ABSTRACT

STAPLETON, JAMES ROBERT. Structures and Singularities in n-Symplectic Geometry. (Under the direction of Larry Norris.)

Several non-standard situations in n-symplectic geometry are analyzed. Non-canonical dynamics introduced by Künzle are generalized to the frame bundle of a manifold M, LM, and subsequently shown to be too restrictive to reproduce similar or generalized results. The n-symplectic potential is altered in a generalization of the charged symplectic potential. Singularities are discovered in the n-symplectic dynamics, and the role of the n-symplectic gauge freedom in these singularities is discussed. Finally, attention is narrowed from the full frame bundle of $\mathbb{R}^n$ to a coordinate slice B_1 which exhibits both symplectic and n-symplectic properties. Tools are developed for working with general observables on B_1. Dynamics not seen on $L\mathbb{R}^n$ or $T^*\mathbb{R}^n$, somewhat natural Kaluza-Klein-type structures, and more singularities are revealed upon the slice.

Structures and Singularities in n-Symplectic Geometry

by
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DEDICATION

I would like to dedicate this to my family, my raccoon, and my kitty.
Mom and Dad, you've always believed in me and encouraged me. Thank you for being there.
My coon and my kitty, your love and support have always been invaluable. I only hope I can return the gifts you give to me.

BIOGRAPHY

James Robert Stapleton, who has always gone by the name Rob, is the son of Tony and Debbie Stapleton of Chester, South Carolina. Rob graduated from D.W. Daniel High School in 2001, became the third generation in his family to attend and then graduate from Clemson University in 2005. With a degree in applied mathematics and a love of geometry and pure mathematics, he attended graduate school at North Carolina State University. He obtained his Masters of Science in mathematics in 2007, becoming the first person in his family to achieve a Masters degree.

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Chapter 1

Introduction

Symplectic geometry is a powerful and useful tool for studying Hamiltonian mechanics on a smooth, n-dimensional manifold M. To every smooth (C^∞) function or "observable" on the cotangent bundle, $\vec{f}$, a unique vector field $X_{\vec{f}}$ is assigned using the canonical form $\vec{\theta}$ via the equation

$$\mathrm{d}\vec{f} = -X_{\vec{f}} \lrcorner \, \mathrm{d}\vec{\theta}.$$

The integral curves of the Hamiltonain vector field $X_{\vec{f}}$ are interpreted as the equations of motion (or the dynamics) of a particle moving under the influence of the forces described by $\vec{f}$. n-symplectic geometry is a generalization of symplectic geometry introduced by Larry Norris [11], where analysis takes place on the bundle of linear frames $\pi : LM \to M$ over a smooth manifold M. The $\mathbb{R}^n$-valued, tensorial soldering form $\hat{\theta} = \theta^i \hat{r}_i$ (where the $\hat{r}_i$ form the natural basis for $\mathbb{R}^n$) takes the place of canonical form $\vec{\theta}$ and Hamiltonian vector fields $X_{\hat{f}}$ are assigned to smooth functions $\hat{f}$ by

$$\mathrm{d}\hat{f} = -X_{\hat{f}} \lrcorner \, \mathrm{d}\hat{\theta}. \tag{1.1}$$

Crucial differences between symplectic and n-symplectic geometry emerge from these structure equations. First, the observables $\hat{f}$ of n-symplectic geometry are $\mathbb{R}^n$-valued functions, whereas observables $\vec{f}$ of symplectic geometry are $\mathbb{R}$-valued functions. Second, not every smooth $\mathbb{R}^n$-valued function $\hat{f}$ can be assigned a Hamiltonian vector field via equation (1.1). Only certain vector fields can satisfy equation (1.1), and these vector fields pick out only certain polynomial observables $\hat{f}$ [11]. To obtain a full polynomial algebra, the structure equation is expanded to allow for $\bigotimes_s^k \mathbb{R}^n$-valued functions, where $\otimes_s$ is the symmetric tensor product, and

$$\bigotimes_s^k \mathbb{R}^n = \underbrace{\mathbb{R}^n \otimes_s \mathbb{R}^n \otimes_s \ldots \otimes_s \mathbb{R}^n}_{k \text{ copies}}.$$

This introduces not only a $\mathbb{Z}_+$ grading on the algebra of n-symplectic observables, but a non-trivial kernel when determining Hamiltonian vector fields for the $\bigotimes_s^k \mathbb{R}^n$-valued functions where $k > 1$. This kernel leads to studying equvalence classes of Hamiltonian vector fields, from which a graded Poisson algebra of observables can be constructed.

We will begin by expanding this quick review of n-symplectic geometry in the following section in order to make precise the structures with which we will be working. In this review, we will also lay out notation that we will be using throughout the remaining chapters. We will move from there to studying some features of non-canonical n-symplectic geometry. In Chapter 2, we will generalize the work of Künzle from [8], and examine some of the new dynamics that occur when the soldering form $\hat{\theta} = \theta^i \hat{r}_i$ is replaced with

$$\hat{\phi} = \phi^i \hat{r}_i = \left(\theta^i + \gamma_a^{bi} \omega_b^a\right) \hat{r}_i$$

where ω_b^a are the 1-forms of a connection on LM and γ_a^{bi} are constants. The addition of the connection eliminates certain symmetries of the canonical n-symplectic potential. This restricts the allowable observables further, and the related dynamics do not display a natural contribution of spin as those of Künzle do.

In Chapter 3, we consider the charged n-symplectic observable in flat space and, equivalently, the charged n-symplectic form in flat space

$$\hat{\theta}_{\hat{c}} = \theta_{\hat{c}}^i \hat{r}_i = \left(\theta^i + \eta_{aj} A^{ia} \mathrm{d}x^j\right) \hat{r}_i.$$

This definition is similar to the standard charged symplectic form in flat space [2]. Certain standard and reasonable choices lead to the standard symplectic dynamics of a charged particle, but with additional equations of motion that show a factor of $\frac{1}{2}$ difference in equations of motion in the momentum space [2]. We will show that these choices also lead to singularities in the equations of motion that are able to be controlled by initial conditions. After a study of the gauge freedom, the choice of representative from an equivalence class of Hamiltonian vector fields, we will also show that the difference by a factor of $\frac{1}{2}$ can be eliminated by proper choice of gauge.

Finally in Chapter 4, we will shift our focus from non-canonical n-symplectic geometry to the study of n-symplectic geometry restricted to a special submanifold $B_1 \subset LM$. The manifold B_1 is a coordinate slice, chosen so that it is both a symplectic and an n-symplectic manifold; that is, it has a natural symplectic structure, and the soldering form $\hat{\theta}$ pulls back onto this submanifold. Being both symplectic and n-symplectic, the n-symplectic dynamics can be very readily compared to standard symplectic dynamics on B_1 itself. Being a submanifold of LM, the algebra of observables turns out to be more limited. We will discuss how to overcome these limitations on observables on B_1, examine new dynamics not seen in symplectic geometry, show that there exist natural Kaluza-Klein-type structures encoded into the dynamics on B_1, and finally show some limitations of this slice by exploring the existence of singularities in the equations of motion.

1.1 Review of n-Symplectic Geometry

This section is intended to provide only an overview of n-symplectic geometry and an introduction to much of the terminology used in this document The reader is referred to the literature ([11], [12], [13], [2], [3], [4], [1]) for the details. Let M be a smooth, n-dimensional manifold, We denote by n the dimension of the manifold M if it has not been specified. Also we denote by LM the bundle of linear frames (the frame bundle) over M. Every point $u \in LM$ is a pair (p, e_i), where $p \in M$ and $e_i := (e_1, e_2, \ldots, e_n)$ is a linear frame at the point p (a basis for the tangent space at p, T_pM). LM is a principal fibre bundle with projection map $\pi : LM \to M$ defined by $\pi(p, e_i) = p$, and structure group the general linear group $GL(n)$. If (U, x^i) is a coordinate chart on M, then we define local coordinates (x^i, π^j_k) on $\pi^{-1}(U) \subset LM$ by

$$\begin{aligned} x^i(u) &= x^i(p, e_j) := x^i(p) \\ \pi^j_k(u) &= \pi^j_k(p, e_i) := e^j \left(\frac{\partial}{\partial x^k}\Big|_p \right) \end{aligned} \tag{1.2}$$

where $e^j := (e^1, e^2, \ldots, e^n)$ denotes the coframe dual to the linear frame e_j. In these local coordinates, vectors Y have the form

$$Y = Y^s \partial_s + Y^r_s \frac{\partial}{\partial \pi^r_s} \tag{1.3}$$

Throughout, we will be using the notation ∂_s for $\frac{\partial}{\partial x^s}$. Also, following convention, x^i will refer to coordinates both on the base manifold M and the frame bundle LM (specifically, on $U \subset M$ and $\pi^{-1}(U) \subset LM$). Note that, since the e_i is a linear frame, the local coordinates $\pi^j_k(u)$ form a nonsingular matrix for every $u \in LM$. This will be an important fact when discussing singularities in Chapters 3 and 4. In Chapter 2, we will also make use of coordinate functions v^k_m dual to π^j_k in the sense that

$$\begin{aligned} (v^k_m \pi^j_k)(u) &= \delta^j_m(u) \\ (v^r_s \pi^k_r)(u) &= \delta^k_s(u) \end{aligned} \tag{1.4}$$

for every $u \in LM$, where δ^j_m is the Kronecker delta function.

For each point $u \in LM$, we may also define [7] a linear isomorphism $u : \mathbb{R}^n \to T_{\pi(u)}M$ by

$u(\xi^i \hat{r}_i) = (p, e_j)(\xi^i \hat{r}_i) := \xi^i e_i$, with inverse

$$u^{-1}(Y) = (p, e_i)^{-1}(Y) = e^i(Y)\hat{r}_i \tag{1.5}$$

where the $\hat{r}_i$ form the standard basis for $\mathbb{R}^n$. For any vector $Y \in T_u LM$, we define the soldering form $\hat{\theta}$ by

$$\hat{\theta}(Y) := u^{-1}(\mathrm{d}\pi Y) \tag{1.6}$$

In local coordinates,

$$\hat{\theta} = \theta^i \hat{r}_i = \pi^i_k \mathrm{d}x^k \hat{r}_i \tag{1.7}$$

The soldering form is a globally-defined, $\mathbb{R}^n$-valued 1-form on LM, comparable to the $\mathbb{R}$-valued canonical 1-form $\vec{\theta} = p_k \mathrm{d}q^k$ on the cotangent bundle T^*M. Symplectic geometry is based upon using $\vec{\theta}$ to assign a unique Hamiltonian vector field $X_{\vec{f}}$ to each observable function $\vec{f} : T^*M \to \mathbb{R}$ by the equation $\mathrm{d}\vec{f} = -X_{\vec{f}} \lrcorner\, \mathrm{d}\vec{\theta}$. As a generalization to this construction, n-symplectic geometry uses

$$\mathrm{d}\hat{f} = -X_{\hat{f}} \lrcorner\, \mathrm{d}\hat{\theta} \tag{1.8}$$

as the basic equation which assigns Hamiltonian vector fields $X_{\hat{f}}$ on LM to observables $\hat{f} : LM \to \mathbb{R}^n$ in a manner independent of coordinates. A key difference between symplectic and n-symplectic observables is that not every smooth (C^∞) function is compatible with equation (1.8). We will describe these observables shortly.

Equation (1.8) can also be extended and generalized to allow $\bigotimes_s^k \mathbb{R}^n$-valued functions on LM, where $\bigotimes_s^k \mathbb{R}^n$ is the totally-symmetric tensor product of k copies of $\mathbb{R}^n$.

$$\bigotimes_s^k \mathbb{R}^n = \underbrace{\mathbb{R}^n \otimes_s \mathbb{R}^n \otimes_s \ldots \otimes_s \mathbb{R}^n}_{k\ copies} \tag{1.9}$$

These $\bigotimes_s^k \mathbb{R}^n$-valued functions are determined by their coordinate functions in a basis for $\bigotimes_s^k \mathbb{R}^n$. We write $\hat{f} = \hat{f}^{i_1 i_2 \ldots i_k} \hat{r}_{i_1} \otimes_s \hat{r}_{i_2} \otimes_s \ldots \otimes_s \hat{r}_{i_k}$. The generalized n-symplectic equation is

$$\mathrm{d}\hat{f}^{i_1 i_2 \ldots i_k} \hat{r}_{i_1} \otimes_s \ldots \otimes_s \hat{r}_{i_k} = -\left(k! X_{\hat{f}}^{i_1 i_2 \ldots i_{k-1}} \lrcorner\, \mathrm{d}\theta^{i_k}\right) \hat{r}_{i_1} \otimes_s \ldots \otimes_s \hat{r}_{i_k} \tag{1.10}$$

Or in terms of the coordinate functions,

$$\mathrm{d}\hat{f}^{(i_1 i_2 \ldots i_k)} = -k! X_{\hat{f}}^{(i_1 i_2 \ldots i_{k-1}} \lrcorner\, \mathrm{d}\theta^{i_k)}, \tag{1.11}$$

the round brackets (parentheses) around indices indicating symmetrization on those indices[1]. The functions $\hat{f} = \hat{f}^{i_1 i_2 \ldots i_k} \hat{r}_{i_1} \otimes_s \hat{r}_{i_2} \otimes_s \ldots \otimes_s \hat{r}_{i_k}$ are called rank k observables.

The generalized n-symplectic equation defines not just one but multiple Hamiltonian vector fields for rank $k > 1$ observables. Only one Hamiltonian vector field is desired for the analysis of dynamics, so there arises the question of how to choose one vector field from many. This choice is made by first considering the distribution that is spanned by the multiple Hamiltonian vector fields of a rank $k > 1$ observable, and then choosing a nonzero vector field from this distribution. For this reason, we will refer to the Hamiltonian vector fields defined by a rank $k > 1$ observable primarily as its Hamiltonian distribution.

The assignment of Hamiltonian distributions to rank $k > 1$ observables is, however, not unique; the kernel of the right-hand side of equation (1.11) is nontrivial. The following lemma shows this non-uniqueness and extends the similar result in [11].

Lemma 1.1 *The set of equations $Y^{(i_1 i_2 \ldots i_{k-1}} \rfloor d\theta^{i_k)} = 0$, where θ^i are the coordinate functions of the soldering form on LM, has as its solution vectors fields*

$$Y^{i_1 i_2 \ldots i_{k-1}} = Y^{s i_1 i_2 \ldots i_{k-1}} \partial_s + Y_s^{r i_1 i_2 \ldots i_{k-1}} \frac{\partial}{\partial \pi^r_s}$$

where $Y^{s(i_1 i_2 \ldots i_{k-1})} = Y_s^{(r i_1 i_2 \ldots i_{k-1})} = 0$ but are otherwise arbitrary.

Proof: As noted in the lemma, we write, we write $Y^{i_1 i_2 \ldots i_{k-1}} = Y^{s i_1 i_2 \ldots i_{k-1}} \partial_s + Y_s^{r i_1 i_2 \ldots i_{k-1}} \frac{\partial}{\partial \pi^r_s}$, as well as $\theta^i = \pi^i_k \mathrm{d}x^k$ in our local coordinates (x^i, π^j_k) on LM [7]. Evaluating the interior product $Y^{(i_1 i_2 \ldots i_{k-1}} \rfloor \mathrm{d}\theta^{i_k)}$ in these local coordinates, we see that

$$Y_s^{(i_k i_1 i_2 \ldots i_{k-1})} \mathrm{d}x^s - Y^{s(i_1 i_2 \ldots i_{k-1}} \delta^{i_k)}_r \mathrm{d}\pi^r_s = 0 \tag{1.12}$$

The linear independence of the 1-forms $\mathrm{d}x^s$ and $\mathrm{d}\pi^r_s$ allows us to separate this set of equations into two.

$$\begin{aligned} Y_s^{(i_k i_1 i_2 \ldots i_{k-1})} &= 0 \\ Y^{s(i_1 i_2 \ldots i_{k-1}} \delta^{i_k)}_r &= 0 \end{aligned} \tag{1.13}$$

Contraction on indices r and s in the second set of equations completes the proof. □

This non-uniqueness of solution to equation (1.11), the vector fields described in the above lemma, is interpreted as a gauge freedom on the bundle LM. For rank 2 observables, the gauge

[1] The symmetrization in equation (1.11) comes about due to considering our observables only to be $\bigotimes^k_s \mathbb{R}^n$-valued functions on LM. With similar results and only slight changes to certain proofs, n-symplectic geometry can also be built from functions which are $\bigotimes^k_a \mathbb{R}^n$-valued. Here, $\bigotimes^k_a \mathbb{R}^n$ denotes the totally anti-symmetric tensor product of k copies of $\mathbb{R}^n$ [11]. Anti-symmetric n-symplectic observables have not been as well-studied as the symmetric observables we are considering.

vector field is purely vertical, $T^i = T_k^{[ji]}\frac{\partial}{\partial \pi_k^j}$ (the square brackets indicating anti-symmetrization on those indices). This is not the case for rank $k > 2$ observables. Usually, objects such as $\hat{X}_{\hat{f}} = X_{\hat{f}}^{i_1 i_2 \ldots i_{k-1}} \hat{r}_{i_1} \otimes_s \hat{r}_{i_2} \otimes_s \ldots \otimes_s \hat{r}_{i_{k-1}}$ or $\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k} X_{\hat{f}}^{i_1 i_2 \ldots i_{k-1}}$ (for some constants α_j) are what are studied. This symmetry on the indices of the vector fields removes the non-uniqueness from the horizontal portion, and so the gauge is typically considered to be a vertical vector field for any rank $k > 1$ observable [11]. This gauge freedom also means that each rank $k > 2$ observable defines an equivalence class of Hamiltonian distributions. Two distributions are defined to be equivalent in this sense if they both satisfy equation (1.11), or equivalently if their difference is a gauge vector field.

The local coordinate formula for the most general rank k observable $\hat{F}$ is a polynomial of degree at most k in the π_j^i coordinate functions, and whose coefficients are smooth (C^∞) functions on M [11]. Explicitly,

$$\begin{aligned} \hat{F}^{(i_1 i_2 \ldots i_k)} = f^{a_1 a_2 \ldots a_k} \pi_{a_1}^{i_1} \pi_{a_2}^{i_2} \ldots \pi_{a_k}^{i_k} + B_1^{a_1 a_2 \ldots a_{k-1} (i_1} \pi_{a_1}^{i_2} \pi_{a_2}^{i_3} \ldots \pi_{a_{k-1}}^{i_k)} \\ + B_2^{a_1 \ldots a_{k-2} (i_1 i_2} \pi_{a_1}^{i_3} \ldots \pi_{a_{k-2}}^{i_k)} + \ldots + B_k^{(i_1 i_2 \ldots i_k)} \end{aligned} \tag{1.14}$$

where $f^{a_1 a_2 \ldots a_k} = f^{(a_1 a_2 \ldots a_k)}$, and f and each function B_i is a function of x^s alone. Sometimes, this dependence is written as $f(u) = f(x^i) := (f \circ \pi)(u)$ for $u \in LM$, but we will suppress this composition with the projection $\pi : LM \to M$. In contrast to convention, we do allow the leading coefficient function $f^{a_1 a_2 \ldots a_k}$ to be identically zero. It is cumbersome or not illustrative to write out the Hamiltonian distribution defined by $\hat{F}$, so we will now review some less complicated but important observables in order to demonstrate the Hamiltonian distributions.

Kobayashi and Nomizu defined the natural lift of a vector field on M to LM in [7]. This definition was generalized by Norris in [11] to the natural lift of a symmetric tensor field on M to LM. This natural lift to LM of any totally symmetric rank k contravariant tensor field $\vec{t}$ on M is a vector field corresponding (by equation (1.11)) to a rank k tensorial observable $\hat{t}$. If $\vec{t} = t^{a_1 \ldots i_k} \partial_{i_1} \otimes_s \ldots \otimes_s \partial_{i_k}$, then

$$\hat{t} = t^{a_1 \ldots a_k} \pi_{a_1}^{i_1} \ldots \pi_{a_k}^{i_k} \hat{r}_{i_1} \otimes_s \ldots \otimes_s \hat{r}_{i_k}. \tag{1.15}$$

The corresponding Hamiltonian distribution, written without gauge terms, is then given by[2]

$$k! X_{\hat{t}}^{i_1 \ldots i_{k-1}} = k t^{a_1 \ldots a_{k-1} s} \pi_{a_1}^{i_1} \ldots \pi_{a_{k-1}}^{i_{k-1}} \partial_s - t_{,s}^{a_1 \ldots a_k} \pi_{a_1}^{i_1} \ldots \pi_{a_{k-1}}^{i_{k-1}} \pi_{a_k}^r \frac{\partial}{\partial \pi_s^r} \tag{1.16}$$

This means, in particular, that a metric tensor $\vec{g} = g^{ab} \partial_a \otimes_s \partial_b$ on M defines a rank 2 observable

[2]We follow a standard notation, denoting partial differentiation with respect to x^i with a comma before the index. As two examples, $F_{,i}^k := \frac{\partial}{\partial x^i} F^k = \partial_i F^k$ and $F_{,ij}^k := \partial_i \partial_j F^k$.

$\hat{g} = g^{ab}\pi^i_a\pi^j_b\hat{r}_i \otimes_s \hat{r}_j$ on LM. Its Hamiltonian distribution is

$$X^i_{\hat{g}} = g^{as}\pi^i_a\partial_s - \frac{1}{2}\left(g^{ab}_{,k}\pi^i_a\pi^j_b + T^{ji}_k\right)\frac{\partial}{\partial\pi^j_k}. \tag{1.17}$$

The T^{ji}_k term is the gauge freedom; it is a function where $T^{(ji)}_k = 0$ and is otherwise arbitrary. Any choice of torsion-free linear connection will select a single Hamiltonian distribution from the equivalence class containing $X^i_{\hat{g}}$; i.e. the choice of gauge T^{ij}_k will be fixed globally [13]. The choice of the Levi-Civita connection, in fact, will select a distribution that yields standard geodesic motion plus parallel transport of the momentum frame [11]. Three other notably useful observables are $\hat{q}^a_b := q^a\hat{r}_b$, $\hat{\pi}_b := \pi^a_b\hat{r}_a$, and $\hat{r}_a$. These are all rank 1 observables, and their respective Hamiltonian vector fields can be found at the end of this section in Table 1.2

In order to cut down on the preponderance of indices that comes with n-symplectic observables and their Hamiltonian vector fields or distributions, we will often make use of multi-index notation. Unless otherwise noted, capital Latin indices will represent multiple indices, and they may be subscripted with an indication of how many indices they represent. Examples can be seen in Table 1.1.

When all ranks of observables are considered, they form a graded algebra; addition is carried out component-wise on observables of the same rank and the product is the symmetric tensor product, usually written as juxtaposition. If $\hat{F}$ is an observable of rank k and $\hat{G}$ is an observable of rank m, then we may define a bracket of these two observables by

$$\{\hat{F}, \hat{G}\} = k! X^{(I_{k-1}}_{\hat{F}}(\hat{G}^{J_m)})\hat{r}_{I_{k-1}}\hat{r}_{J_m} = -m! X^{(J_{m-1}}_{\hat{G}}(\hat{F}^{I_k)})\hat{r}_{J_{m-1}}\hat{r}_{I_k} \tag{1.18}$$

This bracket is independent of gauge and choice of local coordinates, and it satisfies all the properties of a Poisson bracket [11], making the space of all observables a graded Poisson

Table 1.1: Examples of Multi-Index Notation

Multi-Index	Standard Notation
X^{I_k}	$X^{i_1 i_2 \ldots i_k}$
$f^{(I_k)}$	$f^{(i_1 i_2 \ldots i_k)}$
$\hat{r}_{I_k}$	$\hat{r}_{i_1} \otimes_s \hat{r}_{i_2} \otimes_s \ldots \otimes_s \hat{r}_{i_k}$
$\hat{r}_{I_k}\hat{r}_{J_m}$	$\hat{r}_{i_1} \otimes_s \ldots \otimes_s \hat{r}_{i_k} \otimes_s \hat{r}_{j_1} \otimes_s \ldots \otimes_s \hat{r}_{j_m}$
$X^{(I_{k-1} \lrcorner\, d\theta^{i_k})}$	$X^{(i_1 i_2 \ldots i_{k-1} \lrcorner\, d\theta^{i_k})}$
$f^{J_k}\pi^{I_k}_{J_k}$	$f^{j_1 j_2 \ldots j_k}\pi^{i_1}_{j_1}\pi^{i_2}_{j_2}\ldots\pi^{i_k}_{j_k}$
$\alpha_I X^I$	$\alpha_{i_1}\alpha_{i_2}\ldots X^{i_1 i_2 \ldots}$

algebra. The Poisson bracket of a rank k observable and a rank m observable is seen to be a rank $k+m-1$ observable. Furthermore, if $\hat{F}$ and $\hat{G}$ are the natural lifts of contravariant tensor fields $\vec{F}$ and $\vec{G}$, then we use equations (1.15) and (1.16) to write explicitly

$$\{\hat{F}, \hat{G}\}^{I_{k+m-1}} = \begin{pmatrix} kF^{s(a_2a_3\dots a_k}G^{cb_2b_3\dots b_m)}_{,s} \\ -mG^{s(b_2b_3\dots b_m}F^{ca_2a_3\dots a_k)}_{,s} \end{pmatrix} \pi_c^{(i_1}\pi_{a_2}^{i_2}\dots\pi_{a_k}^{i_k}\pi_{b_2}^{i_{k+1}}\dots\pi_{b_m}^{i_{k+m-1})} \tag{1.19}$$

This $\bigotimes_s^{k+m-1}\mathbb{R}^n$-valued function on LM corresponds to (is the natural lift of) the differential concomitant of $\vec{F}$ and $\vec{G}$ on M, as given by Schouten and Nijenhuis [10][15].

Finally, n-symplectic Hamiltonian vector fields and distributions can be mapped to the cotangent bundle in a very direct way, and in some cases be shown to then be equivalent to related symplectic Hamiltonian vector fields. Consider T^*M as the associated bundle $LM \times_{GL(n)} \mathbb{R}^{n*}$ as follows. Let the $\hat{r}^i$ form the standard basis for $\mathbb{R}^{n*}$. For any point $u \in LM$ and $\alpha = \alpha_i\hat{r}^i \in \mathbb{R}^{n*} \setminus \{0\}$, the pair $[u, \alpha]$ is a point in $LM \times_{GL(n)} \mathbb{R}^{n*}$ (specifically, it is a representative of an equivalence class of points, hence the square brackets). For any arbitrary but fixed $\alpha \in \mathbb{R}^{n*}$, we define the map $\psi_\alpha : LM \to T^*M \setminus \tilde{S}_0$, where $\tilde{S}_0$ is the zero section of T^*M, by

$$\psi_\alpha(u) := [u, \alpha] \tag{1.20}$$

We also note that $\alpha_i\pi_j^i(u) = \alpha_i e^i(\partial_j|_{\pi(u)}) = p_j(\alpha_i e^i)$, where the $p_j := (p_1, p_2, \dots, p_n)$ are the standard momentum coordinates on T^*M defined by the local coordinates x^i on M. It is easy to show, then, that

$$\begin{aligned} \psi_{\alpha *}(\partial_s) &= \partial_s \\ \psi_{\alpha *}\left(\frac{\partial}{\partial \pi_s^r}\right) &= \alpha_s \frac{\partial}{\partial p_r} \end{aligned} \tag{1.21}$$

Consider a rank k observable $\hat{F}$. We may map vectors in its Hamiltonian distribution $X_{\hat{F}}^{I_{k-1}}$ to vectors $X_{\psi_\alpha(\hat{F})}$ on $T^*M \setminus \tilde{S}_0$ as follows.

$$X_{\psi_\alpha(\hat{F})}(\psi_\alpha(u)) := k!\psi_{\alpha *}(\alpha_{I_{k-1}} X_{\hat{F}}^{I_{k-1}}(u)), \quad u \in LM \tag{1.22}$$

$X_{\psi_\alpha(\hat{F})}$ is a smooth vector field on $T^*M \setminus \tilde{S}_0$. Furthermore, if $\hat{F}$ is a tensorial observable, then $X_{\psi_\alpha(\hat{F})}$ is equal to the symplectic Hamiltonian vector field $X_{\vec{F}}$ [12], where

$$\vec{F}([u, \alpha]) := \alpha_{I_k}\hat{F}^{I_k}(u) \tag{1.23}$$

n-symplectic geometry has been studied extensively in the literature. As a few examples, L. K. Norris developed the field of study [11][12][13], D. Cartin studied the charged particle in an n-symplectic setting [2], M. McLean investigated n-symplectic geometry's relation to other generalizations of symplectic geometry [3], and J. K. Lawson and J. D. Brown studied aspects of quantization using n-symplectic geometry [4][1].

Table 1.2: Useful n-symplectic Observables and their Hamiltonian Vector Fields

Observable	Hamiltonian Vector Fields
$\hat{r}_a$	$X_{\hat{r}_a} = 0$
$\hat{x}^a_b$	$X_{\hat{x}^a_b} = \frac{\partial}{\partial \pi^b_a}$
$\hat{\pi}_b$	$X_{\hat{\pi}_b} = -\partial_b$
$\hat{g} = g^{ab}\hat{\pi}_a\hat{\pi}_b$	$X^i_{\hat{g}} = g^{as}\pi^i_a\partial_s - \frac{1}{2}g^{ab}_{,k}\pi^i_a\pi^j_b\frac{\partial}{\partial \pi^j_k}$

Chapter 2

Künzle's non-canonical spin

Consider a system with nonzero and fixed rest mass m and fixed spin magnitude s. Künzle found in 1972 that if one chooses $\theta = m\vec{\theta}^1 + s\omega_{34}$ as a presymplectic potential on the Lorentz bundle over space-time, where ω_{ab} are the 1-forms of a connection without torsion, then the equations of motion are equivalent to those of Souriau and Dixon for a massive particle with spin [8]. We will attempt to copy this construction and generalize it upon the frame bundle. Instead of either using a presymplectic potential or defining and using a pre-n-symplectic potential, we will add a general piece of a torsion-free linear connection to the soldering form to create a new, non-canonical n-symplectic potential. In order to study the dynamics produced by this new potential, we will find the Hamiltonian vector fields that are now allowable in the modified structure equation. This will show what modifications have been made to the algebra of n-symplectic observables, if any, and the combination of observables and Hamiltonian vector fields will show the possible dynamics in this new scheme. In particular, we are concerned with the allowable rank 2 observables and their Hamiltonian distributions, since the metric observables are observables of this rank. It turns out that even the simple cases, where we add in only one piece of a connection as Künzle did, produces results general enough from which to draw strong conclusions.

2.1 Generalizing Künzle's work on the Frame Bundle

Künzle essentially changed the symplectic potential $\vec{\theta}$ by adding a piece of a connection. Let ω^a_b be the 1-forms of a torsion-free linear connection on LM and let γ^{bi}_a be constants. We define 1-forms

$$\phi^i := \theta^i + \gamma^{bi}_a \omega^a_b \tag{2.1}$$

It is important to note that one should not choose a connection ω and constants γ^{bi}_a such that $\gamma^{bi}_a \omega^a_b = 0$ identically.

Theorem 2.1 *The $\mathbb{R}^n$-valued 2-form $d\phi^i\hat{r}_i$, with ϕ^i as defined above, is non-degenerate.*

Proof: For a torsion-free connection on LM, we know that [7]

$$\begin{aligned} \mathrm{d}\theta^i &= \theta^j \wedge \omega^i_j \\ \mathrm{d}\omega^a_b &= \omega^k_b \wedge \omega^a_k + \Omega^a_{bcd}\theta^c \wedge \theta^d \end{aligned}$$

where Ω^a_{bcd} is the curvature tensor of the connection ω. This gives us the basic formula

$$\mathrm{d}\phi^i = \mathrm{d}\theta^i + \gamma^{bi}_a \mathrm{d}\omega^a_b = \theta^j \wedge \omega^i_j + \gamma^{bi}_a \omega^k_b \wedge \omega^a_k + \gamma b i_a \Omega^a_{bcd}\theta^c \wedge \theta^d \tag{2.2}$$

Contracting with an arbitrary vector field X, we obtain

$$\begin{aligned} X \lrcorner\ \mathrm{d}\phi^i &= \theta^j(X)\omega^i_j - \omega^i_j(X)\theta^j + \gamma^{bi}_a \left(\omega^k_b(X)\omega^a_k - \omega^a_k(X)\omega^k_b + 2\Omega^a_{bcd}\theta^c(X)\theta^d \right) \\ &= \left(2\gamma^{bi}_a \Omega^a_{bcd}\theta^c(X) - \omega^i_d(X) \right)\theta^d + \left(\theta^s(X)\delta^i_r + \gamma^{bi}_r \omega^s_b(X) - \gamma^{si}_a \omega^a_r(X) \right)\omega^r_s \end{aligned}$$

Setting $X \lrcorner\ \mathrm{d}\phi^i = 0$ and using the linear independence of the 1-forms θ^i and ω^a_b, we obtain the two sets of equations

$$2\gamma^{bi}_a \Omega^a_{bcd}\theta^c(X) - \omega^i_d(X) = 0 \tag{2.3}$$

$$\theta^s(X)\delta^i_r + \gamma^{bi}_r \omega^s_b(X) - \gamma^{si}_a \omega^a_r(X) = 0 \tag{2.4}$$

Contracting equation (2.4) on s and r gives $\theta^s(X) = 0$. Using this in equation (2.3), we find $\omega^i_j(X) = 0$. Thus, $X = 0$ and $\mathrm{d}\phi^i\hat{r}_i$ is non-degenerate. $\square$

We saw that the 2-form $\mathrm{d}\theta^i\hat{r}_i$ has a non-zero kernel when contracted with higher-rank vector fields X^I. Let us contract the 2-forms $\mathrm{d}\phi^j$ with arbitrary vector fields X^I and symmetrize on the upper indices. We obtain

$$\begin{aligned} X^{(I} \lrcorner\ \mathrm{d}\phi^{j)} = &\left(2\Omega^a_{bcd}\theta^c(\gamma^{b(j}_a X^{I)}) - \omega^{(j}_d(X^{I)}) \right)\theta^d \\ &+ \left(\theta^s(X^{(I}\delta^{j)}_r) + \omega^s_b(\gamma^{b(j}_r X^{I)}) - \omega^a_r(\gamma^{s(j}_a X^{I)}) \right)\omega^r_s \end{aligned}$$

Again, setting this equal to 0 nets

$$2\Omega^a_{bcd}\theta^c(\gamma^{b(j}_a X^{I)}) - \omega^{(j}_d(X^{I)}) = 0 \tag{2.5}$$

$$\theta^s(X^{(I}\delta^{j)}_r) + \omega^s_b(\gamma^{b(j}_r X^{I)}) - \omega^a_r(\gamma^{s(j}_a X^{I)}) = 0 \tag{2.6}$$

Contracting (2.6) on s and r gives $\theta^{(j}(X^{I)}) = 0$. Writing $X^I = X^{sI}\partial_s + X^{rI}_s \frac{\partial}{\partial \pi^r_s}$, we see that $\theta^{(j}(X^{I)}) = X^{k(I}\pi^{j)}_k = 0$. Already, we see that we have a similar but more complicated kernel condition than the standard case shown in Lemma 1.1. For our current purposes, the kernel does not need to be calculated explicitly; the invariant parts of the vector fields determined by the n-symplectic equation

$$\mathrm{d}\hat{F} = -k! X^I_{\hat{F}} \lrcorner\ \mathrm{d}\phi^j \hat{r}_I \hat{r}_j \tag{2.7}$$

will be sufficient. We begin by looking at a very simple case.

Theorem 2.2 *Let ϕ^i be as defined in (2.1). If ω^a_b are the 1-forms of a torsion-free flat linear connection and if $\gamma^{bi}_a = \delta^b_3 \delta^2_a \delta^i_1$, then rank k observables $\hat{F} = F^{I_k} \hat{r}_{I_k}$ that have the form*

$$F^{(I_k)} = f^{J_k} \pi^{I_k}_{J_k} + B_1^{J_{k-1}(i_k} \pi^{I_{k-1})}_{J_{k-1}} + B_2^{J_{k-2}(i_{k-1} i_k} \pi^{I_{k-2})}_{J_{k-2}} + \ldots + B_k^{(I_k)} + \gamma^{b(i_k}_a \pi^{I_{k-1})}_{J_{k-1}} \upsilon^m_b \pi^a_c f^{cJ_{k-1}}_{,m}$$

where each B_i is constant, f^{J_k} are functions of the x^i alone and linear in x^i, $f^{J_k} = f^{(J_k)}$ satisfy (2.7), and υ^m_b are coordinate functions dual to π^b_k as defined in equation (1.4). In particular, the only allowable rank 1 observables are

$$\hat{F} = \left(f^k \pi^i_k + \gamma^{bi}_a \upsilon^m_b \pi^a_c f^c_{,m} + \xi^i \right) \hat{r}_i$$

and the only allowable rank 2 observables are

$$\hat{F} = \left(f^{ab} \pi^i_a \pi^j_b + h^{d(i} \pi^{j)}_d + k^{(ij)} + \gamma^{b(i}_a \pi^{j)}_d \upsilon^m_b \pi^a_c f^{cd}_{,m} \right) \hat{r}_i \hat{r}_j .$$

It can be checked directly that there exists a collection of vector fields $X^I_{\hat{F}}$ corresponding to any observable $\hat{F}$ with the form given in the theorem such that equation (2.7) is satisfied, and that equation (2.7) cannot be satisfied with an observable $\hat{F}$ of the general form given in equation (1.14). The proofs of the specific claims of the rank 1 and rank 2 observables are long, and are relegated to Appendix A. What is of particular importance to note from this theorem is that the only tensor fields on the base manifold M that can be lifted to allowable observables (particularly rank 2 observables) are constant tensor fields. This drastically restricts the possible classic Hamiltonians (or observables) that can be studied in this non-canonical setting; furthermore, appendix equation (A.112) shows that the Hamiltonian distribution corresponding to this rank 2 tensorial observable with constant coefficients would be no different than the canonical Hamiltonian distribution given by equation (1.17) for the same observable.

Thinking that, perhaps, the n-symplectic form $\phi^i \hat{r}_i$ defined by a flat connection is too restrictive, we consider a simple case when $\phi^i \hat{r}_i$ is defined using a general connection.

Theorem 2.3 *Let ϕ^i be as in (2.1) and γ^{bi}_a as in Theorem 2.2. If ω^a_b are the 1-forms of a torsion-free linear connection, then there is no non-trivial rank 2 observable $\hat{F}$ that can satisfy (2.7) in general.*

The long proof can be found in Appendix B. The theorem and its proof are restricted to rank 2 observables in this case because rank 2 observables are where the standard equations of motion are to be found, and the techniques used in the proof generalize readily to any rank of observable. The main difference between each rank of observable is the level of complexity and number of equations to check, as can be seen in the two proofs of Appendix A.

We are left with the negative result that the non-canonical construction of Künzle cannot be lifted directly to the frame bundle LM. The n-symplectic 2-form $\mathrm{d}\phi^i \hat{r}_i$ defined by $\phi^i = \theta^i + \gamma_a^{bi}\omega_b^a$ is non-degenerate, but it is too restrictive when it comes to determining motions of rank k observables by the equation $\mathrm{d}\hat{F} = -k! X_{\hat{F}}^I \lrcorner\, \mathrm{d}\phi^j \hat{r}_I \hat{r}_j$; the only non-trivial solutions come about from special values of the curvature of the torsion-free connection ω.

Chapter 3

The Charged Particle and the Charged Hamiltonian

Throughout our discussion of the n-symplectic charged particle, we will only concern ourselves with the case of flat space-time; the metric used throughout will be the Minkowski metric $\eta = \eta^{ab}\partial_a \otimes \partial_b$, where $\eta^{ab} = \eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ in inertial coordinates on $\mathbb{R}^4$. The corresponding rank 2 tensorial observable on the frame bundle has the local coordinate form $\hat{\eta} = \eta^{ab}\hat{\pi}_a\hat{\pi}_b$.

We begin with the rank 2 observable

$$\hat{c} = \left(\eta^{ab}\pi^i_a\pi^j_b - 2A^{ia}\pi^j_a + 2B^{ij}\right)\hat{r}_i\hat{r}_j. \tag{3.1}$$

We will interpret the functions A^{ia} to be a collection of n vector potentials so as to create a generalized Maxwell field tensor,

$$F^i_{ab} := \eta_{cb}A^{ic}_{,a} - \eta_{ca}A^{ic}_{,b}, \tag{3.2}$$

thus we will require that the A^{ia} not to be identically zero. The Hamiltonian vector fields defined by $\hat{c}$ are easily calculated to be, up to gauge freedom,

$$X^i_{\hat{c}} = \left(\eta^{as}\pi^i_a - A^{is}\right)\partial_s + \left(A^{ia}_{,k}\pi^j_a - B^{ij}_{,k}\right)\frac{\partial}{\partial \pi^j_k}. \tag{3.3}$$

We would like to study a single vector field in the distribution spanned by the $X^i_{\hat{c}}$ so that we may study its integral curves, from which we obtain equations of motion. This is accomplished by choosing an arbitrary but fixed $\alpha = \alpha_i\hat{r}^i \in \mathbb{R}^{n*} \setminus \{0\}$ to select the vector field $\alpha_i X^i_{\hat{c}}$. Using this same α we are able to map the observable $\hat{c}$ and its Hamiltonian distribution $X^i_{\hat{c}}$ to an observable

$\vec{c} := \psi_\alpha(\alpha_i \alpha_j \hat{c}^{ij})$ and a symplectic Hamiltonian vector field[1] $X_{\vec{c}}$ on the cotangent bundle via the ϕ_α map described at the end of §1.1. Before we do that, however, we introduce some notation that will be helpful throughout our discussion of the n-symplectic charged particle.

$$\begin{aligned} \pi_s &:= \alpha_i \pi^i_s \\ A^a &:= \alpha_i A^{ia} \\ F_{ab} &:= \alpha_i F^i_{ab}. \end{aligned}$$

Later, we will also make use of the momentum rest space. This is the vertical space orthogonal to π_s in the following sense: It was noted in §1.1 that $\pi_s(u) = p_s(\alpha_i e^i)$ for $u = (p, e_i) \in LM$ and p_s the standard momentum coordinates on M defined by the coordinates x^i on M. This is a single cotangent vector at $\pi(u) \in M$, and there are $n-1$ more cotangent vectors at this point that span the space of possible cotangent vectors at this point. The momentum rest space is this $n-1$ dimensional space at $u \in LM$ linearly independent from $\pi_s(u)$. The functions $\perp^j_k := \delta^j_k - \frac{1}{\alpha^2}\alpha^j \alpha_k$, where $\alpha^2 = \alpha^a \alpha_a = \eta^{ab}\alpha_a \alpha_b$, act as a projection operator on the coordinate functions π^k_m, projecting onto this momentum rest space. We now introduce the related notations

$$\begin{aligned} \perp \pi^j_k &:= \perp^j_a \pi^a_k \\ \perp F^j_{ab} &:= \perp^j_c F^c_{ab}. \end{aligned}$$

Now, we return to the distribution in equation (3.3) and apply $\psi_{\alpha *}$ to see that the corresponding (symplectic) Hamiltonian vector field on the cotangent bundle is

$$\psi_{\alpha *}(\alpha_i X^i_{\hat{c}}) = X_{\vec{c}} = (\eta^{as}\pi_a - A^s)\,\partial_s + \left(A^a_{,k}\pi_a - \alpha_i \alpha_j B^{ij}_{,k}\right)\frac{\partial}{\partial \pi_k} \tag{3.4}$$

We are using the notation $\pi_k = p_k$ as they are, essentially, interchangeable so long as we are careful about their domains. The integral curves of this vector field are governed by the equations

$$\dot{x}^s = \eta^{as}\pi_a - A^s \tag{3.5}$$

$$\dot{\pi}_k = A^a_{,k}\pi_a - \alpha_i \alpha_j B^{ij}_{,k} \tag{3.6}$$

These equations combine into

$$\ddot{x}_k = F_{kc}\dot{x}^c + A^a_{,k}A_a - \alpha_i \alpha_j B^{ij}_{,k} \tag{3.7}$$

[1] Although we use the notation $\vec{c}$ and $X_{\vec{c}}$, we cannot say that $X_{\vec{c}}$ is the symplectic Hamiltonian vector field associated with $\vec{c}$ due to the nontensorial nature of $\hat{c}$ [12].

If we were to choose the particular form $B^{ij} = \frac{1}{2}\eta_{ab}A^{ia}A^{jb}$ instead of leaving the functions B^{ij} arbitrary, then equation (3.7) simplifies to

$$\ddot{x}_k = F_{kc}\dot{x}^c \tag{3.8}$$

which is the standard Lorentz Force Law in flat space-time [5]. Our observable

$$\hat{c} = \left(\eta^{ab}\pi^i_a\pi^j_b - 2A^{ia}\pi^j_a + \eta_{ab}A^{ia}A^{jb}\right)\hat{r}_i\hat{r}_j = \eta^{ab}\left(\pi^i_a - A^{ik}\eta_{ka}\right)\left(\pi^j_b - A^{jm}\eta_{mb}\right)\hat{r}_i\hat{r}_j \tag{3.9}$$

can be seen to be a straighforward generalization of

$$H = \frac{1}{2m}(p_i - eA_i)^2 \tag{3.10}$$

the standard Hamiltonian (or observable) for a massive, charged[2] particle in flat space-time [9]. We will call $\hat{c}$ the charged n-symplectic Hamiltonian observable. It should be noted that the analysis and the end observable are nearly the same when curved space-time is considered (c.f. [2]).

Similarly, it has been shown [6] that one may instead modify the symplectic form in such a way that the dynamics of the free particle are equivalent to the dynamics of the charged particle under the standard symplectic form. We follow this method and define the charged n-symplectic form to be

$$\hat{\theta}_{\hat{c}} = \theta^i_{\hat{c}}\hat{r}_i = \left(\pi^i_j + \eta_{aj}A^{ia}\right)\mathrm{d}x^j\hat{r}_i \tag{3.11}$$

where, as above, the A^{ia} are functions on the base manifold. The charged n-symplectic form has the same non-degeneracy conditions as the standard n-symplectic form [2], and the Hamiltonian vector fields defined by the corresponding equation

$$\mathrm{d}\hat{\eta} = -2X^{(i}_{\hat{\eta}} \lrcorner\ \mathrm{d}\theta^{j)}_{\hat{c}}\hat{r}_i\hat{r}_j \tag{3.12}$$

are

$$X^i_{\hat{\eta}} = \eta^{as}\pi^i_a\partial_s + \left(\eta^{ab}\pi^{(i}_a F^{j)}_{kb} + T^{ij}_k\right)\frac{\partial}{\partial\pi^j_k} \tag{3.13}$$

Here, we have included the gauge term T^{ij}_k, as it will be useful to consider in the next section.

[2]In our discussions, we make no mention of the mass of the particle m nor of its electric charge e. We are considering particles in flat space-time with constant mass and constant electric charge. If these values are constant, the analysis and conclusions are virtually identical to what is presented here. One may interpret these results as being exactly correct for a particle in flat space-time with constant mass and electric charge, and unit charge-to-mass ratio $\frac{e}{m}$. A change in the charge-to-mass ratio will only change some results by a multiplicative factor proportional to $\frac{e}{m}$, and thus we choose to suppress mass and electric charge in our discussion.

Mapping this distribution to the cotangent bundle gives the vector field

$$\psi_{\alpha *}(\alpha_i X^i_{\hat{\eta}}) = X_{\vec{\eta}} = \eta^{as}\pi_a \partial_s + \eta^{ab} F_{kb}\pi_a \frac{\partial}{\partial \pi_k} \tag{3.14}$$

whose integral curves are given by

$$\dot{x}^s = \eta^{as}\pi_a \tag{3.15}$$

$$\dot{\pi}_k = \eta^{ab} F_{kb}\pi_a \tag{3.16}$$

which again combine to give the Lorentz Force Law. As the two methods provide equivalent results, we will primarily use the charged n-symplectic form in the calculations to follow.

3.1 General Equations of Motion

We would like to study the dynamics of the charged particle on the frame bundle to see what additional information is available to us. We begin by choosing some constant $\alpha = \alpha_i \hat{r}^i \in \mathbb{R}^{n*} \setminus \{0\}$. Then, $\alpha_i X^i_{\hat{\eta}}$ is an arbitrary vector field in the Hamiltonian distribution. The integral curves of this vector field are, from (3.13)

$$\dot{q}^s = \eta^{as}\pi_a \tag{3.17}$$

$$\dot{\pi}^j_k = \alpha_i \left(\eta^{ab}\pi^{(i}_a F^{j)}_{kb} + T^{ij}_k \right) \tag{3.18}$$

In order to allow these differential equations to combine as in the standard analysis and produce a Lorentz Force-type equation, we must single out the linear combination of equations $\dot{\pi}_k = \alpha_j \dot{\pi}^j_k$ from equation (3.18). This leaves $n-1$ equations from the set to be determined in order to fully describe the integral curves. These equations must lie in the space orthogonal to π_k, and so can be calculated by $\perp^j_a \dot{\pi}^a_k = \perp \dot{\pi}^j_k$. Even though the numbering index j takes values $1, 2, \ldots, n$, the set $\{\perp \dot{\pi}^1_k, \ldots, \perp \dot{\pi}^{(n)}_k\}$ only spans an $(n-1)$-dimensional space for any fixed k. So, the set of equations describing the $n^2 + n$ degrees of freedom for the integral curves of the Hamiltonian vector field $\alpha_i X^i_{\hat{\eta}}$ are

$$\begin{aligned} \dot{x}^s &= \eta^{as}\pi_a \\ \dot{\pi}_k &= \eta^{ab} F_{kb}\pi_a \\ \perp \dot{\pi}^r_s &= \perp^r_j \alpha_i \left(\eta^{ab}\pi^{(i}_a F^{j)}_{sb} + T^{ij}_s \right) \end{aligned} \tag{3.19}$$

In practice, one is free to choose an appropriate basis of $\perp \dot{\pi}^r_s$ in order to calculate the integral curves. The $\dot{\pi}$ equations, in whatever linearly independent combination is chosen, define the motion of legs of the momentum frame. The leg defined by the $\dot{\pi}_k$ equations will be known as the *primary leg* of the momentum frame, and those defined by $\perp \dot{\pi}^r_s$ will be known as *secondary legs* or the *momentum rest space.*

The $\dot{x}$ and $\dot{\pi}_a$ equations combine naturally in the same way as equations (3.16) to form a Lorentz Force-type law on LM. The primary difference between equations (3.19) on LM and equations (3.16) on T^*M, then, is the equations defining the motion of the momentum rest space. The equations exhibit two key features that we will explore: Gauge freedom, and an explicit symmetry in the terms. The gauge term T^{ij}_s cannot be removed (i.e. a global choice of gauge cannot be enforced) in the way shown in [11] or [13]; even though our observable $\hat{\eta}$ is tensorial, the charged n-symplectic form has caused the Hamiltonian distribution defined by $\hat{\eta}$ not to have the correct transformation property in order to be the distribution of a connection on the frame bundle. This can be seen most clearly in an examination of equation (3.12): The left-hand side transforms tensorially, so in order for the right-hand side to transform tensorially the vector fields $X^i_{\hat{\eta}}$ must transform in a way to offset the non-tensorial transformation of the charged n-symplectic form. This assures a non-tensorial transformation property of $X^i_{\hat{\eta}}$.

As we will see in §3.3.2, the persistence of gauge freedom in the momentum rest space is a natural property of n-symplectic geometry. A particular choice of gauge will be explored in §3.3.3. The explicit symmetry, on the other hand, leads quickly to very interesting features in the n-symplectic dynamics. The following section is devoted to a single example following [2] exploring the effects of this symmetry.

3.2 The Oscillotron

In this section, we will study special features of the general charged n-symplectic equations of motion (3.19). First, we will specify a generalized Maxwell field tensor F^i_{ab} and direction α such that the equations naturally lead one to the hopes of building a classical theory of spin $\frac{1}{2}$ particles. Then, we will specialize from a general manifold to $\mathbb{R}^4$ in order to show explicitly the major obstruction to such a theory; however, this obstruction also reveals interesting dynamics not seen on the cotangent bundle. We have dubbed the particle or observer that follows these motions an oscillotron. The dynamics of the oscillotron will be studied and some classical (non-relativistic) conclusions drawn. In §3.3.3, we will revisit the oscillotron in order to show that it may not be a realistic model.

For a simple example, we begin by choosing our generalized Maxwell field tensor F^i_{ab} such that it takes on non-zero values only when $i = 1$. The natural choice of direction within the distribution is now $\alpha = \hat{r}^1$. This means that the primary momentum leg is π^1_k, and a natural basis for the momentum rest space is $\{\pi^A_s | A = 2, \ldots, n\}$. Our vector field in the Hamiltonian distribution is

$$\alpha_i X^i_{\hat{\eta}} = \eta^{as}\pi^1_a\partial_s + \alpha_i\left(\eta^{ab}\pi^{(i}_a\delta^{j)}_1 F^1_{kb} + T^{ij}_k\right) \tag{3.20}$$

The general equations of motion, equations (3.19), become

$$\begin{aligned}
\dot{x}^s &= \eta^{as}\pi^1_a \\
\dot{\pi}^1_k &= \eta^{ab}F^1_{kb}\pi^1_a \\
\dot{\pi}^A_s &= \delta^A_j\delta^1_i\eta^{ab}\pi^{(i}_a\delta^{j)}_1 F^1_{sb} + \alpha_i T^{iA}_s, \quad A = 2,\dots,n
\end{aligned} \tag{3.21}$$

In order to examine the momentum rest space, we need to have some definite value for the gauge term. A convenient choice is $T^{ij}_k = 0$. We emphasize that this choice is merely for convenience, and not made following the methods described in [11] or [13]. We then expand the symmetry in the momentum rest frame on the indices i and j to see that

$$\dot{\pi}^A_s = \frac{1}{2}\eta^{ab}\pi^1_a F^A_{sb} + \frac{1}{2}\eta^{ab}\pi^A_a F^1_{sb} = \frac{1}{2}\eta^{ab}\pi^A_a F^1_{sb}. \tag{3.22}$$

The equations defining the primary and secondary legs of the momentum frame now have the same form save the difference by a factor of $\frac{1}{2}$. This difference is intrinsic in the n-symplectic geometry; it arises naturally from the symmetrization on two indices, which itself is due to the use of a rank 2 observable defining the Hamiltonian vector field. We will also see in §3.3.1 that the factor $\frac{1}{2}$ cannot be changed arbitrarily via gauge choice.

The momentum rest space is shown to move with velocity naturally one half that of the primary leg. Such a strong result in a simple but general sample calculation would lend creedance to the idea that a classical theory of spin $\frac{1}{2}$ particles is encoded in n-symplectic dynamics. To further this thought and pursuit, let us consider the simple circular motion of a charged particle in a constant magnetic field, with underlying manifold $\mathbb{R}^4$. For this, we simply choose $F^1_{23} = B$ where B is the strength of the magnetic field, and the remaining undetermined values of F^i_{ab} to be zero. In other words, we have

$$F^1_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.23}$$

The equations of motion are now

$$\begin{aligned}\dot{q}^s &= \eta^{as}\pi_a^1 \\ \dot{\pi}_k^1 &= \eta^{ab}F_{kb}^1\pi_a^1 \\ \dot{\pi}_s^A &= \frac{1}{2}\eta^{ab}\pi_a^A F_{sb}^1, \quad A = 2,3,4\end{aligned} \tag{3.24}$$

The only non-trivial $\dot{\pi}$ equations are $\dot{\pi}_2^1 = B\pi_3^1$, $\dot{\pi}_3^1 = -B\pi_2^1$, $\dot{\pi}_2^A = \frac{1}{2}B\pi_3^A$, and $\dot{\pi}_3^A = -\frac{1}{2}B\pi_2^A$, leading to solutions

$$\pi(s) = \begin{pmatrix} \pi_1^1(0) & \pi_2^1(0)\cos(Bs) + \pi_3^1(0)\sin(Bs) & \pi_3^1(0)\cos(Bs) - \pi_2^1(0)\sin(Bs) & \pi_4^1(0) \\ \pi_1^2(0) & \pi_2^2(0)\cos\left(\frac{Bs}{2}\right) + \pi_3^2(0)\sin\left(\frac{Bs}{2}\right) & \pi_3^2(0)\cos\left(\frac{Bs}{2}\right) - \pi_2^2(0)\sin\left(\frac{Bs}{2}\right) & \pi_4^2(0) \\ \pi_1^3(0) & \pi_2^3(0)\cos\left(\frac{Bs}{2}\right) + \pi_3^3(0)\sin\left(\frac{Bs}{2}\right) & \pi_3^3(0)\cos\left(\frac{Bs}{2}\right) - \pi_2^3(0)\sin\left(\frac{Bs}{2}\right) & \pi_4^3(0) \\ \pi_1^4(0) & \pi_2^4(0)\cos\left(\frac{Bs}{2}\right) + \pi_3^4(0)\sin\left(\frac{Bs}{2}\right) & \pi_3^4(0)\cos\left(\frac{Bs}{2}\right) - \pi_2^4(0)\sin\left(\frac{Bs}{2}\right) & \pi_4^4(0) \end{pmatrix} \tag{3.25}$$

We see explicitly that the primary and secondary legs of the momentum frame move along the same circular path with velocities differing by a factor of $\frac{1}{2}$. This would mean that after one orbit, the momentum rest space would be in a "negative" or "opposite" orientation to that in which it began, and it would require an additional orbit in order to return to its original orientation. Such an effect is often an illustration of or analogy for a spin $\frac{1}{2}$ particle. This intrinsic difference of $\frac{1}{2}$ has a more sinister consequence, as the next lemma and its corollary will show.

Lemma 3.1 *The determinant of the π matrix in* (3.25) *can be written as $C + A\cos\left(\frac{Bs}{2} + \phi\right)$, where C, A, and ϕ are constants depending only on the initial conditions $\pi_k^j(0)$.*

Proof: Let $M_{i,j}(K)$ be the $(i,j)^{\text{th}}$ minor of matrix K (the determinant of the matrix left after removing row i and column j of matrix K). Expanding by minors along the first row, we have

$$\det(\pi(s)) = \pi_1^1(s)M_{1,1}(\pi(s)) - \pi_2^1(s)M_{1,2}(\pi(s)) + \pi_3^1(s)M_{1,3}(\pi(s)) - \pi_4^1(s)M_{1,4}(\pi(s)).$$

A quick calculation shows that $\pi_1^1(s)M_{1,1}(\pi(s)) = \pi_1^1(0)M_{1,1}(\pi(0))$ and $\pi_4^1(s)M_{1,4}(\pi(s)) = \pi_4^1(0)M_{1,4}(\pi(0))$, whereas

$$M_{1,2}(\pi(s)) = M_{1,2}(\pi(0))\cos\left(\frac{Bs}{2}\right) - M_{1,3}(\pi(0))\sin\left(\frac{Bs}{2}\right) = \bar{A}\cos\left(\frac{Bs}{2} + \psi\right), \text{and}$$

$$M_{1,3}(\pi(s)) = M_{1,3}(\pi(0))\cos\left(\frac{Bs}{2}\right) + M_{1,2}(\pi(0))\sin\left(\frac{Bs}{2}\right) = \bar{A}\sin\left(\frac{Bs}{2} + \psi\right).$$

Also, $\pi^1_2(s) = \pi^1_2(0)\cos(Bs) + \pi^1_3(0)\sin(Bs) = \bar{D}\cos(Bs+\delta)$ and $\pi^1_3(s) = \pi^1_3(0)\cos(Bs) - \pi^1_2(0)\sin(Bs) = \bar{D}\sin(Bs+\delta)$. It is then clear that $\det(\pi(s)) = C + \bar{A}\bar{D}\cos\left(\frac{Bs}{2}+\phi\right)$, where

$$\begin{aligned}
C &= \pi^1_1(0)M_{1,1}(\pi(0)) - \pi^1_4(0)M_{1,4}(\pi(0)),\\
\bar{A} &= \sqrt{M_{1,2}(\pi(0))^2 + M_{1,3}(\pi(0))^2},\\
\bar{D} &= \sqrt{\pi^1_2(0)^2 + \pi^1_3(0)^2}, \text{and}\\
\phi &= \psi - \delta.
\end{aligned}$$

The only undetermined constants that appear above are the initial conditions $\pi^i_k(0)$; $M_{a,b}(\pi(0))$ is a function of the $\pi^i_j(0)$, and both ψ and δ are functions of $M_{a,b}$. $\square$

Corollary 3.2 *There exist initial conditions for which the path of a charged particle or observer in a constant magnetic field leaves the frame bundle $L\mathbb{R}^4$ at a finite time.*

3.3 Power of gauge freedom

In this section, we will review properties of totally symmetric indexed functions in order to gain a better understanding of the equivalence classes of Hamiltonian distributions.

3.3.1 How to arrange indices

The n-symplectic observables of rank 2 or greater define certain totally symmetric parts of their Hamiltonain distributions by (1.11), and the not-totally-symmetric parts are arbitrary. An equivalence relation on Hamiltonian distributions is then defined by two distributions being equal if these totally symmetric parts are equal. We are primarily interested in how these arbitrary terms interact with and can affect the uniquely-defined terms of the vector fields, thus characterizing different representatives of the same equivalence class.

We begin with a few lemmas.

Lemma 3.3 *Let f^{I_k} and T^{I_k} be indexed collections of functions (using multi-index notation, $I_k = i_1 i_2 \ldots i_k$) on a common, arbitrary domain $\mathcal{D}$ with $k \geq 2$, such that $f^{I_k} = f^{(I_k)}$ and $T^{(I_k)} = 0$. For any point $u \in \mathcal{D}$, the equation $f^{I_k}(u) + T^{I_k}(u) = \gamma f^{I_k}(u)$ for some real number γ implies either $\gamma = 1$ or $T^{I_k} = 0$.*

Proof: This equation can be rewritten as $T^{I_k}(u) = (\gamma-1)f^{I_k}(u)$. Let $u \in \mathcal{D}$ be a point such that $f^{I_k}(u) \neq 0$. Since $T^{(I_k)} = 0$, we are left with $(\gamma-1)f^{I_k}(u) = 0$, and thus $\gamma = 1$. If, on the other hand, u is chosen such that $f^{I_k}(u) = 0$, then immediately $T^{I_k}(u) = 0$. $\square$

This lemma shows us that the "intrinsic $\frac{1}{2}$ difference" seen in the equations of motion for the charged particle (equations (3.21) and (3.22)) is dependent upon the geometry and not the gauge freedom. The factor of $\frac{1}{2}$ appears primarily due to our choices of F^i_{ab} and α, along with the symmetry inherent in the vertical term. Between equations (3.21) and equation (3.22), a choice of gauge was made. Lemma 3.3 shows that the $\frac{1}{2}$ in equation (3.22) would remain $\frac{1}{2}$ no matter what choice of gauge was made. This also underscores the choice of gauge being made out of convenience; it was convenient to let the gauge term disappear and the factor of $\frac{1}{2}$ stand alone.

Working with totally symmetric observables which are polynomial in the πs, symmetry is enforced on each term individually. This symmetry, which is passed to the corresponding Hamiltonian vector fields, can be affected by choice of gauge. The following lemma explains.

Lemma 3.4 *Let f^{I_k} and T^{I_k} be indexed collections of functions on a common domain $\mathcal{D}$ (using multi-index notation, $I_k = i_1 i_2 \dots i_k$) with $k \geq 2$, such that $T^{(I_k)} = 0$. For σ any permutation of $\{1, 2, \dots, k\}$, there is a choice of functions T^{I_k} such that $f^{(I_k)} + T^{I_k} = f^{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(k)}}$.*

Proof: Let σ' be another permutation of $\{1, 2, \dots, k\}$. Then

$$T^{I_k} = \frac{1}{k!}\left(f^{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(k)}} - f^{i_{\sigma'(1)} i_{\sigma'(2)} \cdots i_{\sigma'(k)}}\right) \tag{3.26}$$

satisfies the condition $T^{(I_k)} = 0$. The term $f^{(I_k)}$ can be written as the sum

$$\frac{1}{k!}\sum_{\tau} f^{i_{\tau(1)} i_{\tau(2)} \cdots i_{\tau(k)}} \tag{3.27}$$

where τ ranges over all permutations of $\{1, 2, \dots, k\}$. Adding T^{I_k} (as above) to this sum has the effect of replacing the term $f^{i_{\sigma'(1)} i_{\sigma'(2)} \cdots i_{\sigma'(k)}}$ with the term $f^{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(k)}}$. Repeating this process for every choice of σ' (or, equivalently, choosing $T^{I_k} = \frac{1}{k!}\sum_{\tau}\left(f^{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(k)}} - f^{i_{\tau(1)} i_{\tau(2)} \cdots i_{\tau(k)}}\right)$) leaves in the sum $k!$ copies of the single term $f^{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(k)}}$. Thus, $f^{(I_k)} + T^{I_k} = f^{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(k)}}$ as desired. $\square$

Another way to view this lemma is that it makes concrete the idea presented previously that any f^{I_k} can be decomposed into a totally-symmetric term $f^{(I_k)}$ plus a term whose totally symmetric part is zero, T^{I_k}. These sums are exactly the ones encountered when dealing with this n-symplectic gauge freedom.

3.3.2 Where gauge freedom exists, where it is limited

It has been discussed thoroughly that the structure equation (1.11) does not uniquely define Hamiltonian vector fields $X^I_{\hat{f}}$ for observables $\hat{f}$ of rank 2 or greater. The gauge freedom can be removed by moving to the cotangent bundle by means of the ψ_α map [12]. Remaining on the frame bundle, it has been shown [13] that the choice of gauge for the Hamiltonian distribution of a tensorial observable can be fixed globally by the choice of a torsion-free linear connection. If such a choice cannot be made, then there will always remain some level of gauge freedom in the equations of motion on the frame bundle defined by the Hamiltonian vector fields of an observable of rank 2 or greater.

Take, for example, the rank 3 tensorial observable $\hat{f} = f^{abc}\hat{\pi}_a\hat{\pi}_b\hat{\pi}_c$. The Hamiltonian vector fields defined by this observable are

$$X^{ij}_{\hat{f}} = \frac{1}{2}\left(f^{abs}\pi^i_a\pi^j_b + T^{ijs}\right)\partial_s - \frac{1}{3!}\left(f^{abc}_{,s}\pi^i_a\pi^j_b\pi^r_c + T^{ijr}_s\right)\frac{\partial}{\partial\pi^r_s} \tag{3.28}$$

Selecting an arbitrary $\alpha = \alpha_i r^i \in \mathbb{R}^{n*} \setminus \{0\}$, we then look at the integral curves of $\alpha_i\alpha_j X^{ij}_{\hat{f}}$; they are defined by the equations

$$2\dot{x}^s = f^{abs}\pi_a\pi_b \tag{3.29}$$

$$3!\dot{\pi}^r_s = -f^{abc}_{,s}\pi_a\pi_b\pi^r_c - \alpha_i\alpha_j T^{ijr}_s \tag{3.30}$$

The second set of equations can be split into two more sets of equations,

$$\begin{aligned} 3!\dot{\pi}_s &= -f^{abc}_{,s}\pi_a\pi_b\pi_c \\ 3! \perp \dot{\pi}^k_s &= -f^{abc}_{,s}\pi_a\pi_b \perp \pi^k_c - \alpha_i\alpha_j \perp^k_r T^{ijr}_s, \end{aligned} \tag{3.31}$$

where again $\perp^k_r$ is the projection operator into the orthogonal subspace to π_s, and we continue to use the shorthand $\pi_a = \pi^i_a\alpha_i$ and $\perp \pi^k_c = \perp^k_r \pi^r_c$. The second set of these equations represents $n-1$ sets of equations; the exact equations depend upon what basis for the momentum rest space is chosen. The sum $\alpha_i\alpha_j \perp^k_r T^{ijr}_s$ in these equations is nonzero for many choices of gauge T^{ijr}_s. This may be more clearly seen by writing $\perp^k_r = \delta^k_r - \frac{1}{\alpha^2}\alpha^k\alpha_r$, and then

$$\alpha_i\alpha_j \perp^k_r T^{ijr}_s = \alpha_i\alpha_j T^{ijk}_s - \frac{1}{\alpha^2}\alpha^k\alpha_i\alpha_j\alpha_r T^{ijr}_s = \alpha_i\alpha_j T^{ijk}_s \tag{3.32}$$

since $T^{(ijr)}_s = 0$.

This process can be carried out for any observable of rank 2 or greater with the same result. Notice that if we had chosen two arbitrary constants $\alpha, \beta \in \mathbb{R}^{n*} \setminus \{0\}$ such that $\alpha \neq \beta$, then there would be a gauge term included in the $\dot{q}^s$ and $\dot{\pi}_s$ equations of motion of the vector field $\alpha_i\beta_j X^{ij}_{\hat{f}}$ in the Hamiltonian distribution. Furthermore, if we had separated equation (3.30) in any different manner than equations (3.31), the gauge term would have remained in every equation. Therefore, we see again what was seen in the construction of the ψ_α map: The classic equations of motion (those of x^s and $\pi_s = p_s$ on the cotangent bundle) are independent of this n-symplectic gauge freedom. Moreover, we see that without some global choice of gauge, the arbitrariness will persist in the equations of motion on the frame bundle (specificially, in the momentum rest space).

Where this n-symplectic gauge freedom exists, the ability to manipulate the other (symmetric) terms is limited. Lemma 3.4 shows how one may use the gauge term to fix an order of indices in these symmetric terms. The greatest limitation to the ability to arrange indices comes from inherent symmetries in the functions to be considered. For any observable of rank $k \geq 2$, the kth degree term has a totally symmetric component function, and no assumption of symmetry in the lower-degree component functions is made *a priori*. Consider the general rank 2 observable $\hat{f} = \left(f^{ab}\pi^i_a\pi^j_b + 2A^{ai}\pi^j_a + 2B^{ij} \right) \hat{r}_i\hat{r}_j$ whose Hamiltonian distribution is

$$X^i_{\hat{f}} = \left(f^{as}\pi^i_a + A^{si} \right) \partial_s - \frac{1}{2} \left(f^{ab}_{,k}\pi^i_a\pi^j_b + 2A^{a(i}_{,k}\pi^{j)}_a + 2B^{(ij)}_{,k} + T^{ij}_k \right) \frac{\partial}{\partial \pi^j_k} \tag{3.33}$$

Lemma 3.4 shows us that we can choose T^{ij}_k such that the vertical portion is equal to

$$-\frac{1}{2} \left(f^{ab}_{,k}\pi^i_a\pi^j_b + 2A^{ai}_{,k}\pi^j_a + 2B^{(ij)}_{,k} \right) \quad \text{or to} \quad -\frac{1}{2} \left(f^{ab}_{,k}\pi^i_a\pi^j_b + 2A^{aj}_{,k}\pi^i_a + 2B^{ij}_{,k} \right).$$

The fact that $f^{ab} = f^{ba}$ precludes us from arranging that term's indices with a choice of gauge T^{ij}_k for two reasons: One, the terms $f^{ab}_{,k}\pi^i_a\pi^j_b$ and $f^{ab}_{,k}\pi^j_a\pi^i_b$ are equal. And two, the choice of T^{ij}_k required by the lemma is equal to 0. It would appear that the highest-degree term (the tensorial term) remains invariant under the n-symplectic gauge, but this is not entirely true. Notice first that the sum $f^{ab}\pi^i_a\pi^j_b$ can be written equivalently as $f^{ab}\pi^c_a\pi^d_b\delta^i_c\delta^j_d$. Next, let's write these terms out as the explicit sum

$$f^{ab}\pi^1_a\pi^1_b\delta^i_1\delta^j_1 + \ldots + f^{ab}\pi^1_a\pi^{(n)}_b\delta^i_1\delta^j_{(n)} + f^{ab}\pi^2_a\pi^1_b\delta^i_2\delta^j_1 + \ldots + f^{ab}\pi^{(n)}_a\pi^{(n)}_b\delta^i_{(n)}\delta^j_{(n)}. \tag{3.34}$$

While the entire sum is symmetric in the indices i and j, most of the individual terms in the sum (e.g. $f^{ab}\pi^1_a\pi^2_b\delta^i_1\delta^j_2$) are not. Then by Lemma 3.4, most of the terms in the sum can have the positions of the i and j index swapped by gauge choice. The only terms in this sum which are wholly independent of gauge are the terms $f^{ab}\pi^{(k)}_a\pi^{(k)}_b\delta^i_{(k)}\delta^j_{(k)}$ for a fixed but arbitrary k. The expansion (3.34) can be applied to any degree term of any rank observable, and can be useful whenever there are symmetries inherent in the component functions. This does not mean that the other terms can be entirely gauged away, only that their indices can be manipulated by gauge choice; for example, $f^{ab}\pi^1_a\pi^2_b\delta^i_1\delta^j_2 + f^{ab}\pi^1_a\pi^2_b\delta^i_2\delta^j_1$ can be gauged to $2f^{ab}\pi^1_a\pi^2_b\delta^i_1\delta^j_2$ or $2f^{ab}\pi^1_a\pi^2_b\delta^i_2\delta^j_1$.

3.3.3 Effect of arranging indices, charged particle example

We have just seen how and when we can arrange indices of terms in the n-symplectic Hamiltonian vector fields using the gauge freedom. Now, we will explore some of the consequences of choices of orders of indices.

This first example has limited use and scope in the standard study of n-symplectic geometry, but is illustrative nonetheless. Consider again the rank 3 tensorial observable with Hamiltonian vector fields (3.28). Expand $f^{abs}\pi^i_a\pi^j_b$ and $f^{abc}_{,s}\pi^i_a\pi^j_b\pi^r_c$ in the same way as (3.34), and choose as gauge terms $T^{ijs} = f^{abs}\pi^1_a\pi^2_b\left(\delta^i_2\delta^j_1 - \delta^i_1\delta^j_2\right)$ and $T^{ijr}_s = f^{abc}_{,s}\pi^1_a\pi^2_b\pi^r_c\left(\delta^i_2\delta^j_1 - \delta^i_1\delta^j_2\right)$. Then, $X^{12}_{\hat{f}} = 0$. There are three important things to note about this construction: First, only one vector field in the distribution has been gauged to zero. The entire distribution cannot simultaneously be gauged to zero in general. Second, it requires two upper indices on the Hamiltonian vector field, and thus it cannot be performed for observables of rank 1 or 2. And third, the values for the two upper indices must be different. This is not normally encountered by itself; in applications, a single $\alpha \in \mathbb{R}^{n*} \setminus \{0\}$ is chosen in order to select a vector field from the distribution, and $\alpha_i\alpha_j X^{ij}_{\hat{f}} \neq X^{12}_{\hat{f}}$.

In §3.2, we showed that certain choices bring to light an intrinsic difference of a factor of $\frac{1}{2}$ in the motion of the primary and secondary legs of the momentum frame of the charged n-symplectic observable in flat space. While such a difference seems promising for the development of a classical theory of spin, it also leads directly to singularities in the motion of the particle/observer. Returning to equation (3.20) with our choice of the generalized Maxwell field tensor given in equation (3.23), we use Lemma 3.4 to choose a gauge T^{ji}_k other than zero such that

$$\alpha_i X^i_{\hat{\eta}} = \eta^{as}\pi^1_a\partial_s + \pi^j_a F^1_{kb}\eta^{ab}\frac{\partial}{\partial\pi^j_k} \tag{3.35}$$

This choice of gauge eliminates the difference of $\frac{1}{2}$ between the equations of motion of the primary momentum π^1_s and the momentum rest space π^A_s. Furthermore, this choice of gauge has eliminated the possibility of singularities in the motion on the frame bundle; the motion in the π^a_b is given by the solutions

$$\pi(s) = \begin{pmatrix} \pi^1_1(0) & \pi^1_2(0)\cos(Bs) + \pi^1_3(0)\sin(Bs) & \pi^1_3(0)\cos(Bs) - \pi^1_2(0)\sin(Bs) & \pi^1_4(0) \\ \pi^2_1(0) & \pi^2_2(0)\cos(Bs) + \pi^2_3(0)\sin(Bs) & \pi^2_3(0)\cos(Bs) - \pi^2_2(0)\sin(Bs) & \pi^2_4(0) \\ \pi^3_1(0) & \pi^3_2(0)\cos(Bs) + \pi^3_3(0)\sin(Bs) & \pi^3_3(0)\cos(Bs) - \pi^3_2(0)\sin(Bs) & \pi^3_4(0) \\ \pi^4_1(0) & \pi^4_2(0)\cos(Bs) + \pi^4_3(0)\sin(Bs) & \pi^4_3(0)\cos(Bs) - \pi^4_2(0)\sin(Bs) & \pi^4_4(0) \end{pmatrix}$$

The determinant of this matrix is constant. On the other hand, we may use Lemma 3.4 to choose a gauge such that

$$\alpha_i X^i_{\hat{\eta}} = \eta^{as}\pi^1_a\partial_s + \pi^1_a F^j_{kb}\eta^{ab}\frac{\partial}{\partial \pi^j_k} \tag{3.36}$$

The equations of motion on the frame bundle now become rather interesting again. The differential equations are

$$\begin{aligned} \dot{q}^s &= \eta^{as}\pi^1_a \\ \dot{\pi}^1_s &= \pi^1_a F^1_{sb}\eta^{ab} \\ \dot{\pi}^A_s &= 0, A = 2, 3, 4. \end{aligned} \tag{3.37}$$

The solution $\pi(s)$ matrix is

$$\pi(s) = \begin{pmatrix} \pi^1_1(0) & \pi^1_2(0)\cos(Bs) + \pi^1_3(0)\sin(Bs) & \pi^1_3(0)\cos(Bs) - \pi^1_2(0)\sin(Bs) & \pi^1_4(0) \\ \pi^2_1(0) & \pi^2_2(0) & \pi^2_3(0) & \pi^2_4(0) \\ \pi^3_1(0) & \pi^3_2(0) & \pi^3_3(0) & \pi^3_4(0) \\ \pi^4_1(0) & \pi^4_2(0) & \pi^4_3(0) & \pi^4_4(0) \end{pmatrix},$$

and it is apparent once again that there can exist singularities in the equations of motion. As an example, choose $\pi^1_3(0) = \pi^2_2(0) = \pi^3_2(0) = \pi^4_2(0) = 0$, and $\pi^1_2(0) = 1$ for a singularity to occur at $s = \frac{\pi}{2B}$. With singularities so closely tied to both gauge freedom and initial conditions, the oscillotron would seem to be an appropriate model only in very specific instances.

Chapter 4

A Symplectic Submanifold of LM

In this chapter, we will explore a special subbundle $B_1 \subset LM$. The importance of this subbundle was first mentioned in [14], where it was shown that B_1 is both a symplectic and an n-symplectic manifold. B_1 is, in general, a local slice of LM. In order to realize B_1 as a coordinate slice, we shall now restrict our manifold M to be an n-dimensional Euclidean space $\mathbb{R}^n$. After a brief review of the definition of B_1 and its algebra of n-symplectic observables $\mathfrak{b}_1$, we will show how one can examine the dynamics upon B_1 of n-symplectic observables not in the algebra $\mathfrak{b}_1$, examine these dynamics to discover and interpret new structures not seen in symplectic geometry, discover multiple Kaluza-Klein-type theories encoded naturally into the structure of the Hamiltonian distributions on B_1, and discover that standard motions on B_1 develop singularities in much the same way as we saw with the charged n-symplectic observable.

4.1 Definition of B_1 and its Algebra of Observables

Norris and Brown defined B_1 as the coordinate slice on $L\mathbb{R}^n$ given by

$$\pi^A_b = \delta^A_b, \quad A = 2, 3, \ldots, n \tag{4.1}$$

That is, all the points in $u \in L\mathbb{R}^n$ such that $\pi^A_b(u) = \delta^A_b$ for $A = 2, \ldots, n$. The slice B_1 is not unique. There are $n-1$ other slices B_k that can be defined similarly. Also, in the notation of the previous sections, for any choice of $\alpha \in \mathbb{R}^{n*} \setminus \{0\}$ we can define the slice B_α as all the points of $L\mathbb{R}^n$ such that $\bar{\pi}^A_b = \delta^A_b$, where the $\bar{\pi}^A_b$ form a basis for $\perp \pi^a_b$, the vertical space orthogonal to $\pi_s = \alpha_i \pi^i_s$. Any of these slices will behave equivalently to B_1; the only notable feature separating B_1 from the other similar slices is the fact that we have chosen to use it explicitly.

Recall that every point $u \in LM$ is written as a pair (p, e_i) where the e_i form a linear frame for the tangent space at $p \in M$. This means that $\pi^i_j(u)$ is non-degenerate as a matrix.

By identifying π^1_k with the standard momentum coordinates p_k on $T^*\mathbb{R}^n$, we see that B_1 is naturally isomorphic to $T^*\mathbb{R}^n$ minus all points where $p_1 = 0$. This means that B_1 is a manifold of dimension $2n$, and thus B_1 has a natural symplectic structure. The soldering form $\hat{\theta}$ pulls back to and $\mathrm{d}\hat{\theta}$ is still non-degenerate on B_1, so B_1 is an n-symplectic submanifold of $L\mathbb{R}^n$ as well. It will be this combination of symplectic and n-symplectic character that produces rich and interesting structures on B_1.

As B_1 is isomorphic to $T^*\mathbb{R}^n$ minus the points where $p_1 = 0$, we will not need to make use of the ψ_α map described in and around equation (1.20) in order to examine our Hamiltonian vector fields upon $T^*\mathbb{R}^n$. We still will want to select a single vector field from a Hamiltonian distribution, so we can accomplish this by choosing constants C_I and examining the vector field $C_I X^I_{\hat{f}}$. Often, however, we will forego mentioning C_I and simply describe the vector field by choices of indices (e.g. $X^{112}_{\hat{f}}$).

A vector tangent to $B_1 \subset L\mathbb{R}^n$ must have the form $X = X^s\partial_s + X^1_s\frac{\partial}{\partial\pi^1_s}$ in our local coordinates (x^i, π^r_s). We define $\mathfrak{b}_1$ to be the algebra of observables of B_1; that is, all the n-symplectic observables defined on B_1 such that for every Hamiltonian vector field defined by the observable, some member of the equivalence class of that Hamiltonian vector field is tangent to B_1 (or, in the case of rank 1 observables, the single Hamiltonian vector field is tangent to B_1). This algebra does not contain every n-symplectic observable. A special choice was made in defining B_1, and only certain observables can have their Hamiltonian distributions gauged to be tangent to the slice.

The Hamiltonian vector fields of the observables $\hat{r}_1, \hat{x}^a_1$, and $\hat{\pi}_b$ are easily seen to be tangent to B_1 (see Table 1.2), so they are in $\mathfrak{b}_1$. Norris and Brown [14] use these three observables as the basis of a polynomial algebra of observables they called b_1, a "basic algebra" for quantization, and then showed that every observable in b_1 can have its Hamiltonian distribution gauged to be tangent to B_1. This basic algebra contains many observables whose Hamiltonian distributions have useful properties, so we will generally restrict our consideration of observables in $\mathfrak{b}_1$ to those in b_1.

4.2 Reduction To and Recovery From B_1

The advantages of an explicit symplectic structure and a simplified algebra of observables make working on the slice B_1 an appealing option to working on all of $L\mathbb{R}^n$, but it becomes clear quickly that $\mathfrak{b}_1$ is a very restrictive algebra. An important observable in n-symplectic geomtery is the metric observable, the rank 2 tensorial observable $\hat{g} = g^{ab}\hat{\pi}_a\hat{\pi}_b$ on $L\mathbb{R}^n$ determined by the metric tensor $\vec{g}$ on $\mathbb{R}^n$. $\hat{g}$ will not, however, be in the subalgebra $\mathfrak{b}_1$ unless the component functions g^{ab} are constant. In this section, we will show how to reproduce the effects of $\hat{g}$ upon B_1 using observables in $\mathfrak{b}_1$. Several methods will be investigated, and utility of the corresponding Hamiltonian vector fields and with the Poisson bracket will be compared.

Consider a smooth metric tensor $\vec{g}$ whose component functions g^{ab} have a (convergent) power series expansion $g^{ab} = \lambda^{ab} + \lambda^{ab}_{c}x^c + \lambda^{ab}_{cd}x^c x^d + \ldots$, where each $\lambda^{ab}_{I_r}$ are real constants. In practice, the vast majority of metrics considered are at least locally somewhere analytic (as opposed to being nowhere analytic), so the methods described in this section will be generally useful in practice, at least locally. We define

$$\begin{aligned} g_0 &:= \lambda^{ab}\hat{\pi}_a\hat{\pi}_b \\ g_1 &:= \lambda^{ab}_{c}x^c\hat{\pi}_a\hat{\pi}_b \\ g_2 &:= \lambda^{ab}_{cd}x^c x^d\hat{\pi}_a\hat{\pi}_b \\ &\quad etc. \end{aligned}$$

We also define the components[1] $g^{ab}_r := \lambda^{ab}_{I_r}x^{I_r}$ so that we may write[2] $\hat{g} = \sum\limits_r g_r = \sum\limits_r g^{ab}_r\hat{\pi}_a\hat{\pi}_b$. To each of the rank 2 observables g_r, there is a corresponding rank $r+2$ observable $\hat{g}_r \in \mathfrak{b}_1$:

$$\begin{aligned} \hat{g}_0 &:= \lambda^{ab}\hat{\pi}_a\hat{\pi}_b \\ \hat{g}_1 &:= \lambda^{ab}_{c}\hat{x}^c_1\hat{\pi}_a\hat{\pi}_b \\ \hat{g}_2 &:= \lambda^{ab}_{cd}\hat{x}^c_1\hat{x}^d_1\hat{\pi}_a\hat{\pi}_b \\ &\quad etc. \end{aligned}$$

Notice that we may also write $\hat{g}_k = g_k\hat{r}_{1_k}$ (so $\hat{g}_0 = g_0$, $\hat{g}_1 = g_1\hat{r}_1$, $\hat{g}_2 = g_2\hat{r}_1\hat{r}_1$, etc.), and that the two sequences of observables behave similarly under the Poisson bracket.

Lemma 4.1 *Let $\hat{h}$ be an arbitrary but fixed observable, and let g_k and $\hat{g}_k$ be defined as above. Then, $\{\hat{g}_k, \hat{h}\} = \{g_k, \hat{h}\}\hat{r}_{1_k}$.*

Proof: Since the Poisson bracket is a derivation[11], we may write

$$\{\hat{g}_k, \hat{h}\} = \{g_k\hat{r}_{1_k}, \hat{h}\} = \{g_k, \hat{h}\}\hat{r}_{1_k} + g_k\{\hat{r}_{1_k}, \hat{h}\}$$

The Poisson bracket of $\hat{r}_1$ with any observable is identically zero, so $\{\hat{g}_k, \hat{h}\} = \{g_k, \hat{h}\}\hat{r}_{1_k}$ as desired. □

If $\hat{h}$ is a rank k observable, then $\{\hat{g}_r, \hat{h}\}$ is an observable of rank $k+r+1$ and $\{g_r, \hat{h}\}$ is an observable of rank $k+1$. By the lemma above, the only difference between the totally

[1] The g^{ab}_r are referred to as components as they serve a similar purpose to the $\hat{f}^{I_r}$ when we write $\hat{f} = \hat{f}^{I_r}\hat{r}_{I_r}$ for a general rank r observable.

[2] We leave the summations unadorned when applicable to note that the expansion $g^{ab} = \lambda^{ab} + \lambda^{ab}_{c}x^c + \ldots$ may be finite or infinite.

symmetric functions $\{\hat{g}_r, \hat{h}\}^{I_{k+r+1}}$ and $\{g_r, \hat{h}\}^{I_{k+1}}$ is r Kroneker δ's. Explicitly,

$$\{\hat{g}_r, \hat{h}\}^{i_1 i_2 \ldots i_{k+r+1}} = \{g_r, \hat{h}\}^{(i_1 i_2 \ldots i_{k+1}} \delta_1^{i_{k+2}} \ldots \delta_1^{i_{k+r+1})}$$

This means that each $\{\hat{g}_r, \hat{h}\}^{I_{k+1} 1_r} = \{\hat{g}_r, \hat{h}\}^{I_{k+1} 11\ldots 1}$ is proportional to $\{g_r, \hat{h}\}^{I_{k+1}}$; however, the constant of proportionality is different for different values of the multi-index $I_{k+1} = (i_1, i_2, \ldots, i_{k+1})$. As an example to show the difference,

$$\begin{aligned} \{\hat{g}_r, \hat{h}\}^{1_{k+r+1}} &= \{g_r, \hat{h}\}^{1_{k+1}} \\ \{\hat{g}_r, \hat{h}\}^{21_{k+r}} = \{\hat{g}_r, \hat{h}\}^{211\ldots 1} &= \frac{k+1}{k+r+1} \{g_r, \hat{h}\}^{21_k} \end{aligned} \tag{4.2}$$

If we write $\{\hat{g}_r, \hat{h}\}^{I_{k+1} 1_r} = \gamma \{g_r, \hat{h}\}^{1_{k+1}}$, then the constant of proportionality γ depends on the values of r and the multi-index I_{k+1}, and it is a straight-forward counting argument to calculate each $\gamma(r, I_{k+1})$. All of this leads to the following lemma.

Lemma 4.2 *For an arbitrary observable $\hat{h}$ and the metric observable $\hat{g}$ with a series expansion, we are able to calculate the action $\{\hat{g}, \hat{h}\}$ using either the actions $\{\hat{g}_r, \hat{h}\}$ or $\{g_r, \hat{h}\}$ of observables $\hat{g}_r$ and g_r in $\mathfrak{b}_1$.*

Proof: From the fact that $\hat{g} = \sum_r g_r$ and the linearity of the Poisson bracket, we have $\{\hat{g}, \hat{h}\} = \sum_r \{g_r, \hat{h}\}$. The fact that $\{\hat{g}_r, \hat{h}\}^{I_{k+1} 1_r} = \gamma(r, I_{k+1}) \{g_r, \hat{h}\}^{1_{k+1}}$ allows us to write

$$\{\hat{g}, \hat{h}\} = \sum_r \frac{1}{\gamma(r, I_{k+1})} \{\hat{g}_r, \hat{h}\}^{I_{k+1} 1_r} \hat{r}_{I_{k+1}}$$

thus completing the proof[3].□

[3] One argument about this proof is that it seems to "sweep under the rug" the fact that the series expansion of g^{ab} may be infinite, not a finite sum or polynomial expansion, and the Poisson bracket is not explicitly required to be countably linear in its arguments. Assume g^{ab} has an infinite series expansion $\sum_{r=0}^{\infty} \lambda_{I_r}^{ab} x^{I_r}$. By definition of the Poisson bracket,

$$\begin{aligned} \{\hat{g}, \hat{h}\} &= -k! X_{\hat{h}}^{I_{k-1}} (\pi_a^{j_1} \pi_b^{j_2} g^{ab}) \hat{r}_{I_{k-1} J_2} \\ &= -k! X_{\hat{h}}^{I_{k-1}} \left(\sum_{r=0}^{\infty} \pi_a^{j_1} \pi_b^{j_2} \lambda_{I_r}^{ab} x^{I_r} \right) \hat{r}_{I_{k-1} J_2} \\ &= \left(\sum_{r=0}^{\infty} -k! X_{\hat{h}}^{I_{k-1}} (\pi_a^{j_1} \pi_b^{j_2} \lambda_{I_r}^{ab} x^{I_r}) \right) \hat{r}_{I_{k-1} J_2} \\ &= \sum_{r=0}^{\infty} \{g_r, \hat{h}\} \end{aligned}$$

The Hamiltonian vector fields $X^{I_{r+1}}_{\hat{g}_r}$ of each observable $\hat{g}_r \in \mathfrak{b}_1$ are easily calculated to be

$$X^i_{\hat{g}_0} = \lambda^{as}\pi^i_a\partial_s \tag{4.3}$$

$$(r+2)!X^{I_{r+1}}_{\hat{g}_r} = 2g^{as}_r\delta^{(I_r}_{1_{r-1}}\pi^{i_{r+1})}_a\partial_s - \frac{\partial g^{ab}_r}{\partial q^s}\delta^{(I_{r-1}}_{1_r}\pi^{i_r}_a\pi^{i_{r+1})}_b\frac{\partial}{\partial\pi^1_s} \tag{4.4}$$

Pieces of the (series expansion of) the metric $\hat{g}$ and its Hamiltonian vector fields can be seen in these vector fields, but in order to recover the dynamics of $\hat{g}$ we must combine all of these distributions in the proper way. We can relate the sum of the observables $\hat{g}_0$ through $\hat{g}_r$ for some finite r to a sum of their Hamiltonian vector fields via the structure equation (1.11) if each observable is raised to rank $r+2$. This is accomplished by tensoring in (multiplying) observables $\hat{r}_1 \in \mathfrak{b}_1$. The resulting sum of structure equations is

$$\begin{aligned}\mathrm{d}(\hat{g}_0\hat{r}_{1_r}&+\hat{g}_1\hat{r}_{1_{r-1}} + \ldots + \hat{g}_r)\\ &= -(r+2)!\left(X_{\hat{g}_r} + \frac{(r+1)!}{(r+2)!}X_{\hat{g}_{r-1}} + \ldots + \frac{2}{(r+2)!}X_{\hat{g}_0}\right) \lrcorner\ \mathrm{d}\theta \end{aligned}\tag{4.5}$$

$$\begin{aligned}\mathrm{d}\Big(\big[g^{ab}_0 &+ g^{ab}_1 + \ldots + g^{ab}_r\big]\delta^{(I_r}_{1_r}\pi^{i_{r+1}}_a\pi^{i_{r+2})}_b\Big)\\ &= -\Big((r+2)!X^{(I_{r+1}}_{\hat{g}_r} + (r+1)!X^{(I_r}_{\hat{g}_{r-1}}\delta^{i_{r+1}}_1 + \ldots + 2X^{(i_1}_{\hat{g}_0}\delta^{I_r}_{1_r}\Big) \lrcorner\ \mathrm{d}\theta^{i_{r+2})}\end{aligned}\tag{4.6}$$

From this, it would seem easier to create a single new rank $r+2$ observable

$$\tilde{g}_r := \sum_{t=0}^{r}\hat{g}_t\hat{r}_{1_{r-t}} \tag{4.7}$$

with components $\tilde{g}^{ab}_r = \sum\limits_{t=0}^{r} g^{ab}_t$. Using equations (4.6) and (4.4), the related Hamiltonian distribution (without gauge terms) is

$$(r+2)!X^{I_{r+1}}_{\tilde{g}_r} = 2\tilde{g}^{ab}_r\delta^{(I_r}_{1_{r-1}}\pi^{i_{r+1})}_a\partial_s - \frac{\partial\tilde{g}^{ab}_r}{\partial x^s}\delta^{(I_{r-1}}_{1_r}\pi^{i_r}_a\pi^{i_{r+1})}_b\frac{\partial}{\partial\pi^1_s}. \tag{4.8}$$

We might be led to conclude that the Hamiltonian distribution $X^{I_{r+1}}_{\tilde{g}_r}$ is proportional to $X^i_{\hat{g}}$ using an argument similar to the discussion around equations (4.2). This is only true when the gauge terms are neglected; if $r > 0$, then $X^{I_{r+1}}_{\tilde{g}_r}$ will have a horizontal gauge term that $X^i_{\hat{g}}$ lacks.

The vector field $X^{I_{k-1}}_{\hat{h}}$ is a differential operator, and it can be moved inside the infinite sum because a power series is absolutely convergent in its radius of convergence. The proof follows.

Explicitly,

$$(r+2)!X_{\tilde{g}_r}^{I_{r+1}} = 2\left(\tilde{g}_r^{ab}\delta_{1_{r-1}}^{(I_r}\pi_a^{i_{r+1})} + T^{I_{r+1}s}\right)\partial_s - \left(\frac{\partial\tilde{g}_r^{ab}}{\partial x^s}\delta_{1_r}^{(I_{r-1}}\pi_a^{i_r}\pi_b^{i_{r+1})} + T_s^{1I_{r+1}}\right)\frac{\partial}{\partial\pi_s^1}. \quad (4.9)$$

Lemma 4.2 and its proof show us that, nevertheless, this single observable behaves under the Poisson bracket very much like $\hat{g}$. The difference between $\{\hat{g},\hat{h}\}$ and $\{\tilde{g},\hat{h}\}$ will decrease as the rank r of $\tilde{g}$ increases.

The single observable $\tilde{g} \in \mathfrak{b}_1$ contains all the information the (finite collection of) observables $\hat{g}_r \in \mathfrak{b}_1$ contain, and either can be used to approximate $\hat{g}$ on B_1 and $\mathfrak{b}_1$ to any desired degree of accuracy. At the moment, the primary advantage to using $\tilde{g}$ to approximate $\hat{g}$ on B_1 as opposed to the collection of observables $\{\hat{g}_r\}$ is that there is only one observable, so any information is gleaned from a single source and not decoded from the sum of multiple calculations. We will see in the next section that the Hamiltonian distribution of $\tilde{g}$ reproduces the dynamics of $\hat{g}$ to any desired degree of accuracy, prompting us to use $\tilde{g}$ in order to study dynamics on B_1.

4.3 Gauging to B_1

We have shown how to work with the polynomial algebra of observables $b_1 \subset \mathfrak{b}_1$ in order to calculate the dynamics of many observables not in the algebra. We are able to accomplish this because polynomials are dense in the set of smooth functions; however, working with large and possibly infinite sums can become rather bothersome in practice. In this section, we present two methods for starting with an observable not in $\mathfrak{b}_1$ and being able to produce Hamiltonian vector fields tangent to B_1.

Our first method is more limited in application, as it generally provides only a single Hamiltonian vector field tangent to B_1. Let $\hat{g}$ be the rank 2 tensorial metric observable, $\hat{g} = g^{ab}\hat{\pi}_a\hat{\pi}_b$, with Hamiltonian distribution

$$X_{\hat{g}}^i = g^{as}\pi_a^i\partial_s - \frac{1}{2}\left(g_{,k}^{ab}\pi_a^i\pi_b^j + T_k^{ij}\right)\frac{\partial}{\partial\pi_k^j}. \quad (4.10)$$

Expand the term $g_{,k}^{ab}\pi_a^i\pi_b^j$ as in (3.34), and let

$$T_k^{ij} = 2g_{,k}^{ab}\pi_a^1\pi_b^2\delta_1^{[j}\delta_2^{i]} + 2g_{,k}^{ab}\pi_a^1\pi_b^3\delta_1^{[j}\delta_3^{i]} + \ldots + 2g_{,k}^{ab}\pi_a^1\pi_b^{(n)}\delta_1^{[j}\delta_{(n)}^{i]}. \quad (4.11)$$

This choice of gauge does not leave the entire distribution spanned by $X_{\hat{g}}^i$ tangent to B_1, but we have

$$X_{\hat{g}}^1 = g^{as}\pi_a^1\partial_s - \frac{1}{2}g_{,k}^{ab}\pi_a^1\pi_b^1\frac{\partial}{\partial\pi_k^1} \quad (4.12)$$

which is tangent to B_1. This same process can be repeated for any rank $k > 1$ tensorial observable. For a non-tensorial observable, this process is expanded to first remove the symmetry forced upon the lower-degree terms appearing in the vertical portion of the Hamiltonian distribution, and then each lower-degree term is expanded and gauge-altered as the tensorial (highest-degree) term was.

Our second method is more general, but requires us to create a new observable again. Let $\hat{F}$ be a general rank 2 observable, $\hat{F} = \left(F^{ab}\pi_a^i\pi_b^j + G^{ai}\pi_a^j + H^{ij}\right)\hat{r}_{ij}$. We define the rank 3 observable

$$\hat{F}_+ := \hat{F}\hat{r}_1 = \left(F^{ab}\pi_a^i\pi_b^j + G^{ai}\pi_a^j + H^{ij}\right)\hat{r}_{ij1} \tag{4.13}$$

As the previous method primarily uses tensorial observables, we should note that no observable formed this way will be tensorial. The Hamiltonian distribution defined by $\hat{F}_+$ is given by

$$\begin{aligned} 3! X^{ij}_{\hat{F}_+} &= \left(2F^{as}\pi_a^{(i}\delta_1^{j)} + G^{s(i}\delta_1^{j)} + T^{ijs}\right)\partial_s \\ &\quad - \left(F^{ab}_{,s}\pi_a^{(i}\pi_b^j\delta_1^{r)} + G^{a(i}_{,s}\pi_a^j\delta_1^{r)} + H^{(ij}_{,s}\delta_1^{r)} + T^{ijr}_s\right)\frac{\partial}{\partial \pi_s^r} \end{aligned} \tag{4.14}$$

By Lemma 3.4 we are able to select a gauge T^{ijr} to fix the index r on the Kroneker δ in every term in the vertical part of the distribution, thus making the entire distribution tangent to the slice B_1. This method generalizes: For any rank k observable $\hat{G}$, the rank $k+1$ observable $\hat{G}_+ = \hat{G}\hat{r}_1$ can be gauged to be tangent to B_1. We chose to present this method using a general rank 2 observable so as to easily and explicitly show the Hamiltonian distribution.

4.4 New Dynamics on B_1

Now, let us restrict our attention to a quadratic approximation of $\hat{g}$ upon B_1,

$$\tilde{g}_2 = (\lambda^{ab}\hat{r}_1\hat{r}_1 + \lambda^{ab}_c\hat{x}_1^c\hat{r}_1 + \lambda^{ab}_{cd}\hat{x}_1^c\hat{x}_1^d)\hat{\pi}_a\hat{\pi}_b, \tag{4.15}$$

with Hamiltonian distribution

$$4! X^{ijk}_{\tilde{g}_2} = 2\left(\tilde{g}_2^{as}\pi_a^{(i}\delta_1^j\delta_1^{k)} + T^{ijks}\right)\partial_s - \left(\tilde{g}^{ab}_{2,s}\pi_a^{(i}\pi_b^j\delta_1^{k)} + T^{1ijk}_s\right)\frac{\partial}{\partial \pi_s^1}, \tag{4.16}$$

By varying the values of indices i, j, and k, the Hamiltonian distribution can be seen to split naturally into four groups[4] with gauge terms suppressed.

$$\begin{aligned}
4!X_{\tilde{g}_2}^{ABC} &= 0 && (4.17)\\
4!X_{\tilde{g}_2}^{AB1} &= -\frac{2}{3!}\tilde{g}_{2,s}^{AB}\frac{\partial}{\partial \pi_s^1} && (4.18)\\
4!X_{\tilde{g}_2}^{A11} &= 2\frac{2}{3!}\tilde{g}_2^{As}\partial_s - \frac{4}{3!}\tilde{g}_{2,s}^{Ab}\pi_b^1\frac{\partial}{\partial \pi_s^1} && (4.19)\\
4!X_{\tilde{g}_2}^{111} &= 2\tilde{g}_2^{as}\pi_a^1\partial_s - \tilde{g}_{2,s}^{ab}\pi_a^1\pi_b^1\frac{\partial}{\partial \pi_s^1} && (4.20)
\end{aligned}$$

where captial Roman indices here are single indices that do not take the value 1. There is only one vector field in the final group, and it is independent of gauge[5]. The integral curves of this vector field follow from the equations

$$\begin{aligned}
4!\dot{x}^s &= 2\tilde{g}_2^{as}\pi_a^1\\
4!\dot{\pi}_s^1 &= -\tilde{g}_{2,s}^{ab}\pi_a^1\pi_b^1\\
\dot{\pi}_s^A &= 0, \qquad A = 2, \ldots, n
\end{aligned} \tag{4.21}$$

Let's assume the existence of a series approximation of g_{ab}, leading to the quadratic observable $\tilde{g}_{ab}^2 \in \mathfrak{b}_1$ such that $\tilde{g}_2^{ab}\tilde{g}_{bc}^2 \approx \delta_c^a$. Then, the first two integral curve equations combine as usual to give an approximation to the geodesic equation

$$\ddot{x}^s \approx -\Gamma_{ab}^s\dot{x}^a\dot{x}^b. \tag{4.22}$$

Unlike the standard case (the equations of motion of $\hat{g}$ on LM [11]), the remaining $\dot{\pi}$ equations in (4.21) do not show parallel transport of the remaining legs of the momentum frame. We have recovered the symplectic dynamics of our metric observable $\hat{g}$ after specializing it to the slice B_1, but not the full n-symplectic dynamics; this is due to the symplectic nature of the slice B_1.

The vector fields in the remaining groups[6] (and their integral curves) do not correspond to the dynamics of the free particle observable in symplectic geometry; they are new, internal dyamics. Specifically, the vector fields in the second group (equation (4.18)) produce purely

[4] Note that choosing the gauge so as to fix an order of the indices i, j, and k has but two effects. First, it removes the numerical factor introduced by expanding the symmetrized terms to see which survive and which are removed; choosing the gauge so that $4!X_{\tilde{g}_2}^{ijk} = 2\tilde{g}_2^{as}\pi_a^{(i}\delta_1^j\delta_1^{k)}\partial_s - \tilde{g}_{2,s}^{ab}\pi_a^i\pi_b^j\delta_1^k\frac{\partial}{\partial\pi_s^1}$ gives $4!X_{\tilde{g}_2}^{AB1} = -\tilde{g}_{2,s}^{AB}\frac{\partial}{\partial\pi_s^1}$. And second, it sets the order of the indices to define each group; choosing the gauge so that $4!X_{\tilde{g}_2}^{ijk} = 2\tilde{g}_2^{as}\pi_a^{(i}\delta_1^j\delta_1^{k)}\partial_s - \tilde{g}_{2,s}^{ab}\pi_a^i\pi_b^k\delta_1^j\frac{\partial}{\partial\pi_s^1}$ gives $4!X_{\tilde{g}_2}^{AB1} = 0$ but $4!X_{\tilde{g}_2}^{A1B} = -\tilde{g}_{2,s}^{AB}\frac{\partial}{\partial\pi_s^1}$.

[5] It is, in fact, the only vector field in these groups to be explicitly independent of gauge

[6] This includes the trivial group described by equation (4.17), because gauge freedom still exists in those vector fields.

internal motion, a change in momentum without a change in position. The integral curves are simple to write down:

$$\begin{aligned} \dot{x}^s = \dot{\pi}_s^A &= 0 \\ 4!\dot{\pi}_s^1 &= -\frac{2}{3!}\tilde{g}_{2,s}^{AB} \end{aligned} \tag{4.23}$$

So, we see that the momentum ($\pi_s^1 = p_s$) is linear in time, while the rest of the momentum frame as well as the position remain constant. Consider a particle or observer in free-fall (travelling along a geodesic) for a time, then changing momentum, and then resuming its previous free-fall motion (along a new geodesic depending on the position and new momentum when it resumes). Classically, such a change in momentum could come from some external event such as a collision or an internal event such as particle decay, situations in which the change would occur (nearly) instantaneously. Groups of vector fields in the n-symplectic Hamiltonian distribution $X_{\tilde{g}_2}^{ijk}$ contain these disparate dynamics, so we may model this change in motion not by changing our observable over time but by changing over time the vector field chosen from the Hamiltonian distribution, $C_{ijk}X_{\tilde{g}_2}^{ijk}$. Instead of choosing constants as was mentioned in §4.1, allow the C_{ijk} to be piecewise constant in time (the parameter along an integral curve). By being piecewise constant in time, none of the dynamics (the integral curves, particularly from equations (4.21) and (4.23)) are changed by C_{ijk}, but the curve would be allowed to "follow different paths" as it develops. This piecewise motion could allow the particle or observer to seem to stop suddenly, change momentum over time without changing position, and then continue moving along its new "natural" path. This does not follow what is seen in particle decay, so we are lead to declare that these internal motions, seen only on the frame bundle, should then be interpreted as following a different time parameter; a parameter to measure travel along these vertical curves, but that is separate from and does not contribute to proper time. Proper time should always be measured along the curves that can be mapped invariantly (namely the fourth group, equation (4.20)).

Next, we should attempt to interpret the remaining nontrivial group of vector fields, (4.19). We look again to the integral curves

$$\begin{aligned} 4!\dot{x}^s &= 2\frac{2}{3!}\tilde{g}_2^{As} \\ 4!\dot{\pi}_s^1 &= -\frac{4}{3!}\tilde{g}_{2,s}^{Ab}\pi_b^1 \\ \dot{\pi}_s^A &= 0 \end{aligned} \tag{4.24}$$

In contrast to the second group, motion on the base (i.e. change in the x^s) is now possible. The dynamics of this group cannot be considered as purely internal. Let us consider an example similar to that in the previous paragraph: We choose constants C_{ijk} piecewise constant in time such that we travel along a geodesic (equations (4.21)) for a time, change to a curve defined by this group of Hamiltonian vector fields, and then resume travel along some geodesic. The time spent along an integral curve from this group would not contribute to proper time, so the particle would appear to vanish from one point and then reappear at another. Such motion could be used to model wormholes or quantum teleportation.

Finally, the trivial group $X^{ABC}_{\tilde{g}_2}$ deserves some attention. These vector fields will be equal to zero (up to gauge freedom) no matter the choice of metric. Any choice of C_{ijk} that selects only Hamiltonian vector fields from this group would, therefore, be equivalent to the choice $C_{ijk} = 0$. This stands in contradistinction to the choice of ψ_α map on $L\mathbb{R}^n$; we have no reason a priori to require that $C_{ijk} \neq 0$ like our choice of α for the ψ_α map. If we continue to demand that motion along the constant integral curves of the vector fields in this group not contribute to proper time, then this group would have no measurable effect upon a particle's motion. Looking only on the frame bundle, a choice of C_{ijk} that begins non-zero and then, at some time, becomes and stays zero would would show motion for a certain amount of time and then suddenly coming to a stop.

The decision to examine only a quadratic approximation $\tilde{g}$ of $\hat{g}$ on B_1 was made merely for convenience. For an order $r \geq 2$ approximation, the Hamiltonian vector fields $X^{I_{r+1}}_{\tilde{g}_r}$ will still fall into four groups depending on the values of the indices I_{r+1}: At least 3 indices not equal to 1, two indices not equal to 1, one index not equal to 1, and no index not equal to 1, given by

$$(r+2)!X^{ABCI_{r-2}}_{\tilde{g}_r} = 0 \tag{4.25}$$

$$(r+2)!X^{AB1_{r-1}}_{\tilde{g}_r} = -2\frac{(r-1)!}{(r+1)!}\tilde{g}^{AB}_{r,s}\frac{\partial}{\partial \pi^1_s} \tag{4.26}$$

$$(r+2)!X^{A1_r}_{\tilde{g}_r} = 2\frac{r!}{(r+1)!}\left(\tilde{g}^{As}_r\partial_s - \tilde{g}^{Ab}_{r,s}\pi^1_b\frac{\partial}{\partial \pi^1_s}\right) \tag{4.27}$$

$$(r+2)!X^{1_{r+1}}_{\tilde{g}_r} = 2\tilde{g}^{as}_r\pi^1_a\partial_s - \tilde{g}^{ab}_{r,s}\pi^1_a\pi^1_b\frac{\partial}{\partial \pi^1_s} \tag{4.28}$$

The fourth group, the single vector field $X^{1_{r+1}}_{\tilde{g}_r} = X^{111\ldots1}_{\tilde{g}_r}$, will be the only group whose vector fields are free of gauge. The integral curves of this Hamiltonian vector field will be approximately the geodesics of the metric on M. This approximation will become more accurate as r increases. The other three groups have the same features as the corresponding groups of the quadratic approximation, namely purely internal motion, spatial and internal motion, and being trivial.

4.5 Kaluza-Klein Structures

We have shown that we can approximate the metric observable $\hat{g}$ on B_1 (really, any observable with a series expansion) to an arbitrary level of accuracy. We will now refer to working with the metric observable $\hat{g}$ on B_1 directly, meaning that we can approximate these results to a desired level of accuracy.

In the three non-zero groups of Hamiltonian vector fields for $\hat{g}$ on B_1 the metric appears in three forms: The matrix g^{ab}, the vectors g^{Ab}, and the scalars g^{AB}. One might be inclined to rearrange these terms into a Kaluza-Klein-type theory, creating an $(n+1)$-dimensional metric

$$\mathcal{G}^{\alpha\beta} = \begin{pmatrix} g^{11} & g^{12} & \dots & g^{1n} & -g^{1A} \\ g^{21} & g^{22} & \dots & g^{2n} & -g^{2A} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g^{n1} & g^{n2} & \dots & g^{nn} & -g^{nA} \\ -g^{A1} & -g^{A2} & \dots & -g^{An} & g^{AB} \end{pmatrix} \tag{4.29}$$

with inverse

$$\mathcal{G}_{\alpha\beta} = \begin{pmatrix} g_{11} + k\delta^1_A\delta^1_A & g_{12} + k\delta^1_A\delta^2_A & \dots & g_{1n} + k\delta^1_A\delta^n_A & k\delta^1_A \\ g_{21} + k\delta^2_A\delta^1_A & g_{22} + k\delta^2_A\delta^2_A & \dots & g_{2n} + k\delta^2_A\delta^n_A & k\delta^2_A \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n1} + k\delta^n_A\delta^1_A & g_{n2} + k\delta^n_A\delta^2_A & \dots & g_{nn} + k\delta^n_A\delta^n_A & k\delta^n_A \\ k\delta^1_A & k\delta^2_A & \dots & k\delta^n_A & k \end{pmatrix} \tag{4.30}$$

where $g^{AB} = \frac{1}{k} + g^{AA}$. Each choice of A and B gives a different Kaluza-Klein-type metric, but when A and B take the same value the metric is singular, so in the end we have $(n-1)(n-2)$ different Kaluza-Klein-type metrics from which to choose. The main difference between these metrics and a standard Kaluza-Klein metric is that the scalar k will not be constant in general. Interestingly enough, the choice of index B makes no difference in the geodesic equation. In 4 dimensions, the 4-dimensional part of the 5-dimensional geodesic equation becomes

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = \frac{1}{2}g^{ab}k_{,b}(\dot{x}^5\dot{x}^5 - 2\dot{x}^A\dot{x}^5) \tag{4.31}$$

where $a, b, c = 1, 2, 3, 4$. The right-hand side contains only a term with a derivative on k (which would be 0 in the standard theory) and no reference to a Maxwell field tensor. This is due to the fact that the vector potential term A^a is built from the metric; $A^a = g^{Aa}$, so $A_a = \delta^A_a$ and $F_{ab} = A_{b,a} - A_{a,b} = 0$.

If instead we arrange the metric pieces as

$$\mathcal{G}_{\alpha\beta} = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} & -g_{1A} \\ g_{21} & g_{22} & \cdots & g_{2n} & -g_{2A} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} & -g_{nA} \\ -g_{A1} & -g_{A2} & \cdots & -g_{An} & g_{AB} \end{pmatrix} \tag{4.32}$$

with inverse

$$\mathcal{G}^{\alpha\beta} = \begin{pmatrix} g^{11} + k\delta^1_A\delta^1_A & g^{12} + k\delta^1_A\delta^2_A & \cdots & g^{1n} + k\delta^1_A\delta^n_A & k\delta^1_A \\ g^{21} + k\delta^2_A\delta^1_A & g^{22} + k\delta^2_A\delta^2_A & \cdots & g^{2n} + k\delta^2_A\delta^n_A & k\delta^2_A \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g^{n1} + k\delta^n_A\delta^1_A & g^{n2} + k\delta^n_A\delta^2_A & \cdots & g^{nn} + k\delta^n_A\delta^n_A & k\delta^n_A \\ k\delta^1_A & k\delta^2_A & \cdots & k\delta^n_A & k \end{pmatrix} \tag{4.33}$$

and again $g_{AB} = \frac{1}{k} + g_{AA}$, then the Maxwell field tensor is non-trivial (where the potential $A_j = g_{jA}$). If we were then to require k to be a constant, the metric above conforms exactly to a standard Kaluza-Klein-type metric. This requirement restricts the possible choices of metric g_{ab}, and the implications of these restrictions is an avenue for futher research.

4.6 Singularities on B_1

In this section, we are motivated by the existence of singularities in the dynamics of the charged n-symplectic observable to question whether or not similar singularities exist in the dynamics of the free particle observable on B_1. It was noted in Section § 4.1 that for any point $u = (p, e_i) \in B_1$, it is required that $\pi^1_1(u) \neq 0$. If the integral curve of a Hamiltonian vector field on B_1 were to travel along a path such that $\pi^1_1 \to 0$, then not only would it leave the slice but it would leave the entire space LM, just as was the case in §3.2. We will examine the dynamics of the free particle observable upon B_1 first with a specific metric, examining the three nontrivial classes of Hamiltonian vector fields to show the explicit existence of singularities in the dynamics. Then, we will discuss briefly the existence of singularities in the dynamics given by a general metric.

Consider the metric $g^{ab} = \mathrm{diag}(1, 1 + (x^1)^2, 1, 1)$. Since it is a quadratic metric it can be represented exactly upon B_1 by $\tilde{g} := \tilde{g}_2 = (\delta^{ab}\hat{r}_1\hat{r}_1 + \delta^a_2\delta^b_2\hat{x}^1_1\hat{x}^1_1)\hat{\pi}_a\hat{\pi}_b$. The vector fields spanning the Hamiltonian distribution on B_1 are, by equation (4.16),

$$4!X^{ijk}_{\tilde{g}} = (\delta^{as} + \delta^a_2\delta^s_2(x^1)^2)\pi^{(i}_a\delta^j_1\delta^{k)}_1\partial_s - 2x^1\pi^{(i}_2\pi^j_2\delta^{k)}_1\frac{\partial}{\partial\pi^1_1} \tag{4.34}$$

The only nontrivial vertical components of these vector fields occur when i, j, and k take values 1 or 2. We will use σ for the parameter along the integral curve γ of any of vector field in this distribution. The explicit mention of γ will usually be suppressed, writing $\pi^i_j(\sigma)$ in place of $\pi^i_j(\gamma(\sigma))$, etc.

First, we examine the internal motions from the second group (equation (4.18)). The only nontrivial vector field in this group is $X^{221}_{\tilde{g}}$, and the only nontrivial differential equation defining γ is

$$4!\frac{\mathrm{d}\pi^1_1}{\mathrm{d}\sigma} = -\frac{4}{3!}x^1 \tag{4.35}$$

If the inital values of x^1 and π^1_1 have the same sign (e.g. $\pi^1_1(0) = 4, x^1(0) = 4! \cdot 3!$), then this curve will lead to (a point such that) $\pi^1_1(\sigma_s) = 0$ at some finite time σ_s (in the example, $\sigma_s = 1$).

Next, we consider the vector field $X^{211}_{\tilde{g}}$ from group (4.19). The relevant differential equations are

$$4!\frac{\mathrm{d}\pi^1_1}{\mathrm{d}\sigma} = -\frac{8}{3!}x^1\pi^1_2 \tag{4.36}$$

$$4!\frac{\mathrm{d}x^1}{\mathrm{d}\sigma} = 4!\frac{\mathrm{d}\pi^1_2}{\mathrm{d}\sigma} = 0 \tag{4.37}$$

The motion along π^1_1 is, again, linear in time. If the initial values of π^1_2, x^1, and π^1_1 are chosen such that the product $\pi^1_2 x^1$ has the same sign as $\pi^1_1(0)$, there will be a finite time σ_s such that $\pi^1_1(\sigma_s) = 0$. For an explicit example, let $\pi^1_2 = -3!$, $x^1 = -4!$, and $\pi^1_1(0) = 8$, leading to $\sigma_s = 1$.

This only leaves the standard geodesic motion of the vector field $X^{111}_{\tilde{g}}$. The relevant differential equations are

$$\begin{aligned}
4!\frac{\mathrm{d}x^1}{\mathrm{d}\sigma} &= \pi^1_1 \\
4!\frac{\mathrm{d}\pi^1_1}{\mathrm{d}\sigma} &= -2x^1(\pi^1_2)^2 \\
4!\frac{\mathrm{d}\pi^1_2}{\mathrm{d}\sigma} &= 0
\end{aligned} \tag{4.38}$$

Once again, π^1_2 is constant in time. This leads to sinusoidal motion in x^1 and π^1_1. Specifically,

$$\begin{aligned}
x^1(\sigma) &= x^1(0)\cos(a\sigma) + \frac{\pi^1_1(0)}{4!a}\sin(a\sigma) \\
\pi^1_1(\sigma) &= \pi^1_1(0)\cos(a\sigma) - ax^1(0)\sin(a\sigma) \\
a &= \frac{\pi^1_2}{4!}\sqrt{2}
\end{aligned} \tag{4.39}$$

Thus, any choice of initial conditions (so long as $\pi^1_1(0) \neq 0$) will lead to $\pi^1_1(\sigma_s) = 0$ at multiple finite times σ_s. As an explicit example: $x^1(0) = 0$, $\pi^1_2 = \frac{1}{\sqrt{2}}$, and $\pi^1_1(0) \neq 0$ leads to singularities at $\sigma_s = 4!(2n+1)\pi$ for all $n \in \mathbb{Z}$.

We have shown the existence of singularities in the dynamics of a quadratic metric observable on B_1. Are we then able to determine necessary or sufficient criteria for the existence of singularities for a general metric g^{ab}? The general formulae for the groups of vector fields are given by equations (4.25)-(4.28). Only from the second group can we discern both necessary and sufficient conditions. We see again that $\frac{\mathrm{d}\pi^1_1}{\mathrm{d}\sigma}$ is constant. If and only if a choice of indices A and B (each not equal to 1) can be made such that $g^{AB}_{,1} \neq 0$, one can choose initial conditions $\pi^1_1(0) \neq 0$ and $x^k(0)$ such that $\pi^1_1(\sigma_s) = -2\frac{(r-1)!}{(r+1)!\cdot(r+2)!} g^{AB}_{,1} \sigma_s + \pi^1_1(0) = 0$. The ability to choose indices A and B in this way is a condition on the metric g^{ab}.

The condition that $g^{ab}_{,1} \neq 0$ for some choice of a and b is necessary in all three non-trivial groups of vector fields for there to be a singularity. The third and fourth groups of Hamiltonian vector fields (equations (4.27) and (4.28)), however, do not yield any more information as to which metrics lead to singularities and which do not. Take the third group for example: If $g^{21} = x^1$ and $g^{22} = g^{23} = g^{24} = 0$, then $\pi^1_1(\sigma)$ will be exponential in σ and there will be no singularity in finite time. On the other hand, $g^{22} = x^1$, and $g^{21} = g^{23} = g^{24} = 0$ leads to π^1_1 being linear in σ and $\pi^1_1(\sigma_s) = 0$ for some finite σ_s. More complicated metrics only compound the analysis, and the fourth group is even more obtuse[7].

[7] The metrics mentioned in this example would only be valid (nondegenerate) locally, but nonetheless serve to illustrate the point.

Chapter 5

Conclusion and Future Work

We have examined the structures and dynamics in a number of non-canonical and non-standard situations in n-symplectic geometry. We began by attempting to lift Künzle's work to the frame bundle in a general way by adding some combination of connection 1-forms ω^a_b to the soldering form. Even in the simplest of cases, however, the resulting n-symplectic dynamics were too restrictive. The allowable observables are very simple, and the associated motions do not contain a contribution from spin or that can be called spin. We then examined the charged n-symplectic form obtained by adding a generalized electromagnetic vector potential to the soldering form. While the noted $\frac{1}{2}$ difference in the momentum rest space seemed to be indicative of a natural setting in which to build a classical theory of spin-$\frac{1}{2}$ particles, we were able to show that the same reasonable choices that lead to this difference also lead to the possibility of introducing singularities in the equations of motion. We discussed the gauge freedom inherent in many n-symplectic Hamiltonian vector fields that had largely been swept aside in previous studies in favor of gauge-invariant techniques. It became clear, then, that with the proper choice of gauge, the $\frac{1}{2}$ difference and the associated singularities can be removed. This conclusion does not invalidate Lemma 3.3, as the $\frac{1}{2}$ difference only appears (or disappears) after a particular choice of α for the mapping of the Hamiltonian distribution to the cotangent bundle. From non-canonical n-symplectic geometry we moved to the non-standard, investigating n-symplectic geometry on the symplectic submanifold B_1. The algebra $\mathfrak{b}_1$ of observables on B_1 is more limited than the algebra of observables on LM. We primarily concerned ourselves with $b_1 \subset \mathfrak{b}_1$, a polynomial algebra. As polynomials are dense in the space of smooth functions, we were able to demonstrate a number of methods by which many observables on LM can be represented by observables in b_1. With this detail out of the way, we calculated the n-symplectic Hamiltonian vector fields associated with the free particle to discover motions not seen in symplectic geometry. These motions were then classified into four groups: One trivial, one purely internal (change in momentum, no change in position), one that presents change in momentum and position, and the classical motion. The two non-trivial, non-classical groups of motions present motions that are not normally seen classically by free particles, and are not inherently independent of gauge. We assert that there must be some sort of impetus for a particle to travel along these integral curves. Furthermore, in order for these motions to be reconcilled with classical observations, we assert that the time spent along these integral curves not contribute to proper time. This would cause the change in position or momentum to occur instantly, providing possible models of particle decay or wormholes. In the classification of these new motions, we noted that specific pieces of the metric appear in such a way as to be collected nicely to form Kaluza-Klein-type theories on B_1. To finish our discussion of B_1, we noted that singularities can exist even in the motions of the free particle.

Each of the three preceeding chapters leaves certain questions unanswered. In Chapter 2, we attempted to duplicate Künzle's work on the entire frame bundle, whereas Künzle only examined motions on the Lorentz subbundle. Would considering the Lorentz- or some other subbundle of LM bring to light new symmetries that would allow for more interesting dynamics? Also, we were able to determine the allowable Hamiltonian vector fields (and, thus, the allowable observables) explicitly only in very simple cases. These gave general results, but the results are not necessarily exhaustive. New methods of analyzing the n-symplectic structure equation are needed in order to rule out the possibility of observables not covered in Theorem 2.2 or Theorem 2.3.

In Chapter 3, we make use of gauge freedom to change what is seen on the frame bundle without changing the classical motions seen on the cotangent bundle via the ψ_α map. One is led to question how "physical" these gauge terms are. We chose a gauge term in order to remove the $\frac{1}{2}$ difference in the motion of the momentum rest space, and we showed that this choice of gauge is not unique. Is it useful or even possible to classify gauge terms by the dynamics they produce?

In Chapter 4, we considered using a time-dependent combination of Hamiltonian vector fields $C_I X^I$ in order to produce a piecewise-smooth integral curve using more than one group of motions on B_1. Working on B_1 affords us this luxury; normally, we must use the ψ_α map in order to bring the n-symplectic motions to the cotangent bundle. Different choices of α for the ψ_α map amount to linear changes of the p_s coordinates on the cotangent bundle. This is seen most easily by noting that there is a $GL(n)$ matrix $\overline{g}^a_b$ such that $\overline{g}^a_b \alpha_i \pi^i_a = \beta_i \pi^i_b$, for $\alpha, \beta \in \mathbb{R}^{n*} \setminus \{0\}$. If one were to consider a piecewise constant $\alpha \in \mathbb{R}^{n*} \setminus \{0\}$, then a particle following the path of an integral curve of the vector field $\psi_{\alpha*}(\alpha_I X^I)$ on T^*M would essentially appear to change its orientation from time to time. These dynamics would not be particularly interesting. If the α were allowed to continuously change, however, rich new structures seem to appear. Take, for example, the n-symplectic observable for the free particle in flat space, $\hat{\eta} = \eta^{ab} \hat{\pi}_a \hat{\pi}_b$. Its Hamiltonian distribution is $X^i_{\hat{\eta}} = \eta^{as} \pi^i_a \partial_s$. We naïvely map this distribution to the vector field $\psi_{\alpha *}(\alpha_i X^i_{\hat{\eta}}) = \eta^{as}\pi^i_a \alpha_i \partial_s$. The integral curves of this vector field are given by the differential equations

$$\dot{x}^s = \eta^{as}\pi^i_a \alpha_i$$
$$\dot{p}_s = 0$$

These equations even combine to produce the standard result of $\ddot{x}^s = 0$. The key difference lies in the momentum coordinates: We have $\alpha_i \pi^i_s = p_s$, and so the momentum coordinates are changing over time, but they remain constant along the path of this particle. The particle would appear to be a point of calm in this sea of momentum flux. Further research is necessary to make sense of a variable α and what new dynamics it could bring.

REFERENCES

[1] J. D. Brown. *N-symplectic Quantization*. PhD thesis, North Carolina State University, 2008.

[2] D. Cartin. Some aspects of geometric pre-quantization on the affine frame bundle. Master's thesis, North Carolina State University, 1995.

[3] M. de León, M. McLean, L. K. Norris, A. Rey-Roca, and M. Salgado. Geometric Structures in Field Theory. *ArXiv Mathematical Physics e-prints*, August 2002.

[4] R. O. Fulp, J. K. Lawson, and L. K. Norris. Geometric Prequantization on the Spin Bundle Based on N-symplectic Geometry: The Dirac Equation. *Intl. J. Theor. Phys.*, 33(5):1011–1028, 1994.

[5] J. D. Jackson. *Classical Electrodynamics*. John Wiley & Sons, Inc., New Jersey, USA, 3 edition, 1991.

[6] A. Kheyfets and L. K. Norris. P(4) affine and superhamiltonain formulations of charged particle dynamics. *Int. J. Theor. Phys.*, 27(2):159–182, 1988.

[7] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*, volume 1. Interscience Publishers, 1963.

[8] H. P. Künzle. Canonical dynamics of spinning particles in gravitational and electromagnetic fields. *J. Math. Phys.*, 13(5):739–744, 1972.

[9] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. W. H. Freeman and Company, San Francisco, California, USA, 1973.

[10] A. Nijenhuis. Jacobi-type identities for bilinear differential concomitants of certain tensor fields. *Indag. Math.*, 17:390–403, 1955.

[11] L. K. Norris. Generalized symplectic geometry on the frame bundle of a manifold. *Proc. Symp. Pure Math.*, 54(Part 2):435–465, 1993.

[12] L. K. Norris. Symplectic geometry on TM derived from n-symplectic geometry on LM. *J. Geom. Phys.*, 13:51–78, 1994.

[13] L. K. Norris. Schouten-Nijenhuis Brackets. *J. Math. Phys.*, 38:2694–2709, 1997.

[14] L. K. Norris and J. D. Brown. n-symplectic quantization on lrn. *ArXiv e-prints*, November 2009.

[15] J. A. Schouten. Uber differentialkomitanten zweier kontravarianter grossen. *Proc. Kon. Ned. Akad. Wet. Amsterdam*, 43:449–452, 1940.

APPENDICES

Appendix A

Proof of Theorem 2.2

We will prove the two specific claims made about observables in Theorem 2.2.

Recall that, in the local coordinates x^i and π^i_j we have $\theta^i = \pi^i_j \mathrm{d}x^j$ and $\omega^a_b = \pi^a_c \mathrm{d}v^c_b$. The coordinates v^j_k are dual to the π^i_j in the sense that

$$v^j_k \pi^i_j = \delta^i_k$$
$$v^j_k \pi^k_m = \delta^j_m$$

Choosing $\gamma^{bi}_a = \delta^b_3 \delta^2_a \delta^i_1$, equation (2.1) becomes

$$\phi^i = \pi^i_k \mathrm{d}x^k + \delta^i_1 \pi^2_c \mathrm{d}v^c_3 \tag{A.1}$$

and we calculate

$$\begin{aligned}
\mathrm{d}\phi^i &= \mathrm{d}\pi^i_k \wedge \mathrm{d}x^k + \gamma^{bi}_a \mathrm{d}\pi^a_c \wedge \mathrm{d}v^c_b \\
&= \mathrm{d}\pi^i_k \wedge \mathrm{d}x^k - \delta^i_1 v^c_l v^m_3 \mathrm{d}\pi^2_c \wedge \mathrm{d}\pi^l_m
\end{aligned} \tag{A.2}$$

In the course of both proofs, we will let the use the Greek letters α, β, and γ to represent indices whose value is never 1, 2, or 3, respectively. These Greek indices will be used to represent when a choice of value for a particular Latin index is made. If the index is not to be summed, it will be placed in parentheses. For example, $X^\alpha = X^{(\alpha)}$ and $v^a_\gamma \pi^\gamma_b = \delta^a_b - v^a_3 \pi^3_b \neq v^a_{(\gamma)} \pi^{(\gamma)}_b$.

A.1 Rank 1

Contracting $\mathrm{d}\phi^i$ with a general vector field $X = X^s \partial_s + X^r_s \frac{\partial}{\partial \pi^r_s}$ yields

$$X \lrcorner\, \mathrm{d}\phi^i = X^i_k \mathrm{d}x^k - X^s \mathrm{d}\pi^i_s + \Sigma^{si}_r \mathrm{d}\pi^r_s \tag{A.3}$$

where $\Sigma_r^{si} = v_l^c v_3^m \delta_1^i \left(X_m^l \delta_r^2 \delta_c^s - X_c^2 \delta_r^l \delta_m^s \right)$. Taking the exterior derivative of both sides of equation (2.7) leaves $\mathrm{d}(X \lrcorner\ \mathrm{d}\phi^i) = 0$, and we calculate

$$\begin{aligned} \mathrm{d}(X \lrcorner\ \mathrm{d}\phi^i) = X^i_{k,l} \mathrm{d}x^l \wedge \mathrm{d}x^k - \left(\frac{\partial X^i_k}{\partial \pi^r_s} + \delta^i_r X^s_{,k} - \Sigma^{si}_{r,k} \right) \mathrm{d}x^k \wedge \mathrm{d}\pi^r_s \\ - \delta^i_r \frac{\partial X^s - \Sigma^{si}_r}{\partial \pi^p_q} \mathrm{d}\pi^p_q \wedge \mathrm{d}\pi^r_s = 0 \end{aligned} \tag{A.4}$$

Using the linear independence of the 1-forms $\mathrm{d}x^k$ and $\mathrm{d}\pi^r_s$, we can separate this into three sets of equations:

$$0 = X^i_{k,l} \tag{A.5}$$

$$0 = \frac{\partial X^i_k}{\partial \pi^r_s} + \delta^i_r X^s_{,k} - \Sigma^{si}_{r,k} \tag{A.6}$$

$$0 = \delta^i_r \frac{\partial X^s - \Sigma^{si}_r}{\partial \pi^p_q} \tag{A.7}$$

Equations (A.5) and (A.7) can be rewritten taking advantage of their natural antisymmetry. Notice that $A_{[bc]} = 0 \Rightarrow A_{bc} = A_{cb}$.

$$X^i_{k,l} = X^i_{l,k} \tag{A.8}$$

$$\delta^i_r \frac{\partial X^s - \Sigma^{si}_r}{\partial \pi^p_q} = \delta^i_p \frac{\partial X^q - \Sigma^{qi}_p}{\partial \pi^r_s} \tag{A.9}$$

We begin our analysis with equations (A.9). Choosing the free index $i = \alpha \neq 1$ gives

$$\delta^\alpha_l \frac{\partial}{\partial \pi^j_k} X^m = \delta^\alpha_j \frac{\partial}{\partial \pi^l_m} X^k \tag{A.10}$$

In equation (A.10), l is a free index. Choosing $l = \alpha$, we get

$$\frac{\partial}{\partial \pi^j_k} X^m = \delta^{(\alpha)}_j \frac{\partial}{\partial \pi^{(\alpha)}_m} X^k \tag{A.11}$$

Notice that the choice of α was free; choosing $\alpha = 2$ would imply

$$\frac{\partial}{\partial \pi^\beta_k} X^m = 0 \tag{A.12}$$

Then choosing $\alpha = 3$, equation (A.11) would imply

$$\frac{\partial}{\partial \pi^2_k} X^m = 0 \tag{A.13}$$

Together, equations (A.12) and (A.13) imply $\frac{\partial}{\partial \pi^j_k} X^m = 0$. Therefore, X^m is a function of the x^i alone, so we can write

$$X^m = f^m(x) \tag{A.14}$$

though the variable dependence of functions will be suppressed in general. Now choosing $i = \alpha$ in equation (A.6), we see that

$$\frac{\partial}{\partial \pi^k_l} X^\alpha_j = -\delta^\alpha_k f^l_{,j} \tag{A.15}$$

Differentiating, we see that $\frac{\partial^2}{\partial \pi^a_b \partial \pi^k_l} X^\alpha_j = 0$, which implies that X^α_j is linear in the π^r_s. We can write $X^\alpha_j = C^{\alpha l}_{jk}(x)\pi^k_l + g^\alpha_j(x)$. Substituting this into equation (A.15), we see that $C^{(\alpha)l}_{j\ \ \ \ k} = -\delta^{(\alpha)}_k f^l_{,j}$. This implies

$$X^\alpha_j = g^\alpha_j(x) - \pi^\alpha_b f^b_{,j} \tag{A.16}$$

Then by defining $g^1_j := X^1_j + \pi^1_b f^b_{,j}$, we are able to write

$$X^i_j = g^i_j - \pi^i_b f^b_{,j} \tag{A.17}$$

It is not known at this point if g^1_j is a function of the x^k or π^r_s alone.

Using equation (A.17), we can rewrite equation (A.6) in the following way

$$\frac{\partial X^i_j}{\partial \pi^k_l} = -\delta^i_k f^l_{,j} - \delta^i_1 v^c_k v^l_3 X^2_{c,j} + \delta^i_1 \delta^2_k v^l_c v^m_3 g^c_{m,j} - \delta^i_1 \delta^2_k v^m_3 f^l_{,mj} \tag{A.18}$$

Letting $i = 1$, we find

$$\frac{\partial X^1_j}{\partial \pi^k_l} = -\delta^1_k f^l_{,j} - v^c_k v^l_3 X^2_{c,j} + \delta^2_k v^l_c v^m_3 g^c_{m,j} - \delta^2_k v^m_3 f^l_{,mj} \tag{A.19}$$

This will be useful in the future.

The term $v^m_c v^d_3 X^c_d$ occurs in Σ^{si}_r. Expanding this using equation (A.17) gives

$$v^m_c v^d_3 X^c_d = v^d_3(v^m_b g^b_d - v^m_a \pi^a_b f^b_{,d}) = v^d_3(v^m_b g^b_d - f^m_{,d}) \tag{A.20}$$

Now equations (A.14), (A.17), and (A.20) allow us to rewrite equation (A.9) as

$$\frac{\partial}{\partial \pi_k^j}\left[v_l^d v_3^m X_d^2 - \delta_l^2 v_3^d (v_b^m g_d^b - f_{,d}^m)\right] = \frac{\partial}{\partial \pi_m^l}\left[v_j^d v_3^k X_d^2 - \delta_j^2 v_3^d (v_b^k g_d^b - f_{,d}^k)\right] \tag{A.21}$$

Choosing $l \neq 2$ and $j = 2$ gives

$$\frac{\partial}{\partial \pi_k^2}\left[v_{(\beta)}^d v_3^m X_d^2\right] = \frac{\partial}{\partial \pi_m^{(\beta)}}\left[v_j^d v_3^k X_d^2 - v_3^d (v_b^k g_d^b - f_{,d}^k)\right] \tag{A.22}$$

Expanding both derivatives, the equation reduces to

$$v_{(\beta)}^d v_3^m v_b^k g_d^b = v_3^d \left(v_b^k \frac{\partial g_d^b}{\partial \pi_m^{(\beta)}} - v_{(\beta)}^k v_b^m g_d^b \right) \tag{A.23}$$

Using equation (A.18) and the fact that $\frac{\partial g_d^b}{\partial \pi_m^a} = \delta_1^b \frac{\partial g_d^1}{\partial \pi_m^a}$, we expand

$$\begin{aligned} \frac{\partial g_d^b}{\partial \pi_m^a} &= \delta_1^b \left(\frac{\partial X_d^1}{\partial \pi_m^a} + \delta_a^1 f_{,d}^m \right) \\ &= \delta_1^b \left(-v_a^c v_3^m X_{c,d}^2 + \delta_a^2 v_c^m v_3^r g_{r,d}^c - \delta_a^2 v_3^r f_{,rd}^m \right) \end{aligned} \tag{A.24}$$

in order to rewrite equation (A.23) as

$$v_{(\beta)}^d v_3^m v_b^k g_d^b = -v_3^d v_1^k v_{(\beta)}^c v_3^m X_{c,d}^2 - v_3^d v_{(\beta)}^k v_b^m g_d^b \tag{A.25}$$

Next, we multiply through by $\pi_k^a \pi_m^b$ to obtain

$$v_{(\beta)}^d \delta_3^b g_d^a = -v_3^d \delta_1^a v_{(\beta)}^c \delta_3^b X_{c,d}^2 - v_3^d \delta_{(\beta)}^a g_d^b. \tag{A.26}$$

Choosing $b \neq 3$ gives

$$v_3^d g_d^{(\gamma)} = 0, \tag{A.27}$$

and choosing $a = 2$ in (A.26) gives

$$v_{(\beta)}^d g_d^2 = 0. \tag{A.28}$$

Taking the derivative of equation (A.28) with respect to π_s^r and multiplying by π_a^r yields

$$g_c^2 = 0. \tag{A.29}$$

Using equations (A.27) and (A.29), we can rewrite equation (A.26) as

$$\delta_3^b \delta_{(\beta_2)}^a v_{(\beta_1)}^d g_d^{(\beta_2)} = \delta_1^a \delta_3^b v_3^d v_{(\beta_1)}^c \pi_r^2 f_{,cd}^r - \delta_{(\beta_1)}^a \delta_3^b v_3^d g_d^3 \tag{A.30}$$

where β_1 and β_2 are two arbitrary but fixed numbers not equal to 2. No assumption is made a priori as to whether or not $\beta_1 = \beta_2$. Our calculations will simplify if we let $\beta_1 = \beta_2 = \beta$, which we are free to do. This leaves

$$\delta^a_{(\beta)} v^d_{(\beta)} g^{(\beta)}_d = \delta^a_1 v^d_3 v^c_{(\beta)} \pi^2_r f^r_{,cd} - \delta^a_{(\beta)} v^d_3 g^3_d \tag{A.31}$$

Selecting $a = \beta$ and $\beta = 3$ reduces equation (A.31) to $v^d_3 g^3_d = 0$, which along with equation (A.27) gives us $v^d_3 g^a_d = 0$. For any choice $a = \alpha$, differentiating and multiplying by π leads to

$$g^\alpha_c = 0 \tag{A.32}$$

When $a = 1$, we have $v^d_3 g^1_d = 0$. Differentiating and expanding gives

$$\frac{\partial v^d_3 g^1_d}{\partial \pi^k_l} = v^d_3 \left(v^c_k v^l_3 \pi^2_r f^r_{,cd} - \delta^2_k v^r_3 f^l_{,rd} \right) - v^l_3 v^d_k g^1_d = 0 \tag{A.33}$$

Multiplying the right-hand equation by $\pi^3_l \pi^k_s$ yields

$$g^1_s = v^d_3 \left(\pi^2_r f^r_{,sd} - v^r_3 \pi^2_s \pi^3_l f^l_{,rd} \right) \tag{A.34}$$

So by the definition of g^1_s,

$$X^1_s = v^d_3 \pi^2_r f^r_{,sd} - v^r_3 v^d_3 \pi^2_s \pi^3_l f^l_{,rd} - \pi^1_l f^l_{,s} \tag{A.35}$$

Now,

$$\begin{aligned} \frac{\partial X^1_j}{\partial \pi^k_l} &= \delta^2_k v^d_3 f^l_{,jd} - v^d_k v^l_3 \pi^2_r f^r_{,jd} + v^r_k v^l_3 v^d_3 \pi^2_j \pi^3_s f^s_{,rd} + v^r_3 v^d_k v^l_3 \pi^2_j \pi^3_s f^s_{,rd} \\ &\quad - \delta^2_k \delta^l_j v^r_3 v^d_3 \pi^3_s f^s_{,rd} - \delta^3_k v^r_3 v^d_3 \pi^2_j f^l_{,rd} - \delta^1_k f^l_{,j} \end{aligned} \tag{A.36}$$

and equation (A.19) are both equations for $\frac{\partial X^1_j}{\partial \pi^k_l}$. Equating them and simplifying, we see that

$$2v^d_k v^l_3 \pi^2_r f^r_{,dj} - 2\delta^2_k v^d_3 f^l_{,dj} = (v^r_k v^l_3 v^d_3 \pi^2_j \pi^3_s + v^r_3 v^d_k v^l_3 \pi^2_j \pi^3_s - \delta^2_k \delta^l_j v^r_3 v^d_3 \pi^3_s - \delta^3_k \delta^l_s v^r_3 v^d_3 \pi^2_j) f^s_{,rd} \tag{A.37}$$

Multiplying equation (A.37) by π^γ_l, this reduces to

$$2\delta^2_k v^d_3 \pi^\gamma_l f^l_{,dj} = (\delta^2_k \pi^\gamma_j v^r_3 v^d_3 \pi^3_s - \delta^3_k v^r_3 v^d_3 \pi^2_j \pi^\gamma_l) f^s_{,rd} \tag{A.38}$$

Choosing $k = 3$ gives

$$v^r_3 v^d_3 \pi^{(\gamma)}_l f^l_{,rd} = 0, \tag{A.39}$$

and choosing $k = 2$ in (A.38) gives

$$v_3^r v_3^d \pi_s^3 f^s_{,rd} = 2 v_3^d v^j_{(\gamma)} \pi_l^{(\gamma)} f^l_{,dj}, \tag{A.40}$$

which add together to become

$$v_3^r v_3^d \pi_s^a f^s_{,rd} = 2 \delta_3^a v_3^d v^r_{(\gamma)} \pi_l^{(\gamma)} f^l_{,rd}. \tag{A.41}$$

Multiplying by v_a^c, equation (A.41) becomes

$$v_3^r v_3^d f^c_{,rd} = 2 v_3^c v_3^d v^r_{(\gamma)} \pi_l^{(\gamma)} f^l_{,rd} \tag{A.42}$$

Differentiating with respect to v_b^a leads to

$$\delta_3^b v_3^r f^c_{,ra} = \delta_a^c \delta_3^b v_3^d v^r_{(\gamma)} \pi_l^{(\gamma)} f^l_{,rd} + \delta_3^b v_3^c v^r_{(\gamma)} \pi_l^{(\gamma)} f^l_{,ra} + \delta^b_{(\gamma)} v_3^c v_3^r \pi_l^{(\gamma)} f^l_{,ar} - v_3^c v_3^d v^r_{(\gamma)} \pi_a^{(\gamma)} \pi_l^b f^l_{,rd} \tag{A.43}$$

Choosing $b \neq \{\gamma, 3\}$, equation (A.43) becomes

$$0 = v_3^d v^c_{(\gamma)} \pi_a^{(\gamma)} \pi_l^{(b \neq \{\gamma,3\})} f^l_{,cd} \tag{A.44}$$

Again, the choice of $\gamma \neq 3$ was free. If we had chosen $\gamma = 2$, then equation (A.44) would become

$$0 = v_3^d v^c_{(\gamma)} \pi_a^{(\gamma)} \pi_l^{(b>3)} f^l_{,cd} \tag{A.45}$$

$$0 = v_3^d v^c_{(\gamma)} \pi_a^{(\gamma)} \pi_l^1 f^l_{,cd} \tag{A.46}$$

Additionally, $\gamma = 1$ would give us

$$0 = v_3^d v^c_{(\gamma)} \pi_a^{(\gamma)} \pi_l^2 f^l_{,cd} \tag{A.47}$$

These last three equations can be written together as

$$0 = v_3^d v^c_{(\gamma)} \pi_a^{(\gamma)} \pi_l^{(\gamma)} f^l_{,cd} \tag{A.48}$$

Choosing $b = \gamma$, equation (A.43) becomes

$$v_3^d \pi_l^{(\gamma)} f^l_{,ad} = v_3^d v^c_{(\gamma)} \pi_a^{(\gamma)} \pi_l^{(\gamma)} f^l_{,cd} = 0 \tag{A.49}$$

Using equations (A.48) and (A.49), we can rewrite equation (A.43) as

$$v_3^r f^c_{,ra} = \delta_a^c v_3^d v^r_{(\gamma)} \pi_l^{(\gamma)} f^l_{,rd} + v_3^c v^r_{(\gamma)} \pi_l^{(\gamma)} f^l_{,ra} - v_3^c \pi_a^{(\gamma)} v_3^d v^r_{(\gamma)} \pi_l^3 f^l_{,rd} \tag{A.50}$$

Differentiating with respect to v^p_γ gives

$$0 = v^c_3 \pi^{(\gamma)}_l f^l_{,pa} - v^c_3 v^r_{(\gamma)} \pi^{(\gamma)}_p \pi^{(\gamma)}_l f^l_{,ra} + v^c_3 \pi^{(\gamma)}_p \pi^{(\gamma)}_a v^d_3 v^r_{(\gamma)} \pi^3_l f^l_{,rd} - v^c_3 \pi^{(\gamma)}_a v^d_3 \pi^3_l f^l_{,pd} \tag{A.51}$$

Contracting equation (A.51) on $c = p$, we get

$$0 = v^p_3 v^d_3 \pi^3_l f^l_{,pd} \tag{A.52}$$

This lets us rewrite equation (A.41) as

$$v^r_3 v^d_3 \pi^a_s f^s_{,rd} = 0 \tag{A.53}$$

Which then allows us to rewrite equation (A.42) as

$$v^r_3 v^d_3 f^a_{,rd} = 0 \tag{A.54}$$

and upon differentiating twice, we see that

$$f^a_{,bc} = 0. \tag{A.55}$$

This now says that $g^1_c = 0$, and so

$$g^a_c = 0 \tag{A.56}$$

We are now able to write

$$\begin{aligned} X^i &= f^i(x) \\ X^i_j &= -\pi^i_k f^k_{,j}(x) \end{aligned} \tag{A.57}$$

Thus, equation (A.3) becomes

$$X \lrcorner \, \mathrm{d}\phi^i = -\pi^i_k f^k_{,j} \mathrm{d}x^j + \left[-\delta^i_j f^k + \delta^i_1 v^c_j v^k_3 \pi^2_b f^b_{,c} - \delta^i_1 \delta^2_j v^m_3 f^k_{,m} \right] \mathrm{d}\pi^j_k \tag{A.58}$$

And we are able to solve the equation $\mathrm{d}\hat{F}^i = -X_{hook} \mathrm{d}\phi^i$ for the rank 1 observable $\hat{F} = \hat{F}^i \hat{r}_i$.

$$\begin{aligned} \mathrm{d}\hat{F}^i &= \hat{F}^i_{,j} \mathrm{d}x^j + \frac{\partial \hat{F}^i}{\partial \pi^j_k} \\ &= \pi^i_k f^k_{,j} \mathrm{d}x^j + \left[\delta^i_j f^k - \delta^i_1 v^c_j v^k_3 \pi^2_b f^b_{,c} + \delta^i_1 \delta^2_j v^m_3 f^k_{,m} \right] \mathrm{d}\pi^j_k \end{aligned} \tag{A.59}$$

Using linear independence of the 1-forms, we have the two sets of equations

$$\hat{F}^i_{,j} = \pi^i_k f^k_{,j} \tag{A.60}$$

$$\frac{\partial \hat{F}^i}{\partial \pi^j_k} = \delta^i_j f^k - \delta^i_1 v^c_j v^k_3 \pi^2_b f^b_{,c} + \delta^i_1 \delta^2_j v^m_3 f^k_{,m} \tag{A.61}$$

Equation (A.60) can be integrated, giving

$$\hat{F}^i = \pi^i_k f^k + h^i(\pi) \tag{A.62}$$

Differentiating equation (A.62) and comparing to equation (A.61), we see that

$$\frac{\partial h^i(\pi)}{\partial \pi^j_k} = -\delta^i_1 v^c_j v^k_3 \pi^2_b f^b_{,c} + \delta^i_1 \delta^2_j v^c_3 f^k_{,c} = \frac{\partial}{\partial \pi^j_k}\left(\delta^i_1 v^c_3 \pi^2_b f^b_{,c}\right) \tag{A.63}$$

Thus the only rank 1 observables $\hat{F} = \hat{F}^i \hat{r}_i$ that can satisfy equation (2.7) are

$$\hat{F}^i = \pi^i_k f^k + \gamma^{bi}_a v^m_b \pi^a_c f^c_{,m} + \xi^i \tag{A.64}$$

where f^k are arbitrary linear functions of the x^i, and ξ^i are constant functions. □

A.2 Rank 2

Before we begin the proof of the statement made concerning rank observables, we prove two helpful lemmas.

Lemma A.1 *If* $\frac{\partial}{\partial \pi^e_g} F^{ab}_{cd} = 0$ *and* $v^d_b \pi^c_a F^{ab}_{cd} = 0$*, then* $F^{ab}_{cd} = 0$*.*

Proof: This statement can be proved directly by differentiating multiple times with respec to π, multiplying through by multiple πs, differentiating multiple more times with respect to π, and finally contracting many of the indices and resubstituting. A more concise way to prove this lemma is to take a Taylor series expansion of v^d_b in terms of πs at a point u_0 where $v^d_b(u_0) = \delta^d_b$. This gives

$$v^d_b = \delta^d_b + \frac{1}{2}\pi^d_k \pi^k_b + \dots$$

Distributing the multiplication through this sum, we have

$$v^d_b \pi^c_a F^{ab}_{cd} = \pi^c_a F^{ad}_{cd} + \frac{1}{2}\pi^c_a F^{ab}_{cd} \pi^d_k \pi^k_b + \dots = 0$$

By the linear independence of the π^a_b coordinates, each term in the sum must, individually, be equal to 0. First, $\pi^c_a F^{ad}_{cd} = 0$. Differentiating with respect to π^r_s gives

$$F^{sd}_{rd} = 0$$

If, instead, we had taken a Taylor series expansion of π^c_a in terms of the vs, the first term would have yielded $F^{ds}_{dr} = 0$. Next, we have $\pi^c_a \pi^d_k \pi^k_b F^{ab}_{cd} = 0$. Differentiating with respect to π^r_s, π^t_u, and then π^v_w gives

$$\delta^u_v F^{sw}_{rt} + \delta^w_t F^{su}_{rv} + \delta^s_v F^{uw}_{tr} + \delta^s_t F^{wu}_{vr} + \delta^w_r F^{us}_{tv} + \delta^u_r F^{ws}_{vt} = 0 \tag{A.65}$$

Contracting on u and v gives

$$nF^{sw}_{rt} + F^{sw}_{tr} + \delta^s_t F^{wu}_{ur} + \delta^w_r F^{us}_{tu} + F^{ws}_{rt} = 0$$

Contracting this equation on s and t then w and r shows $F^{ab}_{ba} = 0$. Returning to (A.65) and contracting on s and t then w and r leaves

$$F^{wu}_{vw} + F^{uw}_{wv} = 0$$

Returning again to equation (A.65) and contracting on u and t gives

$$F^{sw}_{rv} + F^{ws}_{vr} = 0$$

These identities now reduce (**??**) to

$$F^{sw}_{rt} = 0$$

the desired result. □

Lemma A.2 *Consider a collection of functions F^{bqsA}_{ap} where the lower-case indices take on values $1, \ldots, n$ and upper-case indices take on values $2, \ldots, n$. If F^{bqsA}_{ap} satisfies*

1. $F^{bqs(A}_{ap}\delta^{B)}_r = F^{bsq(A}_{ar}\delta^{B)}_p$
2. $F^{bqsA}_{ap} = F^{qbsA}_{pa}$

then $n > 2 \Rightarrow F^{bqsA}_{ap} = 0$.

Proof: We have $F^{bqs(A}_{ap}\delta^{B)}_r = F^{bsq(A}_{ar}\delta^{B)}_p$. First, notice that $p = 1 \Rightarrow F^{bqsA}_{a1} = 0$; similarly, $a = 1 \Rightarrow F^{bqsA}_{1p} = F^{qbsA}_{p1} = 0$. Now, sum all the terms in which r and A take on the same values.

$$nF^{bqsB}_{ap} = \delta^B_p F^{bsqA}_{aA} + \delta^A_p F^{bsqB}_{aA}.$$

If we were simply to write $\delta_p^A F_{aA}^{bsqB} = F_{ap}^{bsqB}$, we would be incorrect; A cannot take on the value 1 and p can. The fact that $F_{a1}^{bsqB} = 0$ allows us to perform this index trickery and write

$$nF_{ap}^{bqsB} = \delta_p^B F_{aA}^{bsqA} + F_{ap}^{bsqB}. \tag{A.66}$$

Form the sum of the terms in which B and p take on the same values.

$$nF_{aB}^{bqsB} = nF_{aA}^{bsqA} \Rightarrow F_{aB}^{bqsB} = F_{aB}^{bsqB}. \tag{A.67}$$

Return to equation (A.66) and sum the terms in which B and a take on the same values to get

$$nF_{Bp}^{bqsB} = F_{pA}^{bsqA} + F_{Bp}^{bsqB}. \tag{A.68}$$

Again, we are able to use the fact that $F_{1A}^{bsqA} = 0$ to allow us to write $\delta_p^B F_{BA}^{bsqA} = F_{pA}^{bsqA}$. This and equation (A.67) allow us to write

$$nF_{Bp}^{bsqB} = F_{pB}^{bsqB} + F_{Bp}^{bqsB}. \tag{A.69}$$

Combine equations (A.68) and (A.69) to see that

$$\begin{aligned} n^2 F_{Bp}^{bqsB} &= nF_{pB}^{bsqB} + nF_{Bp}^{bsqB} = (n+1)F_{pB}^{bsqB} + F_{Bp}^{bqsB} \\ (n^2-1)F_{Bp}^{bqsB} &= (n+1)F_{pB}^{bsqB} \\ (n-1)F_{Bp}^{bqsB} &= F_{pB}^{bsqB} \end{aligned} \tag{A.70}$$

As F_{pB}^{bsqB} is symmetric in s and q, F_{Bp}^{bqsB} must be as well. Now, let's manipulate equation (A.70) using the symmetries of F:

$$\begin{aligned} (n-1)F_{Bp}^{bqsB} &= F_{pB}^{bsqB} = F_{Bp}^{sbqB} = F_{Bp}^{sqbB} = F_{pB}^{qsbB} = F_{pB}^{qbsB} \\ &= F_{Bp}^{bqsB}. \end{aligned}$$

So long as $n \neq 2$, this implies $F_{Bp}^{bqsB} = 0 = F_{pB}^{bqsB}$. This allows equation (A.66) to be written

$$nF_{ap}^{bqsB} = F_{ap}^{bsqB}; \tag{A.71}$$

which tells us that

$$nF_{ap}^{bsqB} = F_{ap}^{bqsB}. \tag{A.72}$$

Combining these last two equations (as before) shows

$$n^2 F^{bqsB}_{ap} = F^{bqsB}_{ap} \Rightarrow F^{bqsB}_{ap} = 0$$

as desired. $\square$

Returning now to equation (A.2), we contract this 2-form with the vector fields $X^i = X^{si}\partial_s + X^{ji}_k \frac{\partial}{\partial \pi^j_k}$ and symmetrize on the upper indices

$$X^{(i} \lrcorner\, \mathrm{d}\phi^{j)} = X^{(ij)}_k \mathrm{d}x^k - \left[X^{s(i}\delta^{j)}_r - \Sigma^{s(ij)}_r\right] \mathrm{d}\pi^r_s \tag{A.73}$$

where $\Sigma^{sij}_r = v^c_l v^m_3 \delta^j_1 \left(X^{li}_m \delta^2_r \delta^s_c - X^{2i}_c \delta^l_r \delta^s_m\right)$. As before, we calculate the exterior derivative of equation (2.7) and use the linear independence of the anti-symmetric 2-forms to arrive at three sets of equations

$$X^{(ij)}_{k,l} = X^{(ij)}_{l,k} \tag{A.74}$$

$$\frac{\partial X^{(ij)}_k}{\partial \pi^r_s} = -X^{s(i}_{,k}\delta^{j)}_r + \Sigma^{s(ij)}_{r,k} \tag{A.75}$$

$$\frac{\partial X^{s(i}\delta^{j)}_r - \Sigma^{s(ij)}_r}{\partial \pi^p_q} = \frac{\partial X^{q(i}\delta^{j)}_p - \Sigma^{q(ij)}_p}{\partial \pi^r_s} \tag{A.76}$$

Notice that equations (A.76) can be expanded and rewritten as follows:

$$\frac{\partial X^{s(i}\delta^{j)}_r - \delta^2_r v^s_l v^m_3 X^{l(i}_m \delta^{j)}_1}{\partial \pi^p_q} + v^m_r v^s_3 \frac{\partial X^{2(i}_m \delta^{j)}_1}{\partial \pi^p_q} = \frac{\partial X^{q(i}\delta^{j)}_p - \delta^2_p v^q_l v^m_3 X^{l(i}_m \delta^{j)}_1}{\partial \pi^r_s} + v^m_p v^q_3 \frac{\partial X^{2(i}_m \delta^{j)}_1}{\partial \pi^r_s} \tag{A.77}$$

Choosing $i = \alpha_1$ and $j = \alpha_2$, where α_1 and α_2 are independent values other than 1, this then collapses to

$$\frac{\partial X^{s(\alpha_1}\delta^{\alpha_2)}_r}{\partial \pi^p_q} = \frac{\partial X^{q(\alpha_1}\delta^{\alpha_2)}_p}{\partial \pi^r_s} \tag{A.78}$$

Taking the derivative of both sides, we get

$$\frac{\partial}{\partial \pi^a_b} \frac{\partial X^{s(\alpha_1}\delta^{\alpha_2)}_r}{\partial \pi^p_q} = \frac{\partial}{\partial \pi^a_b} \frac{\partial X^{q(\alpha_1}\delta^{\alpha_2)}_p}{\partial \pi^r_s}$$

By Lemma A.2, $F^{bqs\alpha}_{ap} = \frac{\partial^2 X^{s\alpha}}{\partial \pi^a_b \partial \pi^c_d} = 0$. We write $X^{s\alpha} = A^{bs\alpha}_a \pi^a_b + g^{s\alpha}$. Now, equation (A.78) becomes

$$A^{qs(\alpha_1}_p \delta^{\alpha_2)}_r = A^{sq(\alpha_1}_r \delta^{\alpha_2)}_p \tag{A.79}$$

Choosing $\alpha_1 = \alpha_2 = \alpha$, we can then vary r and p to see the following:

$$\begin{aligned} r = 1 &\Rightarrow A_1^{qs\alpha} = 0 \\ r = \alpha, p = \alpha &\Rightarrow A_{(\alpha)}^{qs(\alpha)} = A_{(\alpha)}^{sq(\alpha)} \\ r = \alpha &\Rightarrow A_p^{qs\alpha} = \delta_p^{(\alpha)} A_{(\alpha)}^{qs(\alpha)} \end{aligned}$$

Returning then to equation (A.79) and choosing $r = \alpha_1$, $p = \alpha_2$, and $\alpha_1 \neq \alpha_2$, we see that

$$A_{(\alpha_1)}^{qs(\alpha_1)} = A_{(\alpha_2)}^{qs(\alpha_2)}$$

This allows us now to write $A_a^{qs\alpha} = f^{qs}\delta_a^{\alpha}$, where $f^{qs} = f^{sq}$. Thus,

$$X^{s\alpha} = f^{as}\pi_a^{\alpha} + g^{s\alpha} \tag{A.80}$$

Next, return to equations (A.75) let $i = \alpha_1$ and $j = \alpha_2$ as before. We have

$$\begin{aligned} \frac{\partial X_k^{(\alpha_1\alpha_2)}}{\partial \pi_s^r} &= -X_{,k}^{s(\alpha_1}\delta_r^{\alpha_2)} \\ &\Rightarrow \frac{\partial^3 X_k^{(\alpha_1\alpha_2)}}{\partial \pi_b^a \partial \pi_d^c \partial \pi_s^r} = 0. \end{aligned} \tag{A.81}$$

Write $X_k^{(\alpha_1\alpha_2)} = B_{ack}^{bd(\alpha_1\alpha_2)}\pi_b^a\pi_d^c + 2C_{ak}^{b(\alpha_1\alpha_2)}\pi_b^a + D_k^{(\alpha_1\alpha_2)}$. Noticing that $B_{ack}^{bd(\alpha_1\alpha_2)} = B_{cak}^{db(\alpha_1\alpha_2)}$, equation (A.81) can be written as

$$2B_{rak}^{sb(\alpha_1\alpha_2)}\pi_b^a + 2C_{rk}^{s(\alpha_1\alpha_2)} = -f_{,k}^{bs}\pi_b^{(\alpha_1}\delta_r^{\alpha_2)} - g_{,k}^{s(\alpha_1}\delta_r^{\alpha_2)}$$

Taking a derivative of this equation with respect to π_d^c we see that $B_{ack}^{bd(\alpha_1\alpha_2)} = -\frac{1}{2}f_{,k}^{bd}\delta_a^{(\alpha_1}\delta_c^{\alpha_2)}$, which says that $C_{ak}^{b(\alpha_1\alpha_2)} = -\frac{1}{2}g_{,k}^{s(\alpha_1}\delta_r^{\alpha_2)}$. Now, we can write

$$X_k^{(\alpha_1\alpha_2)} = -\frac{1}{2}\left(f_{,k}^{ab}\pi_a^{\alpha_1}\pi_b^{\alpha_2} + 2g_{,k}^{b(\alpha_1}\pi_b^{\alpha_2)} + 2h_k^{(\alpha_1\alpha_2)}\right)$$

Return to equation (A.77), and choosing $i = 1$ and $j = \alpha$ gives

$$\frac{\partial X^{s1}\delta_r^{\alpha} + X^{s\alpha}\delta_r^1 - \delta_r^2 v_l^s v_3^m X_m^{l\alpha}}{\partial \pi_q^p} + v_r^m v_3^s \frac{\partial X_m^{2\alpha}}{\partial \pi_q^p} = \frac{\partial X^{q1}\delta_p^{\alpha} + X^{q\alpha}\delta_p^1 - \delta_p^2 v_l^q v_3^m X_m^{l\alpha}}{\partial \pi_s^r} + v_p^m v_3^q \frac{\partial X_m^{2\alpha}}{\partial \pi_s^r} \tag{A.82}$$

Noticing that $\frac{\partial}{\partial \pi_q^1} X_m^{2\alpha} = \frac{\partial}{\partial \pi_q^1} X^{s\alpha} = 0$, choose $p = 1$ and the equation becomes

$$\frac{\partial X^{s1}\delta_r^\alpha + X^{s\alpha}\delta_r^1 - \delta_r^2 v_l^s v_3^m X_m^{l\alpha}}{\partial \pi_q^1} = \frac{\partial X^{q\alpha}}{\partial \pi_s^r} + v_1^m v_3^q \frac{\partial X_m^{2\alpha}}{\partial \pi_s^r}$$
$$\frac{\partial X^{s1}\delta_r^\alpha + X^{s\alpha}\delta_r^1 - \delta_r^2 v_l^s v_3^m X_m^{l\alpha}}{\partial \pi_q^1} = \frac{\partial X^{q\alpha}}{\partial \pi_s^r} - \frac{1}{2} v_1^m v_3^q \left(X_{,m}^{s2}\delta_r^\alpha + X_{,m}^{s\alpha}\delta_r^2\right) \tag{A.83}$$

Now, choosing $r = \alpha$ (the same α that was chosen for j) gives

$$\frac{\partial X^{s1} - \delta_{(\alpha)}^2 v_l^s v_3^m X_m^{l(\alpha)}}{\partial \pi_q^1} = \frac{\partial X^{q(\alpha)}}{\partial \pi_s^{(\alpha)}} - \frac{1}{2} v_1^m v_3^q \left(X_{,m}^{s2} + X_{,m}^{s(\alpha)}\delta_{(\alpha)}^2\right) \tag{A.84}$$

The index α can be any value other than 1. Choosing α to be any value other than 2, the equation becomes

$$\frac{\partial X^{s1}}{\partial \pi_q^1} = f^{sq} - \frac{1}{2} v_1^m v_3^q X_{,m}^{s2} \tag{A.85}$$

This allows us to write

$$X^{s1} = f^{bs}\pi_b^1 + \frac{1}{2} v_3^m X_{,m}^{s2} + G^{s1}, \tag{A.86}$$

where $\frac{\partial}{\partial \pi_q^1} G^{s1} = 0$. Using equation (B.15), we simplify equation (A.84) to

$$-\frac{\partial v_l^s v_3^m X_m^{l\alpha}}{\partial \pi_q^1} = -\frac{1}{2} v_1^m v_3^q X_{,m}^{s\alpha} \tag{A.87}$$

Expanding the derivative on the left-hand side and multiplying through by π_s^c gives

$$v_3^m \frac{\partial X_m^{c\alpha}}{\partial \pi_q^1} - \delta_1^c v_l^q v_3^m X_m^{l\alpha} - v_1^m v_3^q X_m^{c\alpha} = \frac{1}{2} v_1^m v_3^q X_{,m}^{s\alpha}\pi_s^c. \tag{A.88}$$

Choosing $c = \alpha_1$ and noticing that $\frac{\partial}{\partial \pi_q^1} X_m^{\alpha_1\alpha} = 0$, this simplifies to

$$v_1^m \left(g_{,m}^{b\alpha_1}\pi_b^\alpha + 2h_m^{\alpha_1\alpha}\right) = 0 \tag{A.89}$$

Taking the derivative with respect to v_c^a, we see that

$$\delta_1^c \pi_b^\alpha g_{,a}^{b\alpha_1} + 2\delta_1^c h_a^{\alpha_1\alpha} - v_1^m \pi_a^\alpha \pi_b^c g_{,m}^{b\alpha_1} = 0.$$

Multiplying by v_c^a gives

$$v_1^a h_a^{\alpha_1\alpha} = 0$$

Since $\frac{\partial}{\partial \pi_c^b} h_a^{\alpha_1 \alpha} = \frac{\partial}{\partial v_c^b} h_a^{\alpha_1 \alpha} = 0$, this implies

$$h_a^{\alpha_1 \alpha} = 0$$

Which by equation (A.89) and Lemma A.1 implies

$$g_{,r}^{s\alpha} = 0$$

Return now to equations (A.75). Choosing $i = 1$ and $j = \alpha$ gives

$$2\frac{\partial X_k^{1\alpha}}{\partial \pi_s^r} = -X_{,k}^{s1}\delta_r^\alpha - X_{,k}^{s\alpha}\delta_r^1 + v_l^c v_3^m \left[X_{m,k}^{l\alpha}\delta_r^2\delta_c^s - X_{c,k}^{2\alpha}\delta_r^l\delta_m^s\right] \tag{A.90}$$

Then, choosing $r = 1$ and multiplying through by v_3^k, we see that

$$2v_3^k \frac{\partial X_k^{1\alpha}}{\partial \pi_s^1} = -v_3^k X_{,k}^{s\alpha} - v_3^s v_1^m v_3^k X_{m,k}^{2\alpha} \tag{A.91}$$

Choosing $c = 1$ in equation (A.88) gives another expression for $v_3^k \frac{\partial X_k^{1\alpha}}{\partial \pi_s^1}$,

$$v_3^k \frac{\partial X_k^{1\alpha}}{\partial \pi_s^1} = v_l^s v_3^k X_k^{l\alpha} + v_1^k v_3^s X_k^{c\alpha} + \frac{1}{2} v_1^k v_3^s X_{,k}^{d\alpha}\pi_d^c. \tag{A.92}$$

Equating these last to expressions and multiplying through by π_s^γ gives

$$2v_3^k X_k^{\gamma\alpha} = -v_3^k f_{,k}^{bs}\pi_b^\alpha\pi_s^\gamma \tag{A.93}$$

In particular, $\gamma = 1$ allows us to write

$$X_k^{i\alpha} = -\frac{1}{2} f_{,k}^{bs}\pi_s^i\pi_b^\alpha + \chi_k^{i\alpha}, \tag{A.94}$$

where $\chi_k^{i\alpha} = \delta_1^i \chi_k^{1\alpha}$ and $v_3^k \chi_k^{1\alpha} = 0$. Using this form, we can rewrite equation (A.90) as

$$2\frac{\partial \chi_k^{1\alpha}}{\partial \pi_s^r} = \delta_r^\alpha \left(\frac{1}{2} v_3^m X_{,mk}^{s2} + G_{,k}^{s1}\right) - v_l^c v_3^m \left[X_{m,k}^{l\alpha}\delta_r^2\delta_c^s - X_{c,k}^{2\alpha}\delta_r^l\delta_m^s\right] \tag{A.95}$$

Notice that

$$\frac{\partial v_3^k \chi_k^{1\alpha}}{\partial \pi_s^r} = 0 = v_3^k \frac{\partial \chi_k^{1\alpha}}{\partial \pi_s^r} - v_r^k v_3^s \chi_k^{1\alpha};$$

therefore,

$$\delta_3^d \chi_c^{1\alpha} = \pi_s^d \pi_c^r v_3^k \frac{\partial \chi_k^{1\alpha}}{\partial \pi_s^r}.$$

So we may use this to expand equation (A.95) to

$$4\delta_3^d\chi_c^{1\alpha} = \pi_a^d\pi_c^\alpha v_3^k v_3^m f_{,mk}^{ab}\pi_b^2 + \pi_c^2 v_3^k v_3^m f_{,mk}^{ab}\pi_a^d\pi_b^\alpha - \delta_3^d v_3^k f_{,ck}^{ab}\pi_a^2\pi_b^\alpha + 2\pi_s^d\pi_c^\alpha v_3^k G_{,k}^{s1} \tag{A.96}$$

Choosing $d = \gamma$ and multiplying by v_r^c gives

$$0 = \delta_r^\alpha\pi_a^\gamma v_3^k v_3^m f_{,mk}^{ab}\pi_b^2 + \delta_r^2 v_3^k v_3^m f_{,mk}^{ab}\pi_a^\gamma\pi_b^\alpha + 2\delta_r^\alpha\pi_s^\gamma v_3^k G_{,k}^{s1}.$$

Choosing $r = 2$ and $\alpha = 3$ shows

$$v_3^k v_3^m f_{,mk}^{ab}\pi_a^\gamma\pi_b^3 = 0, \tag{A.97}$$

and choosing $r = 3 = \alpha$ we see that

$$\pi_a^\gamma v_3^k v_3^m f_{,mk}^{ab}\pi_b^2 + 2\pi_s^\gamma v_3^k G_{,k}^{s1} = 0. \tag{A.98}$$

These two identities allow us to simplify equation (A.96) after choosing $d = 3$ and multiplying by v_3^c.

$$v_3^k v_3^m f_{,mk}^{sb}\pi_b^2 + 2v_3^k G_{,k}^{s1} = 0 \tag{A.99}$$

This simplifies equation (A.96) to

$$4\chi_c^{1\alpha} = v_3^m v_3^k f_{,mk}^{ab}\pi_a^3\pi_b^\alpha\pi_c^2 - v_3^k f_{,ck}^{ab}\pi_a^2\pi_b^\alpha. \tag{A.100}$$

Equation (A.82) now reduces all the way down to

$$\delta_r^\alpha\frac{\partial G^{s1}}{\partial\pi_q^p} = \delta_p^\alpha\frac{\partial G^{q1}}{\partial\pi_s^r} \tag{A.101}$$

Choosing $r = \alpha = 2$, we get

$$\frac{\partial G^{s1}}{\partial\pi_q^p} = \delta_p^2\frac{\partial G^{q1}}{\partial\pi_s^2}$$

Additionally choosing $p = 2$ gives

$$\frac{\partial G^{s1}}{\partial\pi_q^2} = \frac{\partial G^{q1}}{\partial\pi_s^2},$$

which implies

$$\frac{\partial G^{s1}}{\partial\pi_q^p} = \delta_p^2\frac{\partial G^{s1}}{\partial\pi_q^2}.$$

Choosing $p = 2$ and $r = \alpha = 3$ in equation (A.101) gives

$$\frac{\partial G^{s1}}{\partial\pi_q^2} = 0 \Rightarrow \frac{\partial G^{s1}}{\partial\pi_q^p} = 0 \tag{A.102}$$

Looking back to equations (A.99) and (A.100), notice that

$$\begin{aligned} v_3^c \chi_c^{1\alpha} = 0 &= -v_3^c v_3^k f^{ab}_{,ck} \pi_a^2 \pi_b^\alpha; \\ 0 &= \pi_s^\alpha \left(v_3^k v_3^m f^{sb}_{,mk} \pi_b^2 + 2 v_3^k G^{s1}_{,k} \right) \\ &= \pi_s^\alpha v_3^k G^{s1}_{,k}. \end{aligned}$$

By Lemma A.1, $G^{s1}_{,k} = 0$. Now equation (A.99) takes on the form

$$v_3^k v_3^m f^{sb}_{,mk} \pi_b^2 = 0$$

Taking the derivative with respect to v_c^a gives

$$2\delta_3^c v_3^m f^{sb}_{,ma} \pi_b^2 = v_3^k v_3^m f^{sb}_{,mk} \pi_a^2 \pi_b^c. \tag{A.103}$$

Contracting on a and s, choosing $c = 3$, and equation (A.98) give $v_3^m f^{bs}_{,ms} \pi_b^2 = 0$. Using Lemma A.1, this says that

$$f^{bs}_{,ms} = 0.$$

Next, taking the derivative of equation (A.103) with respect to v_i^h and multiplying through by v_c^j, we see that

$$2v_3^j \left(\delta_3^i f^{sb}_{,ha} \pi_b^2 - v_3^m f^{sb}_{,ma} \pi_h^2 \pi_b^i \right) = 2\delta_3^i v_3^k f^{sj}_{,hk} \pi_a^2 - \left(\pi_h^2 \pi_a^i \delta_b^j + \delta_h^j \pi_a^2 \pi_b^i \right) v_3^k v_3^m f^{sb}_{,mk}. \tag{A.104}$$

Contracting on j and h and comparing to equation (A.103) gives

$$v_3^k v_3^m f^{sb}_{,mk} \pi_a^2 \pi_b^i = -n v_3^k v_3^m f^{sb}_{,mk} \pi_a^2 \pi_b^i, \tag{A.105}$$

thus $v_3^k v_3^m f^{sb}_{,mk} \pi_a^2 \pi_b^i = 0$, which reduces equation (A.103) to $v_3^m f^{sb}_{,ma} \pi_b^2 = 0$, and Lemma A.1 shows us that

$$f^{sb}_{,ma} = 0. \tag{A.106}$$

Returning to equations (A.75), choose $i = j = 1$.

$$\frac{\partial X_k^{11}}{\partial \pi_s^r} = -\delta_r^1 f^{bs}_{,k} \pi_b^1 + \delta_r^2 v_l^s v_3^m X^{l1}_{m,k} \tag{A.107}$$

Noticing that $X^{l1}_{m,k} = \delta_1^l X^{11}_{m,k}$, we can write

$$\frac{\partial X_k^{11}}{\partial \pi_s^r} = -\delta_r^1 f^{bs}_{,k} \pi_b^1 + \delta_r^2 v_1^s v_3^m X^{11}_{m,k} \tag{A.108}$$

Choosing $r=1$ gives $\frac{\partial}{\partial \pi_s^1} X_k^{11} = -f_{,k}^{bs}\pi_b^1$. Write

$$X_k^{ij} = -\frac{1}{2} f_{,k}^{ab}\pi_a^i\pi_b^j + \chi_k^{ij}, \tag{A.109}$$

where $\chi_k^{ij} = \delta_1^i\delta_1^j\chi_k^{11}$ and $\frac{\partial}{\partial \pi_s^1}\chi_k^{11} = 0$. This form changes equation (A.108) into

$$\frac{\partial \chi_k^{11}}{\partial \pi_s^r} = \delta_r^2 v_1^s v_3^m \chi_{m,k}^{11} \tag{A.110}$$

Choosing $i=j=1$, we can now reduce equation (A.77) to

$$\delta_p^2\left(v_r^q v_1^s v_3^m + v_1^q v_r^m v_3^s\right)\chi_m^{11} = \delta_r^2\left(v_p^s v_1^q v_3^m + v_1^s v_p^m v_3^q\right)\chi_m^{11} \tag{A.111}$$

Multiplying by $\pi_q^\alpha \pi_s^b$, we get

$$\delta_p^2\delta_r^\alpha\delta_1^b v_3^m\chi_m^{11} = \delta_r^2\delta_1^b\delta_3^\alpha v_p^m\chi_m^{11}$$

When $p=\beta$, we see that $v_\beta^m\chi_m^{11}=0$. Choosing $p=\alpha=2$ and $r=3$, we see that $v_2^m\chi_m^{11}=0$. Thus $v_a^m\chi_m^{11}=0$, and $\chi_b^{11}=0$.

Finally, we can fully describe the vector fields $X^i = X^i j\frac{\partial}{\partial x^j} + X_k^{ij}\frac{\partial}{\partial \pi_k^j}$.

$$\begin{aligned} X^{ij} &= f^{ai}\pi_a^j + g^{ij} \\ X_k^{ij} &= -\frac{1}{2} f_{,k}^{ab}\pi_a^i\pi_b^j \end{aligned} \tag{A.112}$$

Where f^{ai} and $g^{s\alpha}$ are functions on the base (functions of the x^k alone), f^{ai} is linear in x^k, $g^{s\alpha}$ is constant, and $g^{s1} = \frac{1}{2}v_3^m f_{,m}^{as}\pi_a^2 + G^{s1}$ where G^{s1} is constant. It will be easier momentarily if we write $g^{si} = \frac{1}{2}(\delta_1^i v_3^m \pi_a^2 f_{,m}^{as} + h^{si})$ where the h^{si} are all constant. We can also write

$$\Sigma_r^{s(ij)} = -\frac{1}{2}\pi_b^{(i}\delta_1^{j)}\frac{\partial(v_3^m f_{,m}^{ab}\pi_a^2)}{\partial \pi_s^r}.$$

As before, to solve the equation $\mathrm{d}\hat{F}^{(ij)} = -2X^{(i} \lrcorner\, \mathrm{d}\phi^{j)}$ for the unknown rank 2 observable $\hat{F} = \hat{F}^{ij}\hat{r}_i\hat{r}_j$, we begin by direct calculation.

$$\begin{aligned} \mathrm{d}\hat{F}^{(ij)} &= \hat{F}_k^{(ij)}\mathrm{d}x^k + \frac{\partial \hat{F}^{(ij)}}{\partial \pi_s^r}\mathrm{d}\pi_s^r \\ &= -2X_k^{(ij)}\mathrm{d}x^k + 2\left[X^{s(i}\delta_r^{j)} - \Sigma_r^{s(ij)}\right]\mathrm{d}\pi_s^r \\ &= f_{,k}^{ab}\pi_a^i\pi_b^j\mathrm{d}x^k + 2\left[f^{as}\pi_a^{(i}\delta_r^{j)} + g^{s(i}\delta_r^{j)} - \Sigma_r^{s(ij)}\right]\mathrm{d}\pi_s^r \end{aligned} \tag{A.113}$$

The linear independence of the 1-forms produces two sets of equations

$$\hat{F}^{(ij)}_{,k} = f^{ab}_{,k}\pi^i_a\pi^j_b \tag{A.114}$$

$$\frac{\partial \hat{F}^{(ij)}}{\partial \pi^r_s} = 2f^{as}\pi^{(i}_a\delta^{j)}_r + 2g^{s(i}\delta^{j)}_r - 2\Sigma^{s(ij)}_r \tag{A.115}$$

Integrating equation (A.114), we get

$$\hat{F}^{(ij)} = f^{ab}\pi^i_a\pi^j_b + \xi^{(ij)}(\pi)$$

and (A.115) tells us that

$$\frac{\partial \xi}{\partial \pi^r_s} = 2g^{s(i}\delta^{j)}_r - 2\Sigma^{s(ij)}_r$$

Notice that

$$\frac{\partial g^{b(i}\pi^{j)}_b}{\partial \pi^r_s} = g^{s(i}\delta^{j)}_r + \frac{1}{2}\pi^{(j}_b\delta^{i)}_1\frac{\partial v^m_3 f^{ab}_{,m}\pi^2_b}{\partial \pi^r_s} = g^{s(i}\delta^{j)}_r - \Sigma^{s(ij)}_r$$

So, $\xi^{(ij)} = 2g^{b(i}\pi^{j)}_b + k^{(ij)}$ where $k^{(ij)}$ are constant, and the only allowable rank 2 observables are

$$\hat{F}^{(ij)} = f^{ab}\pi^i_a\pi^j_b + h^{d(i}\pi^{j)}_d + k^{(ij)} + \gamma^{b(i}_a\pi^{j)}_d v^m_b\pi^a_c f^{cd}_{,m} \tag{A.116}$$

□

Appendix B

Proof of Theorem 2.3

We will let the use the Greek letters α, β, and γ to represent indices whose value is never 1, 2, or 3, respectively, as in the proof of Theorem 2.2. In the local coordinates x^i and π^i_j on LM, we have $\theta^i = \pi^i_j \mathrm{d}x^j$ and $\omega^a_b = \pi^a_c \left(\mathrm{d}v^c_b + \Gamma^c_{de} v^e_b \mathrm{d}x^d\right)$. The coordinates v^j_k are dual to the π^i_j in the sense that

$$
\begin{aligned}
v^j_k \pi^i_j &= \delta^i_k \\
v^j_k \pi^k_m &= \delta^j_m
\end{aligned}
$$

Choosing $\gamma^{bi}_a = \delta^b_3 \delta^2_a \delta^i_1$, we have

$$
\mathrm{d}\phi^j = \mathrm{d}\pi^j_k \wedge \mathrm{d}x^k + \left(\delta^j_1 v^c_l v^m_3\right) \mathrm{d}\pi^l_m \wedge \mathrm{d}\pi^2_c + \left(\frac{\partial H^j_k}{\partial \pi^l_m}\right) \mathrm{d}\pi^l_m \wedge \mathrm{d}x^k + \left(H^j_{k,l}\right) \mathrm{d}x^l \wedge \mathrm{d}x^k
$$

where $H^j_k = \delta^j_1 H_k = \delta^j_1 \left(\pi^2_c \Gamma^c_{km} v^m_3\right)$. Contracting this 2-form with the vector fields $X^i = X^{si}\partial_s + X^{ri}_s \frac{\partial}{\partial \pi^r_s}$ and symmetrizing, we have

$$
\begin{aligned}
X^{(i} \lrcorner\, \mathrm{d}\phi^{j)} = &\left[X^{(ij)}_k + X^{l(i}\left(H^{j)}_{k,l} - H^{j)}_{l,k}\right) + X^{l(i}_m \frac{\partial H^{j)}_k}{\partial \pi^l_m}\right] \mathrm{d}x^k \\
&- \left[X^{s(i}\delta^{j)}_r - \Sigma^{s(ij)}_r + X^{k(i}\frac{\partial H^{j)}_k}{\partial \pi^r_s}\right] \mathrm{d}\pi^r_s
\end{aligned} \tag{B.1}
$$

where $\Sigma^{sij}_r = v^c_l v^m_3 \delta^j_1 \left(X^{li}_m \delta^2_r \delta^s_c - X^{2i}_c \delta^l_r \delta^s_m\right)$. From equation (2.7) we know that $\mathrm{d}(X^i \lrcorner\, \mathrm{d}\phi^j) = 0$, and as before the linear independence of the 2-forms allows us to write out three sets of equations:

$$X^{(ij)}_{k,l} + \left[X^{m(i}(H^{j)}_{k,m} - H^{j)}_{m,k}) + X^{a(i}_b \frac{\partial H^{j)}_k}{\partial \pi^a_b}\right]_{,l} = X^{(ij)}_{l,k} + \left[X^{m(i}(H^{j)}_{l,m} - H^{j)}_{m,l}) + X^{a(i}_b \frac{\partial H^{j)}_l}{\partial \pi^a_b}\right]_{,k} \tag{B.2}$$

$$\frac{\partial\left[X^{(ij)}_k + X^{m(i}(H^{j)}_{k,m} - H^{j)}_{m,k}) + X^{a(i}_b \frac{\partial H^{j)}_k}{\partial \pi^a_b}\right]}{\partial \pi^r_s} = -X^{s(i}_{,k}\delta^{j)}_r + \Sigma^{s(ij)}_{r,k} - \left[X^{m(i}\frac{\partial H^{j)}_m}{\partial \pi^r_s}\right]_{,k} \tag{B.3}$$

$$\frac{\partial\left[X^{s(i}\delta^{j)}_r - \Sigma^{s(ij)}_r + X^{k(i}\frac{\partial H^{j)}_k}{\partial \pi^r_s}\right]}{\partial \pi^p_q} = \frac{\partial\left[X^{q(i}\delta^{j)}_p - \Sigma^{q(ij)}_p + X^{k(i}\frac{\partial H^{j)}_k}{\partial \pi^p_q}\right]}{\partial \pi^r_s} \tag{B.4}$$

Notice that equation (B.4) can be expanded and rewritten as follows:

$$\begin{aligned}\frac{\partial X^{s(i}\delta^{j)}_r}{\partial \pi^p_q} - \delta^2_r \frac{\partial v^s_l v^m_3 X^{l(i}_m \delta^{j)}_1}{\partial \pi^p_q} + v^m_r v^s_3 \frac{\partial X^{2(i}_m \delta^{j)}_1}{\partial \pi^p_q} + \frac{\partial H_k}{\partial \pi^r_s}\frac{\partial X^{k(i}\delta^{j)}_1}{\partial \pi^p_q} \\ = \frac{\partial X^{q(i}\delta^{j)}_p}{\partial \pi^r_s} - \delta^2_p \frac{\partial v^q_l v^m_3 X^{l(i}_m \delta^{j)}_1}{\partial \pi^r_s} + v^m_p v^q_3 \frac{\partial X^{2(i}_m \delta^{j)}_1}{\partial \pi^r_s} + \frac{\partial H_k}{\partial \pi^p_q}\frac{\partial X^{k(i}\delta^{j)}_1}{\partial \pi^r_s}\end{aligned} \tag{B.5}$$

Choosing $i = \alpha_1$ and $j = \alpha_2$, where α_1 and α_2 are independent values other than 1, this then collapses to

$$\frac{\partial X^{s(\alpha_1}\delta^{\alpha_2)}_r}{\partial \pi^p_q} = \frac{\partial X^{q(\alpha_1}\delta^{\alpha_2)}_p}{\partial \pi^r_s} \tag{B.6}$$

Taking the derivative of both sides, we get

$$\frac{\partial}{\partial \pi^a_b}\frac{\partial X^{s(\alpha_1}\delta^{\alpha_2)}_r}{\partial \pi^p_q} = \frac{\partial}{\partial \pi^a_b}\frac{\partial X^{q(\alpha_1}\delta^{\alpha_2)}_p}{\partial \pi^r_s} \tag{B.7}$$

Lemma A.2 tells us that $F^{bqs\alpha}_{ap} = \frac{\partial^2 X^{s\alpha}}{\partial \pi^a_b \partial \pi^p_q} = 0$, and so we write

$$X^{s\alpha} = f^{as}\pi^\alpha_a + g^{s\alpha} \tag{B.8}$$

Next, return to equations (B.3) let $i = \alpha_1$ and $j = \alpha_2$ as before. We have

$$\begin{aligned}\frac{\partial X^{(\alpha_1\alpha_2)}_k}{\partial \pi^r_s} &= -X^{s(\alpha_1}_{,k}\delta^{\alpha_2)}_r \\ \Rightarrow \frac{\partial^3 X^{(\alpha_1\alpha_2)}_k}{\partial \pi^a_b \partial \pi^c_d \partial \pi^r_s} &= 0.\end{aligned} \tag{B.9}$$

Write $X^{(\alpha_1\alpha_2)}_k = B^{bd(\alpha_1\alpha_2)}_{ack}\pi^a_b\pi^c_d + 2C^{b(\alpha_1\alpha_2)}_{ak}\pi^a_b + D^{(\alpha_1\alpha_2)}_k$. Noticing that $B^{bd(\alpha_1\alpha_2)}_{ack} = B^{db(\alpha_1\alpha_2)}_{cak}$, equation (B.9) can be written as

$$2B_{rak}^{sb(\alpha_1\alpha_2)}\pi_b^a + 2C_{rk}^{s(\alpha_1\alpha_2)} = -f_{,k}^{bs}\pi_b^{(\alpha_1}\delta_r^{\alpha_2)} - g_{,k}^{s(\alpha_1}\delta_r^{\alpha_2)} \tag{B.10}$$

Taking a derivative of this equation with respect to π_d^c we see that $B_{ack}^{bd(\alpha_1\alpha_2)} = -\frac{1}{2}f_{,k}^{bd}\delta_a^{(\alpha_1}\delta_c^{\alpha_2)}$, which says that $C_{ak}^{b(\alpha_1\alpha_2)} = -\frac{1}{2}g_{,k}^{s(\alpha_1}\delta_r^{\alpha_2)}$. Now, we can write

$$X_k^{(\alpha_1\alpha_2)} = -\frac{1}{2}\left(f_{,k}^{ab}\pi_a^{\alpha_1}\pi_b^{\alpha_2} + 2g_{,k}^{b(\alpha_1}\pi_b^{\alpha_2)} + 2h_k^{(\alpha_1\alpha_2)}\right) \tag{B.11}$$

Return to equation (B.5), and choosing $i = 1$ and $j = \alpha$ gives

$$\begin{aligned}\frac{\partial X^{s1}\delta_r^\alpha + X^{s\alpha}\delta_r^1}{\partial \pi_q^p} - \delta_r^2\frac{\partial v_l^s v_3^m X_m^{l\alpha}}{\partial \pi_q^p} + v_r^m v_3^s\frac{\partial X_m^{2\alpha}}{\partial \pi_q^p} + \frac{\partial H_k}{\partial \pi_s^r}\frac{\partial X^{k\alpha}}{\partial \pi_q^p} \\ = \frac{\partial X^{q1}\delta_p^\alpha + X^{q\alpha}\delta_p^1}{\partial \pi_s^r} - \delta_p^2\frac{\partial v_l^q v_3^m X_m^{l\alpha}}{\partial \pi_s^r} + v_p^m v_3^q\frac{\partial X_m^{2\alpha}}{\partial \pi_s^r} + \frac{\partial H_k}{\partial \pi_q^p}\frac{\partial X^{k\alpha}}{\partial \pi_s^r}\end{aligned} \tag{B.12}$$

Noticing that $\frac{\partial}{\partial \pi_q^1}X_m^{2\alpha} = \frac{\partial}{\partial \pi_q^1}X^{s\alpha} = 0$, choose $p = 1$ and the equation becomes

$$\begin{aligned}\delta_r^\alpha\frac{\partial X^{s1}}{\partial \pi_q^1} - \delta_r^2\frac{\partial v_l^s v_3^m X_m^{l\alpha}}{\partial \pi_q^1} = \delta_r^\alpha f^{sq} + \delta_r^\alpha\frac{\partial H_k}{\partial \pi_q^1}f^{sk} \\ + \frac{1}{2}\frac{\partial v_3^m}{\partial \pi_q^1}(\delta_r^2 f_{,m}^{as}\pi_a^\alpha + \delta_r^\alpha f_{,m}^{as}\pi_a^2 + \delta_r^2 g_{,m}^{s\alpha} + \delta_r^\alpha g_{,m}^{s2})\end{aligned} \tag{B.13}$$

Choosing $r = \alpha$ gives

$$\frac{\partial X^{s1}}{\partial \pi_q^1} - \delta_{(\alpha)}^2\frac{\partial v_l^s v_3^m X_m^{l\alpha}}{\partial \pi_q^1} = f^{sq} + \frac{1}{2}\frac{\partial v_3^m}{\partial \pi_q^1}(\delta_{(\alpha)}^2 f_{,m}^{as}\pi_a^\alpha + f_{,m}^{as}\pi_a^2 + \delta_{(\alpha)}^2 g_{,m}^{s\alpha} + g_{,m}^{s2}) + \frac{\partial H_k}{\partial \pi_q^1}f^{sk} \tag{B.14}$$

The index α can be any value other than 1. Choosing α to be any value other than 2, the equation becomes

$$\frac{\partial X^{s1}}{\partial \pi_q^1} = f^{sq} + \frac{1}{2}\frac{\partial v_3^m}{\partial \pi_q^1}(f_{,m}^{as}\pi_a^2 + g_{,m}^{s2}) + \frac{\partial H_k}{\partial \pi_q^1}f^{sk} \tag{B.15}$$

This allows us to write

$$X^{s1} = f^{as}\pi_a^1 + \frac{1}{2}v_3^m X_{,m}^{s2} + f^{sk}H_k + G^{s1} \tag{B.16}$$

where $\frac{\partial}{\partial \pi_q^1} G^{s1} = 0$. We may also rewrite equation (B.13) as

$$\begin{aligned} -\frac{\partial v_l^s v_3^m X_m^{l\alpha}}{\partial \pi_q^1} &= \frac{1}{2}\frac{\partial v_3^m}{\partial \pi_q^1}(f_{,m}^{as}\pi_a^\alpha + g_{,m}^{s\alpha}) \\ \frac{\partial v_l^s v_3^m X_m^{l\alpha}}{\partial \pi_q^1} &= -\frac{1}{2}\frac{\partial v_3^m}{\partial \pi_q^1} X_{,m}^{s\alpha} \end{aligned} \tag{B.17}$$

Expanding the derivative on the left-hand side and multiplying through by π_s^c gives

$$v_3^m \frac{\partial X_m^{c\alpha}}{\partial \pi_q^1} - \delta_1^c v_l^q v_3^m X_m^{l\alpha} - v_1^m v_3^q X_m^{c\alpha} = \frac{1}{2} v_1^m v_3^q X_{,m}^{s\alpha}\pi_s^c. \tag{B.18}$$

Choosing $c = \alpha_1$ and noticing that $\frac{\partial}{\partial \pi_q^1} X_m^{\alpha_1 \alpha} = 0$, are able to simplify down to

$$v_1^m \left(g_{,m}^{b\alpha_1} \pi_b^\alpha + 2h_m^{\alpha_1\alpha} \right) = 0 \tag{B.19}$$

Taking the derivative with respect to v_c^a, we see that

$$\delta_1^c \pi_b^\alpha g_{,a}^{b\alpha_1} + 2\delta_1^c h_a^{\alpha_1\alpha} - v_1^m \pi_a^\alpha \pi_b^c g_{,m}^{b\alpha_1} = 0. \tag{B.20}$$

Multiplying by v_c^a gives

$$v_1^a h_a^{\alpha_1\alpha} = 0 \tag{B.21}$$

Since $\frac{\partial}{\partial \pi_c^b} h_a^{\alpha_1\alpha} = \frac{\partial}{\partial v_c^b} h_a^{\alpha_1\alpha} = 0$, this implies

$$h_a^{\alpha_1\alpha} = 0 \tag{B.22}$$

Which by equation (B.19) and Lemma A.1 implies

$$g_{,r}^{s\alpha} = 0 \tag{B.23}$$

Choosing $c = 1$ in equation (B.18) gives

$$v_3^m \frac{\partial X_m^{1\alpha}}{\partial \pi_q^1} = -\frac{1}{2}\frac{\partial v_3^m}{\partial \pi_q^1} X_{,m}^{s\alpha}\pi_s^1 + v_l^q v_3^m X_m^{l\alpha} + v_1^m v_3^q X_m^{1\alpha}. \tag{B.24}$$

Return now to equations (B.3). Choosing $i = 1$ and $j = \alpha$ gives

$$
\begin{aligned}
2\frac{\partial X^{1\alpha}_k}{\partial \pi^r_s} + (H_{k,m} - H_{m,k})\frac{\partial X^{m\alpha}}{\partial \pi^r_s} + X^{m\alpha}\frac{\partial H_{k,m}}{\partial \pi^r_s} + \frac{\partial X^{a\alpha}_b}{\partial \pi^r_s}\frac{\partial H_k}{\partial \pi^a_b} + X^{a\alpha}_b\frac{\partial^2 H_k}{\partial \pi^r_s \partial \pi^a_b} \\
= -X^{s1}_{,k}\delta^\alpha_r - X^{s\alpha}_{,k}\delta^1_r + v^c_l v^m_3\left(X^{l\alpha}_{m,k}\delta^2_r\delta^s_c - X^{2\alpha}_{c,k}\delta^l_r\delta^s_m\right) - X^{m\alpha}_{,k}\frac{\partial H_m}{\partial \pi^r_s}
\end{aligned}
\tag{B.25}
$$

Choosing $r = 1$ and multiplying through by v^k_3, we can write

$$
\begin{aligned}
2v^k_3\frac{\partial X^{1\alpha}_k}{\partial \pi^1_s} + v^k_3 X^{m\alpha}\frac{\partial H_{k,m}}{\partial \pi^1_s} + v^k_3\frac{\partial X^{a\alpha}_b}{\partial \pi^1_s}\frac{\partial H_k}{\partial \pi^a_b} + v^k_3 X^{a\alpha}_b\frac{\partial^2 H_k}{\partial \pi^1_s \partial \pi^a_b} \\
= -v^k_3 X^{s\alpha}_{,k} - v^k_3 v^m_1 v^s_3 X^{2\alpha}_{m,k} - v^k_3 X^{m\alpha}_{,k}\frac{\partial H_m}{\partial \pi^1_s}
\end{aligned}
\tag{B.26}
$$

Using equation (B.24), this becomes

$$
\begin{aligned}
v^k_1 v^s_3 X^{m\alpha}_{,k}\pi^1_m - v^k_3\pi^2_c\Gamma^c_{km}v^m_1\left(\frac{1}{2}v^b_1 v^s_3 X^{e\alpha}_{,b}\pi^1_e + v^s_l v^b_3 X^{l\alpha}_b + v^b_1 v^s_3 X^{1\alpha}_b\right) + 2v^s_l v^k_3 X^{l\alpha}_k + 2v^k_1 v^s_3 X^{1\alpha}_k \\
+ v^k_3 X^{m\alpha}\frac{\partial H_{k,m}}{\partial \pi^1_s} + v^k_3 X^{a\alpha}_b\frac{\partial^2 H_k}{\partial \pi^1_s \partial \pi^a_b} = -v^k_3 X^{s\alpha}_{,k} - v^k_3 v^m_1 v^s_3 X^{2\alpha}_{m,k} - v^k_3 X^{m\alpha}_{,k}\frac{\partial H_m}{\partial \pi^1_s}
\end{aligned}
\tag{B.27}
$$

Multiplying through by π^γ_s gives

$$
2v^k_3 X^{\gamma\alpha}_k = -\pi^\gamma_s v^k_3 X^{s\alpha}_{,k} \tag{B.28}
$$

In particular, $\gamma = 1$ allows us to write

$$
X^{i\alpha}_k = -\frac{1}{2}f^{ab}_{,k}\pi^i_a\pi^\alpha_b + J^{i\alpha}_k, \tag{B.29}
$$

where $J^{i\alpha}_k = \delta^i_1 J^{1\alpha}_k$ and $v^k_3 J^{1\alpha}_k = 0$. We may now rewrite and simplify equation (B.12) as

$$
\delta^\alpha_r\frac{\partial G^{s1}}{\partial \pi^p_q} = \delta^\alpha_p\frac{\partial G^{q1}}{\partial \pi^r_s} \tag{B.30}
$$

Choosing $r = \alpha = 2$, we get

$$
\frac{\partial G^{s1}}{\partial \pi^p_q} = \delta^2_p\frac{\partial G^{q1}}{\partial \pi^2_s} \tag{B.31}
$$

Additionally choosing $p = 2$ gives

$$
\frac{\partial G^{s1}}{\partial \pi^2_q} = \frac{\partial G^{q1}}{\partial \pi^2_s}, \tag{B.32}
$$

which implies

$$\frac{\partial G^{s1}}{\partial \pi^p_q} = \delta^2_p \frac{\partial G^{s1}}{\partial \pi^2_q}. \tag{B.33}$$

Choosing $p = 2$ and $r = \alpha = 3$ in equation (B.30) gives

$$\frac{\partial G^{s1}}{\partial \pi^2_q} = 0 \Rightarrow \frac{\partial G^{s1}}{\partial \pi^p_q} = 0 \tag{B.34}$$

We next return to equation (B.25), which becomes

$$\begin{aligned}
2\frac{\partial J^{1\alpha}_k}{\partial \pi^r_s} + \delta^\alpha_r \pi^2_c \Gamma^c_{kl,m} v^l_3 f^{sm} + (f^{bm}\pi^\alpha_b + g^{m\alpha})\left(\delta^2_r \Gamma^s_{kl,m} v^l_3 - \pi^2_c \Gamma^c_{kl,m} v^l_r v^s_3\right) - v^b_3 \frac{\partial J^{1\alpha}_b}{\partial \pi^r_s} \pi^2_d \Gamma^d_{km} v^m_1 \\
-\frac{1}{2}\left(\Gamma^b_{km} v^m_3 f^{cs}_{,b} \delta^2_r \pi^\alpha_c + \Gamma^b_{km} v^m_3 f^{cs}_{,b} \delta^\alpha_r \pi^2_c\right) + \frac{1}{2}\delta^\alpha_r \pi^2_l \Gamma^l_{km} v^b_3 f^{ms}_{,b} \\
+ J^{d\alpha}_l \pi^2_c \Gamma^c_{km} v^m_d v^l_r v^s_3 - \frac{1}{2} f^{mb}_{,l} \pi^\alpha_b \pi^2_c \Gamma^c_{km} v^l_r v^s_3 + \frac{1}{2}\delta^2_r f^{mb}_{,l} \pi^\alpha_b \Gamma^s_{km} v^l_3 \\
+ \frac{1}{2} f^{ab}_{,l} \pi^2_a \pi^\alpha_b \Gamma^l_{km} v^m_r v^s_3 = -\frac{1}{2}\delta^\alpha_r v^m_3 f^{bs}_{,mk} \pi^2_b - \delta^\alpha_r f^{sm} \pi^2_c \Gamma^c_{ml,k} v^l_3 - \delta^\alpha_r G^{s1}_{,k} \\
-\frac{1}{2}\delta^2_r v^m_3 f^{sb}_{,mk} \pi^\alpha_b + \frac{1}{2} v^m_r v^s_3 f^{ab}_{,mk} \pi^2_a \pi^\alpha_b - \delta^2_r f^{bm}_{,k} \pi^\alpha_b \Gamma^s_{ml} v^l_3 + \pi^2_c \Gamma^c_{ml} v^l_r v^s_3 f^{bm}_{,k} \pi^\alpha_b
\end{aligned} \tag{B.35}$$

Using the fact that

$$\frac{\partial v^b_3 J^{1\alpha}_b}{\partial \pi^r_s} = 0 = v^b_3 \frac{\partial J^{1\alpha}_b}{\partial \pi^r_s} - v^b_r v^s_3 J^{1\alpha}_b$$

reduces this to

$$\begin{aligned}
2\frac{\partial J^{1\alpha}_k}{\partial \pi^r_s} + \delta^\alpha_r \pi^2_c \Gamma^c_{kl,m} v^l_3 f^{sm} + (f^{bm}\pi^\alpha_b + g^{m\alpha})\left(\delta^2_r \Gamma^s_{kl,m} v^l_3 - \pi^2_c \Gamma^c_{kl,m} v^l_r v^s_3\right) \\
-\frac{1}{2}\left(\Gamma^b_{km} v^m_3 f^{cs}_{,b} \delta^2_r \pi^\alpha_c + \Gamma^b_{km} v^m_3 f^{cs}_{,b} \delta^\alpha_r \pi^2_c\right) + \frac{1}{2}\delta^\alpha_r \pi^2_l \Gamma^l_{km} v^b_3 f^{ms}_{,b} \\
- \frac{1}{2} f^{mb}_{,l} \pi^\alpha_b \pi^2_c \Gamma^c_{km} v^l_r v^s_3 + \frac{1}{2}\delta^2_r f^{mb}_{,l} \pi^\alpha_b \Gamma^s_{km} v^l_3 \\
+ \frac{1}{2} f^{ab}_{,l} \pi^2_a \pi^\alpha_b \Gamma^l_{km} v^m_r v^s_3 = -\frac{1}{2}\delta^\alpha_r v^m_3 f^{bs}_{,mk} \pi^2_b - \delta^\alpha_r f^{sm} \pi^2_c \Gamma^c_{ml,k} v^l_3 - \delta^\alpha_r G^{s1}_{,k} \\
-\frac{1}{2}\delta^2_r v^m_3 f^{sb}_{,mk} \pi^\alpha_b + \frac{1}{2} v^m_r v^s_3 f^{ab}_{,mk} \pi^2_a \pi^\alpha_b - \delta^2_r f^{bm}_{,k} \pi^\alpha_b \Gamma^s_{ml} v^l_3 + \pi^2_c \Gamma^c_{ml} v^l_r v^s_3 f^{bm}_{,k} \pi^\alpha_b
\end{aligned} \tag{B.36}$$

Multiplying through by $v_3^k \pi_s^d$ gives

$$\begin{aligned}
2\delta_3^d v_r^k J_k^{1\alpha} + \delta_r^\alpha \pi_c^2 \Gamma^c_{kl,m} v_3^k v_3^l f^{sm} \pi_s^d + (f^{bm}\pi_b^\alpha + g^{m\alpha}) \left(\delta_r^2 \pi_s^d \Gamma^s_{kl,m} v_3^k v_3^l - \delta_3^d \pi_c^2 \Gamma^c_{kl,m} v_3^k v_r^l\right) \\
-\frac{1}{2}\left(\Gamma^b_{km} v_3^k v_3^m f^{cs}_{,b} \pi_s^d \delta_r^2 \pi_c^\alpha + \Gamma^b_{km} v_3^k v_3^m f^{cs}_{,b} \pi_s^d \delta_r^\alpha \pi_c^2\right) + \frac{1}{2}\delta_r^\alpha \pi_l^2 \Gamma^l_{km} v_3^k v_3^b f^{ms}_{,b} \pi_s^d \\
-\frac{1}{2}\delta_3^d f^{mb}_{,l} \pi_b^\alpha \pi_c^2 \Gamma^c_{km} v_3^k v_r^l + \frac{1}{2}\delta_r^2 f^{mb}_{,l} \pi_b^\alpha \pi_s^d \Gamma^s_{km} v_3^k v_3^l \\
+\frac{1}{2}\delta_3^d f^{ab}_{,l} \pi_a^2 \pi_b^\alpha \Gamma^l_{km} v_3^k v_r^m = -\frac{1}{2}\delta_r^\alpha v_3^m v_3^k f^{bs}_{,mk} \pi_s^d \pi_b^2 - \delta_r^\alpha f^{sm} \pi_s^d \pi_c^2 \Gamma^c_{ml,k} v_3^k v_3^l - \delta_r^\alpha \pi_s^d G^{s1}_{,k} v_3^k \\
-\frac{1}{2}\delta_r^2 v_3^m v_3^k f^{sb}_{,mk} \pi_s^d \pi_b^\alpha + \frac{1}{2}\delta_3^d v_r^m v_3^k f^{ab}_{,mk} \pi_a^2 \pi_b^\alpha - \delta_r^2 v_3^k f^{bm}_{,k} \pi_b^\alpha \pi_s^d \Gamma^s_{ml} v_3^l + \delta_3^d \pi_c^2 \Gamma^c_{ml} v_r^l v_3^k f^{bm}_{,k} \pi_b^\alpha
\end{aligned} \quad \text{(B.37)}$$

Differentiating with respect to v_z^y and then multiplying by v_z^y reduces this to

$$2\delta_3^d v_r^k J_k^{1\alpha} = -g^{m\alpha}\left(\delta_r^2 \pi_s^d \Gamma^s_{kl,m} v_3^k v_3^l - \delta_3^d \pi_c^2 \Gamma^c_{kl,m} v_3^k v_r^l\right) \quad \text{(B.38)}$$

Choosing $d = \gamma$, we see that

$$\pi_s^\gamma \Gamma^s_{kl,m} g^{m\alpha} v_3^k v_3^l = 0 \quad \text{(B.39)}$$

After a series of derivatives, contractions, and resubstitutions similar to those used in the proof of Lemma A.1, we end up with

$$\Gamma^s_{kl,m} g^{m\alpha} = 0; \quad \text{(B.40)}$$

Thus,

$$J_k^{1\alpha} = 0. \quad \text{(B.41)}$$

Then equation (B.37) reduces enough to use Lemma A.1 to show that

$$G^{s1}_{,k} = 0. \quad \text{(B.42)}$$

So, we will write $g^{s1} = G^{s1}$, and $X^{si} = f^{bs}\pi_b^i + g^{si} + \gamma_a^{bi}\left(\frac{1}{2} v_3^m f^{bs}_{,m} \pi_b^2 + f^{bs}\pi_c^2 \Gamma^c_{bl} v_b^l\right)$. Looking to equation (B.3) and choosing $i = j = 1$, we have

$$\frac{\partial X^{11}_k}{\partial \pi^r_s} + H_{k,m}\left(\delta^1_r f^{sm} + \frac{1}{2}\delta^2_r f^{sm}_{,l} v^l_3 - \frac{1}{2} v^l_r v^s_3 f^{bm}_{,l}\pi^2_b + \delta^2_r f^{bm}\Gamma^s_{bl} v^l_3 - f^{bm}\pi^2_c \Gamma^c_{bl} v^l_r v^s_3\right)$$
$$-H_{m,k}\left(\frac{1}{2}\delta^2_r f^{sm}_{,l} v^l_3 - \frac{1}{2} v^l_r v^s_3 f^{bs}_{,l}\pi^2_b + \delta^2_r f^{bm}\Gamma^s_{bl} v^l_3 - f^{bm}\pi^2_c \Gamma^c_{bl} v^l_r v^s_3\right)$$
$$+X^{m1}\frac{\partial H_{k,m}}{\partial \pi^r_s} + \frac{\partial H_k}{\partial \pi^a_b}\frac{\partial X^{a1}_b}{\partial \pi^r_s} + X^{a1}_b \frac{\partial^2 H_k}{\partial \pi^r_s \partial \pi^a_b}$$
$$= -\delta^1_r\left(f^{bs}_{,k}\pi^1_b + \frac{1}{2} v^m_3 f^{bs}_{,mk}\pi^2_b + f^{sb}_{,k}\pi^2_c\Gamma^c_{bl} v^l_3\right)$$
$$+v^c_l v^m_3\left(X^{l1}_{m,k}\delta^2_r\delta^s_c + \frac{1}{2} f^{ab}_{,ck}\pi^2_a\pi^1_b\delta^l_r\delta^s_m\right)$$

If we write $X^{11}_k = -\frac{1}{2} f^{ab}_{,k}\pi^1_a\pi^1_b + J^{11}_k$, then this becomes

$$\frac{\partial J^{11}_b}{\partial \pi^r_s}\left(\delta^b_k + \frac{\partial H_k}{\partial \pi^1_b}\right) = -\frac{\partial X^{m1}}{\partial \pi^r_s} H_{k,m} - X^{m1}\frac{\partial H_{k,m}}{\partial \pi^r_s}$$
$$+H_{m,k}\frac{\partial \frac{1}{2} v^l_3 f^{bm}_{,l}\pi^2_b + f^{bm} H_b}{\partial \pi^r_s} - X^{a1}_b\frac{\partial^2 H_k}{\partial \pi^r_s \partial \pi^a_b} - \delta^1_r\left(\frac{1}{2} v^m_3 f^{bs}_{,mk}\pi^2_b + f^{sb}_{,k}\pi^2_c\Gamma^c_{bl} v^l_3\right)$$
$$+\delta^2_r v^s_l v^m_3 X^{l1}_{m,k} + \frac{1}{2} v^m_r v^s_3 f^{ab}_{,mk}\pi^2_a\pi^1_b + \frac{1}{2}\frac{\partial f^{ab}_{,m}\pi^l_a\pi^1_b}{\partial \pi^r_s}\frac{\partial H_k}{\partial \pi^l_m} \tag{B.43}$$

We will return to this equation. Looking now at equation (B.5), we select $i = j = 1$ to see that

$$-2\delta^2_r\frac{\partial v^s_1 v^m_3 J^{11}_m}{\partial \pi^p_q} + \left(\delta^2_r\Gamma^s_{km} v^m_3 - \pi^2_c\Gamma^c_{km} v^m_r v^s_3\right)\left(\delta^2_p f^{qk}_{,l} v^l_3 - v^l_p v^q_3 f^{bk}_{,l}\pi^2_b\right)$$
$$= -2\delta^2_p\frac{\partial v^q_1 v^m_3 J^{11}_m}{\partial \pi^r_s} + \left(\delta^2_p\Gamma^q_{km} v^m_3 - \pi^2_c\Gamma^c_{km} v^m_p v^q_3\right)\left(\delta^2_r f^{sk}_{,l} v^l_3 - v^l_r v^s_3 f^{bk}_{,l}\pi^2_b\right) \tag{B.44}$$

or

$$\delta^2_r\frac{\partial v^s_1 v^m_3 J^{11}_m - \frac{1}{2}\frac{\partial H_k}{\partial \pi^r_s} v^m_3 f^{bk}_{,m}\pi^2_b}{\partial \pi^p_q} = \delta^2_p\frac{\partial v^q_1 v^m_3 J^{11}_m - \frac{1}{2}\frac{\partial H_k}{\partial \pi^p_q} v^m_3 f^{bk}_{,m}\pi^2_b}{\partial \pi^r_s} \tag{B.45}$$

$$\delta^2_r\frac{\partial v^s_1 v^m_3 J^{11}_m}{\partial \pi^p_q} - \frac{1}{2}\frac{\partial H_k}{\partial \pi^r_s}\frac{\partial v^m_3 f^{bk}_{,m}\pi^2_b}{\partial \pi^p_q} = \delta^2_p\frac{\partial v^q_1 v^m_3 J^{11}_m}{\partial \pi^r_s} - \frac{1}{2}\frac{\partial H_k}{\partial \pi^p_q}\frac{\partial v^m_3 f^{bk}_{,m}\pi^2_b}{\partial \pi^r_s} \tag{B.46}$$

Choosing $r = p = 2$ in equation (B.45) shows a symmetry in the s and q. Choosing, instead,

$r = 2$ and $p = \beta$, we see

$$\frac{\partial v_1^s v_3^m J_m^{11} - \frac{1}{2}\frac{\partial H_k}{\partial \pi_s^2} v_3^m f_{,m}^{bk} \pi_b^2}{\partial \pi_q^\beta} = 0. \tag{B.47}$$

Thus,

$$\begin{aligned} v_1^s v_3^m \frac{\partial J_m^{11}}{\partial \pi_q^\beta} &= \left(v_\beta^s v_1^q v_3^m + v_1^s v_\beta^m v_3^q\right) J_m^{11} - \frac{1}{2}\Gamma_{kl}^s f_{,m}^{bk} \pi_b^2 \left(v_\beta^l v_3^q v_3^m + v_3^l v_\beta^m v_3^q\right) \\ &+ \frac{1}{2}\pi_c^2 \Gamma_{kl}^c f_{,m}^{bk} \pi_b^2 \left(v_\beta^l v_2^q v_3^s v_3^m + v_2^l v_\beta^s v_3^q v_3^m + v_2^l v_3^s v_\beta^m v_3^q\right) \end{aligned} \tag{B.48}$$

Multiplying by $\pi_q^3 \pi_s^2$ and choosing $\beta = 3$ gives

$$0 = \pi_s^2 \Gamma_{kl}^s f_{,m}^{bk} \pi_b^2 v_3^l v_3^m$$

Again, a number of derivatives, contractions, and resubstitutions leads to

$$\Gamma_{kl}^s f_{,m}^{bk} = 0 \tag{B.49}$$

This now reduces equations (B.48) and (B.45) to

$$v_1^s v_3^m \frac{\partial J_m^{11}}{\partial \pi_q^\beta} = \left(v_\beta^s v_1^q v_3^m + v_1^s v_\beta^m v_3^q\right) J_m^{11} \tag{B.50}$$

$$\delta_r^2 \frac{\partial v_1^s v_3^m J_m^{11}}{\partial \pi_q^p} = \delta_p^2 \frac{\partial v_1^q v_3^m J_m^{11}}{\partial \pi_s^r}. \tag{B.51}$$

As well as equation (B.43) to

$$\begin{aligned} \frac{\partial J_b^{11}}{\partial \pi_s^r}\left(\delta_k^b - \pi_c^2 \Gamma_{kl}^c v_1^l v_3^b\right) &= -\frac{\partial X^{m1}}{\partial \pi_s^r} H_{k,m} - X^{m1}\frac{\partial H_{k,m}}{\partial \pi_s^r} \\ &+ H_{m,k} f^{bm} \frac{\partial H_b}{\partial \pi_s^r} - X_b^{a1} \frac{\partial^2 H_k}{\partial \pi_s^r \partial \pi_b^a} - \frac{1}{2}\delta_r^1 v_3^m f_{,mk}^{bs} \pi_b^2 \\ &+ \delta_r^2 v_l^s v_3^m X_{m,k}^{l1} + \frac{1}{2} v_r^m v_3^s f_{,mk}^{ab} \pi_a^2 \pi_b^1 + \frac{1}{2}\frac{\partial f_{,m}^{ab} \pi_a^l \pi_b^1}{\partial \pi_s^r}\frac{\partial H_k}{\partial \pi_m^l} \end{aligned} \tag{B.52}$$

Multiplying this equation by $v_1^q v_3^k$, we see that

$$\begin{aligned} v_1^q v_3^k \frac{\partial J_k^{11}}{\partial \pi_s^r} - v_1^q v_3^b \frac{\partial J_b^{11}}{\partial \pi_s^r} \pi_c^2 \Gamma_{kl}^c v_1^l v_3^k &= -v_1^q v_3^k \frac{\partial X^{m1}}{\partial \pi_s^r} H_{k,m} - v_1^q v_3^k X^{m1} \frac{\partial H_{k,m}}{\partial \pi_s^r} \\ &+ v_1^q v_3^k H_{m,k} f^{bm} \frac{\partial H_b}{\partial \pi_s^r} - v_1^q v_3^k X_b^{a1} \frac{\partial^2 H_k}{\partial \pi_s^r \partial \pi_b^a} - \frac{1}{2} \delta_r^1 v_1^q v_3^k v_3^m f_{,mk}^{bs} \pi_b^2 \\ &+ \delta_r^2 v_1^q v_3^k v_l^s v_3^m X_{m,k}^{l1} + \frac{1}{2} v_1^q v_3^k v_r^m v_3^s f_{,mk}^{ab} \pi_a^2 \pi_b^1 + \frac{1}{2} v_1^q v_3^k \frac{\partial f_{,m}^{ab} \pi_a^l \pi_b^1}{\partial \pi_s^r} \frac{\partial H_k}{\partial \pi_m^l} \end{aligned} \quad \text{(B.53)}$$

Choosing $r = \beta$, we may use equation (B.50) to reduce this to

$$\begin{aligned} \left(v_\beta^q v_1^s v_3^m + v_1^q v_\beta^m v_3^s\right) J_m^{11} - \left(v_\beta^q v_1^s v_3^m + v_1^q v_\beta^m v_3^s\right) J_m^{11} \pi_c^2 \Gamma_{kl}^c v_1^l v_3^k & \\ = -v_1^q v_3^k \frac{\partial X^{m1}}{\partial \pi_s^\beta} H_{k,m} - v_1^q v_3^k X^{m1} \frac{\partial H_{k,m}}{\partial \pi_s^\beta} & \\ + v_1^q v_3^k H_{m,k} f^{bm} \frac{\partial H_b}{\partial \pi_s^\beta} - v_1^q v_3^k X_b^{a1} \frac{\partial^2 H_k}{\partial \pi_s^\beta \partial \pi_b^a} - \frac{1}{2} \delta_\beta^1 v_1^q v_3^k v_3^m f_{,mk}^{bs} \pi_b^2 & \\ + \frac{1}{2} v_1^q v_3^k v_\beta^m v_3^s f_{,mk}^{ab} \pi_a^2 \pi_b^1 + \frac{1}{2} v_1^q v_3^k \frac{\partial f_{,m}^{ab} \pi_a^l \pi_b^1}{\partial \pi_s^\beta} \frac{\partial H_k}{\partial \pi_m^l} & \end{aligned} \quad \text{(B.54)}$$

Choose $\beta = 3$ and multiply through by π_q^3 to show

$$v_1^s v_3^m J_m^{11} - v_1^s v_3^m J_m^{11} \pi_c^2 \Gamma_{kl}^c v_1^l v_3^k = v_1^s v_3^m J_m^{11} \left(1 - \pi_c^2 \Gamma_{kl}^c v_1^l v_3^k\right) = 0 \quad \text{(B.55)}$$

As $1 - \pi_c^2 \Gamma_{kl}^c v_1^l v_3^k = 0$ is never satisfied for any Γ_{jk}^i, we are left to conclude $v_3^m J_m^{11} = 0$. This renders equation (B.51) a triviality, and reduces equation (B.54) to

$$\begin{aligned} v_\beta^m v_3^s J_m^{11} (1 - \pi_c^2 \Gamma_{kl}^c v_1^l v_3^k) = & \\ -\left(\delta_\beta^1 f^{sm} - \frac{1}{2} v_\beta^l v_3^s f_{,l}^{bm} \pi_b^2 - f^{bm} \pi_c^2 \Gamma_{bl}^c v_\beta^l v_3^s\right) v_3^k H_{k,m} + v_3^k X^{m1} \pi_c^2 \Gamma_{kl,m}^c v_\beta^l v_3^s & \\ -v_3^k H_{m,k} f^{bm} \pi_c^2 \Gamma_{bl}^c v_\beta^l v_3^s + v_3^k X_b^{a1} \left(\delta_a^2 \Gamma_{kl}^b v_\beta^l v_3^s - \pi_c^2 \Gamma_{kl}^c [v_a^l v_\beta^b v_3^s + v_\beta^l v_a^s v_3^b]\right) & \\ -\frac{1}{2} \delta_\beta^1 v_3^k v_3^m f_{,mk}^{bs} \pi_b^2 + \frac{1}{2} v_3^k v_\beta^m v_3^s f_{,mk}^{ab} \pi_a^2 \pi_b^1 + \frac{1}{2} v_3^k f_{,m}^{ab} \left(\delta_\beta^l \delta_a^s \pi_b^1 + \delta_\beta^1 \delta_b^s \pi_a^l\right) \frac{\partial H_k}{\partial \pi_m^l} & \end{aligned} \quad \text{(B.56)}$$

Multiplying by π_s^γ further simpifies this to

$$0 = v_3^k v_3^l \pi_s^\gamma \pi_c^2 \left(2 f^{sm} \Gamma_{kl,m}^c + f_{,kl}^{cs} - \Gamma_{kl}^m f_{,m}^{sc}\right)$$

which, after differentiation, contraction, and resubstitution, yields

$$f^{ab}_{,kl} = \Gamma^m_{kl} f^{ab}_{,m} - 2f^{am}\Gamma^b_{kl,m} \tag{B.57}$$

If both equation (B.49) and equation (B.57) are to hold for a general torsion-free connection, then $f^{ab} = 0$. This then makes the vector fields X^i trivial, and thus there are no non-trivial solutions to equation (2.7). $\square$