ABSTRACT

WANG, YINGYING. Supply Risk in a Multiple-sourcing Supply System. (Under the direction of Dr. Russell E. King and Dr. Donald P. Warsing.)

In a single-period, single-item, single-site inventory system, we address the issue of selecting suppliers from multiple unreliable suppliers and allocating orders among them to satisfy uncertain demand and minimize total cost from the view of retailers. The suppliers may have different fixed order costs, item costs, and/or restrictions on minimum and maximum order sizes. Supplier reliability is modeled as the probability of on-time delivery, which implies that with a certain probability, the supplier fails to fulfill the entire order. Total cost consists of item costs (proportional to delivered quantities), end-of-period costs (including disposal and penalty costs), and in some of our models, fixed ordering costs.

In the supply-risk literature, most multiple-sourcing inventory models make the assumption that fixed ordering costs are incurred and demand is deterministic, or that there is no fixed cost and demand is stochastic. However, in practice, demand is stochastic and there is also overhead associated with generating or receiving an order. We solve the problem of determining the optimal order policy under such circumstances. In most of our models, fixed cost is incurred when an order is placed, which complicates the problem in that order cost is not proportional to the order quantity. Fixed cost is not directly incurred in some models, but there are minimum order size constraints for those problems, which reflect fixed costs indirectly. Suppliers’ binomial delivery probability complicates the problem because the optimal order quantity allocation is more difficult to determine compared with order quantity allocation problems with perfectly reliable suppliers or unreliable suppliers delivering a random percentage of an order (random yield).

Due to fixed cost and/or minimum order size constraints, the expected total cost is nonconvex, which is analyzed by considering several cases separately. Within each case, the cost function is proven to be convex and can be optimized. These cases are compared with each other to determine optimal policies. Properties of this problem are examined so that some comparisons are unnecessary, which makes the process more efficient computationally. We consider three types of models based on different
assumptions about the number of suppliers and the initial inventory level.

Model I assumes two unreliable suppliers with zero initial inventory level. The optimal policy structures as a function of reliability level are derived for the identical-supplier model with different fixed cost scenarios. The effects of cost parameters, reliability levels, and demand distributions on optimal policy structures and order quantities are investigated. Models with different assumptions about item costs or reliability levels are also discussed and the corresponding optimal policy structures are provided. An approximation method to compute the optimal order quantities and corresponding computational results are presented.

Model II lies between the single-period model (Model I) and a multiple-period model by making the assumption of an arbitrary initial inventory level. We prove that there are ten possible optimal policy structures as a function of initial inventory level. Two special models are considered. One assumes that the more expensive supplier is perfectly reliable and the other assumes that the two suppliers are identical. Numerical experiments are carried out to compare the costs of implementing optimal single-period policies in three-period problems with the optimal costs resulting from a Markov Decision Process formulation. The conditions under which the optimal single-period policies are good approximations for multiple-period problems are investigated. A heuristic method is proposed to reduce the approximation errors.

Model III extends Model I by assuming that the number of suppliers is more than two. Fixed order costs are dropped from this model. Instead, order size constraints are added to represent minimum order size requirements and suppliers’ capacities. This model is divided into two problems: (1) the master problem which involves selection of a set of suppliers and (2) a subproblem which involves determination of the order quantity allocation to a given set of suppliers. Five methods are proposed to solve subproblems and four methods are proposed to solve master problems. Combinations of these methods are discussed and implemented in numerical experiments to show their performance. The effects of various problem parameters on the supplier selection decision are presented.
Supply Risk in a Multiple-sourcing Supply System

by

Yingying Wang

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APPROVED BY:

Dr. S. Sebnem Ahiska
Dr. Ana R. Vila-Parrish
Dr. Russell E. King
Co-chair of Advisory Committee
Dr. Donald P. Warsing
Co-chair of Advisory Committee
DEDICATION

To

my dear parents

and

loving memory of my grandma
Yingying Wang was born in Hangzhou, China on October 30, 1985. She attended Zhejiang University (Hangzhou, China) during 2004-2008 and graduated with a Bachelor of Science degree in Automatic Control. She joined the PhD program of Industrial and Systems Engineering at North Carolina State University (Raleigh, NC) in 2008. Since then she has been conducting research on supply chain system modeling and optimization under the direction of Dr. Russell King and Dr. Donald Warsing. She was awarded Provost’s Fellowship in 2008 and Edward P. Fitts Graduate Fellowship in 2009, and received a Master of Industrial Engineering degree “en route” in 2010. Upon graduation, she will join LLamasoft as a supply chain design consultant to continue to pursue her interest in the area of supply chain.
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Chapter 1

Introduction

In a single-period, single-item, single-site inventory system, we address the issue of selecting suppliers from multiple unreliable suppliers and allocating orders among them to satisfy uncertain demand and minimize total cost from the view of retailers. The suppliers may have different fixed order costs, item costs, and/or restrictions on minimum and maximum order sizes. Supplier reliability is modeled as the probability of on-time delivery, which implies that with a certain probability, the supplier fails to fulfill the entire order. Total cost consists of item costs (proportional to delivered quantities), end-of-period costs (including disposal and penalty costs), and in some of our models, fixed ordering costs.

In the supply-risk literature, most multiple-sourcing inventory models make the assumption that fixed ordering costs are incurred and demand is deterministic, or that there is no fixed cost and demand is stochastic. However, in practice, demand is stochastic and there is also overhead associated with generating or receiving an order. We solve the problem of determining the optimal order policy under such circumstances. In most of our models, fixed cost is incurred when an order is placed, which complicates the problem in that order cost is not proportional to the order quantity. Fixed cost is not directly incurred in some models, but there are minimum order size constraints for those problems, which reflect fixed costs indirectly. Suppliers’ binomial delivery probability complicates the problem because the optimal order quantity allocation is more difficult to determine compared with order quantity allocation problems with perfectly reliable suppliers or unreliable suppliers delivering a random percentage of an order
Due to fixed cost and/or minimum order size constraints, the expected total cost is nonconvex, which is analyzed by considering several cases separately. Within each case, the cost function is proven to be convex and can be optimized. These cases are compared with each other to determine optimal policies. Properties of this problem are examined so that some comparisons are unnecessary, which makes the process more efficient computationally. We consider three types of models based on different assumptions about the number of suppliers and the initial inventory level.

In Chapter 2, we assume two unreliable suppliers with zero initial inventory level. Two suppliers have different reliability levels, different item costs and different fixed ordering costs. There are two types of fixed costs: overall fixed cost and individual fixed costs. The former one is incurred only once, no matter how many suppliers are ordered from, while the latter ones are associated with orders from individual suppliers. Fixed costs may be incurred upon order placement or after order delivery. When the two suppliers are different, the optimal policy and quantity allocation can be determined by comparing the costs of four cases, which are ordering nothing, ordering only from the less reliable supplier, ordering only from the more reliable supplier, and ordering from both suppliers. Rules that can be used to simplify the cost comparison process are proposed.

Optimal policy structures as a function of reliability level are derived for the identical-supplier models with different fixed cost scenarios. The effects of cost parameters, reliability levels, and demand distributions on optimal policy structures and order quantities are investigated. Based on the identical-supplier models, models with more realistic assumptions, where two suppliers have different reliability levels or provide different item costs, are also discussed and the corresponding optimal policy structures are determined.

The model we consider in Chapter 3 extends the model in Chapter 2 by allowing the initial inventory level to be arbitrary instead of zero. This model is still single-period but becomes closer to a multiple-period problem where excess inventory is carried forward and unsatisfied demand is backlogged. Individual fixed cost incurred upon order placement is included in the model. For two different suppliers, we prove that there are ten possible optimal policy structures as a function of initial inven-
tory level, although extensive numerical experiments show that some complicated policies rarely occur. Two special models are considered. One assumes that the more expensive supplier is perfectly reliable, and the other assumes that the two suppliers are identical. The numbers of optimal policies for the two special models reduce to six and two, respectively.

Optimal policies of multiple-period problems are very difficult to determine analytically, and time-consuming to solve numerically, especially when the number of periods are relatively large. In order to test whether we can use single-period optimal policies to approximate multiple-period ones, numerical experiments are carried out to compare the costs of implementing single-period optimal policies in three-period problems with the optimal costs of resulting from a Markov Decision Process formulation. By using Analysis of Variance (ANOVA) to analyze the numerical experiment results, the conditions under which the optimal single-period policies are good approximations for multiple-period problems are investigated. A heuristic method is proposed to reduce the approximation errors.

In Chapter 4, we extend the model in Chapter 2 by considering more than two suppliers. Fixed order costs are dropped from this model. Instead, order size constraints are added to represent minimum order size requirements and suppliers’ capacities. When there is no minimum order size constraint, we prove that sourcing from more suppliers cannot lead to increased cost. The general model is divided into two parts: (1) the master problem which involves selection of a set of suppliers and (2) a subproblem which involves determination of the order quantity allocation to a given set of suppliers. Five methods are proposed to solve subproblems, of which the Primal Affine Scaling algorithm and Free Variable Reduction are used to solve the continuous formulations of subproblems, while Restricted Local Search, Unrestricted Search and Nelder-Mead method are used to obtain integer solutions. Enumeration approaches and Branch and Bound methods are proposed to solve master problems. Appropriate combinations of these methods are discussed and implemented in numerical experiments to show their performance. The effects of various problem parameters on the supplier selection decision are presented.
Chapter 2

Optimal Policy and Effects of Parameters for a Dual-sourcing Supply System

2.1 Introduction and Literature Review

Suppliers are not perfectly reliable: when an order is placed, the supplier may fail to fulfill the order within the specified length of time. Supply disruptions result from various reasons: manufacturers’ equipment breakdowns, labor strikes, transportation accidents, or other unpredictable situations. For instance, BMW AG, the world’s largest maker of luxury cars, halted production at three factories in Germany during the Icelandic volcanic ash crisis in 2010 due to supply shortage. The temporary manufacturing shutdown, caused by a shortage of interior and electronic parts, which are usually delivered by air, delayed production of about 7,000 vehicles [3]. One common approach to mitigating the uncertainties in supply chains is to source from multiple suppliers.

A multiple-sourcing strategy, especially dual sourcing, has been studied by many researchers (see [27] for a detailed review). Literature on the area is focused on selection of suppliers and quantity allocation among suppliers. Some papers in this field mainly focus on supplier delivery performance rather than supply disruptions. One example is the problem related to the choice of supply modes: whether a product should be transported via air with short lead times but high costs, or by sea with long
lead times but low shipping costs. Here we are concerned with employing a multiple-sourcing strategy to deal with supply uncertainty. The related literature can be classified into the following categories according to how uncertainty in the supply process is modeled; namely supply with random yield, all-or-nothing delivery, supplier unavailability, and random lead times. Each is discussed below.

**Random yield:** In this case, the quantity received from the supplier may differ from what was ordered, or a fraction of the received order may be defective. This assumption is practical in many application areas such as electronics fabrication and assembly as well as chemical processes. A comprehensive review of yield uncertainty is offered by Yano and Lee [44]. Other papers have appeared since their review. For example, random yield is modeled as random capacities of suppliers in [13]. Dada et al. [11] consider selection of unreliable suppliers in a newsvendor problem and conclude that low item cost is the order qualifier, while high reliability level is the order winner. Federgruen and Yang [15] also investigate an $N$-supplier-selection problem, where the item cost is identical for all the suppliers. Two approximation methods based on a large-deviations technique and the central limit theorem, respectively, are developed to determine the optimal set of suppliers and order quantity allocation among suppliers. Gurnani et al. [21] focus on determining the discount rate provided by the unreliable supplier to make dual-sourcing a better policy. Quantity allocation between the reliable supplier and the unreliable supplier is also investigated. Other work in the area of yield uncertainty includes [16], [43], and also Model II of [1].

**All-or-nothing Delivery:** In this case, either the order quantity is delivered in full with a given probability, or nothing arrives. All-or-nothing can be regarded as a special case of random yield where the fraction of the quantity received is either 0 (nothing) or 1 (all). This assumption may be valid when, for example, the order cannot arrive at the buyer’s warehouse because of accidents, or when perishable products are delivered to destinations later than the expiration date due to transportation delays, which renders them unsalable. Model I of [1] falls into this category. They determine the structure of the optimal policy for sourcing from two different unreliable suppliers under continuously distributed demand. Both single period and multiple-period problems are analyzed, and the following policy is shown to be optimal in period $n$: when the inventory level $x > \tilde{u}^n$, order nothing; when $\tilde{v}^n < x \leq \tilde{u}^n$,
order from the less reliable supplier, who is also cheaper; and when $x < \bar{v}^m$, order from both suppliers. Swaminathan and Shanthikumar [41] consider the same problem under discrete demand and obtain a different result: ordering from the more expensive and more reliable supplier alone can be optimal for some cases. Babich et al. [5] extend the work of Anupindi and Akella [1] by assuming the reliability levels of suppliers are correlated and suppliers control the wholesale prices.

**Supplier Unavailability:** Suppliers are down from time to time and the available and unavailable periods of the suppliers are random. Parlar and Perry [38] analyze order-quantity/reorder-point models with two or more identical unreliable suppliers in a continuous-review context. Renewal processes are used to develop a long-run average cost objective function, and the available/unavailable status of suppliers is modeled as a continuous-time Markov chain, where the durations of the ON/OFF periods are modeled as exponential distributions. Optimal values of the parameters of a predetermined ordering policy were provided, but the optimality of the policy itself has not been proved. Güler and Parlar [20] extend the work of Parlar and Perry [38] by proposing the case of Erlang-k distributed availability durations and general unavailability durations. Tomlin [42] incorporates volume flexibility in a dual-sourcing inventory system, and Zhou et al. [46] consider the problem under transportation circumstances (i.e., the selection of a logistics company and allocation of shipments).

**Lead-time uncertainty:** Unreliability of suppliers can be characterized by lead-time distributions with large means and variances. Similar to [21], Ganeshan et al. [17] examine a supply system consisting of two suppliers: a reliable one and an unreliable one, where sourcing from the unreliable one may be preferable due to a price advantage. They propose a simple heuristic and sample exchange curves, which show the relations between discount rates and costs under different quantity allocation scenarios, to determine the discount rate that the unreliable supplier should offer to make multiple-sourcing worthwhile, and the fraction of stock ordered from each supplier. Multiple-sourcing from suppliers with different lead-times is also discussed in [30] and [31]. A similar model of lead-time differentiation, with delayed delivery, is considered by Anupindi and Akella [1] in their Model III, which assumes that if the order is not delivered by the unreliable supplier during the current period, it will definitely arrive next period.
Compared with multiple-sourcing, single-unreliable-supplier models are discussed more extensively in the literature. They mainly focus on determining the optimal order policy, and provide insights into the multiple-supplier problem. Here we also classify papers according to the way they model unreliability. The assumption of all-or-nothing is made by Özekici and Parlar [34]. They show how the random environment that affects the demand, supply, and all cost parameters. Then they prove the optimality of an environment-dependent, base-stock policy when the order cost is linear in the order quantity, and the optimality of a two-parameter, environment-dependent \((s, S)\) policy under reasonable conditions. Güllü et al. [19] not only consider the all-or-nothing case, but also model the demand distribution as a three-point probability mass function, where the supply is either completely available, partially available, or unavailable. Henig and Gerchak [22], and Arreola-Risa and DeCroix [2] model the unreliable-supplier problem as a random-yield problem. Finally, there are several papers modeling supplier unavailability with random time durations [36, 39, 37, 29, 35, 28, 40].

We contribute to the dual-sourcing literature by incorporating fixed cost and stochastic demand into a single model. A majority of studies that investigate order policies that account for multiple suppliers assume that there is no fixed cost incurred when an order is placed [1, 41]. Those who consider fixed costs usually assume deterministic demand [38]. The model proposed by Federgruen and Yang [15] includes both fixed cost and stochastic demand assumptions, but they mainly focus on selecting the optimal set of suppliers from \(N\) suppliers who provide the identical item cost while a given service level is guaranteed. Due to the difficulty of service level constraint, they propose two approximation methods to determine the optimal set of suppliers and the corresponding order quantity allocation. In this paper, we address the issue of determining the structure of the optimal order policy and the corresponding quantity allocation between two unreliable suppliers. A single-period, single-item, single-site inventory system is considered. Hence we assume zero initial inventory, i.e., there is no initial backordered demand or incoming inventory to start with. The reliability of suppliers is represented by a delivery probability \(\alpha\). With probability \(\alpha\), a supplier delivers the order successfully at the beginning of the selling season, otherwise they fail to deliver the order at all. Failed orders are canceled. The demand is stochastic and continuously distributed with a known distribution. Our model is similar to Model I of [1], but with zero
initial inventory level and fixed order costs. Fixed cost complicates the problem in that the total cost of the order (fixed cost and item cost) is not directly proportional to the order quantity. We analyze the nonlinear expected total cost function by considering four cases separately: ordering nothing, sourcing from the less reliable supplier, sourcing from the more reliable supplier and dual-sourcing. Within each case, the cost function is convex and can be optimized. These cases are compared with each other to derive the cost difference functions and the threshold levels. By investigating the properties the cost difference functions and the threshold levels, the optimal policy structure as a function of reliability level can be determined. Some general comparison rules are proposed to make the process more efficient.

The remainder of this paper is organized as follows. Section 2.2 describes the general model with two unreliable suppliers. Models with different fixed-cost scenarios are proposed and the corresponding optimal policies are determined given the reliability levels of the two suppliers. In Section 2.3, we mainly focused on deriving the optimal policy structure as a function of reliability level for some specially structured problems, such as the identical-supplier models with various fixed order costs. Properties of the optimal order quantities and the cost functions are presented. The effects of parameters, such as item cost, and the effects of demand distributions on optimal policy structures and order quantities are discussed.

### 2.2 Model Formulation

In this section, we formulate the general model. Without loss of generality, supplier 1 is assumed to be no more reliable than supplier 2, i.e., \( \alpha_1 \leq \alpha_2 \). It is not necessary that the item cost of supplier 1 is lower than that of supplier 2. The optimal order policy and quantity allocation can be determined by comparing the following:

- **Case 0**: order nothing
- **Case 1**: order only from supplier 1
- **Case 2**: order only from supplier 2
• Case 3: order from both suppliers

The cost comparison process can be simplified for certain fixed-cost scenarios. Table 2.1 shows the notation used in this model. Note that the penalty cost is assumed to be greater than the item cost, otherwise ordering nothing is always the best policy.

In practice, fixed costs can be incurred in a variety of ways. There can be an overall fixed cost $K_o$, which is incurred only once, no matter how many suppliers are ordered from. This reflects the overhead associated with generating orders, and is incurred upon order placement. There can also be fixed cost associated with orders from individual suppliers. This can be composed of two portions: the cost incurred to make an order on supplier $i$, $K_{pi}$; and cost incurred upon receipt of an order (e.g., logistics cost) from supplier $i$, $K_{ri}$.

The sequence of the events is as follows: before the selling season begins, an order is made on $n$ of the suppliers, $n \in \{0, 1, 2\}$. Overall fixed cost $K_o$ is incurred if one or more orders is made. Individual fixed cost $K_{pi}$ is also incurred upon order placement if we order from supplier $i$. The ordered stock from supplier $i$, if any, arrives right before the beginning of the selling season with probability $\alpha_i$. If supplier $i$ delivers the order, the item cost ($c_i$/item) and individual fixed cost $K_{ri}$ are incurred, otherwise they are not. Then, stock on-hand is used to satisfy random demand during the season. A disposal cost ($h$/item) is incurred for any stock left at the end of the season after satisfying all the demand, and a penalty cost ($p$/item) is incurred for any unsatisfied demand. At most one order can be placed on each supplier, regardless of whether the delivery is successful or not.

The objective of this sourcing problem is to minimize the single-period cost function $M(Q_1, Q_2)$, which consists of fixed order costs associated with order placement and receipt, variable costs that are proportional to the order quantity delivered successfully, and the expected end-of-period costs. Denoting the inventory level after the order arrives by $y$, the expected end-of-period cost can be written as:

$$L(y) = \int_0^y h(y - \xi) f(\xi) d\xi + \int_y^\infty p(\xi - y) f(\xi) d\xi \quad (2.1)$$
Table 2.1: Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>Inventory level on hand after ordering (order-up-to level)</td>
</tr>
<tr>
<td>$P(y)$</td>
<td>Expected penalty cost with initial inventory level $y$</td>
</tr>
<tr>
<td>$D(y)$</td>
<td>Expected disposal cost with initial inventory level $y$</td>
</tr>
<tr>
<td>$L(y)$</td>
<td>Expected end-of-period cost, $L(y) = P(y) + D(y)$</td>
</tr>
<tr>
<td>$K_o$</td>
<td>Overall fixed cost</td>
</tr>
<tr>
<td>$K_{ri}$</td>
<td>Individual fixed cost incurred upon order receipt from supplier $i$, $i \in {1, 2}$</td>
</tr>
<tr>
<td>$K_{pi}$</td>
<td>Individual fixed cost incurred upon order placement from supplier $i$, $i \in {1, 2}$</td>
</tr>
<tr>
<td>$K^t$</td>
<td>Threshold level of fixed cost between optimal policy structures</td>
</tr>
<tr>
<td>$c_i$</td>
<td>Per item cost from supplier $i$, $i \in {1, 2}$</td>
</tr>
<tr>
<td>$\alpha_i$</td>
<td>Probability that the order on supplier $i$ will be delivered, $i \in {1, 2}$</td>
</tr>
<tr>
<td>$\bar{\alpha}_i$</td>
<td>$1 - \alpha_i$, $i \in {1, 2}$</td>
</tr>
<tr>
<td>$h$</td>
<td>Per item disposal cost</td>
</tr>
<tr>
<td>$p$</td>
<td>Per item penalty cost</td>
</tr>
<tr>
<td>$c_{ui}$</td>
<td>Underage cost of supplier $i$, $c_{ui} = p - c_i$</td>
</tr>
<tr>
<td>$c_{oi}$</td>
<td>Overage cost of supplier $i$, $c_{oi} = h + c_i$</td>
</tr>
<tr>
<td>$\gamma_i$</td>
<td>Newsvendor critical fractile for supplier $i$, $\gamma_i = \frac{p - c_i}{h + p}$</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Demand</td>
</tr>
<tr>
<td>$f(\xi)$</td>
<td>Density function of demand</td>
</tr>
<tr>
<td>$F(\xi)$</td>
<td>Cumulative distribution function of demand</td>
</tr>
<tr>
<td>$l$</td>
<td>Lower limit of the demand distribution</td>
</tr>
<tr>
<td>$u$</td>
<td>Upper limit of the demand distribution</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Demand mean</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Demand standard deviation</td>
</tr>
<tr>
<td>$Q_i$</td>
<td>Order quantity from supplier $i$</td>
</tr>
<tr>
<td>$Q_{si}^*$</td>
<td>Optimal order quantity when single-sourcing from supplier $i$</td>
</tr>
<tr>
<td>$Q_{di}^*$</td>
<td>Optimal order quantity when dual-sourcing from supplier $i$</td>
</tr>
<tr>
<td>$M(Q_1, Q_2)$</td>
<td>Single period cost function</td>
</tr>
<tr>
<td>$M_k$</td>
<td>Optimal cost function of case $k$, $k \in {0, 1, 2, 3}$</td>
</tr>
<tr>
<td>$\Delta M_{kq}$</td>
<td>Optimal cost difference between case $k$ and case $q$, $\Delta M_{kq} = M_k - M_q$</td>
</tr>
<tr>
<td>$x \to z^+$</td>
<td>$x$ approaches the value $z$ from above (right)</td>
</tr>
<tr>
<td>$x \to z^-$</td>
<td>$x$ approaches the value $z$ from below (left)</td>
</tr>
</tbody>
</table>
Lemma 2.1 shows the properties of the expected end-of-period cost function, which are important in deriving conclusions and proving propositions in this paper.

**Lemma 2.1.** The properties of $L(y)$ are as follows. Although for our problem $y \geq 0$, the following results are true for $y \in (-\infty, +\infty)$.

(a) The expected end-of-period cost function $L(y)$ is convex in $y$.

(b) $L(y_2) - L(y_1) \geq L'(y_1)(y_2 - y_1)$.

(c) $L(y_2) - L(y_1) \leq L'(y_2)(y_2 - y_1)$.

(d) Let $y_1 \leq y_2 \leq y_3 \leq y_4$ with $y_1 + y_4 = y_2 + y_3$, then $L(y_1) + L(y_4) \geq L(y_2) + L(y_3)$.

**Proof.** See Appendix.

The single-period cost function is given by

$$M(Q_1, Q_2) = K_o(\max(Q_1, Q_2)) + K_{p1}(Q_1) + K_{p2}(Q_2) + \alpha_1 K_{r1}(Q_1) + \alpha_2 K_{r2}(Q_2) + \alpha_1 c_1 Q_1$$

$$+ \alpha_2 c_2 Q_2 + \alpha_1 \alpha_2 L(Q_1 + Q_2) + \alpha_1 \tilde{\alpha}_2 L(Q_1) + \tilde{\alpha}_1 \alpha_2 L(Q_2) + \tilde{\alpha}_1 \tilde{\alpha}_2 L(0)$$

(2.2)

where $K_j(Q) = \begin{cases} 
K_j & \text{if } Q > 0, j \in \{o, p1, p2, r1, r2\} \\
0 & \text{if } Q = 0.
\end{cases}$

The first three terms represent the overall fixed cost and individual fixed costs incurred upon order placement. The next four terms are the expected order cost, which includes individual fixed costs incurred upon order receipt and item costs. The last four terms represent the expected end-of-period costs based on the possible outcomes of supplier deliveries: orders from both suppliers are delivered, only order from supplier 1 is delivered, only order from supplier 2 is delivered, and nothing is delivered.

Obviously, $M(Q_1, Q_2)$ is not a convex function because of the fixed costs. The four cases mentioned at the beginning of this section are considered: order nothing (Case 0), source only from supplier 1 (Case 1), source only from supplier 2 (Case 2), and dual-source (Case 3). Within a given case, the total fixed cost incurred is not a function of the order quantities, thus, the cost function of each case is convex.
and the optimal cost can be found. By comparing the optimal costs of the four cases, we can determine
the overall optimal cost and order policy. The cost of ordering nothing is the end-of-period cost with
\( y = 0 \). It is easy to verify that when we single-source, the optimal policy is to order up to the level
\( Q^{*}_{s_i} = F^{-1}\left( \frac{p - c_i}{h + p} \right) \), which is the newsvendor critical fractile (service level). The dual-sourcing optimal
order quantities are defined as \( Q^{*}_{d_1} \) from supplier 1 and \( Q^{*}_{d_2} \) from supplier 2. The overall optimal cost
function can thus be given by:

\[
\min_{Q_1, Q_2} M(Q_1, Q_2) = \min\{ M(0, 0), \min_{Q_1 > 0, Q_2 = 0} M(Q_1, 0), \min_{Q_1 = 0, Q_2 > 0} M(0, Q_2), \min_{Q_1 > 0, Q_2 > 0} M(Q_1, Q_2) \} = \min\{ L(0),
K_o + K_{p1} + \alpha_1 K_{r1} + \alpha_1 c_1 Q^{*}_{s1} + \alpha_1 L(Q^{*}_{s1}) + \bar{\alpha}_1 L(0),
K_o + K_{p2} + \alpha_2 K_{r2} + \alpha_2 c_2 Q^{*}_{s2} + \alpha_2 L(Q^{*}_{s2}) + \bar{\alpha}_2 L(0),
K_o + K_{p1} + K_{p2} + \alpha_1 K_{r1} + \alpha_2 K_{r2} + \alpha_1 c_1 Q^{*}_{d1} + \alpha_2 c_2 Q^{*}_{d2} + \alpha_1 \alpha_2 L(Q^{*}_{d1} + Q^{*}_{d2}) + \bar{\alpha}_1 \alpha_2 L(Q^{*}_{d1}) + \bar{\alpha}_1 \bar{\alpha}_2 L(0) \}.
\]

where \( Q^{*}_{s1}, Q^{*}_{s2}, Q^{*}_{d1} \) and \( Q^{*}_{d2} \) can be obtained from the first-order optimality conditions (or Karush-
Kuhn-Tucker conditions). The specific forms are \( Q^{*}_{s1} = F^{-1}\left( \frac{p - c_1}{h + p} \right) \), \( Q^{*}_{s2} = F^{-1}\left( \frac{p - c_2}{h + p} \right) \), and \( Q^{*}_{d1} \) and \( Q^{*}_{d2} \) satisfying the following constraints:

\[
\alpha_2 F(Q^{*}_{d1} + Q^{*}_{d2}) + (1 - \alpha_2) F(Q^{*}_{d1}) = \frac{p - c_1}{h + p}
\]

\[
\alpha_1 F(Q^{*}_{d1} + Q^{*}_{d2}) + (1 - \alpha_1) F(Q^{*}_{d2}) = \frac{p - c_2}{h + p}
\]

\[
Q^{*}_{d1} > 0, Q^{*}_{d2} > 0
\]

From the above equations and constraints we can see that the goal is to determine the order quantities
that make the expected service level equal the newsvendor critical fractile. The optimal order quantity
of single-sourcing is not influenced by the reliability of the supplier because the probability of order
delivery is independent of the order quantity. It only leads to higher cost by deviating from the optimal
service level. The optimal order quantities of dual-sourcing, however, are affected by reliability levels because the expected service level is a function of $\alpha_i, i \in \{1, 2\}$ (see (2.3) and (2.4)). The reliability levels also influence the way the total order quantity is split between the two suppliers: the more reliable supplier gets a larger share.

**Proposition 2.1.**

(a) If $\alpha_1 \geq \frac{p - c_2}{p - c_1}$, single-sourcing from supplier 1 is always better than dual-sourcing whether there are fixed costs or not, and it is optimal when there is no fixed cost.

(b) If $\alpha_2 \geq \frac{p - c_1}{p - c_2}$, single-sourcing from supplier 2 is always better than dual-sourcing whether there are fixed costs or not, and it is optimal when there is no fixed cost.

**Proof.** When there is no fixed cost, the first-order optimality conditions are $\frac{\partial M}{\partial Q_i} \geq 0$ and $(\frac{\partial M}{\partial Q_i})Q_i = 0$ where

$$\frac{\partial M}{\partial Q_1} = \alpha_1 c_1 + \alpha_1 \alpha_2 (h + p) F(Q_1^* + Q_2^*) + \alpha_1 (1 - \alpha_2) (h + p) F(Q_1^*) - \alpha_1 p \quad (2.6)$$

$$\frac{\partial M}{\partial Q_2} = \alpha_2 c_2 + \alpha_1 \alpha_2 (h + p) F(Q_1^* + Q_2^*) + \alpha_2 (1 - \alpha_1) (h + p) F(Q_2^*) - \alpha_2 p \quad (2.7)$$

When $Q_1 > 0$ and $Q_2 = 0$, from $\frac{\partial M}{\partial Q_i} \geq 0$ we can obtain $\frac{\partial M}{\partial Q_1} = 0$ and $\frac{\partial M}{\partial Q_2} \geq 0$. The two equations can be solved to obtain $\alpha_1 \geq \frac{p - c_2}{p - c_1}$. Therefore, if $\alpha_1 \geq \frac{p - c_2}{p - c_1}$, single-sourcing from supplier 1 (Case 1) is always better than dual-sourcing. Since dual-sourcing always incurs no less fixed costs than single-sourcing does, the conclusion also holds when fixed costs are charged. Note that since $\alpha_1 \leq 1$, $\alpha_1 \geq \frac{p - c_2}{p - c_1}$ does not hold if $c_2 < c_1$. The latter part can be proved by comparing Case 1 with Case 0 and Case
The last inequality follows because $\alpha_1 = 0 \leq \alpha_1 \leq \alpha = 0$. 

(b) can be proved in the same manner.

If the conditions in Proposition 1 do not hold, the dual-sourcing case can be optimal. The optimal quantities of dual-sourcing can be obtained from implementing the iterative algorithm proposed by Anupindi and Akella [1]. However, the algorithm is relatively complex. We improve the algorithm in the following two ways.

1. The explicit expressions for optimal order quantities exist under some circumstances if the demand distribution is bounded. When the demand is distributed between $l$ and $u$, the quantities are computed as follows. Let $Q_1 = F^{-1} \left( \frac{F(Q^*_1) - \alpha_2}{1 - \alpha_2} \right)$, and $Q_2 = F^{-1} \left( \frac{F(Q^*_2) - \alpha_1}{1 - \alpha_1} \right)$. If $Q_1 + Q_2 \geq u$, then $Q_{d_1}^* = Q_1$, $Q_{d_2}^* = Q_2$. Otherwise, the algorithm in [1] should be implemented.

2. $Q_1 = \frac{Q^*_1 - \alpha_2 Q^*_2}{1 - \alpha_1 \alpha_2}$ and $Q_2 = \frac{Q^*_2 - \alpha_1 Q^*_1}{1 - \alpha_1 \alpha_2}$ serve as better starting points for the algorithm. They are obtained by assuming a uniform demand distribution and always lead to convergence in our numerical experiments while the original ones described in [1] might fail to achieve the optimal solution.

Proposition 2.2. If there is no fixed cost and the conditions in Proposition 1 do not hold, then dual-sourcing is optimal.
Proof. It can be proved by comparing the four cases, i.e., showing that $\Delta M_{10}, \Delta M_{20}, \Delta M_{31}, \Delta M_{32} \leq 0$. Here $\Delta M_{32} \leq 0$ is proved as an example. The other three inequalities can be proved in the same way.

$$
\Delta M_{32}(K = 0) = \alpha_1 c_1 Q_{d1}^* + \alpha_2 c_2 Q_{d2}^* + \alpha_1 \alpha_2 L(Q_{d1}^* + Q_{d2}^*) + \alpha_1 \bar{\alpha}_2 L(Q_{d1}^*) + \bar{\alpha}_1 \alpha_2 L(Q_{d2}^*)
$$

$$
+ \bar{\alpha}_1 \bar{\alpha}_2 L(0) - [\alpha_2 c_2 Q_{s2}^* + \alpha_2 L(Q_{s2}^*) + (1 - \alpha_2) L(0)]
$$

$$
= \alpha_1 c_1 Q_{d1}^* + \alpha_2 c_2 (Q_{d2}^* - Q_{s2}^*) + \alpha_1 \alpha_2 [L(Q_{d1}^* + Q_{d2}^*) - L(Q_{s2}^*)]
$$

$$
+ \bar{\alpha}_1 \alpha_2 [L(Q_{d2}^*) - L(Q_{s2}^*)] + \alpha_1 \bar{\alpha}_2 [L(Q_{d1}^*) - L(0)]
$$

$$
\leq \alpha_1 c_1 Q_{d1}^* + \alpha_2 c_2 (Q_{d2}^* - Q_{s2}^*) + \alpha_1 \alpha_2 (Q_{d1}^* + Q_{d2}^* - Q_{s2}^*) L'(Q_{d1}^* + Q_{d2}^*)
$$

$$
+ \bar{\alpha}_1 \alpha_2 (Q_{d2}^* - Q_{s2}^*) L'(Q_{d2}^*) + \alpha_1 \bar{\alpha}_2 Q_{d1}^* L'(Q_{d1}^*)
$$

$$
= \alpha_1 c_1 Q_{d1}^* + \alpha_2 c_2 Q_{d2}^* - \alpha_2 c_2 Q_{s2}^* + \alpha_2 (Q_{d2}^* - Q_{s2}^*) (-c_2) + \alpha_1 Q_{d1}^* (-c_1)
$$

$$
= 0
$$

The inequality follows because of the properties of $L(y)$ shown in Lemma 2.1.

Now we discuss the optimal policy under different fixed-cost scenarios based on the two Propositions above.

2.2.1 $K_o$ Only

Overall fixed cost is incurred only if at least one order is placed. Therefore, it only influences the decision of whether or not to place orders. Based on Proposition 1 and Proposition 2, we can obtain the following results.

- If $\alpha_1 \geq \frac{p - c_2}{p - c_1}$, the optimal policy is either to order nothing (Case 0) or to single-source from supplier 1 (Case 1), depending on the value of $K_o$.

- If $\alpha_2 \geq \frac{p - c_1}{p - c_2}$, the optimal policy is either to order nothing (Case 0) or to single-source from supplier 2 (Case 2).
• The case where $\alpha_1 > \frac{p-c_2}{p-c_1}$ and $\alpha_2 > \frac{p-c_1}{p-c_2}$ is impossible because at least one of $\frac{p-c_1}{p-c_2}$ and $\frac{p-c_2}{p-c_1}$ must be greater than or equal to 1 whereas $\alpha_i \leq 1$.

• The case $\alpha_1 = \frac{p-c_2}{p-c_1}$ and $\alpha_2 = \frac{p-c_1}{p-c_2}$ holds only when $c_1 = c_2$ and $\alpha_1 = \alpha_2 = 1$. In this situation, the optimal costs of Case 1, 2, 3 are the same, but whether they are better than Case 0 depends on the value of $K_o$.

• If $\alpha_1 < \frac{p-c_2}{p-c_1}$ and $\alpha_2 < \frac{p-c_1}{p-c_2}$, the costs of dual-sourcing and order-nothing should be compared to determine the optimal policy. Single-sourcing (Cases 1 and 2) cannot be optimal in this case.

2.2.2 $K_p$ Only

When individual fixed costs incurred upon order placement are the only source of fixed ordering costs, the dual-sourcing case need not be considered under certain conditions.

**Proposition 2.3.** If ordering nothing (Case 0) is no worse than either or both single-sourcing cases (Cases 1 and 2), then the dual-sourcing case cannot be optimal.

**Proof.** First we prove that if $M_0 \leq M_1$, then $M_2 < M_3$. From $M_1 \leq M_2$, we have $L(0) \leq K_{p1} + \alpha_1 c_1 Q_{s1}^* + \alpha_1 L(Q_{s1}^*) + (1 - \alpha_1)L(0)$, and it can be derived that $K_{p1}/\alpha_1 + c_1 Q_{s1}^* + L(Q_{s1}^*) \geq L(0)$. The inequality $K_{p2} + \alpha_2 c_2 Q_{s2}^* + \alpha_2 L(Q_{s2}^*) + (1 - \alpha_2)L(0) < K_{p2} + \alpha_2 c_2 Q_{d2}^* + \alpha_2 L(Q_{d2}^*) + (1 - \alpha_2)L(0)$ holds because of the optimality of $Q_{s2}^*$. Therefore,
Similarly, if \( M_0 \leq M_2 \), then \( M_1 < M_3 \).

Proposition 2.3 can be interpreted from the view of marginal benefit. The cost difference between ordering nothing and single-sourcing from supplier \( i \) is the largest marginal benefit of supplier \( i \), which
can be proved as follows:

\[ M_0 - M_2 - (M_1 - M_3) = M_0 + M_3 - M_1 - M_2 \]
\[ = L(0) + \alpha_1 c_1 Q_{d1} + \alpha_2 c_2 Q_{d2}^* + \alpha_1 \alpha_2 L(Q_{d1}^* + Q_{d2}^*) + \alpha_1(1 - \alpha_2)L(Q_{d1}^*) \]
\[ + (1 - \alpha_1)\alpha_2 L(Q_{d2}^*) + (1 - \alpha_1)(1 - \alpha_2)L(0) - [\alpha_1 c_1 Q_{s1}^* + \alpha_2 c_2 Q_{s2}^* \]
\[ + \alpha_1 L(Q_{s1}^*) + \alpha_2 L(Q_{s2}^*) + (1 - \alpha_1)L(0) + (1 - \alpha_2)L(0)] \]
\[ = \alpha_1 c_1 Q_{d1}^* + \alpha_1 L(Q_{d1}^*) + (1 - \alpha_1)L(0) - [\alpha_1 c_1 Q_{s1}^* + \alpha_1 L(Q_{s1}^*) \]
\[ + (1 - \alpha_1)L(0)] + \alpha_2 c_2 Q_{d2}^* + \alpha_2 L(Q_{d2}^*) + (1 - \alpha_2)L(0) \]
\[ - [\alpha_2 c_2 Q_{s2}^* + \alpha_2 L(Q_{s2}^*) + (1 - \alpha_2)L(0)] \]
\[ + \alpha_1 \alpha_2[L(Q_{d1}^* + Q_{d2}^*) - L(Q_{d1}^*) - L(Q_{d2}^*) + L(0)] \]
\[ > 0 \]

The inequality follows because of the optimality of \( Q_{s1}^* \) and \( Q_{s2}^* \), and the convex properties of \( L(y) \). If nothing can be gained by switching from ordering nothing to single-sourcing from supplier \( i \), then the marginal benefit of supplier \( i \) is negative; that is, including supplier \( i \) cannot lead to a better result.

**Optimal Procedure.** The optimal cost and policy can be obtained by the following procedure:

1. Calculate the costs of Cases 0, 1 and 2 according to (2.2).

2. If \( M_0 \leq M_1 \) or \( M_0 \leq M_2 \), then the optimal cost is \( \min(M_0, M_1, M_2) \), and the optimal policy is the one with the minimum cost.

3. If \( M_0 > \max(M_1, M_2) \), i.e., order-nothing is the worst among Cases 0-2, then the cost of dual-sourcing \( M_3 \) needs to be calculated. The optimal cost is \( \min(M_1, M_2, M_3) \), and the optimal policy is the one with the minimum cost.

**Numerical Example.** This procedure can be illustrated by a specific case of uniformly distributed demand. Let \( K_{p1} = 15, K_{p2} = 30, \alpha_1 = 0.85, \alpha_2 = 0.9, c_1 = 1, c_2 = 1.2, h = 0.2, p = 2.5 \), and let demand be uniform distributed between \([0,100]\). From (2.2), we can obtain \( M_0 = 125.0 \),
$Q_{s1}^* = 55.6$, $Q_{s2}^* = 48.1$, $M_1 = 104.6$, and $M_2 = 126.83$. Because Case 0 is not the worst case, Case 3 need not be considered and the optimal policy is to order 55.6 units from supplier 1. If the fixed cost $K_{f2}$ is reduced from 30 to 10, then $M_2 = 106.8$. Since $M_0 > M_1$ and $M_0 > M_2$, $M_3$ needs to be computed, which is $M_3 = 114.5$. Comparing it with $M_1$ and $M_2$, it is easy to see that Case 1 is still optimal.

2.2.3 $K_r$ Only

Here we consider the case where only individual fixed cost incurred upon order delivery is included.

- When $c_1 < c_2$ and $\alpha_1 > \frac{p - c_2}{p - c_1}$, selecting supplier 2 cannot be optimal so only Case 0 and Case 1 need be compared. Then $\Delta M_{10} = \alpha_1 K_{r1} + \alpha_1 c_1 Q_{s1}^* + \alpha_1 L(Q_{s1}^*) + (1 - \alpha_1)L(0) - L(0)$ yields the result that Case 0 is better than Case 1 if $K_{r1} > L(0) - [c_1 Q_{s1}^* + L(Q_{s1}^*)]$, otherwise Case 1 is optimal.

- When $c_1 > c_2$ and $\alpha_2 > \frac{p - c_1}{p - c_2}$, selecting supplier 1 cannot be optimal so only Case 0 and Case 2 need be compared. Thus, Case 0 is optimal if $K_{r2} > L(0) - [c_2 Q_{s2}^* + L(Q_{s2}^*)]$, otherwise Case 2 is optimal.

- For any other situations, the optimal policy can be determined by comparing all of the four cases. It can be proved that Proposition 2.3 still holds in this scenario so the Optimal Procedure can be applied to simplify the cost-comparing process.

2.2.4 $K_o + K_p$

In this scenario, the optimal policy can be determined from combining the methods in Section 2.2.1 and 2.2.2. First, $K_o$ can be dropped from the model and a potential optimal policy can be obtained from the Optimal Procedure in Section 2.2.2. If ordering nothing is selected, then it is optimal. Otherwise $K_o$ should be added back to the cost of the selected policy, and the resulting total cost compared with that of ordering nothing to determine the best solution.

For other fixed-cost scenarios, all four cases must be compared to obtain the optimal policy.
2.3 Results and Insights for Specially-structured Problems

In practice, the reliability levels of suppliers are not necessarily known nor constant. Therefore, it is useful to know over what range the current optimal policy still holds. Expressing the optimal policy structure as a function of the reliability levels serves this function well. In this section, we focus on deriving the optimal policy structures for some specially-structured problems, such as the identical-supplier models. The optimal policy structure as a function of the reliability level $\alpha$ is determined by investigating the cost difference functions. Models with different assumptions about fixed costs are investigated and compared. For the identical-supplier models, we consider three scenarios involving fixed costs, (1) only overall fixed cost $K_o$ is incurred, (2) only individual fixed cost $K_p$ is incurred upon order placement, and (3) only individual fixed cost $K_r$ is incurred upon order receipt. Models with more complex fixed cost scenarios, such as $K_o + K_p$, $K_o + K_r$, $K_p + K_r$, or even $K_o + K_r + K_p$, can be derived by the same method, but in the interest of brevity, are not included here. Interesting results and properties of optimal order quantities are shown. Moreover, we study the effects of various parameters, such as item cost, fixed cost, disposal cost and penalty cost, as well as the effects of the demand distribution, on the optimal policy structure and the optimal order quantities. The optimal policy structures are also developed for the non-identical-supplier models, which are models with the identical-item-cost constraint or the identical-reliability-level constraint relaxed based on the identical-supplier model with the fixed cost $K_p$ only.

2.3.1 Optimal Order Quantities of Identical-supplier Models

Because the optimal quantities are not functions of the fixed costs, we discuss the properties of the optimal quantities and the effects of the parameters on them before the models with different fixed cost scenarios are investigated.

When the two suppliers are identical, i.e., $c_1 = c_2 = c$, and $\alpha_1 = \alpha_2 = \alpha$, the subscripts for the purpose of differentiating supplier 1 and supplier 2 can be omitted. Case 1 (single-sourcing from supplier 1) and Case 2 (single-sourcing from supplier 2) are equivalent and can be referred to as the single-sourcing case ($s$). For the purpose of having consistent notation, we refer to Case 0 and Case 3
as the order-nothing case \((n)\) and dual-sourcing case \((d)\), respectively.

In this model, the optimal quantity for single-sourcing is \(Q^*_s = F^{-1}\left(\frac{p-c}{h+p}\right)\), and for dual-sourcing we obtain \(Q^*_d\) by solving

\[
\bar{\alpha} F(Q^*_d) + \alpha F(2Q^*_d) = \frac{p-c}{h+p}, \quad \text{where } Q^*_d > 0. \tag{2.8}
\]

Proposition 2.4. The properties of optimal quantities are as follows:

(a) \(Q^*_s\) is not a function of \(\alpha\);

(b) \(Q^*_d\) is monotonically increasing as the reliability level \(\alpha\) decreases;

(c) \(Q^*_d < 2Q^*_s \) for \(\alpha \in (0, 1]\), and the equality holds when \(\alpha = 1\).

Proof.

(a). This is obvious given the expression of \(Q^*_s = \frac{p-c}{h+p}\).

(b). By taking the first-order derivative of (2.8) with respect to \(\alpha\), we have

\[
-F(Q^*_d) + F(2Q^*_d) + (1-\alpha)f(Q^*_d) \frac{\partial Q^*_d}{\partial \alpha} + 2\alpha f(2Q^*_d) \frac{\partial Q^*_d}{\partial \alpha} = 0 \tag{2.9}
\]

Since \(F(2Q^*_d) \geq F(Q^*_d), f(Q^*_d) \geq 0\) and \(f(2Q^*_d) \geq 0\), we have

\[
\frac{\partial Q^*_d}{\partial \alpha} = \frac{F(Q^*_d) - F(2Q^*_d)}{(1-\alpha)f(Q^*_d) + 2\alpha f(2Q^*_d)} \leq 0 \tag{2.10}
\]

(c). From (2.8) we can see that \(2Q^*_s = F^{-1}\left(\frac{p-c}{h+p}\right) = Q^*_s\) when \(\alpha = 1\), and \(Q^*_d \rightarrow \left(F^{-1}\left(\frac{p-c}{h+p}\right)\right)^- = (Q^*_s)^-\) when \(\alpha \rightarrow 0^+\). Because \(Q^*_d\) is monotonically increasing when \(\alpha\) decreases, we can conclude that \(Q^*_d < Q^*_s \leq 2Q^*_s\) on \(\alpha \in (0, 1]\) (see Figure 2.1). \(\square\)

If only one supplier is chosen, the optimal order quantity is not influenced by the reliability level of that supplier while the optimal order quantities of dual-sourcing are a function of \(\alpha\). When suppliers are highly reliable, the order quantity from each supplier is close to \(\frac{Q^*_s}{2}\) since the probability of receiving
Figure 2.1: The Optimal Order Quantities for Single-sourcing and Dual-sourcing. The parameters used are: item cost $c = 1$, penalty cost $p = 3$, disposal cost $h = 0.3$, demand is exponentially distributed with mean $\mu = 100$.

both orders is high and the desirable order-up-to level is $Q^*_s$. As suppliers become less reliable, the order quantity monotonically increases from $\frac{Q^*_s}{2}$ to $Q^*_s$. When the suppliers are almost completely unreliable ($\alpha \to 0^+$), the probability of receiving the order quantity from both suppliers ($\alpha^2$) is near zero, while receiving stock from either but not both ($2\alpha(1 - \alpha)$) is more likely. Thus, the order quantity from each supplier goes to $Q^*_s$. The trivial situation where suppliers are completely unreliable is not included in the proposition because order quantity is irrelevant in this case.

Obviously, within each case, fixed cost has no effect on the optimal order quantities. The effects of penalty cost, item cost and disposal cost on the optimal order quantities can be obtained from the first-order derivatives of $Q^*_s$ and $Q^*_d$. If the derivatives are positive, we tend to order more stock as the parameters increase; otherwise, we tend to order less. Table 2.2 summarizes the effects of penalty cost $p$, disposal cost $h$, and item cost $c$. When penalty cost increases, it is desirable to order more stock to avoid being penalized. $Q^*_s$ and $Q^*_d$ are monotonically increasing as the penalty cost increases. The effects of item cost and disposal cost are the opposite to those of penalty cost. The specific mathematical derivation process can be found in the Appendix.

The demand distributions can have a significant impact on order quantity allocation as well. We
Table 2.2: The effects of penalty cost $p$, disposal cost $h$ and item cost $c$

<table>
<thead>
<tr>
<th></th>
<th>$\Delta M_{sn}$</th>
<th>$\Delta M_{dn}$</th>
<th>$\Delta M_{ds}$</th>
<th>Order quantities</th>
</tr>
</thead>
<tbody>
<tr>
<td>penalty cost $p$ increases</td>
<td>decreases</td>
<td>decreases</td>
<td>decreases</td>
<td>increases</td>
</tr>
<tr>
<td>item cost $c$ increases</td>
<td>increases</td>
<td>increases</td>
<td>increases</td>
<td>increases</td>
</tr>
<tr>
<td>disposal cost $h$ increases</td>
<td>increases</td>
<td>increases</td>
<td>may decrease under some situations</td>
<td>decreases</td>
</tr>
</tbody>
</table>

Focus on studying the effects of demand mean $\mu$ and standard deviation $\sigma$. Since different distributions may exert different influences, we consider two specific demand distributions. Uniform demand distributions are considered for their simplicity, and we also model demand as a triangular distribution since we would expect demand to often exhibit a central tendency. The effects can be proved by expressing $Q^*_s$, $Q^*_d$ as functions of $\mu$ and $\sigma$, and then taking the first-order derivatives with respect to $\mu$ and $\sigma$. The specific mathematical derivation process can be found in the Appendix. This method can be generalized to be applied to other distributions.

**Uniform distributions between \([l, u]\)**

- When $\mu$ increases, the optimal quantities $Q^*_s$ and $Q^*_d$ increase linearly.

- When $\sigma$ increases, the change in optimal quantities depends on the relation between underage cost $c_u = p - c$ and overage cost $c_o = h + c$.
  - If underage cost is relatively large, the optimal quantities increase when $\sigma$ increases.
  - If overage cost is relatively large, the optimal quantities decrease when $\sigma$ increases.

**Symmetric triangular distributions between \([l, u]\)**

- The effects of $\sigma$ on $Q^*_d$ depend on other parameters such as the underage cost, overage cost, reliability level, demand mean, and sometimes $\sigma$ itself.

- The other effects are the same with those of the uniform distributions.
2.3.2 Optimal Policy Structures of Identical-supplier Models

The optimal policy structure as a function of the reliability level $\alpha$ is determined by investigating the cost functions and the cost difference functions. Because the two suppliers are identical, the conditions in Proposition 2.1 ($\alpha_1 > \frac{p-c_2}{p-c_1}$ and $\alpha_2 > \frac{p-c_1}{p-c_2}$) are never satisfied, so dual-sourcing is always optimal for this scenario when there is no fixed cost.

**Proposition 2.5.** When there is no fixed ordering cost in the two-identical-supplier model, ordering from both suppliers is always better because the savings in penalty cost is always no less than the extra cost of disposal and item costs.

**Proof.** See Appendix.

It can be proved that sourcing from more suppliers always incurs no more penalty cost, but no less item cost. Clearly, single-sourcing results in no less disposal cost than doing nothing. Usually, dual-sourcing also leads to more disposal cost than single-sourcing, although it is possible that dual-sourcing results in less disposal cost. However, such an exception rarely occurs in our numerical experiments. As the proof of Proposition 2.5 shows, the savings in penalty cost is always no less than the extra spent on disposal and item costs. Therefore, sourcing from more suppliers is always preferred. However, if individual fixed cost is incurred, the savings on penalty cost may not be enough to cover the extra spent on disposal cost, item cost and fixed cost. So we need to compare the three cases to determine the optimal policy structure as a function of reliability level $\alpha$.

Next, we can obtain the properties of the optimal costs of dual-sourcing and single-sourcing. Recall that the cost of ordering nothing is a constant (the penalty cost of the expected demand, i.e., $P(0) = \int_0^{\infty} p\xi f(\xi)d\xi = pE(\xi)$).

**Proposition 2.6.** For the single-sourcing and dual-sourcing cases, as the suppliers become more reliable, the optimal cost decreases.

**Proof.** It is proved by showing that $M_{d(s)}(\alpha)$ is monotonically decreasing on $(0, 1]$. See Appendix.

Therefore, although the reliability level does not impact the order quantity of single-sourcing, it does
have influence on the cost of single-sourcing.

Now we discuss the optimal policy structure of the models with different fixed cost scenarios. Although low reliability levels may not be realistic, the whole range $\alpha \in (0, 1]$ is considered for the sake of simplicity in the optimal policy structure derivation process unless the optimal policy structure becomes complicated for low reliability levels under certain conditions.

### 2.3.2.1 $K_o$ Only

When there is only an overall fixed cost $K_o$, $\Delta M_{ds}$ is the same as the case when there is no fixed cost, which implies that dual-sourcing is always better than single-sourcing. As a result, we only need to consider the cost difference between ordering-nothing and dual-sourcing to determine the optimal policy, i.e.

$$
\Delta M_{dn}(\alpha) = K_o + 2\alpha cQ_d^* + \alpha^2 L(2Q_d^*) + 2\alpha \alpha L(Q_d^*) + \alpha^2 L(0) - L(0) \tag{2.11}
$$

which is a decreasing function of $\alpha$ on $(0, 1]$ as indicated by Proposition 2.6. So $\Delta M_{dn}(\alpha) = 0$ has at most one root, which is denoted as $\alpha_{nd}$. When $\alpha_{nd} \in (0, 1]$, the optimal policy is to order nothing for $\alpha < \alpha_{nd}$ and dual-source for $\alpha \geq \alpha_{nd}$. When $\alpha_{nd} > 1$, $\Delta M_{dn}(\alpha) > 0$ on the interval $\alpha \in (0, 1]$ so ordering nothing is always optimal. The threshold level of $K_o$, represented by $K_o^t$, is the value which satisfies $\Delta M_{dn}(\alpha = 1) = 0$ so can be computed as follows:

$$
K_o^t = L(0) - 2cQ_d^* - L(2Q_d^*) \tag{2.12}
$$

The optimal policy structure can then be summarized as follows (see Figure 2.2.a):

i) When $K_o \leq K_o^t$, solve $\Delta M_{dn}(\alpha) = 0$ for $\alpha_{nd}$; order nothing for $\alpha < \alpha_{nd}$; otherwise, dual-source;

ii) When $K_o > K_o^t$, $\alpha_{nd}$ does not exist on $\alpha \in (0, 1]$, then ordering nothing is optimal.

This result is intuitive since the overall fixed cost only impacts the decision of whether or not to order; once it has been decided to order something, more suppliers are always better. There is no explicit
expression for \( \alpha_{nd} \), but it can be solved quickly from \( \Delta M_{dn}(\alpha) = 0 \) using numerical methods.

When \( K_o \) increases, the range of values of \( \alpha \) where ordering nothing is preferred increases because \( \alpha_{nd} \) increases. The effects of penalty cost, item cost and disposal cost on optimal policies can be obtained from the first-order derivatives of \( \Delta M_{dn} \) (see Table 2.2). The specific mathematical derivation process can be found in the Appendix. The derivatives of item cost and disposal cost are positive, which implies that ordering nothing is desirable if item cost and/or disposal cost increase; otherwise, dual-sourcing is preferred. When penalty cost increases, it is desirable to dual-source to avoid penalties.
2.3.2.2 K<sub>p</sub> Only and General Demand Distributions

When individual fixed costs are incurred upon order placement, from Proposition 2.3 we can derive the special case for the identical-supplier model: when ordering nothing costs less than single-sourcing, it also costs less than dual-sourcing. Therefore, we only need to consider the cost difference between single-sourcing and ordering nothing, i.e., $\Delta M_{sn}$ and the cost difference between dual-sourcing and single-sourcing, i.e., $\Delta M_{ds}$, to obtain the optimal policy structure. When $\Delta M_{sn} > 0$, it is optimal to order nothing; otherwise $\Delta M_{ds}$ needs to be calculated. In this case, if $\Delta M_{ds} > 0$, single-sourcing is optimal; otherwise dual-sourcing is optimal. The general expressions of the cost differences are as follows:

$$\Delta M_{sn} = K_p + \alpha cQ_s^* + \alpha L(Q_s^*) - \alpha L(0), \quad (2.13)$$

$$\Delta M_{ds} = K_p + \alpha c(2Q_d^* - Q_s^*) + \alpha^2 L(2Q_d^*) + 2\alpha \bar{\alpha} L(Q_d^*) - \alpha \bar{\alpha} L(0) - \alpha L(Q_s^*) \quad (2.14)$$

**Proposition 2.7.** The cost differences $\Delta M_{sn}(\alpha)$ and $\Delta M_{ds}(\alpha)$ have the following properties:

(a) $\Delta M_{sn}(\alpha \to 0^+) = \Delta M_{ds}(\alpha \to 0^+) = \Delta M_{ds}(\alpha = 1) = K_p$;

(b) $\Delta M_{sn}(\alpha)$ is monotonically decreasing as $\alpha$ increases from 0 to 1;

(c) $\Delta M_{ds}(\alpha)$ is monotonically increasing as $\alpha$ increases from 0.5 to 1;

(d) $\Delta M_{sn}(\alpha) < \Delta M_{ds}(\alpha)$.

**Proof.**

(a). It can be easily verified by substituting $\alpha \to 0^+$ and $\alpha = 1$ into $\Delta M_{sn}$ and $\Delta M_{ds}$.

(b). Taking the first-order derivative of $\Delta M_{sn}$ with respect to $\alpha$, we have $\frac{\partial \Delta M_{sn}}{\partial \alpha} = cQ_s^* + L(Q_s^*) - L(0) \leq cQ_s^* - Q_s^* L'(Q_s^*) = 0$. The inequality follows due to Proposition 2.1. The last equality follows due to that fact that $L'(Q_s^*) = -c$.

(c). Taking the first-order derivative of $\Delta M_{ds}$ with respect to $\alpha$, we have $\frac{\partial \Delta M_{ds}}{\partial \alpha} = 2cQ_d^* + 2\alpha [L(2Q_d^*) - L(Q_d^*)] + 2(1 - \alpha)[L(Q_d^*) - L(0)] - [cQ_s^* + L(Q_s^*) - L(0)]$. Now we investigate when the inequality $\frac{\partial \Delta M_{ds}}{\partial \alpha} \geq 0$ holds. Setting $\frac{\partial \Delta M_{ds}}{\partial \alpha} \geq 0$ and rearranging it, we get $\alpha \geq \frac{cQ_s^* + L(Q_s^*) + L(0) - 2L(Q_d^*) - 2cQ_d^*}{2[L(2Q_d^*) - 2L(Q_d^*) + L(0)]}$. Let
\[ \text{RHS}(\alpha) = \frac{cQ_s^* + L(Q_s^*) + L(0) - 2L(Q_s^*) - 2\alpha Q_s^*}{2L(2Q_d^*) - 2L(Q_s^*) + L(0)}. \]

If \( \alpha \geq \text{RHS}(\alpha) \) holds for certain range of \( \alpha \), then \( \frac{\partial \text{RHS}}{\partial \alpha} \geq 0 \) also holds on that range. When \( \alpha = 1 \), \( Q_s^* = 2Q_d^* \), so \( \text{RHS}(\alpha = 1) = \frac{cQ_s^* + L(Q_s^*) + L(0) - 2L(Q_s^*) - 2\alpha Q_s^*}{2L(2Q_d^*) - 2L(Q_s^*) + L(0)} = \frac{cQ_s^* + L(Q_s^*) + L(0) - 2L(Q_s^*/2) - 2\alpha Q_s^*}{2L(2Q_d^*) - 2L(Q_s^*/2) + L(0)} = \frac{1}{2}. \]

If \( \frac{\partial \text{RHS}}{\partial \alpha} \geq 0 \), i.e., \( \text{RHS}(\alpha) \) is monotonically increasing as \( \alpha \) increases, then \( \text{RHS}(\alpha) \leq \frac{1}{2} \) for \( \alpha \leq 1 \). Therefore, \( \alpha \geq \text{RHS}(\alpha) \) holds for \( \alpha \in [0.5, 1] \). \( \frac{\partial \text{RHS}}{\partial \alpha} \geq 0 \) is proved as follows.

\[
\frac{\partial \text{RHS}}{\partial \alpha} = \frac{1}{4[L(2Q_d^*) - 2L(Q_s^*) + L(0)]^2} \left\{ 2\left[ -2L'(Q_d^*) \frac{\partial Q_d^*}{\partial \alpha} - 2c \frac{\partial Q_d^*}{\partial \alpha} \right] [L(2Q_d^*) - 2L(Q_s^*) + L(0)] \\
- 2\left[ 2L'(Q_d^*) \frac{\partial Q_d^*}{\partial \alpha} - 2L'(Q_s^*) \frac{\partial Q_d^*}{\partial \alpha} \right] [cQ_s^* + L(Q_s^*) + L(0) - 2L(Q_d^*) - 2cQ_d^*] \right\} \\
= \frac{1}{[L(2Q_d^*) - 2L(Q_s^*) + L(0)]^2} \frac{\partial Q_d^*}{\partial \alpha} \left\{ \alpha [L'(2Q_d^*) - L'(Q_s^*)] [L(2Q_d^*) - 2L(Q_d^*) + L(0)] \\
- [L'(2Q_d^*) - L'(Q_s^*)] [cQ_s^* + L(Q_s^*) + L(0) - 2L(Q_d^*) - 2cQ_d^*] \right\} \\
= \frac{L'(2Q_d^*) - L'(Q_s^*)}{\alpha [L(2Q_d^*) - 2L(Q_s^*) + L(0)]^2} \frac{\partial Q_d^*}{\partial \alpha} \left\{ 2\alpha Q_s^* + \alpha^2 L(2Q_d^*) + 2\alpha L(Q_d^*) \\
+ \bar{a}^2 L(0) - [\alpha cQ_s^* + \alpha L(Q_s^*) + \bar{a} L(0)] \right\} \\
= \frac{L'(2Q_d^*) - L'(Q_s^*)}{\alpha [L(2Q_d^*) - 2L(Q_s^*) + L(0)]^2} \frac{\partial Q_d^*}{\partial \alpha} \Delta M_{ds}(K = 0) \geq 0,
\]

Therefore, \( \frac{\partial M_{ds}}{\partial \alpha} \geq 0 \) holds for \( \alpha \in [0.5, 1] \).

(d).

\[
\Delta M_{ds} - \Delta M_{sn} = K_p + \alpha c(2Q_d^* - Q_s^*) + \alpha^2 L(2Q_d^*) + 2\alpha \bar{a} L(Q_s^*) - \alpha \bar{a} L(0) \\
- \alpha L(Q_s^*) - [K_p + \alpha cQ_s^* + \alpha L(Q_s^*) - \alpha L(0)] \\
= 2\alpha c(Q_d^* - Q_s^*) + \alpha^2 [L(2Q_d^*) - 2L(Q_d^*) + L(0)] + 2\alpha [L(Q_d^*) - L(Q_s^*)] \\
\geq 2\alpha c(Q_d^* - Q_s^*) + \alpha^2 [L(2Q_d^*) - 2L(Q_d^*) + L(0)] + 2\alpha [Q_d^* - Q_s^*] L'(Q_s^*) \\
= \alpha^2 [L(2Q_d^*) - 2L(Q_d^*) + L(0)] > 0
\]
First, we consider the cost difference between single-sourcing and ordering nothing. Proposition 2.7.a states that the cost difference is $K_p$ when suppliers are almost perfectly unreliable ($\alpha \to 0^+$). In this case, penalty costs on expected demand are incurred when nothing is ordered, while both penalty and fixed costs are incurred if something is ordered; thus, doing nothing is a better choice. Proposition 2.7.b implies that there is at most one point at which we are indifferent between the two choices ($\Delta M_{sn}(\alpha) = 0$) as reliability improves from almost perfectly unreliable to perfectly reliable. This threshold level is denoted as $\alpha_{ns}$. For $\alpha < \alpha_{ns}$, ordering nothing is optimal, otherwise single-sourcing is better. If $\alpha_{ns} > 1$, then $\Delta M_{sn}$ is always positive, i.e., ordering nothing is always optimal. The threshold level $\alpha_{ns}$ is calculated as:

$$\alpha_{ns} = \frac{K_p}{cQ^*_s + L(Q^*_s) - L(0)} \quad (2.15)$$

Second, we discuss the cost difference function $\Delta M_{ds}(\alpha)$. When suppliers are perfectly reliable or almost perfectly unreliable, the costs of dual-sourcing and single-sourcing are the same except that dual-sourcing incurs the fixed cost twice. $\Delta M_{ds}(\alpha)$ might not be a unimodal function on $\alpha \in (0, 1]$ for general demand distributions. When the reliability level is below 0.5, it is possible that multiple threshold levels exist, although we show that the function is better behaved for certain special distributions later. From a practical point of view, it is very likely that a supplier who delivers less than half of the time will not be considered by customers. Therefore, we derive the optimal policy structure for general demand distributions only for $\alpha \in [0.5, 1]$ since $\Delta M_{ds}(\alpha)$ is monotonically increasing on $\alpha \in [0.5, 1]$. Similar to $\Delta M_{sn}(\alpha)$, there is at most one reliability level ($\alpha_{ds}$) where $\Delta M_{ds} = 0$ on the interval $[0.5, 1]$. Since $\Delta M_{ds}(\alpha = 1) = K_p > 0$, $\alpha_{ds}$ exists if $\Delta M_{ds}(\alpha = 0.5) < 0$. For $0.5 \leq \alpha < \alpha_{ds}$, dual-sourcing is better, otherwise single-sourcing is preferred. There is no explicit expression for $\alpha_{ds}$, but it can be solved from $\Delta M_{ds}(\alpha) = 0$ using numerical methods. Our numerical experiments show that the False Position method and Bisection method [8] can usually obtain $\alpha_{ds}$ to three decimal places in less than a second. If $\Delta M_{ds}(\alpha = 0.5) > 0$, then $\alpha_{ds}$ does not exist and single-sourcing is always better than dual-sourcing on $\alpha = [0.5, 1]$. 

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From Proposition 2.7.d we know that $\Delta M_{ds}(\alpha = 0.5) > \Delta M_{sn}(\alpha = 0.5)$. In addition, $\Delta M_{ds}(\alpha)$ is increasing on $[0.5, 1]$ and $\Delta M_{sn}(\alpha)$ is decreasing on this range. Thus, at most only one of the threshold levels $\alpha_{ns}$ and $\alpha_{ds}$ exists on $[0.5, 1]$. The optimal policy structure on the range of $[0.5, 1]$ can be summarized as follows.

i) When $\Delta M_{ds}(\alpha = 0.5) < 0$, single-sourcing is always better than ordering nothing and only $\alpha_{ds}$ exists. Dual-source if $\alpha < \alpha_{ds}$ and single-source if $\alpha \geq \alpha_{ds}$.

ii) When $\Delta M_{ds}(\alpha = 0.5) \geq 0$ and $\Delta M_{sn}(\alpha = 0.5) < 0$, no threshold level exists. Single-source for any $\alpha$.

iii) When $\Delta M_{sn}(\alpha = 0.5) > 0$ and $\Delta M_{sn}(\alpha = 1) < 0$, single-sourcing is always better than dual-sourcing and only $\alpha_{ns}$ exists. Order nothing if $\alpha < \alpha_{ns}$ and single-source if $\alpha \geq \alpha_{ns}$.

iv) When $\Delta M_{sn}(\alpha = 1) \geq 0$, no threshold level exists. Order nothing for any $\alpha$.

When $K_p$ increases, then clearly the order cost of sourcing from more suppliers (one and two suppliers) increases, thus the range of $\alpha$ values when order-nothing is optimal increases. The effects of penalty cost, item cost and disposal cost on optimal policies can be obtained from the first-order derivatives of $\Delta M_{ds}$ and $\Delta M_{sn}$ (see Table 2.2).

• When penalty cost increases, sourcing from more suppliers becomes better since there is a higher probability of receiving the order and avoiding penalties.

• The effects of item cost are the opposite to those of penalty cost.

• When disposal cost increases, ordering nothing is more likely to be preferred over the single-source case; and most of the time, single-sourcing is more likely to become better than dual-sourcing. Although $\frac{\partial \Delta M_{ds}}{\partial h} < 0$ is possible in theory, that is, dual-sourcing becomes better when disposal cost increases, such an exception rarely occurs in our numerical experiments.
2.3.2.3  \( K_p \) Only and Special Demand Distributions

For certain demand distributions, such as uniform distributions and exponential distributions, the optimal policy structure can be extended to the interval of \( \alpha \in (0, 1] \) due to unimodality of \( \Delta M_{ds}(\alpha) \).

**Proposition 2.8.** On the range of \( \alpha \in (0, 1] \),

(a) \( \Delta M_{ds}(\alpha) \) is a unimodal function when demand is uniformly distributed between \([0, u]\).

(b) \( \Delta M_{ds}(\alpha) \) is a convex function when demand is exponentially distributed with demand mean \( \mu \).

**Proof.** See Appendix.

Because \( \Delta M_{ds}(\alpha) \) is a unimodal function on \((0, 1]\) for uniform demand, there are at most two threshold reliability levels (see Figure 2.2.b), which are denoted as \( \alpha_{sd} \) and \( \alpha_{ds} \), where \( \alpha_{sd} \leq \alpha_{ds} \). When \( \alpha_{sd} \leq \alpha < \alpha_{ds} \), dual-sourcing is better; otherwise single-sourcing costs less. If \( \alpha_{sd} = \alpha_{ds} \), at this point we are indifferent between the two choices. One of the threshold levels of \( K_p \), represented by \( K_{t1}^p \), is the value which satisfies \( \Delta M_{ds}(\alpha = \alpha_{sd} = \alpha_{ds}) = 0 \). Under this condition, \( \alpha_{sd} \) and \( \alpha_{ds} \) are the value which minimizes \( \Delta M_{ds} \). We use \( \alpha_{sds} \) to represent the value since \( \alpha_{sd} = \alpha_{ds} \). Therefore, \( K_{t1}^p \) can be computed as follows:

\[
K_{t1}^p = -[\alpha_{sds}c(2Q^*_d - Q^*_s) + \alpha_{sds}^2L(2Q^*_d) + 2\alpha_{sds}\alpha_{sds}L(Q^*_d) - \alpha_{sds}\alpha_{sds}L(0) - \alpha_{sds}L(Q^*_s)]. \tag{2.16}
\]

If \( \alpha_{sd} \) and \( \alpha_{ds} \) do not exist, single-sourcing is always a better choice than dual-sourcing. We consider \( \Delta M_{sn}(\alpha) \) to determine the optimal policy. The threshold level \( \alpha_{ns} \) can be obtained by solving \( \Delta M_{sn}(\alpha) = 0 \). When \( \alpha_{ns} \in (0, 1] \), the optimal policy is to order nothing if \( 0 \leq \alpha < \alpha_{ns} \), and single-source otherwise (see Figure 2.2.c). When \( \alpha_{ns} > 1, \Delta M_{sn}(\alpha) > 0 \) always holds on the interval \( \alpha \in (0, 1] \) so ordering nothing is always optimal. One of the threshold levels of \( K_p \), represented by \( K_{t2}^p \), is the value which satisfies \( \Delta M_{sn}(\alpha = 1) = 0 \) so \( K_{t2}^p \) can be computed as follows:

\[
K_{t2}^p = L(0) - cQ^*_s - L(Q^*_s). \tag{2.17}
\]
From Proposition 2.7 we can infer the relationship between the threshold reliability levels.

(a) Given that \( \alpha_{ns}, \alpha_{sd} \) and \( \alpha_{ds} \) exist, for \( K_p > 0 \), \( \alpha_{ns} < \alpha_{sd} \).

(b) If \( \alpha_{ns} \) does not exist, then \( \alpha_{sd} \) and \( \alpha_{ds} \) cannot exist, while the converse is not necessarily true.

From \( \Delta M_{sn} \) and \( \Delta M_{ds} \) we can see that fewer suppliers are preferred when \( K_p \) increases so the relation between \( K_p^{t1} \) and \( K_p^{t2} \) is \( K_p^{t1} < K_p^{t2} \).

Consequently, the optimal policy structure on \( \alpha \in (0, 1] \) is as follows.

i) When \( \alpha_{ns}, \alpha_{sd} \) and \( \alpha_{ds} \) exist (\( K_p \leq K_p^{t1} \)), order nothing if \( 0 \leq \alpha < \alpha_{ns} \), dual-source if \( \alpha_{sd} \leq \alpha < \alpha_{ds} \), and single source otherwise. (Figure 2.2.b)

ii) When only \( \alpha_{ns} \) exists (\( K_p^{t1} < K_p \leq K_p^{t2} \)), order nothing if \( 0 \leq \alpha < \alpha_{ns} \), and single-source otherwise. (Figure 2.2.c)

iii) When none of \( \alpha_{ns}, \alpha_{sd} \) and \( \alpha_{ds} \) exists (\( K_p > K_p^{t2} \)), order nothing.

The effects of demand distributions (we consider demand means \( \mu \) and standard deviations \( \sigma \)) on the optimal ordering policy can be obtained by expressing \( \Delta M_{sn} \) and \( \Delta M_{ds} \) as functions of \( \mu \) and \( \sigma \), and then taking the first-order derivatives with respect to \( \mu \) and \( \sigma \). The specific mathematical derivation process can be found in the Appendix.

**Uniform Distributions between \([l, u]\)**

- When \( \mu \) increases, \( \Delta M_{sn} \) decreases so that single-sourcing is more likely to be preferred. When \( Q^*_d \geq l \), \( \Delta M_{ds} \) decreases if \( \mu \) increases. When \( Q^*_d < l \), the demand mean has no influence on the choice between single-sourcing and dual-sourcing.

- When \( \sigma \) increases, \( \Delta M_{sn} \) increases, which implies that order nothing may become a better policy. When \( Q^*_d \geq l \), the effects of \( \sigma \) on \( \Delta M_{ds} \) depends on other parameters. When \( Q^*_d < l \), larger \( \sigma \) leads to dual-sourcing compared with single-sourcing.
Symmetric Triangular Distributions Between $[l, u]$

- The effects of $\sigma$ on $\Delta M_{ds}$ depend on other parameters such as the underage cost, overage cost, reliability level, demand mean, and sometimes $\sigma$ itself.
- The other effects are the same with those of the uniform distributions.

2.3.2.4 $K_r$ Only and General Demand Distributions

When individual cost is paid only when the order is delivered successfully, intuitively, order placement is more likely to occur. Now the cost differences become

$$\Delta M_{sn} = \alpha K_r + \alpha cQ_s^* + \alpha L(Q_s^*) - \alpha L(0)$$

and

$$\Delta M_{ds} = 2\alpha K_r + 2\alpha cQ_d^* + \alpha^2 L(2Q_d^*) + 2\alpha L(Q_d^*) + \alpha^2 L(0) - \left[\alpha K_r + \alpha cQ_s^* + \alpha L(Q_s^*) + (1-\alpha)L(0)\right]$$

Note that proposition 2.3 still holds in this scenario, thus $\Delta M_{dn}$ may not need to be considered to determine the overall optimal policy. The function $\Delta M_{sn}$ can be linearly increasing if $K_r + cQ_s^* + L(Q_s^*) - L(0) > 0$ or linearly decreasing if $K_r + cQ_s^* + L(Q_s^*) - L(0) < 0$ on $(0, 1]$. Since $\Delta M_{sn}(\alpha \to 0^+) = 0$, if $\Delta M_{sn}(\alpha = 1) > 0$, we always prefer ordering nothing to single-sourcing. If $\Delta M_{sn}(\alpha = 1) \leq 0$, then single-sourcing is preferred.

For general distributions, it can be easily verified that proposition 2.7.(c) and 2.7.(d) also hold for this fixed cost scenario. The optimal policy structure on the range $\alpha \in [0.5, 1]$ is as follows.

i) When $K_r + cQ_s^* + L(Q_s^*) - L(0) \geq 0$, order nothing.

ii) When $K_r + cQ_s^* + L(Q_s^*) - L(0) < 0$, solve $\Delta M_{ds} = 0$ for $\alpha_{ds}$. If $\alpha_{ds}$ does not exist on the interval $[0.5, 1]$, single-source.

iii) When $K_r + cQ_s^* + L(Q_s^*) - L(0) < 0$, solve $\Delta M_{ds} = 0$ for $\alpha_{ds}$. If $\alpha_{ds}$ exists on the interval $[0.5, 1]$, dual-source for $\alpha < \alpha_{ds}$ and single-source otherwise.
2.3.2.5  

K_r Only and Special Demand Distributions

For uniform and exponential demand distributions, because $\Delta M_{ds}(\alpha \to 0^+) = \Delta M_{sn}(\alpha \to 0^+) = 0$, $\Delta M_{ds}(\alpha = 1) = K_r > 0$ and $\Delta M_{ds}$ is a unimodal function, only one positive critical threshold level $\alpha_{ds}$ exists on $[0, 1]$ at most. From

$$\frac{\partial M_{ds}}{\partial \alpha} |_{\alpha \to 0^+} = \frac{2cQ_d^* + 2\alpha[2L(Q_d^*) - L(Q_d^*)] + 2(1 - \alpha)[L(Q_d^*) - L(0)] - [cQ_s^* + L(Q_s^*) - L(0)]}{\partial \Delta M_{sn}}$$

we know that if $\frac{\partial \Delta M_{sn}}{\partial \alpha} \leq 0$, which is equivalent to $\Delta M_{sn}(\alpha = 1) \leq 0$, then $\alpha_{ds}$ exists on $(0, 1]$; otherwise $M_{sn} > 0$ and $M_{ds} > 0$ so ordering nothing is always optimal. The threshold level of fixed cost $K_r^t$ is the value when $M_{sn}(\alpha = 1) = 0$ so

$$K_r^t = L(0) - cQ_s^* - L(Q_s^*)$$

(2.18)

The optimal policy structure can be summarized as follows (see Figure 2.2.d).

i) When $K_r > K_r^t$, order nothing.

ii) When $K_r \leq K_r^t$, dual-source for $\alpha < \alpha_{ds}$ and single-source otherwise.

2.3.3  

Comparison of Identical-supplier Models

Now we compare the optimal policy structures of the three models on $\alpha \in (0, 1]$ for uniform and exponential demand distributions, of which the conclusions on the range $\alpha \in [0.5, 1]$ can be applied to general demand distributions. For the model with $K_p$ only, there are three possible optimal policy structures, as described in Section 2.3.2.3. Each of them is compared with the corresponding optimal policies of the other two models, with the same values of parameters but different types of fixed order
costs. The cost difference functions have the following relations:

\[ \Delta M_{sn}(K_p = K) = \Delta M_{sn}(K_o = K) = \Delta M_{sn}(K_r = K) - \alpha K + K \]

\[ \Delta M_{ds}(K_p = K) = \Delta M_{ds}(K_o = K) + K = \Delta M_{ds}(K_r = K) - \alpha K + K \]

\[ \Delta M_{dn}(K_p = K) = \Delta M_{dn}(K_o = K) + K = \Delta M_{dn}(K_r = K) - 2\alpha K + 2K \]

**Scenario 1:** \( \alpha_{ns}, \alpha_{sd} \) and \( \alpha_{ds} \) exist for the model with \( K_p \) only. This is the case where the optimal policy for the model with \( K_p \) only changes from ordering nothing, single-sourcing, dual-sourcing, and then to single-sourcing as the suppliers’ reliability goes from perfectly unreliable to perfectly reliable, as shown in Figure 2.2.b. We use superscripts to differentiate the threshold values in different models. For example, \( \alpha_{ns}^p \) is the threshold value between ordering nothing and single-sourcing in the \( K_p \) model.

- The optimal policy of the model with \( K_o \) only is to order nothing then dual-source, as shown in Figure 2.2.a. The range of dual-sourcing is larger than the range of non-zero ordering (dual-sourcing and single-sourcing) in the \( K_p \) model, because \( \Delta M_{dn}(K_o = K, \alpha = 0) = pE(\xi) + K > 0 \), and \( \Delta M_{dn}(K_p = K, \alpha = \alpha_{nd}^o) = \Delta M_{dn}(K_o = K, \alpha = \alpha_{nd}^o) + K = K > 0 \) so that \( \alpha_{nd}^o < \alpha_{ns}^p \).

- The optimal policy of the \( K_r \) model is to dual-source then single-source (Figure 2.2.d), based on the facts that \( \Delta M_{sn}(K_r = K, \alpha \to 0^+) = \Delta M_{ds}(K_r = K, \alpha \to 0^+) = 0 \), \( \Delta M_{sn}(K_r = K, \alpha = 1) = \Delta M_{sn}(K_p = K, \alpha = 1) < 0 \), \( \Delta M_{ds}(K_r = K, \alpha = 1) = \Delta M_{ds}(K_p = K, \alpha = 1) > 0 \), and \( \Delta M_{sn} \) is linear.

**Scenario 2:** Only \( \alpha_{ns} \) exists for the model with \( K_p \) only. This is the case where the optimal policy for the \( K_p \) only model is to order nothing then single-source as suppliers change from perfectly unreliable to perfectly reliable, as shown in Figure 2.2.c. The optimal policy of the model with \( K_o \) only is to order nothing then dual-source, and the range of dual-sourcing is larger than the single-sourcing range in the \( K_p \) model for the same reason mentioned in the first case. By the same token, the
optimal policy of the $K_r$ model is to dual-source then single-source.

**Scenario 3: No threshold level exists for the $K_p$ model.** This is the case where ordering nothing is always optimal for the $K_p$ only model, which implies that $\Delta M_{sn}(K_p = K) \geq 0$. The optimal policy is also to order nothing for the other two models since $\Delta M_{sn}(K_o = K) = \Delta M_{sn}(K_p = K) \geq 0$ and $\Delta M_{sn}(K_r = K) \geq 0$ based on the facts that $\Delta M_{sn}(K_r = K, \alpha \to 0^+) = 0$, $\Delta M_{sn}(K = K_r, \alpha = 1) = \Delta M_{sn}(K_p = K, \alpha = 1) \geq 0$, and $\Delta M_{sn}$ is linear.

Overall, we tend to order from more suppliers if fixed cost is charged upon order delivery instead of order placement, or if overall fixed cost is charged instead of individual fixed cost. The optimal policy structures of models with more complex fixed cost scenarios, such as $K_o + K_p$, $K_o + K_r$, $K_p + K_r$, or even $K_o + K_r + K_p$, can be derived by comparing the functions $\Delta M_{sn}$, $\Delta M_{ds}$ and $\Delta M_{dh}$. Proposition 2.3 does not apply to the models that includes $K_o$.

### 2.3.4 Optimal Policy Structures of Non-identical-supplier Models

In this section, we consider two models where the identical-supplier assumption is relaxed. Suppliers with different item costs or different reliability levels are investigated, and the corresponding optimal policy structures are derived.

#### 2.3.4.1 Different Item Costs

In this model, the two suppliers have different item costs but identical reliability levels $\alpha$ and individual fixed costs. Without loss of generality, we assume that the item cost of supplier 1 is less than that of supplier 2, i.e., $c_1 < c_2$. Recall that in Table 2.1, the notation $\Delta M_{kq}$ stands for the the optimal cost difference between Case $k$ and Case $q$, and $\alpha_{kq}$ is the value of $\alpha$ at which the optimal policy changes from Case $k$ to Case $q$, where $k, q \in \{0, 1, 2, 3\}$ since the two suppliers are different in this section.
From

\[ \Delta M_{21} = M_2 - M_1 = K_o + K_p + \alpha K_r + \alpha c_2 Q^*_s + \alpha L(Q^*_s) + (1 - \alpha)L(0) \]

\[ - [K_o + K_p + \alpha K_r + \alpha c_1 Q^*_{s1} + \alpha L(Q^*_s) + (1 - \alpha)L(0)] \]

\[ > \alpha[c_1(Q^*_s - Q^*_s) + L(Q^*_s) - L(Q^*_s)] \]

\[ > \alpha[c_1(Q^*_s - Q^*_s) + L'(Q^*_s)(Q^*_s - Q^*_s)] \]

\[ = 0, \]

we can conclude that single-sourcing from supplier 1 is preferred compared with single-sourcing from supplier 2. Therefore, only \( \Delta M_{31} \) and \( \Delta M_{10} \) are required to obtain the optimal policy structure when \( K_o \) is not incurred. Case 0, Case 1 and Case 3 need to be compared with each other when \( K_o \) is included since Proposition 2.3 does not apply to these scenarios.

**Proposition 2.9.** The properties of the optimal quantities of dual-sourcing are as follows:

(a) \( Q^*_{d1} > Q^*_{d2} \);

(b) \( Q^*_{d2} \) is monotonically decreasing as suppliers become more reliable; \( Q^*_{d1} \) may increase as reliability improves.

**Proof.** See Appendix.

For the model with \( K_p \) only and general demand distributions, the optimal policy structure on \( \alpha \in \left[ \frac{p-c_2}{2(p-c_1)} ; 1 \right] \) can be summarized as follows (proof can be found in the Appendix):

i) When \( \Delta M_{31} (\alpha = \frac{p-c_2}{2(p-c_1)}) < 0 \), only \( \alpha_{31} \) exists. Dual-source if \( \alpha < \alpha_{31} \) and single-source from supplier 1 (Case 1) if \( \alpha \geq \alpha_{31} \);

ii) When \( \Delta M_{31} (\alpha = \frac{p-c_2}{2(p-c_1)}) \geq 0 \) and \( \Delta M_{10} (\alpha = \frac{p-c_2}{2(p-c_1)}) < 0 \), no threshold level exists. Single-source from supplier 1 (Case 1);

iii) When \( \Delta M_{10} (\alpha = \frac{p-c_2}{2(p-c_1)}) > 0 \) and \( \Delta M_{10} (\alpha = 1) < 0 \), only \( \alpha_{01} \) exists, order nothing (Case 0) if \( \alpha < \alpha_{01} \) and order from supplier 1 (Case 1) if \( \alpha \geq \alpha_{01} \);
iv) When $\Delta M_{10}(\alpha = 1) \geq 0$, no threshold level exists, order nothing.

Optimal policy structures of models with other fixed order cost scenarios can be easily obtained based on the conclusions in this section and the discussion in the previous sections.

2.3.4.2 Different Reliability Levels

Now we consider the situation in which the two suppliers have different reliability levels, but all the other parameters are the same. Without loss of generality, we assume that supplier 2 is more reliable, i.e., $\alpha_1 \leq \alpha_2$. The cost difference between sourcing from supplier 1 (Case 1) and ordering from supplier 2 (Case 2) is $\Delta M_{21} = (\alpha_2 - \alpha_1)[cQ_1^* + L(Q_1^*) - L(0)] < 0$. Therefore, single-sourcing from supplier 2 is always better than single-sourcing from supplier 1. Only $\Delta M_{32}$ and $\Delta M_{20}$ are required to obtain the optimal policy structure when $K_o$ is not incurred. Case 0, Case 2 and Case 3 need to be compared with each other when $K_o$ is incurred since Proposition 2.3 does not apply to these scenarios.

When $K_o$ is not incurred, the optimal policy structure is as follows and the proof can be found in the Appendix:

![Optimal policy structure for suppliers with different reliability levels](image)

Figure 2.3: Optimal policy structure of suppliers with different reliability levels. The parameters used are: individual fixed cost $K_{p1} = K_{p2} = 24$, item cost $c_1 = c_2 = 1$, penalty cost $p = 6$, disposal cost $h = 0.5$, demand is exponentially distributed with mean $\mu = 100$. 

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For a given value of $\alpha_2$,

i) If the optimal policy is to order nothing when $\alpha_1 = \alpha_2$, then it is also optimal for any other level of $\alpha_1 \leq \alpha_2$.

ii) If the optimal policy is to single-source from supplier 2 when $\alpha_1 = \alpha_2$, then it is also optimal for any other level of $\alpha_1 \leq \alpha_2$.

iii) If the optimal policy is to dual-source when $\alpha_1 = \alpha_2$, then there exists a threshold level $\alpha_{32}$ such that dual-sourcing is optimal for $\alpha_1 \geq \alpha_{32}$ while single-sourcing from supplier 2 is optimal for other levels of $\alpha_1$.

Figure 2.3 graphically shows these results when demand is exponentially distributed and only fixed cost $K_p$ is incurred. When $K_o$ is incurred, $\Delta M_{32}$ does not change but $\Delta M_{30}$ is increased by $K_o$, and $\Delta M_{30} = \Delta M_{32} + \Delta M_{20}$. The derivation process in the proof can be modified accordingly to determine the corresponding optimal policy structure.

### 2.4 Conclusion

In this chapter, we consider a dual-sourcing supply system with two unreliable suppliers. Various types of fixed order costs may be incurred, including overall fixed cost and individual fixed cost incurred upon order placement, and individual fixed cost incurred upon order delivery. Fixed order costs and stochastic demand result in a nonlinear expected total cost function. Four cases are investigated: ordering nothing, single-sourcing from the less reliable supplier, single-sourcing from the more reliable supplier and dual-sourcing. Within each case, the cost function is linear, so that the optimal policy can be determined by comparing the four cases. For models with certain fixed cost scenarios, methods are proposed to simplify the cost comparison process. If there is no fixed cost, we prove that dual-sourcing is always optimal as long as the two suppliers have the same item costs, but not necessarily the same reliability level. If overall fixed cost $K_o$ is incurred only, single-sourcing is never optimal and thus can be eliminated from comparing. If individual fixed cost is incurred only, we prove that is ordering nothing is no worse than the single-sourcing cases, dual-sourcing cannot be optimal.
We develop the properties of cost difference functions and determine the optimal policy structure as a function of the reliability level $\alpha$, i.e., on-time delivery probability, for the two-identical-supplier model. The optimal policies of the models with different fixed cost scenarios are summarized in Table 2.3.

### Table 2.3: Summary of the Optimal Policies for the Models with Different Fixed Cost Scenarios

<table>
<thead>
<tr>
<th>Models</th>
<th>$K_o$ only</th>
<th>$K_r$ only</th>
<th>$K_p$ only</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order nothing</td>
<td>$K_o &gt; K_o^t$</td>
<td>$K_r &gt; K_r^t$</td>
<td>$K_p &gt; K_p^{12}$</td>
</tr>
<tr>
<td></td>
<td>$K_o \leq K_o^t$ and $\alpha \leq \alpha_{nd}$</td>
<td>$K_r &lt; K_r^t$ and $\alpha &gt; \alpha_{ds}$</td>
<td>$K_p \leq K_p^{12}$ and $\alpha &lt; \alpha_{ns}$</td>
</tr>
<tr>
<td>Single-sourcing</td>
<td>never</td>
<td>$K_r \leq K_r^t$ and $\alpha &gt; \alpha_{ds}$</td>
<td>$K_p \leq K_p^{12}$ and $\alpha_{ns} &lt; \alpha \leq \alpha_{sd}$ or $\alpha &gt; \alpha_{ds}$</td>
</tr>
<tr>
<td>Dual-sourcing</td>
<td>$K_o &lt; K_o^t$ and $\alpha &gt; \alpha_{nd}$</td>
<td>$K_r \leq K_r^t$ and $\alpha \leq \alpha_{ds}$</td>
<td>$K_p \leq K_p^{12}$ and $\alpha_{sd} &lt; \alpha \leq \alpha_{ds}$</td>
</tr>
</tbody>
</table>

When only overall fixed cost $K_o$ is incurred, the optimal policy is either to order nothing or dual-source. When only individual fixed cost upon order receipt $K_r$ is incurred, the optimal policy is either to order nothing no matter how reliable the suppliers are, or to dual-source when suppliers are relatively unreliable and single-source when suppliers are relatively reliable. For general demand distributions with individual fixed cost incurred upon order placement $K_p$ only, the optimal policy as $\alpha$ goes from 0.5 to 1 has four possible structures, which are (1) ordering nothing, (2) ordering nothing then single-sourcing, (3) single-sourcing, and (4) single-sourcing then dual-sourcing, depending on the specific parameters such as penalty cost or item cost. The effects of parameters, including item cost $c$, penalty cost $p$ and disposal cost $h$, on the optimal policy structure and optimal order quantities are studied. When item cost and disposal cost increase, we also tend to order from fewer suppliers and the order quantities are decreasing as well. Penalty cost influences the optimal policy structure and order quantities in the opposite way.

The optimal policy structures for models with different reliability levels or item costs are also investigated. If there is no fixed cost, we prove that dual-sourcing is always optimal as long as the two suppliers have the same item costs, but not necessarily the same reliability level. If individual fixed cost
is incurred in this situation, for certain level of $\alpha_2$, the case $\alpha_1 = \alpha_2$ is considered first. According to the optimal policy for $\alpha_1 = \alpha_2$, the optimal policy for other levels of $\alpha_1$ can be determined. If the two suppliers have different item costs but the same reliability level, the cheaper supplier is always chosen when we single-source.

Overall, we contribute to the dual-sourcing literature by extending the work of [1] and assuming fixed costs and zero initial inventory level. By investigating the properties the cost difference functions and the threshold levels, the optimal policy structures as a function of reliability level and fixed order costs are determined.
Chapter 3

Optimal Policy Structure for Arbitrary Initial Inventory Level

3.1 Introduction

The model we consider in this chapter extends the model in Chapter 2 by allowing the initial inventory level to be arbitrary instead of zero. This model is still single-period but becomes closer to a multiple-period problem where excess inventory is carried forward and unsatisfied demand is backlogged. Individual fixed cost incurred upon order placement is included in the model. For two different suppliers, we prove that there are ten possible optimal policy structures as a function of initial inventory level, although extensive numerical experiments show that some complicated policies rarely occur. Two special models are considered. One assumes that the more expensive supplier is perfectly reliable, and the other assumes that the two suppliers are identical. The numbers of optimal policies for the two special models reduce to six and two, respectively.

Optimal policies of multiple-period problems are very difficult to determine analytically, and time-consuming to solve numerically, especially when the number of periods are relatively large. In order to test whether we can use single-period optimal policies to approximate multiple-period ones, numerical experiments are carried out to compare the costs of implementing single-period optimal policies in
three-period problems with the optimal costs of resulting from a Markov Decision Process formulation. By using Analysis of Variance (ANOVA) to analyze the numerical experiment results, the conditions under which the optimal single-period policies are good approximations for multiple-period problems are investigated. A heuristic method is proposed to reduce the approximation error.

3.2 Model formulation

In this section, the model is formulated and some preliminary results are presented. Table 3.1 shows the notation used in the model.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>Initial inventory level at the beginning of a period</td>
</tr>
<tr>
<td>$y$</td>
<td>Inventory level on hand after ordering (order-up-to level)</td>
</tr>
<tr>
<td>$K_i$</td>
<td>Individual fixed order cost incurred upon order placement from supplier $i$</td>
</tr>
<tr>
<td>$c_i$</td>
<td>Per unit ordering cost from supplier $i$</td>
</tr>
<tr>
<td>$\alpha_i$</td>
<td>Probability that the order will be delivered</td>
</tr>
<tr>
<td>$\bar{\alpha}_i$</td>
<td>$1 - \alpha_i$</td>
</tr>
<tr>
<td>$h$</td>
<td>Per unit disposal cost</td>
</tr>
<tr>
<td>$p$</td>
<td>Per unit penalty cost</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Demand</td>
</tr>
<tr>
<td>$f(\xi)$</td>
<td>Density function of demand</td>
</tr>
<tr>
<td>$F(\xi)$</td>
<td>Cumulative distribution function of demand</td>
</tr>
<tr>
<td>$u$</td>
<td>Upper limit of demand distribution</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Demand mean</td>
</tr>
<tr>
<td>$S_i$</td>
<td>Optimal order-up-to level when ordering only from supplier $i$</td>
</tr>
<tr>
<td>$s_{ij}$</td>
<td>The threshold level between Case $i$ and $j$ (Case $i$ is better than $j$ if $x &lt; s_{ij}$)</td>
</tr>
<tr>
<td>$Q_i$</td>
<td>Order quantity from supplier $i$</td>
</tr>
<tr>
<td>$Q^*_i$</td>
<td>Optimal order quantity from supplier $i$</td>
</tr>
<tr>
<td>$M(Q_1, Q_2</td>
<td>x)$</td>
</tr>
<tr>
<td>$M_i(x)$</td>
<td>Optimal cost for Case $i$ given an initial inventory level $x$</td>
</tr>
</tbody>
</table>

The sequence of the events is as follows: At the beginning of a period, the initial inventory level $x$ is reviewed and an order may be made on neither, both, or either of the suppliers. An individual fixed
cost $K_i$ is incurred if we order from supplier $i$. Orders arrive instantaneously if there are any, and the corresponding item cost is charged for each unit ordered. If supplier $i$ fails to deliver the order, the item costs are not paid though the fixed cost is still incurred, reflecting the overhead associated with generating an order. Then, stock on-hand is used to satisfy random demand during this period. Disposal cost $sh$/unit is incurred for any stock left at the end of the period after satisfying all the demand, and penalty cost $sp$/unit for the excess amount of demand over available stock during the period. At most one order can be placed from each supplier, whether the delivery is successful or not.

Without loss of generality, we assume that supplier 1 is no more reliable than supplier 2, i.e., $\alpha_1 \leq \alpha_2$, but $c_1 \leq c_2$ does not necessarily hold. Given that the inventory level after order delivery is $y$, the expected end-of-period cost is as follows:

$$L(y) = \begin{cases} 
\int_0^y h(y - \xi)f(\xi)d\xi + \int_{y}^{\infty} p(\xi - y)f(\xi)d\xi, & y \geq 0 \\
\int_{\infty}^{\infty} p(\xi - y)f(\xi)d\xi, & y < 0 
\end{cases}$$

(3.1)

**Proposition 3.1.** The expected end-of-period cost $L(y)$ is convex in $y$.

**Proof.** See Appendix.

The objective is to minimize the single-period cost function $M(Q_1, Q_2|x)$ given by

$$M(Q_1, Q_2|x) = K_1(Q_1) + K_2(Q_2) + \alpha_1 c_1 Q_1 + \alpha_2 c_2 Q_2 + \alpha_1 \alpha_2 L(x + Q_1 + Q_2) + \alpha_1 \bar{\alpha}_2 L(x + Q_1) + \bar{\alpha}_1 \alpha_2 L(x + Q_2) + \bar{\alpha}_1 \bar{\alpha}_2 L(x)$$

(3.2)

where $K_i(Q_i) = \begin{cases} 
K_i, & \text{if } Q_i > 0 \\
0, & \text{if } Q_i = 0.
\end{cases}$

The first two terms represent individual fixed order costs incurred upon order placement. The next two terms are the expected item costs. The last four terms represent the expected disposal and penalty costs of possible outcomes of supplier deliveries: orders from both suppliers are delivered, only the order from supplier 1 is delivered, only the order from supplier 2 is delivered, and nothing is delivered.
Obviously, \( M(Q_1, Q_2|x) \) is not a convex function because of the fixed costs. However, we can consider the following four cases: ordering nothing, single-sourcing from supplier 1, single-sourcing from supplier 2, and dual-sourcing. Within each case, the total fixed cost incurred is a constant and does not change as the order policy changes. The cost function of each case is convex since it is a summation of constant values and convex functions (Proposition 3.1). By comparing the optimal costs of the four cases, we can obtain the overall optimal cost and order policy. The overall optimal cost function can be written as:

\[
M^*(x) = \min_{Q_1, Q_2} M(Q_1, Q_2|x) \\
= \min\{M(0, 0|x), \min_{Q_1 > 0} M(Q_1, 0|x), \min_{Q_2 > 0} M(0, Q_2|x), \min_{Q_1 > 0, Q_2 > 0} M(Q_1, Q_2|x)\} \\
= \min\{M_0(x), M_1(x), M_2(x), M_3(x)\}
\]

(3.3)

Now we consider the four cases separately and show the optimal cost function and order policy for each case.

**Case 0:** \( Q_1 = Q_2 = 0 \). It is easy to see that the expected cost is

\[
M_0(x) = M(0, 0|x) = L(x)
\]

(3.4)

**Case 1:** \( Q_1 > 0, Q_2 = 0 \).

\[
M_1(x) = \min_{Q_1 > 0} M(Q_1, 0|x) = \min_{Q_1 > 0} \{K_1 + \alpha_1 c_1 Q_1 + \alpha_1 L(x + Q_1) + \bar{\alpha}_1 L(x)\}
\]

(3.5)

According to the first-order optimality conditions:

\[
\frac{\partial M}{\partial Q_1} = \alpha_1 [c_1 + L'(x + Q_1)] = 0
\]

(3.6)

Solving for \( Q_1 \) we can get the minimizer \( Q_1^* = S_1 - x \), and \( S_1 \) can be obtained by the following
equation:
\[ S_1 = F^{-1} \left( \frac{p - c_1}{h + p} \right) \] (3.7)

It is optimal to order up to \( S_1 \) when we choose to order from supplier 1 only. The expected cost is

\[ M_1(x) = \min_{Q_1 > 0} M(Q_1, 0|x) = M(S_1 - x, 0, x) = K_1 + \alpha_1 c_1 (S_1 - x) + \alpha_1 L(S_1) + \bar{\alpha}_1 L(x) \] (3.8)

**Case 2:** \( Q_1 = 0, Q_2 > 0 \).

\[ M_2(x) = \min_{Q_2 > 0} M(0, Q_2|x) = \min_{Q_2 > 0} \left\{ K_2 + \alpha_2 c_2 Q_2 + \alpha_2 L(x + Q_2) + \bar{\alpha}_2 L(x) \right\} \] (3.9)

Similar to Case 1, it is optimal to order up to \( S_2 \) when we choose to order from supplier 2 only, where

\[ S_2 = F^{-1} \left( \frac{p - c_2}{h + p} \right) \] (3.10)

Correspondingly, the expected cost is

\[ M_2(x) = \min_{Q_2 > 0} M(0, Q_2|x) = M(0, S_2 - x|x) = K_2 + \alpha_2 c_2 (S_2 - x) + \alpha_2 L(S_2) + \bar{\alpha}_2 L(x) \] (3.11)

**Case 3:** \( Q_1 > 0, Q_2 > 0 \).

\[ M_3(x) = \min_{Q_1 > 0, Q_2 > 0} M(Q_1, Q_2|x) = \min_{Q_1 > 0, Q_2 > 0} \left\{ K_1 + K_2 + \alpha_1 c_1 Q_1 + \alpha_2 c_2 Q_2 + \alpha_1 \alpha_2 L(x + Q_1 + Q_2) + \alpha_1 \bar{\alpha}_2 L(x + Q_1) + \bar{\alpha}_1 \alpha_2 L(x + Q_2) + \bar{\alpha}_1 \bar{\alpha}_2 L(x) \right\} \] (3.12)

According to first order optimality conditions:

\[ \frac{\partial M}{\partial Q_1} = \alpha_1 \left[ c_1 + \bar{\alpha}_2 L'(x + Q_1) + \alpha_2 L'(x + Q_1 + Q_2) \right] = 0 \] (3.13)
\[ \frac{\partial M}{\partial Q_2} = \alpha_2 [c_2 + \bar{\alpha}_1 L'(x + Q_2) + \alpha_1 L'(x + Q_1 + Q_2^\dagger)] = 0 \quad (3.14) \]

Then the optimal solution \((Q_1^\star, Q_2^\star)\) can be computed using the algorithm in [1]. Substituting \((Q_1^\star, Q_2^\star)\) into (3.12) we can get the expected optimal cost of Case 3.

Proposition 2.1, 2.2 and 2.3 still hold for the model with arbitrary initial inventory level (the corresponding proofs can be easily modified), which make the cost comparison process more efficient computationally.

### 3.3 Optimal Policy Structure

In this section, we first derive the optimal policy for \(x = -\infty\). Then according to the properties of cost difference functions and the relations between threshold levels, we can determine the optimal policy as a function of initial inventory level.

#### 3.3.1 The optimal policy for \(x = -\infty\)

The optimal policies of single-sourcing cases (Case 1 and Case 2) are order-up-to policies as shown in the previous section. This implies that for every unit decrease in on-hand inventory, the order quantity from the supplier increases by one, or to put it in another way, the extra ordering placed from the supplier is a constant \((\Delta Q = 1)\). However, this is not the case for dual-sourcing. When dual-sourcing is adopted, Anupindi and Akella [1] prove that for a unit decrease in on-hand-inventory, the order quantity from either supplier increases by no more than one, but total order quantity always increases by no less than one. According to Theorem 3.2 in [1], when \(x\) is relatively low, taking \(x = -\infty\) as an extreme case, optimal order quantities of dual-sourcing have a regular pattern: when penalty cost is relatively low, any extra ordering is done only from the supplier who has a higher marginal benefit; the order quantity from the supplier with lower marginal benefit becomes a constant; when penalty cost is relatively high, the extra ordering is placed from both suppliers. Although fixed cost is not considered in their work, this theorem still applies to our model for the reason that extra ordering is the first-order derivative information and does not involve fixed cost.
Here we restate Theorem 3.2 in [1] as a reminder. Since Anupindi and Akella [1] do not provide the proof, we prove it and the proof can be found in the Appendix.

**Proposition 3.2. (Theorem 3.2 [1])** Assume \( c_1 \neq c_2 \) and the support of the demand distribution is bounded. Then one of the following three holds:

(a). If \( p > p_{\text{critical}} \), then

\[
\lim_{x \to -\infty} \frac{\partial Q^*_1}{\partial x} = -1, \quad \text{and} \quad \lim_{x \to -\infty} \frac{\partial Q^*_2}{\partial x} = -1,
\]

where

\[
p_{\text{critical}} = \begin{cases} 
\frac{\alpha_1 h + c_1}{1 - \alpha_2}, & \text{if } \alpha_1(p - c_1) < \alpha_2(p - c_2). \\
\frac{\alpha_1 h + c_2}{1 - \alpha_1}, & \text{otherwise}.
\end{cases}
\] (3.15)

Otherwise (i.e., \( p < p_{\text{critical}} \)):

(b). If \( \alpha_1(p - c_1) > \alpha_2(p - c_2) \), then

\[
\lim_{x \to -\infty} \frac{\partial Q^*_1}{\partial x} = -1, \quad \text{and} \quad \lim_{x \to -\infty} \frac{\partial Q^*_2}{\partial x} = 0,
\]

(c). If \( \alpha_1(p - c_1) < \alpha_2(p - c_2) \), then

\[
\lim_{x \to -\infty} \frac{\partial Q^*_1}{\partial x} = 0, \quad \text{and} \quad \lim_{x \to -\infty} \frac{\partial Q^*_2}{\partial x} = -1.
\]

**Proof.** See Appendix. \( \square \)

Then we conclude that optimal quantities of dual-sourcing can be computed from the following formulae when \( x \) is relatively low.

**Proposition 3.3.** (a). If an extra order is placed from both suppliers,

\[
Q^*_1 = F^{-1}[\frac{p - (\alpha_2 h + c_1)/\bar{\alpha}_2}{h + p}] - x
\] (3.16)
\[ Q_2^* = F^{-1}\left[ \frac{p - (\alpha_1 h + c_2)/\tilde{\alpha}_1}{h + p} \right] - x. \]  
(3.17)

(b). If an extra order is placed from supplier 1 only,

\[ Q_1^* = F^{-1}\left[ \frac{\alpha_1(p - c_1) - \alpha_2(p - c_2)}{\alpha_2\alpha_1(h + p)} \right] - x \]  
(3.18)

\[ Q_2^* = F^{-1}\left[ \frac{p - c_2}{\alpha_1(h + p)} \right] - F^{-1}\left[ \frac{\alpha_1(p - c_1) - \alpha_2(p - c_2)}{\alpha_2\alpha_1(h + p)} \right]. \]  
(3.19)

(c). If an extra order is placed from supplier 2 only,

\[ Q_1^* = F^{-1}\left[ \frac{p - c_1}{\alpha_2(h + p)} \right] - F^{-1}\left[ \frac{\alpha_2(p - c_2) - \alpha_1(p - c_1)}{\alpha_1\alpha_2(h + p)} \right] \]  
(3.20)

\[ Q_2^* = F^{-1}\left[ \frac{\alpha_2(p - c_2) - \alpha_1(p - c_1)}{\alpha_1\alpha_2(h + p)} \right] - x. \]  
(3.21)

Obviously, when the extra order is placed from both suppliers, the optimal policy is dual-sourcing. For (b) and (c), costs need to be compared to determine the optimal policy.

When an extra order is placed from supplier 1 only, the possible optimal policies are single-sourcing from supplier 1 and dual-sourcing. Therefore, we need to compare Case 1 and Case 3 to determine the optimal policy.

\[ \Delta M_{31}(-\infty) = M_3(-\infty) - M_1(-\infty) \]

\[ = K_1 + K_2 + \alpha_1 c_1 Q_1^* + \alpha_2 c_2 Q_2^* + \alpha_1 \alpha_2 L(x + Q_1^* + Q_2^*) + \alpha_1 \tilde{\alpha}_2 L(x + Q_1^*) \]

\[ + \tilde{\alpha}_1 \alpha_2 L(x + Q_2^*) + \tilde{\alpha}_1 \tilde{\alpha}_2 L(x) - [K_1 + \alpha_1 c_1 (S_1 - x) + \alpha_1 L(S_1) + \tilde{\alpha}_1 L(x)] \]

\[ = K_2 + \alpha_1 c_1 (Q_1^* + x - S_1) + \alpha_2 c_2 Q_2^* + \alpha_1 \alpha_2 L(x + Q_1^* + Q_2^*) + \alpha_1 \tilde{\alpha}_2 L(x + Q_1^*) \]

\[ - \tilde{\alpha}_1 \alpha_2 p(x + Q_2^*) + \tilde{\alpha}_1 \alpha_2 px - \alpha_1 L(S_1) \]

\[ = K_2 + \alpha_1 c_1 (Q_1^* + x - S_1) + \alpha_2 c_2 Q_2^* + \alpha_1 \alpha_2 L(x + Q_1^* + Q_2^*) + \alpha_1 \tilde{\alpha}_2 L(x + Q_1^*) \]

\[ - \tilde{\alpha}_1 \alpha_2 p Q_2^* - \alpha_1 L(S_1) \]  
(3.22)
When (3.22) > 0, the optimal policy is to order from supplier 1 only. Otherwise, dual-sourcing is optimal.

By the same token, we need to compare the costs of dual-sourcing and single-sourcing from supplier 2 when an extra ordering is placed from supplier 2.

\[
\Delta M_{32}(-\infty) = M_3(-\infty) - M_2(-\infty) = K_1 + \alpha_2 c_2(Q_2^* + x - S_2) + \alpha_1 c_1 Q_1^* + \alpha_1 \alpha_2 L(x + Q_1^* + Q_2^*) - \alpha_1 \bar{\alpha}_2 p Q_2^* - \alpha_2 L(S_2) \quad (3.23)
\]

If (3.23) > 0, the optimal policy is to order from supplier 2 only. Otherwise, dual-sourcing is optimal.

### 3.3.2 Properties of Optimal Cost and Cost Differences

Now we consider the optimal policy structure when the initial inventory level is arbitrary. By comparing the optimal costs of the four cases, we can determine the optimal case.

1. **Case 0 vs. Case 1.** This is an \((s_{10}, S_1)\) policy: when the initial inventory level \(x < s_{10}\), order up to \(S_1\) from supplier 1; otherwise do not order. \(S_1\) can be obtained from (3.7), and the threshold level \(s_{10}\) is the largest root of the following equation:

\[
L(s_{10}) = K_1/\alpha_1 + c_1(S_1 - s_{10}) + L(S_1) \quad (3.24)
\]

2. **Case 0 vs. Case 2.** By the same token, this is an \((s_{20}, S_2)\) policy: when the initial inventory level \(x < s_{20}\), order up to \(S_2\) from supplier 2; otherwise do not order. \(S_2\) can be obtained from (3.10), and the threshold level \(s_{20}\) is the largest root of the following equation:

\[
L(s_{20}) = K_2/\alpha_2 + c_2(S_2 - s_{20}) + L(S_2) \quad (3.25)
\]
(3). Case 0 vs. Case 3. The threshold level \( s_{30} \) is the largest root of the following equation:

\[
L(s_{30}) = K_1 + K_2 + \alpha_1 c_1 Q^*_1 + \alpha_2 c_2 Q^*_2 + \alpha_1 \alpha_2 L(s_{30} + Q^*_1 + Q^*_2) \\
+ \alpha_1 \bar{\alpha}_2 L(s_{30} + Q^*_1) + \bar{\alpha}_1 \alpha_2 L(s_{30} + Q^*_2) + \bar{\alpha}_1 \bar{\alpha}_2 L(s_{30})
\] (3.26)

If \( x < s_{30} \), dual-sourcing is better than ordering nothing. Otherwise ordering nothing is preferred.

Proposition 3.4 and 3.5 show how the initial inventory level impacts the optimal costs of Case 1-3 and their cost differences. We only need to consider the range \([−∞, s_{i0}]\) since case 0 is optimal on \((s_{i0}, +∞)\) according to Proposition 2.3.

**Proposition 3.4.** The optimal costs of Case 1, 2 and 3 are monotonically increasing as \( x \) decreases.

**Proof.** See Appendix.

**Proposition 3.5.** \( \Delta M_{31} \) and \( \Delta M_{32} \) are monotonically increasing as \( x \) increases. If \( \alpha_1 (p - c_1) \geq \alpha_2 (p - c_2) \), \( \Delta M_{12} \) is also monotonically increasing as \( x \) increases. Otherwise, \( \Delta M_{12} \) is a convex function and \( \Delta M_{12} > 0 \) for \( x = −∞ \).

**Proof.** See Appendix.

Therefore, there is at most one point at which \( \Delta M_{31}(x) = 0 \) and \( \Delta M_{32}(x) = 0 \). Denote the threshold levels as \( s_{31} \) and \( s_{32} \), respectively. If \( \alpha_1 (p - c_1) \geq \alpha_2 (p - c_2) \), i.e., ordering from supplier 1 has larger marginal benefit, there is at most one point at which \( \Delta M_{12}(x) = 0 \), denote it as \( s_{12} \). If \( \alpha_1 (p - c_1) < \alpha_2 (p - c_2) \), i.e., ordering from supplier 2 has larger marginal benefit, a minimizer of \( \Delta M_{12}(x) \); denoted as \( x^*_{12} \), can be obtained from \( \frac{\partial \Delta M_{12}}{\partial x} = 0 \):

\[
x^*_{12} = F^{-1} \left[ p + (\alpha_2 c_2 - \alpha_1 c_1)/(\alpha_1 - \alpha_2) \right] / (p + h)
\] (3.27)

Because \( \Delta M_{12} \) is a convex function, there are at most two points at which \( \Delta M_{12} = 0 \). Let them be \( s_{21} \) and \( s_{12} \), and \( s_{21} < s_{12} \).

The relations between threshold levels can help determine the optimal policy structure.
Proposition 3.6. (a) \( s_{30} \) is smaller than at least one of the other two levels \( s_{10} \) and \( s_{20} \), i.e., at least one of the following inequalities is true: \( s_{10} > s_{30} \), \( s_{20} > s_{30} \).

(b) \( s_{31} < s_{20} \).

(c) \( s_{32} < s_{10} \).

From (a) We can infer that if \( s_{10} > s_{20} \), then \( s_{10} > s_{30} \); if \( s_{20} \geq s_{10} \), then \( s_{20} > s_{30} \).

Proof. See Appendix.

3.3.3 When Case 1 is optimal for \( x = -\infty \)

We know that when \( p < p_{\text{critical}} \), \( \alpha_1 (p - c_1) > \alpha_2 (p - c_2) \) and (3.22) > 0, ordering from supplier 1 only is optimal for \( x = -\infty \).

Proposition 3.7. When ordering from supplier 1 only is optimal for \( x = -\infty \), there are two possible optimal ordering policies:

1. If \( s_{20} < s_{10} \), order up to \( S_1 \) from supplier 1 when \( x < s_{10} \), otherwise do not order.
2. If \( s_{10} \leq s_{20} \), order up to \( S_1 \) from supplier 1 when \( x \leq s_{12} \); order up to \( S_2 \) from supplier 2 when \( s_{12} < x < s_{20} \); otherwise do not order.

Proof. Because \( \Delta M_{31}(x) \) increases as \( x \) increases, and \( \Delta M_{31}(-\infty) > 0 \), Case 1 is always better than Case 3. Therefore dual-sourcing is never optimal.

When \( s_{20} < s_{10} \), then for \( x \in [s_{20}, s_{10}) \), Case 1 is better than Case 0, while Case 0 is better than Case 2. This makes Case 1 optimal for \( x \in [s_{20}, s_{10}) \). From Proposition 3.5, we know that Case 3 is never optimal. The optimal policy is to order up to \( S_1 \) from supplier 1 when \( x < s_{10} \), otherwise do not order (see Figure 3.1.a).

When \( s_{20} \geq s_{10} \), then for \( x \in [s_{10}, s_{20}) \), Case 2 is optimal. So \( s_{12} \) exists on the range \( x \in (-\infty, s_{20}) \). The optimal policy is to order up to \( S_1 \) from supplier 1 if \( x \leq s_{12} \), order up to \( S_2 \) from supplier 2 if \( s_{12} < x < s_{20} \) and do nothing otherwise.

\( \square \)
(a) Optimal Policy Structure Case 1 - Case 0: beta distribution $\beta(2, 2) \times 30$, $\alpha_1 = 0.95$, $\alpha_2 = 0.98$, $c_1 = 1$, $c_2 = 1.5$, $K_1 = 10$, $K_2 = 20$, $p = 2$, $h = 0.3$

(b) Optimal Policy Structure Case 2 - Case 0: beta distribution $\beta(2, 2) \times 30$, $\alpha_1 = 0.5$, $\alpha_2 = 0.95$, $c_1 = 1$, $c_2 = 1.2$, $K_1 = 10$, $K_2 = 5$, $p = 2$, $h = 0.5$

(c) Optimal Policy Structure Case 2 - Case 1 - Case 0: beta distribution $\beta(2, 2) \times 30$, $\alpha_1 = 0.9$, $\alpha_2 = 0.95$, $c_1 = 1$, $c_2 = 1.1$, $K_1 = 0$, $K_2 = 5$, $p = 2$, $h = 0.3$

(d) Optimal Policy Structure Case 2 - Case 1 - Case 2 - Case 0: beta distribution $\beta(2, 2) \times 30$, $\alpha_1 = 0.95$, $\alpha_2 = 0.98$, $c_1 = 1$, $c_2 = 1.5$, $K_1 = 10$, $K_2 = 0$, $p = 4$, $h = 0.3$

(e) Optimal Policy Structure Case 3 - Case 1 - Case 0: beta distribution $\beta(2, 2) \times 30$, $\alpha_1 = 0.5$, $\alpha_2 = 0.85$, $c_1 = 1$, $c_2 = 1.2$, $K_1 = 0$, $K_2 = 5$, $p = 2$, $h = 0.3$

(f) Optimal Policy Structure Case 3 - Case 2 - Case 0: beta distribution $\beta(2, 2) \times 30$, $\alpha_1 = 0.7$, $\alpha_2 = 0.9$, $c_1 = 1$, $c_2 = 1.2$, $K_1 = 10$, $K_2 = 5$, $p = 3$, $h = 0.3$

Figure 3.1: Optimal Policy Structure Examples
3.3.4 When Case 2 is optimal for \( x = -\infty \)

We know that when \( p < p_{\text{critical}} \), \( \alpha_1(p - c_1) < \alpha_2(p - c_2) \) and (3.23) > 0, ordering from supplier 2 only is optimal for \( x = -\infty \).

**Proposition 3.8.** When ordering from supplier 2 only is optimal for \( x = -\infty \), the optimal policy is as follows.

1. If \( s_{20} < s_{10} \), order up to \( S_2 \) from supplier 2 when \( x \leq s_{21} \); order up to \( S_1 \) from supplier 1 when \( s_{21} < x < s_{10} \); otherwise do nothing.

2. If \( s_{10} \leq s_{20} \), and \( x^*_{12} > s_{20} \) or \( \Delta M_{12}(x^*_{12}) \geq 0 \), order up to \( S_2 \) from supplier 2 when \( x \leq s_{20} \); order nothing otherwise.

3. If \( s_{10} \leq s_{20} \), and \( \Delta M_{12}(x^*_{12}) < 0 \), order up to \( S_2 \) from supplier 2 when \( x \leq s_{21} \); order up to \( S_1 \) from supplier 1 when \( s_{21} < x \leq s_{12} \); order up to \( S_2 \) from supplier 2 when \( s_{12} < x \leq s_{20} \); order nothing otherwise.

**Proof.** Because \( \Delta M_{32}(x) \) increases as \( x \) increases, and \( \Delta M_{32}(-\infty) > 0 \), ordering from supplier 2 is always better than ordering from both suppliers. Therefore Case 3 is never optimal.

When \( s_{20} < s_{10} \), then for \( x \in [s_{20}, s_{10}) \), Case 1 is optimal. Because \( \Delta M_{12}(s_{20}) < 0 \), \( s_{21} \) exists on the range \( x \in (-\infty, s_{20}) \). The optimal policy is to order up to \( S_2 \) from supplier 2 if \( x \leq s_{21} \); order up to \( S_1 \) from supplier 1 if \( s_{21} < x < s_{10} \); otherwise order nothing (see Figure 3.1.c).

When \( s_{10} \leq s_{20} \), then for \( x \in [s_{10}, s_{20}) \), ordering from supplier 2 is optimal. If the minimizer \( x^*_{12} > s_{20} \), combined with the fact that \( \Delta M_{12}(s_{20}) > 0 \), we know that \( \Delta M_{12}(x) > 0 \) on \( x \in (-\infty, s_{20}) \). As a result, the optimal policy is to order up to \( S_2 \) from supplier 2 when \( x \leq s_{20} \); order nothing otherwise (see Figure 3.1.b).

If the minimizer is \( x^*_{12} \leq s_{20} \), then we need to check if Case 2 costs less than Case 3 at \( x^*_{12} \). If \( \Delta M_{12}(x^*_{12}) \geq 0 \), the optimal policy is the same as stated above. If \( \Delta M_{12}(x^*_{12}) < 0 \), then \( s_{21} \) and \( s_{12} \) exist. The optimal policy is to order up to \( S_2 \) from supplier 2 when \( x \leq s_{21} \); order up to \( S_1 \) from supplier 1 when \( s_{21} < x \leq s_{12} \); order up to \( S_2 \) from supplier 2 when \( s_{12} < x \leq s_{20} \); order nothing otherwise (see Figure 3.1.d).
3.3.5 When Case 3 is optimal for $x = -\infty$

We know that in the following scenarios, dual-sourcing is optimal for $x = -\infty$:

1. $p \geq p_{\text{critical}}$.
2. $p < p_{\text{critical}}$, $\alpha_1(p - c_1) > \alpha_2(p - c_2)$ and (3.22) $< 0$.
3. $p < p_{\text{critical}}$, $\alpha_1(p - c_1) < \alpha_2(p - c_2)$ and (3.23) $< 0$.

Proposition 3.9. When dual-sourcing is optimal for $x = -\infty$, the optimal policy is as follows.

1. If $s_{20} < s_{10}$ and $\alpha_1(p - c_1) \geq \alpha_2(p - c_2)$, or if $s_{20} < s_{10}$, $\alpha_1(p - c_1) < \alpha_2(p - c_2)$ and $s_{32} < s_{31}$, dual-source when $x \leq s_{31}$; single-source from supplier 1 up to $S_1$ when $s_{31} < x \leq s_{10}$; order nothing otherwise.
2. If $s_{20} < s_{10}$, $\alpha_1(p - c_1) < \alpha_2(p - c_2)$ and $s_{31} \leq s_{32}$, dual-source when $x \leq s_{32}$; order up to $S_2$ from supplier 2 when $s_{32} < x \leq s_{21}$; order up to $S_1$ from supplier 1 when $s_{21} < x \leq s_{10}$; order nothing otherwise.
3. If $s_{20} \geq s_{10}$, $\alpha_1(p - c_1) \geq \alpha_2(p - c_2)$ and $s_{12} < s_{32}$, or if $s_{20} \geq s_{10}$, $\alpha_1(p - c_1) < \alpha_2(p - c_2)$, and $x^*_{12} > s_{20}$, or if $s_{20} \geq s_{10}$, $\alpha_1(p - c_1) < \alpha_2(p - c_2)$, $x^*_{12} \leq s_{20}$, and $\Delta M_{12}(x^*_{12}) \geq 0$, or if $s_{20} \geq s_{10}$, $\alpha_1(p - c_1) < \alpha_2(p - c_2)$, $x^*_{12} \leq s_{20}$, $\Delta M_{12}(x^*_{12}) < 0$, and $s_{32} > s_{12}$, dual-source when $x \leq s_{32}$, order up to $S_2$ from supplier 2 when $s_{32} < x \leq s_{20}$, order nothing otherwise.
4. If $s_{20} \geq s_{10}$, $\alpha_1(p - c_1) \geq \alpha_2(p - c_2)$ and $s_{32} \leq s_{12}$, or if $s_{20} \geq s_{10}$, $\alpha_1(p - c_1) < \alpha_2(p - c_2)$, $x^*_{12} \leq s_{20}$, $\Delta M_{12}(x^*_{12}) < 0$, and $s_{32} \in (s_{21}, s_{12})$, dual-source when $x \leq s_{31}$, order up to $S_1$ from supplier 1 when $s_{31} < x \leq s_{12}$, ordering up to $S_2$ from supplier 2 when $s_{12} < x \leq s_{20}$, otherwise order nothing.
5. If $s_{20} \geq s_{10}$, $\alpha_1(p - c_1) < \alpha_2(p - c_2)$, $x^*_{12} \leq s_{20}$, $\Delta M_{12}(x^*_{12}) < 0$, and $s_{32} < s_{21}$, dual-source when $x \leq s_{32}$, order up to $S_2$ from supplier 2 when $s_{32} < x \leq s_{21}$, order up to $S_1$ from supplier 1 when $s_{21} < x \leq s_{12}$, order up to $S_2$ from supplier 2 when $s_{12} < x \leq s_{20}$, otherwise order nothing.

Proof. Scenario (a): $s_{20} < s_{10}$

(i) $\alpha_1(p - c_1) \geq \alpha_2(p - c_2)$

If dual-sourcing is not considered, from Section 3.3.3 we know that the optimal policy is: for $s_{20} < s_{10}$,
order up to $S_1$ from supplier 1 when $x < s_{10}$, otherwise do not order. Now it is known that dual-sourcing is optimal for $x = -\infty$, Case 3 should be compared with Case 1 to derive the policy structure.

According to Proposition 3.6, $s_{30} < s_{10}$, so that single-sourcing from supplier 1 is optimal on the range $[s_{30}, s_{10})$. The optimal policy is to order $Q_1^*$ from supplier 1 and $Q_2^*$ from supplier 2 when $x \leq s_{31}$; single-source from supplier 1 up to $S_1$ when $s_{31} < x \leq s_{10}$; do nothing otherwise (see Figure 3.1.c).

(ii) $\alpha_1(p - c_1) < \alpha_2(p - c_2)$

If dual-sourcing is not under consideration, from Section 3.3.4 we know that the optimal policy is to order up to $S_2$ from supplier 2 when $x \leq s_{21}$; order up to $S_1$ from supplier 1 when $s_{21} < x < s_{10}$; otherwise order nothing.

Now we show the optimal policy structure when Case 3 is included.

If $s_{32} < s_{31}$, the optimal policy is the same as stated in (i).

If $s_{31} \leq s_{32}$, the optimal policy is to dual-source when $x \leq s_{32}$; order up to $S_2$ from supplier 2 when $s_{32} < x \leq s_{21}$; order up to $S_1$ from supplier 1 when $s_{21} < x \leq s_{10}$; order nothing otherwise.

Scenario (b): $s_{20} \geq s_{10}$

(i) $\alpha_1(p - c_1) \geq \alpha_2(p - c_2)$

If dual-sourcing is not considered, from Section 3.3.3 we know that the optimal policy is to order up to $S_1$ from supplier 1 when $x \leq s_{12}$; order up to $S_2$ from supplier 2 when $s_{12} < x < s_{20}$; otherwise order nothing. Now it is known that dual-sourcing is optimal for $x = -\infty$, Case 3 should be compared with Case 1 and Case 2 to determine the policy structure.

If $s_{12} < s_{32}$, the optimal policy is to dual-source when $x \leq s_{32}$, order up to $S_2$ from supplier 2 when $s_{32} < x \leq s_{20}$, do nothing otherwise (see Figure 3.1.f).

If $s_{32} \leq s_{12}$, the optimal policy is to dual-source when $x \leq s_{31}$, order up to $S_1$ from supplier 1 when $s_{31} < x \leq s_{12}$, ordering up to $S_2$ from supplier 2 when $s_{12} < x \leq s_{20}$, otherwise order nothing.

(ii) $\alpha_1(p - c_1) < \alpha_2(p - c_2)$

If the minimizer $x_{12}^* > s_{20}$, the optimal policy is to dual-source when $x \leq s_{32}$, order up to $S_2$ from supplier 2 when $x \leq s_{20}$, do nothing otherwise.

If the minimizer is $x_{12}^* \leq s_{20}$, and $\Delta M_{12}(x_{12}^*) \geq 0$, the optimal policy is the same as stated above.

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If the minimizer is $x_{12}^* \leq s_{20}$, and $\Delta M_{12}(x_{12}^*) < 0$, then $s_{21}$ and $s_{12}$ exist.

- if $s_{32} > s_{12}$
  
  Dual-source when $x \leq s_{32}$, order up to $S_2$ from supplier 2 when $s_{32} < x \leq s_{20}$, order nothing otherwise.

- if $s_{32} \in (s_{21}, s_{12})$
  
  Dual-source when $x \leq s_{31}$, order up to $S_1$ from supplier 1 when $s_{31} < x \leq s_{12}$, order up to $S_2$ from supplier 2 when $s_{12} < x \leq s_{20}$, otherwise order nothing.

- if $s_{32} < s_{21}$
  
  Dual-source when $x \leq s_{32}$, order up to $S_2$ from supplier 2 when $s_{32} < x \leq s_{21}$, order up to $S_1$ from supplier 1 when $s_{12} < x \leq s_{21}$, order up to $S_2$ from supplier 2 when $s_{12} < x \leq s_{20}$, otherwise order nothing.

Figure 3.2 and 3.3 summarize the ten optimal policy structures.

### 3.4 Two Special Models

#### 3.4.1 Supplier 2 is Perfectly Reliable

Here we consider a special case where supplier 2 is perfectly reliable, i.e., $\alpha_2 = 1$.

Substituting $\alpha_2 = 1$ into (3.13), we can conclude that the order-up-to level for dual-sourcing is also $S_1$, i.e., $x + Q_1^* + Q_2^* = S_1$. 
The conditions under which the optimal policy is single-sourcing when $x = -\infty$

$p < p_{\text{critical}}$

$\alpha_1(p - c_1) > \alpha_2(p - c_2)$

& $\Delta M_{13}(-\infty) < 0$

$\alpha_1(p - c_1) < \alpha_2(p - c_2)$

& $\Delta M_{13}(-\infty) < 0$

See Figure 3.3

Figure 3.2: Optimal Policy Structures When Single-sourcing is Optimal for $x = -\infty$
The conditions under which the optimal policy is dual-sourcing when \( x = -\infty \):

- \( p < p_{\text{critical}} \):
  - \( \alpha_1(p - c_1) > \alpha_2(p - c_2) \) and \( \Delta M_{31}(-\infty) < 0 \),
  - \( \alpha_1(p - c_1) < \alpha_2(p - c_2) \) and \( \Delta M_{32}(-\infty) < 0 \).

- \( p \geq p_{\text{critical}} \):
  
See Figure 3.2

The conditions for comparing reorder points are as follows:

- \( \alpha(p - c_1) < \alpha(p - c_2) \)
- \( \alpha(p - c_1) \geq \alpha(p - c_2) \)
- \( \alpha(p - c_1) \geq \alpha(p - c_2) \)
- \( \alpha(p - c_1) < \alpha(p - c_2) \)

Figure 3.3: Optimal Policy Structures When Dual-sourcing is Optimal for \( x = -\infty \)
3.4.1.1 Higher Marginal Benefit for Supplier 1 When $x = -\infty$

If $\alpha_1(p - c_1) > p - c_2$, single-sourcing from supplier 2 cannot be optimal for $x = -\infty$. In addition,

$$
\Delta M_{31}(-\infty) = M_3(-\infty) - M_1(-\infty) \\
= K_1 + K_2 + \alpha_1 c_1 Q_1^* + \alpha_2 c_2 Q_2^* + \alpha_1 \alpha_2 L(x + Q_1^* + Q_2^*) + \alpha_1 \tilde{\alpha}_2 L(x + Q_1^*) \\
+ \alpha_1 \alpha_2 L(x + Q_2^*) + \tilde{\alpha}_1 \tilde{\alpha}_2 L(x) - [K_1 + \alpha_1 c_1 (S_1 - x) + \alpha_1 L(S_1) + \tilde{\alpha}_1 L(x)] \\
= K_2 + \alpha_1 c_1 (Q_1^* + x - S_1) + c_2 Q_2^* + \alpha_1 L(S_1) - \tilde{\alpha}_1 \alpha_2 p Q_2^* - \alpha_1 L(S_1) \\
= K_2 - \alpha_1 c_1 Q_2^* + \alpha_2 c_2 Q_2^* - \tilde{\alpha}_1 p Q_2^* \\
= K_2 + Q_2^* (c_2 - \tilde{\alpha}_1 p - \alpha_1 c_1) \\
> 0,
$$

so single-sourcing from supplier 1 is optimal for $x = -\infty$.

If $s_{20} < s_{10}$, then $\Delta M_{12}(x = s_{20}) < 0$, Case 2 is never optimal so the optimal policy is to order from supplier 1 when $x < s_{10}$, and order nothing otherwise.

If $s_{10} \leq s_{20}$, the optimal policy is to order from supplier 1 when $x < s_{12}$, order from supplier 2 when $s_{12} \leq x < s_{20}$, and order nothing otherwise.

3.4.1.2 Higher Marginal Benefit for Supplier 2 When $x = -\infty$

If $\alpha_1(p - c_1) < p - c_2$, then $p_{critical} = +\infty$. Therefore, the extra order is always placed from supplier 2 rather than from both suppliers.

(i). If $\Delta M_{32}(x = -\infty) > 0$

In this case, single-sourcing from supplier 2 is optimal for $x = -\infty$.

When $s_{20} < s_{10}$, then $\Delta M_{12}(x = s_{20}) < 0$, and the optimal policy is to order from supplier 2 when $x < s_{21}$, order from supplier 1 when $s_{21} \leq x < s_{10}$, and order nothing otherwise.

When $s_{10} \leq s_{20}$, then for $x \in [s_{10}, s_{20})$, sourcing from supplier 2 is optimal. If the minimizer $x_{12}^* > s_{20}$, then from $\Delta M_{12}(s_{20}) > 0$, we know that $\Delta M_{12}(x) > 0$ on $x \in (-\infty, s_{20})$. As a result, the
optimal policy is to order up to $S_2$ from supplier 2 when $x \leq s_{20}$; order nothing otherwise.

If the minimizer $x_{12}^* \leq s_{20}$ and $\Delta M_{12}(x_{12}^*) \geq 0$, the optimal policy is the same as stated above.

If $x_{12}^* \leq s_{20}$ and $\Delta M_{12}(x_{12}^*) < 0$, the optimal policy is to order up to $S_2$ from supplier 2 when $x \leq s_{21}$; order up to $S_1$ from supplier 1 when $s_{21} < x \leq s_{12}$; order up to $S_2$ from supplier 2 when $s_{12} < x \leq s_{20}$; order nothing otherwise.

(ii). If $\Delta M_{32}(x = -\infty) \leq 0$ in this case, dual-sourcing is optimal for $x = -\infty$. This is true because

$$\frac{\partial \Delta M_{32}}{\partial x} = \frac{\partial M_3}{\partial x} - \frac{\partial M_2}{\partial x}$$

$$= -\alpha_1 c_1 - \alpha_2 c_2 - \alpha_1 \alpha_2 L'(x + Q_1^* + Q_2^*) + \bar{\alpha}_1 \bar{\alpha}_2 L'(x) - [-\alpha_2 c_2 + (1 - \alpha_2)L'(x)]$$

$$= -\alpha_1 c_1 - c_2 - \alpha_1 L'(S_1) - [-c_2]$$

$$= -\alpha_1 c_1 - c_2 + \alpha_1 c_1 + c_2$$

$$= 0,$$  

(3.28)

which implies that the difference between Case 3 and Case 2 is a constant, $\Delta M_{32} < 0$ and ordering from supplier 2 only is never optimal. At $x = s_{30}$, the cost of dual-sourcing is the same with that of ordering nothing, but better than the cost of Case 2. Consequently, $s_{20} < s_{30}$, and from Proposition 3.6 we can conclude that $s_{30} < s_{10}$. The optimal policy is to dual-source when $x < s_{31}$, order from supplier 1 when $s_{31} \leq x < s_{10}$, and order nothing otherwise.

3.4.2 Two Identical Suppliers

In this section, we consider the optimal policy structure when the two suppliers are identical. So case 1 and case 2 are the same single-sourcing case. The subscript $s$ is used to denote the single-sourcing case and $d$ is used to denote the dual-sourcing case. In order to keep the notation consistent, $n$ is used to represent the case of ordering nothing.

From Proposition 3.2, we can conclude that for $x = -\infty$, when $p > p_{critical}$, an extra ordering is placed from both suppliers, otherwise it is placed from only one supplier, and $p_{critical} = \frac{\alpha h + c}{1 - \alpha}$. When
the extra ordering is placed from either supplier, the optimal costs of dual-sourcing and single-sourcing need to be compared.

\[
\Delta M_{ds}(-\infty) = M_d(-\infty) - M_s(-\infty) = 2K + 2\alpha cQ^* + \alpha^2 L(x + 2Q^*) + 2\alpha \bar{\alpha} L(x + Q^*) + \bar{\alpha}^2 L(x) - [K + \alpha c (S - x) + \alpha L(S) + \bar{\alpha} L(x)] = K + \alpha c (2Q^* + x - S) + \alpha^2 L(x + 2Q^*) + \alpha \bar{\alpha} L(x + Q^*) - \bar{\alpha} \alpha p Q^* - \alpha L(S)
\]

Although \(L(x + Q^*)\) and \(pQ^*\) are not constants when \(x = -\infty\), the first-order derivative of \(L(x + Q^*)\) \(- pQ^*\) is 0, which implies that \(L(x + Q^*) - pQ^*\) is a constant. The other terms of the above equation are also constants, so does the cost difference \(\Delta M_{ds}(-\infty)\). When \(\Delta M_{ds}(-\infty) < 0\), dual-sourcing is optimal; otherwise single-sourcing is optimal.

Generalizing from Proposition 3.6, we know that \(s_{dn} < s_{sn}\). There are only two possible optimal policy structures:

- If \(p < p_{critical}\) and \(\Delta M_{ds}(-\infty) > 0\), order up to \(S\) when \(x < s_{sn}\), otherwise order nothing.
- If \(p \geq p_{critical}\), or \(p < p_{critical}\) and \(\Delta M_{ds}(-\infty) \leq 0\), order \(Q^*\) from each supplier when \(x < s_{ds}\), order up to \(S\) when \(s_{ds} \leq x < s_{sn}\), otherwise order nothing.

### 3.5 Single-period versus Multiple-period: Numerical Experiments

#### 3.5.1 The Effects of Parameters on Approximation Errors

The optimal policy structures of the multiple-period problem, especially when the number of periods is relatively large, is very difficult to determine analytically, and time-consuming to solve numerically. For example, if Markov Decision Process (MDP) method is used to obtain the optimal policy of an infinite horizon problem, both the state space (a state is defined as an inventory level) and the number of scenarios can be extremely large since the suppliers are unreliable and uncapacitated, and backorders are allowed. One purpose of deriving the optimal policy structures of the single-period model with arbitrary
initial inventory levels is to approximate the optimal policy structures of the multiple-period problem. Therefore, in our numerical experiment, we compare the costs of implementing the optimal policies obtained from single-period problems in three-period problems, with the optimal costs of the three-period problems solved by MDP. In our analytical results, the optimal policy structures are derived under continuous demand distributions, while MDP can only solve discrete problems. In order to distinguish whether the differences in the results are caused by different time horizons (single-period or three-period) or by different demand distributions (discrete or continuous), the costs of implementing the optimal policies obtained from the corresponding single-period discrete models in three-period models are also evaluated and compared.

The single-period discrete model formulation is as follows:

\[ M(Q_1, Q_2|x) = K_1(Q_1) + K_2(Q_2) + \alpha_1 c_1 Q_1 + \alpha_2 c_2 Q_2 + \alpha_1 \alpha_2 L_{dis}(x + Q_1 + Q_2) \]
\[ + \alpha_1 \bar{\alpha}_2 L_{dis}(x + Q_1) + \bar{\alpha}_1 \alpha_2 L_{dis}(x + Q_2) + \bar{\alpha}_1 \bar{\alpha}_2 L_{dis}(x), \quad (3.29) \]

where

\[ L_{dis}(y) = \sum_{\omega=0}^{y-1} h(y - \omega) Pr(\omega) + \sum_{\omega=y+1}^{u} p(\omega - y) Pr(\omega), \]

\( \omega \) is random discrete demand, and \( Pr(\omega) \) is the corresponding probability mass function.

The multiple-period discrete model formulation can be written as follows:

\[ M_t(Q_{1,t}, Q_{2,t}|x_t) = K_1(Q_{1,t}) + K_2(Q_{2,t}) + \alpha_1 c_1 Q_{1,t} + \alpha_2 c_2 Q_{2,t} + \alpha_1 \alpha_2 G_t(x_t + Q_{1,t} + Q_{2,t}) \]
\[ + \alpha_1 \bar{\alpha}_2 G_t(x_t + Q_{1,t}) + \bar{\alpha}_1 \alpha_2 G_t(x_t + Q_{2,t}) + \bar{\alpha}_1 \bar{\alpha}_2 G_t(x_t), \quad (3.30) \]

where

\[ G_t(y_t) = h \sum_{\omega=0}^{y_t} (y_t - \omega) Pr(\omega) + p \sum_{\omega=y_t}^{u} (\omega - y_t) Pr(\omega) + \sum_{\omega=0}^{u} M_{t-1}(y_t - \omega, Q_{1,t-1}, Q_{2,t-1}) Pr(\omega) \]
\[ = L_{dis,t}(y_t) + \sum_{\omega=0}^{u} M_{t-1}(y_t - \omega, Q_{1,t-1}, Q_{2,t-1}) Pr(\omega), \]
\( t \) is the number of periods remaining, and \( M_0 \equiv 0 \).

Overall, three types of policies are considered:

- **Policy 1**: Optimal policy obtained by minimizing the cost of a three-period problem with a discrete demand distribution (model (3.30) with \( t = 3 \) and the initial inventory level \( x_3 = 0 \)).

- **Policy 2**: Single-period optimal policy under continuous demand distribution (model (3.2)) applied in the three-period model (model (3.30) with \( t = 3 \)).

- **Policy 3**: Single-period optimal policy under discrete demand distribution (model (3.29)) applied in the three-period model (model (3.30) with \( t = 3 \)).

For discrete problems, beta demand distributions are discretized as follows.

\[
Pr(\omega) = F(\omega + 0.5) - F(\omega - 0.5), \quad \omega = 0, 1, 2, ..., u
\]

Two factorial experiments are designed to test the effects of parameters on the approximation cost errors. Each experiment includes 7 factors and 2 levels of each parameter (see Table 3.2).

<table>
<thead>
<tr>
<th>p</th>
<th>h</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>( K_1 )</th>
<th>( K_2 )</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>u</th>
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<tr>
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<td>9</td>
<td>0.3</td>
<td>{0.9,1.1}</td>
<td>{1.2,1.4}</td>
<td>{0.5}</td>
<td>{0.5}</td>
<td>{0.7,0.83}</td>
<td>{0.87,0.94}</td>
</tr>
<tr>
<td>Experiment 2</td>
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<td>{0.3,0.7}</td>
<td>0.9</td>
<td>1.1</td>
<td>{0.5}</td>
<td>{0.5}</td>
<td>{0.7,0.83}</td>
<td>{0.87,0.94}</td>
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</tbody>
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<table>
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<th>p</th>
<th>h</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>( K_1 )</th>
<th>( K_2 )</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>u</th>
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<tbody>
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<td>0.8258</td>
<td>0.027</td>
<td>0</td>
<td>0.0022</td>
<td>0</td>
<td>0.6845</td>
<td></td>
<td></td>
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<tr>
<td>Experiment 2</td>
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<td>0</td>
<td>0.0035</td>
<td>0</td>
<td>0.0091</td>
<td>0</td>
<td>0.0671</td>
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</table>
Table 3.4: p-values of Different Parameters Levels When Policy 3 is Implemented

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<th>h</th>
<th>c₁</th>
<th>c₂</th>
<th>K₁</th>
<th>K₂</th>
<th>α₁</th>
<th>α₂</th>
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</thead>
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<tr>
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<td>0.3534</td>
<td>0.4115</td>
<td>0.1223</td>
<td>0.036</td>
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</tr>
<tr>
<td>Experiment 2</td>
<td>0.1768</td>
<td>0.0031</td>
<td>0.5576</td>
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</tr>
</tbody>
</table>

Figure 3.4: Significance of Different Parameters in Experiment 1: X₁ ≡ c₁, X₂ ≡ c₂, X₃ ≡ K₁, X₄ ≡ K₂, X₅ ≡ α₁, X₆ ≡ α₂, X₇ ≡ u

N-way (N = 7) ANOVA tests are carried out to analyze the $2^7$ factorial experiment results. Main effects and interaction effects between two parameters (factors) are included in the ANOVA tests (see Figure 3.4 and 3.5). The p-values of the main effects are shown in Table 3.3 and 3.4. In Experiment 1, when Policy 2 is implemented in the three-period problems, differences in errors can be found in different levels of $c_1$, $K_1$, $K_2$, $α_1$, $α_2$ with a significance < 0.05. The null hypothesis cannot be
rejected for $c_2$ and $u$. When Policy 3 is implemented in the three-period problems, the null hypothesis can be rejected for $K_2$, $\alpha_2$ and $u$. In Experiment 2, the null hypothesis cannot be rejected only for $u$ when Policy 2 is adopted, and it cannot be rejected for $K_1$ and $u$ when Policy 3 is adopted. Overall, each parameter, except $c_2$, has impact on errors in at least one set of parameters. For different sets of parameters, some factors’ influences on errors are dominated by those of the other parameters, such as $K_2$ and $\alpha_2$. So the main effects of these parameters seem to be not significant. If those dominant factors are eliminated from the factorial experiment, i.e., only the data with the same level of $K_2$ and $\alpha_2$ are examined, $p$-values of the originally insignificant factors become $< 0.05$. It can be shown that different levels of all the factors have impact on errors if proper sets of parameter values are chosen. Therefore, it can be concluded that service levels, fixed costs, reliability levels and demand upper bounds affect approximation errors.

Now we investigate how these parameters impact approximation errors. From numerical experiments, we find that when service levels are relatively high (high $p$, low $h$, low $c$), and/or fixed cost $K$ is
relatively high, and/or demand is relatively large (high $u$), and/or suppliers are relatively unreliable (low $\alpha$), we tend to order to cover multiple periods. Hence larger approximation errors.

### 3.5.2 A Heuristic Method

From the numerical results of the previous section we can see that the approximation errors are usually not acceptable, as the average errors are always greater than 10% (the continuous columns and discrete columns in Table 3.5 and 3.6). As we concluded in the previous section, when service levels are relatively high (high $p$, low $h$, low $c$), and/or fixed cost $K$ is relatively high, and/or demand is relatively large (high $u$), and/or suppliers are relatively unreliable (low $\alpha$), we tend to order to cover multiple periods and approximation errors tend to grow. Therefore, errors are usually caused by under-ordering when implementing single-period optimal policies in multiple-period problems. This issue can be resolved by either inflating order quantities, or deflating inventory levels. The latter one is adopted in our heuristic method, which implements the single-period optimal policy corresponding to a lower inventory level than the actual inventory level in the multiple period problem. According to the effects of parameters on error, demand mean $\mu$ must be included when we compute the amount of inventory deflation since orders are more likely to be aggregated and carried forward to the next period. The terms $F^{-1}(\frac{p-c_i}{h+p})/u$ are used to incorporate not only the service levels but also the shapes of demand distributions. The effects of reliability levels are on the contrary to those of service levels and demand so they are in the denominator. As a result, we empirically generalize the following formula:

$$\Delta = \text{round}\left(\frac{F^{-1}(\frac{p-c_1}{h+p})/u)(F^{-1}(\frac{p-c_2}{h+p})/u)}{\alpha_1\alpha_2}\mu\right)$$

(3.31)

When implementing single-period policies, current inventory level is deflated by $\Delta$ when the current period is not the last period. For example, when the initial inventory level of period $t$ is $x_t = 5$, the single-period policy corresponding to $x_t = 5 - \Delta$ is adopted when $t \neq 1$.

Computation time depends heavily on the the number of states (inventory levels), which is decided by $u$ and the suppliers’ capacities $Q_{i}^{max}$. Although there is no maximum order size constraint in the
analytical model, numerical experiments cannot be done without it, otherwise the lower limit of states cannot be determined. For computation purposes, a supply capacity of $Q_{max} = 2u$ is assumed for each supplier. If any order quantity in the results reaches the capacity, then we can extend the capacities and re-run the experiments to ensure the uncapacitated supply assumption is correctly simulated. However, this does not happen in our experiments. To be specific about the state range, given zero initial inventory level at period 1, the possible states for period 2 are from $-u$ (no supplier delivers order and demand is at the upper limit) to $2Q_{max}$ (both suppliers deliver orders and no demand occurs). At the end of the three-period horizon, the states are from $-3u$ to $6Q_{max}$. In order to implement single-period policies with deflated inventory levels, the policies for states from $-3u - \Delta$ to $6Q_{max} - \Delta$ need to be computed. Although the number of states are the same, the policies for inventory levels from $-3u - \Delta$ to $-3u - 1$ are usually dual-sourcing, which take longer to compute than the policies for inventory levels from $6Q_{max} - \Delta + 1$ to $6Q_{max}$, which are usually to order nothing.

The same sets of parameters from the previous section are used to test the performance of the heuristic method. Table 3.5 and 3.6 shows the computation times and errors of implementing single-period policies, and the computation time of implementing the optimal policies obtained from MDP. The results for two levels of $u$ are displayed separately for the $2 \times 2^7$ problems, since there are significant differences in computation time. Figure 3.6 exhibits the scatter plots of running time versus approximation errors. The average computation times of MDP are shown as the vertical lines in the Figure 3.6 for comparison.

From Table 3.5 and 3.6 we can see that the computation times of implementing policies from discrete models (three-period models and single-period models with discrete demand distributions) are highly consistent with the number of states, while those of continuous models are not. The heuristic method takes a little longer than implementing the single-period policies without inventory deflation due to calculation of policies for states from $-3u - \Delta$ to $-3u - 1$, but it reduces errors significantly.
Table 3.5: Computation Time and Errors of Implementing Single-period Policies When \( u = 10 \)

<table>
<thead>
<tr>
<th>Error (%)</th>
<th>Computation Time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Continuous</td>
</tr>
<tr>
<td>Average</td>
<td>15.64</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>6.46</td>
</tr>
<tr>
<td>Minimum</td>
<td>3.51</td>
</tr>
<tr>
<td>Maximum</td>
<td>33.27</td>
</tr>
</tbody>
</table>

Table 3.6: Computation Time and Errors of Implementing Single-period Policies When \( u = 15 \)

<table>
<thead>
<tr>
<th>Error (%)</th>
<th>Computation Time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Continuous</td>
</tr>
<tr>
<td>Average</td>
<td>15.99</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>6.08</td>
</tr>
<tr>
<td>Minimum</td>
<td>5.26</td>
</tr>
<tr>
<td>Maximum</td>
<td>29.42</td>
</tr>
</tbody>
</table>

3.6 Conclusions

In this chapter, we extend the model in Chapter 2 by allowing arbitrary initial inventory level and only including individual fixed cost incurred upon order placement. For two different suppliers, we prove that there are ten possible optimal policy structures as a function of initial inventory level (see below), although extensive numerical experiments show that some complicated policies rarely occur.

1. 

\[
(Q_1, Q_2) = \begin{cases} 
(S_1 - x, 0), & x \leq s_{21} \\
(0, 0), & x > s_{21} 
\end{cases}
\]

2. 

\[
(Q_1, Q_2) = \begin{cases} 
(S_1 - x, 0), & x \leq s_{23} \\
(0, S_2 - x), & s_{23} < x \leq s_{31} \\
(0, 0), & x > s_{31} 
\end{cases}
\]
(a) Implement Policies from Continuous Models

(b) Implement Policies from Discrete Models

Figure 3.6: Computation Time and Errors of Implementing Shifted Single-period Policies

3.

\[
(Q_1, Q_2) = \begin{cases}
(0, S_2 - x), & x \leq s_{31} \\
(0, 0), & x > s_{31}
\end{cases}
\]

4.

\[
(Q_1, Q_2) = \begin{cases}
(0, S_2 - x), & x \leq s_{32} \\
(S_1 - x, 0), & s_{32} < x \leq s_{21} \\
(0, 0), & x > s_{21}
\end{cases}
\]

5.

\[
(Q_1, Q_2) = \begin{cases}
(0, S_2 - x), & x \leq s_{32} \\
(S_1 - x, 0), & s_{32} < x \leq s_{23} \\
(0, S_2 - x), & s_{23} < x \leq s_{31} \\
(0, 0), & x > s_{31}
\end{cases}
\]

6.

\[
(Q_1, Q_2) = \begin{cases}
(Q_1^*, Q_2^*), & x \leq s_{42} \\
(S_1 - x, 0), & s_{42} < x \leq s_{21} \\
(0, 0), & x > s_{21}
\end{cases}
\]
Two special models are considered. One assumes that the more expensive supplier is perfectly reliable, and the other assumes that the two suppliers are identical. The numbers of optimal policies for the two special models reduce to six and two, respectively.
We also investigate the possibility of using single-period optimal policies to approximate those of multiple-period problems. The approximation errors of implementing single-period optimal policies directly are unacceptable as the numerical experiments show. But from Analysis of Variance (ANOVA) of the results, we conclude that when service levels are relatively high (high $p$, low $h$, low $c$), and/or fixed cost $K$ is relatively high, and/or demand is relatively large (high $u$), and/or suppliers are relatively unreliable (low $\alpha$), we tend to order to cover multiple periods and approximation errors tend to grow. Therefore, errors are usually caused by under-ordering when implementing single-period optimal policies in multiple-period problems. A heuristic method, which implements the single-period optimal policy corresponding to a lower inventory level than the actual inventory level in the multiple period problem, is proposed to reduce approximation error. Based on the effects of parameters on error, a formula is empirically generated to compute the deflation amount of inventory level. We found that the heuristic method takes a little longer than implementing the single-period optimal policies without inventory deflation, but it reduces errors significantly.
Chapter 4

Unreliable Supplier Selection and Order Quantity Allocation under Order Size Constraints

4.1 Introduction

During recent years, companies are allocating more resources to their core competencies and increasingly outsourcing their non-core activities. However, suppliers are not perfectly reliable: when an order is placed, the supplier may fail to fulfill the order within the specified length of time. Supply disruptions result from various reasons: manufacturers’ equipment breakdowns, labor strikes, transportation accidents, or other unpredictable situations. For instance, BMW AG, the world’s largest maker of luxury cars, halted production at three factories in Germany during the Icelandic volcanic ash crisis in 2010 due to supply shortage. The temporary manufacturing shutdown, caused by a shortage of interior and electronic parts, which are usually delivered by air, delayed production of about 7,000 vehicles [3]. One common approach to mitigating the uncertainties in supply chains is to source from multiple suppliers. Therefore, effective supplier selection becomes significantly important.

The supplier selection problem has drawn considerable attention from both industry and academe.
Ho et al. [23] review the literature on the multi-criteria decision making approaches for supplier evaluation and selection from 2000 to 2008. Given that our paper addresses sourcing decisions under supply disruptions, we mainly review prior work which is related to this issue. Papers not discussed in [23] are included as well. Overall, the related literature can be classified into two categories according to the assumptions about the reliability of suppliers: perfectly reliable or unreliable.

For perfectly reliable supplier selection, models where suppliers offer order quantity discounts are considered by Goossens et al. [18] and Benton [6]. Chauhan and Proth [10] and Awasthi et al. [4] both investigate the problem with concave purchase cost and ordering quantities constraints, and propose heuristic methods for their models. The former one considers deterministic demand, while the latter one assumes stochastic demand. Based on the model proposed by Awasthi et al. [4], Zhang and Zhang [45] solve the selection problem with fixed cost.

In the area of unreliable supplier selection, papers can be categorized further by how uncertainty in the supply process is modeled, namely supply with random yield, all-or-nothing delivery, supplier unavailability, and random lead times. Random yield means the quantity received from the supplier may differ from what was ordered, or a fraction of the received order may be defective. This assumption is practical in many application areas such as electronics fabrication and assembly as well as chemical processes. A comprehensive review of yield uncertainty is offered by Yano and Lee [44]. Other papers have appeared since their review. For example, Erdem et al. [13] model random yield as random capacities of suppliers. Dada et al. [11] consider selection of unreliable suppliers in a newsvendor problem and conclude that low item cost is the order qualifier, while high reliability level is the order winner. Federgruen and Yang [15] also investigate an $N$-supplier-selection problem, where the item cost is identical for all the suppliers. Burke et al. [9] study the selection and purchase problem under stochastic demand with limitation on minimum order size and proposed an optimal approach for the model. Other work in the area of yield uncertainty includes [16], [43], and also Model II in [1].

All-or-nothing delivery can be regarded as a special case of random yield where the fraction of the quantity received is either 0 (nothing) or 1 (all). Usually it is used to model on-time delivery performance. For example, perishable products may be delivered to destinations later than the expiration date.
due to transportation delays, which renders them unsalable. Model I considered by Anupindi and Akella [1] falls into this category. However, they mainly focus on determining the optimal policy structure of sourcing from two different unreliable suppliers instead of selecting the optimal set of suppliers from several potential suppliers. The same problem is considered by Swaminathan and Shanthikumar [41] under discrete demand, while Babich et al. [5] extend the work of Anupindi and Akella [1] by assuming the reliability levels of suppliers are correlated and suppliers control the wholesale prices.

Supplier unavailability means suppliers are down from time to time and the available and unavailable periods of the suppliers are random. Research work focusing on this topic include [38, 20, 42, 46]. Lead-time uncertainty describes models where unreliability of suppliers can be characterized by lead-time distributions with large means and variances [30, 17, 31]. Multiple-period models have to be formulated to develop lead-time uncertainty.

In our model, the interaction among suppliers makes it difficult to solve the order quantity allocation problem. In comparison, the quantity allocation of some models with different assumptions about reliability levels can be determined relatively easily. When suppliers are all perfectly reliable with everything else remaining the same, Awasthi et al. [4] prove that in an optimal solution, at most one supplier may not satisfy the following $Q^*_i \in \{0, Q^{min}_i, Q^{max}_i\}$. Based on this proposition, Zhang and Zhang [45] prove that if supplier $i_0$ is the supplier which does not satisfy $Q^*_i \in \{0, Q^{min}_i, Q^{max}_i\}$ (the corresponding item cost is indicated by $c_{i_0}$), then $Q^*_i \in \{0, Q^{max}_i\}$ for $c_i < c_{i_0}$ and $Q^*_i \in \{0, Q^{min}_i\}$ for $c_i > c_{i_0}$.

When unreliability is modeled as random yield, for example, Burke et al. [9] use the historical percentage $r$ of good units received from a supplier to represent supplier reliability. The optimal order quantity needed $q$ can be determined in a similar way of the perfectly reliable supplier model, then $Q^* = q/r$ should be ordered from an unreliable supplier. However, this is not the case when all-or-nothing delivery is assumed. In our problem, more than one supplier may not satisfy $Q^*_i \in \{0, Q^{min}_i, Q^{max}_i\}$, and the problem must be resolved to obtain the optimal solution every time the selected set of suppliers changes.

We contribute to the literature by solving the supplier selection problem under the assumption of all-or-nothing delivery. The survey conducted by Kannan and Tan [24] shows that among thirty supplier selection criteria, the four most important ones are the ability to meet delivery due dates, commitment
to quality, technical expertise and price of materials, parts and services, respectively. In this paper, we investigate the importance of on-time delivery performance and item price in the supplier selection problem from an analytical perspective. Our model is similar to the one considered by Burke et al. [9] where supplier reliability is modeled as random yield. Random yield implies quality issues of suppliers, which is the second important criterion in supplier selection according to the survey by Kannan and Tan [24].

In this paper, on-time delivery performance is modeled as the on-time delivery probability $\alpha$, which implies that with probability $1 - \alpha$, the supplier fails to fulfill the entire order. Failed orders are canceled. This problem is addressed under the newsvendor setting: a single-period, single-item, single-site inventory system and stochastic demand. Suppliers may have different item prices, reliability levels, minimum order size requirements, and supply capacities. We develop a discrete model with the objective of minimizing the expected total cost, including item costs, disposal and penalty costs. There are two sets of decision variables associated with this problem. First, for each potential supplier, a binary variable is used to indicate whether the supplier is selected or not. Second, integer variables are used to represent order quantities. The general model is divided into two parts: (1) master problem: to select a set of suppliers. Five methods are proposed to solve subproblems and four methods are developed for master problems; (2) subproblem: to determine the order quantity allocation given a set of selected suppliers. Appropriate combinations of these methods are discussed and implemented in numerical experiments to show their performance. Conclusions are drawn at the end.

4.2 Model Formulation

A buyer firm (manufacturer or retailer) wants to procure a seasonal product from a group of potential suppliers to satisfy stochastic demand in a single selling season. The sequence of the events is as follows: before the selling season begins, supplier selection decisions are made based on a potential set of $N$ suppliers, who are indexed by $i$ and the full set is indicated by $\mathcal{U}$. Then orders are made on the selected set of suppliers, which is represented by $\mathcal{P}$ and $\mathcal{P} \subseteq \mathcal{U}$. If an order is made on supplier $i$, the supplier selection indicator $z_i = 1$ for $i \in \mathcal{P}$, otherwise $z_i = 0$ for $i \notin \mathcal{P}$. The order quantity from supplier $i$
is $Q_i \in [Q_i^{\text{min}}, Q_i^{\text{max}}]$ if $i \in P$ (which is equivalent to $z_i = 1$), and $Q_i = 0$ and $z_i = 0$ otherwise. The ordered stock from supplier $i$, if any, arrives right before the beginning of the selling season with probability $\alpha_i$. If supplier $i$ delivers the order, the item costs ($c_i$/item) are incurred, otherwise they are not. Then, stock on-hand is used to satisfy random demand $\omega$ during this period, which follows a known discrete probability distribution with a probability mass function $Pr(\omega)$. A disposal cost ($h$/item) is incurred for any stock left at the end of the period after satisfying all the demand, and a penalty cost ($p$/item) is incurred for any unsatisfied demand. We assume that $p > c_i$ for all $i$ because any supplier with $c_i > p$ will definitely not be selected and thus is not included in the set $U$. The penalty cost can be interpreted as loss of revenue since our objective is to minimize cost instead of maximize profit. Loss of goodwill may also be included in penalty cost. At most one order can be placed from each supplier, regardless of the delivery result.

The objective of this sourcing problem is to minimize the expected single-period cost function $M(z, Q)$, which consists of variable costs that are proportional to the order quantity delivered successfully, and the expected end-of-period cost.

Denoting the inventory level after the order arrives by $y$, the expected end-of-period cost can be written as:

$$L(y) = \sum_{\omega=0}^{y-1} h(y - \omega) Pr(\omega) + \sum_{\omega=y+1}^{\omega} p(\omega - y) Pr(\omega)$$

(4.1)

We use $t \in \{0, 1\}^N$ to indicate on-time order delivery, so $t_i = \begin{cases} 
0, & Q_i \text{ is not delivered by supplier } i \\
1, & Q_i \text{ is delivered by supplier } i.
\end{cases}$

The decision variables are the order placement indicator $z_i$, where $z_i = \begin{cases} 
0, & \text{supplier } i \text{ is not selected} \\
1, & \text{supplier } i \text{ is selected},
\end{cases}$

and order quantity allocation $Q_i, i \in U$, which are integer values. Therefore, the expected total cost function can be written as:

$$M(z, Q) = \sum_{i=1}^{N} \alpha_i c_i z_i Q_i + \sum_{t} \left\{ \prod_{i=1}^{N} [\alpha_i t_i + (1 - \alpha_i)(1 - t_i)] L(\sum_{i=1}^{N} t_i z_i Q_i) \right\}$$

(4.2)
### Table 4.1: Notation

<table>
<thead>
<tr>
<th>Decision variables:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_i$</td>
<td>Indicator of whether supplier $i$ is selected, $i \in {1, 2, \cdots, N}$</td>
</tr>
<tr>
<td>$Q_i$</td>
<td>Order quantity from supplier $i$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>Number of suppliers</td>
</tr>
<tr>
<td>$u$</td>
<td>Upper limit of demand, i.e., demand is distributed between $[0, u]$</td>
</tr>
<tr>
<td>$a, b$</td>
<td>Shape parameters of beta demand distributions</td>
</tr>
<tr>
<td>$\alpha_i$</td>
<td>Probability of on-time delivery (reliability level), $i \in {1, 2, \cdots, N}$</td>
</tr>
<tr>
<td>$c_i$</td>
<td>Per item ordering cost from supplier $i$, $i \in {1, 2, \cdots, N}$</td>
</tr>
<tr>
<td>$Q_i^{\text{min}}, Q_i^{\text{max}}$</td>
<td>Minimum and maximum constraints of $Q_i$, $i \in {1, 2, \cdots, N}$</td>
</tr>
<tr>
<td>$h$</td>
<td>Per item disposal cost</td>
</tr>
<tr>
<td>$p$</td>
<td>Per item penalty cost</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Others</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_i$</td>
<td>Indicator of whether the order $Q_i$ is delivered from supplier $i$, $i \in {1, 2, \cdots, N}$</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Random continuous demand</td>
</tr>
<tr>
<td>$f(\xi)$</td>
<td>Density function of demand</td>
</tr>
<tr>
<td>$F(\xi)$</td>
<td>Cumulative distribution function of demand</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Random discrete demand</td>
</tr>
<tr>
<td>$Pr(\omega)$</td>
<td>Probability mass function of demand $\omega$</td>
</tr>
<tr>
<td>$y$</td>
<td>Inventory level on-hand after ordering (order-up-to level)</td>
</tr>
<tr>
<td>$L(y)$</td>
<td>Expected end-of-period cost with a discrete demand distribution</td>
</tr>
<tr>
<td>$L_c(y)$</td>
<td>Expected end-of-period cost with a continuous demand distribution</td>
</tr>
<tr>
<td>$M(z, Q)$</td>
<td>Single-period cost function with a discrete demand distribution</td>
</tr>
<tr>
<td>$M_c(z, Q)$</td>
<td>Single-period cost function with a continuous demand distribution</td>
</tr>
<tr>
<td>$U$</td>
<td>The full set of $N$ potential suppliers</td>
</tr>
<tr>
<td>$P$</td>
<td>The set of selected suppliers</td>
</tr>
<tr>
<td>$m$</td>
<td>The number of selected suppliers</td>
</tr>
</tbody>
</table>
Correspondingly, the problem can be represented by the following model:

\[
\begin{align*}
\min \quad & M(z, Q) \tag{4.3} \\
\text{s.t.} \quad & z_i Q_i^{\min} \leq Q_i \leq z_i Q_i^{\max}, \quad i \in \{1, 2, \ldots, N\} \\
& z_i \in \{0, 1\}, \quad Q_i \in \mathbb{Z}, \quad i \in \{1, 2, \ldots, N\} 
\end{align*}
\]  

**Proposition 4.1.** Both problem (4.3) and its continuous formulation are NP-hard.

**Proof.** We show the continuous formulation of problem (4.3) and prove that it is NP-hard. The continuous formulation involves a known continuous demand distribution with the probability density function \(f(\xi)\) and the cumulative density function \(F(\xi)\). The continuous expected end-of-period cost is:

\[
L_c(y) = \int_0^y h(y - \xi) f(\xi) d\xi + \int_y^u p(\xi - y) f(\xi) d\xi \tag{4.6}
\]

Order quantity allocation \(Q_i, i \in \mathcal{U}\) are now continuous variables. Therefore, the expected total cost function can be written as:

\[
M_c(z, Q) = \sum_{i=1}^N \alpha_i c_i z_i Q_i + \sum_{i=1}^N \left\{ \prod_{t=1}^N \left[ \alpha_t t_i + (1 - \alpha_t)(1 - t_i) \right] L_c\left( \sum_{i=1}^N t_i z_i Q_i \right) \right\}
\]

\[
= pE(\xi) + \sum_{i=1}^N \left\{ \prod_{t=1}^N \left[ \alpha_t t_i + (1 - \alpha_t)(1 - t_i) \right] (h + p) \left[ \sum_{i=1}^N t_i z_i Q_i F\left( \sum_{i=1}^N t_i z_i Q_i \right) \right] \right\}
\]

\[
- \int_0^{\sum_{i=1}^N t_i z_i Q_i} \xi f(\xi) d\xi - \sum_{i=1}^N \left\{ \frac{p - c_i}{h + p} t_i z_i (Q_i) \right\} \tag{4.7}
\]

Correspondingly, the problem can be represented by the following mixed-integer nonlinear programming model:

\[
\begin{align*}
\min \quad & M_c(z, Q) \tag{4.8} \\
\text{s.t.} \quad & z_i Q_i^{\min} \leq Q_i \leq z_i Q_i^{\max}, \quad i \in \{1, 2, \ldots, N\} \\
& z_i \in \{0, 1\}, \quad Q_i \in \mathbb{R}, \quad i \in \{1, 2, \ldots, N\}
\end{align*}
\]
The problem described by Awasthi et al. [4], where all the suppliers are perfectly reliable, is proved to be NP-hard even when they quote the same item cost and minimum order quantity equals to supply capacity for each supplier. Since it is a special case of our problem, problem (4.8) is also NP-hard. The explicit mode formulation (4.3) can be proved to be NP-hard in the same way since demand distributions are not involved in the proof in [4].

4.3 Order Quantity Allocation: Subproblems

The supplier selection problem consists of two parts: select a set of suppliers (master problem) and determine the order quantity allocation among them (subproblem). In this section, we consider the order allocation problem when a selected set of suppliers $\mathcal{P}$ is given. For a given set of suppliers $\mathcal{P}$, the supplier indicator vector $z$ can be determined accordingly and (4.2) can rewritten as the following formulation by dropping supplier $i$ where $i \in \mathcal{U}\setminus\mathcal{P}$

$$M_{\mathcal{P}}(\mathbf{Q}_{\mathcal{P}}) = \sum_{i \in \mathcal{P}} \alpha_i c_i Q_i + \sum_{t \in \mathcal{P}} \{ \prod_{i \in \mathcal{P}} [\alpha_i t_i + (1 - \alpha_i)(1 - t_i)] L(\sum_{i \in \mathcal{P}} t_i Q_i) \}$$  \hspace{1cm} (4.9)

where $\mathcal{P}$ has cardinality $m$ and $\mathbf{Q}_{\mathcal{P}}$ is the vector after eliminating zeros in $\mathbf{Q}$ in (4.2). Model (4.9) is the explicit (or discrete) formulation of subproblem.

4.3.1 Methods Used to Solve Continuous Formulations of Subproblems

Although fractional demand and order quantities are rare in reality, continuous and differentiable demand distributions usually simplifies the analysis. Models with continuous demand distributions are relatively easier to solve and derive analytical insights. Optimal solutions obtained from the continuous model are usually very close approximations of good implementable solutions, or serve as starting points in the methods used to provide integer solutions. Therefore, continuous formulation of subproblem is discussed first to obtain analytical insights, and two methods are proposed to produce continuous solutions. The continuous formulation of subproblem can be written similarly to (4.9) with $L(y)$ replaced
by $L_c(y)$:

$$\min M_{cP}(Q_P) = \sum_{i \in P} \alpha_i c_i Q_i + \sum_{t \in \mathbb{P}} \{ \prod_{i \in P} [\alpha_i t_i + (1 - \alpha_i)(1 - t_i)] L_c(\sum_{i \in P} t_i Q_i) \} \quad (4.10)$$

s.t. $Q_i^{\min} \leq Q_i \leq Q_i^{\max}, \; Q_i \in \mathbb{R}, \; i \in \mathbb{P}$

**Lemma 4.1.** $M_{cP}(Q_P)$ is convex because the Hessian matrix of $M_{cP}(Q_P)$ is positive semi-definite.

**Proof.**

$$\frac{\partial M_{cP}(Q_P)}{\partial Q_j} = \alpha_j (h + p) \{ \sum_{t \in \mathbb{P}} \prod_{i \in P \setminus j} [\alpha_i t_i + (1 - \alpha_i)(1 - t_i)] F(Q_j + \sum_{i \in P \setminus j} t_i Q_i) - \frac{p - c_j}{h + p} \} \quad (4.11)$$

Let $H_M$ be the Hessian matrix of $M_P(Q)$.

$$H_M(j, k) = \frac{\partial^2 M_{cP}(Q_P)}{\partial Q_j \partial Q_k} = \alpha_j \alpha_k (h + p) \{ \sum_{t \in \mathbb{P}} \prod_{i \in P \setminus j,k} [\alpha_i t_i + (1 - \alpha_i)(1 - t_i)] f(Q_j + Q_k + \sum_{i \in P \setminus j,k} t_i Q_i) \} \quad (4.12)$$

$$H_M(j, j) = \frac{\partial^2 M_{cP}(Q_P)}{\partial Q_j^2} = \alpha_j (h + p) \{ \sum_{t \in \mathbb{P}} \prod_{i \in P \setminus j} [\alpha_i t_i + (1 - \alpha_i)(1 - t_i)] f(Q_j + \sum_{i \in P \setminus j} t_i Q_i) \} \quad (4.13)$$

For example, when there are three suppliers

$$H_M(1, 1) = (h + p) \alpha_1 [\alpha_2 \alpha_3 f(Q_1 + Q_2 + Q_3) + \alpha_2 (1 - \alpha_3) f(Q_1 + Q_2) + \alpha_2 (1 - \alpha_3) f(Q_1 + Q_3) + (1 - \alpha_2)(1 - \alpha_3) f(Q_1)]$$

$$H_M(1, 2) = (h + p) \alpha_1 \alpha_2 [\alpha_3 f(Q_1 + Q_2 + Q_3) + (1 - \alpha_3) f(Q_1 + Q_2)]$$

$H_M(1, 3), H_M(2, 1), H_M(2, 2), H_M(2, 3), H_M(3, 1), H_M(3, 2),$ and $H_M(3, 3)$ can be written according to (4.12) and (4.13).

For $N$ suppliers, there are $2^N - 1$ terms of $(h + p) \prod_{i \in P} [\alpha_i t_i + (1 - \alpha_i)(1 - t_i)] f(\sum_{i \in P} t_i Q_i)$. It appears in the $j$th row and the $k$th column if $t_j = 1, t_k = 1$, and it appears in the $j$th row and the $j$th column (the diagonal) if $t_j = 1, t_{i \neq j} = 0$. $H_M$ can be written as the sum of the $2^N - 1$ matrices, where
one matrix only has one such term. For example, when \( N = 2 \),

\[
H_M = (h + p) \begin{bmatrix}
\alpha_1 \alpha_2 f(Q_1 + Q_2) + \alpha_1(1 - \alpha_2) f(Q_1) & \alpha_1 \alpha_2 f(Q_1 + Q_2) \\
\alpha_1 \alpha_2 f(Q_1 + Q_2) & \alpha_1 \alpha_2 f(Q_1 + Q_2) + (1 - \alpha_1) \alpha_2 f(Q_2)
\end{bmatrix}
\]

\[
= (h + p) \alpha_1 \alpha_2 f(Q_1 + Q_2) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (h + p) \alpha_1 (1 - \alpha_2) f(Q_1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
+ (h + p)(1 - \alpha_1) \alpha_2 f(Q_2) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

The values in the \( j \)th row and \( j \)th column are zeros if \( t_j = 0 \) in the matrix with the coefficient \((h + p) \prod_{i \in \mathcal{P}} [\alpha_i t_i + (1 - \alpha_i)(1 - t_i)] f(\sum_{i \in \mathcal{P}} t_i Q_i)\). The other values in the matrix are \((h + p) \prod_{i \in \mathcal{P}} [\alpha_i t_i + (1 - \alpha_i)(1 - t_i)] f(\sum_{i \in \mathcal{P}} t_i Q_i)\). Through elementary matrix operations, which do not affect the positive semi-definite property of a matrix, we can always obtain the matrix in the following format:

\[
(h + p) \prod_{i \in \mathcal{P}} [\alpha_i t_i + (1 - \alpha_i)(1 - t_i)] f(\sum_{i \in \mathcal{P}} t_i Q_i) \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}
\]

where \( E \) is defined as the matrix with all the values equal to 1. The principal minors of the above matrix are always zeros. Based on the theorem that the sum of positive semi-definite matrices must be positive semi-definite, \( M_c(\mathcal{P}) \) is positive semi-definite.

Therefore, Karush-Kuhn-Tucker (KKT) conditions can be used to determine the optimal solution. We assume that the dual variable associated with \( Q_i - Q_i^{\max} \leq 0 \) is \( \lambda_i \), and the one with \( Q_i^{\min} - Q_i \leq 0 \)
is $\mu_i$. We have

\begin{align*}
\text{(Stationarity)} \quad \frac{\partial M_{i,p}}{\partial Q_i} + \lambda_i - \mu_i &= 0, \quad \text{for } i \in \mathcal{P} \quad (4.14) \\
\text{(Primal feasibility)} \quad Q_i - Q_i^{\text{max}} &\leq 0, \quad Q_i^{\text{min}} - Q_i &\leq 0, \quad \text{for } i \in \mathcal{P} \quad (4.15) \\
\text{(Dual feasibility)} \quad \lambda_i &\geq 0, \quad \mu_i &\geq 0, \quad \text{for } i \in \mathcal{P} \quad (4.16) \\
\text{(Complementary slackness)} \quad \lambda_i(Q_i - Q_i^{\text{max}}) &= 0, \quad \mu_i(Q_i^{\text{min}} - Q_i) = 0, \quad \text{for } i \in \mathcal{P} \quad (4.17)
\end{align*}

where $\frac{\partial M_{i,p}}{\partial Q_i} = (4.11)$.

The optimality conditions can be interpreted as follows. The fraction $\frac{p - c_i}{h + p}$ is the service level that minimizes the total cost if the item cost is $c_i$, so it is referred to as the optimal service level. Since item costs are different in our problem, we have multiple “optimal service levels”. If the order size constraints (4.15) are not active ($Q_i^{\text{min}} < Q_i < Q_i^{\text{max}}$), then $\lambda_i = \mu_i = 0$ according to (4.16) and (4.17). From (4.14) and (4.11) we have

$$\sum_{t=1}^{m-1} \prod_{i \in \mathcal{P} / j} [\alpha t_i + (1 - \alpha t_i)(1 - t_i)] F(Q_j + \sum_{i \in \mathcal{P} / j} t_i Q_i) \frac{p - c_i}{h + p},$$

which implies that for each supplier, if its order is delivered successfully, but the delivery results of others are unknown, then the expected service level should equal to its optimal service level to achieve the minimum cost. If the order size constraints are active ($Q_i = Q_i^{\text{min}}$ or $Q_i = Q_i^{\text{max}}$), then $\lambda_i > 0$ or $\mu_i > 0$, and the expected service level deviates from the optimal service level (higher than the optimal service level because the minimum order constraint is enforced, or lower due to supply capacity). The quantity allocation is a function of disposal cost, penalty cost, demand distribution, all the suppliers’ reliability levels, item costs and order size constraints. Now we introduce methods that can be used to solve the optimality conditions.
4.3.1.1 The Primal Affine Scaling Algorithm

Since (4.10) is a linearly constrained convex programming problem, the Primal Affine Scaling (PAS) algorithm [14] can be applied to solve it. This algorithm is intended to solve quadratic programming problems and can be extended to solve general convex programming problems with linear constraints because a convex function can always be locally approximated by a quadratic function. The standard form of a quadratic programming problem is

\[
\begin{align*}
\text{(Primal)} & \quad \min & & \frac{1}{2} x^T D x + c^T x \\
\text{s.t.} & & A x = b, \ x \geq 0 \\
\text{(Dual)} & \quad \max & & -\frac{1}{2} x^T D x + b^T w \\
\text{s.t.} & & -Dx + A^T w + s = c, \ s \geq 0
\end{align*}
\]

with \( x \) being primal variables, \( w \) being dual variables and \( s \) being slack variables. The subproblem can be written in the standard form, where \( x = [Q; \phi; \varphi] = [Q; Q^{\text{max}} - Q; Q - Q^{\text{min}}] \), \( D + c = \nabla M(x) \), \( D = \nabla^2 M(x) \), \( b = [Q^{\text{max}}; Q^{\text{min}}] \), \( X = \text{diag}(x) \), \( A = \begin{bmatrix} I_n & I_n & 0 \\ -I_n & 0 & I_n \end{bmatrix} \).

The step-by-step implementation procedure is as follows.

- **Step 1 (initialization)** Set \( x^0 = [Q^{\text{min}} + Q^{\text{max}}]/2 \); Choose \( \epsilon_1, \epsilon_2, \epsilon_3 \) to be three sufficiently small positive numbers, and \( \beta < 1 \) to be a positive constant.

- **Step 2 (compute dual estimates)** Compute \( H_k = (\nabla^2 M(x) + X_k^{-2})^{-1} \), dual variables \( w^k = [AH_k A^T]^{-1} AH_k \nabla M(x^k) \), and dual slackness \( s^k = \nabla M(x^k) - A^T w^k \).

- **Step 3 (check for optimality)** If \( \frac{\|Ax^k - b\|}{\|b\| + 1} \leq \epsilon_1 \), and either \( s^k \geq 0 \) or \( \frac{\|s^k\|}{\|\nabla M(x^k)\| + 1} \leq \epsilon_2 \), and \( (x^k)^T s^k \leq \epsilon_3 \), then stop with an optimal solution \( x^k \) (to an accuracy of \( \epsilon_3 \)). Otherwise, go to step 4.

- **Step 4 (compute the translation direction)** \( d_k = -H_k s^k \).

- **Step 5 (compute step-length)** The maximum step size without violating the nonnegativity of \( x \) is
  \[
  \beta_k^1 = \min \left\{ \frac{\beta x_k^i}{d_k^i} \mid d_k^i < 0 \right\}.
  \]
  To compute the maximum step size without increasing the objective
value, use line search to find $\beta^2_k$ on approximating the minimizer of $M(\beta^2_k) = M(x^k + \beta^2_k d^k_x)$.

Set $\beta_k = \min\{\beta^1_k, \beta^2_k\}$.

- Step 6 (move to a new solution) Perform the translation $x^{k+1} = x^k + \beta_k d^k_x$. Set $k = k + 1$ and go to step 2.

The starting point $x^0$ can be any other feasible solution besides $[Q^\min + Q^\max]/2$. But note that the PAS algorithm is an interior point method, so a feasible initial solution on the border of feasible region might lead to a non-stationary solution. The accuracy of solutions obtained by the PAS algorithm is decided by the termination criteria $\epsilon = [\epsilon_1, \epsilon_2, \epsilon_3]$. Theoretically, the optimal solution can be obtained if $\epsilon$ is infinitely small. While in practice, the absolute optimal solution is impossible because the PAS algorithm terminates within a certain positive tolerance. Thus in the rest of the paper, results obtained by such algorithm are referred to as $\epsilon$-optimal solutions/costs. In the PAS algorithm, different starting points, different parameters such as $\epsilon$ and different step sizes in the line search method all have the possibility of leading to different $\epsilon$-optimal solutions/costs.

In step 5, a backtracking line search [7] is used to find the appropriate step length. It is an inexact line search method that is quite simple and effective. Using exact step length search methods in our problem would require more computational effort. The backtracking line search algorithm works as follows: given a descent direction $\triangle x$ for $M$, $r \in (0, 0.5)$, $v \in (0, 1)$, and a initial length $\beta$, we reduce the length by a factor $v$, i.e., $\beta = v\beta$, until the stopping condition $M(x + \beta \triangle x) \leq M(x) + r\beta \nabla M(x)^T \triangle x$ holds. The constant $r$ can be interpreted as the fraction of the decrease in $M$ predicted by linear extrapolation that we will accept. The parameter $r$ is typically chosen between 0.01 and 0.3, meaning that we accept a decrease in $M$ between 1% and 30% of the prediction based on a linear extrapolation. The parameter $v$ is often chosen to be between 0.1 (which corresponds to a very crude search) and 0.8 (which corresponds to a less crude search) [7].

The initial length $\beta^2$ is usually set to be $\beta^1$, which is the maximum step size without violating the nonnegativity of $x$ as described in step 5. The value of constant $v$ is usually chosen to be between 0.3 and 0.6 in our numerical experiments.
4.3.1.2 Free Variable Reduction

The basic idea of the heuristic Free Variable Reduction algorithm is to solve the unconstrained formulation, then fix the variable that deviates from the constraints most at the value of its corresponding constraint. This solve-and-fix process is repeated until $Q \in [Q_{\text{min}}, Q_{\text{max}}]$. We use this method because unconstrained continuous convex programming problems are usually easier and faster to solve than the corresponding constrained ones.

- Step 1: set $k = m$ and solve the unconstrained problem to obtain $Q$ using Newton’s method (see below).

- Step 2: let $\Delta u = \max(Q_i - Q_{i}^{\text{max}})$ and $\Delta l = \max(Q_{i}^{\text{min}} - Q_i)$. If $\Delta u \geq \Delta l$ and $\Delta u > 0$, go to step 3; If $\Delta u < \Delta l$ and $\Delta l > 0$, go to step 4; otherwise no constraint is violated, stop.

- Step 3: set the corresponding $Q_i$ to $Q_i^{\text{max}}$, if $k = 0$, stop; otherwise let $k = k - 1$, resolve the problem and go to step 2.

- Step 4: set the corresponding $Q_i$ to $Q_{i}^{\text{min}}$, if $k = 0$, stop; otherwise let $k = k - 1$, resolve the problem and go to step 2.

Newton’s method [7] is employed in step 1 to solve the unconstrained problems and the step-by-step implementation is as follows.

- Step 1.1 (initialization) Set $x^0 = [Q_{i}^{\text{min}} + Q_{i}^{\text{max}}]/2$; Choose $\epsilon$ to be a sufficiently small positive number.

- Step 1.2 (compute the direction and decrement) Compute the direction $d_x^k = -\nabla^2 M(x)^{-1}\nabla M(x^k)$, and the decrement $\lambda^2 = \nabla M(x^k)^T \nabla^2 M(x)^{-1} \nabla M(x^k)$.

- Step 1.3 (check for optimality) If $\lambda^2/2 \leq \epsilon$, then stop with an optimal solution $x^k$ (to an accuracy of $\epsilon$). Otherwise, go to step 4.
• Step 1.4 (compute step-length) To compute the maximum step size without increasing the objective value, use backtracking line search to find $\beta^k$ on approximating the minimizer of $M(\beta^k) = M(x^k + \beta^k d_x)$.

• Step 1.5 (move to a new solution) Update $x^{k+1} = x^k + \beta^k d_x$. Set $k = k + 1$ and go to step 2.

### 4.3.2 Methods Used to Obtain Discrete Solutions

Now we introduce three methods used to obtain integer solutions since fractional order quantities are not practical in reality.

#### 4.3.2.1 Restricted Local Search

The most intuitive method to obtain integer solutions is to search the immediate neighboring integer points of the continuous optimal solution, evaluate those points in the model with the corresponding discrete demand distribution, and regard the one with the minimum cost as the best solution. It works most of the time because continuous demand distributions are usually very good approximation of the discrete demand distributions, especially when the number of possible demand values is large. However, the optimal integer solution is not necessarily in the neighborhood of the optimal continuous solution even when the objective function is kept unchanged, let alone when it is discretized.

#### 4.3.2.2 Unrestricted Search

The shortcoming of the restricted local search method is that the optimal integer solution is not necessarily in the neighborhood of the optimal continuous solution. It can be improved by multiple local search, or unrestricted search. The basic idea is to start with a feasible integer solution, local search the feasible integer points around this given one, choose the best one among these points as the new starting point, and repeat this step until the solution converges, i.e., the new starting point is the same with the last one. If the objective function is not discrete convex, different initial solutions may lead to different local optimums. We know that the continuous objective function is convex from Lemma 4.1, but proof of discrete convexity of the explicit formulation of subproblem for $N > 1$ is very difficult. Our numerical
experiment of 3400 random subproblems shows that for each subproblem, four different random starting points always lead to the same solution. Therefore, single starting point is adopted when compared with the other methods. We can start this algorithm with a random feasible solution, or use a solution generated from the continuous problem. The computation time of the latter one is usually shorter. The corresponding continuous problem solved by the PAS algorithm with a relatively large stopping criterion $\epsilon$ usually quickly leads to a good enough initial solution for the unrestricted search method. Then integer feasible solutions around this point are evaluated and compared to determine the best one, which serves as the new initial solution. However, the algorithm may take a long time to converge if only the immediate neighborhood is searched every time, especially when the starting point is far away from the final solution and the number of suppliers is relatively large, since higher-dimension problems result in more local points. For example, $w^m$ points need to be evaluated for $m$ selected suppliers and $w$ integer order quantities from each supplier. Under such circumstances, the distance of the feasible solutions from the starting point is increased, which is equivalent to longer step size $\beta$ in the PAS algorithm. In numerical experiments, we start with a distance of $\beta = m = |P|$, which is the number of selected suppliers. For example, with the starting point $(20, 10)$ and the step size $\beta = 5$, eight points $(20, 15)$, $(20, 5)$, $(25, 10)$, $(15, 10)$, $(25, 15)$, $(25, 5)$, $(15, 15)$ and $(15, 5)$ are generated. Infeasible points are eliminated by comparing those generated points with the order size constraints. The algorithm terminates when the solution converges with $\beta = 1$, otherwise the search continues with the step size reducing to $\beta = \lceil \frac{\beta}{2} \rceil$. The detailed algorithm is described as follows:

- **Step 1 (pick a feasible starting point)** Let the starting point $x_0 = \lfloor \frac{Q^{\text{max}} + Q^{\text{min}}}{2} \rfloor$, or $x_0 = Q^*$, where $Q^*$ is the optimal solution of the corresponding continuous problem. Let the step size $\beta = m = |P|$.

- **Step 2 (enumerate feasible points)** Generate all the points $x$ which satisfy $\|x - x_0\| = \beta$ and $Q^{\text{min}} \leq x \leq Q^{\text{max}}$.

- **Step 3 (evaluate and determine the best solution among given points)** Let $x^* = \arg\min_x M(Q = x)$. If $x^* \neq x_0$, then let $x_0 = x^*$ and go to step 2. If $x^* = x_0$ and $\beta > 1$, then let $\beta = \lceil \frac{\beta}{2} \rceil$ and go to
step 2. Otherwise stop.

4.3.2.3 The Nelder-Mead Method

The Nelder-Mead algorithm proposed by [33], one of the most popular algorithms for multidimensional optimization problems, uses only function values and does not require derivative information, which makes it suitable for problems with non-smooth functions, such as ours. For problems with $n$ dimensions, it begins with a set of $n + 1$ points that are regarded as the vertices of a non-degenerate working simplex $S$ (convex hull), i.e., the $n + 1$ points must not lie in the same hyperplane. The corresponding set of function values at the vertices are evaluated and a sequence of transformations of the working simplex $S$ are performed to decrease the function values at its vertices. This process is repeated until the working simplex $S$ becomes sufficiently small, or when the function values are close enough to satisfy stopping criteria.

The Nelder-Mead Method is intended to solve continuous problems, but our problem is only defined on integer points. We can extend it to the continuous region by assigning any continuous solution the same function value of its floor-rounding integer solution:

$$M_{exp}(Q_P) = M_P(\lfloor Q_P \rfloor), \text{ for } Q_i^{\text{min}} \leq Q_i \leq Q_i^{\text{max}}, \ i = 1, 2, ..., N$$  \hspace{1cm} (4.18)

$M_{exp}(Q_P)$ is a continuous non-smooth function so it can be solved by the Nelder-Mead Method, whereas most algorithms used to solve optimization problems need gradient information, or try to form an approximate gradient at the point being evaluated.

One of the advantages of the Nelder-Mead method is that it frequently shows significant improvements in the first few iterations and quickly produces quite satisfactory results. In addition, it typically requires only one or two function evaluations per iteration, which is important to our problem since the function evaluation is not cheap due to the computation of an exponentially increasing number of delivery events. Therefore, this method is often faster than the other methods, especially the unrestricted search method which requires at least $2^m$ function evaluations per iteration when $m$ suppliers are se-
lected. However, the lack of convergence theory is often reflected in practice as a numerical breakdown of the algorithm, even for smooth and well-behaved functions. It can take an enormous number of iterations with negligible improvement in function value, despite being nowhere near a minimum, which usually results in premature termination of iterations [32]. In our implementation, we find that such cases happen when the starting point is assigned randomly, so that multiple random starting points are employed to restart the algorithm several times to deal with those cases. This, however, appears to be unnecessary when a relatively good starting point is given, such as the rounded solution from the continuous formulation.

The original Nelder-Mead algorithm is designed for unconstrained problems. With bounded variables, the vertices can leave the feasible region after certain transformation operations. Problems with constraints can be converted to unconstrained problems by introducing Lagrange multipliers, so that penalties related to constraint violations are added to the objective function. Or we can simply perform a projection to the bound when any value is outside of its bounds [26]. The latter approach is adopted in our implementation. The step-by-step implementation can be found in the appendix.

4.4 Determination of the Optimal Set of Suppliers: Master Problem

For a set of \( N \) potential suppliers, there are \( 2^N \) sets of selected suppliers (subproblems). In this section, we introduce the approaches which are used to determine the optimal set of suppliers with the subproblems solved by the algorithms proposed in the previous section. These approaches can be divided into two types: (1) the quantity allocation step and the supplier selection step are separate: the costs of different sets of suppliers are compared to determine the best set after the subproblems are solved, such as enumeration methods; (2) the quantity allocation and the selection of suppliers are combined as a two-step iterative algorithm: a initial set of suppliers is chosen and modified as order quantities are allocated and projected to the feasible region, such as the branch and bound methods.

Before these algorithms are discussed in detail, the continuous formulation is analyzed to get insights into changes of order allocation in different sets of suppliers.

Proposition 4.2. When suppliers are identical with no order size constraints, the total optimal order
quantity is not necessarily an increasing function of $N$, but the optimal order quantity from each supplier is a decreasing function of $N$, i.e., $Q_N^* < Q_{N-1}^*$ (here $Q_N^*$ is the order quantity from each supplier when there are $N$ identical suppliers.)

**Proof.** We know that

$$
\sum_{j=0}^{N-2} \frac{(N-2)!}{(N-j-2)!j!} \alpha^{N-j-2}(1-\alpha)^j F[(N-j)Q_{N-1}^*] \\
= \sum_{j=0}^{N-1} \frac{(N-1)!}{(N-j-1)!j!} \alpha^{N-j-1}(1-\alpha)^j F[(N-j)Q_N^*] = \frac{p-c}{h+p}
$$

If $Q_N^* \geq Q_{N-1}^*$,

$$
\sum_{j=0}^{N-1} \frac{(N-1)!}{(N-j-1)!j!} \alpha^{N-j-1}(1-\alpha)^j F[(N-j)Q_N^*] \\
= \sum_{j=0}^{N-2} \frac{(N-2)!}{(N-j-2)!j!} \alpha^{N-j-2}(1-\alpha)^j F[(N-j)Q_{N-1}^*] + (1-\alpha)F[(N-j-1)Q_N^*] \\
\geq \sum_{j=0}^{N-2} \frac{(N-2)!}{(N-j-2)!j!} \alpha^{N-j-2}(1-\alpha)^j F[(N-j)Q_{N-1}^*] + (1-\alpha)F[(N-j-1)Q_{N-1}^*] \\
> \sum_{j=0}^{N-2} \frac{(N-2)!}{(N-j-2)!j!} \alpha^{N-j-2}(1-\alpha)^j F[(N-j-1)Q_{N-1}^*]
$$

which contradicts with the previous equation. Therefore, it must be true that $Q_N^* < Q_{N-1}^*$.

When suppliers are different, i.e., they have different item costs and/or reliability levels, adding suppliers to the current set of selected suppliers could lead to increased individual order quantities. For example, when $\alpha = [0.82, 0.85, 0.9]$, $c = [1.2, 1.3, 1.4]$, $h = 0.5$, $p = 15$, beta demand distribution $a = 8$, $b = 2$, $u = 300$, $Q_{\min} = 0$ and $Q_{\max} = \infty$, the optimal order quantities of sourcing from supplier 2 and supplier 3 are $Q_2^* = 70.0$ and $Q_3^* = 223.0$ while the optimal order quantities of sourcing from all three suppliers are $Q_1^* = 124.1$, $Q_2^* = 119.4$ and $Q_3^* = 133.5$. Adding supplier 1 to the set $\mathcal{P} = \{2, 3\}$ leads to increasing in order quantity from supplier 2.
As a result, for one supplier in different sets, the optimal order quantities are usually different, and even the changing trend cannot be determined without knowing the parameters of other suppliers. The problem must be resolved to obtain the optimal solution every time the selected set of suppliers changes.

4.4.1 Enumeration of Subproblems

4.4.1.1 Total Enumeration

The most direct method to determine the optimal set of suppliers is to enumerate, compute and compare the costs of sourcing from the $2^N$ sets of selected suppliers for a set of $N$ potential suppliers. If the subproblem can be solved optimally, total enumeration approach also yields an optimal solution. When there is no minimum order constraint, only the subproblem of sourcing from all the suppliers, instead of $2^N$ subproblems, needs to be solved, although some $Q_i^*$ may turn out to be zero in the optimal solution.

**Proposition 4.3.** Adding supplier $j$ to the selected set of supplier $\mathcal{P}$ cannot lead to increased cost when $Q_j^\text{min} = 0$.

**Proof.** Suppose the optimal cost of sourcing from $\mathcal{P}$ is $M_\mathcal{P}(Q^*_\mathcal{P})$ and the optimal cost of the new set is $M_{\mathcal{P}'}(Q'^*_\mathcal{P}')$. If $Q'^*_j > 0$, then $M_{\mathcal{P}'}(Q'^*_\mathcal{P}') < M_\mathcal{P}(Q^*_\mathcal{P})$ must hold. Otherwise, the solution $Q'^*_i = Q^*_i$ for $i \in \mathcal{P}$ and $Q'^*_j = 0$, which leads to $M_{\mathcal{P}'}(Q'^*_\mathcal{P}') = M_\mathcal{P}(Q^*_\mathcal{P})$, would be a better solution. A more
rigorous proof is as follows:

\[ M_{P+j} - M_P = \sum_t \left\{ \prod_{i \in P+j} [\alpha_t (1 + (1 - \alpha_t)(1 - t_i)] \sum_{i \in P+j} t_i Q_i^* F(\sum_{i \in P+j} t_i Q_i^*) \right. \]

\[ - \int_0^{\sum_{i \in P+j} t_i Q_i^*} \xi f(\xi) d\xi - \sum_{i \in P+j} \frac{p - c_i}{h + p} t_i Q_i^* \} \right\} + pE(\xi) - \left\{ \sum_{i \in P} \sum_{t \in P} [\alpha_t (1 - \alpha_t)(1 - t_i)](h + p) \sum_{i \in P} t_i Q_i^* F(\sum_{i \in P} t_i Q_i^*) \right. \]

\[ + \int_0^{\sum_{i \in P} t_i Q_i^*} \xi f(\xi) d\xi \]

\[ - \sum_{i \in P} \frac{p - c_i}{h + p} t_i Q_i^* \}

\[ = (1 - \alpha_j) \left\{ \sum_t \left\{ \prod_{i \in P} [\alpha_t t_i + (1 - \alpha_t)(1 - t_i)] \sum_{i \in P} t_i Q_i^* F(\sum_{i \in P} t_i Q_i^*) \right. \]

\[ - \sum_{i \in P} t_i Q_i^* F(\sum_{i \in P} t_i Q_i^*) - \int_0^{\sum_{i \in P} t_i Q_i^*} \xi f(\xi) d\xi + \int_0^{\sum_{i \in P} t_i Q_i^*} \xi f(\xi) d\xi \]

\[ - \sum_{i \in P} \frac{p - c_i}{h + p} t_i Q_i^* \}

\[ \leq (1 - \alpha_j) \left\{ \sum_t \left\{ \prod_{i \in P} [\alpha_t t_i + (1 - \alpha_t)(1 - t_i)] \sum_{i \in P} t_i Q_i^* - \sum_{i \in P} t_i Q_i^* \right. \]

\[ - \sum_{i \in P} \frac{p - c_i}{h + p} t_i Q_i^* \right\} + \sum_{i \in P} \sum_{t \in P} \alpha_j \{ \prod_{i \in P} [\alpha_t t_i + (1 - \alpha_t)(1 - t_i)] \}

\[ \sum_{i \in P} t_i Q_i^* - \sum_{i \in P} t_i Q_i^* \right\} F(\sum_{i \in P} t_i Q_i^*) \]

\[ + \frac{p - c_i}{h + p} t_i Q_i^* \}

\[ = \sum_{i \in P} \alpha_i (Q_i^* - Q_i^*) \left\{ \prod_{k \in P+j-i} \{\alpha_k t_k + (1 - \alpha_k)(1 - t_k)\} F(\sum_{k \in P+j-i} t_k Q_k^* + Q_i^*) \right. \]

\[ - \frac{p - c_i}{h + p} \}

\[ + \alpha_j Q_j^* \} \sum_{i \in P} \left\{ \prod_{i \in P} [\alpha_t t_i + (1 - \alpha_t)(1 - t_i)] \}

\[ \sum_{i \in P} t_i Q_i^* + Q_j^* \right\} F(\sum_{i \in P} t_i Q_i^*) \]

\[ = \sum_{i \in P} \alpha_i (Q_i^* - Q_i^*) \frac{\partial M_{P+j}}{\partial Q_i^*} + \alpha_j Q_j^* \frac{\partial M_{P+j}}{\partial Q_j^*} \]

\[ = 0 \]
The first inequality follows due to the property of convex functions. The last equality is based on the optimality conditions $Q_i^* \frac{\partial M_{P_i}}{\partial Q_i} |_{Q_i^*} = 0.

Therefore, sourcing from more suppliers does not result in increased cost when $Q_{\min} = 0$ and solving the subproblem of sourcing from all the suppliers can obtain the optimal solution.

4.4.1.2 Partial Enumeration: A Cutting Rule

Although $N$ should not be a very large number in reality, solving $2^N$ nonlinear problems can be time-consuming even for a realistic value of $N$. For example, the average computation time of each problem with $N = 9$ is about 20 minutes when the subproblem is solved by the PAS algorithm with $\epsilon = 0.001$ in our numerical experiment. It takes significantly longer when the unrestricted search is employed for subproblems. As a substitution, partial enumeration approach can be applied.

For those problems where $Q_{\min} > 0$ for at least one supplier, a cutting rule is proposed to reduce the number of subproblems that need to be computed. Through extensive numerical experiments we find that if adding supplier $j$ to a set of suppliers $P$ leads to increased cost, then adding this supplier to any superset $\overline{P} \supset P$ usually results in increased cost as well. Since exceptions rarely occur, we employ this result as a cutting rule in a partial enumeration approach. As long as sourcing from the set $P$ costs more than from any subset $P$, sourcing from any superset $\overline{P} \supset P$ usually costs more than sourcing from $P$. Consequently, all the subproblems of sourcing from the supersets $\overline{P}$ can be cut without computation. The basic idea of the partial enumeration method is to first create a candidate problem list by enumerating all $2^N$ sets of suppliers, then begin with the candidate problems with the fewest number of suppliers. For each candidate problem, if a discarded set of suppliers is a subset of the current set, the current set is also discarded (no computation required). Otherwise the candidate problem is solved. If it costs more than sourcing from any of its subsets, it is discarded. The partial enumeration is a heuristic method since the optimal set of suppliers may be the one having been eliminated.

When the number of selected suppliers $m \in \{0, 1\}$, the algorithms used to solve subproblems do
Create \( Lst_k \) which has \( \nu = \binom{C}{k} \) sets and let \( j = 1 \).

\[ k = 0 \quad M^* = \infty \quad Lst_d = \emptyset \]

\[ \exists P \text{ s.t. } P \subset S^j_k \quad \text{and } P \in Lst_d \]

Solve the subproblem of sourcing from the set \( S^j_k \) and obtain the optimal policy \( Q^j_k \) and cost \( M^j_k \).

If \( M^* > M^j_k \), then \( M^* = M^j_k \) and \( Q^* = Q^j_k \).

\[ \exists S^k_{k-1} \subset S^j_k \quad \text{and } M^j_k \geq M^k_{k-1} ? \]

\[ j = j + 1 \]

\[ j = \nu ? \]

\[ k = k + 1 \]

\[ k > N ? \]

Stop

\[ \exists P \text{ s.t. } P \subset S^j_k \quad \text{and } P \in Lst_d \]

\[ j = j + 1 \]

Put \( S^j_k \) in \( Lst_d \)

Figure 4.1: Flow Chart of the Enumeration with the Cutting Rule Algorithm
not need to be implemented to obtain the optimal cost and quantity since the explicit expressions exist:

\[ M_0^* = L(0) = pE(\omega) \]  \hspace{1cm} (4.19)

and

\[ M_1^* = \alpha_ic_iq_i^* + \alpha_iL(q_i^*) + (1-\alpha)L(0), \]  \hspace{1cm} (4.20)

where

\[ q_i^* = \arg \min_{Q_i} \left\{ Q_i = \max\{Q_i^{\min}, \min\{\left\lfloor F^{-1}\left(\frac{p-c_i}{n+p}\right)\right\rfloor, Q_i^{\max}\}\} \right\} \]

\[ Q_i = \max\{Q_i^{\min}, \min\{\left\lceil F^{-1}\left(\frac{p-c_i}{n+p}\right)\right\rceil, Q_i^{\max}\}\} \]

The best case of this method is to compute the \( N+1 \) subproblems with \( m \in \{0, 1\} \) by (4.19) and (4.20), and all the single-source cases cost more than ordering nothing so the other subproblems are discarded without computation. The worst case is to compute all the other \( 2^N - N - 1 \) subproblems, just like total enumeration.

The step-by-step implementation procedure is as follows and the flow chart is shown in Figure 4.1:

- **Step 1** (initialization) Set \( m = 1 \), the optimal cost \( M^* = pE(\omega) \), the optimal order policy \( Q^* = 0 \), the discarded problem list \( Lst_d = \emptyset \).

- **Step 2** (generate candidate problems) If \( m > N \), stop. Otherwise create a candidate problem list \( Lst_m \) which has \( v = \frac{N!}{m!(N-m)!} \) subproblems and let \( j = 1 \).

- **Step 3** (discard or solve current subproblem) The current set of suppliers is \( S_m^j \). If \( \exists \mathcal{P} \text{ s.t. } \mathcal{P} \subset S_m^j \) and \( \mathcal{P} \subset Lst_d \), then put \( S_m^j \) in \( Lst_d \) and go to step 5. Otherwise solve the subproblem to obtain the solution \( Q_m^j \) and \( M_m^j \).

- **Step 4** (update optimal solution or discard set) If \( M_m^j < M^* \), then \( M^* = M_m^j \) and \( Q^* = Q_m^j \). If \( \exists S_{m-1}^j \text{ s.t. } S_{m-1}^j \subset S_m^j \) and \( M_{m-1}^j < M_m^j \), put \( S_m^j \) in \( Lst_d \).

- **Step 5** (move to the next subproblem) \( j = j + 1 \). If \( j > v \), let \( m = m + 1 \) and go to step 2. Otherwise go to step 3.
4.4.2 Branch and Bound

Another common way to solve integer programming problems is branch and bound (B&B). Two branch and bound methods are proposed in this paper. We branch either the decision variable $z_i$, or the feasible region of $Q_i$. Bounds are provided by relaxation subproblems. If relaxation subproblems can be solved optimally, branch and bound approaches also yield an optimal solution.

4.4.2.1 Branch and Bound with $z$

In this branch and bound algorithm, we branch the decision variable $z$, and bounds are provided by solving relaxation problem with binary variable $z$ relaxed. The relaxation problem is as follows:

$$\begin{align*}
\min & \quad M_c(z, Q) \\
\text{s.t.} & \quad z_iQ_i^{\min} \leq Q_i \leq z_iQ_i^{\max}, \quad i \in \{1, 2, \ldots, N\} \\
& \quad 0 \leq z_i \leq 1, \quad Q_i \in \mathbb{R}, \quad i \in \{1, 2, \ldots, N\}
\end{align*}$$

(4.21)

Binary branching strategy and best-bound candidate problem selection rule are used in the hope of minimizing the number of candidate problems. The branching variable selection rule is to choose the least fractional variable $z_c = \arg\min_{i \in p} [z_i, 1 - z_i]$.

The step-by-step implementation procedure is as follows:

- **Step 1** (initialize and solve the root problem) Let $k = 0$. Solve the relaxation problem (4.21). The solution is $M^k$ and $x^k = [z^k, Q^k]$. Let $M^* = M^k$, $z^* = z^k$ and $Q^* = Q^k$. If $z_i \in \{0, 1\}$ for $i \in \{1, 2, \ldots, N\}$, stop. Otherwise choose the $z_c = z_i$ where $\arg\min_{i \in p} [z_i, 1 - z_i]$ and put two relaxation problems $RP_{k0}$ with $z_c = 0$ and $RP_{k1}$ with $z_c = 1$ in the candidate problem list $Lst_c$.

- **Step 2** (solve child node problems) solve $RP_{k0}$ and $RP_{k1}$ to obtain the optimal values $M^{k0}$ and $M^{k1}$, and delete them from $Lst_c$.

- **Step 3** (prune or branch) For $v \in \{k0, k1\}$,
  - If $RP_v$ is infeasible, prune the branch by infeasibility.
If \( z_i \in \{0, 1\} \) for \( i \in \{1, 2, \cdots, N\} \), update incumbent \( Q^* = Q^v \) and prune the branch by optimality.

If there is any \( z_i \notin \{0, 1\} \), \( M_P(x^v) \) serves as the lower bound of current branch. If \( M_P(Q^v) > M^* \), prune by bound. Otherwise \( k = k + 1 \), choose the \( z_c = z_i \) where \( \arg\min_{i \in P} [z_i, 1 - z_i] \) and put the problems \( RP_{k, 0} \) with \( z_c = 0 \), and \( RP_{k, 1} \) with \( z_c = 1 \) in \( Lst_c \).

- Step 4 (terminate or select a new node to solve) If \( Lst_c = \emptyset \), stop and incumbent \( Q^* \) is optimal. Otherwise choose the node \( k \) with the best bound and unsolved child nodes and go to step 2.

Figure 4.2 is an example with the number of suppliers \( N = 3 \).

Figure 4.2: A Numerical Example of Branch and Bound with \( z \). Parameters are as follows: beta demand distribution \( \text{Beta}(2, 4) \times 100 \), reliability levels \( \alpha = [0.8, 0.85, 0.9] \), item costs \( c = [1, 1.2, 1.4] \), disposal cost \( h = 0.3 \), penalty cost \( p = 7 \), minimum order size constraints \( Q^{min} = [20, 30, 40] \), supply capacities \( Q^{max} = [87, 90, 100] \).

Note that the root node always has \( 2N \) variables, and any node with \( k_1 \) fixed \( z_i = 1 \) and \( k_2 \) fixed \( z_i = 0 \) has \( 2N - k_1 - 2k_2 \) variables. The computational time of relaxation problems is usually longer as the number of variables increases. The values of the decision variables have significant influence on
the order size constraints, hence the values of order quantities. As a result, more candidate problems need to be solved since the lower bounds provided by the relaxation problems are not very tight. The decision variables $z$ can be removed so that computational time can be reduced and bound errors can be eliminated, which leads to the second branch and bound method.

4.4.2.2 Branch and Bound without $z$

In this method, $Q$ is branched and bounds are provided by solving the relaxation problem that expands the feasible region of $Q$. In the original problem, $Q$ has two feasible regions: 0 and $[Q_{\text{min}}, Q_{\text{max}}]$. The binary variable $z$ is used to indicate which feasible region $Q$ belongs to. The relaxation problem is formed by extending $Q$ to $[0, +\infty]$, and lower bounds are obtained by solving the relaxation problem. The best-bound strategy is still used as the candidate problem selection rule in the hope of minimizing the number of candidate problems. Then we branch the selected $Q_i$ to the two feasible regions. Let $P_u$ and $P_s$ be the set of unselected suppliers and the set of selected suppliers, respectively. If $i \in P_u$, then $Q_i = 0$; If $i \in P_s$, then the constraint $Q_{i,\text{min}} \leq Q_i \leq Q_{i,\text{max}}$ is added to the relaxation problem; Otherwise $Q_i \geq 0$. The relaxation problem is as follows:

$$\begin{align*}
\min & \quad M_c(Q) \\
\text{s.t.} & \quad Q_{i,\text{min}} \leq Q_i \leq Q_{i,\text{max}} \quad \text{for} \quad i \in P_s \\
& \quad Q_i = 0 \quad \text{for} \quad i \in P_u \\
& \quad Q_i \geq 0 \quad \text{for} \quad i \notin P_u, \ i \notin P_s, i \in U
\end{align*}$$

(4.22)

The basic idea of this method is to solve the problem with minimum constraints first ($P_u = P_s = \emptyset$), then compare the solution with $Q_{\text{min}}$ and $Q_{\text{max}}$. If the solution satisfies all the order size constraints, it is optimal. Otherwise we branch the $Q_i$ that deviates most from the constraints, which leads to two nodes (binary branching): one with the additional constraint $Q_i = 0$ (supplier $i$ is not selected) and the other one with the additional constraint $Q_{i,\text{min}} \leq Q_i \leq Q_{i,\text{max}}$ (supplier $i$ is selected). These steps are repeated until the optimal solution is found. The PAS algorithm is again employed to solve the relaxation
The step-by-step implementation procedure is as follows:

- **Step 1** (initialize and solve the root problem) Let $\mathcal{P}_u = \emptyset$, and $k = 0$. Solve the relaxation problem. If infeasible, stop. Otherwise obtain the solution $M^0$ and $Q^0$. Let $M^* = M^0$ and $Q^* = Q^0$. If $Q^0_{\min} \leq Q^* \leq Q^0_{\max}$, stop. Otherwise choose the $Q_i$ where $\argmax_{i \in \mathcal{U}} \{Q_i - Q_i^{\max}, \min\{Q_i, Q_i^{\min} - Q_i\}\}$ and put two relaxation problems $RP_{k0}$ with $i \in \mathcal{P}_u^k$ and $RP_{k1}$ with $i \in \mathcal{P}_s^k$ in the candidate problem list $Lst_c$.

- **Step 2** (solve child node problems) solve $RP_{k0}$ and $RP_{k1}$ to obtain the optimal values $M^{k0}$ and $M^{k1}$, and delete them from $Lst_c$.

- **Step 3** (prune or branch) For $v \in \{k0, k1\}$,
  
  - If $RP_v$ is infeasible, prune the branch by infeasibility.
  
  - If all the order size constraints are satisfied, i.e., $Q_i^{\min} \leq Q_i^v \leq Q_i^{\max}$ or $Q_i^v = 0$, and $M_P(Q^v) < M^*$, update incumbent $Q^* = Q^v$ and prune the branch by optimality.
– If order size constraints are not satisfied, \( M_P(Q^v) \) serves as the lower bound of current branch. If \( M_P(Q^v) > M^* \), prune by bound. Otherwise \( k = k + 1 \), choose the \( Q_i \) where 
\[
\arg\max_{i \in \mathcal{U}} \{Q_i - Q_i^{max}, \min\{Q_i, Q_i^{min} - Q_i\}\}
\]
and put the problems \( RP_{k0} \) with \( P_u^k = \{i, P^u_i\}, P_s^k = P^v_s \), and \( RP_{k1} \) with \( P_u^k = \{i, P^v_i\}, P_s^k = P^v_u \) in \( Lst_c \).

• Step 4 (terminate or select a new node to solve) If \( Lst_c = \emptyset \), stop and incumbent \( Q^* \) is optimal. Otherwise choose the node \( k \) with the best bound and unsolved child nodes and go to step 2.

Figure 4.3 is an example with the number of suppliers \( N = 3 \).

### 4.5 Master Problem-Subproblem Methods

Now we discuss the combinations of master problem methods and subproblem methods to obtain integer solutions. Integer solutions can be obtained at two levels: the subproblem level, or the master problem level. In the former, an integer solution is obtained for each subproblem. In the latter, optimal or near-optimal continuous solutions are obtained from the continuous formulation (4.8), and then refined to be integer solutions.

Unrestricted search method (US) and the Nelder-Mead method (NM) can be used to acquire integer solutions at the subproblem level. The starting points of these two methods can be provided by either the Primal Affine Scaling (PAS) algorithm with very relaxed termination criteria, or random generation. It usually takes longer to use random generated starting points. NM is more likely to have large deviations if the starting point is not good so multiple starting points (MSP) may be needed to restart the algorithm several times. Restricted local search (RLS) can be used to acquire integer solutions at the subproblem level, but it requires a very good starting point and does not guarantee optimality since it only searches very limited local area around a given point. Thus, RLS is usually used to obtain integer solutions at the master problem level, with near-optimal continuous solutions provided by either PAS or free variable reduction method (FVR). RLS is near-optimal because it does not change the selection statuses of suppliers. To put in another way, if one supplier is not selected in the optimal solution of the continuous problem, the order quantity from this supplier is still 0 in the searched points. Nevertheless, the optimal
integer solution may be in another set of suppliers.

The enumeration methods can be combined with all the subproblem methods mentioned above. In branch and bound methods, continuous formulations of relaxation problems (subproblems) are employed because explicit formulations are not defined on continuous regions then relaxation problems do not exist. The continuous formulations of subproblems can be solved by PAS or FVR. PAS is better than FVR since FVR is a heuristic method. Acquiring integer solutions at the subproblem level in B&B with $z$ is impossible since $z$ is continuous in relaxation problems. Acquiring integer solutions at the subproblem level in B&B without $z$ is not recommended because integer solutions are evaluated in explicit formulations, which may cause wrongly selected branches due to deviations in bounds. As a result, integer solutions can only be obtained at the master problem level.

Table 4.2 summarizes the combinations of methods that may work well.

<table>
<thead>
<tr>
<th>Master Problem Solution</th>
<th>Starting Point Solution</th>
<th>Integer Solution at the Subproblem Level</th>
<th>Integer Solution at the Master Problem Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE</td>
<td>PAS</td>
<td>US</td>
<td>-</td>
</tr>
<tr>
<td>TE</td>
<td>PAS</td>
<td>NM</td>
<td>-</td>
</tr>
<tr>
<td>TE</td>
<td>MSP</td>
<td>NM</td>
<td>-</td>
</tr>
<tr>
<td>TE</td>
<td>PAS</td>
<td>-</td>
<td>RLS</td>
</tr>
<tr>
<td>TE</td>
<td>FVR</td>
<td>-</td>
<td>RLS</td>
</tr>
<tr>
<td>PE</td>
<td>PAS</td>
<td>US</td>
<td>-</td>
</tr>
<tr>
<td>PE</td>
<td>PAS</td>
<td>NM</td>
<td>-</td>
</tr>
<tr>
<td>PE</td>
<td>MSP</td>
<td>NM</td>
<td>-</td>
</tr>
<tr>
<td>PE</td>
<td>PAS</td>
<td>-</td>
<td>RLS</td>
</tr>
<tr>
<td>PE</td>
<td>FVR</td>
<td>-</td>
<td>RLS</td>
</tr>
<tr>
<td>B&amp;B with $z$</td>
<td>PAS</td>
<td>-</td>
<td>RLS</td>
</tr>
<tr>
<td>B&amp;B without $z$</td>
<td>PAS</td>
<td>-</td>
<td>RLS</td>
</tr>
</tbody>
</table>
4.6 Numerical Experiments

Numerical experiments were carried out to evaluate and compare the performance of the methods described in the previous section. They were coded in MATLAB R2010a and run on a computer with an Intel(R) Core(TM) i7 CPU 870 2.93GHz processor, 8.00GB RAM, and 64-bit Windows 7 operating system.

In this comparison study, three cases with \( N \in \{3, 5, 7\} \) were tested, and 100 problems were generated randomly for each case. Table 4.3 shows the values of the parameters used to generate the numerical experiments. The specific parameters for each problem were randomly sampled from these values. The reliability level is no lower than 0.5 since a retailer would not consider a supplier whose historical delivery performance is so poor in practice. For continuous formulations, demand follows beta distributions, i.e., \( \xi \sim Beta(a, b) \times u \). For discrete formulations, beta demand distributions are discretized as follows.

\[
P_r(\omega) = F(\omega + 0.5) - F(\omega - 0.5), \quad \omega = 0, 1, 2, ..., u
\]

Table 4.3: The Parameter Pool for Numerical Experiments

<table>
<thead>
<tr>
<th>( a, b \in {1, 2, 4, 8, 12, 16} )</th>
<th>( u = {100 : 100 : 1100} )</th>
<th>( h = {0.5 : 0.5 : 3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_i = {0.5 : 0.05 : 1} )</td>
<td>( p = {4 : 3 : 25} )</td>
<td>( c_i = {0.8 : 0.1 : 2} )</td>
</tr>
</tbody>
</table>

The order size constraints \( Q_{\text{min}} \) and \( Q_{\text{max}} \) are obtained by sampling values from beta distributions, i.e., \( Q_{\text{min}} \sim Beta(2, 7) \times u \) and \( Q_{\text{max}} \sim Beta(2, 2) \times u \) (Figure 4.4 shows the sample distributions with \( u = 100 \)). The distribution from which \( Q_{\text{min}} \) is sampled is right-skewed so that relatively small values have higher probabilities to be sampled. Relatively small order sizes are more reasonable constraints because the order is expected to be spread among multiple suppliers. High maximum order size constraints may never be active and high minimum order size constraints may lead to the selection of very few suppliers. A matrix with two rows and \( N \) columns is generated, with the first row sampled from \( Beta(2, 7) \times u \) and the second row sampled from \( Beta(2, 2) \times u \). These numbers are rounded to
integers, and sorted by column to guarantee that the value in the first row is greater or equal to the value in the second row in each column. Then the first row is assigned as the maximum order size constraints and the second row the minimum order size constraints.

![Figure 4.4: The Beta Distribution Used to Generate $Q_{\text{min}}$ and $Q_{\text{max}}$.](image)

Part of the methods listed in Table 4.2 are tested. TE-FVR-RLS and PE-MSP-NM are not included because from PE-FVR-RLS and TE-MSP-NM we can have a general idea about their results. If the deviations are acceptable in PE-FVR-RLS, the results of TE-FVR-RLS must be no worse but the computation time is larger. Compared with TE-MSP-NM, PE-MSP-NM has the same or larger deviations but faster.

- **TE-PAS-US**: Total enumeration of $2^N$ sets of suppliers with the unrestricted search method embedded to solve the subproblems. The starting point of each subproblem is obtained by using the PAS algorithm to solve the corresponding continuous subproblem with termination criteria $\epsilon = 1$.

- **TE-PAS-NM**: Total enumeration of $2^N$ sets of suppliers with the Nelder-Mead method embedded to solve the subproblems. The starting point of each subproblem is obtained by using the PAS algorithm to solve the corresponding continuous subproblem with termination criteria $\epsilon = 1$.

- **TE-MSP-NM**: Total enumeration of $2^N$ sets of suppliers with the Nelder-Mead method embedded
to solve the subproblems. Each subproblem is solved twice using two randomly generated starting points.

- **TE-PAS-RLS:** Restricted local search the integer solutions around the $\epsilon$-optimal solution obtained by the corresponding continuous problem using TE. The PAS algorithm is used to solve the continuous subproblems with $\epsilon = 0.001$.

- **PE-PAS-US:** Partial enumeration of the subproblems using the cutting rule. The rest is the same with TE-PAS-US.

- **PE-PAS-NM:** Partial enumeration of the subproblems using the cutting rule. The rest is the same with TE-PAS-NM.

- **PE-PAS-RLS:** Partial enumeration of the subproblems using the cutting rule. The rest is the same with TE-PAS-RLS.

- **PE-FVR-RLS:** Restricted local search the integer solutions around the best solution obtained by the corresponding continuous problem using PE. The Free Variable Reduction method is used to solve the continuous subproblems with $\epsilon = 0.001$.

- **B&B w/ z-PAS-RLS:** Restricted local search the integer solutions around the $\epsilon$-optimal solution obtained by the corresponding continuous formulation using the branch and bound with $z$. The PAS algorithm is used to solve the continuous relaxation problems with $\epsilon = 0.001$.

- **B&B w/o z-PAS-RLS:** Restricted local search the integer solutions around the $\epsilon$-optimal solution obtained by the corresponding continuous formulation using the branch and bound without $z$. The PAS algorithm is used to solve the continuous relaxation problems with $\epsilon = 0.001$.

Computation time of the methods used to solve subproblems, relaxation problems or unconstrained problems not only depends on specific parameters, but also the number of variables and the step length computation method. Table 4.4 and Figure 4.5, 4.6, 4.7 show the numerical experiment results.

Table 4.4 shows that TE-PAS-US always obtain the best solution among all the methods as expected since there is no negative value in the deviation columns. TE-PAS-US usually takes longer than the
Table 4.4: Results of Integer Models with Integer Solutions

<table>
<thead>
<tr>
<th>Deviation from TE-PAS-US (%)</th>
<th>Avg.</th>
<th>Std.</th>
<th>Min</th>
<th>Max</th>
<th>Avg.</th>
<th>Std.</th>
<th>Min</th>
<th>Max</th>
<th>Avg.</th>
<th>Std.</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE-PAS-NM</td>
<td>0.045</td>
<td>0.243</td>
<td>0.000</td>
<td>2.294</td>
<td>0.066</td>
<td>0.314</td>
<td>0.000</td>
<td>2.675</td>
<td>0.098</td>
<td>0.340</td>
<td>0.000</td>
<td>1.872</td>
</tr>
<tr>
<td>TE-MSP-NM</td>
<td>0.127</td>
<td>0.682</td>
<td>0.000</td>
<td>6.017</td>
<td>0.044</td>
<td>0.180</td>
<td>0.000</td>
<td>1.553</td>
<td>0.060</td>
<td>0.213</td>
<td>0.000</td>
<td>1.907</td>
</tr>
<tr>
<td>PE-PAS-US</td>
<td>0.000</td>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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<td>0.000</td>
<td>2.675</td>
<td>0.098</td>
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<td>0.000</td>
<td>1.872</td>
</tr>
<tr>
<td>PE-FVR-RLS</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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<td>0.000</td>
</tr>
<tr>
<td>B&amp;B w/o z-PAS-RLS</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.044</td>
<td>0.435</td>
<td>0.000</td>
<td>4.355</td>
<td>0.098</td>
<td>0.340</td>
<td>0.000</td>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.044</td>
<td>0.435</td>
<td>0.000</td>
<td>4.355</td>
<td>0.098</td>
<td>0.340</td>
<td>0.000</td>
<td>1.872</td>
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<th>Std.</th>
<th>Min</th>
<th>Max</th>
<th>Avg.</th>
<th>Std.</th>
<th>Min</th>
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<th>Avg.</th>
<th>Std.</th>
<th>Min</th>
<th>Max</th>
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<td>0.25</td>
<td>0.24</td>
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<td>9.00</td>
<td>6.87</td>
<td>3.53</td>
<td>40.67</td>
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<td>9.14</td>
<td>3.43</td>
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<td>0.76</td>
<td>0.45</td>
<td>3.62</td>
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<td>6.87</td>
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<td>297.55</td>
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<td>0.19</td>
<td>0.17</td>
<td>1.13</td>
<td>2.89</td>
<td>2.94</td>
<td>0.17</td>
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<td>11.69</td>
<td>12.57</td>
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</tr>
<tr>
<td>B&amp;B w/ z-PAS-RLS</td>
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<td>0.09</td>
<td>0.79</td>
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<td>2.43</td>
<td>0.12</td>
<td>12.20</td>
<td>8.33</td>
<td>10.84</td>
<td>0.18</td>
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other methods for a relatively large $N$. PE-PAS-US is much faster since a considerable number of sets are eliminated, which may lead to a solution worse than that of TE-PAS-US once in a while (see the scatter plots: Figure 4.5, 4.6 and 4.7). PE, as a general supplier selection method, may be used if $Q_{min}$ and $Q_{max}$ are relatively higher so fewer suppliers may be selected and more cases may be cut. For the unrestricted search method, the best solution of the subproblem can be obtained regardless of the starting point, although a random starting point usually results in longer computation time.

TE-PAS-NM is slower than TE-PAS-US for $N = 3$, but faster when the number of suppliers $N$ increases. It obtains near-best solutions sometimes and the worst case deviation is 2.675%. PE-PAS-NM is even faster, but the worst case also becomes 4.8%. TE-MSP-NM takes the longest time among all the methods, but it has no requirement about the starting points. The running time of the search method is significantly longer than that of the Nelder-Mead method if random starting points are given. Through numerical experiments we find that a good starting point usually not only shortens computation time but also leads to a better solution when using the Nelder-Mead method. The maximum deviation is 2.675% when the starting point is given by the PAS algorithm, while it becomes 6.017% when two
random initial solutions are used to start the algorithm. A single random starting point shortens the average computation time by about half, but the deviation becomes even bigger.

As we mentioned in the previous section, RLS does not guarantee optimality since local search does not change the selection statuses of suppliers, and a better solution may exist with another set of suppliers. TE-FVR-RLS and PE-PAS-RLS are consistently better than the other 8 methods for \( N \in \{3, 5, 7\} \). They take less time, the maximum deviation is less than 5% and the best solution can be obtained most of the time (see the scatter plots: Figure 4.5, 4.6 and 4.7). The performance of the B&B-PAS-RLS methods and TE-PAS-RLS lie in the middle among all the methods.

Fewer suppliers may be selected as \( Q^{\text{min}} \) increases, and more suppliers may be selected to satisfy demand when \( Q^{\text{max}} \) decreases. Therefore, when the order size constraints are relatively relaxed, the B&B methods, which start with solving problem of all the suppliers selected, may be faster than PE. Under such cases, the cutting rule may not take effect, so few cases may be eliminated in PE, whereas B&B may solve fewer relaxation problems and unconstrained problems.

Overall, when the number of suppliers is relatively small, TE-PAS-US can be used since it is quick and always lead to the best solutions. For a relatively large pool of suppliers, PE-FVR-RLS, PE-PAS-RLS and PE-PAS-NM, are able to produce a good enough solution fast. When a good starting point is relatively expensive to obtain, TE-MSP-NM is an ideal method to choose.

### 4.7 Conclusions

In this chapter, we address the issue of selecting suppliers from multiple unreliable suppliers and allocating orders among them to satisfy stochastic demand and minimize total cost in the newsvendor setting. The suppliers may have different item costs, requirements on minimum order sizes and limited supply capacities. Supplier reliability is modeled as delivery probability. We formulate this problem as a discrete model and prove it to be NP-hard.

The general model is divided into two parts: (1) master problem: to select a set of suppliers; (2) subproblem: to determine the order quantity allocation given a set of selected suppliers. Five methods are proposed to solve subproblems, of which the Primal Affine Scaling algorithm and Free Variable Re-
duction are used to solve the continuous formulations of subproblems, while Restricted Local Search, Unrestricted Search and Nelder-Mead method are used to obtain integer solutions. Theoretically, PAS can guarantee optimality, but the termination criteria cannot be infinitely small so that only $\epsilon$-optimal solutions can be obtained. FVR, RLS and NM are heuristic methods. US always obtain the best solution empirically, but proof of optimality is very difficult. Enumeration approaches and Branch and Bound methods are proposed to solve master problems. Given optimal solutions of subproblems, Total Enumeration and the Branch and Bound methods can lead to optimal solutions for master problems, while Partial Enumeration may produce near-optimal solutions.

Appropriate combinations of these methods are discussed and implemented in numerical experiments to show their performances in solving this model. We conclude that when the number of supplier is relatively small, TE-PAS-US can be used since it is quick and always lead to the best solutions. For a relatively large pool of suppliers, PE-FVR-RLS, PE-PAS-RLS and PE-PAS-NM, are able to produce a good enough solution fast. When a good starting point is relatively expensive to obtain, TE-MSP-NM is an ideal method to choose.
Figure 4.5: Numerical Results: Scatter Plot of Computation Time and Deviation from TE-PAS-US for $N = 3$
Figure 4.6: Numerical Results: Scatter Plot of Computation Time and Deviation from TE-PAS-US for $N = 5$
Figure 4.7: Numerical Results: Scatter Plot of Computation Time and Deviation from TE-PAS-US for $N = 7$
Chapter 5

Conclusions and Future Work

Our research extends the literature on multiple-sourcing under supply risk by incorporating stochastic demand and fixed order costs, or order size constraints, in a single model. Supplier reliability is modeled as the probability of delivery, which implies that with a certain probability, the supplier fails to fulfill the entire order. Suppliers’ binomial delivery probability complicates the problem because the optimal order quantity allocation is more difficult to determine compared with order quantity allocation problems with perfectly reliable suppliers or unreliable suppliers delivering a random percentage of an order (random yield). Stochastic demand is represented by a known distribution. In most of our models, fixed cost is incurred when an order is placed, which complicates the problem in that order cost is not proportional to the order quantity. Minimum order size constraints, which reflect fixed costs indirectly, are included in the models where fixed cost is not incurred. Due to fixed cost and/or minimum order size constraints, the expected total cost is nonconvex, which is analyzed by considering several cases separately. Within each case, the cost function is proven to be convex and can be optimized. These cases are compared with each other to determine optimal policies.

In Chapter 2, we focus on deriving optimal policy structures as a function of reliability level for dual-sourcing models with different assumptions about fixed cost, item cost and reliability level. The impacts of cost parameters, reliability levels, and demand distributions on optimal policy structures and order quantities are investigated. We assume zero initial inventory level, two different unreliable suppliers,
and various types of fixed order costs: overall fixed cost and individual fixed cost incurred upon order placement, and individual fixed cost incurred upon order delivery.

Four cases are investigated: ordering nothing, single-sourcing from the less reliable supplier, single-sourcing from the more reliable supplier, and dual-sourcing. Within each case, the cost function is linear, so that the optimal policy can be determined by comparing the four cases. For models with certain scenarios, methods are proposed to simplify the cost comparison process. If there is no fixed cost, we prove that dual-sourcing is always optimal as long as the two suppliers have the same item costs, but not necessarily the same reliability level. If overall fixed cost \( K_o \) is incurred only, single-sourcing is never optimal and thus can be eliminated. If only individual fixed cost is incurred, we prove that if ordering nothing is no worse than the single-sourcing cases, dual-sourcing cannot be optimal.

For the two-identical-supplier model, the optimal policies of the models with different fixed cost scenarios are summarized in Table 2.3. When only overall fixed cost \( K_o \) is incurred, the optimal policy is either (1) to order nothing no matter how reliable the suppliers are, or (2) to order nothing when suppliers are relatively unreliable and dual-source when suppliers are relatively reliable. For models with individual fixed costs only, optimal policy structures can be derived analytically on the interval \( \alpha \in [0.5, 1] \) for general demand distributions, and \( \alpha \in (0, 1] \) for special demand distributions such as uniform distributions or exponential distributions. Here we summarize for the second scenario and the conclusions on the interval \( \alpha \in [0.5, 1] \) apply to models with general demand distributions. When only individual fixed cost \( K_r \) is incurred upon order receipt, the optimal policy is either (1) to order nothing no matter how reliable the suppliers are, or (2) to dual-source when suppliers are relatively unreliable and single-source when suppliers are relatively reliable. When only individual fixed cost \( K_p \) is incurred upon order placement, the optimal policy have three possible structures, which are (1) ordering nothing, (2) ordering nothing then single-sourcing, (3) ordering nothing, single-sourcing, dual-sourcing, and then single-sourcing again as suppliers change from perfectly unreliable to perfectly reliable.

Based on the identical-supplier model, optimal policy structures are obtained with relaxed assumptions about reliability levels or item costs. When item costs are different, the structures are very similar to those of identical-supplier model except that sourcing from the less expensive supplier is always pre-
ferred when single-sourcing. When reliability levels are different ($\alpha_1 \leq \alpha_2$) and overall fixed cost is not incurred, the optimal policy structures are determined by considering the scenario $\alpha_1 = \alpha_2$ first. For a given value of $\alpha_2$,

i) If the optimal policy is to order nothing when $\alpha_1 = \alpha_2$, then it is also optimal for any other level of $\alpha_1 \leq \alpha_2$.

ii) If the optimal policy is to single-source from supplier 2 when $\alpha_1 = \alpha_2$, then it is also optimal for any other level of $\alpha_1 \leq \alpha_2$.

iii) If the optimal policy is to dual-source when $\alpha_1 = \alpha_2$, then there exists a threshold level $\alpha_{32}$ such that dual-sourcing is optimal for $\alpha_1 \geq \alpha_{32}$ while single-sourcing from supplier 2 is optimal for other levels of $\alpha_1$.

The effects of parameters, including item cost $c$, penalty cost $p$ and disposal cost $h$, on the optimal policy structure and optimal order quantities are studied and concluded as follows: when item cost and disposal cost increase, we usually tend to order from fewer suppliers and order quantities decrease; penalty cost influences the optimal policy structure and order quantities in the opposite way.

The model we consider in Chapter 3 extends the model in Chapter 2 by allowing the initial inventory level to be arbitrary. For two different suppliers, we prove that there are ten possible optimal policy structures as a function of initial inventory level, although extensive numerical experiments show that some complicated policies rarely occur. For special models where the more expensive supplier is perfectly reliable, or the two suppliers are identical, the numbers of optimal policies reduce to six and two, respectively.

Optimal policies of multiple-period problems are very difficult to determine analytically, and time-consuming to solve numerically, especially when the number of periods are relatively large. A heuristic method, which implements the single-period optimal policy corresponding to a lower inventory level than the actual inventory level in the multiple period problem, is proposed to approximate multiple-period optimal policies. The formula used to compute the deflation amount of inventory level, is empirically generated from the conclusions of the effects of parameters on single-period optimal policy.
approximation error: when service levels are relatively high (high $p$, low $h$, low $c$), and/or fixed cost $K$ is relatively high, and/or demand is relatively large (high $u$), and/or suppliers are relatively unreliable (low $\alpha$), we tend to order to cover multiple periods and approximation errors tend to grow. We found that the heuristic method takes a little longer than implementing the single-period optimal policies without inventory deflation, but it reduces errors significantly.

In Chapter 4, we focus on the issue of selecting suppliers from multiple different unreliable suppliers and allocating orders among them to satisfy stochastic demand and minimize total cost in the newsvendor setting. Fixed order costs are dropped from this model. Instead, order size constraints are added to represent minimum order size requirements and suppliers’ capacities. The suppliers may have different item costs, requirements on minimum order sizes and limited supply capacities. We formulate this problem as a discrete model and prove it to be NP-hard.

The general model is divided into two parts: (1) the master problem which involves selection of a set of suppliers and (2) a subproblem which involves determination of the order quantity allocation to a given set of suppliers. When there is no minimum order size constraint, we prove that sourcing from more suppliers cannot lead to increased cost so only the subproblem of sourcing from all the suppliers need to be considered. When there are minimum order size requirements, five methods are proposed to solve subproblems, of which the Primal Affine Scaling (PAS) algorithm and Free Variable Reduction (FVR) are used to solve the continuous formulations of subproblems, while Restricted Local Search (RLS), Unrestricted Search (US) and Nelder-Mead (NM) method are used to obtain integer solutions. Theoretically, PAS can guarantee optimality, but the termination criteria cannot be infinitely small so that only $\epsilon$-optimal solutions can be obtained. FVR, RLS and NM are heuristic methods. US always obtain the best solution empirically, but proof of optimality is very difficult.

Enumeration approaches and Branch and Bound methods are proposed to solve master problems. Given optimal solutions of subproblems, Total Enumeration (TE) and the Branch and Bound methods (B&B with $z$ and B&B without $z$) can lead to optimal solutions for master problems, while Partial Enumeration (PE) may produce near-optimal solutions.

Appropriate combinations of these methods are discussed and implemented in numerical experi-
ments to show their performances. We conclude that when the number of supplier is relatively small, TE-PAS-US can be used since it is quick and always lead to the best solutions. For a relatively large pool of suppliers, PE-FVR-RLS, PE-PAS-RLS and PE-PAS-NM, are able to produce a good enough solution fast. When a good starting point is relatively expensive to obtain, TE-MSP-NM is an ideal method to choose. If order size constraints are relatively relaxed (minimum order size constraints are low and maximum order size constraints are high), the Branch and Bound methods, which start with solving problem of all the suppliers selected, may be faster than PE. Under such cases, the cutting rule may not take effect so few cases may be eliminated by PE, whereas B&B may solve fewer relaxation problems and unconstrained problems.

In all of our models, the reliability levels and item costs of suppliers are assumed to be independent, which may not be the case in reality. For example, some suppliers may be geographically close and subject to the same natural disaster that could interrupt regular production and supply processes. Thus, their reliability levels are correlated. Another possibility is that suppliers may have competitive relationships so their item costs are not independent from each other. Future work could involve such correlations among suppliers when we model supply risk in the supplier selection and order quantity allocation problem.
REFERENCEs


Appendix A

Proofs of Chapter 2

Lemma 2.1

Proof. (a) Because the demand distribution is assumed to be continuous, \( L(y) \) is continuous and differentiable. The first order derivative of \( L(y) \) is: \( L'(y) = (h + p)F(y) - p \). The second order derivative of \( L(y) \) is: \( L''(y) = (h + p)f(y) \geq 0 \). As a result, \( L'(y) \) is non-decreasing and \( L(y) \) is convex.

(b) It can be obtained according to the property of convex functions.

(c) It can be obtained according to the property of convex functions.

(d) If \( y_1 = y_4 \), the equality is active. Suppose \( y_1 \neq y_4 \) and let \( y_2 = \frac{y_4 - y_2}{y_4 - y_1} y_1 + \frac{y_4 - y_1}{y_4 - y_1} y_4 \), \( y_3 = \frac{y_4 - y_3}{y_4 - y_1} y_1 + \frac{y_4 - y_1}{y_4 - y_1} y_4 \). Because \( L(y) \) is convex, we have

\[
L(y_2) + L(y_3) \leq \frac{y_4 - y_2}{y_4 - y_1} L(y_1) + \frac{y_2 - y_1}{y_4 - y_1} L(y_4) + \frac{y_4 - y_3}{y_4 - y_1} L(y_1) + \frac{y_3 - y_1}{y_4 - y_1} L(y_4)
= 2\frac{y_4 - y_2 - y_3}{y_4 - y_1} L(y_1) + \frac{y_2 + y_3 - 2y_1}{y_4 - y_1} L(y_4)
= 2\frac{y_4 - y_1 - y_4}{y_4 - y_1} L(y_1) + \frac{y_1 + y_4 - 2y_1}{y_4 - y_1} L(y_4)
= L(y_1) + L(y_4)
\]

\[
\square
\]

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Proposition 2.5

**Proof.** This result can also be proved by dividing the cost function into three parts: item cost, expected disposal cost $D(y) = \int_0^y h(y - \xi) f(\xi) d\xi$ and expected penalty cost $P(y) = \int_y^\infty p(\xi - y) f(\xi) d\xi$. Both $D(y)$ and $P(y)$ are convex functions. The expected disposal cost $D(y)$ monotonically increases as $y$ increases, and the expected penalty cost $P(y)$ monotonically decreases as $y$ increases.

\[
D'(y) = hF(y) \geq 0
\]

\[
D''(y) = hf(y) \geq 0
\]

\[
P'(y) = pF(y) - p \leq 0
\]

\[
P''(y) = pf(y) \geq 0
\]

The expected penalty cost difference between single-sourcing and doing nothing.

\[
\Delta P_{sn} = \alpha [P(Q^*_s) - P(0)] \leq 0
\]
The expected penalty cost difference between dual-sourcing and single-sourcing.

\[
\Delta P_{ds} = \alpha^2 P(2Q^*_d) + 2\alpha\bar{\alpha} P(Q^*_d) + \bar{\alpha}^2 P(0) - \alpha P(Q^*_s) - \bar{\alpha} P(0)
\]
\[
= \alpha[\alpha P(2Q^*_d) + \bar{\alpha} P(Q^*_d) - P(Q^*_s)] + \bar{\alpha}[\alpha P(Q^*_d) + \bar{\alpha} P(0) - P(0)]
\]
\[
= \alpha^2[P(2Q^*_d) - P(Q^*_s)] + \alpha\bar{\alpha}[P(Q^*_d) - P(Q^*_s)] + \bar{\alpha}\alpha[P(0) - P(0)]
\]
\[
\leq \alpha^2(2Q^*_d - Q^*_s)P'(2Q^*_d) + \alpha\bar{\alpha}(Q^*_d - Q^*_s)P'(Q^*_d) + \bar{\alpha}\alpha Q^*_d P'(Q^*_d)
\]
\[
= \alpha Q^*_d[\alpha P'(2Q^*_d) + \bar{\alpha} P'(Q^*_d)] + \alpha(Q^*_d - Q^*_s)[\alpha P'(2Q^*_d) + \bar{\alpha} P'(Q^*_d)]
\]
\[
= \alpha(2Q^*_d - Q^*_s)[\alpha P'(2Q^*_d) + \bar{\alpha} P'(Q^*_d)]
\]
\[
= \alpha(2Q^*_d - Q^*_s)p[\alpha F'(2Q^*_d) + \bar{\alpha} F'(Q^*_d) - 1]
\]
\[
= \alpha(2Q^*_d - Q^*_s)p[F(Q^*_d) - 1]
\]
\[
\leq 0
\]

The expected disposal cost difference between single-sourcing and doing nothing.

\[
\Delta D_{sn} = \alpha[D(Q^*_s) - D(0)] \geq 0
\]

The expected disposal cost difference between dual-sourcing and single-sourcing.

\[
\Delta D_{ds} = \alpha^2 D(2Q^*_d) + 2\alpha\bar{\alpha} D(Q^*_d) + \bar{\alpha}^2 D(0) - \alpha D(Q^*_s) - \bar{\alpha} D(0)
\]
\[
= \alpha[\alpha D(2Q^*_d) + \bar{\alpha} D(Q^*_d) - D(Q^*_s)] + \bar{\alpha}[\alpha D(Q^*_d) + \bar{\alpha} D(0) - D(0)]
\]
\[
= \alpha^2[D(2Q^*_d) - D(Q^*_s)] + \alpha\bar{\alpha}[D(Q^*_d) - D(Q^*_s)] + \bar{\alpha}\alpha[Q^*_d - D(0)]
\]
\[
\geq \alpha^2(2Q^*_d - Q^*_s)D'(Q^*_s) + \alpha\bar{\alpha}(Q^*_d - Q^*_s)D'(Q^*_s) + \bar{\alpha}\alpha D(Q^*_d)
\]
\[
\Delta L_{sn} = \Delta P_{sn} + \Delta H_{sn} = \alpha[L(Q^*_s) - L(0)] \leq \alpha Q^*_s L'(Q^*_s) = -\alpha Q^*_s \leq 0
\]
\[
\Delta L_{ds} = \Delta P_{ds} + \Delta H_{ds}
\]
\[
= \alpha^2 L(2Q_d^*) + 2\alpha \tilde{\alpha} L(Q_d^*) + \tilde{\alpha}^2 L(0) - \alpha L(Q_s^*) - \tilde{\alpha} L(0)
\]
\[
= \alpha [\alpha L(2Q_d^*) + \tilde{\alpha} L(Q_d^*) - L(Q_s^*)] + \tilde{\alpha} [\alpha L(Q_d^*) + \tilde{\alpha} L(0) - L(0)]
\]
\[
\leq \alpha^2 (2Q_d^* - Q_s^*) L'(2Q_d^*) + \alpha \tilde{\alpha} (Q_d^* - Q_s^*) L'(Q_d^*) + \alpha \tilde{\alpha} Q_d^* L'(Q_d^*)
\]
\[
= \alpha Q_d^* [\alpha L'(2Q_d^*) + \tilde{\alpha} L'(Q_d^*)] + \alpha (Q_d^* - Q_s^*) [\alpha L'(2Q_d^*) + \tilde{\alpha} L'(Q_d^*)]
\]
\[
= \alpha (2Q_d^* - Q_s^*) [\alpha L'(2Q_d^*) + \tilde{\alpha} L'(Q_d^*)]
\]
\[
= \alpha (2Q_d^* - Q_s^*) L'(Q_s^*)
\]
\[
= -\alpha (2Q_d^* - Q_s^*) c
\]
\[
\leq 0
\]

\[
\Delta M_{sn}(K = 0) = \Delta L_{sn} + \alpha c Q_s^* \leq \alpha Q_s^* L'(Q_s^*) + \alpha c Q_s^* = 0
\]

\[
\Delta M_{ds}(K = 0) = \Delta L_{ds} + \alpha (2Q_d^* - Q_s^*) c \leq -\alpha (2Q_d^* - Q_s^*) c + \alpha (2Q_d^* - Q_s^*) c = 0
\]

\[\square\]

**Proposition 2.6**

**Proof.** We need to show that \( \frac{\partial M_s}{\partial \alpha} \leq 0 \) and \( \frac{\partial M_d}{\partial \alpha} \leq 0 \).

\[
\frac{\partial M_s}{\partial \alpha} = c Q_s^* + L(Q_s^*) - L(0) \leq c Q_s^* + (Q_s^* - 0) L'(Q_s^*) = 0
\]
\[
\frac{\partial M_d}{\partial \alpha} = 2cQ^*_d + 2\alpha \frac{\partial Q^*_d}{\partial \alpha} + 2\alpha L(2Q^*_d) + 2L(Q^*_d) - 4\alpha L(Q^*_d) - 2(1 - \alpha)L(0) + [2\alpha^2 L'(2Q^*_d) \\
+ 2(1 - \alpha)\alpha L'(Q^*_d)\frac{\partial Q^*_d}{\partial \alpha}]
\]

\[
= 2cQ^*_d + 2L(Q^*_d) - 2L(0) + 2\alpha[L(2Q^*_d) - 2L(Q^*_d) + L(0)] \\
+ 2\alpha \frac{\partial Q^*_d}{\partial \alpha}[c + \alpha L'(2Q^*_d) + (1 - \alpha)L'(Q^*_d)]
\]

\[
= 2cQ^*_d + 2\alpha[L(2Q^*_d) - L(Q^*_d)] + 2(1 - \alpha)[L(Q^*_d) - L(0)] \\
+ 2\alpha \frac{\partial Q^*_d}{\partial \alpha}[c + \alpha L'(2Q^*_d) + (1 - \alpha)L'(Q^*_d)]
\]

\[
= 2cQ^*_d + 2\alpha[L(2Q^*_d) - L(Q^*_d)] + 2(1 - \alpha)[L(Q^*_d) - L(0)] \\
\leq 2cQ^*_d + 2\alpha Q^*_d L'(2Q^*_d) + 2(1 - \alpha)Q^*_d L'(Q^*_d)
\]

\[
= 0
\]

\[\Box\]

**Proposition 2.8**

**Proof.** (a) When \(2Q^*_d \leq u\),

\[
\frac{\partial^2 \Delta M_{ds}}{\partial \alpha^2} = 2\alpha \frac{\partial Q^*_d}{\partial \alpha} + 2L'(Q^*_d) \frac{\partial Q^*_d}{\partial \alpha} + 2[L(2Q^*_d) - 2L(Q^*_d) + L(0)]
\]

\[
+ 4\alpha \frac{\partial Q^*_d}{\partial \alpha}[L'(2Q^*_d) - L'(Q^*_d)]
\]

\[
= 2\alpha \frac{\partial Q^*_d}{\partial \alpha}[L'(2Q^*_d) - L'(Q^*_d)] + 2[L(2Q^*_d) - 2L(Q^*_d) + L(0)]
\]

\[
= 2\alpha \frac{F(Q^*_d) - F(2Q^*_d)}{(1 - \alpha)f(Q^*_d) + 2\alpha f(2Q^*_d)}(h + p)[F(2Q^*_d) - F(Q^*_d)]
\]

\[
+ 2(h + p)\{2Q^*_d[F(2Q^*_d) - F(Q^*_d)] - \int^{2Q^*_d} Q^*_d \xi f(\xi) d\xi + \int^{Q^*_d} Q^*_d \xi f(\xi) d\xi\}
\]

\[
= 2(h + p)\{- \frac{Q^*_d}{u} + \frac{2Q^*_d}{u} - \int^{2Q^*_d} Q^*_d \xi d\xi + \int^{Q^*_d} Q^*_d \xi d\xi\}
\]

\[
= 2(h + p)\{\frac{Q^*_d^2}{u} - \frac{3Q^*_d^2}{2u} + \frac{Q^*_d^2}{2u}\}
\]

\[
= 0,
\]

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which implies that $\Delta M_{ds}$ is a linear function.

When $2Q_d^* \geq u$, $Q_d^* = \frac{u - c - \alpha (h + p)}{(1 - \alpha)(h + p)} u$ and $\frac{u - c}{h + p} \geq \frac{1 + \alpha}{2}$.

$$\frac{\partial^2 \Delta M_{ds}}{\partial \alpha^2} = 2\alpha \frac{F(Q_d^*) - F(2Q_d^*)}{(1 - \alpha)f(Q_d^*)} + 2\alpha f(2Q_d^*)(h + p)[F(2Q_d^*) - F(Q_d^*)],$$

$$+ 2(h + p)\{2Q_d^*[F(2Q_d^*) - F(Q_d^*)] - \int_{Q_d^*}^{2Q_d^*} \xi f(\xi)d\xi + \int_{Q_d^*}^Q f(\xi)d\xi\}$$

$$= 2(h + p)\{-\frac{\alpha u}{1 - \alpha}(1 - \frac{Q_d^*}{u})^2 + 2Q_d^*(1 - \frac{Q_d^*}{u}) - \frac{u^2 - Q_d^{*2}}{2u} + \frac{Q_d^{*2}}{2u}\}$$

$$= 2(h + p)\{-\frac{\alpha u}{1 - \alpha}(1 - \frac{Q_d^*}{u})^2 - u(1 - \frac{Q_d^*}{u})^2 + \frac{u}{2}\}$$

$$= \frac{u}{2} - u(1 - \frac{Q_d^*}{u})^2 \frac{1}{1 - \alpha} = \frac{u}{2} - u \frac{(h + c)^2}{(h + p)^2(1 - \alpha)^2}$$

When $0 < \alpha \leq 0.5$, we have $\frac{u}{1 - \alpha} \leq 2$. Because $\frac{u - c}{h + p} \geq \frac{1 + \alpha}{2} \iff \frac{1 - \alpha}{2} \iff \frac{(h + c)^2}{(h + p)^2(1 - \alpha)^2} \leq \frac{1}{4} \iff \frac{(h + c)^2}{(h + p)^2(1 - \alpha)^2} \cdot \frac{1}{1 - \alpha} \leq \frac{1}{4} \cdot 2$, we have $\frac{\partial^2 \Delta M_{ds}}{\partial \alpha^2} \geq 0$.

It is easy to prove that when $\alpha = \frac{2Q_d^* - u}{u}$, $2Q_d^* = u$. Denote it as $\alpha_t$. When $\alpha < \alpha_t$, $2Q_d^* > u$ and when $\alpha > \alpha_t$, $2Q_d^* < u$. If $\alpha_t < 0.5$, $\Delta M_{ds}$ is a linearly increasing function on $\alpha \in [\alpha_t, 1]$ and convex on $\alpha \in (0, \alpha_t]$. If $\alpha_t > 0.5$, $\Delta M_{ds}$ is increasing on $\alpha \in [0.5, 1]$ and convex on $\alpha \in (0, 0.5]$. Therefore, it is a unimodal function on $(0, 1)$. 

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Because \( F \) is the first-order derivative of the optimality conditions of order quantities, we can write:

\[
\frac{\partial^2 \Delta M_{ds}}{\partial \alpha^2} = 2\alpha \frac{F(Q^*_d) - F(2Q^*_d)}{(1 - \alpha)f(Q^*_d) + 2\alpha f(2Q^*_d)} (h + p)[F(2Q^*_d) - F(Q^*_d)]
\]

\[
+ 2(h + p) \{2Q^*_d[F(2Q^*_d) - F(Q^*_d)] - \int_{Q^*_d}^{2Q^*_d} \xi f(\xi) d\xi + \int_{Q^*_d}^{2Q^*_d} \xi f(\xi) d\xi\}
\]

\[
= 2(h + p)\left\{\frac{e^{-Q^*_d/\mu} - e^{-Q^*_d/\mu}}{(1 - \alpha)e^{-Q^*_d/\mu} + 2\alpha e^{-2Q^*_d/\mu} / \mu} \right. \frac{e^{-Q^*_d/\mu} - e^{-2Q^*_d/\mu}}{2} + 2Q^*_d(e^{-Q^*_d/\mu} - e^{-2Q^*_d/\mu})
\]

\[
+ 2Q^*_d(e^{-Q^*_d/\mu} - e^{-2Q^*_d/\mu} - 2\mu e^{-Q^*_d/\mu} - 2\mu e^{-Q^*_d/\mu} + \mu)\right\}
\]

\[
= 2(h + p)\left\{\mu(e^{-Q^*_d/\mu} - 2e^{-Q^*_d/\mu} + 1) - \frac{e^{-Q^*_d/\mu} - e^{-Q^*_d/\mu}}{(1 - \alpha)e^{-Q^*_d/\mu} + 2\alpha e^{-2Q^*_d/\mu} / \mu}\right\}
\]

\[
= 2(h + p)\left\{\frac{2h + p}{2} \right\} \left\{\frac{1 - \alpha}{(1 - \alpha)e^{Q^*_d/\mu} + 2\alpha} \right\}
\]

\[
\geq 0
\]

\[\square\]

Proposition 2.9

**Proof.** From the optimality conditions of order quantities \( \alpha F(Q^*_d + Q^*_d) + \bar{\alpha} F(Q^*_d) = \frac{\nu - c_1}{h + p} \) and \( \alpha F(Q^*_d + Q^*_d) + \bar{\alpha} F(Q^*_d) = \frac{\nu - c_2}{h + p} \), as well as the fact \( c_1 < c_2 \), we can show that \( Q^*_d > Q^*_d \). The first-order derivative of the optimality conditions of order quantities are

\[
\alpha f(Q^*_d + Q^*_d) \left( \frac{\partial Q^*_d}{\partial \alpha} + \frac{\partial Q^*_d}{\partial \alpha} \right) + \bar{\alpha} f(Q^*_d) \frac{\partial Q^*_d}{\partial \alpha} + F(Q^*_d + Q^*_d) - F(Q^*_d) = 0 \quad (A.1)
\]

\[
\alpha f(Q^*_d + Q^*_d) \left( \frac{\partial Q^*_d}{\partial \alpha} + \frac{\partial Q^*_d}{\partial \alpha} \right) + \bar{\alpha} f(Q^*_d) \frac{\partial Q^*_d}{\partial \alpha} + F(Q^*_d + Q^*_d) - F(Q^*_d) = 0 \quad (A.2)
\]

Because \( F(Q^*_d + Q^*_d) - F(Q^*_d) \geq 0, F(Q^*_d + Q^*_d) - F(Q^*_d) \geq 0, f(Q^*_d + Q^*_d) \geq 0, f(Q^*_d + Q^*_d) \geq 0 \)

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and \( f(Q_{d2}^*) \geq 0 \), at least one of \( \frac{\partial Q_{d1}^*}{\partial \alpha} \) and \( \frac{\partial Q_{d2}^*}{\partial \alpha} \) are non-positive. From (A.1)-(A.2) we get \( \alpha [f(Q_{d1}^*) \frac{\partial Q_{d1}^*}{\partial \alpha} - f(Q_{d2}^*) \frac{\partial Q_{d2}^*}{\partial \alpha}] \) \( + \) \( F(Q_{d2}^*) - F(Q_{d1}^*) = 0 \). Since \( F(Q_{d2}^*) - F(Q_{d1}^*) < 0 \), we know that \( f(Q_{d1}^*) \frac{\partial Q_{d1}^*}{\partial \alpha} - f(Q_{d2}^*) \frac{\partial Q_{d2}^*}{\partial \alpha} \geq 0 \). Therefore, \( \frac{\partial Q_{d2}^*}{\partial \alpha} \leq 0 \).

Proof of the optimal policy in Section 2.3.4.1

**Proof.** According to Proposition 2.1, single-sourcing from supplier 1 is always better than dual-sourcing on the range \( \alpha \in \left[ \frac{p-c_2}{p-c_1}, 1 \right] \). Now we prove that \( \Delta M_{31} \) is monotonically increasing on \( \left( \frac{p-c_2}{2(p-c_1)}, \frac{p-c_2}{p-c_1} \right) \).

\[
\frac{\partial \Delta M_{31}}{\partial \alpha} = c_1 Q_{d1}^* + c_2 Q_{d2}^* + 2\alpha L(Q_{d1}^* + Q_{d2}^*) + (1 - 2\alpha)[L(Q_{d1}^*) + L(Q_{d2}^*)] - 2(1 - \alpha)L(0) - c_1 Q_{s1}^* - L(Q_{s1}^*) + L(0) + \alpha c_1 \frac{\partial Q_{d1}^*}{\partial \alpha} + \alpha c_2 \frac{\partial Q_{d2}^*}{\partial \alpha} + \alpha L'(Q_{d1}^* + Q_{d2}^*) \left( \frac{\partial Q_{d1}^*}{\partial \alpha} + \frac{\partial Q_{d2}^*}{\partial \alpha} \right)
\]

\[
= c_1 Q_{d1}^* + c_2 Q_{d2}^* + 2\alpha L(Q_{d1}^* + Q_{d2}^*) + (1 - 2\alpha)[L(Q_{d1}^*) + L(Q_{d2}^*)] - 2(1 - \alpha)L(0) - c_1 Q_{s1}^* - L(Q_{s1}^*) + L(0)
\]

Set \( \frac{\partial \Delta M_{31}}{\partial \alpha} \geq 0 \) and rearrange it we can get \( \alpha \geq \frac{c_1 Q_{s1}^* + L(Q_{s1}^*) + L(0) - L(Q_{d1}^*) - L(Q_{d2}^*) - c_1 Q_{d1}^* - c_2 Q_{d2}^*}{2[L(Q_{d1}^* + Q_{d2}^*) - L(Q_{d1}^*) - L(Q_{d2}^*) + L(0)]} \). Let
\[\text{RHS}(\alpha) = \frac{c_1 Q_{s1} + L(Q_{s2}) + L(0) - L(Q_{d1}) - L(Q_{d2}) - c_1 Q_{d1} - c_2 Q_{d2}}{2[L(Q_{d1} + Q_{d2}) - L(Q_{d1}) - L(Q_{d2}) + L(0)]},\]

\[
\frac{\partial \text{RHS}}{\partial \alpha} = \frac{1}{4[L(Q_{d1} + Q_{d2}) - L(Q_{d1}) - L(Q_{d2}) + L(0)]^2} \{2[-L'(Q_{d1}) \frac{\partial Q_{d1}^*}{\partial \alpha} - L'(Q_{d2}) \frac{\partial Q_{d2}^*}{\partial \alpha}] \\
- c_1 \frac{\partial Q_{d1}^*}{\partial \alpha} - c_2 \frac{\partial Q_{d2}^*}{\partial \alpha}][L(Q_{d1} + Q_{d2}) - L(Q_{d1}) - L(Q_{d2}) + L(0)] \\
- 2[L'(Q_{d1} + Q_{d2}) (\frac{\partial Q_{d1}^*}{\partial \alpha} + \frac{\partial Q_{d2}^*}{\partial \alpha}) - L'(Q_{d1}) \frac{\partial Q_{d1}^*}{\partial \alpha}] \\
- L'(Q_{d2}) \frac{\partial Q_{d2}^*}{\partial \alpha}[c_1 Q_{s1} + L(Q_{s1}) + L(0) - L(Q_{d1}) - L(Q_{d2}) - c_1 Q_{d1} - c_2 Q_{d2}]\}
\]

\[
= \frac{1}{2[L(Q_{d1} + Q_{d2}) - L(Q_{d1}) - L(Q_{d2}) + L(0)]^2} \{\alpha[L'(Q_{d1} + Q_{d2}) (\frac{\partial Q_{d1}^*}{\partial \alpha} + \frac{\partial Q_{d2}^*}{\partial \alpha}) \\
- L'(Q_{d1}) \frac{\partial Q_{d1}^*}{\partial \alpha} - L'(Q_{d2}) \frac{\partial Q_{d2}^*}{\partial \alpha}] [\alpha L(Q_{d1} + Q_{d2}) - \alpha L(Q_{d1}) - \alpha L(Q_{d2}) + \alpha L(0)] \\
- [c_1 Q_{s1} + L(Q_{s1}) + L(0) - L(Q_{d1}) - L(Q_{d2}) - c_1 Q_{d1} - c_2 Q_{d2}]\}
\]

\[
= \frac{1}{2\alpha[L(Q_{d1} + Q_{d2}) - L(Q_{d1}) - L(Q_{d2}) + L(0)]^2} \{L'(Q_{d1} + Q_{d2}) (\frac{\partial Q_{d1}^*}{\partial \alpha} + \frac{\partial Q_{d2}^*}{\partial \alpha}) \\
- L'(Q_{d1}) \frac{\partial Q_{d1}^*}{\partial \alpha} - L'(Q_{d2}) \frac{\partial Q_{d2}^*}{\partial \alpha}] [\alpha c_1 Q_{s1} + \alpha c_2 Q_{s2} + \alpha^2 L(Q_{d1} + Q_{d2}) \\
- \alpha \bar{\alpha}[L(Q_{d1}^*) + L(Q_{d2}^*)] + \alpha^2 L(0) - [\alpha c_1 Q_{s1} + \alpha L(Q_{s1}) + \alpha L(0)]\}
\]

\[
= \frac{1}{2\alpha[L(Q_{d1} + Q_{d2}) - L(Q_{d1}) - L(Q_{d2}) + L(0)]^2} \{L'(Q_{d1} + Q_{d2}) (\frac{\partial Q_{d1}^*}{\partial \alpha} + \frac{\partial Q_{d2}^*}{\partial \alpha}) \\
- L'(Q_{d1}) \frac{\partial Q_{d1}^*}{\partial \alpha} - L'(Q_{d2}) \frac{\partial Q_{d2}^*}{\partial \alpha}] \Delta M_{31}(K = 0) \\
+ \alpha L(Q_{d1} + Q_{d2}) \frac{\partial Q_{d2}^*}{\partial \alpha} \Delta M_{31}(K = 0)\}
\]

If \(\frac{\partial Q_{d1}^*}{\partial \alpha} \leq 0\), it is easy to see that \([F(Q_{d1} + Q_{d2}) - F(Q_{d1})] \frac{\partial Q_{d1}^*}{\partial \alpha} + [F(Q_{d1} + Q_{d2}) - F(Q_{d2})] \frac{\partial Q_{d2}^*}{\partial \alpha}\) \leq 0. If \(\frac{\partial Q_{d1}^*}{\partial \alpha} > 0\), in (A.1) we can infer from \(F(Q_{d1} + Q_{d2}) - F(Q_{d1}) \geq 0\) and \(\bar{\alpha} f(Q_{d2}^*) \geq 0\) that
\[ \left| \frac{\partial Q^*_d}{\partial \alpha} \right| \leq \left| \frac{\partial Q^*_s}{\partial \alpha} \right|. \]

Combined with the fact that \( F(Q_{d1}^* + Q_{d2}^*) - F(Q_{d1}^*) < F(Q_{d1}^* + Q_{d2}^*) - F(Q_{d2}^*) \), it can also be concluded that \( F(Q_{d1}^* + Q_{d2}^*) - F(Q_{d1}^*) \left| \frac{\partial Q^*_d}{\partial \alpha} \right| + \left| \frac{\partial Q^*_s}{\partial \alpha} \right| \}
\leq 0.
Therefore, \( \frac{\partial \text{RHS}}{\partial \alpha} \geq 0. \)

Let \( \text{RHS} = \frac{g(\alpha)}{h(\alpha)} \)

Because \( \lim_{\alpha \to (\frac{p-c_2}{p-c_1})} h(\alpha) = \lim_{\alpha \to (\frac{p-c_1}{p-c_1})} c_1Q_{s1}^* + L(Q_{s1}^*) + L(0) - L(Q_{d1}^*) - L(Q_{d2}^*) - c_1Q_{d1}^* - c_2Q_{d2}^* = c_1Q_{s1}^* + L(Q_{s1}^*) + L(0) - L(Q_{s1}^*) - L(0) - c_1Q_{s1}^* = 0, \)
and \( \lim_{\alpha \to (\frac{p-c_1}{p-c_1})} h(\alpha) = \lim_{\alpha \to (\frac{p-c_1}{p-c_1})} -2L(Q_{d1}^*) - L(Q_{d1}^*) - L(Q_{d2}^*) + L(Q_{s1}^*) - L(Q_{s1}^*) - L(0) + L(0) = 0, \)
by L'Hopital's rule we can get
\[
\lim_{\alpha \to (\frac{p-c_1}{p-c_1})} \frac{g(\alpha)}{h(\alpha)} = \lim_{\alpha \to (\frac{p-c_1}{p-c_1})} \frac{g'(\alpha)}{h'(\alpha)}
\]
\[
= \lim_{\alpha \to (\frac{p-c_1}{p-c_1})} \frac{-L'(Q_{d1}^*) \frac{\partial Q^*_d}{\partial \alpha} - L'(Q_{d2}^*) \frac{\partial Q^*_s}{\partial \alpha} - c_1 \frac{\partial Q^*_d}{\partial \alpha} - c_2 \frac{\partial Q^*_s}{\partial \alpha}}{-2L'(Q_{s1}^*) + L'(0) \frac{\partial Q^*_s}{\partial \alpha} - L'(Q_{d1}^*) \frac{\partial Q^*_d}{\partial \alpha} - L'(Q_{d2}^*) \frac{\partial Q^*_s}{\partial \alpha}}
\]
\[
= \frac{-L'(Q_{s1}^*) \frac{\partial Q^*_d}{\partial \alpha} - L'(0) \frac{\partial Q^*_s}{\partial \alpha} - c_1 \frac{\partial Q^*_d}{\partial \alpha} - c_2 \frac{\partial Q^*_s}{\partial \alpha}}{-2L'(Q_{s1}^*) + L'(0) \frac{\partial Q^*_s}{\partial \alpha} - L'(Q_{d1}^*) \frac{\partial Q^*_d}{\partial \alpha} - L'(Q_{d2}^*) \frac{\partial Q^*_s}{\partial \alpha}}
\]
\[
= \frac{-L'(0) \frac{\partial Q^*_d}{\partial \alpha} - c_2 \frac{\partial Q^*_s}{\partial \alpha}}{-2L'(Q_{s1}^*) + L'(0) \frac{\partial Q^*_s}{\partial \alpha} - L'(Q_{d1}^*) \frac{\partial Q^*_d}{\partial \alpha} - L'(Q_{d2}^*) \frac{\partial Q^*_s}{\partial \alpha}}
\]
\[
= \frac{p - c_2}{2(p - c_1)}
\]

Consequently, \( \frac{\partial M_{31}}{\partial \alpha} \geq 0 \) for \( \alpha \in \left[ \frac{p-c_2}{2(p-c_1)}, \frac{p-c_2}{p-c_1} \right) \), which implies that \( \Delta M_{31} \) is monotonically increasing on \( \alpha \in \left[ \frac{p-c_2}{2(p-c_1)}, \frac{p-c_2}{p-c_1} \right) \). There is at most one point on the range \( \alpha \in \left[ \frac{p-c_2}{2(p-c_1)}, \frac{p-c_2}{p-c_1} \right) \) at which we are indifferent between dual-sourcing and single-sourcing. Denote it as \( \alpha_{31} \).

From \( \Delta M_{10} = K_p + \alpha c_1 Q_{s1}^* + \alpha L(Q_{s1}^*) - \alpha L(0) \), we can obtain the threshold level between
single-sourcing and doing nothing $\alpha_{01} = \frac{K_p}{c_1 q_{s1}^* + L(Q_{s1}^*) - L(0)}$. Because

$$
\Delta M_{31} - \Delta M_{10} = \alpha c_1 Q_{d1}^* + \alpha c_2 Q_{d2}^* + \alpha^2 L(Q_{d1}^* + Q_{d2}^*) + \alpha \bar{\alpha}[L(Q_{d1}^*) + L(Q_{d2}^*)] + \bar{\alpha}^2 L(0) + L(0) - 2[\alpha c_1 Q_{s1}^* + 2\alpha L(Q_{s1}^*) + (1 - \alpha)L(0)]
$$

$$
= \alpha[c_1 Q_{d1}^* + c_2 Q_{d2}^* - 2c_1 Q_{s1}^*] + \alpha^2[L(Q_{d1}^* + Q_{d2}^*) - L(Q_{d1}^*) - L(Q_{d2}^*) + L(0)]
$$

$$
+ \alpha[L(Q_{d1}^*) + L(Q_{d2}^*) - 2L(Q_{s1}^*)]
$$

$$
\geq \alpha[c_1 Q_{d1}^* + c_2 Q_{d2}^* - 2c_1 Q_{s1}^*] + \alpha^2[L(Q_{d1}^* + Q_{d2}^*) - L(Q_{d1}^*) - L(Q_{d2}^*) + L(0)]
$$

$$
+ \alpha[L(Q_{d1}^*) + Q_{d2}^* - 2Q_{s1}^*]L'(Q_{s1}^*)
$$

$$
= \alpha(c_2 - c_1)Q_{d2}^* + \alpha^2[L(Q_{d1}^* + Q_{d2}^*) - L(Q_{d1}^*) - L(Q_{d2}^*) + L(0)]
$$

$$
> 0,
$$

at most one of $\alpha_{31}$ and $\alpha_{01}$ exists on the range $\alpha \in \left[\frac{p - c_2}{2(p - c_1)}, 1\right]$.

The optimal policy in Section 2.3.4.2

**Proof.** From $M_{20} = K_p + \alpha_2 K_r + \alpha_2 c Q_s^* + \alpha_2 L(Q_s^*) + (1 - \alpha_2)L(0) - L(0)$, we can get that when $\alpha_2 \leq \frac{K_p}{L(0) - K_r - c Q_s^* + L(Q_s^*)} = \alpha_{ns}$, it is better to order nothing than single-sourcing from supplier 1. Otherwise, sourcing from supplier 2 is preferred. According to Proposition 2.3, ordering nothing is optimal for $\alpha_2 \leq \alpha_{ns}$ no matter what the value of $\alpha_1$ is.

Now we prove that for a certain level of $\alpha_2$, cost of dual-sourcing is increasing as $\alpha_1$ decreases.
Assume that $\alpha'_1 \geq \alpha_1$, $M_d$ and $Q_{d1}^*$ correspond to $\alpha_1$ and $M'_d$ and $\hat{Q}_{d1}^*$ correspond to $\alpha'_1$.

\[
M_d - M'_d = 2K_p + (\alpha_1 + \alpha_2)K_r + \alpha_1 cQ_{d1}^* + \alpha_2 cQ_{d2}^* + \alpha_1 \alpha_2 L(Q_{d1}^* + Q_{d2}^*) + \alpha_1 \bar{\alpha}_2 L(Q_{d1}^*) \\
+ \alpha_2 \bar{\alpha}_1 L(Q_{d2}^*) + \bar{\alpha}_1 \bar{\alpha}_2 L(0) - [2K_p + (\alpha_1 + \alpha_2)K_r + \alpha'_1 c\hat{Q}_{d1}^* + \alpha_2 c\hat{Q}_{d2}^* \\
+ \alpha'_1 \alpha_2 L(\hat{Q}_{d1}^* + \hat{Q}_{d2}^*) + \alpha'_1 \bar{\alpha}_2 L(\hat{Q}_{d1}^*) + \alpha_2 \bar{\alpha}_1 L(\hat{Q}_{d2}^*) + \bar{\alpha}_1 \bar{\alpha}_2 L(0)] \\
= c(\alpha_1 Q_{d1}^* + \alpha_2 Q_{d2}^* - \alpha'_1 \hat{Q}_{d1}^* - \alpha_2 \hat{Q}_{d2}^*) + \alpha_1 \alpha_2 [L(Q_{d1}^* + Q_{d2}^*) - L(\hat{Q}_{d1}^* + \hat{Q}_{d2}^*)] \\
+ \alpha_2(\alpha_1 - \alpha'_1) L(\hat{Q}_{d1}^* + \hat{Q}_{d2}^*) + \alpha_1 \bar{\alpha}_1 [L(Q_{d2}^*) - L(\hat{Q}_{d2}^*)] + \alpha_2(\bar{\alpha}_1 - \bar{\alpha'}_1) L(\hat{Q}_{d2}^*) \\
+ \bar{\alpha}_2 \alpha_1 [L(Q_{d1}^*) - L(\hat{Q}_{d1}^*)] + \bar{\alpha}_2(\alpha_1 - \alpha'_1) L(\hat{Q}_{d1}^*) + \bar{\alpha}_2 L(0)[\bar{\alpha}_1 - \bar{\alpha'}_1] \\
\geq c(\alpha_1 Q_{d1}^* + \alpha_2 Q_{d2}^* - \alpha'_1 \hat{Q}_{d1}^* - \alpha_2 \hat{Q}_{d2}^*) + \alpha_1 \alpha_2 [Q_{d1}^* + Q_{d2}^* - \hat{Q}_{d1}^* - \hat{Q}_{d2}^*] L'(\hat{Q}_{d1}^* + \hat{Q}_{d2}^*) \\
+ \alpha_1 \bar{\alpha}_2 [Q_{d1}^* - \hat{Q}_{d1}^*] L'(\hat{Q}_{d1}^*) + \alpha_2(\hat{Q}_{d2}^* - \hat{Q}_{d2}^*) L'(\hat{Q}_{d2}^*) + \alpha_2(\alpha_1 - \alpha'_1) [L(\hat{Q}_{d1}^* + \hat{Q}_{d2}^*) - L(\hat{Q}_{d1}^*)] \\
- L(\hat{Q}_{d2}^*) + \bar{\alpha}_2(\alpha_1 - \alpha'_1) [L(\hat{Q}_{d1}^*) - L(0)] \\
= c(\alpha_1 Q_{d1}^* + \alpha_2 Q_{d2}^* - \alpha'_1 \hat{Q}_{d1}^* - \alpha_2 \hat{Q}_{d2}^*) + \alpha_1 (\alpha_1 - \alpha'_1) \hat{Q}_{d1}^* L'(\hat{Q}_{d2}^*) + \bar{\alpha}_2(\alpha_1 - \alpha'_1) \hat{Q}_{d1}^* L'(0) \\
\geq c \hat{Q}_{d1}^*(\alpha_1 - \alpha'_1) + \alpha_2(\alpha_1 - \alpha'_1) \hat{Q}_{d1}^* L'(\hat{Q}_{d2}^*) + \bar{\alpha}_2(\alpha_1 - \alpha'_1) \hat{Q}_{d1}^* L'(0) \\
= c \hat{Q}_{d1}^*(\alpha_1 - \alpha'_1) + (\alpha_1 - \alpha'_1) \hat{Q}_{d1}^*[\alpha_2 L'(\hat{Q}_{d2}^*) + \bar{\alpha}_2 L'(0)] \\
\geq 0
\]

In addition, the cost of single-sourcing from supplier 2 is not influenced by the change of supplier 1’s reliability level. Therefore, for a certain level of $\alpha_2$, if single-sourcing is better than dual-sourcing when $\alpha_1 = \alpha_2$, single-sourcing is also better than dual-sourcing for other levels of $\alpha_1$. If dual-sourcing is optimal for $\alpha_1 = \alpha_2$, then there is a threshold point $\alpha_{ds}$ where the cost of dual-sourcing is the same as that of single-sourcing. When $\alpha_1 \leq \alpha_{ds}$, it is optimal to dual-source. Otherwise, single-sourcing from supplier 2 is the best policy. □
Appendix B

Proofs of Chapter 3

Proposition 3.1

Proof. Because the demand distribution is assumed to be continuous, \( L(y) \) is continuous and differentiable. The first order derivative of \( L(y) \) is:

\[
L'(y) = (h + p)F(y) - p
\]  
(B.1)

The second order derivative of \( L(y) \) is:

\[
L''(y) = (h + p)f(y) \geq 0
\]  
(B.2)

As a result, \( L'(y) \) is non-decreasing and \( L(y) \) is convex.

Proposition 3.2 and 3.3

Proof. From (B.1) we know that

\[
-p \leq L'(y) \leq h
\]  
(B.3)

Note that \( L'(-\infty) = -p \) and \( L'(+\infty) = h \).
Because \( x \leq x + Q^*_1, x + Q^*_1 \leq x + Q^*_1 + Q^*_2 \), and \( L'(y) \) is non-decreasing, we have

\[
L'(x) \leq L'(x + Q^*_1), L'(x + Q^*_2) \leq L'(x + Q^*_1 + Q^*_2)
\]  
(B.4)

It is obvious that \( \lim_{x \to -\infty} \frac{\partial Q^*_i}{\partial x} < -1 \), i.e., ordering more than one unit for every unit decreased in initial inventory level, and \( \lim_{x \to -\infty} \frac{\partial Q^*_i}{\partial x} > 0 \), i.e., ordering fewer stock as \( x \) decreases, are impossible. So we only need to consider the range \((0, 1]\).

If \( -1 < \lim_{x \to -\infty} \frac{\partial Q^*_1}{\partial x} < 0 \) and \( -1 < \lim_{x \to -\infty} \frac{\partial Q^*_2}{\partial x} < 0 \), then \( x + Q^*_1 = -\infty \) and \( x + Q^*_2 = -\infty \). There is no solution for (3.13) and (3.14).

If \( \lim_{x \to -\infty} \frac{\partial Q^*_1}{\partial x} = 0 \), then \( x + Q^*_1 = -\infty \) and

\[
x + Q^*_1 + Q^*_2 = F^{-1} \left[ \frac{p - c_1}{\alpha_2(h + p)} \right]
\]  
(B.5)

\[
x + Q^*_2 = F^{-1} \left[ \frac{\alpha_2(p - c_2) - \alpha_1(p - c_1)}{\alpha_1 \alpha_2(h + p)} \right]
\]  
(B.6)

When \( \alpha_2(p - c_2) > \alpha_1(p - c_1) \), \( x + Q^*_2 \) is a constant, which implies that the order-up-to level from supplier 2 is fixed so that \( \lim_{x \to -\infty} \frac{\partial Q^*_2}{\partial x} = -1 \). This corresponds to case (c) where an extra ordering is placed from supplier 2 only.

If \( \lim_{x \to -\infty} \frac{\partial Q^*_1}{\partial x} = 0 \), then \( x + Q^*_2 = -\infty \) and

\[
x + Q^*_1 + Q^*_2 = F^{-1} \left[ \frac{p - c_2}{\alpha_1(h + p)} \right]
\]  
(B.7)

\[
x + Q^*_1 = F^{-1} \left[ \frac{\alpha_1(p - c_1) - \alpha_2(p - c_2)}{\alpha_2 \alpha_1(h + p)} \right]
\]  
(B.8)

When \( \alpha_1(p - c_1) > \alpha_2(p - c_2) \), \( x + Q^*_1 \) is a constant, which implies that the order-up-to level from supplier 1 is fixed so that \( \lim_{x \to -\infty} \frac{\partial Q^*_1}{\partial x} = -1 \). This corresponds to case (b) where an extra ordering is placed from supplier 1 only.

If \( \lim_{x \to -\infty} \frac{\partial Q^*_1}{\partial x} = -1 \) and \( -1 < \lim_{x \to -\infty} \frac{\partial Q^*_2}{\partial x} < 0 \), then \( x + Q^*_1 + Q^*_2 = +\infty \) and \( x + Q^*_2 = -\infty \). The optimal condition (3.14) can not be satisfied. By the same token, the case where
−1 < \lim_{x \to -\infty} \frac{\partial Q_1^*}{\partial x} < 0 \text{ and } \lim_{x \to -\infty} \frac{\partial Q_2^*}{\partial x} = -1 \text{ is not possible, either.}

So the only case left is \( \lim_{x \to -\infty} \frac{\partial Q_1^*}{\partial x} = -1 \) and \( \lim_{x \to -\infty} \frac{\partial Q_2^*}{\partial x} = -1 \), i.e., an extra ordering is placed from both suppliers. The order-up-to level \( x + Q_1^* + Q_2^* \) increases as the initial inventory level decreases, while \( x + Q_1^* \) is a constant and so does \( x + Q_2^* \). As a result, \( L'(x + Q_1^* + Q_2^*) = h \) when \( x = -\infty \). Substituting this into optimality conditions (3.13) and (3.14), yields:

\[
L'(x + Q_1^*) = -\frac{\alpha_2 h + c_1}{1 - \alpha_2} \geq -p \quad \text{(B.9)}
\]

\[
L'(x + Q_2^*) = -\frac{\alpha_1 h + c_2}{1 - \alpha_1} \geq -p \quad \text{(B.10)}
\]

From (B.9) and (B.10) we can get the value of \( p_{\text{critical}} \), i.e., (3.15). Besides, the order-up-to level \( x + Q_1^* \) and \( x + Q_2^* \) can also be obtained:

\[
x + Q_1^* = F^{-1}\left[\frac{p - (\alpha_2 h + c_1)/\alpha_2}{h + p}\right] \quad \text{(B.11)}
\]

\[
x + Q_2^* = F^{-1}\left[\frac{p - (\alpha_1 h + c_2)/\alpha_1}{h + p}\right] \quad \text{(B.12)}
\]

The results show that if the penalty cost is not too high, an extra ordering is placed from the supplier which has a lower marginal cost. Otherwise we need to source from both suppliers to increase the chance of receiving the order to avoid penalty.

\[
\Box
\]

Proposition 3.4

\[
\frac{\partial M_1}{\partial x} = -\alpha_1 c_1 + \tilde{\alpha}_1 L'(x) < -\alpha_1 c_1 + \tilde{\alpha}_1 L'(S_1) = -c_1 < 0 \quad \text{(B.13)}
\]

\[
\frac{\partial M_2}{\partial x} = -\alpha_2 c_2 + \tilde{\alpha}_2 L'(x) < -\alpha_2 c_2 + \tilde{\alpha}_2 L'(S_2) = -c_2 < 0 \quad \text{(B.14)}
\]
The optimality conditions (3.13) and (3.14) are used in the above proof. \hfill \square

**Proposition 3.5**

**Proof.**

\[
\frac{\partial \Delta M_{31}}{\partial x} = \frac{\partial M_3}{\partial x} - \frac{\partial M_1}{\partial x} = -\alpha_1 c_1 - \alpha_2 c_2 - \alpha_1 \alpha_2 L'(x + Q^*_1 + Q^*_2) + \bar{\alpha}_1 \bar{\alpha}_2 L'(x) - [-\alpha_1 c_1 + (1 - \alpha_1) L'(x)] \\
= -\alpha_2 c_2 - \alpha_1 \alpha_2 L'(x + Q^*_1 + Q^*_2) + \bar{\alpha}_1 \bar{\alpha}_2 L'(x) - (1 - \alpha_1) L'(x) \\
= -\alpha_2 [c_2 + \alpha_1 L'(x + Q^*_1 + Q^*_2) + \bar{\alpha}_1 L'(x)] \\
> -\alpha_2 [c_2 + \alpha_1 L'(x + Q^*_1 + Q^*_2) + \bar{\alpha}_1 L'(x + Q^*_2)] \\
= -\alpha_2 [c_2 - c_2] \\
= 0
\]

(B.15)
By the same token, \( \frac{\partial \Delta M_{12}}{\partial x} > 0 \).

\[
\frac{\partial \Delta M_{12}}{\partial x} = \frac{\partial M_1}{\partial x} - \frac{\partial M_2}{\partial x} = -\alpha_1 c_1 + \tilde{\alpha}_1 L'(x) - [-\alpha_2 c_2 + \tilde{\alpha}_2 L'(x)]
= \alpha_2 c_2 - \alpha_1 c_1 + (\alpha_2 - \alpha_1)L'(x) = \alpha_2 c_2 - \alpha_1 c_1 + (\alpha_2 - \alpha_1)[(h + p)F(x) - p] \tag{B.16}
\]

\[
\frac{\partial^2 \Delta M_{12}}{\partial x^2} = (\alpha_2 - \alpha_1) L''(x) = (\alpha_2 - \alpha_1)(h + p)f(x) \geq 0 \tag{B.17}
\]

If \( \alpha_1(p - c_1) \geq \alpha_2(p - c_2) \), i.e., ordering from supplier 1 has larger marginal benefit, substitute

\[
p \leq \frac{\alpha_2 c_2 - \alpha_1 c_1}{\alpha_2 - \alpha_1} \tag{B.18}
\]

into (B.16):

\[
\frac{\partial \Delta M_{12}}{\partial x} = \alpha_2 c_2 - \alpha_1 c_1 + (\alpha_2 - \alpha_1)[(h + p)F(x) - p]
= \alpha_2 c_2 - \alpha_1 c_1 + (\alpha_2 - \alpha_1)hF(x) - (\alpha_2 - \alpha_1)p[1 - F(x)]
\geq \alpha_2 c_2 - \alpha_1 c_1 + (\alpha_2 - \alpha_1)hF(x) - (\alpha_2 - \alpha_1)\frac{\alpha_2 c_2 - \alpha_1 c_1}{\alpha_2 - \alpha_1}[1 - F(x)]
= (\alpha_2 - \alpha_1)hF(x) + (\alpha_2 c_2 - \alpha_1 c_1)F(x)
= F(x)[\alpha_2(h + c_2) - \alpha_1(h + c_1)]
\geq 0 \tag{B.19}
\]

which implies that there is at most one point at which \( \Delta M_{12}(x) = 0 \), denote it as \( s_{12} \). Besides, Case 1 costs less than Case 2 for \( x = -\infty \).

If \( \alpha_1(p - c_1) < \alpha_2(p - c_2) \), i.e., ordering from supplier 2 has larger marginal benefit, a minimizer of \( \Delta M_{12}(x) \), denoted as \( x^*_{12} \), can be obtained from (B.16) = 0:

\[
x^*_{12} = F^{-1}\left[\frac{p + (\alpha_2 c_2 - \alpha_1 c_1)/(\alpha_1 - \alpha_2)}{p + h}\right] \tag{B.20}
\]
Besides, when \( x = -\infty \)
\[
\frac{\partial \Delta M_{12}}{\partial x} = \alpha_2 c_2 - \alpha_1 c_1 - (\alpha_2 - \alpha_1)p < 0
\]
which implies that Case 1 costs more than Case 2 for \( x = -\infty \). Because \( \Delta M_{12} \) is a convex function, there are at most two points at which \( \Delta M_{12} = 0 \). \( \square \)

Proposition 3.6

Proof. (a). After dividing (3.26) by \( \alpha_1 \alpha_2 \) and rearranging the terms, it turns out to be:

\[
\frac{K_1/\alpha_1 + c_1 Q_1^* + L(s_{30} + Q_1^*) - L(s_{30})}{\alpha_2} + \frac{K_2/\alpha_2 + c_2 Q_2^* + L(s_{30} + Q_2^*) - L(s_{30})}{\alpha_1}
\]
\[
+ L(s_{30} + Q_1^* + Q_2^*) - L(s_{30} + Q_1^*) - L(s_{30} + Q_2^*) + L(s_{30}) = 0
\]

(B.22)

Because \( L(x) \) is convex, \( L(s_{30} + Q_1^* + Q_2^*) - L(s_{30} + Q_1^*) - L(s_{30} + Q_2^*) + L(s_{30}) > 0 \). Therefore, at least one of the first two terms is negative, i.e., \( K_1/\alpha_1 + c_1 Q_1^* + L(s_{30} + Q_1^*) < L(s_{30}) \) and/or \( K_2/\alpha_2 + c_2 Q_2^* + L(s_{30} + Q_2^*) < L(s_{30}) \). Because \( S_1 \) and \( S_2 \) are the optimal order-up-to levels, \( K_1/\alpha_1 + c_1(S_1 - s_{30}) + L(S_1) < K_1/\alpha_1 + c_1 Q_1^* + L(s_{30} + Q_1^*) < L(s_{30}) \) and/or \( K_2/\alpha_2 + c_2(S_2 - s_{30}) + L(S_2) < K_2/\alpha_2 + c_2 Q_2^* + L(s_{30} + Q_2^*) < L(s_{30}) \). Compared with (3.24) and (3.25), we can conclude that \( s_{10} > s_{30} \) and/or \( s_{20} > s_{30} \).

(b). After dividing \( \Delta M_{31}(x) \) by \( \alpha_1 \alpha_2 \) and rearranging the terms:

\[
\frac{\Delta M_{31}(x)}{\alpha_1 \alpha_2} = \frac{K_1/\alpha_1 + c_1 Q_1^*}{\alpha_2} + \frac{K_2/\alpha_2 + c_2 Q_2^*}{\alpha_1} + L(x + Q_1^* + Q_2^*) + \frac{(1 - \alpha_2)L(x + Q_1^*)}{\alpha_2}
\]
\[
+ \frac{(1 - \alpha_1)L(x + Q_2^*)}{\alpha_1} - \frac{1 - \alpha_1 - \alpha_2 + \alpha_1 \alpha_2}{\alpha_1 \alpha_2} L(x)
\]
\[
- \frac{K_1/\alpha_1 + c_1(S_1 - x) + L(S_1)}{\alpha_2} - \frac{1 - \alpha_1}{\alpha_1 \alpha_2} L(x)
\]
\[
= \frac{K_1/\alpha_1 + c_1 Q_1^* + L(x + Q_1^*)}{\alpha_2} + \frac{K_2/\alpha_2 + c_2 Q_2^* + L(x + Q_2^*) - L(x)}{\alpha_1}
\]
\[
- \frac{K_1/\alpha_1 + c_1(S_1 - x) + L(S_1)}{\alpha_2} + L(x + Q_1^* + Q_2^*)
\]
\[
- L(x + Q_1^*) - L(x + Q_2^*) + L(x)
\]

(B.23)
By definition we know that

$$
\frac{\Delta M_{31}(s_{31})}{\alpha_1 \alpha_2} = \frac{M_3(s_{31}) - M_1(s_{31})}{\alpha_1 \alpha_2} = \frac{K_1}{\alpha_1} + c_1 Q_1^* + L(s_{31} + Q_1^*) + \frac{K_2}{\alpha_2} + c_2 Q_2^* + L(s_{31} + Q_2^*) - L(s_{31})
$$

$$
- \frac{K_1}{\alpha_1} + c_1 (S_1 - s_{31}) + L(S_1) + L(s_{31} + Q_1^* + Q_2^*) - L(s_{31} + Q_1^* + Q_2^*)
$$

$$
- L(s_{31} + Q_2^*) + L(s_{31})
$$

$$
= 0 \quad \text{(B.24)}
$$

Because $L(x)$ is convex, $L(s_{31} + Q_1^* + Q_2^*) - L(s_{31} + Q_1^*) - L(s_{31} + Q_2^*) + L(s_{31}) > 0$. Furthermore, $S_1$ is the optimal order-up-to level when order is placed only from supplier 1, so $\frac{K_1}{\alpha_1} + c_1 Q_1^* + L(s_{31} + Q_1^*) - K_1/\alpha_1 + c_1 (S_1 - s_{31}) + L(S_1) > 0$. Therefore, $\frac{K_2}{\alpha_2} + c_2 Q_2^* + L(s_{31} + Q_2^*) - L(s_{31}) < 0$, rearrange the terms:

$$
L(s_{31}) > \frac{K_2}{\alpha_2} + c_2 Q_2^* + L(s_{31} + Q_2^*) > \frac{K_2}{\alpha_2} + c_2 (S_2 - s_{31}) + L(S_2) \quad \text{(B.25)}
$$

As a result, $s_{31} < s_{20}$.

By the same token, if the threshold initial inventory level $s_{32}$ exists on the range $(-\infty, s_{20}]$, then $s_{32} < s_{10}$. $\square$
Appendix C

Appendix of Chapter 4

C.1 The Nelder-Mead Method: Step-by-Step Algorithm

The step-by-step algorithm used in our implementation is based on the \textit{fminsearch} function in Matlab [25], the work by DErrico [12] and Luersen et al. [26].

- (Initialization): Set the simplex transformation parameters $\rho = 1$ for reflection, $\psi = 0.5$ for contraction, $\chi = 2$ for expansion, $\tau = 0.5$ for shrinkage; compute the objective function values on an initial simplex of $n + 1$ vertices $v_i^k$, where $i = 0, 1, ..., n$.

- (Centroid and Transformation) Perform the following steps for each iteration $k$ until termination criterion $\epsilon = 1e - 4$ is met:

  Step 1 (Centroid): Label the $n + 1$ vertices so that $f(v_0^k) \leq f(v_2^k) \leq ... \leq f(v_n^k)$

  - Set $\bar{v}^k = \frac{1}{n} \sum_{i=0}^{n-1} v_i^k$.

  Step 2 (Reflection): Set $vr^k = (1 + \rho)\bar{v}^k - \rho v_n^k$

  - If $vr^k$ is out of the feasible region, it is projected on the bounds by setting $vr_i^k = v_i^{min}$ if $vr_i < v_i^{min}$ and $vr_i^k = v_i^{max}$ if $vr_i > v_i^{max}$

  - If $f(v_0^k) \leq f(vr^k) < f(v_{n-1}^k)$, replace $v_n^k$ with $vr^k$, and return to step 1
Else if \( f(vr^k) < f(v_0^k) \), go to step 3

Else go to step 4

Step 3 (Expansion): Set \( ve^k = \psi \times vr^k + (1 - \psi)\bar{v}^k \)

- If \( ve^k \) is out of the feasible region, it is projected on the bounds by setting \( ve_i^k = v_i^{min} \) if \( ve_i^k < v_i^{min} \), and \( ve_i^k = v_i^{max} \) if \( ve_i^k > v_i^{max} \)

- If \( f(ve^k) \leq f(vr^k) \), replace \( v_0^k \) with \( ve^k \), and return to step 1

Step 4 (Contraction): Set \( vt^k = v_n^k \)

- If \( f(vr^k) \leq f(vt^k) \), set \( vt^k = vt^k \)

- \( vc^k = \chi vt^k + (1 - \chi)\bar{v}^k \)

- If \( f(vc^k) \leq f(v_n^k) \), replace \( v_n^k \) with \( vt^k \), and return to step 1

Step 5 (Shrinkage): For \( i = 1, \ldots, n \), do:

- \( vs_i^k = v_0^k + \tau(v_i^k - v_0^k) \)

- Set \( v_i^k = vs_i^k \)

- If \( i = n \), the loop ends and return to step 1
Appendix D

Supplement of Chapter 2

D.1 Optimal Policy Structures of Identical-supplier Models

D.1.1 $K_o + K_p$

In this case, both overall and individual fixed costs are incurred when an order is placed, no matter if it arrives successfully or not.

Obviously, $\Delta M_{ds}$ is equivalent to (2.14) but $\Delta M_{sn}$ has an additional term $K_o$ compared with (2.13). The overall fixed cost impacts the decision of whether to place an order, but it does not influence the choice between dual-sourcing and single-sourcing. Compare with the threshold levels of the $K_p$ only model, $\alpha_{ds}$ and $\alpha_{sd}$ do not change, although $\alpha_{ns}$ increases when $K_o$ is incurred.

\[
\alpha_{ns} = \frac{K_o + K_p}{cQ^*_s + L(Q^*_s) - L(0)} \tag{D.1}
\]

For uniform and exponential distributions, $\alpha_{ns} < \alpha_{sd}$ does not hold in this scenario. If $\alpha_{ns} \geq \alpha_{sd}$, when $\alpha \leq \alpha_{sd}$, $\Delta M_{sn} > 0$ and $\Delta M_{ds} > 0$ so that it is optimal to do nothing. If $\alpha_{sd} < \alpha_{ns} \leq \alpha_{ds}$, when $\alpha_{sd} < \alpha \leq \alpha_{ns}$, $\Delta M_{sn} > 0$ and $\Delta M_{ds} < 0$ so $\Delta M_{dn}$ should be considered. From $\Delta M_{dn} = \Delta M_{ds} + \Delta M_{sn}$, $\Delta M_{dn}(\alpha = \alpha_{sd}) > 0$ and $\Delta M_{dn}(\alpha = \alpha_{ns}) < 0$, we can conclude that there exists a critical threshold level $\alpha_{nd}$ and $\alpha_{sd} < \alpha_{nd} < \alpha_{ns}$. As a result, the optimal policy is to order nothing.
when $\alpha \leq \alpha_{nd}$ and dual-source when $\alpha_{nd} < \alpha \leq \alpha_{ds}$. If $\alpha_{ds} < \alpha < \alpha_{ns}$, ordering nothing is optimal due to $\Delta M_{sn} > 0$ and $\Delta M_{ds} > 0$. Since $\Delta M_{dn}$ is a decreasing function of $\alpha$, ordering nothing is also optimal for $\alpha \leq \alpha_{ds}$. When $\alpha \geq \alpha_{ns}$, single-sourcing is optimal because of $\Delta M_{sn} < 0$ and $\Delta M_{ds} > 0$.

Overall, the optimal policy structure on $\alpha \in (0,1]$ can be summarized as follows:

i) When $\alpha_{ns}$, $\alpha_{sd}$ and $\alpha_{ds}$ exist:

- If $\alpha_{ns} < \alpha_{sd}$, order nothing for $\alpha \leq \alpha_{ns}$, dual-source for $\alpha_{sd} \leq \alpha < \alpha_{ds}$ and single-source otherwise.
- If $\alpha_{sd} \leq \alpha_{ns} < \alpha_{ds}$, order nothing for $\alpha \leq \alpha_{nd}$, dual-source for $\alpha_{nd} \leq \alpha < \alpha_{ds}$ and single-source otherwise.
- If $\alpha_{ds} \leq \alpha_{ns}$, single-source for $\alpha > \alpha_{ns}$ and order nothing otherwise.

ii) When $\alpha_{ns}$ does not exist. Order nothing.

iii) When $\alpha_{ns}$ exists only. Single-source for $\alpha > \alpha_{ns}$ and order nothing otherwise.

Compared with the model where only $K_p$ is incurred, we can see that the range of single-sourcing $[\alpha_{ns}, \alpha_{sd}]$ might disappear when an overall fixed cost being charged.

For general demand distributions, $\alpha_{ds}$ exists only when $\Delta M_{ds}(\alpha = 0.5) \leq 0$, and $\alpha_{nd}$ exists on the interval $[0.5,1]$ only when $\Delta M_{dn}(\alpha = 0.5) \geq 0$ and $\Delta M_{dn}(\alpha = 1) < 0$. The optimal policy structure on $[0.5,1]$ can be summarized as follows:

i) When $\alpha_{ns}$ exists on $[0.5,1]$ but $\alpha_{ds}$ does not exist, or $\alpha_{ns} \geq \alpha_{ds}$, order nothing for $\alpha \in [0.5,\alpha_{ns}]$ and single-source for $\alpha \in (\alpha_{ns},1]$.

ii) When $\alpha_{ns}$ exists on $(0,0.5)$, but $\alpha_{ds}$ does not exist, single-source.

iii) When $\alpha_{ns}$ does not exist, order nothing.

iv) When $\alpha_{nd} \leq 0.5 \leq \alpha_{ds} \leq 1$, dual-source for $\alpha \in [0.5,\alpha_{ds}]$ and single-source otherwise.
v) When $0.5 \leq \alpha_{nd} \leq \alpha_{ds} \leq 1$, order nothing for $\alpha \in [0.5, \alpha_{nd}]$, dual-source for $\alpha \in (\alpha_{nd}, \alpha_{ds}]$ and single-source for $\alpha \in (\alpha_{ds}, 1]$.

### D.1.2 $K_o + K_r$

In this scenario, the overall fixed cost is incurred when an order is placed, but the individual cost is paid only when the order is delivered successfully. Compare with the model where $K_r$ incurred only, the function $\Delta M_{ds}$ does not change. When $K_o + \alpha K_r + cQ_s^* + L(Q_s^*) - L(0) < 0$, if $\Delta M_{sn}(\alpha = 1) \leq 0$, the threshold level $\alpha_{ns}$ exists. Therefore, doing nothing is better for $\alpha < \alpha_{ns}$, and single-sourcing is preferred for $\alpha \geq \alpha_{ns}$.

For general distributions, it can be proved that $\Delta M_{ds}$ is still monotonically increasing on $[0.5, 1]$. The optimal policy structure on the range $\alpha \in [0.5, 1]$ is the same as that of the model with $K_o + K_p$, although values of the threshold reliability levels are different.

For uniform and exponential distributions, the optimal policy structure can be summarized as follows.

i) When $\alpha_{ns} < \alpha_{ds}$, order nothing for $\alpha \leq \alpha_{nd}$, dual-source for $\alpha_{nd} \leq \alpha < \alpha_{ds}$ and single-source otherwise.

ii) When $\alpha_{ns} \geq \alpha_{ds}$, order nothing for $\alpha \leq \alpha_{ns}$, and single-source otherwise.

iii) When $\alpha_{ns}$ does not exist. Order nothing.

### D.2 Mathematical Derivation of the Effects of Parameters on Optimal Policies and Order Quantities

Based on the identical-supplier model, we investigate the effects of parameters on the optimal policy structure and order quantities. Because the optimal policy is determined by the threshold levels, which are computed from cost difference functions, we only need to know the effects of parameters on the cost difference functions.
Proposition D.1. When $\Delta M_{sn}$, $\Delta M_{dn}$ and $\Delta M_{ds}$ increase, the critical threshold reliability levels $\alpha_{ns}$, $\alpha_{nd}$ and $\alpha_{sd}$ increase and $\alpha_{ds}$ decreases if all of them still exist. Therefore we tend to order from fewer suppliers; however, the trend is opposite when $\Delta M_{sn}$, $\Delta M_{dn}$ and $\Delta M_{ds}$ decrease.

Proof. Assume that $\Delta M_{kq} < \Delta M'_{kq}$, $\alpha_{kq}$ is associated with $\Delta M_{kq}$ while $\alpha'_{kq}$ is associated with $\Delta M'_{kq}$. Therefore, $\Delta M_{sn}(\alpha = \alpha_{ns}) = 0$, $\Delta M'_{sn}(\alpha = \alpha_{ns}) > 0$ and $\Delta M'_{sn}(\alpha = \alpha'_{ns}) = 0$. In addition to the fact that $\Delta M_{sn}$ is a linearly decreasing function, it can be derived that $\alpha_{ns} < \alpha'_{ns}$. By the same token, we can conclude that $\alpha_{nd} < \alpha'_{nd}$, $\alpha_{sd} < \alpha'_{sd}$ and $\alpha_{ds} > \alpha'_{ds}$. If $\alpha'_{sd}$ and $\alpha'_{ds}$ do not exist, then the range of dual-sourcing disappear.

The effects of penalty cost, item cost and disposal cost on optimal policies can be obtained from the first-order derivatives of $\Delta M_{sn}$, $\Delta M_{dn}$ and $\Delta M_{ds}$. If the derivatives are positive, ordering from fewer suppliers is desirable as the parameters increase; otherwise, sourcing from more suppliers is preferred. By the same token, the effects on optimal quantities are obtained from the first-order derivatives of $Q^*_s$ and $Q^*_d$. If the derivatives are positive, we tend to order more stock as the parameters increase; otherwise, we tend to order less stock.

D.2.1 Penalty Cost

We know that $Q^*_s$ is solved from the following equation:

$$F(Q^*_s) = \frac{p - c}{h + p}. \quad (D.2)$$

Taking the first-order derivative of the following equation with respect to $p$ by both sides and rearranging it, we have

$$\frac{\partial Q^*_s}{\partial p} = \frac{h + c}{(h + p)^2} \cdot \frac{1}{f(Q^*_s)} > 0.$$ 

Taking the first-order derivative of (2.8) with respect to $p$ by both sides, we obtain

$$\alpha f(Q^*_d) \frac{\partial Q^*_d}{\partial p} + 2 \alpha f(Q^*_d) \frac{\partial Q^*_d}{\partial p} = \frac{h + c}{(h + p)^2},$$
and rearranging this, we have

\[ \frac{\partial Q_d^*}{\partial p} = \frac{h + c}{(h + p)^2} \cdot \frac{1}{\alpha f(Q_d^*) + 2\alpha f(2Q_d^*)} > 0. \]

Substituting

\[ \frac{\partial L(y)}{\partial p} = \int_y^\infty (\xi - y)f(\xi)d\xi + \frac{\partial L(y)}{\partial y} \cdot \frac{\partial y}{\partial p} \]

into

\[ \frac{\partial \Delta M_{sn}}{\partial p} = \alpha \frac{\partial Q_s^*}{\partial p} + \alpha \left[ \int_{Q_s^*}^{\infty} (\xi - Q_s^*)f(\xi)d\xi + L'(Q_s^*)\frac{\partial Q_s^*}{\partial p} \right] - \alpha \int_0^\infty \xi f(\xi)d\xi \]

\[ = \alpha \int_{Q_s^*}^{\infty} (\xi - Q_s^*)f(\xi)d\xi - \alpha \int_0^\infty \xi f(\xi)d\xi < 0 \]

and

\[ \frac{\partial \Delta M_{ds}}{\partial p} = 2\alpha \frac{\partial Q_d^*}{\partial p} + \alpha^2 \left[ 2L'(2Q_d^*)\frac{\partial Q_d^*}{\partial p} + \int_{2Q_d^*}^{\infty} (\xi - 2Q_d^*)f(\xi)d\xi + 2\alpha L'(Q_d^*)\frac{\partial Q_d^*}{\partial p} \right] \]

\[ + \alpha \int_{Q_d^*}^{\infty} (\xi - Q_d^*)f(\xi)d\xi + \tilde{\alpha}^2 \int_0^\infty \xi f(\xi)d\xi - \alpha \frac{\partial Q_d^*}{\partial p} - \alpha L'(Q_d^*)\frac{\partial Q_d^*}{\partial p} \]

\[ - \tilde{\alpha} \int_{Q_d^*}^{\infty} (\xi - Q_d^*)f(\xi)d\xi - \tilde{\alpha} \int_0^\infty \xi f(\xi)d\xi \]

\[ = \alpha \frac{\partial Q_d^*}{\partial p} \left[ 2\alpha L'(2Q_d^*) + 2\tilde{\alpha} L'(Q_d^*) + 2\alpha \frac{\partial Q_d^*}{\partial p} \right] - \alpha \frac{\partial Q_d^*}{\partial p} \left[ L'(Q_d^*) + c \right] + \frac{\Delta P_{ds}}{p} \]

\[ = \frac{\Delta P_{ds}}{p} \leq 0 \]

\[ \frac{\partial \Delta M_{dn}}{\partial p} = \frac{\partial \Delta M_{ds}}{\partial p} + \frac{\partial \Delta M_{sn}}{\partial p} < 0 \]

When penalty cost increases, it is desirable to order more stock to avoid penalty. Optimal quantities of single-sourcing and dual-sourcing are monotonically increasing as penalty cost increases, and sourcing from more suppliers becomes better since it could increase the chance of receiving the order.
D.2.2 Item Cost

Taking first-order derivative of (D.2) with respect to \( c \) by both sides and rearrange it we can obtain

\[
\frac{\partial Q_s^*}{\partial c} = -\frac{1}{h + p} \cdot \frac{1}{f(Q_s^*)} < 0
\]

Optimal quantity of single-sourcing is monotonically decreasing as item cost increases.

Taking first-order derivative of (2.8) with respect to \( h \) by both sides, we have

\[
\bar{\alpha} f(Q_d^*) \frac{\partial Q_d^*}{\partial c} + 2\alpha f(2Q_d^*) \frac{\partial Q_d^*}{\partial c} = -\frac{1}{h + p}
\]

and rearranging this, we have

\[
\frac{\partial Q_d^*}{\partial c} = -\frac{1}{h + p} \cdot \frac{1}{\bar{\alpha} f(Q_d^*) + 2\alpha f(2Q_d^*)} < 0
\]

Optimal quantity of dual-sourcing is monotonically decreasing as item cost increases.

\[
\frac{\partial \Delta M_{sn}}{\partial c} = \alpha \frac{\partial Q_s^*}{\partial c} + \alpha Q_s^* + \alpha L'(Q_s^*) \frac{\partial Q_s^*}{\partial c} = \alpha Q_s^* > 0
\]

Ordering nothing may be preferred as item cost increases.

\[
\frac{\partial \Delta M_{ds}}{\partial c} = 2\alpha \frac{\partial Q_d^*}{\partial c} + 2\alpha Q_d^* + 2\alpha^2 L'(2Q_d^*) \frac{\partial Q_d^*}{\partial c} + 2\alpha \bar{\alpha} L'(Q_d^*) \frac{\partial Q_d^*}{\partial c} - \alpha c \frac{\partial Q_d^*}{\partial c} - \alpha Q_s^* - \alpha L'(Q_s^*) \frac{\partial Q_s^*}{\partial c}
\]

\[
= \alpha \frac{\partial Q_d^*}{\partial c} [2\alpha L'(2Q_d^*) + 2\alpha L'(Q_d^*) + 2c] - \alpha \frac{\partial Q_s^*}{\partial c} [L'(Q_s^*) + c] + \alpha (2Q_d^* - Q_s^*)
\]

\[
> 0
\]

\[
\frac{\partial \Delta M_{dn}}{\partial c} = \frac{\partial \Delta M_{ds}}{\partial c} + \frac{\partial \Delta M_{sn}}{\partial c} > 0
\]

The effects of item cost are the opposite to those of penalty cost. Optimal quantities of single-sourcing
and dual-sourcing are monotonically decreasing as item cost increases, and sourcing from fewer suppliers becomes better.

### D.2.3 Disposal Cost

Taking first-order derivative of (D.2) with respect to $h$ by both sides and rearranging this, we can obtain

$$\frac{\partial Q^*_s}{\partial h} = -\frac{p - c}{(h + p)^2} \cdot \frac{1}{f(Q^*_s)} < 0$$

Optimal quantity of single-sourcing is monotonically decreasing as disposal cost increases.

Taking first-order derivative of (2.8) with respect to $h$ by both sides, we have

$$\bar{\alpha} f(Q^*_d) \frac{\partial Q^*_d}{\partial h} + 2 \alpha f(2Q^*_d) \frac{\partial Q^*_d}{\partial h} = -\frac{p - c}{(h + p)^2}$$

and rearranging this, we have

$$\frac{\partial Q^*_d}{\partial h} = -\frac{p - c}{(h + p)^2} \cdot \frac{1}{\bar{\alpha} f(Q^*_d) + 2 \alpha f(2Q^*_d)} < 0$$

Optimal quantity of dual-sourcing is monotonically decreasing as disposal cost increases.

Substituting

$$\frac{\partial L(y)}{\partial h} = \int_0^y (y - \xi) f(\xi) d\xi + \frac{\partial L(y)}{\partial y} \cdot \frac{\partial y}{\partial h}$$

into

$$\frac{\partial \Delta M_{sn}}{\partial h} = \alpha c \frac{\partial Q^*_s}{\partial h} + \alpha \int_0^{Q^*_s} (Q^*_s - \xi) f(\xi) d\xi + L'(Q^*_s) \frac{\partial Q^*_s}{\partial h} = \alpha \int_0^{Q^*_s} (Q^*_s - \xi) f(\xi) d\xi > 0$$
\[
\frac{\partial \Delta M_{ds}}{\partial h} = 2 \alpha_c \frac{\partial Q^*_d}{\partial h} + \alpha^2 [2L'(2Q^*_d) \frac{\partial Q^*_d}{\partial h} + \int_0^{2Q^*_d} (2Q^*_d - \xi) f(\xi) d\xi] + 2 \alpha \alpha_c [L'(Q^*_d) \frac{\partial Q^*_d}{\partial h}]
\]
\[
\quad + \int_0^{Q^*_d} (Q^*_d - \xi) f(\xi) d\xi] - \alpha_c \frac{\partial Q^*_s}{\partial h} - \alpha [L'(Q^*_s) \frac{\partial Q^*_s}{\partial h} + \int_0^{Q^*_s} (Q^*_s - \xi) f(\xi) d\xi]
\]
\[
= \alpha \frac{\partial Q^*_s}{\partial h} [2\alpha L'(2Q^*_d) + 2\alpha L'(Q^*_d) + 2c] - \alpha \frac{\partial Q^*_s}{\partial h} [L'(Q^*_s) + c] + \frac{\Delta D_{ds}}{h}
\]
\[
= \frac{\Delta D_{ds}}{h}
\]

\[
\frac{\partial \Delta M_{dn}}{\partial h} = \frac{\partial \Delta M_{ds}}{\partial h} + \frac{\partial \Delta M_{sa}}{\partial h} = \alpha^2 \int_0^{2Q^*_d} (2Q^*_d - \xi) f(\xi) d\xi + 2\alpha \alpha_c \int_0^{Q^*_d} (Q^*_d - \xi) f(\xi) d\xi > 0
\]

When disposal cost increases, the optimal quantities of dual-sourcing and single-sourcing are also monotonically decreasing. Ordering nothing may be preferred compared with the single-sourcing case; and most of the time, single-sourcing is more likely to become better than dual-sourcing. Although \(\frac{\partial \Delta M_{ds}}{\partial h} < 0\) is possible in theory, that is, dual-sourcing becomes better when disposal cost increases, such an exception rarely occurs in our numerical experiments.

### D.3 Mathematical Derivation of the Effects of Demand Distributions on Optimal Policies and Order Quantities

Demand distributions have great impact on the optimal policy and order quantity allocation. We focus on studying the effects of demand mean \(\mu\) and standard deviation \(\sigma\).

#### D.3.1 Uniform demand distribution

When demand is uniformly distributed between \([l \triangleq \mu - d, u \triangleq \mu + d]\). The mean is \(\mu\) and the standard deviation is \(\sigma = \frac{d}{\sqrt{3}}\). The optimal quantities can be written as follows:

\[
Q^*_s = \mu + \frac{p - c - (h + c)}{h + p} \sqrt{3} \sigma = \mu + \frac{c_u - c_o}{h + p} \sqrt{3} \sigma
\]  
(D.3)
When \( u > 2l \),
\[
Q_d^* = \begin{cases} 
\frac{\mu}{1+\alpha} + \frac{c_u - c_o}{(h+p)(1+\alpha)} \sqrt{3}\sigma, & \text{if } \frac{al}{2d} \leq \gamma < \frac{(1+\alpha)u/2-l}{2d} \quad (Q_d^* \geq l, 2Q_d^* \leq u) \\
\mu + \sqrt{3}\sigma \frac{(1+\alpha)c_u - (1+\alpha)c_o}{\alpha(h+p)}, & \text{if } \gamma \geq \frac{(1+\alpha)u/2-l}{2d} \quad (Q_d^* \geq l, 2Q_d^* > u) \\
\frac{\mu}{2} + \sqrt{3}\sigma \frac{(2-\alpha)c_u - \alpha c_o}{2\alpha(h+p)}, & \text{if } \gamma < \frac{al}{2d} \quad (Q_d^* < l, l < 2Q_d^* \leq u) 
\end{cases}
\]  

(D.4)

When \( u \leq 2l \),
\[
Q_d^* = \begin{cases} 
\frac{\mu}{1+\alpha} + \frac{c_u - c_o}{(h+p)(1+\alpha)} \sqrt{3}\sigma, & \text{if } \gamma \geq \alpha \quad (Q_d^* \geq l, 2Q_d^* > u) \\
\mu + \sqrt{3}\sigma \frac{\alpha c_u - \alpha c_o}{\alpha(h+p)}, & \text{if } \gamma < \alpha \quad (Q_d^* < l, l < 2Q_d^* \leq u) \\
\text{any value as long as } Q_d^* < l \text{ and } 2Q_d^* \geq u, & \text{if } \gamma = \alpha \quad (Q_d^* < l, 2Q_d^* \geq u) 
\end{cases}
\]  

(D.5)

where \( \gamma = \frac{p-c}{h+p} \). The derivation of (D.3), (D.4) and (D.5) is attached below. From these two equations we can conclude that when demand mean increases, optimal quantities \( Q_s^* \) and \( Q_d^* \) increase linearly.

When the standard deviation of demand increases, the change of optimal quantities depends on the relation between underage cost and overage cost. Specifically, if underage cost is relatively large, optimal quantities increase when \( \sigma \) increases; if overage cost is relatively large, optimal quantities decrease when \( \sigma \) increases.

Now we consider the effects of \( \mu \) and \( \sigma \) on the cost difference functions \( \Delta M_{sn} \), \( \Delta M_{dn} \) and \( \Delta M_{ds} \).
\[
\Delta M_{sn} = K_o + K_p + \alpha K_r - \alpha c_u \mu + \alpha \sqrt{3}\sigma \frac{c_u c_o}{c_u + c_o}  
\]  

(D.6)

The derivation of (D.6) is attached below. When demand mean \( \mu \) increases, \( \Delta M_{sn} \) decreases so that we tend to single-source. When the standard deviation increases, \( \Delta M_{sn} \) increases, which implies that doing nothing may become a better policy.

For \( \Delta M_{ds} \), the following conclusions can be drawn and the proof is attached below.

- When \( Q_d^* \geq l, 2Q_d^* \leq u \), we have \( \frac{\partial \Delta M_{ds}}{\partial \mu} < 0 \), that is, if demand mean increases, we tend to dual-source. \( \frac{\partial \Delta M_{ds}}{\partial \sigma} \) depends on reliability level, penalty cost, disposal cost, item cost, demand
mean and standard deviation itself.

- When \( Q_d^* \geq l, 2Q_d^* < u \), if demand mean increases, we tend to dual-source. The effects of \( \sigma \) depends on reliability level, penalty cost, holding cost and item cost.

- When \( Q_d^* < l, l < 2Q_d^* \leq u \) and \( Q_d^* < l, 2Q_d^* \geq u \), the demand mean has no influence on the choice between single-sourcing and dual-sourcing. If the standard deviation increases, we tend to dual-source.

The effects on \( \Delta M_{dn} \) can be obtained by combining the results of \( \Delta M_{sn} \) and \( \Delta M_{ds} \) since \( \Delta M_{dn} = \Delta M_{sn} + \Delta M_{ds} \).

Derivation of (D.3), (D.4) and (D.5) The end of period cost can be written as follows:

\[
L(y) = \begin{cases} (y - \mu)h, & y > u \\ \frac{h(y-l)^2 + p(u-y)^2}{4d}, & l \leq y \leq u \\ (\mu - y)p, & y < l \end{cases}
\]

\[
L'(y) = \begin{cases} h, & y > u \\ \frac{h(y-l) - p(u-y)}{2d}, & l \leq y \leq u \\ -p, & y < l \end{cases}
\]

For single-sourcing,

\[
Q_s^* = F^{-1}(\frac{p - c}{h + p}) = F^{-1}(\frac{c_u}{c_u + c_o}) = l + \frac{c_u}{c_u + c_o} \cdot (u - l) = \mu + \frac{c_u - c_o}{c_u + c_o} \cdot \frac{d}{(h + p)(1 + \alpha)} \cdot \sqrt{3\sigma}
\]

For dual-sourcing, when \( u > 2l \),

1. if \( Q_d^* \geq l \) and \( 2Q_d^* \leq u \), substituting \( F(Q_d^*) = \frac{Q_d^* - l}{2d} \) and \( F(2Q_d^*) = \frac{2Q_d^* - l}{2d} \) into (2.8) we get

\[
Q_d^* = \frac{\mu}{1 + \alpha} + \frac{c_u - c_o}{(h + p)(1 + \alpha)} \cdot \frac{d}{(h + p)(1 + \alpha)} \cdot \sqrt{3\sigma}
\] (D.7)
Therefore, \( Q_d^* \geq l \) and \( 2Q_d^* \leq u \) are equivalent to the condition \( \frac{a \gamma}{2d} \leq \gamma < \frac{(1+\alpha)u/2-l}{2d} \).

2. if \( Q_d^* \geq l \) and \( 2Q_d^* > u \), substituting \( F(Q_d^*) = \frac{Q_d^*-l}{2d} \) and \( F(2Q_d^*) = 1 \) into (2.8) we get

\[
Q_d^* = \mu + \frac{\alpha c_u - (1+\alpha)c_o}{\bar{\alpha}(h+p)} = \mu + \sqrt{3\sigma} \frac{\bar{\alpha}c_u - (1+\alpha)c_o}{\bar{\alpha}(h+p)} \tag{D.8}
\]

Therefore, \( Q_d^* \geq l \) and \( 2Q_d^* > u \) are equivalent to the conditions \( \gamma \geq \alpha \) and \( \gamma \geq \frac{(1+\alpha)u/2-l}{2d} \).

Because \( \alpha < \frac{(1+\alpha)u/2-l}{2d} \), the condition for this situation can be simplified as \( \gamma \geq \frac{(1+\alpha)u/2-l}{2d} \).

3. if \( Q_d^* < l \) and \( l < 2Q_d^* \leq u \), substituting \( F(Q_d^*) = 0 \) and \( F(2Q_d^*) = \frac{2Q_d^*-l}{2d} \) into (2.8) we get

\[
Q_d^* = \frac{\mu}{2} + \frac{(2-\alpha)c_u - \alpha c_o}{2\alpha(h+p)} = \frac{\mu}{2} + \sqrt{3\sigma} \frac{(2-\alpha)c_u - \alpha c_o}{2\alpha(h+p)} \tag{D.9}
\]

Therefore, \( Q_d^* < l \) and \( l < 2Q_d^* \leq u \) are equivalent to the conditions \( \gamma < \frac{a \gamma}{2d} \) and \( \gamma < \alpha \). Because \( \alpha > \frac{a \gamma}{2d} \), the condition for this situation can be simplified as \( \gamma < \frac{a \gamma}{2d} \).

when \( u \leq 2l \),

1. if \( Q_d^* \geq l \) and \( 2Q_d^* \leq u \), \( Q_d^* \) is the same as (D.8). \( Q_d^* \geq l \) and \( 2Q_d^* > u \) are equivalent to the conditions \( \gamma \geq \alpha \) and \( \gamma \geq \frac{(1+\alpha)u/2-l}{2d} \). Because \( \alpha > \frac{(1+\alpha)u/2-l}{2d} \), the condition for this situation can be simplified as \( \gamma \geq \alpha \).

2. if \( Q_d^* \geq l \) and \( 2Q_d^* > u \), \( Q_d^* \) is the same as (D.9). \( Q_d^* < l \) and \( l < 2Q_d^* \leq u \) are equivalent to the conditions \( \gamma < \frac{a \gamma}{2d} \) and \( \gamma < \alpha \). Because \( \alpha < \frac{a \gamma}{2d} \), the condition for this situation can be simplified as \( \gamma < \alpha \).

3. if \( Q_d^* < l \) and \( l < 2Q_d^* \leq u \), substituting \( F(Q_d^*) = 0 \) and \( F(2Q_d^*) = 1 \) into (2.8) we can conclude that \( Q_d^* \) can be any value as long as \( Q_d^* < l \) and \( l < 2Q_d^* \leq u \) when \( \gamma = \alpha \).
Derivation of (D.6)

\[ M_s = K_o + K_p + \alpha K_r + \alpha c Q^*_s + \alpha L(Q^*_s) + \tilde{\alpha} L(0) \]

\[ = K_o + K_p + \alpha K_r + \alpha c \left[ \mu + \sqrt{3}\sigma \frac{c_u - c_o}{c_u + c_o} \right] + \alpha \frac{h(Q^*_s - l)^2 + p(u - Q^*_s)^2}{4d} + \tilde{\alpha} L(0) \]

\[ = K_o + K_p + \alpha K_r + \alpha c \left[ \mu + \sqrt{3}\sigma \frac{c_u - c_o}{c_u + c_o} \right] + \alpha \frac{h(\sqrt{3}\sigma \frac{c_u - c_o}{c_u + c_o} + \sqrt{3}\sigma)^2 + p(\sqrt{3}\sigma - \sqrt{3}\sigma \frac{c_u - c_o}{c_u + c_o})^2}{4\sqrt{3}\sigma} \]

\[ + \tilde{\alpha} L(0) \]

\[ = K_o + K_p + \alpha K_r + \alpha c \mu + \alpha \sqrt{3}\sigma \left[ \frac{c_u - c_o}{c_u + c_o} \right] + \alpha \frac{h(\frac{c_u - c_o}{c_u + c_o} + 1)^2 + p(1 - \frac{c_u - c_o}{c_u + c_o})^2}{4} + \tilde{\alpha} L(0) \]

\[ = K_o + K_p + \alpha K_r + \alpha c \mu + \frac{\alpha \sqrt{3}\sigma}{4} \left[ 2(c_o - c_u) \frac{c_u - c_o}{c_u + c_o} + (c_u + c_o) + (c_u + c_o) \frac{c_u - c_o}{c_u + c_o} \right] \]

\[ + \tilde{\alpha} L(0) \]

\[ = K_o + K_p + \alpha K_r + \alpha c \mu + \frac{\alpha \sqrt{3}\sigma}{4} \left( 4c_u c_o \frac{c_u - c_o}{c_u + c_o} \right) + \tilde{\alpha} L(0) \]

Proof of the uniform demand effects for \( \Delta M_{ds} \)

\[ \Delta M_{ds} = K_p + \alpha K_r + 2\alpha c Q^*_s + \alpha^2 L(2Q^*_s) + 2\alpha \tilde{\alpha} L(Q^*_s) + \tilde{\alpha}^2 L(0) - \alpha c Q^*_s - \alpha L(Q^*_s) - \tilde{\alpha} L(0) \]

\[ = K_p + \alpha K_r + 2\alpha c (\mu + A\sqrt{3}\sigma) + \alpha^2 (2\mu + 2A\sqrt{3}\sigma - \mu) h \]

\[ + 2\alpha^2 \sqrt{3}\sigma \frac{h(A + 1)^2 + p(1 - A)^2}{4\sqrt{3}\sigma} - \alpha \tilde{\alpha} (p\mu) - \alpha c \mu - \alpha \sqrt{3}\sigma \frac{c_u c_o}{c_u + c_o} \]

\[ = K_p + \alpha K_r + \alpha (\alpha c_o - \tilde{\alpha} c_u) \mu + \sqrt{3}\alpha \sigma \left\{ 2\alpha c_o + \tilde{\alpha} (c_o - c_u) \right\} A + \frac{\tilde{\alpha}^2}{2} (c_o + c_u) (1 + A^2) \]

\[ - \frac{c_u c_o}{c_u + c_o} \}

\[ = K_p + \alpha K_r + \alpha (\alpha c_o - \tilde{\alpha} c_u) \mu + \sqrt{3}\alpha \sigma \frac{2c_o (\tilde{\alpha} c_u - 2\alpha c_o)}{2\tilde{\alpha} (c_u + c_o)}, \]

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where $A = \frac{c_u - c_o}{c_u + c_o}$. When $u > 2l$, i.e., $3d \geq \mu$, from

$$\gamma = \frac{c_u}{c_o + c_u} \geq \frac{(1 + \alpha)u/2 - l}{2d} = \frac{(\alpha - 1)\mu + (3 + \alpha)d}{4d} \geq \frac{(\alpha - 1)3d + (3 + \alpha)d}{4d} = \alpha$$

we can conclude that $\alpha c_o - \bar{\alpha}c_u \leq 0$. By the same token, when $u \leq 2l$, the same conclusion can be drawn. Therefore, when demand mean increases, we tend to dual-source. When standard deviation increases, we tend to dual-source if underage cost is relatively high, and single-sourcing is preferred if overage cost is relatively high.

When $Q_d^* = \frac{\mu}{2} + \sqrt{3\sigma} \frac{(2-\alpha)c_u - \alpha c_o}{2\alpha(h+p)}$, we have

$$\Delta M_{ds} = K_p + \alpha K_r + 2\alpha cQ_d^* + \alpha^2 L(2Q_d^*) + 2\alpha\bar{\alpha}L(Q_d^*) + \bar{\alpha}^2 L(0) - \alpha cQ_s^* - \alpha L(Q_s^*) - \bar{\alpha} L(0)$$

$$= K_p + \alpha K_r + \alpha c(\mu + 2B\sqrt{3}\sigma) + \alpha^2 \frac{3\sigma^2[2(2B + 1)^2 + p(1 - 2B)^2]}{4\sqrt{3}\sigma} + 2\alpha\bar{\alpha}p(\frac{\mu}{2} - \sqrt{3}\sigma B)$$

$$- \alpha\bar{\alpha}(p\mu) - \alpha c\mu - \alpha\sqrt{3}\sigma \frac{c_u c_o}{c_u + c_o}$$

$$= K_p + \alpha K_r - \sqrt{3}\alpha\sigma \frac{4\alpha c_o^2}{c_u + c_o},$$

where $B = \frac{(2-\alpha)c_u - \alpha c_o}{2\alpha(h+p)}$. In this case, demand mean has no influence on the choice between single-sourcing and dual-sourcing. When standard deviation increases, we tend to dual-source.

When $Q_d^* < l$ and $2Q_d^* \geq u$,

$$\Delta M_{ds} = K_p + \alpha K_r + 2\alpha cQ_d^* + \alpha^2 L(2Q_d^*) + 2\alpha\bar{\alpha}L(Q_d^*) + \bar{\alpha}^2 L(0) - \alpha cQ_s^* - \alpha L(Q_s^*) - \bar{\alpha} L(0)$$

$$= K_p + \alpha K_r + 2\alpha cQ_d^* + \alpha^2 (2Q_d^* - \mu)h + 2\alpha\bar{\alpha}p(\mu - Q_d^*) - \alpha\bar{\alpha}p\mu - \alpha c\mu - \alpha\sqrt{3}\sigma \frac{c_u c_o}{c_u + c_o}$$

$$= K_p + \alpha K_r + 2\alpha cQ_d^* + 2\alpha^2 Q_d^*h - 2\alpha\bar{\alpha}pQ_d^* + \alpha(c_u - c_o) - \alpha\sqrt{3}\sigma \frac{c_u c_o}{c_u + c_o}$$

$$= K_p + \alpha K_r - \alpha\sqrt{3}\sigma \frac{c_u c_o}{c_u + c_o}$$

In this case, demand mean has no influence on the choice between single-sourcing and dual-sourcing.

When standard deviation increases, we tend to dual-source. 

\[\square\]
D.3.2 Triangular demand distribution

Now we consider the effects of a central tendency triangular demand distribution, which has lower limit $l$, upper limit $u$ and mode (or mean) $\mu = \frac{l+u}{2}$.

$$Q_s^* = \begin{cases} 
\mu + [\sqrt{12}\gamma - \sqrt{6}]\sigma, & \text{if } l \leq Q_s^* \leq \mu \quad (0 \leq \gamma = \frac{c_u}{c_u+c_o} \leq 0.5) \\
\mu + [\sqrt{6} - \sqrt{12(1-\gamma)}]\sigma, & \text{if } \mu < Q_s^* \leq u \quad (0.5 < \gamma = \frac{c_u}{c_u+c_o} \leq 1)
\end{cases}$$

$$\Delta M_{sn} = \begin{cases} 
K_o + K_p + \alpha K_r - \alpha c_u \mu + \alpha c_u (\sqrt{6} - \frac{2}{3}\sqrt{12\gamma})\sigma, & \text{if } l \leq Q_s^* \leq \mu \quad (0 \leq \gamma \leq 0.5) \\
K_o + K_p + \alpha K_r - \alpha c_u \mu + \alpha c_o (\sqrt{6} - \frac{2}{3}\sqrt{12(1-\gamma)}), & \text{if } \mu < Q_s^* \leq u \quad (0.5 < \gamma \leq 1)
\end{cases}$$

When $\mu$ increases, $Q_s^*$ increases and we tend to single-source. If $c_u < c_o$, then $l \leq Q_s^* \leq \mu$, and $Q_s^*$ decreases as $\sigma$ increases. If $c_u \geq c_o$, then $\mu < Q_s^* \leq u$, and $Q_s^*$ increases as $\sigma$ increases. Compared with single-sourcing, we tend to do nothing when $\sigma$ increases.

The optimal quantities can be summarized as follows:

- When $Q \leq l, l \leq 2Q_d^* \leq \mu$,
  $$Q_d^* = \frac{1}{2} \{ \mu + [4\sqrt{3\gamma} - \sqrt{6}]\sigma \} \quad (D.10)$$

- When $Q_d^* \leq l, \mu \leq 2Q_d^* \leq u$,
  $$Q_d^* = \frac{1}{2} \{ \mu + [\sqrt{6} - 4\sqrt{3(1-\gamma)}]\sigma \} \quad (D.11)$$

- When $Q_d^* \leq l, u \leq 2Q_d^*, Q_d^*$ can be any value.

- When $l < Q_d^* \leq \mu, \mu < 2Q_d^* \leq u$,
  $$\frac{\partial Q_d^*}{\partial \mu} = \frac{\alpha(u-2Q_d^*) + \alpha(Q_d^*-l)}{2\alpha(u-2Q_d^*) + \alpha(Q_d^*-l)} > 0 \quad (D.12)$$
\[
\frac{\partial Q^*_d}{\partial \sigma} = \frac{\sigma(\gamma - \alpha)/12 + \sqrt{6}\alpha(u - 2Q^*_d) + \sqrt{6}\bar{\alpha}(Q^*_d - l)}{2\alpha(u - 2Q^*_d) + \bar{\alpha}(Q^*_d - l)} \tag{D.13}
\]

- When \( l < Q^*_d \leq \mu, u \leq 2Q^*_d \),

\[
Q^*_d = \mu + \left[2\sqrt{\frac{3(\gamma - \alpha)}{(1 - \alpha)} - \sqrt{6}\sigma}\right] \tag{D.14}
\]

- When \( \mu < Q^*_d \leq u, u \leq 2Q^*_d \),

\[
Q^*_d = \mu + \left[\sqrt{6} - 2\sqrt{\frac{3(\bar{\alpha} - \gamma)}{(1 - \alpha)}}\right] \sigma \tag{D.15}
\]

When demand mean increases, order quantity also increases. However, the effects of \( \sigma \) depend on other parameters such as underage cost, overage cost, reliability level, demand mean, and sometimes \( \sigma \) itself.

For the cost difference functions \( \Delta M_{dn} \) and \( \Delta M_{ds} \), the explicit expressions as a function of \( \mu \) and \( \sigma \) are very complicated. However, it can be proved that \( \frac{\partial \Delta M_{dn}}{\partial \mu} = 0 \) when \( Q^*_d < l \), \( \frac{\partial \Delta M_{dn}}{\partial \mu} < 0 \) otherwise, and \( \frac{\partial \Delta M_{ds}}{\partial \mu} < 0 \). Therefore, \( \mu \) has no influence on the choice between dual-sourcing and single-sourcing when \( Q^*_d \) is relatively small, but dual-sourcing might be preferred when \( \mu \) increases if \( Q^*_d \) is relatively large. Dual-sourcing is preferred when \( \mu \) increases compared with ordering nothing. The effects of \( \sigma \), as those on \( Q^*_d \), depend on other parameters such as underage cost, overage cost, reliability level, demand mean, and sometimes \( \sigma \) itself.
Appendix E

Supplement of Chapter 4

E.1 Numerical Experiment: Continuous Formulations

We propose six methods to solve the continuous model formulation, i.e., Total Enumeration (TE), Partial Enumeration (PE), Branch and Bound without $z$ (B&B w/o $z$), Branch and Bound with $z$ (B&B w/ $z$), Free Variable Reduction without $K$ (FVR w/o $K$) and Free Variable Reduction with $K$ (FVR w/ $K$). The Primal Affine Scaling (PAS) algorithm is embedded in the first four methods to solve subproblems or relaxation problems. Newton’s method is embedded in the last two methods to solve unconstrained subproblems. The number of subproblems, relaxation problems and unconstrained subproblems are referred to as the number of nodes hereafter. Computation time of the PAS algorithm and Newton’s method not only depends on specific parameters, but also the number of variables and the step length computation method. Table E.1, Figure E.1, E.2, E.3 and E.4 show the numerical experiment results.

Theoretically, TE and the B&B methods are exact methods. However, in Table E.1, there are non-zeros deviations in the rows of the B&B methods, and negative deviations exist in the $min$ columns, implying that for some problems, TE and the B&B methods do not obtain the best solution. The solution obtained by FVR, which is a heuristic method, may be better than that obtained by TE or the B&B methods. The reason is that the PAS algorithm and Newton’s method can only obtain the optimal solution when the stopping criteria $\epsilon \to 0$, while when the algorithms are implemented, the stopping cri-
terion cannot be small enough to guarantee optimality. In fact, $\epsilon = 0.001$ is used in our experiments for both methods. Small termination criteria not only lead to longer computation time, but also may cause near-singular matrix, which is likely to result in inaccurate solutions since matrix inversion is involved in the PAS algorithm. In addition, in the PAS algorithm and Newton’s method, different starting points, different parameters such as $\epsilon$ and different line search methods in choosing step size all have the possibility of leading to different optimal solutions/costs within tolerance. Therefore, negative deviations, very small though, are possible in numerical experiments.

From Figure E.4 and Table E.1 we can see that TE usually takes longer than the other methods when the number of suppliers is relatively large. It is better than the other two exact methods, the B&B methods, for a relatively small pool of suppliers. PE, as a heuristic method, usually leads to the same result with TE in these experiments, and is much faster than TE since more cases have been eliminated. Com-

<table>
<thead>
<tr>
<th>Deviation from TE (%)</th>
<th>3 Suppliers</th>
<th>5 Suppliers</th>
<th>7 Suppliers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Avg.</td>
<td>Std.</td>
<td>Min</td>
</tr>
<tr>
<td>PE</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>B&amp;B w/o $z$</td>
<td>0.000</td>
<td>0.001</td>
<td>-0.008</td>
</tr>
<tr>
<td>B&amp;B w/ $z$</td>
<td>0.000</td>
<td>0.001</td>
<td>-0.008</td>
</tr>
<tr>
<td>FVR w/o K</td>
<td>1.138</td>
<td>5.571</td>
<td>-0.008</td>
</tr>
<tr>
<td>FVR w/ K</td>
<td>0.000</td>
<td>0.001</td>
<td>-0.008</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Computation time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE</td>
</tr>
<tr>
<td>PE</td>
</tr>
<tr>
<td>B&amp;B w/o $z$</td>
</tr>
<tr>
<td>B&amp;B w/ $z$</td>
</tr>
<tr>
<td>FVR w/o K</td>
</tr>
<tr>
<td>FVR w/ K</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of Nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE</td>
</tr>
<tr>
<td>PE</td>
</tr>
<tr>
<td>B&amp;B w/o $z$</td>
</tr>
<tr>
<td>B&amp;B w/ $z$</td>
</tr>
<tr>
<td>FVR w/o K</td>
</tr>
<tr>
<td>FVR w/ K</td>
</tr>
</tbody>
</table>

1Negative deviations are caused by the insufficiently small termination criterion $\epsilon$. There are 3 problems in $N = 3$, 3 problems in $N = 5$, and 2 problems in $N = 7$, which have absolute deviations greater than 0.001%. For those exact methods, the values in the minimum deviation columns become less than 0.001% if $\epsilon$ is set to 0.0001 for the 8 outliers.
putation time of subproblems increases as the number of suppliers \( N \) increases in the PAS algorithm. PE always solves subproblems with the fewest variables (suppliers) first \( \binom{N}{k} \) subproblems with \( k \) variables, \( 0 \leq k \leq N \), and is likely to avoid computing subproblems with more variables, which usually takes longer, by using the cutting rule. This method may be used if \( Q_{\text{min}} \) and \( Q_{\text{max}} \) are relatively higher so fewer suppliers may be selected and more cases may be cut under such circumstances. Optimality cannot be guaranteed when PE is employed as we can see a non-zero deviation exists in the max column of Table E.1 when \( N = 7 \). The worst case we have seen so far is the problem with \( N = 3 \), scale = 500,
Figure E.2: Numerical Results: Scatter Plot of Computation Time and Deviation from TE for $N = 5$

\[ a = 16, \ b = 4, \ \alpha = [0.8, 0.65, 0.9], \ c = [1, 1, 1.6], \ h = 1.5, \ p = 22, \ Q^{\text{min}} = [202, 117, 63], \]
\[ Q^{\text{max}} = [405, 410, 500]. \]

The optimal solution within tolerance is $Q^* = [207.67, 191.48, 242.70]$ and $M^* = 1380.2$, while the solution of PE is $Q = [0, 0, 447.44]$ and $M = 1612.9$, leading to a deviation of 16.86%.

As two exact methods, B&B methods reduce the number of nodes and take less time than TE does, especially for a relatively large pool of suppliers. When compared with PE, the average computation time is longer even though the number of nodes is usually smaller. The main reason is that computa-
tion time of nodes increases as the number of suppliers $N$ increases in the PAS algorithm. PE always solves subproblems with fewer variables and is likely to avoid time-consuming subproblems with more variables by using the cutting rule. B&B, however, always starts with $N$ (without $z$) or $2N$ variables (with $z$). For B&B w/o $z$, the number of variables reduces by one in the branch $Q_i = 0$ and it remains the same in the branch $Q_{i,\min} \leq Q_i \leq Q_{i,\max}$. For B&B w/ $z$, the number of variables reduces by two in the branch $z_i = 0$, which leads to the corresponding $Q_i = 0$, and it reduces by one in the $z_i = 1$ branch. As a result, it may be better to use the B&B methods for a relatively large number of suppliers.
Figure E.4: Numerical Results: Computation Time

and optimal solution within tolerance is required. TE may be used for a small $N$ and PE may be used when near-optimal solution is allowed.

Moreover, fewer suppliers may be selected as $Q_{\text{min}}$ increases, and more suppliers may be selected to satisfy demand when $Q_{\text{max}}$ decreases. Therefore, when the order size constraints are relatively relaxed, the B&B methods, which start with solving problem of all the suppliers selected, may be faster than PE. Under such cases, the cutting rule may not take effect so few cases may be eliminated in PE, whereas B&B methods may solve fewer relaxation problems.

FVR w/o $K$ has a large variance in computation time, and the highest deviations from TE among these methods, which may not be a good methods to choose for continuous problems. FVR w/ $K$ is quick and has small deviations, and is usually appropriate to choose when the order size constraints are relatively relaxed and the requirement for optimality is not strict.

Since fractional order quantities are not practical in real world, we can evaluate the immediate neighboring points around the optimal solutions and compare them to determine the implementable best solutions. The computation time shown in Table E.2 is the time of obtaining the optimal continuous
solution within tolerance plus the local search time. The results are compared with the best solutions obtained by the corresponding discrete models. The absolute average deviations are less than 0.4% and the maximum deviation is about 10%, which implies that continuous demand distributions are usually very good approximations of discrete ones.

Table E.2: Comparison of the Six Methods: Continuous Models with Integer Solutions

<table>
<thead>
<tr>
<th></th>
<th>3 Suppliers</th>
<th>5 Suppliers</th>
<th>7 Suppliers</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Computation time (seconds)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TE</td>
<td>0.63</td>
<td>0.15</td>
<td>1.03</td>
</tr>
<tr>
<td>PE</td>
<td>0.45</td>
<td>0.19</td>
<td>1.04</td>
</tr>
<tr>
<td>B&amp;B w/o z</td>
<td>0.70</td>
<td>0.33</td>
<td>2.50</td>
</tr>
<tr>
<td>B&amp;B w/ z</td>
<td>1.45</td>
<td>0.54</td>
<td>3.27</td>
</tr>
<tr>
<td>FVR w/o K</td>
<td>0.26</td>
<td>0.08</td>
<td>0.46</td>
</tr>
<tr>
<td>FVR w/ K</td>
<td>0.30</td>
<td>0.14</td>
<td>0.75</td>
</tr>
<tr>
<td><strong>Absolute Deviation from the Best Solution of the Discrete Model (%)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TE</td>
<td>0.35</td>
<td>1.32</td>
<td>10.62</td>
</tr>
<tr>
<td>PE</td>
<td>0.35</td>
<td>1.32</td>
<td>10.62</td>
</tr>
<tr>
<td>B&amp;B w/o z</td>
<td>0.35</td>
<td>1.32</td>
<td>10.62</td>
</tr>
<tr>
<td>B&amp;B w/ z</td>
<td>0.35</td>
<td>1.32</td>
<td>10.62</td>
</tr>
<tr>
<td>FVR w/o K</td>
<td>0.33</td>
<td>1.30</td>
<td>10.62</td>
</tr>
<tr>
<td>FVR w/ K</td>
<td>0.35</td>
<td>1.32</td>
<td>10.62</td>
</tr>
</tbody>
</table>

E.2 Effects of Parameters on Supplier Selection

When there is no order size constraints nor fixed order cost, our problem is a special case of the problem described by Dada et al. [11], who model the reliability level in a more general way. The quantity delivered is defined to be the minimum of the order quantity and the production capability, which is a random distribution may or may not related to the order quantity. Our definition of reliability level can be regarded as a two-point distribution, where the production capability of supplier \( i \) is 0 with probability \( 1 - \alpha_i \) and \( \infty \) (or a large enough number) with probability \( \alpha_i \).

Dada et al. [11] prove that low item cost is the “order qualifier” while high reliability level is the “order winner”. To be more specific, reliability of a given supplier does not influence the decision of
whether it is selected or not, although it does affect the order share of this supplier. In an optimal solution, a given supplier is selected only if all less-expensive suppliers are selected, regardless of the given supplier’s reliability level, and no more than one perfectly reliable supplier need be active. However, when order size constraints are active and fixed order costs are incurred, these conclusions do not apply anymore. We show that at most four factors can affect the selection decision of a given supplier. Note that our discussion is from the view of a supplier, so it is limited to the parameters of one’s own, that is, a given supplier’s fixed order cost \( K_i \), item cost \( c_i \), reliability level \( \alpha_i \), supply capacity \( Q_i^{\max} \) and minimum order size constraint \( Q_i^{\min} \).

Now we analyze the effects of the five parameters of a particular supplier on its selection decision. Suppose that \( Q^* \) is the optimal solution of sourcing from the set \( P + j \) and \( Q^* \) is the optimal solution of sourcing from the set \( P \). Given that \( M_P(Q^*) \geq M_{P+j}(Q^*) \), we investigate the changes to the parameters of supplier \( j \) that would cause it to be dropped from the set \( P \), i.e., \( M_P(Q^*) \leq M_{P+j}(Q^*) \). The optimal costs of sourcing from the set \( P + j \) can be written as

\[
M_{P+j}(Q^{*}) = \sum_{i \in P+j} K_i + pE(\xi) + (1 - \alpha_j)\{\sum_{t \in P} (\prod_{t \in P} [\alpha_t(1 - t) + (1 - \alpha_t)(1 - t)])(h + p)\sum_{i \in P} t_i Q_i^{*}F(\sum_{i \in P} t_i Q_i^{*}) - \int_0^{\sum_{t \in P} t_i Q_i^{*}} \xi f(\xi)d\xi - \sum_{i \in P} t_i Q_i^{*}\}
\]

Note that \( \sum_{i \in P} K_i + W_1 + pE(\xi) = M_P(Q^*) \geq M_P(Q^*) \) due to the non-optimality of \( Q^* \) when sourcing from the set \( P \). Obviously, when \( K_j \) increases to a certain value, which equals to the marginal benefit of supplier \( j \), supplier \( j \) is dropped.

We know that \( M_{P+j}(Q^*) = M_P(Q^*) \) when \( Q_j^{*} = Q_j^{max} = 0 \), then \( M_{P+j}(Q^*) \leq M_P(Q^*) \) always holds due to \( \frac{\partial M_{P+j}(Q_j^{*})}{\partial Q_j^{*}} \mid Q_j^{*} = Q_j^{max} \leq 0 \) based on KKT conditions. Therefore, \( Q_j^{max} \) does not influence the selection of supplier \( j \) unless it becomes zero. However, as \( Q_j^{min} \) increases to a certain value,
supplier $j$ would be dropped because $\frac{\partial M_{P+j}}{\partial Q_j}|_{Q_j=Q_j^{\min}} \geq 0$ based on KKT conditions.

Now we consider the effects of $\alpha_j$. Substitute (4.14) into (4.17) we have

$$(\frac{\partial M_{P+j}}{\partial Q_i} + \lambda_i)(Q_i^{\min} - Q_i) = 0, \quad (\mu_i - \frac{\partial M_{P+j}}{\partial Q_i})(Q_i - Q_i^{\max}) = 0 \quad (E.1)$$

Taking derivative with respect to $\alpha_j$ we can obtain

$$-\frac{\partial Q_i}{\partial \alpha_j}(\frac{\partial M_{P+j}}{\partial Q_i} + \lambda_i) + \frac{\partial^2 M_{P+j}}{\partial Q_i\partial \alpha_j}(Q_i^{\min} - Q_i) = 0, \quad \frac{\partial Q_i}{\partial \alpha_j}(\mu_i - \frac{\partial M_{P+j}}{\partial Q_i}) - \frac{\partial^2 M_{P+j}}{\partial Q_i\partial \alpha_j}(Q_i - Q_i^{\max}) = 0 \quad (E.2)$$

Therefore, if $Q_i = Q_i^{\max}$, then $\frac{\partial M_{P+j}}{\partial Q_i} + \lambda_i = 0$ based on (E.1), and $\frac{\partial^2 M_{P+j}}{\partial Q_i\partial \alpha_j} = 0$ and $\frac{\partial Q_i}{\partial \alpha_j} = 0$ based on (E.2). If $Q_i = Q_i^{\min}$, then $\frac{\partial M_{P+j}}{\partial Q_i} - \mu_i = 0$ based on (E.1), and $\frac{\partial^2 M_{P+j}}{\partial Q_i\partial \alpha_j} = 0$ and $\frac{\partial Q_i}{\partial \alpha_j} = 0$ based on (E.2). If $Q_i^{\min} < Q_i < Q_i^{\max}$, then $\frac{\partial M_{P+j}}{\partial Q_i} = \lambda_i = \mu_i = 0$. Consequently,

$$\frac{\partial M_{P+j}}{\partial \alpha_j} = -W_1 + W_2 + \sum_{j \in P^+} \frac{\partial M_{P+j}}{\partial Q_i} \frac{\partial Q_i}{\partial \alpha_j} = -W_1 + W_2 \quad (E.3)$$

When $Q_j^{\min} = 0$, $M_{P+j}(Q^*) \leq M_P(Q^*)$ always holds based on Proposition 4.3, so the selection of supplier $j$ is not affected by its reliability level. For some $Q_j^{\min} > 0$, if supplier $j$ is in the optimal set when $\alpha_j = 1$, which implies $\frac{\partial M_{P+j}}{\partial \alpha_j}|_{\alpha_j=1} = -W_1 + W_2 \leq 0$, so cost increases as $\alpha_j$ decreases. When $\alpha_j$ decreases to a certain level, say $\alpha_j^d$, $M_{P+j}(Q^*) \geq M_P(Q^*)$ and supplier $j$ is dropped. As $\alpha_j$ decreases further, we have $\frac{\partial M_{P+j}}{\partial \alpha_j} = -W_1 + W_2 \geq 0$ until $\alpha_j$ decreases to 0. Obviously, $M_{P+j}(Q^*) = M_P(Q^*)$ when $\alpha_j = 0$. Overall, the optimal cost increases and then decreases to $M_P(Q^*)$ as supplier $j$ changes from perfectly reliable to perfectly unreliable. Supplier $j$ is in the optimal set when $\alpha_j^d \leq \alpha_j \leq 1$, and dropped otherwise.

As for the item cost $c_j$, we have $\frac{\partial M_{P+j}}{\partial c_j} = \alpha_j Q_j' + \sum_{j \in P^+} \frac{\partial M_{P+j}}{\partial Q_i} \frac{\partial Q_i}{\partial c_j} = \alpha_j Q_j'$. Therefore, as $c_j$ increases to a certain value, supplier $j$ is dropped. The effects of these parameters are summarized in table E.3.

For the full set of suppliers $U$, where the cardinality is $|U| = N$, we have $2^{N-1}$ subsets with supplier $j$, and a best subset without $j$ among all the $2^{N-1}$ subsets without $j$, whose cost is a constant
since it is not affected by the change of supplier $j$’s parameters. The costs of all the subsets with $j$ have the same changing trends as the parameters vary, so the effects of parameters on when supplier $j$ is included in the optimal sourcing set also follow the rules we summarize in table E.3. Therefore, when fixed costs are incurred and order size constraints are enforced, item cost is no longer the only factor in deciding whether a given supplier is selected or not. Fixed cost and minimum order size constraint also have impact. The effect of reliability level, however, depends on the value of the minimum order requirement. More than one perfectly reliable supplier could be selected due to limited supply capacities.

Table E.3: Effects of Supplier $j$’s Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Supplier $j$ is dropped if</th>
<th>Parameters</th>
<th>Supplier $j$ is dropped if</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_j$</td>
<td>increases to a certain level</td>
<td>$c_j$</td>
<td>increases to a certain level</td>
</tr>
<tr>
<td>$Q_{j}^{\text{max}}$</td>
<td>has no influence on the selection decision</td>
<td>$Q_{j}^{\text{min}}$</td>
<td>increases to a certain level</td>
</tr>
<tr>
<td>$\alpha_j$</td>
<td>has no influence for low $Q_{j}^{\text{min}}$, and decreases to a certain level for some high $Q_{j}^{\text{min}}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>