

ABSTRACT

VISWANATHAN, GOPAL. The $1/k$ -Eulerian Polynomials. (Under the direction of Dr. Carla D. Savage.)

The *Eulerian polynomials* that were discovered by Leonhard Euler have applications in combinatorics. Apart from combinatorially counting a single statistic over permutations, these polynomials satisfy multiple recurrences and are enumerated by an exponential generating function. Equally interesting are their coefficients called the *Eulerian numbers*. These numbers satisfy several identities and have several applications, one of the most famous being the factorial expansion of t^n , for positive integers t and n .

There have been numerous works in the past that have given various generalizations of these polynomials. In this thesis we generalize these polynomials for each positive integer k , using a new technique that uses geometry. This geometric approach depends on the theory of *lecture hall partitions* and *Ehrhart theory*. We call our generalization the $1/k$ -*Eulerian polynomials* and show them to be combinatorially counting a single statistic over a generalization of permutations. Several identities and recurrences are derived for both the polynomials and their coefficients, all of which reduce to the properties of the Eulerian polynomials and the Eulerian numbers when $k = 1$.

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The $1/k$ -Eulerian Polynomials

by
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DEDICATION

To the Omnipresent, parents, sister and my advisor.

BIOGRAPHY

Gopal Viswanathan was born in 1987 and brought up in Mumbai, India. Prior to coming to N.C State University, he obtained his Bachelor's degree in Computer engineering from the University of Mumbai.

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Chapter 1

Introduction

The *Eulerian polynomials* were first introduced by Leonhard Euler in his book “Institutiones calculi differentialis cum eius usu in analysi finitorum ac Doctrina serierum” (chap. VII), in 1755. According to [12], Euler introduced these polynomials en route to deriving a closed form for the sums $\sum_{k=1}^m k^{2p}(-1)^k$ and $\sum_{k=1}^m k^{2p+1}(-1)^k$, for positive integers k, p and m . He defines these polynomials $A_n(x)$ inductively as

$$A_n(x) = \sum_{i=0}^{n-1} \binom{n}{i} A_i(x)(x-1)^{n-1-i} \quad (1.1)$$

with initial condition $A_0(x) = 1$.

The coefficients of these polynomials of degree $n - 1$ are called the *Eulerian numbers*.

The focus of this thesis is to generalize these polynomials and their coefficients for each positive integer k . Our goal is to emulate their closed forms and all of their well known identities using the theory of *lecture hall partitions*. We call these generalizations the $1/k$ -*Eulerian polynomials*, $A_n^{(k)}(x)$ and $1/k$ -*Eulerian numbers*, $a_{n,j}^{(k)}$ respectively. When $k = 1$, $1/k$ -Eulerian polynomials reduce to the Eulerian polynomials. We now define our generalizations.

1.1 Definitions of the Eulerian polynomials and the $1/k$ -Eulerian polynomials

We give the following three equivalent definitions of the Eulerian polynomials. Each definition is accompanied by its generalization.

Definition 1

$$\sum_{t \geq 0} (t+1)^n x^t = \frac{A_n(x)}{(1-x)^{n+1}}. \quad (1.2)$$

$$\sum_{t \geq 0} \binom{t-1+1/k}{t} (kt+1)^n x^t = \frac{A_n^{(k)}(x)}{(1-x)^{n+1/k}}. \quad (1.3)$$

Definition 2

$$\sum_{n \geq 0} A_n(x) \frac{z^n}{n!} = \left(\frac{(1-x)}{e^{z(x-1)} - x} \right) \quad (1.4)$$

$$\sum_{n \geq 0} A_n^{(k)}(x) \frac{z^n}{n!} = \left(\frac{(1-x)}{e^{kz(x-1)} - x} \right)^{1/k} \quad (1.5)$$

Definition 3

$$A_{n+1}(x) = x(1-x) \frac{d}{dx} A_n(x) + (1+nx)A_n(x) \quad (1.6)$$

with initial condition $A_0(x) = 1$.

$$A_{n+1}^{(k)}(x) = kx(1-x) \frac{d}{dx} A_n^{(k)}(x) + (1+knx)A_n^{(k)}(x) \quad (1.7)$$

with initial condition $A_0^{(k)}(x) = 1$.

Equations (1.4), (1.6) and (1.1) can be elegantly derived from equation (1.2). According to [10], Euler gave equation (1.4) in his memoir.

We will prove in later chapters that equations (1.3), (1.5) and (1.7) are equivalent.

Clearly, setting $k = 1$ in equations (1.3), (1.5) and (1.7) give equations (1.2), (1.4) and (1.6).

The name $1/k$ -Eulerian polynomials was coined because equation (1.5) is the k th root of a k -generalization of (1.4).

We generalize equation (1.2) using previous work by Katie Bright and Carla Savage in [3] and Michael Schuster and Carla Savage in [16] in order to develop our generalization. We interpret the term $(t+1)^n$ in (1.2) geometrically as the number of lattice points in a polytope which we will see in later chapters. We then generalize this polytope to arrive at a new counting

term that generalizes $(t + 1)^n$. This term is the cornerstone of this thesis, since it helps us to define our generalization.

1.2 Definitions of the Eulerian numbers and the $1/k$ -Eulerian numbers

The Eulerian numbers can be defined in any of the following ways. Along with each definition we give the corresponding generalization.

Definition 1

$$a_{n,j} = (n - j)a_{n-1,j-1} + (j + 1)a_{n-1,j} \quad j, n > 0, \quad (1.8)$$

with initial conditions $a_{n,0} = 1$ and $a_{0,j} = 1$ iff $j = 0$.

$$a_{n+1,j}^{(k)} = (kj + 1)a_{n,j}^{(k)} + k(n + 1 - j)a_{n,j-1}^{(k)} \quad (1.9)$$

with initial conditions $a_{n,0}^{(k)} = 1$ for $n > 0$ and $a_{n,j}^{(k)} = 0$ for $j > n$.

Definition 2

$$a_{n,j} = \sum_{t=0}^j (-1)^{j-t} \binom{n+1}{j-t} (t+1)^n. \quad (1.10)$$

$$a_{n,j}^{(k)} = \sum_{t=0}^j (-1)^{j-t} \binom{n+1/k}{j-t} \binom{t-1+1/k}{t} (kt+1)^n \quad (1.11)$$

with initial conditions $a_{n,0} = 1$ and $a_{0,j} = 1$ iff $j = 0$.

Definition 3

$$t^n = \sum_j a_{n,j} \binom{t+j}{n} \quad (\text{Worpitzky's identity}) \quad (1.12)$$

$$\binom{t-1+1/k}{t} (kt+1)^n = \sum_j a_{n,j}^{(k)} \binom{t+1/k+n-j-1}{t-j} \quad (1.13)$$

Clearly, setting $k = 1$ in equations (1.9), (1.11), (1.13) give equations (1.8), (1.10), (1.12).

We will prove in later chapters that equations (1.9), (1.11) and (1.13) are equivalent.

We now state a combinatorial definition of the Eulerian polynomials. We will see in the later chapters how we generalize this definition for the $1/k$ -Eulerian polynomials.

1.3 Combinatorial definition of the Eulerian polynomials

We need the following terminologies before we get to the combinatorial definition.

Let S_n denote the set of permutations of $\pi : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$. A *descent* of a permutation $\pi \in S_n$ is a position i , ($1 \leq i \leq n - 1$) such that $\pi(i) > \pi(i + 1)$. For every permutation $\pi \in S_n$, we define $DES(\pi)$ to be the set of descent positions in π and $des(\pi) = |DES(\pi)|$

Example:

Let $n = 3$. Then $S_3 = \{123, 213, 231, 132, 312, 321\}$.

We have,

$$DES(123) = \{\} \text{ and } des(123) = 0,$$

$$DES(213) = \{1\} \text{ and } des(213) = 1,$$

$$DES(231) = \{2\} \text{ and } des(231) = 1,$$

$$DES(132) = \{2\} \text{ and } des(132) = 1,$$

$$DES(312) = \{1\} \text{ and } des(312) = 1,$$

$$DES(321) = \{1, 2\} \text{ and } des(321) = 2.$$

According to paper [10], Riordan showed that the Eulerian numbers have the following combinatorial definition.

$$a_{n,j} = \# \{ \pi \in S_n \mid des(\pi) = j \}$$

Motivated by this definition, the Eulerian polynomials $A_n(x)$ are combinatorially defined as,

$$A_n(x) = \sum_{\pi \in S_n} x^{des(\pi)}. \tag{1.14}$$

Clearly when $n = 3$ from the above discussion,

$$A_3(x) = x^2 + 4x + 1 = \sum_{i=0}^2 a_{n,i} x^i.$$

Hence we define the Eulerian numbers $a_{n,i}$ as the number of permutations in S_n having exactly i descents. We will see in the later chapter how we generalize permutations to define our generalization.

1.4 Why is this Generalization Different?

There have been a few generalizations of the Eulerian polynomials in the past. A few of these are listed below.

1. In Carlitz and Hoggat's [7], the authors define the following generalization using a positive integer p .

$$\sum_{t \geq 0} \binom{t+p-1}{p} x^t = \frac{G_n^p(x)}{(1-x)^{p+1}}$$

where $G_n^p(x) = \sum_{k=1}^{pn-p+1} G_{n,k}^p x^k$ is the generalization. The authors prove recurrences and closed forms for both the generating function and its coefficients. They also provide a combinatorial interpretation of the generalization using the *extended Eulerian numbers*.

2. In Carlitz and Scoville's [6], the authors define a generalization $A(r, s|\alpha, \beta)$ by the following equation.

$$\sum_{r,s=0}^{\infty} A(r, s|\alpha, \beta) \frac{x^r y^s}{(r+s)!} = (1 + xF(x, y))^\alpha (1 + yF(x, y))^\beta$$

where α, β are unrestricted and

$$F(x, y) = \frac{e^x - e^y}{xe^y - ye^x}.$$

A recurrence and a closed form is derived. Combinatorial applications of this generalization is also discussed.

3. In Koutras's [15], the author defines a generalization which he calls the p_n -associated Eulerian numbers $A_{n,k}$ by,

$$p_n(t) = \sum_k A_{n,k} \binom{t+n-k}{n}$$

where $p_n(t)$ for a whole number n is a polynomial in t of degree n with $p_0(t) = 1$. When $p_n(t) = t^n$ we have,

$$t^n = \sum_k A_{n,k} \binom{t+n-k}{n}$$

Clearly the above is a variation of the Worpitzky identity given in equation (1.12) and this is how he defines the Eulerian numbers $A_{n,k}$ in his paper.

4. In Haglund and Visontai's [13], the authors make use of the fact that the Eulerian polynomials have real roots. They then define a multivariate generalization of the Eulerian polynomials, such that the generalization has real roots, since it leads to several nice properties of the coefficients of the polynomial.

In our paper we follow a completely different approach that uses the theory of lecture hall partitions. In addition to using lecture hall theory, we interpret equation (1.2) geometrically to come up with a new generalization. Also as given in our publication [17], our generalization has a combinatorial interpretation in terms of the statistics “excedance” and “number of cycles” on permutations. The excedance statistic of a permutation π which is denoted by “ $exc(\pi)$ ” is equal to $\#\{i \mid \pi(i) > i\}$. The number of cycles of a permutation π is denoted by “ $\#cyc(\pi)$ ”. We have,

$$A_n^{(k)}(x) = \sum_{\pi \in S_n} x^{exc(\pi)} k^{n-\#cyc(\pi)}$$

where As discussed in [17], it is a special case of a bivariate generalization of the Eulerian polynomials $F_n(x, y)$, with $y = 1/k$ for a positive integer k . For an integer y , $F_n(x, y)$ is a special case in the work of Carlitz's [5]. For real y , Dillon and Roselle in [9] define the reciprocal of $F_n(x, y)$ using its exponential generating function defined in [9]. The results that we derive in section 4.1 are all special cases of the results derived in [5] and [9].

1.5 Outline of the Thesis

The Thesis is organized as follows:

1. In chapter 2, we will review lecture hall theory and *Ehrhart* theory and connect them to our generalizations.
2. In chapter 3, we derive a closed form for the $1/k$ -Eulerian polynomials.
3. In chapter 4, we extend well known identities for the Eulerian polynomials to the $1/k$ -Eulerian polynomials.
4. In chapter 5, we propose some open questions that come out this thesis.

Chapter 2

Geometric definition of the $1/k$ -Eulerian polynomials through Ehrhart theory

In this chapter, we first review lecture hall partition theory and Ehrhart theory, since they are the main tools used to develop our generalization. We then define the $1/k$ -Eulerian polynomials geometrically.

The main tool we use to prove various results about our generalizations, is the relationship between the $1/k$ -Eulerian polynomials and the *generalized lecture hall polytope*. We will first review lecture hall theory.

2.1 Review Of Lecture Hall Theory

A *composition* of an integer $N > 0$, into n nonnegative parts, is an n -tuple of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, such that $N = \lambda_1 + \lambda_2 + \dots + \lambda_n$. The only composition of 0 is $\lambda = ()$. The paper by Bousquet-Mélou and Eriksson [1] introduced a family L_n of n -tuples of integers called the set of lecture hall partitions such that,

$$L_n = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_n) \mid 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \dots \leq \frac{\lambda_n}{n} \right\}.$$

We refer here to $[1, 2, \dots, n]$ as the *constraint sequence* where the i th term ($1 \leq i \leq n$) corresponds to the the denominator of λ_i in the constraint. The family derives its name due to the

following argument. Consider an auditorium with n rows, where the speaker and the listeners are of negligible height and the n rows are located at distances $1, 2, \dots, n$ respectively, from the speaker at heights $\lambda_1, \lambda_2, \dots, \lambda_n$, then the constraint,

$$0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \dots \leq \frac{\lambda_n}{n}$$

guarantees that every listener seated on any row gets a clear view of the speaker. The paper [1] also proves the lecture hall theorem which states that the number of lecture hall partitions of length n is equal to the number of partitions into odd parts, where the odd parts belong to the set $\{1, 3, \dots, 2n - 1\}$. The paper by Bousquet-Mélou and Eriksson [2] provides the generating function for a more general version of the lecture hall partitions where the constraint sequence under consideration was a special *polynomic sequence* called the (k, ℓ) -sequence. The constraint sequence for the lecture hall partitions is a special case of this sequence. In this thesis, we attempt to generalize the generating function of the lecture hall partitions but with a different constraint sequence of which, the constraint sequence for the lecture hall partitions is a special case.

The paper by Corteel, Lee and Savage [8] provides closed form expressions for the family of lecture hall partitions and a particularly interesting result being a closed form that enumerates the family of lecture hall partitions with first part bounded. The following result was proved in [8].

Theorem 1. *For nonnegative integers j, i satisfying $i \leq n$, the number of lecture hall partitions in L_n with first part bounded as $\lambda_n \leq nj + i$ is given by*

$$(j + 1)^{n-i} (j + 2)^i. \tag{2.1}$$

The thesis considers the problem of enumerating the family of nonnegative integer sequences $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ that are constrained by the ratio of consecutive parts with first part bounded. That is we consider the set of n -tuple of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ such that

$$0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{k+1} \leq \dots \leq \frac{\lambda_n}{(n-1)k+1} \text{ and } \lambda_n \leq \eta$$

where $\eta \in \{\mathbb{N}\} \cup \{0\}$, $n \geq 0$, $k \in \mathbb{N}$. The set of lecture hall partitions are nothing but a special case of the above family with $k = 1$. Thus this thesis generalizes Theorem 1. The paper [8] gives a recurrence that enumerates *any* family of nonnegative integer sequences $(\lambda_1, \lambda_2, \dots, \lambda_n)$

that are constrained by the ratio of consecutive parts. Even though the recurrence equips us with a tool to enumerate any family by programming a computational machine, the recurrence in itself is not trivially solvable for any family. We refer to this recurrence in this thesis and construct a simpler form of this recurrence for our family that turns out to be solvable. Before we continue further, we need some background of Ehrhart theory.

2.2 Background Of Ehrhart Theory And The Generalized Lecture Hall Polytope

In chapter 1, we gave equation (1.2). One of the main observations is that, the term $(t + 1)^n$ is the number of lattice points in the n -dimensional cube of width t . Katie Bright and Carla Savage showed in [3] that, the lattice points in the n -dimensional cube of width t , Q_n^t are in bijection with the lattice points of the *lecture hall polytope* given by

$$L_n^t = \left\{ (\lambda_1, \dots, \lambda_n) \mid 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \dots \leq \frac{\lambda_n}{n} \leq t \right\} \subseteq \mathbb{R}^n.$$

Here

$$Q_n^t = \{ (x_1, \dots, x_n) \in \mathbb{Z}_{\geq 0}^n \mid 0 \leq x_i \leq t, 1 \leq i \leq n \}.$$

We obtain a generalization of the Eulerian polynomials by generalizing the lecture hall polytope, adopting the approach in [16].

We define the *generalized lecture hall polytope* $\mathbb{P}_{n,k}$ as follows.

$$\mathbb{P}_{n,k} = \left\{ \lambda \mid 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{k+1} \leq \dots \leq \frac{\lambda_n}{(n-1)k+1} \leq 1 \right\} \subseteq \mathbb{R}^n.$$

Let $t\mathbb{P}_{n,k}$ represent the number of points in the t th dilation of $\mathbb{P}_{n,k}$ where t is a positive integer. Its *Ehrhart polynomial* is given by

$$i(\mathbb{P}_{n,k}, t) = |(t\mathbb{P}_{n,k} \cap \mathbb{Z}^n)|.$$

Thus

$$i(\mathbb{P}_{n,k}, t) = \left| \left\{ \lambda \in \mathbb{Z}^n \mid 0 \leq \frac{\lambda_1}{1} \leq \dots \leq \frac{\lambda_{n-1}}{(n-2)k+1} \leq \frac{\lambda_n}{(n-1)k+1} \leq t \right\} \right|.$$

We now define $G_{n,k}^{(j,d,r)}$. Let $G_{n,k}^{(j,d,r)}$ denote the number of n -tuple of integers $(\lambda_1, \lambda_2, \dots, \lambda_n)$

satisfying the constraint,

$$0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{k+1} \leq \dots \leq \frac{\lambda_n}{(n-1)k+1} \text{ and } \lambda_n \leq j((n-1)k+1) + (n-1)d + r \quad (2.2)$$

where $k \in \mathbb{N}$, $n, j \in \{\mathbb{N}\} \cup \{0\}$, $d \in \{0, 1, \dots, k-1\}$, $r \in \{0, 1, \dots, n-1\}$. Thus we have

$$i(\mathbb{P}_{n,k}, t) = G_{n,k}^{(t,0,0)}.$$

2.3 Combinatorial definition of the $1/k$ -Eulerian polynomials

We first need a few definitions for setting things up.

2.3.1 Terminology And Definitions

For a positive integer k , a k -inversion sequence of length n (≥ 0), is an n -tuple (e_1, e_2, \dots, e_n) , such that $0 \leq e_i \leq (i-1)k$ for $i = 1, 2, \dots, n$. When $k = 1$, it has been proved that there exists a bijection between the set of inversion sequences and the set of permutations of the first n positive integers. Richard A. Brualdi's book [4] is a good reference for the proof of this. We use the method of generalizing permutations of the first n positive integers by k -inversion sequences of length n , borrowing the idea of generalizing permutations using s -inversion sequences in [16].

Let $\mathbb{I}_{n,k}$ be the set of all k -inversion sequences of length n .

For $e \in \mathbb{I}_{n,k}$, an *ascent* of e is a position j ($1 \leq j < n$), such that,

$$\frac{e_j}{(j-1)k+1} < \frac{e_{j+1}}{jk+1}.$$

We use the following two statistics adopted from [16].

Let $Asc(e)$ be the set of all ascents in a k -inversion sequence e .

Let $asc(e)$ denote $|Asc(e)|$.

We now present a combinatorial definition of the $1/k$ -Eulerian polynomials that will be used subsequently.

2.3.2 The Definition

The following result is proved in [16].

$$\sum_{\pi \in S_n} x^{\text{des}(\pi)} = \sum_{e \in \mathbb{I}_{n,1}} x^{\text{asc}(e)}.$$

Hence,

$$A_n(x) = \sum_{e \in \mathbb{I}_{n,1}} x^{\text{asc}(e)}. \quad (2.3)$$

This establishes the equivalence of equation (2.3) with equation (1.14).

The $1/k$ -Eulerian polynomials are defined by,

$$A_n^{(k)}(x) = \sum_{e \in \mathbb{I}_{n,k}} x^{\text{asc}(e)} = \sum_{j=0}^{n-1} a_{n,j}^{(k)} x^j. \quad (2.4)$$

We now see how the k -inversion sequences generalize the Eulerian Polynomials.

In the next section we see how to use the combinatorial definition of the $1/k$ -Eulerian polynomials, to give them a geometric flavor through Ehrhart theory.

2.4 Ehrhart Series of The Generalized Lecture Hall Polytope

The following theorem is a special case of a theorem from [16]. For completeness we have adapted the proof in [16] for our special case.

Theorem 2. *For a fixed positive integer k , and for nonnegative integer n ,*

$$\sum_{t \geq 0} i(\mathbb{P}_{n,k}, t) x^t = \frac{\sum_{i=0}^{n-1} a_{n,i}^{(k)} x^i}{(1-x)^{n+1}}. \quad (2.5)$$

Proof. The proof uses the notion of barred inversion sequences, adapted from the method of barred permutations from [11]. A *barred k -inversion sequence* is a sequence $e \in I_{n,k}$ in which one or more vertical bars are inserted before and/or after elements e_i , with the stipulation that if i is an ascent of e , then there is at least 1 bar at position i , the space immediately preceding e_{i+1} .

We show that both sides of (2.5) count barred k -inversion sequences of length n . First, fix the inversion sequence, e , and sum over all $e \in \mathbb{I}_{n,k}$. Second, fix the number of bars, t , and sum over $t \geq 0$.

The first way is easy: a barred k -inversion sequence $e \in \mathbb{I}_{n,k}$ must have at least one bar following each ascent position, but additional bars may be distributed among all of the $n + 1$ “spaces” of e . The number of ways to place j identical bars into $n + 1$ spaces is the coefficient of x^j in $1/(1-x)^{n+1}$, so, the right-hand side of (2.5) counts the number of barred k -inversion sequences in $\mathbb{I}_{n,k}$.

To count barred k -inversion sequences with t bars, we describe a bijection between them and $\lambda \in (t\mathbb{P}_{n,k} \cap \mathbb{Z}^n)$ counted by $i(\mathbb{P}_{n,k}, t)$. As discussed previously,

$$i(\mathbb{P}_{n,k}, t) = G_{n,k}^{(t,0,0)}.$$

Let e be a barred k -inversion sequence with t bars. For $1 \leq i \leq n$, let b_i be the total number of bars preceding e_i in any position. Then $b_1 \leq b_2 \leq \dots \leq b_n$. Define $\lambda = (\lambda_1, \dots, \lambda_n)$ by

$$\lambda_i = ((i-1)k+1)b_i - e_i.$$

We show that $\lambda \in (t\mathbb{P}_{n,k} \cap \mathbb{Z}^n)$. First note that $\lambda_i \geq 0$, since $e_i < (i-1)k+1$ and if $b_i = 0$, then no position $j < i$ is an ascent, so $e_1 = \dots = e_i = 0$. Since e is a k -inversion sequence, $\frac{e_j}{(j-1)k+1} < 1$ for all j . Now, if $i \in \mathbf{Asc}(e)$, there is at least 1 bar between e_i and e_{i+1} , so $b_i < b_{i+1}$ and

$$\frac{\lambda_i}{(i-1)k+1} = b_i - \frac{e_i}{(i-1)k+1} \leq b_i \leq b_{i+1} - 1 \leq b_{i+1} - \frac{e_{i+1}}{ik+1} = \frac{\lambda_{i+1}}{ik+1}.$$

On the other hand, if $i \notin \mathbf{Asc}(e)$, then

$$\frac{e_i}{(i-1)k+1} \geq \frac{e_{i+1}}{ik+1},$$

so

$$\frac{\lambda_i}{(i-1)k+1} = b_i - \frac{e_i}{(i-1)k+1} \leq b_{i+1} - \frac{e_{i+1}}{ik+1} = \frac{\lambda_{i+1}}{ik+1}.$$

To complete the proof we have,

$$\frac{\lambda_n}{(n-1)k+1} = b_n - \frac{e_n}{(n-1)k+1} \leq t.$$

To prove that this is a bijection, we define the inverse. If $\lambda \in (t\mathbb{P}_{n,k} \cap \mathbb{Z}^n)$, let

$$b = \left(\left\lceil \frac{\lambda_1}{1} \right\rceil, \dots, \left\lceil \frac{\lambda_n}{(n-1)k+1} \right\rceil \right) = (b_1, \dots, b_n).$$

Then $b_n \leq t$. Let $e = (e_1, \dots, e_n)$, where

$$e_i = ((i-1)k+1)b_i - \lambda_i.$$

Then $e \in \mathbb{I}_{n,k}$ and we “bar” it by placing b_1 bars before e_1 , $b_i - b_{i-1}$ bars before e_i for $2 \leq i \leq n$, and $t - b_n$ bars after e_n . Since $\lambda \in (t\mathbb{P}_{n,k} \cap \mathbb{Z}^n)$, $1 \leq i < n$ and there is no bar after e_i , then $b_i = b_{i+1}$ and therefore

$$\frac{e_i}{(i-1)k+1} = b_i - \frac{\lambda_i}{(i-1)k+1} \geq b_i - \frac{\lambda_{i+1}}{ik+1} = b_{i+1} - \frac{\lambda_{i+1}}{ik+1} = \frac{e_{i+1}}{ik+1}.$$

Thus $i \notin \mathbf{Asc}(e)$.

This completes the proof of Theorem 2. □

Corollary 1. *The Ehrhart series of the generalized lecture hall polytope is*

$$\frac{\sum_{t \geq 0} a_{n,t}^{(k)} x^t}{(1-x)^{n+1}} = \frac{A_n^{(k)}(x)}{(1-x)^{n+1}}.$$

Hence we have,

$$A_n^{(k)}(x) = (1-x)^{n+1} \sum_{t \geq 0} i(\mathbb{P}_{n,k}, t) x^t.$$

In the next chapter, we evaluate $i(\mathbb{P}_{n,k}, t)$. And using this, we prove various identities for $A_n^{(k)}(x)$.

Chapter 3

Computation Of The Ehrhart Polynomial Of The Generalized Lecture Hall Polytope

We find a closed form for $G_{n,k}^{(j,d,r)}$ defined in section 2.2 in chapter 2 in order to evaluate $G_{n,k}^{(j,0,0)}$. Our approach is to devise a recurrence for $G_{n,k}^{(j,d,r)}$ and solve it. This computation is the principle one in this thesis, since it directly leads us to the closed form of the $1/k$ -Eulerian polynomials.

However we would first need the following lemma which we will prove.

3.1 Lemma

Lemma 1. *If $r > 0$ and $0 \leq d \leq k - 1$ then*

$$\left\lfloor \frac{(n-2)k+1}{(n-1)k+1}((n-1)d+r) \right\rfloor = (n-2)d+r-1.$$

Proof. It suffices to prove that

$$\frac{(n-2)k+1}{(n-1)k+1}((n-1)d+r) - 1 < (n-2)d+r-1 \leq \frac{(n-2)k+1}{(n-1)k+1}((n-1)d+r)$$

For the first inequality,

$$((n-1)k+1)((n-2)d+r) - ((n-2)k+1)((n-1)d+r) = kr - d > 0.$$

The last line follows because $r > 0$ and $0 \leq d \leq k-1$.

For the second inequality,

$$((n-2)k+1)((n-1)d+r) - ((n-1)k+1)((n-2)d+r-1) = k(n-1-r) + d + 1 \geq 0.$$

The last line follows since $0 \leq r \leq n-1$ and $0 \leq d \leq k-1$. This completes the proof of the lemma. \square

3.2 The Recurrence

Theorem 3. For integers $n \geq 0$, $k \geq 1$, $j \geq 0$ and nonnegative integers d, r satisfying $d = r = 0$ if $n = 0$ and otherwise $r \leq n-1$ and $(n-1)d+r < (n-1)k+1$,

$$G_{n,k}^{(j,d,r)} = \begin{cases} 1 & \text{if } n = 0 \text{ or } j = d = r = 0, \text{ else} \\ G_{n,k}^{(j-1,k,0)} + G_{n-1,k}^{(j,0,0)} & \text{if } d = r = 0, \text{ else} \\ G_{n,k}^{(j,d-1,n-1)} & \text{if } r = 0, \text{ else} \\ G_{n,k}^{(j,d,r-1)} + G_{n-1,k}^{(j,d,r-1)} & \text{otherwise.} \end{cases}$$

Proof. Let $S_{n,k}^{(j,d,r)}$ denote the set counted by $G_{n,k}^{(j,d,r)}$ and let $\lambda \in S_{n,k}^{(j,d,r)}$. When $n = 0$, or when $j = d = r = 0$, only the empty sequence satisfies the constraints. Otherwise, $n > 0$, with $j + d + r > 0$.

If $d = r = 0$, then $j > 0$ and $\lambda_n \leq j((n-1)k+1)$. We have two disjoint cases here. Either

$$\lambda_n = j((n-1)k+1)$$

or

$$\lambda_n \leq j((n-1)k+1) - 1 = (j-1)((n-1)k+1) + (n-1)k + 0.$$

Clearly in the latter case $(\lambda_1, \dots, \lambda_n) \in S_{n,k}^{(j-1,k,0)}$. In the former case we have by definition of $S_{n,k}^{(j,d,r)}$,

$$\lambda_{n-1} \leq \frac{((n-2)k+1)j((n-1)k+1)}{((n-1)k+1)}.$$

This implies that

$$\lambda_{n-1} \leq j((n-2)k+1).$$

Hence $(\lambda_1, \dots, \lambda_n) \in S_{n,k}^{(j,0,0)}$ iff $(\lambda_1, \dots, \lambda_{n-1}) \in S_{n-1,k}^{(j,0,0)}$.

Otherwise, if $r = 0$, but $d > 0$, then $\lambda_n \leq j((n-1)k+1) + (n-1)d$ implying that

$$\lambda_n \leq j((n-1)k+1) + (n-1)(d-1) + n-1.$$

Therefore $(\lambda_1, \dots, \lambda_n) \in S_{n,k}^{(j,d,0)}$ iff $(\lambda_1, \dots, \lambda_n) \in S_{n,k}^{(j,d-1,n-1)}$.

Otherwise, $r > 0$ and $n > 0$. Again this case gives rise to two disjoint cases. Either

$$\lambda_n = j((n-1)k+1) + (n-1)d + r$$

or

$$\lambda_n \leq (j-1)((n-1)k+1) + (n-1)d + r - 1.$$

Clearly in the latter case $(\lambda_1, \dots, \lambda_n) \in S_{n,k}^{(j,d,r-1)}$. In the former case we have by definition of $S_{n,k}^{(j,d,r)}$,

$$\lambda_{n-1} \leq \frac{((n-2)k+1)(j((n-1)k+1) + (n-1)d + r)}{((n-1)k+1)}.$$

This implies that

$$\lambda_{n-1} \leq j((n-2)k+1) + \left\lfloor \frac{((n-2)k+1)((n-1)d+r)}{((n-1)k+1)} \right\rfloor.$$

We use Lemma 1 and the fact that $n \geq 2$ and $r > 0$ to conclude that

$$\lambda_{n-1} \leq j((n-2)k+1) + (n-2)d + r - 1 \text{ and } (n-2)d + r - 1 \geq 0.$$

Therefore $(\lambda_1, \dots, \lambda_n) \in S_{n,k}^{(j,d,r)}$ iff $(\lambda_1, \dots, \lambda_{n-1}) \in S_{n-1,k}^{(j,d,r-1)}$. This completes the proof of Theorem 3. \square

3.3 Vandermonde's Convolution

We state the following formula known as Vandermonde's convolution that will be used in deriving the closed form. The formula and its proof can be found in [12].

For $m, n \in \mathbb{Z}$ and $r, s \in \mathbb{R}$,

$$\sum_k \binom{r}{m+k} \binom{s}{n-k} = \binom{r+s}{m+n}$$

3.4 Closed Form For $G_{n,k}^{(j,d,r)}$

We now proceed to derive a closed form for $G_{n,k}^{(j,d,r)}$.

3.4.1 Proof Of The Closed Form

Theorem 4. For integers $n \geq 0, k \geq 1, j \geq 0$ and nonnegative integers d, r satisfying $d = r = 0$ if $n = 0$ and otherwise $r \leq n - 1$ and $(n - 1)d + r < (n - 1)k + 1$,

$$G_{n,k}^{(j,d,r)} = (-1)^j \sum_{p=0}^j \binom{\frac{1}{k} - 1}{j-p} \binom{\frac{-1}{k}}{p} (kp+1)(kp+d+1)^{n-1-r} (kp+d+2)^r. \quad (3.1)$$

Proof. Let $f_{n,k}^{(j,d,r)}$ denote the expression on the right-hand side of (3.1). We prove that $f_{n,k}^{(j,d,r)}$ satisfies the same recurrence as Theorem 3.

When $n = 0, d = r = 0$, by using Vandermonde's convolution it is easy to verify that $f_{0,k}^{(j,0,0)} = 1$. When $j = d = r = 0$, clearly $f_{n,k}^{(0,0,0)} = 1$. Otherwise, $n > 0$ with $j + d + r > 0$.

If $d = r = 0$, then $j > 0$ and

$$\begin{aligned}
f_{n,k}^{(j-1,k,0)} &+ f_{n-1,k}^{(j,0,0)} \\
&= (-1)^{j-1} \sum_{p=0}^{j-1} \binom{\frac{1}{k}-1}{j-1-p} \binom{\frac{-1}{k}}{p} (kp+1)(kp+k+1)^{n-1} \\
&+ (-1)^j \sum_{p=0}^j \binom{\frac{1}{k}-1}{j-p} \binom{\frac{-1}{k}}{p} (kp+1)(kp+1)^{n-2} \\
&= (-1)^j \sum_{p=1}^j \binom{\frac{1}{k}-1}{j-p} \binom{\frac{-1}{k}}{p} kp(kp+1)^{n-1} \\
&+ (-1)^j \binom{\frac{1}{k}-1}{j} + (-1)^j \sum_{p=1}^j \binom{\frac{1}{k}-1}{j-p} \binom{\frac{-1}{k}}{p} (kp+1)^{n-1} \\
&= (-1)^j \sum_{p=0}^j \binom{\frac{1}{k}-1}{j-p} \binom{\frac{-1}{k}}{p} (kp+1)(kp+1)^{n-1} \\
&= f_{n,k}^{(j,0,0)}.
\end{aligned}$$

Otherwise, if $r = 0$, but $d > 1$, then

$$\begin{aligned}
f_{n,k}^{(j,d-1,n-1)} &= (-1)^j \sum_{p=0}^j \binom{\frac{1}{k}-1}{j-p} \binom{\frac{-1}{k}}{p} (kp+1)(kp+d-1+1)^{n-1-(n-1)}(kp+d-1+2)^{n-1} \\
&= (-1)^j \sum_{p=0}^j \binom{\frac{1}{k}-1}{j-p} \binom{\frac{-1}{k}}{p} (kp+1)(kp+d+1)^{n-1} \\
&= f_{n,k}^{(j,d,0)}.
\end{aligned}$$

Otherwise, $r > 0$ and $n > 0$, and

$$\begin{aligned}
f_{n,k}^{(j,d,r-1)} &+ f_{n-1,k}^{(j,d,r-1)} \\
&= (-1)^j \sum_{p=0}^j \binom{\frac{1}{k}-1}{j-p} \binom{\frac{-1}{k}}{p} (kp+1)(kp+d+1)^{(n-1)-(r-1)}(kp+d+2)^{(r-1)} \\
&+ (-1)^j \sum_{p=0}^j \binom{\frac{1}{k}-1}{j-p} \binom{\frac{-1}{k}}{p} (kp+1)(kp+d+1)^{(n-1)-1-(r-1)}(kp+d+2)^{(r-1)} \\
&= (-1)^j \sum_{p=0}^j \binom{\frac{1}{k}-1}{j-p} \binom{\frac{-1}{k}}{p} (kp+1)(kp+d+1)^{(n-1-r)}(kp+d+2)^{(r-1)}[kp+d+1+1] \\
&= f_{n,k}^{(j,d,r)}.
\end{aligned}$$

This completes the proof of Theorem 4. □

Corollary 2. *For integers $n \geq 0, k \geq 1, t \geq 0$,*

$$G_{n,k}^{(t,0,0)} = (-1)^t \sum_{p=0}^t \binom{\frac{1}{k}-1}{t-p} \binom{\frac{-1}{k}}{p} (kp+1)^n. \quad (3.2)$$

As dicussed in the previous chapters, $G_{n,k}^{(t,0,0)}$ is the Ehrhart polynomial of the generalized lecture hall polytope $\mathbb{P}_{n,k}$ defined in section 2.2 in chapter 2 and the above corollary gives a closed form for it.

The following chapter uses this Ehrhart polynomial to prove various identities for the $1/k$ -Eulerian polynomials.

Chapter 4

Identities for our Generalization

4.1 Emulation of Identities of the Eulerian Polynomials

In chapter 1, we have discussed a closed form for the Eulerian polynomials, their inductive definition, their exponential generating function and a differential recurrence for them. We also defined the corresponding generalizations in chapter 1. We prove all of these generalizations in this chapter. We also generalize a differential operator characterization of the Eulerian polynomials that we did not discuss in chapter 1.

4.1.1 Closed form for our Generalization

Theorem 5. *For a fixed positive integer k , and for a nonnegative integer n ,*

$$A_n^{(k)}(x) = (1-x)^{n+1/k} \sum_{p \geq 0} \binom{p-1+1/k}{p} (kp+1)^n x^p. \quad (4.1)$$

Proof.

As discussed in earlier chapters,

$$A_n^{(k)}(x) = (1-x)^{n+1} \sum_{t \geq 0} i(\mathbb{P}_{n,k}, t) x^t = \sum_{t \geq 0} G_{n,k}^{(t,0,0)} x^t.$$

Hence,

$$\begin{aligned}
A_n^{(k)}(x) &= (1-x)^{n+1} \sum_{t \geq 0} (-1)^t \sum_{p=0}^t \binom{\frac{1}{k}-1}{t-p} \binom{-1}{p} (kp+1)^n x^t \\
&= (1-x)^{n+1} \sum_{p \geq 0} \sum_{t \geq p} (-1)^t \binom{\frac{1}{k}-1}{t-p} \binom{-1}{p} (kp+1)^n x^t \\
&= (1-x)^{n+1} \sum_{p \geq 0} \binom{-1}{p} (kp+1)^n \sum_{t \geq 0} (-1)^{t+p} \binom{\frac{1}{k}-1}{t} x^{t+p} \\
&= (1-x)^{n+1/k} \sum_{p \geq 0} \binom{p-1+1/k}{p} (kp+1)^n x^p.
\end{aligned}$$

This completes the proof of the Theorem.

4.1.2 Exponential generating function of the k -Eulerian polynomials

Theorem 6. For a fixed positive integer k ,

$$\sum_{n \geq 0} A_n^{(k)}(x) \frac{z^n}{n!} = \left(\frac{(1-x)}{e^{kz(x-1)} - x} \right)^{1/k} \quad (4.2)$$

Proof.

$$\begin{aligned}
\sum_{n \geq 0} A_n^{(k)}(x) \frac{z^n}{n!} &= \sum_{n \geq 0} (1-x)^{n+1/k} \sum_{p \geq 0} \binom{p-1+1/k}{p} (kp+1)^n x^p \frac{z^n}{n!} \\
&= (1-x)^{1/k} \sum_{p \geq 0} \binom{p-1+1/k}{p} x^p \sum_{n \geq 0} (1-x)^n (kp+1)^n \frac{z^n}{n!} \\
&= (1-x)^{1/k} \sum_{p \geq 0} \binom{p-1+1/k}{p} x^p e^{z(1-x)(kp+1)} \\
&= (1-x)^{1/k} e^{z(1-x)} \sum_{p \geq 0} (-1)^p \binom{-1/k}{p} x^p e^{z(1-x)kp} \\
&= \left(\frac{(1-x)}{e^{kz(x-1)} - x} \right)^{1/k}
\end{aligned}$$

This completes the proof of Theorem 4.2. □

4.1.3 Two-term Differential recurrence for our Generalization

Theorem 7. For a positive integers k and n ,

$$A_{n+1}^{(k)}(x) = kx(1-x)\frac{d}{dx}A_n^{(k)}(x) + (1+knx)A_n^{(k)}(x) \quad (4.3)$$

with initial condition $A_0^{(k)}(x) = 1$.

Proof.

By Theorem 5 we have,

$$\begin{aligned} A_{n+1}^{(k)}(x) &= (1-x)^{n+1+1/k} \sum_{p \geq 0} \binom{p-1+1/k}{p} (kp+1)^{n+1} x^p \\ &= (1-x)^{n+1+1/k} \left(\sum_{p \geq 0} \binom{p-1+1/k}{p} kp(kp+1)^n x^p \right) \\ &\quad + (1-x)^{n+1+1/k} \left(\sum_{p \geq 0} \binom{p-1+1/k}{p} (kp+1)^n x^p \right) \\ &= kx(1-x) \left((1-x)^{n+1/k} \sum_{p \geq 0} \binom{p-1+1/k}{p} (kp+1)^n p x^{p-1} \right) + (1-x)A_n^{(k)}(x) \end{aligned}$$

We also have,

$$\frac{d}{dx}A_n^{(k)}(x) = (1-x)^{n+1/k} \sum_{p \geq 0} \binom{p-1+1/k}{p} (kp+1)^n p x^{p-1} - \frac{(n+1/k)A_n^{(k)}(x)}{1-x}$$

Combining the above two calculations completes the proof of the theorem. \square

4.1.4 Inductive Definition of the k -Eulerian Polynomials

In the following theorem, we introduce an inductive definition of the $1/k$ -Eulerian polynomials. Before heading to the theorem we need a few definitions and a lemma.

We first define a generalizaion of the binomial coefficient that is crucial part of our theorem.

For a positive integer k , we define the k -binomial coefficient $\binom{n}{i}_k$ by

$$\binom{n}{i}_k = \binom{n-1}{i}_k + k \binom{n-1}{i-1}_k$$

with initial conditions $\binom{0}{0}_k = k, \binom{n}{0}_k = 1$ for $n > 0$ and $\binom{n}{i}_k = 0$ for $i > n$. Trivially $\binom{n}{i}_1 = \binom{n}{i}$.

We also need the following lemma that was proved by Dr. Savage in [17].

Lemma 2. *For a positive integer n ,*

$$\sum_{i=0}^n \binom{n}{i}_k w^i = (1 + kw)(1 + w)^{n-1} \quad (4.4)$$

(The sum is equal to k if $n = 0$)

Proof.

The claim is trivially true when $n = 1$. Let us assume it holds for all integers $< n$. Then,

$$\begin{aligned} \sum_{i=0}^n \binom{n}{i}_k w^i &= 1 + \sum_{i=1}^{n-1} \binom{n-1}{i}_k w^i + \sum_{i=1}^{n-1} \binom{n-1}{i-1}_k w^i + kw^n \\ &= (1 + kw)(1 + w)^{n-2} + w(1 + kw)(1 + w)^{n-2} = (1 + kw)(1 + w)^{n-1}. \end{aligned}$$

This completes the proof of the lemma.

The Inductive Definition

Theorem 8. *For a positive integer k and for a nonnegative integer n ,*

$$A_n^{(k)}(x) = \sum_{i=0}^{n-1} \binom{n}{i}_k A_i^{(k)}(x) (k(x-1))^{n-1-i} \quad (4.5)$$

with initial condition $A_0^{(k)}(x) = 1$.

Proof. Let $S_n^{(k)}(x) = \frac{A_n^{(k)}(x)}{(1-x)^{n+1/k}}$. We will show equivalently that,

$$(1-x)S_n^{(k)}(x) = \sum_{i=0}^{n-1} \binom{n}{i}_k A_i^{(k)}(x) (-k)^{n-1-i}.$$

We use the series expansion for $S_n^{(k)}(x)$ given by theorem 5, equate the coefficient of x^j and reduce the proof to showing that,

$$\begin{aligned} & \sum_{i=0}^{n-1} \binom{n}{i}_k \binom{j-1+1/k}{j} (kj+1)^i (-k)^{n-1-i} \\ &= \binom{j-1+1/k}{j} (kj+1)^n - \binom{j-2+1/k}{j-1} (k(j-1)+1)^n \end{aligned}$$

or equivalently we need to prove that

$$\sum_{i=0}^{n-1} \binom{n}{i}_k (kj+1)^i (-k)^{n-1-i} = (kj+1)^n - kj(k(j-1)+1)^{n-1}$$

Setting $w = -(j-1/k)$ in lemma 2 we have,

$$\sum_{i=0}^n \binom{n}{i}_k (-1)^i (j+1/k)^i = (1-k(j+1/k))(1-j-1/k)^{n-1}.$$

Which gives

$$\sum_{i=0}^n \binom{n}{i}_k (-k)^{n-1-i} (kj+1)^i = -kj(k(j-1)+1)^{n-1}$$

Subtracting the $i = n$ term from both sides completes the proof of the theorem.

4.1.5 Differential operator definition for our generalization

A lesser known characterization of the Eulerian Polynomials that me and Dr. Savage found in Sloane's encyclopedia of integer sequences states the following.

Define the operator D by $D = x \frac{d}{dx}$. Then for a nonnegative integer n ,

$$D^n \left(\frac{1}{(1-x)} \right) = \frac{x A_n(x)}{(1-x)^{n+1}}. \quad (4.6)$$

We generalize this below. Notice that when we insert $k = 1$ in the following theorem and use the fact that

$$A_n(x) = x^{n-1}A_n(1/x)$$

we get equation (4.6).

Theorem 9. *For a positive integer k , define the operator D_k by $D_k = kx \frac{d}{dx}$. Then for a nonnegative integer n ,*

$$D_k^n \left(\frac{1}{(1-x)^{1/k}} \right) = \frac{x^n A_n^{(k)}(1/x)}{(1-x)^{n+1/k}}. \quad (4.7)$$

Proof.

The proof is by induction on n . When $n = 0$, the theorem is clearly true. Let our hypothesis be that the claim holds for all $n < m + 1$.

We have,

$$D_k^{m+1} \left(\frac{1}{(1-x)^{1/k}} \right) = kx \frac{d}{dx} \left(\frac{x^m A_m^{(k)}(1/x)}{(1-x)^{m+1/k}} \right).$$

Applying the chain rule to theorem 7, we have,

$$\frac{d}{dx} A_m^{(k)}(1/x) = \frac{1}{k(1-x)} \left(A_{m+1}^{(k)}(1/x) - (1 + km/x) A_m^{(k)}(1/x) \right)$$

Now,

$$\begin{aligned} kx \frac{d}{dx} \left(\frac{x^m A_m^{(k)}(1/x)}{(1-x)^{m+1/k}} \right) &= kx \frac{x^m}{(1-x)^{m+1/k}} \frac{d}{dx} A_m^{(k)}(1/x) + kx A_m^{(k)} \frac{d}{dx} \left(\frac{x^m}{(1-x)^{m+1/k}} \right) \\ &= \frac{x^{m+1} A_m^{(k)}(1/x)}{(1-x)^{m+1/k+1}} - \frac{x^{m+1} (1 + km/x) A_m^{(k)}(1/x)}{(1-x)^{m+1/k+1}} \\ &\quad + kx A_m^{(k)}(1/x) \frac{d}{dx} \left(\frac{x^m}{(1-x)^{m+1/k}} \right) \end{aligned}$$

The above calculation completes the proof of the theorem. □

4.2 Emulation of Identities of the Eulerian Numbers

4.2.1 Expression for the $1/k$ -Eulerian numbers

Theorem 10. *For a positive integer k and for a nonnegative integer n ,*

$$a_{n,j}^{(k)} = \sum_{t=0}^j (-1)^{j-t} \binom{n+1/k}{j-t} \binom{t-1+1/k}{t} (kt+1)^n \quad (4.8)$$

Proof. We have,

$$A_n^{(k)}(x) = \sum_{j=0}^{n-1} a_{n,j}^{(k)} x^j.$$

From theorem 5 we have,

$$\begin{aligned} A_n^{(k)}(x) &= (1-x)^{n+1/k} \sum_{p \geq 0} \binom{p-1+1/k}{p} (kp+1)^n x^p \\ &= \sum_{p \geq 0} \sum_{j \geq 0} \binom{p-1+1/k}{p} (kp+1)^n x^p \binom{n+1/k}{j} (-x)^j \\ &= \sum_{j \geq 0} \sum_{t=0}^j \binom{t-1+1/k}{t} (kt+1)^n \binom{n+1/k}{j-t} (-1)^{j-t} x^j. \end{aligned}$$

Equating the coefficient of x^j completes the proof of the theorem. □

4.3 A Generalized Worpitzky Identity

Theorem 11. *For a positive integer k and for a nonnegative integer n ,*

$$\binom{t-1+1/k}{t} (kt+1)^n = \sum_j a_{n,j}^{(k)} \binom{t+1/k+n-j-1}{t-j} \quad (4.9)$$

Proof. From theorem 5 we have,

$$\begin{aligned}
\sum_{t \geq 0} \binom{t-1+1/k}{t} (kt+1)^n x^t &= (1-x)^{-(n+1/k)} A_n^{(k)}(x) \\
&= \sum_{t \geq 0} \binom{t+n+1/k-1}{t} x^t \sum_{j=0}^{n-1} a_{n,j}^{(k)} x^j \\
&= \sum_{t \geq 0} \sum_j \binom{t+n+1/k-1-j}{t-j} a_{n,j}^{(k)} x^t
\end{aligned}$$

Equating the coefficient of x^t completes the proof of the theorem . □

4.4 A Recurrence for the $1/k$ -Eulerian Numbers

Theorem 12. For a positive integer k and for a nonnegative integers n, j ,

$$a_{n+1,j}^{(k)} = (kj+1)a_{n,j}^{(k)} + k(n+1-j)a_{n,j-1}^{(k)} \quad (4.10)$$

with initial conditions $a_{n,0}^{(k)} = 1$ for $n \geq 0$ and $a_{n,j}^{(k)} = 0$ for $j > n$.

Proof. Using Theorem 7 we have,

$$\begin{aligned}
\sum_{j=0}^n a_{n+1,j}^{(k)} x^j &= kx(1-x) \frac{d}{dx} \sum_{j=0}^{n-1} a_{n,j}^{(k)} x^j + (1+knx) \sum_{j=0}^{n-1} a_{n,j}^{(k)} x^j \\
&= \sum_{j=0}^{n-1} kja_{n,j}^{(k)} x^j - \sum_{j=0}^{n-1} kja_{n,j}^{(k)} x^{j+1} + \sum_{j=0}^{n-1} a_{n,j}^{(k)} x^j + \sum_{j=0}^{n-1} kna_{n,j}^{(k)} x^{j+1} \\
&= \sum_{j=0}^{n-1} (kj+1)a_{n,j}^{(k)} + k(n+1-j)a_{n,j-1}^{(k)} x^j + ka_{n,n-1}^{(k)} x^n \\
&= \sum_{j=0}^n (kj+1)a_{n,j}^{(k)} + k(n+1-j)a_{n,j-1}^{(k)} x^j
\end{aligned}$$

This completes the proof of the theorem. □

In the next chapter we see a combinatorial proof of the above recurrence. We also pose some open questions.

Chapter 5

Some Combinatorics and Open Questions

In this chapter we first view a combinatorial proof of equation (4.10) for $k = 1$ and give a proof for $k = 2$. We leave it as an open question for $k > 2$. We also pose some other open questions.

We first present the following lemma which we will need to complete the proofs.

Lemma 3. *For integers $i \geq 1$, $0 \leq e_i \leq (i - 1)2$, and $k = 1, 2$,*

$$\frac{e_i}{k(i-1)+1} < \frac{e_{i+1}}{ki+1} \iff \frac{e_i+1}{ki+1} < \frac{e_{i+1}+1}{k(i+1)+1} \quad (5.1)$$

Proof. The proof is trivial when $k = 1$. We move on to the $k = 2$ case.

When $e_i = 0$, $e_{i+1} \geq 1$ and the lemma clearly holds. Otherwise $e_i \geq 1$ and we start off by proving that $e_i + \frac{2e_i}{2(i-1)+1}$ and $e_i + \frac{2(e_i+1)}{2i+1}$ lie between the same two integers.

From the given conditions clearly, $0 < \frac{e_i}{2(i-1)+1} < 1$ and $0 < \frac{e_i+1}{2i+1} < 1$.

Suppose

$$0 < \frac{e_i}{2(i-1)+1} < \frac{1}{2}$$

It follows that,

$$0 < e_i + 1 < i + \frac{1}{2}.$$

Implying that,

$$0 < \frac{e_i + 1}{2i + 1} < \frac{1}{2}.$$

Now suppose

$$\frac{1}{2} \leq \frac{e_i}{2i - 1}.$$

It follows that,

$$i + \frac{1}{2} \leq e_i + 1.$$

Implying that,

$$\frac{e_i + 1}{2i + 1} \geq \frac{1}{2}.$$

The converse is similar. Thus $e_i + \frac{2e_i}{2i-1}$ and $e_i + \frac{2(e_i+1)}{2i+1}$ lie between the same two integers.

We have,

$$\frac{e_i}{(i-1)2+1} < \frac{e_{i+1}}{(i)2+1} \implies e_{i+1} \geq \left\lceil e_i + \frac{2e_i}{2i-1} \right\rceil > \left\lfloor e_i + \frac{2(e_i+1)}{2i+1} \right\rfloor.$$

We also have,

$$\frac{e_i}{(i-1)2+1} \geq \frac{e_{i+1}}{(i)2+1} \implies e_{i+1} \leq \left\lfloor e_i + \frac{2e_i}{2i-1} \right\rfloor < \left\lceil e_i + \frac{2(e_i+1)}{2i+1} \right\rceil.$$

This completes the proof of the lemma. □

5.1 A Combinatorial Proof When $k = 1$ (the proof of the recurrence of the Eulerian numbers)

When $j = n = 0$, we have only the null string satisfying the given condition. When $j = 0$ and only the string $0, \dots, 0$ of length n satisfies the given constraint. When $j > 0$ and $n = 0$ there exists no such string. Otherwise we have to prove that,

$$a_{n,j}^{(1)} = (j+1)a_{n-1,j}^{(1)} + (n-j)a_{n-1,j-1}^{(1)}.$$

Let $e = e_1 \dots e_{n-1} \in \mathbb{I}_{n-1,1}$ and let $i \in \{0, \dots, n-1\}$.

We define a function $\Theta : \mathbb{I}_{n-1,1} \times \{0, \dots, n-1\} \rightarrow \mathbb{I}_{n,1}$ as follows:

$$\Theta(e, i) = e'$$

where $e' = e_1 \dots e_i 0 e_{i+1} + 1 \dots e_{n-1} + 1$.

Under Θ we claim that,

1. If $i = n - 1$ or $i \in \text{Asc}(e)$, then $\text{asc}(e) = \text{asc}(e')$.
2. Otherwise, $\text{asc}(e') = \text{asc}(e) + 1$.

For 1, if $i = n - 1$, then $e' = e0$, so $\text{asc}(e) = \text{asc}(e')$. If $i < n - 1$ and $i \in \text{Asc}(e)$, then $i \notin \text{Asc}(e')$, but $i + 1 \in \text{Asc}(e')$. For $l < i$, $l \in \text{Asc}(e')$ iff $l \in \text{Asc}(e)$. Using lemma 3, if $l > i$, then $l \in \text{Asc}(e')$ iff $l \in \text{Asc}(e)$. Thus $\text{asc}(e) = \text{asc}(e')$.

For 2, if $i \notin \text{Asc}(e)$, then $i \notin \text{Asc}(e')$, but $i + 1 \in \text{Asc}(e')$. For $l \in \text{Asc}(e)$, $l \neq i$, using lemma 3, $l \in \text{Asc}(e) \iff (l < i \text{ and } l \in \text{Asc}(e')) \text{ or } (l > i \text{ and } l + 1 \in \text{Asc}(e'))$. Thus $\text{asc}(e') = \text{asc}(e) + 1$.

So, every $e \in \mathbb{I}_{n-1,1}$ gives rise to $\text{asc}(e) + 1$ sequences $e' \in \mathbb{I}_{n,1}$ with $\text{asc}(e') = \text{asc}(e)$ and $n - \text{asc}(e)$ elements $e' \in \mathbb{I}_{n,1}$ with $\text{asc}(e') = \text{asc}(e) + 1$.

We complete the proof by showing that every $e' \in \mathbb{I}_{n,1}$ has a unique preimage $(e, i) \in \mathbb{I}_{n-1,1}$. We construct the inverse of e' by the following algorithm:

Let t be the position of the last 0 in $e' = e_1 \dots e_{t-1} 0 e_{t+1} \dots e_n$.

Then e' arises from $(e_1 \dots e_{t-1} e_{t+1} - 1 \dots e_n - 1, t - 1)$.

This completes the combinatorial proof for the $k = 1$ case. □

5.2 A Combinatorial Proof When $k = 2$

When $j = n = 0$, we have only the null string satisfying the given condition. When $j = 0$, then only the string $0, \dots, 0$ of length n satisfies the given constraint. When $j > 0$ and $n = 0$ there exists no such string. Otherwise we have to prove that,

$$a_{n,j}^{(2)} = (2j + 1)a_{n-1,j}^{(2)} + 2(n - j)a_{n-1,j-1}^{(2)}.$$

Let $e = e_1 \dots e_{n-1} \in \mathbb{I}_{n-1,k}$ and let $i \in \{0, \dots, n - 1\}$.

We define a function $\Theta : \mathbb{I}_{n-1,2} \times \{0, \dots, n - 1\} \rightarrow \mathbb{I}_{n,2}$ as follows:

$$\Theta(e, i) = e'$$

where $e' = e_1 \dots e_i 0 e_{i+1} + 1 \dots e_{n-1} + 1$.

Let $j \in \{1, \dots, n-1\}$.

We define another function $\Phi : \mathbb{I}_{n-1,2} \times \{1, \dots, n-1\} \rightarrow \mathbb{I}_{n,2}$ as follows:

$$\Phi(e, i) = e'$$

where $e' = e_1 \dots e_j 2j e_{j+1} + 1 \dots e_{n-1} + 1$.

Under Θ we claim that,

1. If $i = n-1$ or $i \in \text{Asc}(e)$, then $\text{asc}(e) = \text{asc}(e')$.
2. Otherwise, $\text{asc}(e') = \text{asc}(e) + 1$.

For 1, if $i = n-1$, then $e' = e0$, so $\text{asc}(e) = \text{asc}(e')$. If $i < n-1$ and $i \in \text{Asc}(e)$, then $i \notin \text{Asc}(e')$, but $i+1 \in \text{Asc}(e')$. For $l < i$, $l \in \text{Asc}(e')$ iff $l \in \text{Asc}(e)$. Using lemma 3, if $l > i$, then $l \in \text{Asc}(e')$ iff $l \in \text{Asc}(e)$. Thus $\text{asc}(e) = \text{asc}(e')$.

For 2, if $i \notin \text{Asc}(e)$, then $i \notin \text{Asc}(e')$, but $i+1 \in \text{Asc}(e')$. For $l \in \text{Asc}(e)$, $l \neq i$, using lemma 3, $l \in \text{Asc}(e) \iff (l < i \text{ and } l \in \text{Asc}(e')) \text{ or } (l > i \text{ and } l+1 \in \text{Asc}(e'))$. Thus $\text{asc}(e') = \text{asc}(e) + 1$.

Under Φ we claim that,

1. If $j \in \text{Asc}(e)$, then $\text{asc}(e) = \text{asc}(e')$.
2. Otherwise, $\text{asc}(e') = \text{asc}(e) + 1$.

The proof is similar to the Θ case and is left as an exercise to the reader.

So, every $e \in \mathbb{I}_{n-1,2}$ gives rise to $2 \text{asc}(e) + 1$ sequences $e' \in \mathbb{I}_{n,2}$ with $\text{asc}(e') = \text{asc}(e)$ and $2(n - \text{asc}(e))$ elements $e' \in \mathbb{I}_{n,2}$ with $\text{asc}(e') = \text{asc}(e) + 1$.

We complete the proof by showing that every $e' \in \mathbb{I}_{n,2}$ has a unique preimage $(e, i) \in \mathbb{I}_{n-1,2}$. We construct the inverse of e' by the following algorithm:

If possible, let $t \geq 2$ be the position of the last $2(t-1)$ in $e' = e_1 \dots e_{t-1} 2(t-1) e_{t+1} \dots e_n$.

Then e' arises from $(e_1 \dots e_{t-1} e_{t+1} - 1 \dots e_n - 1, t-1)$.

Else let t be the position of the last 0 in $e' = e_1 \dots e_{t-1} 0 e_{t+1} \dots e_n$.

Then e' arises from $(e_1 \dots e_{t-1} e_{t+1} - 1 \dots e_n - 1, t-1)$.

This completes the combinatorial proof for the $k = 2$ case. □

5.3 Open Questions

1. What could be the combinatorial proof of equation (4.10) for $k > 2$?
2. The q -analog of an expression, is a generalization of an expression in terms of a parameter q , that reduces to the original expression as $q \rightarrow 1$. For example the q -analog of a positive integer n is

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

Also the q -analog of $n!$ is

$$[n]_q! = (1 + q)(1 + q^2) \dots (1 + q + \dots + q^{n-1}). \quad (5.2)$$

The q -analogs of expressions are interesting when they have a combinatorial interpretation. That is the parameter q counts some statistic of some set S of interest. Equation (5.2) is interesting because it has been shown that,

$$[n]_q! = \sum_{\pi \in S_n} q^{\text{inv}(\pi)}.$$

Here $\text{inv}(\pi) = \#\{(i, j) \mid i < j \text{ and } \pi(i) > \pi(j)\}$ is called the *inversion number* of π .

Coming to the major purpose of this discussion, the q -analog of the Eulerian polynomials is given by

$$A_n(x, q) = (x; q)_{n+1} \sum_{t \geq 0} ([t + 1]_q)^n x^t.$$

Here $(x; q)_{n+1}$ denotes $\prod_{i=0}^n (1 - xq^i)$.

This q -analog is of interest because it has a combinatorial interpretation given by

$$\sum_{\pi \in S_n} x^{\text{des}(\pi)} q^{\text{maj}(\pi)}.$$

Here $\text{maj}(\pi) = \sum_{i \in \text{des}(\pi)} i$ is called the *major index* of π . According to [10] the above combinatorial interpretation was due to Carlitz.

Our question: Is there a meaningful q -analog of the $1/k$ -Eulerian polynomials that has a combinatorial meaning? If so what statistic would q represent?

3. The *Stirling numbers of the second kind* are special numbers in combinatorics that

count the number of partitions of a set. The numbers $S(n, k)$ denote the Stirling numbers of the second kind and count the number of partitions of the set $\{1, 2, \dots, n\}$ into k disjoint sets. Apart from their combinatorial definition, these numbers satisfy a recurrence and have a closed form for their exponential generating function. The Eulerian numbers are connected to these special numbers by the following equation.

$$\sum_i a_{n,i} \binom{i}{n-q} = S(n, q)q! \quad (5.3)$$

Our question: Can we define a generalization of the Stirling numbers that relates to the $1/k$ -Eulerian polynomials? The generalized Stirling numbers should have a combinatorial interpretation that generalizes the one, the Stirling numbers of the second kind have. Also if such a relation exists it should reduce to equation (5.3) when $k = 1$.

4. As discussed before, the Worpitzky identity that gives the factorial expansion of t^n in terms of the Eulerian numbers is given by

$$t^n = \sum_j a_{n,j} \binom{t+j}{n}. \quad (5.4)$$

[14] gives a combinatorial proof of the above equation using the set of sequences $a_1 \dots a_n$, where $1 \leq a_i \leq t$.

We have derived in section 4 the generalized Worpitzky identity given by

$$\binom{t-1+1/k}{t} (kt+1)^n = \sum_j a_{n,j}^{(k)} \binom{t+1/k+n-j-1}{t-j}. \quad (5.5)$$

Our question: Does there exist a combinatorial proof of the above equation? Could we generalize the approach used in [14] to achieve the same?

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