

## ABSTRACT

ERBACHER, CHRISTINA ELIZABETH. Root Multiplicities of the Indefinite Kac-Moody Algebra  $HD_4^{(3)}$ .  
(Under the direction of Dr. Kailash Misra.)

Victor Kac and Robert Moody (independently) generalized the concept of finite dimensional semisimple Lie algebras to the infinite dimensional case in 1968. We call this class of Lie algebras Kac-Moody algebras. There are many applications of Kac-Moody algebras in physics and mathematics.

An important problem concerning Kac-Moody algebras is finding their root multiplicities. Each Kac-Moody algebra is determined by what we call a *Dynkin diagram* which may be used to construct a *generalized Cartan matrix* (GCM). Every indecomposable symmetrizable GCM is one of three types: *finite*, *affine*, or *indefinite*. The root multiplicities are known for finite and affine type Kac-Moody algebras. For indefinite type Kac-Moody algebras, though, determining root multiplicities is still an open problem.

In this thesis we study the root multiplicities of the hyperbolic indefinite type Kac-Moody algebra,  $HD_4^{(3)}$ . Using a well known construction (see, for example, [Benkart, Kang & Misra, 1993]) we realize  $\mathfrak{g} = HD_4^{(3)}$  as a  $\mathbb{Z}$ -graded Lie algebra with local part  $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$  where  $\mathfrak{g}_0$  is the affine Kac-Moody algebra  $D_4^{(3)}$ . With this realization of  $HD_4^{(3)}$  we then use Kang's multiplicity formula (1994) which involves finding weight multiplicities of certain  $D_4^{(3)}$ -modules. To compute such multiplicities we use crystal base theory for  $D_4^{(3)}$ -modules.

Let  $\{\alpha_0, \alpha_1, \alpha_2\}$  and  $\{\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2\}$  denote the simple roots of  $D_4^{(3)}$  and  $HD_4^{(3)}$ , respectively. Let  $\delta$  be the canonical imaginary root of  $D_4^{(3)}$ . We calculate multiplicities of some  $HD_4^{(3)}$  roots of the form  $-l\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2 - n\delta$  for  $1 \leq l \leq 3$ ,  $k, m_i \in \mathbb{Z}_{\geq 0}$ . In each case we see that Frenkel's conjectured bound for the root multiplicities holds. We also find the *energy function* for the level 1  $D_4^{(3)}$  *perfect crystal* and use this function to generalize our results for level 1 roots of the form  $-\alpha_{-1} - k\delta$ .

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Root Multiplicities of the Indefinite Kac-Moody Algebra  $HD_4^{(3)}$

by  
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North Carolina State University  
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## BIOGRAPHY

Christina Erbacher was born on May 10, 1985 near Buffalo, NY and grew up just outside of Buffalo in the Village of Williamsville. Math was her best and favorite subject throughout middle school and high school. When Christina began her undergraduate career at SUNY Geneseo in Geneseo, NY she was a business major, then an accounting major, then a marketing major and started to consider sociology and even psychology. Needless to say, she was interested in many subjects and couldn't decide what she wanted to be when she grew up. But, she still had yet to consider mathematics.

In the spring of Christina's sophomore year at Geneseo, she took an introductory physics course and found herself doing mathematics for the first time in almost two years. It was exciting for her to feel confident in her understanding of the material and to be able to help her friends in the course. Helping her friends was so rewarding, in fact, that she began to consider teaching as a possible career. For as long as she could remember, her mom had said she should be a teacher. As is the case with many daughters, Christina had proceeded to think of anything but teaching as a possibility. As is also the case with many mothers, Christina's mom had known best all along.

At the end of her sophomore year, Christina became a math major. Soon after, she was hired at the Geneseo math department's walk-in tutoring center and was absolutely thrilled with the tutoring experience. Being able to guide fellow students to understanding, while deepening her own understanding of mathematics through teaching was exhilarating. Teaching was surely the career Christina would pursue.

After finishing her bachelor's, Christina was not yet ready to stop studying mathematics. So, following the advice of her professors, she applied to graduate school and accepted a teaching assistantship at North Carolina State University. She then moved to Raleigh, NC and began working on her PhD in mathematics.

While at NC State, Christina has earned her Master's in Mathematics as well as her Master's of Education in Math Education. She has met and befriended many wonderful mathematicians including her (now) fiancé, Ryan Therkelsen. She will defend her PhD thesis in May 2012 and move to be with Ryan in the Kentucky/Southern Ohio region at the end of May. Christina is very excited to begin a position as an Educator Assistant Professor in the Mathematics Department at the University of Cincinnati in August in 2012. She and Ryan will be married on December 29, 2012.

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Graduate school has been an incredible experience for me. There were times during this experience that I never wanted it to end and times where I thought there was no possible way I could finish. As I near the end of this journey and begin to embark on the next, it is essential that I thank the people around me who teach me to be patient with myself, to believe in my capacity for success, and to appreciate every step of the way.

Many of the graduate students in our department at NC State can attest to Dr. Misra's exceptional wisdom and patience. I have been extremely fortunate to have had Dr. Misra as my thesis advisor and I am not sure I would be here today if it weren't for his unfailing patient guidance. I would like to thank Dr. Misra for pushing me to reach my potential as a student of mathematics. Throughout a few years' worth of weekly meetings, Dr. Misra has seen me through countless highs and lows of academic, personal, and social experiences and he still swears that he is not (yet) sick of me. It is difficult for me to express the depth of my gratitude for Dr. Misra's advisement - his words of advice and his patience both as a teacher and an advisor will be with me throughout my life and career. Thank you, Dr. Misra.

One of the most wonderful parts of graduate school has been meeting my (now) fiancé, Ryan Therkelsen. On a related note, one of the most difficult parts of graduate school was staying in Raleigh while Ryan moved to Louisville, KY in 2010. I am not sure if I can even count how many times I have considered leaving NC State to be with Ryan, but I am sure that without Ryan's patient encouragement and belief in me I would have left much sooner. Ryan, with you by my side I feel like I could jump over the moon. Thank you for being my favorite. :)

It is without a doubt that I will never be able to thank my parents enough. Throughout all of my years of school my mom and dad have supported and believed in me and all of my dreams. They have taught me (by example) to be enthusiastic about life and to both have and pursue high expectations for myself. To Emily, Anthony, and Daniel as well - I love you so much, thank you!!

Next, I must thank Melissa Tolley (Melly Mel) for living with me these past two years. Mel is an unbelievably wonderful friend and has been put through what I am sure has been a true test of patience and love while dealing with me these past two difficult years. She has set a supreme example of friendship and has been essential to my graduate school experience. Mel - I am not sure I could have done it without you. I will miss having such a lovely roommate and officemate, thank you for being my friend. :)

There are so many more people I need to thank and I am afraid of forgetting someone. With this in mind, I still must mention a few more specific people.

I would like to thank the members of my committee: Dr. Stitz, Dr. Lada, and Dr. Putcha. They are not only members of my committee but have played integral roles in making my graduate experience at NC State so rewarding both as mathematicians and advisors. Dr. Griggs, Dr. Fenn, and Dr. Lee you have inspired me to embrace the world of teaching mathematics - and not only in the classroom - thank you. To the many other wonderful math teachers I have had - including Mr. Krom at Williamsville South High School and Dr.

Tang at SUNY Geneseo - thank you.

Without the friends I have made at NC State, graduate school would not have been as much fun. Thank you to all of them. Especially, though - thank you to Justin for sharing an office with me this past year; thank you to Catherine and Becca for sharing an office with me last year - and especially to Becca for being a really great mentor; thank you to Emma for intense brie, tomato soup, and brownies study sessions during our first year; I am going to stop there because I will be sure to miss someone if I keep listing names, but you all know who you are and I love you :)

While living in Raleigh one of my oldest friends, Maria, moved to Chapel Hill to go to law school. Thank you Maria (and Kendra!) for being there to enjoy these two cities with me when we were not hard at work ;)

Thank you to my other oldest friend, Jes for coming to visit Maria and I and for the much-needed phone calls over the years. Thank you to Emshee, Cait, Car, Kriss-y, and Lauren for your love, friendship, laughter, and for making me feel truly happy to be alive - as only wonderful friends can do.

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# Chapter 1

## Introduction

Lie algebras and Lie groups are closely related to the study of symmetries in nature and are naturally related with numerous physical phenomenon. The finite dimensional simple Lie algebras were completely classified around 1894 by Cartan. In 1968 Victor Kac and Robert Moody defined the infinite dimensional analog of finite dimensional semisimple Lie algebras, now known as Kac-Moody Lie algebras. An important class of infinite dimensional Kac-Moody Lie algebras are the affine Lie algebras which are known to physicists as current algebras.

The representation theory of affine Lie algebras has been an important direction of mathematical research during the last three decades, with numerous connections to other areas of mathematics such as number theory and combinatorics. The representation theory of affine Lie algebras also has many connections to physics, such as in conformal field theory, quantum physics, and statistical mechanics. Many of these interactions have been possible due to a clear understanding of the combinatorial properties of affine Lie algebras and their representations.

Each Kac-Moody algebra is determined by its Dynkin diagram, from which we may construct the corresponding generalized Cartan matrix (GCM). Each GCM is one of three types: finite, affine, or indefinite and the corresponding Kac-Moody algebra is classified in the same way.

A fundamental problem in the study of Kac-Moody algebras is determining root multiplicities. For Kac-Moody algebras of finite and affine type, the root multiplicities are well known (see [6]). For infinite dimensional, non-affine Kac-Moody algebras of indefinite type, determining root multiplicities is still an open problem (for example, see [1], [12], [5].)

In this thesis we study the root multiplicities for the Kac-Moody algebra of indefinite type,  $HD_4^{(3)}$ . In Chapter 2 we review some of the fundamental definitions and theories of Kac-Moody algebras, including the definition of a root and its multiplicity. We also include specific examples of concepts pertaining to our study.

A key approach to our study is to utilize a construction of  $HD_4^{(3)}$  (as in [1], [12]) that begins with the affine Kac-Moody algebra  $D_4^{(3)}$ , and realizes  $HD_4^{(3)}$  as a  $\mathbb{Z}$ -graded Lie algebra with local part  $V(\Lambda_0) \oplus$

$D_4^{(3)} \oplus V^*(\Lambda_0)$ , where  $V(\Lambda_0)$  is the  $D_4^{(3)}$  highest weight module with weight  $\Lambda_0$  and  $V^*(\Lambda_0)$  is its finite dual. We detail this construction in Chapter 3. This construction allows us to determine root multiplicities of  $HD_4^{(3)}$  with Kang's multiplicity formula ([8]), which requires counting the multiplicities of weights in certain  $D_4^{(3)}$  integrable highest weight modules. We also review Kang's formula in Chapter 3.

To count the multiplicities of  $D_4^{(3)}$  weights we use the crystal base theory for the quantum affine algebra  $U_q(D_4^{(3)})$  that was introduced by Kashiwara ([10]) and Lusztig ([13]) (independently) in 1991. The crystal base for an integrable module of the affine Lie algebra  $D_4^{(3)}$  has a nice combinatorial structure which is encoded in a directed graph called a crystal graph. In this process, determining the root multiplicities reduces to counting the vectors of certain weights on the crystal graph. We review quantum affine algebras and crystal base theory, as well as examples pertaining to our work, in Chapter 4.

Let  $\{\alpha_0, \alpha_1, \alpha_2\}$  and  $\{\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2\}$  denote the simple roots of  $D_4^{(3)}$  and  $HD_4^{(3)}$ , respectively. Let  $\delta$  be the canonical imaginary root of  $D_4^{(3)}$ . In Chapter 5 we detail the implication of our strategy involving Kang's multiplicity formula and crystal base theory to calculate calculate multiplicities of some  $HD_4^{(3)}$  roots of the form  $-l\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2 - n\delta$  for  $1 \leq l \leq 3$ ,  $k, m_i \in \mathbb{Z}_{\geq 0}$ . We also find the *energy function* for the level 1  $D_4^{(3)}$  perfect crystal and use this information to generalize our results for level 1 roots of the form  $-\alpha_{-1} - k\delta$ .

# Chapter 2

## Kac-Moody Lie Algebras

In this chapter we provide an introduction to one of the important mathematical objects we use in our study, Kac-Moody Lie Algebras. For more detailed information see, for example, [6], [4], and [8].

### 2.1 Definitions and Examples

**Definition.** Let  $\mathfrak{g}$  be a vector space over  $\mathbb{C}$  with an anti-symmetric, bilinear operation  $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  (called the bracket). Then  $\mathfrak{g}$  is a Lie algebra if the bracket satisfies what is called the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \text{ for all } x, y, z \in \mathfrak{g}.$$

**Definition.** Let  $\mathfrak{g}$  be a Lie algebra. The universal enveloping algebra of  $\mathfrak{g}$  is the pair  $(U(\mathfrak{g}), i)$ , where  $U(\mathfrak{g})$  is an associative algebra over  $\mathbb{C}$  and  $i : \mathfrak{g} \rightarrow U(\mathfrak{g})$  is a Lie algebra homomorphism defined by:

$$i([x, y]) = i(x)i(y) - i(y)i(x) \text{ for all } x, y \in \mathfrak{g},$$

satisfying the universal property.

**Theorem 2.1. Poincare-Birkhoff-Witt Theorem, [4].** Let  $(U(\mathfrak{g}), i)$  be the universal enveloping algebra of  $\mathfrak{g}$ . Then, the map  $i : \mathfrak{g} \rightarrow U(\mathfrak{g})$  is injective and for an ordered  $\mathfrak{g}$ -basis  $\{x_l | l \in I\}$ ,  $\{x_{l_1}x_{l_2} \cdots x_{l_n} | l_1 \leq \cdots l_n, n \geq 0\}$  forms a basis for  $U(\mathfrak{g})$ .

**Definition.** Let  $\mathfrak{g}$  be a Lie algebra and  $V$  be a vector space over  $\mathbb{C}$ . A representation of  $\mathfrak{g}$  on  $V$  is a Lie algebra homomorphism  $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ . A vector space  $V$  is called a  $\mathfrak{g}$ -module if there is a bilinear map  $\mathfrak{g} \times V \rightarrow V$ , denoted by  $(x, v) \mapsto x \cdot v$ , satisfying

$$[x, y] \cdot v = x \cdot (y \cdot v) \text{ for } x, y \in \mathfrak{g}, v \in V.$$

A representation  $\phi$  of  $\mathfrak{g}$  on  $V$  defines a  $\mathfrak{g}$ -module structure on  $V$ , and vice versa.

A representation of a Lie algebra  $\mathfrak{g}$  naturally extends to a representation of its universal enveloping algebra,  $U(\mathfrak{g})$ . Conversely, a representation of  $U(\mathfrak{g})$  is also a representation of  $\mathfrak{g}$ . Therefore, the representation theory of a Lie algebra and its universal enveloping algebra are essentially equivalent and we can study representations of  $\mathfrak{g}$  by studying representations of  $U(\mathfrak{g})$ .

**Definition.** A matrix  $A = (a_{ij})_{i,j \in I}$  for a finite index set  $I$  is called a generalized Cartan matrix (GCM) if it satisfies the following conditions:

$$a_{ii} = 2 \text{ for all } i \in I$$

$$a_{ij} \in \mathbb{Z}_{\leq 0} \text{ for all } i \neq j \in I$$

$$a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \text{ for all } i, j \in I$$

**Definition.** A matrix  $A$  is symmetrizable if there exists a diagonal matrix  $D = \text{diag}(s_i \mid i \in I)$  with all  $s_i \in \mathbb{Z}_{>0}$  such that  $DA$  is symmetric. A matrix  $A$  is indecomposable if for every pair of non empty subsets  $I_1, I_2 \subset I$  with  $I_1 \cup I_2 = I$ , there exists some  $i \in I_1$  and  $j \in I_2$  such that  $a_{ij} \neq 0$ .

**Definition.** An indecomposable GCM,  $A$ , is said to be of

- finite type if there exists  $u > 0$  such that  $Au > 0$ ,
- affine type if there exists a  $u > 0$  such that  $Au = 0$ ,
- indefinite type if there exists a  $u > 0$  such that  $Au < 0$ .

Let  $A = (a_{ij})_{i,j \in I=\{1,\dots,n\}}$  be a symmetrizable GCM of rank  $l$ . The *Cartan datum* of  $A$  is the quintuple,  $(A, \Pi, \check{\Pi}, P, \check{P})$ . We call  $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \check{P}$  the *Cartan subalgebra* where the set  $\check{P}$  is a free abelian group of rank  $2|I| - l$  with  $\mathbb{Z}$ -basis  $\{h_i \mid i \in I\} \cup \{d_s \mid s = 1, \dots, |I| - l\}$ , called the *dual weight lattice*. Then, define the set  $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(\check{P}) \subset \mathbb{Z}\}$  to be the *weight lattice*.

Call the set  $\check{\Pi} = \{h_i \mid i \in I\}$  the set of *simple coroots*, and choose a linearly independent subset:  $\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$  with

$$\alpha_j(h_i) = a_{ij}, \quad \alpha_j(d_s) = 0 \text{ or } 1 \text{ for } i, j \in I, s = 1, \dots, |I| - l.$$

Call these elements *simple roots*. Then, define the *fundamental weights* to be the linear functionals  $\Lambda_i \in \mathfrak{h}^*$  ( $i \in I$ ), given by:

$$\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d_s) = 0 \text{ for } j \in I, s = 1, \dots, |I| - l.$$

**Definition.** The Kac-Moody algebra  $\mathfrak{g}$  associated with a Cartan datum  $(A, \Pi, \check{\Pi}, P, \check{P})$  is the Lie algebra on generators  $e_i, f_i$  ( $i \in I$ ) and  $h \in \check{P}$  with the following six relations:

1.  $[h, h'] = 0$  for  $h, h' \in \check{P}$ ,

2.  $[e_i, f_j] = \delta_{ij} h_i$ ,
3.  $[h, e_i] = \alpha_i(h) e_i$  for  $h \in \check{P}$ ,
4.  $[h, f_i] = -\alpha_i(h) f_i$  for  $h \in \check{P}$ ,
5.  $(ade_i)^{1-a_{ij}} e_j = 0$  for  $i \neq j$ ,
6.  $(adf_i)^{1-a_{ij}} f_j = 0$  for  $i \neq j$ .

The generators  $e_i, f_i$ , ( $i \in I$ ) are called the Chevalley generators.

**Example 2.2.** The algebra  $\mathfrak{g} = D_4^{(3)}$  is the affine Kac-Moody algebra  $\mathfrak{g}(A)$  associated with the generalized Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}.$$

**Example 2.3.**  $HD_4^{(3)}$  is the Kac-Moody algebra of indefinite type associated with the generalized Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{pmatrix},$$

So,  $HD_4^{(3)} = \mathfrak{g}(A)$  where  $\mathfrak{g}(A)$  is the Lie algebra on generators  $\{e_i, f_i\}_{i=-1}^2$  and  $\mathfrak{h}$  with defining relations:

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_i, \quad i = -1, 0, 1, 2 \\ [h, h'] &= 0, \quad h, h' \in \mathfrak{h} \\ [h, e_i] &= \langle \alpha_i, h \rangle e_i, \quad i = -1, 0, 1, 2, \quad h \in \mathfrak{h} \\ [h, f_i] &= -\langle \alpha_i, h \rangle f_i, \quad i = -1, 0, 1, 2, \quad h \in \mathfrak{h} \end{aligned} \tag{2.1}$$

Where,  $\langle \alpha_j, h_i \rangle = \alpha_j(h_i) = a_{ij}$  for  $i, j = -1, 0, 1, 2$ .

**Definition.** To each generalized Cartan matrix  $A = (a_{ij})_{i,j \in I}$ , we associate an oriented graph called the Dynkin diagram of  $A$  with the following structure:

1. There are  $|I|$  nodes.
2. The  $i^{th}$  and  $j^{th}$  node are connected by  $a_{ij} \cdot a_{ji}$  edges.
3. If  $|a_{ij}| < |a_{ji}|$ , then the edges from the  $i^{th}$  node to the  $j^{th}$  node are directed.

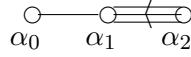


Figure 2.1: Dynkin diagram of  $D_4^{(3)}$ .

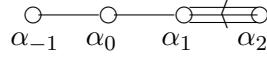


Figure 2.2: Dynkin diagram of  $HD_4^{(3)}$ .

**Example 2.4.** Figure 2.1 is the Dynkin diagram of  $D_4^{(3)}$ .

**Example 2.5.** Figure 2.2 is the Dynkin diagram of  $HD_4^{(3)}$

Notice we obtain the Dynkin diagram of  $HD_4^{(3)}$  from the Dynkin diagram of  $D_4^{(3)}$  by adding a node.

## 2.2 Roots and Weights of Kac-Moody Algebras

**Definition.** For the simple roots  $\alpha_i$ ,  $i \in I$ , of a Kac-Moody Lie algebra  $\mathfrak{g}$ , the free abelian group  $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$  is called the root lattice and  $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$  is called the positive root lattice. For each  $\alpha \in Q$ , let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

If  $\alpha \neq 0$  and  $\mathfrak{g}_\alpha \neq 0$ , then  $\alpha$  is called a root of  $\mathfrak{g}$  and  $\mathfrak{g}_\alpha$  is called the root space attached to  $\alpha$ . The dimension of  $\mathfrak{g}_\alpha$  is called the root multiplicity of  $\alpha$ .

Note that  $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$  and  $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$ . The Kac-Moody algebra  $\mathfrak{g}$  has the root space decomposition:

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha,$$

and all roots are either positive ( $\alpha \in Q_+$ ) or negative ( $\alpha \in Q_- = -Q_+$ ). We let  $\Delta$ ,  $\Delta^+$ , and  $\Delta^-$  represent the set of all roots, positive roots and negative roots, respectively. Then define  $\mathfrak{g}_+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$  and  $\mathfrak{g}_- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$ , and we have the triangular decomposition:

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+.$$

**Definition.** For each  $i \in I$ , define the simple reflection  $r_i$  on  $\mathfrak{h}^*$  by

$$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i.$$

The Weyl group,  $W$ , is the subgroup of  $\text{End}(\mathfrak{h}^*)$  generated by all simple reflections. Any  $w \in W$  may be expressed as a product of simple reflections,  $w = \prod_{k=1}^t r_{ik}$ . If it is minimal amongst all such expressions, we say  $w$  is a reduced expression and call  $t$  the length of  $w$ , denoted  $l(w)$ .

The following results are proven in [6]. First, we have that the *Chevalley involution*,  $\zeta : \mathfrak{g} \rightarrow \mathfrak{g}$ , given by

$$\zeta(e_i) = -f_i, \quad \zeta(f_i) = -e_i, \quad \zeta(h) = -h$$

is an automorphism. Then, since  $\zeta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ , we have that  $\text{mult}(\alpha) = \text{mult}(-\alpha)$  for all  $\alpha \in \Delta$ . Next, if  $\alpha$  is a root of a Kac-Moody algebra of finite type,  $\mathfrak{g}$ , then there exists a  $w \in W$  such that  $w(\alpha) = \alpha_i$  for some  $i \in I$ ,  $\text{mult}(\alpha) = 1$ , and  $k\alpha$  is a root if and only if  $k = \pm 1$ . On the other hand, if  $\alpha$  is a root of a Kac-Moody algebra of affine or indefinite type, we say  $\alpha$  is a *real root* if  $\alpha$  is  $w$ -conjugate to a simple root. Hence, real roots all have multiplicity 1. If  $\alpha$  is not a real root we say  $\alpha$  is an imaginary root. If  $\alpha$  is an imaginary root of an affine Kac-Moody algebra  $\mathfrak{g}$ , then  $k\alpha$  is also an imaginary root of  $\mathfrak{g}$  for all  $k \in \mathbb{Z}$ , and the multiplicity of  $\alpha$  is equal to the rank of the Cartan matrix associated with  $\mathfrak{g}$ . For Kac-Moody algebras of indefinite type, determining root multiplicities is still an open problem.

Now, for a Kac-Moody algebra  $\mathfrak{g}$ , we discuss the weights (and associated weight spaces) of a  $\mathfrak{g}$ -module  $V$ .

**Definition.** For any  $\lambda \in \mathfrak{h}^*$ , the  $\lambda$ -weight space  $V_\lambda$  is the set

$$V_\lambda = \{v \in V \mid h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

If  $V_\lambda \neq 0$ , we call  $\lambda$  a weight of  $V$  and the dimension of  $V_\lambda$  is called the weight multiplicity of  $\lambda$  in  $V$ .

**Definition.** A module  $V$  is called a weight module if it admits a weight space decomposition:

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu.$$

When all weight spaces,  $V_\mu$ , of a weight module  $V$  are finite dimensional, we define the character of  $V$  as

$$chV = \sum_{\mu} \dim(V_\mu) e^\mu,$$

where  $e^\mu$  are formal basis elements of the group algebra  $\mathbb{C}[\mathfrak{h}^*]$  with multiplication defined by  $e^\lambda e^\mu = e^{\lambda+\mu}$ .

**Definition.** Let  $\text{wt}(V)$  be the set of all weights of the module  $V$  and define the category  $\mathcal{O}$ , its objects consisting of weight modules  $V$  over  $\mathfrak{g}$  with finite dimensional weight spaces for which there exists a finite number of elements  $\lambda_1, \lambda_2, \dots, \lambda_s \in \mathfrak{h}^*$  such that

$$\text{wt}(V) \subset \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda_1\} \cup \dots \cup \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda_s\},$$

where  $\mu \geq \lambda$  if and only if  $\mu - \lambda \in Q_+ = \sum_{i \in I} \mathbb{Z}_+ \alpha_i$ .

Of particular significance are the  $\mathfrak{g}$ -modules in the category  $\mathcal{O}$  called highest weight modules.

**Definition.** A weight module  $V$  is a highest weight module of highest weight  $\lambda \in \mathfrak{h}^*$  if there exists a nonzero vector  $v_\lambda \in V$ , called a highest weight vector, such that

$$\begin{aligned} e_i \cdot v_\lambda &= 0 \text{ for all } i \in I, \\ h \cdot v_\lambda &= \lambda(h)v_\lambda \text{ for all } h \in \mathfrak{h}, \\ V &= U(\mathfrak{g})v_\lambda. \end{aligned}$$

If we let  $\mathfrak{g}$  be any Kac-Moody algebra and  $V$  be a highest weight  $\mathfrak{g}$ -module with highest weight  $\lambda$ , then  $V = \bigoplus_{\mu \leq \lambda} V_\mu$ , where each  $V_\mu$  is finite dimensional and  $\dim(V_\lambda) = 1$ .

**Definition.** A  $\mathfrak{g}$ -module  $M(\lambda)$  with highest weight  $\lambda$  is a Verma module if every other  $\mathfrak{g}$ -module with highest weight  $\lambda$  is a quotient of  $M(\lambda)$ .

For every  $\lambda \in \mathfrak{h}^*$ , there exists a unique (up to isomorphism) Verma module,  $M(\lambda)$ , with unique maximal proper submodule,  $M'(\lambda)$ . For  $L(\lambda) = M(\lambda)/M'(\lambda)$ ,  $L(\lambda)$  is an irreducible  $\mathfrak{g}$ -module with highest weight  $\lambda$ . Every irreducible  $\mathfrak{g}$ -module in the category  $\mathcal{O}$  is isomorphic to  $L(\lambda)$  for some  $\lambda \in \mathfrak{h}^*$ .

We say  $V$  is an *integrable*  $\mathfrak{g}$ -module if all  $e_i$  and  $f_i$  for  $i \in I$ , are locally nilpotent on  $V$ . A weight  $\lambda$  of  $V$  is called *integral* if  $\lambda(h_i) \in \mathbb{Z}$  for  $i \in I$ . The set of all integral weights is known as the *weight lattice*, denoted by  $P$ . Define  $P^+ = \{\lambda \in P \mid \lambda(h_i) \geq 0 \text{ for all } i \in I\}$ , and  $P^{++} = \{\lambda \in P \mid \lambda(h_i) > 0 \text{ for all } i \in I\}$

**Definition.** The category  $\mathcal{O}_{int}$  consists of integrable  $\mathfrak{g}$ -modules  $V$  in the category  $\mathcal{O}$  such that all weights of  $V$  are integral weights.

Every  $\mathfrak{g}$ -module in the category  $\mathcal{O}_{int}$  is completely reducible and every irreducible  $\mathfrak{g}$ -module in  $\mathcal{O}_{int}$  is isomorphic to a highest weight module  $L(\lambda)$  with  $\lambda \in P^+$ . Lastly, we note the action of the Weyl group for modules in this category. Let  $V \in \mathcal{O}_{int}$ ,  $\lambda \in wt(V)$ , and  $w \in W$ . Then we have:  $\dim(V_\lambda) = \dim(V_{w\lambda})$ .

# Chapter 3

## Construction of $HD_4^{(3)}$ and Kang's Multiplicity Formula

We wish to use Kang's (1994) multiplicity formula to compute root multiplicities of  $HD_4^{(3)}$ . To use Kang's formula, we construct  $HD_4^{(3)}$  as in [1] and [2]. This construction is detailed in Section 3.1. Then, in Section 3.2 we review Kang's formula.

### 3.1 Construction of $HD_4^{(3)}$

We begin by considering the Lie algebra  $\mathfrak{g}_0 = D_4^{(3)}$ . We construct two  $\mathfrak{g}_0$ -modules,  $V$  and  $V^*$ , and a  $\mathfrak{g}_0$ -module homomorphism  $\psi : V^* \otimes V \rightarrow \mathfrak{g}_0$ . With these four ingredients, we build a graded Lie algebra  $\mathfrak{g}$  which intersects its local part  $V \oplus \mathfrak{g}_0 \oplus V^*$  trivially. Finally, we show  $\mathfrak{g} \cong HD_4^{(3)}$ .

The algebra  $\mathfrak{g}_0 = D_4^{(3)}$  is the affine Kac-Moody algebra  $\mathfrak{g}_0(A_0)$  associated with the generalized Cartan matrix

$$A_0 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix},$$

obtained from the Dynkin diagram

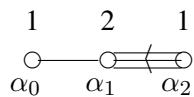


Figure 3.1: Dynkin diagram of  $D_4^{(3)}$  labeled with  $\delta$  coordinates.

The numerical labels are the coordinates of the unique vector  $\delta = (a_0, a_1, a_2)^t$  such that  $A_0\delta = 0$  and the  $a_i$  are positive relatively prime integers.

$A_0$  has the realization  $\{\mathfrak{h}_0, \Pi = \{\alpha_0, \alpha_1, \alpha_2\}, \check{\Pi} = \{H_0, H_1, H_2\}\}$ . The center of  $\mathfrak{g}_0$  is one dimensional and is spanned by  $K = H_0 + 2H_1 + 3H_2 \in \mathfrak{h}_0$ . Let  $d \in \mathfrak{h}_0$  be such that

$$\begin{aligned}\langle \alpha_0, d \rangle &= 1, \\ \langle \alpha_1, d \rangle &= 0, \\ \langle \alpha_2, d \rangle &= 0.\end{aligned}$$

Then,  $\{d, H_0, H_1, H_2\}$  form a basis for  $\mathfrak{h}_0$ . The algebra  $\mathfrak{g}_0$  is generated by the elements  $\{E_i\}_{i=0}^2, \{F_i\}_{i=0}^2$ , and the Cartan subalgebra,  $\mathfrak{h}_0$ .

Let  $\Lambda_0 \in \mathfrak{h}_0^*$  be such that

$$\langle \Lambda_0, d \rangle = 0, \langle \Lambda_0, h_0 \rangle = 1, \langle \Lambda_0, h_1 \rangle = 0, \langle \Lambda_0, h_2 \rangle = 0,$$

and let  $V = V(\Lambda_0)$  be the irreducible highest weight module over  $\mathfrak{g}_0$  with highest weight  $\Lambda_0$ . Let  $v_0$  represent the highest weight vector of weight  $\Lambda_0$  in  $V(\Lambda_0)$ . Then we have the following relations for  $i = 0, 1, 2$ :

$$\begin{aligned}H_i \cdot v_0 &= \delta_{0,i} \cdot v_0, \\ d \cdot v_0 &= 0, \\ E_i \cdot v_0 &= 0\end{aligned}$$

Using the module action given by:

$$\langle g \cdot v^*, v \rangle = -\langle v^*, g \cdot v \rangle \text{ for all } v^* \in V^*(\Lambda_0), v \in V(\Lambda_0), g \in \mathfrak{g}_0,$$

the restricted dual  $V^* = V^*(\Lambda_0)$  is an irreducible lowest weight  $\mathfrak{g}_0$ -module with lowest weight  $-\Lambda_0$ . Let  $v_0^*$  represent the lowest weight vector of  $V^*(\Lambda_0)$ . The following relations hold for  $i = 0, 1, 2$ :

$$\begin{aligned}H_i \cdot v_0^* &= -\delta_{0,i} \cdot v_0^*, \\ d \cdot v_0^* &= 0, \\ F_i \cdot v_0^* &= 0.\end{aligned}$$

The last ingredient we need for our construction is a homomorphism from  $V^* \otimes V$  to  $\mathfrak{g}_0$ . This homomorphism makes use of the standard invariant nondegenerate symmetric bilinear form:  $(\cdot | \cdot)_{\mathfrak{h}_0}$ , given

by:

$$\begin{aligned}(H_i|H_j) &= a_j \check{a}_j^{-1} a_{ij} \quad i = 0, 1, 2 \\ (d|d) &= 0 \\ (H_i|d) &= \delta_{i0} \quad i = 0, 1, 2\end{aligned}$$

The  $a_0, a_1, a_2$  are the numerical labels of the Dynkin diagram for  $D_4^{(3)}$ . Whereas,  $\check{a}_0, \check{a}_1, \check{a}_2$  the labels of the Dynkin diagram of the dual algebra obtained by reversing the directions of all arrows and keeping the same enumeration of vertices as shown in Figure 2.2 (see [6]). Thus, we have the relations:

$$\begin{aligned}(H_0|d) &= \delta_{0,0} = 1, \\ (H_0|H_0) &= 1(\frac{1}{1})(2) = 2, \\ (H_0|H_1) &= 2(\frac{1}{2})(-1) = -1, \\ (H_0|H_2) &= 1(\frac{1}{3})(0) = 0, \\ (H_1|d) &= \delta_{1,0} = 0, \\ (H_1|H_0) &= 1(\frac{1}{1})(-1) = -1, \\ (H_1|H_1) &= 2(\frac{1}{2})(2) = 2, \\ (H_1|H_2) &= 1(\frac{1}{3})(-3) = -1, \\ (H_2|d) &= \delta_{2,0} = 0, \\ (H_2|H_0) &= 1(\frac{1}{1})(0) = 0, \\ (H_2|H_1) &= 2(\frac{1}{2})(-1) = -1, \\ (H_2|H_2) &= 1(\frac{1}{3})(2) = \frac{2}{3}.\end{aligned}$$

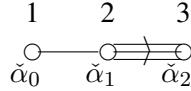


Figure 3.2: Dynkin diagram of  $G_2^{(1)}$ .

The standard bilinear form on  $\mathfrak{h}_0$  may be uniquely extended to an invariant, nondegenerate, symmetric,

bilinear form on all of  $\mathfrak{g}_0$ , which we will denote  $(\cdot | \cdot)_{\mathfrak{g}_0}$ . Then define the map  $\Psi : V^* \otimes V \rightarrow \mathfrak{g}_0$  by  $\Psi(v^* \otimes v) = -\sum_{i \in I} \langle v^*, x_i \cdot v \rangle x_i - 2\langle v^*, v \rangle K$ , where  $\{x_i \mid i \in I\}$  is an orthonormal basis for  $\mathfrak{g}_0$  with respect to the bilinear form  $(\cdot | \cdot)_{\mathfrak{g}_0}$ .

Now, we begin our construction of  $\mathfrak{g}$ . The space  $V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V^*(\Lambda_0)$  has a local Lie algebra structure with the bracket defined for all  $v \in V(\Lambda_0)$ ,  $v^* \in V^*(\Lambda_0)$ , and  $g \in \mathfrak{g}_0$ , such that:

$$\begin{aligned}[v^*, v] &= \Psi(v^* \otimes v), \\ [g, v] &= g \cdot v, \\ [g, v^*] &= g \cdot v^*. \end{aligned}$$

For  $j \geq 1$  define  $\hat{\mathfrak{g}}_{-j}$  to be the space spanned by all products of  $j$  vectors from  $V(\Lambda_0)$ , and define  $\hat{\mathfrak{g}}_j$  to be the space spanned by all products of  $j$  vectors from  $V^*(\Lambda_0)$ . Then,

$$\begin{aligned}\hat{\mathfrak{g}}_{-1} &= V(\Lambda_0), \\ \hat{\mathfrak{g}}_0 &= \mathfrak{g}_0, \\ \hat{\mathfrak{g}}_1 &= V(\Lambda_0)^*,\end{aligned}$$

and  $\hat{\mathfrak{g}} = \bigoplus_{j \in \mathbb{Z}} \hat{\mathfrak{g}}_j$  is the maximal graded Lie algebra with local part  $V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V^*(\Lambda_0)$ . Notice,  $\hat{\mathfrak{g}}_- = \bigoplus_{j \geq 1} \hat{\mathfrak{g}}_{-j}$  and  $\hat{\mathfrak{g}}_+ = \bigoplus_{j \geq 1} \hat{\mathfrak{g}}_{+j}$  are the free Lie algebras generated by  $V(\Lambda_0)$  and  $V^*(\Lambda_0)$ , respectively.

For all  $k > 1$ , define the subspaces:

$$\begin{aligned}J_k &= \{x \in \hat{\mathfrak{g}}_k \mid [y_1, [\dots [y_{k-1}, x]] \dots] = 0 \quad \forall y_1, \dots, y_{k-1} \in V(\Lambda_0)\} \\ J_{-k} &= \{x \in \hat{\mathfrak{g}}_{-k} \mid [y_1, [\dots [y_{k-1}, x]] \dots] = 0 \quad \forall y_1, \dots, y_{k-1} \in V^*(\Lambda_0)\} \\ J_{\pm} &= \sum_{k>1} J_{\pm k} \\ J &= J_- \oplus J_+.\end{aligned}$$

$J_{\pm}$  are ideals of  $\hat{\mathfrak{g}}$  and  $J$  is the largest graded ideal of  $\hat{\mathfrak{g}}$  which intersects the local part of  $\hat{\mathfrak{g}}$  trivially.

Then, we define

$$\begin{aligned}\mathfrak{g} &= \hat{\mathfrak{g}}/J \\ &= (\bigoplus_{i>1} \mathfrak{g}_{-i}) \oplus V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V^*(\Lambda_0) \oplus (\bigoplus_{i>1} \mathfrak{g}_i),\end{aligned}$$

where for  $i > 1$ ,  $\mathfrak{g}_{\pm i} = \hat{\mathfrak{g}}_{\pm i}/J_{\pm i}$ . This is the minimal graded Lie algebra with local part  $V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V^*(\Lambda_0)$ . In addition to the fact that in our construction each subspace  $\mathfrak{g}_{-j}$  (respectfully  $\mathfrak{g}_{+j}$ ) is a direct sum of irreducible highest weight (respectfully lowest weight) modules over  $\mathfrak{g}_0$ , our proof of the following theorem allows us to view the root spaces of  $HD_4^{(3)}$  as weight spaces of  $\mathfrak{g}_0$ -modules. Therefore, we may

then use the representation theory of the affine Kac-Moody algebra  $\mathfrak{g}_0 = D_4^{(3)}$  to find the root multiplicities of the indefinite Kac-Moody algebra  $HD_4^{(3)}$ .

**Theorem 3.1.** *Consider the map  $\phi$  from  $HD_4^{(3)} \rightarrow \mathfrak{g}$  given by:*

$$\begin{aligned} e_{-1} &\mapsto v_0^*, & f_{-1} &\mapsto v_0, & h_{-1} &\mapsto -d - 2K, \\ e_i &\mapsto E_i, & f_i &\mapsto F_i, & h_i &\mapsto H_i \end{aligned}$$

for  $i = 0, 1, 2$ , where  $\{e_{-1}, e_0, e_1, e_2\}, \{f_{-1}, f_0, f_1, f_2\}$ , and  $\mathfrak{h} = \text{span}\{h_{-1}, h_0, h_1, h_2\}$  are the Chevalley generators and Cartan subalgebra of  $HD_4^{(3)}$ , respectively. The map  $\phi : HD_4^{(3)} \rightarrow \mathfrak{g}$  is an isomorphism.

*Proof.* Recall,  $HD_4^{(3)} = \mathfrak{g}(A)$  where

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Also,  $\mathfrak{g}(A)$  is the Lie algebra on generators  $\{e_{-1}, e_0, e_1, e_2\}, \{f_{-1}, f_0, f_1, f_2\}$ , and  $\mathfrak{h}$  with defining relations stated in equation 2.1. Since  $J$  is the maximal graded ideal of  $\hat{\mathfrak{g}}$  which intersects the local part of  $\hat{\mathfrak{g}}$  trivially and  $\phi$  is clearly linear and bijective, we need only show the following relations hold:

$$\begin{aligned} [\phi(e_i), \phi(f_j)] &= \delta_{ij}\phi(h_i) \quad (i, j = -1, 0, 1, 2) \\ [\phi(h), \phi(h')] &= 0 \quad (h, h' \in \mathfrak{h}) \\ [\phi(h), \phi(e_i)] &= \langle \alpha_i, h \rangle \phi(e_i) \quad (i = -1, 0, 1, 2) \\ [\phi(h), \phi(f_i)] &= -\langle \alpha_i, h \rangle \phi(f_i) \quad (i = -1, 0, 1, 2) \end{aligned}$$

Notice, that for  $i, j = 0, 1, 2$ , the above statements follow directly from the two facts:

1.  $\phi(e_i) = E_i, \phi(f_i) = F_i$ , and  $\phi(h_i) = H_i$  for  $i = 0, 1, 2$ .
2. The generalized Cartan matrix for  $\mathfrak{g}_0 = D_4^{(3)}$  is identical to the generalized Cartan matrix for  $HD_4^{(3)}$  with column (-1) and row (-1) removed.

Thus, we have reduced our task to showing the following:

$$[\phi(e_{-1}), \phi(f_{-1})] = \phi(h_{-1}) \quad (3.1)$$

$$[\phi(e_{-1}), \phi(f_i)] = 0 \quad (i = 0, 1, 2) \quad (3.2)$$

$$[\phi(e_i), \phi(f_{-1})] = 0 \quad (i = 0, 1, 2) \quad (3.3)$$

$$[\phi(h), \phi(h')] = 0 \quad (h, h' \in \mathfrak{h}) \quad (3.4)$$

$$[\phi(h), \phi(e_{-1})] = \langle \alpha_{-1}, h \rangle \phi(e_{-1}) \quad (h \in \mathfrak{h}) \quad (3.5)$$

$$[\phi(h), \phi(f_{-1})] = -\langle \alpha_{-1}, h \rangle \phi(f_{-1}) \quad (h \in \mathfrak{h}) \quad (3.6)$$

First, we show (3.1). Note that,

$$[\phi(e_{-1}), \phi(f_{-1})] = [v_0^*, v_0] = \Psi(v_0^* \otimes v_0) = - \sum_{i \in I} \langle v_0^*, x_i \cdot v_0 \rangle x_i - 2 \langle v_0^*, v_0 \rangle K,$$

where  $\{x_i \mid i \in I\}$  is an orthonormal basis for  $\mathfrak{g}_0$  with respect to the bilinear form  $(\cdot \mid \cdot)_{\mathfrak{g}_0}$ , and  $K = H_0 + 2H_1 + 3H_2$ . Then, let  $y_{-1} = \frac{1}{\sqrt{-2}}(d - K)$  and  $y_0 = \frac{1}{\sqrt{2}}(d + K)$ . We show that  $y_{-1}, y_0$  are orthonormal:

$$\begin{aligned} (y_{-1}|y_{-1}) &= \frac{1}{-2}(d - K \mid d - K) \\ &= \frac{1}{-2}((d|d) - (d|K) - (K|d) + (K|K)) \\ &= \frac{1}{-2}(0 - 1 - 1 + 0) \\ &= 1 \end{aligned}$$

$$\begin{aligned} (y_0|y_0) &= \frac{1}{2}(d + K \mid d + K) \\ &= \frac{1}{2}((d|d) + (d|K) + (K|d) + (K|K)) \\ &= \frac{1}{2}(0 + 1 + 1 + 0) \\ &= 1 \end{aligned}$$

$$\begin{aligned}
(y_{-1}|y_0) &= \frac{1}{\sqrt{-4}}(d - K | d + K) \\
&= \frac{1}{\sqrt{-4}}((d|d) + (d|K) - (K|d) - (K|K)) \\
&= \frac{1}{\sqrt{-4}}(0 + 1 - 1 - 0) \\
&= 0
\end{aligned}$$

Also, define  $\{y_1, y_2\}$  as the orthonormal basis for the space  $\mathbb{H} = \text{span}\{H_1, H_2\}$ . We know such a basis exists because the bilinear form described above, restricted to  $\mathbb{H}$ , is an inner product. Then, the set  $\{y_{-1}, y_0, y_1, y_2\}$  forms an orthonormal basis for  $\mathfrak{h}_0$ .

Next, let  $\{x_i \mid i \in I\}$  be the orthonormal basis for  $\mathfrak{g}_0$  ordered in such a way that  $x_i = y_i$  for  $i = -1, 0, 1, 2$ , and we have:

$$\begin{aligned}
[\phi(e_{-1}), \phi(f_{-1})] &= \Psi(v_0^* \otimes v_0) \\
&= - \sum_{i \in I} \langle v_0^*, x_i \cdot v_0 \rangle x_i - 2 \langle v_0^*, v_0 \rangle K \\
&= - \langle v_0^*, \frac{1}{\sqrt{-2}}(d - K) \cdot v_0 \rangle (\frac{1}{\sqrt{-2}}(d - K)) \\
&\quad - \langle v_0^*, \frac{1}{\sqrt{2}}(d + K) \cdot v_0 \rangle (\frac{1}{\sqrt{2}}(d + K)) \\
&\quad - \sum_{i=1}^2 \langle v_0^*, x_i \cdot v_0 \rangle x_i - 2 \langle v_0^*, v_0 \rangle K
\end{aligned}$$

This last equality follows from the following relations:

$$\begin{aligned}
\langle g \cdot v^*, v \rangle &= -\langle v^*, g \cdot v \rangle, \\
E_i \cdot v_0 &= 0, \\
F_i \cdot v_0^* &= 0.
\end{aligned}$$

So we can rewrite each term where  $x_i \neq y_i$  so it is 0, and we are left to sum only over  $i = -1, 0, 1, 2$ .

Next, consider:  $-\sum_{i=1}^2 \langle v_0^*, x_i \cdot v_0 \rangle x_i$ . Since  $x_1, x_2 \in \mathbb{H}$ , and we know that each  $H_i \cdot v_0 = \delta_{0i}v_0$ , then,  $x_1 \cdot v_0 = 0$  and  $x_2 \cdot v_0 = 0$  as well. Thus, we are left with:

$$\begin{aligned}
[\phi(e_{-1}), \phi(f_{-1})] &= - \langle v_0^*, \frac{1}{\sqrt{-2}}(d - K) \cdot v_0 \rangle (\frac{1}{\sqrt{-2}}(d - K)) \\
&\quad - \langle v_0^*, \frac{1}{\sqrt{2}}(d + K) \cdot v_0 \rangle (\frac{1}{\sqrt{2}}(d + K)) \\
&\quad - 2 \langle v_0^*, v_0 \rangle K
\end{aligned}$$

We may then use the following:

$$\begin{aligned}
\langle v_0^*, v_0 \rangle &= 1, \\
H_i \cdot v_0 &= \delta_{0i} v_0, \\
K &= H_0 + 2H_1 + 3H_2, \\
\langle g \cdot v^*, v \rangle &= -\langle v^*, g \cdot v \rangle, \\
d \cdot v_0^* &= 0
\end{aligned}$$

to show:

$$\begin{aligned}
[\phi(e_{-1}), \phi(f_{-1})] &= -\frac{1}{\sqrt{-2}}(\langle d \cdot v_0^*, v_0 \rangle + \langle v_0^*, -K \cdot v_0 \rangle) \frac{1}{\sqrt{-2}}(d - K) \\
&\quad - \frac{1}{\sqrt{2}}(\langle d \cdot v_0^*, v_0 \rangle + \langle v_0^*, K \cdot v_0 \rangle) \frac{1}{\sqrt{2}}(d + K) - 2K \\
&= \frac{1}{\sqrt{-2}}\langle v_0^*, v_0 \rangle \frac{1}{\sqrt{-2}}(d - K) - \frac{1}{\sqrt{2}}(\langle v_0^*, v_0 \rangle) \frac{1}{\sqrt{2}}(d + K) - 2K \\
&= \frac{1}{-2}(d - K) - \frac{1}{2}(d + K) - 2K \\
&= \frac{1}{-2}d + \frac{1}{2}K - \frac{1}{2}d - \frac{1}{2}K - 2K \\
&= -d - 2K \\
&= \phi(h_{-1})
\end{aligned}$$

Next we show (3.2) and (3.3). For  $i = 0, 1, 2$  we have:

$$\begin{aligned}
[\phi(e_{-1}), \phi(f_i)] &= [v_0^*, F_i] \\
&= -[F_i, v_0^*] \\
&= -F_i \cdot v_0^* \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
[\phi(e_i), \phi(f_{-1})] &= [E_i, v_0] \\
&= E_i \cdot v_0 \\
&= 0.
\end{aligned}$$

Then, for (3.4) we first note that  $[H, H'] = 0$  for all  $H, H' \in \mathfrak{h}_0$ . Then notice

$$\phi(h_{-1}) = -d - 2K = -d - 2H_0 - 4H_1 - 6H_2 \in \mathfrak{h}_0$$

and  $\phi(h_i) = H_i \in \mathfrak{h}_0$  for  $i = 0, 1, 2$ . Since  $h$  and  $h'$  in  $\mathfrak{h}$  are linear combinations of  $\{h_{-1}, h_0, h_1, h_2\}$ ,  $\phi(h)$  and  $\phi(h')$  must both be elements of  $\mathfrak{h}_0$ . Thus,  $[\phi(h), \phi(h')] = 0$ .

Lastly, we show (3.5) and (3.6). For arbitrary  $c_{-1}, c_0, c_1, c_2 \in \mathbb{C}$ , let  $h = c_{-1}h_{-1} + c_0h_0 + c_1h_1 + c_2h_2 \in \mathfrak{h}$ . Then, using the relations  $H_i \cdot v_0^* = -\delta_{0i}v_0^*$  and  $d \cdot v_0^* = 0$ :

$$\begin{aligned} [\phi(h), \phi(e_{-1})] &= [c_{-1}(-d - 2K) + c_0H_0 + c_1H_1 + c_2H_2, v_0^*] \\ &= (2c_{-1} - c_0)v_0^* \end{aligned}$$

And, with  $\langle \alpha_j, h_i \rangle = a_{ij}$ ,

$$\begin{aligned} \langle \alpha_{-1}, h \rangle \phi(e_{-1}) &= \langle \alpha_{-1}, h \rangle v_0^* \\ &= \langle \alpha_{-1}, c_{-1}h_{-1} + c_0h_0 + c_1h_1 + c_2h_2 \rangle v_0^* \\ &= (c_{-1}a_{-1,-1} + c_0a_{0,-1} + c_1a_{1,-1} + c_2a_{2,-1})v_0^* \\ &= (2c_{-1} - c_0 + 0 + 0)v_0^* \\ &= (2c_{-1} - c_0)v_0^*, \end{aligned}$$

as desired. Similarly,

$$\begin{aligned} [\phi(h), \phi(f_{-1})] &= [c_{-1}(-d - 2K) + c_0H_0 + c_1H_1 + c_2H_2, v_0] \\ &= (-2c_{-1} + c_0)v_0 \end{aligned}$$

$$\begin{aligned} -\langle \alpha_{-1}, h \rangle \phi(f_{-1}) &= -\langle \alpha_{-1}, h \rangle v_0 \\ &= -\langle \alpha_{-1}, c_{-1}h_{-1} + c_0h_0 + c_1h_1 + c_2h_2 \rangle v_0 \\ &= -(c_{-1}a_{-1,-1} + c_0a_{0,-1} + c_1a_{1,-1} + c_2a_{2,-1})v_0^* \\ &= -(2c_{-1} - c_0 + 0 + 0)v_0 \\ &= (-2c_{-1} + c_0)v_0. \end{aligned}$$

We have now shown all relations (3.1) through (3.6) hold, proving our assertion, and may now identify  $E_i, F_i$ , and  $H_i$  with  $e_i, f_i$ , and  $h_i$ , respectively.  $\square$

## 3.2 Multiplicity Formula

Now that we have realized  $HD_4^{(3)}$  as a minimal graded Lie algebra with local part  $V(\Lambda_0) \oplus D_4^{(3)} \oplus V^*(\Lambda_0)$  in Section 3.1 (also, as in [1]), we may use Kang's multiplicity formula to determine root multiplicities of  $HD_4^{(3)}$  using certain integrable highest weight  $D_4^{(3)}$  modules. We recall Kang's formula here; for more details about this material see, for example [8].

**Theorem 3.2.** [8] Let  $\mathfrak{g} = HD_4^{(3)}$ ,  $\mathfrak{g}_0 = D_4^{(3)}$ , and let  $V_0(\lambda)$  be the highest weight  $\mathfrak{g}_0$ -module with highest weight  $\lambda$ . Let  $\Delta^\pm$  be the set of positive (respectively, negative) roots of  $\mathfrak{g}$ ,  $\Delta_S^\pm$  the set of positive (respectively, negative) roots of  $\mathfrak{g}_0$ , and  $\Delta^\pm(S) = \Delta^\pm / \Delta_S^\pm$ . Then, for  $\alpha \in \Delta^-(S)$ :

$$\dim(\mathfrak{g}_\alpha) = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \left(\frac{\tau}{\alpha}\right) \mathcal{B}(\tau)$$

where:

$\mu$  is the classical Möbius function such that  $\mu(n) = (-1)^s$  if  $n$  is a product of  $s$  distinct primes,

$$\mu(1) = 1, \text{ and } \mu(n) = 0 \text{ otherwise};$$

$$\tau|\alpha \text{ if } \alpha = k\tau \text{ for some positive integer } k, \text{ and then } \frac{\alpha}{\tau} = k \text{ and } \frac{\tau}{\alpha} = 1/k;$$

$$W(S) = \{w \in W \mid w\Delta^- \cap \Delta^+ \subseteq \Delta^+(S)\};$$

$$\text{let } V = \sum_{\substack{w \in W(S) \\ l(w) \geq 1}} (-1)^{l(w)+1} V_0(w\rho - \rho);$$

$\{\tau_i \mid i = 1, 2, \dots\}$  is an enumeration of the negative weights of  $V$  such that  $\dim V_{\tau_i} \neq 0$ ;

$$T(\tau) = \{(n_i, \tau_i) \mid n_i \in \mathbb{Z}_{\geq 0}, \sum n_i \tau_i = \tau\},$$

the finite set of all partitions of  $\tau$  into a sum of  $\tau_i$ 's with  $\tau_i \leq \tau$  in the enumeration;

$$K_{\tau_i} = \sum_{w \in W(S)} (-1)^{l(w)+1} \dim V_0(w\rho - \rho)_{\tau_i}, \text{ the alternating direct sum of } \mathfrak{g}_0\text{-modules};$$

$$\text{and } \mathcal{B}(\tau) = \sum_{(n_i, \tau_i) \in T(\tau)} \frac{(\sum_i n_i - 1)!}{\prod (n_i!)} \prod K_{\tau_i}^{n_i}.$$

Essentially, given a root  $\alpha$  of  $HD_4^{(3)}$ , to determine  $\dim(\mathfrak{g}_\alpha)$  we first determine what negative roots  $\tau$  divide  $\alpha$  and sum over these  $\tau$ . Both  $\mu(\frac{\alpha}{\tau})$  and  $(\frac{\tau}{\alpha})$  are clearly defined above. So, consider how to compute  $\mathcal{B}(\tau)$ . Note that  $\mathcal{B}(\tau)$  sums over the partitions of each  $\tau$ , but that we need to only consider the partitions  $(n_i, \tau_i)$  that result in  $0 \neq \dim V_0(w\rho - \rho)_{\tau_i}$ , where  $V_0(w\rho - \rho)$  is an integrable highest weight  $D_4^{(3)}$ -module of highest weight  $w\rho - \rho$ . Thus, we determine which  $\tau_i$  are actually weights of  $D_4^{(3)}$ -modules of the form

$V_0(w\rho - \rho)$ . To find  $w\rho - \rho$  we consider the set  $W(S)$ , consisting of Weyl group elements that, acting on the roots of  $HD_4^{(3)}$ , result in a root not exclusively in the set of roots of  $D_4^{(3)}$ ; that is, a root containing an “ $\alpha_{-1}$ ”-term. This entire process is detailed when used in Sections 5.4, and 5.5.

To determine the multiplicities of weights of  $D_4^{(3)}$ -modules, we use the crystal base theory for the quantum affine group  $U_q(D_4^{(3)})$  which we discuss in Chapter 4.

## Chapter 4

# Quantum Affine Algebras

Kang's multiplicity formula (see Section 3.2) involves counting weight multiplicities for certain  $D_4^{(3)}$ -modules. To do this, we may use crystal basis theory for the quantum group  $U_q(D_4^{(3)})$ .

In this chapter we introduce the *quantum deformation*  $U_q(\mathfrak{g})$  of the universal enveloping algebra  $U(\mathfrak{g})$  of a Kac-Moody algebra  $\mathfrak{g}$ . Hence, we consider the *quantum affine algebra*  $U_q(\mathfrak{g})$  of an *affine* Kac-Moody algebra  $\mathfrak{g}$ . We then review the ideas of *crystal basis theory* for  $U_q(\mathfrak{g})$ -modules. For more details on these topics, see [4].

### 4.1 Definitions and Examples

First we introduce some notation. Given  $n \in \mathbb{Z}$  and any symbol  $x$ , define

$$[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}},$$

with  $[0]_x! = 1$ ,  $[n]_x! = [n]_x[n-1]_x \cdots [1]_x$  for  $n \in \mathbb{Z}_{>0}$ . Thus, for nonnegative integers  $m$  and  $n$ , we have

$$\left[ \begin{array}{c} m \\ n \end{array} \right]_x = \frac{[m]_x!}{[n]_x![m-n]_x!}.$$

Then, fix an indeterminate  $q$ , and we have elements of the field  $\mathbb{C}(q)$ :

$$[n]_q \text{ and } \left[ \begin{array}{c} m \\ n \end{array} \right]_q$$

called  *$q$ -integers* and  *$q$ -binomial coefficients*, respectively. Next, let  $A = (a_{ij})_{i,j \in I}$  be a symmetrizable generalized Cartan matrix with symmetrizing matrix  $D = \text{diag}(s_i \in \mathbb{Z}_{>0} \mid i \in I)$  and let  $(A, \Pi, \check{\Pi}, P, \check{P})$  be the Cartan datum associated with  $A$ . We define the following.

**Definition.** The quantum group or the quantized universal enveloping algebra,  $U_q(\mathfrak{g})$ , associated with a Cartan datum  $(A, \Pi, \check{\Pi}, P, \check{P})$  is the associative algebra over  $\mathbb{C}(q)$  with unity generated by the elements  $e_i, f_i$  for  $i \in I$  and  $q^h$  for  $h \in \check{P}$  with defining relations:

1.  $q^0 = 1, q^h q^{h'} = q^{h+h'},$  for  $h, h' \in \check{P},$
2.  $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i,$  for  $h \in \check{P},$
3.  $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i,$  for  $h \in \check{P},$
4.  $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$  for  $i, j \in I,$
5.  $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0,$  for  $i \neq j,$
6.  $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0,$  for  $i \neq j.$

Here,  $q_i = q^{s_i}$  and  $K_i = q^{s_i h_i}.$

Notice, as  $q \rightarrow 1, U_q(\mathfrak{g}) \rightarrow U(\mathfrak{g}).$  Now, let  $A = (a_{ij})_{i,j \in I}$  be a generalized Cartan matrix of affine type. Recall the dual weight lattice  $\check{P} = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d$  and Cartan subalgebra  $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \check{P}.$  Also recall the actions of the simple roots  $(\alpha_i)$ , simple coroots  $(h_i)$ , and fundamental weights  $(\Lambda_i):$

$$\begin{aligned} \alpha_i(h_j) &= a_{ji}, \quad \alpha(d) = \delta_{0i}, \\ \Lambda_i(h_j) &= \delta_{ij}, \quad \Lambda_i(d) = 0 \quad (i, j \in I). \end{aligned}$$

The *affine weight lattice* is the set  $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(\check{P}) \subset \mathbb{Z}\}$  and we call  $(A, \Pi, \check{\Pi}, P, \check{P})$  an *affine Cartan datum*, to which we associate the *affine Kac-Moody algebra*  $\mathfrak{g}.$  The center of  $\mathfrak{g}$  is one-dimensional and is generated by the *canonical central element*  $K = k_0 h_0 + \cdots + k_n h_n,$  and the imaginary roots of  $\mathfrak{g}$  are nonzero integral multiples of the *null root*,  $\delta = d_0 \alpha_0 + \cdots + d_n \alpha_n.$  The coefficients  $k_i$  and  $d_i, i \in I,$  are the nonnegative integers given in [6]. Using the fundamental weights and the null root, the affine weight lattice can be written as  $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\frac{1}{d_0}\delta.$  We call the elements of  $P$  *affine weights*, and the *affine dominant integral weights* are the elements of the set  $P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}.$  The *level* of an affine dominant integral weight  $\lambda$  is the nonnegative integer  $\lambda(K).$

**Example 4.1.** For the affine Kac-Moody algebra  $D_4^{(3)}$

$$K = h_0 + 2h_1 + 3h_2,$$

$$\text{and } \delta = \alpha_0 + 2\alpha_1 + \alpha_2.$$

**Definition.** The quantum affine algebra  $U_q(\mathfrak{g})$  is the quantum group associated with the affine Cartan datum  $(A, \Pi, \check{\Pi}, P, \check{P})$ . The subalgebra of  $U_q(\mathfrak{g})$  generated by  $e_i$ ,  $f_i$ , and  $K_i^{\pm 1}$  for  $i \in I$  is denoted by  $U'_q(\mathfrak{g})$  and is (also) called the quantum affine algebra.

Let  $\bar{P}^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n$  and  $\bar{\mathfrak{h}} = \mathbb{C} \otimes_{\mathbb{Z}} \bar{P}^\vee$ . We also set  $\bar{P} = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n$  and consider  $\alpha_i$  and  $\Lambda_i$  as linear functionals on  $\bar{\mathfrak{h}}$ . The elements of  $\bar{P}$  are called classical weights and  $(A, \Pi, \check{\Pi}, \bar{P}, \bar{P}^\vee)$  is called the classical Cartan datum. The quantum affine algebra  $U'_q(\mathfrak{g})$  can be regarded as the quantum group associated with the classical Cartan datum.

Denote the projection  $cl : P \rightarrow \bar{P}$  by  $\lambda \mapsto \bar{\lambda}$  and fix an embedding  $af : \bar{P} \rightarrow P$  such that  $cl \circ af = id$  and  $af \circ cl(\alpha_i) = \alpha_i$  for  $i \neq 0$ . Then we define the set of *classical dominant integral weights* to be the set  $\bar{P}^+ = cl(P^+) = \{\lambda \in \bar{P} \mid \lambda(h_i) \geq 0 \text{ for all } i \in I\}$ . A classical dominant integral weight is said to have *level*  $l \geq 0$  if  $\lambda(c) = l$ ; this is the same level as its affine counterpart.

The representation theory of quantized universal enveloping algebras is essentially parallel to that of Kac-Moody algebras. For instance, we have the following theorem.

**Theorem 4.2.** [4] If  $\lambda \in P^+$  and  $L^q(\lambda)$  is the irreducible highest weight  $U_q(\mathfrak{g})$ -module with highest weight  $\lambda$ , then  $L^1(\lambda)$  is isomorphic to the irreducible highest weight module  $L(\lambda)$  over  $U(\mathfrak{g})$  with highest weight  $\lambda$ . Hence, the character of  $L^q(\lambda)$  over  $U_q(\mathfrak{g})$  is the same as the character of  $L(\lambda)$  over  $U(\mathfrak{g})$ .

With this theorem we may use crystal basis theory for  $U_q(\mathfrak{g})$ -modules to obtain results for  $U(\mathfrak{g})$ -modules, and hence  $\mathfrak{g}$ -modules. We review crystal basis theory in the following section.

## 4.2 Crystal Bases

A crystal base of a  $U_q(\mathfrak{g})$ -module,  $V$ , can be viewed as a basis at  $q = 0$ . Crystal bases depict the internal structure of integrable representations of quantum groups through their combinatorial characteristics. By finding combinatorial descriptions of crystal bases we may determine the weight multiplicities in highest weight  $U_q(D_4^{(3)})$ -modules. As the character of  $L^q(\lambda)$ , a highest weight  $U_q(\mathfrak{g})$ -module with highest weight  $\lambda$ , is equal to the character of  $L(\lambda)$ , a highest weight  $U(\mathfrak{g})$ -module (equivalently,  $\mathfrak{g}$ -module) with highest weight  $\lambda$ , the multiplicities in highest weight  $U_q(D_4^{(3)})$ -modules determine the weight multiplicities in highest weight  $D_4^{(3)}$ -modules. We will use these weight multiplicities to determine the root multiplicities of  $HD_4^{(3)}$  with Kang's root multiplicity formula.

We begin our discussion of crystal base theory by reviewing some essential definitions. Let  $F = \mathbb{C}(q)$  be the field of rational functions in  $q$  with coefficients in  $\mathbb{C}$ . Define the subring of  $F$ ,  $A$ , as follows:

$$A = \left\{ \frac{f(q)}{g(q)} \mid f, g \in \mathbb{C}[q] \text{ and } g(0) \neq 0 \right\}.$$

The evaluation map  $\xi : A \text{ mod } qA \rightarrow \mathbb{C}$ , given by

$$\begin{aligned}\xi(f(q) + q \cdot g(q)) &= f(0) + 0 \cdot g(0) \\ &= f(0),\end{aligned}$$

is an isomorphism. We define the following:

**Definition.** Let  $V$  be a vector space over  $F$ . A local base at  $q = 0$  of  $V$  is a pair  $(\mathcal{L}, \mathcal{B})$ , where

1.  $\mathcal{L}$  is a free  $A$ -module such that  $V$  is generated by  $\mathcal{L}$  as a vector space over  $F$  (that is,  $V \cong F \otimes_A \mathcal{L}$ ).
2.  $\mathcal{B}$  is a base of the vector space  $\mathcal{L} \text{ mod } q\mathcal{L}$  over  $F$ .

Now, let  $\mathfrak{g} = \mathfrak{g}(A)$  be a symmetrizable Kac-Moody Lie algebra with realization

$\{\mathfrak{h}, \Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*, \check{\Pi} = \{h_i \mid i \in I\} \subset \mathfrak{h}\}$ . Let  $P$  be the weight lattice of  $\mathfrak{g}$ ,  $U_q(\mathfrak{g})$  be the quantized universal enveloping algebra of  $\mathfrak{g}$ , and  $U_q(\mathfrak{g}_{(i)})$  be the subalgebra of  $U_q(\mathfrak{g})$  generated by  $\{e_i, f_i, t_i^\pm\}$ . We have that for each  $i \in I$ ,  $U_q(\mathfrak{g}_{(i)})$  is isomorphic to  $U_q(sl(2, \mathbb{C}))$ .

Let  $M$  be an integrable  $U_q(\mathfrak{g})$ -module. Then, by  $U_q(sl(2, \mathbb{C}))$  representation theory, for each  $i \in I$ , any element  $u \in M_\lambda$  can be uniquely written as  $u = \sum_{k \geq 0} f_i^{(k)} u_k$ , where  $u_k \in \ker(e_i) \cap M_{\lambda+k\alpha_i}$ . The endomorphisms  $\tilde{e}_i$  and  $\tilde{f}_i$  on  $M$ , below, are called the *Kashiwara operators*:

$$\begin{aligned}\tilde{e}_i(u) &= \sum_{k \geq 0} f_i^{(k-1)} u_k \\ \tilde{f}_i(u) &= \sum_{k \geq 0} f_i^{(k+1)} u_k\end{aligned}$$

for  $u \in M_\lambda$ .

**Definition.** A crystal base of an integrable  $U_q(\mathfrak{g})$ -module,  $M$  is a pair  $(\mathcal{L}, \mathcal{B})$ , such that the following hold:

1.  $(\mathcal{L}, \mathcal{B})$  is a local base at  $q = 0$ .
2.  $\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda$ , where  $\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda$ .
3.  $\mathcal{B} = \bigcup_{\lambda \in P} \mathcal{B}_\lambda$ , where  $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{L}_\lambda / q\mathcal{L}_\lambda)$ .
4.  $\tilde{e}_i \mathcal{L} \subset \mathcal{L}$ ,  $\tilde{f}_i \mathcal{L} \subset \mathcal{L}$ .
5.  $\tilde{e}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$ ,  $\tilde{f}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$ .
6. For  $b, b' \in \mathcal{B}$ ,  $b' = \tilde{f}_i b$  if and only if  $b = \tilde{e}_i b'$ .

The existence and uniqueness of crystal bases for integrable  $U_q(\mathfrak{g})$ -modules follows from the following theorem due to Kashiwara:

**Theorem 4.3.** Let

$$\mathcal{L}(\lambda) = \sum_{\substack{i \geq 0 \\ i_1, \dots, i_l}} A \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda,$$

and let

$$\mathcal{B}(\lambda) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda \bmod q\mathcal{L}(\lambda) \mid l \geq 0, i_1, \dots, i_l \in I \right\} / \{0\}.$$

Then,  $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$  is a crystal base of  $L(\lambda)$ , the irreducible  $U_q(\mathfrak{g})$ -module with highest weight  $\lambda$ .

Each crystal base is uniquely associated with a crystal graph, as described by the following definition.

**Definition.** Let  $M$  be an integrable  $U_q(\mathfrak{g})$ -module with crystal basis  $(\mathcal{L}, \mathcal{B})$ . The crystal graph of  $M$  has  $\mathcal{B}$  as its set of vertices. For each  $i \in I$ , we join  $b \in \mathcal{B}$  to  $b' \in \mathcal{B}$  with an  $i$ -colored arrow  $(b \xrightarrow{i} b')$  if and only if  $b' = \tilde{f}_i b$  (equivalently, if and only if  $b = \tilde{e}_i b'$ ).

Now, let  $\mathcal{B}$  be a crystal graph of an integrable  $U_q(\mathfrak{g})$ -module,  $M = \bigoplus_{\lambda \in P} (M_\lambda)$ . For  $b \in \mathcal{B}_\lambda$ , we say  $b$  is of weight  $\lambda$  and write  $wt(b) = \lambda$ .

Define the maps  $\epsilon_i$  and  $\varphi_i$  as follows:

$$\begin{aligned} \epsilon_i(b) &= \max \left\{ n \mid \tilde{e}_i^{(n)} b \neq 0 \right\} \\ \varphi_i(b) &= \max \left\{ n \mid \tilde{f}_i^{(n)} b \neq 0 \right\}. \end{aligned}$$

We see that  $\epsilon_i(b)$  gives the number of  $i$ -colored arrows coming into the vertex  $b$ , and  $\varphi_i(b)$  gives the number of  $i$ -colored arrows coming out of the vertex  $b$ . Thus,  $\varphi_i(b) + \epsilon_i(b)$  gives the length of the  $i$ -string through  $b$ . Notice that  $\varphi_i(b) - \epsilon_i(b) = \langle h_i, wt(b) \rangle = \lambda(h_i)$ .

One of the most important combinatorial characteristics of crystal bases is their stability under the tensor product.

**Theorem 4.4. Tensor Product Rule,** [4]. Let  $M_j$  be an integrable  $U_q(\mathfrak{g})$ -module and let  $(\mathcal{L}_j, \mathcal{B}_j)$  be a crystal basis of  $M_j$  for  $j = 1, 2$ . Let  $\mathcal{L} = \mathcal{L}_1 \otimes_A \mathcal{L}_2$  and  $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$ . Then,  $(\mathcal{L}, \mathcal{B})$  is a crystal basis of  $M_1 \otimes_{\mathbb{F}} M_2$  and

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \epsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \epsilon_i(b_2) \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \epsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \epsilon_i(b_2). \end{cases} \end{aligned}$$

So, we have:

$$\begin{aligned} \text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2), \\ \epsilon_i(b_1 \otimes b_2) &= \max(\epsilon_i(b_1), \epsilon_i(b_2) - \langle h_i, \text{wt}(b_1) \rangle), \\ \varphi_i(b_1 \otimes b_2) &= \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \text{wt}(b_2) \rangle). \end{aligned}$$

There is a combinatorial rule using this theorem to determine the action of Kashiwara operators on multi-fold tensor products of crystal graphs. That is, let  $M_j$  be an integrable  $U_q(\mathfrak{g})$ -module with crystal basis  $(\mathcal{L}_j, \mathcal{B}_j)$  for  $j = 1, \dots, n$ . Fix  $i \in I$  and let  $b = b_1 \otimes b_2 \otimes \dots \otimes b_n \in \mathcal{B}_1 \otimes \dots \otimes \mathcal{B}_n$ . We assign a sequence of plusses and minuses to  $b$  as follows: For each  $b_j$  assign  $\epsilon_i(b_j)$  minuses and  $\varphi_i(b_j)$  plusses, so that

$$\begin{aligned} b &= b_1 \otimes b_2 \otimes \dots \otimes b_N \\ &\mapsto (\underbrace{-, \dots, -}_{\epsilon_i(b_1)}, \underbrace{+, \dots, +}_{\varphi_i(b_1)}, \dots, \underbrace{-, \dots, -}_{\epsilon_i(b_n)}, \underbrace{+, \dots, +}_{\varphi_i(b_n)}). \end{aligned}$$

Then, we cancel out all of the pairs of the form  $(+, -)$  to obtain a sequence of  $-$ 's followed by a sequence of  $+$ 's, called the  $i$ -signature of  $b$ . Then  $\tilde{e}_i$  acts on  $b_j$  corresponding to the right-most minus in the  $i$ -signature and  $\tilde{f}_i$  acts on  $b_k$  corresponding to the left-most plus in the  $i$ -signature:

$$\tilde{e}_i b = b_1 \otimes \dots \otimes \tilde{e}_i b_j \otimes \dots \otimes b_n,$$

$$\tilde{f}_i b = b_1 \otimes \dots \otimes \tilde{f}_i b_k \otimes \dots \otimes b_n.$$

### 4.3 Path Realizations

To discuss path realizations, we first must define a *perfect crystal* and recall a fundamental crystal isomorphism theorem, then we may obtain a *path realization* of crystal graphs for irreducible highest weight modules over quantum affine algebras. From the previous section, we recall the maps (as in [4]):

$$\text{wt} : \mathcal{B} \rightarrow P, b \in \mathcal{B}_\lambda \mapsto \text{wt}(b) = \lambda,$$

$$\tilde{e}_i, \tilde{f}_i : \mathcal{B} \rightarrow \mathcal{B} \cup \{0\},$$

$$\epsilon_i, \varphi_i : \mathcal{B} \rightarrow \mathbb{Z},$$

satisfying:

$$\varphi_i(b) = \epsilon_i(b) + \langle h_i, \text{wt}(b) \rangle,$$

$$\tilde{e}_i \mathcal{B}_\lambda \subset \mathcal{B}_{\lambda+\alpha_i} \cup \{0\}, \quad \tilde{f}_i \mathcal{B}_\lambda \subset \mathcal{B}_{\lambda-\alpha_i} \cup \{0\},$$

$$\begin{aligned}\epsilon_i(\tilde{e}_i b) &= \epsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1 \text{ if } \tilde{e}_i b \in \mathcal{B}, \\ \epsilon_i(\tilde{f}_i b) &= \epsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1 \text{ if } \tilde{f}_i b \in \mathcal{B}, \\ \tilde{f}_i b &= b' \text{ if and only if } b = \tilde{e}_i b' \text{ for all } i \in I \text{ and } b, b' \in \mathcal{B}.\end{aligned}$$

These maps characterize crystal bases  $(\mathcal{L}, \mathcal{B})$  of  $U_q(\mathfrak{g})$ -modules in the category  $\mathcal{O}_{int}^q$  and we use these maps to define a *crystal*.

**Definition.** Let  $U_q(\mathfrak{g})$  be the quantized universal enveloping algebra of a Kac-Moody algebra associated with realization  $(\mathfrak{h}, \Pi, \check{\Pi})$  and weight lattice  $P$ . A  $U_q(\mathfrak{g})$ -crystal associated with  $\mathfrak{g}$  is a  $P$ -weighted set,  $\mathcal{B}$  together with the maps  $wt : \mathcal{B} \rightarrow P$ ,  $\tilde{e}_i, \tilde{f}_i : \mathcal{B} \rightarrow \mathcal{B} \cup \{0\}$ , and  $\epsilon_i, \varphi_i : \mathcal{B} \rightarrow \mathbb{Z} \cup \{-\infty\}$ ,  $i \in I$ , satisfying the following properties:

1.  $\varphi_i(b) = \epsilon_i(b) + \langle h_i, wt(b) \rangle$  for all  $i \in I$ ,
2.  $wt(\tilde{e}_i b) = wt(b) + \alpha_i$  if  $\tilde{e}_i b \in \mathcal{B}$ ,
3.  $wt(\tilde{f}_i b) = wt(b) - \alpha_i$  if  $\tilde{f}_i b \in \mathcal{B}$ ,
4.  $\epsilon_i(\tilde{e}_i b) = \epsilon_i(b) - 1$ ,  $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$  if  $\tilde{e}_i b \in \mathcal{B}$ ,
5.  $\epsilon_i(\tilde{f}_i b) = \epsilon_i(b) + 1$ ,  $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$  if  $\tilde{f}_i b \in \mathcal{B}$ ,
6.  $\tilde{f}_i b = b'$  if and only if  $b = \tilde{e}_i b'$  for  $b, b' \in \mathcal{B}$ ,  $i \in I$ ,
7. if  $\varphi_i(b) = -\infty$  for  $b \in \mathcal{B}$ , then  $\tilde{e}_i b = \tilde{f}_i b = 0$ .

An affine crystal is a crystal associated with a quantum affine algebra  $U_q(\mathfrak{g})$ , and a classical crystal is a crystal associated with a quantum affine algebra  $U'_q(\mathfrak{g})$  (that is associated with a classical Cartan datum).

For two crystals  $\mathcal{B}_1, \mathcal{B}_2$  we define the tensor product  $\mathcal{B}_1 \otimes \mathcal{B}_2$  to have the underlying set  $\mathcal{B}_1 \times \mathcal{B}_2$  and write  $b_1 \otimes b_2$  for  $(b_1, b_2)$ . We understand  $b_1 \otimes 0 = 0 = 0 \otimes b_2$  and the action of  $\tilde{e}_i$  and  $\tilde{f}_i$  are analogous to the action defined in Theorem 4.4. If  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are  $P$ -weighted crystals, then so is  $\mathcal{B}_1 \times \mathcal{B}_2$ , and for  $b_1 \otimes b_2 \in \mathcal{B}_1 \otimes \mathcal{B}_2$ ,  $wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2)$ .

Recall the set of classical dominant integral weights,  $\bar{P}^+$  from Section 4.1 and, for a positive integer  $l$ , let  $\bar{P}_l^+ = \{\lambda \in \bar{P}^+ \mid \lambda(K) = l\}$ . Let  $\mathcal{B}$  be a  $U'_q(\mathfrak{g})$ -crystal (classical crystal), and for  $b \in \mathcal{B}$ , define:

$$\epsilon(b) = \sum_i \epsilon_i(b) \Lambda_i \text{ and } \varphi(b) = \sum_i \varphi_i(b) \Lambda_i.$$

Note,  $wt(b) = \varphi(b) - \epsilon(b)$ . We may now define a *perfect crystal*.

**Definition.** A perfect crystal of level  $l$ ,  $\mathcal{B}_l$ , is a finite  $U'_q(\mathfrak{g})$ -crystal that satisfies the following:

1. there exists a finite dimensional  $U'_q(\mathfrak{g})$ -module with a crystal basis whose crystal graph is isomorphic to  $\mathcal{B}$ ,
2.  $\mathcal{B}_l \otimes \mathcal{B}_l$  is connected,
3. there exists a classical weight  $\lambda_0 \in \bar{P}$ , with  $\dim(\mathcal{B}_{\lambda_0}) = 1$  such that  $\text{wt}(\mathcal{B}) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \alpha_i$ ,
4. for any  $b \in \mathcal{B}_l$ ,  $\langle K, \epsilon(b) \rangle \geq l$ ,
5. for each  $\lambda \in \bar{P}_l^+$ , there exists unique vectors  $b^\lambda \in \mathcal{B}_l$  and  $b_\lambda \in \mathcal{B}_l$  such that

$$\epsilon(b^\lambda) = \lambda \text{ and } \varphi(b_\lambda) = \lambda.$$

In this thesis we use the perfect crystal outlined in the following example.

**Example 4.5.** We recall the  $U_q(D_4^{(3)})$  perfect crystal  $\mathcal{B}_l$  of level  $l \geq 1$  from [11]. This perfect crystal is a set

$$\begin{aligned} \mathcal{B}_l = \{ b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in \mathbb{Z}_{\geq 0}^6 \mid s(b) = x_1 + x_2 + \frac{x_3 + \bar{x}_3}{2} + \bar{x}_2 + \bar{x}_1 \leq l \\ \text{and } (x_3 + \bar{x}_3) \equiv 0 \pmod{2} \}. \end{aligned}$$

with the actions of the Kashiwara operators  $\tilde{e}_i$  and  $\tilde{f}_i$  for  $i = 0, 1, 2$  given as follows:

for  $b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in \mathcal{B}_l$  :

$$\begin{aligned} \tilde{e}_1(b) &= \begin{cases} (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2 + 1, \bar{x}_1 - 1) & \text{if } \bar{x}_2 - \bar{x}_3 \geq (x_2 - x_3)_+, \\ (x_1, x_2, x_3 + 1, \bar{x}_3 - 1, \bar{x}_2, \bar{x}_1) & \text{if } \bar{x}_2 - \bar{x}_3 < 0 \leq x_3 - x_2, \\ (x_1 + 1, x_2 - 1, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \text{if } (\bar{x}_2 - \bar{x}_3)_+ < x_2 - x_3, \end{cases} \\ \tilde{f}_1(b) &= \begin{cases} (x_1 - 1, x_2 + 1, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \text{if } (\bar{x}_2 - \bar{x}_3)_+ \leq x_2 - x_3, \\ (x_1, x_2, x_3 - 1, \bar{x}_3 + 1, \bar{x}_2, \bar{x}_1) & \text{if } \bar{x}_2 - \bar{x}_3 \leq 0 < x_3 - x_2, \\ (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2 - 1, \bar{x}_1 + 1) & \text{if } \bar{x}_2 - \bar{x}_3 > (x_2 - x_3)_+, \end{cases} \\ \tilde{e}_2(b) &= \begin{cases} (x_1, x_2, x_3, \bar{x}_3 + 2, \bar{x}_2 - 1, \bar{x}_1) & \text{if } \bar{x}_3 \geq x_3, \\ (x_1, x_2 + 1, x_3 - 2, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \text{if } \bar{x}_3 < x_3, \end{cases} \\ \tilde{f}_2(b) &= \begin{cases} (x_1, x_2 - 1, x_3 + 2, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \text{if } \bar{x}_3 \leq x_3, \\ (x_1, x_2, x_3, \bar{x}_3 - 2, \bar{x}_2 + 1, \bar{x}_1) & \text{if } \bar{x}_3 > x_3, \end{cases} \end{aligned}$$

$$\tilde{e}_0(b) = \begin{cases} \mathcal{E}_1(b) = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1 + 1) & \text{if } (E_1), \\ \mathcal{E}_2(b) = (x_1, x_2, x_3 - 1, \bar{x}_3 - 1, \bar{x}_2, \bar{x}_1 + 1) & \text{if } (E_2), \\ \mathcal{E}_3(b) = (x_1, x_2, x_3 - 2, \bar{x}_3, \bar{x}_2 + 1, \bar{x}_1) & \text{if } (E_3), \\ \mathcal{E}_4(b) = (x_1, x_2 - 1, x_3, \bar{x}_3 + 2, \bar{x}_2, \bar{x}_1) & \text{if } (E_4), \\ \mathcal{E}_5(b) = (x_1 - 1, x_2, x_3 + 1, \bar{x}_3 + 1, \bar{x}_2, \bar{x}_1) & \text{if } (E_5), \\ \mathcal{E}_6(b) = (x_1 - 1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \text{if } (E_6), \end{cases}$$

$$\tilde{f}_0(b) = \begin{cases} \mathcal{F}_1(b) = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1 - 1) & \text{if } (F_1), \\ \mathcal{F}_2(b) = (x_1, x_2, x_3 + 1, \bar{x}_3 + 1, \bar{x}_2, \bar{x}_1 - 1) & \text{if } (F_2), \\ \mathcal{F}_3(b) = (x_1, x_2, x_3 + 2, \bar{x}_3, \bar{x}_2 - 1, \bar{x}_1) & \text{if } (F_3), \\ \mathcal{F}_4(b) = (x_1, x_2 + 1, x_3, \bar{x}_3 - 2, \bar{x}_2, \bar{x}_1) & \text{if } (F_4), \\ \mathcal{F}_5(b) = (x_1 + 1, x_2, x_3 - 1, \bar{x}_3 - 1, \bar{x}_2, \bar{x}_1) & \text{if } (F_5), \\ \mathcal{F}_6(b) = (x_1 + 1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \text{if } (F_6), \end{cases}$$

The conditions  $F_1 - F_6$  follow. Conditions  $E_1 - E_6$  are obtained by interchanging the inequalities  $>$  (resp.  $<$ ) with  $\geq$  (resp.  $\leq$ ) in the respective conditions  $F_1 - F_6$ .

$$(F_1) \begin{cases} 2\bar{x}_1 + 2\bar{x}_2 + \bar{x}_3 - x_3 - 2x_2 - 2x_1 > 0 \\ 2\bar{x}_1 + 3\bar{x}_3 - x_3 - 2x_2 - 2x_1 > 0 \\ \bar{x}_1 + x_3 - x_2 - x_1 > 0 \\ \bar{x}_1 - x_1 > 0 \end{cases}$$

$$(F_2) \begin{cases} 2\bar{x}_1 + 2\bar{x}_2 + \bar{x}_3 - x_3 - 2x_2 - 2x_1 \leq 0 \\ 2\bar{x}_2 + \bar{x}_3 - 3x_3 \leq 0 \\ \bar{x}_2 - \bar{x}_3 \leq 0 \\ \bar{x}_1 - x_1 > 0 \end{cases}$$

$$(F_3) \begin{cases} 2\bar{x}_1 + 3\bar{x}_3 - x_3 - 2x_2 - 2x_1 \leq 0 \\ 3\bar{x}_3 - x_3 - 2x_2 \leq 0 \\ \bar{x}_3 - x_3 \leq 0 \\ \bar{x}_2 - \bar{x}_3 > 0 \\ \bar{x}_1 + \bar{x}_2 - \bar{x}_3 - x_1 > 0 \end{cases}$$

$$(F_4) \begin{cases} 2\bar{x}_1 + 2\bar{x}_2 + \bar{x}_3 - 3x_3 - 2x_1 > 0 \\ 2\bar{x}_2 + \bar{x}_3 - 3x_3 > 0 \\ \bar{x}_3 - x_3 > 0 \\ x_3 - x_2 \leq 0 \\ \bar{x}_1 + \bar{x}_3 - x_2 - x_1 \leq 0 \end{cases}$$

$$(F_5) \begin{cases} 2\bar{x}_1 + 2\bar{x}_2 + \bar{x}_3 - x_3 - 2x_2 - 2x_1 > 0 \\ 3\bar{x}_3 - x_3 - 2x_2 > 0 \\ x_3 - x_2 > 0 \\ \bar{x}_1 - x_1 \leq 0 \end{cases}$$

$$(F_6) \begin{cases} 2\bar{x}_1 + 2\bar{x}_2 + \bar{x}_3 - x_3 - 2x_2 - 2x_1 \leq 0 \\ 2\bar{x}_1 + 2\bar{x}_2 + \bar{x}_3 - 3x_3 - 2x_1 \leq 0 \\ \bar{x}_1 + \bar{x}_2 - \bar{x}_3 - x_1 \leq 0 \\ \bar{x}_1 - x_1 \leq 0 \end{cases}$$

From [11], we also have the following formulas for  $\phi_i$  and  $\epsilon_i$  ( $i \in \{0, 1, 2\}$ ):

$$\begin{aligned} \phi_1(b) &= x_1 + (x_3 - x_2 + (\bar{x}_2 - \bar{x}_3)_+)_+, \\ \phi_2(b) &= x_2 + \frac{1}{2}(\bar{x}_3 - x_3)_+, \\ \epsilon_1(b) &= \bar{x}_1 + (\bar{x}_3 - \bar{x}_2 + (x_2 - x_3)_+)_+ \\ \epsilon_2(b) &= \bar{x}_2 + \frac{1}{2}(x_3 - \bar{x}_3)_+. \end{aligned}$$

Then, set

$$z_1 = \bar{x}_1 - x_1, \quad z_2 = \bar{x}_2 - \bar{x}_3, \quad z_3 = x_3 - x_2, \quad z_4 = (\bar{x}_3 - x_3)/2,$$

and

$$A = (0, z_1, z_1 + z_2, z_1 + z_2 + 3z_4, z_1 + z_2 + z_3 + 3z_4, 2z_1 + z_2 + z_3 + 3z_4);$$

then we have

$$\begin{aligned} \phi_0(b) &= l - s(b) + \max A, \\ \epsilon_0(b) &= l - s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4). \end{aligned}$$

For  $l = 1$  we have  $\mathcal{B}_1 = \{b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$  such that

$$\begin{aligned} b_0 &= (0, 0, 0, 0, 0, 0), \quad b_1 = (1, 0, 0, 0, 0, 0) \\ b_2 &= (0, 1, 0, 0, 0, 0), \quad b_3 = (0, 0, 2, 0, 0, 0) \\ b_4 &= (0, 0, 1, 1, 0, 0), \quad b_5 = (0, 0, 0, 2, 0, 0) \\ b_6 &= (0, 0, 0, 0, 1, 0), \quad b_7 = (0, 0, 0, 0, 0, 1) \end{aligned}$$

**Example 4.6.** Figure 4.1 is the crystal graph of  $\mathcal{B}_1$  with the relations defined in Example 4.5.

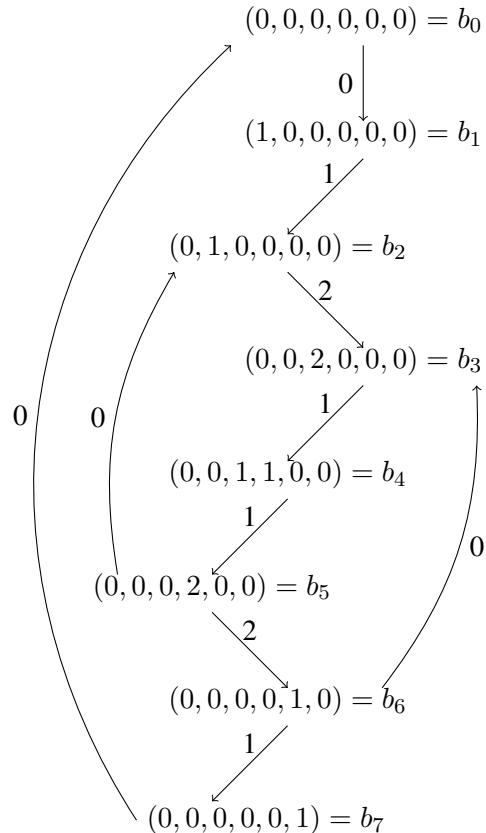


Figure 4.1: Crystal Graph of the Perfect Crystal  $\mathcal{B}_1$  outlined in Example 4.5

To complete this chapter, now we develop the tools we may use to realize the combinatorial structure of a crystal, obtaining what is called the *path realization* of a crystal graph. For the remainder of this chapter, assume:  $\mathfrak{g}$  is an affine algebra,  $l \in \mathbb{Z}_{>0}$ ,  $\mathcal{B}_l$  is a perfect crystal of level  $l$ ,  $\lambda \in \bar{P}_l^+$ ,  $b_\lambda$  is the unique vector in  $\mathcal{B}_l$  such that  $\varphi(b_\lambda) = \lambda$ ,  $L(\lambda)$  is the highest weight  $U_q(\mathfrak{g})$ -module of highest weight  $\lambda$ , and  $\mathcal{B}(\lambda)$  is the crystal

graph of  $L(\lambda)$ . To begin, we introduce the following isomorphism theorem (see [4]), which is instrumental in developing a path realization for  $\mathcal{B}(\lambda)$ .

**Theorem 4.7.** *The map*

$$\Psi : \mathcal{B}(\lambda) \rightarrow \mathcal{B}(\epsilon(b_\lambda)) \otimes \mathcal{B}_l \text{ given by } u_\lambda \mapsto u_{\epsilon(b_\lambda)} \otimes b_\lambda$$

where  $u_\lambda$  and  $u_{\epsilon(b_\lambda)}$  are the highest weight vectors of  $\mathcal{B}(\lambda)$  and  $\mathcal{B}(\epsilon(b_\lambda))$ , respectively, is a strict isomorphism of crystals.

With repeated application of Theorem 4.7, we may obtain a sequence of isomorphisms:

$$\Psi_1 : \mathcal{B}(\lambda) \rightarrow \mathcal{B}(\lambda_1) \otimes \mathcal{B}_l \text{ given by } u_\lambda \mapsto u_{\lambda_1} \otimes b_\lambda$$

$$\Psi_2 : \mathcal{B}(\lambda) \rightarrow \mathcal{B}(\lambda_2) \otimes \mathcal{B}_l \otimes \mathcal{B}_l \text{ given by } u_\lambda \mapsto u_{\lambda_2} \otimes b_{\lambda_1} \otimes b_\lambda$$

⋮

$$\Psi_k : \mathcal{B}(\lambda) \rightarrow \mathcal{B}(\lambda_k) \otimes \mathcal{B}_l^{\otimes k} \text{ given by } u_\lambda \mapsto u_{\lambda_k} \otimes b_{\lambda_{k-1}} \otimes \cdots \otimes b_{\lambda_1} \otimes b_\lambda$$

⋮

where  $\lambda_{k+1} = \epsilon(b_{\lambda_k})$  with  $\lambda_0 = \lambda$ . This sequence of isomorphisms yields two infinite sequences:

$$w_\lambda = (\lambda_k)_{k=0}^\infty = (\dots, \lambda_{k+1}, \lambda_k, \dots, \lambda_1, \lambda_0) \in (\bar{P}_l^+)^{\infty} \quad (4.1)$$

$$p_\lambda = (b_{\lambda_k})_{k=0}^\infty = (\dots, b_{\lambda_{k+1}}, b_{\lambda_k}, \dots, b_{\lambda_1}, b_{\lambda_0}) \in \mathcal{B}_l^{\otimes \infty} \quad (4.2)$$

Since the sets  $\bar{P}_l^+$  and  $\mathcal{B}_l$  are both finite, there must exist an  $N > 0$  such that  $\lambda_N = \lambda_0$ ; and, since the maps  $\varphi$  and  $\epsilon$  are bijective, the following equalities hold:

$$\begin{aligned}
b_{\lambda_0} &= \varphi^{-1}(\lambda_0) = \varphi^{-1}(\lambda_N) = b_{\lambda_N} \\
\lambda_1 &= \epsilon(b_{\lambda_0}) = \epsilon(b_{\lambda_N}) = \lambda_{N+1} \\
b_{\lambda_1} &= \varphi^{-1}(\lambda_1) = \varphi^{-1}(\lambda_{N+1}) = b_{\lambda_{N+1}} \\
&\vdots \\
\lambda_j &= \lambda_{j+N} \\
b_{\lambda_j} &= b_{\lambda_{j+N}} \\
&\vdots \\
\lambda_{N-1} &= \lambda_{2N-1} \\
b_{\lambda_{N-1}} &= b_{\lambda_{2N-1}}
\end{aligned}$$

Hence,  $w_\lambda$  and  $p_\lambda$  defined in 4.1 and 4.2, respectively, are periodic with the same period  $N > 0$ .

**Definition.** The path  $p_\lambda = (\cdots \otimes b_{\lambda_k} \otimes \cdots \otimes b_{\lambda_1} \otimes b_{\lambda_0})$ , with  $\lambda_{k+1} = \epsilon(b_{\lambda_k})$  is called the ground state path of weight  $\lambda$

**Example 4.8.** The ground state path of weight  $\Lambda_0$  in  $\mathcal{B}_1$ , the  $U'_q(D_4^{(3)})$  perfect crystal of level one given in Example 4.5, is the following:

$$p_{\Lambda_0} = (\cdots \otimes b_0 \otimes b_0)$$

where  $b_0 = (0, 0, 0, 0, 0, 0) \in \mathcal{B}_1$  as in Example 4.5.

**Definition.** A  $\lambda$ -path in  $\mathcal{B}_l$  is a sequence  $p = (\cdots \otimes p_k \otimes \cdots \otimes p_1 \otimes p_0)$  with  $p_k \in \mathcal{B}_l$  such that  $p_k = b_{\lambda_k}$  for all  $k \gg 0$ .

We let  $\mathcal{P}(\lambda)$  be the set consisting of all  $\lambda$ -paths in  $\mathcal{B}_l$ . The following two theorems give  $\mathcal{P}(\lambda)$  a  $U'_q(\mathfrak{g})$ -crystal structure and then show this structure is a realization of the crystal base  $\mathcal{B}(\lambda)$  for the highest weight  $U'_q(\mathfrak{g})$ -module,  $L(\lambda)$ .

**Theorem 4.9.** Let  $p \in \mathcal{P}(\lambda)$ ,  $b$  be the  $\lambda$ -ground state path, and  $N > 0$  be the smallest positive integer such that  $p_k = b_{\lambda_k}$  for all  $k \geq N$ . Then, the following maps (for  $i \in I$ ) define a crystal structure on  $\mathcal{P}(\lambda)$ :

$$\bar{wt} : \mathcal{P}(\lambda) \rightarrow \bar{P} \text{ such that}$$

$$\bar{wt}(p) = \lambda_N + \sum_{k=0}^{N-1} \bar{wt}(p_k),$$

$$\tilde{e}_i, \tilde{f}_i : \mathcal{P}(\lambda) \rightarrow \mathcal{P}(\lambda) \cup \{0\} \text{ such that}$$

$\tilde{e}_i(p) = \cdots \otimes p_{N+1} \otimes \tilde{e}_i(p_N \otimes \cdots \otimes p_0)$  and

$$\tilde{f}_i(p) = \cdots \otimes p_{N+1} \otimes \tilde{f}_i(p_N \otimes \cdots \otimes p_0),$$

$\epsilon_i, \varphi_i : \mathcal{P}(\lambda) \rightarrow \mathbb{Z}$  such that

$$\epsilon_i(p) = [\epsilon_i(p_{N-1} \otimes \cdots \otimes p_0) - \varphi_i(b_{\lambda_N})]_+ \text{ and}$$

$$\varphi_i(p) = \varphi_i(p_{N-1} \otimes \cdots \otimes p_0) + [\varphi_i(b_{\lambda_N}) - \epsilon_i(p_{N-1} \otimes \cdots \otimes p_0)]_+.$$

At the end of this chapter we provide two detailed examples of (partial) path realizations. But first, we note the following theorem.

**Theorem 4.10.** *Let  $\mathfrak{g}$  be an affine algebra,  $\lambda \in \bar{P}_l^+$ ,  $L(\lambda)$  a highest weight  $U'_q(\mathfrak{g})$ -module of highest weight  $\lambda$ , and  $\mathcal{B}(\lambda)$  the crystal base of  $L(\lambda)$ . Then  $\mathcal{B}(\lambda)$  and  $\mathcal{P}(\lambda)$  are isomorphic as  $U'_q(\mathfrak{g})$ -crystals.*

**Example 4.11.** *In Figure 4.2 we provide the (partial) path realization of  $V(\Lambda_0)$ , a  $U'_q(D_4^{(3)})$ -module, see Example 4.5 for notation.*

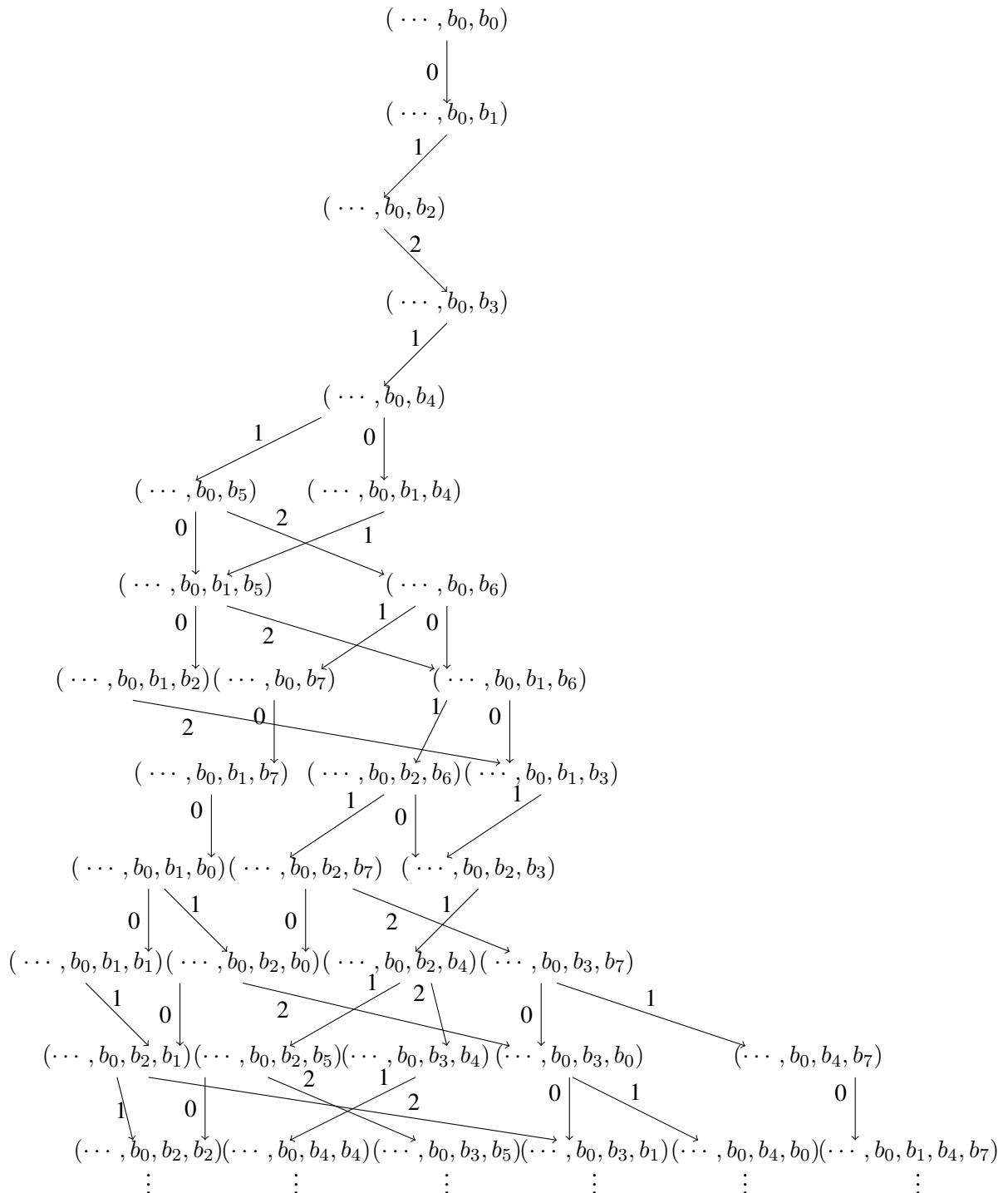


Figure 4.2: Partial path realization of  $V(\Lambda_0)$ , a  $U'_q(D_4^{(3)})$ -module.

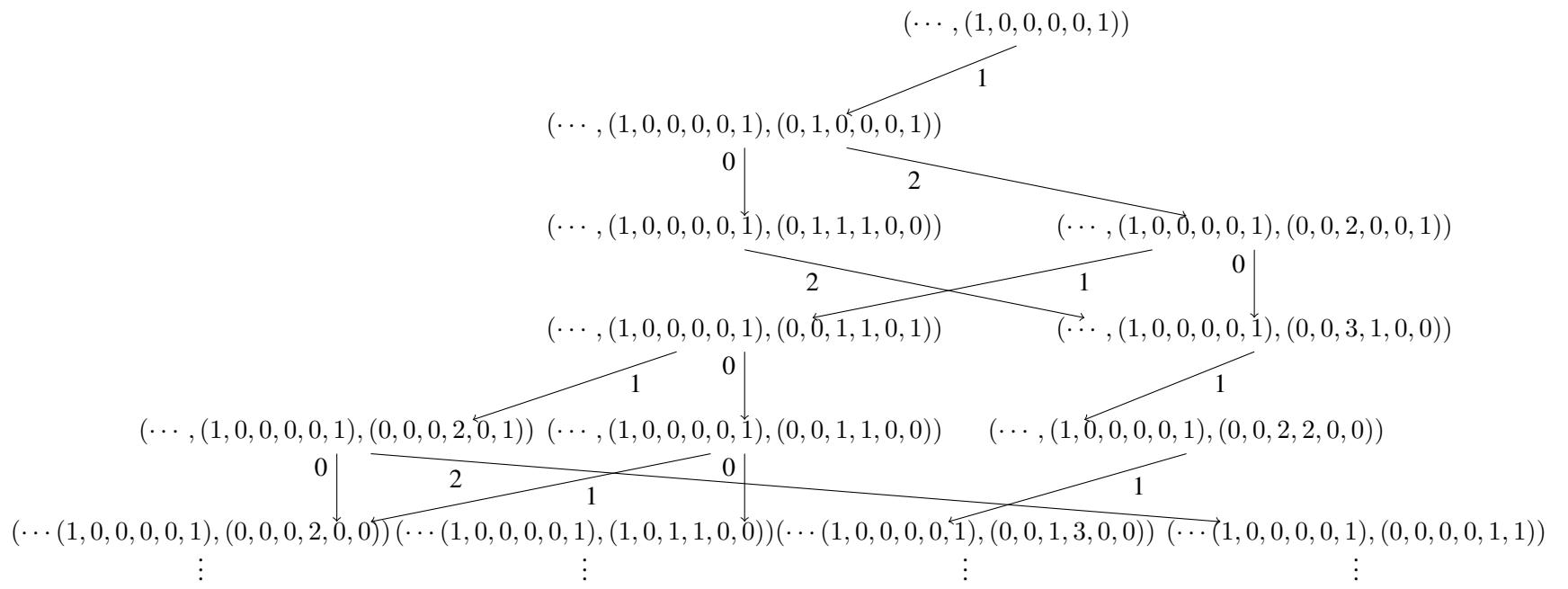


Figure 4.3: Partial path realization of  $V(\Lambda_1)$ , a  $U'_q(D_4^{(3)})$ -module.

**Example 4.12.** In Figure 4.3 we provide the (partial) path realization of  $V(\Lambda_1)$ , a  $U'_q(D_4^{(3)})$ -module, where the perfect crystal basis is  $\mathcal{B}_2$ , constructed as in Example 4.5.

# Chapter 5

## Root Multiplicities of $HD_4^{(3)}$

In this chapter we use our first result, Theorem 3.1, Kang's multiplicity formula, and the path realizations for irreducible highest weight  $D_4^{(3)}$ -modules to determine root multiplicities of  $HD_4^{(3)}$ . First, though, we find some general results concerning root multiplicities of  $HD_4^{(3)}$ .

### 5.1 General Results

**Theorem 5.1.** Let  $\mathfrak{g} = HD_4^{(3)}$ ,  $\mathfrak{g}_0 = D_4^{(3)}$ , and let  $\alpha = -l\alpha_{-1} - k\delta$ .

Then

if  $0 \neq l = k$ :

$$\dim \mathfrak{g}_\alpha = \begin{cases} 2 & \text{if } k \equiv 0 \pmod{3} \\ 1 & \text{if } k \not\equiv 0 \pmod{3} \end{cases}$$

and if  $0 \neq k < l$ :

$$\dim \mathfrak{g}_\alpha = 0$$

*Proof.* Let  $\mathfrak{g} = HD_4^{(3)}$  and let  $k \in \mathbb{Z}$ . Recall  $r_i : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ , where  $r_i(\beta) = \beta - \beta(h_i)\alpha_i$ . Observe:

$$\begin{aligned} r_{-1}(-k\alpha_{-1} - k\delta) &= -k\alpha_{-1} - k\delta - (-k\alpha_{-1} - k\delta)(h_{-1})\alpha_{-1} \\ &= -k\alpha_{-1} - k\delta - (-k\alpha_{-1} - k\alpha_0 - 2k\alpha_1 - k\alpha_2)(h_{-1})\alpha_{-1} \\ &= -k\alpha_{-1} - k\delta - (-k(2) - k(-1) - 2k(0) - k(0))\alpha_{-1} \\ &= -k\alpha_{-1} - k\delta + 2k\alpha_{-1} - k\alpha_{-1} \\ &= -k\delta \end{aligned}$$

Thus, we have

$$\begin{aligned} \dim \mathfrak{g}_{-k\alpha_{-1}-k\delta} &= \dim \mathfrak{g}_{r_{-1}(-k\alpha_{-1}-k\delta)} \\ &= \dim \mathfrak{g}_{-k\delta}. \end{aligned}$$

Since  $-k\delta$  is an imaginary root of  $\mathfrak{g}_0 = D_4^{(3)}$ , and then by Corollary 8.3 in [6]:

$$\begin{aligned} \dim(\mathfrak{g}_{-k\delta}) &= \dim(\mathfrak{g}_0)_{-k\delta} \\ &= \begin{cases} 2 & \text{if } k \equiv 0 \pmod{3} \\ 1 & \text{if } k \not\equiv 0 \pmod{3} \end{cases} \end{aligned}$$

Next, since  $\dim \mathfrak{g}_{-k\delta} \neq 0$ ,  $-k\delta$  is a root of  $\mathfrak{g}$  for all  $k$ . Then consider the  $\alpha_{-1}$ -string through  $-k\delta$ . This string is of length  $-k\delta(h_{-1}) = k$ :

$$-k\delta - r\alpha_{-1}, \dots, -k\delta, \dots, -k\delta + q\alpha_{-1}.$$

Since the coefficients of the simple roots in  $\alpha_{-1} - k\delta$  have mixed signs,  $\alpha_{-1} - k\delta$  must not be a root of  $\mathfrak{g}$ . Thus our string consists of  $\{-l\alpha_{-1} - k\delta \mid 0 < l \leq k\}$ . Hence,  $\dim \mathfrak{g}_{-l\alpha_{-1}-k\delta} = 0$  for  $l > k$ .  $\square$

**Theorem 5.2.** Let  $\mathfrak{g} = HD_4^{(3)}$  and  $\mathfrak{g}_0 = D_4^{(3)}$ . Let  $\Delta$  and  $\Delta^\pm$  be the roots, and positive (respectively, negative) roots of  $\mathfrak{g}$ . Then let  $\Delta_S$  and  $\Delta_S^\pm$  be the roots, and positive (respectfully, negative) roots of  $\mathfrak{g}_0$ . For  $l, k, \in \mathbb{Z}_{>0}$  and  $\alpha = m_1\alpha_1 + m_2\alpha_2 \in \Delta_S^+$ ,

$$\text{mult}(-l\alpha_{-1} - \alpha - k\delta) = \text{mult}(-(k-l)\alpha_{-1} - \alpha - k\delta).$$

*Proof.* Let  $i, j \in I = \{-1, 0, 1, 2\}$ . Recall the simple reflections  $r_j$ , given by  $r_j(\alpha_i) = \alpha_i(h_j)\alpha_j$ , generate the Weyl group for  $HD_4^{(3)}$ . Thus, for any  $\alpha \in \Delta$  and  $j \in I$ ,  $\text{mult}(\alpha) = \text{mult}(r_j\alpha)$ . Consider:

$$\begin{aligned} r_{-1}(-l\alpha_{-1} - m_1\alpha_1 - m_2\alpha_2 - k\delta) &= -l\alpha_{-1} - (-l\alpha_{-1})(h_{-1})\alpha_{-1} \\ &\quad - m_1\alpha_1 - (-m_1\alpha_1)(h_{-1})\alpha_{-1} - m_2\alpha_2 - (-m_2\alpha_2)(h_{-1})\alpha_{-1} \\ &\quad - k\alpha_0 - (-k\alpha_0)(h_{-1})\alpha_{-1} - 2k\alpha_1 - (-2k\alpha_1)(h_{-1})\alpha_{-1} \\ &\quad - k\alpha_2 - (-k\alpha_2)(h_{-1})\alpha_{-1} \\ &= -l\alpha_{-1} + 2l\alpha_{-1} - k\alpha_{-1} - \alpha - k\delta \\ &= -(k-l)\alpha_{-1} - \alpha - k\delta. \end{aligned}$$

$\square$

**Corollary 5.3.** For  $l \in \mathbb{Z}_{>0}$ ,  $\alpha = m_1\alpha_1 + m_2\alpha_2 \in \Delta_S^+$ , we have  $\text{mult}(-l\alpha_{-1} - \alpha - l\delta) = 1$ .

*Proof.* By the above,  $\text{mult}(-l\alpha_{-1} - \alpha - l\delta) = \text{mult}(-\alpha - l\delta)$ . Since  $-\alpha - l\delta$  is a real root of  $D_4^{(3)}$  (as all imaginary roots must be of the form  $n\delta$  for  $n \in \mathbb{Z}$ ), we have that  $\text{mult}(-\alpha - l\delta) = 1$ .  $\square$

**Corollary 5.4.**  $\text{mult}(-(k-1)\alpha_{-1} - k\delta) = \text{mult}(-\alpha_{-1} - k\delta)$ .

*Proof.* This follows immediately from the above theorem as  $-(k - (k-1)) = -1$ .  $\square$

## 5.2 Frenkel's Conjecture

For each root multiplicity we determine, we also wish to consider Frenkel's conjectured bound ([3]). In 1985, Frenkel conjectured that for any hyperbolic Kac-Moody algebra  $\mathfrak{g}$  with Cartan matrix  $A$  and standard invariant bilinear form  $(\cdot|\cdot)$ :

$$\text{mult}(\alpha) \leq p^{(\text{rank}(A)-2)} \left( 1 - \frac{(\alpha|\alpha)}{2} \right),$$

where

$$\sum_{k=0}^{\infty} p^2(k)q^k = \prod_{i=1}^{\infty} (1 - q^i)^{-2}.$$

For a Kac-Moody algebra with symmetrizable GCM  $A = (a_{ij})_{i,j \in I}$ , let  $D = \text{diag}(s_i \mid i \in I)$  be such that  $DA$  is a symmetric matrix. Then we may define  $(\cdot|\cdot)$  on the simple roots by

$$(\alpha_i|\alpha_j) = s_i a_{ij},$$

which is unique up to a scalar multiple (see, for example [4], [6]). We choose our symmetrizing matrix to be  $D = \text{diag}(1, 1, 1, 3)$  and so:

$$\begin{aligned} (\alpha_{-1}|\alpha_{-1}) &= 2 & (\alpha_0|\alpha_{-1}) &= -1 & (\alpha_1|\alpha_{-1}) &= 0 & (\alpha_2|\alpha_{-1}) &= 0 \\ (\alpha_{-1}|\alpha_0) &= -1 & (\alpha_0|\alpha_0) &= 2 & (\alpha_1|\alpha_0) &= -1 & (\alpha_2|\alpha_0) &= 0 \\ (\alpha_{-1}|\alpha_1) &= 0 & (\alpha_0|\alpha_1) &= -1 & (\alpha_1|\alpha_1) &= 2 & (\alpha_2|\alpha_1) &= -3 \\ (\alpha_{-1}|\alpha_2) &= 0 & (\alpha_0|\alpha_2) &= 0 & (\alpha_1|\alpha_2) &= -3 & (\alpha_2|\alpha_2) &= 6 \end{aligned}$$

Then, for  $\mathfrak{g} = HD_4^{(3)}$  and  $\alpha = -l\alpha_{-1} - k\delta$ , with  $(\cdot|\cdot)$  as defined above, Frenkel's conjectured bound becomes:

$$\begin{aligned} \text{mult}(\alpha) &\leq p^{(\text{rank}(A)-2)} \left( 1 - \frac{(\alpha|\alpha)}{2} \right) \\ &\leq p^2 \left( 1 - \frac{2l^2 - 2lk}{2} \right) \\ &\leq p^2 (1 + l(k-l)) \end{aligned}$$

Consider this conjecture in terms of real and imaginary roots. A root  $\alpha \in \Delta$  is called *real* if there exists a Weyl group element  $w \in W$  such that  $w(\alpha)$  is a simple root. A root  $\alpha$  which is not real is called *imaginary*. By Proposition 5.2(c) in [6], for a Kac-Moody algebra with a symmetrizable GCM  $A$  and a standard invariant bilinear form  $(\cdot|\cdot)$ , we have that a root  $\alpha$  is an imaginary root if and only if  $(\alpha|\alpha) \leq 0$  and hence,  $\left(1 - \frac{(\alpha|\alpha)}{2}\right) > 0$ . Thus, if  $\alpha$  is a real root,  $(\alpha|\alpha) > 0$  and hence  $\left(1 - \frac{(\alpha|\alpha)}{2}\right) \leq 0$ . If  $\alpha$  is a real root,  $\text{mult}(\alpha) = 1$ . So, for simplicity, in our calculations of Frenkel's bound we define  $p^2(m) = 1$  for  $m \leq 0$ .

In the following sections, each time we find the multiplicity for a root, we also calculate  $\frac{(\alpha|\alpha)}{2}$  and include Frenkel's conjectured bound. We will do this for the remainder of the thesis.

### 5.3 Level 1 Root Multiplicities

Now, we consider level 1 roots of the form:  $-\alpha_{-1} - k\delta$ , where  $\delta = \alpha_0 + 2\alpha_1 + \alpha_2$  is the null root (imaginary root). Note that since

$$-\alpha_{-1}(h_0, h_1, h_2) = \Lambda_0(h_0, h_1, h_2),$$

the simple root  $\alpha_{-1}$  in  $HD_4^{(3)}$  corresponds to the weight  $-\Lambda_0$  for  $D_4^{(3)}$  and by the construction in Chapter 3, level 1 roots of  $HD_4^{(3)}$  are weights in the  $D_4^{(3)}$ -module  $V(\Lambda_0)$ .

Using crystal base theory, counting the multiplicity of such a weight consists of counting the number of corresponding paths in the path realization of  $\mathcal{P}(\Lambda_0)$ . Recall the  $U'_q(D_4^{(3)})$  perfect crystal  $B_l$  of level  $l \geq 1$  as outlined in Example 4.5. For  $l = 1$  we have  $B_1 = \{b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$  such that

$$\begin{aligned} b_0 &= (0, 0, 0, 0, 0, 0), \\ b_1 &= (1, 0, 0, 0, 0, 0), \\ b_2 &= (0, 1, 0, 0, 0, 0), \\ b_3 &= (0, 0, 2, 0, 0, 0), \\ b_4 &= (0, 0, 1, 1, 0, 0), \\ b_5 &= (0, 0, 0, 2, 0, 0), \\ b_6 &= (0, 0, 0, 0, 1, 0), \\ b_7 &= (0, 0, 0, 0, 0, 1). \end{aligned}$$

Further recall the path realization of the  $U'_q(D_4^{(3)})$ -module  $V(\Lambda_0)$  given in Figure 4.2. We determine the following level one root multiplicities by counting the appropriate paths in the path realization of  $V(\Lambda_0)$ . These paths can be counted in Figure 4.2 for roots of the form  $-\alpha_{-1} - k\delta$  with  $k = 1, 2, 3, 4$ . We then used the MATLAB program `lev1mult` (see appendix) to determine the multiplicities of level 1 roots of the form  $-\alpha_{-1} - k\delta$  for values of  $k$  greater than 4. These results are collected in Table 5.1, Table 5.2, Table 5.3, and Table 5.4.

Table 5.1: Root Multiplicities for  $-\alpha_{-1} - k\delta$ ,  $1 \leq k \leq 6$ 

$\alpha$	Multiplicity	Paths in $\mathcal{P}(\Lambda_0)$ of weight $\alpha$	$p^2 \left(1 - \frac{(\alpha \alpha)}{2}\right)$
$-\alpha_{-1} - \delta$	1	$\cdots \otimes b_0 \otimes b_4$	$p^2(1) = 1$
$-\alpha_{-1} - 2\delta$	2	$\cdots \otimes b_0 \otimes b_1 \otimes b_7$ $\cdots \otimes b_0 \otimes b_2 \otimes b_6$	$p^2(2) = 5$
$-\alpha_{-1} - 3\delta$	4	$\cdots \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_0$ $\cdots \otimes b_0 \otimes b_1 \otimes b_4 \otimes b_7$ $\cdots \otimes b_0 \otimes b_0 \otimes b_3 \otimes b_5$ $\cdots \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_4$	$p^2(3) = 10$
$-\alpha_{-1} - 4\delta$	6	$\cdots \otimes b_0 \otimes b_1 \otimes b_7 \otimes b_0$ $\cdots \otimes b_0 \otimes b_2 \otimes b_6 \otimes b_0$ $\cdots \otimes b_0 \otimes b_1 \otimes b_5 \otimes b_6$ $\cdots \otimes b_0 \otimes b_0 \otimes b_5 \otimes b_3$ $\cdots \otimes b_0 \otimes b_0 \otimes b_6 \otimes b_2$ $\cdots \otimes b_0 \otimes b_0 \otimes b_7 \otimes b_1$	$p^2(4) = 20$
$-\alpha_{-1} - 5\delta$	9	$\cdots \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_0 \otimes b_0$ $\cdots \otimes b_0 \otimes b_1 \otimes b_4 \otimes b_7 \otimes b_0$ $\cdots \otimes b_0 \otimes b_0 \otimes b_2 \otimes b_3 \otimes b_7$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_6 \otimes b_5$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_7 \otimes b_4$ $\cdots \otimes b_0 \otimes b_0 \otimes b_2 \otimes b_6 \otimes b_4$ $\cdots \otimes b_0 \otimes b_0 \otimes b_2 \otimes b_7 \otimes b_3$ $\cdots \otimes b_0 \otimes b_0 \otimes b_3 \otimes b_7 \otimes b_2$ $\cdots \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_7 \otimes b_1$	$p^2(5) = 36$
$-\alpha_{-1} - 6\delta$	16	$\cdots \otimes b_0 \otimes b_1 \otimes b_7 \otimes b_0 \otimes b_0$ $\cdots \otimes b_0 \otimes b_2 \otimes b_6 \otimes b_0 \otimes b_0$ $\cdots \otimes b_0 \otimes b_0 \otimes b_3 \otimes b_5 \otimes b_0$ $\cdots \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_4 \otimes b_0$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_0 \otimes b_7$ $\cdots \otimes b_0 \otimes b_1 \otimes b_4 \otimes b_4 \otimes b_7$ $\cdots \otimes b_0 \otimes b_0 \otimes b_2 \otimes b_0 \otimes b_6$ $\cdots \otimes b_0 \otimes b_0 \otimes b_2 \otimes b_4 \otimes b_6$ $\cdots \otimes b_0 \otimes b_0 \otimes b_3 \otimes b_0 \otimes b_5$ $\cdots \otimes b_0 \otimes b_0 \otimes b_3 \otimes b_4 \otimes b_5$ $\cdots \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_0 \otimes b_4$ $\cdots \otimes b_0 \otimes b_1 \otimes b_4 \otimes b_7 \otimes b_4$ $\cdots \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_4 \otimes b_4$ $\cdots \otimes b_0 \otimes b_0 \otimes b_5 \otimes b_0 \otimes b_3$ $\cdots \otimes b_0 \otimes b_0 \otimes b_6 \otimes b_0 \otimes b_2$ $\cdots \otimes b_0 \otimes b_0 \otimes b_7 \otimes b_0 \otimes b_1$	$p^2(6) = 65$

Table 5.2: Root Multiplicities for  $-\alpha_{-1} - k\delta$ ,  $k = 7$

$\alpha$	Multiplicity	Paths in $\mathcal{P}(\Lambda_0)$ of weight $\alpha$	$p^2 \left(1 - \frac{(\alpha \alpha)}{2}\right)$
$-\alpha_{-1} - 7\delta$	22	$\cdots \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_0 \otimes b_0 \otimes b_0$ $\cdots \otimes b_0 \otimes b_1 \otimes b_4 \otimes b_7 \otimes b_0 \otimes b_0$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_5 \otimes b_6 \otimes b_0$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_4 \otimes b_0 \otimes b_7$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_3 \otimes b_2 \otimes b_7$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_1 \otimes b_7$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_5 \otimes b_0 \otimes b_6$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_4 \otimes b_5 \otimes b_6$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_2 \otimes b_6$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_5 \otimes b_1 \otimes b_6$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_6 \otimes b_0 \otimes b_5$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_6 \otimes b_1 \otimes b_5$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_7 \otimes b_0 \otimes b_4$ $\cdots \otimes b_0 \otimes b_0 \otimes b_2 \otimes b_6 \otimes b_0 \otimes b_4$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_3 \otimes b_5 \otimes b_4$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_7 \otimes b_1 \otimes b_4$ $\cdots \otimes b_0 \otimes b_0 \otimes b_2 \otimes b_7 \otimes b_0 \otimes b_3$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_5 \otimes b_3$ $\cdots \otimes b_0 \otimes b_0 \otimes b_3 \otimes b_7 \otimes b_0 \otimes b_2$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_6 \otimes b_2$ $\cdots \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_7 \otimes b_0 \otimes b_1$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_5 \otimes b_6 \otimes b_1$	$p^2(7) = 110$

Table 5.3: Root Multiplicities for  $-\alpha_{-1} - k\delta$ ,  $k = 8$ 

$\alpha$	Multiplicity	Paths in $\mathcal{P}(\Lambda_0)$ of weight $\alpha$	$p^2 \left(1 - \frac{(\alpha \alpha)}{2}\right)$
$-\alpha_{-1} - 8\delta$	33	$\cdots \otimes b_0 \otimes b_1 \otimes b_7 \otimes b_0 \otimes b_0 \otimes b_0$ $\cdots \otimes b_0 \otimes b_2 \otimes b_6 \otimes b_0 \otimes b_0 \otimes b_0$ $\cdots \otimes b_0 \otimes b_0 \otimes b_2 \otimes b_3 \otimes b_7 \otimes b_0$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_5 \otimes b_3 \otimes b_0$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_6 \otimes b_2 \otimes b_0$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_7 \otimes b_1 \otimes b_0$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_0 \otimes b_0 \otimes b_7$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_2 \otimes b_6 \otimes b_7$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_5 \otimes b_3 \otimes b_7$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_6 \otimes b_2 \otimes b_7$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_7 \otimes b_1 \otimes b_7$ $\cdots \otimes b_0 \otimes b_0 \otimes b_2 \otimes b_6 \otimes b_1 \otimes b_7$ $\cdots \otimes b_0 \otimes b_0 \otimes b_2 \otimes b_0 \otimes b_0 \otimes b_6$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_7 \otimes b_2 \otimes b_6$ $\cdots \otimes b_0 \otimes b_0 \otimes b_2 \otimes b_6 \otimes b_2 \otimes b_6$ $\cdots \otimes b_0 \otimes b_0 \otimes b_2 \otimes b_7 \otimes b_1 \otimes b_6$ $\cdots \otimes b_0 \otimes b_0 \otimes b_3 \otimes b_0 \otimes b_0 \otimes b_5$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_4 \otimes b_6 \otimes b_5$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_3 \otimes b_5$ $\cdots \otimes b_0 \otimes b_0 \otimes b_3 \otimes b_7 \otimes b_1 \otimes b_5$ $\cdots \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_0 \otimes b_0 \otimes b_4$ $\cdots \otimes b_0 \otimes b_1 \otimes b_4 \otimes b_7 \otimes b_0 \otimes b_4$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_5 \otimes b_6 \otimes b_4$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_5 \otimes b_3 \otimes b_4$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_6 \otimes b_2 \otimes b_4$ $\cdots \otimes b_0 \otimes b_0 \otimes b_4 \otimes b_7 \otimes b_1 \otimes b_4$ $\cdots \otimes b_0 \otimes b_0 \otimes b_5 \otimes b_0 \otimes b_0 \otimes b_3$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_5 \otimes b_7 \otimes b_3$ $\cdots \otimes b_0 \otimes b_0 \otimes b_0 \otimes b_7 \otimes b_2 \otimes b_3$ $\cdots \otimes b_0 \otimes b_0 \otimes b_6 \otimes b_0 \otimes b_0 \otimes b_2$ $\cdots \otimes b_0 \otimes b_0 \otimes b_1 \otimes b_6 \otimes b_7 \otimes b_2$ $\cdots \otimes b_0 \otimes b_0 \otimes b_7 \otimes b_0 \otimes b_0 \otimes b_1$ $\cdots \otimes b_0 \otimes b_0 \otimes b_2 \otimes b_6 \otimes b_7 \otimes b_1$	$p^2(8) = 185$

Table 5.4: Root Multiplicities for  $-\alpha_{-1} - k\delta$ ,  $9 \leq k \leq 20$

$\alpha$	Multiplicity	$p^2 \left(1 - \frac{(\alpha \alpha)}{2}\right)$
$-\alpha_{-1} - 9\delta$	50	$p^2(9) = 300$
$-\alpha_{-1} - 10\delta$	70	$p^2(10) = 481$
$-\alpha_{-1} - 11\delta$	98	$p^2(11) = 752$
$-\alpha_{-1} - 12\delta$	143	$p^2(12) = 1165$
$-\alpha_{-1} - 13\delta$	193	$p^2(13) = 1770$
$-\alpha_{-1} - 14\delta$	266	$p^2(14) = 2665$
$-\alpha_{-1} - 15\delta$	368	$p^2(15) = 3956$
$-\alpha_{-1} - 16\delta$	493	$p^2(16) = 5822$
$-\alpha_{-1} - 17\delta$	659	$p^2(17) = 8470$
$-\alpha_{-1} - 18\delta$	892	$p^2(18) = 12230$
$-\alpha_{-1} - 19\delta$	1170	$p^2(19) = 17490$
$-\alpha_{-1} - 20\delta$	1543	$p^2(20) = 24842$

Note here that, taking into account an index shifting, the multiplicities we found for  $\alpha_{-1} - 2\delta$ ,  $\alpha_{-1} - 3\delta$ , and  $\alpha_{-1} - 4\delta$  coincide with the multiplicities found for the same roots using a different technique in [5]; in addition, we note that we recovered all the multiplicities found in [5] corresponding to level 1 roots.

We wish to generalize these results for arbitrary  $k$ . To do this we make use of Theorem 5.5, (see, [9]) that uses the following definition of an *energy function*.

**Theorem 5.5.** [9] Let

$$p = (\cdots \otimes p_k \otimes \cdots \otimes p_1 \otimes p_0)$$

be a  $\lambda$ -path in  $B(\lambda)$ , and let

$$b = (\cdots \otimes b_{\lambda_k} \otimes \cdots \otimes b_{\lambda_1} \otimes b_{\lambda_0})$$

be the ground state path of weight  $\lambda$ . Then, the affine weight of  $p$  is given by the formula:

$$wt(p) = \lambda + \sum_{k=0}^{\infty} (\bar{wt}(p_k) - \bar{wt}(b_{\lambda_k})) - \left( \sum_{k=0}^{\infty} (k+1)(H(p_{k+1} \otimes p_k) - H(b_{\lambda_{k+1}} \otimes b_{\lambda_k})) \right) \delta$$

**Definition.** Let  $V$  be a finite dimensional  $U'_q(\mathfrak{g})$ -module, and let  $(\mathcal{L}, \mathcal{B})$  be a crystal basis for  $V$ . An energy function on  $\mathcal{B}$  is a function  $H : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathbb{Z}$  such that for all  $i \in I$  and  $b_1 \otimes b_2$  in  $\mathcal{B} \otimes \mathcal{B}$  such that

$\tilde{e}_i(b_1 \otimes b_2) \in \mathcal{B} \otimes \mathcal{B}$ ,

$$H(\tilde{e}_i(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2) & \text{if } i \neq 0 \\ H(b_1 \otimes b_2) + 1 & \text{if } i = 0 \text{ and } \phi_0(b_1) \geq \epsilon_0(b_2) \\ H(b_1 \otimes b_2) - 1 & \text{if } i = 0 \text{ and } \phi_0(b_1) < \epsilon_0(b_2) \end{cases}$$

Let  $B_1$  be the  $U'_q(D_4^{(3)})$  perfect crystal of level 1 described in Example 4.5. We determined the energy function of  $B_1$ ,  $H : \mathcal{B}_1 \otimes \mathcal{B}_1 \rightarrow \mathbb{Z}$ , given by Table 5.5. With this energy function now defined, we use the formula for the affine weight of a path in  $\mathcal{P}(\Lambda_0)$ , given by Theorem 5.5, to generalize our findings for the multiplicity of a root of the form  $-\alpha_{-1} - k\delta$  with Theorem 5.6. To do this we make use of the following classical weights of the elements of  $B_1 = \{b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$  (recall,  $\bar{wt}(b_k) = \varphi(b_k) - \epsilon(b_k)$ ):

$$\begin{aligned} \bar{wt}(b_0) &= 0, \\ \bar{wt}(b_1) &= \delta - \alpha_0, \\ \bar{wt}(b_2) &= \alpha_1 + \alpha_2, \\ \bar{wt}(b_3) &= \alpha_1, \\ \bar{wt}(b_4) &= 0, \\ \bar{wt}(b_5) &= -\alpha_1, \\ \bar{wt}(b_6) &= -\alpha_1 - \alpha_2, \\ \bar{wt}(b_7) &= \alpha_0 - \delta. \end{aligned}$$

**Theorem 5.6.** For  $p \in \mathcal{P}(\Lambda_0)$  with  $wt(p) = -\alpha_{-1} - k\delta$ ,

$$p = (\cdots \otimes b_0 \otimes x_m \otimes x_{m-1} \otimes \cdots \otimes x_0),$$

where  $m$  is the smallest positive integer such that  $x_j = b_0$  for all  $j > m$ , and

$$m \leq \begin{cases} k-1 & \text{for } k = 1, 2, 3 \\ k-2 & \text{for } k \geq 4 \end{cases}$$

*Proof.* As  $k$  is finite, we know only finitely many elements of the tensor  $p$  may be altered from the ground state path. Let  $m$  be the smallest positive integer such that  $x_j = b_0$  for all  $j > m$ .

Table 5.5: Energy Function of  $B_1$ 

$H(b \otimes b')$	$b \otimes b'$	$H(b \otimes b')$	$b \otimes b'$
0	$b_0 \otimes b_0$		$b_1 \otimes b_1$
	$b_1 \otimes b_4$		$b_2 \otimes b_1$
	$b_1 \otimes b_5$		$b_2 \otimes b_2$
	$b_1 \otimes b_6$		$b_3 \otimes b_1$
	$b_1 \otimes b_7$		$b_3 \otimes b_2$
	$b_2 \otimes b_6$		$b_3 \otimes b_3$
	$b_2 \otimes b_7$		$b_4 \otimes b_1$
	$b_3 \otimes b_7$		$b_4 \otimes b_2$
	$b_4 \otimes b_7$		$b_4 \otimes b_3$
			$b_5 \otimes b_1$
1	$b_1 \otimes b_0$		$b_5 \otimes b_2$
	$b_2 \otimes b_0$		$b_5 \otimes b_3$
	$b_3 \otimes b_0$		$b_5 \otimes b_4$
	$b_4 \otimes b_0$		$b_5 \otimes b_5$
	$b_5 \otimes b_0$		$b_6 \otimes b_1$
	$b_6 \otimes b_0$		$b_6 \otimes b_2$
	$b_7 \otimes b_0$		$b_6 \otimes b_3$
			$b_6 \otimes b_4$
	$b_0 \otimes b_1$		$b_6 \otimes b_5$
	$b_0 \otimes b_2$		$b_6 \otimes b_6$
	$b_0 \otimes b_3$		$b_7 \otimes b_1$
	$b_0 \otimes b_4$		$b_7 \otimes b_2$
	$b_0 \otimes b_5$		$b_7 \otimes b_3$
	$b_0 \otimes b_6$		$b_7 \otimes b_4$
2	$b_0 \otimes b_7$		$b_7 \otimes b_5$
			$b_7 \otimes b_6$
	$b_1 \otimes b_2$		$b_7 \otimes b_7$
	$b_1 \otimes b_3$		
	$b_2 \otimes b_3$		
	$b_2 \otimes b_4$		
	$b_2 \otimes b_5$		
	$b_3 \otimes b_4$		
	$b_3 \otimes b_5$		
	$b_3 \otimes b_6$		
	$b_4 \otimes b_4$		
	$b_4 \otimes b_5$		
	$b_4 \otimes b_6$		
	$b_5 \otimes b_6$		

Without loss of generality, suppose  $m = k$ . Then:

$$\begin{aligned}-\alpha_{-1} - k\delta &= wt(\cdots \otimes b_0 \otimes x_k \otimes \cdots \otimes x_0) \\&= \Lambda_0 + \bar{wt}(x_k) + \cdots + \bar{wt}(x_0) - \\&\quad ((k+1)H(b_0 \otimes x_k) + kH(x_k \otimes x_{k-1}) + \cdots + H(x_1 \otimes x_0))\delta\end{aligned}$$

Since  $H(b_i \otimes b_l) \geq 0$  for all  $i, l \in \{0, 1, \dots, 7\}$ ,  $H(b_0 \otimes x_k)$  must be 0 to preserve the above equality. So,  $x_k$  must be  $b_0$  by the definition of the energy function. So, for arbitrary  $k$  we have that  $m \leq k-1$  and:

$$\begin{aligned}-\alpha_{-1} - k\delta &= wt(\cdots \otimes b_0 \otimes x_{k-1} \otimes \cdots \otimes x_0) \\&= \Lambda_0 + \bar{wt}(x_{k-1}) + \cdots + \bar{wt}(x_0) - (kH(b_0 \otimes x_{k-1}) + \cdots + H(x_1 \otimes x_0))\delta\end{aligned}$$

Suppose  $x_{k-1} \neq b_0$ . Then:

$$\begin{aligned}-\alpha_{-1} - k\delta &= wt(\cdots \otimes b_0 \otimes x_{k-1} \otimes \cdots \otimes x_0) \\&= \Lambda_0 + \bar{wt}(x_{k-1}) + \cdots + \bar{wt}(x_0) - (k(1) + (k-1)H(x_{k-1} \otimes x_{k-2}) + \cdots + H(x_1 \otimes x_0))\delta \\&= \Lambda_0 + \bar{wt}(x_{k-1}) + \cdots + \bar{wt}(x_0) - k\delta - ((k-1)H(x_{k-1} \otimes x_{k-2}) + \cdots + H(x_1 \otimes x_0))\delta\end{aligned}$$

by the definition of the energy function. So, in order for this equality to hold, we must have:

$$\bar{wt}(x_{k-1}) + \cdots + \bar{wt}(x_0) = 0, \text{ and}$$

$$H(x_{k-1} \otimes x_{k-2}) = \cdots = H(x_1 \otimes x_0) = 0.$$

We consider the definition of the energy function and see that this implies (for  $x_{k-1} \neq b_0$ ),

$$x_{k-1} = b_4, b_3, b_2, \text{ or } b_1.$$

If  $k = 1$ , then we have  $p = (\cdots \otimes b_0 \otimes x_0)$ , with  $\bar{wt}(x_0) = 0$ . Thus,  $x_0$  must be  $b_4$  and this is the only possibility. So  $mult(-\alpha_{-1} - \delta) \geq 1$ , and we have part of our desired result: for  $k = 1$ ,  $m \leq k-1$ .

If  $k = 2$ , then we have  $p = (\cdots \otimes b_0 \otimes x_1 \otimes x_0)$ , with  $x_1 \neq b_0$ . Then we consider each case for  $x_1$  (using the definition of the energy function):

If  $x_1 = b_4$  then  $x_2 = b_7$ , but  $\bar{wt}(b_4) + \bar{wt}(b_7) = \alpha_0 - \delta \neq 0 \Rightarrow \Leftarrow$

If  $x_1 = b_3$  then  $x_2 = b_7$ , but  $\bar{wt}(b_3) + \bar{wt}(b_7) = \alpha_1 + \alpha_0 - \delta \neq 0 \Rightarrow \Leftarrow$

If  $x_1 = b_2$  then  $x_2 = b_7$ , but  $\bar{wt}(b_2) + \bar{wt}(b_7) = \alpha_1 + \alpha_2 + \alpha_0 - \delta \neq 0 \Rightarrow \Leftarrow$

or,  $x_2 = b_6$ , and  $\bar{wt}(b_2) + \bar{wt}(b_6) = 0 \checkmark$

If  $x_1 = b_1$  then  $x_2 = b_7$ , and  $\bar{wt}(b_1) + \bar{wt}(b_7) = 0 \checkmark$

or,  $x_2 = b_6$ , but  $\bar{wt}(b_1) + \bar{wt}(b_6) = \delta - \alpha_0 - \alpha_1 - \alpha_2 \neq 0 \Rightarrow \Leftarrow$

or,  $x_2 = b_5$ , but  $\bar{wt}(b_1) + \bar{wt}(b_5) = \delta - \alpha_0 - \alpha_1 \neq 0 \Rightarrow \Leftarrow$

or,  $x_2 = b_4$ , but  $\bar{wt}(b_1) + \bar{wt}(b_4) = \delta - \alpha_0 \neq 0 \Rightarrow \Leftarrow$

So there are two cases that are possible, hence  $mult(-\alpha_{-1} - 2\delta) \geq 2$ , and we have another part of our desired result: for  $k = 2, m \leq k - 1$ .

If  $k = 3$ , then we have  $p = (\cdots \otimes b_0 \otimes x_2 \otimes x_1 \otimes x_0)$ , with  $x_1 \neq b_0$ . Then we consider each case for  $x_2$  (using the definition of the energy function):

If  $x_2 = b_4$  then  $x_1 = b_7$ , but  $H(x_1 \otimes x_0) = H(b_7 \otimes x_0) \neq 0 \Rightarrow \Leftarrow$

If  $x_2 = b_3$  then  $x_1 = b_7$ , but  $H(x_1 \otimes x_0) = H(b_7 \otimes x_0) \neq 0 \Rightarrow \Leftarrow$

If  $x_2 = b_2$  then  $x_1 = b_7$ , but  $H(x_1 \otimes x_0) = H(b_7 \otimes x_0) \neq 0 \Rightarrow \Leftarrow$

or,  $x_1 = b_6$ , but  $H(x_1 \otimes x_0) = H(b_6 \otimes x_0) \neq 0 \Rightarrow \Leftarrow$

If  $x_2 = b_1$  then  $x_1 = b_7$ , but  $H(x_1 \otimes x_0) = H(b_7 \otimes x_0) \neq 0 \Rightarrow \Leftarrow$

or,  $x_1 = b_6$ , but  $H(x_1 \otimes x_0) = H(b_6 \otimes x_0) \neq 0 \Rightarrow \Leftarrow$

or,  $x_1 = b_5$ , but  $H(x_1 \otimes x_0) = H(b_5 \otimes x_0) \neq 0 \Rightarrow \Leftarrow$

or,  $x_1 = b_4$ , and  $x_0 = b_7 \rightarrow p = (\cdots \otimes b_0 \otimes b_1 \otimes b_4 \otimes b_7) \checkmark$

So, there is one case that is possible, hence  $mult(-\alpha_{-1} - 3\delta) \geq 1$ , and we have that for  $k = 3, m \leq k - 1$ .

If  $k \geq 4$ , then  $p = (\cdots \otimes b_0 \otimes x_{k-1} \otimes \cdots \otimes x_0)$ , with  $x_{k-1} \neq b_0$  and  $x_{k-1} = b_4, b_3, b_2$ , or  $b_1$ . Observe from the case  $k = 3$ , that the only possibility that arises is  $x_{k-1} = b_1, x_{k-2} = b_4$ , and  $x_{k-3} = b_7$ . Then,  $H(x_{k-3} \otimes x_{k-4}) = H(b_7 \otimes x_{k-4}) \neq 0$ , so for  $k \geq 4, x_{k-1}$  must be  $b_0$  and  $m \leq k - 2$ .  $\square$

## 5.4 Level 2 Root Multiplicities

In this section, we consider multiplicities of roots of the form  $\alpha = -2\alpha_{-1} - k\delta$  of  $HD_4^{(3)}$ . That is, we want to find  $\dim(\mathfrak{g}_{-2\alpha_{-1} - k\delta})$ . To do this, we recall Kang's multiplicity formula from Section 3.2:

$$\dim(\mathfrak{g}_\alpha) = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \left(\frac{\tau}{\alpha}\right) \mathcal{B}(\tau).$$

Consider:  $\tau$  such that  $\tau|\alpha$ . So,  $\alpha|\alpha$  always. We then consider any other possible  $\tau$  that divides  $\alpha = -2\alpha_{-1} - k\delta$ . Note that  $\tau = \frac{\alpha}{2}$  is possible, but only if  $k$  is even. By the Theorem 5.1, we have:

$$\dim \mathfrak{g}_{-2\alpha_{-1}-\delta} = 0,$$

$$\dim \mathfrak{g}_{-2\alpha_{-1}-2\delta} = 1.$$

Now, consider  $k = 3$ , or

$$\dim \mathfrak{g}_{-2\alpha_{-1}-3\delta}$$

As  $k$  is odd we have:

$$\begin{aligned} \dim(\mathfrak{g}_{-2\alpha_{-1}-3\delta}) &= \mu\left(\frac{-2\alpha_{-1}-3\delta}{-2\alpha_{-1}-3\delta}\right) \left(\frac{-2\alpha_{-1}-3\delta}{-2\alpha_{-1}-3\delta}\right) \mathcal{B}(-2\alpha_{-1}-3\delta) \\ &= \mathcal{B}(-2\alpha_{-1}-3\delta) \end{aligned}$$

Consider  $T(\tau)$  in this case. Notice that the formula for  $\mathcal{B}(\tau)$  is very interconnected; that is, only certain  $\tau_i$  of  $T(\tau)$  will give  $0 \neq \dim V(w\rho - \rho)_{\tau_i}$  (and hence,  $0 \neq K_{\tau_i}$ ). So, first we determine  $w \in W(S)$ , and hence,  $V(w\rho - \rho)$ . Then, we find which  $\tau_i$  are sufficient so that  $V(w\rho - \rho)_{\tau_i}$  has nonzero dimension.

To find  $w \in W(S)$  we use the following lemma:

**Lemma 5.7.** [8] For  $w = w'r_j$  and  $l(w) = l(w') + 1$ ,

$$w \in W(S) \Leftrightarrow w' \in W(S) \text{ and } w'(\alpha_j) \in \Delta^+(S).$$

And recall the reflections:

$$r_i : \mathfrak{h}^* \rightarrow \mathfrak{h}^* \text{ such that } r_i(\beta) = \beta - \beta(h_i)\alpha_i.$$

So, we start with  $w = r_{-1}$ , as it is the only  $w \in W(S)$  of length 1 that gives  $w'(\alpha_j) \in \Delta^+(S)$ :

$$\begin{aligned} w &= r_{-1} = 1 \cdot r_{-1} \text{ and } l(w) = l(r_{-1}) = l(1) + 1 = 0 + 1 \\ w &= r_{-1} \in W(S) \text{ since } w' = 1 \in W(S) \text{ and} \\ 1(\alpha_{-1}) &= \alpha_{-1} \in \Delta^+(S) \end{aligned}$$

$$\begin{aligned}
w &= r_{-1}r_0 \text{ and } l(w) = l(r_{-1}r_0) = l(r_{-1}) + 1 = 2 \\
w &= r_{-1}r_0 \in W(S) \text{ since } w' = r_{-1} \in W(S) \text{ and} \\
r_{-1}(\alpha_0) &= \alpha_0 - \alpha_0(h_{-1})\alpha_{-1} = \alpha_0 - (-1)\alpha_{-1} = \alpha_0 + \alpha_{-1} \in \Delta^+(S)
\end{aligned}$$

Notice:

$w = r_{-1}r_i$  will result in  $\alpha_i(h_{-1}) = 0$  for  $i = 1, 2$  and hence, the corresponding  $w'(\alpha_j) \notin \Delta^+(S)$   
So,  $w = r_{-1}r_0$  is the only  $w \in W(S)$  of length 2.

Before organizing these results, we determine how to represent the fundamental weights of  $D_4^{(3)}$ ,  $\{\Lambda_0, \Lambda_1, \Lambda_2, \}$ , in terms of the simple roots  $\{\alpha_0, \alpha_1, \alpha_2\}$ . Consider

$$\begin{pmatrix} \Lambda_0 & \Lambda_1 & \Lambda_2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \end{pmatrix}$$

So we have:

$$\begin{aligned}
\tilde{\alpha}_0 &= 2\Lambda_0 - \Lambda_1 \\
\tilde{\alpha}_1 &= -\Lambda_0 + 2\Lambda_1 - \Lambda_2 \\
\tilde{\alpha}_2 &= -3\Lambda_1 + 2\Lambda_2
\end{aligned}$$

But, recall that for  $d \in \mathfrak{h}$  we have the following relations:

$$\begin{aligned}
\langle \alpha_0, d \rangle &= 1 = \delta(d), \quad \langle \alpha_1, d \rangle = 0, \quad \langle \alpha_2, d \rangle = 0 \text{ and} \\
\Lambda_i(d) &= 0 \text{ for } i = 0, 1, 2.
\end{aligned}$$

So, to account for these relations, we must add  $\delta$  to  $\tilde{\alpha}_0$  and obtain the following:

$$\begin{aligned}
\alpha_0 &= 2\Lambda_0 - \Lambda_1 + \delta \\
\alpha_1 &= -\Lambda_0 + 2\Lambda_1 - \Lambda_2 \\
\alpha_2 &= -3\Lambda_1 + 2\Lambda_2
\end{aligned}$$

We organize these findings in Table 5.6, also using  $\rho \in \mathfrak{h}^*$  such that  $\rho(h_i) = 1$ . Note that with any root of the form  $-2\alpha_{-1} - k\delta = 2\Lambda_0 - k\delta$ , since  $2\Lambda_0(K) = 2\Lambda_0(h_0 + 2h_1 + 3h_2) = 2$ , we only need to consider  $w\rho - \rho$  up to level 2.

With these results, we now consider partitions  $\tau_i$  of  $-2\alpha_{-1} - 3\delta$  such that  $\dim V(\Lambda_0)_{\tau_i}$  and  $\dim V(\Lambda_1)_{\tau_i}$

Table 5.6:  $w\rho - \rho$  for  $w \in W(S)$

length $k$	$w \in W(S),$ $l(w) = k$	$w\rho - \rho,$ $\alpha\text{-basis}$	$w\rho - \rho,$ $\Lambda\text{-basis}$	level
1	$r_{-1}$	$\begin{aligned} r_{-1}\rho - \rho \\ = \rho - \rho(h_{-1})\alpha_{-1}\rho \\ = -\alpha_{-1} \end{aligned}$	$\begin{aligned} -\alpha_{-1} = \Lambda_0 \\ \end{aligned}$	$\begin{aligned} \Lambda_0(h_0 + 2h_1 + 3h_2) \\ = 1 \end{aligned}$
2	$r_{-1}r_0$	$\begin{aligned} r_{-1}r_0\rho - \rho \\ = r_{-1}(\rho - \rho(h_0)\alpha_0) - \rho \\ = r_{-1}(\rho - \alpha_0) - \rho \\ = r_{-1}\rho - r_{-1}(\alpha_0) - \rho \\ = \rho - \rho(h_{-1})\alpha_{-1} \\ - (\alpha_0 - \alpha_0(h_{-1})\alpha_{-1}) \\ - \rho \\ = \rho - \alpha_{-1} - \alpha_0 - \alpha_{-1} - \rho \\ = -2\alpha_{-1} - \alpha_0 \end{aligned}$	$\begin{aligned} -2\alpha_{-1} - \alpha_0 \\ = 2\Lambda_0 - 2\Lambda_0 \\ + \Lambda_1 - \delta \\ = \Lambda_1 - \delta \end{aligned}$	$\begin{aligned} (\Lambda_1 - \delta)(h_0 + 2h_1 + 3h_2) \\ = 2 \end{aligned}$

$\delta)_{\tau_i}$  are nonzero. For  $V(\Lambda_0)_{\tau_i}$ ,  $\tau_i$  must be of the form:

$$\Lambda_0 - \sum_{i=0}^2 m_i \alpha_i = -\alpha_{-1} - \sum_{i=0}^2 m_i \alpha_i,$$

where  $m_i \in \mathbb{Z}_{\geq 0}$ ,  $i \in \{0, 1, 2\}$ . With the MATLAB program `mult_parts1` (see appendix), we use the path realization  $\mathcal{P}(\Lambda_0)$  to determine which partitions  $\tau_i$  are weights in  $V(\Lambda_0)$  (and hence,  $\dim V(\Lambda_0)_{\tau_i} \neq 0$ ), and their corresponding weight multiplicities, these weights and their mulitplicities are listed in Table 5.7.

Table 5.7:  $\tau_i \in T(-2\alpha_{-1} - 3\delta)$  that are weights in  $V(\Lambda_0)$

$\alpha = \lambda_i + \tau_i$	$\lambda_i$	$\dim V(\Lambda_0)_{\lambda_i}$	$\tau_i$	$\dim V(\Lambda_0)_{\tau_i}$
$-2\alpha_{-1} - 3\delta$	$-\alpha_{-1}$	1	$-\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 3\alpha_2$	4
	$-\alpha_{-1} - \alpha_0$	1	$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 3\alpha_2$	1
	$-\alpha_{-1} - \alpha_0 - \alpha_1$	1	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 3\alpha_2$	1
	$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 2\alpha_2$	1
	$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 2\alpha_2$	2
	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 2\alpha_2$	1
	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - \alpha_2$	1
	$-\alpha_{-1} - \alpha_0 - 4\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - \alpha_2$	1

For  $V(\Lambda_1 - \delta)_{\tau_i}$ ,  $\tau_i$  must be of the form:

$$\Lambda_1 - \delta - \sum_{i=0}^2 m_i \alpha_i = -2\alpha_{-1} - \alpha_0 - \sum_{i=0}^2 m_i \alpha_i,$$

where  $m_i \in \mathbb{Z}_{\geq 0}$ ,  $i \in \{0, 1, 2\}$ . So,  $\alpha = -2\alpha_{-1} - 3\delta$  cannot be partitioned because each weight of  $V(\Lambda_1 - \delta)$  must have a “ $-2\alpha_{-1}$ ” term. So we have now:

$$\begin{aligned} \dim(\mathfrak{g}_{-2\alpha_{-1}-3\delta}) &= \mathcal{B}(-2\alpha_{-1} - 3\delta) \\ &= \sum_{(n_i, \tau_i) \in T(-2\alpha_{-1}-3\delta)} \frac{(\sum n_i - 1)!}{\prod (n_i!)} \Pi K_{\tau_i}^{n_i}, \\ K_{\tau_i} &= \sum_{w \in W(S)} (-1)^{l(w)+1} \dim V(w\rho - \rho)_{\tau_i}. \end{aligned}$$

For  $V(\Lambda_0)$ , each partition consists of two  $n_i$ 's, each equal to 1 (see how  $\tau_i$  must be defined above). Hence, for each  $i$ ,

$$\frac{(\sum n_i - 1)!}{\prod (n_i!)} = \frac{(2-1)!}{1} = 1.$$

For  $V(\Lambda_1 - \delta)$ , there is only one  $n_i = 1$ , so we have:

$$\frac{(\sum n_i - 1)!}{\prod (n_i!)} = \frac{(1-1)!}{1} = 1.$$

So,

$$\begin{aligned} \dim(\mathfrak{g}_{-2\alpha_{-1}-3\delta}) &= \sum_{(n_i, \tau_i) \in T(-2\alpha_{-1}-3\delta)} \Pi K_{\tau_i}^{n_i} \\ &= \sum_{\substack{\lambda_i + \tau_i = -2\alpha_{-1}-3\delta \\ \lambda_i < \tau_i}} ((-1)^2 \dim V(\Lambda_0)_{\lambda_i}) ((-1)^2 \dim V(\Lambda_0)_{\tau_i}) \\ &\quad + (-1)^3 \dim V(\Lambda_1 - \delta)_{-2\alpha_{-1}-3\delta} \\ &= \sum_{\substack{\lambda_i + \tau_i = -2\alpha_{-1}-3\delta \\ \lambda_i < \tau_i}} (\dim V(\Lambda_0)_{\lambda_i}) (\dim V(\Lambda_0)_{\tau_i}) - \dim V(\Lambda_1 - \delta)_{-2\alpha_{-1}-3\delta} \\ &= 12 - \dim V(\Lambda_1 - \delta)_{-2\alpha_{-1}-3\delta}, \end{aligned}$$

using Table 5.7. Now consider  $V(\Lambda_1 - \delta)_{-2\alpha_{-1}-3\delta}$ . We know that, for the one-dimensional module,  $V(\delta)$ ,  $V(\Lambda_1 - \delta) \otimes V(\delta) \cong V(\Lambda_1)$ . So, we can consider weights of  $V(\Lambda_1)$  by counting paths in the path realization  $\mathcal{P}(\Lambda_1)$ . Thus, we wish translate  $\alpha = -2\alpha_{-1} - 3\delta$  into a weight of  $V(\Lambda_1)$ . To do this, we must account for

the tensor product and add a  $\delta$ :

$$\dim V(\Lambda_1 - \delta)_{-2\alpha_{-1} - 3\delta} = \dim V(\Lambda_1)_{-2\alpha_{-1} - 2\delta}.$$

Then to count the multiplicity of this weight by counting paths in the path realization  $\mathcal{P}(\Lambda_1)$ , we rewrite  $\alpha = -2\alpha_{-1} - 2\delta$  as  $\Lambda_1 - \sum_{i=0}^2 m_i \alpha_i$  for some  $m_i \in \mathbb{Z}_{\geq 0}$ ,  $i \in \{0, 1, 2\}$ :

$$\begin{aligned} -2\alpha_{-1} - 2\delta &= 2\Lambda_0 - 2\delta \\ &= \Lambda_1 - (\Lambda_1 - 2\Lambda_0) - 2\delta \\ &= \Lambda_1 - (-\alpha_0 + \delta) - 2\delta \\ &= \Lambda_1 + \alpha_0 - 3\alpha_0 - 6\alpha_1 - 3\alpha_2 \\ &= \Lambda_1 - 2\alpha_0 - 6\alpha_1 - 3\alpha_2 \end{aligned}$$

So, we must find the number of paths with two 0-arrows, six 1-arrows, and three 2-arrows in  $\mathcal{P}(\Lambda_1)$ . We use the MATLAB program perfectcrystalB (see appendix) to find the perfect crystal of level 2 for  $D_4^{(3)}, \mathcal{B}_2$ . We are then able to find the ground state path of weight  $\Lambda_1$ :

$$p_{\Lambda_1} = (\dots, (1, 0, 0, 0, 0, 1), (1, 0, 0, 0, 0, 1)).$$

Using this ground state path with the MATLAB program lev2mult (see appendix) we then construct the path realization  $\mathcal{P}(\Lambda_1)$  (see Figure 4.3) to count these arrows. And we have:

$$\begin{aligned} \dim(\mathfrak{g}_{-2\alpha_{-1} - 3\delta}) &= 12 - \dim V(\Lambda_1 - \delta)_{-2\alpha_{-1} - 3\delta} \\ &= 12 - 8 \\ &= 4 \end{aligned}$$

Next we consider  $k = 4$ , or

$$\dim \mathfrak{g}_{-2\alpha_{-1} - 4\delta}.$$

Again,

$$\dim(g_\alpha) = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \binom{\tau}{\alpha} \mathcal{B}(\tau),$$

and again, we consider  $\tau$  such that  $\tau|\alpha: \alpha|\alpha$  always, and here  $\tau = \frac{\alpha}{2} = -\alpha_{-1} - 2\delta$  divides  $\alpha$  as well. So,

$$\begin{aligned} \dim(\mathfrak{g}_{-2\alpha_{-1}-4\delta}) &= \mu\left(\frac{-2\alpha_{-1}-4\delta}{-2\alpha_{-1}-4\delta}\right)\left(\frac{-2\alpha_{-1}-4\delta}{-2\alpha_{-1}-4\delta}\right)\mathcal{B}(-2\alpha_{-1}-4\delta) \\ &\quad + \mu\left(\frac{-2\alpha_{-1}-4\delta}{-2\alpha_{-1}-2\delta}\right)\left(\frac{-2\alpha_{-1}-2\delta}{-2\alpha_{-1}-4\delta}\right)\mathcal{B}(-2\alpha_{-1}-2\delta) \\ &= \mathcal{B}(-2\alpha_{-1}-4\delta) + \mu(2)\left(\frac{1}{2}\right)\mathcal{B}(-\alpha_{-1}-2\delta) \\ &= \mathcal{B}(-2\alpha_{-1}-4\delta) - \frac{1}{2}\mathcal{B}(-\alpha_{-1}-2\delta) \end{aligned}$$

As before, we consider partitions  $(n_i, \tau_i)$  of  $-2\alpha_{-1}-4\delta$  such that  $\dim V(\Lambda_0)_{\tau_i} \neq 0$  and the partitions  $(n_i, \tau_i)$  such that  $\dim V(\Lambda_1-\delta)_{\tau_i} \neq 0$ . For  $V(\Lambda_0)_{\tau_i}$ ,  $\tau_i$  must be of the form:

$$\Lambda_0 - \sum_{i=0}^2 m_i \alpha_i = -\alpha_{-1} - \sum_{i=0}^2 m_i \alpha_i.$$

For  $V(\Lambda_1-\delta)_{\tau_i}$ ,  $\tau_i$  must be of the form:

$$\Lambda_1 - \delta - \sum_{i=0}^2 m_i \alpha_i = -2\alpha_{-1} - \alpha_0 - \sum_{i=0}^2 m_i \alpha_i.$$

Hence, we have:

$$\begin{aligned} \dim(\mathfrak{g}_{-2\alpha_{-1}-4\delta}) &= \mathcal{B}(-2\alpha_{-1}-4\delta) - \frac{1}{2}\mathcal{B}(-\alpha_{-1}-2\delta) \\ &= \sum_{\substack{\lambda_i+\tau_i=-2\alpha_{-1}-4\delta \\ \lambda_i < \tau_i}} (\dim V(\Lambda_0)_{\lambda_i})(\dim V(\Lambda_0)_{\tau_i}) + \frac{1}{2}(\dim V(\Lambda_0)_{-\alpha_{-1}-2\delta})^2 \\ &\quad - \dim V(\Lambda_1-\delta)_{-2\alpha_{-1}-4\delta} - \frac{1}{2}\dim V(\Lambda_0)_{-\alpha_{-1}-2\delta} \end{aligned}$$

We use the path realization  $\mathcal{P}(\Lambda_0)$  to determine which partitions  $(n_i, \tau_i)$  of  $\alpha = -2\alpha_{-1} - 4\delta$  result in weights of  $V(\Lambda_0)$  (and hence,  $\dim V(\Lambda_0)_{\tau_i} \neq 0$ ), and their corresponding multiplicities. These weights are collected in Table 5.8.

Table 5.8:  $\tau_i \in T(-2\alpha_{-1} - 4\delta)$  that are weights in  $V(\Lambda_0)$

$\alpha = \lambda_i + \tau_i$	$\lambda_i$	$\dim V(\Lambda_0)_{\lambda_i}$	$\tau_i$	$\dim V(\Lambda_0)_{\tau_i}$
$2\alpha_{-1} - 4\delta$	$-\alpha_{-1}$	1	$-\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 4\alpha_2$	6
	$-\alpha_{-1} - \alpha_0$	1	$-\alpha_{-1} - 3\alpha_0 - 8\alpha_1 - 4\alpha_2$	2
	$-\alpha_{-1} - \alpha_0 - \alpha_1$	1	$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 4\alpha_2$	2
	$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 3\alpha_2$	2
	$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 3\alpha_2$	4
	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 3\alpha_2$	2
	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 2\alpha_2$	2
	$-\alpha_{-1} - \alpha_0 - 4\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$	2
	$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 1\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 3\alpha_2$	1
	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 1\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 3\alpha_2$	1
	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 2\alpha_2$	1
	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 2\alpha_2$	2	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 2\alpha_2$	2

With the results from Table 5.8 we then have:

$$\begin{aligned}
\dim(\mathfrak{g}_{-2\alpha_{-1}-4\delta}) &= \sum_{\substack{\lambda_i + \tau_i = -2\alpha_{-1}-4\delta \\ \lambda_i < \tau_i}} (\dim V(\Lambda_0)_{\lambda_i})(\dim V(\Lambda_0)_{\tau_i}) + \frac{1}{2}(\dim V(\Lambda_0)_{-\alpha_{-1}-2\delta})^2 \\
&\quad - \dim V(\Lambda_1 - \delta)_{-2\alpha_{-1}-4\delta} - \frac{1}{2}\dim V(\Lambda_0)_{-\alpha_{-1}-2\delta} \\
&= 25 + \frac{1}{2}(2)^2 - \dim V(\Lambda_1 - \delta)_{-2\alpha_{-1}-4\delta} - \frac{1}{2}\dim V(\Lambda_0)_{-\alpha_{-1}-2\delta} \\
&= 27 - \dim V(\Lambda_1 - \delta)_{-2\alpha_{-1}-4\delta} - \frac{1}{2}\dim V(\Lambda_0)_{-\alpha_{-1}-2\delta}
\end{aligned}$$

Now consider  $V(\Lambda_1 - \delta)_{-2\alpha_{-1}-4\delta}$ . Again, for the one-dimensional module,  $V(\delta)$ ,  $V(\Lambda_1 - \delta) \otimes V(\delta) \cong V(\Lambda_1)$ . So, we can consider weights of  $V(\Lambda_1)$  by counting paths in the path realization  $\mathcal{P}(\Lambda_1)$ . Thus, we wish translate  $\alpha = -2\alpha_{-1} - 4\delta$  into a weight of  $V(\Lambda_1)$ . To do this, we must account for the tensor product and add a  $\delta$ :

$$\dim V(\Lambda_1 - \delta)_{-2\alpha_{-1}-4\delta} = \dim V(\Lambda_1)_{-2\alpha_{-1}-3\delta}.$$

Then to count the multiplicity of this weight by counting paths in the path realization  $\mathcal{P}(\Lambda_1)$ , we rewrite

$\alpha = -2\alpha_{-1} - 3\delta$  as  $\Lambda_1 - \sum_{i=0}^2 m_i \alpha_i$  for some  $m_i \in \mathbb{Z}_{\geq 0}$ ,  $i \in \{0, 1, 2\}$ :

$$\begin{aligned} -2\alpha_{-1} - 3\delta &= 2\Lambda_0 - 3\delta \\ &= \Lambda_1 - (\Lambda_1 - 2\Lambda_0) - 3\delta \\ &= \Lambda_1 - (-\alpha_0 + \delta) - 3\delta \\ &= \Lambda_1 + \alpha_0 - 4\alpha_0 - 8\alpha_1 - 4\alpha_2 \\ &= \Lambda_1 - 3\alpha_0 - 8\alpha_1 - 4\alpha_2 \end{aligned}$$

So, we must find the number of paths with three 0-arrows, eight 1-arrows, and four 2-arrows in  $\mathcal{P}(\Lambda_1)$ . We use the MATLAB program lev2mult (see appendix) to count these arrows. Note that we also have  $\dim V(\Lambda_0)_{-\alpha_{-1}-2\delta} = 2$  from Section 5.3. Thus:

$$\begin{aligned} \dim(\mathfrak{g}_{-2\alpha_{-1}-4\delta}) &= 27 - \dim V(\Lambda_1 - \delta)_{-2\alpha_{-1}-2\delta} - \frac{1}{2} \dim V(\Lambda_0)_{-\alpha_{-1}-2\delta} \\ &= 27 - 17 - \frac{1}{2}(2) \\ &= 9 \end{aligned}$$

In general, we have:

$$\begin{aligned} \dim(\mathfrak{g}_{-2\alpha_{-1}-k\delta}) &= \sum_{\substack{\lambda_i + \tau_i = -2\alpha_{-1}-k\delta \\ \lambda_i < \tau_i}} (\dim V(\Lambda_0)_{\lambda_i})(\dim V(\Lambda_0)_{\tau_i}) + \frac{\delta_{(0,k \bmod 2)}}{2} (\dim V(\Lambda_0)_{-\alpha_{-1}-\frac{k}{2}\delta})^2 \\ &\quad - \dim V(\Lambda_1 - \delta)_{-2\alpha_{-1}-k\delta} - \frac{\delta_{(0,k \bmod 2)}}{2} (\dim V(\Lambda_0)_{-\alpha_{-1}-\frac{k}{2}\delta}). \end{aligned}$$

We use this and the above techniques to find root multiplicities for level 2 roots of the form  $-\alpha_{-1} - k\delta$ ,  $1 \leq k \leq 8$ , collected in Table 5.9. Recall from Section 5.2 Frenkel's conjectured bound  $\text{mult}(\alpha) \leq p^2 \left(1 - \frac{(\alpha|\alpha)}{2}\right)$ ; we include  $\frac{(\alpha|\alpha)}{2}$  and  $p^2 \left(1 - \frac{(\alpha|\alpha)}{2}\right)$  with our root multiplicity calculations. Also recall from Section 5.2, if  $\alpha$  is a real root,  $(\alpha|\alpha) > 0$  and hence  $\left(1 - \frac{(\alpha|\alpha)}{2}\right) \leq 0$ . If  $\alpha$  is a real root, it is known  $\text{mult}(\alpha) = 1$ . So, for simplicity, in our calculations of Frenkel's conjectured bound we define  $p^2(k) = 1$  for  $k < 0$ .

Note here that, taking into account an index shifting, the multiplicities we found for  $-2\alpha_{-1} - 4\delta$ ,  $-2\alpha_{-1} - 5\delta$ , and  $-2\alpha_{-1} - 7\delta$  coincide with the multiplicities found for the same roots using a different technique in [5]. We also note that we recovered all the level 2 root multiplicities found in [5] and detail these results in Table 5.10.

In the interest of providing a more complete list of level 2 roots, we supplement the [5] results with Table 5.11, Table 5.12, Table 5.13, Table 5.14, Table 5.15, and Table 5.16 (note: If multiplicity is 0,  $\alpha$  is not a root). To explore patterns in these additional mutliplicities we factor out multiples of  $\delta$  from the roots

Table 5.9: Root Multiplicities for  $-2\alpha_{-1} - k\delta$ ,  $1 \leq k \leq 15$

$\alpha$	Multiplicity	$\frac{(\alpha \alpha)}{2}$	$p^2 \left(1 - \frac{(\alpha \alpha)}{2}\right)$
$-2\alpha_{-1} - 2\delta$	1	0	2
$-2\alpha_{-1} - 3\delta$	$12 - 8 = 4$	-2	10
$-2\alpha_{-1} - 4\delta$	$25 + \frac{1}{2}(2)^2 - 17 - \frac{1}{2}(2) = 9$	-4	36
$-2\alpha_{-1} - 5\delta$	$59 + 0 - 37 - 0 = 22$	-6	110
$-2\alpha_{-1} - 6\delta$	$112 + \frac{1}{2}(4)^2 - 68 - \frac{1}{2}(4) = 50$	-8	300
$-2\alpha_{-1} - 7\delta$	$224 + 0 - 125 - 0 = 99$	-10	752
$-2\alpha_{-1} - 8\delta$	$411 + \frac{1}{2}(6)^2 - 229 - \frac{1}{2}(6) = 197$	-12	1770
$-2\alpha_{-1} - 9\delta$	$767 + 0 - 390 - 0 = 377$	-14	3956
$-2\alpha_{-1} - 10\delta$	$1303 + \frac{1}{2}(9)^2 - 658 - \frac{1}{2}(9) = 681$	-16	8470
$-2\alpha_{-1} - 11\delta$	$2320 + 0 - 1101 - 0 = 1219$	-18	17490
$-2\alpha_{-1} - 12\delta$	$3782 + \frac{1}{2}(16)^2 - 1774 - \frac{1}{2}(16) = 2128$	-20	35002
$-2\alpha_{-1} - 13\delta$	$6435 + 0 - 2832 - 0 = 3603$	-22	68150
$-2\alpha_{-1} - 14\delta$	$10287 + \frac{1}{2}(22)^2 - 4482 - \frac{1}{2}(22) = 6036$	-24	129512
$-2\alpha_{-1} - 15\delta$	$16864 + 0 - 6930 - 0 = 9934$	-25	240840

we have found and combine this reorganized data with relevant results from Table 5.9 in Table 5.17 and Table 5.18.

Table 5.10: Recovered Level 2 Root Multiplicities from [5]

$\alpha$	Multiplicity	$\frac{(\alpha \alpha)}{2}$	$p^2 \left(1 - \frac{(\alpha \alpha)}{2}\right)$
$-2\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 3\alpha_2$	$19 + 0 - 13 - 0 = 6$	-3	20
$-2\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 4\alpha_2$	$5 + \frac{1}{2} - 4 - \frac{1}{2} = 1$	0	2
$-2\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - 4\alpha_2$	$19 + 0 - 13 = 6$	-3	20
$-2\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 4\alpha_2$	$41 + 0 - 25 - 0 = 16$	-5	65
$-2\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 5\alpha_2$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 5\alpha_2$	$12 + 0 - 8 - 0 = 4$	-2	10
$-2\alpha_{-1} - 4\alpha_0 - 9\alpha_1 - 5\alpha_2$	$19 + 0 - 13 - 0 = 6$	-3	20
$-2\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 5\alpha_2$	$41 + 0 - 25 - 0 = 16$	-5	65
$-2\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 5\alpha_2$	$59 + 0 - 37 - 0 = 22$	-6	110
$-2\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 5\alpha_2$	$82 + 0 - 49 - 0 = 33$	-7	185
$-2\alpha_{-1} - 4\alpha_0 - 10\alpha_1 - 6\alpha_2$	$5 + \frac{1}{2} - 4 - \frac{1}{2} = 1$	0	2
$-2\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 6\alpha_2$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - 4\alpha_0 - 11\alpha_1 - 6\alpha_2$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 6\alpha_2$	$19 + 0 - 13 - 0 = 6$	-3	20
$-2\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 6\alpha_2$	$25 + 2 - 17 - 1 = 9$	-4	36
$-2\alpha_{-1} - 5\alpha_0 - 11\alpha_1 - 6\alpha_2$	$41 + 0 - 25 - 0 = 16$	-5	65
$-2\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 6\alpha_2$	$82 + 0 - 49 - 0 = 33$	-7	185
$-2\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 6\alpha_2$	$112 + 8 - 68 - 2 = 50$	-8	300
$-2\alpha_{-1} - 7\alpha_0 - 12\alpha_1 - 6\alpha_2$	$167 + 0 - 96 - 0 = 71$	-9	481
$-2\alpha_{-1} - 5\alpha_0 - 11\alpha_1 - 7\alpha_2$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 7\alpha_2$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 5\alpha_0 - 12\alpha_1 - 7\alpha_2$	$12 + 0 - 8 - 0 = 4$	-2	10
$-2\alpha_{-1} - 5\alpha_0 - 13\alpha_1 - 7\alpha_2$	$19 + 0 - 13 - 0 = 6$	-3	20
$-2\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 7\alpha_2$	$41 + 0 - 25 - 0 = 16$	-5	65
$-2\alpha_{-1} - 7\alpha_0 - 12\alpha_1 - 7\alpha_2$	$59 + 0 - 37 - 0 = 22$	-6	110
$-2\alpha_{-1} - 6\alpha_0 - 13\alpha_1 - 7\alpha_2$	$82 + 0 - 49 - 0 = 33$	-7	185
$-2\alpha_{-1} - 7\alpha_0 - 13\alpha_1 - 7\alpha_2$	$167 + 0 - 96 - 0 = 71$	-9	481
$-2\alpha_{-1} - 7\alpha_0 - 14\alpha_1 - 7\alpha_2$	$224 + 0 - 125 - 0 = 99$	-10	752
$-2\alpha_{-1} - 8\alpha_0 - 14\alpha_1 - 7\alpha_2$	$314 + 0 - 169 - 0 = 145$	-11	1165
$-2\alpha_{-1} - 6\alpha_0 - 14\alpha_1 - 8\alpha_2$	$25 + 2 - 17 - 1 = 9$	-4	36
$-2\alpha_{-1} - 6\alpha_0 - 15\alpha_1 - 8\alpha_2$	$41 + 0 - 25 - 0 = 16$	-5	65
$-2\alpha_{-1} - 7\alpha_0 - 14\alpha_1 - 8\alpha_2$	$82 + 0 - 49 - 0 = 33$	-7	185
$-2\alpha_{-1} - 7\alpha_0 - 15\alpha_1 - 8\alpha_2$	$167 + 0 - 96 - 0 = 71$	-9	481
$-2\alpha_{-1} - 8\alpha_0 - 15\alpha_1 - 8\alpha_2$	$314 + 0 - 169 - 0 = 145$	-11	1165
$-2\alpha_{-1} - 9\alpha_0 - 16\alpha_1 - 8\alpha_2$	$568 + 0 - 296 - 0 = 272$	-13	2665
$-2\alpha_{-1} - 5\alpha_0 - 14\alpha_1 - 8\alpha_2$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - 6\alpha_0 - 13\alpha_1 - 8\alpha_2$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 7\alpha_0 - 13\alpha_1 - 8\alpha_2$	$19 + 0 - 13 - 0 = 6$	-3	20
$-2\alpha_{-1} - 8\alpha_0 - 14\alpha_1 - 8\alpha_2$	$112 + 8 - 68 - 2 = 50$	-8	300
$-2\alpha_{-1} - 8\alpha_0 - 16\alpha_1 - 8\alpha_2$	$411 + 18 - 229 - 3 = 197$	-12	1770

Table 5.11: Additional Level 2 Root Multiplicities (Part 1)

$\alpha$	Multiplicity
$-2\alpha_{-1} - \alpha_0$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - \alpha_0 - \alpha_1$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 2\alpha_0 - \alpha_1$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 2\alpha_0 - \alpha_1 - \alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 2\alpha_0 - 2\alpha_1$	$0 + \frac{1}{2} - 0 - \frac{1}{2} = 0$
$-2\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - \alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 2\alpha_2$	$0 + \frac{1}{2} - 0 - \frac{1}{2} = 0$
$-2\alpha_{-1} - 2\alpha_0 - 3\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - \alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 2\alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 3\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 2\alpha_0 - 4\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - \alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - \alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - \alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 2\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 2\alpha_1 - \alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 2\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 3\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 3\alpha_1 - \alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 3\alpha_0 - 3\alpha_1 - 2\alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 3\alpha_0 - 3\alpha_1 - 3\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 4\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - \alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$	$7 + 0 - 5 - 0 = 2$
$-2\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - 3\alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - 4\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 5\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 2\alpha_2$	$7 + 0 - 5 - 0 = 2$
$-2\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 3\alpha_2$	$7 + 0 - 5 - 0 = 2$
$-2\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 4\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 5\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 6\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 2\alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 4\alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 5\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 6\alpha_2$	$0 + 0 - 0 - 0 = 0$

Table 5.12: Additional Level 2 Root Multiplicities (Part 2)

$\alpha$	Multiplicity
$-2\alpha_{-1} - 4\alpha_0 - \alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - \alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 2\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 2\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 2\alpha_1 - 2\alpha_2$	$0 + \frac{1}{2} - 0 - \frac{1}{2} = 0$
$-2\alpha_{-1} - 4\alpha_0 - 3\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 3\alpha_1 - \alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 3\alpha_1 - 2\alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 3\alpha_1 - 3\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 4\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 4\alpha_1 - \alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 4\alpha_1 - 2\alpha_2$	$5 + \frac{1}{2} - 4 - \frac{1}{2} = 1$
$-2\alpha_{-1} - 4\alpha_0 - 4\alpha_1 - 3\alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 4\alpha_1 - 4\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 5\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 5\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 5\alpha_1 - 2\alpha_2$	$7 + 0 - 5 - 0 = 2$
$-2\alpha_{-1} - 4\alpha_0 - 5\alpha_1 - 3\alpha_2$	$7 + 0 - 5 - 0 = 2$
$-2\alpha_{-1} - 4\alpha_0 - 5\alpha_1 - 4\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 5\alpha_1 - 5\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 6\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 2\alpha_2$	$5 + \frac{1}{2} - 4 - \frac{1}{2} = 1$
$-2\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 3\alpha_2$	$19 + 0 - 13 - 0 = 6$
$-2\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 4\alpha_2$	$5 + \frac{1}{2} - 4 - \frac{1}{2} = 1$
$-2\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 5\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 6\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 7\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - 2\alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - 3\alpha_2$	$19 + 0 - 13 - 0 = 6$
$-2\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - 4\alpha_2$	$19 + 0 - 13 - 0 = 6$
$-2\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - 5\alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - 6\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - 7\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 8\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 3\alpha_2$	$7 + 0 - 5 - 0 = 2$
$-2\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 5\alpha_2$	$7 + 0 - 5 - 0 = 2$

Table 5.13: Additional Level 2 Root Multiplicities (Part 3)

$\alpha$	Multiplicity
$-2\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 6\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 7\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 8\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - \alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - \alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 2\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 2\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 2\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 3\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 3\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 3\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 3\alpha_1 - 3\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 4\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 4\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 4\alpha_1 - 2\alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 4\alpha_1 - 3\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 4\alpha_1 - 4\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 5\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 5\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 5\alpha_1 - 2\alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 5\alpha_0 - 5\alpha_1 - 3\alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 5\alpha_0 - 5\alpha_1 - 4\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 5\alpha_1 - 5\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 6\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 6\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 6\alpha_1 - 2\alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 5\alpha_0 - 6\alpha_1 - 3\alpha_2$	$12 + 0 - 8 - 0 = 4$
$-2\alpha_{-1} - 5\alpha_0 - 6\alpha_1 - 4\alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 5\alpha_0 - 6\alpha_1 - 5\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 6\alpha_1 - 6\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 7\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 7\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 7\alpha_1 - 2\alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 7\alpha_1 - 3\alpha_2$	$19 + 0 - 13 - 0 = 6$
$-2\alpha_{-1} - 5\alpha_0 - 7\alpha_1 - 4\alpha_2$	$19 + 0 - 13 - 0 = 6$
$-2\alpha_{-1} - 5\alpha_0 - 7\alpha_1 - 5\alpha_2$	$1 + 0 - 1 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 7\alpha_1 - 6\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 7\alpha_1 - 7\alpha_2$	$0 + 0 - 0 - 0 = 0$

Table 5.14: Additional Level 2 Root Multiplicities (Part 4)

$\alpha$	Multiplicity
$-2\alpha_{-1} - 5\alpha_0 - 8\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 3\alpha_2$	$12 + 0 - 8 - 0 = 4$
$-2\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 4\alpha_2$	$41 + 0 - 25 - 0 = 16$
$-2\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 5\alpha_2$	$12 + 0 - 8 - 0 = 4$
$-2\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 6\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 7\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 8\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 9\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 3\alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 4\alpha_2$	$41 + 0 - 25 - 0 = 16$
$-2\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 5\alpha_2$	$41 + 0 - 25 - 0 = 16$
$-2\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 6\alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 7\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 8\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 9\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 10\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 3\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 4\alpha_2$	$19 + 0 - 13 - 0 = 6$
$-2\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 6\alpha_2$	$19 + 0 - 13 - 0 = 6$
$-2\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 7\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 8\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 9\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 10\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - \alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - \alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 2\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 2\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 2\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 3\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 3\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 3\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 3\alpha_1 - 3\alpha_2$	$0 + 0 - 0 - 0 = 0$

Table 5.15: Additional Level 2 Root Multiplicities (Part 5)

$\alpha$	Multiplicity
$-2\alpha_{-1} - 6\alpha_0 - 4\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 4\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 4\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 4\alpha_1 - 3\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 4\alpha_1 - 4\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 5\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 5\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 5\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 5\alpha_1 - 3\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 5\alpha_1 - 4\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 5\alpha_1 - 5\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 6\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 6\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 6\alpha_1 - 2\alpha_2$	$0 + \frac{1}{2} - 0 - \frac{1}{2} = 0$
$-2\alpha_{-1} - 6\alpha_0 - 6\alpha_1 - 3\alpha_2$	$3 + 0 - 2 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 6\alpha_1 - 4\alpha_2$	$0 + \frac{1}{2} - 0 - \frac{1}{2} = 0$
$-2\alpha_{-1} - 6\alpha_0 - 6\alpha_1 - 5\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 6\alpha_1 - 6\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 7\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 7\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 7\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 7\alpha_1 - 3\alpha_2$	$7 + 0 - 5 - 0 = 2$
$-2\alpha_{-1} - 6\alpha_0 - 7\alpha_1 - 4\alpha_2$	$7 + 0 - 5 - 0 = 2$
$-2\alpha_{-1} - 6\alpha_0 - 7\alpha_1 - 5\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 7\alpha_1 - 6\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 7\alpha_1 - 7\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 8\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - 3\alpha_2$	$7 + 0 - 5 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - 4\alpha_2$	$25 + \frac{1}{2}(2)^2 - 17 - \frac{1}{2}(2) = 9$
$-2\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - 5\alpha_2$	$7 + 0 - 5 - 0 = 2$
$-2\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - 6\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - 7\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - 8\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 9\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 3\alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 4\alpha_2$	$41 + 0 - 25 - 0 = 16$

Table 5.16: Additional Level 2 Root Multiplicities (Part 6)

$\alpha$	Multiplicity
$-2\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 5\alpha_2$	$41 + 0 - 25 - 0 = 16$
$-2\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 6\alpha_2$	$3 + 0 - 2 - 0 = 1$
$-2\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 7\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 8\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 9\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 10\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 3\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 4\alpha_2$	$25 + \frac{1}{2}(2)^2 - 17 - \frac{1}{2}(2) = 9$
$-2\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 5\alpha_2$	$82 + 0 - 49 - 0 = 33$
$-2\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 6\alpha_2$	$25 + \frac{1}{2}(2)^2 - 17 - \frac{1}{2}(2) = 9$
$-2\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 7\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 8\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 9\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 10\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 11\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 3\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 4\alpha_2$	$7 + 0 - 5 - 0 = 2$
$-2\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 5\alpha_2$	$82 + 0 - 49 - 0 = 33$
$-2\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 6\alpha_2$	$82 + 0 - 49 - 0 = 33$
$-2\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 7\alpha_2$	$7 + 0 - 5 - 0 = 2$
$-2\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 8\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 9\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 10\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 11\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 12\alpha_1$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - \alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 2\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 3\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 4\alpha_2$	$0 + \frac{1}{2} - 0 - \frac{1}{2} = 0$
$-2\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 5\alpha_2$	$41 + 0 - 25 - 0 = 16$
$-2\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 7\alpha_2$	$41 + 0 - 25 - 0 = 16$
$-2\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 8\alpha_2$	$0 + \frac{1}{2} - 0 - \frac{1}{2} = 0$
$-2\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 9\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 10\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 11\alpha_2$	$0 + 0 - 0 - 0 = 0$
$-2\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 12\alpha_2$	$0 + 0 - 0 - 0 = 0$

Table 5.17:  $\delta$ -factored Level 2 Root Multiplicities (Part 1)

$\alpha$	Multiplicity	$\frac{(\alpha \alpha)}{2}$	$p^2 \left(1 - \frac{(\alpha \alpha)}{2}\right)$
$-2\alpha_{-1} - \delta - \alpha_0$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - \delta - \alpha_0 - \alpha_1$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - \delta - \alpha_0 - \alpha_1 - \alpha_2$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - \delta - 2\alpha_0 - \alpha_1$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - \delta - 2\alpha_0 - \alpha_1 - \alpha_2$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - 2\delta$	1	0	2
$-2\alpha_{-1} - 2\delta - \alpha_0$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 2\delta - \alpha_0 - \alpha_1$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 2\delta - \alpha_0 - \alpha_1 - \alpha_2$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 2\delta - \alpha_0 - 2\alpha_1$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - 2\delta - 2\alpha_0$	$5 + \frac{1}{2} - 4 - \frac{1}{2} = 1$	0	2
$-2\alpha_{-1} - 2\delta - 2\alpha_0 - \alpha_1$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 2\delta - 2\alpha_0 - \alpha_1 - \alpha_2$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 2\delta - 2\alpha_0 - 2\alpha_1$	$5 + \frac{1}{2} - 4 - \frac{1}{2} = 1$	0	2
$-2\alpha_{-1} - 2\delta - 3\alpha_0 - \alpha_1$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - 2\delta - 3\alpha_0 - \alpha_1 - \alpha_2$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - 2\delta - 3\alpha_0 - 2\alpha_1$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - 3\delta$	$12 + 0 - 8 - 0 = 4$	-2	10
$-2\alpha_{-1} - 3\delta - \alpha_2$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - 3\delta - \alpha_0$	$19 + 0 - 13 - 0 = 6$	-3	20
$-2\alpha_{-1} - 3\delta - \alpha_0 - \alpha_2$	$5 + \frac{1}{2} - 4 - \frac{1}{2} = 1$	0	2
$-2\alpha_{-1} - 3\delta - \alpha_0 - \alpha_1$	$19 + 0 - 13 - 0 = 6$	-3	20
$-2\alpha_{-1} - 3\delta - \alpha_0 - \alpha_1 - \alpha_2$	$19 + 0 - 13 - 0 = 6$	-3	20
$-2\alpha_{-1} - 3\delta - \alpha_0 - 2\alpha_1$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 3\delta - 2\alpha_0$	$12 + 0 - 8 - 0 = 4$	-2	10
$-2\alpha_{-1} - 3\delta - 2\alpha_0 - \alpha_2$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - 3\delta - 2\alpha_0 - \alpha_1$	$19 + 0 - 13 - 0 = 6$	-3	20
$-2\alpha_{-1} - 3\delta - 2\alpha_0 - \alpha_1 - \alpha_2$	$19 + 0 - 13 - 0 = 6$	-3	20
$-2\alpha_{-1} - 3\delta - 2\alpha_0 - 2\alpha_1$	$12 + 0 - 8 - 0 = 4$	-2	10
$-2\alpha_{-1} - 3\delta - 2\alpha_0 - 3\alpha_1$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - 3\delta - 3\alpha_0 - \alpha_1$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 3\delta - 3\alpha_0 - \alpha_1 - \alpha_2$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 3\delta - 3\alpha_0 - 3\alpha_1$	$3 + 0 - 2 - 0 = 1$	1	1

Table 5.18:  $\delta$ -factored Level 2 Root Multiplicities (Part 2)

$\alpha$	Multiplicity	$\frac{(\alpha \alpha)}{2}$	$p^2 \left(1 - \frac{(\alpha \alpha)}{2}\right)$
$-2\alpha_{-1} - 4\delta$	$25 + \frac{1}{2}(2)^2 - 17 - \frac{1}{2}(2) = 9$	-4	36
$-2\alpha_{-1} - 4\delta - \alpha_2$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 4\delta - \alpha_0$	$41 + 0 - 25 - 0 = 16$	-5	65
$-2\alpha_{-1} - 4\delta - \alpha_0 - \alpha_2$	$12 + 0 - 8 - 0 = 4$	-2	10
$-2\alpha_{-1} - 4\delta - \alpha_0 - \alpha_1$	$41 + 0 - 25 - 0 = 16$	-5	65
$-2\alpha_{-1} - 4\delta - \alpha_0 - \alpha_1 - \alpha_2$	$41 + 0 - 25 - 0 = 16$	-5	65
$-2\alpha_{-1} - 4\delta - \alpha_0 - \alpha_1 - 2\alpha_2$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - 4\delta - \alpha_0 - 2\alpha_1$	$19 + 0 - 13 - 0 = 6$	-3	20
$-2\alpha_{-1} - 4\delta - 2\alpha_0$	$25 + \frac{1}{2}(2)^2 - 17 - \frac{1}{2}(2) = 9$	-4	36
$-2\alpha_{-1} - 4\delta - 2\alpha_0 - \alpha_2$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 4\delta - 2\alpha_0 - \alpha_1$	$41 + 0 - 25 - 0 = 16$	-5	65
$-2\alpha_{-1} - 4\delta - 2\alpha_0 - \alpha_1 - \alpha_2$	$41 + 0 - 25 - 0 = 16$	-5	65
$-2\alpha_{-1} - 4\delta - 2\alpha_0 - \alpha_1 - 2\alpha_2$	$3 + 0 - 2 - 0 = 1$	1	1
$-2\alpha_{-1} - 4\delta - 2\alpha_0 - 2\alpha_1$	$25 + \frac{1}{2}(2)^2 - 17 - \frac{1}{2}(2) = 9$	-4	36
$-2\alpha_{-1} - 4\delta - 2\alpha_0 - 2\alpha_1 - \alpha_2$	$82 + 0 - 49 - 0 = 33$	-7	185
$-2\alpha_{-1} - 4\delta - 2\alpha_0 - 3\alpha_1$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 5\delta$	$59 + 0 - 37 - 0 = 22$	-6	110
$-2\alpha_{-1} - 5\delta - \alpha_2$	$19 + 0 - 13 - 0 = 6$	-3	20
$-2\alpha_{-1} - 5\delta - \alpha_0 - \alpha_2$	$25 + \frac{1}{2}(2)^2 - 17 - \frac{1}{2}(2) = 9$	-4	36
$-2\alpha_{-1} - 5\delta - \alpha_0 - \alpha_1$	$82 + 0 - 49 - 0 = 33$	-7	185
$-2\alpha_{-1} - 5\delta - \alpha_0 - \alpha_1 - \alpha_2$	$82 + 0 - 49 - 0 = 33$	-7	185
$-2\alpha_{-1} - 5\delta - \alpha_0 - \alpha_1 - 2\alpha_2$	$7 + 0 - 5 - 0 = 2$	-1	5
$-2\alpha_{-1} - 5\delta - \alpha_0 - 2\alpha_1$	$41 + 0 - 25 - 0 = 16$	-5	65
$-2\alpha_{-1} - 6\delta$	$112 + \frac{1}{2}(4)^2 - 68 - \frac{1}{2}(4) = 50$	-8	300
$-2\alpha_{-1} - 6\delta - \alpha_2$	$41 + 0 - 25 - 0 = 16$	-5	65

## 5.5 Level 3 Root Multiplicities

By our Theorem 5.1, we have the following:

$$\dim \mathfrak{g}_{-3\alpha_1 - \delta} = 0$$

$$\dim \mathfrak{g}_{-3\alpha_1 - 2\delta} = 0$$

$$\dim \mathfrak{g}_{-3\alpha_1 - 3\delta} = 2$$

Now, consider  $k = 4$ , or

$$\dim \mathfrak{g}_{-3\alpha_1 - 4\delta}$$

As with level 2 multiplicities, we again use Kang's multiplicity formula:

$$\dim(\mathfrak{g}_\alpha) = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \left(\frac{\tau}{\alpha}\right) \mathcal{B}(\tau).$$

Consider  $\tau$  such that  $\tau|\alpha$ . Here, we cannot use  $\tau = \frac{\alpha}{2}$  because  $-\frac{3}{2}\alpha_1 - \frac{k}{2}\delta$  is not a root. Note that  $\tau = \frac{\alpha}{3}$  is possible, but only if  $k$  is divisible by 3. So, for  $k = 4$  we only need to consider  $\tau = \alpha$ :

$$\begin{aligned} \dim \mathfrak{g}_{-3\alpha_1 - 4\delta} &= \mu\left(\frac{-3\alpha_1 - 4\delta}{-3\alpha_1 - 4\delta}\right) \left(\frac{-3\alpha_1 - 4\delta}{-3\alpha_1 - 4\delta}\right) \mathcal{B}(-3\alpha_1 - 4\delta) \\ &= \mathcal{B}(-3\alpha_1 - 4\delta) \\ &= \sum_{(n_i, \tau_i) \in T(-3\alpha_1 - 4\delta)} \frac{(\sum_i n_i - 1)!}{\prod (n_i!)} \prod K_{\tau_i}^{n_i} \end{aligned}$$

With,

$$\begin{aligned} T(-3\alpha_1 - 4\delta) &= \{(n_i, \tau_i) \mid n_i \in \mathbb{Z}_{\geq 0}, \sum n_i \tau_i = -3\alpha_1 - 4\delta\}, \\ K_{\tau_i} &= \sum_{w \in W(S)} (-1)^{l(w)+1} \dim V(w\rho - \rho)_{\tau_i} \end{aligned}$$

Table 5.19: Level 3  $w\rho - \rho$  for  $w \in W(S)$

length $k$	$w \in W(S),$ $l(w) = k$	$w\rho - \rho,$ $\alpha\text{-basis}$	$w\rho - \rho,$ $\Lambda\text{-basis}$	level
3	$r_{-1}r_0r_1$	$\begin{aligned} & r_{-1}r_0r_1\rho - \rho \\ &= r_{-1}r_0(\rho - \alpha_1) - \rho \\ &= r_{-1}(r_0\rho - r_0\alpha_1) - \rho \\ &= r_{-1}(\rho - \rho(h_0)\alpha_0 - \alpha_1 \\ &\quad + \alpha_1(h_0)\alpha_0) - \rho \\ &= r_{-1}(\rho - \alpha_0 - \alpha_1 \\ &\quad - \alpha_0) - \rho \\ &= r_{-1}\rho - 2r_{-1}\alpha_0 \\ &\quad - r_{-1}\alpha_1 - \rho \\ &= \rho - \rho(h_{-1})\alpha_{-1} \\ &\quad - 2\alpha_0 + 2\alpha_0(h_{-1})\alpha_{-1} \\ &\quad - \alpha_1 + \alpha_1(h_{-1})\alpha_{-1} - \rho \\ &= -\alpha_{-1} - 2\alpha_0 - 2\alpha_{-1} - \alpha_1 \\ &= -3\alpha_{-1} - 2\alpha_0 - \alpha_1 \end{aligned}$	$\begin{aligned} & -3\alpha_{-1} - 2\alpha_0 - \alpha_1 \\ &= 3\Lambda_0 \\ &= -2(2\Lambda_0 - \Lambda_1 + \delta) \\ &= -(-\Lambda_0 + 2\Lambda_1 - \Lambda_2) \\ &= 3\Lambda_0 - 4\Lambda_0 + 2\Lambda_1 \\ &= -2\delta + \Lambda_0 - 2\Lambda_1 + \Lambda_2 \\ &= \Lambda_2 - 2\delta \end{aligned}$	$(\Lambda_2 - 2\delta)(K)$ $= 3$

In addition to those  $w \in W(S)$  (and the corresponding  $V(w\rho - \rho)$ ) determined for the level 2 case, we also find  $w \in W(S)$  such that the level of  $(w\rho - \rho)$  is 3:

$$\begin{aligned}
w &= r_{-1}r_0r_1 \text{ and } l(w) = l(r_{-1}r_0r_1) = l(r_{-1}r_0) + 1 = 3 \\
w &= r_{-1}r_0r_1 \in W(S) \text{ since } w' = r_{-1}r_0 \in W(S) \text{ and} \\
r_{-1}r_0(\alpha_1) &= r_{-1}(\alpha_1 - \alpha_1(h_0)\alpha_0) \\
&= r_{-1}(\alpha_1 - (-1)\alpha_0) \\
&= r_{-1}(\alpha_1 + \alpha_0) \\
&= \alpha_1 - \alpha_1(h_{-1})\alpha_{-1} + \alpha_0 - \alpha_0(h_{-1})\alpha_{-1} \\
&= \alpha_1 - 0\alpha_{-1} + \alpha_0 - (-\alpha_{-1}) \\
&= \alpha_1 + \alpha_0 + \alpha_{-1} \in \Delta^+(S)
\end{aligned}$$

Notice for  $w = r_{-1}r_0r_{-1}$ ,  $r_{-1}r_0(\alpha_{-1}) \notin \Delta^+(S)$ , and for  $w = r_{-1}r_0r_2$ ,  $r_{-1}r_0(\alpha_2) \notin \Delta^+(S)$ , so  $w = r_{-1}r_0r_1$  is the only  $w \in W(S)$  of length 3. We then add this  $w$  to the findings collected in Table 5.6 from the previous section; see Table 5.19.

We also note, as in the level 2 case, for partitions  $\tau_i$  of  $3\alpha_{-1} - 4\delta$  to be possible weights of  $V(\Lambda_0)$ ,  $\tau_i$

must be of the form:

$$\Lambda_0 - \sum_{i=0}^2 m_i \alpha_i = -\alpha_{-1} - \sum_{i=0}^2 m_i \alpha_i,$$

where  $m_i \in \mathbb{Z}_{\geq 0}$ ,  $i \in \{0, 1, 2\}$ . For  $\tau_i$  to be a possible weight of  $V(\Lambda_1 - \delta)$ ,  $\tau_i$  must be of the form:

$$\Lambda_1 - \delta - \sum_{i=0}^2 m_i \alpha_i = -2\alpha_{-1} - \alpha_0 - (m_0 - 1)\alpha_0 - m_1 \alpha_1 - m_2 \alpha_2,$$

where  $(m_0 - 1), m_1, m_2 \in \mathbb{Z}_{\geq 0}$ . Lastly, for  $\tau_i$  to be a possible weight of  $V(\Lambda_2 - 2\delta)$ ,  $\tau_i$  must be of the form:

$$\Lambda_2 - 2\delta - \sum_{i=0}^2 m_i \alpha_i = -3\alpha_{-1} - 2\alpha_0 - \alpha_1 - (m_0 - 2)\alpha_0 - (m_1 - 1)\alpha_1 - m_2 \alpha_2,$$

where  $(m_0 - 2), (m_1 - 1), m_2 \in \mathbb{Z}_{\geq 0}$ .

So we now have the following: for  $m_i \in \mathbb{Z}_{\geq 0}$ ,  $i \in \{0, 1, 2\}$ ,  $\tilde{m}_0 \in \mathbb{Z}_{\geq 1}$ , and  $\hat{m}_0 \in \mathbb{Z}_{\geq 2}$ ,  $\hat{m}_1 \in \mathbb{Z}_{\geq 1}$ :

$$\begin{aligned} \dim \mathfrak{g}_{-3\alpha_{-1}-4\delta} &= \sum_{(n_i, \tau_i) \in T(-3\alpha_{-1}-4\delta)} \frac{(\sum_i n_i - 1)!}{\prod(n_i!)} \Pi K_{\tau_i}^{n_i} \\ &= \sum_{(n_i, \tau_i) \in T(-3\alpha_{-1}-4\delta)} \frac{(\sum_i n_i - 1)!}{\prod(n_i!)} \Pi \left( \sum_{w \in W(S)} (-1)^{l(w)+1} \dim V(w\rho - \rho)_{\tau_i} \right)^{n_i} \\ &= \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1}-4\delta) \\ \tau_i = -\alpha_{-1} - \sum_{i=0}^2 m_i \alpha_i}} \frac{(\sum_i n_i - 1)!}{\prod(n_i!)} \Pi (\dim V(\Lambda_0)_{\tau_i} - \dim V(\Lambda_1 - \delta)_{\tau_i} + \dim V(\Lambda_2 - 2\delta)_{\tau_i})^{n_i} \\ &\quad + \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1}-4\delta) \\ \tau_1 = -\alpha_{-1} - \sum_{i=0}^2 m_i \alpha_i \\ \tau_2 = -2\alpha_{-1} - \tilde{m}_0 \alpha_0 - m_1 \alpha_1 - m_2 \alpha_2}} (\dim V(\Lambda_0)_{\tau_1} - \dim V(\Lambda_1 - \delta)_{\tau_1} + \dim V(\Lambda_2 - 2\delta)_{\tau_1}) \\ &\quad \cdot (\dim V(\Lambda_0)_{\tau_2} - \dim V(\Lambda_1 - \delta)_{\tau_2} + \dim V(\Lambda_2 - 2\delta)_{\tau_2}) \\ &\quad + \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1}-4\delta) \\ \tau_1 = -3\alpha_{-1} - \hat{m}_0 \alpha_0 - \hat{m}_1 \alpha_1 - m_2 \alpha_2}} (\dim V(\Lambda_0)_{\tau_1} - \dim V(\Lambda_1 - \delta)_{\tau_1} + \dim V(\Lambda_2 - 2\delta)_{\tau_1}) \\ &= \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1}-4\delta) \\ \tau_i = -\alpha_{-1} - \sum_{i=0}^2 m_i \alpha_i}} \frac{(\sum_i n_i - 1)!}{\prod(n_i!)} \Pi (\dim V(\Lambda_0)_{\tau_i})^{n_i} \\ &\quad + \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1}-4\delta) \\ \tau_1 = -\alpha_{-1} - \sum_{i=0}^2 m_i \alpha_i \\ \tau_2 = -2\alpha_{-1} - \tilde{m}_0 \alpha_0 - m_1 \alpha_1 - m_2 \alpha_2}} (\dim V(\Lambda_0)_{\tau_1}) (-\dim V(\Lambda_1 - \delta)_{\tau_2}) \end{aligned}$$

$$+ \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1} - 4\delta) \\ \tau_1 = -3\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2}} (\dim V(\Lambda_2 - 2\delta)_{\tau_1})$$

We use the path realizations of  $\mathcal{P}(\Lambda_0)$ ,  $\mathcal{P}(\Lambda_1)$ , and  $\mathcal{P}(\Lambda_2)$  to determine which partitions  $\tau_i$  are weights in  $V(\Lambda_0)$ ,  $V(\Lambda_1 - \delta)$ , and  $V(\Lambda_2 - 2\delta)$ , respectively. That is,  $\dim V(\Lambda_0)_{\tau_i} \neq 0$ ,  $V(\Lambda_1 - \delta)_{\tau_i} \neq 0$ , and  $V(\Lambda_2 - 2\delta)_{\tau_i} \neq 0$ . We begin first by collecting the partitions  $\tau_i \in T(-3\alpha_{-1} - 4\delta)$  that are weights in  $V(\Lambda_0)$  with the MATLAB program sumparts (see appendix); these results are organized in Table 5.20.

Table 5.20:  $\tau_i \in T(-3\alpha_{-1} - 4\delta)$  that are weights in  $V(\Lambda_0)$ 

$\tau_1$	$dimV(\Lambda_0)_{\tau_1}$	$\tau_2$	$dimV(\Lambda_0)_{\tau_2}$	$\tau_3$	$dimV(\Lambda_0)_{\tau_3}$
$-\alpha_{-1}$	1	$-\alpha_{-1}$	1	$-\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 4\alpha_2$	6
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0$	1	$-\alpha_{-1} - 3\alpha_0 - 8\alpha_1 - 4\alpha_2$	2
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - \alpha_1$	1	$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 4\alpha_2$	2
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 3\alpha_2$	2
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 3\alpha_2$	4
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 3\alpha_2$	2
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 2\alpha_2$	2
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - 4\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$	2
$-\alpha_{-1}$	1	$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 3\alpha_2$	1
$-\alpha_{-1}$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 3\alpha_2$	1
$-\alpha_{-1}$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 2\alpha_2$	1
$-\alpha_{-1}$	1	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 2\alpha_2$	2	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 2\alpha_2$	2
$-\alpha_{-1} - \alpha_0$	1	$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 3\alpha_2$	1
$-\alpha_{-1} - \alpha_0$	1	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 3\alpha_2$	1
$-\alpha_{-1} - \alpha_0$	1	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 2\alpha_2$	1
$-\alpha_{-1} - \alpha_0$	1	$-\alpha_{-1} - \alpha_0 - 4\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 2\alpha_2$	2
$-\alpha_{-1} - \alpha_0 - \alpha_1$	1	$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 3\alpha_2$	1
$-\alpha_{-1} - \alpha_0 - \alpha_1$	1	$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 3\alpha_2$	1
$-\alpha_{-1} - \alpha_0 - \alpha_1$	1	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 2\alpha_2$	2
$-\alpha_{-1} - \alpha_0 - \alpha_1$	1	$-\alpha_{-1} - \alpha_0 - 4\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 2\alpha_2$	1
$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	1	$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 2\alpha_2$	1
$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	1	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 2\alpha_2$	2
$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	1	$-\alpha_{-1} - \alpha_0 - 4\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - \alpha_2$	1
$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 2\alpha_2$	2
$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 2\alpha_2$	1
$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - \alpha_2$	1
$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - \alpha_0 - 4\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - \alpha_2$	1
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - \alpha_2$	1	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - \alpha_2$	1

Now, we consider each  $\tau_i$  that are weights of  $V(\Lambda_0)$  and subtract each from  $\tau = -3\alpha_{-1} - 4\delta$  to obtain  $\tau_i$  of the form:

$$-2\alpha_{-1} - \tilde{m}_0\alpha_0 - \sum_{i=1}^2 m_i\alpha_i$$

So, given a  $\tau_i$  that is a weight of  $V(\Lambda_0)$ , if  $(-3\alpha_{-1} - 4\delta) - \tau_i = -2\alpha_{-1} - \tilde{m}_0\alpha_0 - \sum_{i=1}^2 m_i\alpha_i$ , to count the multiplicity of  $-2\alpha_{-1} - \tilde{m}_0\alpha_0 - \sum_{i=1}^2 m_i\alpha_i$  in  $V(\Lambda_1 - \delta)$  we must determine the number of paths with  $(\tilde{m}_0 - 1)$  0-arrows,  $m_1$  1-arrows and  $m_2$  2-arrows in  $\mathcal{P}(\Lambda_1)$ . We use the MATLAB program sumparts (see appendix) to find such mutiplicities and collect these results in Table 5.21.

Lastly, we must determine  $\dim V(\Lambda_2 - 2\delta)_{\tau_i}$ . Since each weight of  $\dim V(\Lambda_2 - 2\delta)$  must have a “ $-3\alpha_{-1}$ ” term,  $\tau = -3\alpha_{-1} - 4\delta$  cannot be partitioned further. Thus, we must simply determine  $\dim V(\Lambda_2 - 2\delta)_{-3\alpha_{-1}-4\delta}$ . We note that for the one-dimensional module  $V(2\delta)$ ,  $V(\Lambda_2 - 2\delta) \otimes V(2\delta) \cong V(\Lambda_2)$ . So, we can consider weights of  $V(\Lambda_2)$  by counting paths in the path realization  $\mathcal{P}(\Lambda_2)$ . Thus, we wish translate  $\alpha = -3\alpha_{-1} - 4\delta$  into a weight of  $V(\Lambda_2)$ . To do this, we must account for the tensor product and add  $2\delta$ :

$$\dim V(\Lambda_2 - 2\delta)_{-3\alpha_{-1}-4\delta} = \dim V(\Lambda_2)_{-3\alpha_{-1}-2\delta}.$$

Then to count the multiplicity of this weight by counting paths in the path realization  $\mathcal{P}(\Lambda_2)$ , we rewrite  $\alpha = -3\alpha_{-1} - 2\delta$  as  $\Lambda_2 - \sum_{i=0}^2 m_i\alpha_i$  for some  $m_i \in \mathbb{Z}_{\geq 0}$ ,  $i \in \{0, 1, 2\}$ :

$$\begin{aligned} -3\alpha_{-1} - 2\delta &= 3\Lambda_0 - 2\delta \\ &= \Lambda_2 - (\Lambda_2 - 3\Lambda_0) - 2\delta \\ &= \Lambda_2 - (-3\alpha_{-1} - 2\alpha_0 - \alpha_1 + 2\delta + 3\alpha_{-1}) - 2\delta \\ &= \Lambda_2 + 2\alpha_0 + \alpha_1 - 4\delta \\ &= \Lambda_2 - 2\alpha_0 - 7\alpha_1 - 4\alpha_2 \end{aligned}$$

So, we must find the number of paths with two 0-arrows, seven 1-arrows, and four 2-arrows in  $\mathcal{P}(\Lambda_2)$ . We use the MATLAB program perfectcrystalB to find the perfect crystal of level 3 of  $D_4^{(3)}$ ,  $\mathcal{B}_3$ , and use this information find the ground state path of weight  $\Lambda_2$ :

$$p_{\Lambda_2} = (\dots, (0, 1, 1, 1, 1, 0), (0, 1, 1, 1, 1, 0)).$$

Using this ground state path with the MATLAB program lev3mult (see appendix) we then construct the path realization  $\mathcal{P}(\Lambda_2)$  to count these arrows and find

$$\dim V(\Lambda_2)_{-3\alpha_{-1}-2\delta} = \dim V(\Lambda_2 - 2\delta)_{-3\alpha_{-1}-4\delta} = 11.$$

Table 5.21:  $\tau_i \in T(-3\alpha_{-1} - 4\delta)$  such that  $\tau_1 + \tau_2 = -3\alpha_{-1} - 4\delta$  and  $\tau_1 = -\alpha_{-1} - \sum_{i=0}^2 m_i \alpha_i$  is a weight of  $V(\Lambda_0)$ , and  $\tau_2 = -2\alpha_{-1} - \sum_{i=0}^2 \check{m}_i \alpha_i$  is a weight of  $V(\Lambda_1 - \delta)$

$\tau_1$	$dimV(\Lambda_0)_{\tau_1}$	$\tau_2$	$dimV(\Lambda_1 - \delta)_{\tau_2}$
$-\alpha_{-1}$	1	$-2\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 4\alpha_2$	17
$-\alpha_{-1} - \alpha_0$	1	$-2\alpha_{-1} - 3\alpha_0 - 8\alpha_1 - 4\alpha_2$	5
$-\alpha_{-1} - \alpha_0 - \alpha_1$	1	$-2\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 4\alpha_2$	5
$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 3\alpha_2$	5
$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 3\alpha_2$	8
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - \alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 3\alpha_2$	5
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 2\alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 2\alpha_2$	5
$-\alpha_{-1} - \alpha_0 - 4\alpha_1 - 2\alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$	5
$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - \alpha_2$	1	$-2\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 3\alpha_2$	2
$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - \alpha_2$	1	$-2\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 3\alpha_2$	2
$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 2\alpha_2$	1	$-2\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 2\alpha_2$	2
$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 2\alpha_2$	2	$-2\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 2\alpha_2$	4
$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 2\alpha_2$	1	$-2\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 2\alpha_2$	2
$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 3\alpha_2$	1	$-2\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - \alpha_2$	2
$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 3\alpha_2$	1	$-2\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - \alpha_2$	2
$-\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$	2	$-2\alpha_{-1} - \alpha_0 - 4\alpha_1 - 2\alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 2\alpha_2$	2	$-2\alpha_{-1} - \alpha_0 - 3\alpha_1 - 2\alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 3\alpha_2$	2	$-2\alpha_{-1} - \alpha_0 - 3\alpha_1 - \alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 3\alpha_2$	4	$-2\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 3\alpha_2$	2	$-2\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 4\alpha_2$	2	$-2\alpha_{-1} - \alpha_0 - \alpha_1$	1
$-\alpha_{-1} - 3\alpha_0 - 8\alpha_1 - 4\alpha_2$	2	$-2\alpha_{-1} - \alpha_0$	1

With this last result and the results organized in Table 5.20 and in Table 5.21 we have:

$$\begin{aligned}
& \dim \mathfrak{g}_{-3\alpha_{-1}-4\delta} \\
&= \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1}-4\delta) \\ \tau_i = -\alpha_{-1} - \sum_{i=0}^2 m_i \alpha_i}} \frac{(\sum_i n_i - 1)!}{\prod(n_i!)} \prod (\dim V(\Lambda_0)_{\tau_i})^{n_i} \\
&+ \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1}-4\delta) \\ \tau_1 = -\alpha_{-1} - \sum_{i=0}^2 m_i \alpha_i \\ \tau_2 = -2\alpha_{-1} - \tilde{m}_0 \alpha_0 - m_1 \alpha_1 - m_2 \alpha_2}} (\dim V(\Lambda_0)_{\tau_1}) (-\dim V(\Lambda_1 - \delta)_{\tau_2}) \\
&+ \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1}-4\delta) \\ \tau_1 = -3\alpha_{-1}-4\delta}} (\dim V(\Lambda_2 - 2\delta)_{\tau_1}) \\
&= 86 - 91 + 11 \\
&= 6.
\end{aligned}$$

Note that this result is in accordance with Corollary 5.3

We may use the same process to determine

$$\begin{aligned}
& \dim \mathfrak{g}_{-3\alpha_{-1}-5\delta} \\
&= \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1}-5\delta) \\ \tau_i = -\alpha_{-1} - \sum_{i=0}^2 m_i \alpha_i}} \frac{(\sum_i n_i - 1)!}{\prod(n_i!)} \prod (\dim V(\Lambda_0)_{\tau_i})^{n_i} \\
&+ \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1}-5\delta) \\ \tau_1 = -\alpha_{-1} - \sum_{i=0}^2 m_i \alpha_i \\ \tau_2 = -2\alpha_{-1} - \tilde{m}_0 \alpha_0 - m_1 \alpha_1 - m_2 \alpha_2}} (\dim V(\Lambda_0)_{\tau_1}) (-\dim V(\Lambda_1 - \delta)_{\tau_2}) \\
&+ \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1}-5\delta) \\ \tau_1 = -3\alpha_{-1}-5\delta}} (\dim V(\Lambda_2 - 2\delta)_{\tau_1}) \\
&= 243 - 254 + 33 \\
&= 26.
\end{aligned}$$

Note that this result is in accordance with Corollary 5.2.

For  $\dim \mathfrak{g}_{-3\alpha_{-1}-6\delta}$ , since 6 is a multiple of 3, we have the following:

$$\dim \mathfrak{g}_{-3\alpha_{-1}-6\delta}$$

$$\begin{aligned}
&= \mu \left( \frac{-3\alpha_{-1} - 4\delta}{-3\alpha_{-1} - 4\delta} \right) \left( \frac{-3\alpha_{-1} - 4\delta}{-3\alpha_{-1} - 4\delta} \right) \mathcal{B}(-3\alpha_{-1} - 4\delta) + \mu \left( \frac{-3\alpha_{-1} - 6\delta}{-\alpha_{-1} - 2\delta} \right) \left( \frac{-\alpha_{-1} - 2\delta}{-3\alpha_{-1} - 6\delta} \right) \mathcal{B}(-\alpha_{-1} - 2\delta) \\
&\quad = \mathcal{B}(-3\alpha_{-1} - 4\delta) - \frac{1}{3} \mathcal{B}(-\alpha_{-1} - 2\delta) \\
&= \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1} - 5\delta) \\ \tau_i = -\alpha_{-1} - \sum_{i=0}^2 m_i \alpha_i}} \frac{(\sum_i n_i - 1)!}{\prod(n_i!)} \prod (\dim V(\Lambda_0)_{\tau_i})^{n_i} \\
&\quad + \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1} - 5\delta) \\ \tau_1 = -\alpha_{-1} - \sum_{i=0}^2 m_i \alpha_i \\ \tau_2 = 2\alpha_{-1} - \tilde{m}_0 \alpha_0 - m_1 \alpha_1 - m_2 \alpha_2}} (\dim V(\Lambda_0)_{\tau_1}) (-\dim V(\Lambda_1 - \delta)_{\tau_2}) \\
&\quad + \sum_{\substack{(n_i, \tau_i) \in T(-3\alpha_{-1} - 5\delta) \\ \tau_1 = -3\alpha_{-1} - 6\delta}} (\dim V(\Lambda_2 - 2\delta)_{\tau_1}) \\
&\quad - \frac{1}{3} \dim V(\Lambda_0)_{-\alpha_{-1} - 2\delta} \\
&= 604 + \frac{2}{3!} (2)^3 - 594 + 65 - \frac{1}{3} (2) \\
&= 77.
\end{aligned}$$

Repeating this strategy we continue to find the multiplicity of  $HD_4^{(3)}$  roots of the form  $-3\alpha_{-1} - k\delta$ ,  $7 \leq k \leq 15$  and collect our results in Table 5.22. Recall from Section 5.2 Frenkel's conjectured bound  $\text{mult}(\alpha) \leq p^2 \left(1 - \frac{(\alpha|\alpha)}{2}\right)$ ; we include  $\frac{(\alpha|\alpha)}{2}$  and  $p^2 \left(1 - \frac{(\alpha|\alpha)}{2}\right)$  with our root multiplicity calculations. Also recall from Section 5.2, if  $\alpha$  is a real root,  $(\alpha|\alpha) > 0$  and hence  $\left(1 - \frac{(\alpha|\alpha)}{2}\right) \leq 0$ . If  $\alpha$  is a real root, it is known  $\text{mult}(\alpha) = 1$ . So, for simplicity, in our calculations of Frenkel's conjectured bound we define  $p^2(k) = 1$  for  $k < 0$ .

Note here that, taking into account an index shifting, the multiplicities we found for  $-3\alpha_{-1} - 6\delta$ ,  $-3\alpha_{-1} - 7\delta$ , and  $-3\alpha_{-1} - 8\delta$  coincide with the multiplicities found for the same roots using a different technique in [5]. We also note that we recovered all the level 3 root multiplicities found in [5] and detail these results in Table 5.23.

In the interest of providing a more complete list of level 3 roots, we supplement the results in [5] with Table 5.24, Table 5.25, Table 5.26, Table 5.27, and Table 5.28 (note: If multiplicity is 0,  $\alpha$  is not a root). To explore patterns in these additional mutliplicities we factor out any multiples of  $\delta$  from the roots we have found and combine this reorganized data with some of Table 5.22 in Table 5.30 and Table 5.31.

Table 5.22: Root Multiplicities for  $-3\alpha_{-1} - k\delta$ ,  $1 \leq k \leq 15$ 

$\alpha$	Multiplicity	$\frac{(\alpha \alpha)}{2}$	$p^2 \left(1 - \frac{(\alpha \alpha)}{2}\right)$
$-3\alpha_{-1} - 3\delta$	$30 + \frac{2}{3!}(1)^3 - 32 + 4 - \frac{1}{3}(1) = 2$	0	2
$-3\alpha_{-1} - 4\delta$	$86 - 91 + 11 = 8$	-3	20
$-3\alpha_{-1} - 5\delta$	$243 - 254 + 33 = 22$	-6	110
$-3\alpha_{-1} - 6\delta$	$604 + \frac{2}{3!}(2)^3 - 594 + 65 - \frac{1}{3}(2) = 77$	-9	481
$-3\alpha_{-1} - 7\delta$	$1397 - 1341 + 141 = 197$	-12	1770
$-3\alpha_{-1} - 8\delta$	$3159 - 2950 + 301 = 510$	-15	5822
$-3\alpha_{-1} - 9\delta$	$6718 + \frac{2}{3!}(4)^3 - 6061 + 570 - \frac{1}{3}(4) = 1247$	-18	17490
$-3\alpha_{-1} - 10\delta$	$13832 - 12107 + 1079 = 2804$	-21	49010
$-3\alpha_{-1} - 11\delta$	$27747 - 23663 + 2032 = 6116$	-24	129512
$-3\alpha_{-1} - 12\delta$	$53891 + \frac{2}{3!}(6)^3 - 44623 + 3596 - \frac{1}{3}(6) = 12934$	-27	326015
$-3\alpha_{-1} - 13\delta$	$102235 - 82405 + 6336 = 26166$	-30	786814
$-3\alpha_{-1} - 14\delta$	$190269 - 149539 + 11022 = 51752$	-33	1831065
$-3\alpha_{-1} - 15\delta$	$346389 + \frac{2}{3!}(9)^3 - 265311 + 18586 - \frac{1}{3}(9) = 99664$	-36	4126070

Table 5.23: Recovered Level 3 Root Multiplicities from [5]

$\alpha$	Multiplicity	$\frac{(\alpha \alpha)}{2}$	$p^2 \left(1 - \frac{(\alpha \alpha)}{2}\right)$
$-3\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 4\alpha_2$	$243 - 246 + 29 = 26$	-6	110
$-3\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 5\alpha_2$	$243 - 246 + 29 = 26$	-6	110
$-3\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 5\alpha_2$	$446 - 448 + 52 = 50$	-8	300
$-3\alpha_{-1} - 7\alpha_0 - 11\alpha_1 - 5\alpha_2$	$594 - 588 + 65 = 71$	-9	481
$-3\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 6\alpha_2$	$30 + \frac{2}{3!}(1)^3 - 32 + 4 - \frac{1}{3}(1) = 2$	0	2
$-3\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 6\alpha_2$	$175 - 181 + 22 = 16$	-5	65
$-3\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 6\alpha_2$	$446 - 448 + 52 = 50$	-8	300
$-3\alpha_{-1} - 7\alpha_0 - 11\alpha_1 - 6\alpha_2$	$594 - 588 + 65 = 71$	-9	481
$-3\alpha_{-1} - 7\alpha_0 - 12\alpha_1 - 6\alpha_2$	$1070 - 1037 + 112 = 145$	-11	1165
$-3\alpha_{-1} - 8\alpha_0 - 13\alpha_1 - 6\alpha_2$	$1417 - 1371 + 151 = 197$	-12	1770
$-3\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 7\alpha_2$	$62 - 67 + 9 = 4$	-2	10
$-3\alpha_{-1} - 7\alpha_0 - 11\alpha_1 - 7\alpha_2$	$86 - 91 + 11 = 6$	-3	20
$-3\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 7\alpha_2$	$243 - 246 + 29 = 26$	-6	110
$-3\alpha_{-1} - 7\alpha_0 - 12\alpha_1 - 7\alpha_2$	$446 - 448 + 52 = 50$	-8	300
$-3\alpha_{-1} - 6\alpha_0 - 13\alpha_1 - 7\alpha_2$	$446 - 448 + 52 = 50$	-8	300
$-3\alpha_{-1} - 7\alpha_0 - 13\alpha_1 - 7\alpha_2$	$1070 - 1037 + 112 = 145$	-11	1165
$-3\alpha_{-1} - 8\alpha_0 - 13\alpha_1 - 7\alpha_2$	$1417 - 1371 + 151 = 197$	-12	1770
$-3\alpha_{-1} - 8\alpha_0 - 14\alpha_1 - 7\alpha_2$	$42420 - 2275 + 233 = 378$	-14	3956
$-3\alpha_{-1} - 9\alpha_0 - 15\alpha_1 - 7\alpha_2$	$3159 - 2922 + 289 = 526$	-15	5822
$-3\alpha_{-1} - 7\alpha_0 - 12\alpha_1 - 8\alpha_2$	$20 - 22 + 3 = 1$	1	1
$-3\alpha_{-1} - 6\alpha_0 - 13\alpha_1 - 8\alpha_2$	$62 - 67 + 9 = 4$	-2	10
$-3\alpha_{-1} - 6\alpha_0 - 14\alpha_1 - 8\alpha_2$	$175 - 181 + 22 = 16$	-5	65
$-3\alpha_{-1} - 8\alpha_0 - 14\alpha_1 - 8\alpha_2$	$1070 - 1037 + 112 = 145$	-11	1165
$-3\alpha_{-1} - 9\alpha_0 - 15\alpha_1 - 8\alpha_2$	$3159 - 2922 + 289 = 526$	-15	5822
$-3\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 8\alpha_2$	$9 - 9 + 1 = 1$	3	1
$-3\alpha_{-1} - 7\alpha_0 - 13\alpha_1 - 8\alpha_2$	$175 - 181 + 22 = 16$	-5	65
$-3\alpha_{-1} - 8\alpha_0 - 13\alpha_1 - 8\alpha_2$	$243 - 254 + 33 = 22$	-6	110
$-3\alpha_{-1} - 6\alpha_0 - 15\alpha_1 - 8\alpha_2$	$243 - 246 + 29 = 26$	-6	110
$-3\alpha_{-1} - 7\alpha_0 - 14\alpha_1 - 8\alpha_2$	$594 - 588 + 65 = 71$	-9	481
$-3\alpha_{-1} - 7\alpha_0 - 15\alpha_1 - 8\alpha_2$	$1070 - 1037 + 112 = 145$	-11	1165
$-3\alpha_{-1} - 8\alpha_0 - 15\alpha_1 - 8\alpha_2$	$2420 - 2275 + 233 = 378$	-14	3956
$3\alpha_{-1} - 9\alpha_0 - 16\alpha_1 - 8\alpha_2$	$5247 - 4778 + 460 = 929$	-17	12230
$-3\alpha_{-1} - 10\alpha_0 - 17\alpha_1 - 8\alpha_2$	$6696 - 6039 + 569 = 1226$	-18	17490

Table 5.24: Additional Level 3 Root Multiplicities (Part 1)

$\alpha$	Multiplicity
$-3\alpha_{-1} - 2\alpha_0 - \alpha_1$	$2 - 3 + 1 = 0$
$-3\alpha_{-1} - 2\alpha_0 - \alpha_1 - \alpha_2$	$2 - 3 + 1 = 0$
$-3\alpha_{-1} - 2\alpha_0 - 2\alpha_1$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - \alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 2\alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 2\alpha_0 - 3\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - \alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 2\alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 3\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 2\alpha_0 - 4\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - \alpha_2$	$2 - 3 + 1 = 0$
$-3\alpha_{-1} - 3\alpha_0 - \alpha_1$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 3\alpha_0 - \alpha_1 - \alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 2\alpha_1$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 2\alpha_1 - \alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 2\alpha_1 - 2\alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 3\alpha_1$	$0 + \frac{2}{3!}(1)^3 - 0 + 0 - \frac{1}{3}(1) = 0$
$-3\alpha_{-1} - 3\alpha_0 - 3\alpha_1 - \alpha_2$	$9 - 9 + 1 = 1$
$-3\alpha_{-1} - 3\alpha_0 - 3\alpha_1 - 2\alpha_2$	$9 - 9 + 1 = 1$
$-3\alpha_{-1} - 3\alpha_0 - 3\alpha_1 - 3\alpha_2$	$0 + \frac{2}{3!}(1)^3 - 0 + 0 - \frac{1}{3}(1) = 0$
$-3\alpha_{-1} - 3\alpha_0 - 4\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - \alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - 2\alpha_2$	$20 - 22 + 3 = 1$
$-3\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - 3\alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - 4\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 5\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - \alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 2\alpha_2$	$20 - 22 + 3 = 1$
$-3\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 3\alpha_2$	$20 - 22 + 3 = 1$
$-3\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 4\alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 5\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 6\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 2\alpha_2$	$9 - 9 + 1 = 1$
$-3\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 4\alpha_2$	$9 - 9 + 1 = 1$
$-3\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 5\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 6\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - \alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - \alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 2\alpha_1$	$0 - 0 + 0 = 0$

Table 5.25: Additional Level 3 Root Multiplicities (Part 2)

$\alpha$	Multiplicity
$-3\alpha_{-1} - 4\alpha_0 - 2\alpha_1 - \alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 2\alpha_1 - 2\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 3\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 3\alpha_1 - \alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 3\alpha_1 - 2\alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 3\alpha_1 - 3\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 4\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 4\alpha_1 - \alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 4\alpha_1 - 2\alpha_2$	$20 - 22 + 3 = 1$
$-3\alpha_{-1} - 4\alpha_0 - 4\alpha_1 - 3\alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 4\alpha_1 - 4\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 5\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 5\alpha_1 - \alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 5\alpha_1 - 2\alpha_2$	$27 - 30 + 4 = 1$
$-3\alpha_{-1} - 4\alpha_0 - 5\alpha_1 - 3\alpha_2$	$27 - 30 + 4 = 1$
$-3\alpha_{-1} - 4\alpha_0 - 5\alpha_1 - 4\alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 5\alpha_1 - 5\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 6\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 2\alpha_2$	$20 - 22 + 3 = 1$
$-3\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 3\alpha_2$	$62 - 67 + 9 = 4$
$-3\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 4\alpha_2$	$20 - 22 + 3 = 1$
$-3\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 5\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 6\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 7\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - 2\alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - 3\alpha_2$	$62 - 67 + 9 = 4$
$-3\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - 4\alpha_2$	$62 - 67 + 9 = 4$
$-3\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - 5\alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - 6\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 7\alpha_1 - 7\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 8\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 2\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 3\alpha_2$	$27 - 30 + 4 = 1$
$-3\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 5\alpha_2$	$27 - 30 + 4 = 1$
$-3\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 6\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 7\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 8\alpha_2$	$0 - 0 + 0 = 0$

Table 5.26: Additional Level 3 Root Multiplicities (Part 3)

$\alpha$	Multiplicity
$-3\alpha_{-1} - 5\alpha_0 - \alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - \alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 2\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 2\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 2\alpha_1 - 2\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 3\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 3\alpha_1 - \alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 3\alpha_1 - 2\alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 3\alpha_1 - 3\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 4\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 4\alpha_1 - \alpha_2$	$2 - 3 + 1 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 4\alpha_1 - 2\alpha_2$	$9 - 11 + 2 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 4\alpha_1 - 3\alpha_2$	$2 - 3 + 1 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 4\alpha_1 - 4\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 5\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 5\alpha_1 - \alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 5\alpha_1 - 2\alpha_2$	$20 - 22 + 3 = 1$
$-3\alpha_{-1} - 5\alpha_0 - 5\alpha_1 - 3\alpha_2$	$20 - 22 + 3 = 1$
$-3\alpha_{-1} - 5\alpha_0 - 5\alpha_1 - 4\alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 5\alpha_1 - 5\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 6\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 6\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 6\alpha_1 - 2\alpha_2$	$20 - 22 + 3 = 1$
$-3\alpha_{-1} - 5\alpha_0 - 6\alpha_1 - 3\alpha_2$	$62 - 67 + 9 = 4$
$-3\alpha_{-1} - 5\alpha_0 - 6\alpha_1 - 4\alpha_2$	$20 - 22 + 3 = 1$
$-3\alpha_{-1} - 5\alpha_0 - 6\alpha_1 - 5\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 6\alpha_1 - 6\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 7\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 7\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 7\alpha_1 - 2\alpha_2$	$9 - 11 + 2 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 7\alpha_1 - 3\alpha_2$	$91 - 99 + 14 = 6$
$-3\alpha_{-1} - 5\alpha_0 - 7\alpha_1 - 4\alpha_2$	$91 - 99 + 14 = 6$
$-3\alpha_{-1} - 5\alpha_0 - 7\alpha_1 - 5\alpha_2$	$9 - 11 + 2 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 7\alpha_1 - 6\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 7\alpha_1 - 7\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 8\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 2\alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 3\alpha_2$	$62 - 67 + 9 = 4$
$-3\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 4\alpha_2$	$175 - 181 + 22 = 16$

Table 5.27: Additional Level 3 Root Multiplicities (Part 4)

$\alpha$	Multiplicity
$-3\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 5\alpha_2$	$62 - 67 + 9 = 4$
$-3\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 6\alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 7\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 8\alpha_1 - 8\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 9\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 2\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 3\alpha_2$	$20 - 22 + 3 = 1$
$-3\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 4\alpha_2$	$175 - 181 + 22 = 16$
$-3\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 5\alpha_2$	$175 - 181 + 22 = 16$
$-3\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 6\alpha_2$	$20 - 22 + 3 = 1$
$-3\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 7\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 8\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 9\alpha_1 - 9\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 10\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 2\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 3\alpha_2$	$2 - 3 + 1 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 4\alpha_2$	$91 - 99 + 14 = 6$
$-3\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 6\alpha_2$	$91 - 99 + 14 = 6$
$-3\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 7\alpha_2$	$2 - 3 + 1 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 8\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 9\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 5\alpha_0 - 10\alpha_1 - 10\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - \alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - \alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 2\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 2\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 2\alpha_1 - 2\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 9\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 2\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 3\alpha_2$	$30 + \frac{2}{3!}(1)^3 - 32 + 4 - \frac{1}{3}(1) = 2$
$-3\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 4\alpha_2$	$243 - 246 + 29 = 26$
$-3\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 5\alpha_2$	$243 - 246 + 29 = 26$
$-3\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 6\alpha_2$	$30 + \frac{2}{3!}(1)^3 - 32 + 4 - \frac{1}{3}(1) = 2$
$-3\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 7\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 8\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 9\alpha_1 - 9\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 3\alpha_1$	$0 - 0 + 0 = 0$

Table 5.28: Additional Level 3 Root Multiplicities (Part 5)

$\alpha$	Multiplicity
$-3\alpha_{-1} - 6\alpha_0 - 3\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 3\alpha_1 - 2\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 3\alpha_1 - 3\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 4\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 4\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 4\alpha_1 - 2\alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 4\alpha_1 - 3\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 4\alpha_1 - 4\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 5\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 5\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 5\alpha_1 - 2\alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 5\alpha_1 - 3\alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 5\alpha_1 - 4\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 5\alpha_1 - 5\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 6\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 6\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 6\alpha_1 - 2\alpha_2$	$9 - 9 + 1 = 1$
$-3\alpha_{-1} - 6\alpha_0 - 6\alpha_1 - 3\alpha_2$	$30 + \frac{2}{3!}(1)^3 - 32 + 4 - \frac{1}{3}(1) = 2$
$-3\alpha_{-1} - 6\alpha_0 - 6\alpha_1 - 4\alpha_2$	$9 - 9 + 1 = 1$
$-3\alpha_{-1} - 6\alpha_0 - 6\alpha_1 - 5\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 6\alpha_1 - 6\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 7\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 7\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 7\alpha_1 - 2\alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 7\alpha_1 - 3\alpha_2$	$62 - 67 + 9 = 4$
$-3\alpha_{-1} - 6\alpha_0 - 7\alpha_1 - 4\alpha_2$	$62 - 67 + 9 = 4$
$-3\alpha_{-1} - 6\alpha_0 - 7\alpha_1 - 5\alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 7\alpha_1 - 6\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 7\alpha_1 - 7\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 8\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - 2\alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - 3\alpha_2$	$62 - 67 + 9 = 4$
$-3\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - 4\alpha_2$	$175 - 181 + 22 = 16$
$-3\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - 5\alpha_2$	$62 - 67 + 9 = 4$
$-3\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - 6\alpha_2$	$1 - 1 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - 7\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 8\alpha_1 - 8\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 10\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$

Table 5.29: Additional Level 3 Root Multiplicities (Part 6)

$\alpha$	Multiplicity
$-3\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 2\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 3\alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 4\alpha_2$	$175 - 181 + 22 = 16$
$-3\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 5\alpha_2$	$446 - 448 + 52 = 50$
$-3\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 6\alpha_2$	$175 - 181 + 22 = 16$
$-3\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 7\alpha_2$	$5 - 6 + 1 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 8\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 9\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 10\alpha_1 - 10\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 11\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 2\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 3\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 4\alpha_2$	$62 - 67 + 9 = 4$
$-3\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 5\alpha_2$	$446 - 448 + 52 = 50$
$-3\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 6\alpha_2$	$446 - 448 + 52 = 50$
$-3\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 7\alpha_2$	$62 - 67 + 9 = 4$
$-3\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 8\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 9\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 10\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 11\alpha_1 - 11\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 12\alpha_1$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - \alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 2\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 3\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 4\alpha_2$	$9 - 9 + 1 = 1$
$-3\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 5\alpha_2$	$243 - 246 + 29 = 26$
$-3\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 7\alpha_2$	$243 - 246 + 29 = 26$
$-3\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 8\alpha_2$	$9 - 9 + 1 = 1$
$-3\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 9\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 10\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 11\alpha_2$	$0 - 0 + 0 = 0$
$-3\alpha_{-1} - 6\alpha_0 - 12\alpha_1 - 12\alpha_2$	$0 - 0 + 0 = 0$

Table 5.30:  $\delta$ -factored Level 3 Root Multiplicities (Part 1)

$\alpha$	Multiplicity	$\frac{(\alpha \alpha)}{2}$	$p^2 \left(1 - \frac{(\alpha \alpha)}{2}\right)$
$-3\alpha_{-1} - \delta - 2\alpha_0 - \alpha_1$	$9 - 9 + 1 = 1$	3	1
$-3\alpha_{-1} - \delta - 2\alpha_0 - \alpha_1 - \alpha_2$	$9 - 9 + 1 = 1$	3	1
$-3\alpha_{-1} - 2\delta - \alpha_0$	$20 - 22 + 3 = 1$	1	1
$-3\alpha_{-1} - 2\delta - \alpha_0 - \alpha_1$	$20 - 22 + 3 = 1$	1	1
$-3\alpha_{-1} - 2\delta - \alpha_0 - \alpha_1 - \alpha_2$	$20 - 22 + 3 = 1$	1	1
$-3\alpha_{-1} - 2\delta - \alpha_0 - 2\alpha_1$	$9 - 9 + 1 = 1$	3	1
$-3\alpha_{-1} - 2\delta - 2\alpha_0$	$20 - 22 + 3 = 1$	1	1
$-3\alpha_{-1} - 2\delta - 2\alpha_0 - \alpha_1$	$27 - 30 + 4 = 1$	0	2
$-3\alpha_{-1} - 2\delta - 2\alpha_0 - \alpha_1 - \alpha_2$	$27 - 30 + 4 = 1$	0	2
$-3\alpha_{-1} - 2\delta - 2\alpha_0 - 2\alpha_1$	$20 - 22 + 3 = 1$	1	1
$-3\alpha_{-1} - 2\delta - 3\alpha_0 - \alpha_1$	$20 - 22 + 3 = 1$	1	1
$-3\alpha_{-1} - 2\delta - 3\alpha_0 - \alpha_1 - \alpha_2$	$20 - 22 + 3 = 1$	1	1
$-3\alpha_{-1} - 2\delta - 3\alpha_0 - 2\alpha_1$	$20 - 22 + 3 = 1$	1	1
$-3\alpha_{-1} - 2\delta - 4\alpha_0 - 2\alpha_1$	$9 - 9 + 1 = 1$	3	1
$-3\alpha_{-1} - 3\delta$	$30 + \frac{2}{3!}(1)^3 - 32 + 4 - \frac{1}{3}(1) = 2$	0	2
$-3\alpha_{-1} - 3\delta - \alpha_2$	$9 - 9 + 1 = 1$	3	1
$-3\alpha_{-1} - 3\delta - \alpha_0$	$62 - 67 + 9 = 4$	-2	10
$-3\alpha_{-1} - 3\delta - \alpha_0 - \alpha_2$	$20 - 22 + 3 = 1$	1	1
$-3\alpha_{-1} - 3\delta - \alpha_0 - \alpha_1$	$62 - 67 + 9 = 4$	-2	10
$-3\alpha_{-1} - 3\delta - \alpha_0 - \alpha_1 - \alpha_2$	$62 - 67 + 9 = 4$	-2	10
$-3\alpha_{-1} - 3\delta - \alpha_0 - 2\alpha_1$	$27 - 30 + 4 = 1$	0	2
$-3\alpha_{-1} - 3\delta - 2\alpha_0$	$62 - 67 + 9 = 4$	-2	10
$-3\alpha_{-1} - 3\delta - 2\alpha_0 - \alpha_2$	$20 - 22 + 3 = 1$	1	1
$-3\alpha_{-1} - 3\delta - 2\alpha_0 - \alpha_1$	$91 - 99 + 14 = 6$	-3	20
$-3\alpha_{-1} - 3\delta - 2\alpha_0 - \alpha_1 - \alpha_2$	$91 - 99 + 14 = 6$	-3	20
$-3\alpha_{-1} - 3\delta - 2\alpha_0 - 2\alpha_1$	$62 - 67 + 9 = 4$	-2	10
$-3\alpha_{-1} - 3\delta - 2\alpha_0 - 3\alpha_1$	$20 - 22 + 3 = 1$	1	1
$-3\alpha_{-1} - 3\delta - 3\alpha_0 - 3\alpha_1$	$30 + \frac{2}{3!}(1)^3 - 32 + 4 - \frac{1}{3}(1) = 2$	0	2
$-3\alpha_{-1} - 3\delta - 3\alpha_0$	$30 + \frac{2}{3!}(1)^3 - 32 + 4 - \frac{1}{3}(1) = 2$	0	2
$-3\alpha_{-1} - 3\delta - 3\alpha_0 - \alpha_2$	$9 - 9 + 1 = 1$	3	1
$-3\alpha_{-1} - 3\delta - 3\alpha_0 - \alpha_1$	$62 - 67 + 9 = 4$	-2	10
$-3\alpha_{-1} - 3\delta - 3\alpha_0 - \alpha_1 - \alpha_2$	$62 - 67 + 9 = 4$	-2	10
$-3\alpha_{-1} - 3\delta - 3\alpha_0 - 2\alpha_1$	$62 - 67 + 9 = 4$	-2	10

Table 5.31:  $\delta$ -factored Level 3 Root Multiplicities (Part 2)

$\alpha$	Multiplicity	$\frac{(\alpha \alpha)}{2}$	$p^2 \left(1 - \frac{(\alpha \alpha)}{2}\right)$
$-3\alpha_{-1} - 4\delta$	$86 - 91 + 11 = 8$	-3	20
$-3\alpha_{-1} - 4\delta - \alpha_2$	$27 - 30 + 4 = 1$	0	2
$-3\alpha_{-1} - 4\delta - \alpha_0$	$175 - 181 + 22 = 16$	-5	65
$-3\alpha_{-1} - 4\delta - \alpha_0 - \alpha_2$	$62 - 67 + 9 = 4$	-2	10
$-3\alpha_{-1} - 4\delta - \alpha_0 - \alpha_1$	$175 - 181 + 22 = 16$	-5	65
$-3\alpha_{-1} - 4\delta - \alpha_0 - \alpha_1 - \alpha_2$	$175 - 181 + 22 = 16$	-5	65
$-3\alpha_{-1} - 4\delta - \alpha_0 - \alpha_1 - 2\alpha_2$	$20 - 22 + 3 = 1$	1	1
$-3\alpha_{-1} - 4\delta - \alpha_0 - 2\alpha_1$	$91 - 99 + 14 = 6$	-3	20
$-3\alpha_{-1} - 4\delta - 2\alpha_0 - \alpha_1$	$243 - 246 + 29 = 26$	-6	110
$-3\alpha_{-1} - 4\delta - 2\alpha_0 - \alpha_1 - \alpha_2$	$243 - 246 + 29 = 26$	-6	110
$-3\alpha_{-1} - 4\delta - 2\alpha_0 - \alpha_1 - 2\alpha_2$	$30 + \frac{2}{3!}(1)^3 - 32 + 4 - \frac{1}{3}(1) = 2$	0	2
$-3\alpha_{-1} - 4\delta - 2\alpha_0$	$175 - 181 + 22 = 16$	-5	65
$-3\alpha_{-1} - 4\delta - 2\alpha_0 - \alpha_2$	$62 - 67 + 9 = 4$	-2	10
$-3\alpha_{-1} - 4\delta - 2\alpha_0 - 2\alpha_1$	$175 - 181 + 22 = 16$	-5	65
$-3\alpha_{-1} - 4\delta - 2\alpha_0 - 3\alpha_1$	$62 - 67 + 9 = 4$	-2	10
$-3\alpha_{-1} - 4\delta - 2\alpha_0 - 4\alpha_1$	$9 - 9 + 1 = 1$	3	1
$-3\alpha_{-1} - 5\delta$	$243 - 254 + 33 = 22$	-6	110
$-3\alpha_{-1} - 5\delta - \alpha_2$	$91 - 99 + 14 = 6$	-3	20
$-3\alpha_{-1} - 5\delta - \alpha_0$	$446 - 448 + 52 = 50$	-8	300
$-3\alpha_{-1} - 5\delta - \alpha_0 - \alpha_2$	$175 - 181 + 22 = 16$	-5	65
$-3\alpha_{-1} - 5\delta - \alpha_0 - \alpha_1$	$446 - 448 + 52 = 50$	-8	300
$-3\alpha_{-1} - 5\delta - \alpha_0 - \alpha_1 - \alpha_2$	$446 - 448 + 52 = 50$	-8	300
$-3\alpha_{-1} - 5\delta - \alpha_0 - \alpha_1 - 2\alpha_2$	$62 - 67 + 9 = 4$	-2	10
$-3\alpha_{-1} - 5\delta - \alpha_0 - 2\alpha_1$	$243 - 246 + 29 = 26$	-6	110
$-3\alpha_{-1} - 6\delta$	$604 + \frac{2}{3!}(2)^3 - 594 + 65 - \frac{1}{3}(2) = 26$	-9	481
$-3\alpha_{-1} - 6\delta - \alpha_2$	$243 - 246 + 29 = 26$	-6	110
$-3\alpha_{-1} - 6\delta - 2\alpha_2$	$9 - 9 + 1 = 1$	3	1

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## **APPENDIX**

# Appendix A

## MATLAB Code

### A.1 lev1mult.m

```
function [c]=lev1mult(w)
%%%%%%%%%%%%%
%finds the multiplicity of the user-inputted level 1 root
% Inputs:
%     w is the column vector representing number of \alpha_0s, \alpha_1s,
%           \alpha_2s in the level 1 root of interest
%
%Outputs:
%   c (mult(alpha) in HD_4^{(3)}) 
%   can output C: the paths which will be listed as rows
%                   read from left to right(instead of right to left), and in
%                   terms of actual tuples as defined by B_l originally -- see
%                   original definitions; should have at least one repition of
%                   ground state to right
%%%%%%%%%%%%%
root=[1;w];
%adding a 1 to the top of the vector to represent the 1 from -1\alpha_{-1}
A=zeros(1,6);
highest_weight_vector=zeros(4,1);
highest_weight_vector(1)=1;
arrows=root-highest_weight_vector;
if arrows(1)==0
arrows=arrows(2:4);
arrows=arrows';
[C,E]=crystal_path(A,arrows,1);
B=organize(C);
%
```

```

c=size(E,1);
else
c=-25;
end;

```

```

function [A,B]=crystal_path(pattern,goalarrows,lev)
%%%%%%%%%%%%%
%called by lev1mult
% Inputs:
%   pattern: ground state path (for level 1 this is all zeros)
%   goalarrows: determined by the original input w
%   lev: 1, 2, or 3
%
% Outputs:
%   returns A the paths horizontally concatenated as a matrix and
%   B the arrows as rows of a matrix.
%%%%%%%%%%%%%
prev_path=zeros(1,3);
[A,B]=one_move(pattern,pattern,prev_path,goalarrows,lev);
%just acted on the ground state for the first time.
for i=1:sum(goalarrows)-1;
    % sum(goalarrows) implies we will go to that row in the crystal graph.
    % for example -2alpha_0-3alpha_1 would give us a sum(goalarrows)=5,
    % so we look up to the fifth row.
    [A,B]=next_move(A,B,pattern,goalarrows,lev);
end;

```

```

function [paths,arrows]=one_move(A,pattern,prev_path,goalarrows,lev)
%%%%%%%%%%%%%
%called by crystal_path
% Inputs:
%   A: ground state path
%   pattern: ground state path
%   prev_path: row of length 3 to count how many 0 arrows, 1 arrows and 2 arrows
%               found
%   goalarrows: determined by user input
%   lev: 1, 2, or 3
%
% Outputs:
%   [results,path] where results is the emanating paths from the path A,
%   and path tells which f-arrow leads to which emanating path.
%%%%%%%%%%%%%

```

```

arrows=zeros(0,3);
y=length(A);
b=length(pattern);
paths=zeros(0,y+b);
% space for two elements of a path, one the ground state -
% this will be the rightmost entries
for k=1:3
    c=zeros(1,3);
    c(1,k)=1;
    if prev_path(k)<goalarrows(k)
        B=A;
        % element of crystal path you want to act on
        num=identify_action(tensorrule(k-1,A,lev));
        % notice it sends k-1, so we find phi and eps from 0 to 2
        if num ~= 0.1;
            B((6*(num-1)+1):(6*num))=ftilda(k-1,A(6*(num-1)+1:6*(num)));
            % which of the basic paths to act on, where off to the right will be
            % repeating ground state paths
            add=[B,pattern];
            % we act on paths from the left, and to the right represents the
            % infinitely repeating pattern of the ground state
            paths=[paths;add];
            arrows=[arrows;c];
            % c is a row vector with a one in the k'th column, indicating that we just
            % acted f_{k-1} (so a one in first column means we acted f_0, etc..)
        end;
    end;
end;

```

```

function B=tensorrule(index, A, lev)
%%%%%%%%%%%%%
% called by identify_action which is called by one_move
% Inputs:
% index: counter 0,1,2 if prevpath(counter)<goalarrows(counter), the index for
%         which you wish to compute the phi_i and epsilon_i
% A: the element of the crystal path on which you wish to act on, starting with
%     the ground state path
%
% Outputs:
%     returns phi_index of the ith element in the sequence in B(i,2) and
%     epsilon_index of the ith element in the tensor product in B(i,2)
%%%%%%%%%%%%%
y=length(A);
B=zeros(y/(6),2);

```

```

% determine how many pieces of a path you are dealing with, each is a perfect
% crystal basis element and has length 6,
% y grows as the end of he path changes more from the original ground state
% 2 columns for each piece, one for phi, one for eps
for i=1:y/(6)
    B(i,1)=phi(index,A(6*(i-1)+1:(6*i)),lev);
    % for A of length 6, we have i=1:1, so A(1:6)
    % for A of length 12, we are concerned with rightmost two basis elements of
    % this crystal path, i=1:2, so A(1:6), then A(7:12);
    % you are finding phi and eps of each of right most basic elements of the
    % tensor and for each basic element, phi and eps will be stored on a line
    B(i,2)=epsilon(index, A(6*(i-1)+1:(6*i)),lev);
    %Builds matrix B that will be sent to identify action, will have phi and eps for
    % each basic elemnt of the tensor stored on each row
end;

```

```

function action=identify_action(A)
%%%%%%%%%%%%%
% Inputs:
%   A: phi and eps for each basic elemnt of the tensor stored on each row with
%       ph_i in first column and epsilon_i in second column (tensorrule output)
%
% Outputs:
%   Gives the row number of the element in the sequence of the crystal path that
%   A represents which should be acted upon by ftilda_i.
%   If there should be no action, it returns 0.1.
%%%%%%%%%%%%%
[a,b]=size(A);
% a is how many basic elements we are concerned with (b should always be 2),
% that is, a is how many p_i's we are dealing with, up to p_{N-1} (see Chapter 3)
for j=1:a
    [x,y]=size(A);
    A=[A;0,0];
    A(:,2)=[0;A(1:x,2)];
    % all rows, eps col, shift orig 2nd col down and add a zero to the top
    for i=1:x+1
        num=A(i,1)-A(i,2);
        %we are looking at the signature of the path from right to left by
        %finding the difference: phi_i(p_k)-eps_i(p_{k-1})
        %we determine how much the signature collapses, which elements are left
        %with plusses under them after cancelling with the minuses
        if num >= 0
            A(i,1)=num;
            A(i,2)=0;

```

```

% which p_k's have plusses over them, i.e. which ones can f_i act on
else
    A(i,1)=0;
    A(i,2)=-1*num;
        %left only with minuses
end;
end;
j=j+1;
end;
A=A(:,1);
%all rows, just concerned with col 1
test=find(A>0);
%returns the row numbers of positive entries (in order)
[a,b]=size(test);
if a==0
    action=0.1;
else action=test(a,b);
%returns the highest row number with a positive entry because f_i must act on
%the elements pointed to by the i-sequence from right to left
end;

```

```

function [paths,arrows]= next_move(olddpaths,oldarrows,pattern,goalarrows,lev)
%%%%%%%%%%%%%
% called by crystal_path
% this function will act for a count of 1:sum(goalarrows)-1, as we have already gone
% down one row of the path realization with one_move, this will work down the
% remaining sum(goalarrows)-1 rows
%
% Inputs:
% oldpaths: (output from one_move at first, then recursively given by next_move)
%           the emanating paths from the element of the crystal path on
%           which you originally wanted to act
% oldarrows: tells which f-arrow leads to which emanating path.
% pattern: the repeating ground state path
% goalarrows: col vector determined by original input
% lev: level of the highest weight module on which you are counting
%
% Outputs:
% [paths,arrows]: where paths is the emanating paths from the ground state,
%                 and arrows tells which f-arrow leads to which path
%%%%%%%%%%%%%
y=length(pattern);
[x2,y2]=size(olddpaths);
newpaths=zeros(0,y2+y);

```

```

newarrows=zeros(0,3);
% possible arrows are 0,1,2 so we need 3 spaces
for i=1:x2;
    A=oldpaths(i,:);
    current_arrows=oldarrows(i,:);
    [paths,arrows]= one_move(A,pattern,current_arrows,goalarrows,lev);
    [c,d]=size(arrows);
    mult=ones(c,1);
    addition=mult*current_arrows;
    newarrows=[newarrows;(addition+arrows)];
    %at first just fills in an empty row, then adds on to the matrix
    newpaths=[newpaths;paths];
end;
test=[pattern;newpaths(:,y2+1-6:y2+y-6)];
test=sum(abs(diff(test)));
%diff(X) returns a matrix of row differences
if test==0
    newpaths=newpaths(:,1:y2);
end;
% some of rows in newpaths may be repeats, so we remove the repeated paths
[a,b]=size(newpaths);
if a>1
    [sorted,index]=sortrows(newpaths);
    D=sum(abs(diff(sorted)),2);
    f=find(D~=0);
    rows=[index(f);index(a)];
    paths=newpaths(rows,:);
    arrows=newarrows(rows,:);
else
    paths=newpaths;
    arrows=newarrows;
end;

```

## A.2 mult\_parts1.m

```

function [ P ] = mult_parts1(w)
%%%%%%%%%%%%%
%finds multiplicities of 2 level1 weights of V(\Lambda_0) that sum up to
%the user-inputted level2 root
% Inputs:
%      w is the column vector representing number of \alpha_0s, \alpha_1s,
%      \alpha_2s in the level 2 root of interest

```

```

%
% Outputs:
%   P: matrix where columns 1-4 are the coefficients of \alpha_{-1}, \alpha_0,
%       \alpha_1, and \alpha_2 of a level 1 root, column 5 its multiplicity;
%   then columns 6-9 is the corresponding level 1 root
%   (column 10 is its multiplicity)
%   such that columns 1-4 and 6-9 are a partition the user-inputted level 2 root
%   Column 11 is the product of the two multiplicities and the lower right-hand
%   corner of the matrix is the sum of all such products
%   (see multiplicity formula)
%%%%%%%%%%%%%%%
D=lev1multupto(w);
% determines which partitions of w are level 1 weights and their multiplicities
root=[2;w];
[m,n]=size(D);
P=zeros(0,11);
for i=1:m
    for j=i:m
        trial1=D(i,1:4);
        trial2=D(j,1:4);
        if [trial1+trial2]' == root
            if trial1==trial2
                continue
            else
                mult=D(i,5)*D(j,5);
            end
            P=[P;trial1,D(i,5),trial2,D(j,5),mult];
        else
            continue
        end
    end
end
[j,k]=size(P);
sum=0;
for i=1:j
    prod=P(i,11);
    sum=sum+prod;
end
P=[P; [zeros(1,10),sum]];
end

```

```

function [Z]=lev1multupto(w)
%%%%%%%%%%%%%%%
% takes in a root and determines all weights of V(\Lambda_0) less than or equal to

```

```

% it; we use this to simplify the process of partitioning roots by getting rid of all
% cases that are not even weights of  $V(\Lambda_0)$ 
% Inputs:
%   w=[a;b;c]
%     a is number of 0-arrows
%     b is number of 1-arrows
%     c is number of 2-arrows
%
% Outputs:
%   Z: list of roots of  $V(\Lambda_0)$  up to w with their corresponding multiplicities
%%%%%%%%%%%%%
root=w;
D=[1,0,0,0,1];
%since  $-\alpha_{-1}$  counts trivially as a weight with mult=1
root=[1;root];
%adding a 1 to the top of the vector (to represent the 1 from  $1*\alpha_{-1}$ )
R=upto(root);
%finds the potential weights
A=[0,0,0,0,0,0];
%the repeating ground state path for  $V(\lambda_0)$ 
highest_weight_vector=zeros(4,1);
%spots for  $\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2$ 
highest_weight_vector(1)=1;
%highest_weight_vector now has a 1 at the top
[m,n]=size(R);
for i=1:n
    root=R(:,i);
    %considers each potential weight less than the input root
arrows=root-highest_weight_vector;
if arrows(1)==0
arrows=arrows(2:4);
arrows=arrows';
[C,E]=crystal_path(A,arrows,1);
c=size(E,1);
else
c=-25;
end;
if c<= 0
    continue
else
D=[D;root',c];
end
end
Z=D;
end

```

```

function [A]=upto(root)
%%%%%%%%%%%%%
% Inputs:
%   root: column vector with coefficients of \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2
%
% Outputs:
%   A: column i of matrix A + column i of matrix B = root
%%%%%%%%%%%%%
m=length(root);
root=root(2:m);
n=length(root);
A=zeros(n,0);
for i=1:(root(1))+1;
    for j=1:(root(2))+1;
        for k=1:(root(3))+1;
            x=[i-1;j-1;k-1];
            A=[A,x];
        %matrix where each column is potential weight less than that of the input root
        end;
    end;
end;
[m,n]=size(A);
add=ones(1,n);
A=[add;A];
%add a row of ones, so each column will be 1, a, b, c from:
%-1*\alpha_{-1}-a*\alpha_0-b*\alpha_1-c*\alpha_2

```

### A.3 perfectcrystalB.m

```

function [ B ] = perfectcrystalB( index,x,lev )
%%%%%%%%%%%%%
% Inputs:
%   index: for the f_index desired
%   x: a known basis element of the perfect crystal
%   lev: level of desired perfect crystal
%
% Outputs:
%   B: given an index, B contains resulting perfect crystal basis elements if
%       after acting f_index on x as many times as possible
%%%%%%%%%%%%%
B=zeros(0,6);

```

```

B=[x;B];
n=length(x)/2;
A=[x(1:n);fliplr(x(n+1:2*n))];
% takes second half of tensor an flips left to right
% tensor begins as: x_1,x_2,x_3,bar{x}_3,bar{x}_2,bar{x}_1
%so row 1 is x_1,x_2,x_3 and
%      row 2 is bar{x}_1,bar{x}_2,bar{x}_3
%this simplifies the below procedures
y=0;
%y will be equal to phi_index
if index == 0
    while A(1,1)>=0 && A(1,2)>=0 && A(1,3)>=0 && A(2,1)>=0 && A(2,2)>=0 &&
    A(2,3)>=0 && A(1,1)+A(1,2)+((A(1,3)+A(2,3))/2)+A(2,2)+A(2,1) <= lev &&
    mod(A(1,3)+A(2,3),2)==0
        %while resulting f_0(b) is in B_1, we can try to still act f_0 on it.
        %how many times we can act f_0 and still result in a basis element is phi_0
        %F_1:
        if 2*A(2,1)+2*A(2,2)+A(2,3)-A(1,3)-2*A(1,2)-2*A(1,1)>0 &&
        2*A(2,1)+3*A(2,3)-A(1,3)-2*A(1,2)-2*A(1,1)>0 &&
        A(2,1)+A(1,3)-A(1,2)-A(1,1)>0 && A(2,1)-A(1,1)>0
            %if b satisfies (F_1), then f_0 acts as this:
            A=A+[0,0,0;-1,0,0];
        %F_2:
        elseif 2*A(2,1)+2*A(2,2)+A(2,3)-A(1,3)-2*A(1,2)-2*A(1,1)<=0 &&
        2*A(2,2)+A(2,3)-3*A(1,3)<=0 &&
        A(2,2)-A(2,3)<=0 && A(2,1)-A(1,1)>0
            %if b satisfies (F_2), then f_0 acts as this:
            A=A+[0,0,1;-1,0,1];
        %F_3:
        elseif 2*A(2,1)+3*A(2,3)-A(1,3)-2*A(1,2)-2*A(1,1)<=0 &&
        3*A(2,3)-A(1,3)-2*A(1,2)<=0 && A(2,3)-A(1,3)<=0 && A(2,2)-A(2,3)>0 &&
        A(2,1)+A(2,2)-A(2,3)-A(1,1)>0
            %if b satisfies (F_3), then f_0 acts as this:
            A=A+[0,0,2;0,-1,0];
        %F_4:
        elseif 2*A(2,1)+2*A(2,2)+A(2,3)-3*A(1,3)-2*A(1,1)>0 &&
        2*A(2,2)+A(2,3)-3*A(1,3)>0 && A(2,3)-A(1,3)>0 && A(1,3)-A(1,2)<=0 &&
        A(2,1)+A(1,3)-A(1,2)-A(1,1)<=0
            %if b satisfies (F_4), then f_0 acts as this:
            A=A+[0,1,0;0,0,-2];
        %F_5:
        elseif 2*A(2,1)+2*A(2,2)+A(2,3)-A(1,3)-2*A(1,2)-2*A(1,1)>0 &&
        3*A(2,3)-A(1,3)-2*A(1,2)>0 && A(1,3)-A(1,2)>0 && A(2,1)-A(1,1)<=0
            %if b satisfies (F_5), then f_0 acts as this:
            A=A+[1,0,-1;0,0,-1];

```

```

%F_6:
elseif 2*A(2,1)+2*A(2,2)+A(2,3)-A(1,3)-2*A(1,2)-2*A(1,1)<=0 &&
2*A(2,1)+2*A(2,2)+A(2,3)-3*A(1,3)-2*A(1,1)<=0 &&
A(2,1)+A(2,2)-A(2,3)-A(1,1)<=0 && A(2,1)-A(1,1)<=0
%if b satisfies (F_6), then f_0 acts as this:
A=A+[1,0,0;0,0,0];
else
    break;
end
if A(1,1)>=0 && A(1,2)>=0 && A(1,3)>=0 && A(2,1)>=0 && A(2,2)>=0 &&
A(2,3)>=0 && A(1,1)+A(1,2)+(A(1,3)+A(2,3))/2+A(2,2)+A(2,1) <= lev &&
mod(A(1,3)+A(2,3),2)==0
z=[A(1,:),fliplr(A(2,:))];
B=[B;z];
y=y+1;
end
if y>10
    break;
end
end
elseif index == 1
while A(1,1)>=0 && A(1,2)>=0 && A(1,3)>=0 && A(2,1)>=0 && A(2,2)>=0 &&
A(2,3)>=0 && A(1,1)+A(1,2)+((A(1,3)+A(2,3))/2)+A(2,2)+A(2,1) <= lev &&
mod(A(1,3)+A(2,3),2)==0
%while resulting f_1(b) is in B_l, we can try to still act f_1 on it.
%how many times we can act f_1 and still result in a basis element is phi_1
if subplus(A(2,2)-A(2,3)) <= A(1,2)-A(1,3)
    A=A+[-1,1,0;0,0,0];
elseif A(2,2)-A(2,3) <= 0 && A(1,3)-A(1,2) > 0
    A=A+[0,0,-1;0,0,1];
elseif A(2,2)-A(2,3) > subplus(A(1,2)-A(1,3))
    A=A+[0,0,0;1,-1,0];
else
    break;
end
if A(1,1)>=0 && A(1,2)>=0 && A(1,3)>=0 && A(2,1)>=0 && A(2,2)>=0 &&
A(2,3)>=0 && A(1,1)+A(1,2)+(A(1,3)+A(2,3))/2+A(2,2)+A(2,1) <= lev &&
mod(A(1,3)+A(2,3),2)==0
z=[A(1,:),fliplr(A(2,:))];
B=[B;z];
y=y+1;
end
if y>10
    break;
end

```

```

end

elseif index == 2
    while A(1,1)>=0 && A(1,2)>=0 && A(1,3)>=0 && A(2,1)>=0 && A(2,2)>=0 &&
A(2,3)>=0 && A(1,1)+A(1,2)+((A(1,3)+A(2,3))/2)+A(2,2)+A(2,1) <= lev &&
mod(A(1,3)+A(2,3),2)==0
        %while resulting f_2(b) is in B_l, we can try to still act f_2 on it.
        %how many times we can act f_2 and still result in a basis element is phi_2
        if A(2,3) <= A(1,3)
            A=A+[0,-1,2;0,0,0];
        elseif A(2,3) > A(1,3)
            A=A+[0,0,0;0,1,-2];
        else
            break;
        end
        if A(1,1)>=0 && A(1,2)>=0 && A(1,3)>=0 && A(2,1)>=0 && A(2,2)>=0 &&
A(2,3)>=0 && A(1,1)+A(1,2)+(A(1,3)+A(2,3))/2+A(2,2)+A(2,1) <= lev &&
mod(A(1,3)+A(2,3),2)==0
            z=[A(1,:),fliplr(A(2,:))];
            B=[B;z];
            y=y+1;
        end
        if y>10
            break;
        end
    end
end

```

## A.4 lev2mult.m

```

function [c]=lev2mult(w)
%%%%%%%%%%%%%
% This program takes in a weight
% -2\alpha_{-1}-a\alpha_0-b\alpha_1-c\alpha_2
% and gives its multiplicity in V(\lambda_1-\delta) by counting paths in
% path realization P(\lambda_1).
% Inputs:
% w: weight vector (of arrows), so w=[a,b,c]
%
% Outputs:
% c: multiplicity
%%%%%%%%%%%%%

```

```

root=w-[1;0;0];
%to count the mult of a level 2 weight by using the path realiz of V(lambda1)
%we must account for V(lambda1-delta), and add delta; we then only
%count number of paths with (a-1) 0-arrows than stated here:
%-2\alpha_{-1}-a\alpha_0-b\alpha_1-c\alpha_2
if root(1)<0
    c=-25;
else
    root=[1;root];
%adding a 1 to the top of the vector (to represent the 1 from 1lambda_1)
A=[1,0,0,0,0,1];
%A is repeating ground state path for path realization of V(lambda_1)
highest_weight_vector=zeros(4,1);
highest_weight_vector(1)=1;
arrows=root-highest_weight_vector;
if arrows(1)==0
    arrows=arrows(2:4);
    arrows=arrows';
    [C,E]=crystal_path(A,arrows,2);
% 2= level --> Perfect Crystal B_2, highest weight module V(lambda_1)
    B=organize2(C);
    c=size(E,1);
else
    c=-25;
end;
end;

```

## A.5 sumparts.m

```

function [ P ] = sumparts( w,lev )
%%%%%%%%%%%%%
%finds multiplicities of 2 level1 weights of V(\Lambda_0) that sum up to
%the user-inputed level2 root
% Inputs:
%     w: the column vector representing number of \alpha_0s, \alpha_1s,
%         \alpha_2s in the level 3 root of interest
%     lev: if lev=1 then will find multiplicities of 3 level1 weights of
%           V(\Lambda_0) that sum up to level3 root w
%           if lev=2 then will find multiplicities of 1 level2 weight of
%           V(\Lambda_1-\delta) and 1 level1 weight of V(\Lambda_0) that sum
%           to the level3 root w
%
```

```

% Outputs:
%      P: matrix of partitions of the level3 root and corresponding multiplicities
%%%%%%%%%%%%%%%
if lev==1
    D=lev1multupto(w);
    %list of roots of  $V(\Lambda_0)$  up to w with their corresponding multiplicities
    root=[3;w];
    [m,n]=size(D);
    P=zeros(0,16);
    for i=1:m
        for j=i:m
            for k=j:m
                trial1=D(i,1:4);
                trial2=D(j,1:4);
                trial3=D(k,1:4);
                if [trial1+trial2+trial3]' == root
                    if trial1== trial2
                        mult=D(i,5)*D(j,5)*D(k,5);
                    elseif trial1==trial3
                        mult=D(i,5)*D(j,5)*D(k,5);
                    elseif trial2==trial3
                        mult=D(i,5)*D(j,5)*D(k,5);
                    else
                        mult=D(i,5)*D(j,5)*D(k,5)*2;
                        %see Kang's formula
                    end
                    P=[P;trial1,D(i,5),trial2,D(j,5),trial3,D(k,5),mult];
                else
                    continue
                end
            end
        end
    end
    [j,k]=size(P);
    sum=0;
    for i=1:j
        prod=P(i,16);
        sum=sum+prod;
    end
    P=[P;[zeros(1,15),sum]];
elseif lev==2
    D=lev1multupto(w);
    %list of roots of  $V(\Lambda_0)$  up to w with their corresponding multiplicities
    P=lev2multupto(D,w);
    %list of partitions of level3 root into level1 weight and level2 weight

```

```
end
```

```
function [Z]=lev2multupto(D,w)
%%%%%%%%%%%%%
% Inputs:
% D: matrix, lists all level 1 partitions of the level 3 root that are weights
%      of V(\Lambda_0)
% w: weight vector (of arrows), so w=[a,b,c]
%
% Outputs:
% Z: matrix with partitions of the original lev3 root where each
%      row is a partition. the first part of the row will be the lev2 part of the
%      partition followed by its mult; second part of the row will be the lev1
%      corresponding part of the partition followed by its mult
%%%%%%%%%%%%%
root=w;
root=[3;root];
%adding a 3 to the top of the vector (to represent the 1 from 3\alpha_{-1})
[m,n]=size(D);
C=zeros(0,10);
for i=1:m
    lev1part=D(i,1:4);
    triallev2part=root-lev1part';
    if triallev2part==[2;1;0;0]
        c=1;
        C=[C;triallev2part',c,D(i,1:5)];
    else
        c=lev2mult(triallev2part(2:4));
        if c<=0
            continue
        else
            C=[C;triallev2part',c,D(i,1:5)];
        end
    end
end
[j,k]=size(C);
sum=0;
for i=1:j
    prod=C(i,5)*C(i,10);
    sum=sum+prod;
end
Z=[C; [0,0,0,0,0,0,0,0,sum]];
end
```

## A.6 lev3mult.m

```

function [c]=lev3mult(w)
%%%%%%%%%%%%%
% Inputs:
% w: for level3 weight -3\alpha_{-1}-a\alpha_0-b\alpha_1-c\alpha_2,
%     w=[a;b;c]
%
% Outputs:
% c: multiplicity of the level 3 weight in V(\Lambda_2-2\delta)
%%%%%%%%%%%%%
root=w-[2;1;0];
%to count the mult of a level 3 root by using the path realiz of V(\Lambda_2)
%we account for V(\Lambda_2-2\delta), and add 2\delta and then count
%paths with (a-2) 0-arrows and (b-1) 1-arrows than stated here:
%-3\alpha_{-1}-a\alpha_0-b\alpha_1-c\alpha_2
if root(1)<0
    c=-25;
else
    root=[1;root];
%adding a 1 to the top of the vector (to represent the 1 from \lambda_2)
    A=[0,1,1,1,0];
%repeating ground state path for V(\lambda_2)
    highest_weight_vector=zeros(4,1);
    highest_weight_vector(1)=1;
    arrows=root-highest_weight_vector;
    if arrows(1)==0
        arrows=arrows(2:4);
        arrows=arrows';
%transpose
    [C,E]=crystal_path(A,arrows,3);
% 3= level --> Perfect crystal B_3, V(\lambda_2)
    c=size(E,1);
    else
        c=-25;
    end;
end;

```