ABSTRACT

JAYNE, REBECCA LINN. Maximal Dominant Weights of Some Integrable Modules for the Special Linear Affine Lie Algebras and Their Multiplicities. (Under the direction of Kailash Misra.)

Kac-Moody algebras were discovered by Victor Kac and Robert Moody in the 1960's. A particular class of infinite dimensional Kac-Moody algebras, known as affine Kac-Moody algebras, have been an important area of study. In particular, mathematicians study the irreducible integrable highest weight modules of Kac-Moody algebras. A useful tool in studying Kac-Moody algebras are crystal bases. The notion of a crystal base was introduced by Masaki Kashiwara and George Lusztig in the study of quantum groups. In crystal base theory, we use combinatorial objects to study highest weight modules of Kac-Moody algebras.

In this thesis, we study a particular affine Kac-Moody algebra, $\widehat{sl}(n)$, an infinite dimensional analog of the Lie algebra of traceless $n \times n$ matrices. In particular, we consider irreducible integrable highest weight $\widehat{sl}(n)$-modules. In the weight structure of these modules, the maximal weights form something of a roof. That is, all other weights occur on strings stemming from the maximal weights. We determine the maximal dominant weights of certain $\widehat{sl}(n)$-modules. In addition, we study the multiplicity of particular maximal dominant weights using crystal base theory. We find a relationship between the multiplicity of these weights and avoiding permutations.

Maximal Dominant Weights of Some Integrable Modules for the Special Linear Affine Lie Algebras and Their Multiplicities

by
Rebecca Linn Jayne

A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh, North Carolina

2011

APPROVED BY:

Bojko Bakalov

Thomas Lada

Ernest Stitzinger

Kailash Misra
Chair of Advisory Committee

BIOGRAPHY

Rebecca Linn Jayne spent most of her youth on the Eastern Shore of Maryland. In 2002, she graduated as valedictorian of her Queen Anne's County High School class. That fall, she matriculated at McDaniel College in Westminster, Maryland. There, she majored in Mathematics and minored in Computer Science and Spanish. She graduated *summa cum laude* and as a College Scholar in May 2006. She then moved to Raleigh to study Mathematics at North Carolina State University, where she received her Master of Science degree in May 2008. Jayne will soon be returning to the Eastern Shore as she has recently accepted a tenure-track assistant professor position at Washington College in Chestertown, Maryland.

ACKNOWLEDGEMENTS

I begin by thanking my advisor, Kailash Misra, for his constant support, advice, and encouragement. I would also like to thank the other members of my committee and the Department of Mathematics for their roles in helping me to achieve my goals.

My family members also deserve my thanks. My parents, grandmothers, and brother have always had faith in me and my ability to succeed.

Next, I thank my support system of friends from various times in my life. In particular, I thank Bill Knouse for helping me realize my goals. I also thank my officemates from the last two years - Mindy Capaldi, Catherine Buell, and Christina Erbacher - for their encouragement and for always believing in me.

I also would like to acknowledge partial support from NSA grant H98230-08-1-0080.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

Introduction

Lie algebras were originally discovered by their namesake, Sophus Lie, during the nineteenth century. Lie was studying symmetries of solutions to differential equations at the time. Since then, Lie algebras have been studied quite extensively outside of the context of differential equations.

By the end of the nineteenth century, Wilhelm Killing and Élie Cartan had classified simple finite dimensional Lie algebras (c.f. [5]). In the 1960's, Victor Kac [8] and Robert Moody [18] independently discovered Kac-Moody algebras, generalizing finite dimensional semisimple Lie algebras. In general, they constructed a Kac-Moody algebra by beginning with an $n \times n$ integer matrix (generalized Cartan matrix) with special properties. They then constructed Kac-Moody algebras using $3n$ generators and relations dependent upon the entries of the symmetric generalized Cartan matrices [10].

Requiring additional restrictions on the Cartan matrix gives an affine Kac-Moody algebra. These affine Kac-Moody algebras are infinite dimensional and can be constructed using finite dimensional simple Lie algebras. In this thesis, we will primarily consider $\widehat{sl}(n)$, the affine Kac-Moody algebra constructed from the finite dimensional $sl(n)$, which is the simple Lie algebra of traceless $n \times n$ matrices.

An important area of study is that of integrable highest weight modules of Kac-Moody algebras, which were introduced by Kac [9] in the 1970's. It can be shown that for a dominant weight Λ and a Kac-Moody algebra $\mathfrak{g}$, there is a unique, up to isomorphism, irreducible integrable highest weight module, which we will denote by $V(\Lambda)$. An area of interest is the weight structure of these modules. The words "highest weight" in the description of the module are motivated by the fact that every other weight of the module is less than the highest weight, Λ, under a particular ordering. Given a weight, we can often consider other weights by adding or subtracting a null root, δ. There are certain weights, called maximal weights, to which we can add δ and no longer have a weight. That is to say, these maximal weights form something like a ceiling over all other weights. In this thesis, we determine the set of maximal dominant weights for certain $\widehat{sl}(n)$ modules.

Another area of interest in the study of integrable highest weight modules is the study of the dimensions of the weight spaces of $V(\Lambda)$, known as weight multiplicities. As a tool to determine these weight multiplicities, we use crystal base theory, which comes from the study of quantum groups. In the 1980's, Vladimir Drinfel'd [2] and Michio Jimbo [6] introduced the notion of a quantum group. We can view quantum groups as neither groups nor Lie algebras, but as Hopf algebras. Quantum groups are deformations of universal enveloping algebras of Kac-Moody algebras. For a symmetrizable generalized Cartan matrix, a Kac-Moody algebra $\mathfrak{g}$ can be constructed. Associated with $\mathfrak{g}$, a quantum group $U_q(\mathfrak{g})$, where q is an indeterminate, can be constructed. A very useful fact is that the representation theory of $U_q(\mathfrak{g})$ is parallel to the representation theory of $\mathfrak{g}$ [13]. In particular, dimensions of weight spaces remain invariant under q-deformations.

In the early 1990's, Masaki Kashiwara [11], [12] and George Lusztig [14] independently developed crystal base theory. In this theory, we can use combinatorial objects to study the weight multiplicities of $U_q(\mathfrak{g})$-modules and hence $\mathfrak{g}$-modules. Around this same time, Kailash Misra and Tetsuji Miwa [16] introduced the notion of combinatorial objects known as extended Young diagrams to give an explicit realization of the crystal basis for level one representations of $U_q(\widehat{sl}(n))$. Jimbo, Misra, Miwa, and Masato Okado [7] later generalized this theory to arbitrary integrable highest weight $U_q(\widehat{sl}(n))$-modules.

In this thesis, we use and build on this work. In Chapter 2, we will review a more detailed construction of Kac-Moody algebras and quantum groups. Then in Chapter 3, we will discuss crystal base theory and the combinatorial realization of crystals in the form of extended Young diagrams. Next, in Chapter 4, we determine the maximal dominant weights of specific $\widehat{sl}(n)$-modules. Finally, in Chapter 5, we consider the multiplicity of certain maximal dominant weights using crystal base theory. Specifically, we conjecture a relationship between the multiplicity of these weights and pattern avoiding permutations and provide some evidence to justify our conjecture.

Chapter 2

Kac-Moody Algebras and Quantum Groups

In this chapter, we describe Lie algebras and then focus specifically on a particular class of Lie algebras known as Kac-Moody algebras. We discuss the construction of Kac-Moody algebras as well as their representation theory. In addition, we describe quantum groups.

Unless otherwise noted, we will consider our underlying field to be $\mathbb{C}$.

2.1 Lie Algebras

Definition 2.1.1 *A* **Lie algebra** $\mathfrak{g}$ *is a vector space over a field* $\mathbb{C}$*, upon which is defined a product:* $[,]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ *such that for all* $x, y, z \in \mathfrak{g}$ *and* $a, b \in \mathbb{C}$*,*

1. $[ax+by,z]=a[x,z]+b[y,z]$ *and* $[x,ay+bz]=a[x,y]+b[x,z]$ *(*$\mathbb{C}$*-bilinear)*

2. $[x,x]=0$

3. $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$. *(Jacobi identity)*

Many examples of important finite dimensional Lie algebras are Lie algebras of matrices. In these Lie algebras, we define the product as the commutator $[A,B]=AB-BA$ via matrix multiplication. For example, particularly important are Lie algebras of trace zero $n \times n$ matrices. We will first look at the case of trace zero 2×2 matrices.

Example 2.1.2 *We denote the Lie algebra of 2×2 traceless matrices over $\mathbb{C}$ by $sl(2)$. A basis of $sl(2)$ is given by:*

$$\left\{ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

We see that $[h, e] = 2e, [h, f] = -2f, [e, f] = h$.

For general n, we denote the Lie algebra of trace zero $n \times n$ matrices over $\mathbb{C}$ by $sl(n)$. A basis of $sl(n)$ is given by: $\{h_i = E_{ii} - E_{i+1,i+1}, E_{jk} \mid 1 \leq i \leq n-1, 1 \leq j \neq k \leq n\}$, where E_{ij} is an $n \times n$ matrix of all zeros except for a one in the i-th row and j-th column. Notice that as in Example 2.1.2, diagonal elements are denoted with an h, though in the $n \times n$ case, we must index these elements.

Definition 2.1.3 *An* **associative algebra** *A over $\mathbb{C}$ is a ring A which can also be viewed as a vector space over $\mathbb{C}$, where the underlying addition and 0 element are the same in the ring and the vector space and $a(x \cdot y) = (ax) \cdot y = x \cdot (ay)$, for all $a \in \mathbb{C}, x, y \in A$.*

The set of all $n \times n$ matrices over a field $\mathbb{C}$ is an associative algebra. Additionally, if we consider an associative algebra A over $\mathbb{C}$, we can define a bracket on A such that $[x, y] = x \cdot y - y \cdot x$ and can now view A as a Lie algebra.

We define a **Lie subalgebra** to be a subset of a Lie algebra that is itself a Lie algebra. Note that for $sl(n)$, the set of all diagonal elements is a Lie subalgebra of $sl(n)$. This is a particularly important subalgebra called the Cartan subalgebra, which we will denote $\mathfrak{h}$.

We are also concerned with subspaces of Lie algebras called ideals. A subspace M of a Lie algebra $\mathfrak{g}$ is an **ideal** if $[x, y] \in M$ for all $x \in \mathfrak{g}, y \in M$. A Lie algebra is **simple** if it is nonabelian and its only ideals are $\{0\}$ and itself. Another important class of Lie algebras are semisimple Lie algebras. To define semisimple, we will consider the derived series of $\mathfrak{g}$,

$$\mathfrak{g} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \mathfrak{g}^{(3)} \supseteq \dots,$$

where $\mathfrak{g}^{(m)} = [\mathfrak{g}^{(m-1)}, \mathfrak{g}^{(m-1)}]$. If $\mathfrak{g}^{(m)} = \{0\}$ for some m, then $\mathfrak{g}$ is said to be **solvable**. A Lie algebra is **semisimple** if it has no nonzero solvable ideals. Equivalently, a Lie algebra is semisimple if it can be written as a direct sum of simple ideals.

As in the study of groups and rings, in the study of Lie algebras we develop the notion of homomorphisms. In this thesis, we will be especially interested in a special type of homomorphism known as a representation.

Definition 2.1.4 *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras. A linear transformation $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a* **Lie algebra homomorphism** *if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}_1$.*

Definition 2.1.5 *A homomorphism $\varphi : \mathfrak{g} \to gl(V)$ is a* **Lie algebra representation** *of the Lie algebra $\mathfrak{g}$ on the vector space V, where $gl(V)$ is the set of all linear transformations from V to V.*

Representations are very closely related to modules.

Definition 2.1.6 *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. A vector space V over $\mathbb{C}$ is a* **$\mathfrak{g}$-module** *if there is an operation $\cdot : \mathfrak{g} \times V \to V$ such that for all $x, y \in \mathfrak{g}$, $u, v \in V$, and $a, b \in \mathbb{C}$:*

1. $x \cdot (au + bv) = a(x \cdot u) + b(x \cdot v)$,
2. $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$,
3. $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$.

It is important to notice that modules for a Lie algebra and Lie algebra representations are equivalent. Suppose V is a $\mathfrak{g}$-module with an operation we will denote $\cdot$. If we define $\varphi\colon \mathfrak{g} \to gl(V)$ by $\varphi(x)v = x \cdot v$, it can be shown that φ is a Lie algebra representation.

Similarly, we can reverse this process to obtain a module from a Lie algebra representation. For this reason, we will often use either term to refer to the same construction, though representation will refer to the map and module will refer to the vector space.

We consider various properties of modules. For a $\mathfrak{g}$-module V, a subspace U of V is a **submodule** if $x \cdot u \in U$ for all $x \in \mathfrak{g}$ and $u \in U$. A nonzero $\mathfrak{g}$-module V is **irreducible** if V has no proper nonzero submodules.

An important representation is the adjoint representation. In the adjoint representation, we view the Lie algebra $\mathfrak{g}$ as a representation onto itself; that is, $\mathfrak{g}$ is itself a $\mathfrak{g}$-module. We also consider the adjoint action on subalgebras of a Lie algebra $\mathfrak{g}$. For the following examples, note that when we look at the module action on the Cartan subalgebra, we find behavior similar to eigenvalue and eigenvector pairs.

Definition 2.1.7 *The* **adjoint representation** *for a Lie algebra* $\mathfrak{g}$ *is* $\Psi\colon \mathfrak{g} \to gl(\mathfrak{g})$*, defined by* $\Psi(x)(y) = ad_x(y) = [x, y]$ *for all* $x, y \in \mathfrak{g}$.

To see that Ψ does preserve the bracket, consider the following:

$$\begin{aligned}
[\Psi(x), \Psi(y)](z) &= \Psi(x)\Psi(y)(z) - \Psi(y)\Psi(x)(z) \\
&= \Psi(x)[y, z] - \Psi(y)[x, z] \\
&= [x, [y, z]] - [y, [x, z]] \\
&= [x, [y, z]] + [y, [z, x]] \\
&= -[z, [x, y]] \\
&= [[x, y], z] \\
&= \Psi([x, y])(z)
\end{aligned}$$

Example 2.1.8 *Consider the adjoint representation of the Cartan subalgebra of* $sl(2)$ *on* $sl(2)$.

- $\Psi(h)(e) = [h, e] = 2e$
- $\Psi(h)(f) = [h, f] = -2f$

Notice that e and f are behaving like eigenvectors and 2 and -2, respectively, are the corresponding eigenvalues.

Now let us consider the adjoint representation of the Cartan subalgebra of $sl(3)$ on $sl(3)$.

Example 2.1.9 *We consider the adjoint representation of the Cartan subalgebra on $sl(3)$. Recall that the Cartan subalgebra is $\mathfrak{h} = \{h_1 = E_{11} - E_{22}, h_2 = E_{22} - E_{33}\}$. We obtain the relationships in Table 2.1.*

Table 2.1: Adjoint Representation of $\mathfrak{h}$ on $sl(3)$

z	$\Psi(h_1)(z)$	$\Psi(h_2)(z)$
E_{12}	$2E_{12}$	$-E_{12}$
E_{13}	E_{13}	E_{13}
E_{23}	$-E_{23}$	$2E_{23}$
E_{21}	$-2E_{21}$	E_{21}
E_{31}	$-E_{31}$	$-E_{31}$
E_{32}	E_{32}	$-2E_{32}$

Note that if we define α_1 and $\alpha_2 \in \mathfrak{h}^*$ such that,

$$\alpha_1(h_1) = 2 \quad \alpha_1(h_2) = -1$$
$$\alpha_2(h_1) = -1 \quad \alpha_2(h_2) = 2,$$

we can rewrite the table above as in Table 2.2. Here, α_1 and α_2 are called roots.

Definition 2.1.10 *Let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ \forall x \in \mathfrak{g}\}$ for $\alpha \in \mathfrak{h}^*$*

1. *For $\alpha \neq 0$, if $\mathfrak{g}_\alpha$ is nonzero, α is a* **root** *of $\mathfrak{g}$.*

2. *For $\alpha \neq 0$, if $\mathfrak{g}_\alpha$ is nonzero, $\mathfrak{g}_\alpha$ is called the* **root space** *of α in $\mathfrak{g}$*

Consider our example of $\mathfrak{g} = sl(3)$. The space $\{ad_h \mid h \in \mathfrak{h}\}$ is a space of simultaneously diagonalizable linear operators on $\mathfrak{g}$ and we can write $\mathfrak{g}$ as a direct sum of its root spaces

Table 2.2: Adjoint Representation of $\mathfrak{h}$ on $sl(3)$ with Roots

z	$\Psi(h_1)(z)$	$\Psi(h_2)(z)$
E_{12}	$\alpha_1(h_1)E_{12}$	$\alpha_1(h_2)E_{12}$
E_{13}	$(\alpha_1+\alpha_2)(h_1)E_{13}$	$(\alpha_1+\alpha_2)(h_2)E_{13}$
E_{23}	$\alpha_2(h_1)E_{23}$	$\alpha_2(h_2)E_{23}$
E_{21}	$-\alpha_1(h_1)E_{21}$	$-\alpha_1(h_2)E_{21}$
E_{31}	$-(\alpha_1+\alpha_2)(h_1)E_{31}$	$-(\alpha_1+\alpha_2)(h_2)E_{31}$
E_{32}	$-\alpha_2(h_1)E_{32}$	$-\alpha_2(h_2)E_{32}$

(including $\mathfrak{g}_0 = \mathfrak{h}$). That is,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-(\alpha_1+\alpha_2)}.$$

We call this the **root space decomposition** of $\mathfrak{g}$.

Often when we consider a module, we will not be using the adjoint representation. An example of another representation follows.

Example 2.1.11 *Let $\mathfrak{g} = sl(3)$ and let $V = \mathbb{C}^3$. We will define the representation by $\phi(x)(v) = xv$ for $x \in \mathfrak{g}, v \in V$, where the juxtaposition here is just the usual matrix-vector multiplication. Note that $\phi(h)\colon V \to V$ is diagonalizable for all $h \in \mathfrak{h}$. Since $0 = \phi([h,h']) = [\phi(h), \phi(h')]$ for all $h, h' \in \mathfrak{h}$, the set $\{\phi(h) \mid h \in \mathfrak{h}\}$ is a space of simultaneously diagonalizable linear operators on V. Note that a basis of V is*

$$\left\{ v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We have the following relationships:

$h_1 \cdot v_1 = v_1 \qquad h_2 \cdot v_1 = 0$

$h_1 \cdot v_2 = -v_2 \qquad h_2 \cdot v_2 = v_2$

$h_1 \cdot v_3 = 0 \qquad h_2 \cdot v_3 = -v_3$

Definition 2.1.12 *Let* $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid h \cdot v = \lambda(h)x \ \forall x \in \mathfrak{g}\}$ *for* $\lambda \in \mathfrak{h}^*$.

1. *If* $\mathfrak{g}_\lambda$ *is nonzero,* λ *is a* **weight**.

2. *If* $\mathfrak{g}_\lambda$ *is nonzero,* $\mathfrak{g}_\lambda$ *is called the* λ**-weight space**.

Example 2.1.13 *Continuing Example 2.1.11, if we define* $\lambda_1, \lambda_2, \lambda_3 \in \mathfrak{h}^*$, *by the following:*

$$\begin{array}{ll} \lambda_1(h_1) = 1 & \lambda_1(h_2) = 0 \\ \lambda_2(h_1) = -1 & \lambda_2(h_2) = 1 \\ \lambda_3(h_1) = 0 & \lambda_3(h_2) = -1. \end{array}$$

We obtain the weight space decomposition: $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3}$

Notice that in all of these examples, we were able to give a weight space or root space decomposition of the given module. This was related to our ability to create a set of simultaneously diagonalizable linear operators. We guarantee this by our selection of Lie algebra. In fact, Weyl's Theorem states that when $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra, any $\mathfrak{g}$-module can be written as a direct sum of irreducible submodules.

2.2 Kac-Moody Algebras

We discuss the construction of finite and affine Kac-Moody algebras. We begin with an examination of specific matrices called generalized Cartan matrices.

Definition 2.2.1 *A* **generalized Cartan matrix (GCM)** *is an* $(n-1) \times (n-1)$ *matrix* $A = (a_{ij})_{i,j=1}^{n-1}$ *such that*

1. $a_{ii} = 2$ *for* $i = 1, \ldots n-1$,

2. a_{ij} *is a nonpositive integer for* $i \neq j$,

3. $a_{ij} = 0 \implies a_{ji} = 0$

If A *is positive definite as well,* A *is called a* **Cartan matrix**.

A GCM is **decomposable** if there exists some $\sigma \in S_{n-1}$ such that $\sigma(A) = (a_{\sigma(i),\sigma(j)})_{i,j=1}^{n-1}$ is in block diagonal form. Otherwise, A is **indecomposable**. In this sense, we need only consider indecomposable GCM's. We can classify these indecomposable GCM's according to the following theorem.

Theorem 2.2.2 *[10] Let A be an indecomposable GCM. Then one and only one of the the following is true.*

1. *(Finite) det(A) $\neq 0$, $\exists u > 0$ such that $Au > 0$, $Av \geq 0 \implies v > 0$ or $v = 0$,*
2. *(Affine) corank(A) = 1, $\exists u > 0$ such that $Au = 0$, $Av \geq 0 \implies Av = 0$,*
3. *(Indefinite) $\exists u > 0$ such that $Au < 0$, $Av \geq 0, v \geq 0 \implies v = 0$.*

We take u and v to be column vectors in $\mathbb{Z}^{n-1}$ and take $u > 0$ to mean that $u_i > 0$ for $i = 1, 2, \ldots n-1$.

It is worth noting that the GCM's can be further described by associating each with a Dynkin diagram. For an indecomposable $(n-1) \times (n-1)$ GCM, the Dynkin diagram is a connected graph with $n-1$ vertices connected according to the following conditions: [10]

- If $a_{ij}a_{ji} \leq 4$, we connect vertices i and j by $\max\{|a_{ij}|, |a_{ji}|\}$ lines with an arrow pointing toward i if $|a_{ij}| > 1$.
- If $a_{ij}a_{ji} > 4$, we connect vertices i and j by a bold line marked with $(|a_{ij}|, |a_{ji}|)$.

Example 2.2.3 *The following GCM and Dynkin diagram are in correspondence.*

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

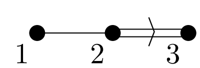


Figure 2.1: A Corresponding GCM and Dynkin diagram

A GCM is called **symmetrizable** if there exists a matrix $S = diag(s_1, s_2, \ldots, s_{n-1})$ such that the product SA is a symmetric matrix (i.e. $(SA)^T = SA$). From now on, we will consider only symmetrizable GCM's. We wish to construct Lie algebras using information from GCM's. More specifically, we wish to construct finite and affine Kac-Moody Lie algebras. Though the process of constructing these Lie algebras is very similar for the finite case and the affine case, we will examine them separately.

We begin with an $(n-1) \times (n-1)$ GCM, A, of finite type. We define a **realization** of A to be a triple $(\mathring{\mathfrak{h}}, \Pi, \Pi^\vee)$. Here, $\mathring{\mathfrak{h}}$ is a vector space of dimension $n-1$; $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} \subset \mathring{\mathfrak{h}}^*$; $\Pi^\vee = \{h_1, h_2, \ldots, h_{n-1}\} \subset \mathring{\mathfrak{h}}$. Π and $\Pi^\vee$ are linearly independent sets such that $\alpha_j(h_i) = a_{ij}$ for $i, j = 1, 2, \ldots n-1$. The elements of Π are the **simple roots** and those of $\Pi^\vee$ are called the **simple coroots**. For convenience, we will use $I = \{1, 2, \ldots, n-1\}$ as an index set.

Using this notation, we can construct a finite dimensional Lie algebra as follows.

Theorem 2.2.4 *(Serre) [5] Let A be a Cartan matrix and $(\mathring{\mathfrak{h}}, \Pi, \Pi^\vee)$ a realization of A. Then $\mathring{\mathfrak{g}}$, the Lie algebra generated by $\{e_i, f_i, h_i \mid i \in I\}$ satisfying the following relations, is a* ***finite dimensional Kac-Moody Lie algebra****.*

1. $[h_i, h_j] = 0$ $(i, j \in I)$
2. $[e_i, f_j] = \delta_{ij} h_i$ $(i, j \in I)$,
3. $[h_i, e_j] = a_{ij} e_j$ $(i, j \in I)$,
4. $[h_i, f_j] = -a_{ij} f_j$ $(i, j \in I)$,
5. $(ad\ e_i)^{1-a_{ij}} e_j = 0,\ i \neq j$,
6. $(ad\ f_i)^{1-a_{ij}} f_j = 0,\ i \neq j$.

For more information about the proof of this theorem, refer to [5]. $\mathring{\mathfrak{h}}$ is the **Cartan subalgebra**. We call e_i, f_i $(i \in I)$ the **Chevalley generators**. Note that if we let $\mathring{\mathfrak{n}}_+$ be the subalgebra of $\mathring{\mathfrak{g}}$ generated by the $e_i's$ and $\mathring{\mathfrak{n}}_-$ that generated by the $f_i's$, we obtain the decomposition:

$$\mathring{\mathfrak{g}} = \mathring{\mathfrak{n}}_+ \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}_-.$$

We call $Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \ldots \oplus \mathbb{Z}\alpha_{n-1}$ the **root lattice**. We define a partial ordering on elements on $\mathfrak{h}^*$ by stating that $\lambda \geq \mu$ when $\lambda - \mu \in Q^+ = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \mathbb{Z}_{\geq 0}\alpha_2 \oplus \ldots \oplus \mathbb{Z}_{\geq 0}\alpha_{n-1}$.

In addition, every root is an integral linear combination of elements of Π. It can be shown that to be a root, coefficients of the α_i's in a particular linear combination are either all positive or all negative. If they are all positive, we say that a root $\alpha > 0$. If they are all negative, we say that $\alpha < 0$. For a root α, it can also be shown that if $\alpha > 0, \mathfrak{g}_\alpha \in \mathring{\mathfrak{n}}_+$ and $\mathfrak{g}_\alpha \in \mathring{\mathfrak{n}}_-$ for $\alpha < 0$.

Example 2.2.5 *Consider the generalized Cartan matrix of finite type and its corresponding Dynkin diagram below.*

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & -1 & 2 \end{pmatrix}$$

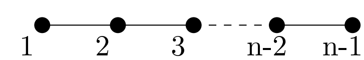


Figure 2.2: GCM and Dynkin diagram for $sl(n)$

The Kac-Moody Lie algebra associated with A is $\mathring{\mathfrak{g}} = sl(n)$. As we discussed in the first section, we can have many roots of $\mathring{\mathfrak{g}} = sl(n)$. We find that the highest root is $\theta = \sum_{i=1}^{n-1} \alpha_i$. That is to say, if we add to θ some α_i, $i \in I$, $\alpha_i + \theta$ is not a root.

Now, we wish to associate a Lie algebra with each symmetrizable affine GCM. By definition, an affine GCM has corank 1. Let A be an $n \times n$ symmetrizable GCM and $S = diag(s_0, s_1, \ldots, s_{n-1})$ be a diagonal matrix such that SA is symmetric. Note that we are indexing from 0 to $n-1$ and we will let $I = \{0, 1, \ldots, n-1\}$. Though we could define a realization as in the finite case, we may be interested in more information and describe the following.

- Let $P^\vee$ be the free abelian group of rank $n+1$ with basis $\{h_0, h_1, \dots, h_{n-1}, d\}$, where d is a derivation. Then $P^\vee = \left(\bigoplus_{i=0}^{n-1} \mathbb{Z}h_i\right) \bigoplus (\mathbb{Z}d)$. $P^\vee$ is known as the **dual weight lattice**.

- Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee = \mathrm{span}_{\mathbb{C}}\{h_0, h_1, \dots h_{n-1}, d\}$. This space $\mathfrak{h}$ is known as the **Cartan subalgebra**.

- Define $P = \mathrm{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z}) = \{\lambda \in \mathfrak{h} \mid \lambda(P^\vee) \subset \mathbb{Z}\}$. P is called the **weight lattice**.

- Let $\Pi^\vee = \{h_0, h_1, \dots, h_{n-1}\} \subset P^\vee$. The elements of $\Pi^\vee$ are the **simple coroots**.

- Define $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ where the α_j's are linearly independent and defined so that $\alpha_j(h_i) = a_{ij}$ $(i, j \in I)$. The elements of Π are the **simple roots**.

The set $(A, P^\vee, P, \Pi^\vee, \Pi\}$ is called the Cartan datum associated with A.

Definition 2.2.6 *[4] Let A be a symmetrizable $n \times n$ affine GCM. Then the* **affine Kac-Moody Lie algebra**, $\mathfrak{g}$, *associated with the Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi\}$ is the Lie algebra generated by $\{e_i, f_i \mid i \in I\} \cup P^\vee$ satisfying the following relations:*

1. $[h_i, h_j] = 0$ $(i, j \in I)$,

2. $[e_i, f_j] = \delta_{ij} h_i$ $(i, j \in I)$,

3. $[h_i, e_j] = a_{ij} e_j$ $(i, j \in I)$,

4. $[h_i, f_j] = -a_{ij} f_j$ $(i, j \in I)$,

5. $(ad\ e_i)^{1-a_{ij}} e_j = 0,\ i \neq j$

6. $(ad\ f_i)^{1-a_{ij}} f_j = 0,\ i \neq j$

As in the finite case, e_i, f_i $(i, j \in I)$ are the **Chevalley generators**. Note that we used $\mathring{\mathfrak{g}}$ to describe the finite dimensional Kac-Moody algebra, but we removed the circle in the affine case. We will continue to use this distinction in notation when relevant. For example, we obtain the decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ for the affine Kac-Moody Lie algebra as an affine analog of the statement $\mathring{\mathfrak{g}} = \mathring{\mathfrak{n}}_+ \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}_-$. In addition, similar to the finite case, $Q = \sum_{i=0}^{n-1} \mathbb{Z}\alpha_i$ is the root lattice.

Example 2.2.7 *Consider the generalized Cartan matrix of affine type and its associated Dynkin diagram below.*

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & & 0 \\ 0 & -1 & 2 & -1 & & & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & & & 2 & -1 \\ -1 & 0 & 0 & & & -1 & 2 \end{pmatrix}$$

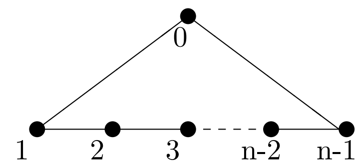


Figure 2.3: GCM and Dynkin diagram for $\widehat{sl}(n)$

The GCM A is associated with the affine Kac-Moody Lie algebra $\mathfrak{g} = \widehat{sl}(n)$. Note the similarities between this matrix and the matrix in Example 2.2.5. In particular, we have taken the Cartan matrix for $\mathring{\mathfrak{g}} = sl(n)$ and appended a row at the top and a column at the left. Just as the GCM's for $\mathring{\mathfrak{g}}$ and $\mathfrak{g}$ are closely related, so are the Lie algebras.

In fact, we can state the following relationship between $\mathring{\mathfrak{g}} = sl(n)$ and $\mathfrak{g} = \widehat{sl}(n)$:

$$\mathfrak{g} = \mathring{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d, \text{where}$$

- c is a central element,
- d is the derivation $1 \otimes t\frac{d}{dt}$,
- $[x \otimes t^i, y \otimes t^j] = [x, y] \otimes t^{i+j} + i\delta_{i+j,0}tr(xy)c$,
- $[c, x \otimes t^i] = 0$,
- $[d, c] = 0$, and

- $[d, x \otimes t^i] = i(x \otimes t^i)$.

Also note that $\mathfrak{h} = \mathring{\mathfrak{h}} \otimes 1 \oplus \mathbb{C}c \oplus \mathbb{C}d = \text{span}\{h_0, h_1, h_2, \dots h_{n-1}, d\}$ is the Cartan subalgebra. In addition, we define an element of Q to be the null root. For $\mathfrak{g} = \widehat{sl}(n), \delta = \sum_{i=0}^{n-1} \alpha_i$ is the null root.

We also wish to define a nondegenerate, symmetric, invariant bilinear form on $\mathfrak{h}$ and $\mathfrak{h}^*$ for $\mathfrak{g} = \widehat{sl}(n)$. We begin by defining the bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$ by

$$(h_i|h_j) = a_{ij}, i, j \in I \qquad (h_i|d) = \alpha_i(d), i \in I.$$

Now, we wish to extend this form to $\mathfrak{h}^*$. Consider $\nu \colon \mathfrak{h} \to \mathfrak{h}^*$ such that $\nu(h)(h') = (h|h')\ \forall h \in \mathfrak{h}$. ν is a vector space isomorphism. For $\alpha, \beta \in \mathfrak{h}^*$, define $(\alpha|\beta) = (\nu^{-1}(\alpha)|\nu^{-1}(\beta))$. We can see that $\nu(h_i) = \alpha_i$ and so

$$(\alpha_i|\alpha_j) = a_{ij}.$$

Whether we consider the form on $\mathfrak{h}$ or on $\mathfrak{h}^*$, we call this nondegenerate, symmetric, invariant bilinear form and denote it by $(\cdot|\cdot)$.

2.3 The Universal Enveloping Algebra

In this section, we wish to construct the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, denoted $U(\mathfrak{g})$. We first define a tensor algebra by defining a sequence of spaces:

$$T^0(\mathfrak{g}) = \mathbb{C}, \qquad T^1(\mathfrak{g}) = \mathfrak{g}, \qquad T^2(\mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g}, \qquad \dots, \qquad T^n(\mathfrak{g}) = \underbrace{\mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{n}.$$

Now define

$$T(\mathfrak{g}) = \bigoplus_{n \geq 0} T^n(\mathfrak{g}).$$

$T(\mathfrak{g})$ is a free associative algebra and for the inclusion map i, any associative algebra A and any linear map ϕ, there exists a map ψ such that the following diagram commutes.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "i"] \arrow[dr, "\phi"'] & T(\mathfrak{g}) \arrow[d, "\exists!\ \psi"] \\ & A \end{tikzcd}$$

Consider the ideal $J =< i(x) \otimes i(y) - i(y) \otimes i(x) - i([x, y]) \mid x, y \in \mathfrak{g} >$ of $T(\mathfrak{g})$. We define the **universal enveloping algebra** of $\mathfrak{g}$ to be the Lie algebra $U(\mathfrak{g}) = T(\mathfrak{g})/J$. Now define $j\colon \mathfrak{g} \to U(\mathfrak{g})$ to be $j(x) = i(x) + J$; j is a Lie algebra homomorphism. For any associative algebra A and any linear map $\phi\colon \mathfrak{g} \to A$ satisfying $\phi([x, y]) = \phi(x)\phi(y) - \phi(y)\phi(x)$, there exists a unique homomorphism $\psi\colon U(\mathfrak{g}) \to A$ such that the following diagram commutes.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "j"] \arrow[dr, "\phi"'] & U(\mathfrak{g}) \arrow[d, "\exists!\ \psi"] \\ & A \end{tikzcd}$$

Alternately, we can construct the universal enveloping algebra of a Kac-Moody algebra $\mathfrak{g}$ with generators and relations.

Theorem 2.3.1 *[4] $U(\mathfrak{g})$ is the associative algebra over $\mathbb{C}$ generated by e_i, f_i $(i \in I)$ and $\mathfrak{h}$ subject to the following:*

1. $hh' = h'h$ *for* $h, h' \in \mathfrak{h}$,
2. $e_i f_j - f_j e_i = \delta_{ij} h_i$ *for* $i, j \in I$,
3. $he_i - e_i h = \alpha(h) e_i$ *for* $h \in \mathfrak{h}, i \in I$,
4. $hf_i - f_i h = -\alpha(h) f_i$ *for* $h \in \mathfrak{h}, i \in I$,
5. $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ *for* $i \neq j$,
6. $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ *for* $i \neq j$.

We wish to understand the universal enveloping algebra a bit more deeply. To do this, we will consider the Poincaré-Birkhoff-Witt Theorem.

Theorem 2.3.2 (c.f. [5])

1. *The map* $j\colon \mathfrak{g} \to U(\mathfrak{g})$ *is injective.*

2. *Let* $\{x_\alpha \mid \alpha \in \Omega\}$ *be an ordered basis of* $\mathfrak{g}$*. Then* $\{x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k} \mid k \geq 0, \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k\}$ *is a basis for* $U(\mathfrak{g})$*.*

That is to say, we have a basis for $U(\mathfrak{g})$ and we can view $\mathfrak{g}$ as a subspace of $U(\mathfrak{g})$.

Example 2.3.3 *Consider* $\mathring{\mathfrak{g}} = sl(2) = span\{f, h, e\}$*. Then a basis for* $U(\mathring{\mathfrak{g}})$ *is*

$$\{f^\ell h^k e^m | \ell, k, m \in \mathbb{Z}_{\geq 0}\}.$$

We also have the following proposition.

Proposition 2.3.4 *Let* $\mathfrak{g}$ *be a Lie algebra and let* $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_k$ *be subalgebras of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k$*. Then* $U(\mathfrak{g}) = U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2) \otimes \ldots \otimes U(\mathfrak{g}_k)$*.*

Note that if we have a triangular decomposition of a Kac-Moody algebra $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, we can write $U(\mathfrak{g}) = U(\mathfrak{n}_+) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_-)$.

One reason that the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is important to us concerns representations. Consider a $\mathfrak{g}$-module V. Since elements of $\mathfrak{g}$ generate $U(\mathfrak{g})$, we can define the action of $U(\mathfrak{g})$ on V using the module action of $\mathfrak{g}$ on V as follows:

$$(x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k}) \cdot v = x_{\alpha_1} \cdot (x_{\alpha_2} \cdots (x_{\alpha_k} \cdot v)).$$

Thus, we can extend a representation of $\mathfrak{g}$ to a representation of $U(\mathfrak{g})$. By the Poincaré-Birkhoff-Witt Theorem, we think of $\mathfrak{g}$ as a subspace of $U(\mathfrak{g})$ and so a representation of $U(\mathfrak{g})$ is a representation of $\mathfrak{g}$ as well. It is important to realize that when we study the representations of a Lie algebra $\mathfrak{g}$, we are also studying the representations of $U(\mathfrak{g})$ and vice versa.

2.4 Integrable Highest Weight Modules

In a previous section, we developed the notion of a $\mathfrak{g}$-module for a Lie algebra $\mathfrak{g}$. We wish to study specific $\mathfrak{g}$-modules called integrable highest weight $\mathfrak{g}$-modules, where $\mathfrak{g}$ is a Kac-Moody algebra. In this section, we will show the existence and uniqueness (up to isomorphism) of integrable highest weight modules of Kac-Moody algebras.

A $\mathfrak{g}$-module V is a **weight module** if we can write V as a direct sum of its weight spaces, i.e.

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda.$$

We are often concerned with the dimension of a weight space V_λ. We call this the **multiplicity of** λ. We call an element x of $\mathfrak{g}$ **locally nilpotent** on V if for each $v \in V$, there exists a positive integer N such that $x^N(v) = 0$. When all e_i, f_i of a weight module are locally nilpotent on V, the module is called **integrable**.

An important class of modules are highest weight modules. We define a highest weight module as follows.

Definition 2.4.1 *A weight module V is a* **highest weight module with highest weight** Λ *if there exists a nonzero $v_\Lambda \in V$ such that*

1. $e_i(v_\Lambda) = 0$ *for all* $i \in I$*,*
2. $h(v_\Lambda) = \Lambda(h)v_\Lambda$ *for all* $h \in \mathfrak{h}$ *(ie Λ is a weight of V),*
3. $U(\mathfrak{g})(v_\Lambda) = V$*.*

It follows that every other weight λ of V is less than Λ and the multiplicity of each weight is finite.

Now, we consider Verma modules, a specific type of highest weight modules. If a highest weight $\mathfrak{g}$-module $M(\Lambda)$ with highest weight Λ is such that every highest weight $\mathfrak{g}$-module with highest weight Λ is a quotient of $M(\Lambda)$, we call $M(\Lambda)$ a **Verma module**. To show that for a given Λ, the Verma module $M(\Lambda)$ exists, consider again the universal enveloping algebra $U(\mathfrak{g})$ and take the left ideal $J(\Lambda)$ of $U(\mathfrak{g})$ generated by the e_i's and $h - \Lambda(h)$ for $h \in \mathfrak{h}$. Then we can take

$$M(\Lambda) = U(\mathfrak{g})/J(\Lambda).$$

Since this construction of $M(\Lambda)$ is a highest weight module and any other highest weight module with highest weight Λ can be written as a quotient of $M(\Lambda)$, $M(\Lambda)$ is a Verma module. In addition, there exists a unique proper maximal submodule, $N(\Lambda)$, of $M(\Lambda)$. Thus, there is a unique, up to isomorphism, irreducible highest weight module with highest weight Λ:

$$V(\Lambda) = M(\Lambda)/N(\Lambda).$$

Now we have constructed a unique irreducible highest weight module for a particular highest weight. Recall that we want to study modules that have these properties and in addition are integrable. We need to determine a condition to ensure that $V(\Lambda)$, as constructed above, is also integrable. Consider the following sets.

Definition 2.4.2 *[10] Let $\mathfrak{g}$ be a Kac-Moody algebra and $\mathfrak{h}$ its Cartan subalgebra.*

1. *$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}, i \in I\}$ is the weight lattice. Elements in this set are known as* ***integral weights****.*

2. *$P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}, i \in I\}$ is the set of* ***dominant integral weights****.*

Proposition 2.4.3 *[10] The $\mathfrak{g}$-module $V(\Lambda)$ is integrable if and only if $\Lambda \in P^+$.*

Thus, we now have constructed unique, irreducible, integrable highest weight modules for every dominant integral weight. From now on, when we consider $V(\Lambda)$ we will assume that $\Lambda \in P^+$, unless explicitly stated otherwise. Also, we will denote the set of weights of $V(\Lambda)$ by $P(\Lambda)$. Note that $P(\Lambda) \subset P$.

Now, take $\mathfrak{g} = \widehat{sl}(n)$. We denote the canonical central element $c = h_0 + h_1 + \ldots + h_{n-1}$. Note that c commutes with other elements of $\mathfrak{g}$. We define the **level of $\Lambda \in \mathfrak{h}^*$ or of the**

module $V(\Lambda)$ to be

$$k = \sum_{i=0}^{n-1} \Lambda(h_i).$$

When $\Lambda \in P^+$, the level of Λ is a positive integer. We define the **fundamental weights** $\Lambda_i \in \mathfrak{h}^*$ for $i \in I$ to be such that $\Lambda_i(h_j) = \delta_{ij}$ and $\Lambda_i(d) = 0$.

2.5 Quantum Groups

In this section, we discuss the quantum deformations of the universal enveloping algebras of Kac-Moody algebras. These quantum deformations are also known as quantum groups and are denoted $U_q(\mathfrak{g})$. There is an important relationship between the representation theory of Kac-Moody algebras and the representation theory of quantum groups.

Let q be any indeterminate. For any integer n, we define the following:

- $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}} \in \mathbb{Z}[q, q^{-1}]$ are q**-integers**;
- $[0]_q! = 1, [n]_q! = [n]_q[n-1]_q[n-2]_q \cdots [1]_q$ for $n \in \mathbb{Z}_{>0}$;
- $\begin{bmatrix} m \\ n \end{bmatrix}_q = \dfrac{[m]_q!}{[n]_q![m-n]_q!}$ for $m, n \in \mathbb{Z}_{\geq 0}, m \geq n$ are q**-binomial coefficients**.

As usual, let A be a symmetrizable GCM with $D = \mathrm{diag}(s_i \in \mathbb{Z}_{>0} | i \in I)$.

Definition 2.5.1 *[4] The* **quantum group** *or the* **quantized universal enveloping algebra** $U_q(\mathfrak{g})$ *associated with Cartan datum* $(A, \Pi, \Pi^\vee, P^\vee, P)$ *is the associative algebra over* $\mathbb{C}(q)$ *with unity generated by* e_i, f_i $(i \in I)$ *and* q^h $(h \in P^\vee)$ *subject to the following relations.*

1. $q^0 = 1, q^h q^{h'} = q^{h+h'}$ *for* $h, h' \in P^\vee$,
2. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ *for* $h \in P^\vee, i \in I$,
3. $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ *for* $h \in P^\vee, i \in I$,
4. $e_i f_j - f_j e_i = \delta_{ij} \dfrac{q^{s_i h_i} - q^{-s_i q_i}}{q^{s_i} - q^{-s_i}}$ *for* $i, j \in I$,

5. $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{s_i}} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ *for* $i \neq j$,

6. $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{s_i}} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ *for* $i \neq j$.

Since the above relations are homogeneous, $U_q(\mathfrak{g})$ has the following root space decomposition:

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} U_q(\mathfrak{g})_\alpha,$$

where $U_q(\mathfrak{g})_\alpha = \{u \in U_q(\mathfrak{g}) \mid q^h u q^{-h} = q^{\alpha(h)} u \ \forall h \in P^\vee\}$.

Define $U_q(\mathfrak{g})^+$ to be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements e_i, $U_q(\mathfrak{g})^-$ to be that generated by the f_i's, $U_q(\mathfrak{g})^0$ to be the subalgebra generated by q^h for $h \in P^\vee$. Then, as we have had triangular decompositions of other spaces, we also have the following triangular decomposition for $U_q(\mathfrak{g})$. (c.f. [4].)

$$U_q(\mathfrak{g}) \cong U_q(\mathfrak{g})^+ \otimes U_q(\mathfrak{g})^0 \otimes U_q(\mathfrak{g})^-.$$

Quantum groups are neither groups nor Lie algebras. They have the structure of a Hopf algebra.

Proposition 2.5.2 *[4] $U_q(\mathfrak{g})$ has a Hopf algebra structure with comultiplication Δ, counit ϵ, and antipode S as follows*

1. $\Delta(q^h) = q^h \otimes q^h$, $\Delta(e_i) = e_i \otimes q^{-s_i h_i} + 1 \otimes e_i$, $\Delta(f_i) = f_i \otimes 1 + q^{s_i h_i} \otimes f_i$,

2. $\epsilon(q^h) = 1$, $\epsilon(e_i) = \epsilon(f_i) = 0$,

3. $S(q^h) = q^{-h}, S(e_i) = -e_i q^{s_i h_i}, S(f_i) = q^{-s_i h_i} f_i$.

We now will discuss the representation theory of quantum groups. It is important to notice that there are many similarities to the representation theory of Kac-Moody algebras. To distinguish between the theories, we will often include a superscript or subscript q to emphasize that we are discussing $U_q(\mathfrak{g})$-modules, not $U(\mathfrak{g})$-modules. We call a $U_q(\mathfrak{g})$-module V^q a weight module if it can be written

$$V^q = \bigoplus_{\lambda \in P} V^q_\lambda, \qquad \text{where } V^q_\lambda = \{v \in V^q \mid q^h v = q^{\lambda(h)} v \ \forall\ h \in P^\vee\}.$$

As might be expected, if V^q_λ is nonzero, λ is called a **weight** of V^q and V^q_λ is the **weight space** associated with $\lambda \in P$. The dimension of the weight space is called the **weight multiplicity** of λ.

Definition 2.5.3 *A* **highest weight module with highest weight** $\Lambda \in P$ *is a weight module* V^q *such that*

1. $e_i v_\Lambda = 0$ *for all* $i \in I$*,*
2. $q^h v_\Lambda = q^{\Lambda(h)} v_\Lambda$ *for all* $h \in P^\vee$
3. $V^q = U_q(\mathfrak{g}) v_\Lambda$*,*

and v_Λ *is called the* **highest weight vector***.*

Again, it follows that the multiplicity of any weight is finite and all weights are less than Λ.

As in the representation theory for Kac-Moody algebras, we wish to study unique irreducible, integrable highest weight modules. We follow a similar procedure to construct modules of this form. Let $\Lambda \in P$ and let e_i $(i \in I)$ and $q^h - q^{\Lambda(h)}$ generate the left ideal $J^q(\Lambda)$. As before, we define the **Verma module**

$$M^q(\Lambda) = U_q(\mathfrak{g})/J^q(\Lambda).$$

It can be shown that $M^q(\Lambda)$ is a highest weight module and that every highest weight $U_q(\mathfrak{g})$-module is a homomorphic image of $M^q(\Lambda)$. Additionally, $M^q(\Lambda)$ has a unique maximal submodule $N^q(\Lambda)$ and we can obtain an irreducible highest weight module

$$V^q(\Lambda) = M^q(\Lambda)/N^q(\Lambda).$$

As before, $V^q(\Lambda)$ is integrable if all e_i and f_i are locally nilpotent on V^q.

Proposition 2.5.4 *[4] $V^q(\Lambda)$ is integrable if and only if $\Lambda \in P^+$.*

Thus, we are able to construct a unique irreducible integrable highest weight $U_q(\mathfrak{g})$-module for each $\Lambda \in P^+$. Again, we must emphasize the relationship between $\mathfrak{g}$-modules and $U_q(\mathfrak{g})$-modules.

Proposition 2.5.5 *The multiplicity of λ in $V^q(\Lambda)$ is equal to the multiplicity of λ in $V(\Lambda)$.*

Chapter 3

Crystal Base

Crystal base theory is useful as a tool for studying representations of Kac-Moody algebras. By using a combinatorial description of crystal bases, the multiplicity of weights for particular representations can be found.

3.1 Crystal Base

Let $V^q = \bigoplus_{\lambda \in P} V^q_\lambda$ be an integrable $U_q(\mathfrak{g})$-module such that all weight multiplicities are finite and the set of all weights of V^q is contained in $D(\lambda_1) \cup \cdots \cup D(\lambda_s)$, where $\lambda_1, \ldots, \lambda_s \in P$ and $D(\lambda) = \{\mu \in P \mid \mu \leq \lambda\}$. Let λ be a weight of V^q so that V^q_λ is nonzero. Then for each i, $v \in V^q_\lambda$, it is known that v can be written as

$$v = v_0 + f_i^{(1)} v_1 + f_i^{(2)} v_2 + \ldots + f_i^{(N)} v_N,$$

where N is a nonnegative integer, $v_k \in V^q_{\lambda + k\alpha_i} \cap \ker e_i$, and $f_i^{(k)} = \dfrac{1}{[k]_q!} f_i^k$. (c.f. [4])

We define the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i \colon V^q \to V^q$ such that:

$$\tilde{e}_i(v) = \sum_{k=1}^{N} f_i^{(k-1)} v_k,$$
$$\tilde{f}_i(v) = \sum_{k=0}^{N} f_i^{(k+1)} v_k.$$

Note that for all $v \in V_\lambda^q$, $\tilde{e}_i(v) \in V_{\lambda+\alpha_i}^q$ and $\tilde{f}_i(v) \in V_{\lambda-\alpha_i}^q$.

Now, consider the integral domain $\mathbb{A}_0$ with fraction field $\mathbb{C}(q)$ defined as follows:

$$\mathbb{A}_0 = \left\{ \frac{g(q)}{h(q)} \middle| \, g(q), h(q) \in \mathbb{C}[q], h(0) \neq 0 \right\}.$$

We have the following definitions:

Definition 3.1.1 *[4] A free $\mathbb{A}_0$-submodule L of V^q is a* **crystal lattice** *if*

1. *L generates V^q as a vector space over $\mathbb{C}(q)$,*
2. *$L = \bigoplus_{\lambda \in P} L_\lambda$, where $L_\lambda = L \cap V_\lambda^q$, and*
3. *$\tilde{e}_i(L) \subset L$, $\tilde{f}_i(L) \subset L$ for all i.*

Note that $< q >$ is a maximal ideal of $\mathbb{A}_0$. There is a field isomorphism from $\mathbb{A}_0 / < q >$ onto $\mathbb{C}$, given by $f(q) + < q > \mapsto f(0)$. Thus, we have $\mathbb{C} \otimes_{\mathbb{A}_0} L \cong L/qL$.

Definition 3.1.2 *[4] A pair (L, B) is a* **crystal base** *for V^q if*

1. *L is a crystal lattice,*
2. *B is a $\mathbb{C}$ basis for L/qL,*
3. *$B = \bigcup_{\lambda \in P} B_\lambda$, where $B_\lambda = B \cap (L_\lambda / qL_\lambda)$,*
4. *$\tilde{e}_i(B), \tilde{f}_i(B) \subset B \cup \{0\}$, and*

5. For $b, b' \in B$, $\tilde{f}_i(b) = b' \iff b = \tilde{e}_i(b')$.

Here, we call B a **crystal**.

As we are particularly interested in irreducible highest weight modules, we wish to describe a crystal base for this type of module. Let Λ be a dominant integral weight and $V^q(\Lambda)$ the irreducible integrable $U_q(\mathfrak{g})$-module with highest weight Λ and highest weight vector v_Λ. Then the following theorem holds.

Theorem 3.1.3 *[17] $(L(\Lambda), B(\Lambda))$ is a crystal base of $V^q(\Lambda)$, where*

$$L(\Lambda) = \sum_{\substack{\ell \geq 0, \\ i_1, i_2, \ldots i_\ell \in I}} \mathbb{A}_0 \tilde{f}_{i_1} \tilde{f}_{i_2} \ldots \tilde{f}_{i_\ell} v_\Lambda,$$

$$B(\Lambda) = \left\{ \tilde{f}_{i_1} \tilde{f}_{i_2} \ldots \tilde{f}_{i_\ell} \bmod qL(\Lambda) \mid \ell \geq 0, i_1, i_2, \ldots, i_\ell \in I \right\}$$

An important utility of crystal bases and crystal base theory lies in the following theorem.

Theorem 3.1.4 *Let $V^q(\Lambda)$ be an integrable $U_q(\mathfrak{g})$-module and let (L, B) be a crystal base of V^q. Then for all $\lambda \in P$,*

$$\textit{mult } \lambda = \# B_\lambda.$$

That is, to find the multiplicity of a weight λ, we need only determine the number of elements in B_λ. To do this, we often use combinatorial objects to realize the crystal. For our purposes, we will use objects known as extended Young diagrams.

3.2 Extended Young Diagrams

The crystal base structures that we will need can be realized through extended Young diagrams. We will define these objects as constructed in [7] and as described in [17], [1].

Definition 3.2.1 *An* **extended Young diagram** $Y = (y_k)_{k \geq 0}$ *is a weakly increasing sequence* *with integer entries such that there exists some fixed y_∞ with $y_k = y_\infty$ for $k \gg 0$. y_∞ is called the* **charge** *of Y.*

With each sequence $(y_k)_{k \geq 0}$, we can draw a unique diagram, Y, in the $\mathbb{Z} \times \mathbb{Z}$ right half plane. For each element y_k of the sequence, we draw a column with depth y_∞ - y_k, aligned so the top of the column is at $y = y_\infty$. We fill in square boxes for all columns from the depth to the charge. That is, for the k-th sequence element, there will be a column of height $y_\infty - y_k$ filled in with boxes. We obtain a diagram with a finite number of boxes.

Example 3.2.2 *$Y = (-4, -4, -3, -2, -2, 0, 0, 0, \ldots)$ is an extended Young diagram. The associated diagram is given in Figure 3.1.*

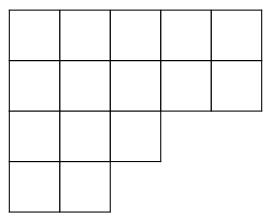


Figure 3.1: Extended Young Diagram in Diagram Form

Note that we will always have columns that decrease in height as we move from left to right. Given an extended Young diagram, Y, we assign a color to each box. There are n possible colors, indexed $i = 0, 1, \ldots, n-1$. For ease of notation, we often refer to color $(n-j)$ by $-j$. To determine the color of any box, consider the coordinates (a, b) of the lower right corner of the box. We color a box color i when $(a+b) \equiv i \bmod n$. Note that boxes on the same diagonal (upper left to lower right) will have the same color.

We define the weight $wt(Y)$ of an extended Young diagram Y of charge j to be $wt(Y) = \Lambda_j - \sum_{i=0}^{n-1} c_i \alpha_i$, where c_i is the number of boxes of color i in the diagram.

Example 3.2.3 *The diagram* $Y = (y_k)_{k \geq 0} = (-4, -4, -3, -2, -2, 0, 0, 0, \ldots)$ *with colored boxes is given in Figure 3.2.* $wt(Y) = \Lambda_0 - 3\alpha_0 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_{n-3} - 2\alpha_{n-2} - 2\alpha_{n-1}$.

0	1	2	3	4
−1	0	1	2	3
−2	−1	0		
−3	−2			

Figure 3.2: Extended Young Diagram in Tableau Form With Colored Boxes

The weight of a k-tuple $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_k)$ is the sum of the weights of its elements, i.e. $wt(\mathcal{Y}) = \displaystyle\sum_{i=1}^{k} wt(Y_i)$. Let $\Lambda = \Lambda_{\gamma_1} + \Lambda_{\gamma_2} + \ldots + \Lambda_{\gamma_k}$, where $0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_k \leq n-1$. Then $\mathcal{Y}(\Lambda) = \{\mathbf{Y} = (Y_j)_{1 \leq j \leq k} \mid Y_j \text{ has charge } \gamma_j\}$. We also define $Y[n] = (y_k + n)_{k \geq 0}$, which can be visualized as a vertical shift by n units of the Young diagram Y.

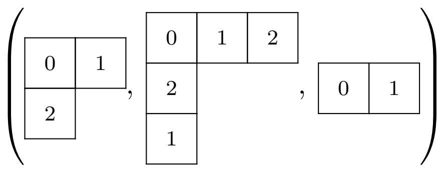
The following theorem shows the relationship between extended Young diagrams and the crystal base $B(\Lambda)$.

Theorem 3.2.4 *[7] Let* $\mathfrak{g} = \widehat{sl}(n, F)$*, and let* $\Lambda = \Lambda_{\gamma_1} + \cdots + \Lambda_{\gamma_k}$, $0 \leq \gamma_1 \leq \cdots \leq \gamma_k \leq n-1$. *Then,* $B(\Lambda) = \{\mathbf{Y} = (Y_1, \ldots, Y_k) \in \mathcal{Y}(\Lambda) | Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_k \supseteq (Y_{k+1} = Y_1[n]) \supseteq (Y_{k+2} = Y_2[n]) \supseteq \cdots \supseteq (Y_{2k} = Y_k[n])$*, and for each* $j \geq 0, \exists\ i \geq 1$ *s.t.* $(Y_{i+1})_j > (Y_i)_{j+1}\}$

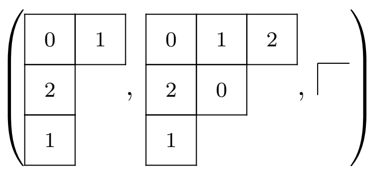
Notice that when $i > k$, $Y_i = Y_{i \pmod n}[n]$. Note that when comparing two diagrams of equal charge, the containment condition may be easier to visualize with the diagrams associated with sequences. However, when comparing diagrams of unequal charge, it may be easier to consider the sequences. The inequality condition is typically more understandable when viewing sequences as well.

Example 3.2.5 *We will let* $n = 3$ *and* $\Lambda = 3\Lambda_0$*. Let us look specifically at the elements of* $B(\Lambda)$ *of weight* $\mu = \Lambda - 3\alpha_0 - 4\alpha_1 - 3\alpha_2$*. That is, we must consider 3-tuples of extended*

Young diagrams containing 3 boxes of color 0, 4 boxes of color 1, and 3 boxes of color 2. These boxes must be oriented correctly to match the color rules. For example, the upper left box in each extended Young diagram must be of color 0. We wish to find the diagrams that fit the conditions of Theorem 3.2.4. To best illustrate the conditions of the theorem, we give examples of diagrams that fail at least one condition. For each example, we include each extended Young diagram both in diagram form and in sequence form. We then give several diagrams that meet all of the conditions.

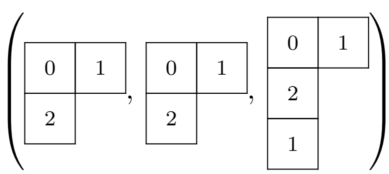


$$Y_1 = (\text{-}2, \text{-}1, 0, 0, 0, 0, \ldots) \quad Y_2 = (\text{-}3, \text{-}1, \text{-}1, 0, 0, 0, \ldots) \quad Y_3 = (\text{-}1, \text{-}1, 0, 0, 0, 0, \ldots) \quad Y_4 = (1, 2, 3, 3, 3, 3, \ldots)$$

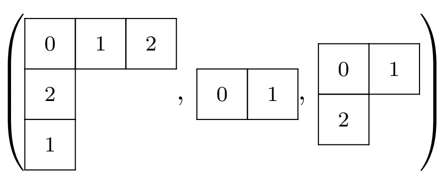


$$Y_1 = (\text{-}3, \text{-}1, 0, 0, 0, 0, \ldots) \quad Y_2 = (\text{-}3, \text{-}2, \text{-}1, 0, 0, 0, \ldots) \quad Y_3 = (0, 0, 0, 0, 0, 0, \ldots) \quad Y_4 = (0, 2, 3, 3, 3, 3, \ldots)$$

Figure 3.3: Diagrams That Fail $Y_1 \supseteq Y_2$

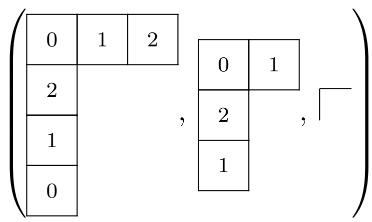


$$Y_1 = (\text{-}2, \text{-}1, 0, 0, 0, 0, \ldots) \quad Y_2 = (\text{-}2, \text{-}1, 0, 0, 0, 0, \ldots) \quad Y_3 = (\text{-}3, \text{-}1, 0, 0, 0, 0, \ldots) \quad Y_4 = (1, 2, 3, 3, 3, 3, \ldots)$$

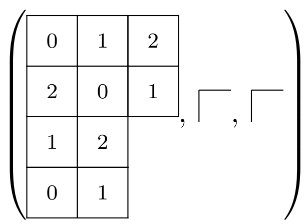


$$Y_1 = (\text{-}3, \text{-}1,\text{-}1, 0, 0, 0, \ldots) \quad Y_2 = (\text{-}1, \text{-}1, 0, 0, 0, 0, \ldots) \quad Y_3 = (\text{-}2, \text{-}1, 0, 0, 0, 0, \ldots) \quad Y_4 = (0, 2, 2, 3, 3, 3, \ldots)$$

Figure 3.4: Diagrams That Fail $Y_2 \supseteq Y_3$

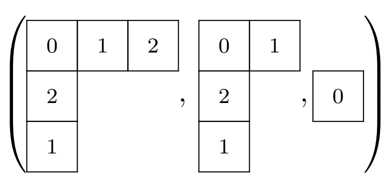


$Y_1 = $ (-4, -1, -1, 0, 0, 0, ...)
$Y_2 = $ (-3, -1, 0, 0, 0, 0, ...)
$Y_3 = $ (0, 0, 0, 0, 0, 0, ...)
$Y_4 = $ (-1, 2, 2, 3, 3, 3, ...)

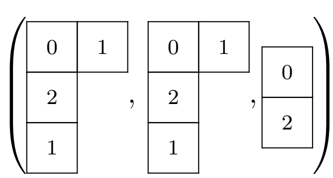


$Y_1 = $ (-4, -4, -2, 0, 0, 0, ...)
$Y_2 = $ (0, 0, 0, 0, 0, 0, ...)
$Y_3 = $ (0, 0, 0, 0, 0, 0, ...)
$Y_4 = $ (-1, -1, 1, 3, 3, 3, ...)

Figure 3.5: Diagrams That Fail $Y_3 \supseteq Y_4 = Y_1[3]$

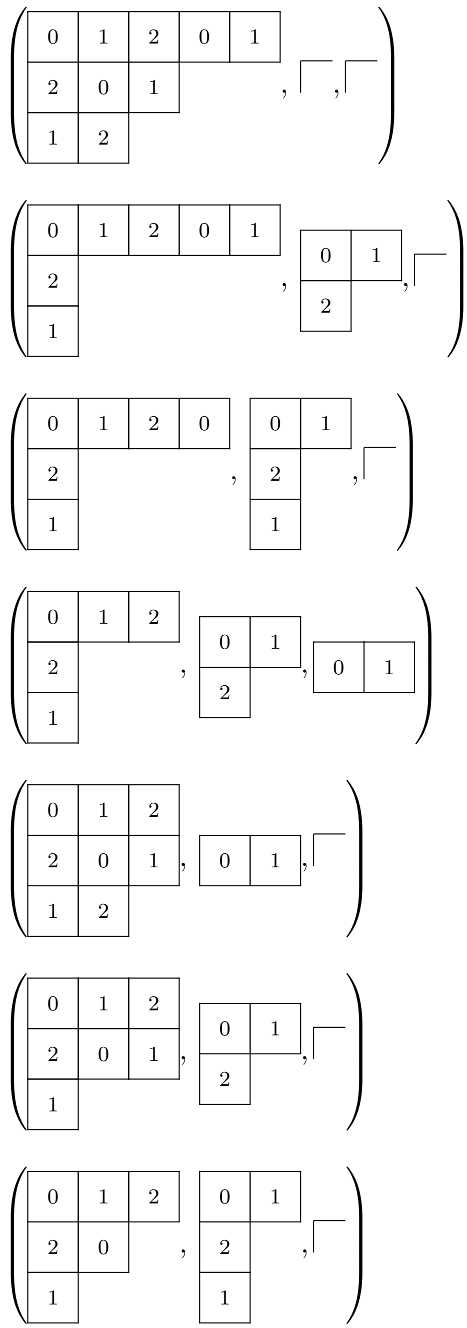


$Y_1 = $ (-3, -1, -1, 0, 0, 0, ...)
$Y_2 = $ (-3, -1, 0, 0, 0, 0, ...)
$Y_3 = $ (-1, 0, 0, 0, 0, 0, ...)
$Y_4 = $ (0, 2, 2, 3, 3, 3, ...)



$Y_1 = $ (-3, -1, 0, 0, 0, 0, ...)
$Y_2 = $ (-3, -1, 0, 0, 0, 0, ...)
$Y_3 = $ (-2, 0, 0, 0, 0, 0, ...)
$Y_4 = $ (0, 2, 3, 3, 3, 3, ...)

Figure 3.6: Diagrams That Fail $\forall\ j \geq 0, \exists\ i \geq 1$ s.t. $(Y_{i+1})_j > (Y_i)_{j+1}$



$Y_1 = (\text{-}3, \text{-}3, \text{-}2, \text{-}1, \text{-}1, 0, \dots)$
$Y_2 = (0, 0, 0, 0, 0, 0, \dots)$
$Y_3 = (0, 0, 0, 0, 0, 0, \dots)$
$Y_4 = (0, 0, 1, 2, 2, 3, \dots)$

$Y_1 = (\text{-}3, \text{-}1, \text{-}1, \text{-}1, \text{-}1, 0, \dots)$
$Y_2 = (\text{-}2, \text{-}1, 0, 0, 0, 0, \dots)$
$Y_3 = (0, 0, 0, 0, 0, 0, \dots)$
$Y_4 = (0, 2, 2, 2, 2, 3, \dots)$

$Y_1 = (\text{-}3, \text{-}1, \text{-}1, \text{-}1, 0, 0, \dots)$
$Y_2 = (\text{-}3, \text{-}1, 0, 0, 0, 0, \dots)$
$Y_3 = (0, 0, 0, 0, 0, 0, \dots)$
$Y_4 = (0, 2, 2, 2, 3, 3, \dots)$

$Y_1 = (\text{-}3, \text{-}1, \text{-}1, 0, 0, 0, \dots)$
$Y_2 = (\text{-}2, \text{-}1, 0, 0, 0, 0, \dots)$
$Y_3 = (\text{-}1, \text{-}1, 0, 0, 0, 0, \dots)$
$Y_4 = (0, 2, 2, 3, 3, 3, \dots)$

$Y_1 = (\text{-}3, \text{-}3, \text{-}2, 0, 0, 0, \dots)$
$Y_2 = (\text{-}1, \text{-}1, 0, 0, 0, 0, \dots)$
$Y_3 = (0, 0, 0, 0, 0, 0, \dots)$
$Y_4 = (0, 0, 1, 3, 3, 3, \dots)$

$Y_1 = (\text{-}3, \text{-}2, \text{-}2, 0, 0, 0, \dots)$
$Y_2 = (\text{-}2, \text{-}1, 0, 0, 0, 0, \dots)$
$Y_3 = (0, 0, 0, 0, 0, 0, \dots)$
$Y_4 = (0, 1, 1, 3, 3, 3, \dots)$

$Y_1 = (\text{-}3, \text{-}2, \text{-}1, 0, 0, 0, \dots)$
$Y_2 = (\text{-}3, \text{-}1, 0, 0, 0, 0, \dots)$
$Y_3 = (0, 0, 0, 0, 0, 0, \dots)$
$Y_4 = (0, 1, 2, 3, 3, 3, \dots)$

Figure 3.7: Diagrams That Satisfy Theorem 3.2.4

Chapter 4

Maximal Dominant Weights

In this chapter, we determine the maximal dominant weights of $V(k\Lambda_0)$ and of $V((k-1)\Lambda_0+\Lambda_s)$, where $1 \le s \le \lfloor \frac{n}{2} \rfloor$.

4.1 Preliminaries

From now on, we will take $\mathfrak{g} = \widehat{sl}(n)$. The Cartan datum is $(A, \check{P}, P, \check{\Pi}, \Pi)$, where

- $A = (a_{ij})_{i,j=0}^{n-1}$, with $a_{ii} = 2, a_{ij} = -1$ for $|i-j| = 1, a_{0,n-1} = a_{n-1,0} = -1$, and $a_{ij} = 0$ otherwise,
- $\check{P} = \left(\bigoplus_{i=0}^{n-1} \mathbb{Z} h_i \right) \oplus \mathbb{Z} d$,
- $P = \mathrm{Hom}_{\mathbb{Z}}(\check{P}, \mathbb{Z})$,
- $\check{\Pi} = \{h_0, h_1, \dots, h_{n-1}\}$, and
- $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$.

A is of affine type and $\mathfrak{h} = \{h_0, h_1, \dots, h_{n-1}, d\}$ is the Cartan subalgebra. Let $\mathring{A} = (a_{ij})_{i,j=1}^{n-1}$ be the matrix A with the first row and column deleted. Note that $\mathring{A}$ is the Cartan matrix for $sl(n)$.

Consider the integrable highest weight module $V(\Lambda)$ where $\Lambda \in P^+$ and let $P(\Lambda)$ denote the set of weights of $V(\Lambda)$. Although the number of weights of modules of this type is infinite, there are only finitely many maximal dominant weights. The maximal weights of $V(\Lambda)$ are those weights $\lambda \in P(\Lambda)$ such that $\lambda + \delta \notin P(\Lambda)$. We denote the set of maximal weights of $V(\Lambda)$ by $\max(\Lambda)$. A proposition from [10] states that

$$P(\Lambda) = \bigcup_{\lambda \in \max(\Lambda)} \{\lambda - n\delta | n \in \mathbb{Z}_{\geq 0}\}.$$

Thus, the maximal weights are those that, when we consider the weight structure, form something like a roof; other weights occur on strings stemming from the maximal weights. Now, we wish to determine the maximal dominant weights of various highest weight modules $V(\Lambda)$. To do this, we can use the following proposition found in [10].

Proposition 4.1.1 *The map* $\lambda \mapsto \overline{\lambda}$ *is a bijection from* $\max(\Lambda) \cap P^+$ *to* $kC_{af} \cap (\overline{\Lambda} + \overline{Q})$, *where* k *is the level of* Λ. *In particular, the set of dominant maximal weights of* $V(\Lambda)$ *is finite.*

To understand this proposition we need to define the following:

- $^-\colon \mathfrak{h}^* \to \mathring{\mathfrak{h}}^*$, where $\lambda \mapsto \overline{\lambda} = \lambda - \lambda(c)\Lambda_0 - (\lambda|\Lambda_0)\delta$
- $\mathring{\mathfrak{h}}^*_{\mathbb{R}} = span_{\mathbb{R}}\{h_1, h_2, \ldots, h_{n-1}\} \subset \mathfrak{h}$.
- $kC_{af} = \{\lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{R}} \mid \lambda(h_i) \geq 0, (\lambda|\theta) \leq k\}$

Note that Proposition 4.1.1 shows that the set of maximal dominant weights of $V(\Lambda)$ is finite. Since the set $kC_{af} \cap (\overline{\Lambda} + \overline{Q})$ is known, we can use information about the map $^-$ to determine the set of maximal dominant weights explicitly.

Example 4.1.2 *It is useful to examine the map* $^-$ *in a little more depth.*

$$\overline{\alpha_0} = \alpha_0 - \alpha_0(c)\Lambda_0 - (\alpha_0|\Lambda_0)\delta$$

$$= \alpha_0 - 0 - \sum_{i=0}^{\ell} \alpha_i$$

$$= -\sum_{i=1}^{\ell} \alpha_i$$

$$i \neq 0 : \overline{\alpha_i} = \alpha_i - \alpha_i(c)\Lambda_i - (\alpha_i|\Lambda_0)\delta$$

$$= \alpha_i - 0 - 0$$

$$= \alpha_i$$

$$\overline{\Lambda_0} = \Lambda_0 - \Lambda_0(c)\Lambda_0 - (\Lambda_0|\Lambda_0)\delta$$

$$= \Lambda_0 - \Lambda_0 - 0$$

$$= 0$$

$$i \neq 0 : \overline{\Lambda_i} = \Lambda_i - \Lambda_i(c)\Lambda_0 - (\Lambda_i|\Lambda_0)\delta$$

$$= \Lambda_i - \Lambda_0 - 0$$

$$= \Lambda_i - \Lambda_0$$

Recall that the Dynkin diagram for $\mathfrak{g}$ is as in Figure 4.1.

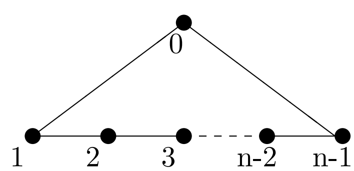


Figure 4.1: Dynkin Diagram for $\mathfrak{g}$.

That is to say, the diagram has a cycle. We will first be considering the maximal dominant weights of $V(k\Lambda_0)$. Notice that our results can just be shifted around the diagram to give the maximal dominant weights of $V(k\Lambda_i)$ for $(i \in I)$. Next, we will examine the maximal dominant weights of $V((k-1)\Lambda_0 + \Lambda_s)$, where $1 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor$. Because of the cyclic nature of the diagram, we need not consider values of s greater than $\left\lfloor \frac{n}{2} \right\rfloor$. In addition, finding the maximal dominant weights of $V((k-1)\Lambda_i + \Lambda_s)$ for $i \neq 0$ can be done by just shifting indices.

In [21], Tsuchioka examines the maximal dominant weights for the $\widehat{sl}(p)$-modules $V(2\Lambda_0)$ and $V(\Lambda_0+\Lambda_s)$, $1 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor$. He stipulates that p be prime, a restriction we have removed here. In the same paper, he finds the multiplicity of the maximal dominant weights as well. These results become a special case of the results that follow in this thesis.

4.2 Maximal dominant weights of $V(k\Lambda_0)$

Theorem 4.2.1 *Let $n \geq 2$, $k \geq 1$, $\lambda \in \max(k\Lambda_0) \cap P^+$. Then $\lambda = k\Lambda_0 - \sum_{i=0}^{n-1}(\ell - x_i)\alpha_i$, where $\ell, x_i \in \mathbb{Z}_{\geq 0}$, $\ell - x_i \geq 0$, $x_0 = 0$,*

Then the elements of $(x_1, x_2, \ldots, x_{n-1})$ are as follows:

- $(x_1, x_2, \ldots x_p) =$

$$\begin{aligned}
&(x_1, 2x_1, \ldots, x_1 j_{x_1},\\
&x_1 j_{x_1} + (x_1 - 1), x_1 j_{x_1} + 2(x_1 - 1), \ldots, x_1 j_{x_1} + (x_1 - 1) j_{x_1 - 1}\\
&\vdots\\
&1 + \sum_{m=0}^{x_1-2}(x_1 - m) j_{x_1 - m}, 2 + \sum_{m=0}^{x_1-2}(x_1 - m) j_{x_1 - m}, \ldots, \ell - 1)
\end{aligned}$$

- $x_{p+1} = x_{p+2} = \ldots = x_{q-1} = \ell$

- $(x_q, x_{q+1}, \ldots x_{n-1}) =$

$$\begin{aligned}
&(\ell - 1, \ell - 2, \ldots, 1 + \sum_{m=2}^{x_{n-1}} m i_m,\\
&\sum_{m=2}^{x_{n-1}} m i_m, \sum_{m=2}^{x_{n-1}} (m i_m) - 2, \ldots, 2 + \sum_{m=3}^{x_{n-1}} m i_m,\\
&\vdots\\
&x_{n-1} i_{x_{n-1}}, x_{n-1} i_{x_{n-1}} - x_{n-1}, \ldots, x_{n-1}),
\end{aligned}$$

satisfying

– $x_1 + x_{n-1} \leq k$

– *Let* $\beta = \dfrac{x_1 x_{n-1}}{x_1 + x_{n-1}}$*. Then* $0 \leq \ell \leq (\lfloor \beta n \rfloor - \min\{\lfloor \beta n \rfloor \bmod x_1, \lfloor \beta n \rfloor \bmod x_{n-1}\})$.

– $p = \ell - 1 - \sum_{m=2}^{x_1} (m-1) j_m$

– $q = n - (\ell - 1) + \sum_{m=2}^{x_{n-1}} (m-1) i_m$

– $0 \leq j_m \leq \left\lfloor \frac{\ell}{m} \right\rfloor, 1 \leq m \leq x_1$

– $0 \leq i_m \leq \left\lfloor \frac{\ell}{m} \right\rfloor, 1 \leq m \leq x_{n-1}$

– $\sum_{m=1}^{x_{n-1}} m i_m = \sum_{m=1}^{x_1} m j_m = \ell$

– $\max\{2\ell - n, 0\} \leq \sum_{m=2}^{k-1} (m-1)(i_m + j_m)$

Explanation of Terms: We let i_m be the number of times that we decrease by m in consecutive coefficients of the x_i's. In this count, we add one if x_{n-1} is equal to m. Similarly, j_m will be the number of times that we increase by m in consecutive coefficients of the x_i's and we add one to j_m if x_1 is equal to m.

As previously mentioned, a special case of this theorem was proved by Tsuchioka in [21]. This matches Theorem 4.2.1 when $k = 2$ and n is prime. The weights listed here will be important as we continue our study.

Corollary 4.2.2 *[21] Let n be prime and $n \geq 2, \Lambda = 2\Lambda_0$. Then* $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - \gamma_\ell \mid 1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor\}$*, where*

$$\begin{aligned} \gamma_\ell = \ell\alpha_0 + &(\ell-1)\alpha_1 + (\ell-2)\alpha_2 + \ldots + \alpha_{\ell-1} \\ &+ \alpha_{n-\ell+1} + 2\alpha_{n-\ell+2} + \ldots + (\ell-1)\alpha_{n-1}. \end{aligned}$$

Corollary 4.2.3 *For fixed n, if* $(2\Lambda_0 - \gamma) \in \max(2\Lambda_0) \cap P^+$*, then* $(k\Lambda_0 - \gamma) \in \max(k\Lambda_0) \cap P^+$ *for* $k \geq 2$.

Proof Suppose $(2\Lambda_0 - \gamma) \in \max(2\Lambda_0) \cap P^+$. Then γ is associated with a specific x_1, x_{n-1}, and ℓ. All but one boundary condition depends only on these values and not on k, and so are easily satisfied. We need only recognize that since $x_1 + x_{n-1} \leq 2$, we also have $x_1 + x_{n-1} \leq k$. Thus, γ satisfies the required conditions for $(k\Lambda_0 - \gamma) \in \max(k\Lambda_0) \cap P^+$.

This result will allow us to look at certain types of weights that appear in $V(k\Lambda_0)$ for various k. We wish to prove Theorem 4.2.1, but to do so, there is more groundwork to be done.

Remark 4.2.4 *The following facts about ceiling and floor functions are well known, but may not be familiar to the reader. Assume $n, m \in \mathbb{Z}, x, y \in \mathbb{R}$, n positive. (c.f. [3])*

1. $\left\lfloor \dfrac{x+m}{n} \right\rfloor = \left\lfloor \dfrac{\lfloor x \rfloor + m}{n} \right\rfloor$
2. $\left\lceil \dfrac{x+m}{n} \right\rceil = \left\lceil \dfrac{\lceil x \rceil + m}{n} \right\rceil$
3. $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x+y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$
4. $\lceil x \rceil + \lceil y \rceil - 1 \leq \lceil x+y \rceil \leq \lceil x \rceil + \lceil y \rceil$
5. $\left\lceil \dfrac{m}{n} \right\rceil = \left\lfloor \dfrac{m-1}{n} \right\rfloor + 1$
6. $\left\lfloor \dfrac{m}{n} \right\rfloor = \left\lceil \dfrac{m+1}{n} \right\rceil - 1$

Lemma 4.2.5 *Let $x, a, j \in \mathbb{N}$, $a \equiv j \mod x$, $0 \leq q \leq (j-1)$. Then* $\left\lceil \dfrac{x-q}{a} \right\rceil = \left\lfloor \dfrac{x}{a} \right\rfloor + 1.$

Proof. $\left\lceil \dfrac{x-q}{a} \right\rceil = \left\lfloor \dfrac{x-q-1}{a} \right\rfloor + 1 = \left\lfloor \dfrac{x}{a} \right\rfloor + 1.$

Lemma 4.2.6 *Let $a, b, n \in \mathbb{N}$, $n \geq 2$. Let ℓ be the maximum integer satisfying* $\left\lceil \dfrac{\ell}{a} \right\rceil + \left\lceil \dfrac{\ell}{b} \right\rceil \leq n$. *Then* $\ell = \left\lfloor \dfrac{abn}{a+b} \right\rfloor - \min\left\{ \left\lfloor \dfrac{abn}{a+b} \right\rfloor \mod a, \left\lfloor \dfrac{abn}{a+b} \right\rfloor \mod b \right\}$

Proof. Without loss of generality, assume $j = \min\left\{ \left\lfloor \frac{abn}{a+b} \right\rfloor \mod a, \left\lfloor \frac{abn}{a+b} \right\rfloor \mod b \right\} = \left\lfloor \frac{abn}{a+b} \right\rfloor \mod a$.

First, we will show that $\ell = \left\lfloor \frac{abn}{a+b} \right\rfloor - j$ satisfies $\left\lceil \frac{\ell}{a} \right\rceil + \left\lceil \frac{\ell}{b} \right\rceil \leq n$ and then we will show that ℓ is the maximum such integer.

$$\begin{aligned}
\left\lceil \frac{\ell}{a} \right\rceil + \left\lceil \frac{\ell}{b} \right\rceil &= \left\lceil \frac{\lfloor \frac{abn}{a+b} \rfloor - j}{a} \right\rceil + \left\lceil \frac{\lfloor \frac{abn}{a+b} \rfloor - j}{b} \right\rceil \\
&= \left\lfloor \frac{\lfloor \frac{abn}{a+b} \rfloor - j}{a} \right\rfloor + \left\lfloor \frac{\lfloor \frac{abn}{a+b} \rfloor - (j+1)}{b} \right\rfloor + 1 \\
&= \left\lfloor \frac{\frac{abn}{a+b} - j}{a} \right\rfloor + \left\lfloor \frac{\frac{abn}{a+b} - (j+1)}{b} \right\rfloor + 1 \\
&\leq \left\lfloor \frac{abn - j(a+b)}{a(a+b)} + \frac{abn - (j+1)(a+b)}{b(a+b)} \right\rfloor + 1 \\
&= \left\lfloor \frac{ab^2n - jb(a+b) + a^2bn - a(j+1)(a+b)}{ab(a+b)} \right\rfloor + 1 \\
&= \left\lfloor n - \frac{jb(a+b) + a(j+1)(a+b)}{ab(a+b)} \right\rfloor + 1 \\
&= n + \left\lfloor - \frac{a(j+1) + jb}{ab} \right\rfloor + 1 \\
&\leq n
\end{aligned}$$

Now suppose we add 1 to the value of ℓ, i.e.

$$\begin{aligned}
\left\lceil \frac{\ell+1}{a} \right\rceil + \left\lceil \frac{\ell+1}{b} \right\rceil &= \left\lceil \frac{\lfloor \frac{abn}{a+b} \rfloor - (j-1)}{a} \right\rceil + \left\lceil \frac{\lfloor \frac{abn}{a+b} \rfloor - (j-1)}{b} \right\rceil \\
&= \left\lfloor \frac{\left\lfloor \frac{abn}{a+b} \right\rfloor}{a} \right\rfloor + 1 + \left\lfloor \frac{\left\lfloor \frac{abn}{a+b} \right\rfloor}{b} \right\rfloor + 1 \text{ (Lemma 4.2.5)} \\
&= \left\lfloor \frac{abn}{a(a+b)} \right\rfloor + 1 + \left\lfloor \frac{abn}{b(a+b)} \right\rfloor + 1 \\
&\geq \left\lfloor \frac{abn}{a(a+b)} + \frac{abn}{b(a+b)} \right\rfloor + 1 \\
&= n + 1, \text{ and } \ell + 1 \text{ is too large, as desired.}
\end{aligned}$$

Now let $m \in \mathbb{Z}_{>1}$, as follows

$$\begin{aligned}
\left\lceil \frac{\ell+m}{a} \right\rceil + \left\lceil \frac{\ell+m}{b} \right\rceil &= \left\lceil \frac{\lfloor \frac{abn}{a+b} \rfloor - j + m}{a} \right\rceil + \left\lceil \frac{\lfloor \frac{abn}{a+b} \rfloor - j + m}{b} \right\rceil \\
&\geq \left\lceil \frac{\lfloor \frac{abn}{a+b} \rfloor - (j-1)}{a} \right\rceil + \left\lceil \frac{m-1}{a} \right\rceil - 1 + \left\lceil \frac{\lfloor \frac{abn}{a+b} \rfloor - (j-1)}{b} \right\rceil + \left\lceil \frac{m-1}{b} \right\rceil - 1 \\
&= \left\lfloor \frac{\left\lfloor \frac{abn}{a+b} \right\rfloor}{a} \right\rfloor + \left\lceil \frac{m-1}{a} \right\rceil + \left\lfloor \frac{\left\lfloor \frac{abn}{a+b} \right\rfloor}{b} \right\rfloor + \left\lceil \frac{m-1}{b} \right\rceil \quad \text{(Lemma 4.2.5)} \\
&\geq n - 1 + \left\lceil \frac{m-1}{a} \right\rceil + \left\lceil \frac{m-1}{b} \right\rceil \\
&\geq n + 1 \text{ when } m > 1.
\end{aligned}$$

Thus, $\ell = \left\lfloor \frac{abn}{a+b} \right\rfloor - j$ is the value we desired.

Lemma 4.2.7 *For $n \geq 2$,*
$\{\boldsymbol{x} = (x_1, x_2, \ldots, x_{n-1})^T | x_1 = a \in \mathbb{Z}_{>0}, x_{n-1} = b \in \mathbb{Z}_{>0}$ with $(\mathring{A}\boldsymbol{x})_i \geq 0$, and $x_i \in \mathbb{Z}$ for $i = 1, 2, \ldots n-1\}$ is equal to the following set.

$$\begin{aligned}
\{\gamma = (&a, 2a, \ldots, aj_a, aj_a + (a-1), aj_a + 2(a-1), \ldots, aj_a + (a-1)j_{a-1} \\
&\vdots \\
&(a-j) + \sum_{m=0}^{j-1} (a-m)j_{a-m}, 2(a-j) + \sum_{m=0}^{j-1} (a-m)j_{a-m}, \ldots, \sum_{m=0}^{j} (a-m)j_{a-m}, \\
&\vdots \\
&1 + \sum_{m=0}^{a-2} (a-m)j_{a-m}, 2 + \sum_{m=0}^{a-2} (a-m)j_{a-m}, \ldots, \ell - 1, \\
&\ell^{<n-2\ell+1+\sum_{m=2}^{k-1}(m-1)(i_m+j_m)>}, \\
&\ell - 1, \ell - 2, \ldots, 1 + \sum_{m=2}^{b} mi_m, \sum_{m=2}^{b} mi_m, \sum_{m=2}^{b} (mi_m) - 2, \ldots, 2 + \sum_{m=3}^{b} mi_m, \\
&\vdots
\end{aligned}$$

$$\sum_{m=j}^{b} mi_m, \sum_{m=j}^{b}(mi_m) - j, \ldots, j + \sum_{m=j+1}^{b} mi_m,$$

$$\vdots$$

$$bi_b, bi_b - b, \ldots, b)\}$$

where $\ell^{<m>}$ *is defined as* $\underbrace{\ell, \ell, \ldots, \ell}_{m}$ *and the following bounds are satisfied:*

- *Let* β *be the fraction* $\dfrac{ab}{a+b}$*. Then*

$$\max\{a, b\} \leq \ell \leq (\lfloor \beta n \rfloor - \min\{(\lfloor \beta n \rfloor \bmod a), (\lfloor \beta n \rfloor \bmod b)\}),$$

- $0 \leq i_m \leq \left\lfloor \dfrac{\ell}{m} \right\rfloor, 1 \leq m \leq b,\ 1 \leq i_b,$ *with* $i_m \in \mathbb{Z}_{\geq 0}$,
- $0 \leq j_m \leq \left\lfloor \dfrac{\ell}{m} \right\rfloor, 1 \leq m \leq a,\ 1 \leq j_a,$*with* $j_m \in \mathbb{Z}_{\geq 0}$,
- $\max\{2\ell - n, 0\} \leq \displaystyle\sum_{m=2}^{\max\{a,b\}} (m-1)(i_m + j_m),$
- $\displaystyle\sum_{m=1}^{b} mi_m = \sum_{m=1}^{a} mj_m = \ell.$

Proof. Since $\mathring{A}$ is a Cartan Matrix of finite type, $x_i \geq 0$ for $i = 1, 2, \ldots, n-1$.

Claim 4.2.8 *If* $x_{k-1} > x_k$*, then* $x_k > x_{k+1}$*.*

Proof. $x_{k-1} > x_k \implies x_k + x_{k-1} > 2x_k \implies x_k > 2x_k - x_{k-1}$.

$0 \leq -x_{k-1} + 2x_k - x_{k+1} < x_k - x_{k+1}$, and thus $x_k > x_{k+1}$.

Claim 4.2.9 *If* $x_k = x_{k-1}$*, then* $x_j \leq x_k$*,* $k \leq j \leq n-1$*.*

Proof. Let $x_k = x_{k-1}$. Then $0 \leq -x_{k-1} + 2x_k - x_{k+1} = x_k - x_{k+1} \implies x_{k+1} \leq x_k$. Using Claim 4.2.8, we obtain the desired result.

From these two claims and using the symmetry of $\mathring{A}$, we see that as the index k increases, the values of x_k increase for each successive x_k, attaining a maximum value which we will call ℓ and which may repeat. Then the values of x_k decrease (if necessary) to $x_{n-1} = b$.

Claim 4.2.10 *If $x_k - x_{k-1} \leq m$, then $x_{k+1} - x_k \leq m$, where $m \in \mathbb{Z}$.*

Proof. Suppose $x_k - x_{k-1} \leq m$. Then

$$0 \leq -x_{k-1} + 2x_k - x_{k+1} \leq m + x_k - x_{k+1} \implies x_{k+1} - x_k \leq m.$$

Claim 4.2.10 describes how the values of x_k increase and decrease. Moving from left to right, we can increase some number of times by a, then by $a-1$, $a-2$, and so on. By symmetry, if we decrease, we decrease some number of times (possibly zero) by 1, then by 2, and so on until we decrease by b from x_{n-1} to $x_n = 0$. Thus the coefficients of the x_k's follow the pattern stated in the lemma.

Now that we have determined the coefficients, we must describe the bounds.

- For a particular k and n, the maximum value of ℓ will be attained in the $n-1$ tuple that increases by a and decreases by b as many times as possible. Otherwise, the maximum value of ℓ would not be attained. By this reasoning, we make the relation $\lceil \frac{\ell-a}{a} \rceil + 1 + \lceil \frac{\ell-b}{b} \rceil \leq n-1$, which is equivalent to $\lceil \frac{\ell}{a} \rceil + \lceil \frac{\ell}{b} \rceil \leq n$. Thus the upperbound for ℓ must satisfy this relation. By the Lemma 4.2.6, we see that the upperbound for ℓ is $\left\lfloor \dfrac{abn}{a+b} \right\rfloor - \min\left\{ \left\lfloor \dfrac{abn}{a+b} \right\rfloor \bmod a, \left\lfloor \dfrac{abn}{a+b} \right\rfloor \bmod b \right\}$
- If we were to increase by m more than $\lfloor \frac{\ell}{m} \rfloor$ times, then ℓ would no longer be the maximum value attained. Similar for decreasing by j_m. This gives us the conditions involving i_m and j_m.
- Since $2\ell - \displaystyle\sum_{m=2}^{\max\{a,b\}} (m-1)(i_m + j_m) - 1 \leq n-1$, and each i_m and j_m must be nonnegative, we obtain $\max\{2\ell - n, 0\} \leq \displaystyle\sum_{m=2}^{\max\{a,b\}} (m-1)(i_m + j_m)$.

- Consider $x_0 = 0$. Then from x_0 to x_1, we increase by a one time. Each time we increase by any number, we are getting closer to ℓ. Thus ℓ is the summation of the number of times we increase by each value multiplied by that value. We obtain $\sum_{m=1}^{b} m i_m = \sum_{m=1}^{a} m j_m = \ell$. $\square$

Proof of Theorem 4.2.1

By Proposition 4.1.1, $\max(\Lambda) \cap P^+$ is bijective to $kC_{af} \cap \overline{Q}$ via the map $\lambda \mapsto \overline{\lambda}$. We will find all $\lambda \in kC_{af} \cap \overline{Q}$ and use the inverse of this bijective map to list all elements of $\max(\Lambda) \cap P^+$. If $\lambda = \Lambda + \sum_{j=0}^{n-1} q_j \alpha_j \in \max(\Lambda) \cap P^+$ (with $q_j \in \mathbb{Z}_{\leq 0}, 1 \leq j \leq n-1$) maps to $\overline{\lambda} = \sum_{j=1}^{n-1} x_j \alpha_j \in kC_{af} \cap \overline{Q}$ via the above bijection, we can use this map to obtain the relation $x_j = q_j - q_0,\ 1 \leq j \leq n-1$. (c.f. [21].)

By definition,

$$kC_{af} \cap \overline{Q} = \left\{ \lambda = \sum_{j=1}^{n-1} x_j \alpha_j \;\middle|\; \lambda(h_j) \geq 0,\ 1 \leq j < n, (\lambda|\theta) \leq k \right\}.$$

Thus the elements $\lambda \in kC_{af} \cap \overline{Q}$ satisfy the following conditions:

$$\begin{cases} \lambda(h_1) = 2x_1 - x_2 & \geq 0, \\ \lambda(h_2) = -x_1 + 2x_2 - x_3 & \geq 0, \\ \vdots & \vdots \\ \lambda(h_{n-2}) = -x_{n-3} + 2x_{n-2} - x_{n-1} & \geq 0, \\ \lambda(h_{n-1}) = -x_{n-2} + 2x_{n-1} & \geq 0, \\ (\lambda|\theta) = x_1 + x_{n-1} & \leq k. \end{cases}$$

This is equivalent to the set of all $n-1$ tuples $(x_1, x_2, \ldots, x_{n-1})$ such that $x_1 + x_{n-1} \leq k$ and $\mathring{A}\mathbf{x} = 0$. Since $\mathring{A}$ is of finite type, $\lambda(h_j) \geq 0, j \in \{1, 2, \ldots, n-1\}$ implies that $x_j \geq 0, j \in \{1, 2, \ldots, n-1\}$. It is easy to see that $x_1 = 0 \iff x_2 = 0 \iff \ldots \iff x_{n-1} = 0$, so that either $x_1 = x_2 = \ldots = x_{n-1} = 0$ or $x_1, x_{n-1} \in \mathbb{Z}_{>0}$ and $x_1 + x_{n-1} \leq k$.

From Lemma 4.2.7, we are able to define the set of $(n-1)$-tuples $(x_1, x_2, \ldots, x_{n-1})$ satisfying these conditions. Since $x_j = q_j - q_0$ and thus $q_j = x_j + q_0$, we have

$$(q_1, q_2, \ldots, q_{n-1}) = (x_1 + q_0, x_2 + q_0, \ldots, (\ell + q_0)^{<n-2\ell+1+\sum_{m=2}^{k-1}(m-1)(i_m+j_m)>}, \\ \ldots, x_{n-2} + q_0, x_{n-1} + q_0).$$

$q_0 \leq 0$ implies that $q_0 = -\ell - r$ for some $r \in \mathbb{Z}_{\geq 0}$. Then $\lambda = \tilde{\lambda} - r\delta$, where

$$\tilde{\lambda} = \Lambda - \ell\alpha_0 - (\ell - x_1)\alpha_1 - (\ell - x_2)\alpha_2 - \ldots \alpha_{\ell-1-\sum_{m=2}^{k-1}(m-1)j_m} \\ - \alpha_{n-(\ell-1)+\sum_{m=2}^{k-1}(m-1)i_m} - \ldots - (\ell - x_{n-1})\alpha_{n-1}.$$

We claim that $r = 0$. Suppose $r \neq 0$, i.e. $r \geq 1$. Then $\lambda + \delta \leq \Lambda$ and $\lambda + \delta \in P^+$. This implies that $\lambda + \delta \in P(\Lambda)$, which contradicts that $\lambda \in \max(\Lambda)$. Therefore, $\lambda = \tilde{\lambda}$. $\square$

Example 4.2.11 *For $k = 3$, $n = 8$, we find all elements of $\max(\Lambda) \cap P^+$, where $\Lambda = 3\Lambda_0$. We must consider all cases for which $x_1 + x_7 \leq k$.*

- *Case: $x_1 + x_7 = 0$. Here, $x_1 = x_7 = 0$. Thus, we have $\ell = 0$ and we contribute the weight Λ.*

- *Case: $x_1 + x_7 = 1$. Then we must have either $x_1 = 0$ and $x_7 = 1$ or $x_1 = 1$ and $x_7 = 0$, neither of which can happen, so we contribute no weights.*

- *Case: $x_1 + x_7 = 2$. If either x_1 or x_7 is 2, the other must be 0, which cannot occur. So we just have the case $x_1 = x_7 = 1$. Now, $1 \leq \ell \leq 4$. See Table 4.1.*

  In this table, the column labeled $-\alpha_i$ gives the negative coefficient of α_i in the weight.

- *Case: $x_1 + x_7 = 3$. We have two viable new cases here: $x_1 = 1$ and $x_7 = 2$ and the reverse: $x_1 = 2$ and $x_7 = 1$. We will first examine $x_1 = 1$ and $x_7 = 2$ as in Table 4.2.*

Table 4.1: Maximal Dominant Weights: $k = 3$, $n = 8$, $x_1 = x_7 = 1$

ℓ	i_1	j_1	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
1	1	1	1	0	0	0	0	0	0	0
2	2	2	2	1	0	0	0	0	0	1
3	3	3	3	2	1	0	0	0	1	2
4	4	4	4	3	2	1	0	1	2	3

Table 4.2: Maximal Dominant Weights: $k = 3$, $n = 8$, $x_1 = 1, x_7 = 2$

ℓ	i_1	i_2	j_1	j_2	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
2	0	1	2	0	2	1	0	0	0	0	0	0
3	1	1	3	0	3	2	1	0	0	0	0	1
4	2	1	4	0	4	3	2	1	0	0	1	2
4	0	2	4	0	4	3	2	1	0	0	0	2
5	1	2	5	0	5	4	3	2	1	0	1	3

By symmetry, we can obtain the weights when $x_1 = 2$ and $x_7 = 1$. The weights are as in Table 4.3.

Table 4.3: Maximal Dominant Weights: $k = 3$, $n = 8$, $x_1 = 2, x_7 = 1$

ℓ	i_1	i_2	j_1	j_2	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
2	2	0	0	1	2	0	0	0	0	0	0	1
3	3	0	1	1	3	1	0	0	0	0	1	2
4	4	0	2	1	4	2	1	0	0	1	2	3
4	4	0	0	2	4	2	0	0	0	1	2	3
5	5	0	1	2	5	3	1	0	1	2	3	4

We have satisfied all cases and have thus listed the elements of $\max(\Lambda) \cap P^+$.

4.3 Maximal dominant weights of $V((k-1)\Lambda_0 + \Lambda_s)$

We will now consider the maximal dominant weights of $V((k-1)\Lambda_0 + \Lambda_s)$, where $1 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor$. Recall that we include the stipulation that $1 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor$ not because other values of s are invalid, but because other values of s are redundant. The Dynkin diagram for $\widehat{sl}(n)$ is cyclic and we can move around the diagram shifting indices if necessary to find the maximal dominant weights for different weights of the same form.

Theorem 4.3.1 *Let $n \geq 2$, $\Lambda = (k-1)\Lambda_0 + \Lambda_s$, $k \geq 2$, $1 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor$. Then* $\max(\Lambda) \cap P^+ = \left\{ \Lambda - \sum_{i=0}^{n-1} (\ell - x_i)\alpha_i \right\}$ *where the possible values of x_i are described in the following cases. Note that in every case, $x_i \in \mathbb{Z}_{\geq 0}, \ell - x_i \geq 0$. We take $x_1 = a, x_{n-1} = b$ and stipulate that $a + b \leq k-1$.*

Case 1 *$a = b = 0$: $\{(0, 0, \dots, 0)\}, \ell = 0$.*

Case 2 *$n = 2$: $\{(x_1) | 1 \leq x_1 \leq \left\lfloor \frac{k-1}{2} \right\rfloor\}, \ell = x_1$.*

Case 3 *$a = 0, b > 0, n \geq 3$*

$$
\begin{aligned}
\{(0, 0, \dots, \overset{s}{0}, 1, 2, \dots, \ell^{< n-2\ell-s+1+\sum_{m=2}^{b}(m-1)\bar{i}_m >},& \\
&\ell - 1, \ell - 2, \dots, 1 + \sum_{m=2}^{b} m\bar{i}_m, \\
&\sum_{m=2}^{b} m\bar{i}_m, \sum_{m=2}^{b} (m\bar{i}_m) - 2, \dots, 2 + \sum_{m=3}^{b} m\bar{i}_m, \\
&\vdots \\
&\sum_{m=j}^{b} m\bar{i}_m, \sum_{m=j}^{b} (m\bar{i}_m) - j, \dots, j + \sum_{m=j+1}^{b} m\bar{i}_m, \\
&\vdots \\
&b\bar{i}_b, b\bar{i}_b - b, \dots, b)\},
\end{aligned}
$$

satisfying:

- $b \leq \ell \leq \left\lfloor \frac{b(n-s)}{1+b} \right\rfloor$,
- $0 \leq \bar{i}_m \leq \left\lfloor \dfrac{\ell}{m} \right\rfloor, 1 \leq m \leq b, 1 \leq \bar{i}_b,$ *with* $\bar{i}_m \in \mathbb{Z}_{\geq 0}$,
- $\max\{2\ell - (n-s), 0\} \leq \sum_{m=2}^{b} (m-1)(\bar{i}_m)$,

- $\displaystyle\sum_{m=1}^{b} m\bar{i}_m = \ell.$

Case 4 $a > 0, b = 0, n \geq 3$

$$\begin{aligned}\{(&a, 2a, \ldots, aj_a, aj_a + (a-1), aj_a + 2(a-1), \ldots, aj_a + (a-1)j_{a-1}\\ &\vdots\\ &(a-j) + \sum_{m=0}^{j-1}(a-m)j_{a-m}, 2(a-j) + \sum_{m=0}^{j-1}(a-m)j_{a-m}, \ldots, \sum_{m=0}^{j}(a-m)j_{a-m},\\ &\vdots\\ &1 + \sum_{m=0}^{a-2}(a-m)j_{a-m}, 2 + \sum_{m=0}^{a-2}(a-m)j_{a-m}, \ldots, \ell - 1,\\ &\ell^{<n-2l-s+1+\sum_{m=2}^{a}(m-1)j_m>},\\ &\ell-1, \ell-2, \ldots, 1, \overset{s}{0}, 0, \ldots 0)\},\end{aligned}$$

satisfying

- $a \leq \ell \leq \left\lfloor \frac{as}{a+1} \right\rfloor,$
- $0 \leq j_m \leq \left\lfloor \frac{\ell}{m} \right\rfloor, 1 \leq m \leq a, 1 \leq j_a,$ *with* $j_m \in \mathbb{Z}_{\geq 0},$
- $\max\{2\ell - s, 0\} \leq \displaystyle\sum_{m=2}^{a}(m-1)(j_m),$
- $\displaystyle\sum_{m=1}^{a} mj_m = \ell.$

Case 5 $a > 0, b > 0$ *and the leftmost* x_i *that holds the maximum value* (ℓ) *occurs to the left of the* s^{th} *position,* $n \geq 3$.

$$\{(a, 2a, \ldots, aj_a, aj_a + (a-1), aj_a + 2(a-1), \ldots, aj_a + (a-1)j_{a-1}$$

$$\vdots$$

$$(a-j)+\sum_{m=0}^{j-1}(a-m)j_{a-m}, 2(a-j)+\sum_{m=0}^{j-1}(a-m)j_{a-m}, \ldots, \sum_{m=0}^{j}(a-m)j_{a-m},$$

$$\vdots$$

$$1+\sum_{m=0}^{a-2}(a-m)j_{a-m}, 2+\sum_{m=0}^{a-2}(a-m)j_{a-m}, \ldots, \ell-1,$$

$$\ell^{<s-2\ell+\bar{\ell}+\sum_{m=2}^{\max\{a,b\}}(m-1)(i_m+j_m)+q>}$$

$$\ell-1, \ell-2, \ldots, \ell-i_1, \ell-i_1-2, \ell-i_1-4, \ldots, \ell-i_1-2i_2,$$

$$\vdots$$

$$\ell-\sum_{m=1}^{m=j-1} i_m - j, \ell-\sum_{m=1}^{m=j-1} i_m - 2j, \ldots, \ell-\sum_{m=1}^{m=j-1} i_m$$

$$\vdots$$

$$\bar{\ell}+zi_z, \bar{\ell}+z(i_z-1), \ldots, \bar{\ell}+z,$$

$$\overset{s}{\bar{\ell}}, \bar{\ell}^{<n-s-\bar{\ell}+\sum_{m=2}^{b}\bar{i}_m>},$$

$$\bar{\ell}-q, \bar{\ell}-2q, \ldots, \bar{\ell}-q\bar{i}_q,$$

$$\bar{\ell}-q\bar{i}_q-(q+1), \bar{\ell}-q\bar{i}_q-2(q+1), \ldots, \bar{\ell}-q\bar{i}_q-(q+1)\bar{i}_{q+1},$$

$$\vdots$$

$$b\bar{i}_b, b\bar{i}_b-b, \ldots, b)\},$$

satisfying

- $\max\{a,b\} \leq \ell \leq \min\{(s-1)a, \left\lfloor \frac{a(bn+s)}{a+b+1} \right\rfloor - j\}$, *where*
  - $bn+s \equiv m \bmod (a+b+1)$,
  - *if* $m \leq b, j = m - \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil$,
  - *if* $m > b, j = b+1-\left\lceil \frac{m(b+1)}{a+b+1} \right\rceil$,
- $\max\{b, \lceil \frac{\ell}{a} \rceil + \ell - s\} \leq \bar{\ell} \leq \min\{\ell, b(n-s)\}$,

- $0 \le j_m \le \left\lfloor \frac{\ell}{m} \right\rfloor, 1 \le m \le a, 1 \le j_a, j_m \in \mathbb{Z}_{\ge 0}$,
- $0 \le \bar{i}_m \le \left\lfloor \frac{\bar{\ell}}{m} \right\rfloor, 1 \le m \le b, 1 \le \bar{i}_b, \bar{i}_m \in \mathbb{Z}_{\ge 0}$,
- $(z = \min\{m | \bar{i}_m > 0\} + 1) \le i_m \le \left\lfloor \frac{\ell - \bar{\ell}}{m} \right\rfloor, 1 \le m \le b, i_m \in \mathbb{Z}_{\ge 0}$,
- $\displaystyle\sum_{m=1}^{b} m(i_m + \bar{i}_m) = \sum_{m=1}^{a} m j_m = \ell$,
- *$q = z - 1$ if $\bar{\ell}$ is not repeated and is equal to 1 otherwise.*

Case 6 *The leftmost x_i that holds the maximum value (ℓ) occurs to the right of or at the s^{th} position, $n \ge 3$. We will denote the value of x_s by $\bar{\ell}$ and let $t = x_s - x_{s-1}$.*

$$
\begin{aligned}
\{(a, 2a, \ldots, & a j_a, a j_a + (a-1), a j_a + 2(a-1), \ldots, a j_a + (a-1) j_{a-1} \\
& \vdots \\
& \bar{\ell} - t j_t, \bar{\ell} - t(j_t - 1), \ldots, \bar{\ell} - t, \\
& \bar{\ell}^{< s - \bar{\ell} + \sum_{m=2}^{a} (m-1) j_m >}, \overset{s}{\bar{\ell}} \\
& \bar{\ell} + (t+1), \bar{\ell} + 2(t+1), \ldots, \bar{\ell} + \bar{j}_{t+1}(t+1), \\
& \bar{\ell} + \bar{j}_{t+1}(t+1) + t, \bar{\ell} + \bar{j}_{t+1}(t+1) + 2t, \ldots, \bar{\ell} + \bar{j}_{t+1}(t+1) + \bar{j}_t t, \\
& \vdots \\
& \ell - \bar{j}_1, \ell - \bar{j}_1 + 1, \ldots, \ell - 1, \\
& \ell^{< n - s - 1 - \ell - (\ell - \bar{\ell}) + \sum_{m=2}^{\max\{a+1, b\}} (m-1)(i_m + \bar{j}_m) >}, \\
& \ell - 1, \ell - 2, \ldots, 1 + \sum_{m=2}^{b} m \bar{i}_m, \sum_{m=2}^{b} m \bar{i}_m, \sum_{m=2}^{b} (m \bar{i}_m) - 2, \ldots, 2 + \sum_{m=3}^{b} m \bar{i}_m, \\
& \vdots \\
& \sum_{m=j}^{b} m \bar{i}_m, \sum_{m=j}^{b} (m \bar{i}_m) - j, \ldots, j + \sum_{m=j+1}^{b} m \bar{i}_m, \\
& \vdots \\
& b \bar{i}_b, b \bar{i}_b - b, \ldots, b)\},
\end{aligned}
$$

satisfying

- $\max\{a, b-n+s+1\} \leq \bar{\ell} \leq \min\{as, b(n-s)\}$,
- $\bar{\ell} \leq \ell \leq \left\lfloor \frac{b((t+1)(n-s)+\bar{\ell})}{t+b+1} \right\rfloor - j$, *where*
  - $((t+1)(n-s)+\bar{\ell}) \equiv m \bmod (t+b+1)$,
  - *if* $m \leq t, j = m - \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil$,
  - *if* $m > t, j = t+1 - \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil$,
- $0 \leq j_m \leq \left\lfloor \frac{\bar{\ell}}{m} \right\rfloor, \max\{t,1\} \leq m \leq a, 1 \leq j_a, j_m \in \mathbb{Z}_{\geq 0}$,
- $0 \leq \bar{j}_m \leq \left\lfloor \frac{\ell-\bar{\ell}}{m} \right\rfloor, 1 \leq m \leq t+1, \bar{j}_m \in \mathbb{Z}_{\geq 0}$,
- $0 \leq \bar{i}_m \leq \left\lfloor \frac{\ell}{m} \right\rfloor, 1 \leq m \leq b, 1 \leq i_b, i_m \in \mathbb{Z}_{\geq 0}$,
- $\displaystyle\sum_{m=1}^{a+1} m(j_m + \bar{j}_m) = \sum_{m=1}^{b} m\bar{i}_m = \ell$,
- $\displaystyle\sum_{m=t}^{a} mj_m = \bar{\ell}$.

Explanation of Terms: As in Theorem 4.2.1, we have that $x_i = \ell$ for some i, allowing the coefficient of at least one α_i to be zero. This ℓ is the maximum value of any x_i and sometimes repeats. Because we now have s as a contributing factor, we now have another value which may repeat. This value occurs in position s and will be denoted $x_s = \bar{\ell}$. To accommodate this, we modify our use of indices a bit. We use j_m to denote the number of times we increase by m in consecutive x_i's to the left of position s. If there is an additional increase to the right of position s, we use $\bar{j}_m$ for the number of times we increase by m to the right of position s. Similarly, we use $\bar{i}_m$ to denote any decreases that occur to the right of position s and i_m for any that occur to the left of position s.

In order to prove this theorem, we need several lemmas, which follow.

Lemma 4.3.2 *The largest value of ℓ satisfying*

$$\left\lceil \frac{\ell}{a} \right\rceil + \left\lceil \frac{\ell - b(n-s)}{b+1} \right\rceil \leq s \tag{4.3.1}$$

is $\left\lfloor \frac{a(bn+s)}{a+b+1} \right\rfloor - j$*, where*

- $bn + s \equiv m \bmod (a+b+1)$
- *if* $m \leq b+1, j = m - \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil$
- *if* $m > b+1, j = b+1 - \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil$

Proof: Let q be such that $bn+s = (a+b+1)q+m$. Then $\ell = \left\lfloor \frac{a(bn+s)}{a+b+1} \right\rfloor - j = \left\lfloor \frac{a((a+b+1)q+m)}{a+b+1} \right\rfloor - j = aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - j$

Case 1 $m \leq b$

We wish to show that $\ell = aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - m + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil$ *satisfies (4.3.1) and is the maximum integer that does so.*

$$\begin{aligned}
&\left\lceil \frac{\ell}{a} \right\rceil + \left\lceil \frac{\ell - b(n-s)}{b+1} \right\rceil = \\
&\quad = \left\lceil \frac{aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - m + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{a} \right\rceil \\
&\qquad\qquad + \left\lceil \frac{aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - m + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil - b(n-s)}{b+1} \right\rceil \\
&\quad = q + \left\lceil \frac{\left\lfloor \frac{am}{a+b+1} \right\rfloor - m + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{a} \right\rceil \\
&\qquad\qquad + \left\lceil \frac{aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - m + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil - bn + b(q(a+b+1)+m-bn)}{b+1} \right\rceil \\
&\quad = q + \left\lceil \frac{\left\lfloor \frac{am}{a+b+1} \right\rfloor - m + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{a} \right\rceil
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left\lceil \frac{q(a+b)(b+1) - b(b+1)n + (b-1)m + \left\lfloor \frac{am}{a+b+1} \right\rfloor + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{b+1} \right\rceil \\
&= q + \left\lceil \frac{\left\lfloor \frac{am}{a+b+1} \right\rfloor - m + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{a} \right\rceil + q(a+b) - bn + \left\lceil \frac{(b-1)m + \left\lfloor \frac{am}{a+b+1} \right\rfloor + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{b+1} \right\rceil \\
&= q(a+b+1) - bn + \left\lceil \frac{\left\lfloor \frac{am}{a+b+1} \right\rfloor - m + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{a} \right\rceil + \left\lceil \frac{(b-1)m + \left\lfloor \frac{am}{a+b+1} \right\rfloor + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{b+1} \right\rceil \\
&= q(a+b+1) - bn + \left\lceil \frac{\frac{am+1}{a+b+1} - 1 - m + \frac{m(b+1)}{a+b+1}}{a} \right\rceil + \left\lceil \frac{(b-1)m + \frac{am+1}{a+b+1} - 1 + \frac{m(b+1)}{a+b+1}}{b+1} \right\rceil \\
&= q(a+b+1) - bn + \left\lceil \frac{am + 1 - (a+b+1)(1+m) + m(b+1)}{a(a+b+1)} \right\rceil \\
&\quad + \left\lceil \frac{(b-1)m(a+b+1) + am + 1 - (a+b+1) + m(b+1)}{(a+b+1)(b+1)} \right\rceil \\
&= q(a+b+1) - bn + \left\lceil \frac{-a-b}{a(a+b+1)} \right\rceil + \left\lceil \frac{mb(a+b+1) - a - b}{(a+b+1)(b+1)} \right\rceil \\
&= q(a+b+1) - bn + 0 + \left\lceil \frac{mb}{b+1} + \frac{-(a+b)}{(b+1)(a+b+1)} \right\rceil \\
&= q(a+b+1) - bn + m \\
&= s
\end{aligned}
$$

Thus ℓ as described above satisfies 4.3.1. Suppose we add a positive integer, r, to the value of ℓ given. That is, suppose $\ell = aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - m + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil + r$. Then, following similar steps as above, we obtain the following:

$$
\begin{aligned}
&\left\lceil \frac{\ell}{a} \right\rceil + \left\lceil \frac{\ell - b(n-s)}{b+1} \right\rceil = \\
&\quad = \left\lceil \frac{aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - m + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil + r}{a} \right\rceil + \left\lceil \frac{aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - m + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil + r - b(n-s)}{b+1} \right\rceil \\
&\quad = q(a+b+1) - bn + \left\lceil \frac{-a-b+r(a+b+1)}{a(a+b+1)} \right\rceil + \left\lceil \frac{mb(a+b+1) - a - b + r(a+b+1)}{(a+b+1)(b+1)} \right\rceil
\end{aligned}
$$

$$\geq q(a+b+1) - bn + 1 + m$$

$$> s$$

Case 2 $m > b$

We wish to show that $\ell = aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - (b+1) + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil$ *satisfies 4.3.1 and is the maximum integer that does so.*

$$\left\lceil \frac{\ell}{a} \right\rceil + \left\lceil \frac{\ell - b(n-s)}{b+1} \right\rceil =$$

$$= \left\lceil \frac{aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - (b+1) + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{a} \right\rceil + \left\lceil \frac{aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - (b+1) + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil - b(n-s)}{b+1} \right\rceil$$

$$= q + \left\lceil \frac{\left\lfloor \frac{am}{a+b+1} \right\rfloor - (b+1) + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{a} \right\rceil$$

$$+ \left\lceil \frac{aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - (b+1) + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil - bn + b(q(a+b+1) + m - bn)}{b+1} \right\rceil$$

$$= q + \left\lceil \frac{\left\lfloor \frac{am}{a+b+1} \right\rfloor - (b+1) + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{a} \right\rceil$$

$$+ \left\lceil \frac{q(a+b)(b+1) - b(b+1)n + bm - (b+1) + \left\lfloor \frac{am}{a+b+1} \right\rfloor + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{b+1} \right\rceil$$

$$= q + \left\lceil \frac{\left\lfloor \frac{am}{a+b+1} \right\rfloor - (b+1) + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{a} \right\rceil + q(a+b) - bn - 1 + \left\lceil \frac{bm + \left\lfloor \frac{am}{a+b+1} \right\rfloor + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{b+1} \right\rceil$$

$$= q(a+b+1) - bn - 1 + \left\lceil \frac{\left\lfloor \frac{am}{a+b+1} \right\rfloor - (b+1) + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{a} \right\rceil + \left\lceil \frac{bm + \left\lfloor \frac{am}{a+b+1} \right\rfloor + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil}{b+1} \right\rceil$$

$$= q(a+b+1) - bn - 1 + \left\lceil \frac{\frac{am+1}{a+b+1} - 1 - (b+1) + \frac{m(b+1)}{a+b+1}}{a} \right\rceil + \left\lceil \frac{bm + \frac{am+1}{a+b+1} - 1 + \frac{m(b+1)}{a+b+1}}{b+1} \right\rceil$$

$$= q(a+b+1) - bn - 1 + \left\lceil \frac{am + 1 - (a+b+1)(b+2) + m(b+1)}{a(a+b+1)} \right\rceil$$

$$
\begin{aligned}
&\qquad\qquad + \left\lceil \frac{bm(a+b+1)+am+1-(a+b+1)+m(b+1)}{(a+b+1)(b+1)} \right\rceil \\
&= q(a+b+1) - bn - 1 + \left\lceil \frac{m-(b+2)}{a} + \frac{1}{a(a+b+1)} \right\rceil + m + \left\lceil \frac{-a-b}{(a+b+1)(b+1)} \right\rceil \\
&\leq q(a+b+1) - bn - 1 + 1 + m + 0 \\
&= q(a+b+1) - bn + m \\
&= s
\end{aligned}
$$

Thus ℓ as described above satisfies 4.3.1. As in the previous case, suppose we add a positive integer, r, to the value of ℓ given. That is, suppose $\ell = aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - (b+1) + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil + r$. Then, following similar steps as above, we obtain the following:

$$
\begin{aligned}
&\left\lceil \frac{\ell}{a} \right\rceil + \left\lceil \frac{\ell - b(n-s)}{b+1} \right\rceil = \\
&= \left\lceil \frac{aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - (b+1) + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil + r}{a} \right\rceil \\
&\qquad\qquad + \left\lceil \frac{aq + \left\lfloor \frac{am}{a+b+1} \right\rfloor - (b+1) + \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil + r - b(n-s)}{b+1} \right\rceil \\
&= q(a+b+1) - bn - 1 + \left\lceil \frac{m+r-(b+2)}{a} + \frac{1}{a(a+b+1)} \right\rceil + m + \left\lceil \frac{r(a+b+1)-a-b}{(a+b+1)(b+1)} \right\rceil \\
&\geq q(a+b+1) - bn - 1 + 1 + m + 1 \\
&> s
\end{aligned}
$$

Lemma 4.3.3 *The largest value of ℓ satisfying*

$$
\left\lceil \frac{\ell - \bar{\ell}}{t+1} \right\rceil + \left\lceil \frac{\ell}{b} \right\rceil \leq n - s \tag{4.3.2}
$$

is: $\left\lfloor \frac{b((t+1)(n-s)+\bar{\ell})}{t+b+1} \right\rfloor - j$, *where*

- $((t+1)(n-s)+\bar{\ell}) \equiv m \bmod (t+b+1)$
- *if* $m \leq t, j = m - \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil$
- *if* $m > t, j = t+1 - \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil$

Proof: Let q be such that $(t+1)(n-s)+\bar{\ell} = (t+b+1)q+m$. Then $\ell = \left\lfloor \frac{b((t+1)(n-s)+\bar{\ell})}{t+b+1} \right\rfloor - j = \left\lfloor \frac{b((t+b+1)q+m)}{t+b+1} \right\rfloor - j = bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - j.$

Case 1 $m \leq t$

We wish to show that $\ell = bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - m + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil$ *satisfies 4.3.2 and is the maximum integer that does so.*

$$
\begin{aligned}
&\left\lceil \frac{\ell - \bar{\ell}}{t+1} \right\rceil + \left\lceil \frac{\ell}{b} \right\rceil \\
&\quad = \left\lceil \frac{bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - m + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil - \bar{\ell}}{t+1} \right\rceil + \left\lceil \frac{bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - m + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil}{b} \right\rceil \\
&\quad = \left\lceil \frac{bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - m + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil - ((t+b+1)q + m - (t+1)(n-s))}{t+1} \right\rceil \\
&\qquad\qquad + \left\lceil \frac{bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - m + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil}{b} \right\rceil \\
&\quad = n - s - q + \left\lceil \frac{\left\lfloor \frac{bm}{t+b+1} \right\rfloor - 2m + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil}{t+1} \right\rceil + q + \left\lceil \frac{\left\lfloor \frac{bm}{t+b+1} \right\rfloor - m + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil}{b} \right\rceil \\
&\quad = n - s + \left\lceil \frac{\frac{bm+1}{t+b+1} - 1 - 2m + \frac{m(t+1)}{t+b+1}}{t+1} \right\rceil + \left\lceil \frac{\frac{bm+1}{t+b+1} - 1 - m + \frac{m(t+1)}{t+b+1}}{b} \right\rceil \\
&\quad = n - s + \left\lceil \frac{bm + 1 - (1+2m)(t+b+1) + m(t+1)}{(t+1)(t+b+1)} \right\rceil \\
&\qquad\qquad + \left\lceil \frac{bm + 1 - (m+1)(t+b+1) + m(t+1)}{(t+b+1)b} \right\rceil \\
&\quad = n - s + \left\lceil \frac{-m(t+b+1) - (t+b)}{(t+1)(t+b+1)} \right\rceil + \left\lceil \frac{-(b+t)}{b(t+b+1)} \right\rceil
\end{aligned}
$$

$$= n - s + \left\lceil \frac{-m}{t+1} + \frac{-(t+b)}{(t+1)(t+b+1)} \right\rceil + \left\lceil \frac{-(b+t)}{b(t+b+1)} \right\rceil$$

$$= n - s$$

Thus ℓ as described above satisfies 4.3.2. As in the previous case, suppose we add a positive integer, r, to the value of ℓ given. That is, suppose $\ell = bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - m + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil + r$. Then, following similar steps as above, we obtain the following:

$$\begin{aligned}
&\left\lceil \frac{\ell - \bar{\ell}}{t+1} \right\rceil + \left\lceil \frac{\ell}{b} \right\rceil \\
&\quad = \left\lceil \frac{bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - m + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil + r - \bar{\ell}}{t+1} \right\rceil + \left\lceil \frac{bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - m + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil + r}{b} \right\rceil \\
&\quad = n - s + \left\lceil \frac{bm + 1 - (1+2m)(t+b+1) + r(t+b+1) + m(t+1)}{(t+1)(t+b+1)} \right\rceil \\
&\qquad\qquad + \left\lceil \frac{bm + 1 - (m+1)(t+b+1) + r(t+b+1) + m(t+1)}{(t+b+1)b} \right\rceil \\
&\quad = n - s + \left\lceil \frac{-m(t+b+1) - (t+b) + r(t+b+1)}{(t+1)(t+b+1)} \right\rceil + \left\lceil \frac{r(t+b+1) - (b+t)}{b(t+b+1)} \right\rceil \\
&\quad = n - s + \left\lceil \frac{r-m}{t+1} + \frac{-(t+b)}{(t+1)(t+b+1)} \right\rceil + \left\lceil \frac{r(t+b+1) - (b+t)}{b(t+b+1)} \right\rceil \\
&\quad \geq n - s + 1
\end{aligned}$$

Case 2 $m > t$

We wish to show that $\ell = bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - (t+1) + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil$ satisfies 4.3.2 and is the maximum integer that does so.

$$\begin{aligned}
&\left\lceil \frac{\ell - \bar{\ell}}{t+1} \right\rceil + \left\lceil \frac{\ell}{b} \right\rceil \\
&\quad = \left\lceil \frac{bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - (t+1) + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil - \bar{\ell}}{t+1} \right\rceil + \left\lceil \frac{bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - (t+1) + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil}{b} \right\rceil
\end{aligned}$$

$$
\begin{aligned}
&= \left\lceil \frac{bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - (t+1) + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil - ((t+b+1)q + m - (t+1)(n-s))}{t+1} \right\rceil \\
&\qquad\qquad + \left\lceil \frac{bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - (t+1) + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil}{b} \right\rceil \\
&= n - s - q - 1 + \left\lceil \frac{\left\lfloor \frac{bm}{t+b+1} \right\rfloor - m + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil}{t+1} \right\rceil + q + \left\lceil \frac{\left\lfloor \frac{bm}{t+b+1} \right\rfloor - (t+1) + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil}{b} \right\rceil \\
&= n - s - 1 + \left\lceil \frac{\frac{bm+1}{t+b+1} - 1 - m + \frac{m(t+1)}{t+b+1}}{t+1} \right\rceil + \left\lceil \frac{\frac{bm+1}{t+b+1} - 1 - (t+1) + \frac{m(t+1)}{t+b+1}}{b} \right\rceil \\
&= n - s - 1 + \left\lceil \frac{bm + 1 - (1+m)(t+b+1) + m(t+1)}{(t+1)(t+b+1)} \right\rceil \\
&\qquad\qquad + \left\lceil \frac{bm + 1 - (t+2)(t+b+1) + m(t+1)}{(t+b+1)b} \right\rceil \\
&= n - s - 1 + \left\lceil \frac{1 - (b+t+1)}{(t+1)(t+b+1)} \right\rceil + \left\lceil \frac{(m-t-2)(t+b+1) + 1}{b(t+b+1)} \right\rceil \\
&= n - s - 1 + \left\lceil \frac{-(b+t)}{(t+1)(t+b+1)} \right\rceil + \left\lceil \frac{m-t-2}{b} + \frac{1}{b(t+b+1)} \right\rceil \\
&\leq n - s - 1 + 1 \\
&= n - s
\end{aligned}
$$

Thus ℓ as described above satisfies 4.3.2. As in the previous case, suppose we add a positive integer, r, to the value of ℓ given. That is, suppose $\ell = bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - (t+1) + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil + r$. Then, following similar steps as above, we obtain the following:

$$
\begin{aligned}
&\left\lceil \frac{\ell - \bar{\ell}}{t+1} \right\rceil + \left\lceil \frac{\ell}{b} \right\rceil \\
&\quad = \left\lceil \frac{bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - (t+1) + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil + r - \bar{\ell}}{t+1} \right\rceil \\
&\qquad\qquad + \left\lceil \frac{bq + \left\lfloor \frac{bm}{t+b+1} \right\rfloor - (t+1) + \left\lceil \frac{m(t+1)}{t+b+1} \right\rceil + r}{b} \right\rceil
\end{aligned}
$$

$$
\begin{aligned}
&= n-s-1+\left\lceil \frac{bm+1-(1+m)(t+b+1)+m(t+1)+r(t+b+1)}{(t+1)(t+b+1)} \right\rceil \\
&\qquad\qquad + \left\lceil \frac{bm+1-(t+2)(t+b+1)+m(t+1)+r(t+b+1)}{(t+b+1)b} \right\rceil \\
&= n-s-1+\left\lceil \frac{-(b+t)+r(t+b+1)}{(t+1)(t+b+1)} \right\rceil + \left\lceil \frac{(r+m-t-2)(t+b+1)+1}{b(t+b+1)} \right\rceil \\
&= n-s-1+\left\lceil \frac{-(b+t)+r(t+b+1)}{(t+1)(t+b+1)} \right\rceil + \left\lceil \frac{(r+m-t-2)}{b} + \frac{1}{b(t+b+1)} \right\rceil \\
&\geq n-s+1
\end{aligned}
$$

Lemma 4.3.4 *For* $n \geq 2$*,*
$\{\boldsymbol{x} = (x_1, x_2, \ldots, x_{n-1})^T | x_1 + x_{n-1} \leq (k-1), (A'\boldsymbol{x})_i \geq 0,$ *except* $(A'\boldsymbol{x})_s \geq -1, (1 \leq s \leq \lfloor \frac{n}{2} \rfloor),$ *and* $x_i \in \mathbb{Z}$ *for* $i = 1, 2, \ldots n-1\}$ *is equal to the union of the following sets. For convenience, we will set* $a = x_1$ *and* $b = x_{n-1}$*.*

Case 1 $a = b = 0$*:* $\{(0, 0, \ldots, 0)\}$.

Case 2 $n = 2$*:* $\{(x_1) | 1 \leq x_1 \leq \lfloor \frac{k-1}{2} \rfloor\}$.

Case 3 $a = 0, b > 0, n \geq 3$

$$
\begin{aligned}
&\{(0, 0, \ldots, \overset{s}{0}, 1, 2, \ldots, \ell^{<n-2\ell-s+1+\sum_{m=2}^{b}(m-1)\bar{i}_m>}, \\
&\qquad \ell-1, \ell-2, \ldots, 1+\sum_{m=2}^{b} m\bar{i}_m, \\
&\qquad \sum_{m=2}^{b} m\bar{i}_m, \sum_{m=2}^{b}(m\bar{i}_m)-2, \ldots, 2+\sum_{m=3}^{b} m\bar{i}_m, \\
&\qquad \vdots \\
&\qquad \sum_{m=j}^{b} m\bar{i}_m, \sum_{m=j}^{b}(m\bar{i}_m)-j, \ldots, j+\sum_{m=j+1}^{b} m\bar{i}_m, \\
&\qquad \vdots \\
&\qquad b\bar{i}_b, b\bar{i}_b - b, \ldots, b)\},
\end{aligned}
$$

satisfying:

- $b \leq \ell \leq \left\lfloor \frac{b(n-s)}{1+b} \right\rfloor$,
- $0 \leq \bar{i}_m \leq \left\lfloor \dfrac{\ell}{m} \right\rfloor, 1 \leq m \leq b, 1 \leq \bar{i}_b,$ *with* $\bar{i}_m \in \mathbb{Z}_{\geq 0}$,
- $\max\{2\ell - (n-s), 0\} \leq \sum_{m=2}^{b} (m-1)(\bar{i}_m)$,
- $\sum_{m=1}^{b} m\bar{i}_m = \ell.$

Case 4 $a > 0, b = 0, n \geq 3$

$$
\begin{aligned}
\{(a, 2a, \ldots, &aj_a, aj_a + (a-1), aj_a + 2(a-1), \ldots, aj_a + (a-1)j_{a-1} \\
&\vdots \\
&(a-j) + \sum_{m=0}^{j-1}(a-m)j_{a-m}, 2(a-j) + \sum_{m=0}^{j-1}(a-m)j_{a-m}, \ldots, \sum_{m=0}^{j}(a-m)j_{a-m}, \\
&\vdots \\
&1 + \sum_{m=0}^{a-2}(a-m)j_{a-m}, 2 + \sum_{m=0}^{a-2}(a-m)j_{a-m}, \ldots, \ell - 1, \\
&\ell^{<n-2l-s+1+\sum_{m=2}^{a}(m-1)j_m>}, \\
&\ell - 1, \ell - 2, \ldots, 1, \overset{s}{0}, 0, \ldots 0)\},
\end{aligned}
$$

satisfying

- $a \leq \ell \leq \left\lfloor \frac{as}{a+1} \right\rfloor$,
- $0 \leq j_m \leq \left\lfloor \frac{\ell}{m} \right\rfloor, 1 \leq m \leq a, 1 \leq j_a,$ *with* $j_m \in \mathbb{Z}_{\geq 0}$,
- $\max\{2\ell - s, 0\} \leq \sum_{m=2}^{a} (m-1)(j_m)$,

- $\sum_{m=1}^{a} m j_m = \ell.$

Case 5 *$a > 0, b > 0$ and the leftmost x_i that holds the maximum value (ℓ) occurs to the left of the s^{th} position, $n \geq 3$.*

$$
\begin{aligned}
\{(a, 2a, \ldots, & a j_a, a j_a + (a-1), a j_a + 2(a-1), \ldots, a j_a + (a-1) j_{a-1} \\
& \vdots \\
& (a-j) + \sum_{m=0}^{j-1} (a-m) j_{a-m}, 2(a-j) + \sum_{m=0}^{j-1} (a-m) j_{a-m}, \ldots, \sum_{m=0}^{j} (a-m) j_{a-m}, \\
& \vdots \\
& 1 + \sum_{m=0}^{a-2} (a-m) j_{a-m}, 2 + \sum_{m=0}^{a-2} (a-m) j_{a-m}, \ldots, \ell - 1, \\
& \ell^{<s-2\ell+\bar{\ell}+\sum_{m=2}^{\max\{a,b\}}(m-1)(i_m+j_m)+q>} \\
& \ell - 1, \ell - 2, \ldots, \ell - i_1, \ell - i_1 - 2, \ell - i_1 - 4, \ldots, \ell - i_1 - 2 i_2, \\
& \vdots \\
& \ell - \sum_{m=1}^{m=j-1} i_m - j, \ell - \sum_{m=1}^{m=j-1} i_m - 2j, \ldots, \ell - \sum_{m=1}^{m=j-1} i_m \\
& \vdots \\
& \bar{\ell} + z i_z, \bar{\ell} + z(i_z - 1), \ldots, \bar{\ell} + z, \\
& \overset{s}{\bar{\ell}}, \bar{\ell}^{<n-s-\bar{\ell}+\sum_{m=2}^{b} \bar{i}_m>}, \\
& \bar{\ell} - q, \bar{\ell} - 2q, \ldots, \bar{\ell} - q \bar{i}_q, \\
& \bar{\ell} - q \bar{i}_q - (q+1), \bar{\ell} - q \bar{i}_q - 2(q+1), \ldots, \bar{\ell} - q \bar{i}_q - (q+1) \bar{i}_{q+1}, \\
& \vdots \\
& b \bar{i}_b, b \bar{i}_b - b, \ldots, b)\},
\end{aligned}
$$

satisfying

- $\max\{a,b\} \leq \ell \leq \min\{(s-1)a, \left\lfloor \frac{a(bn+s)}{a+b+1} \right\rfloor - j\}$, *where*
  - $bn + s \equiv m \bmod (a+b+1)$,
  - *if* $m \leq b, j = m - \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil$,
  - *if* $m > b, j = b + 1 - \left\lceil \frac{m(b+1)}{a+b+1} \right\rceil$,
- $\max\{b, \lceil \frac{\ell}{a} \rceil + \ell - s\} \leq \bar{\ell} \leq \min\{\ell, b(n-s)\}$,
- $0 \leq j_m \leq \left\lfloor \frac{\ell}{m} \right\rfloor, 1 \leq m \leq a, 1 \leq j_a, j_m \in \mathbb{Z}_{\geq 0}$,
- $0 \leq \bar{i}_m \leq \left\lfloor \frac{\bar{\ell}}{m} \right\rfloor, 1 \leq m \leq b, 1 \leq \bar{i}_b, \bar{i}_m \in \mathbb{Z}_{\geq 0}$,
- $(z = \min\{m | \bar{i}_m > 0\} + 1) \leq i_m \leq \left\lfloor \frac{\ell - \bar{\ell}}{m} \right\rfloor, 1 \leq m \leq b, i_m \in \mathbb{Z}_{\geq 0}$,
- $\sum_{m=1}^{b} m(i_m + \bar{i}_m) = \sum_{m=1}^{a} m j_m = \ell$,
- $q = z - 1$ *if* $\bar{\ell}$ *is not repeated and is equal to* 1 *otherwise.*

Case 6 *The leftmost* x_i *that holds the maximum value* (ℓ) *occurs to the right of or at the* s^{th} *position,* $n \geq 3$*. We will denote the value of* x_s *by* $\bar{\ell}$ *and let* $t = x_s - x_{s-1}$ *.*

$$
\begin{aligned}
\{(a, 2a, \ldots, &a j_a, a j_a + (a-1), a j_a + 2(a-1), \ldots, a j_a + (a-1) j_{a-1} \\
&\vdots \\
&\bar{\ell} - t j_t, \bar{\ell} - t(j_t - 1), \ldots, \bar{\ell} - t, \\
&\bar{\ell}^{<s - \bar{\ell} + \sum_{m=2}^{a} (m-1) j_m>}, \overset{s}{\bar{\ell}} \\
&\bar{\ell} + (t+1), \bar{\ell} + 2(t+1), \ldots, \bar{\ell} + \bar{j}_{t+1}(t+1), \\
&\bar{\ell} + \bar{j}_{t+1}(t+1) + t, \bar{\ell} + \bar{j}_{t+1}(t+1) + 2t, \ldots, \bar{\ell} + \bar{j}_{t+1}(t+1) + \bar{j}_t t, \\
&\vdots \\
&\ell - \bar{j}_1, \ell - \bar{j}_1 + 1, \ldots, \ell - 1, \\
&\ell^{<n - s - 1 - \ell - (\ell - \bar{\ell}) + \sum_{m=2}^{\max\{a+1,b\}} (m-1)(i_m + \bar{j}_m)>},
\end{aligned}
$$

$$\ell-1,\ell-2,\ldots,1+\sum_{m=2}^{b} m\bar{i}_m,\sum_{m=2}^{b} m\bar{i}_m,\sum_{m=2}^{b}(m\bar{i}_m)-2,\ldots,2+\sum_{m=3}^{b} m\bar{i}_m,$$

$$\vdots$$

$$\sum_{m=j}^{b} m\bar{i}_m,\sum_{m=j}^{b}(m\bar{i}_m)-j,\ldots,j+\sum_{m=j+1}^{b} m\bar{i}_m,$$

$$\vdots$$

$$b\bar{i}_b,b\bar{i}_b-b,\ldots,b)\},$$

satisfying

- $\max\{a,b-n+s+1\}\leq\bar{\ell}\leq\min\{as,b(n-s)\}$,
- $\bar{\ell}\leq\ell\leq\left\lfloor\frac{b((t+1)(n-s)+\bar{\ell})}{t+b+1}\right\rfloor-j,$ *where*
  - $((t+1)(n-s)+\bar{\ell})\equiv m \bmod (t+b+1)$,
  - *if* $m\leq t, j=m-\left\lceil\frac{m(t+1)}{t+b+1}\right\rceil$,
  - *if* $m>t, j=t+1-\left\lceil\frac{m(t+1)}{t+b+1}\right\rceil$,
- $0\leq j_m\leq\left\lfloor\frac{\bar{\ell}}{m}\right\rfloor,\max\{t,1\}\leq m\leq a,1\leq j_a,j_m\in\mathbb{Z}_{\geq 0}$,
- $0\leq\bar{j}_m\leq\left\lfloor\frac{\ell-\bar{\ell}}{m}\right\rfloor,1\leq m\leq t+1,\bar{j}_m\in\mathbb{Z}_{\geq 0}$,
- $0\leq\bar{i}_m\leq\left\lfloor\frac{\ell}{m}\right\rfloor,1\leq m\leq b,1\leq i_b,i_m\in\mathbb{Z}_{\geq 0}$,
- $\displaystyle\sum_{m=1}^{a+1} m(j_m+\bar{j}_m)=\sum_{m=1}^{b} m\bar{i}_m=\ell$,
- $\displaystyle\sum_{m=t}^{a} mj_m=\bar{\ell}$.

Proof We begin by adjusting our previous claims from Lemma 4.2.7 to examine the behavior at and around the s^{th} position, as this is the only condition that has changed.

Claim 4.3.5 *If* $x_{s-1} > x_s$*, then* $x_s \geq x_{s+1}$.

Proof. $x_{s-1} > x_s \implies x_s + x_{s-1} > 2x_s \implies x_s > 2x_s - x_{s-1}$.
$0 \leq 1 - x_{s-1} + 2x_s - x_{s+1} < 1 + x_s - x_{s+1}$, and thus $x_s \geq x_{s+1}$.

Claim 4.3.6 *If* $x_s = x_{s-1}$*, then* $x_{s+1} \leq x_s + 1$.

Proof. Let $x_s = x_{s-1}$. Then $0 \leq 1 - x_{s-1} + 2x_s - x_{s+1} = 1 + x_s - x_{s+1} \implies x_{s+1} \leq x_s + 1$.

Claim 4.3.7 *If* $x_s - x_{s-1} \leq m$*, then* $x_{s+1} - x_s \leq m + 1$*, where* $m \in \mathbb{Z}$.

Proof. Suppose $x_s - x_{s-1} \leq m$. Then

$$0 \leq 1 - x_{s-1} + 2x_s - x_{s+1} \leq 1 + m + x_s - x_{s+1} \implies x_{s+1} - x_s \leq m + 1.$$

Claim 4.3.8 *If* $x_s - x_{s+1} \leq m$*, then* $x_{s-1} - x_s \leq m + 1$.

Proof. Suppose $x_s - x_{s+1} \leq m$. Then

$$0 \leq 1 - x_{s-1} + 2x_s - x_{s+1} \leq 1 - x_{s-1} + x_s + m \implies x_{s-1} - x_s \leq m + 1.$$

Armed with these claims, we can determine the sets as above. It is easy to see that if $x_1 = 0$, then $x_2 = x_3 = \ldots = x_s = 0$. Then by Claim 4.3.7 and symmetry, if $x_{n-1} > 0$, then $x_{s+1} = 1$ and thus we can treat the values of $x_{s+1} = 1, x_{s+2}, \ldots, x_{n-1} (= b > 0)$ as the case $x_1 = 1, x_{n-1} = b$ from Theorem 4.2.1 with $n = n - s$. By this reasoning and by symmetry, we obtain sets in Case 3 ($x_1 = 0, x_{n-1} = b$) and Case 4 ($x_1 = a, x_{n-1} = 0$) . We also obtain Case 1. Case 2 can be found in [21].

The other cases are not quite as simple. Both arise from cases in which both x_1 and x_{n-1} are nonzero. It is convenient to divide up the cases based on whether the maximum x_i value ℓ is attained at or to the right of position s or if ℓ is attained to the left of position s.

Now let us consider Case 5, in which ℓ is attained somewhere to the left of position s. The pattern found in the x_i's is generally familiar. We begin with $x_1 > 0$ and increase as usual as we move to the right, attaining the maximum value, ℓ, in a position less than s. ℓ may repeat, but then may decrease to obtain $x_s = \bar{\ell}$. This decrease from ℓ to $\bar{\ell}$ begins in the usual manner and then by Claim 4.3.5, $\bar{\ell}$ may repeat and then may decrease to $b = x_{n-1}$. We will use i_m to indicate the number of decreases by m between ℓ and $\bar{\ell}$ and $\bar{i}_m$ for decreases to the right of $x_s = \bar{\ell}$. Note that the maximum decrease to the left of $x_s = \bar{\ell}$ is one more than the minimum decrease to the right of $x_s = \bar{\ell}$ if $\bar{\ell}$ does not repeat.

The upperbound for ℓ in this case requires a bit of work. Note that the maximum value of ℓ possible will occur when we increase and decrease by the maximum possible amounts while adhering to our required conditions. Through these conditions, this maximum value of ℓ must satisfy the following equation.

$$\left\lceil \frac{\ell - a}{a} \right\rceil + 1 + \left\lceil \frac{\ell - b(n-s) - (b+1)}{b+1} \right\rceil + 1 + n - 1 - s \leq n - 1$$

- $\left\lceil \frac{\ell - a}{a} \right\rceil$: the number of x_i's needed to increase to ℓ;
- 1: ℓ;
- $\left\lceil \frac{\ell - b(n-s) - (b+1)}{b+1} \right\rceil$: the number of x_i's needed to decrease from ℓ at the fastest rate possible (increments of $b+1$);
- 1: $\bar{\ell}$;
- $n - 1 - s$: positions to the right of s;
- $n - 1$: total number of x_i's.

This equation can be simplified to $\left\lceil \frac{\ell}{a} \right\rceil + \left\lceil \frac{\ell - b(n-s)}{b+1} \right\rceil \leq s$. By Lemma 4.3.2, we obtain the upperbound for ℓ. The other bounds are straightforward.

Now, we will consider the final case, Case 6. In this case, we stipulate that the maximum value ℓ is attained somewhere at or to the right of position s. As expected, we begin with $x_1 > 0$ and increase as we move to the right. When we reach position s, either we have attained

ℓ or we have not. If we have not attained ℓ, we continue to increase, though we may begin by increasing by one more than the difference between x_s and x_{s-1}. We increase in the usual way, by smaller and smaller differences until we attain ℓ. After attaining ℓ, we may repeat and then will decrease as usual to $x_{n-1} = b$. We will use $\bar{j}_m$ to denote the number of times we increase by m between position x_s and the position where we finally attain ℓ. We use t to refer to the difference between x_s and x_{s-1}. We first determine the upper and lower bounds for $\bar{\ell}$ and then fix $\bar{\ell}$ to determine possible values of ℓ. The largest possible value of $\bar{\ell}$ comes from increasing only by a s times. Thus, $\bar{\ell} \leq as$. However, $\bar{\ell}$ cannot always reach this value, as there must be a decrease to $x_{n-1} = b$ according to our pattern. Thus, we note that $\left\lceil \frac{\bar{\ell}-b}{b} \right\rceil \leq n-s-1$, or equivalently, $\bar{\ell} \leq b(n-s)$. To determine the upperbound for ℓ, note that the largest value of ℓ occurs when we both increase and decrease by the largest values possible. Thus, the maximum value of ℓ must satisfy:

$$\left\lceil \frac{\ell - \bar{\ell} - (t+1)}{t+1} \right\rceil + 1 + \left\lceil \frac{\ell - b}{b} \right\rceil \leq n-s-1$$

- $\left\lceil \dfrac{\ell - \bar{\ell} - (t+1)}{t+1} \right\rceil$: The number of x_i's needed to increase from $\bar{\ell}$ to ℓ at the fastest rate possible (increasing by $t+1$);
- 1: ℓ;
- $\left\lceil \dfrac{\ell - b}{b} \right\rceil$: The number of x_i's needed to decrease from ℓ to b;
- $n-s-1$: The number of positions to the right of s.

This equation simplifies to

$$\left\lceil \frac{\ell - \bar{\ell}}{t+1} \right\rceil + \left\lceil \frac{\ell}{b} \right\rceil \leq n-s.$$

From Lemma 4.3.3, we obtain the maximum value of ℓ as described. □

We now wish to prove Theorem 4.3.1. The technique will be similar to that in the proof of Theorem 4.2.1, which involves the maximal dominant weights of $V(k\Lambda_0)$.

Proof of Theorem 4.3.1 By Proposition 4.1.1, $\max(\Lambda) \cap P^+$ is bijective to $kC_{af} \cap (\overline{\Lambda}_s + \overline{Q})$ via the map $\lambda \mapsto \overline{\lambda}$. Note that $\overline{\Lambda}_s$ is equal to $\Lambda_s - \Lambda_0$.

As in the proof of Theorem 4.2.1, we will find all $\lambda \in kC_{af} \cap (\overline{\Lambda}_s + \overline{Q})$ and use the inverse of this bijective map to list all elements of $\max(\Lambda) \cap P^+$. Again, we will map $\lambda = \Lambda + \sum_{j=0}^{n-1} q_j\alpha_j \in \max(\Lambda) \cap P^+$ (with $q_j \in \mathbb{Z}_{\leq 0}, 1 \leq j \leq n-1$) to $\overline{\lambda} = \overline{\Lambda}_s + \sum_{j=1}^{n-1} x_j\alpha_j \in kC_{af} \cap (\overline{\Lambda}_s + \overline{Q})$. We can use this map to obtain the relation $x_j = q_j - q_0,\ 1 \leq j \leq n-1$. (c.f. [21])

Note that

$$kC_{af} \cap (\overline{\Lambda}_s + \overline{Q}) = \left\{ \lambda = \overline{\Lambda}_s + \sum_{j=1}^{n-1} x_j\alpha_j \,\middle|\, \lambda(h_j) \geq 0,\ 1 \leq j < n, (\lambda|\theta) \leq k \right\}.$$

Thus the elements $\lambda \in kC_{af} \cap (\overline{\Lambda}_s + \overline{Q})$ satisfy the following conditions:

$$\begin{cases} \lambda(h_1) = 2x_1 - x_2 & \geq 0 \\ \lambda(h_2) = -x_1 + 2x_2 - x_3 & \geq 0 \\ \vdots & \vdots \\ \lambda(h_{s-1}) = -x_{s-2} + 2x_{s-1} - x_s & \geq 0 \\ \lambda(h_s) = 1 - x_{s-1} + 2x_s - x_{s+1} & \geq 0 \\ \lambda(h_{s+1}) = -x_s + 2x_{s+1} - x_{s+2} & \geq 0 \\ \vdots & \vdots \\ \lambda(h_{n-2}) = -x_{n-3} + 2x_{n-2} - x_{n-1} & \geq 0 \\ \lambda(h_{n-1}) = -x_{n-2} + 2x_{n-1} & \geq 0 \\ (\lambda|\theta) = 1 + x_1 + x_{n-1} & \leq k \end{cases} \tag{4.3.3}$$

Note that as in [21], if $n = 2$ we have

$$\begin{cases} \lambda(h_i) = 1 + 2x_1 & \geq 0 \\ (\lambda|\theta) = 1 + 2x_1 & \leq k \end{cases}$$

Thus, when $n = 2$, $1 \leq x_1 \leq \lfloor \frac{k-1}{2} \rfloor$.

Therefore, we may now assume that $n \geq 3$. By the reasoning in [21], $x_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, 2, \ldots, n-1$. As before, we will examine possible pairs (x_1, x_{n-1}) where $x_1 + x_{n-1} \leq k - 1$. Once we select x_1 and x_{n-1}, we find $x_2, x_3, \ldots, x_{n-2}$ satisfying 4.3.3. Note that this gives the same $(x_1, x_2, \ldots, x_{n-1})$ that satisfy $(\mathring{A}\boldsymbol{x})_i \geq 0$, except $(\mathring{A}\boldsymbol{x})_s \geq -1$. Thus, Lemma 4.3.4 gives exactly the x_i satisfy 4.3.3. As in the proof of Theorem 4.2.1, we obtain $-\ell$ as the coefficient of α_0 and each other α_i has coefficient $-(\ell - x_i)$. $\square$

In [21], Tsuchioka proves the level 2 case when n is prime.

Corollary 4.3.9 *[21] Let $n \geq 2, n$ prime $, 1 \leq s < n$, $\Lambda = \Lambda_0 + \Lambda_s$. Then* $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - \gamma_\ell^s \mid 1 \leq \ell \leq \lfloor \frac{n-s}{2} \rfloor\} \cup \{\Lambda - \mu_\ell^s \mid 1 \leq \ell \leq \lfloor \frac{s}{2} \rfloor\}$*, where*

$$\begin{aligned} \gamma_\ell^s =& \ell\alpha_0 + \ell\alpha_1 + \cdots + \ell\alpha_s \\ & + (\ell-1)\alpha_{s+1} + (\ell-2)\alpha_{s-2} + \cdots + \alpha_{\ell+s-1} \\ & + \alpha_{n-\ell+1} + \cdots + (\ell-2)\alpha_{n-2} + (\ell-1)\alpha_{n-1} \\ \mu_\ell^s =& (\ell-1)\alpha_1 + (\ell-2)\alpha_2 + \cdots + \alpha_{\ell-1} \\ & + \alpha_{s-\ell+1} + \cdots + (\ell-2)\alpha_{s-2} + (\ell-1)\alpha_{s-1} \\ & + \ell\alpha_s + \cdots + \ell\alpha_{n-1}. \end{aligned}$$

Example 4.3.10 *For $k = 3$, $n = 8$, and $s = 3$, we find all elements of* $\max(\Lambda) \cap P^+$*, where $\Lambda = 2\Lambda_0 + \Lambda_3$. We must consider all cases for which $x_1 + x_7 \leq k - 1$.*

- *Case: $x_1 + x_7 = 0$. Here, $x_1 = x_7 = 0$. Thus, we have $\ell = 0$ and we contribute the weight Λ.*

- *Case: $x_1 + x_7 = 1$.* Then we must have either $x_1 = 0$ and $x_7 = 1$ or $x_1 = 1$ and $x_7 = 0$. In the case $x_1 = 0$ and $x_7 = 1$, we apply Case 3. Note that $1 \leq \ell \leq 2$. The results are in Table 4.4.

Table 4.4: Maximal Dominant Weights: $k = 3$, $n = 8$, $s = 3$, $x_1 = 0$, $x_7 = 1$

ℓ	$\bar{i}_1$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
1	1	1	1	1	1	0	0	0	0
2	2	2	2	2	2	1	0	0	1

When $x_1 = 1$ and $x_7 = 0$, we apply Case 4. In this case, we can only have $\ell = 1$. We obtain the results in Table 4.5.

Table 4.5: Maximal Dominant Weights: $k = 3$, $n = 8$, $s = 3$, $x_1 = 1$, $x_7 = 0$

ℓ	j_1	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
1	1	1	0	0	1	1	1	1	1

- *Case: $x_1 + x_7 = 2$.* We have three possibilities: $(x_1, x_7) = (0, 2), (2, 0)$ or $(1, 1)$. In the first case, $(x_1, x_7) = (0, 2)$, we enter Case 3. Here $2 \leq \ell \leq 3$ and we obtain Table 4.6.

Table 4.6: Maximal Dominant Weights: $k = 3$, $n = 8$, $s = 3$, $x_1 = 0$, $x_7 = 2$

ℓ	$\bar{i}_1$	$\bar{i}_2$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
2	0	1	2	2	2	2	1	0	0	0
3	1	1	3	3	3	3	2	1	0	1

When $(x_1, x_7) = (2, 0)$, we apply Case 4. Here, we only have the possibility that $\ell = 2$. Thus, the only possibility is given in Table 4.7.

In the case where $(x_1, x_7) = (1, 1)$, we split into two different subcases for ease of explanation. First, we will refer to Case 5, which is the case in which the maximum x_i falls to

Table 4.7: Maximal Dominant Weights: $k = 3$, $n = 8$, $s = 3$, $x_1 = 2$, $x_7 = 0$

ℓ	j_1	j_2	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
2	0	1	2	0	1	2	2	2	2	2

the left of the s^{th} position. We obtain $1 \leq \ell \leq 2$. When $\ell = 1$, we can only have $\bar{\ell} = 1$. However, when $\ell = 2, 1 \leq \bar{\ell} \leq 2$. We obtain Table 4.8.

Table 4.8: Maximal Dominant Weights: $k = 3$, $n = 8$, $s = 3$, $x_1 = 1$, $x_7 = 1$, ℓ left of s

ℓ	$\bar{\ell}$	j_1	i_1	$\bar{i}_1$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
1	1	1	0	1	1	0	0	0	0	0	0	0
2	2	2	1	1	2	1	0	1	1	1	1	1
2	2	2	0	2	2	1	0	0	0	0	0	1

Now, we consider the case where ℓ occurs for the first time at or to the right of position s. In this case, we have $1 \leq \bar{\ell} \leq 3$. When $\bar{\ell} = 1$, we have $t = 0$ and ℓ ranges from 2 to 3. When $\bar{\ell} = 2$, $\ell = 3$. Finally, when $\bar{\ell} = 3$, we have $3 \leq \ell \leq 4$. We obtain the Table 4.9.

Table 4.9: Maximal Dominant Weights: $k = 3$, $n = 8$, $s = 3$, $x_1 = 1$, $x_7 = 1$, ℓ right of or at s

$\bar{\ell}$	ℓ	j_1	$\bar{j}_1$	$\bar{i}_1$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
1	2	1	1	2	2	1	1	1	0	0	0	1
1	3	1	2	3	3	2	2	2	1	0	1	2
2	3	2	1	3	3	2	1	1	0	0	1	2
3	3	3	0	3	3	2	1	0	0	0	1	2
3	4	3	1	4	4	3	2	1	0	1	2	3

We have satisfied all cases and have thus listed the elements of $\max(\Lambda) \cap P^+$.

Additional examples of Cases 5 and 6 may be helpful to understand these more complex cases.

Example 4.3.11 *Let $k \geq 6, n = 8, s = 3, x_1 = 3, x_{n-1} = 2$ and consider Case 5, in which ℓ occurs for the first time to the left of s. We see that $3 \leq \ell \leq 6$. We will create a separate table* *for each value of ℓ. See Tables 4.10 to 4.13.*

Table 4.10: Maximal Dominant Weights: $k \geq 6$, $n = 8$, $s = 3$, $x_1 = 3$, $x_7 = 2$, $\ell = 3$

$\bar{\ell}$	j_1	j_2	j_3	i_1	$\bar{i}_1$	$\bar{i}_2$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
2	0	0	1	1	0	1	3	0	0	1	1	1	1	1
3	0	0	1	0	1	1	3	0	0	0	0	0	0	1

Table 4.11: Maximal Dominant Weights: $k \geq 6$, $n = 8$, $s = 3$, $x_1 = 3$, $x_7 = 2$, $\ell = 4$

$\bar{\ell}$	j_1	j_2	j_3	i_1	$\bar{i}_1$	$\bar{i}_2$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
3	1	0	1	1	1	1	4	1	0	1	1	1	1	2
4	1	0	1	0	2	1	4	1	0	0	0	0	1	2
4	1	0	1	0	0	2	4	1	0	0	0	0	0	2

Table 4.12: Maximal Dominant Weights: $k \geq 6$, $n = 8$, $s = 3$, $x_1 = 3$, $x_7 = 2$, $\ell = 5$

$\bar{\ell}$	j_1	j_2	j_3	i_1	$\bar{i}_1$	$\bar{i}_2$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
4	0	1	1	1	2	1	5	2	0	1	1	1	2	3
4	0	1	1	1	0	2	5	2	0	1	1	1	1	3
5	0	1	1	0	3	1	5	2	0	0	0	1	2	3
5	0	1	1	0	1	2	5	2	0	0	0	0	1	3

Table 4.13: Maximal Dominant Weights: $k \geq 6$, $n = 8$, $s = 3$, $x_1 = 3$, $x_7 = 2$, $\ell = 6$

$\bar{\ell}$	j_1	j_2	j_3	i_1	$\bar{i}_1$	$\bar{i}_2$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
5	0	0	2	1	3	1	6	3	0	1	1	2	3	4
5	0	0	2	1	1	2	6	3	0	1	1	1	2	4
6	0	0	2	0	4	1	6	3	0	0	1	2	3	4
6	0	0	2	0	2	2	6	3	0	0	0	1	2	4
6	0	0	2	0	0	3	6	3	0	0	0	0	2	4

Example 4.3.12 *We will do the same example, but follow Case 6, in which ℓ occurs for the first time at or to the right of position s. Again, let $k \geq 6, n = 8, s = 3, x_1 = 3, x_{n-1} = 2$. We see that $3 \leq \bar{\ell} \leq 9$. We will create a separate table for each value of $\bar{\ell}$. See Tables 4.14 to 4.20.*

Table 4.14: Maximal Dominant Weights: $k \geq 6$, $n = 8$, $s = 3$, $x_1 = 3$, $x_7 = 2$, $\bar{\ell} = 3$

ℓ	t	j_1	j_2	j_3	$\bar{j}_1$	$\bar{j}_2$	$\bar{i}_1$	$\bar{i}_2$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
4	0	0	0	1	1	0	2	1	4	1	1	1	0	0	1	2
4	0	0	0	1	1	0	0	2	4	1	1	1	0	0	0	2
5	0	0	0	1	2	0	1	2	5	2	2	2	1	0	1	3

Table 4.15: Maximal Dominant Weights: $k \geq 6$, $n = 8$, $s = 3$, $x_1 = 3$, $x_7 = 2$, $\bar{\ell} = 4$

ℓ	t	j_1	j_2	j_3	$\bar{j}_1$	$\bar{j}_2$	$\bar{i}_1$	$\bar{i}_2$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
5	0	1	0	1	1	0	3	1	5	2	1	1	0	1	2	3
5	0	1	0	1	1	0	1	2	5	2	1	1	0	0	1	3
6	0	1	0	1	2	0	0	3	6	3	2	2	1	0	2	4

Table 4.16: Maximal Dominant Weights: $k \geq 6$, $n = 8$, $s = 3$, $x_1 = 3$, $x_7 = 2$, $\bar{\ell} = 5$

ℓ	t	j_1	j_2	j_3	$\bar{j}_1$	$\bar{j}_2$	$\bar{i}_1$	$\bar{i}_2$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
6	0	0	1	1	1	0	2	2	6	3	1	1	0	1	2	4
6	0	0	1	1	1	0	0	3	6	3	1	1	0	0	2	4
5	1	2	0	1	0	0	3	1	5	2	1	0	0	1	2	3
5	1	2	0	1	0	0	1	2	5	2	1	0	0	0	1	3
6	1	2	0	1	1	0	2	2	6	3	2	1	0	1	2	4
6	1	2	0	1	1	0	0	3	6	3	2	1	0	0	2	4
7	1	2	0	1	0	1	1	3	7	4	3	2	0	1	3	5

Table 4.17: Maximal Dominant Weights: $k \geq 6$, $n = 8$, $s = 3$, $x_1 = 3$, $x_7 = 2$, $\bar{\ell} = 6$

ℓ	t	j_1	j_2	j_3	$\bar{j}_1$	$\bar{j}_2$	$\bar{i}_1$	$\bar{i}_2$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
7	0	0	0	2	1	0	1	3	7	4	1	1	0	1	3	5
6	1	1	1	1	0	0	4	1	6	3	1	0	1	2	3	4
6	1	1	1	1	0	0	2	2	6	3	1	0	0	1	2	4
6	1	1	1	1	0	0	0	3	6	3	1	0	0	0	2	4
7	1	1	1	1	1	0	1	3	7	4	2	1	0	1	3	5
8	1	1	1	1	0	1	0	4	8	5	3	2	0	2	4	6

Table 4.18: Maximal Dominant Weights: $k \geq 6$, $n = 8$, $s = 3$, $x_1 = 3$, $x_7 = 2$, $\bar{\ell} = 7$

ℓ	t	j_1	j_2	j_3	$\bar{j}_1$	$\bar{j}_2$	$\bar{i}_1$	$\bar{i}_2$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
7	1	1	0	2	0	0	3	2	7	4	1	0	1	2	3	5
7	1	1	0	2	0	0	1	3	7	4	1	0	0	1	3	5
8	1	1	0	2	1	0	0	4	8	5	2	1	0	2	4	6
7	2	0	2	1	0	0	3	2	7	4	2	0	1	2	3	5
7	2	0	2	1	0	0	1	3	7	4	2	0	0	1	3	5
8	2	0	2	1	1	0	0	4	8	5	3	1	0	2	4	6

Table 4.19: Maximal Dominant Weights: $k \geq 6$, $n = 8$, $s = 3$, $x_1 = 3$, $x_7 = 2$, $\bar{\ell} = 8$

ℓ	t	j_1	j_2	j_3	$\bar{j}_1$	$\bar{j}_2$	$\bar{i}_1$	$\bar{i}_2$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
8	2	0	1	2	0	0	2	3	8	5	2	0	1	2	4	6
8	2	0	1	2	0	0	0	4	8	5	2	0	0	2	4	6

Table 4.20: Maximal Dominant Weights: $k \geq 6$, $n = 8$, $s = 3$, $x_1 = 3$, $x_7 = 2$, $\bar{\ell} = 9$

ℓ	t	j_1	j_2	j_3	$\bar{j}_1$	$\bar{j}_2$	$\bar{i}_1$	$\bar{i}_2$	$-\alpha_0$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$-\alpha_6$	$-\alpha_7$
9	3	0	0	3	0	0	1	4	9	6	3	0	1	3	5	7

Chapter 5

Multiplicity of $k\Lambda_0 - \gamma_\ell$ in $V(k\Lambda_0)$

In this chapter we will consider weights of $V(k\Lambda_0)$ that are of the form $k\Lambda_0 - \gamma_\ell$, where $\gamma_\ell = \ell\alpha_0 + (\ell-1)\alpha_1 + (\ell-2)\alpha_2 + \ldots + \alpha_{\ell-1} + \alpha_{n-\ell+1} + 2\alpha_{n-\ell+2} + \ldots + (\ell-1)\alpha_{n-1}$. We use extended Young diagrams from crystal base theory to determine the multiplicity and to find a relationship between multiplicity of these weights and avoiding permutations.

5.1 Multiplicity of Maximal Dominant Weights of $V(2\Lambda_0)$

We will first consider the weights found in $\max(2\Lambda_0) \cap P^+$ and then examine the weights that carry over to $\max(k\Lambda_0) \cap P^+$ as described in Corollary 4.2.3.

From Theorem 4.2.1, we can write $\max(2\Lambda_0) \cap P^+ = \{2\Lambda_0\} \cup \{2\Lambda_0 - \gamma_\ell | 1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor\}$, where

$$\begin{aligned}\gamma_\ell = \ell\alpha_0 + &(\ell-1)\alpha_1 + (\ell-2)\alpha_2 + \ldots + \alpha_{\ell-1} \\ &+\alpha_{n-\ell+1} + 2\alpha_{n-\ell+2} + \ldots + (\ell-1)\alpha_{n-1}\end{aligned}$$

Note that the multiplicity of $2\Lambda_0$ in $V(2\Lambda_0)$ is known to be one, so we will not mention it again and will focus on the other weights.

We will find a relationship between the multiplicities of these weights and avoiding permutations. A common notation for a permutation of $[n] = \{1, 2, \ldots n\}$ is

$$w = \begin{pmatrix} 1 & 2 & \ldots & n \\ w_1 & w_2 & \ldots & w_n \end{pmatrix},$$

where $1 \mapsto w_1, 2 \mapsto w_2, \ldots, n \mapsto w_n$. We wish to use a slightly different notation and will take the bottom row from the above as a sequence with n terms. That is, $w = w_1 w_2 \ldots w_n$ will mean $1 \mapsto w_1, 2 \mapsto w_2, \ldots, n \mapsto w_n$. A $(j, j-1, \ldots, 1)$-avoiding permutation is a permutation whose sequence does not have a decreasing subsequence of length j. For example, $w = 1342$ is a 321-avoiding permutation because it has no decreasing subsequence of length three. However, $w = 1342$ is not a 21-avoiding permutation because it has decreasing subsequences of length two, namely 32 and 42. In addition, $v = 51342$ is not a 321-avoiding permutation (because 532 and 542 are decreasing subsequences of length three) but is a 4321-avoiding permutation.

Theorem 5.1.1 *The multiplicity of $2\Lambda_0 - \gamma_\ell$ is equal to the number of 321-avoiding permutations of $[\ell]$.*

Remark 5.1.2 *In [21], Tsuchioka finds the multiplicity of $2\Lambda_0 - \gamma_\ell$ to be $\frac{1}{\ell+1}\binom{2\ell}{\ell}$, which is the ℓ^{th} Catalan number. It is known that the ℓ^{th} Catalan number is equal to the number of 321-avoiding permutations of $[\ell]$ (c.f. [20]). We directly exhibit a bijection between the 321-avoiding permutations and the elements of $B(2\Lambda_0)$ of weight $2\Lambda_0 - \gamma_\ell$. As before, Tsuchioka's result depends on n being prime, a restriction removed here.*

Recall that the multiplicity of a weight λ of $V(\Lambda)$ can be found by finding the number of tuples of extended Young diagrams that satisfy the conditions of Theorem 3.2.4.

Proof First, we will show that the multiplicity of $2\Lambda_0 - \gamma_\ell$ for some ℓ is equal to the number of lattice paths in an $\ell \times \ell$ square that can touch but not cross the diagonal line from the lower left corner to the upper right corner of the square. Then, we will show directly that this number of lattice paths is equal to the number of 321-avoiding permutations of $[\ell]$.

Consider the $\ell \times \ell$ square in Figure 5.1. Note that this square has ℓ boxes of color 0, $\ell - 1$ boxes each of color 1 and -1, $\ell - 2$ boxes each of color 2 and -2, and so on, until we have 1 box each of colors $\ell - 1$ and $-(\ell - 1)$. That is, if we consider this square as an extended Young diagram, it has weight $2\Lambda_0 - (\ell\alpha_0 + (\ell-1)\alpha_1 + (\ell-2)\alpha_2 + \ldots + \alpha_{\ell-1} + \alpha_{n-\ell+1} + 2\alpha_{n-\ell+2} + \ldots + (\ell-1)\alpha_{n-1}) = 2\Lambda_0 - \gamma_\ell$.

0	1	2	$\cdots$	$\ell - 1$
-1	0	1	$\cdots$	$\ell - 2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$1 - \ell$	$2 - \ell$	$3 - \ell$	$\cdots$	0

Figure 5.1: $\ell \times \ell$ Extended Young Diagram

We create $\mathbf{Y} = (Y_1, Y_2)$ by drawing a lattice path that does not cross the diagonal in the extended Young diagram above. We then take the boxes below the path, reflect over the diagonal, and place these boxes in Y_2. The square with the boxes removed becomes Y_1. This new ordered pair of extended Young diagrams $\mathbf{Y}$ has weight $2\Lambda_0 - \gamma_\ell$. For example, we have the correspondence in Figure 5.2.

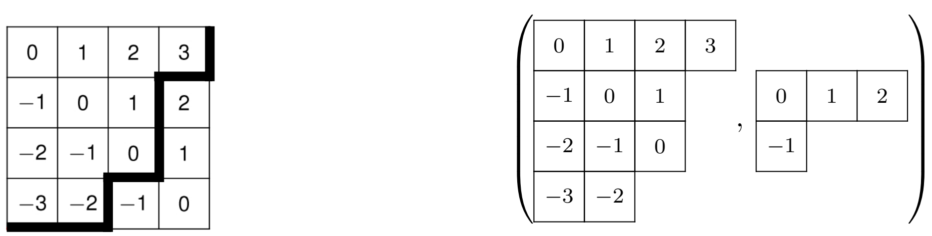


Figure 5.2: Lattice Path to Extended Young Diagram

Every pair $\mathbf{Y} = (Y_1, Y_2)$ in $B(2\Lambda_0)_{2\Lambda_0-\gamma_\ell}$ can be constructed in this manner. Each $\mathbf{Y}$ must have exactly the same number of boxes of each color as the square in Figure 5.1 and thus Y_1 and Y_2 can be combined to make the square. The corresponding lattice path is the path that follows the bottom edge of Y_1.

Now we must show that every such ordered pair of extended Young diagrams $\mathbf{Y}$ is contained in $B(2\Lambda_0)$. To show this, we will use Theorem 3.2.4. We need to show the following:

1. $Y_1 \supseteq Y_2$
2. $Y_2 \supseteq Y_3 = Y_1[n]$
3. For each $j > 0$, there is some i such that $(Y_{i+1})_j > (Y_i)_{j+1}$

By not allowing our path to cross the diagonal, we obtain (1). For (2) and (3), we will think about the diagrams as sequences, rather than as boxes. Since Y_2 has charge zero, the maximum value of any sequence element of Y_2 is zero. The minimum value of any sequence element of Y_1 is $-\ell$. Thus because $n \geq 2\ell$, the minimum value of any element of $Y_3 = Y_1[n]$ is $n - \ell \geq \ell$. We obtain (2). By setting $i = 2$, we satisfy (3) for all j.

Now, we wish to show that the number of lattice paths in an $\ell \times \ell$ square that do not cross the diagonal from the lower left to upper right corner is equal to the number of 321-avoiding permutations of $[\ell]$. (Note: This construction is adapted from [20].)

Let P_ℓ, W_ℓ be the set of all lattice paths that do not cross $y = x$ and 321-avoiding permutations, respectively, described above. We will show that there exists a bijection between P_ℓ and W_ℓ. To make calculations a little neater, we orient the square so that its lower left corner is at the origin. First, define the map $f : P_\ell \to W_\ell$ as follows: let $p \in P_\ell$. Let $L = [\ell] = \{1, 2, \ldots, \ell\}$. As the path moves from $(0, 0)$ to (ℓ, ℓ), let (v_1, j_1) be the coordinates of the point on p at the top of the first vertical move. For $i \in [j_1 - 1]$, let $w_i = i$ and set $L = L \backslash [j_1 - 1]$. Note that since the path must stay below the diagonal, $v_1 + 1 \geq j_1$ and $v_1 + 1 \in L$. Now, set $w_{j_1} = v_1 + 1$ and take $L = L \backslash \{v_1 + 1\}$. We continue traversing the path p. For each vertical move, take (v_k, j_k) to be the point at the top of the move. For $i \in \{j_{k-1} + 1, \ldots, j_k - 1\}$, set $w_i = \min\{L\}, L = L \backslash \{w_i\}$.

Set $w_{j_k} = v_k + 1$ and take $L = L \backslash \{w_{j_k}\}$. We continue this process until we reach the last vertical move, which ends at (ℓ, ℓ). Suppose there have been $s-1$ places already filled. For $i = \{j_s, \ldots, \ell\}$, set $w_i = \min\{L\}, L = L \backslash \{w_i\}$. We obtain a permutation of $\ell, w = w_1 w_2 \ldots w_\ell$.

We must show that w is a 321-avoiding permutation. Suppose w has a decreasing subsequence of length 3; that is, for some $a < b < c$, $w_a > w_b > w_c$. Then, by our construction, $A = (w_a - 1, a)$ and $B = (w_b - 1, b)$ must be points on p. Since $w_a > w_b$, $a < b$, the point A is below and to the right of B. These points cannot be on the same lattice path, so we reach a contradiction. Thus w is a 321-avoiding permutation. It is clear to see that f is injective.

Now, we define a map $g : W_\ell \to P_\ell$ in the following manner. Let $w = w_1 w_2 \ldots w_\ell$. Let $C_i = \{j \mid j > i, w_j < w_i\}, c_i = |C_i|, J = \{j_1, j_2, \ldots, j_r\}_< = \{j \mid c_j > 0\}$. We define a path as in Table 5.1.

Table 5.1: Rules for Drawing Lattice Path

Direction	From	To
Horizontal	$(0, 0)$	$(c_{j_1} + j_1 - 1, 0)$
Vertical	$(c_{j_1} + j_1 - 1, 0)$	$(c_{j_1} + j_1 - 1, j_1)$
Horizontal	$(c_{j_1} + j_1 - 1, j_1)$	$(c_{j_2} + j_2 - 1, j_1)$
Vertical	$(c_{j_2} + j_2 - 1, j_1)$	$(c_{j_2} + j_2 - 1, j_2)$
$\vdots$	$\vdots$	$\vdots$
Vertical	$(c_{j_r} + j_r - 1, j_{r-1})$	$(c_{j_r} + j_r - 1, j_r)$
Horizontal	$(c_{j_r} + j_r - 1, j_r)$	(ℓ, j_r)
Vertical	(ℓ, j_r)	(ℓ, ℓ)

This gives a path p from $(0, 0)$ to (ℓ, ℓ). We must show that p is indeed a lattice path that does not cross the diagonal. By construction, we can see that all ordered pairs have first coordinate greater than or equal to second coordinate, and so the lattice path never crosses the diagonal. Since J is an ordered set, we see that all vertical moves go up. Other than the first and last horizontal moves, it is not as clear that all horizontal moves are to the right, so we want to show that $c_{j_a} + j_a - 1 < c_{j_b} + j_b - 1$ for $j_a < j_b$. This is clear if $c_{j_a} \leq c_{j_b}$ so we need only consider when $c_{j_a} > c_{j_b}$. We have two cases:

Case 1 $w_{j_a} > w_{j_b}$*: Suppose* $j \in C_{j_b}$*. Then* $j > j_b > j_a$ *and* $w_j < w_{j_b} < w_{j_a}$ *is a decreasing subsequence of length 3, a contradiction.*

Case 2 $w_{j_a} < w_{j_b}$*: Let* $j \in C_{j_a}$*. Then* $w_j < w_{j_a} < w_{j_b}$*. If* $j > j_b$*, then* $j \in C_{j_b}$*. If* $j < j_b$*, then* $j \notin C_{j_b}$*. There are at most* $j_b - j_a - 1$ *values* j *such that* $j \in C_{j_a}$ *and* $j \notin C_{j_b}$*. Then* $c_{j_a} \leq c_{j_b} + j_b - j_a - 1$ *and we obtain* $c_{j_a} + j_a - 1 < c_{j_b} + j_b - 1$*.*

It is straightforward to see that g is one-to-one. We have described a bijection between P_ℓ and W_ℓ and thus these sets have the same cardinality. $\square$

Example 5.1.3 *Let* $\ell = 4$*. We begin with a* 4×4 *square. We then draw a lattice path from the lower left to upper right corner that does not cross the diagonal* $y = x$ *as in Figure 5.3.*

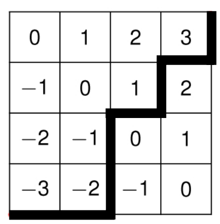


Figure 5.3: Construction of Lattice Path in Square

From the path, we can uniquely obtain a 321-avoiding permutation by using the construction above. Begin with $L = \{1, 2, 3, 4\}$*.*

- *The first vertical move on the path takes us to* $(2, 2)$*. Thus, we let* $w_1 = 1$ *and now* $L = \{2, 3, 4\}$*. Set* $w_2 = 3$ *and* $L = \{2, 4\}$*.*
- *The next vertical move ends at* $(3, 3)$*. Now,* $w_3 = 4$ *and* $L = \{2\}$
- *This leaves* $w_4 = 2$*.*
- $w = 1342$ *is a 321-avoiding permutation.*

Now, suppose we begin with the permutation and want to obtain the path. When we begin with $w = 1342$, we obtain the values of c_i in Table 5.2, which give us the path coordinates in Table 5.3.

Table 5.2: Computation of c_i's for 1342

i	1	2	3	4
C_i	-	4	4	0
c_i	0	1	1	0

Table 5.3: Path Coordinates for 1342

Dir	From	To
H	$(0,0)$	$(2,0)$
V	$(2,0)$	$(2,2)$
H	$(2,2)$	$(3,2)$
V	$(3,2)$	$(3,3)$
H	$(3,3)$	$(4,3)$
V	$(4,3)$	$(4,4)$

This path and permutation correspond to the ordered pair of extended Young diagrams in Figure 5.4.

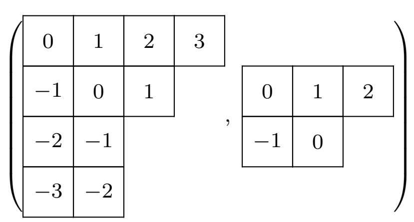


Figure 5.4: Diagram for 1342

Here are the correspondences of diagrams and 321-avoiding permutations for various values of ℓ.

Table 5.4: Diagram/Permutation Correspondence: $\ell = 3$

Row	Diagram	Perm.	Diagram	Perm.
1	$\left(\begin{ytableau} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{ytableau}, \varnothing\right)$	123	$\left(\begin{ytableau} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 \end{ytableau}, \begin{ytableau} 0 \end{ytableau}\right)$	312
2	$\left(\begin{ytableau} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 \end{ytableau}, \begin{ytableau} 0 \\ -1 \end{ytableau}\right)$	213	$\left(\begin{ytableau} 0 & 1 & 2 \\ -1 & 0 \\ -2 & -1 \end{ytableau}, \begin{ytableau} 0 & 1 \end{ytableau}\right)$	132
3	$\left(\begin{ytableau} 0 & 1 & 2 \\ -1 & 0 \\ -2 \end{ytableau}, \begin{ytableau} 0 & 1 \\ -1 \end{ytableau}\right)$	231		

Table 5.5: Diagram/Permutation Correspondence: $\ell = 4$

Row	Diagram	Perm.	Diagram	Perm.
1	$\left(\begin{array}{cccc}0&1&2&3\\-1&0&1&2\\-2&-1&0&1\\-3&-2&-1&0\end{array}, \quad \right)$	1234	$\left(\begin{array}{cccc}0&1&2&3\\-1&0&1&2\\-2&-1&0&1\\-3&-2&-1&\end{array}, \begin{array}{c}0\end{array}\right)$	4123
2	$\left(\begin{array}{cccc}0&1&2&3\\-1&0&1&2\\-2&-1&0&1\\-3&-2&&\end{array}, \begin{array}{c}0\\-1\end{array}\right)$	3124	$\left(\begin{array}{cccc}0&1&2&3\\-1&0&1&2\\-2&-1&0&\\-3&-2&-1&\end{array}, \begin{array}{cc}0&1\end{array}\right)$	1423
3	$\left(\begin{array}{cccc}0&1&2&3\\-1&0&1&2\\-2&-1&0&1\\-3&&&\end{array}, \begin{array}{c}0\\-1\\-2\end{array}\right)$	2134	$\left(\begin{array}{cccc}0&1&2&3\\-1&0&1&\\-2&-1&0&\\-3&-2&-1&\end{array}, \begin{array}{ccc}0&1&2\end{array}\right)$	1243
4	$\left(\begin{array}{cccc}0&1&2&3\\-1&0&1&2\\-2&-1&0&\\-3&-2&&\end{array}, \begin{array}{cc}0&1\\-1&\end{array}\right)$	3412	$\left(\begin{array}{cccc}0&1&2&3\\-1&0&1&2\\-2&-1&&\\-3&-2&&\end{array}, \begin{array}{cc}0&1\\-1&0\end{array}\right)$	1324
5	$\left(\begin{array}{cccc}0&1&2&3\\-1&0&1&2\\-2&-1&0&\\-3&&&\end{array}, \begin{array}{cc}0&1\\-1&\\-2&\end{array}\right)$	2413	$\left(\begin{array}{cccc}0&1&2&3\\-1&0&1&\\-2&-1&0&\\-3&-2&&\end{array}, \begin{array}{ccc}0&1&2\\-1&&\end{array}\right)$	3142
6	$\left(\begin{array}{cccc}0&1&2&3\\-1&0&1&2\\-2&-1&&\\-3&&&\end{array}, \begin{array}{cc}0&1\\-1&0\\-2&\end{array}\right)$	2314	$\left(\begin{array}{cccc}0&1&2&3\\-1&0&1&\\-2&-1&&\\-3&-2&&\end{array}, \begin{array}{ccc}0&1&2\\-1&0&\end{array}\right)$	1342
7	$\left(\begin{array}{cccc}0&1&2&3\\-1&0&1&\\-2&-1&0&\\-3&&&\end{array}, \begin{array}{ccc}0&1&2\\-1&&\\-2&&\end{array}\right)$	2143	$\left(\begin{array}{cccc}0&1&2&3\\-1&0&1&\\-2&-1&&\\-3&&&\end{array}, \begin{array}{ccc}0&1&2\\-1&0&\\-2&&\end{array}\right)$	2341

Table 5.6: Diagram/Permutation Correspondence: $\ell = 5$ Part I

Row	Diagram	Perm.	Diagram	Perm.
1	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&2\\ -3&-2&-1&0&1\\ -4&-3&-2&-1&0 \end{array}, \emptyset\right)$	12345	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&2\\ -3&-2&-1&0&1\\ -4&-3&-2&-1& \end{array}, \begin{array}{c} 0 \end{array}\right)$	51234
2	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&2\\ -3&-2&-1&0&1\\ -4&-3&-2&& \end{array}, \begin{array}{c} 0\\ -1 \end{array}\right)$	41235	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&2\\ -3&-2&-1&0&\\ -4&-3&-2&-1& \end{array}, \begin{array}{cc} 0&1 \end{array}\right)$	15234
3	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&2\\ -3&-2&-1&0&1\\ -4&-3&&& \end{array}, \begin{array}{c} 0\\ -1\\ -2 \end{array}\right)$	31245	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&\\ -3&-2&-1&0&\\ -4&-3&-2&-1& \end{array}, \begin{array}{ccc} 0&1&2 \end{array}\right)$	12534
4	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&2\\ -3&-2&-1&0&\\ -4&-3&-2&& \end{array}, \begin{array}{cc} 0&1\\ -1& \end{array}\right)$	45123	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&2\\ -3&-2&-1&&\\ -4&-3&-2&& \end{array}, \begin{array}{cc} 0&1\\ -1&0 \end{array}\right)$	14235
5	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&2\\ -3&-2&-1&0&1\\ -4&&&& \end{array}, \begin{array}{c} 0\\ -1\\ -2\\ -3 \end{array}\right)$	21345	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&\\ -2&-1&0&1&\\ -3&-2&-1&0&\\ -4&-3&-2&-1& \end{array}, \begin{array}{cccc} 0&1&2&3 \end{array}\right)$	12354
6	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&2\\ -3&-2&-1&0&\\ -4&-3&&& \end{array}, \begin{array}{cc} 0&1\\ -1&\\ -2& \end{array}\right)$	35124	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&\\ -3&-2&-1&0&\\ -4&-3&-2&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&& \end{array}\right)$	41523
7	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&2\\ -3&-2&-1&&\\ -4&-3&&& \end{array}, \begin{array}{cc} 0&1\\ -1&0\\ -2& \end{array}\right)$	34125	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&\\ -3&-2&-1&&\\ -4&-3&-2&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&0& \end{array}\right)$	14523

Table 5.7: Diagram/Permutation Correspondence: $\ell = 5$ Part II

Row	Diagram	Perm.	Diagram	Perm.
8	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&2\\ -3&-2&-1&0&\\ -4&&&& \end{array}, \begin{array}{cc} 0&1\\ -1&\\ -2&\\ -3& \end{array}\right)$	25134	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&\\ -2&-1&0&1&\\ -3&-2&-1&0&\\ -4&-3&-2&& \end{array}, \begin{array}{cccc} 0&1&2&3\\ -1&&& \end{array}\right)$	41253
9	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&\\ -3&-2&-1&0&\\ -4&-3&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&&\\ -2&& \end{array}\right)$	31524	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&\\ -3&-2&-1&&\\ -4&-3&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&0&\\ -2&& \end{array}\right)$	34512
10	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&2\\ -3&-2&&&\\ -4&-3&&& \end{array}, \begin{array}{cc} 0&1\\ -1&0\\ -2&-1 \end{array}\right)$	13245	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&&\\ -3&-2&-1&&\\ -4&-3&-2&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&0&1 \end{array}\right)$	12435
11	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&2\\ -3&-2&-1&&\\ -4&&&& \end{array}, \begin{array}{cc} 0&1\\ -1&0\\ -2&\\ -3& \end{array}\right)$	24135	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&\\ -2&-1&0&1&\\ -3&-2&-1&&\\ -4&-3&-2&& \end{array}, \begin{array}{cccc} 0&1&2&3\\ -1&0&& \end{array}\right)$	14253
12	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&\\ -3&-2&-1&0&\\ -4&&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&&\\ -2&&\\ -3&& \end{array}\right)$	21534	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&\\ -2&-1&0&1&\\ -3&-2&-1&0&\\ -4&-3&&& \end{array}, \begin{array}{cccc} 0&1&2&3\\ -1&&&\\ -2&&& \end{array}\right)$	31254
13	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&\\ -3&-2&&&\\ -4&-3&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&0&\\ -2&-1& \end{array}\right)$	13524	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&&\\ -3&-2&-1&&\\ -4&-3&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&0&1\\ -2&& \end{array}\right)$	31425
14	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&1&2\\ -3&-2&&&\\ -4&&&& \end{array}, \begin{array}{cc} 0&1\\ -1&0\\ -2&-1\\ -3& \end{array}\right)$	23145	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&\\ -2&-1&0&&\\ -3&-2&-1&&\\ -4&-3&-2&& \end{array}, \begin{array}{cccc} 0&1&2&3\\ -1&0&1& \end{array}\right)$	12453

Table 5.8: Diagram/Permutation Correspondence: $\ell = 5$ Part III

Row	Diagram	Perm.	Diagram	Perm.
15	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0&1\\-3&-2&-1\\-4\end{array}, \begin{array}{ccc}0&1&2\\-1&0\\-2\\-3\end{array}\right)$	24513	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2\\-2&-1&0&1\\-3&-2&-1\\-4&-3\end{array}, \begin{array}{cccc}0&1&2&3\\-1&0\\-2\end{array}\right)$	34152
16	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2\\-2&-1&0&1\\-3&-2&-1&0\\-4\end{array}, \begin{array}{cccc}0&1&2&3\\-1\\-2\\-3\end{array}\right)$	21354	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2\\-2&-1&0&1\\-3&-2&-1\\-4\end{array}, \begin{array}{cccc}0&1&2&3\\-1&0\\-2\\-3\end{array}\right)$	24153
17	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2\\-2&-1&0&1\\-3&-2\\-4&-3\end{array}, \begin{array}{cccc}0&1&2&3\\-1&0\\-2&-1\end{array}\right)$	13254	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0\\-3&-2&-1\\-4\end{array}, \begin{array}{ccc}0&1&2\\-1&0&1\\-2\\-3\end{array}\right)$	21435
18	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0&1\\-3&-2\\-4\end{array}, \begin{array}{ccc}0&1&2\\-1&0\\-2&-1\\-3\end{array}\right)$	23514	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2\\-2&-1&0\\-3&-2&-1\\-4&-3\end{array}, \begin{array}{cccc}0&1&2&3\\-1&0&1\\-2\end{array}\right)$	31452
19	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0\\-3&-2\\-4&-3\end{array}, \begin{array}{ccc}0&1&2\\-1&0&1\\-2&-1\end{array}\right)$	13425	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2\\-2&-1&0\\-3&-2\\-4\end{array}, \begin{array}{cccc}0&1&2&3\\-1&0&1\\-2&-1\\-3\end{array}\right)$	23451
20	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2\\-2&-1&0\\-3&-2\\-4&-3\end{array}, \begin{array}{cccc}0&1&2&3\\-1&0&1\\-2&-1\end{array}\right)$	13452	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0\\-3&-2\\-4\end{array}, \begin{array}{ccc}0&1&2\\-1&0&1\\-2&-1\\-3\end{array}\right)$	23415
21	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2\\-2&-1&0&1\\-3&-2\\-4\end{array}, \begin{array}{cccc}0&1&2&3\\-1&0\\-2&-1\\-3\end{array}\right)$	23154	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2\\-2&-1&0\\-3&-2&-1\\-4\end{array}, \begin{array}{cccc}0&1&2&3\\-1&0&1\\-2\\-3\end{array}\right)$	21453

5.2 Multiplicity of Maximal Dominant Weights of $V(k\Lambda_0)$

We now wish to consider the multiplicity of the weights of $V(k\Lambda_0)$ $(k \geq 2)$ denoted $k\Lambda_0 - \gamma_\ell$, where $\gamma_\ell = \ell\alpha_0 + (\ell-1)\alpha_1 + (\ell-2)\alpha_2 + \ldots + \alpha_{\ell-1} + \alpha_{n-\ell+1} + 2\alpha_{n-\ell+2} + \ldots + (\ell-1)\alpha_{n-1}$.

Based on computational evidence, we make the following conjecture about the multiplicity of $k\Lambda_0 - \gamma_\ell$.

Conjecture 5.2.1 $mult_{k\Lambda_0}(k\Lambda_0 - \gamma_\ell) = |\{(k+1)(k)\ldots 21\text{-}avoiding\ permutations\ of\ [\ell]\}|$.

In the previous section, we proved this completely for the $k = 2$ case. Though the proof of the general case is incomplete, calculations from a MATLAB program (see Appendix) provide evidence of the validity of this conjecture for $k = 3$ and $k = 4$. Table 5.9 shows these results and the known permutation counts from [19].

Table 5.9: Evidence of Conjecture 5.2.1 for $k = 3, 4$

Multiplicity of $k\Lambda_0 - \gamma_\ell$ Computed in MATLAB

ℓ	$k=2$	$k=3$	$k=4$
2	2	2	2
3	5	6	6
4	14	23	24
5	42	103	119
6	132	513	694
7	429	2761	4582
8	1430	15767	33324
9	4862	94359	261808

Avoiding permutations of $[\ell]$ Known

ℓ	321	4321	54321
2	2	2	2
3	5	6	6
4	14	23	24
5	42	103	119
6	132	513	694
7	429	2761	4582
8	1430	15767	33324
9	4862	94359	261808

Consider the case that $k = 3$ and $\ell = 3$. We can add an empty diagram in the third position to every 2-tuple in $B(2\Lambda_0)_{2\Lambda_0-\gamma_\ell}$ to obtain a 3-tuple in $B(3\Lambda_0)_{3\Lambda_0-\gamma_\ell}$. For these diagrams, we keep the same correspondence with permutations, as a 321-avoiding permutation is also a 4321-avoiding permutation. We can also create one 3-tuple of extended Young diagrams in $B(3\Lambda_0)_{3\Lambda_0-\gamma_\ell}$ that has a nonempty third position. Since there is only one 4321-avoiding permutation of [3] that is not a 321-avoiding permutation, namely 321, it is reasonable to conjecture that this new diagram and 321 correspond with one another.

Table 5.10: Diagram/Permutation Correspondence: $k = 3$, $\ell = 3$

Diagram	Perm.	Diagram	Perm.
$\left(\begin{matrix}0&1&2\\-1&0&1\\-2&-1&0\end{matrix}, \ulcorner, \ulcorner\right)$	123	$\left(\begin{matrix}0&1&2\\-1&0&1\\-2&-1&\end{matrix}, \begin{matrix}0\end{matrix}, \ulcorner\right)$	312
$\left(\begin{matrix}0&1&2\\-1&0&1\\-2&&\end{matrix}, \begin{matrix}0\\-1\end{matrix}, \ulcorner\right)$	213	$\left(\begin{matrix}0&1&2\\-1&0&\\-2&-1&\end{matrix}, \begin{matrix}0&1\end{matrix}, \ulcorner\right)$	132
$\left(\begin{matrix}0&1&2\\-1&0&\\-2&&\end{matrix}, \begin{matrix}0&1\\-1&\end{matrix}, \ulcorner\right)$	231	$\left(\begin{matrix}0&1&2\\-1&&\\-2&&\end{matrix}, \begin{matrix}0&1\\-1&\end{matrix}, \begin{matrix}0\end{matrix}\right)$	321

We note that we can construct the elements of $B(k\Lambda_0)$ of weight $k\Lambda_0 - \gamma_\ell$ by drawing certain paths in an $\ell \times \ell$ square. Ideally, we will be able to use this notion to develop a bijection between these paths and the avoiding permutations.

Theorem 5.2.2 *The multiplicity of $k\Lambda_0 - \gamma_\ell$ in $V(k\Lambda_0)$ is equal to the number of elements in the set $\mathcal{Y}$, where $\mathcal{Y}$ is the set of k-tuples of extended Young diagrams that can be constructed according to the following conditions:*

1. *Draw a k-tuple of extended Young diagrams, in which $Y_1 = (\overbrace{-\ell, -\ell, -\ell, \ldots, -\ell}^{\ell}, 0, 0, \ldots)$ and $Y_i = (0, 0, 0, \ldots, 0), 2 \leq i \leq k$. Note that Y has weight $k\Lambda_0 - \gamma_\ell$. See Figure 5.5.*

   For ease of notation, we will consider the extended Young diagram Y_1 to have coordinates $(0, 0)$ in the lower left corner.

2. *We draw a sequence of lattice paths from $(0, 0)$ to (ℓ, ℓ). We may only move up and to the right in integral steps. We define $T_i^j, i \geq 2$ to be the number of j-colored boxes between*

$$Y = \left(\begin{array}{|c|c|c|c|} \hline 0 & 1 & \cdots & \ell-1 \\ \hline -1 & 0 & \cdots & \ell-2 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 1-\ell & 2-\ell & \cdots & 0 \\ \hline \end{array}, \quad \ulcorner, \quad \ulcorner, \quad \ulcorner, \ldots, \quad \ulcorner \right)$$

Figure 5.5: k-tuple of Extended Young Diagrams

the $(i-1)^{st}$ path and the $(i-2)^{nd}$ path. (T_i^j will also be the number of boxes of color j in Y_i). We draw the paths according to the following conditions:

(a) The first path must be drawn so that it does not cross the diagonal $y = x$.

(b) For all other paths $i, 2 \leq i \leq k-1$, the following conditions must be satisfied.

i. $T_i^j \leq min \left\{ T_{i-1}^j, \ell - |j| - T_2^j - \sum_{a=2}^{i-1} T_a^j \right\}$

ii. For $j > 0, T_i^j \leq T_i^{j-1} \leq T_i^{j-2} \leq \ldots \leq T_i^1 \leq T_i^0$ and for $j < 0, T_i^j \leq T_i^{j+1} \leq T_i^{j+2} \leq \ldots \leq T_i^{-1} \leq T_i^0$.

3. In sequence, when we draw the $(i-1)^{st}$ path, we will remove the boxes to the right of the path and place them uniquely as a Young diagram in Y_i.

Proof We must show that $B(k\Lambda_0)_{k\Lambda_0-\gamma_\ell} = \mathcal{Y}$. First we will show that $B(k\Lambda_0)_{k\Lambda_0-\gamma_\ell} \subseteq \mathcal{Y}$. Let $X \in B(k\Lambda_0)_{k\Lambda_0-\gamma_\ell}$. Since X must have weight $k\Lambda_0 - \gamma_\ell$, colored boxes can be moved to create Y as in step 1, above. Since $X_2 \subseteq X_1$, if we draw a lattice path from the lower left to upper right corner of this $\ell \times \ell$ box and remove boxes to place in X_2, we know that this path did not cross y=x. Now we will continue to draw paths and remove boxes to create $X_i, 2 \leq i \leq k$. We see that conditions (i) and (ii) must be satisfied as follows.

- $T_i^j \leq T_{i-1}^j$ is satisfied since $X_i \subset X_{i-1}$.

- $T_i^j \leq \ell - |j| - T_2^j - \sum_{a=2}^{i-1} T_a^j$ is satisfied since there are $\ell - |j|$ boxes of color $|j|$ present in X, T_2^j of them must be in X_1, and $\sum_{a=2}^{i-1} T_a^j$ of them are used in preceding $(X_2, X_3, \ldots, X_{i-1})$.

- $j > 0, T_i^j \leq T_i^{j-1} \leq T_i^{j-2} \leq \ldots \leq T_i^1 \leq T_i^0$ and $j < 0, T_i^j \leq T_i^{j+1} \leq T_i^{j+2} \leq \ldots \leq T_i^{-1} \leq T_i^0$ are satisfied since X_i is an extended Young diagram.

Thus $X \in \mathcal{Y}$ and $B(k\Lambda_0)_{k\Lambda_0 - \gamma_\ell} \subseteq \mathcal{Y}$.

Now, we must show that $\mathcal{Y} \subseteq B(k\Lambda_0)_{k\Lambda_0 - \gamma_\ell}$. Suppose $X \in \mathcal{Y}$. Clearly, $wt(X) = k\Lambda_0 - \gamma_\ell$. Now we must show that X meets the conditions of Theorem 3.2.4. That is, we must show the following:

1. Each X_i is an extended Young diagram.

2. $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots \supseteq X_k$

3. For each $s \geq 0, \exists$ some $j \geq 1$ such that $(X_{j+1})_s > (X_j)_{s+1}$

4. $X_k \supseteq X_1[n]$

We will begin with (1). Since X_1 begins as an $\ell \times \ell$ square and we remove boxes by only drawing lattice paths with right and up moves, X_1 remains an extended Young diagram. X_2 is also an extended Young diagram by construction. We now show inductively that $X_3, X_4, \ldots, X_k$ are extended Young diagrams. Assume that all X_i such that $i < q$ are extended Young diagrams. Consider X_q. Construct X_q by removing a finite number of boxes from X_1. We place these colored boxes as if subject to gravity in the upper left corner, but in the appropriate position for their color. Now we must show that there are no "holes" in X_q. That is, there are no slots without boxes that have boxes below them or to the right of them.

- *Case 1:* Suppose there is a hole that should contain a box of color 0. First, suppose that there is a filled in box of color j to the right of the hole. Then $T_{q-1}^j > T_{q-1}^0$, which contradicts condition (ii) above. Now, suppose that there is a filled in box of color $-j$

below the hole, but no filled in boxes to the right. Then $T_{q-1}^{-j} > T_{q-1}^{0}$, a contradiction. Thus, there can be no holes where a box of color 0 should be.

- *Case 2:* Suppose there is a hole that should contain boxes of positive color.
  - *Case a:* Of all the empty slots that should contain boxes of positive color, consider the one that has a filled in box to its right that should have largest color. Suppose the missing box should have color j. Then there is a $j+1$ colored box to its right and $T_{q-1}^{j} < T_{q-1}^{j+1}$, a contradiction. We continue this to see that there are no holes with filled in boxes to the right of them.
  - *Case b:* Consider empty slots that have no boxes to the right, but do have boxes below them. Consider the smallest such j colored box that is missing. Then there are boxes of color $j-1$ below this hole. Since we draw a lattice path in X_1 at every step in our procedure, we could never remove these $j-1$ colored boxes without removing a sufficient number of j colored boxes. We continue moving through this case, possibly returning to case (a) and find that there are no empty slots where a positive color should be.
- *Case 3:* Suppose there is a hole that should contain boxes of negative color.
  - *Case a:* Suppose there is at least one box to the right of the hole. Of all boxes meeting this condition, consider the one such that the color it should be, say j, is maximal. Then there is a box to the right of the hole of color $j+1$. By our construction, we could not have drawn a lattice path in X_1 that would remove this $j+1$ colored box without removing the j colored box, giving a contradiction.
  - *Case b:* Suppose there are no boxes to the right of the hole. That is, there is at least one box below the hole. Of all empty slots fitting this criteria, consider the one of minimal color j. Then we must have $T_{q-1}^{j-1} > T_{q-1}^{j}$, which is a contradiction. We continue in this manner until all potential holes are shown to be filled in.

Now, we must show condition (2), which is the condition involving containment of the diagrams. It is clear that after the first path is drawn, we have $X_1 \supseteq X_2$. As each other path is drawn, we have the condition that $T_i^j \leq \ell - |j| - T_2^j - \sum_{a=2}^{i-1} T_a^j$, which stipulates the the number of boxes of color j removed must be less than the number of boxes of color j accounted for already. This formula double counts the boxes used up in X_2 so that we keep $X_1 \supseteq X_2$. Now consider some diagram X_q given that $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots \supseteq X_{q-1}$. We have the condition that $T_q^j \leq T_{q-1}^j$, guaranteeing that $X_{q-1} \supseteq X_q$. Thus, $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots \supseteq X_k$.

To show condition (3), consider $j = k$ and consider the extended Young diagram X_k written in sequence form. By construction, all values in this sequence are less than or equal to 0. Note that $X_{k+1} = X_1[n]$ and $n \leq 2\ell$. Thus, the smallest values in the sequence for X_{k+1} are all positive and as follows: $(\overbrace{n-\ell, n-\ell, n-\ell, \ldots, n-\ell}^{\ell}, n, n, \ldots)$. Thus, we obtain that $(X_{k+1})_s > (X_k)_{s+1}$ for all values of s.

Condition (4) is clear from proof of condition (3). $\square$

Example 5.2.3 *Figure 5.6 is an example from the case $k = 3$, $\ell = 4$.*

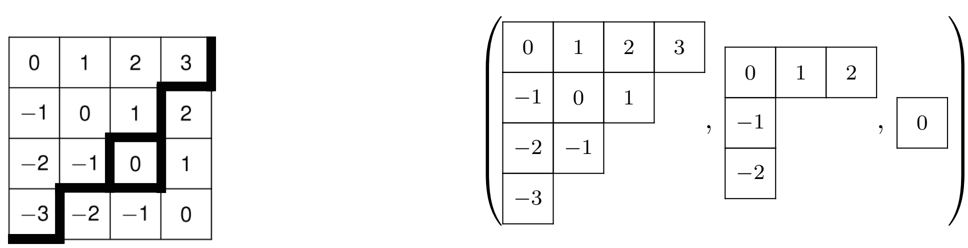


Figure 5.6: Path and Diagram Correspondence $k = 3$

Now, we focus on the $k = 3$ case. As in the $k = 2$ case, let P_ℓ be the set of paths described above and W_ℓ be the set of 4321-avoiding permutations of ℓ. We will primarily consider the map from W_ℓ to P_ℓ. Let $w = w_1 w_2 \ldots w_\ell$ be a 4321-avoiding permutation. Let $C_i = \{j \mid j > i, w_j < w_i\}, c_i = |C_i|, J = \{j_1, j_2, \ldots, j_r\}_< = \{j \mid c_j > 0\}$. Define $D_i = \{(j, q) \mid$

$i < j < q, w_q < w_j < w_i$ and at least one of j, q have not yet been included already$\}$. We let $d_i = |D_i|$ and $K = \{k_1, k_2, \ldots, k_r\}_< = \{k \mid d_k > 0\}$

Example 5.2.4 *Consider the permutation $w = 53412$. We obtain the data in Table 5.11.*

Table 5.11: Data for 53412

i	C_i	c_i	D_i	d_i
1	2,3,4,5	4	24, 25, 34, ~~35~~	3
2	4,5	2	-	0
3	4,5	2	-	0
4	-	0	-	0
5	-	0	-	0

Here, $J = \{1, 2, 3\}$ and $K = \{1\}$. Note that we did not include 35 in D_1 because as we moved from left to right, we had already included both 3 and 5 in pairs.

Though this construction has not yet given us a bijection between P_ℓ and W_ℓ, there is evidence to support the notion that it will. In addition to tables with matching total counts of diagrams and permutations, we can use MATLAB programs to learn more. We can make tables that contain information about the number of diagrams with a certain condition and the number of avoiding permutations with a certain condition. If the numbers are identical, it is reasonable to conjecture that these conditions correspond in the bijection we wish to create. Some of these that we have found are the following:

- The height of the last horizontal move in either path is equal to the maximum value in J.
- The width of the first horizontal move in the either path is equal to $c_{j_1} + j_1 - d_{j_1} - 1$.
- The maximum color removed by the second path is equal to the maximum value in K minus 1.

In the $k = 3, \ell = 4$ case we can take each of the 16 diagrams from the $k = 2$ case and add an empty extended Young diagram as Y_3. These diagrams will correspond to the same 321-

avoiding permutations, as these permutations are also 4321-avoiding. There are nine new tuples of extended Young diagrams, which correspond with the nine 4321-avoiding permutations that are not 321-avoiding permutations. One possible correspondence between these new tuples and permutations is in Table 5.12.

Table 5.12: Diagram/Permutation Correspondence: $k = 3$, $\ell = 4$

Diagram	Perm.	Diagram	Perm.
$\left(\begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & & \\ -3 & -2 & & \end{array}, \begin{array}{cc} 0 & 1 \\ -1 & \end{array}, \begin{array}{c} 0 \end{array}\right)$	4213	$\left(\begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & \\ -2 & -1 & & \\ -3 & -2 & & \end{array}, \begin{array}{ccc} 0 & 1 & 2 \\ -1 & & \end{array}, \begin{array}{c} 0 \end{array}\right)$	4132
$\left(\begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & 0 & & \\ -2 & -1 & & \\ -3 & -2 & & \end{array}, \begin{array}{ccc} 0 & 1 & 2 \\ -1 & & \end{array}, \begin{array}{cc} 0 & 1 \end{array}\right)$	1432	$\left(\begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & & \\ -3 & & & \end{array}, \begin{array}{cc} 0 & 1 \\ -1 & \\ -2 & \end{array}, \begin{array}{c} 0 \end{array}\right)$	4312
$\left(\begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & & & \\ -3 & & & \end{array}, \begin{array}{cc} 0 & 1 \\ -1 & \\ -2 & \end{array}, \begin{array}{c} 0 \\ -1 \end{array}\right)$	3214	$\left(\begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & \\ -2 & -1 & & \\ -3 & & & \end{array}, \begin{array}{ccc} 0 & 1 & 2 \\ -1 & & \\ -2 & & \end{array}, \begin{array}{c} 0 \end{array}\right)$	4231
$\left(\begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & 0 & & \\ -2 & -1 & & \\ -3 & & & \end{array}, \begin{array}{ccc} 0 & 1 & 2 \\ -1 & & \\ -2 & & \end{array}, \begin{array}{cc} 0 & 1 \end{array}\right)$	2431	$\left(\begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & \\ -2 & & & \\ -3 & & & \end{array}, \begin{array}{ccc} 0 & 1 & 2 \\ -1 & & \\ -2 & & \end{array}, \begin{array}{c} 0 \\ -1 \end{array}\right)$	3241
$\left(\begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & 0 & & \\ -2 & & & \\ -3 & & & \end{array}, \begin{array}{ccc} 0 & 1 & 2 \\ -1 & & \\ -2 & & \end{array}, \begin{array}{cc} 0 & 1 \\ -1 & \end{array}\right)$	3421		

When $\ell = 5$, there are 61 new tuples of extended Young diagrams and 61 new permutations. Because the exact correspondence is still unclear, we include a possible correspondence. Within this, we will sort the diagrams and possible corresponding permutations according to the information we have. We will have six tables. Each will exhibit one possible combination of the maximum element of J and the maximum element of K. Recall that we believe these values reflect the maximum color removed by the first path and the second path, respectively. We will label these by $< a, b >$, where a is the maximum color removed by the first path and b is the maximum color removed by the second path. Within these tables, we sort by the length of the first horizontal move. After the $< 1, 0 >$ case, we often have sets of m diagrams and m avoiding permutations that we believe are in correspondence, though the exact correspondence may be unclear.

Table 5.13: Diagram/Permutation Correspondence: $k = 3, \ell = 5$, Type $< 1, 0 >$

Diagram	Perm.	Diagram	Perm.
$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0&1&2\\-3&-2&-1&&\\-4&&&&\end{array},\begin{array}{cc}0&1\\-1&\\-2&\\-3&\end{array},\begin{array}{c}0\end{array}\right)$	54123	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0&1&2\\-3&-2&&&\\-4&&&&\end{array},\begin{array}{cc}0&1\\-1&\\-2&\\-3&\end{array},\begin{array}{c}0\\-1\end{array}\right)$	43125
$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0&1&2\\-3&&&&\\-4&&&&\end{array},\begin{array}{cc}0&1\\-1&\\-2&\\-3&\end{array},\begin{array}{c}0\\-1\\-2\end{array}\right)$	32145	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0&1&2\\-3&-2&-1&&\\-4&-3&&&\end{array},\begin{array}{cc}0&1\\-1&\\-2&\end{array},\begin{array}{c}0\end{array}\right)$	53124
$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0&1&2\\-3&-2&&&\\-4&-3&&&\end{array},\begin{array}{cc}0&1\\-1&\\-2&\end{array},\begin{array}{c}0\\-1\end{array}\right)$	42135	$\left(\begin{array}{ccccc}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0&1&2\\-3&-2&-1&&\\-4&-3&-2&&\end{array},\begin{array}{cc}0&1\\-1&\end{array},\begin{array}{c}0\end{array}\right)$	52134

Table 5.14: Diagram/Permutation Correspondence: $k = 3, \ell = 5$, Type $< 2, 0 >$

Diagram	Diagram	Perm.
$\left(\begin{array}{lllll}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0&1&\\-3&-2&-1&&\\-4&&&&\end{array}, \begin{array}{lll}0&1&2\\-1&&\\-2&&\\-3&&\end{array}, \begin{array}{l}0\end{array}\right)$	$\left(\begin{array}{lllll}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&&&\\-3&-2&&&\\-4&&&&\end{array}, \begin{array}{lll}0&1&2\\-1&0&1\\-2&-1&\\-3&&\end{array}, \begin{array}{l}0\end{array}\right)$	32415, 32514
$\left(\begin{array}{lllll}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0&1&\\-3&-2&&&\\-4&&&&\end{array}, \begin{array}{lll}0&1&2\\-1&&\\-2&&\\-3&&\end{array}, \begin{array}{l}0\\-1\end{array}\right)$	$\left(\begin{array}{lllll}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&&&\\-3&-2&&&\\-4&&&&\end{array}, \begin{array}{lll}0&1&2\\-1&0&1\\-2&&\\-3&&\end{array}, \begin{array}{l}0\\-1\end{array}\right)$	42315, 43512
$\left(\begin{array}{lllll}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0&1&\\-3&&&&\\-4&&&&\end{array}, \begin{array}{lll}0&1&2\\-1&&\\-2&&\\-3&&\end{array}, \begin{array}{l}0\\-1\\-2\end{array}\right)$	$\left(\begin{array}{lllll}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&&&\\-3&&&&\\-4&&&&\end{array}, \begin{array}{lll}0&1&2\\-1&0&1\\-2&&\\-3&&\end{array}, \begin{array}{l}0\\-1\\-2\end{array}\right)$	52413, 53412
$\left(\begin{array}{lllll}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0&1&\\-3&-2&-1&&\\-4&-3&&&\end{array}, \begin{array}{lll}0&1&2\\-1&&\\-2&&\end{array}, \begin{array}{l}0\end{array}\right)$	$\left(\begin{array}{lllll}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&&&\\-3&-2&&&\\-4&-3&&&\end{array}, \begin{array}{lll}0&1&2\\-1&0&1\\-2&-1&\end{array}, \begin{array}{l}0\end{array}\right)$	41325, 42513
$\left(\begin{array}{lllll}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0&1&\\-3&-2&&&\\-4&-3&&&\end{array}, \begin{array}{lll}0&1&2\\-1&&\\-2&&\end{array}, \begin{array}{l}0\\-1\end{array}\right)$	$\left(\begin{array}{lllll}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&&&\\-3&-2&&&\\-4&-3&&&\end{array}, \begin{array}{lll}0&1&2\\-1&0&1\\-2&&\end{array}, \begin{array}{l}0\\-1\end{array}\right)$	51423, 52314
$\left(\begin{array}{lllll}0&1&2&3&4\\-1&0&1&2&3\\-2&-1&0&1&\\-3&-2&-1&&\\-4&-3&-2&&\end{array}, \begin{array}{lll}0&1&2\\-1&&\end{array}, \begin{array}{l}0\end{array}\right)$		51324

Table 5.15: Diagram/Permutation Correspondence: $k = 3, \ell = 5$, Type $< 3, 0 >$

Diagram	Diagram	Perm.
$\left(\begin{array}{lllll} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 \\ -4 \end{array}, \begin{array}{llll} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{array}, \begin{array}{l} 0 \end{array} \right)$	$\left(\begin{array}{lllll} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 \\ -3 & -2 \\ -4 \end{array}, \begin{array}{llll} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 \\ -2 & -1 \\ -3 \end{array}, \begin{array}{l} 0 \end{array} \right)$	32154, 32451
$\left(\begin{array}{lllll} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 \\ -4 \end{array}, \begin{array}{llll} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{array}, \begin{array}{l} 0 \\ -1 \end{array} \right)$	$\left(\begin{array}{lllll} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 \\ -3 & -2 \\ -4 \end{array}, \begin{array}{llll} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 \\ -2 \\ -3 \end{array}, \begin{array}{l} 0 \\ -1 \end{array} \right)$	42351,43152
$\left(\begin{array}{lllll} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 \\ -4 \end{array}, \begin{array}{llll} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{array}, \begin{array}{l} 0 \\ -1 \\ -2 \end{array} \right)$	$\left(\begin{array}{lllll} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 \\ -3 \\ -4 \end{array}, \begin{array}{llll} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 \\ -2 \\ -3 \end{array}, \begin{array}{l} 0 \\ -1 \\ -2 \end{array} \right)$	52341, 53142
$\left(\begin{array}{lllll} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 \\ -4 & -3 \end{array}, \begin{array}{llll} 0 & 1 & 2 & 3 \\ -1 \\ -2 \end{array}, \begin{array}{l} 0 \end{array} \right)$	$\left(\begin{array}{lllll} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 \\ -3 & -2 \\ -4 & -3 \end{array}, \begin{array}{llll} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 \\ -2 & -1 \end{array}, \begin{array}{l} 0 \end{array} \right)$	41352, 42153
$\left(\begin{array}{lllll} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 \\ -4 & -3 \end{array}, \begin{array}{llll} 0 & 1 & 2 & 3 \\ -1 \\ -2 \end{array}, \begin{array}{l} 0 \\ -1 \end{array} \right)$	$\left(\begin{array}{lllll} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 \\ -3 & -2 \\ -4 & -3 \end{array}, \begin{array}{llll} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 \\ -2 \end{array}, \begin{array}{l} 0 \\ -1 \end{array} \right)$	51342, 52143
$\left(\begin{array}{lllll} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 \\ -4 & -3 & -2 \end{array}, \begin{array}{llll} 0 & 1 & 2 & 3 \\ -1 \end{array}, \begin{array}{l} 0 \end{array} \right)$		51243

Table 5.16: Diagram/Permutation Correspondence: $k = 3, \ell = 5$, Type $< 2, 1 >$

Diagram	Diagram
$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&&\\ -3&-2&-1&&\\ -4&&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&&\\ -2&&\\ -3&& \end{array}, \begin{array}{cc} 0&1 \end{array} \right)$	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&&&\\ -3&-2&&&\\ -4&&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&0&\\ -2&-1&\\ -3&& \end{array}, \begin{array}{cc} 0&1 \end{array} \right)$
$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&&\\ -3&-2&&&\\ -4&&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&&\\ -2&&\\ -3&& \end{array}, \begin{array}{cc} 0&1\\ -1& \end{array} \right)$	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&&&\\ -3&-2&&&\\ -4&&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&0&\\ -2&&\\ -3&& \end{array}, \begin{array}{cc} 0&1\\ -1& \end{array} \right)$
$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&&\\ -3&&&&\\ -4&&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&&\\ -2&&\\ -3&& \end{array}, \begin{array}{cc} 0&1\\ -1&\\ -2& \end{array} \right)$	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&&&\\ -3&&&&\\ -4&&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&0&\\ -2&&\\ -3&& \end{array}, \begin{array}{cc} 0&1\\ -1&\\ -2& \end{array} \right)$
Permutations: 24315, 25314, 25413, 45312, 34215, 35214	
$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&&\\ -3&-2&-1&&\\ -4&-3&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&&\\ -2&& \end{array}, \begin{array}{cc} 0&1 \end{array} \right)$	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&&&\\ -3&-2&&&\\ -4&-3&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&0&\\ -2&-1& \end{array}, \begin{array}{cc} 0&1 \end{array} \right)$
$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&&\\ -3&-2&&&\\ -4&-3&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&&\\ -2&& \end{array}, \begin{array}{cc} 0&1\\ -1& \end{array} \right)$	$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&&&\\ -3&-2&&&\\ -4&-3&&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&0&\\ -2&& \end{array}, \begin{array}{cc} 0&1\\ -1& \end{array} \right)$
Permutations: 15423, 35412, 45213, 14325	
$\left(\begin{array}{ccccc} 0&1&2&3&4\\ -1&0&1&2&3\\ -2&-1&0&&\\ -3&-2&-1&&\\ -4&-3&-2&& \end{array}, \begin{array}{ccc} 0&1&2\\ -1&& \end{array}, \begin{array}{cc} 0&1 \end{array} \right)$	Permutation: 15324

Table 5.17: Diagram/Permutation Correspondence: $k = 3, \ell = 5$, Type $< 3, 1 >$

Diagram	Diagram
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 \\ -3 & -2 & -1 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 \end{ytableau} \right)$	$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 \\ -3 & -2 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 & 0 \\ -2 & -1 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 \end{ytableau} \right)$
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 \\ -3 & -2 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 \\ -1 \end{ytableau} \right)$	$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 \\ -3 & -2 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 & 0 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 \\ -1 \end{ytableau} \right)$
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 \\ -3 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 \\ -1 \\ -2 \end{ytableau} \right)$	$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 \\ -3 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 & 0 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 \\ -1 \\ -2 \end{ytableau} \right)$
Permutations: 24351, 25143, 25341, 34251, 35241, 45231	
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 \\ -3 & -2 & -1 \\ -4 & -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \end{ytableau}, \begin{ytableau} 0 & 1 \end{ytableau} \right)$	$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 \\ -3 & -2 \\ -4 & -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 & 0 \\ -2 & -1 \end{ytableau}, \begin{ytableau} 0 & 1 \end{ytableau} \right)$
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 \\ -3 & -2 \\ -4 & -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \end{ytableau}, \begin{ytableau} 0 & 1 \\ -1 \end{ytableau} \right)$	$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 \\ -3 & -2 \\ -4 & -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 & 0 \\ -2 \end{ytableau}, \begin{ytableau} 0 & 1 \\ -1 \end{ytableau} \right)$
Permutations: 14352, 15342, 35142, 45132	
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 \\ -3 & -2 & -1 \\ -4 & -3 & -2 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \end{ytableau}, \begin{ytableau} 0 & 1 \end{ytableau} \right)$	Permutation: 15243

Table 5.18: Diagram/Permutation Correspondence: $k = 3, \ell = 5$, Type $< 3, 2 >$

Diagram	Diagram
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \\ -3 & -2 & -1 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \end{ytableau} \right)$	$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 \\ -2 & -1 \\ -3 & -2 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 & 0 \\ -2 & -1 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \end{ytableau} \right)$
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \\ -3 & -2 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \\ -1 \end{ytableau} \right)$	$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 \\ -2 & -1 \\ -3 & -2 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 & 0 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \\ -1 \end{ytableau} \right)$
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \\ -3 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \\ -1 \\ -2 \end{ytableau} \right)$	$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 \\ -2 & -1 \\ -3 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 & 0 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \\ -1 \\ -2 \end{ytableau} \right)$
Permutations: 21543, 23541, 24531, 32541, 34521, 42531	
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \\ -3 & -2 & -1 \\ -4 & -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \end{ytableau} \right)$	$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 \\ -2 & -1 \\ -3 & -2 \\ -4 & -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 & 0 \\ -2 & -1 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \end{ytableau} \right)$
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \\ -3 & -2 \\ -4 & -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \\ -1 \end{ytableau} \right)$	$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 \\ -2 & -1 \\ -3 & -2 \\ -4 & -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 & 0 \\ -2 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \\ -1 \end{ytableau} \right)$
Permutations: 13542, 14532, 31542, 41532	
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \\ -3 & -2 & -1 \\ -4 & -3 & -2 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \end{ytableau} \right)$	Permutation: 12543

Now, consider the case $k = 4$. After setting Y_4 equal to the empty extended Young diagram and adding to previous results, we get nothing new for $\ell = 3$.

For $\ell = 4$, we add only one tuple of extended Young diagrams and one permutation, shown in Table 5.19.

Table 5.19: Diagram/Permutation Correspondence: $k = 4, \ell = 4$

Diagram	Permutation
$\left(\begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & & & \\ -2 & & & \\ -3 & & & \end{array}, \begin{array}{ccc} 0 & 1 & 2 \\ -1 & & \\ -2 & & \end{array}, \begin{array}{cc} 0 & 1 \\ -1 & \end{array}, \begin{array}{c} 0 \end{array}\right)$	4321

For $\ell = 5$, we add 16 new tuples of extended Young diagrams. A possible grouping based on the length of the first horizontal move and the height of the last horizontal move of the first path is shown below. Though it does not generalize to all ℓ, the same relationship between the first vertical move and $c_{j_1} + j_1 - d_{j_1} - 1$ is present in some sense. There is evidence for many values of ℓ that the notion that the height of the last horizontal move corresponds to the maximum element of J holds here as well. We will label these cases by (a, b), where a the width of the first horizontal move and b is the height of the last horizontal move.

Table 5.20: Diagram/Permutation Correspondence: $k = 4, \ell = 5$, Type $(1, 3)$

$\left(\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 & 3 \\ -2 & -1 & & & \\ -3 & -2 & & & \\ -4 & & & & \end{array}, \begin{array}{ccc} 0 & 1 & 2 \\ -1 & & \\ -2 & & \\ -3 & & \end{array}, \begin{array}{cc} 0 & 1 \\ -1 & \end{array}, \begin{array}{c} 0 \end{array}\right)$	$\left(\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 & 3 \\ -2 & -1 & & & \\ -3 & & & & \\ -4 & & & & \end{array}, \begin{array}{ccc} 0 & 1 & 2 \\ -1 & & \\ -2 & & \\ -3 & & \end{array}, \begin{array}{cc} 0 & 1 \\ -1 & \\ -2 & \end{array}, \begin{array}{c} 0 \end{array}\right)$
$\left(\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 & 3 \\ -2 & & & & \\ -3 & & & & \\ -4 & & & & \end{array}, \begin{array}{ccc} 0 & 1 & 2 \\ -1 & & \\ -2 & & \\ -3 & & \end{array}, \begin{array}{cc} 0 & 1 \\ -1 & \\ -2 & \end{array}, \begin{array}{c} 0 \\ -1 \end{array}\right)$	
Permutations: 43215, 54213, 54312	

Table 5.21: Diagram/Permutation Correspondence: $k = 4, \ell = 5$, Type $(1, 4)$

$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 \\ -3 & -2 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 \\ -1 \end{ytableau}, \begin{ytableau} 0 \end{ytableau}\right)$	$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 \\ -2 & -1 \\ -3 & -2 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \\ -1 \end{ytableau}, \begin{ytableau} 0 & 1 \end{ytableau}\right)$
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 \\ -3 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 \\ -1 \\ -2 \end{ytableau}, \begin{ytableau} 0 \end{ytableau}\right)$	$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 \\ -2 & -1 \\ -3 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \\ -1 \\ -2 \end{ytableau}, \begin{ytableau} 0 \end{ytableau}\right)$
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ -2 \\ -3 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 \\ -1 \\ -2 \end{ytableau}, \begin{ytableau} 0 \\ -1 \end{ytableau}\right)$	$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 \\ -2 & -1 \\ -3 & -2 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \\ -1 \end{ytableau}, \begin{ytableau} 0 \end{ytableau}\right)$
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 \\ -2 & -1 \\ -3 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \\ -1 \\ -2 \end{ytableau}, \begin{ytableau} 0 & 1 \end{ytableau}\right)$	$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 \\ -2 \\ -3 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \\ -1 \\ -2 \end{ytableau}, \begin{ytableau} 0 \\ -1 \end{ytableau}\right)$
$\left(\begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 \\ -2 \\ -3 \\ -4 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 & 3 \\ -1 \\ -2 \\ -3 \end{ytableau}, \begin{ytableau} 0 & 1 & 2 \\ -1 \\ -2 \end{ytableau}, \begin{ytableau} 0 & 1 \\ -1 \end{ytableau}\right)$	
Permutations: 25431, 35421, 43251, 43521, 45321, 53241, 53421, 54132, 54231	

Table 5.22: Diagram/Permutation Correspondence: $k = 4, \ell = 5$, Type $(2,3)$

$\left(\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 & 3 \\ -2 & -1 & & & \\ -3 & -2 & & & \\ -4 & -3 & & & \end{array}, \begin{array}{ccc} 0 & 1 & 2 \\ -1 & & \\ -2 & & \end{array}, \begin{array}{cc} 0 & 1 \\ -1 & \end{array}, \begin{array}{c} 0 \end{array} \right)$	53214

Table 5.23: Diagram/Permutation Correspondence: $k = 4, \ell = 5$, Type $(2,4)$

$\left(\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 & \\ -2 & -1 & & & \\ -3 & -2 & & & \\ -4 & -3 & & & \end{array}, \begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & & & \\ -2 & & & \end{array}, \begin{array}{cc} 0 & 1 \\ -1 & \end{array}, \begin{array}{c} 0 \end{array} \right)$
$\left(\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & & \\ -2 & -1 & & & \\ -3 & -2 & & & \\ -4 & -3 & & & \end{array}, \begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & & & \\ -2 & & & \end{array}, \begin{array}{ccc} 0 & 1 & 2 \\ -1 & & \end{array}, \begin{array}{c} 0 \end{array} \right)$
$\left(\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & & & \\ -2 & -1 & & & \\ -3 & -2 & & & \\ -4 & -3 & & & \end{array}, \begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & & & \\ -2 & & & \end{array}, \begin{array}{ccc} 0 & 1 & 2 \\ -1 & & \end{array}, \begin{array}{cc} 0 & 1 \end{array} \right)$
Permutations: 15432, 51432, 52431

REFERENCES

[1] J.C. Beier. Crystals for Demazure Modules of Special Linear Quantum Affine Algebras. Dissertation. 2008.

[2] V.G. Drinfel'd. Hopf algebras and the quantum Yang-Baxter equation, *Soviet Math Dokl.,* 32: 254-258, 1985.

[3] Ronald L. Graham, Donald E. Knuth, Oren Patashnik. *Concrete Mathematics*, Second edition, Addison-Wesley Publishing Company, New York, 1994.

[4] J. Hong and S-J. Kang. *Introduction to Quantum Groups and Crystal Bases,* Volume 42 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, Rhode Island, 1972.

[5] J.E. Humphreys. *Introduction to Lie Algebras and Representation Theory,* Volume 9 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1972.

[6] M. Jimbo. A q- difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation, *Letters in Mathematical Physics* 10: 63-69, 1985.

[7] M. Jimbo, K. C. Misra, T. Miwa, M. Okado. Combinatorics of representations of $U_q\left(\widehat{sl}(n)\right)$ at $q=0$, *Communications in Mathematical Physics*, 136: 543-566, 1991.

[8] V.G. Kac. Simple irreducible graded Lie algebras of finite growth. *Izvestija AN USSR (ser. mat.)*, 32:1923-1967, 1968.

[9] V.G. Kac. Infinite-dimensional Lie algebras and Dedekind's η-function, *Funk. Analis i ego Prilozh*, 8: 77-78, 1974.

[10] V.G. Kac. *Infinite-dimensional Lie algebras*, Third edition. Cambridge University Press, New York, NY, 1990.

[11] M. Kashiwara. Crystalizing the q-analogue of universal enveloping algebras, *Communications in Mathematical Physics*, 133:249-260, 1990.

[12] M. Kashiwara. On crystal bases of the q analogue of universal enveloping algebras, *Duke Mathematical Journal,* 63:465-516, 1991.

[13] G. Lusztig. Quantum deformation of certain simple modules over enveloping algebras, *Advances in Mathematics*, 70: 237-249, 1988.

[14] G. Lusztig. Canonical bases arising from quantized enveloping algebras. *Journal of the American Mathematical Society*, 3: 447-498, 1990.

[15] K.C. Misra. *Lecture Notes in Lie Algebra*, 2007.

[16] K.C. Misra and T. Miwa. Crystal bases for the basic representation of $U_q(\widehat{(sl)}(n))$. *Communications in Mathematical Physics*, 10: 79-88, 1990.

[17] K. C. Misra, V. Williams. Combinatorics of quantum affine Lie algebra representations. *Contemporary Mathematics*, 343: 167-189, 2004.

[18] R.V. Moody. A new class of Lie algebras, *Journal of Algebra*, 10:211-230, 1968.

[19] The Online Encyclopedia of Integer Sequences. http://oeis.org/.

[20] R.P. Stanley. *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, New York, 1999.

[21] S. Tsuchioka. Catalan numbers and level 2 weight structures of $A^{(1)}_{p-1}$, *RIMS Kŏkyŭroku Bessatsu*, B11: 145-154, 2009.

APPENDIX

Appendix A

MATLAB Code

In this appendix, we share the code used to get the counts of extended Young diagrams in $B(4\Lambda_0)_{4\Lambda_0 - \gamma_\ell}$. This program can be easily modified for the $k = 2$ and $k = 3$ cases. The function we call to obtain the results is `young.m`. Its argument is ℓ and it returns a matrix of the information for the extended Young diagrams for the value of ℓ. For each tuple of extended Young diagrams satisfying the conditions of Theorem 3.2.4, there is a row in the matrix. In this row, the first $2\ell - 1$ entries describe Y_1, the second $2\ell - 1$ describe Y_2, and so on. In order, these columns give the number of boxes in the particular diagram of color $-\ell + 1, -\ell + 2, \ldots, -1, 0, 1, 2, \ldots, \ell - 1$.

```
function Y = young(ℓ)
Y1 = []; Y2 = []; Y3 = []; Y4 = [];
S = zeros(1, 2*ℓ)-1); %Gives number of each color in square
M = S; %Gives max number of each color in Y2
X = S; %Will keep track of current diagram
% These two for loops set up S and M
for i=1:ℓ
    S(1,i) = i;
    M(1,i) = floor(i/2);
end
```

```
for j=(l+1):(2*ℓ−1)
    S(1,j) = 2*ℓ − j;
    M(1,j) = floor(S(1,j) / 2);
end
%sets beginning square (Y1 full and Y2, Y3,Y4 empty)
[Y1,Y2,Y3,Y4] = fillin(S,Y1,Y2,Y3,Y4,X,X,X,ℓ);
%recursion begins
[Y1,Y2,Y3,Y4] = toget2(ℓ,S,M,Y1,Y2,Y3,Y4,X,ℓ);
Y = horzcat(Y1,Y2); Y = horzcat(Y,Y3); Y = horzcat(Y,Y4);
```

The first function called by `young.m` is `fillin.m` which adds the current tuple to the matrix of tuples. Included are some redundant checks to make sure we have containment of diagrams and that Y_1 is still in the form of an extended Young diagram, even after removing various boxes.

```
function [Y1,Y2,Y3,Y4] = fillin(S,Y1,Y2,Y3,Y4,X,Y,Z,ℓ)
J = S−X−Y−Z;  %J represents Y1
if (iscontained(X,J) && iscontained(Y,X) && iscontained(Z,Y)) %tests containment
    if (checkyouth(J,ℓ)==true)
        Y1 = vertcat(Y1, J);
        Y2 = vertcat(Y2, X);
        Y3 = vertcat(Y3, Y);
        Y4 = vertcat(Y4, Z);
    end
end
```

The next two functions given are checks to ensure that we are only counting true tuples of extended Young diagrams. We double check containment in `iscontained.m` and that Y_1 is an extended Young diagram in `checkyouth.m`. These checks are unnecessary now that the program has been fully tested, but were quite useful in the testing process.

```
function contain = iscontained(Y,X)
Z = X − Y;
s = size(Z); c = s(2); i = 1;
contain = true;
while (i <= c) && contain
   if Z(1,i) < 0
       contain = false;
   end
   i = i+1;
end
```

```
function y = checkyouth(X,ℓ)
y = true;
i = 2;
while ((i <= ℓ) && y)
    if(X(1,i−1) > X(1,i))
        y = false;
    end
    i = i+1;
end
while (((i > ℓ) && y) && (i <= 2*ℓ−1))
    if(X(1,i) > X(1,i−1))
        y = false;
    end
    i = i + 1;
end
```

Now, we come to the functions that actually compute the possible tuples that meet the conditions of Theorem 3.2.4. We call `toget2.m` recursively until we have determined a Y_2. Then if there is potential for Y_3 to be nonempty, we move to `toget3.m` to determine a possible Y_3. If it is possible to have Y_4 nonempty, we move to `toget4.m`. Once we have found a tuple that meets the criteria, we call `fillin.m` and recursively begin working with the next possibility. In each function, we begin by filling in the number of boxes of color 0, then 1, then 2, etc. Once we are finished with positive colored boxes, we return to fill in the number of boxes of color -1, then -2, and so on. We use i as a placeholder for where we are in this process.

```
function [Y1,Y2,Y3,Y4] = toget2(l,S,M,Y1,Y2,Y3,Y4,X,i)
% X is our current value of Y2.
% Go to negative colors
if i == (2*ℓ)
    i = ℓ − 1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Finished determining X (Y2) and move onto Y (Y3)
if i == 0
    Y = zeros(1, 2*ℓ−1);  % create 0 vector Y
    k = 1;
    %Move index to first nonzero term in X
    while (X(1,k)==0)
        k = k + 1;
    end
    %a is index of the first nonzero term in X
    a = k;
    k = 2*ℓ−1;
    while (X(1,k)==0)
        k = k − 1;
    end
    % similarly, b is the index of last nonzero term in X
    b = k;
    % If rightmost or leftmost nonzero term is ℓ, go to fillin function, ...
        because Y3 must be empty.
    if ((a == ℓ) || (b == ℓ))
        [Y1,Y2,Y3,Y4] = fillin(S,Y1,Y2,Y3,Y4,X,Y,Y,ℓ); %Y is zero vector here
```

```
    else
        %If not, could have something in Y3
        [Y1,Y2,Y3,Y4] = toget3(ℓ,S,Y1,Y2,Y3,Y4,X,Y,a,b,ℓ);
    end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
elseif i == ℓ
    for j = 1:M(1,ℓ)
        X(1,i) = j;
        [Y1,Y2,Y3,Y4] = toget2(ℓ,S,M,Y1,Y2,Y3,Y4,X,i+1);
    end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
elseif i > ℓ
    if X(1,i-1) == 0
        X(1,i) = 0;
        [Y1,Y2,Y3,Y4] = toget2(ℓ,S,M,Y1,Y2,Y3,Y4,X,i+1);
    elseif X(1,i-1)-1 <= M(1,i)
        for j=(X(1,i-1)-1):(min(X(1,i-1), M(1,i)))
            X(1,i) = j;
            [Y1,Y2,Y3,Y4] = toget2(ℓ,S,M,Y1,Y2,Y3,Y4,X,i+1);
        end
    end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
elseif i < ℓ
    if X(1,i+1) == 0
        X(1,i) = 0;
        [Y1,Y2,Y3,Y4] = toget2(ℓ,S,M,Y1,Y2,Y3,Y4,X,i-1);
    elseif X(1,i+1)-1 <= M(1,i)
        for j=(X(1,i+1)-1):(min(X(1,i+1), M(1,i)))
            X(1,i) = j;
            [Y1,Y2,Y3,Y4] = toget2(ℓ,S,M,Y1,Y2,Y3,Y4,X,i-1);
        end
    end
end
```

```
function [Y1,Y2,Y3,Y4] = toget3(l,S,Y1,Y2,Y3,Y4,X,Y,a,b,i)
% Y will keep track of the current Y3
%if i=b then jump down to negative colors
if i == b
    i = ℓ − 1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% if i=a then we are done constructing Y
if i == a
    Z = zeros(1, 2*ℓ−1);
    k = 1;
    if Z == Y
        [Y1,Y2,Y3,Y4] = fillin(S,Y1,Y2,Y3,Y4,X,Y,Z,l);
    else   %c is first nonzero and d is last nonzero index in Y
        while (Y(1,k)==0)
            k = k + 1;
        end
        c = k;
        k = 2*ℓ−1;
        while (Y(1,k)==0)
        k = k − 1;
        end
        d = k;
        if ((c == ℓ) || (d == ℓ))
        %if rightmost or leftmost nonzero term is 1, Y4 must be empty
        [Y1,Y2,Y3,Y4] = fillin(S,Y1,Y2,Y3,Y4,X,Y,Z,ℓ);
        else %if not, Y4 could be nonempty
        [Y1,Y2,Y3,Y4] = toget4(ℓ,S,Y1,Y2,Y3,Y4,X,Y,Z,a,b,c,d,ℓ);
        end
    end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% If i=ℓ, we range over possible values of Y(ℓ)
% the maximum value of Y(ℓ) is the minimum among:
```

```
% X(1,i) = # boxes of color 0 in Y2
% S(1,i) − 2*X(1,i) = # boxes color 0 unaccounted for
% min(ℓ+1−a, b+1−ℓ) − X(1,ℓ) = # boxes of color 0 that are actually free
elseif i == ℓ
    for j = 0:min(min(X(1,i),S(1,i)−2*X(1,i)), (min(ℓ+1−a,b+1−ℓ)−X(1,ℓ)))
        Y(1,i) = j;
        [Y1,Y2,Y3,Y4] = toget3(ℓ,S,Y1,Y2,Y3,Y4,X,Y,a,b,i+1);
    end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% If i = b−1, this is the largest nonzero entry possible for Y.
% y = max(,) b/c
    % Y(1,i−1) − 1: must have less than or equal to number of boxes of color ...
        b−1 compared to color b−2
    % S(1,i)−X(1,i)−(S(1,i−1)−X(1,i−1)−Y(1,i−1)) because in Y1, need to have ...
        Y1(i) <= Y1(i−1)
% maximum in for loop
    % Y(1,i−1) − number of boxes of color one lower
    % X(1,i) − number of boxes of color in Y2
    % S(1,i) − 2*X(1,i) − number of boxes unaccounted for
    % S(1,i)−X(1,i)−(S(1,i+1)−X(1,i+1)) − need to make sure still keeping Y1 as ...
        a YD.  So we state Y1(i) >= Y1(i+1) and obtain this result.
    % Note that Y(i+1) = 0 since i+1 = b
elseif i == b−1
    if Y(1,i−1) == 0
        Y(1,i) = 0;
        [Y1,Y2,Y3,Y4] = toget3(ℓ,S,Y1,Y2,Y3,Y4,X,Y,a,b,i+1);
    else
        y = max(Y(1,i−1)−1,S(1,i)−X(1,i)−(S(1,i−1)−X(1,i−1)−Y(1,i−1)));
        for j=y:(min(min(min(Y(1,i−1), X(1,i)), ...
            S(1,i)−2*X(1,i)),S(1,i)−X(1,i)−(S(1,i+1)−X(1,i+1))))
            Y(1,i) = j;
            [Y1,Y2,Y3,Y4] = toget3(ℓ,S,Y1,Y2,Y3,Y4,X,Y,a,b,i+1);
        end
```

```
    end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% If i > ℓ and i is not b or b−1.
% Minimum is the the maximum of the following:
    % Y(1,i−1)−1 − number of boxes of previous color − 1
    % S(1,i)−X(1,i)−(S(1,i−1)−X(1,i−1)−Y(1,i−1)) − since Y1(i) <= Y1(i−1)
% Maximum is the minimum of the following:
    % X(1,i) − number of boxes of color i in Y2
    % Y(1,i−1) − number of boxes of color i−1 in Y3
    % S(1,i)−2*X(1,i) − number of boxes available
elseif i > ℓ
    if Y(1,i−1) == 0
        Y(1,i) = 0;
        [Y1,Y2,Y3,Y4] = toget3(ℓ,S,Y1,Y2,Y3,Y4,X,Y,a,b,i+1);
    else
        for j=(max(Y(1,i−1)−1,S(1,i)−X(1,i)−(S(1,i−1)−X(1,i−1)−Y(1,i−1)))):
                (min(min(Y(1,i−1), X(1,i)), S(1,i)−2*X(1,i)))
            Y(1,i) = j;
            [Y1,Y2,Y3,Y4] = toget3(ℓ,S,Y1,Y2,Y3,Y4,X,Y,a,b,i+1);
        end
    end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% If i is a+1, this is the smallest possible nonzero entry
% The minimum is the maximum of
    %Y(1,i+1)−1 − one less than the number of boxes of next biggest color
    %S(1,i)−X(1,i)−(S(1,i+1)−X(1,i+1)−Y(1,i+1)) − condition that keeps Y1 a YD ...
        (b/c Y1(i) <= Y1(i+1))
% The maximum is the minimum of
    % Y(1, i+1) = number of boxes of color one more in Y (Y3)
    % X(1,i) − number of boxes of color in X
    % S(1,i)−2*X(1,i)) − restriction from written proof
```

```
    % S(1,i)-X(1,i)-(S(1,i-1)-X(1,i-1)) - comes from the fact that Y1(i) >=  ...
        Y1(i-1)
elseif i == (a + 1)
    if Y(1,i+1) == 0
        Y(1,i) = 0;
        [Y1,Y2,Y3,Y4] = toget3(ℓ,S,Y1,Y2,Y3,Y4,X,Y,a,b,i-1);
    else
        y = max(Y(1,i+1)-1,S(1,i)-X(1,i)-(S(1,i+1)-X(1,i+1)-Y(1,i+1)));
        for j=y:(min(min(min(Y(1,i+1), X(1,i)), ...
            S(1,i)-2*X(1,i)),S(1,i)-X(1,i)-(S(1,i-1)-X(1,i-1))))
            Y(1,i) = j;
            [Y1,Y2,Y3,Y4] = toget3(ℓ,S,Y1,Y2,Y3,Y4,X,Y,a,b,i-1);
        end
    end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% If i < ℓ and i ~= a+1 or a, we enter this case
% Minimum is the maximum of the following:
    % Y(1,i+1)-1 % One less than the number
    % S(1,i)-X(1,i)-(S(1,i+1)-X(1,i+1)-Y(1,i+1))
% is less than the minimum of the following:
    % X(1,i)
    % S(1,i)-2*X(1,i)
    % Y(1,i+1)
elseif i < ℓ
    if Y(1,i+1) == 0
        Y(1,i) = 0;
        [Y1,Y2,Y3,Y4] = toget3(ℓ,S,Y1,Y2,Y3,Y4,X,Y,a,b,i-1);
    else
        for j=(max(Y(1,i+1)-1,S(1,i)-X(1,i)-(S(1,i+1)-X(1,i+1)-Y(1,i+1)))):
                (min(min(Y(1,i+1), X(1,i)), S(1,i)-2*X(1,i)))
            Y(1,i) = j;
            [Y1,Y2,Y3,Y4] = toget3(ℓ,S,Y1,Y2,Y3,Y4,X,Y,a,b,i-1);
        end
```

```
    end
end
```

```
function [Y1,Y2,Y3,Y4] = toget4(ℓ,S,Y1,Y2,Y3,Y4,X,Y,Z,a,b,c,d,i)
% Z is Y_4
%if i=d then jump down to negative colors
if i == d
    i = ℓ − 1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% if i=c then we are done constructing Y
if i == (c)
    [Y1,Y2,Y3,Y4] = fillin(S,Y1,Y2,Y3,Y4,X,Y,Z,ℓ);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% If i=ℓ, we range over possible values of Y(ℓ).
% The maximum value of Y(ℓ) is the minimum among:
% Y(1,i) = # boxes of color 0 in Y3
% S(1,i) − 2*X(1,i)− Y(1,i) = # boxes color 0 unaccounted for
% min(l+1−a, b+1−l) − X(1,ℓ) = # boxes of color 0 that are actually free
% S(1,i)−X(1,i)−Y(1,i) − (S(1,c)−X(1,c)−Y(1,c))
% S(1,i)−X(1,i)−Y(1,i) − (S(1,d)−X(1,d)−Y(1,d))
elseif i == ℓ
    for j = 0:min(min(min(min(Y(1,i),S(1,i)−2*X(1,i)−Y(1,i)), ...
        (min(ℓ+1−a,b+1−ℓ)−X(1,i))), S(1,i)−X(1,i)−Y(1,i) − ...
        (S(1,c)−X(1,c)−Y(1,c))), S(1,i)−X(1,i)−Y(1,i) − (S(1,d)−X(1,d)−Y(1,d)))
        Z(1,i) = j;
        [Y1,Y2,Y3,Y4] = toget4(ℓ,S,Y1,Y2,Y3,Y4,X,Y,Z,a,b,c,d,i+1);
    end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% If i = d−1, this is the largest nonzero entry possible for Z
% Minimum is the maximum among
    % Z(1,i−1) − 1:  number of boxes of color d−1 compared to color d−2
```

```
    % S(1,i)-X(1,i)-Y(1,i)-(S(1,i-1)-X(1,i-1)-Y(1,i-1)-Z(1,i-1)) because in Y1, ...
        need to have Y1(i) <= Y1(i-1)
% Maximum is minimum among
    % Z(1,i-1) - number of boxes of color one lower
    % Y(1,i) - number of boxes of color in Y3
    % S(1,i)-2*X(1,i) - Y(1,i) - number of boxes unaccounted for
    % S(1,i)-X(1,i) - Y(1,i) -(S(1,i+1)-X(1,i+1)-Y(1,i)) - b/c Y1(i) >= Y1(i+1)
    % Note that Z(i+1) = 0 since i+1 = d
elseif i == d-1
    y = max(Z(1,i-1)-1,S(1,i)-X(1,i)-Y(1,i)-(S(1,i-1)-X(1,i-1)-Y(1,i-1)-Z(1,i-1)));
    if Z(1,i-1) == 0
        Z(1,i) = 0;
        [Y1,Y2,Y3,Y4] = toget4(ℓ,S,Y1,Y2,Y3,Y4,X,Y,Z,a,b,c,d,i+1);
    else
        for j=y:(min(min(min(Z(1,i-1), Y(1,i)), S(1,i)-2*X(1,i)-Y(1,i)),S(1,i)
                    -X(1,i)-Y(1,i)-(S(1,i+1)-X(1,i+1)-Y(1,i+1))))
            Z(1,i) = j;
            [Y1,Y2,Y3,Y4] = toget4(ℓ,S,Y1,Y2,Y3,Y4,X,Y,Z,a,b,c,d,i+1);
        end
    end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% if i > ℓ and i is not d or d-1
% Minimum is maximum among
    % Z(1,i-1)-1 - number of boxes of previous color - 1
    % S(1,i)-X(1,i)-Y(1,i)-(S(1,i-1)-X(1,i-1)-Y(1,i-1)-Z(1,i-1)) - since Y1(i) ...
        <= Y1(i-1)
% Maximum is minimum among
    % Y(1,i) - number of boxes of color i in Y2
    % S(1,i)-2*X(1,i)- Y(i,1) - number of boxes available
    % Z(1, i-1) number of boxes color one less
elseif i > ℓ
    if Z(1,i-1) == 0
        Z(1,i) = 0;
```

```
        [Y1,Y2,Y3,Y4] = toget4(ℓ,S,Y1,Y2,Y3,Y4,X,Y,Z,a,b,c,d,i+1);
    else
        for j=(max(Z(1,i−1)−1,S(1,i)−X(1,i)−Y(1,i)−
                    (S(1,i−1)−X(1,i−1)−Y(1,i−1)−Z(1,i−1)))):
                    (min(min(Z(1,i−1), Y(1,i)), S(1,i)−2*X(1,i)−Y(1,i)))
            Z(1,i) = j;
            [Y1,Y2,Y3,Y4] = toget4(ℓ,S,Y1,Y2,Y3,Y4,X,Y,Z,a,b,c,d,i+1);
        end
    end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% if i is c+1 − this is the smallest possible nonzero entry
% Minimum is maximum among
    %Z(1,i+1)−1 − one less than the number of boxes of next biggest color
    %S(1,i)−X(1,i)−(S(1,i+1)−X(1,i+1)−Y(1,i+1)) − condition that keeps Y1 a YD ...
        (b/c Y1(i) <= Y1(i+1))
% Maximum is minimum among
    % Z(1,i+1) − number of boxes of color one more
    % Y(1,i) − number of boxes of color in X
    % S(1,i)−2*X(1,i)−Y(1,i)) − restriction from written proof
    % S(1,i)−X(1,i)−Y(1,i)−(S(1,i−1)−X(1,i−1)−Y(1,i−1)))) b/c Y1(i) >=  Y1(i−1)
elseif i == (c + 1)
    y = max(Z(1,i+1)−1,S(1,i)−X(1,i)−Y(1,i)−(S(1,i+1)−X(1,i+1)−Y(1,i+1)−Z(1,i+1)));
    if Z(1,i+1) == 0
        Z(1,i) = 0;
        [Y1,Y2,Y3,Y4] = toget4(ℓ,S,Y1,Y2,Y3,Y4,X,Y,Z,a,b,c,d,i−1);
    else
        for j=y:(min(min(min(Z(1,i+1), Y(1,i)), S(1,i)−2*X(1,i)−Y(1,i)),
                    S(1,i)−X(1,i)−Y(1,i)−(S(1,i−1)−X(1,i−1)−Y(1,i−1))))
            Z(1,i) = j;
            [Y1,Y2,Y3,Y4] = toget4(ℓ,S,Y1,Y2,Y3,Y4,X,Y,Z,a,b,c,d,i−1);
        end
    end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% if i < ℓ and i != c+1 or c, we enter this case
% Minimum is maximum among
    % Z(1,i+1)−1 − one less than the number of boxes of next biggest color
    % S(1,i)−X(1,i)−Y(1,i) − (S(1,i+1)−X(1,i+1)−Y(1,i+1)− Z(i,i+1)) − keeps
        %   Y1(i) <= Y1(i+1))
% Maximum is minimum among
    % Z(1,i+1)
    % Y(1,i) − number of boxes of color in X
    % S(1,i)−2*X(1,i)−Y(1,i)) − restriction from written proof
elseif i < l
    if Z(1,i+1) == 0
        Z(1,i) = 0;
        [Y1,Y2,Y3,Y4] = toget4(ℓ,S,Y1,Y2,Y3,Y4,X,Y,Z,a,b,c,d,i−1);
    else
        for j=(max(Z(1,i+1)−1,S(1,i)−X(1,i)−Y(1,i)−
                    (S(1,i+1)−X(1,i+1)−Y(1,i+1)−Z(1,i+1)))):
                (min(min(Z(1,i+1), Y(1,i)), S(1,i)−2*X(1,i)−Y(1,i)))
            Z(1,i) = j;
            [Y1,Y2,Y3,Y4] = toget4(ℓ,S,Y1,Y2,Y3,Y4,X,Y,Z,a,b,c,d,i−1);
        end
    end
end
```