ABSTRACT

YU, FEI. Construction of C^2Q^5 and C^2Q^7 Finite Elements on 2D Rectangular Meshes. (Under the direction of Dr. Zhilin Li).

Finite element method is one of the most powerful and commonly used numerical methods in almost everywhere in mathematics. Finite elements with certain continuity are also becoming more and more popular, for example, splines for 1D interpolation. However, there are not many papers focusing on C^2 finite elements on 2D rectangles. In this thesis, we construct two finite elements, C^2Q^5 and C^2Q^7, on 2D rectangles. We plot the shape functions and do the error analysis. Some numerical results of using C^2Q^5 to interpolate functions on rectangle domains are given.

Construction of C^2Q^5 and C^2Q^7 Finite Elements on 2D Rectangular Meshes

by
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BIOGRAPHY

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1 Introduction

In the mathematical field of numerical analysis, finite element method always plays a very important role by providing a numerical technique for finding approximate solutions to partial differential equations (PDE) and their systems, as well as integral equations [1].

Finite element method is widely used and lots of researches have been done since its invention [2,3]. Then mathematicians started to look at special finite element spaces with certain continuity. For one-dimension, splines, especially cubic splines, are commonly applied to a variety of problems. For instance, In [4], the author described the use of cubic splines for interpolating monotonic data sets, In [5], a way to construct shape-preserving C^2 cubic polynomial splines interpolating convex and/or monotonic data was developed. Also, there was a C^2 interpolation using quartic splines in [6]. However, when it comes to two-dimension, finite elements that are continuously differentiable are relatively difficult to build [7]. Some researches have been done to contruct these C^1 elements on triangles by using polynomial P^k or Q^k, where P^k and Q^k represent polynomials of total degree and separate degree k respectively. For example, the Powell-Sabin P^2-triangle [8] and the Argyris P^5-triangle [9]. As to rectangles, the Boger-Fox-Schmit rectangle [10] is the most famous and commonly used one in C^1. Also, in [7], the author extended the Bogner-Fox-Schmit element to C^1Q^k in 2D and 3D, $k \geq 3$.

In fact, there are quite few papers focusing on C^2 finite elements on 2D rectangles. Nevertheless, some important applications of C^2 finite elements cannot be neglected. For example, these finite elements can be used to solve the fourth order (or even higher) PDEs, like biharmonic equation. Also, Peskin's Immersed Boundary (IB) method [11], which is one of the most popular numerical methods, is associated with discrete delta functions. In order to prove the convergence order of the IB method, we need to find an interpolate function to interpolate the discrete delta functions. Such an interpolation function can be obtained from the C^2 finite element spaces we introduce in this thesis.

We consider the following problem:

Given a quadrilateral mesh (squares), for instance, figure 1, can we find the minimum k such that the piecewise bipolynomial interpolation

$$Q^k(x,y) = \sum_{i=0,j=0}^{i\le k, j\le k} c_{ij} x^i y^j$$

has the following properties:

(a) $Q^k(x,y) \in C^2$;

(b) $D^\alpha(x_m, y_n), 0 \le \|\alpha\|_1 \le 2$(or 3) are given, that is, the function values, and up to all the second (or third) order partial derivatives are given at (x_m, y_n).

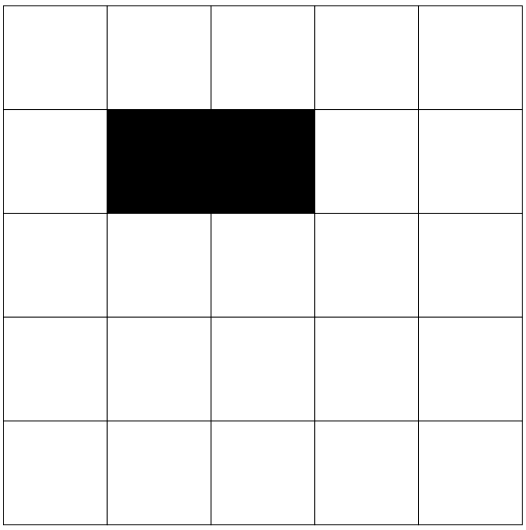


Figure 1: Quadrilateral mesh

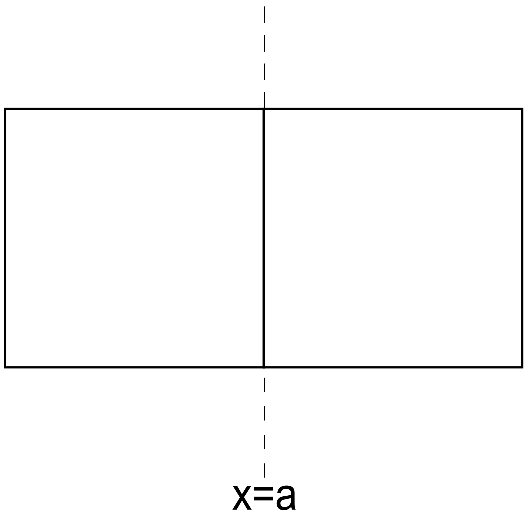


Figure 2: Two neighbor squares (part of figure 1)

Two neighbor squares of the whole mesh are shown in figure 2. In each square, since the interpolation function is a bipolynomial, it belongs to C^∞. However, the major difficulty is how to maintain the continuity along the boundary, for example, the intersection $x = a$ of the two neighbor squares, as shown in figure 2. In order to get the C^2 finite element spaces, we need the function values and all up to the second order derivatives, which are :

$Q^k, Q^k_x, Q^k_y, Q^k_{xx}, Q^k_{xy}, Q^k_{yy}$, where Q^k is the interpolation function and Q^k_x stands for $\frac{\partial Q^k}{\partial x}$.

to be continuous along the 4 sides of each square.

In this thesis, we develop two finite elements, C^2Q^5 and C^2Q^7, that belong to C^2 for 2D rectangles. In section 2, we consider that at each nodal point of the rectangle, up to second order partial derivatives are given and we construct bipolynomials Q^5 that belongs to C^2. In section 3, we consider that at each nodal point of the rectangle, up to third order partial derivatives are given and we construct bipolynomials Q^7 that belongs to C^2. In section 4, we run a numerical interpolation test on $u = \sin(x)\sin(y)$ using C^2Q^5 finite elements. Finally in section 5, we introduce some possible applications of these C^2 finite element spaces.

2 C^2Q^5 finite elements on 2D rectangles

2.1 Construction of C^2Q^5 finite elements

In this section, we discuss the construction of C^2Q^5 finite elements on 2D rectangles. We consider the problem discussed in section 1 when the function values and all up to the second order derivatives are given at each nodal point.

Let Ω be a rectangle domain in 2D. For simplicity, we may let Ω be the unit square with the uniform grid of size. Then after partition we can arbitrarily take one small square,$[a, b] \times [c, d]$(d-c=b-a), as an example as shown in figure 3.

For each point of the square, we already have 6 values, the function values and all up to the second order derivatives. However, in order to get the continuities along the boundaries, we need additional values. Here we use u_{xxy}, u_{xyy} and u_{xxyy} for each point. Then we have

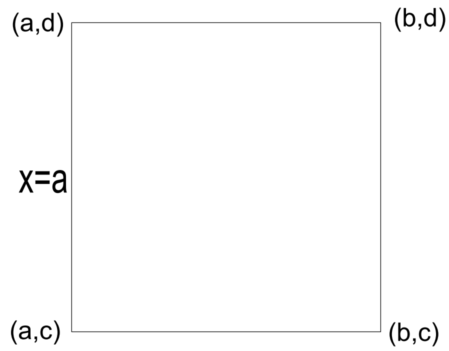


Figure 3: A small square after partition

9 values for the 4 points of a square. The total constrains are $4 \times 9 = 36$, which is exactly the same as the degree of freedom (DOF) of the bipolynomial Q^5. So let k be 5 and now we prove that the interpolation function Q^5 we get indeed belongs to C^2.

First, we introduce a theorem and prove it.

Theorem 2.1. *Let Q^k be a bipolynomial interpolation function with separate degree k on a quadrilateral mesh (in figure 1). For two neighbor squares (in figure 2) Q^k has continuous second order derivatives along the boundary $x = a$ (or $y = b$) if $Q^k|_{x=a}$, $Q^k_x|_{x=a}$ and $Q^k_{xx}|_{x=a}$ (or $Q^k|_{y=b}$, $Q^k_y|_{y=b}$ and $Q^k_{yy}|_{y=b}$) are all continuous, where a, b are constant.*

Proof: We only need to prove the case when the boundary is $x = a$ as shown in figure 2, since the proof is similar for $y = b$ case.

Since $Q^k|_{x=a}$ is continuous, the function value is continuous along $x = a$. Meanwhile, all the tangential (or Y) direction derivatives $\frac{\partial^n Q^k}{\partial y^n}$, $n \geq 1$ are also continuous along this boundary, this is because we can conclude the continuity of $\frac{\partial^n Q^k|_{x=a}}{\partial y^n}$ from the continuity of $Q^k|_{x=a}$ and

$$\frac{\partial^n Q^k|_{x=a}}{\partial y^n} = \frac{\partial^n Q^k}{\partial y^n}|_{x=a}$$

Here, we only use the continuities of up to second order derivatives, which are $Q^k_y|_{x=a}$ and $Q^k_{yy}|_{x=a}$

Similarly, since $Q^k_x|_{x=a}$ is continuous, Q^k_x and Q^k_{xy} are continuous along the boundary. At last, we also have the continuity of Q^k_{xx} along $x = a$.

Thus, the function value and up to second order derivatives of Q^k are all continuous along the boundary $x = a$. This completes the proof of Theorem 2.1.

Remark 2.2. *If for any two neighbor squares in the whole mesh, Q^k always has continuous second order derivatives along the boundary, then $Q^k \in C^2$.*

Then we use the theroem to prove that the interpolation function Q^5 we get indeed belongs to C^2.

Along one side, for instance $x = a$, $Q^5|_{x=a}$ is a fifth polynomial of y, which has the form:

$$Q^5|_{x=a} = \beta_5 y^5 + \beta_4 y^4 + \beta_3 y^3 + \beta_2 y^2 + \beta_1 y + \beta_0$$

Clearly, the DOF of $Q^5|_{x=a}$ is 6. We have the values of u, u_y, u_{yy} at both nodal points, (a, c) and (a, d), which form 6 equations to uniquely determine $Q^5|_{x=a}$.

Then, we take a look at $Q^5_x|_{x=a}$, it is also a fifth polynomial of y, which has the similar form as $Q^5|_{x=a}$ does. It is uniquely determined by the values of u_x, u_{xy}, u_{xyy} at both points.

At last, as to $Q^5_{xx}|_{x=a}$, which is still a fifth polynomial of y, we have the values of u_{xx} u_{xxy} and u_{xxyy} at two points, which are used to uniquely determine $Q^5_{xx}|_{x=a}$.

So far, $Q^5|_{x=a}$, $Q^5_x|_{x=a}$ and $Q^5_{xx}|_{x=a}$ are all continuous. According to Theorem 2.1, we have obtained up to the second order continuity along $x = a$. In fact, the proof is similar for the other side, $x = b$. Meanwhile, the other two sides, $y = c$ and $y = d$, are also taken care of. For example, along $y = c$:

- The values of u, u_x and u_{xx} at both points, (a, c) and (b, c), uniquely determine the fifth polynomial $Q^5|_{y=c}$.

- The values of u_y, u_{xy} and u_{xxy} at both points uniquely determine the fifth polynomial

$Q_y^5|_{y=c}$.

• The values of u_{yy}, u_{xyy} and u_{xxyy} at both points uniquely determine the fifth polynomial $Q_{yy}^5|_{y=c}$.

Hence, along each of the 4 sides, the interpolation function Q^5 always has continuous second order derivatives along the boundary. Since we choose the square arbitrarily, according to Remark 2.2, the interpolation function Q^5 indeed belongs to C^2. We have a system of equations with 36 unknowns and 36 equations. The rank of the coefficient matrix that matlab returns is 36, which indicates that the matrix is non-singular and the system of equations has a unique solution.

2.2 A numerical experiment on C^2Q^5 finite elements

Now, we consider 2D square, $[-1,1] \times [-1,1]$, as an example. We partition it into 4 squares, $[-1,0] \times [0,1]$, $[0,1] \times [0,1]$, $[-1,0] \times [-1,0]$ and $[0,1] \times [-1,0]$ as shown in figure 4.

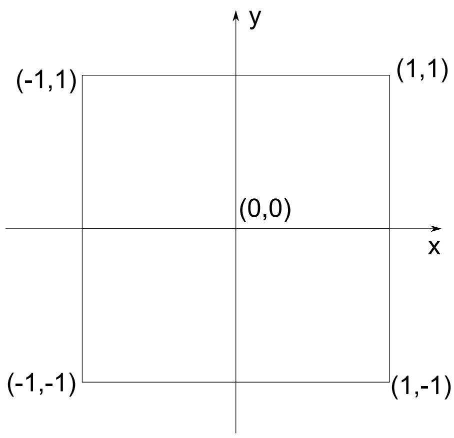


Figure 4: Mesh for shape function

Then we use the C^2Q^5 interpolation on each square to get the interpolation function Q^5. In figure 5-10, The plots show Q^5, Q_x^5, Q_y^5, Q_{xx}^5, Q_{xy}^5 and Q_{yy}^5 of the interpolation function Q^5, which are obtained by imposing function value u of the center point,$(0,0)$, to be 1 and all the rest to be 0. Also, in table 1-2, we calculate the errors in the infinity norm for up to all the second order derivatives along the boundaries $x=0$ and $y=0$ respectively by taking the absolute values of the differences between the values of two sides. Also, each row of the tables indicates that one of the 9 values (all up to the second order derivatives and $u_{xxy},\ u_{xyy},\ u_{xxyy}$) of the center point is imposed to be 1 and the rest are all 0.

According to the figures of plots and the tables of errors, we can also conclude that the interpolation function Q^5 does belong to C^2.

Table 1: Errors($\times 10^{-10}$) of C^2Q^5 in infinity norm along $x=0$

Error($\times 10^{-10}$)	Q^5	Q^5_x	Q^5_y	Q^5_{xx}	Q^5_{xy}	Q^5_{yy}
$u=1$	0.0051	0.0738	0.0146	0.0053	0.1376	0.0470
$u_x=1$	0.0008	0.0403	0.0035	0.0024	0.0737	0.0161
$u_y=1$	0.0031	0.0399	0.0083	0.0027	0.0743	0.0248
$u_{xx}=1$	0.0004	0.0079	0.0010	0.0005	0.0150	0.0037
$u_{xy}=1$	0.0005	0.0217	0.0018	0.0012	0.0395	0.0077
$u_{yy}=1$	0.0005	0.0080	0.0017	0.0006	0.0150	0.0050
$u_{xxy}=1$	0.0002	0.0042	0.0006	0.0003	0.0080	0.0020
$u_{xyy}=1$	0.0001	0.0044	0.0005	0.0002	0.0080	0.0020
$u_{xxyy}=1$	0.0001	0.0008	0.0001	0.0001	0.0016	0.0003

Table 2: Errors($\times 10^{-10}$) of C^2Q^5 in infinity norm along $y=0$

Error($\times 10^{-10}$)	Q^5	Q^5_x	Q^5_y	Q^5_{xx}	Q^5_{xy}	Q^5_{yy}
$u=1$	0.0159	0.0835	0.0021	0.3029	0.0052	0.0050
$u_x=1$	0.0094	0.0476	0.0012	0.1744	0.0034	0.0032
$u_y=1$	0.0098	0.0474	0.0049	0.1919	0.0075	0.0020
$u_{xx}=1$	0.0020	0.0097	0.0005	0.0365	0.0011	0.0005
$u_{xy}=1$	0.0056	0.0268	0.0026	0.1092	0.0037	0.0011
$u_{yy}=1$	0.0017	0.0095	0.0003	0.0343	0.0005	0.0005
$u_{xxy}=1$	0.0011	0.0055	0.0004	0.0220	0.0006	0.0002
$u_{xyy}=1$	0.0010	0.0054	0.0001	0.0194	0.0003	0.0003
$u_{xxyy}=1$	0.0002	0.0010	0.0001	0.0039	0.0001	0.0001

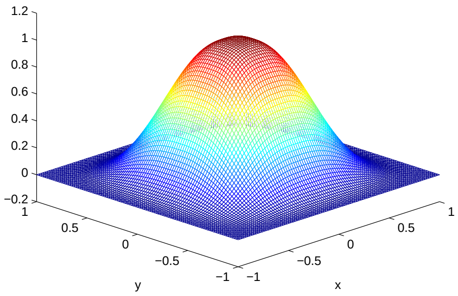


Figure 5: Plot of Q^5 for C^2Q^5 with only $u = 1$ at center point

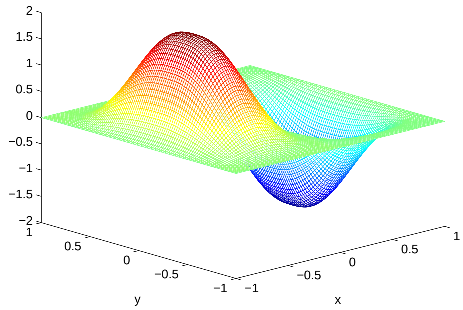


Figure 6: Plot of Q^5_x for C^2Q^5 with only $u = 1$ at center point

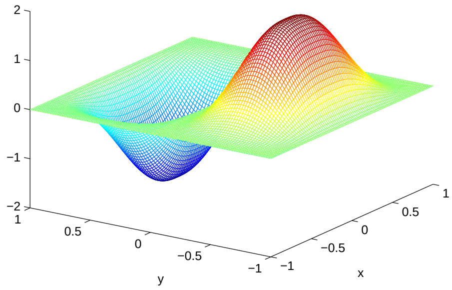


Figure 7: Plot of Q^5_y for C^2Q^5 with only $u = 1$ at center point

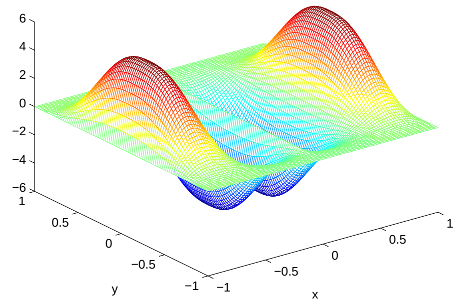


Figure 8: Plot of Q^5_{xx} for C^2Q^5 with only $u = 1$ at center point

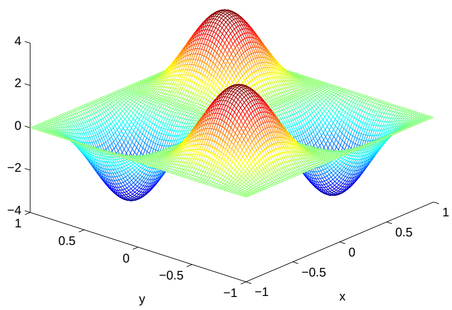


Figure 9: Plot of Q^5_{xy} for C^2Q^5 with only $u = 1$ at center point

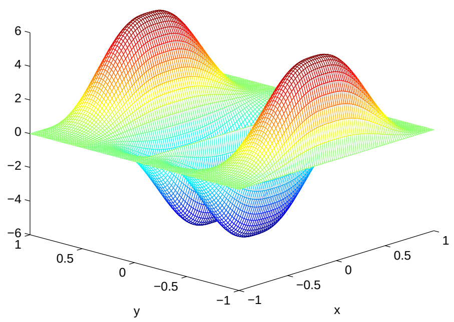


Figure 10: Plot of Q^5_{yy} for C^2Q^5 with only $u = 1$ at center point

3 C^2Q^7 finite elements on 2D rectangles

3.1 Construction of C^2Q^7 finite elements

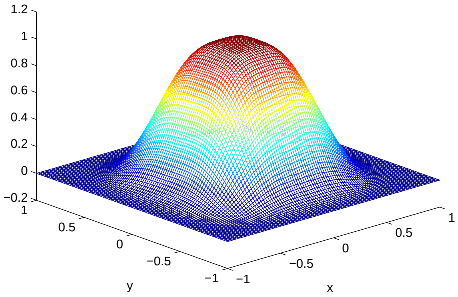
In this section, we construct the C^2Q^7 finite elements on 2D rectangles. The major difference from the problem in section 2 is that this time for each point all up to the third order derivatives instead of second order derivatives are given. However, in order to make the functions specified in Theorem 2.1 continuous along the boundary, we need additional constraints. Now we use the values of u_{xxyy} at each points, also we impose the coefficients of the two higher terms, which are y^7 and y^6, of functions $Q^7_x|_{x=a}$ and $Q^7_{xx}|_{x=a}$ to be zero. There are other ways to impose the additional conditions, too. For instance, the Boger-Fox-Schmit rectangle [10], it has additional points along the boundaries, sometimes also inside the rectangles. In this way, we impose 4 constraints for each side, which make the total additional conditions to be $4 \times 4 = 16$. Also, all up to the third order derivatives (10 values) are given and we impose the values of u_{xxyy} at 4 nodal points. Hence, the total constraints are $16 + 4 \times 10 + 4 = 60$. The total DOF of a Q^7 bi-polynomial is 64. Thus, we have a system of equations with 64 unknowns and 60 equations. The rank of the coefficient matrix that matlab returns is 60, which indicates the matrix has full row rank and the system equations has infinite number of solutions. We suggest to choose the SVD solution as the interpolation function.

Now, we prove the interpolation function Q^7 we get indeed belongs to C^2. We consider the same square, $[a, b] \times [c, d]$, as the one in figure 3. However, along one side, for example $x = a$, instead of 6 we have 8 values, u, u_y, u_{yy}, u_{yyy} at both points. So it at least has to be a seventh order polynomial. Let k be 7, then $Q^7|_{x=a}$ whose DOF is 8 is continuous. Along $x = a$, u_x(or u_{xx}) is still a seventh polynomial. We also have 8 constraints, which are the values of u_x, u_{xy}, u_{xyy} (or $u_{xx}, u_{xxy}, u_{xxyy}$) at both points and the coefficients of two higher terms. Then $Q^7_x|_{x=a}$ (or $Q^7_{xx}|_{x=a}$) is continuous. As to the other three sides, the method works similarly. Thus, by Theorem 2.1, Q^7 has up to the second order continuity along $x = a$. Since we choose the square arbitrarily, according to Remark 2.2, the interpolation function Q^7 indeed belongs to C^2.

3.2 A numerical experiment on C^2Q^7 finite elements

Again, we consider 2D square, $[-1, 1] \times [-1, 1]$, and partition it into the 4 small squares, $[-1, 0] \times [0, 1]$, $[0, 1] \times [0, 1]$, $[-1, 0] \times [-1, 0]$ and $[0, 1] \times [-1, 0]$ as shown in figure 4. Then we use the C^2Q^7 interpolation on each one. In figure 11-16, The plots show $Q^7, Q^7_x, Q^7_y, Q^7_{xx}, Q^7_{xy}$ and Q^7_{yy} of the interpolation function Q^7, which are obtained by imposing function value u of the center point,$(0, 0)$, to be 1 and all the rest to be 0. Also, in table 3-4, we calculate the errors in the infinity norm for up to all the second order derivatives along the boundaries $x = 0$ and $y = 0$ respectively by taking the absolute values of the differences between the values of two sides. Also, each row of the tables indicates that one of the 11 values (all up to the third order derivatives and u_{xxyy}) of the center point is imposed to be 1 and the rest are all 0.

According to the figures of plots and the tables of errors, we can conclude that the interpolation function Q^7 does belong to C^2.



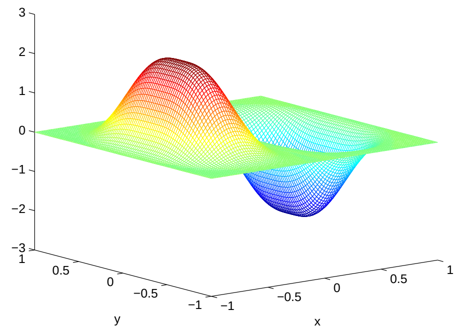
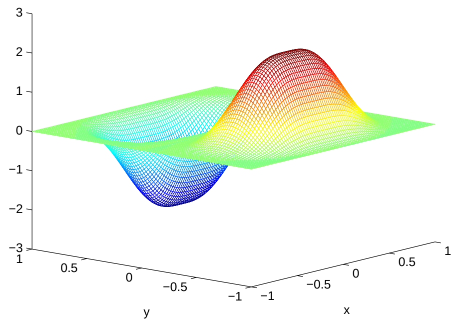


Figure 11: Plot of Q^7 of C^2Q^7 with only $u = 1$ at center point

Figure 12: Plot of Q^7_x of C^2Q^7 with only $u = 1$ at center point



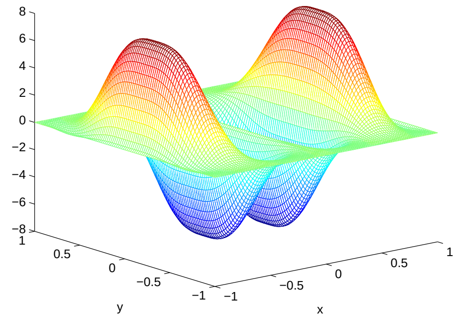
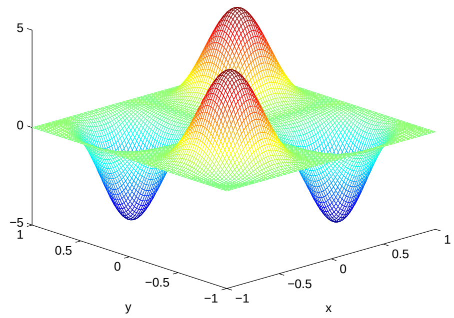


Figure 13: Plot of Q^7_y of C^2Q^7 with only $u = 1$ at center point

Figure 14: Plot of Q^7_{xx} of C^2Q^7 with only $u = 1$ at center point



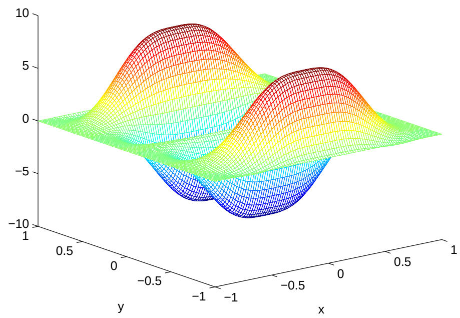


Figure 15: Plot of Q^7_{xy} of C^2Q^7 with only $u = 1$ at center point

Figure 16: Plot of Q^7_{yy} of C^2Q^7 with only $u = 1$ at center point

Table 3: Errors($\times 10^{-10}$) of C^2Q^7 in infinity norm along $x = 0$

Error($\times 10^{-10}$)	Q^7	Q^7_x	Q^7_y	Q^7_{xx}	Q^7_{xy}	Q^7_{yy}
$u = 1$	0.6861	0.0412	3.0140	0.1913	0.1443	15.8418
$u_x = 1$	0.2022	0.0310	0.9185	0.0270	0.0605	4.8801
$u_y = 1$	0.1401	0.0141	0.6033	0.0929	0.0394	2.9158
$u_{xx} = 1$	0.0438	0.0060	0.2001	0.0062	0.0116	1.0607
$u_{xy} = 1$	0.0110	0.0050	0.0525	0.0063	0.0084	0.3015
$u_{yy} = 1$	0.0301	0.0022	0.1302	0.0196	0.0071	0.6371
$u_{xxx} = 1$	0.0045	0.0008	0.0204	0.0008	0.0015	0.1092
$u_{xxy} = 1$	0.0021	0.0010	0.0099	0.0012	0.0020	0.0581
$u_{xyy} = 1$	0.0023	0.0007	0.0109	0.0017	0.0010	0.0602
$u_{yyy} = 1$	0.0032	0.0002	0.0142	0.0020	0.0006	0.0704
$u_{xxyy} = 1$	0.0004	0.0001	0.0020	0.0003	0.0002	0.0111

Table 4: Errors($\times 10^{-10}$) of C^2Q^7 in infinity norm along $y = 0$

Error($\times 10^{-10}$)	Q^7	Q^7_x	Q^7_y	Q^7_{xx}	Q^7_{xy}	Q^7_{yy}
$u = 1$	0.0435	0.0556	3.0623	0.2955	5.7079	0.0774
$u_x = 1$	0.0195	0.0349	0.9168	0.1202	1.7003	0.0209
$u_y = 1$	0.0206	0.0264	0.6079	0.1284	1.1501	0.0369
$u_{xx} = 1$	0.0039	0.0072	0.1997	0.0218	0.3702	0.0047
$u_{xy} = 1$	0.0024	0.0052	0.0522	0.0135	0.0970	0.0018
$u_{yy} = 1$	0.0043	0.0032	0.1290	0.0306	0.2430	0.0037
$u_{xxx} = 1$	0.0005	0.0009	0.0204	0.0019	0.0379	0.0006
$u_{xxy} = 1$	0.0006	0.0013	0.0098	0.0043	0.0182	0.0005
$u_{xyy} = 1$	0.0003	0.0009	0.0111	0.0035	0.0203	0.0006
$u_{yyy} = 1$	0.0004	0.0005	0.0142	0.0030	0.0269	0.0008
$u_{xxyy} = 1$	0.0001	0.0002	0.0021	0.0006	0.0038	0.0001

4 A numerical test on C^2Q^5 interpolation

In this section,we use a numerical test to see the accuracy of the C^2Q^5 interpolation.

First, we introduce an error estimate by theoretical derivation.

Theorem 4.1. *Let Q^k be a piecewise bipolynomial interpolation function on a quadrilateral mesh. If at each nodal point, the function values and up to second order derivatives* *of the true function u are given, then the error in the infinity norm of using C^2Q^k finite elements discussed in section 2 and 3 to interpolate the true function is at least third order convergent.*

Proof: Consider the square shown in figure 4. We use Taylor expansion for both the interpolation function Q^k and the true function u at point (a, c):

$$Q^k(x,y) = Q^k(a,c) + \Delta x Q^k_x(a,c) + \Delta y Q^k_y(a,c)$$

$$+\tfrac{1}{2!}[\Delta^2 x Q^k_{xx}(a,c) + 2\Delta x\Delta y Q^k_{xy}(a,c) + \Delta^2 y Q^k_{yy}(a,c)] + O(\Delta^3 x) + O(\Delta^3 y)$$

$$u(x,y) = u(a,c) + \Delta x u_x(a,c) + \Delta y u_y(a,c)$$

$$+\tfrac{1}{2!}[\Delta^2 x u_{xx}(a,c) + 2\Delta x\Delta y u_{xy}(a,c) + \Delta^2 y u_{yy}(a,c)] + O'(\Delta^3 x) + O'(\Delta^3 y)$$

Let h be the side length of the square. The function values and all up to second order derivatives of the true function at each nodal points are given, which means that:

$$Q^k(a,c) = u(a,c) \quad Q^k_x(a,c) = u_x(a,c) \quad Q^k_y(a,c) = u_y(a,c)$$

$$Q^k_{xx}(a,c) = u_{xx}(a,c) \quad Q^k_{xy}(a,c) = u_{xy}(a,c) \quad Q^k_{yy}(a,c) = u_{yy}(a,c)$$

After substraction we can get the error estimate:

$$\|E_h\|_\infty = \|Q^k(x,y) - u(x,y)\|_\infty = \|O(\Delta^3 x) + O(\Delta^3 y) - O'(\Delta^3 x) - O'(\Delta^3 y)\|_\infty \le O(h^3)$$

This proves that the error is at least third order convergent.

Now, we run a numerical test on C^2Q^5 interpolation.

We consider function: $u = \sin(\pi x)\sin(\pi y)$ on $[0,1]\times[0,1]$. Then by using the C^2Q^5 interpolation introduced in section 2, we can obtain the interpolation function Q^5. Figure 17-22 give the plots of the errors on the whole mesh for different h. Table 5 shows grid refinement analysis and the condition numbers of the coefficient matrices, which are obtained while using the C^2Q^5 interpolation on $[0,h]\times[0,h]$.

From the column $\|E_h\|_\infty/\|E_{h/2}\|_\infty$ in table 5, we can see that the error is about sixth order convergent, which satisfies Theorem 4.1. When h is smaller than 1/64, round-off error dominates.

Table 5: Grid refinement analysis of C^2Q^5 interpolation for $u = \sin(\pi x)\sin(\pi y)$

h	$\|E_h\|_\infty$	$\|E_h\|_\infty / \|E_{h/2}\|_\infty$
1	0.0357	107.5398
1/2	3.3197×10^{-4}	38.4483
1/4	8.6342×10^{-6}	56.9081
1/8	1.5188×10^{-7}	62.1720
1/16	2.4429×10^{-9}	63.5510
1/32	3.8440×10^{-11}	63.5183
1/64	6.0518×10^{-13}	22.1588

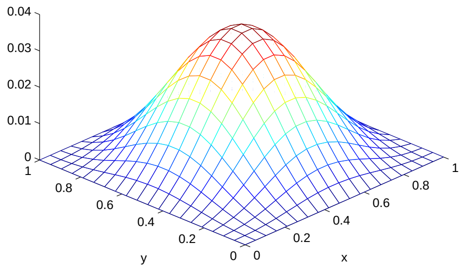


Figure 17: Error of C^2Q^5 interpolation for $u = \sin(\pi x)\sin(\pi y)$ with $h = 1$

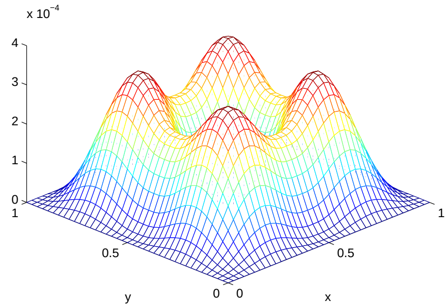


Figure 18: Error of C^2Q^5 interpolation for $u = \sin(\pi x)\sin(\pi y)$ with $h = 1/2$

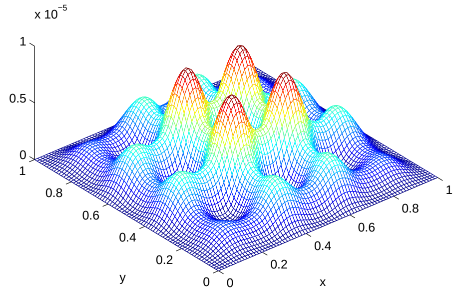


Figure 19: Error of C^2Q^5 interpolation for $u = \sin(\pi x)\sin(\pi y)$ with $h = 1/4$

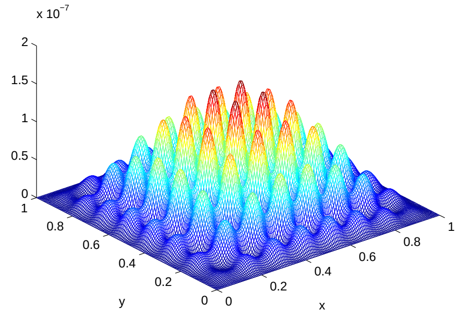


Figure 20: Error of C^2Q^5 interpolation for $u = \sin(\pi x)\sin(\pi y)$ with $h = 1/8$

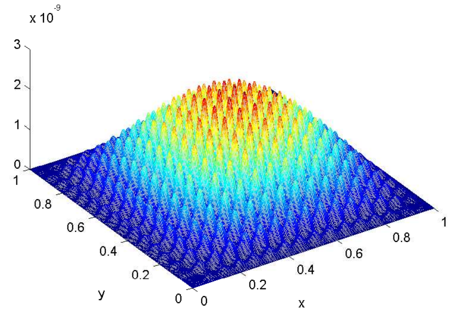


Figure 21: Error of C^2Q^5 interpolation for $u = \sin(\pi x)\sin(\pi y)$ with $h = 1/16$

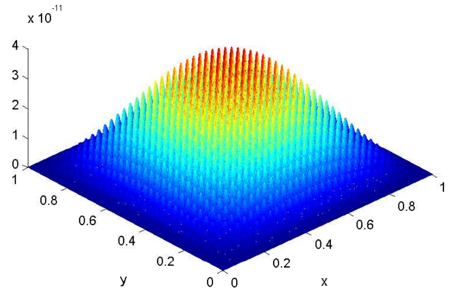


Figure 22: Error of C^2Q^5 interpolation for $u = \sin(\pi x)\sin(\pi y)$ with $h = 1/32$

5 Conclusions and future work

Finite element spaces with certain continuity are more and more commonly used in mathematics. The two C^2 finite elements constructed in section 2 and 3 also can be applied to some problems. They can be used to solve 2D fourth order partial differential equations, for example biharmonic equation.

Also, as mentioned in section 1, in [12], the author gives a proof of the convergence order of the IB method for elliptic interface problems. In the proof, an interpolation function which is used to interpolate the discrete delta function needs to be constructed. The interpolation function is required to be in $C^1 \cap H^2$ and the interpolation needs to be second or higher order accurate. The two finite elements, C^2Q^5 and C^2Q^7 constructed in this thesis, have such properties.

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