ABSTRACT

WILSON, EVAN ANDREW. Root Multiplicities of the Indefinite Type Kac-Moody Algebra $HD_n^{(1)}$. (Under the direction of Kailash C. Misra.)

In 1968, Victor Kac and Robert Moody independently introduced a class of Lie algebras called Kac-Moody algebras, to generalize the concept of finite dimensional semisimple Lie algebras to the infinite dimensional case. There are many applications of Kac-Moody algebras in physics and other areas of mathematics.

Each Kac-Moody algebra is determined by a so-called *generalized Cartan matrix* (GCM). Every indecomposable symmetrizable GCM is one of three kinds: *finite*, *affine*, or *indeterminate* type. A finite type Kac-Moody algebras is a finite dimensional simple Lie algebra, the other types are infinite dimensional.

For indefinite type Kac-Moody algebras an important problem is determining its root multiplicities. For finite and affine type Kac-Moody algebras the root multiplicities are known, but not for a single indefinite type Kac-Moody algebra is this problem completely solved, although certain root multiplicities are known.

In this thesis, we study the root multiplicities of the indefinite type Kac-Moody algebra $HD_n^{(1)}$. We use a construction that realizes $\mathfrak{g} = HD_n^{(1)}$ as a $\mathbb{Z}$-graded Lie algebra with local part $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0$ is the affine type Kac-Moody algebra $D_n^{(1)}$. Using this construction, Kang has given a formula for root multiplicities in terms of weight multiplicities of $\mathfrak{g}_0$-modules. The theory of crystal bases allows us to compute these weight multiplicities. We derive a formula for root multiplicities of the form $-\alpha_{-1} - k\delta$ and $-2\alpha_{-1} - 3\delta$. In particular, we find that they are polynomials in n. We show that $\mathrm{mult}(-k\alpha_{-1} - l\delta) = 0$ if $k > l$ and n if $k = l$. We also give tables of the root multiplicities of the roots $-2\alpha_{-1} - k\delta$ and $-2\alpha_{-1} - \alpha_0 - k\delta$ of $HD_4^{(1)}$ for various k that verifies a conjecture of Frenkel that $\mathrm{mult}(\alpha) \leq p^n\left(1 - \frac{(\alpha|\alpha)}{2}\right)$ for this case (although it has been disproven for type $HC_n^{(1)}$). We also give a conjecture regarding a generating function for degree 2 root multiplicities of $HD_4^{(1)}$.

Root Multiplicities of the Indefinite Type Kac-Moody Algebra $HD_n^{(1)}$

by
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BIOGRAPHY

The author was born in St. Paul, Minnesota, in 1979. He struggled with math until junior high school. He studied computer science at Wheaton College, IL, then received encouragement to add a mathematics major. (He also minored in Biblical Studies). After college, he got his MS in Mathematics from Virginia Commonwealth University, where he stayed on as an adjunct for two years. After receiving encouragement from his friends, he went to NC state to get his PhD in Mathematics, from which he expects to graduate in Spring 2012. During the summer of 2011, he went to Osaka, Japan under NSF's EAPSI fellowship, where he worked with Masato Okado. He also plans to further his studies in a postdoctoral fellowship at St. Paul's University, São Paolo, Brazil, under the mentorship of Vyacheslav Futorny, beginning in May 2012.

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Finally, I would like to thank Tim Wollin,For his friendship and support, and to all who have played some small role in my life before or during my time at NC State–thank you all!

Bender: Ey, bro bot, what's your serial number?

Flexo: 3370318.

Bender: Nooooo waaaaay! Mine's 2716057!

Flexo: BAAAHAHAHA!

Bender: Haw haw haw haw!

Fry: Heh heh. I don't get it.

Bender: [condescendingly] We're both expressable as the sum of two cubes!

Flexo: HWOOOOOOO!

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Chapter 1

Introduction

In 1968, Victor Kac ([15]) and Robert Moody ([33]) independently introduced a class of Lie algebras called Kac-Moody algebras, to generalize the concept of finite dimensional semisimple Lie algebras to the infinite dimensional case. Since then, Kac-Moody algebras have grown into an important field with applications in physics and many areas of mathematics. For example, some Kac-Moody algebras are associated with hyperbolic tesselations of the Poincaré disk (see Figure 1.1 for an example). Each Kac-Moody algebra is determined by a matrix called a generalized Cartan matrix (GCM). Indecomposable, symmetrizable GCMs are classified into three kinds: *finite*, *affine*, and *indefinite* types, and their corresponding Kac-Moody algebras are classifed in the same way. Let $\mathfrak{g}$ be a Kac-Moody algebra. The subspace $\mathfrak{g}_\alpha := \{x_\alpha | [h, x_\alpha] = \langle h, x_\alpha \rangle x_\alpha, h \in \mathfrak{h}\}$, for $\alpha \in Q$, is called the *root space* of $\mathfrak{g}$ corresponding to the *root* α if $\alpha \neq 0$ and $\dim(\mathfrak{g}_\alpha) \neq 0$ where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and Q is the root lattice. If α is a root of $\mathfrak{g}$ then $\dim(\mathfrak{g}_\alpha) < \infty$ (see [16]) and we define $\dim(\mathfrak{g}_\alpha)$ to be the *multiplicity* of α, denoted $\mathrm{mult}(\alpha)$. For a finite type Kac-Moody algebra, $\mathrm{mult}(\alpha) = 1$ for all roots α. If $\mathfrak{g}$ is affine type, then the root multiplicities are also known (see [16]). It is an open and difficult problem to compute the root multiplicities of indefinite type Kac-Moody algebras. This problem has been studied in [8] and [20] for type $HA_1^{(1)}$, [26] and [12] for type $HA_n^{(1)}$, [28] for type $HC_n^{(1)}$, [4] for $HX_n^{(1)}, X = A, B, C, D$, and [18] for $E_{10} = HE_8^{(1)}$. However, there is not a single indefinite type Kac-Moody algebra for which the root multiplicities are known completely.

In this thesis, we study the root multiplicities of the indefinite Kac-Moody algebra

$HD_n^{(1)}, n \geq 4$, which has the GCM:

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix}$$

and the index set $I = \{-1, 0, 1, 2, \ldots, n-2, n-1, n\}$. By restricting to the index set $I \backslash \{-1\}$ we see that the affine type Kac-Moody algebra $D_n^{(1)}$ is a subalgebra of $HD_n^{(1)}$.

In chapter 3 we review the following construction given in [3], (see also [8] and [15]). Let $\mathfrak{g}_0$ be a Lie algebra and let V and V' be two $\mathfrak{g}_0$-modules. Now, let $\psi : V \otimes V' \to \mathfrak{g}_0$ be a $\mathfrak{g}_0$-module homomorphism. We construct the minimal graded Lie algebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ such that $\mathfrak{g}_{-1} = V, \mathfrak{g}_1 = V'$, and no ideal intersects $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ trivially. We remark that $\mathfrak{g}$ is not always a Kac-Moody algebra, unless we set certain conditions on $\mathfrak{g}_0, V, V'$, and ψ. If $\mathfrak{g}_0 = D_n^{(1)}$, $V = V(\Lambda_0)$ is the basic $D_n^{(1)}$-module, $V' = V^*(\Lambda_0)$ is its finite dual, and $\psi : V(\Lambda_0) \otimes V^*(\Lambda_0) \to D_n^{(1)}$ is the $D_n^{(1)}$- module homomorphism such that $\psi(v \otimes w^*) = -\sum_{i \in \mathcal{I}} \langle w^*, x_i \cdot v \rangle - 2\langle w^*, v \rangle c$, where $\{x_i | i \in \mathcal{I}\}$ is a basis of $D_n^{(1)}$ and c spans the one-dimensional center of $D_n^{(1)}$, then $\mathfrak{g} \cong HD_n^{(1)}$.

In chapter 4 we review several results from the homology theory of Lie algebra modules, and a formula of Kang ([19],[22]) that gives root multiplicities in terms of weight multiplicities of certain $\mathfrak{g}_0$-modules. To use this formula, one needs to find the partitions of the desired root, and compute certain weight multiplicities for $D_n^{(1)}$-modules. To do this, we use the theory of quantum groups and crystal bases.

In chapter 5 we review the concepts of quantum groups and crystal bases, and the path realization of $D_n^{(1)}$-modules. In 1985 Drinfel′d ([7]) and Jimbo ([14]) introduced quantum groups as "q-deformations" of universal enveloping algebras of (symmetrizable) Kac-Moody algebras. In 1988, Lusztig ([30]) showed that for generic deformation parameter "q" the representation theory of the quantum group is parallel to that of the underlying Kac-Moody algebra. Around 1990, Kashiwara ([17]) and Lusztig ([31]) introduced the notion of a *crystal base*, which is basis of $V^q(\lambda)$ in the "$q=0$" limit. In [23] and [24] the notion of perfect crystals was introduced to realize the crystal bases of affine algebras. The set $\mathcal{B}(\lambda)$ is called the *crystal* of $V^q(\lambda)$, which can be realized as a semi-infinite tensor product $\mathcal{P}(\lambda) = \cdots \otimes \mathcal{B} \otimes \mathcal{B}$. Here $\mathcal{B}$ is a perfect crystal of level $l = \langle \lambda, c \rangle$. The elements of $\mathcal{P}(\lambda)$ consist of semi-infinite sequences $(\ldots, p_1, p_0)$ satisfying the condition that $p_i = b_i, i \gg 0$ for a certain path $\mathbf{b}_\lambda = (\ldots, b_1, b_0) \in \mathcal{P}(\lambda)$, called the ground state path, corresponding to the highest weight vector. These paths have some applications in mathematical physics (see [32] for example).

In chapter 6, we use the results of previous chapters to compute the multiplicities of certain $HD_n^{(1)}$ roots. In particular, we consider roots of the form $-k\alpha_{-1} - l\delta$. A general result is that $\mathrm{mult}(-k\alpha_{-1} - l\delta) = 0$ if $k > l$ and $\mathrm{mult}(-k\alpha_{-1} - k\delta) = n$. Then we consider roots of degree 1, and 2, where the degree of the root $-k\alpha_{-1} - l\delta$ is defined to be the integer k. Degree 1 root multiplicities are equal to the corresponding weight multiplicities, by Kang's formula. We give an explicit formula for these multiplicities based on a well-known generating series as well as several examples for small l. In particular, we observe that these are all polynomials in n of degree l. In the next section, we consider the degree 2 root $-2\alpha_{-1} - 3\delta$ and compute its multiplicity polynomial. Finally, we discuss a conjecture of Frenkel that states that for a root α of a hyperbolic Kac-Moody algebra of rank $n+2$, $\mathrm{mult}(\alpha) \leq p^n\big(1 - \frac{(\alpha|\alpha)}{2}\big)$, which has been shown not to hold in the $HC_2^{(1)}$ case in [28], [34]. We give a table of some root multiplicities of $HD_4^{(1)}$ based on Peterson's recurrent formula. However, there is no observed contradiction with Frenkel's conjecture in our case. We also conjecture that the multiplicity of any degree 2 root is determined by the integer $1 - \frac{(\alpha|\alpha)}{2}$. This leads to a conjecture regarding a generating function for the degree 2 roots of $HD_4^{(1)}$.

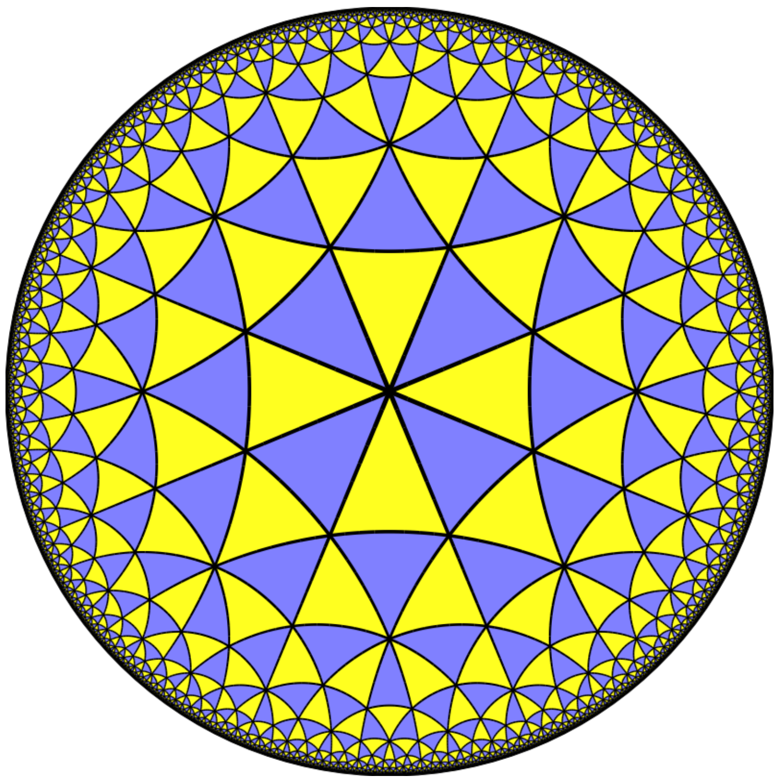


Figure 1.1: Example of a hyperbolic tesselation.

Chapter 2

Kac-Moody Algebras

In this chapter, we review Lie algebras, Kac-Moody algebras, and their representations. We let $k = \mathbb{C}$ denote the field of complex numbers.

2.1 Lie algebras

Definition 1. *A* Lie algebra *is a vector space* $\mathfrak{g}$ *over* k *together with a binary operation called the* bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *which satisfies the following properties:*

1. $[cx + y, z] = c[x, z] + [y, z]$, *and* $[z, cx + y] = c[z, x] + [z, y]$ *for all* $x, y, z \in \mathfrak{g}, c \in k$,
2. $[x, x] = 0$, *for all* $x \in \mathfrak{g}$,
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, *for all* $x, y, z \in \mathfrak{g}$.

Remark: In a Lie algebra,

$$\begin{aligned} [x + y, x + y] &= [x, x] + [x, y] + [y, x] + [y, y] \text{ by Property (1)} \\ &= [x, y] + [y, x] \text{ by Property (2).} \end{aligned}$$

But $[x + y, x + y] = 0$ by Property (2) of the definition of Lie algebra. Therefore, $[x, y] = -[y, x]$.

Remark: Property (3) of the definition of a Lie algebra is called the *Jacobi Identity.* It can also be written in the following equivalent form:

$$\mathrm{ad}_x([y, z]) = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)]$$

where $\mathrm{ad}_x(y) := [x, y]$ is called the *adjoint map.*

Example: Let $\mathfrak{sl}(2,k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in k, b + d = 0 \right\}$ and define $[A, B] = AB - BA$ for all $A, B \in \mathfrak{sl}(2,k)$. Under this bracket, $\mathfrak{sl}(2,k)$ is a Lie algebra. A basis of $\mathfrak{sl}(2,k)$ is

$$\left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

The bracket is given on the basis elements by

$$[e, f] = h, \qquad [h, f] = -2f, \qquad [h, e] = 2e.$$

Example: Let $\mathcal{A}$ be an associative algebra, that is, a vector space over k equipped with an associative bilinear operation $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, (x, y) \mapsto x \cdot y$. Then $\mathcal{A}$ is a Lie algebra with bracket given by $[x, y] = x \cdot y - y \cdot x$, for $x, y \in \mathcal{A}$.

Example: Let V be a vector space over k. Then the vector space $\mathrm{End}(V)$ of invertible linear transformations from V to itself is an associative algebra with product given by function composition. The corresponding Lie algebra is denoted $\mathfrak{gl}(V)$.

The notions of *homomorphism* and *isomorphism* of Lie algebras are fundamental to the study of their structure.

Definition 2. *A* homomorphism *from a Lie algebra* $\mathfrak{g}_1$ *to a Lie algebra* $\mathfrak{g}_2$ *is a map* $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ *satisfying the following properties:*

1. $\phi(cx + y) = c\phi(x) + \phi(y), x, y \in \mathfrak{g}_1, c \in k,$

2. $\phi([x, y]) = [\phi(x), \phi(y)].$

A homomorphism of Lie algebras is called an isomorphism *if it is one-to-one and onto. An isomorphism from a Lie algebra to itself is called an* automorphism*. An* involution *is an automorphism* ω *satisfying* $\omega^2 = \mathrm{id}$ *where* id *denotes the identity map.*

Thus, a homomorphism of Lie algebras is a map preserving both the linear structure and the bracket operation of a Lie algebra.

Example: The *adjoint homorphism* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ given by $\mathrm{ad}(x) = \mathrm{ad}_x$ is a Lie algebra homomorphism.

When studying a Lie algebra, it is often important to understand its *ideals.*

Definition 3. *An* ideal *of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{i}$ of $\mathfrak{g}$ satisfying*

$$[x, i] \in \mathfrak{i}$$

for all $x \in \mathfrak{g}, i \in \mathfrak{i}$.

Example: Let V be a vector space over k. Then the subspace $\mathrm{span}_k\{\mathrm{id}\}$ is an ideal of $\mathfrak{gl}(V)$.

An important construction in Lie algebras is that of a *quotient Lie algebra.*

Definition 4. *Let $\mathfrak{g}$ be a Lie algebra over k and $\mathfrak{i}$ an ideal of $\mathfrak{g}$. Then the* quotient Lie algebra *is defined to be the quotient vector space:*

$$\mathfrak{g}/\mathfrak{i} = \{x + \mathfrak{i} | x \in \mathfrak{g}\}$$

with the bracket

$$[x + \mathfrak{i}, y + \mathfrak{i}] = [x, y] + \mathfrak{i}.$$

In fact, this bracket is well-defined and gives the structure of a Lie algebra to $\mathfrak{g}/\mathfrak{i}$ (see [10]).

Another useful concept is that of a *(universal) enveloping algebra.*

Definition 5.

1. *Let $\mathfrak{g}$ be a Lie algebra. An* enveloping algebra *of $\mathfrak{g}$ is a pair $(\mathcal{A}, \iota)$ where $\mathcal{A}$ is an associative algebra, considered as a Lie algebra with commutator bracket, and $\iota : \mathfrak{g} \to \mathcal{A}$ is a Lie algebra homomorphism.*

2. *The* universal enveloping algebra $(U(\mathfrak{g}), \iota)$ *of $\mathfrak{g}$ is the unique enveloping algebra of $\mathfrak{g}$ satisfying the following universal property: if $(\mathcal{A}, \kappa)$ is another enveloping algebra of $\mathfrak{g}$ then there exists a unique homomorphism of algebras $\phi : U(\mathfrak{g}) \to \mathcal{A}$ such that $\phi \circ \iota = \kappa$, alternately, such that the following diagram commutes:*

$$\begin{tikzcd} & U(\mathfrak{g}) \arrow[dd, dashed, "!\phi"] \\ \mathfrak{g} \arrow[ur, "\iota"] \arrow[dr, "\kappa"'] & \\ & \mathcal{A} \end{tikzcd}$$

The uniqueness of $U(\mathfrak{g})$, provided that it exists, follows from a standard argument. To see that it exists, consider the tensor algebra $T(\mathfrak{g}) := \bigoplus_{i=0}^{\infty} \mathfrak{g}^{\otimes i}$, and let $\mathcal{I}$ be the two-sided ideal of $T(\mathfrak{g})$ generated by the set $\{x \otimes y - y \otimes x - [x,y] | x, y \in \mathfrak{g}\}$. Then the set $U(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{I}$, together with the map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ given by composing the inclusion map of $\mathfrak{g}$ into $T(\mathfrak{g})$ with the quotient map, satisfies the conditions for a universal enveloping algebra.

From the above construction, it is not clear whether $\mathfrak{g}$ is mapped injectively into $U(\mathfrak{g})$ by ι. The Poincaré-Birkhoff-Witt theorem stated below makes it clear that ι is in fact injective, and gives a basis of $U(\mathfrak{g})$ as well.

Theorem 1 (see [10])**.**

1. *The map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is injective.*

2. *Let I be a well-ordered index set and $\{x_i | i \in I\}$ be an ordered basis of $\mathfrak{g}$. Then the set $\{x_{i_1} x_{i_2} \cdots x_{i_k} | i_1 < i_2 < \cdots < i_k\}$ is a basis of $U(\mathfrak{g})$. Here we understand the empty product to be* 1.

2.2 Kac-Moody Algebras

In this section, we define a certain class of possibly infinite dimensional Lie algebras called Kac-Moody algebras and give the basic results and definitions that we will use regarding them.

Every Kac-Moody algebra is determined by a *generalized Cartan matrix* (GCM), which is a matrix $A = (a_{ij})_{i,j \in I}$, where I is a finite index set, satisfying the following conditions:

1. $a_{ii} = 2$,

2. $a_{ij} \leq 0$ if $i \neq j$,

3. $a_{ij} < 0$ if and only if $a_{ji} < 0$.

A GCM is called *symmetrizable* if there exists a diagonal matrix $D = \operatorname{diag}(s_i)_{i \in I}$ such that $s_i \in \mathbb{Q}_{>0}, i \in I$ and DA is a symmetric matrix. The matrix A is called *indecomposable* if for every pair of subsets $I_1, I_2 \subset I$ with $I_1 \cup I_2 = I$, there exists some $i \in I_1$ and $j \in I_2$ such that $a_{ij} \neq 0$. We will consider only symmetrizable GCMs.

For a GCM A with index set I, let I' be a subset of I of cardinality corank(A) and define $\mathfrak{h}$ to be the vector space over $\mathbb{C}$ generated by the set $\{h_i, d_j | i \in I, j \in I'\}$. For $i \in I$, we define $\alpha_i \in \mathfrak{h}^*$ to be the linear functional satisfying $\langle \alpha_i, h_j \rangle = a_{ij}$ for $j \in I$, and $\langle \alpha_i, d_j \rangle = \delta_{ij}$ for $j \in I'$. We define $\Pi = \{\alpha_i | i \in I\}$ to be the set of *simple roots* of $\mathfrak{g}$. The set $\Pi^\vee := \{h_i | i \in I\}$ is defined to be the set of *simple co-roots*.

Definition 6. *Let $A = (a_{ij})_{i,j \in I}$ be a (symmetrizable) GCM and $\Pi, \Pi^\vee$ given sets of simple roots, co-roots. The* Kac-Moody algebra $\mathfrak{g}(A)$ *is the Lie algebra over $\mathbb{C}$ generated by the elements $e_i, f_i, i \in I$, and $\mathfrak{h}$ satisfying the following relations:*

1. $[h, h'] = 0, h, h' \in \mathfrak{h}$
2. $[e_i, f_j] = \delta_{ij} h_i, i, j \in I$
3. $[h, e_i] = \alpha_i(h) e_i, i \in I, h \in \mathfrak{h}$
4. $[h, f_i] = -\alpha_i(h) f_i, i \in I, h \in \mathfrak{h}$
5. $(ad_{e_i})^{1-a_{ij}}(e_j) = 0, (ad_{f_i})^{1-a_{ij}}(f_j) = 0$, *for $i \neq j \in I$.*

Where there is no confusion about A, we write $\mathfrak{g}$ for $\mathfrak{g}(A)$.

Relations (1)-(4) of Definition 6 are called the *Chevalley relations* and the relations in (5) are called the *Serre relations*. We have the following alternate characterization of Kac-Moody algebras:

Theorem 2 (see [16])**.** *Let A be a (symmetrizable)* GCM *and $\mathfrak{g}(A)$ be the Kac-Moody algebra determined by A. Then $\mathfrak{g}(A) = \hat{\mathfrak{g}}/\mathfrak{i}$ where $\hat{\mathfrak{g}}$ is the Lie algebra generated by $\{e_i, f_i, \mathfrak{h}\}$ satisfying relations (1)-(4) of Definition 6 and $\mathfrak{i}$ is the maximal ideal of $\hat{\mathfrak{g}}$ intersecting $\mathfrak{h}$ trivially.*

The subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Cartan subalgebra* of $\mathfrak{g}$. Define $Q := \text{span}_{\mathbb{Z}}(\Pi)$ to be the *root lattice*, $Q^+ := \text{span}_{\mathbb{Z}_{>0}}(\Pi)$ to be the *positive root lattice*, and $Q^- := \text{span}_{\mathbb{Z}_{<0}}(\Pi)$ to be the *negative root lattice* of $\mathfrak{g}$. Finally, define $\mathfrak{g}_\alpha := \{x_\alpha | [h, x_\alpha] = \langle h, x_\alpha \rangle x_\alpha, h \in \mathfrak{h}\}$ for $\alpha \in Q$, to be the *root space* of $\mathfrak{g}$ corresponding to the *root* α if $\alpha \neq 0$ and $\dim(\mathfrak{g}_\alpha) \neq 0$. A root $\alpha \in Q^+$ (resp. Q^-) is called a positive (resp. negative) root. The set of roots of a Kac-Moody algebra is denoted by Δ and the set of positive (resp. negative) roots is denoted by Δ^+ (resp. Δ^-). If α is a root of $\mathfrak{g}$ then $\dim(\mathfrak{g}_\alpha) < \infty$ (see [16]) and we define $\dim(\mathfrak{g}_\alpha)$ to be the *multiplicity* of α, denoted mult(α). We have the following result.

Proposition 1 (see [16]).

1. *(Root space decomposition).* $\mathfrak{g} = \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$.

2. *(Triangular decomposition). Let* $\mathfrak{n}^+$ *(resp.* $\mathfrak{n}^-$*) be the subalgebra of* $\mathfrak{g}$ *generated by* $e_i, i \in I$ *(resp.* $f_i, i \in I$*). Then we have the following:*

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

   and for $\alpha \in \Delta^+$ *we have* $\mathfrak{g}_{\pm\alpha} \subset \mathfrak{n}^\pm$.

3. *(Chevalley involution). There exists an involution* ω *of* $\mathfrak{g}$ *satisfying* $\omega(e_i) = -f_i$ *and* $\omega(h) = -h$ *for* $h \in \mathfrak{h}$.

Remark: From the definition of ω it is clear that $\omega(f_i) = -e_i$.

If $\alpha \in \Delta^+$ is a root of $\mathfrak{g}$ then $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, so we see that $\mathrm{mult}(\alpha) = \mathrm{mult}(-\alpha)$. This fact is important for computing the root multiplicities of Kac-Moody algebras, since by the above proposition every root is either in Δ^+ or Δ^-.

Let $A = (a_{ij})_{i,j\in I}$ be a (symmetrizable) GCM and fix a matrix $D = \mathrm{diag}(s_i)_{i\in I}, s_i \in \mathbb{Q}_{>0}$ such that DA is symmetric. Define the following symmetric bilinear form on $\mathfrak{h}$:

$$\begin{aligned} (h|h_i) &= \langle \alpha_i, h\rangle s_i \text{ for } h \in \mathfrak{h}, i \in I, \\ (d_i|d_j) &= 0 \text{ for } i, j \in I'. \end{aligned}$$

Then, it is possible to extend $(\cdot|\cdot)$ to a symmetric bilinear form on $\mathfrak{g}$ such that the following conditions are satisfied (see [16]):

1. $(\cdot|\cdot)$ is *associative*, that is $([x, y]|z) = (x|[y, z]), x, y, z \in \mathfrak{g}$,

2. $(\cdot|\cdot)$ is non-degenerate on $\mathfrak{g}$ and $\mathfrak{h}$,

3. $(\mathfrak{g}_\alpha|\mathfrak{g}_\beta) = 0$ for all roots α and β unless $\alpha + \beta = 0$

4. $\mathfrak{g}_\alpha$ is non-degererately paired with $\mathfrak{g}_{-\alpha}$ under $(\cdot|\cdot)$ for all roots α.

There is also a corresponding bilinear form, also denoted $(\cdot|\cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$. We start by defining the map $\nu : \mathfrak{h} \to \mathfrak{h}^*$ to be the linear map satisfying $\nu(h)(h') = (h|h')$. This map is one-to-one, since $(\cdot|\cdot)$ is non-degenerate on $\mathfrak{h}$, and therefore bijective since $\dim(\mathfrak{h}) = \dim(\mathfrak{h}^*)$. We then define the form $(\cdot|\cdot)$ on $\mathfrak{h}^*$ by $(\lambda|\mu) = (\nu^{-1}(\lambda)|\nu^{-1}(\mu))$.

For $i \in I$, define a linear transformation $r_i : \mathfrak{h}^* \to \mathfrak{h}^*$ by $r_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i$. Then r_i is its own inverse, hence is an element of the group $\mathrm{GL}(\mathfrak{h}^*)$ of invertible linear transformations of $\mathfrak{h}^*$. We define the *Weyl group* of $\mathfrak{g}$ to be the subgroup W of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the set $\{r_i | i \in I\}$. The *length* of $w \in W$ denoted $\ell(w)$ is the least positive integer t such that $w = r_{i_1} r_{i_2} \cdots r_{i_t}$ for some $i_1, i_2, \ldots i_t \in I$. A root α is called a *real root* if there exists $w \in W$ such that $w(\alpha_i) = \alpha$ for some $i \in I$. Otherwise it is called an *imaginary root*. If α is a real root then $\mathrm{mult}(\alpha) = 1$ (see [16]). However, the multiplicities of imaginary roots are not necessarily equal to 1, so we focus on these roots.

Let I be an index set. For a column vector $v = (v_i)_{i \in I}$ we say $v \geq 0$ if $v_i \geq 0, i \in I$ and similarly $v > 0$ if $v_i > 0, i \in I$. Then we have the following classification of GCMs:

Theorem 3 (see [16])**.** *Let A be an indecomposable* GCM*. Then exactly one of the following three conditions is satisfied for both A and A^T:*

(F) $\det(A) \neq 0$, *there exists $u > 0$ such that $Au > 0$, and $Av \geq 0$ implies $v > 0$ or* $v = 0$,

(A) $\mathrm{corank}(A) = 1$, *there exists $u > 0$ such that $Au = 0$, and $Av \geq 0$ implies $Av = 0$,*

(I) there exists $u > 0$ such that $Au < 0$, and $Av \geq 0$ and $v \geq 0$ imply $v = 0$.

If A is an indecomposable GCM satisfying condition (F) (resp. (A), (I)) above, we call $\mathfrak{g}(A)$ a *finite* (resp. *affine*, *indeterminate*) type Kac-Moody algebra. If $\mathfrak{g}$ is a finite type Kac-Moody algebra, then $\mathfrak{g}$ is a finite-dimensional simple Lie algebra and all its roots are real, so $\mathrm{mult}(\alpha) = 1$ for all roots α.

Let $\mathfrak{g}$ be an affine type Kac-Moody algebra with index set $I = \{0, 1, \ldots, n\}$. There exists a vector $u = (a_0, a_1, \ldots, a_n)^T > 0$ such that $Au = 0$, $a_i \in \mathbb{Z}_{>0}$, $\gcd(a_0, a_1, \ldots, a_n) = 1$. The element $\delta = \sum_{i=0}^n a_i \alpha_i$ is called the *canonical null root*. Dually, there exists a vector $v = (a_0^\vee, a_1^\vee \ldots, a_n^\vee)^T$ such that $A^T v = 0$, $a_i^\vee \in \mathbb{Z}_{>0}$, $\gcd(a_0^\vee, a_1^\vee, \ldots, a_n^\vee) = 1$. The element $c = \sum_{i=0}^n a_i^\vee h_i$ is called the *canonical central element*, and satisfies $[c, x] = 0$ for all $x \in \mathfrak{g}$. In this case, $\mathrm{corank}(A) = 1$, so we take the subset $I' = \{0\} \subset I$ and put $d = d_0$. Furthermore [1],

$$\mathrm{mult}(\alpha) = \begin{cases} 1, & \alpha \text{ real}, \\ n, & \alpha \text{ imaginary}. \end{cases}$$

[1] Strictly this is only true if $\mathfrak{g}$ is an untwisted affine type Kac-Moody algebra (see [16]), which is the only kind we consider.

2.3 Modules and Representations of Lie algebras

We now review the notions of representations and modules of Lie algebras, with a special focus on the results for Kac-Moody algebras which we will use later in the construction of $HD_n^{(1)}$ and in computing its root multiplicities.

Definition 7. *Let $\mathfrak{g}$ be a Lie algebra.*

1. *A* representation *of $\mathfrak{g}$ on a vector space V over k is a Lie algebra homomorphism*

$$\phi : \mathfrak{g} \to \mathfrak{gl}(V).$$

2. *A* $\mathfrak{g}$-module *is a vector space V over k together with an operation $\cdot : \mathfrak{g} \times V \to V$ satisfying the following properties:*

   (a) $(cx + y) \cdot v = c(x \cdot v) + y \cdot v$ *for* $c \in k, x, y \in \mathfrak{g}, v \in V$,

   (b) $x \cdot (cv + w) = c(x \cdot v) + x \cdot w$ *for* $c \in k, x \in \mathfrak{g}, v, w \in V$,

   (c) $x \cdot (y \cdot v) - y \cdot (x \cdot v) = [x, y] \cdot v$ *for* $x, y \in \mathfrak{g}, v \in V$.

A $\mathfrak{g}$-module V is equivalent to a representation ϕ of $\mathfrak{g}$ on V by the following identification: for $x \in \mathfrak{g}, v \in V$,

$$x \cdot v \longleftrightarrow \phi(x)(v).$$

Let V be a $\mathfrak{g}$-module. The action of $\mathfrak{g}$ on V extends to an action of $U(\mathfrak{g})$ on V by defining, for a degree k element $x_1 x_2 \cdots x_k \in U(\mathfrak{g})$, and $v \in V$,

$$(x_1 x_2 \cdots x_k) \cdot v := x_1 \cdot (x_2 \cdot (\cdots (x_k \cdot v) \cdots)).$$

Under the above identification, V is a $U(\mathfrak{g})$-module. Let V be a $\mathfrak{g}$-module. If W is a subspace of V such that $x \cdot W \subset W$ then W is called a $\mathfrak{g}$-*submodule* of V. V is called *irreducible* if it has no submodules other than $\{0\}$ and V. We now give some examples of representations and $\mathfrak{g}$-modules.

Example: As we have seen, the adjoint map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a homomorphism of Lie algebras. Therefore, it is a representation of $\mathfrak{g}$ on itself. In this context, it is called the *adjoint representation.*

Example: Let $V = \text{span}_{\mathbb{C}}\{v_0, v_1, \ldots, v_k\}$ be a vector space and define an action of $\mathfrak{sl}(2, \mathbb{C})$ on V by:

$$\begin{aligned} h \cdot v_j &= (k-2j)v_j \\ f \cdot v_j &= (j+1)v_{j+1} \\ e \cdot v_j &= (k-j+1)v_{j-1} \end{aligned}$$

where v_j is understood to be 0 if $j < 0$ or $j > k$. Then V is an $\mathfrak{sl}(2, \mathbb{C})$-module.

Example: Let V, W be $\mathfrak{g}$-modules. Then $V \otimes W$ can be made into a $\mathfrak{g}$-module by defining $x \cdot (v \otimes w) = x \cdot v \otimes w + v \otimes x \cdot w$.

A $\mathfrak{g}$-module homomorphism, which we define below, is a structure preserving map between two $\mathfrak{g}$-modules.

Definition 8. *Let V, W be two $\mathfrak{g}$-modules. A $\mathfrak{g}$-module* homomorphism *from V to W is a linear map $\phi : V \to W$ satisfying:*

$$\phi(x \cdot v) = x \cdot \phi(v),$$

for all $x \in \mathfrak{g}, v \in V$.

Now we consider the case where $\mathfrak{g}$ is a Kac-Moody algebra. Analogously to how we have defined root spaces, for $\mu \in \mathfrak{h}^*$ we define the set $V_\lambda := \{v \in V | h \cdot v = \langle \mu, h \rangle v, h \in \mathfrak{h}\}$ to be the *weight space* of V of weight λ and $\dim(V_\lambda)$ to be the *(weight) multiplicity* of μ denoted $\text{mult}_V(\mu)$. If all the weight spaces of a $\mathfrak{g}$-module V are finite dimensional, then we define the *character* of V to be the formal sum:

$$\text{ch}(V) = \sum_{\mu \in \mathfrak{h}^*} \text{mult}_V(\mu) e(\mu),$$

where $e(\cdot)$ is the formal exponential satisfying $e(\lambda + \mu) = e(\lambda) \cdot e(\mu)$.

A $\mathfrak{g}$-module V is called a *highest weight module* with highest weight λ if and only if it satisfies the following:

1. There exists $0 \neq v_\lambda \in V$ such that $h \cdot v_\lambda = \langle \lambda, h \rangle v_\lambda$ for all $h \in \mathfrak{h}$,

2. $\mathfrak{n}^+ \cdot v_\lambda = \{0\}$,

3. $U(\mathfrak{g}) \cdot v_\lambda = V$.

We define a partial ordering on $\mathfrak{h}^*$ by $\lambda < \mu$ if and only if $\lambda - \mu \in Q^-$. Any highest weight $\mathfrak{g}$-module V of highest weight λ satisfies the following properties (see [16]):

1. *(Weight Space Decomposition).* $V = \bigoplus_{\mu \le \lambda} V_\mu$,

2. $V_\lambda = \mathbb{C}v_\lambda$,

3. $V_\mu < \infty, \mu \in \mathfrak{h}^*$.

For every $\lambda \in \mathfrak{h}^*$ there exists a unique irreducible highest weight module with highest weight λ (see [16]), which we denote by $V(\lambda)$. Define $P := \{\lambda \in \mathfrak{h}^* | \langle \lambda, h_i \rangle, \langle \lambda, d_j \rangle \in \mathbb{Z}, i \in I, j \in I'\}$ to be the *weight lattice*, $P^\vee := \text{span}_\mathbb{Z}(\{h_i | i \in I\} \cup \{d_j | j \in I'\})$ to be the *coweight lattice* and $P^+ := \{\lambda \in P | \langle \lambda, h_i \rangle \in \mathbb{Z}_{\geq 0}, i \in I\}$ to be the *positive weight lattice*. Elements of P are called *integral weights* and elements of P^+ are called *dominant integral weights*. If $\mathfrak{g}$ is an affine type Kac-Moody algebra, then $P = \text{span}_\mathbb{Z}(\{\Lambda_i | i \in I\} \cup \{a_0^{-1}\delta\})$, where $\Lambda_i \in \mathfrak{h}^*$ is defined by $\Lambda_i(h_j) = \delta_{ij}, i \in I$, $\Lambda_i(d) = 0$. The set of all Λ_i is called the set of *fundamental weights*. For $\lambda \in P^+$, we define the integer $l = \langle \lambda, c \rangle$ to be the *level* of λ. For $l \in \mathbb{Z}_{\geq 0}$ we define the set $P_l^+ := \{\lambda \in P^+ | \langle \lambda, c \rangle = l\}$.

A $\mathfrak{g}$-module is called *integrable* if e_i, f_i act locally nilpotently on V for all $i \in I$. Then we have the following result.

Theorem 4 (see [16]). *The irreducible highest weight $\mathfrak{g}$-module $V(\lambda)$ is integrable if and only if $\lambda \in P^+$.*

2.4 The Indefinite Type Kac-Moody Algebra $HD_n^{(1)}$ and the Affine Type Kac-Moody algebra $D_n^{(1)}$

The Kac-Moody algebra $HD_n^{(1)}, n \geq 4$ is determined by the following GCM:

$$A = (a_{ij})_{i,j=-1}^{n} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix} \tag{2.1}$$

It is an indefinite type Kac-Moody algebra that conatins as a subalgebra the affine type Kac-Moody algebra $D_n^{(1)}$ by deleting the index -1. The canonical null root of $D_n^{(1)}$ is $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ and the canonical central element is $c = h_0 + h_1 + 2h_2 + \cdots + 2h_{n-2} + h_{n-1} + h_n$. In the next chapter, we will describe an explicit construction of $HD_n^{(1)}$ in terms of $D_n^{(1)}$-modules.

Chapter 3

Construction of $HD_n^{(1)}$

The construction of the algebra $HD_n^{(1)}$ has three components:

1. The Lie algebra $\mathfrak{g}_0 = D_n^{(1)}$
2. The $\mathfrak{g}_0$-modules $V(\Lambda_0), V^*(\Lambda_0)$.
3. A $\mathfrak{g}_0$-module homomorphism $\psi : V^*(\Lambda_0) \otimes V(\Lambda_0) \to \mathfrak{g}_0$.

With these components, we construct the graded Lie algebra $\tilde{\mathfrak{g}}$, and $\mathfrak{g}$ as a quotient of $\tilde{\mathfrak{g}}$. We then show that $\mathfrak{g}$ is isomorphic to $HD_n^{(1)}$.

We introduce the following notation:

- $S = \{0, 1, \ldots, n\}$: The index set of $\mathfrak{g}_0 = D_n^{(1)}$.
- Δ_S : The set of roots of $\mathfrak{g}_0$.
- $\Delta_S^{\pm}$: The set of positive (resp. negative) roots of $\mathfrak{g}_0$.
- $\Delta^{\pm}(S) : \Delta^{\pm} \backslash \Delta_S^{\pm}$.
- $W(S) : \{w \in W | w\Delta^- \cap \Delta^+ \subseteq \Delta^+(S)\}$.

3.1 The Homomorphism ψ

Let $\mathfrak{g}_0 = D_n^{(1)}$. Let $\mathfrak{h}_0 = \text{span}_{\mathbb{C}}\{h_0, h_1, \ldots, h_n, d\}$ be the Cartan subalgebra of $\mathfrak{g}_0$. Let $V(\Lambda_0)$ be the irreducible highest weight $\mathfrak{g}_0$-module of highest weight Λ_0. The restricted dual $V^*(\Lambda_0)$ of $V(\lambda)$ is defined to be the subset $\bigoplus_{\mu\in\mathfrak{h}_0^*}(V(\Lambda_0)_\mu)^*$ of $V(\Lambda_0)^*$. Then $V^*(\Lambda_0)$ is a $\mathfrak{g}_0$-module under the action

$$\langle x\cdot w^*, v\rangle = -\langle w^*, x\cdot v\rangle, w^*\in V^*(\Lambda_0) \tag{3.1}$$

In fact, it is a lowest weight module, with lowest weight vector v_0^*, because $\langle f_i\cdot v_0^*, v\rangle = -\langle v_0^*, f_i\cdot v\rangle$ is only non-zero if $f_i\cdot v$ is proportional to v_0. In that case, $\mathrm{wt}(v) = \Lambda_0+\alpha_i$, which is not a weight of $V(\Lambda_0)$. Hence $v=0$, which is a contradiction since $\langle f_i\cdot v_0^*, 0\rangle = 0$. Therefore, $f_i\cdot v_0^* = 0$. Similarly, we can see that $U(\mathfrak{g}_0)\cdot v_0^* = V^*(\Lambda_0)$ and $h\cdot v_0^* = -\Lambda_0(h)v_0^*, h\in\mathfrak{h}_0$.

In order to define $\psi : V^*(\Lambda_0)\otimes V(\Lambda_0)\to\mathfrak{g}_0$, we will make use of the symmetric, associative bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}_0$ which has the following properties:

1. $(\cdot|\cdot)$ is non-degenerate on $\mathfrak{g}_0$
2. $((\mathfrak{g}_0)_\alpha, |(\mathfrak{g}_0)_\beta) = 0$ for all roots α and β unless $\alpha+\beta=0$
3. $(\mathfrak{g}_0)_\alpha$ is non-degererately paired with $(\mathfrak{g}_0)_{-\alpha}$ under $(\cdot|\cdot)$ for all roots α.

For $\alpha\in\Delta_S^+$, let $\{y_{\alpha,1}, y_{\alpha,2},\dots,y_{\alpha,l}\}$ be a basis of $(\mathfrak{g}_0)_\alpha$ and choose a basis

$$\{y_{-\alpha,1}, y_{-\alpha,2},\dots,y_{-\alpha,l}\}$$

of $(\mathfrak{g}_0)_{-\alpha}$ such that $(y_{\alpha,i}|y_{-\alpha,j}) = \delta_{ij}$. Then, the set $B_\alpha := \{x_{\alpha,i} = \frac{1}{\sqrt{\pm2}}(y_{\alpha,i}\pm y_{-\alpha,i})|i = 1,2,\dots,l\}$ is an orthonormal basis of $(\mathfrak{g}_0)_\alpha\oplus(\mathfrak{g}_0)_{-\alpha}$ by properties (2) and (3) above. Set

$$B_0 = \big\{x_i, x_0 = \frac{1}{\sqrt{2}}(c+d), x_{-1} = \frac{1}{\sqrt{-2}}(c-d)\big|i=1,2,\dots,n\big\},$$

where $\{x_i|i=1,2,\dots,n\}$ is an orthonormal basis of $\mathrm{span}_{\mathbb{C}}\{h_1,h_2,\dots,h_n\}$. Then $B = \bigcup_{\alpha\in\Delta_S^+} B_\alpha\cup B_0$ is an orthonormal basis of $\mathfrak{g}_0$. Let $\mathcal{I}$ be an index set of B.

The *structure coefficients* of $\mathfrak{g}_0$ with respect to B are given by:

$$[x_i, x_j] = \sum_{t\in\mathcal{I}} c_{i,j}^t x_t \tag{3.2}$$

We calculate:

$$\begin{aligned}
([x_i, x_j]|x_k) &= \big(\sum_{t\in\mathcal{I}} c_{i,j}^t x_t|x_k\big) \\
&= \sum_{t\in\mathcal{I}} c_{i,j}^t (x_t|x_k) \\
&= c_{i,j}^k
\end{aligned} \tag{3.3}$$

Using associativity of the form we get:

$$\begin{aligned}
([x_i, x_j]|x_k) &= (x_i|[x_j, x_k]) = \big(x_i\big|\sum_{t\in\mathcal{I}} c_{j,k}^t x_t\big) \\
&= \sum_{t\in\mathcal{I}} c_{j,k}^t (x_i|x_t) \\
&= c_{j,k}^i
\end{aligned} \tag{3.4}$$

Therefore

$$c_{i,j}^k = c_{j,k}^i \tag{3.5}$$

for all $i, j, k \in \mathcal{I}$.

Now we define the map $\psi : V^*(\Lambda_0) \otimes V(\Lambda_0) \to \mathfrak{g}_0$ by defining, for every $w^* \in V^*(\Lambda_0)$ and $v \in V(\Lambda_0)$

$$\psi(w^* \otimes v) = -\sum_{i\in\mathcal{I}} \langle w^*, x_i \cdot v\rangle x_i - 2\langle w^*, v\rangle c$$

We wish to show that ψ is a $\mathfrak{g}_0$-module homomorphism from $V^*(\Lambda_0) \otimes V(\Lambda_0)$ to $\mathfrak{g}_0$ considered as a module under the adjoint action. Let $x_i \in B, w^* \in V^*(\Lambda_0), v \in V(\Lambda_0)$ be given. Then

$$\begin{aligned}
x_j \cdot \psi(w^* \otimes v) &= x_j \cdot \big(-\sum_{i\in\mathcal{I}} \langle w^*, x_i \cdot v\rangle x_i - 2\langle w^*, v\rangle c\big) \\
&= \mathrm{ad}_{x_j}\big(-\sum_{i\in\mathcal{I}} \langle w^*, x_i \cdot v\rangle x_i - 2\langle w^*, v\rangle c\big) \\
&= -\sum_{i\in\mathcal{I}} \langle w^*, x_i \cdot v\rangle \mathrm{ad}_{x_j}(x_i) - 2\langle w^*, v\rangle \mathrm{ad}_{x_j}(c) \\
&= -\sum_{i\in\mathcal{I}} \langle w^*, x_i \cdot v\rangle [x_j, x_i] \\
&= \sum_{i\in\mathcal{I}} \langle w^*, x_i \cdot v\rangle [x_i, x_j] \\
&= \sum_{i,k\in\mathcal{I}} \langle w^*, x_i \cdot v\rangle c_{i,j}^k x_k.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\psi(x_j \cdot (w^* \otimes v)) &= \psi(x_j \cdot w^* \otimes v + w^* \otimes x_j \cdot v) \\
&= -\sum_{i\in\mathcal{I}} \langle x_j \cdot w^*, x_i \cdot v\rangle x_i - 2\langle x_j \cdot w^*, v\rangle c \\
&\quad - \sum_{i\in\mathcal{I}} \langle w^*, (x_i \cdot (x_j \cdot v))\rangle x_i - 2\langle w^*, x_j \cdot v\rangle c \\
&= \sum_{i\in\mathcal{I}} \langle w^*, x_j \cdot (x_i \cdot v))\rangle x_i + 2\langle w^*, x_j \cdot v\rangle c \\
&\quad - \sum_{i\in\mathcal{I}} \langle w^*, x_i \cdot (x_j \cdot v))\rangle x_i - 2\langle w^*, x_j \cdot v\rangle c \\
&= \sum_{i\in\mathcal{I}} \langle w^*, [x_j, x_i] \cdot v\rangle x_i \\
&= \sum_{i\in\mathcal{I}} \langle w^*, \big(\sum_{k\in\mathcal{I}} c_{j,i}^k x_k\big) \cdot v\rangle x_i \\
&= \sum_{i,k\in\mathcal{I}} \langle w^*, x_k \cdot v\rangle c_{j,i}^k x_i \\
&= \sum_{i,k\in\mathcal{I}} \langle w^*, x_i \cdot v\rangle c_{j,k}^i x_k.
\end{aligned}$$

Therefore,

$$x_j \cdot \psi(w^* \otimes v) = \psi(x_j \cdot (w^* \otimes v)) \tag{3.6}$$

which proves that ψ is a $\mathfrak{g}_0$-module homomorphism.

3.2 The Construction of $\tilde{\mathfrak{g}}$

Let $\tilde{\mathfrak{g}}_1 = V(\Lambda_0)$, $\tilde{\mathfrak{g}}_{-1} = V^*(\Lambda_0)$, $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0$, and $\tilde{\mathfrak{g}}_-$ and $\tilde{\mathfrak{g}}_+$ be the free Lie algebras generated by $\tilde{\mathfrak{g}}_{-1}$ and $\tilde{\mathfrak{g}}_1$ respectively.

Let $\tilde{\mathfrak{g}}_{\pm i} = \mathrm{span}_{\mathbb{C}}\{[y_1, [y_2, [\ldots, [y_{i-1}, y_i] \ldots]]] | y_1, y_2, \ldots, y_i \in \tilde{\mathfrak{g}}_{\pm 1}\}, i > 0$. We define the map

$$[\cdot,\cdot] : (\tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1) \otimes (\tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1) \to \tilde{\mathfrak{g}}_{-2} \oplus \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_2$$

for $w^* \in \tilde{\mathfrak{g}}_{-1}, v \in \tilde{\mathfrak{g}}_1, x \in \tilde{\mathfrak{g}}_0$ by the following:

$$\begin{aligned}
[w^*, v] &= \psi(w^* \otimes v) \\
[x, v] &= x \cdot v \\
[x, w^*] &= x \cdot w^*.
\end{aligned}$$

The map $[\cdot,\cdot]$ is bilinear, and satisfies the Jacobi identity:

$$\begin{aligned}
[x, [w^*, v]] &= x \cdot \psi(w^* \otimes v) \\
&= \psi(x \cdot (w^* \otimes v)) \\
&= \psi(x \cdot w^* \otimes v + w^* \otimes x \cdot v) \\
&= \psi(x \cdot w^* \otimes v) + \psi(w^* \otimes x \cdot v) \\
&= [[x, w^*], v] + [w^*, [x, v]].
\end{aligned} \tag{3.7}$$

We then extend the bracket operation defined above to the vector space $\tilde{\mathfrak{g}}_- \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_+$ by first defining inductively for each $x \in \tilde{\mathfrak{g}}_0$ a linear map $\mathrm{ad}_x : \mathfrak{g}_i \to \bigoplus_{j=1}^{i} \mathfrak{g}_j$. Using the Jacobi identity in $\tilde{\mathfrak{g}}_+$ each $v \in \tilde{\mathfrak{g}}_i, i > 1$ can be written as a linear combination of elements of the form $[g, h]$ for some $g \in \tilde{\mathfrak{g}}_1, h \in \tilde{\mathfrak{g}}_{i-1}$, so it suffices to define ad_x for elements of that form. Now, set

$$\mathrm{ad}_x([g, h]) = [\mathrm{ad}_x(g), h] + [g, \mathrm{ad}_x(h)] \tag{3.8}$$

and define $[x, v] = \mathrm{ad}_x(v)$ for all $v \in \tilde{\mathfrak{g}}_+$. We can define the linear map $\mathrm{ad}_x : \tilde{\mathfrak{g}}_{-i} \to \bigoplus_{j=1}^{i} \mathfrak{g}_{-j}$ similarly. Because $\tilde{\mathfrak{g}} = \bigoplus_{i \in \mathbb{Z}} \tilde{\mathfrak{g}}_j$, we can linearly extend the map ad_x to all of $\tilde{\mathfrak{g}}$.

It now remains to extend the bracket operation to each $v \in \tilde{\mathfrak{g}}_-$,$w \in \tilde{\mathfrak{g}}_+$. As before, we

define the linear transformations
$\mathrm{ad}_{w^*} : \tilde{\mathfrak{g}}_i \to \bigoplus_{j=0}^{i} \tilde{\mathfrak{g}}_j, \mathrm{ad}_v : \tilde{\mathfrak{g}}_{-i} \to \bigoplus_{j=0}^{i} \tilde{\mathfrak{g}}_{-j}$, for $w^* \in \tilde{\mathfrak{g}}_{-1}, v \in \tilde{\mathfrak{g}}_1$ by

$$\begin{aligned} \mathrm{ad}_{w^*}([g,h]) &= [\mathrm{ad}_{w^*}(g), h] + [g, \mathrm{ad}_{w^*}(h)], g \in \tilde{\mathfrak{g}}_1, h \in \tilde{\mathfrak{g}}_{i-1} \\ \mathrm{ad}_v([g,h]) &= [\mathrm{ad}_v(g), h] + [g, \mathrm{ad}_v(h)], g \in \tilde{\mathfrak{g}}_{-1}, h \in \tilde{\mathfrak{g}}_{-i+1} \end{aligned}$$

and extend linearly to all of $\tilde{\mathfrak{g}}$. We can and do define the Lie algebra homomorphisms $\mathrm{ad} : \tilde{\mathfrak{g}}_\pm \to \mathfrak{gl}(\tilde{\mathfrak{g}})$ inductively by $\mathrm{ad}(v) = \mathrm{ad}_v, \mathrm{ad}(w) = \mathrm{ad}_w$, and, for all $v \in \tilde{\mathfrak{g}}_1, w \in \tilde{\mathfrak{g}}_i$, $x \in \tilde{\mathfrak{g}}_{-1}, y \in \tilde{\mathfrak{g}}_{-i}$

$$\mathrm{ad}([v,w]) = [\mathrm{ad}(v), \mathrm{ad}(w)], \qquad \mathrm{ad}([x,y]) = [\mathrm{ad}(x), \mathrm{ad}(y)], \tag{3.9}$$

where the brackets on the right hand side are the commutator brackets of linear transformations.

For all $v \in \tilde{\mathfrak{g}}_+, w \in \tilde{\mathfrak{g}}_-, x \in \tilde{\mathfrak{g}}$ we define $[v,x] = \mathrm{ad}(v)(x), [w,x] = \mathrm{ad}(w)(x)$. All that remains to show is that the Jacobi identity holds with the bracket so defined. The definitions above prove the Jacobi identity for $v \in \tilde{\mathfrak{g}}_1, x, y \in \tilde{\mathfrak{g}}$ so assume that it holds for all $w \in \bigoplus_{j=0}^{i} \tilde{\mathfrak{g}}_j, x, y, \in \tilde{\mathfrak{g}}$. Then, since the commutator of two derivations is again a derivation, $\mathrm{ad}([v,w]) = [\mathrm{ad}(v), \mathrm{ad}(w)]$ is a derivation, and therefore:

$$\begin{aligned} [[v,w], [x,y]] &= \mathrm{ad}([v,w])[x,y] \\ &= [\mathrm{ad}([v,w])(x), y] + [x, \mathrm{ad}([v,w])(y)] \\ &= [[[v,w],x],y] + [x,[[v,w],y]] \end{aligned} \tag{3.10}$$

which completes the proof the Jacobi identity for all basis elements of $\tilde{\mathfrak{g}}$. Therefore, $\tilde{\mathfrak{g}}$ is a Lie algebra with bracket $[\cdot,\cdot]$.

3.3 The Construction of $\mathfrak{g}$ and isomorphism of $\mathfrak{g}$ with $HD_n^{(1)}$

In this section, we define $\mathfrak{g}$ as a quotient of $\tilde{\mathfrak{g}}$, and then show that it is isomorphic to the Kac-Moody algebra $HD_n^{(1)}$. For all $k > 1$ define the subspaces:

$$\begin{aligned} J_k &= \{x \in \tilde{\mathfrak{g}}_k | [v_1, [v_2, \ldots, [v_{k-1}, x] \ldots]] = 0, \forall v_1, v_2, \ldots, v_{k-1} \in \tilde{\mathfrak{g}}_1\} \\ J_{-k} &= \{x \in \tilde{\mathfrak{g}}_k | [w_1^*, [w_2^*, \ldots, [w_{k-1}^*, x] \ldots]] = 0, \forall w_1^*, w_2^*, \ldots, w_{k-1}^* \in \tilde{\mathfrak{g}}_{-1}\} \end{aligned}$$

Let $J_\pm = \bigoplus_{k>1} J_{\pm k}$ and $J = J_+ \oplus J_-$. Then J_+ and J_- are ideals of $\tilde{\mathfrak{g}}$, and J is the largest graded ideal $\tilde{\mathfrak{g}}$ that intersects $\tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$ trivially (see [3] for a proof). Finally, we define

$$\begin{aligned} \mathfrak{g} &= \tilde{\mathfrak{g}}/J \\ &= \left(\bigoplus_{k<1} \mathfrak{g}_k\right) \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \left(\bigoplus_{k>1} \mathfrak{g}_k\right) \end{aligned} \tag{3.11}$$

where $\mathfrak{g}_{\pm k} = \tilde{\mathfrak{g}}_{\pm k}/J_{\pm k}$ for $k > 1$. Note that since J intersects $\tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$ trivially and is a graded ideal, then

$$\begin{aligned} \mathfrak{g}_{\pm 1} &= \tilde{\mathfrak{g}}_{\pm 1}/(J \cap \tilde{\mathfrak{g}}_{\pm 1}) = \tilde{\mathfrak{g}}_{\pm 1} \\ \mathfrak{g}_0 &= \tilde{\mathfrak{g}}_0/(J \cap \tilde{\mathfrak{g}}_0) = \mathfrak{g}_0 \end{aligned} \tag{3.12}$$

In other words:

$$\mathfrak{g}_1 = V(\Lambda_0), \qquad \mathfrak{g}_{-1} = V^*(\Lambda_0), \qquad \mathfrak{g}_0 = D_n^{(1)} \tag{3.13}$$

a fact that will be important in what follows. In particular, $\mathfrak{g}_0$ is embedded isomorphically in $\mathfrak{g}$, as are the basic representations $V(\Lambda_0)$, and $V^*(\Lambda_0)$. Now we are ready to prove the following main theorem:

Theorem 5. *Let $\{E_i\}_{i=-1}^n, \{F_i\}_{i=-1}^n, \{H_i\}_{i=-1}^n$ be the generators of $HD_n^{(1)}$, and $\{e_i\}_{i=0}^n$, $\{f_i\}_{i=0}^n$, $\mathfrak{h}_0$ be the generators of $D_n^{(1)}$. Then the map $\phi : HD_n^{(1)} \to \mathfrak{g}$, defined on the generators by:*

$$\phi(E_{-1}) = v_0^*, \qquad \phi(F_{-1}) = v_0, \qquad \phi(H_{-1}) = -2c - d$$

$$\phi(E_i) = e_i, \qquad \phi(F_i) = f_i, \qquad \phi(H_i) = h_i, i \in \{0, 1, \ldots, n\} \tag{3.14}$$

is an isomorphism of Lie algebras.

Proof. Recall that $HD_n^{(1)} = \mathfrak{g}(A)$, where A is given in (2.1). Recall also, by Theorem 2, that $\mathfrak{g}(A) = \hat{\mathfrak{g}}(A)/\mathfrak{i}$, where $\hat{\mathfrak{g}}(A)$ is the Lie algebra satisfying relations (1)-(4) of Definition 6, and $\mathfrak{g}(A) = \hat{\mathfrak{g}}(A)/\mathfrak{i}$, and $\mathfrak{i}$ is the maximal ideal of $\hat{\mathfrak{g}}(A)$ intersecting $\mathfrak{h}$ trivially. Since J is the maximal graded ideal of $\tilde{\mathfrak{g}}$ which intersects the local part of $\tilde{\mathfrak{g}}$ trivially, we need only show the following:

$$\begin{aligned} [\phi(H_i), \phi(H_j)] &= 0 && (3.15)\\ [\phi(E_i), \phi(F_j)] &= \delta_{i,j}\phi(H_i) && (3.16)\\ [\phi(H_i), \phi(E_j)] &= a_{ij}\phi(E_j) && (3.17)\\ [\phi(H_i), \phi(F_j)] &= -a_{ij}\phi(F_j) && (3.18) \end{aligned}$$

To show (3.15):

$$[\phi(H_i), \phi(H_{-1})] = [h_i, -2c-d] = 0, i \in \{0, 1, \ldots, n\}$$

To show (3.16):

$$\begin{aligned} [\phi(E_{-1}), \phi(F_{-1})] &= [v_0^*, v_0]\\ &= \psi(v_0^* \otimes v_0)\\ &= -\sum_{i\in\mathcal{I}} \langle v_0^*, x_i \cdot v_0\rangle x_i - 2\langle v_0^*, v_0\rangle c\\ &= -\left\langle v_0^*, \frac{1}{\sqrt{2}}(c+d)\cdot v_0\right\rangle \frac{1}{\sqrt{2}}(c+d)\\ &\quad - \left\langle v_0^*, \frac{1}{\sqrt{-2}}(c-d)\cdot v_0\right\rangle \frac{1}{\sqrt{-2}}(c-d) - 2c\\ &= -\frac{1}{2}(c+d) + \frac{1}{2}(c-d) - 2c\\ &= -2c-d\\ &= \phi(H_{-1})\\ [\phi(E_i), \phi(F_{-1})] &= e_i \cdot v_0 = 0, i \in \{0, 1, \ldots, n\}\\ [\phi(E_{-1}), \phi(F_i)] &= -f_i \cdot v_0^* = 0, i \in \{0, 1, \ldots, n\} \end{aligned}$$

To show (3.17):

$$\begin{aligned}
[\phi(H_i), \phi(E_{-1})] &= [h_i, v_0^*], i \in \{0, 1, \ldots, n\} \\
&= h_i \cdot v_0^* \\
&= -\Lambda_0(h_i) v_0^* \\
&= -\delta_{i,0} v_0^* \\
&= a_{i,-1} \phi(E_{-1}) \\
[\phi(H_{-1}), \phi(E_{-1})] &= [-2c - d, v_0^*] \\
&= -\Lambda_0(-2c - d) v_0^* \\
&= 2v_0^* \\
&= 2\phi(E_{-1})
\end{aligned}$$

and similarly one can show:

$$[\phi(H_i), \phi(F_{-1})] = -a_{i,-1} \phi(F_{-1})$$

□

Since $D_n^{(1)}$ is an affine-type Kac-Moody algebra, the root multiplicities of Δ_S are already known. Because of the Chevalley automorphism $\omega : \mathfrak{n}^- \to \mathfrak{n}^+$, it suffices to consider root multiplicities in $\Delta^+(S)$ or $\Delta^-(S)$. In the next chapter, we will describe a formula for giving the multiplicities of roots in $\Delta^-(S)$, which uses the construction given in this chapter and elements of the theory of homology of $\mathfrak{g}$-modules.

Chapter 4

Multiplicity Formula

In this chapter we review the basic definitions and results of homology of $\mathfrak{g}$-modules, and Kang's multiplicity formula for roots in $\Delta^-(S)$.

Definition 9. *A* chain complex *of $\mathfrak{g}$-modules is a family $\{C_k\}_{k\in\mathbb{Z}}$ of $\mathfrak{g}$-modules together with $\mathfrak{g}$-module homomorphisms $d_k : C_k \to C_{k-1}$ such that $d_k \circ d_{k+1} \equiv 0$. The maps d_k are called* differentials*. The chain complex $\mathcal{C}$ is* admissible *if $\bigcup_{k\in\mathbb{Z}} C_k$ is itself a $\mathfrak{g}$-module.*

Definition 10. *Let $\mathcal{C} = \{C_k\}_{k\in\mathbb{Z}}$ be a chain complex of $\mathfrak{g}$-modules with differentials d_k. The k^{th} homology module of $\mathcal{C}$ is given by*

$$H_k(\mathcal{C}) = ker(d_k)/im(d_{k+1}).$$

Theorem 6 (Euler-Poincaré Principle)**.** *Let $\mathcal{C} = \{C_n\}_{n\in\mathbb{Z}}$ be an admissible chain complex of $\mathfrak{g}$-modules. Then*

$$\sum_{k\in\mathbb{Z}_{\geq 0}} (-1)^k ch(C_k) = \sum_{k\in\mathbb{Z}_{\geq 0}} (-1)^k ch(H_k(\mathcal{C})).$$

Theorem 7 (Kostant's Formula for Kac-Moody algebras [9],[29])**.** *Let $\lambda \in P$ be given. Then*

$$\cdots \to \bigwedge\nolimits^k \mathfrak{g}_- \otimes V(\lambda) \xrightarrow{d_k} \bigwedge\nolimits^{k-1} \mathfrak{g}_- \otimes V(\lambda) \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} \bigwedge\nolimits^0 \mathfrak{g}_- \otimes V(\lambda) \to 0 \to \cdots$$

with,

$$d_k((x_1 \wedge x_2 \wedge \cdots \wedge x_k) \otimes v) = \begin{cases} \sum_{i=1}^{k} (-1)^i (x_1 \wedge \cdots \hat{x}_i \wedge \cdots \wedge x_k) \otimes x_i \cdot v \\ + \sum_{r<t} ([x_r, x_t] \wedge x_1 \wedge \cdots \wedge \hat{x}_r \wedge \cdots \wedge \hat{x}_t \wedge \cdots \wedge x_k) \otimes v \\ \qquad \textit{for } k \geq 1, \\ 0 \ \textit{otherwise}, \end{cases}$$

is a $\mathfrak{g}_0$-module complex. In addition, the homology modules $H_k(\mathfrak{g}_-, V(\lambda))$ of this complex are $\mathfrak{g}_0$-modules and

$$H_k(\mathfrak{g}_-, V(\lambda)) \cong \sum_{\substack{w \in W(S) \\ \ell(w) = k}} V(w(\lambda + \rho) - \rho), \tag{4.1}$$

where $\rho \in \mathfrak{h}^*$ denotes the functional such that $\rho(h_i) = 1, i \in I$.

Now, we consider the case $V(0) \cong \mathbb{C}$, the trivial $\mathfrak{g}$-module. By Theorem 7,

$$\cdots \to \bigwedge^k \mathfrak{g}_- \xrightarrow{d_k} \bigwedge^{k-1} \mathfrak{g}_- \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} \mathbb{C} \to 0 \to \cdots$$

is a $\mathfrak{g}$-module complex where the differential d_k is given by:

$$d_k(x_1 \wedge x_2 \wedge \cdots \wedge x_k) = \begin{cases} \sum_{r<t} [x_r, x_t] \wedge x_1 \wedge \cdots \wedge \hat{x}_r \wedge \cdots \wedge \hat{x}_t \wedge \cdots \wedge x_k \\ \qquad \text{for } k \geq 2, \\ 0 \text{ otherwise.} \end{cases}$$

Applying the Euler-Poincaré principle to this complex gives (omitting the module $\mathbb{C}$):

$$\sum_{k=0}^{\infty} (-1)^k \mathrm{ch}\left(\bigwedge^k \mathfrak{g}_-\right) = \sum_{k=0}^{\infty} (-1)^k \mathrm{ch}(H_k(\mathfrak{g}_-)). \tag{4.2}$$

Consider the left hand side of (4.2):

$$\begin{aligned}
\sum_{k=0}^{\infty}(-1)^k \mathrm{ch}\left(\bigwedge^k \mathfrak{g}_-\right) &= \sum_{k=0}^{\infty}(-1)^k \sum_{\substack{x_{\alpha_1}\wedge x_{\alpha_2}\wedge\cdots\wedge x_{\alpha_k}\\ \alpha_i\in\Delta^-(S)}} e(\alpha_1+\alpha_2+\cdots+\alpha_k)\\
&= \sum_{k=0}^{\infty}(-1)^k \sum_{\substack{x_{\alpha_1}\wedge x_{\alpha_2}\wedge\cdots\wedge x_{\alpha_k}\\ \alpha_i\in\Delta^-(S)}} e(\alpha_1)e(\alpha_2)\cdots e(\alpha_k)\\
&= \prod_{\alpha\in\Delta^-(S)}(1-e(\alpha))^{\dim(\mathfrak{g}_\alpha)}.
\end{aligned}$$

Now, consider the right hand side of (4.2):

$$\begin{aligned}
\sum_{k=0}^{\infty}(-1)^k \mathrm{ch}(H_k(\mathfrak{g}_-)) &= \mathrm{ch}(H_0(\mathfrak{g}_-)) + \sum_{k=1}^{\infty}(-1)^k \mathrm{ch}(H_k(\mathfrak{g}_-))\\
&= 1 - \sum_{k=1}^{\infty}(-1)^{k+1}\mathrm{ch}(H_k(\mathfrak{g}_-))\\
&= 1 - \sum_{\substack{w\in W(S)\\ \ell(w)\geq 1}}(-1)^{\ell(w)+1}\mathrm{ch}(V(w(\rho)-\rho))\\
&= 1 - \sum_{\substack{w\in W(S)\\ \ell(w)\geq 1}}(-1)^{\ell(w)+1}\sum_{\tau\in P}\dim(V(w(\rho)-\rho))_\tau e(\tau)\\
&\qquad \text{(by Kostant's formula (4.1))}\\
&= 1 - \sum_{\tau\in P}\sum_{\substack{w\in W(S)\\ \ell(w)\geq 1}}(-1)^{\ell(w)+1}\dim(V(w(\rho)-\rho))_\tau e(\tau).
\end{aligned}$$

Equating the left and right hand sides of equation (4.2) gives:

$$\prod_{\alpha\in\Delta^-(S)}(1-e(\alpha))^{\dim(\mathfrak{g}_\alpha)} = 1 - \sum_{\tau\in P} K_\tau e(\tau) \tag{4.3}$$

where

$$K_\tau = \sum_{\substack{w\in W(S)\\ \ell(w)\geq 1}}(-1)^{\ell(w)+1}\dim(V(w(\rho)-\rho))_\tau.$$

Taking the logarithm of both sides of (4.3) we obtain

$$\sum_{\alpha\in\Delta^-(S)} \dim(\mathfrak{g}_\alpha)\log(1-e(\alpha)) = \log\left(1-\sum_{\tau\in P} K_\tau e(\tau)\right). \tag{4.4}$$

Using the formal power series expansion $\log(1-x) = -\sum_{k=1}^{\infty}\frac{x^k}{k}$ the left hand side of (4.4) becomes

$$\begin{aligned}
\sum_{\alpha\in\Delta^-(S)} \dim(\mathfrak{g}_\alpha)\log(1-e(\alpha)) &= -\sum_{\alpha\in\Delta^-(S)} \dim(\mathfrak{g}_\alpha)\sum_{k=1}^{\infty}\frac{1}{k}e(\alpha)^k \\
&= -\sum_{\alpha\in\Delta^-(S)}\sum_{k=1}^{\infty}\dim(\mathfrak{g}_\alpha)\frac{1}{k}e(k\alpha).
\end{aligned}$$

The right hand side of (4.4) becomes

$$\begin{aligned}
\log\left(1-\sum_{\tau\in P} K_\tau e(\tau)\right) &= -\sum_{k=1}^{\infty}\frac{1}{k}\left(\sum_{\tau\in P} K_\tau e(\tau)\right)^k \\
&= -\sum_{k=1}^{\infty}\frac{1}{k}\left(\sum_{i=1}^{\infty} K_{\tau_i} e(\tau_i)\right)^k \\
&= -\sum_{k=1}^{\infty}\frac{1}{k}\sum_{\substack{(n_i)\\ \sum n_i=k}}\frac{(\sum n_i)!}{\prod(n_i!)}\prod K_{\tau_i}^{n_i} e\left(\sum n_i\tau_i\right) \\
&\qquad \text{(multinomial expansion)} \\
&= -\sum_{\tau\in P}\left(\sum_{\substack{(n_i)\\ \sum n_i\tau_i=\tau}}\frac{(\sum n_i-1)!}{\prod(n_i!)}\prod K_{\tau_i}^{n_i}\right)e(\tau)
\end{aligned}$$

where $\{\tau_i | i=1,2,\dots\}$ is an enumeration of the elements of P.

Equating the right and left hand sides of (4.4) we see

$$\sum_{\tau\in P} B(\tau)e(\tau) = \sum_{\alpha\in\Delta^-(S)}\sum_{k=1}^{\infty}\dim(\mathfrak{g}_\alpha)\frac{1}{k}e(k\alpha)$$

where

$$B(\tau) = \sum_{\substack{(n_i) \\ \sum n_i \tau_i = \tau}} \frac{(\sum n_i - 1)!}{\prod (n_i!)} \prod K_{\tau_i}^{n_i}.$$

Therefore,

$$\begin{aligned} B(\tau) &= \sum_{\substack{\alpha \in \Delta^-(S) \\ \tau = k\alpha}} \frac{1}{k} \dim(\mathfrak{g}_\alpha) \\ &= \sum_{\substack{\alpha \in \Delta^-(S) \\ \alpha | \tau}} \frac{\alpha}{\tau} \dim(\mathfrak{g}_\alpha) \end{aligned}$$

where the notation $\alpha|\tau$ (α divides τ) means $\tau = k\alpha$ for some $k \in \mathbb{Z}$ and τ/α (resp. α/τ) is equal to k (resp. $1/k$). Using Möbius inversion, we see for $\alpha \in \Delta^-(S)$:

$$\dim(\mathfrak{g}_\alpha) = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\alpha}{\tau} B(\tau),$$

where

$$\mu(n) = \begin{cases} \text{1, if } n \text{ is squarefree with an even number of distinct prime factors,} \\ \text{-1, if } n \text{ is squarefree with an odd number of distinct prime factors,} \\ \text{0, otherwise,} \end{cases}$$

is the classical Möbius function. We then have the following:

Theorem 8 (Kang's Multiplicity Formula [22])**.** *Let $\alpha \in \Delta^-(S)$. Then*

$$dim(\mathfrak{g}_\alpha) = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B(\tau)$$

where,

- $\mu(n) =$ *Classical Möbius Function,*
- $B(\tau) = \sum_{(n_i\tau_i) \in T(\tau)} \frac{(\sum n_i - 1)!}{\prod (n_i!)} \prod K_{\tau_i}^{n_i}$,
- $T(\tau) = \left\{ (n_i\tau_i) \middle| n_i \in \mathbb{Z}_{\geq 0}, \sum n_i\tau_i = \tau, \tau_i \in P \right\}$,

- $K_{\tau_i} = \sum_{\substack{w \in W(S) \\ \ell(w) \geq 1}} (-1)^{\ell(w)+1} dim(V(w\rho - \rho)_{\tau_i})$.

Note that in order to apply this theorem, we must compute weight multiplicities of $D_n^{(1)}$-modules. In the next chapter, we survey the path realization of $D_n^{(1)}$-modules, which uses the theory of quantum groups and crystal bases to give a combinatorial way to compute weight multiplicities.

Chapter 5

The Path Construction of $D_n^{(1)}$-modules

In this chapter, we define quantum groups and crystal bases. In particular, we will realize the crystal bases of integrable modules of $D_n^{(1)}$ using the path realization. We review the necessary notions of perfect crystals and paths. Then we give the data for perfect crystals of $D_n^{(1)}$, which will be used in a later chapter to compute root multiplicities of $HD_n^{(1)}$

We will use the following notation:

- $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$,
- $[n]_q! = [n]_q[n-1]_q!$, where $[0]_q! = 1$,
- $\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}$,
- $e_i^{(k)} = \frac{e_i^k}{[k]_q!}$,
- $f_i^{(k)} = \frac{f_i^k}{[k]_q!}$,
- $\mathbb{A}_0 = \{f/g \mid f, g \in \mathbb{C}[q], g(0) \neq 0\}$.

Recall the sets $\Pi, \Pi^\vee, P, P^\vee$. The tuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ is called a *Cartan datum* (here $P, P^\vee$ can be given subsets of the ones given in chapter 1).

5.1 Quantum Groups and their Modules

A quantum group is an associative algebra that can be seen as a 'q-deformation' of the universal enveloping algebra of a Kac-Moody algebra.

Definition 11. *The* quantum group *or* quantized universal enveloping algebra $U_q(\mathfrak{g})$ *associated with a Cartan datum* $(A, \Pi, \Pi^\vee, P, P^\vee)$ *is the associative algebra over* $\mathbb{C}(q)$ *with 1 generated by the elements* e_i, f_i $(i \in I)$ *and* q^h $(h \in P^\vee)$ *with the following defining relations:*

1. $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$ *for* $h, h' \in P^\vee$,
2. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ *for* $h \in P^\vee$,
3. $q^h f_i q^{-h} = q^{-\alpha_i(h)} e_i$ *for* $h \in P^\vee$,
4. $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ *for* $i, j \in I$,
5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ *for* $i \neq j$,
6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ *for* $i \neq j$.

Where $q_i = q^{s_i}$ *and* $K_i = q^{s_i h_i}$.

Example: The quantum group $U_q(\mathfrak{sl}(2))$ is the associative algebra generated by the set $\{e, f, q^h\}$ satisfying the following relations:

1. $q^h e q^{-h} = q^2 e$,
2. $q^h f q^{-h} = q^{-2} f$,
3. $ef - fe = \frac{q^h - q^{-h}}{q - q^{-1}}$.

Analogous to the definition of a module of a Lie algebra, we have the following:

Definition 12. *A $U_q(\mathfrak{g})$-module is a vector space V^q over $\mathbb{C}(q)$ together with an operation $\cdot : U_q(\mathfrak{g}) \times V^q \to V^q$, which satisfies:*

$$x \cdot (y \cdot v) = (xy) \cdot v$$

for $x, y \in U_q(\mathfrak{g})$, and $v \in V^q$.

Example: Let $V^q = \mathrm{span}_{\mathbb{C}(q)}\{v_0, v_1, \dots, v_k\}$ be a vector space and define an action of $U_q(\mathfrak{sl}(2))$ on V^q by:

$$\begin{aligned} q^h \cdot v_j &= q^{k-2j} v_j \\ f \cdot v_j &= [j+1]_q v_{j+1} \\ e \cdot v_j &= [k-j+1]_q v_{j-1} \end{aligned}$$

where v_j is understood to be 0 if $j < 0$ or $j > k$. Then V^q is a $U_q(\mathfrak{sl}(2))$-module.

Let V^q be a $U_q(\mathfrak{g})$-module. For $\lambda \in \mathfrak{h}^*$ we define the set $V^q_\lambda := \{v \in V^q | q^h \cdot v = q^{\langle \lambda, h \rangle} v, h \in \mathfrak{h}\}$ to be the *weight space* of V^q of weight λ and $\dim(V^q_\lambda)$ to be the *(weight) multiplicity* of λ denoted $\mathrm{mult}_{V^q}(\lambda)$. If all the weight spaces of a $U_q(\mathfrak{g})$-module V^q are finite, we define the *character* of V^q to be the formal sum:

$$\mathrm{ch}(V^q) = \sum_{\mu \in \mathfrak{h}^*} \dim V^q_\mu e(\mu),$$

where $e(\cdot)$ is the formal exponential.

A $U_q(\mathfrak{g})$-module V^q is called a *highest weight module* with highest weight λ if and only if it satisfies the following:

1. There exists $0 \neq v_\lambda \in V$ such that $q^h \cdot v_\lambda = q^{\langle \lambda, h \rangle} v_\lambda$ for all $h \in \mathfrak{h}$,
2. $U_q^+ \cdot v_\lambda = \{0\}$,
3. $U_q(\mathfrak{g}) \cdot v_\lambda = V^q$,

where U_q^+ is the subalgebra of $U_q(\mathfrak{g})$ generated by $\{e_i | i \in I\}$.

Any highest weight $U_q(\mathfrak{g})$-module V^q of highest weight λ satisfies the following properties:

1. *(Weight Space Decomposition).* $V^q = \bigoplus_{\mu \leq \lambda} V^q_\mu$,

2. $V_\lambda^q = \mathbb{C}v_\lambda$,

3. $V_\mu^q < \infty, \mu \in \mathfrak{h}^*$.

For every $\lambda \in \mathfrak{h}^*$ there exists a unique irreducible highest weight module with highest weight λ (see [11]), which we denote by $V^q(\lambda)$.

A large motivation to study quantum groups comes from the following:

Theorem 9 ([30]). *Let $\lambda \in P^+$. Then*

$$ch(V(\lambda)) = ch(V^q(\lambda)).$$

Therefore, in particular

$$\mathrm{mult}_{V(\lambda)}(\mu) = \mathrm{mult}_{V^q(\lambda)}(\mu)$$

.

5.2 Crystal Bases

Before we define crystal bases, we need the notion of the *Kashiwara operators* $\tilde{e}_i, \tilde{f}_i, i \in I$. These are certain modified root vectors for the quantum group $U_q(\mathfrak{g})$. But first, we need a preliminary result:

Lemma 1 ([17]). *Let $\lambda \in P^+$ and $V^q(\lambda)$ be the highest weight $U_q(\mathfrak{g})$-module of highest weight λ. For each $i \in I$, every weight vector $u \in V^q(\lambda)_\mu (\mu \in P)$ may be written in the form*

$$u = u_0 + f_i u_1 + \cdots + f_i^{(N)} u_N,$$

where $N \in \mathbb{Z}_{\geq 0}$ and $u_k \in V^q(\lambda)_{\mu + k\alpha_i} \cap ker\, e_i$ for all $k = 0, 1, \ldots, N$. Here, each u_k in the expression is uniquely determined by u and $u_k \neq 0$ only if $\mu(h_i) + k \geq 0$.

We now have the following:

Definition 13. *Let $\lambda \in P^+$. The* Kashiwara operators *$\tilde{e}_i$ and $\tilde{f}_i (i \in I)$ on $V^q(\lambda)$ are defined by*

$$\tilde{e}_i u = \sum_{k=1}^{N} f_i^{(k-1)} u_k, \qquad \tilde{f}_i u = \sum_{k=0}^{N} f_i^{(k+1)} u_k.$$

We also need an auxiliary definition of a *crystal lattice.*

Definition 14. *Let* $\lambda \in P^+$ *and* $V^q(\lambda)$ *be the highest weight* $U_q(\mathfrak{g})$*-module of highest weight* λ*. A free* $\mathbb{A}_0$*-submodule* $\mathcal{L}$ *of* $V^q(\lambda)$ *is called a* crystal lattice *if*

1. $\mathcal{L}$ *generates* $V^q(\lambda)$ *as a vector space over* $\mathbb{C}(q)$*,*
2. $\mathcal{L} = \bigoplus_{\mu \in P} \mathcal{L}_\lambda$*, where* $\mathcal{L}_\mu = \mathcal{L} \cap V^q(\lambda)_\mu$ *for all* $\lambda \in P$*,*
3. $\tilde{e}_i\mathcal{L} \subset \mathcal{L}, \tilde{f}_i\mathcal{L} \subset \mathcal{L}$ *for all* $i \in I$*.*

Finally, we have the following:

Definition 15. *A* crystal base *of the irreducible highest weight* $U_q(\mathfrak{g})$*-module* $V^q(\lambda), \lambda \in P^+$ *is a pair* $(\mathcal{L}, \mathcal{B})$ *such that*

1. $\mathcal{L}$ *is a crystal lattice of* $V^q(\lambda)$*,*
2. $\mathcal{B}$ *is a* $\mathbb{C}$*-basis of* $\mathcal{L}/q\mathcal{L}$*,*
3. $\mathcal{B} = \bigsqcup_{\mu \in P} \mathcal{B}_\mu$*, where* $\mathcal{B}_\mu = \mathcal{B} \cap (\mathcal{L}_\mu/q\mathcal{L}_\mu)$*,*
4. $\tilde{e}_i\mathcal{B} \subset \mathcal{B} \cup \{0\}$*,* $\tilde{f}_i\mathcal{B} \subset \mathcal{B} \cup \{0\}$ *for all* $i \in I$*,*
5. *for any* $b, b' \in \mathcal{B}$ *and* $i \in I$*, we have* $\tilde{f}_i b = b'$ *if and only if* $b = \tilde{e}_i b'$*.*

The set $\mathcal{B}$ is called the *crystal graph* of $(\mathcal{L}, \mathcal{B})$. This is because $\mathcal{B}$ can be regarded as a colored, oriented graph by defining

$$b \xrightarrow{i} b' \iff \tilde{f}_i b = b'.$$

Proposition 2 ([17], [31])**.**

$$mult_{V^q(\lambda)}(\mu) = \#\mathcal{B}_\mu.$$

Therefore we can turn many questions about weight multiplicities into counting problems on the set $\mathcal{B}$, provided that a crystal base of the corresponding $U_q(\mathfrak{g})$-module exists.

An *(abstract) crystal* is a combinatorial structure that embodies some of the features of a crystal base.

Definition 16. *A* crystal *associated with* $U_q(\mathfrak{g})$ *is a set* $\mathcal{B}$ *together with maps* $wt : \mathcal{B} \to P, \tilde{e}_i, \tilde{f}_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$*, and* $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$*, for* $i \in I$ *satisfying the following properties:*

1. $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b)\rangle$ *for all* $i \in I$*,*

2. $wt(\tilde{e}_i b) = wt(b) + \alpha_i$ *if* $\tilde{e}_i b \in \mathcal{B}$*,*

3. $wt(\tilde{f}_i b) = wt(b) - \alpha_i$ *if* $\tilde{f}_i b \in \mathcal{B}$*,*

4. $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ *if* $\tilde{e}_i b \in \mathcal{B}$*,*

5. $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ *if* $\tilde{f}_i b \in \mathcal{B}$*,*

6. $\tilde{f}_i b = b'$ *if and only if* $b = \tilde{e}_i b'$ *for* $b, b' \in \mathcal{B}, i \in I$*,*

7. if $\varphi_i(b) = -\infty$ *for* $b \in \mathcal{B}$*, then* $\tilde{e}_i b = \tilde{f}_i b = 0$*.*

Then one may easily prove the following:

Proposition 3 ([17])**.** *Let* $(\mathcal{L}, \mathcal{B})$ *be the crystal basis of a* $U_q(\mathfrak{g})$*-module* $V^q(\lambda)$*. Then* $\mathcal{B}$ *is a crystal if we define in addition to* $\tilde{e}_i, \tilde{f}_i$*:*

- $wt(b) = \mu$ *if* $b \in \mathcal{B}_\mu$*,*
- $\varepsilon_i(b) = \max\{k | \tilde{e}_i^k(b) \neq 0\}$*,*
- $\varphi_i(b) = \max\{k | \tilde{f}_i^k(b) \neq 0\}$*.*

The *tensor product* $\mathcal{B}_1 \otimes \mathcal{B}_2$ of crystals $\mathcal{B}_1$ and $\mathcal{B}_2$ is the set $\mathcal{B}_1 \times \mathcal{B}_2$ together with the following maps:

1. $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$,
2. $\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1)\rangle)$,
3. $\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \mathrm{wt}(b_2)\rangle)$,
4. $\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon(b_2), \\ b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon(b_2), \end{cases}$
5. $\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon(b_2), \\ b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon(b_2), \end{cases}$

where we write $b_1 \otimes b_2$ for $(b_1, b_2) \in \mathcal{B}_1 \times \mathcal{B}_2$, and understand $b_1 \otimes 0 = 0 \otimes b_2 = 0$. $\mathcal{B}_1 \otimes \mathcal{B}_2$ is a crystal, as can easily be shown.

5.3 Quantum Affine Algebras and Perfect Crystals

Let $(A, P, P^\vee, \Pi, \Pi^\vee)$ be the Cartan datum of an affine type Kac-Moody algebra $\mathfrak{g}$ with index set I. Then the quantum group $U_q(\mathfrak{g})$ is called a *quantum affine algebra*. Let $U_q'(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{e_i, f_i, K_i^{\pm}|i \in I\}$, also called a quantum affine algebra. Recall that

$$\begin{aligned} P &= \operatorname{span}_{\mathbb{Z}}\{\Lambda_0, \Lambda_1, \ldots, \Lambda_n, \frac{1}{a_0}\delta\}, \\ P^\vee &= \operatorname{span}_{\mathbb{Z}}\{h_0, h_1, \ldots, h_n, d\}, \end{aligned}$$

where Λ_i are the fundamental weights, δ is the standard null root and d is the degree derivation of $\mathfrak{g}$. Similarly, we define the *classical weights*, and *dominant classical weights* to be the sets:

$$\begin{aligned} \bar{P} &= \operatorname{span}_{\mathbb{Z}}\{\Lambda_0, \Lambda_1, \ldots, \Lambda_n\}, \\ \bar{P}^+ &= \{\lambda \in \bar{P}|\langle \lambda, h_i\rangle \geq 0, i \in I\}. \end{aligned}$$

A crystal associated with the Cartan datum $(A, \bar{P}, \bar{P}^\vee, \Pi, \Pi^\vee)$ is called a *classical crystal* (or $U_q'(\mathfrak{g})$*-crystal*).

Remark: The quantum affine algebra $U_q(\mathfrak{g})$ has no finite dimensional modules other than the trivial module. On the other hand, $U_q'(\mathfrak{g})$ can have finite dimensional modules.

The notion of *perfect crystals* was introduced in [23] to realize the $U_q(\mathfrak{g})$-crystal $\mathcal{B}(\lambda), \lambda \in P^+$. Let $\varepsilon(b) = \sum_{i\in I} \varepsilon_i(b)\Lambda_i, \varphi(b) = \sum_{i\in I} \varphi_i(b)\Lambda_i$, and $\bar{P}_l = \{\lambda \in \bar{P}^+|\langle c, \lambda\rangle = l\}$, recalling the canonical central element c of $\mathfrak{g}$.

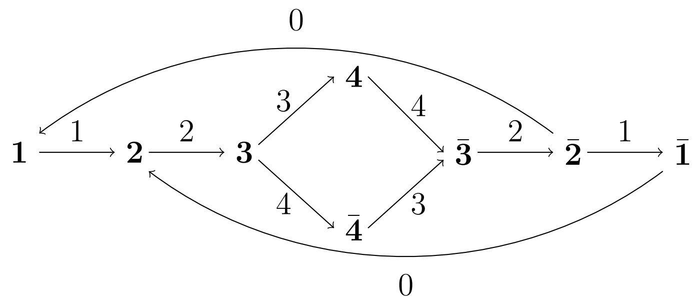
Definition 17. *For a positive integer $l > 0$, we say that a finite classical crystal $\mathcal{B}$ is a* perfect crystal of level l *if it satisfies the following conditions:*

1. *there exists a finite dimensional $U_q'(\mathfrak{g})$-module with a crystal base whose crystal graph is isomorphic to $\mathcal{B}$,*
2. *$\mathcal{B} \otimes \mathcal{B}$ is connected,*
3. *there exists a classical weight $\lambda_0 \in \bar{P}$ such that $wt(\mathcal{B}) \subset \lambda_0 + \sum_{i\neq 0} \mathbb{Z}_{\leq 0}\alpha_i$, and $\#\mathcal{B}_{\lambda_0} = 1$,*
4. *for any $b \in \mathcal{B}$, we have $\langle c, \varepsilon(b)\rangle \geq l$,*

5. *for each* $\lambda \in \bar{P}_l$ *there exist unique* $b^\lambda \in \mathcal{B}$ *and* $b_\lambda \in \mathcal{B}$ *such that* $\varepsilon(b^\lambda) = \lambda, \varphi(b_\lambda) = \lambda$.

Let $\mathcal{B}$ be a perfect crystal of level l. We define $\mathcal{B}^{\min}$ to be the set $\{b \in \mathcal{B} | \langle \mathrm{wt}(b), c\rangle = l\}$.

Example: The following is the crystal graph of a perfect $U_q'(D_4^{(1)})$-crystal.



5.4 Paths, Energy Functions, and Affine Crystals

In this section, we introduce paths and energy functions of perfect crystals then use them to construct the crystal bases of irreducible integrable highest weight modules of quantum affine algebras. Recall that the crystal base of the irreducible integrable highest weight module $V^q(\lambda), \lambda \in P^+$ is denoted by $\mathcal{B}(\lambda)$ and we denote its highest weight vector by u_λ. Then we have the following.

Theorem 10 ([23]). *Fix a positive integer* $l > 0$ *and let* $\mathcal{B}$ *be a perfect crystal of level* l. *For any classical dominant weight* $\lambda \in \bar{P}_l^+$, *there exists a unique crystal isomorphism*

$$\Psi : \mathcal{B}(\lambda) \to \mathcal{B}(\varepsilon(b_\lambda)) \otimes \mathcal{B}$$

given by $u_\lambda \mapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda$, *where* b_λ *is the unique element in* $\mathcal{B}$ *such that* $\varphi(b_\lambda) = \lambda$.

Let $\mathcal{B}$ be as in theorem 10 and define inductively

$$\lambda_0 = \lambda, \lambda_{k+1} = \varepsilon(\lambda_k),$$

$$b_0 = b_\lambda, b_{k+1} = b_{\lambda_{k+1}}.$$

The sequences $\mathbf{b}_\lambda := (b_k)_{k=0}^\infty$ and $\mathbf{w}_\lambda := (\lambda_k)_{k=0}^\infty$ are periodic with the same period. To see this observe that the sets P_l^+ and $\mathcal{B}$ are both finite, and $|P_l^+| = |\mathcal{B}^{\min}|$. This means that, for some integer $k \geq 0$, $b_0 = b_k, \lambda_0 = \lambda_k$, and since $\mathbf{b}_\lambda$ and $\mathbf{w}_\lambda$ are both defined inductively, both sequences must repeat every k iterations. So we take k to be the least such integer, and this is the period of both sequences.

Definition 18. *Let $\mathcal{B}$ be as in Theorem 10 and $(b_k)_{k=0}^{\infty}$ be the sequence defined iteratively above. Then:*

1. *The sequence $\mathbf{b}_\lambda = (b_k)_{k=0}^{\infty}$, is called the* ground-state path *of weight λ.*

2. *A* λ-path *in $\mathcal{B}$ is a sequence $\mathbf{p} = (p_k)_{k=0}^{\infty}$ with $p_k = b_k$ for all $k \gg 0$.*

Example: Consider the level 1 perfect $D_4^{(1)}$-crystal $\mathcal{B}_1$ described in the previous section. Then to find the ground state path of weight Λ_0 we compute:

k	λ_k	b_k
0	Λ_0	$b_{\Lambda_0} = \bar{\mathbf{1}}$
1	$\varepsilon(\bar{\mathbf{1}}) = \Lambda_1$	$b_{\Lambda_1} = \mathbf{1}$
2	$\varepsilon(\mathbf{1}) = \Lambda_0$	$b_{\Lambda_0} = \bar{\mathbf{1}}$

After $k = 2$ the pattern repeats. Therefore, the ground state path of weight Λ_0 for the perfect $D_4^{(1)}$-crystal $\mathcal{B}_1$ is $(\ldots, \bar{\mathbf{1}}, \mathbf{1}, \bar{\mathbf{1}})$.

Let $\mathcal{P}_\lambda, \lambda \in P_+$ be the set of all λ-paths. We seek to define a crystal structure on $\mathcal{P}_\lambda$ such that $\mathcal{P}_\lambda \cong \mathcal{B}(\lambda)$. The idea is to iterate the isomorphism in Theorem 10 and view $\mathcal{B}(\lambda)$ as a semi-infinite tensor product of a perfect crystal of level l:

$$\mathcal{B}(\lambda_0) \cong \mathcal{B}(\lambda_1) \otimes \mathcal{B} \cong \mathcal{B}(\lambda_2) \otimes \mathcal{B} \otimes \mathcal{B} \cong \cdots \cong \mathcal{B}(\lambda_k) \otimes \mathcal{B}^{\otimes k} \cong \cdots \cong \bigotimes_{i=0}^{\infty} \mathcal{B},$$

with

$$u_{\lambda_0} \mapsto u_{\lambda_1} \otimes b_0 \mapsto \cdots \mapsto u_{\lambda_k} \otimes b_{k-1} \otimes b_{k-2} \otimes \cdots \otimes b_0 \mapsto \cdots \mapsto \bigotimes_{k=0}^{\infty} b_k.$$

Therefore it is natural to view the "tail end" of a λ-path as an element of $\mathcal{B}^{\otimes N}$ for sufficiently large N. The explicit $U_q'(\mathfrak{g})$-crystal structure is as follows. Let $\mathbf{p} = (p_k)_{k=0}^{\infty}$ be a λ-path in $\mathcal{B}$ and let $N > 0$ be the smallest positive integer such that $p_k = b_k$ for all $k \geq N$. Let $\mathbf{p}' = p_{N-1} \otimes \cdots \otimes p_1 \otimes p_0$. For each $i \in I$, we define

- $\overline{\mathrm{wt}}\, \mathbf{p} = \lambda + \sum_{k=0}^{N-1} (\overline{\mathrm{wt}}\,(p_k) - \overline{\mathrm{wt}}\,(b_k))$,

- $\tilde{e}_i\mathbf{p} = \begin{cases} \cdots \otimes p_N \otimes \tilde{e}_i(\mathbf{p}') \text{ if } \varphi_i(p_N) < \varepsilon_i(p_{N-1}), \\ 0 \text{ otherwise,} \end{cases}$

- $\tilde{f}_i\mathbf{p} = \cdots \otimes p_{N+1} \otimes \tilde{f}_i(p_N \otimes \mathbf{p}')$,

- $\varepsilon_i(\mathbf{p}) = \max(\varepsilon(\mathbf{p}') - \varphi_i(b_N), 0))$,

- $\varphi_i(\mathbf{p}) = \varphi_i(\mathbf{p}') + \max(\varphi_i(b_N) - \varepsilon_i(\mathbf{p}'), 0)$.

We then have the following:

Theorem 11 ([23])**.** *The maps $\overline{wt}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i$ given above define a $U'_q(\mathfrak{g})$-crystal structure on $\mathcal{P}_\lambda$, and there exists an isomorphism*

$$\Psi : \mathcal{B}(\lambda) \to \mathcal{P}_\lambda$$

given by $u_\lambda \mapsto \mathbf{b}_\lambda$.

Note that the map $\overline{\mathrm{wt}}$ is a map from $\mathcal{P}_\lambda$ to $\bar{P}$ only and not to P. In order to give a $U_q(\mathfrak{g})$-crystal structure to $\mathcal{P}_\lambda$ we need to give the appropriate map $\mathrm{wt} : \mathcal{P}_\lambda \to P$. To do this, we need the following definition:

Definition 19. *Let V be a finite dimensional $U'_q(\mathfrak{g})$-module with crystal $\mathcal{B}$. An* energy function *on $\mathcal{B}$ is a map $H : \mathcal{B} \otimes \mathcal{B} \to \mathbb{Z}$ satisfying the following conditions:*

$$H(\tilde{e}_i(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2), & \textit{if } i \neq 0, \\ H(b_1 \otimes b_2) + 1, & \textit{if } i = 0, \varphi_0(b_1) \geq \varepsilon_0(b_0), \\ H(b_1 \otimes b_2) - 1, & \textit{if } i = 0, \varphi_0(b_1) < \varepsilon_0(b_2), \end{cases}$$

for all $i \in I, b_1 \otimes b_2 \in \mathcal{B} \otimes \mathcal{B}$, with $\tilde{e}_i(b_1 \otimes b_2) \in \mathcal{B} \otimes \mathcal{B}$.

Remark: If $\mathcal{B}$ is perfect, then there is a unique energy function up to translation by an integer.

Example: We give an energy function for the $D_4^{(1)}$-crystal $\mathcal{B}_1$ defined previously. Let

$i, j \in \{1,2,3,4\}$ be given. Then:

$$\begin{aligned}
H(\mathbf{j} \otimes \mathbf{k}) &= \begin{cases} 1, & \text{if } j \geq k \\ 0, & \text{otherwise} \end{cases} \\
H(\bar{\mathbf{j}} \otimes \bar{\mathbf{k}}) &= \begin{cases} 1, & \text{if } j \leq k \\ 0, & \text{otherwise} \end{cases} \\
H(\bar{\mathbf{j}} \otimes \mathbf{k}) &= \begin{cases} 0, & \text{if } j = k = 4 \\ 1, & \text{otherwise} \end{cases} \\
H(\mathbf{j} \otimes \bar{\mathbf{k}}) &= \begin{cases} -1, & \text{if } j = k = 1 \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

So, for example $H(\mathbf{1} \otimes \bar{\mathbf{1}}) = -1$, $H(\bar{\mathbf{1}} \otimes \mathbf{1}) = 1$ and so on. Now we are ready to give the affine weight formula.

Theorem 12 ([23])**.** *Let* $\mathbf{p} \in \mathcal{P}(\lambda)$*. Then the affine weight of* $\mathbf{p}$ *is given by the formula*

$$\begin{aligned}
wt(\mathbf{p}) &= \lambda + \sum_{k=0}^{\infty} (\overline{wt}\, p_k - \overline{wt}\, b_k) \qquad (5.1) \\
&- \left(\sum_{k=0}^{\infty} (k+1)(H(p_{k+1} \otimes p_k) - H(b_{k+1} \otimes b_k)) \right) \delta.
\end{aligned}$$

Example: In the $D_4^{(1)}$-crystal $\mathcal{P}_{\Lambda_0}$, consider $\mathbf{p} = \tilde{f}_0(\ldots, \bar{\mathbf{1}}, \mathbf{1}, \bar{\mathbf{1}}) = (\ldots, \bar{\mathbf{1}}, \mathbf{1}, \mathbf{2})$. Since $\mathrm{wt}(b_{\Lambda_0}) = \Lambda_0$, we expect to have $\mathrm{wt}(\mathbf{p}) = \Lambda_0 - \alpha_0 = \Lambda_2 - \Lambda_0 - \delta$. Indeed, using (5.1) we compute:

$$\begin{aligned}
\mathrm{wt}(\mathbf{p}) &= \Lambda_0 + \sum_{k=0}^{\infty} (\overline{\mathrm{wt}}\, p_k - \overline{\mathrm{wt}}\, b_k) \\
&- \left(\sum_{k=0}^{\infty} (k+1)(H(p_{k+1} \otimes p_k) - H(b_{k+1} \otimes b_k)) \right) \delta \\
&= \Lambda_0 + \overline{\mathrm{wt}}(\mathbf{2}) - \overline{\mathrm{wt}}(\bar{\mathbf{1}}) - (H(\mathbf{1} \otimes \mathbf{2}) - H(\mathbf{1} \otimes \bar{\mathbf{1}}))\delta \\
&= \Lambda_0 + (\Lambda_2 - \Lambda_1 - \Lambda_0) - (\Lambda_0 - \Lambda_1) - (0 - (-1))\delta \\
&= \Lambda_2 - \Lambda_0 - \delta.
\end{aligned}$$

5.5 Perfect Crystal and Energy Function for $D_n^{(1)}$

Let $\mathcal{B}_l := \{b = (x_1, x_2, \ldots, x_n, \bar{x}_n, \bar{x}_{n-1}, \ldots, \bar{x}_1) \in \mathbb{Z}_{\geq 0}^{2n} | s(b) := \sum_{i=1}^n x_i + \sum_{i=1}^n \bar{x}_i = l, x_n = 0 \text{ or } \bar{x}_n = 0\}$ and define

$$\begin{aligned}
\tilde{e}_0 b &= \begin{cases} (x_1, x_2 - 1, \ldots, \bar{x}_2, \bar{x}_1 + 1) \text{ if } x_2 > \bar{x}_2, \\ (x_1 - 1, x_2, \ldots, \bar{x}_2 + 1, \bar{x}_1) \text{ if } x_2 \leq \bar{x}_2, \end{cases} \\
\tilde{e}_n b &= \begin{cases} (x_1, \ldots, x_n + 1, \bar{x}_n, \bar{x}_{n-1} - 1, \ldots, \bar{x}_1) \text{ if } x_n \geq 0, \bar{x}_n = 0, \\ (x_1, \ldots, x_{n-1} + 1, x_n, \bar{x}_n - 1, \ldots, \bar{x}_1) \text{ if } x_n = 0, \bar{x}_n > 0, \end{cases} \\
\tilde{e}_i b &= \begin{cases} (x_1, \ldots, x_i + 1, x_{i+1} - 1, \ldots, \bar{x}_1) \text{ if } x_{i+1} > \bar{x}_{i+1}, \\ (x_1, \ldots, \bar{x}_{i+1} + 1, \bar{x}_i - 1, \ldots, \bar{x}_1) \text{ if } x_{i+1} \leq \bar{x}_{i+1}. \end{cases}
\end{aligned}$$

$$\begin{aligned}
\tilde{f}_0 b &= \begin{cases} (x_1, x_2 + 1, \ldots, \bar{x}_2, \bar{x}_1 - 1) \text{ if } x_2 \geq \bar{x}_2, \\ (x_1 + 1, x_2, \ldots, \bar{x}_2 - 1, \bar{x}_1) \text{ if } x_2 < \bar{x}_2, \end{cases} \\
\tilde{f}_n b &= \begin{cases} (x_1, \ldots, x_n - 1, \bar{x}_n, \bar{x}_{n-1} + 1, \ldots, \bar{x}_1) \text{ if } x_n > 0, \bar{x}_n = 0, \\ (x_1, \ldots, x_{n-1} - 1, x_n, \bar{x}_n + 1, \ldots, \bar{x}_1) \text{ if } x_n = 0, \bar{x}_n \geq 0, \end{cases} \\
\tilde{f}_i b &= \begin{cases} (x_1, \ldots, x_i - 1, x_{i+1} + 1, \ldots, \bar{x}_1) \text{ if } x_{i+1} \geq \bar{x}_{i+1}, \\ (x_1, \ldots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \ldots, \bar{x}_1) \text{ if } x_{i+1} < \bar{x}_{i+1}. \end{cases}
\end{aligned}$$

If $x_i < 0$ or $\bar{x}_i < 0$ in $b' = \tilde{e}_i(b)$ or $\tilde{f}_i(b)$ then b' is understood to be 0.

$$\begin{aligned}
\overline{\mathrm{wt}}(b) &= (\bar{x}_1 - x_1 + \bar{x}_2 - x_2)\Lambda_0 + \sum_{i=1}^{n-2} (x_i - \bar{x}_i + \bar{x}_{i+1} - x_{i+1})\Lambda_i \\
&\quad + (x_{n-1} - \bar{x}_{n-1} + \bar{x}_n - x_n)\Lambda_{n-1} \\
&\quad + (x_{n-1} - \bar{x}_{n-1} + x_n - \bar{x}_n)\Lambda_n, \\
\varphi_0(b) &= \bar{x}_1 + (\bar{x}_2 - x_2)_+, \qquad \varepsilon_0(b) = x_1 + (x_2 - \bar{x}_2)_+, \\
\varphi_i(b) &= x_i + (\bar{x}_{i+1} - x_{i+1})_+ \text{ for } i = 1, \ldots, n-2, \\
\varepsilon_i(b) &= \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+ \text{ for } i = 1, \ldots, n-2, \\
\varphi_{n-1}(b) &= x_{n-1} + \bar{x}_n, \qquad \varepsilon_{n-1}(b) = \bar{x}_{n-1} + x_n, \\
\varphi_n(b) &= x_{n-1} + x_n, \qquad \varepsilon_n(b) = \bar{x}_{n-1} + \bar{x}_n,
\end{aligned}$$

where $(n)_+ := \max(n,0)$. Let

$$
\begin{aligned}
H(b\otimes b') &= \max(\{\theta_j(b\otimes b'), \theta'_j(b\otimes b')|1\le j\le n-2\} \cup \\
&\quad \{\eta_j(b\otimes b'), \eta'_j(b\otimes b')|1\le j\le n\}),
\end{aligned}
$$

where,

$$
\begin{aligned}
\theta_j(b\otimes b') &= \sum_{k=1}^{j}(\bar{x}_k-\bar{x}'_k) \text{ for } j=1,\dots,n-2,\\
\theta'_j(b\otimes b') &= \sum_{k=1}^{j}(x'_k-x_k) \text{ for } j=1,\dots,n-2,\\
\eta_j(b\otimes b') &= \sum_{k=1}^{j}(\bar{x}_k-\bar{x}'_k)+(\bar{x}'_j-x_j) \text{ for } j=1,\dots,n-1,\\
\eta_n(b\otimes b') &= \sum_{k=1}^{n-1}(\bar{x}_k-\bar{x}'_k)+(x_n-\bar{x}'_n),\\
\eta'_j(b\otimes b') &= \sum_{k=1}^{j}(x'_k-x_k)+(x_j-\bar{x}'_j) \text{ for } j=1,\dots,n-1,\\
\eta'_n(b\otimes b') &= \sum_{k=1}^{n-1}(x'_k-x_k)-(x_n-\bar{x}'_n).
\end{aligned}
$$

Then we have the following:

Theorem 13 ([25])**.** *The maps $\tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, wt$ define a $U_q(D_n^{(1)})$-crystal structure on $\mathcal{B}_l$ which is perfect of level l.*

Example: Consider the $D_4^{(1)}$-crystal $\mathcal{B}(\Lambda_0)$. An element $b=(x_1,x_2,\cdots\bar{x}_1)$ of the level 1 crystal $\mathcal{B}_1$ satisfies $s(b)=1$. Therefore, we can more compactly denote b as $\mathbf{i}$ (resp. $\bar{\mathbf{i}}$) if $x_i=1$ (resp. $\bar{x}_i=1$) and the rest of the coordinates are 0. This notation coincides with that of the previous section. The top part of the crystal graph of $\mathcal{B}(\Lambda_0)$ is given in Figure 5.1. Here, only the tail of the each path is given.

Example: We introduce the following notation for the level two crystal $\mathcal{B}_2$. Let the ordered pair $(\mathbf{i},\mathbf{j})$ represent the element $\mathbf{i}+\mathbf{j}$. Here i and j represent integers from 1 to n with or without a bar. The ground state path is $\mathbf{p}=(\dots,(\mathbf{2},\bar{\mathbf{2}}),(\mathbf{2},\bar{\mathbf{2}}))$. The top part of this crystal is given in Figure 5.2, where, as before, only the tail of each path is given.

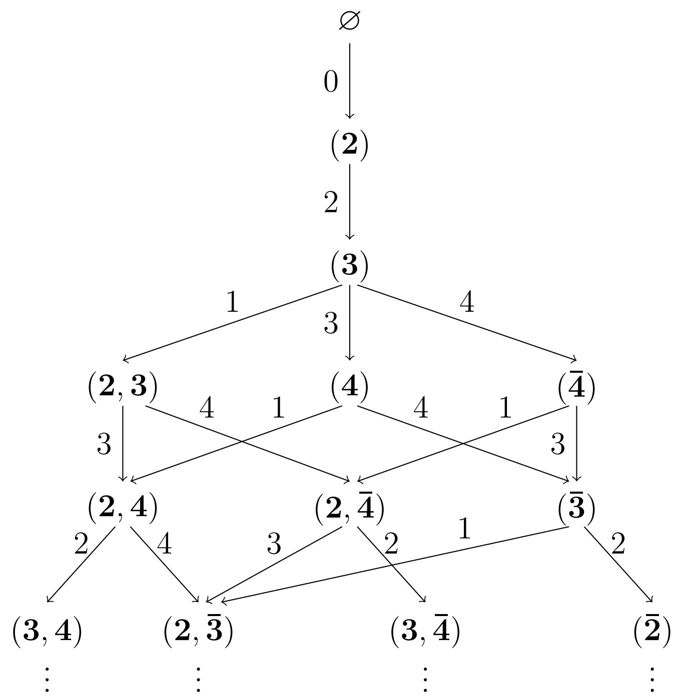


Figure 5.1: Top part of the $D_4^{(1)}$-crystal $\mathcal{B}(\Lambda_0)$

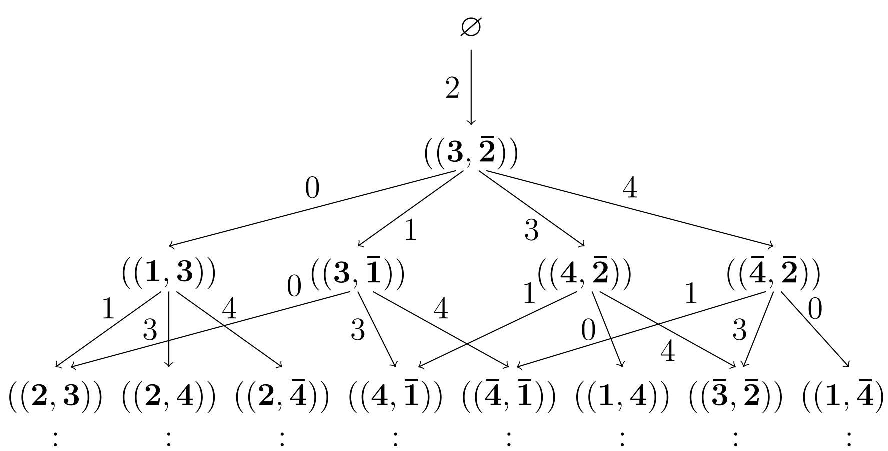


Figure 5.2: Top part of the $D_4^{(1)}$-crystal $\mathcal{B}(\Lambda_2)$.

Chapter 6

Root Multiplicities of $HD_n^{(1)}$

In this chapter, we use the results of previous chapters to compute multiplicities of roots of the form $-l\alpha_{-1} - k\delta$. We use the theory of quantum groups and crystal bases to compute certain weight multiplicities of $D_n^{(1)}$-modules, and hence determine closed formulas for the corresponding root multiplicities.

We begin by proving a fundamental result.

Proposition 4 (Analogous to [28] for $HC_n^{(1)}$)**.** *$-l\alpha_{-1} - k\delta$ is a root of $HD_n^{(1)}$ only if $k \geq l$. Also, mult$(-l\alpha_{-1} - l\delta) = n$.*

Proof. We compute:

$$\begin{aligned} r_{-1}(-l\alpha_{-1} - k\delta) &= -l\alpha_{-1} - k\delta - \langle -l\alpha_{-1} - k\delta, h_{-1}\rangle \alpha_{-1} \\ &= -l\alpha_{-1} - k\delta - (-2l + k)\alpha_{-1} \\ &= (l-k)\alpha_{-1} - k\delta. \end{aligned}$$

If $k < l$ then $(l-k)\alpha_{-1} - k\delta \notin Q^+ \cup Q^-$, hence is not a root of $HD_n^{(1)}$. Therefore:

$$\text{mult}(-l\alpha_{-1} - k\delta) = \text{mult}(r_{-1}(-l\alpha - k\delta)) = 0.$$

Hence, $-l\alpha_{-1} - k\delta$ is not a root.

If $k = l$ then

$$\text{mult}(-l\alpha_{-1} - l\delta) = \text{mult}(r_{-1}(-l\alpha_{-1} - l\delta)) = \text{mult}(-l\delta) = n.$$

$\square$

Recall Kang's multiplicity formula:

$$\dim(\mathfrak{g}_\alpha) = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right)\frac{\tau}{\alpha} B(\tau)$$

where,

$$\begin{aligned}
\mu(n) &= \begin{cases} \text{1, if } n \text{ is squarefree with an even number of distinct prime factors,} \\ \text{-1, if } n \text{ is squarefree with an odd number of distinct prime factors,} \\ \text{0, otherwise,} \end{cases} \\
B(\tau) &= \sum_{(n_i\tau_i)\in T(\tau)} \frac{(\sum n_i - 1)!}{\prod(n_i!)} \prod K_{\tau_i}^{n_i}, \\
T(\tau) &= \left\{ (n_i\tau_i) \middle| n_i \in \mathbb{Z}_{\geq 0}, \sum n_i\tau_i = \tau \right\} \\
K_{\tau_i} &= \sum_{\substack{w\in W(S) \\ \ell(w)\geq 1}} (-1)^{\ell(w)+1} \dim(V(w\rho-\rho)_{\tau_i}).
\end{aligned}$$

The first thing we need to know is which elements of W are in $W(S)$ for a given length ℓ. We use the following result.

Lemma 2 ([19])**.** *$w = w'r_i \in W(S) \iff w' \in W(S), \ell(w) > \ell(w')$, and $w'(\alpha_i) \in \Delta^+(S) = \Delta^+ \backslash \Delta_S^+$.*

Now we can proceed inductively on $\ell(w)$.
Case i: $\ell(w) = 0$. The only length 0 element is the identity, 1.
Case ii: $\ell(w) = 1$. In this case $w = r_i$. However, $\alpha_i \in \Delta^+(S)$ only if $i = -1$.
Case iii: $\ell(w) = 2$. $r_{-1}(\alpha_0) = \alpha_0 + \alpha_1 \in \Delta^+(S)$. Otherwise $r_{-1}(\alpha_i) \notin \Delta^+(S)$.
Consider the restriction of α_{-1} to $\mathfrak{h}_0$:

$$\begin{aligned}
\langle \alpha_{-1}, h_i \rangle &= -\delta_{i0}, i = 0, 1, \ldots, n, \\
\langle \alpha_{-1}, d \rangle &= 0.
\end{aligned}$$

Therefore, $\alpha_{-1}|_{\mathfrak{h}_0} = -\Lambda_0$. We will understand α_{-1} as restricted form, and therefore identify α_{-1} with $-\Lambda_0$. We define the degree of $-l\alpha_{-1} - k\delta$ to be the integer by which it acts on c: namely l.

Table 6.1: Elements of $W(S)$

$\ell(w)$	w	$w\rho - \rho$	level
1	r_{-1}	$-\alpha_{-1} = \Lambda_0$	1
2	$r_{-1}r_0$	$-2\alpha_{-1} - \alpha_0 = \Lambda_2 - \delta$	2

6.1 Degree 1 Roots

In this section, we consider the root $\alpha = -\alpha_{-1} - k\delta$ of $HD_n^{(1)}, n \geq 4, k \geq 1$. By Kang's multiplicity formula, $\text{mult}(\alpha) = \dim(V(\Lambda_0)_\alpha)$. The following result is well-known:

Proposition 5 (See [16])**.**

$$\dim(V(\Lambda_0)_\lambda) = p^n\left(-\frac{(\lambda|\lambda)}{2}\right),$$

where $p^n(k)$ is given by the generating series:

$$\sum_{k=0}^{\infty} p^n(k)q^k = \prod_{i=1}^{\infty}(1-q^i)^{-n}.$$

Using the binomial expansion $(1-q^i)^{-n} = \sum_{j=0}^{\infty}(-1)^j\binom{-n}{j}q^{ij}$, we give a formula for the multiplicity of $-\alpha_{-1} - k\delta$ for any k. We use the notation $n^{(k)} := n(n+1)\cdots(n+k-1)$ for the rising factorial. Now we compute:

$$\begin{aligned}
\sum_{k=0}^{\infty} p^n(k)q^k &= \prod_{i=1}^{\infty}(1-q^i)^{-n} \\
&= \prod_{i=1}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-n}{j}q^{ij} \\
&= \prod_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{n^{(j)}}{j!}q^{ij} \\
&= \sum_{k=0}^{\infty}\left(\sum_{j_1+2j_2+\cdots+lj_l=k}\prod_{i=1}^{l}\frac{n^{(j_i)}}{j_i!}\right)q^k.
\end{aligned}$$

Therefore, $\text{mult}(-\alpha_{-1} - k\delta) = p^n(k) = \sum_{j_1+2j_2+\cdots+lj_l=k}\prod_{i=1}^{l}\frac{n^{(j_i)}}{j_i!}$. Notice, in particular, that it is a polynomial in n of degree k. We give the first few examples in the following table:

Table 6.2: Degree 1 multiplicities.

Root	Multiplicity
$-\alpha_{-1}-\delta$	n
$-\alpha_{-1}-2\delta$	$\frac{n(n+3)}{2}$
$-\alpha_{-1}-3\delta$	$\frac{n(n+1)(n+8)}{6}$
$-\alpha_{-1}-4\delta$	$\frac{n(n+1)(n+3)(n+14)}{24}$
$-\alpha_{-1}-5\delta$	$\frac{n(n+3)(n+6)(n^2+21n+8)}{120}$

6.2 Degree 2 Roots

In this section, we consider the degree 2 root $-2\alpha_{-1}-3\delta$ of $HD_n^{(1)}, n \geq 4$. For $\tau \in P_2^+$ let

$$X(\tau) = \sum_{\lambda \in P} \dim(V(\Lambda_0)_\lambda)\dim(V(\Lambda_0)_{\tau-\lambda}).$$

Then, by Kang's multiplicity formula:

$$\mathrm{mult}(-2\alpha_{-1}-3\delta) = X(2\Lambda_0-3\delta) - \dim(V(\Lambda_2-\delta)_{2\Lambda_0-3\delta}). \tag{6.1}$$

Let $\lambda_i = \Lambda_i - \Lambda_{i-1}$, if $1 \leq i \leq n, i \neq 2, n-1$, and $\lambda_2 = \Lambda_2 - \Lambda_0 - \Lambda_1$, $\lambda_{n-1} = \Lambda_{n-2} - \Lambda_{n-1} - \Lambda_n$. Then we have the following:

Lemma 3. *If $\lambda, \mu \in P(\Lambda_0)$ satisfy $\lambda+\mu = 2\Lambda_0 - 3\delta, \lambda > \mu$, then λ is one of the following:*

$$\Lambda_0, \qquad \Lambda_0 \pm \lambda_i \pm \lambda_j - \delta, i \leq j, (i,j) \neq (1, \bar{1}).$$

Proof. If $\mathbf{p} \in \mathcal{B}(\Lambda_0)$ is such that $\mathrm{wt}(\mathbf{p}) = \Lambda_0 - \sum_{i \in I} a_i\alpha_i$ then a_i is the number of i-colored arrows in a path from u_{Λ_0} to $\mathbf{p}$. If $\mathbf{p} \neq u_{\Lambda_0}$ then $a_0 > 0$, because $\varphi_i(u_{\Lambda_0}) = \Lambda_0(h_i) = \delta_{i0}$. Therefore, we may consider $\mathbf{p}$ to be an element of the D_4-subcrystal of $\mathcal{B}_1 \otimes \mathcal{B}_1$ generated by the element $\mathbf{1} \otimes \mathbf{2}$. Therefore, we see that $b = \mathbf{i} \otimes \mathbf{j}$ with $i < j$, or $b = \mathbf{i} \otimes \bar{\mathbf{j}}$ with $(i,j) \neq (1, \bar{1})$, or $b = \bar{\mathbf{j}} \otimes \bar{\mathbf{i}}, i < j$. The weight of each element may easily be computed by the affine weight formula (5.1). $\square$

We summarize the conditions on λ, along with the dimensions of $V(\Lambda_0)_\lambda$ and $V(\Lambda_0)_{2\Lambda_0-\lambda-3\delta}$ in table 6.3.

Table 6.3: Partitions of λ.

λ	$\dim(V(\Lambda_0)_\lambda)$	$\dim(V(\Lambda_0)_{2\Lambda_0-\lambda-3\delta})$	Count
Λ_0	1	$\frac{n(n+1)(n+8)}{6}$	1
$\Lambda_0 \pm \lambda_i \pm \lambda_j - \delta, i < j$	1	n	$\frac{4n(n-1)}{2}$
$\Lambda_0 - \delta$	n	$\frac{n(n+3)}{2}$	1

Now let us consider the weight multiplicity of $\lambda = 2\Lambda_0 - 3\delta$ of the $D_n^{(1)}$-module $V(\Lambda_2 - \delta) \cong V(\Lambda_2) \otimes V(-\delta) \cong V(\Lambda_2)$ We remark that the weights of $V(\Lambda_2)$ are shifted up by δ under this identification. The ground state path of the path realization of $V(\Lambda_2)$ is $\mathbf{p} = (\ldots, b_g, b_g, b_g)$, where $b_g = (0, 1, 0, \ldots, 0, 1, 0) = (\mathbf{2}, \bar{\mathbf{2}})$.

The only $\tilde{f}_i$ which has non-zero action on b_g is $\tilde{f}_2$. Note that $2\Lambda_0 - 2\delta = \Lambda_2 - (2\alpha_0 + 3\alpha_1 + 6\alpha_2 + \cdots + 6\alpha_{n-2} + 3\alpha_{n-1} + 3\alpha_n)$. Therefore, the only paths we need consider are those for which $\tilde{e}_2^k(\mathbf{p}) = 0, k > 6$, i.e. those of the form $\mathbf{p} = (\ldots, b_g, p_5, p_4, \ldots, p_1, p_0)$. Let $p_k = (x_{1,k}, x_{2,k}, \ldots, x_{n,k}, \bar{x}_{n,k} \ldots, \bar{x}_{1,k}), k = 0, 1, \ldots, 5$. Let $H_k = H(p_{k+1} \otimes p_k)$.

Lemma 4. *Let* $\mathbf{p} \in \mathcal{B}(\Lambda_2)_{2\Lambda_0-2\delta}$. *Then* $\mathbf{p}$ *satisfies exactly one of the following for* $[H_5, H_4, \ldots, H_0]$*:*

Category A: [0,0,0,0,0,2]
Category B: [0,0,0,0,1,0]
Category C: [0,0,0,0,2,-2]
Category D: [0,0,0,1,-1,1]
Category E: [0,0,1,-1,1,-1]

Proof. From the affine weight formula (5.1) we have $6H(b_g \otimes p_5) + 5H(p_5 \otimes p_4) + \cdots + H(p_1 \otimes p_0) = 2$. First, note that $-2 \leq H(b \otimes b') \leq 2$ for all $b, b' \in \mathcal{B}_2$. This follows from the fact that $s(b) = \sum_{i=1}^n x_i + \sum_{i=1}^n \bar{x}_i = 2$ and by observing that all the sums defining the energy function are bounded by $-s(b)$ and $s(b)$. Next, $H_5 \geq 0$ since $H(b \otimes b') \geq \theta_i'(b \otimes b')$, and $\theta_1'((\mathbf{2}, \bar{\mathbf{2}}) \otimes (\mathbf{i}, \mathbf{j})) = 0$ if $i, j \neq \bar{1}$, $\theta_2'((\mathbf{2}, \bar{\mathbf{2}}) \otimes (\mathbf{i}, \bar{\mathbf{1}})) = 0$ if $i \neq \bar{1}, \bar{2}$, and $\theta_1'((\mathbf{2}, \bar{\mathbf{2}}) \otimes (\mathbf{i}, \mathbf{j})) = 0$ otherwise. Furthermore, for $i \geq 0$ it is true that if $H_{i+1} < 0$ then $H_i > 0$. To see this, observe that if $H_{i+1} < 0$ then in particular $\eta_1'(p_{i+2} \otimes p_{i+1}) = x_{1,i+1} - \bar{x}_{1,i+1} < 0$, which implies that $x_{1,i+1} < \bar{x}_{1,i+1}$. Therefore $\eta_1(p_{i+1} \otimes p_i) = \bar{x}_{1,i+1} - x_{1,i+1} > 0$ and we conclude that $H_i > 0$.

Another condition on the H_i's is that if $H_{i+1} = 0$ then $H_i \geq 0$. Suppose, to the contrary, that $H_{i+1} = 0$ and $H_i < 0$. Since $\eta_1(p_{1,i+1} \otimes p_{1,i}) = \bar{x}_{1,i+1} - x_{1,i+1} < 0$, it must be true that $\eta_1'(p_{1,i+2} \otimes p_{1,i+1}) = x_{1,i+1} - \bar{x}_{1,i+1} > 0$, contradicting the hypothesis that $H_{i+1} = 0$. Therefore if $H_{i+1} = 0$ then $H_i \geq 0$.

The final condition on the H_i's is that if $H_i = -2$ then $H_{i+1} = 2$ and, if $i > 0$, then $H_{i-1} = 2$. It must be the case that $\eta_1'(p_{i+1} \otimes p_i) = x_{1,i} - \bar{x}_{1,i} \leq -2$ and $\eta_1(p_{i+1} \otimes p_i) = \bar{x}_{1,i+1} - x_{1,i+1} \leq -2$. But $0 \leq x_{1,i}, \bar{x}_{1,i}, x_{1,i+1}, \bar{x}_{1,i+1} \leq 2$, so $x_{1,i+1} = 2$, $\bar{x}_{1,i+1} = 0$, $\bar{x}_{1,i} = 2$, and $x_{1,i} = 0$, Therefore $\eta_1'(p_{i+2} \otimes p_{i+1}) = x_{1,i+1} - \bar{x}_{i+1} = 2$ so $H_{i+1} = 2$. Furthermore, if $i > 0$ then $\eta_1(p_i \otimes p_{i-1}) = \bar{x}_{1,i} - x_{1,i} = 2$. Summarizing, we have:

(C-I) $\quad -2 \leq H_i \leq 2,$

(C-II) $\quad H_5 \geq 0,$

(C-III) $\quad H_{i+1} < 0 \implies H_i > 0,$

(C-IV) $\quad H_{i+1} = 0 \implies H_i \geq 0,$

(C-V) $\quad H_i = -2 \implies H_{i+1} = 2$ and, if $i > 0, H_{i-1} = 2$.

Now let us see which sequences $[H_5, H_4, \ldots, H_0]$ satisfy the conditions given above and the sum condition: $6H_5 + 5H_4 + \cdots + H_0 = 2$. The following general observation will be helpful in our analysis: if $H_{i+1} > 0$ and $H_i < 0$ then their total contribution to the energy sum must be positive. We analyze the cases in the following table:

Table 6.4: Lemma 4 cases (part 1).

H_{i+1}	H_i	$(i+2)H_{i+1} + (i+1)H_i$
1	-1	1
2	-1	$i+3$
2	-2	2

(Note: the pair $(1, -2)$ does not occur by C-V above). Now, conditions C-II and C-III imply that at most three of the H_i's can be negative. Therefore we divide our search into four cases according to the number of signs.

Case i: None of the H_i's are negative. In this case, there are clearly only two possible energy configurations: $[0,0,0,0,0,2]$, and $[0,0,0,0,1,0]$.

Case ii: One of the H_i's is negative. If H_0 is negative, then H_1 must be positive by C-III and C-IV. By inspection of the table above we see that $H_0 = -2$, and $H_1 = 2$ gives a sum of 2, and the rest of the H_i's are 0. Now suppose $H_i < 0$ for some $i > 0$. Then $H_{i-1}, H_{i+1} > 0$, and the following possible configurations exist:

Table 6.5: Lemma 4 cases (part 2).

H_{i+1}	H_i	H_{i-1}	$(i+2)H_{i+1} + (i+1)H_i + iH_{i-1}$
2	-2	2	$2i+2$
1	-1	1	$i+1$
2	-1	1	$2i+3$
1	-1	2	$2i+1$
2	-1	2	$3i+3$

Of the possible configurations, only the second gives the correct energy, only when $i = 1$. In this case, all of the remaining H_i's must be 0. Summarizing, the possible energy configurations in this case are $[0,0,0,0,2,-2]$, and $[0,0,0,1,-1,1]$.

Case iii: Two of the H_i's are negative. In this case, we must have the sign pattern $+-+-+$ occurring somewhere in the sequence, or $+-++-+$, or else $+-+-$ on the left side of the sequence. First, the sign pattern $+-++-+$ may be ruled out because the two $+-$ pairs already contribute at least 2 to the energy sum. Similarly, we can rule out the sign pattern $+-+-+$. So the only remaining possibility is that there is $+-+-$ at the left side of the sequence. Suppose $H_0 = H_1 = -1$ and $4H_3 + 2H_1 - 4 = 2$. Then $H_3 = H_1 = 1$, and the rest of the H_i's are 0. The following table eliminates the rest of the cases:

Table 6.6: Lemma 4 cases (part 3).

H_3	H_2	H_1	H_0	$4H_3 + \cdots + H_0$
2	-2	2	-2	4
2	-2	2	-1	5
1	-1	2	-2	3
2	-1	2	-2	7

So $[0,0,1,-1,1,-1]$ is the only configuration such that two of the H_i's are negative.

Case iv: Three of the H_i's are negative. This case does not occur. The only allowable sign pattern is $+-+-+-$. Each $+-$ pair contributes at least 1 to the energy sum, which is too large.

□

The following technical lemma will be used several times in the proof of the weight multiplicity formula.

Lemma 5. *If $H_k = H_{k-1} = H_{k-2} = 0$ and $p_{k+1} = b_g$ then $p_k = b_g$.*

Proof. $H_k = H(b_g \otimes p_k) = H((\mathbf{2}, \bar{\mathbf{2}}) \otimes p_k) = 0$. Suppose that $\bar{x}_{k,2} = 0$. Then, in order for H_k to equal 0, we must have $\theta_2(b_g \otimes p_k) = -\bar{x}_{k,1} + 1 \leq 0$, so $\bar{x}_{k,1} \geq 1$. However, $H_{k-1} = 0$ so in particular $\eta_1(p_k \otimes p_{k-1}) = \bar{x}_{k,1} - x_{k,1} \leq 0$, hence $0 < \bar{x}_{k,1} \leq x_{k,1}$. Therefore $p_k = (\mathbf{1}, \bar{\mathbf{1}})$. However, in this case $\theta'_1(b_g, p_k) = 1$, which is a contradiction since H_k is assumed to be 0. Therefore it must be the case that $\bar{x}_{k,2} > 0$. Now, suppose that $x_{k,2} = 0$. Then, $x_{k,1} = 0$ in order for $\theta'_1(b_g \otimes p_k) \leq 0$ to be true. So, by reasoning similar to above, we must have $\bar{x}_{k-1,1} \geq 0$. Since $H_{k-2} = 0$, by reasoning similar to above it must be true that $p_{k-1} = (\mathbf{1}, \bar{\mathbf{1}})$, contradicting the assumption that $H_{k-1} = 0$. So in fact, $p_k = (\mathbf{2}, \bar{\mathbf{2}}) = b_g$. □

Now, we are ready to prove the following:

Lemma 6. *For the $D_n^{(1)}$-module $V(\Lambda_2), n \geq 4$, we have*

$$\dim(V(\Lambda_2))_{2\Lambda_0 - 2\delta} = \frac{n^2(5n-1)}{2}.$$

Proof. Let $s_i = \sum_{k=0}^{5} x_{i,k}$, $\bar{s}_i = \sum_{k=0}^{5} \bar{x}_{i,k}$. Then:

$$\begin{aligned}
\overline{\mathrm{wt}}(\mathbf{p}) &= \Lambda_2 + \sum_{k=0}^{5} \overline{\mathrm{wt}}(p_k) \\
&= \Lambda_2 + \sum_{k=0}^{5} (\bar{x}_{1,k} - x_{1,k} + \bar{x}_{2,k} - x_{2,k})\Lambda_0 \\
&\quad + \sum_{i=1}^{n-2} \sum_{k=0}^{5} (x_{i,k} - \bar{x}_{i,k} + \bar{x}_{i+1,k} - x_{i+1,k})\Lambda_i \\
&\quad + \sum_{k=0}^{5} (x_{n-1} - \bar{x}_{n-1} + \bar{x}_n - x_n)\Lambda_{n-1} \\
&\quad + \sum_{k=0}^{5} (x_{n-1} - \bar{x}_{n-1} + x_n - \bar{x}_n)\Lambda_n \\
&= \Lambda_2 + (\bar{s}_1 - s_1 + \bar{s}_2 - s_2)\Lambda_0 \\
&\quad + \sum_{i=1}^{n-2} (s_i - \bar{s}_i + \bar{s}_{i+1} - s_{i+1})\Lambda_i + (s_{n-1} - \bar{s}_{n-1} + \bar{s}_n - s_n)\Lambda_{n-1} \\
&\quad + (s_{n-1} - \bar{s}_{n-1} + s_n - \bar{s}_n)\Lambda_n,
\end{aligned}$$

so we have, since $\overline{\mathrm{wt}}(\mathbf{p}) = 2\Lambda_0$,

$$\begin{aligned}
2\Lambda_0 - \Lambda_2 &= (\bar{s}_1 - s_1 + \bar{s}_2 - s_2)\Lambda_0 \\
&\quad + \sum_{i=1}^{n-2} (s_i - \bar{s}_i + \bar{s}_{i+1} - s_{i+1})\Lambda_i + (s_{n-1} - \bar{s}_{n-1} + \bar{s}_n - s_n)\Lambda_{n-1} \\
&\quad + (s_{n-1} - \bar{s}_{n-1} + s_n - \bar{s}_n)\Lambda_n.
\end{aligned}$$

By linear independence of the Λ_i's we see that $s_{n-1} - \bar{s}_{n-1} + s_n - \bar{s}_n = 0$ and $s_{n-1} - \bar{s}_{n-1} + \bar{s}_n - s_n = 0$. Therefore, by subtracting these equations, we see $2s_n - 2\bar{s}_n = 0$ so $s_n = \bar{s}_n$. By substitution, we see that $s_{n-1} = \bar{s}_{n-1}$. Therefore, proceeding inductively, we conclude that $s_i = \bar{s}_i$, for $2 < i \leq n$, $s_2 = \bar{s}_2 - 1$, and $s_1 = \bar{s}_1 - 1$.

Category A: The paths in this category have $[H_i]_{i=0}^{5} = [0, 0, 0, 0, 0, 2]$.

For paths in category A, $k = 0$ is the greatest k such that $H_k \neq 0$. Therefore, by Lemma 5, all p_k with $k > 2$ must be equal to b_g. We must have $\bar{x}_{2,2} > 0$. Otherwise, it would be the case that $\bar{x}_{1,2} > 0$ since $\theta_2(b_g \otimes p_2) = -\bar{x}_{1,2} + 1 \leq H(b_g \otimes p_2) = 0$. If $\bar{x}_{1,2} > 0$

and $x_{1,2} = 0$ then $\eta_1(p_3 \otimes p_2) > 0$, contradicting the assumption that $H_2 = 0$. However, $H((\mathbf{2}, \bar{\mathbf{2}}) \otimes (\mathbf{1}, \bar{\mathbf{1}})) = 1$, so we see that $\bar{x}_{2,2} > 0$. Also, $x_{1,2} = 0$ or $H(b_g \otimes p_2) = 1$. Therefore we can choose p_2 to be $(\mathbf{i}, \bar{\mathbf{2}}), i > 1$ or $(\bar{\mathbf{i}}, \bar{\mathbf{2}})$. Every $i > 1$ gives $H(b_g \otimes (\mathbf{i}, \bar{\mathbf{2}})) = 0$ and every i gives $H(b_g \otimes (\bar{\mathbf{i}}, \bar{\mathbf{2}})) = 0$.

Now, it must be the case that $\bar{x}_{1,1} > 0$ or $p_2 = (\mathbf{2}, \bar{\mathbf{2}})$ and $\bar{x}_{2,1} > 0$. For, if $\bar{x}_{1,1} = 0$ then $\theta_1(p_2 \otimes p_1) = \bar{x}_{1,2} \leq H(p_2 \otimes p_1) = 0$ so $\bar{x}_{1,2} = 0$. Therefore $\eta_2(p_2 \otimes p_1) = \bar{x}_{2,2} - x_{2,2} \leq H(p_2 \otimes p_1) = 0$, and $0 < \bar{x}_{2,2} \leq 2, 0 \leq x_{2,2} \leq 1$ so $\bar{x}_{2,2} = x_{2,2} = 1$.

If $\bar{x}_{1,1} = 0$ then $p_2 = (\mathbf{2}, \bar{\mathbf{2}}), p_1 = (\mathbf{i}, \bar{\mathbf{2}}), p_0 = (\bar{\mathbf{i}}, \bar{\mathbf{1}})$, or $p_2 = (\mathbf{2}, \bar{\mathbf{2}}), p_1 = (\bar{\mathbf{i}}, \bar{\mathbf{2}}), p_0 = (\mathbf{i}, \bar{\mathbf{1}}), i > 1$. If $p_1 = (\mathbf{i}, \bar{\mathbf{2}}), p_0 = (\bar{\mathbf{i}}, \bar{\mathbf{1}})$ and $i = 1$ then $H_0 = -1$, otherwise $H_0 = 0$, which rules out this case. If $p_1 = (\bar{\mathbf{i}}, \bar{\mathbf{2}}), p_0 = (\mathbf{i}, \bar{\mathbf{1}}), i > 1$ and $i = n$ then $H_0 = 0$, otherwise $H_0 = 1$, which rules out this case. So for a path to be in category A, we must have $\bar{x}_{1,1} > 0$.

Now suppose it is the case that $p_2 = (\mathbf{i}, \bar{\mathbf{2}}), i > 1$. Then $p_1 = (\bar{\mathbf{i}}, \bar{\mathbf{1}}), p_0 = (\mathbf{j}, \bar{\mathbf{j}})$, or $p_1 = (\mathbf{j}, \bar{\mathbf{1}}), p_0 = (\bar{\mathbf{i}}, \bar{\mathbf{j}})$, or $p_1 = (\bar{\mathbf{j}}, \bar{\mathbf{1}}), p_0 = (\bar{\mathbf{i}}, \mathbf{j})$. If $p_1 = (\bar{\mathbf{i}}, \bar{\mathbf{1}}), p_0 = (\mathbf{j}, \bar{\mathbf{j}}), i > 1$, then $H_1 = 0$ and $H_0 = 2$ if and only if $i \leq j < n$. If $p_1 = (\mathbf{j}, \bar{\mathbf{1}}), p_0 = (\bar{\mathbf{i}}, \bar{\mathbf{j}})$, then $H_0 = 0$ if $j = 1$ and $H_0 = 1$ otherwise, which rules out this case. Finally, if $p_1 = (\bar{\mathbf{j}}, \bar{\mathbf{1}}), p_0 = (\bar{\mathbf{i}}, \mathbf{j})$ then $H_1 = 0$ and $H_0 = 2$ if and only if $1 \leq j < i \leq n$.

Finally, suppose it is the case that $p_2 = (\bar{\mathbf{i}}, \bar{\mathbf{2}})$. Then $H_1 = 0$ if and only if $p_1 = (\bar{\mathbf{j}}, \bar{\mathbf{1}}), 1 \leq j < i \leq n$ or $p_1 = (\mathbf{n}, \bar{\mathbf{1}}), i = n$. However $H_0 = 2$ if and only if it is the case that $p_1 = (\bar{\mathbf{j}}, \bar{\mathbf{1}}), p_0 = (\mathbf{i}, \mathbf{j})$. We summarize the above in the following table.

Table 6.7: Category A cases.

p_2	p_1	p_0	Conditions	Count
$(\mathbf{i}, \bar{\mathbf{2}})$	$(\bar{\mathbf{i}}, \bar{\mathbf{1}})$	$(\mathbf{j}, \bar{\mathbf{j}})$	$1 < i \leq j < n$	$\frac{(n-2)(n-1)}{2}$
$(\mathbf{i}, \bar{\mathbf{2}})$	$(\bar{\mathbf{j}}, \bar{\mathbf{1}})$	$(\bar{\mathbf{i}}, \mathbf{j})$	$1 \leq j < i \leq n$	$\frac{n(n-1)}{2}$
$(\bar{\mathbf{i}}, \bar{\mathbf{2}})$	$(\bar{\mathbf{j}}, \bar{\mathbf{1}})$	$(\mathbf{i}, \mathbf{j})$	$1 \leq j < i \leq n$	$\frac{n(n-1)}{2}$

So the total number of paths in category A is $\frac{(n-1)(3n-2)}{2}$.

Category B: Paths in this category have energy $[0, 0, 0, 0, 1, 0]$. By Lemma 5, it must be the case that $p_k = b_g, k > 4$. First suppose that $p_3 = (\mathbf{i}, \bar{\mathbf{2}}), i > 1$ and $p_2 = (\mathbf{j}, \bar{\mathbf{1}})$. Then $H_2 = 0$ if and only if $i < j \le n$. If $p_1 = (\bar{\mathbf{i}}, \bar{\mathbf{j}})$ and $p_0 = (\mathbf{k}, \bar{\mathbf{k}}), 1 < i < j \le n$ then $H_0 > 0$ which rules out this case. Similarly, if $p_1 = (\bar{\mathbf{i}}, \bar{\mathbf{k}})$ and $p_0 = (\mathbf{k}, \bar{\mathbf{j}}), 1 < i < j \le n$ then $H_0 > 0$. The following table summarizes the remaining cases, and the sufficient and necessary conditions for them to be in category B. They are also distinct, since we make $j \ne k$ in the second case in the table below.

Table 6.8: Category B cases (part 1).

p_3	p_2	p_1	p_0	Conditions	Count
$(\mathbf{i}, \bar{\mathbf{2}})$	$(\mathbf{j}, \bar{\mathbf{1}})$	$(\mathbf{k}, \bar{\mathbf{k}})$	$(\bar{\mathbf{i}}, \bar{\mathbf{j}})$	$1 < i < j \le n$ $i \le k < n$	$\sum_{i=1}^{n-2} i^2 = \frac{(n-1)(n-2)(2n-3)}{6}$
$(\mathbf{i}, \bar{\mathbf{2}})$	$(\mathbf{j}, \bar{\mathbf{1}})$	$(\bar{\mathbf{j}}, \mathbf{k})$	$(\bar{\mathbf{i}}, \bar{\mathbf{k}})$	$1 < i < j \le n$ $1 < k \le n, j \ne k$	$\frac{(n-1)(n-2)^2}{2}$
$(\mathbf{i}, \bar{\mathbf{2}})$	$(\mathbf{j}, \bar{\mathbf{1}})$	$(\bar{\mathbf{j}}, \bar{\mathbf{n}})$	$(\bar{\mathbf{i}}, \mathbf{n})$	$1 < i < j \le n$	$\frac{(n-1)(n-2)}{2}$
$(\mathbf{i}, \bar{\mathbf{2}})$	$(\mathbf{j}, \bar{\mathbf{1}})$	$(\bar{\mathbf{i}}, \mathbf{k})$	$(\bar{\mathbf{j}}, \bar{\mathbf{k}})$	$1 \le k < i < j \le n$	$\frac{n(n-1)(n-2)}{6}$

Now suppose it is the case that $p_3 = (\mathbf{i}, \bar{\mathbf{2}}), i > 1, p_2 = (\bar{\mathbf{j}}, \bar{\mathbf{1}})$. If $p_1 = (\bar{\mathbf{i}}, \mathbf{j})$ and $p_0 = (\mathbf{k}, \bar{\mathbf{k}}), i > 1$ then $H_0 > 0$ unless $i = j \le k$. However, if this is the case then $i = j = n$, or $H_2 > 0$. But if $j = k = n$ then we have $p_1 = p_0 = (\mathbf{n}, \bar{\mathbf{n}})$ which is not in $\mathcal{B}_2$, which rules out this case. If $p_1 = (\mathbf{j}, \bar{\mathbf{k}})$ and $p_0 = (\bar{\mathbf{i}}, \mathbf{k}), i > 1$, then $i = j = k$, or $H_0 > 0$. But then $i = k = n$ or else $H_2 = 2$. Therefore $p_1 = p_0 = (\mathbf{n}, \bar{\mathbf{n}})$, which was already ruled out. Finally, suppose $p_1 = (\bar{\mathbf{i}}, \bar{\mathbf{k}}), i > 1, p_2 = (\mathbf{j}, \mathbf{k})$. Then $i = j = k = n$ or $H_0 > 0$. However, then $H_1 = 2$, which rules out this case.

Now suppose it is the case that $p_2 = (\bar{\mathbf{i}}, \bar{\mathbf{1}}), i > 1$. If $p_1 = (\mathbf{j}, \mathbf{k}), j \le k$ then $i = k = n$, or $H_1 = 2$, which is included in case 3 in the table below. If $p_1 = (\bar{\mathbf{j}}, \bar{\mathbf{k}})$ then $j = k = n$, or $H_0 > 0$. However, if $j = k = n$ then $H_1 = 2$, which rules out this case. The remaining cases are all in category B, and are summarized in the following table, along with the sufficient and necessary conditions for the paths to be in category B. Notice that there is no overlap between each of the cases.

Now suppose it is the case that $p_3 = (\bar{\mathbf{i}}, \bar{\mathbf{2}})$. Then $H_2 = 0$ if and only if $p_2 = (\bar{\mathbf{j}}, \bar{\mathbf{1}}), 1 \le$

Table 6.9: Category B cases (part 2).

p_3	p_2	p_1	p_0	Conditions	Count
$(\mathbf{i},\bar{\mathbf{2}})$	$(\bar{\mathbf{j}},\bar{\mathbf{1}})$	$(\mathbf{k},\bar{\mathbf{k}})$	$(\bar{\mathbf{i}},\mathbf{j})$	$1 < i < k < j \leq n$	$\frac{(n-1)(n-2)(n-3)}{6}$
$(\mathbf{i},\bar{\mathbf{2}})$	$(\bar{\mathbf{j}},\bar{\mathbf{1}})$	$(\bar{\mathbf{i}},\mathbf{k})$	$(\mathbf{j},\bar{\mathbf{k}})$	$1 \leq k \leq i < j \leq n$ $i > 1$	$\sum_{i=1}^{n-2} i(n-i) = \frac{(n-1)(n-2)(n+3)}{6}$
$(\mathbf{i},\bar{\mathbf{2}})$	$(\bar{\mathbf{n}},\bar{\mathbf{1}})$	$(\mathbf{n},\mathbf{k})$	$(\bar{\mathbf{i}},\bar{\mathbf{k}})$	$1 < i \leq n$ $1 < k \leq n$	$(n-1)^2$
$(\mathbf{i},\bar{\mathbf{2}})$	$(\bar{\mathbf{i}},\bar{\mathbf{1}})$	$(\mathbf{j},\bar{\mathbf{k}})$	$(\bar{\mathbf{j}},\mathbf{k})$	$1 \leq j \leq k < i \leq n$	$\sum_{i=1}^{n-1} i(n-i) = \frac{n(n+1)(n-1)}{6}$

$j < i \leq n$ or $p_2 = (\mathbf{n},\bar{\mathbf{1}}), i = n$ as we have previously observed for paths in category A. Suppose $p_2 = (\bar{\mathbf{j}},\bar{\mathbf{1}}), 1 \leq j < i \leq n$. If $p_1 = (\mathbf{k},\mathbf{l}), 1 \leq k, l \leq n$ then $H_1 = 2$ which rules out this case. If $p_1 = (\mathbf{i},\bar{\mathbf{k}}), p_0 = (\mathbf{j},\mathbf{k})$ then $H_0 > 0$ which rules out this case. If $p_1 = (\mathbf{j},\bar{\mathbf{k}}), p_0 = (\mathbf{i},\mathbf{k})$ or $p_1 = (\mathbf{i},\bar{\mathbf{k}}), p_0 = (\mathbf{j},\mathbf{k})$ then $H_0 > 0$ unless $k = n$. However, then $H_1 = 2$, which rules out this case. If $p_1 = (\mathbf{k},\bar{\mathbf{k}}), p_0 = (\mathbf{i},\mathbf{j})$, then $H_0 > 0$, which rules out this case. Therefore, no paths in category B have $p_3 = (\bar{\mathbf{i}},\bar{\mathbf{2}}), p_2 = (\bar{\mathbf{j}},\bar{\mathbf{1}})$.

Now suppose it is the case that $p_3 = (\bar{\mathbf{n}},\bar{\mathbf{2}}), p_2 = (\mathbf{n},\bar{\mathbf{1}})$. If $p_1 = (\mathbf{i},\mathbf{j}), p_0 = (\bar{\mathbf{i}},\bar{\mathbf{j}})$ then $H_1 = 2$ which rules out this case. The remaining cases are all in category B, and are summarized in the following table, along with the sufficient and necessary conditions for the paths to be in category B. Notice that there is no overlap between the two cases.

Table 6.10: Category B cases (part 3).

p_3	p_2	p_1	p_0	Conditions	Count
$(\bar{\mathbf{n}},\bar{\mathbf{2}})$	$(\mathbf{n},\bar{\mathbf{1}})$	$(\mathbf{i},\bar{\mathbf{j}})$	$(\bar{\mathbf{i}},\mathbf{j})$	$1 \leq i \leq j \leq n$ $i \neq n$ or $j \neq n$ $i \neq 1$ or $j \neq n$	$\frac{n^2+n-4}{2}$
$(\bar{\mathbf{n}},\bar{\mathbf{2}})$	$(\mathbf{n},\bar{\mathbf{1}})$	$(\bar{\mathbf{n}},\bar{\mathbf{n}})$	$(\mathbf{n},\mathbf{n})$	None	1

Now suppose it is the case that $p_3 = (\mathbf{2},\bar{\mathbf{2}}), p_2 = (\mathbf{i},\bar{\mathbf{2}})$. Then $i > 1$ since otherwise

$H_2 = 1$. And p_1 cannot be $(\bar{\mathbf{j}}, \bar{\mathbf{1}})$, since otherwise $H_1 = 0$. If $p_1 = (\mathbf{j}, \bar{\mathbf{1}})$ then $j \le i$ since otherwise $H_1 = 0$. But then $p_0 = (\bar{\mathbf{i}}, \mathbf{j})$, which implies that $H_0 > 0$ contrary to hypothesis, which rules out this case.

Now suppose it is the case that $p_2 = (\bar{\mathbf{i}}, \bar{\mathbf{2}})$. We can assume $i > 1$ since we have previously analyzed the case where $p_2 = (\bar{\mathbf{2}}, \bar{\mathbf{1}})$. If $p_1 = (\bar{\mathbf{j}}, \bar{\mathbf{1}})$ then $p_0 = (\mathbf{i}, \mathbf{j})$ which implies that $H_0 > 0$ contrary to hypothesis, which rules out this case. If $p_1 = (\mathbf{j}, \bar{\mathbf{1}})$ then $p_0 = (\mathbf{i}, \bar{\mathbf{j}})$ which implies that $H_0 > 0$ unless $j = 1$ which rules out this case unless $j = 1$. If $p_1 = (\bar{\mathbf{j}}, \bar{\mathbf{1}})$ then $j \ge i$ since otherwise $H_1 = 0$ contrary to hypothesis. In this case $p_0 = (\mathbf{i}, \mathbf{j})$ which implies $H_0 > 0$ contrary to hypothesis, which rules out this case. If $p_1 = (\mathbf{i}, \bar{\mathbf{j}}), 1 < i, i \ne n$ or $j \ne n$ then $p_0 = (\bar{\mathbf{i}}, \mathbf{j})$ which implies that $H_0 > 0$ contrary to hypothesis, which rules out this case. The remaining cases are all in category B, and are summarized in the following table, along with the sufficient and necessary conditions for the paths to be in category B. Notice that there is no overlap among the cases since we make $i \ne j$ in the first line.

Table 6.11: Category B cases (part 4).

p_3	p_2	p_1	p_0	Conditions	Count
$(\mathbf{2}, \bar{\mathbf{2}})$	$(\mathbf{i}, \bar{\mathbf{2}})$	$(\bar{\mathbf{i}}, \mathbf{j})$	$(\bar{\mathbf{j}}, \bar{\mathbf{1}})$	$2 < i \le n, 1 < j \le n$ $i \ne j$	$(n-2)^2$
$(\mathbf{2}, \bar{\mathbf{2}})$	$(\mathbf{i}, \bar{\mathbf{2}})$	$(\bar{\mathbf{i}}, \bar{\mathbf{n}})$	$(\mathbf{n}, \bar{\mathbf{1}})$	$2 < i \le n$	$n-2$
$(\mathbf{2}, \bar{\mathbf{2}})$	$(\mathbf{i}, \bar{\mathbf{2}})$	$(\mathbf{j}, \bar{\mathbf{j}})$	$(\bar{\mathbf{i}}, \bar{\mathbf{1}})$	$1 < i \le n, 1 \le j < n$ $i \ne 2$ or $j \ne 2$	$n(n-2)$
$(\mathbf{2}, \bar{\mathbf{2}})$	$(\bar{\mathbf{i}}, \bar{\mathbf{2}})$	$(\mathbf{j}, \bar{\mathbf{j}})$	$(\mathbf{i}, \bar{\mathbf{1}})$	$1 \le j < i \le n$	$\frac{n(n-1)}{2}$
$(\mathbf{2}, \bar{\mathbf{2}})$	$(\bar{\mathbf{n}}, \bar{\mathbf{2}})$	$(\mathbf{j}, \mathbf{n})$	$(\bar{\mathbf{j}}, \bar{\mathbf{1}})$	$1 < j \le n$	$n-1$

After adding the number of paths in each possible case we get $\frac{3n^3-2n^2-n-2}{2}$ paths in category B.

Category C: Paths in this category have energy $[0, 0, 0, 0, 2, -2]$. Since $H_0 = -2$ we must have $p_1 = (\mathbf{1}, \mathbf{1}), p_0 = (\bar{\mathbf{1}}, \bar{\mathbf{1}})$. Suppose $p_3 = (\mathbf{i}, \bar{\mathbf{2}})$ which implies $p_2 = (\bar{\mathbf{i}}, \bar{\mathbf{1}})$. Then

$i > 1$ or else $H_3 > 0$ contrary to hypothesis. Otherwise, if $p_3 = (\bar{\mathbf{i}}, \bar{\mathbf{2}})$ then $p_2 = (\mathbf{i}, \bar{\mathbf{1}})$ and we must have $i = n$ or else $H_2 > 0$ contrary to hypothesis. All the cases are summarized following table, along with the sufficient and necessary conditions for the paths to be in category C.

Table 6.12: Category C cases.

p_3	p_2	p_1	p_0	Conditions	Count
$(\mathbf{i}, \bar{\mathbf{2}})$	$(\bar{\mathbf{i}}, \bar{\mathbf{1}})$	$(\mathbf{1}, \mathbf{1})$	$(\bar{\mathbf{1}}, \bar{\mathbf{1}})$	$1 < i \leq n$	$n-1$
$(\bar{\mathbf{n}}, \bar{\mathbf{2}})$	$(\mathbf{n}, \bar{\mathbf{1}})$	$(\mathbf{1}, \mathbf{1})$	$(\bar{\mathbf{1}}, \bar{\mathbf{1}})$	None	1

The total number of paths in this category is n.

Category D: Paths in this category have energy $[0, 0, 0, 1, -1, 1]$. Since $H_1 = -1$ we must have $x_{2,1} > 0$ and $\bar{x}_{1,1} > 0$. Suppose that $p_4 = (\mathbf{i}, \bar{\mathbf{2}})$ and $p_3 = (\mathbf{j}, \bar{\mathbf{1}}), i < j$. If $p_0 = (\mathbf{k}, \bar{\mathbf{k}})$ and $p_2 = (\bar{\mathbf{i}}, \mathbf{1})$, $p_1 = (\bar{\mathbf{j}}, \bar{\mathbf{1}})$ then $H_1 = 0$, which rules out this case. If $p_0 = (\bar{\mathbf{i}}, \bar{\mathbf{k}})$ and $p_2 = (\bar{\mathbf{j}}, \mathbf{1})$, $p_1 = (\mathbf{k}, \bar{\mathbf{1}})$ then $H_1 = 0$, which rules out this case. If $p_0 = (\bar{\mathbf{j}}, \mathbf{k})$ and $p_2 = (\bar{\mathbf{i}}, \mathbf{1})$, $p_1 = (\bar{\mathbf{k}}, \bar{\mathbf{1}})$ then $k < i$, since otherwise $H_1 = 0$. But then $k < i < j$, so $H_0 = 2$, which rules out this case. If $p_0 = (\bar{\mathbf{j}}, \mathbf{k})$ and $p_2 = (\bar{\mathbf{k}}, \mathbf{1})$, $p_1 = (\bar{\mathbf{i}}, \bar{\mathbf{1}})$ then $H_0 = 2$ since $i < j$, which rules out this case. If $p_0 = (\bar{\mathbf{j}}, \bar{\mathbf{k}})$ and $p_2 = (\mathbf{k}, \mathbf{1})$, $p_1 = (\bar{\mathbf{i}}, \bar{\mathbf{1}})$ then $j < k$, since otherwise $H_2 = 2$. But then $H_0 = 2$, since $i < j < k$, which rules out this case. If $p_0 = (\bar{\mathbf{j}}, \bar{\mathbf{k}})$ and $p_2 = (\bar{\mathbf{i}}, \mathbf{1})$, $p_1 = (\mathbf{k}, \bar{\mathbf{1}})$ then $H_1 = 0$ unless $i = k = n$, which is counted in line 5 of the table below. The remaining cases are all in category D, and are summarized in the following table, along with the sufficient and necessary conditions for the paths to be in category D. Notice that there is no overlap among the cases.

Now suppose that $p_4 = (\mathbf{i}, \bar{\mathbf{2}})$ and $p_3 = (\bar{\mathbf{j}}, \bar{\mathbf{1}})$. Suppose $p_2 = (\bar{\mathbf{k}}, \mathbf{1}), k \neq i$. Then $j > k$ since otherwise $H_1 = 2$. If $p_1 = (\mathbf{k}, \bar{\mathbf{1}})$ or $(\mathbf{j}, \bar{\mathbf{1}})$ then $H_1 = 0$ unless $k = n$, which is ruled out because $k < j \leq n$, so $p_1 = (\bar{\mathbf{i}}, \bar{\mathbf{1}})$. But in this case $p_0 = (\mathbf{j}, \mathbf{k})$, which implies that $H_0 = 2$. Therefore $p_2 \neq (\bar{\mathbf{k}}, \mathbf{1}), k \neq i$. Now suppose $p_2 = (\bar{\mathbf{i}}, \mathbf{1})$. Then $j > i$ since otherwise $H_1 = 2$. If $p_1 = (\mathbf{k}, \bar{\mathbf{1}})$ or $(\mathbf{j}, \bar{\mathbf{1}})$ then $H_1 = 0$ unless $i = n$, which is ruled out because $i < j \leq n$, so $p_1 = (\bar{\mathbf{k}}, \bar{\mathbf{1}})$. But in this case $p_0 = (\mathbf{j}, \mathbf{k})$, which implies that $H_0 = 2$. Therefore $p_2 \neq (\bar{\mathbf{i}}, \mathbf{1})$. Now suppose $p_2 = (\mathbf{k}, \mathbf{1}), j \neq k$. Then $H_2 = 2$. Therefore $p_2 = (\mathbf{j}, \mathbf{1})$

Table 6.13: Category D cases (part 1).

p_4	p_3	p_2	p_1	p_0	Conditions	Count
$(\mathbf{i},\bar{\mathbf{2}})$	$(\mathbf{j},\bar{\mathbf{1}})$	$(\bar{\mathbf{j}},\mathbf{1})$	$(\bar{\mathbf{i}},\bar{\mathbf{1}})$	$(\mathbf{k},\bar{\mathbf{k}})$	$1 \le k < i < j \le n$	$\frac{n(n-1)(n-2)}{6}$
$(\mathbf{i},\bar{\mathbf{2}})$	$(\mathbf{j},\bar{\mathbf{1}})$	$(\mathbf{k},\mathbf{1})$	$(\bar{\mathbf{k}},\bar{\mathbf{1}})$	$(\bar{\mathbf{i}},\bar{\mathbf{j}})$	$1 < i < j < k \le n$	$\frac{(n-1)(n-2)(n-3)}{6}$
$(\mathbf{i},\bar{\mathbf{2}})$	$(\mathbf{j},\bar{\mathbf{1}})$	$(\mathbf{k},\mathbf{1})$	$(\bar{\mathbf{j}},\bar{\mathbf{1}})$	$(\bar{\mathbf{i}},\bar{\mathbf{k}})$	$1 < i < j < k \le n$	$\frac{(n-1)(n-2)(n-3)}{6}$
$(\mathbf{i},\bar{\mathbf{2}})$	$(\mathbf{j},\bar{\mathbf{1}})$	$(\bar{\mathbf{j}},\mathbf{1})$	$(\bar{\mathbf{k}},\bar{\mathbf{1}})$	$(\bar{\mathbf{i}},\mathbf{k})$	$1 < i < k < j \le n$	$\frac{(n-1)(n-2)(n-3)}{6}$
$(\mathbf{i},\bar{\mathbf{2}})$	$(\mathbf{j},\bar{\mathbf{1}})$	$(\bar{\mathbf{n}},\mathbf{1})$	$(\mathbf{n},\bar{\mathbf{1}})$	$(\bar{\mathbf{i}},\bar{\mathbf{j}})$	$1 < i < j \le n$	$\frac{(n-1)(n-2)}{2}$
$(\mathbf{i},\bar{\mathbf{2}})$	$(\mathbf{j},\bar{\mathbf{1}})$	$(\bar{\mathbf{k}},\mathbf{1})$	$(\bar{\mathbf{j}},\bar{\mathbf{1}})$	$(\bar{\mathbf{i}},\mathbf{k})$	$1 < i < j < k \le n$	$\frac{(n-1)(n-2)(n-3)}{6}$

and $j = n$, since otherwise $H_2 = 2$. If $p_1 = (\mathbf{k}, \bar{\mathbf{1}})$ then $H_1 = 0$.

Now suppose that $p_4 = (\bar{\mathbf{i}}, \bar{\mathbf{2}})$. Then $H_3 = 0$ if and only if $p_3 = (\bar{\mathbf{j}}, \bar{\mathbf{1}}), 1 \le j < i$ or $p_3 = (\mathbf{n}, \bar{\mathbf{1}}), i = n$. Suppose it is the case that $p_3 = (\bar{\mathbf{j}}, \bar{\mathbf{1}}), 1 \le j < i$. If $p_2 = (\mathbf{k}, \mathbf{1}), k \ne n$, then $H_2 = 2$, which rules out this case. If $p_2 = (\mathbf{n}, \mathbf{1})$, then $j = n$, since otherwise $H_2 = 2$. However, this is impossible since $j < i \le n$, which rules out this case. So we must have $p_2 = (\bar{\mathbf{k}}, \mathbf{1})$. Then $k < j$, since otherwise $H_2 = 2$. But then $p_1 = (\mathbf{i}, \bar{\mathbf{1}})$, $(\mathbf{j}, \bar{\mathbf{1}})$, or $(\mathbf{k}, \bar{\mathbf{1}})$, hence $H_1 = 0$ unless $k = n$. But $k < j \le n$, which rules out this case.

Now suppose that $p_4 = (\bar{\mathbf{n}}, \bar{\mathbf{2}})$, and $p_3 = (\mathbf{n}, \bar{\mathbf{1}})$. Then $p_2 = (\bar{\mathbf{j}}, \mathbf{1})$, since otherwise $H_2 = 2$. But then $p_1 = (\mathbf{n}, \bar{\mathbf{1}}), j = n$, since otherwise $H_1 = 0$. The remaining cases are all in category D, and are summarized in the following table, along with the sufficient and necessary conditions for the paths to be in category D. Notice that there is no overlap among the cases.

Table 6.14: Category D cases (part 2).

p_4	p_3	p_2	p_1	p_0	Conditions	Count
$(\mathbf{i},\bar{\mathbf{2}})$	$(\bar{\mathbf{n}},\bar{\mathbf{1}})$	$(\mathbf{n},\mathbf{1})$	$(\bar{\mathbf{k}},\bar{\mathbf{1}})$	$(\bar{\mathbf{i}},\mathbf{k})$	$1 < i < k \le n$	$\frac{(n-1)(n-2)}{2}$
$(\mathbf{i},\bar{\mathbf{2}})$	$(\bar{\mathbf{n}},\bar{\mathbf{1}})$	$(\mathbf{n},\mathbf{1})$	$(\bar{\mathbf{i}},\bar{\mathbf{1}})$	$(\mathbf{k},\bar{\mathbf{k}})$	$1 \le k < i \le n$	$\frac{n(n-1)}{2}$
$(\bar{\mathbf{n}},\bar{\mathbf{2}})$	$(\mathbf{n},\bar{\mathbf{1}})$	$(\bar{\mathbf{n}},\mathbf{1})$	$(\mathbf{n},\bar{\mathbf{1}})$	$(\mathbf{j},\bar{\mathbf{j}})$	$1 \le j < n$	$n-1$

Now suppose that $p_4 = (\mathbf{2}, \bar{\mathbf{2}})$. If $p_2 = (\mathbf{1}, \bar{\mathbf{1}})$ then $H_1 = 0$, which rules out this case. If $p_1 = (\bar{\mathbf{1}}, \bar{\mathbf{1}})$ then $H_0 = 2$, which rules out this case. Therefore $\bar{x}_{1,0} > 0$, in order for $\overline{\mathrm{wt}}(\mathbf{p})$ to be $2\Lambda_0 - \Lambda_2$. Suppose that $p_3 = (\mathbf{i}, \bar{\mathbf{2}})$. If $p_2 = (\bar{\mathbf{i}}, \mathbf{1}), p_1 = (\mathbf{j}, \bar{\mathbf{1}}), p_0 = (\bar{\mathbf{j}}, \bar{\mathbf{1}})$ then $i = j = n$, since otherwise $H_1 = -1$. This case is included in line 5 of the table below. Now suppose that $p_3 = (\bar{\mathbf{i}}, \bar{\mathbf{2}})$. Then $p_2 = (\bar{\mathbf{j}}, \mathbf{1}), j < i$ or $p_2 = (\mathbf{n}, \mathbf{1}), i = n$. If $p_2 = (\bar{\mathbf{j}}, \mathbf{1}), j < i$ then p_1 can be $(\mathbf{i}, \bar{\mathbf{1}})$ or $(\mathbf{j}, \bar{\mathbf{1}})$. But then we must have $j = n$, since otherwise $H_1 = 0$, but that is impossible since $j < i \leq n$, which rules out this case. The remaining cases are all in category D, and are summarized in the following table, along with the sufficient and necessary conditions for the paths to be in category D. Notice that there is no overlap among the cases.

Table 6.15: Category D cases (part 3).

p_4	p_3	p_2	p_1	p_0	Conditions	Count
$(\mathbf{2}, \bar{\mathbf{2}})$	$(\mathbf{i}, \bar{\mathbf{2}})$	$(\mathbf{j}, \mathbf{1})$	$(\bar{\mathbf{j}}, \bar{\mathbf{1}})$	$(\bar{\mathbf{i}}, \bar{\mathbf{1}})$	$1 < i < j \leq n$	$\frac{(n-1)(n-2)}{2}$
$(\mathbf{2}, \bar{\mathbf{2}})$	$(\mathbf{i}, \bar{\mathbf{2}})$	$(\mathbf{j}, \mathbf{1})$	$(\bar{\mathbf{i}}, \bar{\mathbf{1}})$	$(\bar{\mathbf{j}}, \bar{\mathbf{1}})$	$1 < i < j \leq n$	$\frac{(n-1)(n-2)}{2}$
$(\mathbf{2}, \bar{\mathbf{2}})$	$(\mathbf{i}, \bar{\mathbf{2}})$	$(\bar{\mathbf{j}}, \mathbf{1})$	$(\bar{\mathbf{i}}, \bar{\mathbf{1}})$	$(\mathbf{j}, \bar{\mathbf{1}})$	$1 < i < j \leq n$	$\frac{(n-1)(n-2)}{2}$
$(\mathbf{2}, \bar{\mathbf{2}})$	$(\mathbf{i}, \bar{\mathbf{2}})$	$(\bar{\mathbf{i}}, \mathbf{1})$	$(\bar{\mathbf{j}}, \bar{\mathbf{1}})$	$(\mathbf{j}, \bar{\mathbf{1}})$	$1 < j < i \leq n$	$\frac{(n-1)(n-2)}{2}$
$(\mathbf{2}, \bar{\mathbf{2}})$	$(\mathbf{i}, \bar{\mathbf{2}})$	$(\bar{\mathbf{n}}, \mathbf{1})$	$(\mathbf{n}, \bar{\mathbf{1}})$	$(\bar{\mathbf{i}}, \bar{\mathbf{1}})$	$1 < i \leq n$	$n-1$
$(\mathbf{2}, \bar{\mathbf{2}})$	$(\bar{\mathbf{n}}, \bar{\mathbf{2}})$	$(\mathbf{n}, \mathbf{1})$	$(\bar{\mathbf{i}}, \bar{\mathbf{1}})$	$(\mathbf{i}, \bar{\mathbf{1}})$	$1 < i \leq n$	$n-1$

Therefore, the total number of paths in category D is $\frac{(n-1)(5n^2-n+6)}{6}$.

Category E: Paths in this category have energy $[0, 1, -1, 1, -1]$. Therefore, $x_{3,1}, x_{1,1} > 0$ and $\bar{x}_{2,1}, \bar{x}_{0,1} > 0$. Suppose that $p_5 = (\mathbf{i}, \bar{\mathbf{2}}), p_4 = (\mathbf{j}, \bar{\mathbf{1}}), p_5 = (\mathbf{k}, \mathbf{1})$. Then $i < j < k$, since otherwise $H_4 = 1$ or $H_3 = 2$. Then $p_2 = (\bar{\mathbf{k}}, \bar{\mathbf{1}}), p_1 = (\bar{\mathbf{j}}, \mathbf{1}), p_0 = (\bar{\mathbf{i}}, \bar{\mathbf{1}})$, since otherwise $H_0 = 0$ or $H_1 = 2$.

Now suppose that $p_5 = (\mathbf{i}, \bar{\mathbf{2}}), p_4 = (\mathbf{j}, \bar{\mathbf{1}}), p_5 = (\bar{\mathbf{k}}, \mathbf{1})$. Then $i < j$, since otherwise $H_4 = 1$. Then $p_2 = (\mathbf{k}, \bar{\mathbf{1}}), p_1 = (\bar{\mathbf{j}}, \mathbf{1}), p_0 = (\bar{\mathbf{i}}, \bar{\mathbf{1}})$, since otherwise $H_0 = 0$ or $H_1 = 2$. But then $k = n$, since otherwise $H_2 = 0$.

Now suppose that $p_5 = (\mathbf{i}, \bar{\mathbf{2}}), p_4 = (\bar{\mathbf{j}}, \bar{\mathbf{1}})$. If $p_3 = (\bar{\mathbf{k}}, \mathbf{1})$ then $k < j$, since otherwise $H_3 = 2$. Then $p_2 = (\mathbf{k}, \bar{\mathbf{1}}), p_1 = (\mathbf{j}, \mathbf{1}), p_0 = (\bar{\mathbf{i}}, \bar{\mathbf{1}})$, since otherwise $H_0 = 0$ or $H_1 = 2$. But then $k = n$ in order for H_2 to be -1. However $k < j \leq n$, which rules out this case. If $p_5 = (\mathbf{i}, \bar{\mathbf{2}}), p_4 = (\bar{\mathbf{j}}, \bar{\mathbf{1}}), p_3 = (\mathbf{k}, \mathbf{1})$ then $j = k = n$, since otherwise $H_3 = 2$. Then $p_2 = (\bar{\mathbf{n}}, \bar{\mathbf{1}}), p_1 = (\mathbf{n}, \mathbf{1}), p_0 = (\bar{\mathbf{i}}, \bar{\mathbf{1}})$, or $p_2 = (\mathbf{n}, \bar{\mathbf{1}}), p_1 = (\bar{\mathbf{n}}, \mathbf{1}), p_0 = (\bar{\mathbf{i}}, \bar{\mathbf{1}})$ since otherwise $H_0 = 0$ or $H_1 = 2$. However, if $p_2 = (\mathbf{n}, \bar{\mathbf{1}}), p_1 = (\bar{\mathbf{n}}, \mathbf{1}), p_0 = (\bar{\mathbf{i}}, \bar{\mathbf{1}})$ then $H_2 = 0$, which rules out this case. Therefore $p_2 = (\bar{\mathbf{n}}, \bar{\mathbf{1}}), p_1 = (\mathbf{n}, \mathbf{1}), p_0 = (\bar{\mathbf{i}}, \bar{\mathbf{1}})$.

Now suppose that $p_5 = (\bar{\mathbf{i}}, \bar{\mathbf{2}})$. Then $H_4 = 0$ if and only if $p_4 = (\bar{\mathbf{j}}, \bar{\mathbf{1}}), j < i$ or $p_4 = (\mathbf{n}, \bar{\mathbf{1}}), i = n$. Suppose it is the case that $p_4 = (\bar{\mathbf{j}}, \bar{\mathbf{1}}), j < i$. If $p_3 = (\mathbf{k}, \mathbf{1})$ then $j = n$, since otherwise $H_3 = 2$. But $j < i \leq n$, which rules out this case. Now suppose that $p_3 = (\bar{\mathbf{k}}, \mathbf{1})$. Then $k < j$, since otherwise $H_3 = 2$. But then, $p_2 = (\mathbf{i}, \bar{\mathbf{1}}), (\mathbf{j}, \bar{\mathbf{1}})$, or $(\mathbf{k}, \bar{\mathbf{1}})$, which implies that $k = n$ since otherwise $H_2 = 0$. But $k < j \leq n$, which rules out this case. Therefore $p_5 = (\bar{\mathbf{n}}, \bar{\mathbf{2}}), p_4 = (\mathbf{n}, \bar{\mathbf{1}})$. It cannot be the case that $p_3 = (\mathbf{k}, \mathbf{1})$, because in that case $H_3 = 2$. Therefore $p_3 = (\bar{\mathbf{k}}, \mathbf{1})$. But then $k = n$, since otherwise $H_3 = 2$. Now suppose that $p_2 = (\bar{\mathbf{l}}, \mathbf{1})$. Then $l < n$, since otherwise $H_2 = 0$. Then $p_1 = (\mathbf{n}, \mathbf{1})$ or $(\mathbf{l}, \mathbf{1})$. In either case, $H_1 = 0$ since $l < n$, which rules out this case. Therefore $p_2 = (\mathbf{l}, \mathbf{1})$. This implies $l = n$, since otherwise $H_1 = 0$. It cannot be the case that $p_1 = (\mathbf{m}, \mathbf{1})$ since then $H_1 = 2$. So put $p_1 = (\bar{\mathbf{m}}, \mathbf{1}), p_0 = (\mathbf{m}, \bar{\mathbf{1}})$. Then $m = n$, since otherwise $H_0 = 0$. All the cases are summarized following table, along with the sufficient and necessary conditions for the paths to be in category E.

Table 6.16: Category E cases (part 1).

p_5	p_4	p_3	p_2	p_1	p_0	Conditions	Count
$(\mathbf{i}, \bar{\mathbf{2}})$	$(\mathbf{j}, \bar{\mathbf{1}})$	$(\mathbf{k}, \mathbf{1})$	$(\bar{\mathbf{k}}, \bar{\mathbf{1}})$	$(\bar{\mathbf{j}}, \mathbf{1})$	$(\bar{\mathbf{i}}, \bar{\mathbf{1}})$	$1 < i < j < k \leq n$	$\frac{(n-1)(n-2)(n-3)}{6}$
$(\mathbf{i}, \bar{\mathbf{2}})$	$(\mathbf{j}, \bar{\mathbf{1}})$	$(\bar{\mathbf{n}}, \mathbf{1})$	$(\mathbf{n}, \bar{\mathbf{1}})$	$(\bar{\mathbf{j}}, \mathbf{1})$	$(\bar{\mathbf{i}}, \bar{\mathbf{1}})$	$1 < i < j \leq n$	$\frac{(n-1)(n-2)}{2}$
$(\mathbf{i}, \bar{\mathbf{2}})$	$(\bar{\mathbf{n}}, \bar{\mathbf{1}})$	$(\mathbf{n}, \mathbf{1})$	$(\bar{\mathbf{n}}, \bar{\mathbf{1}})$	$(\mathbf{n}, \mathbf{1})$	$(\bar{\mathbf{i}}, \bar{\mathbf{1}})$	$1 < i \leq n$	$n - 1$
$(\bar{\mathbf{n}}, \bar{\mathbf{2}})$	$(\mathbf{n}, \bar{\mathbf{1}})$	$(\bar{\mathbf{n}}, \mathbf{1})$	$(\mathbf{n}, \bar{\mathbf{1}})$	$(\bar{\mathbf{n}}, \mathbf{1})$	$(\mathbf{n}, \bar{\mathbf{1}})$	None	1

Now suppose that $p_5 = (\mathbf{2}, \bar{\mathbf{2}}), p_4 = (\mathbf{i}, \bar{\mathbf{2}})$. If $p_3 = (\mathbf{j}, \mathbf{1})$ then $i < j$, since otherwise

$H_3 = 2$. Then $\bar{x}_{2,1}, \bar{x}_{0,1} = 2$, or $\bar{x}_{1,1} = 1$ in order for $\overline{\mathrm{wt}}(\mathbf{p})$ to be $2\Lambda_0 - \Lambda_2$. It must be the case that $p_2 = (\bar{\mathbf{j}}, \bar{\mathbf{1}}), p_1 = (\bar{\mathbf{i}}, \mathbf{1}), p_0 = (\bar{\mathbf{1}}, \bar{\mathbf{1}})$, since otherwise $H_1 = 2$ or $H_0 = 0$. Now suppose that $p_3 = (\bar{\mathbf{j}}, \mathbf{1})$. Then $j > 1$, since otherwise $H_2 = 0$. Therefore $\bar{x}_{2,1}, \bar{x}_{0,1} = 2$, or $\bar{x}_{1,1} = 1$ in order for $\overline{\mathrm{wt}}(\mathbf{p})$ to be $2\Lambda_0 - \Lambda_2$. Then $p_2 = (\mathbf{j}, \bar{\mathbf{1}}), p_1 = (\bar{\mathbf{i}}, \mathbf{1}), p_0 = (\bar{\mathbf{1}}, \bar{\mathbf{1}})$, since otherwise $H_1 = 2$ or $H_0 = 0$. But then $j = n$, since otherwise $H_2 = 0$.

Now suppose that $p_5 = (\mathbf{2}, \bar{\mathbf{2}}), p_4 = (\bar{\mathbf{i}}, \bar{\mathbf{2}})$. If $p_3 = (\bar{\mathbf{j}}, \mathbf{1})$, then $1 < j < i$ since otherwise $H_3 = 2$ or $H_2 = 2$. Since $j > 1$, it must be the case that $\bar{x}_{2,1}, \bar{x}_{0,1} = 2$, or $\bar{x}_{1,1} = 1$ in order for $\overline{\mathrm{wt}}(\mathbf{p})$ to be $2\Lambda_0 - \Lambda_2$. If $p_2 = (\mathbf{i}, \bar{\mathbf{1}})$ or $(\mathbf{j}, \bar{\mathbf{1}})$ then $H_2 = 0$, since $j < i \leq n$. If $p_2 = (\bar{\mathbf{1}}, \bar{\mathbf{1}})$ then $H_2 = 0$, which rules out this case. Therefore $p_2 = (\mathbf{j}, \mathbf{1})$. But then $i = j = n$ since otherwise $H_3 = 2$. If $p_2 = (\mathbf{k}, \bar{\mathbf{1}})$ then $H_2 = 0$. Therefore $p_2 = (\bar{\mathbf{k}}, \bar{\mathbf{1}})$. But then $k = n$, since otherwise $H_2 = 0$. If $p_1 = (\bar{\mathbf{1}}, \mathbf{1})$, then $p_0 = (\mathbf{n}, \bar{\mathbf{1}})$, which implies that $H_0 = 0$. Therefore $p_1 = (\mathbf{n}, \mathbf{1}), p_0 = (\bar{\mathbf{1}}, \bar{\mathbf{1}})$. All the cases are summarized following table, along with the sufficient and necessary conditions for the paths to be in category E. Therefore the number of paths in category E is $\frac{(n+1)(n^2-n+6)}{6}$.

Table 6.17: Category E cases (part 2).

p_5	p_4	p_3	p_2	p_1	p_0	Conditions	Count
$(\mathbf{2}, \bar{\mathbf{2}})$	$(\mathbf{i}, \bar{\mathbf{2}})$	$(\mathbf{j}, \mathbf{1})$	$(\bar{\mathbf{j}}, \bar{\mathbf{1}})$	$(\bar{\mathbf{i}}, \mathbf{1})$	$(\bar{\mathbf{1}}, \bar{\mathbf{1}})$	$1 < i < j \leq n$	$\frac{(n-1)(n-2)}{2}$
$(\mathbf{2}, \bar{\mathbf{2}})$	$(\mathbf{i}, \bar{\mathbf{2}})$	$(\bar{\mathbf{n}}, \mathbf{1})$	$(\mathbf{n}, \bar{\mathbf{1}})$	$(\bar{\mathbf{i}}, \mathbf{1})$	$(\bar{\mathbf{1}}, \bar{\mathbf{1}})$	$1 < i \leq n$	$n-1$
$(\mathbf{2}, \bar{\mathbf{2}})$	$(\bar{\mathbf{n}}, \bar{\mathbf{2}})$	$(\mathbf{n}, \mathbf{1})$	$(\bar{\mathbf{n}}, \bar{\mathbf{1}})$	$(\mathbf{n}, \mathbf{1})$	$(\bar{\mathbf{1}}, \bar{\mathbf{1}})$	None	1

Combining the results from categories A through E gives the following cubic polynomial for the weight multiplicity of $2\Lambda_0 - 2\delta$ in $V(\Lambda_2)$:

$$\begin{aligned}
\dim(V(\Lambda_2)_{2\Lambda_0-2\delta}) &= \frac{9n^2-15n+6}{6} + \frac{9n^3-6n^2-3n-6}{6} \\
&\quad + \frac{6n}{6} + \frac{5n^3-6n^2+7n-6}{6} + \frac{n^3+5n+6}{6} \\
&= \frac{15n^3-3n^2}{6} \\
&= \frac{n^2(5n-1)}{2}.
\end{aligned}$$

□

Finally, we have the following:

Theorem 14. *The multiplicity of the root $-2\alpha_{-1} - 3\delta$ of the Kac-Moody algebra $HD_n^{(1)}$, $n \geq 4$ is given by the polynomial $\frac{n(n+1)(n+8)}{6}$.*

Proof. We have:

$$\begin{aligned}
X(2\Lambda_0 - 3\delta) &= \frac{n(n+1)(n+8)}{6} + \frac{4n^2(n-1)}{2} + \frac{n^2(n+3)}{2} \\
&= \frac{n(n+1)(n+8)}{6} + \frac{n^2(5n-1)}{2}.
\end{aligned}$$

The multiplicity of $2\Lambda_0 - 3\delta$ in $V(\Lambda_2 - \delta)$ is equal to:

$$\dim(V(\Lambda_2)_{2\Lambda_0-2\delta}) = \frac{n^2(5n-1)}{2}.$$

Therefore, by equation (6.1):

$$\begin{aligned}
\operatorname{mult}(-2\alpha_{-1} - 3\delta) &= X(2\Lambda_0 - 3\delta) - \dim(V(\Lambda_2 - \delta)_{2\Lambda_0-3\delta}) \\
&= \frac{n(n+1)(n+8)}{6} + \frac{n^2(5n-1)}{2} - \frac{n^2(5n-1)}{2} \\
&= \frac{n(n+1)(n+8)}{6}.
\end{aligned}$$

□

6.3 Concluding Remarks

We have so far approached the multiplicities of the $HD_n^{(1)}$ root $-k\alpha_{-1} - l\delta$ by fixing k, l and letting n vary. All the formulas we have seen have been polynomials in n of degree less than l, though this conjecture has not been proven. Another approach would be to fix n, k and letting l vary. It was conjectured by Frenkel that $\mathrm{mult}(\alpha) \le p^n(1 - \frac{(\alpha|\alpha)}{2})$, for a hyperbolic Kac-Moody algebra of rank $n+2$, though this hase been disproven in the case $HC_2^{(1)}$ ([34]). Using the Maple code given in Appendix A, we computed the root multiplicities of the $HD_4^{(1)}$ root $-2\alpha_{-1} - k\delta$, for , and for $-2\alpha_{-1} - \alpha_0 - k\delta$ for various k. We summarize the results in Tables 6.18, 6.19. In our case, there is no observed discrepancy with Frenkel's conjecture.

This data was computed using two methods. For $\alpha \ge -2\alpha_{-1} - 6\delta$, we used procedure `mult` from Appendix A. This approach required computing the multiplicities of all roots $\ge -2\alpha_{-1} - 6\delta$. For this data set, we observed that the root multiplicity depends only on the degree of the root and the integer $1 - \frac{(\alpha|\alpha)}{2}$. This led to a new procedure, `mult_2` in Appendix A, which allowed us to compute the remainder of tables 6.18 and 6.19, and make the following conjecture:

Conjecture 1. *Let α be a root of $HD_4^{(1)}$ of degree 2. Then:*

$$mult(\alpha) = \tilde{p}\left(1 - \frac{(\alpha|\alpha)}{2}\right),$$

where

$$\sum_{k=0}^{\infty} \tilde{p}(k) q^k = \left(\sum_{k=0}^{\infty} p^4(k) q^k\right)(1 - 3q^8 + 7q^{10} - 15q^{12} + 30q^{14} - 54q^{16} + 92q^{18} - 154q^{20} + \cdots)$$

Table 6.18: Multiplicities of roots of the form $-2\alpha_{-1} - k\delta$. This data is conjectural if $k > 6$.

α	$1 - \frac{(\alpha\|\alpha)}{2}$	$\mathrm{mult}(\alpha)$	$p^4(1 - \frac{(\alpha\|\alpha)}{2})$
$-2\alpha_{-1} - 2\delta$	1	4	4
$-2\alpha_{-1} - 3\delta$	3	40	40
$-2\alpha_{-1} - 4\delta$	5	252	252
$-2\alpha_{-1} - 5\delta$	7	1240	1240
$-2\alpha_{-1} - 6\delta$	9	5168	5180
$-2\alpha_{-1} - 7\delta$	11	19116	19208
$-2\alpha_{-1} - 8\delta$	13	64424	64960
$-2\alpha_{-1} - 9\delta$	15	201548	203984
$-2\alpha_{-1} - 10\delta$	17	592692	602348
$-2\alpha_{-1} - 11\delta$	19	1654204	1688400
$-2\alpha_{-1} - 12\delta$	21	4413292	4524760

Table 6.19: Multiplicities of roots of the form $-2\alpha_{-1} - \alpha_0 - k\delta$. This data is conjectural if $k > 5$.

α	$1 - \frac{(\alpha\|\alpha)}{2}$	$\mathrm{mult}(\alpha)$	$p^4(1 - \frac{(\alpha\|\alpha)}{2})$
$-2\alpha_{-1} - \alpha_0 - \delta$	0	1	1
$-2\alpha_{-1} - \alpha_0 - 2\delta$	2	14	14
$-2\alpha_{-1} - \alpha_0 - 3\delta$	4	105	105
$-2\alpha_{-1} - \alpha_0 - 4\delta$	6	574	574
$-2\alpha_{-1} - \alpha_0 - 5\delta$	8	2577	2580
$-2\alpha_{-1} - \alpha_0 - 6\delta$	10	10073	10108
$-2\alpha_{-1} - \alpha_0 - 7\delta$	12	35461	35693
$-2\alpha_{-1} - \alpha_0 - 8\delta$	14	114923	116090
$-2\alpha_{-1} - \alpha_0 - 9\delta$	16	348086	353017
$-2\alpha_{-1} - \alpha_0 - 10\delta$	18	996192	1014580
$-2\alpha_{-1} - \alpha_0 - 11\delta$	20	2716178	2778517

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APPENDIX

Appendix A

Maple Code

In the appendix, we give the Maple code we used to compute root multiplicities of $HD_4^{(1)}$.

Procedure: `height`.
Input: $b \in Q$.
Output: $\mathrm{ht}(b)$. *Note:* The "inline" option is used for greater efficiency.

```
height:=proc(b) option inline;
        add(b[i],i=1..nops(b))
end proc:
```

Procedure: `get_predecessor_greater`.
Input: $a, b \in Q^-, a > b$.
Output: The element $c \in Q^-$ which is a predecessor of a in some well-ordering of $Q^- \cap (b + Q^+)$. I.e., if $Q^- = \{\mu_0, \mu_1, \mu_2, \ldots\}$ is an enumeration of Q^-, and $a = \mu_i$ then the output is $\mu_j < b$ where j is the greatest index $< i$ satisfying this property.

```
get_predecessor_greater:=proc(a,b) local c,n,i,h,j,s,k;
  c:=a;
  n:=nops(c);
  j:=n-1;
  s:=1+c[n];
  while j>0 and c[j]=b[j] or (s>0 and c[j]<>b[j]) do s:=s+c[j];
   j:=j-1 end do;
  for k from n to j+1 by -1 do
```

```
    if s<=b[k] then
      c[k]:=b[k];
      s:=s-b[k]
    else
      c[k]:=s;
      s:=0
    end if
  end do;
  if j<>0 then c[j]:=c[j]-1 end if;
  c
end proc:
```

Procedure: `F`.
Input: $v, w \in \mathbb{C}^6$.
Output: The complex number $\left(\sum_{i=-1}^{4} v_i\alpha_i | \sum_{j=-1}^{4} w_j\alpha_j\right)$.

```
F:=proc(v,w) option inline;
   w[1]*(2*v[1]-v[2])+w[2]*(-v[1]+2*v[2]-v[4])+w[3]*(2*v[3]-v[4])
   +w[4]*(-v[2]-v[3]+2*v[4]-v[5]-v[6])
   +w[5]*(-v[4]+2*v[5])+w[6]*(-v[4]+2*v[6])
end proc:
```

Procedure: `mult`.
Input: $a \in Q^+$.
Output: The multiplicity of a in $HD_4^{(1)}$, if a is a root, 0 otherwise.
Note: The global constant rh is the vector ρ. The table ta is a global variable that stores the previously encountered values of `mult` to minimize unnecessary recursion.

```
ta:=table();
rh:=[-6, -13, -10, -21, -10, -10];
mult:=proc(a) local i,s,b,k,v,r,t,l,d; global ta,rh;
  if type(a,[integer$nops(a)]) then i:=igcd(op(a))
     else return 0
  end if;
  if assigned(ta[a]) then return ta[a] end if;
  if height(a)=1 then ta[a]:=1; return 1 end if;
```

```
  v:=F(a,a-2*rh);
  if v=0 then ta[a]:=0; return 0 end if;
  s:=0;
  b:=-get_predecessor_greater(-a,-a);
  while height(b)>0 do
     r:=igcd(op(b));
     t:=igcd(op(a-b));
     d:=F(b,a-b)*add(mult(b/j)/j,j=1..r)*
            add(mult((a-b)/j)/j,j=1..t);
     s:=s+d;
  b:=-get_predecessor_greater(-b,-a)
  end do;
  s:=s/v;
  if i=1 then ta[a]:=s; return s end if;
  for l from 2 to i do
    s:=s-mult(a/l)/l
  end do;
ta[a]:=s;
s
end proc:
```

Procedure:`mult_2` .
Input:$a \in Q^+$.
*Output:*The multiplicity of a in $HD_4^{(1)}$ assuming that this depends only on the degree of a and $1 - \frac{(a|a)}{2}$, (cf. Conjecture 1).

```
ta:=table();
rh:=[-6, -13, -10, -21, -10, -10];
mult_2:=proc(a) local i,s,b,k,v,r,t,l,d,h; global ta,da,rh;
  if type(a,[integer$nops(a)]) then i:=igcd(op(a))
  else return 0
  end if;
  h:=1-F(a,a)/2;
  if h<0 then return 0 end if;
  if height(a)=1 then ta[a[1],h]:=1; return 1 end if;
```

```
  if assigned(ta[a[1],h]) then return ta[a[1],h] end if;
  v:=F(a,a-2*rh);
  if v=0 then return 0 end if;
  s:=0;
  b:=-get_predecessor_greater(-a,-a);
  while height(b)>0 do
     r:=igcd(op(b));
     t:=igcd(op(a-b));
     d:=F(b,a-b)*add(mult_2(b/j)/j,j=1..r)*
           add(mult_2((a-b)/j)/j,j=1..t);
     s:=s+d;
  b:=-get_predecessor_greater(-b,-a)
  end do;
  s:=s/v;
  if i=1 then ta[a[1],h]:=s; return s end if;
  for l from 2 to i do
    s:=s-mult_2(a/l)/l
  end do;
ta[a[1],h]:=s;
s
end proc:
```