normal distribution is symmetric about its mean and has skewness zero. The negative skewness of log-return distribution means
that the real market is found to be pessimistic in the long run. The kurtosis coefficient is used to describe the flatness of a distribution. We say a distribution is platokurtic or leptokurtic if its kurtosis coefficient is less than or larger than three. We are using normal distribution as a standard and it has kurtosis three. In financial market, the two statistical quantities characterize the fat-tail feature of the log-return distribution when comparing to the normal distribution. We will see an example in the next chapter.

In this chapter, we present an option pricing model which is different from Black-Scholes. In Section 2, we introduce the stock price process with jump and derive some properties which will be used in Chapter 6 when discussing the simulation. In Section 3, we include the bank account process and construct our model. This is a small investor model. In Section 4, we use least square method to estimate the parameters in our model. For the large investor case, we will briefly describe an example in Chapter 7. We follow Hanson’s approach ([21][22][23]) in this chapter.

5.2 Stock price process

Everything we set in the following chapters is based on the complete probability space $(\Omega, F, P)$ and the filtration $\{F_t\}_{t \geq 0}$ which is constructed by appropriate Brownian motion and Poisson process.

Let $X_t$ be the price of a single stock in the financial market and it satisfies

$$\begin{equation}
  dX_t = X_t(\mu_D \, dt + \sigma_D \, dW_t + J(Q)\,dP_t)
\end{equation}$$

where $X(0) = x_0 > 0$ is given, $\mu_D$ is the return appreciation rate, $\sigma_D$ is the return volatility associated with the diffusion part, $W_t$ is a one-dimensional Brownian motion, and $P_t$ is a one-dimensional Poisson Process with rate $\lambda$. We will assume that the parameters are all constant. Suppose the jump amplitude, $J(Q)$, is a function of independent and identically distributed (i.i.d.) normal random variables with mean $\mu_J$ and variance $\sigma^2_J$.

$$Q \sim N(\mu_J, \sigma^2_J)$$

The subscript $J$ indicates that they are associated with the jump. All the processes $W_t$, $P_t$, and $J(Q)$ are mutually independent. $dP_t$ is the number of jumps and $J(Q)\,dP_t$ is the sum of jumps which occur in time interval $dt$. Therefore, the symbol $J(Q)\,dP_t$ in the last term of (1) actually can be defined more rigorously by using a compound Poisson Process or an integral of
Poisson random measure \[23\]. That is,

\[\int_R X_t J(Q) N(dP_t, dQ) = \int_R X_t J(q) \frac{1}{\sqrt{2\pi \sigma^2}} \exp\left(-\frac{(q - \mu)^2}{2\sigma^2}\right) N(dP_t, dq)\]

\[= \sum_{k=1}^{dP_t} X_t J(Q_k)\]

\[\equiv X_t J(Q) dP_t\]

Where \(Q_1, Q_2, ..., Q_k\) are i.i.d. normal random variables. \(N(dP_t, dQ))\) is a Poisson random measure. Just as we mentioned in the introduction, the log-return process can describe the relative change in stock price and investors interested in it. We can find its dynamics by Itos formula,

\[d \ln(X_t) = (\mu_D - 0.5\sigma_D^2) dt + \sigma_D dW_t + \ln(1 + J(Q)) dP_t\]

For simplicity purpose and making the above equation well-defined, we specify the function \(J(q) = e^q - 1\). Then the log-return process becomes

\[d \ln(X_t) = (\mu_D - 5\sigma_D^2) dt + \sigma_D dW_t + Q dP_t\]  \hfill (2)

We will list some important properties of \(X_t\) and \(d \ln(X_t)\) in the rest of this section.

**Proposition 5.2.1.** The equation \((1)\) has explicit solution as

\[X_t = x_0 \exp\left((\mu_D - 5\sigma_D^2) t + \sigma_D W_t + \sum_{k=1}^{dP_t} Q_k\right)\]

**Proof:**
We start from equation (2)

\[ d \ln(X_t) = (\mu_D - 0.5\sigma_D^2) dt + \sigma_D dW_t + QdP_t \]

\[ = (\mu_D - 0.5\sigma_D^2) dt + \sigma_D dW_t + \sum_{k=1}^{dP_t} Q_k \]

\[ \ln(X_t) = \ln(x_0) + \int_0^t (\mu_D - 0.5\sigma_D^2) ds + \int_0^t \sigma_D dW_s + \sum_{k=1}^{P_t} Q_k \]

\[ = \ln(x_0) + (\mu_D - 0.5\sigma_D^2) t + \sigma_D W_t + \sum_{k=1}^{P_t} Q_k \]

\[ d \ln \left( \frac{X_t}{x_0} \right) = (\mu_D - 0.5\sigma_D^2) dt + \sigma_D dW_t + \sum_{k=1}^{dP_t} Q_k \]

Taking exponential on both sides. We obtain

\[ X_t = x_0 \exp \left( (\mu_D - 0.5\sigma_D^2) t + \sigma_D W_t + \sum_{k=1}^{P_t} Q_k \right) \]

The proof is done. □

According to Proposition 2.1 and initial state \( x_0 > 0 \), we have a reasonable corollary that the stock price \( X_t \) is always positive. Moreover, we can find out the probability density function of \( X_t \).

**Proposition 5.2.2.** The probability density function of \( X_t \) is

\[ \phi_{X_t}(a) = \frac{1}{a} \sum_{k=0}^{\infty} \phi(\ln(a) - \ln(x_0); (\mu_D - 0.5\sigma_D^2)t + k\mu_j, \sigma_D^2t + k\sigma_j^2) e^{-\lambda t} \frac{(\lambda t)^k}{k!} \]

for \( a > 0 \) and \( \phi_{X_t}(a) = 0 \) for \( a \leq 0 \). Where \( \phi(a; \mu, \sigma^2) \) is the density of normal distribution with mean \( \mu \) and variance \( \sigma^2 \).

**Proof:**

102
We start from the cumulated density function of $X_t$. For $a > 0$

$$
\Phi_{X_t}(a) = \text{Prob}(X_t \leq a)
\quad = \text{Prob}\left(x_0 \exp\left((\mu_D - .5\sigma_D^2) t + \sigma_D W_t + \sum_{k=1}^{P_t} Q_k\right) \leq a\right)
\quad = \text{Prob}\left(\exp\left((\mu_D - .5\sigma_D^2) t + \sigma_D W_t + \sum_{k=1}^{P_t} Q_k\right) \leq \frac{a}{x_0}\right)
\quad = \text{Prob}\left((\mu_D - .5\sigma_D^2) t + \sigma_D W_t + \sum_{k=1}^{P_t} Q_k \leq \ln(a) - \ln(x_0)\right)
\quad = \sum_{i=0}^{\infty} \text{Prob}\left((\mu_D - .5\sigma_D^2) t + \sigma_D W_t + \sum_{k=1}^{P_t} Q_k \leq \ln(a) - \ln(x_0) \mid P_t = i\right) \text{Prob}(P_t = i)
\quad = \sum_{i=0}^{\infty} \text{Prob}\left((\mu_D - .5\sigma_D^2) t + \sigma_D W_t + \sum_{k=1}^{i} Q_k \leq \ln(a) - \ln(x_0)\right) \text{Prob}(P_t = i)
$$

Notice that $(\mu_D - .5\sigma_D^2) t + \sigma_D W_t + \sum_{k=1}^{i} Q_k$ is a normal distribution with mean $(\mu_D - .5\sigma_D^2) t + i\mu_J$ and variance $\sigma_D^2 t + i\sigma_J^2$. Therefore, the cumulated density becomes

$$
\Phi_{X_t}(a) = \sum_{k=0}^{\infty} \Phi(\ln(a) - \ln(x_0) ; (\mu_D - .5\sigma_D^2)t + k\mu_J, \sigma_D^2 t + k\sigma_J^2) e^{-\frac{\lambda t}{k!}}
$$

Where $\Phi$ is the cumulated density of normal distribution. Then, the density of $X_t$ can be obtained by taking derivative with respect to $a$

$$
\phi_{X_t}(a) = \frac{1}{a} \sum_{k=0}^{\infty} \phi(\ln(a) - \ln(x_0) ; (\mu_D - .5\sigma_D^2)t + k\mu_J, \sigma_D^2 t + k\sigma_J^2) e^{-\frac{\lambda t}{k!}}
$$

Since the $X_t$ is always positive, $\phi_{X_t}(a) = 0$ for all $a < 0$.

The proof is done. \(\square\)

We will have the most interest in the stock price density at terminal time $T$ in Chapter 6, since it is the terminal condition of Bellman’s equation in our option pricing model. The density of $X_T$ is

$$
\phi_{X_T}(a) = \frac{1}{a} \sum_{k=0}^{\infty} \phi(\ln(a) - \ln(x_0) ; (\mu_D - .5\sigma_D^2)T + k\mu_J, \sigma_D^2 T + k\sigma_J^2) e^{-\frac{\lambda T}{k!}}
$$

(3)
Proposition 5.2.3. The probability density function of $d \ln(X_t)$ is

$$
\phi_{d \ln(X_t)}(a) = \sum_{k=0}^{\infty} \phi(a; (\mu_D - .5\sigma_D^2)dt + k\mu_J, \sigma_D^2 dt + k\sigma_J^2) e^{-\lambda dt} \frac{(\lambda dt)^k}{k!}
$$

for $a \in \mathbb{R}$.

Proof:

We start from the cumulated density function of $d \ln(X_t)$.

$$
\Phi_{d \ln(X_t)}(a) = \text{Prob}(d \ln(X_t) \leq a)
$$

$$
= \text{Prob}\left((\mu_D - .5\sigma_D^2) dt + \sigma_D dW_t + \sum_{k=1}^{dP_t} Q_k \leq a\right)
$$

$$
= \sum_{i=0}^{\infty} \text{Prob}\left((\mu_D - .5\sigma_D^2) dt + \sigma_D dW_t + \sum_{k=1}^{dP_t} Q_k \leq a \mid dP_t = i\right) \text{Prob}(dP_t = i)
$$

$$
= \sum_{i=0}^{\infty} \Phi(a; (\mu_D - .5\sigma_D^2)dt + k\mu_J, \sigma_D^2 dt + k\sigma_J^2) e^{-\lambda dt} \frac{(\lambda dt)^k}{k!}
$$

Where $\Phi$ is the cumulated density of normal distribution. Then, the density of $d \ln(X_t)$ can be obtained by making derivative with respect to $a$

$$
\phi_{X_t}(a) = \sum_{k=0}^{\infty} \phi(a; (\mu_D - .5\sigma_D^2)dt + k\mu_J, \sigma_D^2 dt + k\sigma_J^2) e^{-\lambda dt} \frac{(\lambda dt)^k}{k!}
$$

The proof is done. □

The density of log-return $d \ln X_t$ is useful when estimating the parameters. In the next Proposition, we compute the mean and variance of $d \ln(X_t)$.

Proposition 5.2.4. The mean and variance of $d \ln(X_t)$

$$
E[d \ln(X_t)] = (\mu_D - .5\sigma_D^2) dt + \mu_J \lambda dt
$$

$$
Var[d \ln(X_t)] = \sigma_D^2 dt + (\sigma_J^2 + \mu_J^2) \lambda dt
$$
Proof:
The jump term in $d \ln(X_t)$ is the compound Poisson Process $\sum_{k=1}^{dP_t} Q_k$. Firstly, let’s consider its mean and variance.

$$
E \left[ \sum_{k=1}^{dP_t} Q_k \right] = \sum_{i=0}^{\infty} \sum_{k=1}^{i} E \left[ Q_k \right] \text{Prob}(dP_t = i)
$$

$$
= \sum_{i=0}^{\infty} i \mu_J \text{Prob}(dP_t = i)
$$

$$
= \mu_J E[dP_t]
$$

$$
= \mu_J \lambda dt
$$

$$
Var \left[ \sum_{k=1}^{dP_t} Q_k \right] = E \left[ \left( \sum_{k=1}^{dP_t} Q_k \right)^2 \right] - (\mu_J \lambda dt)^2
$$

$$
= \sum_{i=0}^{\infty} E \left[ \left( \sum_{k=1}^{i} Q_k \right)^2 \right] \text{Prob}(dP_t = i) - (\mu_J \lambda dt)^2
$$

Since

$$
\sum_{k=1}^{i} Q_k \sim N(i\mu_J, i\sigma_J^2)
$$

we have

$$
Var \left[ \sum_{k=1}^{dP_t} Q_k \right] = \sum_{i=0}^{\infty} (i\sigma_J^2 + i^2 \mu_J^2) \text{Prob}(dP_t = i) - (\mu_J \lambda dt)^2
$$

$$
= \sigma_J^2 \lambda dt + \mu_J^2 (\lambda dt + \lambda^2 dt^2) - (\mu_J \lambda dt)^2
$$

$$
= \sigma_J^2 \lambda dt + \mu_J^2 \lambda dt
$$

$$
= (\sigma_J^2 + \mu_J^2) \lambda dt
$$

Then, we can obtain the mean and variance of $d \ln(X_t)$. 

105
\[
E [d \ln(X_t)] = E \left[ (\mu_D - .5\sigma_D^2) \, dt + \sigma_D \, dW_t + QdP_t \right] \\
= E [(\mu_D - .5\sigma_D^2) \, dt] + E [\sigma_D \, dW_t] + E [QdP_t] \\
= (\mu_D - .5\sigma_D^2)dt + \mu_J \lambda dt
\]

\[
\text{Var} [d \ln(X_t)] = \text{Var} \left[ (\mu_D - .5\sigma_D^2) \, dt + \sigma_D \, dW_t + QdP_t \right] \\
= \text{Var} [(\mu_D - .5\sigma_D^2) \, dt] + \text{Var} [\sigma_D \, dW_t] + \text{Var} [QdP_t] \\
= \sigma_D^2 dt + (\sigma_J^2 + \mu_J^2) \lambda dt
\]

The proof is done. □

The Skewness and Kurtosis coefficients are defined as follows

Skewness coefficient of \( d \ln(X_t) \)

\[
E \left[ \left( \frac{d \ln(X_t) - E[d \ln(X_t)]}{\sqrt{\text{Var}[d \ln(X_t)]}} \right)^3 \right] \tag{4}
\]

Kurtosis coefficient of \( d \ln(X_t) \)

\[
E \left[ \left( \frac{d \ln(X_t) - E[d \ln(X_t)]}{\sqrt{\text{Var}[d \ln(X_t)]}} \right)^4 \right] \tag{5}
\]

They are the normalized third and fourth moments of random variables. It is well known that the standard normal distribution has skew zero and kurtosis three. In financial market, the skewness is often negative and kurtosis is greater than three.

5.3 The model

We consider a financial market consisting of

1. a risky asset (stock) \( X_t \) with positive price process. Its dynamics is described in previous
sections and we repeat it here

\[ dX_t = X_t(\mu_D \, dt + \sigma_D \, dW_t + J(Q)dP_t) \]

2. a non-risky asset (bank account) \( B_t \) satisfies

\[ dB_t = rB_t dt \]

where \( r \) is a constant and \( B_0 = 1 \). The stochastic process \( u_t \) is the proportion of our wealth invested in the risky asset, \( 0 \leq u_t \leq 1 \). The proportion of our wealth invested in the non-risky asset is \( 1 - u_t \). \( Y_t \) is the amount of wealth we hold at time \( t \) and

\[
\begin{align*}
\frac{dY_t}{Y_t} &= u_t \frac{dX_t}{X_t} + (1 - u_t) \frac{dB_t}{B_t} \\
&= Y_t [u_t (\mu_D \, dt + \sigma_D \, dW_t + J(Q)dP_t) + (1 - u_t) r \, dt] \\
&= Y_t [(u_t \mu_D + (1 - u_t) r) \, dt + u_t \sigma_D dW_t + u_t J(Q)dP_t]
\end{align*}
\]

the initial value of \( Y_t \) needs to be determined in the target problem

\[ Y_0 = y_0 > 0 \]

Suppose we are the seller of a European call option with strike price \( K \) and terminal time \( T \). We need to ask ourselves what is the minimal price we should charge for this option at initial time and what is the hedging strategy for us to promise that at least we will not lose money for sure. We formulate this problem as following

\[
\begin{align*}
\text{minimize} & \quad y_0 \\
\text{subject to} & \\
Y_T & \geq \max(X_T - K, 0) \quad \text{a.s.} \quad (P1) \\
0 & \leq u_t \leq 1, \quad y_0 > 0
\end{align*}
\]

This is a small investor target problem with control \( u_t \) and \( y_0 \). Letting the random variable

\[ Z = \max(X_T - K, 0) \]
According to the techniques we developed in Chapter 4, we fix a \( y > 0 \) and consider the auxiliary control problem

\[
\begin{align*}
\text{minimize} & \quad E[\max(Z - Y_T, 0)] \\
\text{subject to} & \quad 0 \leq u_t \leq 1
\end{align*}
\]

This is a classical stochastic control problem and we have two strategies to solve it numerically, Maximum principle and Dynamic programming. We will choose the second approach. Since our control cannot affect the stock price in this model, the wealth process can completely describe the dynamics and the value function, \( \Phi(t, y) \), has only two arguments \( t \) and \( y \) (without \( x \)).

Following the notation in Section 3.5, we define

\[
F(t, y, u) = -\Phi_t(t, y) - y(u\mu_D + (1-u)r)\Phi_y(t, y) - .5\sigma_D^2 y^2 u^2 \Phi_{yy}(t, y)
\]

\[
-\lambda \int_R (\Phi(t, y + uy(e^q - 1)) - \Phi(t, y)) \frac{1}{\sqrt{2\pi\sigma_J^2}} \exp\left(-\frac{(q - \mu_J)^2}{2\pi\sigma_J^2}\right) dq
\]

and have the HJB equation

\[
\max_{0 \leq u \leq 1} F(t, y, u) = F(t, y, u^*(t, y)) = 0
\]

To simplify the notation, we will discard the function \( F \) in the following context and conclude that \( \Phi(t, y) \) satisfies

\[
\Phi_t(t, y) = \max_{0 \leq u \leq 1} \left\{-yru\Phi_y(t, y) - y(u\mu_D - r)u\Phi_y(t, y) - .5\sigma_D^2 y^2 u^2 \Phi_{yy}(t, y)
\right. \\
-\lambda \int_R (\Phi(t, y + uy(e^q - 1)) - \Phi(t, y)) \frac{1}{\sqrt{2\pi\sigma_J^2}} \exp\left(-\frac{(q - \mu_J)^2}{2\pi\sigma_J^2}\right) dq \}
\]

for all \( 0 < t < T \) and \( y > 0 \). With boundary condition

\[
\Phi(T, y) = E[\max(Z - y, 0)] \quad y > 0;
\]
After characterizing the value function of (P2), the target problem (P1) is equivalent to

\[
\begin{align*}
\text{minimize} & \quad y_0 \\
\text{subject to} & \quad \Phi(0, y_0) = 0 \quad y_0 > 0;
\end{align*}
\]

We will see how to solve it numerically in the next chapter

### 5.4 Model parameter estimation

Firstly, we focus on the parameters which are associated with the stock price process \( X_t \) and let \( \theta \) be the parameter vector.

\[
\theta = [\mu_D, \sigma_D, \mu_J, \sigma_J, \lambda]
\]

The main idea of this section is to find a \( \theta \) that minimizes the distance between empirical and theoretical histogram in the sense of least squares. We separate the procedure in several steps.

**Step 1: Collect data**

We collect 2013 daily closings of Standard and Poors 500 (S&P500) stock index from Jan 1997 to Jan 2005, and denote them by \( X_ı \), for \( ı = 1, 2, ..., 2013 \). The S&P500 is not a single stock; however, by using it, we can avoid extreme behaviours of any one stock. As we mentioned before, investors are interested in the log-return of stock price, so we will use its distribution to find out the parameters in our model. The empirical log-return data are constructed by

\[
d \ln(X_ı) = \ln(X_{ı+1}) - \ln(X_ı) \quad \text{for } ı = 1, 2, ..., 2012
\]

Figure (1) is the scatter plot of the \( d \ln(X_ı) \)'s. The solid red line in Figure (1) is the linear fit and the dash lines are linear fit with three standard deviations. The linear fit is approximately horizontal and the variance of the scatters is uniform, so the constant-parameter assumption is reasonable for the data we are using.

**Step 2: Analyze empirical histogram**

Figure (2) is the histogram of empirical data with 100 bins. We can use MATLAB command to find the first two sample moments and coefficients directly as follows
Figure 5.1: S&P stock index log-return

\[ \text{Mean} = 0.000247 \]
\[ \text{Variance} = 0.000155 \]
\[ \text{Skewness coefficient} = -0.070301 \]
\[ \text{Kurtosis coefficient} = 5.433382 \]

Note that the skewness is negative and the kurtosis is greater than three. As what we expect, the log-return distribution in Figure (2) has the fat-tail feature when comparing to standard normal one. In other words, the log-return tends to favor the negative side rather than the positive one. One explanation of this feature is that the real market is pessimistic in the long run. The height of empirical histogram in each bin is denoted by \( h_i^{ch} \), for \( i = 1, 2, \ldots, 100 \).
Step 3: Analyze theoretical histogram

We have found the density function of $d \ln(X_t)$ in previous sections. Since the density depends on parameters which are unknown, we need to include $\theta$ in its argument and denote it by $\phi_{d \ln(X_t)}(a; \theta)$. We will construct a histogram with the same bins in Step 2 by using $\phi_{d \ln(X_t)}(a; \theta)$. Suppose we have $n$ samples ($n = 2013$ in our case) and the height of the theoretical histogram in each bin is $h_{i}^{th}(\theta)$, for $i = 1, 2, ..., 100$. Then

$$h_{i}^{th}(\theta) = n \int_{B_i} \phi(a; \theta) \, da$$

$$\approx n \phi(b_i; \theta) |B_i|$$
Step 4: The least squares method

For each bin, we have two heights:
1. The height of empirical histogram $h_\text{eh}^i$ which is known explicitly.
2. The height of theoretical histogram $h_\text{th}^i(\theta)$ which is a function of $\theta$.

Consider the least squares problem

$$\min_{\theta} \sum_{i=1}^{100} |h_\text{th}^i(\theta) - h_\text{eh}^i|^2$$

The optimal solution $\theta^*$ is the one we want. To avoid large fitting error, we reduce the dimension of least squares problem from five to three by imposing the first two moments consistent in different histograms. More precisely, by Proposition 2.4, we assume

$$\sigma_D^2 = \frac{SV}{dt} - \lambda (\sigma_J^2 + \mu_J^2)$$
$$\mu_D = \frac{SM}{dt} - \lambda \mu_J + .5\sigma_D^2$$

Where $SM$ is the sample mean and $SV$ is the sample variance. Thus, this is a three dimensional global optimization problem. The solution $\theta^*$ can be found by using MATLAB solver fminsearch with initial guess: $\mu_J = 0, \sigma_J = 0, \lambda = 200$. The final parameter results are

$$\mu_D = 0.205817$$
$$\sigma_D^2 = 0.019209$$
$$\mu_J = -0.001422$$
$$\sigma_J^2 = 0.000139$$
$$\lambda = 100.891219$$

The lambda is pretty large. This is reasonable because the jump rate in an interval needs to be scaled by the length of that interval $dt = 1/243.5$ and the result is a small number.

Step 5: Error discussion

Figure (3) is the theoretical histogram which is constructed by using the parameters we have found in last step. We can also compute some statistical quantities associated with Figure (3).
For the theoretical histogram

\[
\begin{align*}
\text{Mean} &= 0.000247 \\
\text{Variance} &= 0.000155 \\
\text{Skewness coefficient} &= -0.172931 \\
\text{Kurtosis coefficient} &= 4.005704
\end{align*}
\]

The first two moments are exactly the same with empirical data. In step 4, we just impose them to be the same theoretically instead of numerically, so the consistence is still good. Although the skewness and kurtosis coefficients are different, they capture the fat-tail feature of the real market. Figure (4) is the error plot. The height of error in each bin is about one tenth of the empirical histogram. Overall speaking, these parameters are acceptable. If we want to improve the estimation, we can consider time-dependent parameters which are able to capture the slightly change in different rates of the stock price over time. One way to do this is supposing the dynamics of S&P500 from 1997 to 2005 has constant parameters in each year and connects them together, but this is not the case in our scope here.

For the non-risky asset, we choose \( r = 0.0752 \), the average rate of Moody AAA bonds from 1997 to 2005.

\subsection*{5.5 Conclusion}

In this section, we added a compound Poisson Process to the geometric Brownian motion and used it to model the stock price process. We formulated the option pricing problem as a non-classical control problem, and transformed it into a classical one. Parameters of our model was also being estimated here.
Figure 5.3: Theoretical histogram of S&P500 log-return
Figure 5.4: Error plot
Chapter 6

Numerical methods and results

6.1 Introduction

The main purpose of this chapter is to solve the Bellman’s equation with jump in our option pricing model. Such equations are difficult to solve because the unknown is a function instead of a simple vector. If the optimal control can be represented by the state process explicitly, then the Bellman’s equation can be transformed into an integro-partial differential equation which can be solved by finite difference method directly [8]. However, in our case, the appearance of control in the argument of the value function makes the equation implicit. Therefore, we need to do more analysis to simplify this problem before solving it.

This is a two dimensional problem with time and state variables. In order to find the optimal solution, firstly, we discretize the state variable by using a family of basic functions and the Bellman’s equation can be transformed into an ordinary differential equation (ODE). Secondly, we discretize the time variable and use the fourth-order-Runge-Kutta method to solve this ODE, in each step, recording the optimal control. After solving the Bellman’s equation, the result can be used to determine the feasibility of initial price \( y_0 \) in the original target problem (P1) and the smallest feasible \( y_0 \) is the suitable price of our call option.

In Section 2, we briefly introduce the idea of function approximation in numerical analysis. An algorithm for solving the Bellman’s equation without jump is presented in Section 3. In Section 4, we generalize the algorithm to the case with jump process. The numerical results are reported in Section 5. All the implementations we made in this chapter are using MATLAB. We briefly restate our model here.

Suppose we have two assets in a financial market: stock and bank account. The stock price
process satisfies

\[ dX_t = X_t (\mu_D dt + \sigma_D dW_t + J(Q)dP_t) \quad X_0 = x_0 > 0 \quad (1) \]

and the bank account process satisfies

\[ dB_t = rB_t dt \quad B_0 = 1 \quad (2) \]

where \( W_t \) is a one-dimensional Brownian motion, and \( P_t \) is a one-dimensional Poisson Process with rate \( \lambda \). Suppose the jump amplitude, \( J(Q) = \exp(Q) - 1 \), is a function of independent and identically distributed normal random variables with mean \( \mu_J \) and variance \( \sigma_J^2 \).

\[ Q \sim N(\mu_J, \sigma_J^2) \]

All the processes \( W_t, P_t, \) and \( J(Q) \) are mutually independent. The parameters \( (\mu_D, \sigma_D, \mu_J, \sigma_J, \lambda, \) and \( r \) \) have being found in previous chapter. Imagine that an agent wants to find a suitable selling price and a corresponding hedging strategy for a European call option based on the stock price process. To help the agent to make decisions, we formulate the problem as follows

\[
\begin{align*}
\text{minimize} & \quad y_0 \\
\text{subject to} & \quad Y_T \geq \max(X_T - K, 0) \quad a.s. \quad (P1) \\
& \quad 0 \leq u_t \leq 1, \quad y_0 > 0
\end{align*}
\]

Where

\[
\begin{align*}
K &= \text{the strike price of the call option.} \\
T &= \text{the maturity time of the call option.} \\
y_0 &= \text{price of the option at time 0 and } y_0 > 0. \\
u_t &= \text{the proportion of wealth invested in the stock, } 0 \leq u_t \leq 1.
\end{align*}
\]
$Y_t$ is the stochastic process representing the agent’s wealth at time $t$. Satisfies

$$dY_t = Y_t[(u_t\mu_D + (1 - u_t)r)\,dt + u_t\sigma_D\,dW_t + u_tJ(Q)dP_t] \quad (3)$$

with $Y_0 = y_0$. (P1) is an non-classical control problem with controls $y_0$ and $u_t$. The auxiliary problem we used to determine the feasibility of $y_0$ is defined as

$$\begin{align*}
\text{minimize} & \quad E[\max(Z - Y_T, 0)] \\
\text{subject to} & \quad 0 \leq u_t \leq 1
\end{align*} \quad (P2)$$

Where $Z = \max(X_T - K, 0)$. As we derived in Section 5.3, when $y_0$ is fixed, (P2) is classical and its value function, $\Phi(t, y)$, can be characterized by the Bellman’s equation.

$$
\Phi(t, y) = \max_{0 \leq u \leq 1} \left\{ -ry\Phi_y(t, y) - y(\mu_D - r)u\Phi_y(t, y) - \frac{1}{2} \sigma_D^2 y^2 u^2 \Phi_{yy}(t, y) \\
- \lambda \int_R \Phi(t, y + uy(e^q - 1)) - \Phi(t, y) \frac{1}{\sqrt{2\pi\sigma_J^2}} e^{-(q-\mu_J)^2/2\sigma_J^2} dq \right\} \quad (4)
$$

with the terminal condition

$$
\Phi(T, y) = E[\max(Z - y, 0)] \quad (5)
$$

It has been shown that $y_0$ is feasible in (P1) if and only if $\Phi(0, y_0) = 0$. The most difficult part of the simulation is to find the $\Phi(t, y)$ and control $u_t$ in (P2) by solving (4) and (5). After the value function being found, we pick the smallest positive $y_0$ such that $\Phi(0, y_0) = 0$ as the suitable price of the call option and the corresponding control $u_t$ as the hedging strategy.
6.2 Function approximation

We briefly describe the function approximation techniques in this section. For more information, please check references [8]. Suppose \( f(y) \) is the unknown function which we are interested in. The structure of \( f(y) \) may be very complicated or we may have only limited information about this function. One strategy for describing the behaviour of \( f(y) \) is to approximate this function by a linear combination of some known functions called basic functions.

Suppose we have chosen a family of \( n \) linearly independent functions \( \psi_1(y), \psi_2(y), \ldots, \psi_n(y) \) which are known explicitly. For practical reason, we set

\[
\hat{f}(y) = \sum_{i=1}^{n} c_i \psi_i(y)
\]

as the approximation of \( f(y) \). Where the coefficient vector \( c = [c_1, c_2, \ldots, c_n]^T \) needs to be determined.

Consider the example of solving a functional equation. Suppose we want to find a function \( f(y) \), such that

\[
g(y, f(y), f'(y), f''(y)) = 0 \quad (6)
\]

for all \( y \in [y_{\text{min}}, y_{\text{max}}] \). Where the function \( g \) is given. In this case, we approximate \( f(y) \) and its derivatives by

\[
\begin{align*}
    f(y) &\approx \hat{f}(y) = \sum_{i=1}^{n} c_i \psi_i(y), \\
    f'(y) &\approx \hat{f}'(y) = \sum_{i=1}^{n} c_i \psi'_i(y), \\
    f''(y) &\approx \hat{f}''(y) = \sum_{i=1}^{n} c_i \psi''_i(y),
\end{align*}
\]

Then, equation (6) can be written as

\[
g(y, \hat{f}(y), \hat{f}'(y), \hat{f}''(y)) = 0 \quad (7)
\]

for all \( y \in [y_{\text{min}}, y_{\text{max}}] \). Since the unknown \( c \) is an \( n \)-vector, we need \( n \) conditions to quarantine a unique solution. Specify \( n \) points \( y_1, y_2, \ldots, y_n \) in the interval \([y_{\text{min}}, y_{\text{max}}]\) and plug into (7).
We obtain
\[ g(y_j, \hat{f}(y_j), \hat{f}'(y_j), \hat{f}''(y_j)) = 0 \] (8)
for \( j = 1, 2, ..., n \). Therefore, the coefficient vector \( c \) can be found by solving (8) and we take \( \hat{f}(y) \) as the estimated solution of (6). By using the function approximation technique, an infinite dimensional problem has been transformed into an \( n \) dimensional problem. The later one is much easier to solve. The error of this approximation can be measured by
\[
\max_{y \in [y_{\min}, y_{\max}]} |f(y) - \hat{f}(y)|
\]
The magnitude of error may depend on several issues such as

1. The structure of basis functions \( \psi_1(y), \psi_2(y), ..., \psi_n(y) \).
2. The spread of nodes \( y_1, y_2, ..., y_n \).
3. The size of \( n \).

Here are two popular basis function families which will be used to approximate the value function in (P2) later: cubic spline and linear spline. The spline families are piecewise polynomial basic functions. More precisely, an order-\( k \) spline function consists of a series of \( k \)th order polynomial segments and preserve continuity and derivatives of order \( k - 1 \) or less at some pre-specified breakpoints. The explicit structure of spline families can be found in most numerical analysis texts [8] and skipped here. Instead, we plot the cubic and linear spline families in Figure (1) and (2) as examples. Figure (1) is the cubic spline family with four member functions \( (n = 4) \) and two breakpoints \( (p = 2) \) over \([0, 1]\). Figure (2) is the linear spline family with four member functions \( (n = 4) \) and four breakpoints \( (p = 4) \) over \([0, 1]\). People may notice that, in spline families, there is a relation among parameters.

\[
\text{Number of functions in the family (n)} = \text{Order of the spline family (k)} + \text{Number of breaking points (p) - 1.}
\]

In the Bellman’s equation (4), the value function \( \Phi(t, y) \) is pretty smooth except near the boundary time \( T \) where error often occurs. When using the spline functions to approximate \( \Phi(t, y) \), the piecewise structure of spline is able to stop the expending of error from the boundary. This is the reason we choose spline as our basic functions instead of some global polynomial families.
For simplicity purpose, the points $y_1, y_2, ..., y_n$ are chosen to be evenly spaced over $[y_{\text{min}}, y_{\text{max}}]$. When dealing with the jump term in Bellman's equation, this choice lets us easily interpolate quadrature points and reduce the complexity of algorithm implementation.

The number of member functions in the family ($n$) should be moderate. Small $n$ cannot describe $f(y)$ well. On the other hand, a very large $n$ will not only slow down the algorithm but also increase the error for some numerical reason.
6.3 Dynamic programming without jump

The most complicated part of the target problem is handling the jump term in the Bellman's equation (4). For illustration purpose, in this section, we consider the option pricing model without jump first. The discussion of the general case can be found in the next section.

In the simplified model, we remove the compound Poisson process \( J(Q)dP_t \) in (1) and the stock price process becomes a geometric Brownian motion which satisfies

\[
dX_t = X_t (\mu_D \, dt + \sigma_D \, dW_t)
\]  

(9)
Consequently, the dynamics of the wealth $Y_t$ becomes

$$dY_t = Y_t \left[ (u_t \mu_D + (1 - u_t)r) \, dt + u_t \, \sigma_D \, dW_t \right]$$

(10)

The structure of the target problem (P1) and the auxiliary control problem (P2) remain the same. However, the Bellman’s equation of (P2) becomes much easier

$$\Phi_t(t, y) = \max_{0 \leq u \leq 1} \left\{ -ry \Phi_y(t, y) - (\mu_D - r)y u \Phi_y(t, y) - \frac{1}{2} \sigma_D^2 \, y^2 \, u^2 \, \Phi_{yy}(t, y) \right\}$$

(11)

In order to find the suitable price $y_0$ and its hedging strategy $u_t$, we need to solve (11) for the value function first. Equation (11) is a two dimensional problem with time variable $t$ and state variable $y$. In our algorithm, a variation of finite difference method is used on state and time variables separately. Discretize the state to transform (11) into an ODE problem, and then, discretize the time variable to solve this ODE. We separate the algorithm in several steps and focus on the domain $t \in [0, T]$ and $y \in [y_{min}, y_{max}]$.

**Step 1:** Discretize the state variable to form an ODE

Choose a family of basis functions $\psi_1(y), \psi_2(y), ..., \psi_n(y)$ and make approximation of the value function by

$$\Phi(t, y) \approx \sum_{i=1}^{n} c_i(t) \psi_i(y) = \psi(y) \cdot c(t)$$

$$\Phi_t(t, y) \approx \sum_{i=1}^{n} c'_i(t) \psi_i(y) = \psi(y) \cdot c'(t)$$

$$\Phi_y(t, y) \approx \sum_{i=1}^{n} c_i(t) \psi'_i(y) = \psi'(y) \cdot c(t)$$

$$\Phi_{yy}(t, y) \approx \sum_{i=1}^{n} c_i(t) \psi''_i(y) = \psi''(y) \cdot c(t)$$

Where $\psi(y) = [\psi_1(y), \psi_2(y), ..., \psi_n(y)]^T$ and $c(t) = [c_1(t), c_2(t), ..., c_n(t)]^T$ are column vectors of the basis functions and corresponding coefficients, respectively. Note that $c(t)$ is an $n$-vector of time-varying coefficient. After plugging these approximations into (11), the Bellman’s equation...
becomes
\[
\psi(y) \cdot c'(t) = \max_{0 \leq u \leq 1} \left\{ -ry\psi'(y) \cdot c(t) - (\mu_D - r)yu\psi'(y) \cdot c(t) - \frac{1}{2} \sigma_D^2 y^2 u'^2 \psi''(y) \cdot c(t) \right\}
\]
(12)

for all \( y \in [y_{min}, y_{max}] \). The same as before, we need to specify \( n \) points between \( y_{min} \) and \( y_{max} \) to transform the infinite dimensional problem into a finite dimensional one. Choose evenly spaced points and denote the increment by \( \Delta y \). In other words, we discretize \([y_{min}, y_{max}]\) by
\[
y_j = y_{min} + (j - 1)\Delta y \quad \text{for} \quad j = 1, 2, \ldots, n
\]

Plugging the \( y_j \)'s into (12), we obtain \( n \) conditions
\[
\psi(y_j) \cdot c'(t) = \max_{0 \leq u_j \leq 1} \left\{ -ry_j\psi'(y_j) \cdot c(t) - (\mu_D - r)y_ju_j\psi'(y_j) \cdot c(t) - \frac{1}{2} \sigma_D^2 y_j^2 u_j'^2 \psi''(y_j) \cdot c(t) \right\}
\]
(13)

where \( u_j \) is the optimal \( u \)-value associated with \( y_j \), for \( j = 1, 2, \ldots, n \). We can put them together to form the vector equation
\[
\Psi c'(t) = \max_{0 \leq u \leq 1} \left\{ -r diag(y) \Psi_1 c(t) - (\mu_D - r) diag(y) diag(u) \Psi_1 c(t) \right\}
\]
\[
- \frac{1}{2} \sigma_D^2 diag(y^2) diag(u^2) \Psi_2 c(t) \right\}
\]
(14)

Where
\[
\Psi = \begin{bmatrix}
\psi(y_1)^T \\
\psi(y_2)^T \\
\vdots \\
\psi(y_n)^T
\end{bmatrix}, \quad \Psi_1 = \begin{bmatrix}
\psi'(y_1)^T \\
\psi'(y_2)^T \\
\vdots \\
\psi'(y_n)^T
\end{bmatrix}, \quad \Psi_2 = \begin{bmatrix}
\psi''(y_1)^T \\
\psi''(y_2)^T \\
\vdots \\
\psi''(y_n)^T
\end{bmatrix},
\]

124
\[ y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad y^2 = \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_n^2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad u^2 = \begin{bmatrix} u_1^2 \\ u_2^2 \\ \vdots \\ u_n^2 \end{bmatrix}, \]

and \( \text{diag}(y) \) is the \( n \) by \( n \) diagonal matrix generated by vector \( y \), similarly to \( \text{diag}(y^2) \), \( \text{diag}(u) \), and \( \text{diag}(u^2) \).

In (14), the term \( y \) is an abuse of notation; it indicates a known vector instead of an unknown variable. Similarly, \( u \) is not an element in (14); Instead, it represents a vector and needs to be determined. Equation (14) is an ODE with unknown \( n \)-dimensional function \( c(t) \). Notice that the right hand side of (14) is an optimization problem; As long as \( c(t) \) is fixed, it can be separated into \( n \) independent quadratic functions which can be solved easily without any particular package. This good feature will no longer be true when considering the jump diffusion model in the next section.

**Step 2: Find the terminal condition \( c(T) \)**

We will implement a modified fourth-order-Runge-Kutta method to solve the ODE in (14). Before doing so, information about \( c(t) \) on the boundary has to be discussed. At time \( T \), The Bellman’s equation has terminal condition \( \Phi(T, y) = E[\max(Z - y, 0)] \), where \( Z = \max(X_T - K, 0) \). By using the same basis functions and uniform points in step 1, we can rewrite it as

\[ \Psi c(T) = E[\max(Z - y, 0)] \quad (15) \]

The term \( y = [y_1, y_2, ..., y_n]^T \) and \( \Psi \) are defined as before. The right hand side of (15) is another abuse of notation, which really means an \( n \)-dimensional vector with \( i \)th element \( E[\max(Z - y_i, 0)] \). Since the basic matrix \( \Psi \) is invertible, we can rearrange (15) to get

\[ c(T) = \Psi^{-1} E[\max(Z - y, 0)] \quad (16) \]

By Ito’s formula, it is well-known that the explicit solution of (9) at time \( T \) is a log-normal
distribution

\[ X_T = x_0 \exp \left( (\mu_D - \frac{1}{2} \sigma_D^2)T + \sigma_D W_T \right) \]

Note that \((\mu_D - \frac{1}{2} \sigma_D^2)T + \sigma_D W_T\) is a normal random variable with mean \((\mu_D - \frac{1}{2} \sigma_D^2)T\) and variance \(\sigma_D^2 T\). Therefore, the terminal condition \(c(T)\) can be found numerically by using the normal distribution quadrature rule which is specifically designed to compute the expectation of functions of normal random variable. More precisely, suppose the order of approximation is fixed \((qn)\), we choose suitable quadrature nodes \(x_1, x_2, ..., x_qn\) and weights \(w_1, w_2, ..., w_qn\) to estimate the expectation in (16) and

\[
c(T) = \Psi^{-1} \left[ \begin{array}{c} \sum_{k=1}^{qn} w_k \max(z_k - y_1, 0) \\ \sum_{k=1}^{qn} w_k \max(z_k - y_2, 0) \\ \vdots \\ \sum_{k=1}^{qn} w_k \max(z_k - y_n, 0) \end{array} \right], \tag{17}
\]

where \(z_k = \max(x_0 e^{x_k} - K, 0)\). Here is a pseudo-code in MATLAB

```matlab
CT=zeros(n,1);
qn=10;
qmean=(mud-.5*sigmad2)*T;
qvar=sigmad2*T;
[x,w]=qnwnormal(qn,qmean,qvar);
for i=1:n
    CT(i)=w'*(max(x0*exp(x)-K,0)-y(i),0);
end;
CT=Psi\CT;
```

Where `qnwnormal` is a function with three arguments: approximation order, the mean and variance of the normal random variable involved in the expectation and returns the quadrature nodes and weights in column vectors.

**Step 3:** Solve the ODE
The coefficient vector $c(t)$ satisfies the system of ODE (14) with terminal condition (16). There are a lot of good ODE solvers to deal with such problem in MATLAB. They use the same basic idea: discretize the time horizontal and estimate the solution from the starting time to the final time step by step. However, the existing ODE solvers will not record the optimal control $u$ in (14) for us. Therefore, we write a modified fourth-order-Runge-Kutta method which records the optimal $u$'s in each step.

Firstly, define the function $v_{\text{max}}$ to represent the right hand side of (14) by

$$v_{\text{max}}(c) = \max_{0 \leq u \leq 1} \left\{ -r \, \text{diag}(y) \, \Psi_1 c - \left( \mu_D - r \right) \, \text{diag}(y) \, \text{diag}(u) \, \Psi_1 c - \frac{1}{2} \sigma_D^2 \, \text{diag}(y^2) \, \text{diag}(u^2) \, \Psi_2 c \right\} \quad c \in R^{n \times 1}$$

where $y$ indicates a known vector as before. Then, the terminal value problem (14) with (16) becomes

$$\Psi c'(t) = v_{\text{max}}(c(t))$$
$$c(T) = \Psi^{-1}E[\max(Z - y, 0)]$$

Secondly, define variable $\tau = T - t$ to transform above into an initial value problem. We obtain

$$\Psi c'(\tau) = -v_{\text{max}}(c(\tau))$$
$$c(0) = \Psi^{-1}E[\max(Z - y, 0)]$$

Then, we discretize $[0, T]$ in $N$ even subintervals. Suppose $c(\tau) = c$ is given and $h$ is the step size, use the following updating rule to find the function value at the next endpoint $c(\tau + h)$

$$c(\tau + h) = c + (f_1 + f_2 + f_3 + f_4)/3$$
where

\[
\begin{align*}
    f_1 &= -\Psi^{-1}v_{\text{max}}(c) \frac{h}{2} \\
    f_2 &= -\Psi^{-1}v_{\text{max}}(c + f_1) h \\
    f_3 &= -\Psi^{-1}v_{\text{max}}(c + \frac{1}{2} f_2) h \\
    f_4 &= -\Psi^{-1}v_{\text{max}}(c + f_3) \frac{h}{2}
\end{align*}
\]

Record the optimal control \( u \) in each step when computing \( v_{\text{max}} \). At the end of the procedure, we will get an \( n \) by \( N + 1 \) matrix \( C \) and an \( n \) by \( N \) matrix \( U \) which have columns \( c \)'s and \( u \)'s, respectively. For consistency, we need to change the variable \( \tau \) back to \( t \) by using the MATLAB function \texttt{fliplr} which returns \( C \) and \( U \) with columns flipped in the left-right direction. The first column of the resulting matrix \( C \) is the coefficient at time 0 and the last column is the one corresponding to final time \( T \), similarly with \( U \). Here is a pseudo code of the modified fourth-order-Runge-Kutta method:

```matlab
C=zeros(n,N+1);  
C(:,1)=CT;  
U=zeros(n,N);  
c=CT;  
m=vmax(c);  
for i=2:N+1  
    f1=-Phi\m*(h/2);  
    f2=-Phi\vmax(c+f1)*h;  
    f3=-Phi\vmax(c+f2/2)*h;  
    f4=-Phi\vmax(c+f3)*(h/2);  
    c=c+(f1+f2+f3+f4)/3;  
    C(:,i)=c;  
    [m,u]=vmax(c);  
    U(:,i-1)=u;  
end  
C=fliplr(C);  
U=fliplr(U); 
```

Note that the function \( v_{\text{max}} \) which we defined in MATLAB has syntax \([m,u]=vmax(c)\) returns...
two vectors: the maximum value \( m \) and corresponding optimal solution \( u \). The control \( u \) is recorded in each step. After obtaining the coefficient, the value function can be found by using \( \Phi(t, y) = \psi(y) \cdot c(t) \).

**Step 4:** Use the value function to find the suitable price \( y_0 \)

The initial wealth \( y_0 \) is feasible for the target problem (P1) if and only if it satisfies \( \psi(y_0) \cdot c(0) = 0 \) where \( c(0) \) is the first column of the matrix \( C \). In other words, we consider the problem

\[
\begin{align*}
\text{minimize} & \quad y_0 \\
\text{subject to} & \quad \psi(y_0) \cdot c(0) = 0 \\
& \quad y_0 \in [y_{\text{min}}, y_{\text{max}}]
\end{align*}
\]

This problem is straightforward. We divide the interval \([y_{\text{min}}, y_{\text{max}}]\) into pretty small pieces, and then, over the dividing nodes, find the smallest one such that \( |\psi(y_0) \cdot c(0)| \) less than some given tolerance. Here is a sketch of code in MATLAB:

```matlab
c0=C(:,1);
y0=linspace(ymin,ymax,1000);
v0=psi(y0)*c0;
for i=1:length(y0)
    if abs(v0(i))<tolerance
        break;
    end
end
OptPrice=y0(i)
```

The `OptPrice` in above code is the suitable price which we are looking for. We have done the algorithm of the option pricing model without jump. In the next section, we will add the jump process to our model.
6.4 Dynamic programming with jump

In this section, we return to the general case described in the introduction and consider the dynamic programming problem (4) with (5). For convenience of reading, we list them here again

\[
\Phi_t(t, y) = \max_{0 \leq u \leq 1} \left\{ -ry\Phi_y(t, y) - y(\mu_D - r)u\Phi_y(t, y) - \frac{1}{2}\sigma^2_D y^2 u^2 \Phi_{yy}(t, y) \right\} - \lambda \int_{\mathbb{R}} \left[ \Phi(t, y + uy(e^q - 1)) - \Phi(t, y) \right] \frac{1}{\sqrt{2\pi} \sigma^2_J} e^{-(q - \mu_J)^2} dq \]  

(4)

\[
\Phi(T, y) = E[\max(Z - y, 0)] \]  

(5)

The last part in (4) can be written as an expectation by

\[
\int_{\mathbb{R}} \left[ \Phi(t, y + uy(e^q - 1)) - \Phi(t, y) \right] \frac{1}{\sqrt{2\pi} \sigma^2_J} e^{-(q - \mu_J)^2} dq = E \left[ \Phi(t, y + uy(e^Q - 1)) - \Phi(t, y) \right] \]  

(18)

Where \( Q \) is the amplitude of jumps which is a normal distribution with mean \( \mu_J \) and variance \( \sigma^2_J \). Therefore, the integral part in (4) can be computed by using the normal distribution quadrature rule as we mentioned in previous section. The same as before, we will discretize the state variable \( y \) by some chosen basis functions and nodes, and then transform the dynamic programming problem into an ODE problem. However, the appearance of \( u \) in argument of \( \Phi \) in (18) causes the optimization function \( \text{vmax} \) pretty complicated. Moreover, interpolation method has to be employed when approximating the integral. Comparing to the algorithm in previous section, the step 3 and 4 remain the same; Only the first two steps need to be modified. We will use the same notations as before.

**Step 1:** Discretize the state variable to form an ODE

By choosing appropriate basis functions, we can rewrite (4) as
\[
\psi(y) \cdot c'(t) = \max_{0 \leq u \leq 1} \left\{ -ry\psi'(y) \cdot c(t) - (\mu_D - r)yu\psi'(y) \cdot c(t) - \frac{1}{2} \sigma^2_D y^2 u^2 \psi''(y) \cdot c(t) \right. \\
\left. -\lambda \int_R \left[ \psi(y + uy(e^q - 1)) - \psi(y) \right] \cdot c(t) \frac{1}{\sqrt{2\pi\sigma^2_J}} e^{-(q - \mu_J)^2} dq \right\} 
\]

where \( \psi(y) = [\psi_1(y), \psi_2(y), ..., \psi_n(y)]^T \) is the basis function family and \( c(t) = [c_1(t), c_2(t), ..., c_n(t)]^T \) is the coefficient vector which needs to be determined. Discretize the interval \([y_{\text{min}}, y_{\text{max}}]\) by \( n \) evenly spaced nodes \( y_1, y_2, ..., y_n \) and substitute them in (19). We obtain

\[
\Psi'c(t) = \max_{0 \leq u \leq 1} \left\{ -r \text{diag}(y) \Psi_1 c(t) - (\mu_D - r) \text{diag}(y) \text{diag}(u) \Psi_1 c(t) \\
-\frac{1}{2} \sigma^2_D \text{diag}(y^2) \text{diag}(u^2) \Psi_2 c(t) + \lambda \Psi c(t) \\
-\lambda \begin{bmatrix}
\int_R \psi(y_1 + u_1 y_1(e^q - 1))T p(q) dq \\
\int_R \psi(y_2 + u_2 y_2(e^q - 1))T p(q) dq \\
\vdots \\
\int_R \psi(y_n + u_n y_n(e^q - 1))T p(q) dq
\end{bmatrix}
\right\} 
\]

(20)

where

\[
p(q) = \frac{1}{\sqrt{2\pi\sigma^2_J}} e^{-(q - \mu_J)^2}
\]

is the density function of normal random variable with mean \( \mu_J \) and variance \( \sigma^2_J \). In (20), \( y = [y_1, y_2, ..., y_n]^T \), \( u = [u_1, u_2, ..., u_n]^T \), and the notations \( \text{diag}(y) \), \( \text{diag}(y^2) \), \( \text{diag}(u) \), \( \text{diag}(u^2) \) are defined the same with (14). As before, we define a function \( \text{vamx} \) in MATLAB with syntax

\[
[m, u] = \text{vamx}(c)
\]

to represent the embedded optimization problem in (19). That is,
\[
v_{\text{max}}(c) = \max_{0 \leq u \leq 1} \{-r \text{diag}(y)\Psi_1 c - (\mu_D - r) \text{diag}(y) \text{diag}(u)\Psi_1 c \\
- \frac{1}{2} \sigma_D^2 \text{diag}(y^2) \text{diag}(u^2) \Psi_2 c + \lambda \Psi c \\
\begin{bmatrix}
\int_R \psi(y_1 + u_1 y_1 (e^q - 1))^T p(q) dq \\
\int_R \psi(y_2 + u_2 y_2 (e^q - 1))^T p(q) dq \\
\vdots \\
\int_R \psi(y_n + u_n y_n (e^q - 1))^T p(q) dq
\end{bmatrix} \cdot c \}
\]

where \(c \in R^{n \times 1}\). As long as we can evaluate \(v_{\text{max}}\) efficiently, we can use our ODE solver with suitable terminal condition to find the value function and optimal control as before. Comparing to the case without jump, the objective function of \(v_{\text{max}}\) in (21) is no longer a simple quadratic polynomial in \(u\); instead, it includes two parts: a quadratic polynomial and an integral. Basically, we need to find a good \(u\) for the integral part and compare it with the quadratic part to choose the optimal control. The procedure will be pretty complicated because the variable \(u\) involves in the argument of \(\psi\) in the second part. Fortunately, in the application of option pricing model, we can simplify the structure of \(v_{\text{max}}\) by making a reasonable assumption.

In Chapter 5, we have estimated the parameters of our model; Note that the mean of jump amplitude is very small (\(\mu_J = -0.001422\)), so is \(e^q - 1\). Thus, the control \(u\) in the argument of \(\psi\) has pretty small contribution to the objective function when comparing to the quadratic part. For this reason, we ignore the appearance of \(u\) in the integral part and make the following assumption.
\( \text{vmax}(c) \approx \max_{0 \leq u \leq 1} \left\{ -r \ \text{diag}(y) \Psi_1 c - (\mu_D - r) \ \text{diag}(y) \ \text{diag}(u) \Psi_1 c \right. \\
- \frac{1}{2} \sigma_D^2 \ \text{diag}(y^2) \ \text{diag}(u^2) \ \Psi_2 c + \lambda \Psi c \\
\left[ \begin{array}{c} \\
\int_R \psi(y_1 + y_1(e^q - 1))^T p(q) dq \\
\int_R \psi(y_2 + y_2(e^q - 1))^T p(q) dq \\
\int_R \psi(y_n + y_n(e^q - 1))^T p(q) dq \\
\end{array} \right] c \} \quad (22) \)

Hence, we turn to compute \( \text{vmax} \) by using (22) instead of (21). The objective function in (22) is purely a quadratic polynomial and \( u \) can be found easily. The only difficulty left here is evaluating its integral-vector part. In other words, we need to evaluate

\[ \int_R \psi(y_j + y_j(e^q - 1))^T p(q) dq \quad (23) \]

for each \( j = 1, 2, ..., n \).

A common way to estimate such integral is to use the normal distribution quadrature rule. In general, the argument \( y_j + y_j(e^q - 1) \) will not be one of the nodes \( y_1, y_2, ..., y_n \) which we used to discretize the state axis, so the interpolation method has to be employed. Considering the point \( y_j + y_j(e^q - 1) \), let

\[ \epsilon(q) = \frac{y_j(e^q - 1)}{\Delta y} \quad (24) \]

which is the ratio of the extra to the increment of the uniform points \( y_j \)'s. Since \( e^q - 1 \) is small, we can always make \( |\epsilon(q)| \leq 1 \) by choosing appropriate number of uniform points \( n \). By using interpolation, \( \psi(y_j + y_j(e^q - 1)) \) can be approximated by a combination of \( \psi(y_{j-1}), \psi(y_j), \) and \( \psi(y_{j+1}) \). More precisely, if \( q > 0 \), then

\[ \psi(y_j + y_j(e^q - 1)) \approx \epsilon(q) \psi(y_{j+1}) + (1 - \epsilon(q)) \psi(y_j). \]
On the other hand, if $q < 0$, then
\[
\psi(y_j + y_j(e^q - 1)) \approx -\epsilon(q)\psi(y_{j-1}) + (1 + \epsilon(q))\psi(y_j).
\]

Therefore, (23) can be simplified as
\[
\int_R \psi(y_j + y_j(e^q - 1))^T p(q) dq \\
\approx \int_0^\infty [\epsilon(q)\psi(y_{j+1}) + (1 - \epsilon(q))\psi(y_j)]^T p(q) dq + \int_0^0 [-\epsilon(q)\psi(y_{j-1}) + (1 + \epsilon(q))\psi(y_j)]^T p(q) dq \\
= \psi(y_{j+1})^T \int_0^\infty \epsilon(q)p(q) dq + \psi(y_j)^T \left( \int_0^\infty (1 - \epsilon(q))p(q) dq + \int_0^0 (1 + \epsilon(q))p(q) dq \right) \\
+ \psi(y_{j-1})^T \int_{-\infty}^0 -\epsilon(q)p(q) dq \\
= \psi(y_{j+1})^T \int_0^\infty \epsilon(q)p(q) dq + \psi(y_j)^T \left( 1 + \int_0^\infty -\epsilon(q)p(q) dq + \int_0^0 \epsilon(q)p(q) dq \right) \\
+ \psi(y_{j-1})^T \int_0^\infty \epsilon(q)p(q) dq \\
= \psi(y_{j+1})^T \int_0^\infty \epsilon(q)p(q) dq + \psi(y_j)^T \left( 1 - 2 \int_0^\infty \epsilon(q)p(q) dq \right) \\
+ \psi(y_{j-1})^T \int_0^\infty \epsilon(q)p(q) dq \\
= \int_0^\infty \epsilon(q)p(q) dq + \psi(y_j)^T \left( 1 - 2 \psi(y_j)M \right) + \psi(y_{j-1})^T \psi(y_j)M
\] (25)

Use $M$ to represent the value of $\int_0^\infty \frac{e^q - 1}{\Delta y} p(q) dq$ which can be evaluated by normal distribution quadrature rule numerically. Then
\[
\int_0^\infty \epsilon(q)p(q) dq = y_j \int_0^\infty \frac{e^q - 1}{\Delta y} p(q) dq = y_j M \\
(26)
\]

Thus, (25) can be rewrite as
\[
\int_R \psi(y_j + y_j(e^q - 1))^T p(q) dq \approx \psi(y_{j+1})^T y_j M + \psi(y_j)^T (1 - 2y_j M) + \psi(y_{j-1})^T y_j M \\
(27)
\]
Therefore, the last term in (22) can be computed efficiently by the following

\[
\begin{bmatrix}
\int_{\mathbb{R}} \psi(y_1 + y_1(e^q - 1))^T p(q) dq \\
\int_{\mathbb{R}} \psi(y_2 + y_2(e^q - 1))^T p(q) dq \\
\vdots \\
\int_{\mathbb{R}} \psi(y_n + y_n(e^q - 1))^T p(q) dq
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\psi(y_1)^T (1 - y_1 M) + \psi(y_2)^T y_1 M \\
\psi(y_3)^T y_2 M + \psi(y_2)^T (1 - 2y_2 M) + \psi(y_1)^T y_2 M \\
\vdots \\
\psi(y_n)^T y_{n-1} M + \psi(y_{n-1})^T (1 - 2y_{n-1} M) + \psi(y_{n-2})^T y_{n-1} M \\
\psi(y_n)^T (1 - y_n M) + \psi(y_{n-1})^T y_n M
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 - y_1 M \\
1 - 2y_2 M \\
. \\
. \\
1 - 2y_{n-1} M \\
1 - y_n M
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
. \\
. \\
y_{n-1} \\
y_n
\end{bmatrix}
\]

\[
= \text{diag}(\Psi) + \text{diag}(\Psi_D + \Psi_U)(\Psi_D + \Psi_U)
\]

(28)

where

\[
\Psi_D = \begin{bmatrix}
0 \\
\psi(y_1)^T \\
. \\
. \\
\psi(y_{n-2})^T \\
\psi(y_{n-1})^T
\end{bmatrix}, \quad \Psi_U = \begin{bmatrix}
\psi(y_2)^T \\
\psi(y_3)^T \\
. \\
. \\
\psi(y_n)^T \\
0
\end{bmatrix}
\]

In summary, the function \( \text{vmax} \) in which a quadratic optimization problem is embedded can
be evaluated efficiently by
\[
\text{vmax}(c) \approx \max_{0 \leq u \leq 1} \{ -r \text{diag}(y) \Psi_1 c - (\mu_D - r) \text{diag}(y) \text{diag}(u) \Psi_1 c - \frac{1}{2} \sigma_D^2 \text{diag}(y^2) \text{diag}(u^2) \Psi_2 c + \lambda \Psi c \} \\
1 - y_1 M \\
1 - 2y_2 M \\
\vdots \\
1 - 2y_{n-1} M \\
1 - y_n M \\
-\lambda \text{diag}(1 - y_1 M) \Psi - \lambda M \text{diag}(1 - y_1 M) (\Psi_D + \Psi_U)
\]

and the dynamics of \(c(t)\) is
\[
\Psi c'(t) = \text{vmax}(c(t)) \quad (30)
\]

**Step 2: Find the terminal condition \(c(T)\)**

In the case without jump, the stock price at final time \(T\) is a log-normal random variable. In the general case, the structure of \(X_T\) becomes more complicated. Fortunately, we have found its density in Chapter 5 as
\[
\phi_{X_T}(a) = \frac{1}{a} \sum_{k=0}^{\infty} \phi(ln(a) - ln(x_0); (\mu_D - 0.5\sigma_D^2)T + k\mu_J, \sigma_D^2 T + k\sigma_J^2) e^{-\lambda T (\lambda T)^k / k!} \quad (31)
\]
for \(a \geq 0\) and \(\phi_{X_T}(a) = 0\) for \(a < 0\); Where \(\phi(a; \mu, \sigma^2)\) is the density function of normal random variable with mean \(\mu\) and variance \(\sigma^2\). Then, we obtain the terminal condition
\[
c(T) = E [\max(Z - y, 0)] \\
= E [\max(\max(X_T - K, 0) - y, 0)] \\
= \int_R \max(\max(q - K, 0) - y, 0) \phi_{X_T}(q) dq
\]
This is an abuse of notation as before. Actually, we mean

\[
c(T) = \begin{bmatrix}
\int_R \max(\max(q - K, 0) - y_1, 0)\phi_X(q) \, dq \\
\int_R \max(\max(q - K, 0) - y_2, 0)\phi_X(q) \, dq \\
\vdots \\
\int_R \max(\max(q - K, 0) - y_n, 0)\phi_X(q) \, dq
\end{bmatrix}
\]  \tag{31}

It can be evaluated by normal distribution quadrature rule.

At the point, we can solve the ODE (30) with terminal condition (31) by our fourth-order-Runge-Kutta solver and find the smallest \( y_0 \) such that \( \Phi(0, y_0) = 0 \).

### 6.5 Numerical result

There are only two assets in our option pricing model: the stock \( X_t \) and the bank account \( B_t \). In Chapter 5, we choose Standard and Poor (S&P500) stock index as the risky asset \( X_t \) and collect data from Jan 1997 to Jan 2005 to estimate the parameters \( \mu_D, \sigma_D, \mu_J, \sigma_J, \) and \( \lambda \). On the other hand, we choose Moody AAA bonds to represent the non-risky asset \( B_t \) and collect data in the same time period to find the appreciation rate \( r \). As a result, we have

\[
\begin{align*}
\mu_D &= 0.205817 \\
\sigma^2_D &= 0.019209 \\
\mu_J &= -0.001422 \\
\sigma^2_J &= 0.000139 \\
\lambda &= 100.891219 \\
r &= 0.07052
\end{align*}
\]

For simplicity, we consider the normalized problem. Suppose the call option has time to maturity one year, and its strike price equals to its underlying stock price at initial time which is one dollar. That is, we assume
\( T = 1, \quad K = 1, \quad x_0 = 1 \)

By using cubic spline as the basic function family with twenty basis \((n = 20)\), one thousand as the number of steps to solve the ODE in step 3, and \(1 \times 10^{-6}\) as the tolerance to find the suitable price \(y_0\) in step 4, we found the value function of \((P2)\) explicitly and graph it in Figure(3). The suitable price of this option is

\[ y_0 = 0.407204 \]

and the corresponding trading strategy \(u^*\) is graphed in Figure(4).

On the other hand, by using linear spline family with the same settings, we can derive similar results. The value function and optimal trading strategy are recorded in Figure (5) and (6),

---

**Figure 6.3:** Value function \(\Phi(t, y)\) of \((P2)\) by cubic spline family
respectively. The suitable price of this option by using linear spline family is

\[ y_0 = 0.477239 \]

Since \( \Phi(t, y) \) is irregular on its boundary, spline families will be good choices to make approximation. This is because the piecewise structure of spline can often contain the ill effects in a local region, instead of allowing it to propagate over the entire interval.

Comparing the results generated by cubic and linear spline families, although they are slightly different, both of them are reasonable. It is natural to have different results when employing different approximation methods. Note that the value function in Figure (3) is smoother than it in Figure (5) because the cubic polynomials are smoother than linear ones.
In Figure 4 and 6, the optimal strategy $u^*$ are shown, with red indicating height 1 and blue indicating height 0.

6.6 Conclusion

In this chapter, we constructed an algorithm to solve the Bellman’s equation with jump in our option pricing model and found the suitable price and hedging strategy explicitly. Although the model we are dealing with is only for small investors, the same idea also works for large investors. The only difficulty to deal with a large investor model is defining the relation between the investor and the market. As long as it is defined properly, we can follow the same
Figure 6.6: Optimal control by linear spline family

idea in Chapter 5 to estimate model parameters and use the algorithm in Chapter 6 with small adjustments to solve the related Bellman’s equation.
REFERENCES


