

ABSTRACT

BUELL, CATHERINE ANDREA. On Maximal \mathbb{R} -split Tori Invariant under an Involution.
(Under the direction of Aloysius Helminck.)

Symmetric varieties occur in many areas of mathematics. They are defined as the homogeneous spaces G/H with G a reductive algebraic group and H the fixed point group of an involution σ . Similarly, symmetric k -varieties are the homogeneous spaces G_k/H_k where G_k and H_k are the k -points of G and H and k is not necessarily algebraically closed. They occur in many problems in representation theory, geometry, singularity theory, and number theory. Perhaps the best known application is in the representation theory of Lie groups.

To study the representation theory of the symmetric k -varieties over real and local fields much structure of these symmetric k -varieties is needed. For example the orbits of parabolic k -subgroups acting on a symmetric k -variety are of fundamental importance in the study of induced representations. The characterization of these orbits involves conjugacy classes of σ -stable maximal k -split tori and for each of these σ -stable maximal k -split tori a quotient of Weyl groups. This thesis focuses on refining the characterization found in Helminck and Wang [18] and use this to classify the conjugacy classes of σ -stable maximal k -split tori over the real numbers, building on partial results obtained in [9] and [11]. This problem is not only of importance for the representation theory of symmetric spaces but also for several other fields of mathematics and physics.

On Maximal \mathbb{R} -split Tori Invariant under an Involution

by
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A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh, North Carolina

2011

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ACKNOWLEDGEMENTS

Above all others, this work could not have been done without my advisor Dr. Loek Helminck. His knowledge, humor, and personality are perfectly blended to help his students succeed and learn. He has taught me so much and his support in all things is unforgettable. Similarly, I must thank Dr. Andrew Perry for knowing what I could accomplish long before I knew.

Thank you to my family. Mom, Dad, Rebecca, and my beautiful nephew Gustavo, you all inspired me to work hard and helped me to remember that the time spent away from you would be worth it in the end. You were right. Thank you for your love and support.

My mentors, friends, and professors from both North Carolina State University and Springfield College helped me to realize what I could achieve and what I still have left to do. Thank you Dr. Helminck, Dr. Misra, Dr. Lada, Dr. Stitzinger, Dr. Jing, Dr. Baklov, Dr. Martin, Dr. DeJoy, Dr. Perry, Dr. Polito, and Dr. Hu, but most importantly Charlene Wallace and Donna Wisnioski.

Last, there have been wonderful people who have inspired me in every aspect of my life: math, dance, and otherwise. Thank you Luann Pagella and Karen Krolak for inspiring me to be a dancer-mathematician and thank you Herb and Louise Gross for taking me into your family and your mathematics family.

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Chapter 1

Introduction

1.1 Introduction

Symmetric k -varieties have been a topic of interest since the late 1980's. For any field k , reductive group G defined over k and any k -involution of G , we can define a symmetric k -variety. It is the homogeneous space G_k/H_k where H is the fixed point group of the k -involution and G_k and H_k are the k -rational points of G and H , respectively.

The focus of this thesis is for $k = \mathbb{R}$ and symmetric \mathbb{R} -varieties are commonly called real symmetric spaces. The representations of real symmetric spaces have been studied by many mathematicians starting with a study of compact groups and their representations by Cartan, followed by a study of Riemannian symmetric spaces and real Lie groups by Harish Chandra. Mathematicians have begun to generalize these real reductive symmetric spaces to similar spaces over the p -adic numbers and over other base fields. These generalizations play a role in the study of arithmetic subgroups, geometry, singularity theory, the study of Harish Chandra modules and most importantly representation theory.

The primary study of the rationality properties of these spaces over other base fields was published by Helminck and Wang [18]. In the paper they define the symmetric k -variety for any field k with characteristic not equal to two. Similar to the real case, the p -adic symmetric k -varieties are also called p -adic symmetric spaces. In order to study the representations associated with these symmetric k -varieties one needs more information about the decomposition and orbits of the symmetric k -varieties. A thorough understanding of the orbits of parabolic k -subgroups, symmetric subgroups, maximal k -anisotropic (compact) subgroups acting on the symmetric k -varieties are important in the study of these spaces and their representations.

There are descriptions of some of these orbit decompositions in [18], the focus is on the orbits of parabolic k -subgroups, P_k , acting on G_k/H_k or the double coset, $P_k \backslash G_k / H_k$. For a reductive group G and k -involution σ , they had the following theorem.

Theorem 1.1.1 ([18, Proposition 6.10]) *Let $\{A_i \mid i \in I\}$ be the representatives of the H_k -conjugacy classes of σ -stable maximal k -split tori of G .*

$$P_k \backslash G_k / H_k \cong \bigcup_{i \in I} W_{G_k}(A_i) \backslash W_{H_k}(A_i)$$

The goal is to determine the conjugacy classes of σ -stable maximal k -split tori. In the case when $k = \mathbb{R}$, we will simplify the characterization using results from [18] and [10]. First, we will describe each \mathbb{R} -involution, σ , as a pair of commuting involutions defined over \mathbb{C} . The pair (θ, σ) , where θ is a Cartan involution commuting with σ will determine the symmetric pair (G, H) where G is the reductive group and H the fixed point group of σ .

In order to discuss the classification of these tori further, we need to introduce some notation.

1.2 Notation

Definition 1.2.1 A torus, T , is called σ -stable if $\sigma(T) = T$. Then $T = T_\sigma^+ T_\sigma^-$, where

$$T_\sigma^+ = (T \cap H)^0 \text{ and } T_\sigma^- = \{x \in T \mid \sigma(x) = x^{-1}\}$$

Definition 1.2.2 A torus, A , is called σ -split if $\sigma(a) = a^{-1}$ for all $a \in A$.

Note: A (σ, k) -split torus is both σ -split and k -split.

A quasi k -split torus is a torus that is G -conjugate to a k -split torus.

A torus, S , is called σ -fixed if $\sigma(s) = s$ for all $s \in S$.

We will denote all the σ -stable maximal k -split tori of G by \mathcal{A}_k^σ . The set of σ -stable maximal quasi k -split tori will be denoted by \mathcal{A}^σ .

Finally, we say for two tori $A_1, A_2 \in \mathcal{A}_k^\sigma$ (or \mathcal{A}^σ) the pair (A_1, A_2) is standard if $A_1^- \subset A_2^-$ and $A_1^+ \supset A_2^+$. Here, we say that A_1 is standard with respect to A_2 .

On route to the classification, we will have to determine the H -conjugacy classes of σ -stable quasi k -split tori. In order to classify the σ -stable maximal k -split (quasi k -split) tori, we will concentrate on the dimension of the σ -split portion of the torus. We can consider the dimension of the σ -split portion starting at a torus with a maximal σ -split piece down to the maximal torus with the smallest σ -split portion (maximizing the σ -fixed portion of the torus). So for any $A_i \in \mathcal{A}_k^\sigma$ we can organize them by the dimension of the σ -split portion of the torus as seen in Figure 1.1.

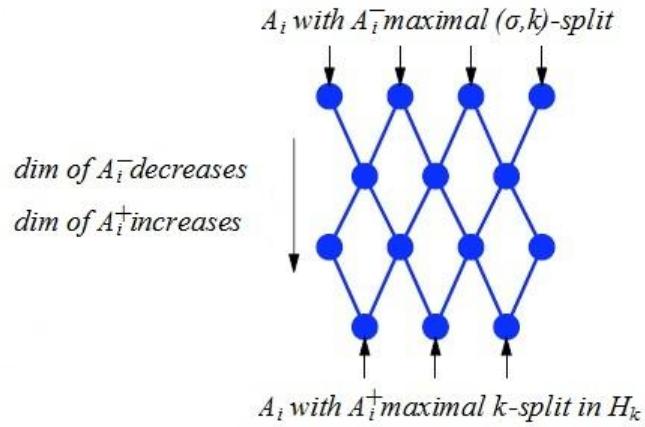


Figure 1.1: Dimensions of σ -stable maximal k -split tori

Once we determine the levels and dimension, we need to determine how many H_k -conjugacy classes are at each level. We will first consider the H -conjugacy classes. However, we may have only one conjugacy class in the top and the bottom and any node in between can split into more cases. It is possible for two tori A_1 and A_2 to be H -conjugate but not H_k -conjugate even though the $\dim((A_1)_\sigma^-) = \dim((A_2)_\sigma^-)$. For example, for $\mathrm{SL}(2, \mathbb{C})$ and the involution $\sigma(A) = (A^T)^{-1}$, we have the follow lattice:

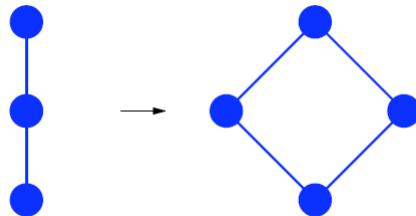


Figure 1.2: Example of H_k -conjugacy class splitting

For $k = \mathbb{R}$, we have simplified the issue of \mathbb{R} -split tori by introducing the Cartan involution, θ , which both commutes with σ and has a maximal \mathbb{R} -anisotropic fixed point group.

It is important to note that when we put tori into standard position there is a Weyl group elements associated with each torus. Then two σ -stable maximal k -split tori in standard position are conjugate when the related element in the Weyl group of the maximal (σ, k) -split torus are conjugate. The last key of the puzzle is to determine which elements of the Weyl group are related to a standard tori and the H -conjugacy classes, as well as, the H_k -conjugacy classes for the k -split and quasi k -split tori.

We call these related Weyl group elements (θ, σ) -singular involutions (k -split) and σ -singular involutions (quasi k -split). These singular involutions are elements of the Weyl group of a maximal (σ, k) -split torus, call it A . First we will consider the roots of the torus, $\Phi(A)$, and the Weyl group of $\Phi(A)$, $W(A)$. Then we will specifically look at $\Phi(A, A_\sigma^-)$ where

$$\Phi(A, A_\sigma^-) = \Phi(A) \cap \Phi(A_\sigma^-) = \{a \in A \mid \sigma(a) = a^{-1}\}$$

It is important to note that we will also discuss the maximal σ -fixed, k -split torus S and $\Phi(S, S_\sigma^+)$. Utilizing both $\Phi(A, A_\sigma^-)$ and $\Phi(S, S_\sigma^+)$ we will learn about the dimensions of the tori inside the maximal k -split torus and dual approaches to finding the singular involutions.

1.3 Summary of Results

Overall, there are 171 cases to consider coming from the pairs of commuting involutions (θ, σ) mentioned earlier and found in [7]. Also, since we are using the Cartan involution, we will view σ as an involution over the algebraic closure and look at classification of

commuting pairs of involutions of G and not \mathbb{R} -involutions of G . To simplify our task, we will use both the pair (θ, σ) and the associated pair $(\theta, \theta\sigma)$. Each pair offers a different, but equivalent approach to the problem. The pairs also give the singular rank, which is the difference between the maximal (σ, \mathbb{R}) -split torus and the maximal \mathbb{R} -split σ -fixed torus. With a few more tools, it is then possible to find the H -conjugacy classes and $H_{\mathbb{R}}$ -conjugacy classes.

Let H be as before and K be the fixed point group of θ , then $H^+ = H \cap K$ and H_k^+ are the k -rational points.

Theorem 1.3.1 ([18, Corollary 12.11]) *Let $\{A_i \mid i \in I\}$ be representatives of the H_k^+ -conjugacy classes of (θ, σ) -stable maximal k -split tori of G . Then*

$$P_k \setminus G_k / H_k^+ \cong \bigcup_{i \in I} W_{G_k}(A_i) \setminus W_{H_k^+}(A_i).$$

All of the 171 cases fall into one of the following categories. We will start by finding the σ -stable maximal quasi \mathbb{R} -split tori which is done in Table 4.2. We will show that finding the singular rank is the next step. Finally, we need to find the σ -singular and (θ, σ) -singular involutions in the proper Weyl groups. A combinatorial approach to these elements in the Weyl group will classify the involutions and we can find the H - and then $H_{\mathbb{R}}$ -conjugacy classes of the tori.

Theorem 1.3.2 (modified from [10, Proposition 8.11]) *If the singular rank is r and the maximal (θ, σ) -singular involution is of type $r \cdot A_1 = A_1 + A_1 + \cdots + A_1$ r times then each involution of lesser dimension is also (θ, σ) -singular.*

Theorem 1.3.3 *If the singular rank is 0 or there are no potential (θ, σ) -singular roots, then there is only one $H_{\mathbb{R}}$ -conjugacy class of σ -stable maximal \mathbb{R} -split tori.*

In the last case, if the singular rank is r and the maximal (θ, σ) -singular involution is of type B_r, C_r, D_r, E_r, F_r , or G_r then we must determine what type of involutions of lesser dimension are (θ, σ) -singular.

Remark We have conjectured that the maximal (θ, σ) -singular involution will never be of type $r \cdot A_1 + X_\ell$. We will show in a later paper that in the 171 cases, the three theorems above will satisfy the results.

Of those four cases above, we will add details for three. Here we summarize them into three parts with extra information derived from observations so far on (θ, σ) -singular involutions. In particular, we learn that the (θ, σ) -singular involutions must be in $W(T)$ where T is a maximal torus of the fixed point group of $\theta\sigma$. We let w_m represent the type of the maximal (θ, σ) -singular involution.

- Case 1: If singular rank is r and $\Phi(A, A_\sigma^-)$ or $\Phi(S, S_\sigma^+)$ is of type $r \cdot A_1$, then all (θ, σ) -stable involutions w with $A_w^- \subset A_{w_m}^-$ are (θ, σ) -singular. We still must verify that w_m of type $r \cdot A_1$ is (θ, σ) -singular. In most cases, $\Phi(T)$ will be type X_ℓ for $A, B, C, D, E_6, E_7, E_8, F_4$, or G_2 . Then our $H_{\mathbb{R}}$ -conjugacy classes will not split from the H -conjugacy classes.
- Case 2: There are no (θ, σ) - or no $(\theta, \theta\sigma)$ - singular roots. We see this in the case when $\Phi(A, A_\sigma^-)$ is empty or when $\Phi(S, S_{\sigma\theta}^-)$ is empty. An empty intersection means there are no candidates to represent a standard pair of tori. Also, if there are no (θ, σ) -singular roots then the singular rank is 0, we can conclude there is only one $H_{\mathbb{R}}$ -conjugacy class of σ -stable maximal \mathbb{R} -split tori of G .
- Case 3: If the singular rank is r and the $\Phi(A, A_\sigma^-)$ or $\Phi(S, S_{\sigma\theta}^-)$ is of type B_r, C_r, D_r, E_r ($r = 6, 7$, or 8), F_r ($r = 4$), or G_r ($r = 2$), then we must determine if other inv-

lutions are (θ, σ) -singular. If w_m is of type $r \cdot A_1$, then we return to case 1. If w_m is of type B_r, C_r, D_r, E_r ($r = 6, 7$, or 8), F_r ($r = 4$), or G_r ($r = 2$), then we need more information. Again, we need to consider the splitting of these involutions in $W(A, H_{\mathbb{R}}^+)$. In fact, in some classes $\Phi(T)$ agrees well and has the same rank; however, we also have $\Phi(T) = \Phi(T_1) + \Phi(T_2) = X_i + Y_j$, where $X, Y = A, B, C, BC$, or E . This will cause the classes to split. In this case, often the singular rank is less than the rank of $\Phi(A, A_\sigma^-)$ or $\Phi(S, S_\sigma^+)$ which complicates matters.

Chapter 2

Preliminaries

2.1 Preliminaries and Definitions

For a majority of the thesis, we will be working with the Lie algebras and use the induced involutions from the group. While an abuse of notation, we will call these induced involutions by σ and θ . The theory in previous papers is equivalent and I will show that the results on the Lie algebra are the same as the results in the group.

We will start with a connected, reductive group G ; however, we will also be in constant connection with the Lie algebra of G , $\mathfrak{g} = \text{Lie}(G)$. We will be considering involutions, commuting involutions, symmetric spaces, and maximal k -split tori of G . Therefore, we require many definitions to explain the connections between the orbits and conjugacy classes mentioned in the introduction.

First, for $x, y \in G$ denote the commutator $xyx^{-1}y^{-1}$ by (x, y) . If X, Y are subgroups of G , the subgroup of G generated by all (x, y) , $x \in X, y \in Y$ will be denoted at $[X, Y]$. Similarly in the Lie algebra of G , \mathfrak{g} , the commutator and bracket is just $xy - yx$ and the above definition will hold.

Definition 2.1.1 A k -involution, σ , is an automorphism of G where $\sigma^2 = id$ and $\sigma \neq id$ and defined over k .

We will denote an automorphism of \mathfrak{g} , induced by σ also by σ .

Definition 2.1.2 For any k -involution τ . Let G_τ be the fixed point group of τ .

$$G_\tau = \{g \in G \mid \tau(g) = g\}$$

To define our symmetric variety, we will use a pairing (G, σ) . Let $H = G_\sigma = \{g \in G \mid \sigma(g) = g\}$ be the fixed point group of σ . Then G/H is a symmetric variety. For a field k , the symmetric k -variety defined by the pair (G, σ) is G_k/H_k , where G_k (resp. H_k) are the k -rational points of G (resp. H). A symmetric \mathbb{R} -variety, $G_{\mathbb{R}}/H_{\mathbb{R}}$, is also called real reductive symmetric space. G_k/H_k is isomorphic to $\{g\sigma(g)^{-1} \mid g \in G_k\}$ and given $x, g \in G_k$, the σ -twisted action associated with G_k is $g * x = gx\sigma(g)^{-1}$.

In Helminck and Wang [18], the double cosets $P_k \setminus G_k/H_k$ are characterized by the quotient of the Weyl groups of A_i where A_i is a representative of an H_k -conjugacy class of σ -stable maximal k -split tori of G . It is important to note that in general, these double cosets are infinite.

For example, $G = \mathrm{SL}(2)$ and $k = \mathbb{Q}$ with $\sigma(x) = (x^T)^{-1}$, then $|P_k \setminus G_k/H_k|$ is not finite. For $k = \mathbb{R}$, the finite condition was verified by both J.Wolf [20] and T. Matsuki [16].

Example Let $G = \mathrm{SL}(2, k)$ and $\sigma(x) = (x^T)^{-1}$. Then

$$H = \{x \in G \mid (x^T)^{-1} = x\} = \{x \in G \mid x^T = x^{-1}\} = \mathrm{SO}(2, k).$$

$$X = G/H = \{g\sigma(g)^{-1} \mid g \in G\} = \{gg^T \mid g \in G\}.$$

Here P are the set upper triangular matrices: $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. Then P acts on G/H by

$$g * x = gx\sigma(g)^{-1} = gxg^T.$$

Upon calculation we find that for $\mathrm{SL}(2, \mathbb{R})$ there is one orbit with representative $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$.

For $\mathrm{SL}(2, \mathbb{C})$ there are 3 orbits with representatives: $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}), (\begin{smallmatrix} 0 & -i \\ -i & 0 \end{smallmatrix})$.

However, we will find that as the dimension increases and the involution changes this brute force calculation of the orbits is unlikely. In such a case, we will use the equivalent quotient of Weyl groups for our calculations. To do so we must first develop the tools to better understand tori, Weyl groups of tori, and the quotient of Weyl groups.

Definition 2.1.3 *In a group, a torus, T , is called σ -stable if $\sigma(T) = T$. Then let $T = T_\sigma^+ T_\sigma^-$, where*

$$T_\sigma^+ = (T \cap H)^0 \text{ and } T_\sigma^- = \{x \in T \mid \sigma(x) = x^{-1}\}^0$$

In a Lie algebra, a torus, \mathfrak{t} , is called σ -stable if $\sigma(\mathfrak{t}) = \mathfrak{t}$. Then let $\mathfrak{t} = \mathfrak{t}_\sigma^+ \oplus \mathfrak{t}_\sigma^-$, where

$$\mathfrak{t}_\sigma^+ = (\mathfrak{t} \cap \mathfrak{h}) \text{ and } \mathfrak{t}_\sigma^- = \{x \in \mathfrak{t} \mid \sigma(x) = -x\}.$$

I discuss both cases because much of the theory is in Lie groups, but we will be looking at the questions in terms of the Lie algebras. I will continue the discussion for the Lie algebra.

Definition 2.1.4 *A torus, \mathfrak{a} , is called σ -split if $\sigma(a) = -a$ for all $a \in \mathfrak{a}$.*

Note: A (σ, k) -split torus is both σ -split and k -split.

A quasi k -split torus is a torus that is \mathfrak{g} -conjugate to a k -split torus.

A torus, \mathfrak{s} , is called σ -fixed if $\sigma(s) = s$ for all $s \in \mathfrak{s}$.

2.2 Weyl group

Recall, the equivalency between the double cosets and the quotients of Weyl groups of tori.

Theorem 2.2.1 ([18, Proposition 6.10]) *Let $\{A_i \mid i \in I\}$ be the representatives of the H_k -conjugacy classes of σ -stable maximal k -split tori of G .*

$$P_k \backslash G_k / H_k \cong \bigcup_{i \in I} W_{G_k}(A_i) \backslash W_{H_k}(A_i)$$

First we must determine the conjugacy classes of σ -stable maximal k -split tori. Then we must understand what the Weyl groups will look like in the Lie group. We will start with understanding the Weyl groups. We will need to understand the Weyl group of a torus, T . The Weyl group of a torus T with respect to H is denoted $W(T, H)$ or $W_H(T) = N_H(T)/Z_H(T)$, where

$$N_H(T) = \{x \in H \mid xTx^{-1} \subset T\} \text{ and}$$

$$Z_H(T) = \{x \in H \mid xt = tx \text{ for all } t \in T\}$$

Typically we view the Weyl group, W , as generated by s_α for all roots α . Recall,

$$s_\alpha(\beta) = \beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}\alpha.$$

Since the goal is to describe the conjugacy classes in terms of both Lie groups and Lie algebras, it is useful to closely follow the previously developed group theory. Therefore, we will use the compact group approach in [5] to describe the Weyl group of a maximal torus of \mathfrak{g} .

Notation \mathfrak{g} is a complex, semisimple Lie algebra which is a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$. \mathfrak{u} is a compact real form of \mathfrak{g} and K is the compact subgroup of $\mathrm{GL}(n, \mathbb{C})$ whose Lie algebra is \mathfrak{u} . \mathfrak{t} is a maximal commutative subalgebra of \mathfrak{u} and we have the associated Cartan subalgebra $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$.

Define

$$Z(\mathfrak{t}) = \{A \in K \mid \text{Ad}_A(H) = H \text{ for all } H \text{ in } \mathfrak{t}\}$$

$$N(\mathfrak{t}) = \{A \in K \mid \text{Ad}_A(H) \subset \mathfrak{t} \text{ for all } H \text{ in } \mathfrak{t}\}$$

Both $Z(\mathfrak{g})$ and $N(\mathfrak{t})$ are subgroups of K . Furthermore, $Z(\mathfrak{g})$ is a normal subgroup of $N(\mathfrak{t})$. Then the Weyl group for \mathfrak{g} is the quotient group $W = N(\mathfrak{t})/Z(\mathfrak{t})$.

We define an action of W on \mathfrak{t} as follows. For each element $w \in W$, choose an element A of the corresponding equivalence class in $N(\mathfrak{t})$. Then for $H \in \mathfrak{t}$ we define the action $w \cdot H$ of w on H by

$$w \cdot H = \text{Ad}_A(H)$$

The action is well defined. Since $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$, the map on \mathfrak{t} extends uniquely to a complex-linear map on \mathfrak{h} . If $w \in W$, then we write $w \cdot H$ for the action of w on an element $H \in \mathfrak{h}$.

Theorem 2.2.2 *For each root α , there exist an element w_α of W such that $w_\alpha \cdot \alpha = -\alpha$ and such that $w_\alpha \cdot H = H$ for all $H \in \mathfrak{h}$ with $(\alpha, H) = 0$.*

Proof Recall for each root α , we can find nonzero elements X_α in \mathfrak{g}_α , Y_α in \mathfrak{g}_α and H_α in \mathfrak{h} which span a subalgebra of \mathfrak{g} isomorphic to $sl(2, \mathbb{C})$. Satisfying:

- $[H_\alpha, X_\alpha] = 2X_\alpha$
- $[H_\alpha, Y_\alpha] = -2Y_\alpha$
- $[X_\alpha, Y_\alpha] = H_\alpha$

Choose $Y_\alpha = X_\alpha^*$, then $X_\alpha - Y_\alpha = X_\alpha - X_\alpha^* = (X_\alpha^1 + iX_\alpha^2) - (-X_\alpha^1 + iX_\alpha^2)$ for some $X_\alpha^1, X_\alpha^2 \in \mathfrak{t}$. Therefore, $X_\alpha - Y_\alpha = 2X_\alpha^1 \subset \mathfrak{t}$. We let A_α be the element of K given by

$A_\alpha = \exp[\frac{\pi}{2}(X_\alpha - Y_\alpha)]$. We want to show that A_α is in $N(\mathfrak{t})$ and that

Ad_{A_α} acts on \mathfrak{h} .

For $H \in \mathfrak{h}$ and that $(\alpha, H) = 0$. Then, $[H, X_\alpha] = (\alpha, H)X_\alpha = 0$ as with Y_α . This means that X_α and Y_α commute with H . Recall that $e^{\text{ad}x} = \text{Ad}(e^X)$ and $e^{\text{ad}x}(Y) = \text{Ad}_{e^X}(Y) = e^X Y e^{-X}$. Therefore, $\text{Ad}_{A_\alpha}(H) = \exp[\frac{\pi}{2}(\text{ad}_{X_\alpha} - \text{ad}_{Y_\alpha})](H) = H$.

Now consider the action of Ad_{A_α} on the one-dimensional subspace of \mathfrak{g} spanned by H_α (or α). As above,

$$\text{Ad}_{A_\alpha}(H_\alpha) = \exp[\frac{\pi}{2}(\text{ad}_{X_\alpha} - \text{ad}_{Y_\alpha})](H_\alpha)$$

We can calculate the right-hand side and we will have $\text{Ad}_{A_\alpha}(H_\alpha) = -H_\alpha$. So Ad_{A_α} acts on H like the identity with $(\alpha, H) = 0$ and Ad_{A_α} acts as minus the identity on the span of α . Therefore, A_α represents the element of the Weyl group, w_α , that was desired.

The following theorem demonstrates the connection with the traditional Lie algebra approach to the Weyl group and the compact group approach.

Theorem 2.2.3 *The Weyl group W is generated by the elements w_α as α ranges over all roots.*

Remark If we let T be the connected Lie subgroup of K with Lie algebra \mathfrak{t} . T is a maximal torus. Then the centralizer of $T = Z(T) = \{A \in K \mid AtA^{-1} = t, \forall t \in T\}$ and the normalizer of $T = N(T) = \{A \in K \mid AtA^{-1} \in T, \forall t \in T\}$ coincide with $Z(\mathfrak{t})$ and $N(\mathfrak{t})$. In fact, for T a maximal torus, $T = Z(T) = Z(\mathfrak{t})$.

We consider the Weyl group in both manners in order to use the established theory of Lie groups and conjugacy classes of tori.

Now, repeating Example 2.1 from earlier in the section, we can now determine the order, $|P_k \setminus G_k/H_k|$, by the equivalent relation with the representative tori and Weyl groups of the tori.

Example Let $G = \mathrm{SL}(2, k)$ and $\sigma(A) = (A^T)^{-1}$

Let $C = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$, where $c^2 + d^2 = 1$.

C is σ -stable because $\sigma(c) \in C$ for all $c \in C$. However, C is not σ -split and C is \mathbb{C} -split and is not \mathbb{R} -split.

Let $D = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ is a σ -stable torus because $\sigma(d) = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \in D$.

D is σ -split because $\sigma(d) = d^{-1}$ for every $d \in D$. D is k -split for any k because it is diagonalizable.

C and D are the representatives of σ -stable maximal \mathbb{C} -split tori of $\mathrm{SL}(2, \mathbb{C})$ and D is the only representative of σ -stable maximal \mathbb{R} -split tori of $\mathrm{SL}(2, \mathbb{R})$.

Our Weyl group will be the permutation matrices.

$$W_G(T) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}. |W_G(T)| = 2.$$

For $k = \mathbb{C}$, there are two maximal σ -stable, k -split tori. The representatives are:

$$D = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ and } B = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}, \text{ where } c^2 + d^2 = 1.$$

Note that C is contained in $H = G_\sigma$, the fixed point group of σ .

$$|W_H(D)| = |N_H(D)/Z_H(D)| = 2$$

$$|W_H(C)| = |N_H(C)/Z_H(C)| = 1$$

So,

$$|\bigcup_{i \in I} W_G(A_i)/W_H(A_i)| = |W_G(D)/W_H(D)| + |W_G(C)/W_H(C)| = 3.$$

For $k = \mathbb{R}$ only D is k -split. So,

$$|\bigcup_{i \in I} W_G(A_i)/W_H(A_i)| = |W_G(D)/W_H(D)| = 1.$$

Again, we would like to be able to consider all involutions, all groups, and all dimensions. We must create tools to approach the problem in more generality.

Chapter 3

Symmetric Spaces and Involutions

3.1 Classification of Symmetric Spaces

In order to classify the $H_{\mathbb{R}}$ -conjugacy classes of σ -stable maximal \mathbb{R} -split tori, it is necessary to first understand the symmetric space determined by σ . The difficulty in the classification arises from the \mathbb{R} -split requirement, but using the classification of the semisimple symmetric spaces, we can reduce the problem to commuting pairs of involutions over the complex numbers. The pair, (θ, σ) described below will aid in the characterization of the tori by guaranteeing that the splitting condition is met.

Let G_0 be a real semisimple Lie group and denote its Lie algebra by \mathfrak{g}_0 . Let $\sigma \in \text{Aut}(G_0)$ be an involution and $(G_0)_{\sigma}$ be the fixed point group of σ . Let H be a closed subgroup of G_0 with Lie algebra \mathfrak{h} such that

$$((G_0)_{\sigma})^0 \subset H \subset (G_0)_{\sigma}$$

The pair (G_0, H) is called a semisimple symmetric pair and $(\mathfrak{g}_0, \mathfrak{h})$ is a semisimple locally symmetric pair. The symmetric space G_0/H is called an affine symmetric space

or a semisimple symmetric space. Similarly, when G_0 is reductive (G_0, H) is called a reductive symmetric pair and $(\mathfrak{g}_0, \mathfrak{h})$ is a reductive locally symmetric pair. We will look at the conjugacy classes in the Lie algebra and later consider the argument at the group level.

We say that two semisimple locally symmetric pairs $(\mathfrak{g}_0, \mathfrak{h}_1)$ and $(\mathfrak{g}_0, \mathfrak{h}_2)$ are isomorphic under an inner automorphism if there exists $\phi \in \text{Aut}(\mathfrak{g}_0)$ such that $\phi(\mathfrak{g}_0) = \mathfrak{g}_0$ and $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$.

Let \mathfrak{g} denote the complexification of \mathfrak{g}_0 . We will show that a semisimple locally symmetric pair (as described above) determines a pair of commuting involutions of \mathfrak{g} . The isomorphism classes of these pairs of commuting involutions will transfer directly to affine symmetric spaces associated with the semisimple locally symmetric pair.

Definition 3.1.1 *An involution $\theta \in \text{Aut}(\mathfrak{g}_0)$ and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the decomposition into the +1 and -1-eigenspaces of θ . Then θ is called a Cartan involution if \mathfrak{k}_0 is a maximal compact subalgebra of \mathfrak{g}_0 .*

Remark

1. By abuse of language we will call a subalgebra compact if the Killing form restricted to \mathfrak{k}_0 is negative definite.
2. Equivalently on the group, a Cartan involution has a maximal k -anisotropic (compact) fixed point group. This will be important because we will use the fact that a torus is θ -split to deem it \mathbb{R} -split.
3. It has been shown that any real, semisimple Lie algebra has a Cartan involution that is unique up to inner automorphism. Therefore, if θ_1 and θ_2 are Cartan involutions of \mathfrak{g}_0 , then there exists a $\phi \in \text{Int}(\mathfrak{g}_0)$ such that $\phi\theta_1\phi^{-1} \cong \theta_2$.

In our discussion, we have a fixed involution σ . We can actually find a Cartan involution that will commute with σ .

Theorem 3.1.2 ([1, Lemma 10.2]) *If \mathfrak{g}_0 is a real semisimple Lie algebra, θ a Cartan involution, and let σ be any involution. Then there exists a $\phi \in \text{Int}(\mathfrak{g}_0)$ such that $\phi\theta\phi^{-1}$ commutes with σ .*

Since $\phi\theta\phi^{-1}$ is still a Cartan involution of \mathfrak{g}_0 , by the above remark, we can find a Cartan involution that commutes with σ . It is an abuse of notation, but we will call this involution θ , where θ is a Cartan involution and $\sigma\theta = \theta\sigma$.

Corollary 3.1.3 *Let $(\mathfrak{g}_0, \mathfrak{h})$ be a semisimple locally symmetric pair. Then there exists a Cartan involution θ of \mathfrak{g}_0 such that $\sigma\theta = \theta\sigma$.*

Definition 3.1.4 *Let \mathfrak{g} be a Lie algebra over \mathbb{C} and the real Lie algebra $\mathfrak{g}^{\mathbb{R}}$. A real form of \mathfrak{g} is a subalgebra, \mathfrak{g}_0 , of $\mathfrak{g}^{\mathbb{R}}$ such that each $Z \in \mathfrak{g}$ can be uniquely written as $Z = X + iY$, where $X, Y \in \mathfrak{g}_0$. Thus \mathfrak{g} is isomorphic to the complexification of \mathfrak{g}_0 . A compact real form \mathfrak{u} is a real form of \mathfrak{g} whose Killing form is negative definite.*

Definition 3.1.5 *For a real form \mathfrak{g}_0 of \mathfrak{g} , there is an conjugation map, τ , of \mathfrak{g} with respect to \mathfrak{g}_0 given by $\tau : X + iY \rightarrow X - iY$ for $(X, Y \in \mathfrak{g}_0)$. The mapping τ has the following properties for $X, Y \in \mathfrak{g}_0, c \in \mathbb{C}$:*

1. $\tau(\tau(X)) = X$
2. $\tau(X + Y) = \tau(X) + \tau(Y)$
3. $\tau(cX) = \bar{c}\tau(X)$
4. $\tau[X, Y] = [\tau(X), \tau(Y)]$

Theorem 3.1.6 ([7, Lemma 10.3]) *Let \mathfrak{g} be a complex semisimple Lie algebra and $\theta_1, \dots, \theta_n$ commuting involutions in $\text{Aut}(\mathfrak{g})$. Then there exists a compact real form \mathfrak{u} of \mathfrak{g} , with conjugation τ , such that $\theta_i\tau = \tau\theta_i$, for $i = 1, \dots, n$.*

Theorem 3.1.7 $\theta \in \text{Aut}(\mathfrak{g})$, $\theta^2 = \text{id}$. *There exists a unique θ -stable compact real form of \mathfrak{g} .*

Proof Let τ be the conjugation of the compact real form \mathfrak{u} . $\theta(\mathfrak{u}) = \mathfrak{u}$ if and only if $\theta\tau = \tau\theta$. Then $\delta = \theta\tau$ is a conjugation. Let $\mathfrak{g}_0 = \mathfrak{g}_\delta = \{g \in \mathfrak{g} | \delta(g) = g\}$, then $\theta|\mathfrak{g}_0$ is a Cartan involution of \mathfrak{g}_0 and $\mathfrak{g}_0 = \mathfrak{u} \cap \mathfrak{g}_0^+ \oplus i\mathfrak{u} \cap \mathfrak{g}_0^-$.

Essentially what the above corollary and theorems suggest is the following. First, suppose $\mathfrak{g} = \mathfrak{g}_0 + i\mathfrak{g}_0$ and \mathfrak{g}_0 a real form with conjugation δ and $\sigma \in \text{Aut}(\mathfrak{g}_0)$. So

$$\delta : \mathfrak{g} \rightarrow \mathfrak{g} \text{ by } \delta(X + iY) = X - iY \text{ for } X, Y \in \mathfrak{g}_0$$

(\mathfrak{g}_0, σ) is a locally symmetric pair. Then there also exists a Cartan involution of \mathfrak{g}_0 such that $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition and $\sigma\theta = \theta\sigma$.

Second, given $\theta, \sigma \in \text{Aut}(\mathfrak{g})$, $\theta^2 = \sigma^2 = \text{id}$, and $\theta\sigma = \sigma\theta$. We can determine a unique compact real form found in the above theorem is both θ and σ -stable because $\sigma\theta = \theta\sigma$. The involutions define a unique semisimple locally symmetric pair.

We want to show that there is a one-to-one correspondence between semisimple locally symmetric pairs $(\mathfrak{g}_0, \mathfrak{h})$ and ordered, commuting pairs of involutions $(\theta, \sigma) \in \text{Aut}(\mathfrak{g})$.

It is important to note that we are dealing with ordered commuting pairs of involutions of \mathfrak{g} . The ordering is important because one involution is defining the symmetric space and the other is the Cartan involution which defines the real form of \mathfrak{g} . We will write the pair as (θ, σ) , where the first involution determines a real form.

Let \mathfrak{u} be a (θ, σ) -stable compact real form of \mathfrak{g} with conjugation τ . Denote $\theta\tau$ by $\bar{\theta}$ and $\sigma\tau$ by $\bar{\sigma}$. Let $\mathfrak{g}_{\bar{\theta}}$ be the fixed points of $\bar{\theta}$ in \mathfrak{g} . Then $(\mathfrak{g}_{\bar{\theta}})_\sigma$ are the set of fixed points of σ in $\mathfrak{g}_{\bar{\theta}}$. Then $(\mathfrak{g}_{\bar{\theta}}, \sigma|_{\mathfrak{g}_{\bar{\theta}}})$ is a locally semisimple symmetric pair corresponding to (θ, σ) . It follows that from [6, Proposition 1.4] that the isomorphism class of $(\mathfrak{g}_{\bar{\theta}}, \sigma|_{\mathfrak{g}_{\bar{\theta}}})$ does not depend on the choice of the (θ, σ) -stable compact real form \mathfrak{u} of \mathfrak{g} .

Remark Pairs of commuting pairs of involutions in $\text{Aut}(\mathfrak{g})$ correspond bijectively to commuting pairs of involutions of \mathfrak{g} .

Theorem 3.1.8 ([7, Theorem 10.6]) *The inner (resp. outer) isomorphism classes of the semisimple locally symmetric pairs $(\mathfrak{g}_0, \mathfrak{h})$ correspond bijectively to the inner (resp. outer) isomorphism classes of ordered pairs of commuting involutions (θ, σ) of \mathfrak{g} or $\text{Aut}(\mathfrak{g})^0$.*

Here \mathfrak{g} is complexification of \mathfrak{g}_0 , \mathfrak{h} is the Lie algebra of $(G_0)_\sigma$, and $\theta|\mathfrak{g}_0$ is a Cartan involution of \mathfrak{g}_0 commuting with σ . These structures have been studied by many including Cartan, Berger, Helminck, Oshima and Sekiguchi. Each mathematician provides new insight into the applications and the specifics of the structure.

In Table 3.1, I have provided the notation from Cartan, Helminck, and Oshima-Sekiguchi corresponding to the case of the pair $(\theta, \theta \text{Int}(\epsilon_i))$ where θ is a Cartan involution. Also included is the pair $(\mathfrak{g}_0, \mathfrak{h})$.

Remark We will be using the notation of Helminck throughout the thesis. It is important to understand the information given by the notation. $X_a^b(\text{Type}, \epsilon_i)$ gives an involution's Cartan type acting on a root system of type X with dimension a . b gives us the dimension of the (-1)-space of the involution.

Last, the ϵ_i is a quadratic element and is part of the classification of pairs of involutions. Further discussion of quadratic elements can be found in Chapter 6 of [7] and in

the last section of this chapter. We will discuss the role of these elements in the context of the conjugacy classes and action on rootspaces.

Some notes on Table 3.1 are helpful in the following cases:

Helminck	Oshima-Sekiguchi	Note
$A_\ell^p(III_a, \epsilon_i), i \neq \ell$	$BC_{p,i}^{2m,2,1}, i \neq \ell$	$m = \ell + 1 - 2p$
$B_\ell^p(I_a, \epsilon_i)$	$B_{p,i}^{m,1}$	$m = 2\ell + 1 - 2p$
$C_\ell^p(II_a, \epsilon_i)$	$BC_{p,i}^{4m,4,3}$	$m = \ell - 2p$
$D_\ell^p(I_a, \epsilon_i)$	$B_{p,i}^{m,1}$	$m = 2\ell - 2p$

Table 3.1: Cartan Involutions

Cartan	Helmink	Oshima-Sekiguchi	\mathfrak{g}	\mathfrak{h}
AI	$A_\ell^\ell(I, \epsilon_i)$	$A_{\ell,i}^\ell$	$sl(\ell, \mathbb{R})$	$so(\ell + 1 - i, i)$
AII	$A_{2\ell+1}^\ell(II, \epsilon_i)$	$A_{\ell,i}^4$	$su^*(2\ell + 2)$	$sp(\ell + 1 - i, i)$
	$A_{2\ell-1}^\ell(III_b, \epsilon_i), i \neq \ell$	$C_{\ell,i}^{2,1}, i \neq \ell$	$su(\ell, \ell)$	$su(\ell - i, i) + su(l - i, i) + so(2)$
$AIII$	$A_{2\ell-1}^\ell(III_b, \epsilon_\ell)$	$C_{\ell,A}^{2,1}$	$su(\ell, \ell)$	$sl(\ell, \mathbb{C}) + \mathbb{R}$
	$A_\ell^p(III_a, \epsilon_i)$	$BC_{p,i}^{2m,2,1}$	$su(\ell - p + 1, p)$	$su(\ell - p + 1 - i) + su(p - i, i) + so(2)$
BI	$B_\ell^p(I_a, \epsilon_i)$	$B_{p,i}^{m,1}$	$so(2\ell + 1 - p, p)$	$so(2\ell + 1 - p - i) + so(p - i, i)$
CI	$C_\ell^\ell(I, \epsilon_i), i \neq \ell$	$C_{\ell,i}^{1,1}, i \neq \ell$	$sp(\ell, \mathbb{R})$	$su(\ell - i, i) + so(2)$
	$C_\ell^\ell(I, \epsilon_\ell)$	$C_{\ell,A}^{1,1}, i \neq \ell$	$sp(\ell, \mathbb{R})$	$sl(\ell, \mathbb{R}) + \mathbb{R}$
	$C_\ell^p(II_a, \epsilon_i)$	$BC_{p,i}^{4m,4,3}$	$sp(\ell - p, p)$	$sp(\ell - p - i, i) + sp(p - i, i)$
CII				

Table 3.1 – Continued

	$C_{2\ell}^\ell(II_b, \epsilon_i), i \neq \ell$	$C_{\ell,i}^{4,3}$	$sp(\ell, \ell)$	$sp(\ell - i, i) + sp(\ell - i, i)$
	$C_{2\ell}^\ell(II_b, \epsilon_\ell)$	$C_{\ell,A}^{4,3}$	$sp(\ell, \ell)$	$sp(\ell, \mathbb{C})$
	$D_\ell^p(I_a, \epsilon_i)$	$B_{p,i}^{m,1}$	$so(2\ell - p, p)$	$so(2\ell - p - i, i) + so(p - i, i)$
<i>DI</i>	$D_\ell^\ell(I_b, \epsilon_i), i \neq \ell$	$D_{\ell,i}^1, i \neq \ell$	$so(\ell, \ell)$	$so(\ell - i, i) + so(\ell - i, i)$
	$D_\ell^\ell(I_b, \epsilon_\ell)$	$D_{\ell,i}^1$	$so(\ell, \ell)$	$su(\ell, \mathbb{C})$
	$D_{2\ell}^\ell(III_a, \epsilon_i), i \neq \ell$	$C_{\ell,i}^{4,1}, i \neq \ell$	$so^*(4\ell)$	$su(2\ell - 2i, 2i) + so(2)$
<i>DIII</i>	$D_{2\ell}^\ell(III_a, \epsilon_\ell)$	$C_{\ell,A}^{4,1}$	$so^*(4\ell)$	$su^*(2\ell) + \mathbb{R}$
	$D_{2\ell+1}^\ell(III_b, \epsilon_i)$	$BC_{\ell,i}^{4,4,1}$	$so^*(4\ell + 2)$	$su^*(2\ell + 1 - 2i, 2i) + so(2)$
<i>EI</i>	$E_6^6(I, \epsilon_1)$	$E_{6,D}^1$	$e_{6(6)}$	$sp(2, 2)$
	$E_6^6(I, \epsilon_2)$	$E_{6,A}^1$	$e_{6(6)}$	$sp(4, \mathbb{R})$
<i>EII</i>	$E_6^4(II, \epsilon_1)$	$F_{4,C}^{2,1}$	$e_{6(2)}$	$su(3, 3) + sl(2, \mathbb{R})$

Table 3.1 – Continued

	$E_6^4(II, \epsilon_4)$	$F_{4,B}^{2,1}$	$e_{6(2)}$	$su(4, 2) + su(2)$
$EIII$	$E_6^2(III, \epsilon_1)$	$BC_{2,A}^{8,6,1}$	$e_{6(-14)}$	$so^*(10) + so(2)$
	$E_6^2(III, \epsilon_2)$	$BC_{2,B}^{8,6,1}$	$e_{6(-14)}$	$so(8, 2) + so(2)$
EIV	$E_6^2(IV, \epsilon_i)$	$A_{2,A}^8$	$e_{6(-26)}$	$f_{4(-20)}$
	$E_7^7(V, \epsilon_1)$	$E_{7,D}^1$	$e_{7(7)}$	$su(4, 4)$
EV	$E_7^7(V, \epsilon_2)$	$E_{7,A}^1$	$e_{7(7)}$	$sl(8, \mathbb{R})$
	$E_7^7(V, \epsilon_7)$	$E_{7,E}^1$	$e_{7(7)}$	$su^*(8)$
$EVII$	$E_7^4(VI, \epsilon_1)$	$F_{4,C}^{4,1}$	$e_{7(-5)}$	$so^*(12) + sl(2, \mathbb{R})$
	$E_7^4(VI, \epsilon_4)$	$F_{4,B}^{4,1}$	$e_{7(-5)}$	$so(8, 4) + su(2)$
$EVII$	$E_7^3(VII, \epsilon_1)$	$C_{3,A}^{8,1}$	$e_{7(-25)}$	$e_{6(-26)} + sl(2, \mathbb{R})$
	$E_7^3(VII, \epsilon_2)$	$C_{3,B}^{8,1}$	$e_{7(-25)}$	$e_{6(-14)} + so(2)$

Table 3.1 – Continued

<i>EVIII</i>	$E_8^8(VIII, \epsilon_1)$	$E_{8,D}^1$	$e_{8(8)}$	$so(8, 8)$
	$E_8^8(VIII, \epsilon_8)$	$E_{8,E}^1$	$e_{8(8)}$	$so^*(16)$
<i>EIX</i>	$E_8^4(IX, \epsilon_1)$	$F_{4,C}^{8,1}$	$e_{8(-24)}$	$e_{7(-25)} + sl(2, \mathbb{R})$
	$E_8^4(IX, \epsilon_2)$	$F_{4,B}^{8,1}$	$e_{8(-24)}$	$e_{7(-5)} + su(2)$
<i>FI</i>	$F_4^4(I, \epsilon_1)$	$F_{4,C}^1$	$f_{4(4)}$	$sp(3, \mathbb{R}) + sl(2, \mathbb{R})$
	$F_4^4(I, \epsilon_4)$	$F_{4,B}^1$	$f_{4(4)}$	$sp(2, 1) + su(2)$
<i>FII</i>	$F_4^1(I, \epsilon_1)$	$BC_{1,A}^{8,7}$	$f_{4(-20)}$	$so(8, 1)$
<i>G</i>	$G_2^2(\epsilon_i)$	G_2^1	$g_{2(2)}$	$sl(2, \mathbb{R}) + sl(2, \mathbb{R})$

The discussion of σ -stable maximal \mathbb{R} -split tori, now takes on another form in the subspaces of $\mathfrak{g}_{\bar{\theta}}$. If we decompose using θ and form the Cartan decomposition of $\mathfrak{g}_{\bar{\theta}}$, we have $\mathfrak{g}_{\bar{\theta}} = \mathfrak{k} \oplus \mathfrak{p}$. Likewise, let $\mathfrak{g}_{\bar{\theta}} = \mathfrak{h} \oplus \mathfrak{q}$ be the decomposition in eigenspaces of $\sigma|_{\mathfrak{g}_{\bar{\theta}}}$. Now θ -split (resp. σ -split and (σ, θ) -split) tori of G correspond to Cartan subspaces t of \mathfrak{p} (resp. \mathfrak{q} and $\mathfrak{p} \cap \mathfrak{q}$). This characterizes the locally semisimple symmetric pairs (\mathfrak{g}_0, σ) in terms of a (σ, θ) -stable Cartan subalgebra \mathfrak{t} of $\mathfrak{g}_{\bar{\theta}}$, such that $t \cap \mathfrak{p}$ (resp. $\mathfrak{t} \cap \mathfrak{q}$ and $\mathfrak{t} \cap \mathfrak{p} \cap \mathfrak{q}$) is maximal abelian in \mathfrak{p} (resp. \mathfrak{q} and $\mathfrak{p} \cap \mathfrak{q}$).

The pairs of commuting involutions of G correspond bijectively with pairs of commuting involutions of \mathfrak{g} . We will again abuse notation and call the pair (θ, σ) to denote the involutions and the lifted involutions.

The most important piece, which completes the discussion is to consider how we can view a σ -stable \mathbb{R} -split tori, $\mathfrak{a} \subset \mathfrak{g}$. We can always find a θ commuting with σ ; therefore, \mathfrak{a} is θ -stable. However, given two commuting involutions we can find a (θ, σ) -stable real form of \mathfrak{g} . Then $\mathfrak{a} = \mathfrak{a}_{\theta}^+ \oplus \mathfrak{a}_{\theta}^-$. But \mathfrak{a}_{θ}^+ is contained in the compact k portion of the decomposition. Therefore, in order for \mathfrak{a} to be \mathbb{R} -split, $\mathfrak{a} = \mathfrak{a}_{\theta}^-$. We can now equate σ -stable maximal \mathbb{R} -split tori with (θ, σ) -stable maximal θ -split tori for commuting involutions (θ, σ) .

Based on this relationship, we can discuss a modification of Helminck and Wang original classification. Let $H^+ = (H \cap K)^0$ where H and K are the fixed point groups of σ and θ , respectively. H_k^+ are the k -rational points of H^+ .

Corollary 3.1.9 ([10, Corollary 3.9]) *Let $\{A_i \mid i \in I\}$ be representatives of the H_k^+ -conjugacy classes of (θ, σ) -stable maximal k -split (ie. θ -split) tori in G . Then*

$$P_k \setminus G_k / H_k^+ \cong \bigcup_{i \in I} W_{G_k}(A_i) \setminus W_{H_k^+}(A_i).$$

We will have to look at each pair $(\mathfrak{g}_{\bar{\theta}}, \sigma|_{\mathfrak{g}_{\bar{\theta}}})$ to determine in each case, the maximal (σ, θ) -stable θ -split tori. We will begin to dissect the conjugacy classes of these tori by looking at the maximal σ -split tori. These tori will build the structure to determine the number of conjugacy classes.

3.2 \mathbb{R} -involutions

In order to look at the tori, we will be looking at the root systems of the reductive groups and associated Lie algebras. It is useful to consider the root datum.

Definition 3.2.1 *A root datum is a quadruple $\Psi = (X, \Phi, \check{X}, \Phi\check{X})$, where X and \check{X} are free abelian groups of finite rank, in duality by a pairing $X \times \check{X} \rightarrow \mathbb{Z}$, denoted by (\cdot, \cdot) , Φ and $\check{\Phi}$ are finite subsets of X and \check{X} with a bijection $\alpha \rightarrow \check{\alpha}$ of Φ onto $\check{\Phi}$. If $\alpha \in \Phi$ we define endomorphisms s_α and $s_{\check{\alpha}}$, of X and \check{X} , respectively, by $s_\alpha(\chi) = \chi - \langle \chi, \check{\alpha} \rangle \alpha$, $s_{\check{\alpha}}(\nu) = \nu - \langle \alpha, \nu \rangle \check{\alpha}$.*

The following two axioms are imposed:

1. If $\alpha \in \Phi$, then $\langle \alpha, \check{\alpha} \rangle = 2$.
2. If $\alpha \in \Phi$, then $s_\alpha(\Phi) \subset \Phi$, $s_{\check{\alpha}}(\check{\Phi}) \subset \check{\Phi}$.

In particular, when we are considering a maximal torus T or maximal θ -split torus of a reductive group or maximal torus \mathfrak{t} of the associated Lie algebra, we are also guaranteed to have the root datum as described above.

We will call $\Phi(T)$ (resp. $\Phi(\mathfrak{t})$) the root system of T (resp. \mathfrak{t}) with associated Weyl group $W(T)$ (resp. $W(\mathfrak{t})$)

So now we have pairs of commuting involutions (θ, σ) and root datum associated with maximal tori. We will discuss the involutions σ and θ on the reductive group or Lie

algebra by considering the action on the root system, by an abuse of notation, we will also call these involutions σ and θ .

The restriction to the common -1-eigenspaces of σ and θ plays an important role in the classification of maximal (θ, σ) -stable k -split tori. Let Φ be the roots system of the maximal torus. We can use the restricted root system to find the root system of the maximal θ -split torus which is equivalent to the maximal \mathbb{R} -split tori. We can complete the classification by looking at σ on this restricted root system. The involution σ will be used to describe the involution of the root system as well as the involution lifted to the Lie algebra.

Let

$$\Phi_0(\theta) = \{\alpha \in \Phi \mid \alpha - \theta(\alpha) = 0\}$$

and

$$\Phi_0(\theta, \sigma) = \{\alpha \in \Phi \mid \alpha - \sigma(\alpha) - \theta(\alpha) + \sigma\theta(\alpha) = 0\}$$

These systems are θ -stable and (θ, σ) -stable respectively. Both are closed subsystems of Φ .

Also, call $\Phi_\theta = \Phi/\Phi_0(\theta)$ and $\Phi_{\theta, \sigma} = \Phi/\Phi_0(\theta, \sigma)$ the restricted roots systems of θ and (θ, σ) respectively. There is a natural project π_θ from Φ to Φ_θ given by

$$\pi_\theta(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha))$$

Similarly, we have a natural projection, $\pi_{\theta, \sigma}$ from Φ to $\Phi_{\theta, \sigma}$ given by

$$\pi_{\theta, \sigma}(\alpha) = \frac{1}{4}(\alpha - \sigma(\alpha) - \theta(\alpha) + \sigma\theta(\alpha))$$

Remark Φ_θ and $\Phi_{\theta, \sigma}$ can be identified with the projections onto the -1-eigenspace of θ and the -1-eigenspaces of σ and θ , respectively.

Example Let Φ be of type A_3 so $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of Φ . Suppose θ acts on roots as follows:

$$\begin{aligned}\theta(\alpha_1) &= \alpha_1 \\ \theta(\alpha_2) &= -(\alpha_1 + \alpha_2 + \alpha_3) \\ \theta(\alpha_3) &= \alpha_3 \\ \theta(\alpha_1 + \alpha_2) &= -(\alpha_2 + \alpha_3) \\ \theta(\alpha_2 + \alpha_3) &= -(\alpha_1 + \alpha_2) \\ \theta(\alpha_1 + \alpha_2 + \alpha_3) &= -\alpha_2\end{aligned}$$

Then $\Phi_0(\theta) = \{\alpha_1, \alpha_3\}$ because both roots are fixed by θ . If we consider the projection from Φ to Φ_θ then we can determine the desired restriction of the basis. We know the roots in $\Phi_0(\theta)$ are sent to 0 and we need only calculate the project for the remaining roots.

$$\begin{aligned}\pi_\theta(\alpha_2) &= \frac{1}{2}(\alpha_2 - \theta(\alpha_2)) = \frac{1}{2}(\alpha_2 + (\alpha_1 + \alpha_2 + \alpha_3)) = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3). \\ \pi_\theta(\alpha_1 + \alpha_2) &= \frac{1}{2}(\alpha_1 + \alpha_2 - \theta(\alpha_1 + \alpha_2)) = \frac{1}{2}(\alpha_1 + \alpha_2 + (\alpha_2 + \alpha_3)) = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3).\end{aligned}$$

Similarly, $\pi_\theta(\alpha_2 + \alpha_3) = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3)$ and $\pi_\theta(\alpha_1 + \alpha_2 + \alpha_3) = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3)$

Then $\Phi_\theta = \{\frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3)\}$ and is of type A_1 .

Now, suppose σ acts on the roots as follows:

$$\begin{aligned}\sigma(\alpha_1) &= -\alpha_3 \\ \sigma(\alpha_2) &= -\alpha_2 \\ \sigma(\alpha_3) &= -\alpha_1 \\ \sigma(\alpha_1 + \alpha_2) &= -(\alpha_2 + \alpha_3) \\ \sigma(\alpha_2 + \alpha_3) &= -(\alpha_1 + \alpha_2) \\ \sigma(\alpha_1 + \alpha_2 + \alpha_3) &= -(\alpha_1 + \alpha_2 + \alpha_3)\end{aligned}$$

First, $\Phi_0(\theta, \sigma) = \Phi_0(\theta)$ which can be seen by direct calculation. It is useful to have a table of the action of $\theta\sigma$ on the roots as well.

$$\begin{aligned}\theta\sigma(\alpha_1) &= -\alpha_3 \\ \theta\sigma(\alpha_2) &= \alpha_1 + \alpha_2 + \alpha_3 \\ \theta\sigma(\alpha_3) &= -\alpha_1 \\ \theta\sigma(\alpha_1 + \alpha_2) &= (\alpha_1 + \alpha_2) \\ \theta\sigma(\alpha_2 + \alpha_3) &= (\alpha_2 + \alpha_3) \\ \theta\sigma(\alpha_1 + \alpha_2 + \alpha_3) &= \alpha_2\end{aligned}$$

Then,

$$\pi_{\theta, \sigma}(\alpha_1) = \frac{1}{4}(\alpha_1 - \sigma(\alpha_1) - \theta_{\alpha_1} + \sigma\theta(\alpha_1)) = \frac{1}{4}(\alpha_1 + \alpha_3 - \alpha_1 - \alpha_3) = 0.$$

A similar calculation verifies our $\Phi_0(\theta, \sigma)$.

$$\begin{aligned}\pi_{\theta, \sigma}(\alpha_2) &= \frac{1}{4}(\alpha_2 - \sigma(\alpha_2) - \theta_{\alpha_2} + \sigma\theta(\alpha_2)) = \frac{1}{4}(\alpha_2 + \alpha_2 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_1 + \alpha_2 + \alpha_3) = \\ &\quad \frac{1}{4}(2\alpha_1 + 4\alpha_2 + 2\alpha_3) = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3).\end{aligned}$$

$\Phi_{\theta, \sigma} = \Phi_\theta$ and $\Phi_{\theta, \sigma} \cap \Phi_\theta = \Phi_\theta$. However, this is not always the case as demonstrated in the next example.

Example Consider Example 3.2, but we will reverse the involutions so now θ acts like σ in Example 3.2 and σ acts like θ in Example 3.2. $\sigma\theta$ has the same action.

First, $\Phi_0(\theta) = \emptyset$ because no root is fixed by θ . Let's determine Φ_θ .

$$\begin{aligned}\pi_\theta(\alpha_1) &= \frac{1}{2}(\alpha_1 - \theta(\alpha_1)) = \frac{1}{2}(\alpha_1 + \alpha_3) = \frac{1}{2}(\alpha_1 + \alpha_3) = \pi_\theta(\alpha_3). \\ \pi_\theta(\alpha_2) &= \frac{1}{2}(\alpha_2 - \theta(\alpha_2)) = \frac{1}{2}(\alpha_2 + \alpha_2) = \alpha_2. \\ \pi_\theta(\alpha_1 + \alpha_2) &= \frac{1}{2}(\alpha_1 + \alpha_2 - \theta(\alpha_1 + \alpha_2)) = \frac{1}{2}(\alpha_1 + \alpha_2 + (\alpha_2 + \alpha_3)) \\ &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3) = \pi_\theta(\alpha_2 + \alpha_3). \\ \pi_\theta(\alpha_1 + \alpha_2 + \alpha_3) &= \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1 + \alpha_2 + \alpha_3) = \alpha_1 + \alpha_2 + \alpha_3\end{aligned}$$

Then $\Phi_\theta = \{\frac{1}{2}(\alpha_1 + \alpha_3), \alpha_2, \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3), \alpha_1 + \alpha_2 + \alpha_3\}$ and is of type C_2 .

$\Phi_{\theta,\sigma}$ is the same as Example 3.2 and $\Phi_{\theta,\sigma} \cap \Phi_\theta = \{\frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3)\}$ and of type A_1 . Different involutions will determine different systems. To describe the involutions more efficiently, it is worthwhile to consider an order to the basis and use an involution diagram.

Definition 3.2.2 *An order \succ on X is called a (θ, σ) -order if it has the following property:*

$$\text{if } \chi \in X, \chi \succ 0, \text{ and } \chi \notin X_0(\theta, \sigma), \text{ then } \sigma(\chi) \succ 0 \text{ and } \theta(\chi) \succ 0$$

Since $\Phi \subset X$, we can have a (θ) -order on Φ which induces an order on $\Phi_0(\theta)$ and Φ_θ and we can have a (θ, σ) -order on Φ which induces an order on $\Phi_0(\theta, \sigma)$ and $\Phi_{\theta,\sigma}$. We can also learn about $\Phi_{\theta,\sigma}$ by considering $\sigma|\Phi_\theta$.

Definition 3.2.3 *A basis Δ of Φ with respect to a θ -order ((θ, σ) -order) will be called a θ -basis ((θ, σ) -basis) of Φ .*

Similarly, we have ordered bases for $\Delta_0(\theta)$, $\Delta_0(\theta, \sigma)$, Δ_θ , and $\Delta_{\theta,\sigma}$ of $\Phi_0(\theta)$, $\Phi_0(\theta, \sigma)$, Φ_θ , and $\Phi_{\theta,\sigma}$.

Lemma 3.2.4 *Let \mathfrak{t} be a θ -stable maximal torus of G , Δ a θ -basis of $\Phi(\mathfrak{t})$, then $\theta = -\theta^* w_0(\theta)$. Where $w_0(\theta)$ is the longest element of $\Phi_0(\theta)$ with respect to a θ -ordered basis Δ_0 and θ^* is the identity or an order two automorphism of the Dynkin diagram of $\Phi(\mathfrak{t})$.*

In [7], this information is used to completely classify the involutions over reductive groups and their associated Lie algebras and all pairs of commuting involutions. Each induced involution on the roots is associated with a diagram. The involution $\theta = -\theta^* w_0(\theta)$

can be retrieved from the diagram. θ^* appears on the diagram with arrows denoting the action. Also, the roots of θ -ordered basis Δ_0 are black dots.

Table 3.2 has the types of involution, diagram representations, restricted roots and type of restricted root system.

Describing the root systems as above will also help in our classification. In particular, we need to find (θ, σ) -singular roots, which live in the intersection of Φ_θ and $\Phi_{\theta, \sigma}$. For each pair of involutions, we have to look at the restricted root systems, projections of the roots of Φ , the action of σ, θ , and $\sigma\theta$, and finally the action on the root spaces. The first three all require a strong knowledge of the above theory. We will utilize the diagram description of the involutions in our classification.

Table 3.2: Involution Diagrams

Type θ	Diagram	Δ_θ	Type Φ_θ
AI			A_l
AII			A_l
$AIII_a$			BC_p
$AIII_b$			C_l
BI			BC_p
CI			C_l
CII_a			BC_p
CII_b			C_l
DI_a			B_p

Table 3.2 – Continued

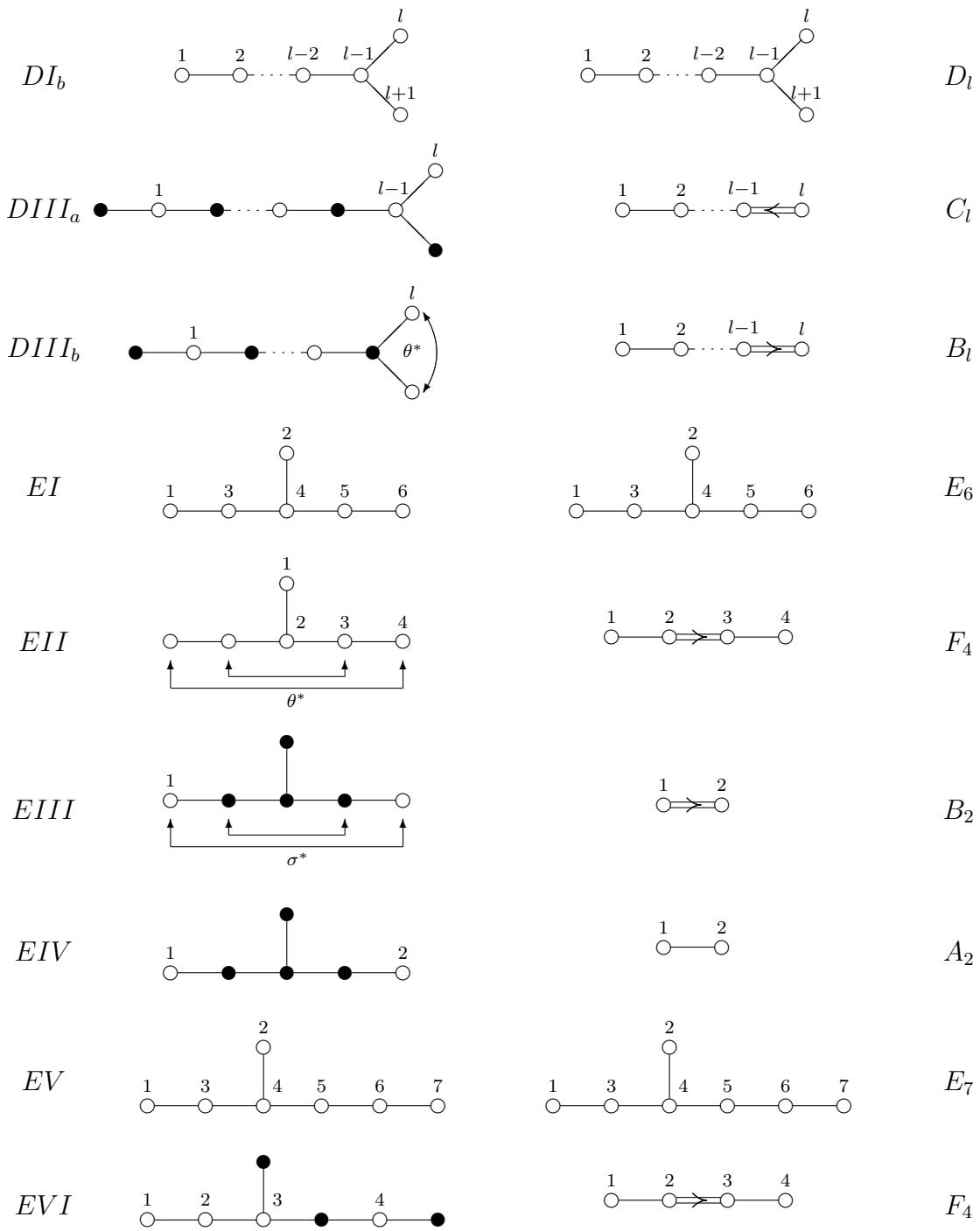
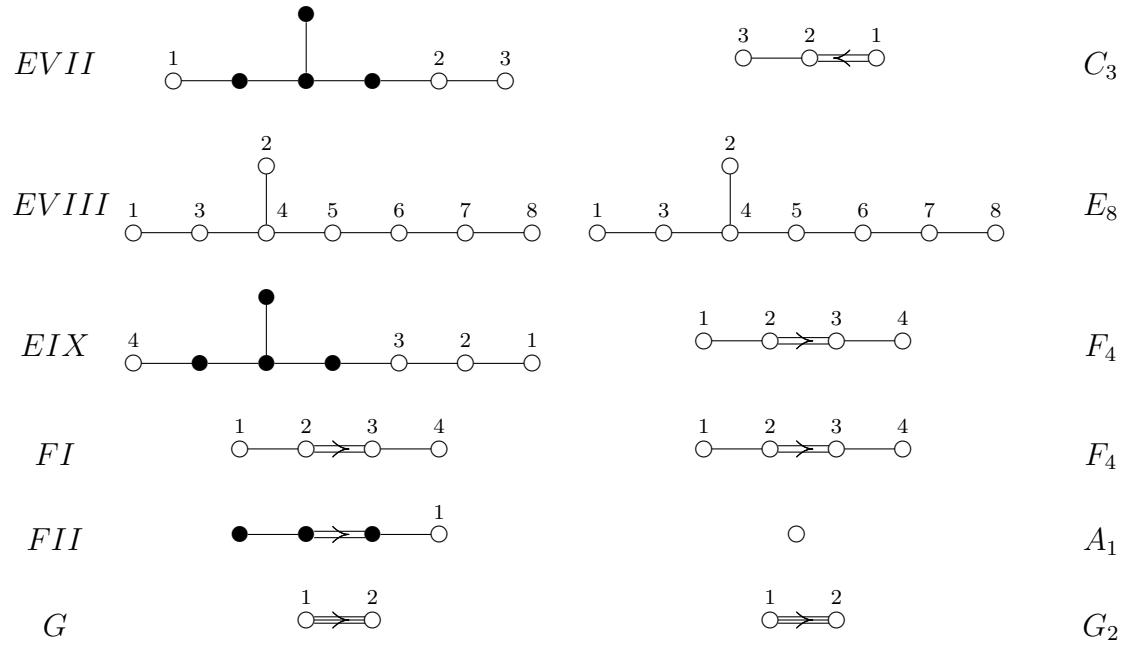


Table 3.2 – Continued

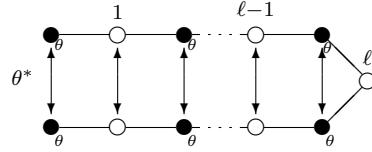


For each of the over 171 pairs we must make the diagram representation of the involutions actions on the root system of the maximal torus. We will use the diagrams in Table 3.2, but with a slight modification. Both the action of σ and θ will be represented on one diagram. Any arrow will be labeled with the appropriate involution. Any black dot means both involutions fix that root. A black dot with an involution attached implies that given involution does not fix the root, but the other one does.

Example Suppose σ is type AII and θ is type $AIII_b$ acting on $A_{4\ell-1}$. First, we will represent both as diagrams.



To represent (θ, σ) the diagram would be:



From the diagram of (θ, σ) we can extract the original involutions.

Remark We will call the maximal (θ, σ) -stable \mathbb{R} -split tori containing the maximal σ -split torus, \mathfrak{a} . $\Phi(\mathfrak{a}) = \Phi_\theta$ will be the type of the \mathfrak{a} determined by θ .

$$\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-) = \Phi(\mathfrak{a}) \cap \Phi(\mathfrak{a}_\sigma^-) = \Phi_{\sigma, \theta} \cap \Phi_\theta$$

are the roots of the maximal (σ, \mathbb{R}) -split torus inside the maximal torus of \mathfrak{g} . Of course, the intersection with Φ_θ is important to ensure that we maintain the \mathbb{R} -split quality which is precisely the θ -split requirement.

3.3 Quadratic Elements and Multiplicities

While quadratic elements will not play an immediate role in our work, it is important to understand how they play a role in the classification of commuting pairs of involutions. For details on multiplicity and quadratic elements please see Chapter 6 in [7].

A quadratic element, ϵ , is an element of a (σ, θ) -stable maximal k -split torus such that $\epsilon^2 = e$. When a quadratic element is used in the classification of the pairs of involutions, we see that σ and $\sigma \text{Int}(\epsilon)$ are two different involutions. The calculation of the quadratic elements was done in [7]. We will be focused on calculating the action of $\text{Int}(\epsilon)$ on the root spaces of our toral roots. In particular, we will focus on the effect on the multiplicity.

Definition 3.3.1 Let \mathfrak{a} be a (θ, σ) -stable \mathbb{R} -split tori with \mathfrak{a}_σ^- maximal. For $\lambda \in \Phi(\mathfrak{a})$ we call $m(\lambda)$ the multiplicity of λ . We can find the multiplicity by counting the number of roots in the original maximal torus that project down to λ (ie. the $\alpha \in \mathfrak{t}$ such that $\frac{1}{2}(\alpha - \theta(\alpha)) = \lambda$).

For $\lambda \in \Phi(\mathfrak{a})$ let $g(\mathfrak{a}, \lambda)$ be the root space (sum of the root spaces projected down to λ). $\sigma\theta(\lambda) = \lambda$ and $\sigma\theta(g(\mathfrak{a}, \lambda)) = g(\mathfrak{a}, \lambda)$. Put

$$g(\mathfrak{a}, \lambda)_{\sigma\theta}^\pm = \{X \in g(\mathfrak{a}, \lambda) \mid \sigma\theta(X) = \pm X\}$$

$$m^\pm(\lambda, \sigma\theta) = \dim(g(\mathfrak{a}, \lambda)_{\sigma\theta}^\pm))$$

The signature of λ will be $(m^+(\lambda, \sigma\theta), m^-(\lambda, \sigma\theta))$

If a particular quadratic element, ϵ , acts on λ then the multiplicities will be flipped. $m^+(\lambda, \sigma\theta) = m^-(\lambda, \sigma\theta \text{Int}(\epsilon))$ and $m^-(\lambda, \sigma\theta), m^+(\lambda, \sigma\theta \text{Int}(\epsilon))$.

Chapter 4

Classification and Standard Tori

4.1 Standard Tori

Recall, the goal is to use the theorem of Helminck and Wang to classify $P_k \setminus G_k/H_k^+$ thereby classifying $P_k \setminus G_k/H_k$. Let $H^+ = (H \cap K)^0$ where H and K are the fixed point groups of σ and θ , respectively.

Corollary 4.1.1 ([18, Corollary 12.11]) *Let $\{A_i \mid i \in I\}$ be representatives of the H_k^+ -conjugacy classes of (θ, σ) -stable maximal k -split tori in G . Then*

$$P_k \setminus G_k/H_k^+ \cong \bigcup_{i \in I} W_{G_k}(A_i) \setminus W_{H_k^+}(A_i).$$

We will discuss the solution in the Lie algebra and then lift it to the Lie group. Let $\mathfrak{A}_k^{(\theta, \sigma)}$ be the (θ, σ) -stable maximal k -split tori of \mathfrak{g} . We will want the \mathfrak{h}_k^+ -conjugacy classes and we will denote this by $\mathfrak{A}_k^{(\theta, \sigma)} / \mathfrak{h}_k^+$.

On route to this classification, we must first find the \mathfrak{h} -conjugacy classes of (θ, σ) -stable maximal quasi k -split tori. We will denote this by $\mathfrak{A}^{(\theta, \sigma)} / \mathfrak{h}$.

Definition 4.1.2 A torus t of \mathfrak{g} is called a quasi k -split torus if t is \mathfrak{g} -conjugate with a k -split torus of \mathfrak{g} .

One last subclass of tori that we will consider are those quasi- k split tori that are \mathfrak{h} -conjugate with a k -split torus of \mathfrak{g} . We will denote this set by $\mathfrak{A}_0^{(\theta,\sigma)}$ and when considering these conjugacy classes, $\mathfrak{A}_0^{(\theta,\sigma)}/\mathfrak{h}$.

The tools to determine $\mathfrak{A}_0^{(\theta,\sigma)}/\mathfrak{h}$, $\mathfrak{A}^{(\theta,\sigma)}/\mathfrak{h}$, and $\mathfrak{A}_{\mathbb{R}}^{(\theta,\sigma)}/\mathfrak{h}_k^+$ are built in the same manner using standard tori and standard involutions. Several refinements and specifications in each case will lead to the individual classification. First, we have a natural map $\zeta :$

$$\zeta : \mathfrak{A}_k^{(\theta,\sigma)}/\mathfrak{h}_k^+ \rightarrow \mathfrak{A}^{(\theta,\sigma)}/\mathfrak{h}$$

sending each \mathfrak{h}_k^+ -conjugacy class of a σ -stable maximal k -split torus onto its \mathfrak{h} -conjugacy class. Its image, $\mathfrak{A}_0^{(\theta,\sigma)}$ consists of the \mathfrak{h} -conjugacy classes of σ -stable maximal quasi- k split tori that are \mathfrak{h} -conjugate to a σ -stable maximal k -split torus.

To classify $\mathfrak{A}_k^{(\theta,\sigma)}/\mathfrak{h}_k^+$ it suffices to classify the image and fibers of ζ . In the case of $k = \mathbb{R}$ this map is actually one-to-one. We start by discussing standard tori.

Definition 4.1.3 For $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{A}_{\mathbb{R}}^{(\theta,\sigma)}$ (or $\mathfrak{A}^{(\theta,\sigma)}$), the pair $(\mathfrak{a}_1, \mathfrak{a}_2)$ is called standard if $\mathfrak{a}_1^- \subset \mathfrak{a}_2^-$ and $\mathfrak{a}_1^+ \supset \mathfrak{a}_2^+$. In this case, we say that \mathfrak{a}_1 is standard with respect to \mathfrak{a}_2 .

As in the case of a single involution, σ , all the elements of $\mathfrak{A}_{\mathbb{R}}^{(\sigma,\theta)}$ and $\mathfrak{A}^{(\sigma,\theta)}$ can be put in standard position. We will omit the involution σ from the notation when there is no confusion. So we will write, \mathfrak{a}_1^- instead of $(\mathfrak{a}_1)_\sigma^-$ for the (-1)-portion of the involution σ on \mathfrak{a}_1 . A standard pair give rise to an involutions of the respective Weyl groups.

Theorem 4.1.4 (adapted from [9, Theorem 3.6]) Let $(\mathfrak{a}_1, \mathfrak{a}_2)$ be a standard pair of (θ, σ) -stable \mathbb{R} -split (or quasi \mathbb{R} -split) tori of \mathfrak{g} . Then we have the following conditions:

1. There exists $g \in Z_G(\mathfrak{a}_1^- \oplus \mathfrak{a}_2^+)$ such that $\text{Ad}_g(\mathfrak{a}_1) = \mathfrak{a}_2$.
2. If $n_1 = \text{Ad}_{(g)^{-1}\tilde{\sigma}(g)}$ and $n_2 = \text{Ad}_{\tilde{\sigma}(g)g^{-1}}$, then $n_1 \in N_G(\mathfrak{a}_1)$ and $n_2 \in N_G(\mathfrak{a}_2)$.
3. Let w_1 and w_2 be the images of n_1 and n_2 in $W(\mathfrak{a}_1)$ and $W(\mathfrak{a}_2)$ respectively. Then $w_1^2 = e, w_2^2 = e$, and $(\mathfrak{a}_1)_{w_1}^+ = (\mathfrak{a}_2)_{w_2}^+ = \mathfrak{a}_1^- \oplus \mathfrak{a}_2^+$ which characterizes w_1 and w_2 .

Proof (a) \mathfrak{a}_1 and \mathfrak{a}_2 are both maximal tori in the group $Z_G(\mathfrak{a}_1^- \oplus \mathfrak{a}_2^+)$. Therefore there exists a $g \in Z_G(\mathfrak{a}_1^- \oplus \mathfrak{a}_2^+)$ such that $\text{Ad}_g(\mathfrak{a}_1) = \mathfrak{a}_2$.

(b) $\mathfrak{a}_2 = \sigma(\mathfrak{a}_2) = \sigma(\text{Ad}_g(\mathfrak{a}_1)) = \text{Ad}_{\tilde{\sigma}(g)}(\sigma(\mathfrak{a}_1)) = \text{Ad}_{\sigma(\tilde{\sigma}(g))}(\mathfrak{a}_1)$ because \mathfrak{a}_1 and \mathfrak{a}_2 are σ -stable. Similarly, $\mathfrak{a}_1 = \text{Ad}_{g^{-1}}\mathfrak{a}_2$. By substitution, $\mathfrak{a}_1 = \text{Ad}_{g^{-1}}\text{Ad}_{\tilde{\sigma}(g)}\mathfrak{a}_1 = \text{Ad}_{g^{-1}\tilde{\sigma}(g)}\mathfrak{a}_1$. Therefore, by definition, $g^{-1}\tilde{\sigma}(g) \in N(\mathfrak{a}_1)$. The argument is similar for \mathfrak{a}_2 .

(c) Since a_2 is σ -stable, $Z_G(\mathfrak{a}_1^- \oplus \mathfrak{a}_2^+)$ is $\tilde{\sigma}$ -stable, hence $\tilde{\sigma}(g) \in Z_G(\mathfrak{a}_1^- \oplus \mathfrak{a}_2^+)$. $\mathfrak{a}_2 = \mathfrak{a}_1^- \oplus \mathfrak{a}_2^+ \oplus (\text{Ad}_g(\mathfrak{a}_1^+) \cap \mathfrak{a}_2^-)$. Let $x \in \mathfrak{a}_1^- \oplus \mathfrak{a}_2^+$. By definition an element $w_1 \in W$, $w_1 \cdot x = \text{Ad}_A(x)$, where A is the element from the corresponding equivalence class in the normalizer. So $w_1 \cdot x = \text{Ad}_{g^{-1}\tilde{\sigma}(g)}(x) = g^{-1}\tilde{\sigma}(g)x\tilde{\sigma}(g)^{-1}g = x$ since g and $\tilde{\sigma}(g) \in Z_G(\mathfrak{a}_1^- \oplus \mathfrak{a}_2^+)$.

Now let $x \in (\text{Ad}_g(\mathfrak{a}_1^+) \cap \mathfrak{a}_2^-)$. Then x can be written as $\text{Ad}_g(A)$ for some $A \in \mathfrak{a}_1^+$. Then $w_1 \cdot x = \text{Ad}_{g^{-1}\tilde{\sigma}(g)}(\text{Ad}_g(A)) = \text{Ad}_{g^{-1}\tilde{\sigma}(g)g}(A)$. Recall g and $\tilde{\sigma}(g)$ commute because they are both in $Z_G(\mathfrak{a}_1^- \oplus \mathfrak{a}_2^+)$. So $w_1 \cdot x = \text{Ad}_{g^{-1}g\tilde{\sigma}(g)}(A) = \text{Ad}_{\tilde{\sigma}(g)}(A) = \text{Ad}_{\tilde{\sigma}(g)}(A) = \sigma(\text{Ad}_g(A)) = \sigma(x) = -x$.

Hence $w_1 = \text{id}$ and $(\mathfrak{a}_2)_{w_1} = \mathfrak{a}_1^- \oplus \mathfrak{a}_2^+$. The argument is similar to show $w_2 = \text{id}$ and $(\mathfrak{a}_1)_{w_2} = \mathfrak{a}_1^- \oplus \mathfrak{a}_2^+$.

In essence, this involution w defines a one dimension piece of the (-1)-eigenspace that can be flipped to add a dimension to the (+1)-eigenspace of σ acting on the torus.

Definition 4.1.5 We will call w_1 (resp. w_2) the \mathfrak{a}_2 -standard involution (resp. \mathfrak{a}_1 -standard involution) of $W(\mathfrak{a}_1)$ (resp. $W(\mathfrak{a}_2)$).

Remark Note, w_1 and w_2 are independent of the choice of $g \in Z_G(\mathfrak{a}_1^- \oplus \mathfrak{a}_2^+)$ such that $\text{Ad}_g(\mathfrak{a}_1) = \mathfrak{a}_2$.

Let θ be a Cartan involution of \mathfrak{g} over k and σ a k -involution with $\sigma\theta = \theta\sigma$ and \mathfrak{h} the fixed point group of σ . Using portions of some propositions from [18] and [10]:

1. (Proposition 11.18) Given any σ -stable maximal k -split torus \mathfrak{a} of \mathfrak{g} , there is a $h \in \mathfrak{h}_k$ such that $\text{ad}(h)(\mathfrak{a})$ is θ -stable.
2. (Lemma 11.5) Any maximal θ -split k torus of \mathfrak{g} is maximal (θ, k) -split.
3. (Proposition 2.14) All maximal (σ, k) -split tori are \mathfrak{h}_k -conjugate.
4. (Proposition 11.3 & 11.4) Any θ -stable maximal k -split torus is θ -split.

Theorem 4.1.6 *There is only one \mathfrak{h}_k -conjugacy class of (σ, θ) -stable maximal (σ, \mathbb{R}) -split tori and one class of (σ, θ) -stable maximal \mathbb{R} -split, σ -fixed tori.*

Proof There is only one \mathfrak{h}_k -conjugacy class of (σ, θ) -stable maximal (σ, \mathbb{R}) -split tori follows from Proposition 2.14 because these tori are maximal in $\mathfrak{g}_{\sigma\theta}$ and therefore (σ, θ) -stable. The maximal σ -fixed, k -split torus is a maximal k -split torus of \mathfrak{h}_k and therefore they are all conjugate.

If we fix an element $\mathfrak{a} \in \mathfrak{A}_{\mathbb{R}}^{(\theta, \sigma)}$ such that \mathfrak{a}^- is a maximal (σ, \mathbb{R}) -split torus of \mathfrak{g} , then any torus in $\mathfrak{A}_{\mathbb{R}}^{(\sigma, \theta)}$ can be put into standard position with \mathfrak{a} . Note, $\mathfrak{A}_{\mathbb{R}}^{(\theta, \sigma)} \subset \mathfrak{A}^{(\theta, \sigma)}$ so $\mathfrak{a} \in \mathfrak{A}^{(\theta, \sigma)}$ and we can put any torus in $\mathfrak{A}^{(\theta, \sigma)}$ into standard position as well.

By Theorem 4.1.6 we know that we can always find a torus that is \mathfrak{h}_k -conjugate to \mathfrak{a} that fits into standard position with the given torus. Therefore, the above theorem can be modified as follows.

Corollary 4.1.7 Let \mathfrak{a}_1 be put in standard position with \mathfrak{a} where \mathfrak{a}^- is a maximal (σ, \mathbb{R}) -split torus of \mathfrak{g} . Then the following hold:

1. There exists $g \in Z_G(\mathfrak{a}_1^- \oplus \mathfrak{a}^+)$ such that $\text{Ad}_g(\mathfrak{a}_1) = \mathfrak{a}$.
2. If $n = \text{Ad}_{\sigma(g)g^{-1}}$, then $n \in N_G(\mathfrak{a})$.
3. Let w be the image of n in $W(\mathfrak{a})$. Then $w^2 = e$, and $(\mathfrak{a})_w^+ = \mathfrak{a}_1^- \oplus \mathfrak{a}^+$ which characterizes w .

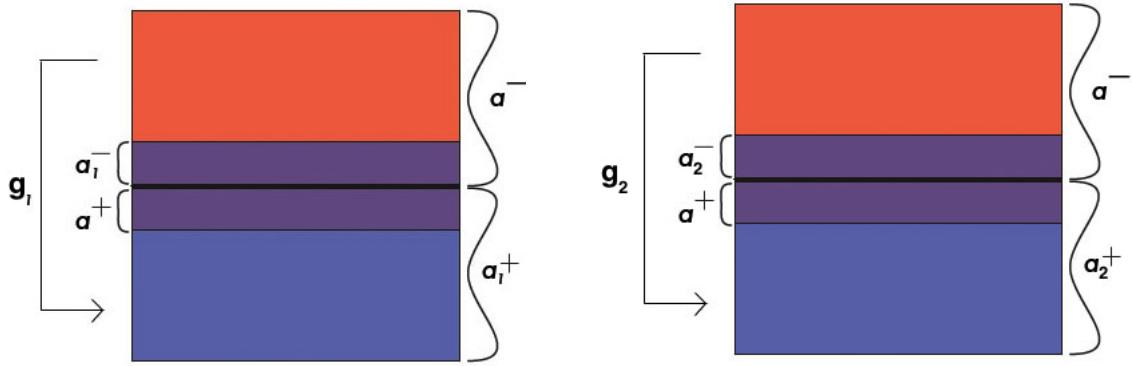


Figure 4.1: Action of g_1 and g_2 on the standard pair in Corollary 4.1.7

At this point, the important fact to note is that any tori $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{A}_k^{(\theta, \sigma)}$ (resp. $\mathfrak{A}^{(\theta, \sigma)}$) can be put into standard position with respect to \mathfrak{a} . Let w_1 and w_2 be the \mathfrak{a}_1 -standard and \mathfrak{a}_2 -standard involutions, respectively, in $W(\mathfrak{a})$. We can now discuss the tori based on these elements of the finite Weyl group.

Proposition 4.1.8 ([18, Proposition 12.6]) Assume that $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{A}_\mathbb{R}^{(\theta, \sigma)}$ are both standard with respect to \mathfrak{a} . Let w_1 and w_2 be the \mathfrak{a}_1 -standard and \mathfrak{a}_2 -standard involutions, respectively, in $W(\mathfrak{a})$. Then

\mathfrak{a}_1 and \mathfrak{a}_2 are $\mathfrak{h}_{\mathbb{R}}^+$ -conjugate if and only if w_1 and w_2 are conjugate under $W(\mathfrak{a}, \mathbb{R}_k^+)$

Proof \Rightarrow Assume $x \in \mathfrak{h}_k^+$ with $\text{Ad}_{\exp(x)}(\mathfrak{a}_1) = \mathfrak{a}_2$ then $\text{Ad}_{\exp(x)}(\mathfrak{a}_1^+) = \mathfrak{a}_2^+$ and $\text{Ad}_{\exp(x)}(\mathfrak{a}_1^-) = \mathfrak{a}_2^-$. Let $m = Z_{\mathfrak{g}}(\mathfrak{a}_2^-)$ then \mathfrak{a} and $\text{Ad}_{\exp(x)}(\mathfrak{a})$ are (σ, θ) -stable maximal k -split tori of m . By [18, Corollary 11.19] there exists a $y \in (m_\sigma \cap m_\theta)_k$ such that $\text{Ad}_{\exp(xy)}(\mathfrak{a}) = \mathfrak{a}$. The image w of xy in $W(\mathfrak{a}, \mathfrak{h}_k^+)$ satisfies $ww_1w^{-1} = w_2$.

\Leftarrow Assume $w \in W(\mathfrak{a}, \mathfrak{h}_k^+)$ and $ww_1w^{-1} = w_2$. Then $\mathfrak{a}_{w_1}^+ = \mathfrak{a}_1^- \oplus \mathfrak{a}^+$ and $\mathfrak{a}_{w_2}^+ = \mathfrak{a}_2^- \oplus \mathfrak{a}^+$ are σ -stable. $\mathfrak{a}_1^- = (\mathfrak{a}_{w_1}^+)^-$ and $\mathfrak{a}_2^- = (\mathfrak{a}_{w_2}^+)^-$, then $ww_1w^{-1} = w_2$ implies that $w(\mathfrak{a}_1) = w_1^{-1}w_2w_1((\mathfrak{a}_{w_1}^+)^-) = (\mathfrak{a}_{w_2}^+)^- = \mathfrak{a}_2^-$ because $\mathfrak{a}_1^+ \supset \mathfrak{a}_2^+$. Let x be the preimage of w in the normalizer. Then $\text{Ad}_{\exp(x)}(\mathfrak{a}_1^-) = \mathfrak{a}_2^-$ and \mathfrak{a}_1 and \mathfrak{a}_2 are \mathfrak{h}_k^+ -conjugate.

Corollary 4.1.9 *Assume that $\mathfrak{a}'_1, \mathfrak{a}'_2 \in \mathfrak{A}^{(\theta, \sigma)}$ are both standard with respect to \mathfrak{a} . Let w'_1 and w'_2 be the \mathfrak{a}'_1 -standard and \mathfrak{a}'_2 -standard involutions, respectively, in $W(\mathfrak{a})$. Then \mathfrak{a}'_1 and \mathfrak{a}'_2 are \mathfrak{h} -conjugate if and only if w_1 and w_2 are conjugate under $W(\mathfrak{a}, \mathfrak{h})$.*

We can now tell when elements in $\mathfrak{A}_{\mathbb{R}}^{(\theta, \sigma)}$ are $\mathfrak{h}_{\mathbb{R}}$ -conjugate using the finite set of involutions in $W(\mathfrak{a}, \mathfrak{h}_{\mathbb{R}}^+)$. The next step is to determine which involutions $w \in W(\mathfrak{a})$ are \mathfrak{a}_i -standard involutions for some $\mathfrak{a}_i \in \mathfrak{A}_{\mathbb{R}}^{(\theta, \sigma)}$ (resp. $\mathfrak{A}^{(\theta, \sigma)}$), what $\Phi(\mathfrak{a}, \mathfrak{h}_{\mathbb{R}}^+)$ (resp. $\Phi(\mathfrak{a}, \mathfrak{h})$) looks like, and finally determine the conjugacy classes of these involutions in $W(\mathfrak{a}, \mathfrak{h}_{\mathbb{R}}^+)$ (resp. $W(\mathfrak{a}, \mathfrak{h})$).

The \mathfrak{a}_i -standard involutions will classify the various families of tori that we are considering. We will be able to discuss the qualities of these involutions in more detail starting with the quasi \mathbb{R} -split case. First we need to discuss the conjugacy classes of elements in the Weyl group and the correspondence to the tori when those Weyl group elements are \mathfrak{a}_i -standard involutions.

4.2 Conjugacy classes in the Weyl group

The complete discussion of conjugacy classes of elements in the Weyl group can be found in [9]. The following is a summary of the classification. First, we want to determine the conjugacy classes of involutions $w \in W(\mathfrak{a})$ with $E(w, -1) \subset E(\sigma, -1)$ because \mathfrak{a}_σ^- is a maximal (σ, \mathbb{R}) -split torus and $\mathfrak{a}_{w_i}^- \subset \mathfrak{a}_\sigma^-$. These involutions determine subsets of a basis Δ_1 of Φ . We will show that we can restrict to looking at the conjugacy classes of these subsets. Let $\Phi = \Phi(\mathfrak{a})$ and $W = W(\mathfrak{a}) = W(\Phi(\mathfrak{a}))$.

Definition 4.2.1 *Let Δ by a basis of Φ .*

1. *Two subsets $\Delta_1, \Delta_2 \subset \Delta$ are called W -conjugate if there exist $w \in W$ such that $w(\Delta_1) = \Delta_2$.*
2. *An involution $w \in W$ is called Δ -standard if Δ is a $(-w)$ -basis of Φ (i.e. $E(w, -1) \cap \Delta = \Delta(w)$ is a basis for $\Phi(w)$)*

Proposition 4.2.2 *Let $\Delta \subset \Phi$ be a fixed basis and $w_1, w_2 \Delta$ -standard involutions in W . Then w_1, w_2 are W -conjugate if and only if $\Delta(w_1), \Delta(w_2)$ are W -conjugate.*

Definition 4.2.3 *Let Δ be a $(-\sigma)$ -basis of Φ and $w(\sigma)$ a fixed σ -maximal involution of $W(\sigma)$, which is Δ -standard. An involution $w \in W$ is called $(\Delta, w(\sigma))$ -standard if w is Δ -standard and $\Delta(w) \subset \Delta(w(\sigma))$.*

Corollary 4.2.4 *Let $\sigma \in \text{Aut}(\Phi)$ be an involution, Δ a $(-\sigma)$ -basis of Φ , $w(\sigma) \in W(\sigma)$ a σ -maximal involution, which is Δ -standard. Then we have the following:*

If $w_1, w_2 \in W$ are $(\Delta, w(\sigma))$ -standard involutions, then w_1, w_2 are W -conjugate if and only if $\Delta(w_1)$ and $\Delta(w_2)$ are W -conjugate.

Next, we must classify all the conjugacy classes of involutions in W . If we then concentrate on the case where σ is split, then classification of conjugacy classes of involutions $w \in W$ with $E(w, -1) \subset E(\sigma, -1)$ reduce to looking at the subsets of $\Phi(w(\sigma))$, which are involutions in W . Where $w(\sigma)$ is a σ -maximal involution.

Lemma 4.2.5 *Let Φ be irreducible and $w \in W$ an involution. Then $\Phi(w)$ is of type $r \cdot A_1 + X_\ell$, where either $X_\ell = \emptyset$ or one of $B_\ell (\ell \geq 1)$, $C_\ell (\ell \geq 1)$, $D_\ell (\ell \geq 1)$, E_7 , E_8 , F_4 , or G_2 , where $r \cdot A_1 = A_1 + A_1 + \cdots + A_1$ r times.*

The conjugacy is based on looking at the orthogonal complements of the basis, $\Delta(w)$ and induction, see [9]. In a majority of classes, the type of $\Phi(w)$ determines the conjugacy class. What is most useful is the interpretation of these classes as a diagram.

Let \mathcal{W} be the set of all W -conjugacy classes of involutions in W . If we define an order $>$ on \mathcal{W} then for $[w_1], [w_2] \in \mathcal{W}$ we have $[w_1] > [w_2]$ if and only if $\Delta(w_1) \subset \Delta(w_2)$ for some representatives w_i of $[w_i] (i = 1, 2)$. We will call these diagrams $\mathcal{L}(\Phi(w))$. Table 4.1 has the diagrams of the conjugacy classes in each type of $w \in W$.

Example Consider $\Phi(w)$ of type B_3 . Then the diagram and the labelled conjugacy classes in $W(\Phi(w))$ are as follows:

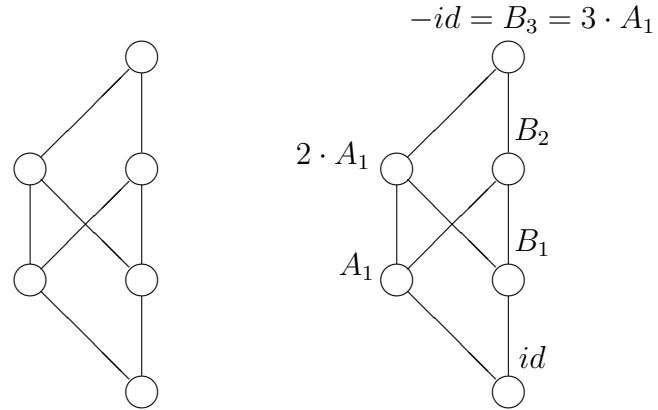


Figure 4.2: W -conjugacy classes for w of Type B_3

Table 4.1: Diagrams of W Conjugacy Classes

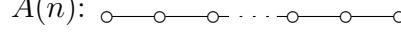
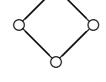
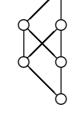
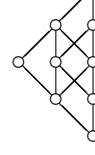
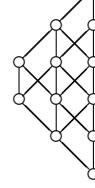
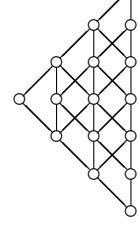
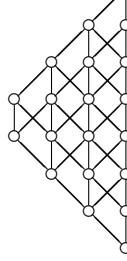
A_ℓ $(l \geq 1)$	$\mathcal{L}(A_\ell) = \mathcal{L}^*(A_\ell) = A(n)$ with $n = [\frac{\ell+1}{2}]$ $A(n):$ 
B_ℓ $(\ell \geq 2)$	$\mathcal{L}(B_\ell) = \mathcal{L}^*(B_\ell) = B(\ell)$ $B(2):$  $B(3):$  $B(4):$  $B(5):$  $B(6):$  $B(7):$  etc.
C_ℓ $(l \geq 2)$	$\mathcal{L}(C_\ell) = \mathcal{L}^*(C_\ell) = B(\ell)$

Table 4.1 – Continued

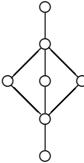
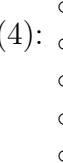
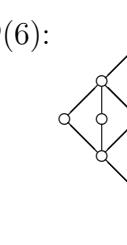
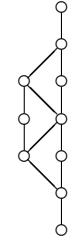
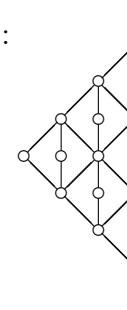
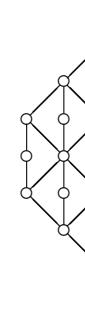
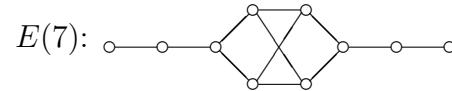
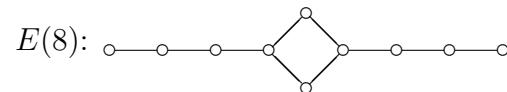
BC_ℓ	$\mathcal{L}(BC_\ell) = \mathcal{L}^*(BC_\ell) = B(\ell)$						
$(\ell \geq 2)$							
$D_{2\ell}$	$\mathcal{L}(D_{2\ell}) = D(2\ell) \quad \mathcal{L}^*(D_{2\ell}) = D^*(2\ell)$						
$(l \geq 2)$							
$D(4):$		$D^*(4):$		$D(6):$		$D^*(6):$	
						$D(8):$	
						$D^*(8):$	
							etc.
$D_{2\ell+1}$	$\mathcal{L}(D_{2\ell+1}) = \mathcal{L}^*(D_{2\ell+1}) = D^*(2\ell)$						
$(\ell \geq 2)$							
E_6	$\mathcal{L}(E_6) = \mathcal{L}^*(E_6) = A(4)$						
E_7	$\mathcal{L}(E_7) = \mathcal{L}^*(E_7) = E(7)$						

Table 4.1 – Continued



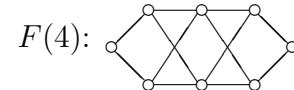
E_8

$$\mathcal{L}(E_8) = \mathcal{L}^*(E_8) = E(8)$$



F_4

$$\mathcal{L}(F_4) = \mathcal{L}^*(F_4) = F(4)$$



G_2

$$\mathcal{L}(G_2) = \mathcal{L}^*(G_2) = B(2)$$

Once we identify the \mathfrak{a}_i -standard involutions in $W(\mathfrak{a})$ we can utilize the diagram to describe the conjugacy classes of tori. In our case, if $w_1, w_2 \in W(\mathfrak{a})$ are a_1 and a_2 -standard involutions of \mathfrak{a}_1 and \mathfrak{a}_2 then

$$\mathfrak{a}_1^- \subset \mathfrak{a}_2^- \iff \mathfrak{a}_{w_1}^- \supset \mathfrak{a}_{w_2}^-$$

Hence

$$[\mathfrak{a}_1] < [\mathfrak{a}_2] \iff [w_1] < [w_2]$$

Finally, we need to determine what the \mathfrak{a}_i standard involutions are going to be in each case. We will start with the quasi \mathbb{R} -split case.

4.3 Classification of quasi \mathbb{R} -split tori

Again before we tackle the larger problem, we must consider when two tori in $\mathfrak{A}^{(\theta, \sigma)}$ are \mathfrak{h} -conjugate based on finding the appropriate Weyl group elements in $W(\mathfrak{a})$ with representatives in \mathfrak{h} . This topic was introduced in [9].

Remark A k -involution τ of \mathfrak{m} is called k -split if there exists a τ -split maximal k -split torus of \mathfrak{m} . Let $\mathfrak{a} \in \mathfrak{A}_k^{(\sigma, \theta)}$, $w \in W(\mathfrak{a})$ where $w^2 = e$ and $w\sigma = \sigma w$ and n is the pre-image of $w \in N_{\mathfrak{g}}(\mathfrak{a})$. Set $\mathfrak{g}_w = Z(\mathfrak{a}_w^+)$ then $n \in Z(\mathfrak{a}_w^+)$ and $\mathfrak{a}_w^- \cap Z(\mathfrak{g}_w)$ is finite and \mathfrak{a}_w^- is a (θ, σ) -stable maximal k -split torus of $[\mathfrak{g}_w, \mathfrak{g}_w]$.

Definition 4.3.1 Let $\mathfrak{a} \in \mathfrak{A}^{(\sigma, \theta)}$, $w \in W(\mathfrak{a})$ and $\mathfrak{g}_w = Z(\mathfrak{a}_w^+)$. Then w is called σ -singular if

$$1. \quad w^2 = e$$

$$2. \quad \sigma w = w\sigma$$

3. $\sigma|[\mathfrak{g}_w, \mathfrak{g}_w]$ is k -split.
4. $[\mathfrak{g}_w, \mathfrak{g}_w] \cap \mathfrak{h}$ contains a maximal quasi k -split torus of $[\mathfrak{g}_w, \mathfrak{g}_w]$.

A root $\alpha \in \Phi(\mathfrak{a})$ is called σ -singular if the corresponding reflection $s_\alpha \in W(\mathfrak{a})$ is σ -singular. A root $\alpha \in \Phi(\mathfrak{a})$ with $\sigma(\alpha) = \pm\alpha$ is called σ -singular if $[\mathfrak{g}_{s_\alpha}, \mathfrak{g}_{s_\alpha}] \not\subset \mathfrak{h}$. These two are equivalent.

We mentioned the root definition of σ -singular roots in two contexts because we have much information from the involution diagram about the roots of the various tori. Alas, these singular involutions will help to classify the \mathfrak{h} -conjugacy classes of quasi \mathbb{R} -split tori.

Proposition 4.3.2 (modified from [11, Proposition 8.6]) *Let $\mathfrak{a} \in \mathfrak{A}_k^{\theta, \sigma}$ with \mathfrak{a}_σ^- maximal. Then there is a one to one correspondence between the $W(\mathfrak{a}, \mathfrak{h})$ -conjugacy classes of a_i -standard involutions in $W(\mathfrak{a})$ and the $W(\mathfrak{a}, \mathfrak{h})$ -conjugacy classes of σ -singular involutions in $W(\mathfrak{a})$*

Using Proposition 4.3.2 and Table 4.1, all we require in the classification is $W(\mathfrak{a})$ and the σ -singular involutions. In each of the 171 cases, we can determine $\Phi(\mathfrak{a})$ from the diagram of the Cartan involution, θ , when projected to the (-1)-eigenspace. This process was described in the \mathbb{R} -involutions section. Once we have $\Phi(\mathfrak{a}) = \Phi_\theta$ we must find the σ -singular roots.

Lemma 4.3.3 (modified from [9, Theorem 4.6]) *Let \mathfrak{a} be a (θ, σ) -stable \mathbb{R} -split torus of \mathfrak{g} with \mathfrak{a}_σ^- a maximal (σ, \mathbb{R}) -split torus of \mathfrak{g} and $w \in W(\mathfrak{a})$, $w^2 = e$. Then the following are equivalent:*

1. w is σ -singular

$$2. \quad \mathfrak{a}_w^- \subset \mathfrak{a}_\sigma^-$$

Proposition 4.3.4 $\alpha \in \Phi(\mathfrak{a})$ is a σ -singular root if and only if $\alpha \in \Phi(\mathfrak{a}) \cap \Phi(\mathfrak{a}_\sigma^-)$.

Proof (\implies) α is a σ -singular root then by Lemma 4.3.3 $\mathfrak{a}_{s_\alpha}^- \subset \mathfrak{a}_\sigma^-$. Therefore, $\alpha \in \Phi(\mathfrak{a}_\sigma^-)$.

Since $\alpha \in \Phi(\mathfrak{a})$ then $\alpha \in \Phi(\mathfrak{a}) \cap \Phi(\mathfrak{a}_\sigma^-)$.

(\Leftarrow) $\alpha \in \Phi(\mathfrak{a}) \cap \Phi(\mathfrak{a}_\sigma^-)$, then $\alpha \in \Phi(\mathfrak{a})$ and $w = s_\alpha$ is a reflection $W(\mathfrak{a})$ so $w^2 = e$.

Since $\alpha \in \Phi(\mathfrak{a}_\sigma^-)$, $\mathfrak{a}_{s_\alpha}^- \subset \mathfrak{a}_\sigma^-$. By Lemma 4.3.3, s_α is σ -singular and α is a σ -singular root.

Theorem 4.3.5 Let $\mathfrak{a} \in \mathfrak{A}_k^{(\theta, \sigma)}$ with \mathfrak{a}_σ^- maximal. Then there is a one to one correspondence between the $W(\mathfrak{a})$ -conjugacy classes of σ -singular involutions in $W(\mathfrak{a})$ and the $W(\mathfrak{a})$ -conjugacy classes of elements in $W(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ where $W(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ is the Weyl group of $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-) = \Phi(\mathfrak{a}) \cap \Phi(\mathfrak{a}_\sigma^-)$.

So for each commuting pair (θ, σ) we can determine $\Phi(\mathfrak{a})$ and $W(\mathfrak{a})$ from the diagram.

If we then restrict σ to $\Phi(\mathfrak{a})$, we can determine the roots of the maximal σ -split torus and the maximal σ -split torus in $\Phi(\mathfrak{a})$. Combined with Table 4.1 we can find the conjugacy classes of elements in $W(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ inside $W(\mathfrak{a})$. Determining the maximum involution in $W(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ tells us where in the diagram of $W(\mathfrak{a})$ to gain the structure of the classes.

Example The following occurs in only four of the 171 cases.

Type (θ, σ)	Type Φ_θ $\Phi(\mathfrak{a})$	Type $\Phi_{\sigma, \theta} \cap \Phi_\theta$ $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-)$	max. involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$
$A_{2\ell+1}^{2\ell+1, \ell}$ (I, II)	$A_{2\ell+1}$	\emptyset	id

In this case, $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-) = \emptyset$ and $W(\mathfrak{a}, \mathfrak{a}_\sigma^-) = \text{id}$. There is only one $W(\mathfrak{a})$ -conjugacy class of σ -singular roots; therefore, there is only one \mathfrak{h} -conjugacy class (σ, θ) -stable maximal quasi \mathbb{R} -split tori.

Example The follow example occurs in a similar manner in about 26 of the 171 cases.

Type (θ, σ)	Type Φ_θ	Type $\Phi_{\sigma, \theta} \cap \Phi_\theta$	max. involution
$\Phi(\mathfrak{a})$	$\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-)$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	
$A_{4\ell-1}^{2\ell, 2\ell-1}(\text{III}_b, \text{II}, \epsilon_0)$	$C_{2\ell}$	$\ell \cdot A_1$	$\ell \cdot A_1$

In this case, $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-) = \ell \cdot A_1$ and $\Phi(\mathfrak{a}) = C_{2\ell}$. If we consider the case when $\ell = 2$, then for the pair $A_7^{4,3}(\text{III}_b, \text{II}, \epsilon_0)$ $\Phi(\mathfrak{a}) = C_4$ and $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-) = 2 \cdot A_1$. Below are the W -conjugacy classes of $W(\mathfrak{a})$ and then the $W(\mathfrak{a})$ -conjugacy classes of $W(\mathfrak{a}, \mathfrak{a}_\sigma^-)$.

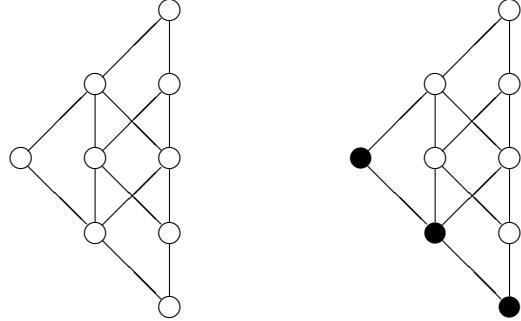


Figure 4.3: Conjugacy classes for Example 4.3

There is only one $W(\mathfrak{a})$ -conjugacy class of σ -singular roots at each dimension; therefore, there is only one \mathfrak{h} -conjugacy class (σ, θ) -stable maximal quasi \mathbb{R} -split tori for each

dimension. We will see that in the $\mathfrak{h}_{\mathbb{R}}$ -conjugacy classes of maximal \mathbb{R} -split tori often these classes can split. Also, we see that only two one-dimensional pieces of the maximal (σ, \mathbb{R}) -split torus is going to flip. We quickly learn that the torus inside the maximal \mathbb{R} -split torus with maximal portion in \mathfrak{h} still has a σ -split portion because not all parts will flip from the (-1) to the $(+1)$ -eigenspace of σ .

Example The remaining of the 171 pairs are similar to the following example.

Type (θ, σ)	Type Φ_θ	Type $\Phi_{\sigma, \theta} \cap \Phi_\theta$	max. involution
$\Phi(\mathfrak{a})$	$\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-)$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	
$B_\ell^{q,p}(I_a, I_a, \epsilon_i)$	B_q	B_p	B_p
$1 \leq p < q \leq \ell$			
$0 \leq i \leq p$			

In this case, $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-) = B_p$ and $\Phi(\mathfrak{a}) = B_q$. If we consider the case when $\ell = 5$, then for the pair $B_5^{4,3}(I_a, I_a, \epsilon_i)$, $\Phi(\mathfrak{a}) = B_4$ and $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-) = B_3$. Below are the conjugacy classes of $W(\mathfrak{a})$ and then the $W(\mathfrak{a})$ -conjugacy classes of $W(\mathfrak{a}, \mathfrak{a}_\sigma^-)$.

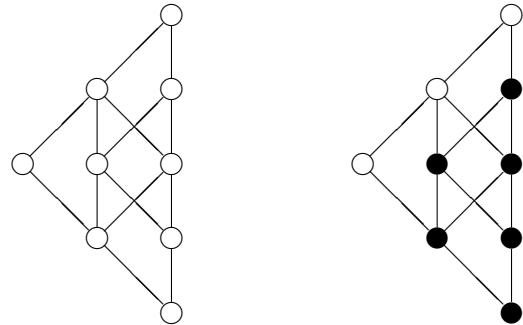


Figure 4.4: Conjugacy classes for Example 4.3

We can see the $W(\mathfrak{a})$ -conjugacy classes of σ -singular roots at each dimension from the diagram. Also, we can count the \mathfrak{h} -conjugacy classes (σ, θ) -stable maximal quasi \mathbb{R} -split tori for each dimension.

The following table consists of the 171 irreducible types to consider, diagrams, and roots types of all the players in the classification of (θ, σ) -stable maximal quasi \mathbb{R} -split tori with representatives in \mathfrak{h} .

Table 4.2: Conjugacy classes of Quasi \mathbb{R} -split tori in \mathfrak{h}

Type (θ, σ)	Diagram (θ, σ)	Type Φ_θ $\Phi(A)$	Diagram $\sigma \Phi_\theta$	Type $\Phi_{\sigma, \theta} \cap \Phi_\theta$ $\Phi(A, A_\sigma^-)$	max. involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$
$A_{2\ell+1}^{2\ell+1, \ell}(\text{I}, \text{II})$		$A_{2\ell+1}$		\emptyset	id
$A_{4\ell-1}^{2\ell-1, 4\ell-1}(\text{II}, \text{I})$		$A_{2\ell-1}$		$A_{2\ell-1}$	$\ell \cdot A_1$
$A_{4\ell+1}^{2\ell, 4\ell+1}(\text{II}, \text{I})$		$A_{2\ell}$		$A_{2\ell}$	$\ell \cdot A_1$
$A_{2\ell-1}^{2\ell-1, \ell}(\text{I}, \text{III}_b, \epsilon_0)$		$A_{2\ell-1}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$A_{4\ell-1}^{2\ell, 4\ell-1}(\text{III}_b, \text{I}, \epsilon_0)$		$C_{2\ell}$		$C_{2\ell}$	$C_{2\ell}$

Table 4.2 – Continued

$A_{4\ell+1}^{2\ell+1, 4\ell+1}(\text{III}_b, \text{I}, \epsilon_0)$		$C_{2\ell+1}$		$C_{2\ell+1}$	$C_{2\ell+1}$
$A_{4\ell-1}^{2\ell-1, 2\ell}(\text{II}, \text{III}_b, \epsilon_\ell)$		$A_{2\ell-1}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$A_{4\ell+1}^{2\ell, 2\ell+1}(\text{II}, \text{III}_b)$		$A_{2\ell}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$A_{4\ell-1}^{2\ell, 2\ell-1}(\text{III}_b, \text{II}, \epsilon_\ell)$		$C_{2\ell}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$A_{4\ell+1}^{2\ell+1, 2\ell}(\text{III}_b, \text{II})$		$C_{2\ell+1}$		$\ell \cdot A_1$	$\ell \cdot A_1$

Table 4.2 – Continued

$A_{2\ell-1}^{2\ell-1,\ell}(\text{I}, \text{III}_b, \epsilon_\ell)$		$A_{2\ell-1}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$A_{2\ell-1}^{\ell,2\ell-1}(\text{III}_b, \text{I}, \epsilon_\ell)$		C_ℓ		C_ℓ	C_ℓ
$A_{2\ell-1}^{2\ell-1}(\text{I}, \epsilon_\ell)$		$A_{2\ell-1}$		$A_{2\ell-1}$	$\ell \cdot A_1$
$A_{4\ell-1}^{2\ell-1,2\ell}(\text{II}, \text{III}_b, \epsilon_0)$		$A_{2\ell-1}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$A_{4\ell-1}^{2\ell,2\ell-1}(\text{III}_b, \text{II}, \epsilon_0)$		$C_{2\ell}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$A_{4\ell-1}^{2\ell-1}(\text{II}, \epsilon_\ell)$		$A_{2\ell-1}$		$A_{2\ell-1}$	$\ell \cdot A_1$

Table 4.2 – Continued

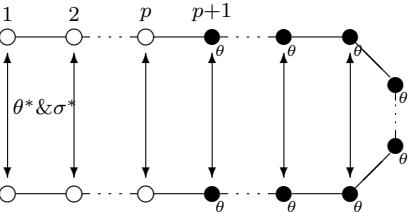
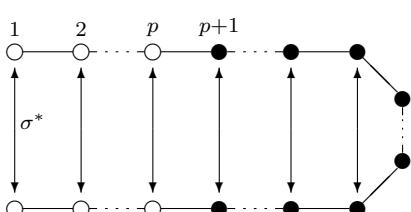
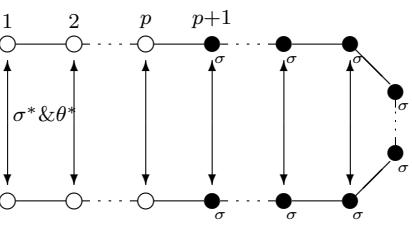
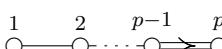
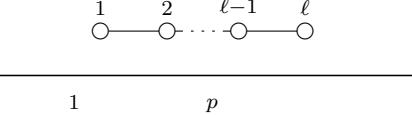
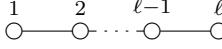
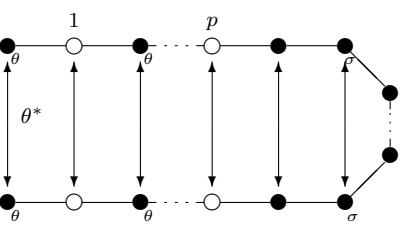
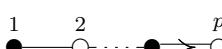
$A_\ell^{\ell,p}(\text{I}, \text{III}_a)$ $1 \leq 2p \leq \ell$		A_ℓ		$p \cdot A_1$	$p \cdot A_1$
$A_\ell^{p,\ell}(\text{III}_a, \text{I})$ $1 \leq 2p \leq \ell$		BC_p		BC_p	BC_p
$A_\ell^\ell(\text{I}, \epsilon_p)$ $1 \leq 2p \leq \ell$		A_ℓ		A_ℓ	$\lfloor \frac{\ell+1}{2} \rfloor \cdot A_1$
$A_{4\ell-1}^{2p,2\ell-1}(\text{III}_a, \text{II})$ $1 \leq 4p \leq 4\ell$		BC_{2p}		\emptyset	identity

Table 4.2 – Continued

$A_{4\ell+1}^{2p,2\ell}(\text{III}_a, \text{II})$ $1 \leq 4p \leq 4\ell$		BC_{2p}		\emptyset	identity
$A_{4\ell-1}^{2\ell-1,2p}(\text{II}, \text{III}_a)$ $1 \leq 4p \leq 4\ell + 2$		$A_{2\ell-1}$		$p \cdot A_1$	$p \cdot A_1$
$A_{4\ell+1}^{2\ell,2p}(\text{II}, \text{III}_a)$ $1 \leq 4p \leq 4\ell + 2$		$A_{2\ell}$		$p \cdot A_1$	$p \cdot A_1$
$A_{2\ell+1}^\ell(\text{II}, \epsilon_p)$ $1 \leq 2p \leq \ell + 1$		A_ℓ		A_ℓ	$\lfloor \frac{\ell+1}{2} \rfloor \cdot A_1$

Table 4.2 – Continued

$A_{2\ell-1}^{\ell,\ell}(III_b, \epsilon_\ell)$		C_ℓ		C_ℓ	C_ℓ
$A_\ell^{p,p}(III_a, \epsilon_p)$ $1 \leq p < \ell$ $0 \leq i \leq p-1$		BC_p		BC_p	BC_p
$A_{2\ell-1}^{\ell,p}(III_b, III_a, \epsilon_i)$ $1 \leq p < \ell$ $0 \leq i \leq p-1$		C_ℓ		C_p	C_p
$A_{2\ell-1}^{p,\ell}(III_a, III_b, \epsilon_i)$ $1 \leq p < \ell$ $0 \leq i \leq p-1$		BC_p		BC_p	BC_p

Table 4.2 – Continued

$A_\ell^{q,p}(\text{III}_a, \text{III}_a, \epsilon_i)$ $1 \leq p < q \leq \frac{1}{2}(\ell + 1)$ $0 \leq i \leq p$		BC_q 	BC_p
$A_\ell^{p,q}(\text{III}_a, \text{III}_a, \epsilon_i)$ $1 \leq p < q \leq \frac{1}{2}(\ell + 1)$ $0 \leq i \leq p$		BC_p 	BC_p
$B_\ell^{p,p}(\text{I}_a, \epsilon_i)$		B_p 	B_p
$B_\ell^{q,p}(\text{I}_a, \text{I}_a, \epsilon_i)$ $1 \leq p < q \leq \ell$ $0 \leq i \leq p$		B_q 	B_p
$B_\ell^{p,q}(\text{I}_a, \text{I}_a, \epsilon_i)$		B_p 	B_p

Table 4.2 – Continued

$C_\ell^{\ell,p}(\text{I}, \text{II}_a)$ ($2p \leq \ell$)		C_ℓ		$p \cdot A_1$	$p \cdot A_p$
$C_\ell^{p,\ell}(\text{II}_a, \text{I})$ ($2p \leq \ell$)		BC_p		BC_p	BC_p
$C_\ell^\ell(\text{I}, \epsilon_p)$ ($2p \leq \ell$)		C_ℓ		C_ℓ	C_ℓ
$C_{2\ell}^{2\ell,\ell}(\text{I}, \text{II}_b, \epsilon_0)$		$C_{2\ell}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$C_{2\ell}^{\ell,2\ell}(\text{II}_b, \text{I}, \epsilon_0)$		C_ℓ		C_ℓ	C_ℓ
$C_{2\ell}^{\ell,\ell}(\text{II}_b, \epsilon_\ell)$		C_ℓ		C_ℓ	C_ℓ
$C_{2\ell}^{2\ell,\ell}(\text{I}, \text{II}_b, \epsilon_\ell)$		$C_{2\ell}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$C_{2\ell}^{\ell,2\ell}(\text{II}_b, \text{I}, \epsilon_\ell)$		C_ℓ		C_ℓ	C_ℓ
$C_{2\ell}^{2\ell,2\ell}(\text{I}, \epsilon_\ell)$		$C_{2\ell}$		$C_{2\ell}$	$C_{2\ell}$

Table 4.2 – Continued

$C_\ell^{p,p}(\Pi_a, \epsilon_p)$		BC_p		BC_p	BC_p
$C_\ell^{q,p}(\Pi_a, \Pi_a, \epsilon_i)$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$		BC_q		BC_p	BC_p
$C_\ell^{p,q}(\Pi_a, \Pi_a, \epsilon_i)$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$		BC_p		BC_p	BC_p
$C_{2\ell}^{\ell,p}(\Pi_b, \Pi_a, \epsilon_i)$ $0 \leq i \leq p-1$ $1 \leq p \leq l$		C_ℓ		C_p	C_p
$C_{2\ell}^{p,\ell}(\Pi_a, \Pi_b, \epsilon_i)$ $0 \leq i \leq p-1$ $1 \leq p \leq l$		BC_p		BC_p	BC_p
$D_\ell^{q,p}(\Pi_a, \Pi_a, \epsilon_i)$ $1 \leq p < q \leq \ell-1$ $(0 \leq i \leq p-1)$		B_q		B_p	B_p

Table 4.2 – Continued

$D_\ell^{p,q}(\text{I}_a, \text{I}_a, \epsilon_i)$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p - 1)$		B_p		B_p	B_p
$D_\ell^{\ell,p}(\text{I}_b, \text{I}_a, \epsilon_i)$ $0 \leq i \leq p - 1$ $1 \leq p < \ell$		D_ℓ		D_p	D_p
$D_\ell^{p,\ell}(\text{I}_a, \text{I}_b, \epsilon_i)$ $0 \leq i \leq p - 1$ $0 \leq p < \ell$		B_p		B_p	B_p
$D_{2\ell}^{\ell,4p}(\text{III}_a, \text{I}_a)$ $(1 \leq 2p \leq \ell - 1)$		C_ℓ		C_{2p}	C_{2p}
$D_{2\ell}^{2p,\ell}(\text{I}_a, \text{III}_a)$ $(1 \leq 2p \leq 2\ell - 1)$		B_{2p}		$p \cdot A_1$	$p \cdot A_1$
$D_{2\ell}^{\ell,\ell}(\text{III}_a, \epsilon_p)$ $(1 \leq 2p \leq \ell - 1)$		C_ℓ		C_ℓ	C_ℓ

Table 4.2 – Continued

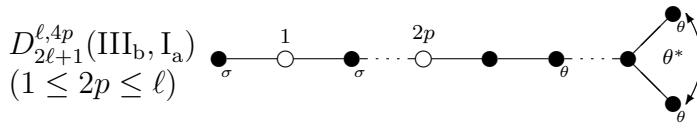
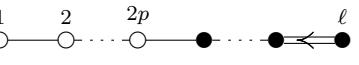
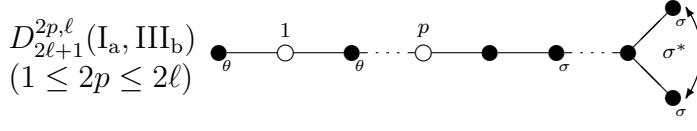
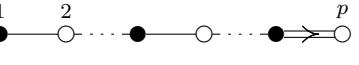
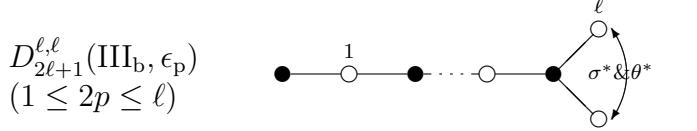
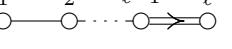
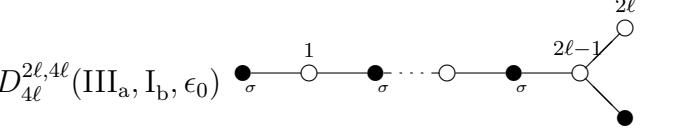
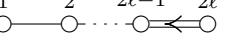
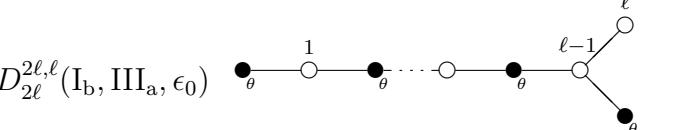
$D_{2\ell+1}^{\ell,4p}(\text{III}_b, \text{I}_a)$ $(1 \leq 2p \leq \ell)$		BC_ℓ		BC_{2p}	BC_{2p}
$D_{2\ell+1}^{2p,\ell}(\text{I}_a, \text{III}_b)$ $(1 \leq 2p \leq 2\ell)$		B_{2p}		$p \cdot A_1$	$p \cdot A_1$
$D_{2\ell+1}^{\ell,\ell}(\text{III}_b, \epsilon_p)$ $(1 \leq 2p \leq \ell)$		BC_ℓ		BC_ℓ	BC_ℓ
$D_{4\ell}^{2\ell,4\ell}(\text{III}_a, \text{I}_b, \epsilon_0)$		$C_{2\ell}$		$C_{2\ell}$	$C_{2\ell}$
$D_{2\ell}^{2\ell,\ell}(\text{I}_b, \text{III}_a, \epsilon_0)$		$D_{2\ell}$		$\ell \cdot A_1$	$\ell \cdot A_1$

Table 4.2 – Continued

$D_{4\ell}^{2\ell,2\ell}(\text{III}_a, \epsilon_\ell)$		$C_{2\ell}$		$C_{2\ell}$	$C_{2\ell}$
$D_{2\ell}^{\ell,2\ell}(\text{III}_a, \text{I}_b, \epsilon_\ell)$		C_ℓ		C_ℓ	C_ℓ
$D_{2\ell}^{2\ell,\ell}(\text{I}_b, \text{III}_a, \epsilon_\ell)$		$D_{2\ell}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$D_{2\ell}^{2\ell,2\ell}(\text{I}_b, \epsilon_\ell)$		$D_{2\ell}$		$D_{2\ell}$	$D_{2\ell}$
$D_{2\ell+1}^{\ell,2\ell+1}(\text{III}_b, \text{I}_b)$		BC_ℓ		BC_ℓ	BC_ℓ
$D_{2\ell+1}^{2\ell+1,\ell}(\text{I}_b, \text{III}_b)$		$D_{2\ell+1}$		$\ell \cdot A_1$	$\ell \cdot A_1$

Table 4.2 – Continued

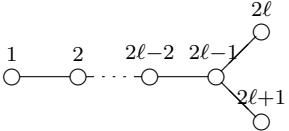
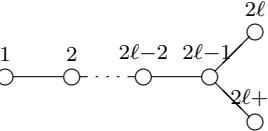
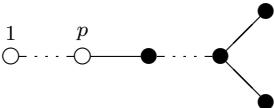
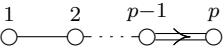
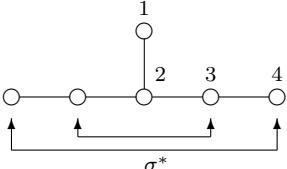
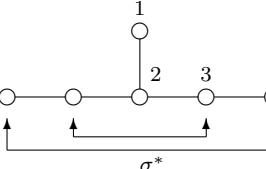
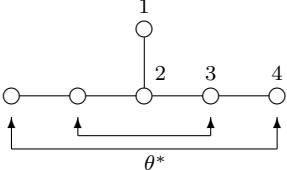
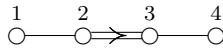
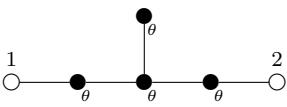
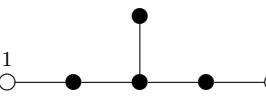
$D_{2\ell+1}^{2\ell+1,2\ell+1}(I_b, \epsilon_\ell)$		D_{2l+1}		$D_{2\ell+1}$	$D_{2\ell+1}$
$D_\ell^{p,p}(I_a, \epsilon_p)$		B_p		B_p	B_p
$E_6^{6,4}(I, II, \epsilon_0)$		E_6		D_4	D_4
$E_6^{4,6}(II, I, \epsilon_0)$		F_4		F_4	F_4
$E_6^{6,2}(I, IV)$		E_6		\emptyset	id

Table 4.2 – Continued

$E_6^{2,6}(\text{IV}, \text{I})$		A_2		A_2	A_1
$E_6^{4,2}(\text{II}, \text{IV})$		F_4		A_1	A_1
$E_6^{2,4}(\text{IV}, \text{II})$		A_2		A_1	A_1
$E_6^{6,4}(\text{I}, \text{II}, \epsilon_1)$		E_6		D_4	D_4
$E_6^{4,6}(\text{II}, \text{I}, \epsilon_1)$		F_4		F_4	F_4

Table 4.2 – Continued

$E_6^{6,6}(I, \epsilon_i)$		E_6		E_6	D_4
$E_6^{6,2}(I, III)$		E_6		$2 \cdot A_1$	$2 \cdot A_1$
$E_6^{2,6}(III, I)$		BC_2		BC_2	BC_2
$E_6^{4,2}(II, III, \epsilon_0)$		F_4		$2 \cdot A_1$	$2 \cdot A_1$
$E_6^{2,4}(III, II, \epsilon_0)$		BC_2		BC_2	BC_2

Table 4.2 – Continued

$E_6^{2,2}(\text{III}, \epsilon_i)$		BC_2		BC_2	BC_2
$E_6^{4,2}(\text{II}, \text{III}, \epsilon_1)$		F_4		C_2	C_2
$E_6^{2,4}(\text{III}, \text{II}, \epsilon_i)$		BC_2		B_2	B_2
$E_6^{4,4}(\text{II}, \epsilon_i)$		F_4		F_4	F_4

Table 4.2 – Continued

$E_6^{2,2}(\text{III}, \text{IV})$		BC_2		A_1	A_1
$E_6^{2,2}(\text{IV}, \text{III})$		A_2		A_1	A_1
$E_6^{2,2}(\text{IV}, \epsilon_1)$		A_2		A_2	A_1
$E_7^{7,4}(\text{V}, \text{VI}, \epsilon_0)$		E_7		F_4	F_4
$E_7^{4,7}(\text{VI}, \text{V}, \epsilon_0)$		F_4		F_4	F_4
$E_7^{7,3}(\text{V}, \text{VII})$		E_7		C_3	C_3

Table 4.2 – Continued

$E_7^{3,7}(\text{VII}, \text{V})$		C_3		C_3	C_3
$E_7^{4,3}(\text{VI}, \text{VII}, \epsilon_1)$		F_4		C_2	C_2
$E_7^{3,4}(\text{VII}, \text{VI}, \epsilon_1)$		C_3		C_2	C_2
$E_7^{7,4}(\text{V}, \text{VI}, \epsilon_1)$		E_7		F_4	F_4
$E_7^{4,4}(\text{VI}, \epsilon_i) i = 1, 4$		F_4		F_4	F_4
$E_7^{4,7}(\text{VI}, \text{V}, \epsilon_1)$		F_4		F_4	F_4
$E_7^{7,7}(\text{V}, \epsilon_i), i = 1, 2, 7$		E_7		E_7	E_7

Table 4.2 – Continued

$E_7^{7,3}(\text{V}, \text{VII}, \epsilon_1)$		E_7		C_3	C_3
$E_7^{3,7}(\text{VII}, \text{V}, \epsilon_1)$		C_3		C_3	C_3
$E_7^{3,4}(\text{VII}, \text{VI})$		C_3		C_2	C_2
$E_7^{4,3}(\text{VI}, \text{VII})$		F_4		C_2	C_2
$E_7^{3,3}(\text{VII}, \epsilon_3)$		C_3		C_3	C_3
$E_8^{8,4}(\text{VIII}, \text{IX}, \epsilon_0)$		E_8		D_4	D_4
$E_8^{4,8}(\text{IX}, \text{VIII}, \epsilon_0)$		F_4		F_4	F_4

Table 4.2 – Continued

$E_8^{4,4}(\text{IX}, \epsilon_i), i = 1, 4$		F_4		F_4	F_4
$E_8^{8,4}(\text{VIII}, \text{IX}, \epsilon_1)$		E_8		D_4	D_4
$E_8^{4,8}(\text{IX}, \text{VIII}, \epsilon_1)$		F_4		F_4	F_4
$E_8^{8,8}(\text{VIII}, \epsilon_8)$		E_8		E_8	E_8
$F_4^{4,1}(\text{I}, \text{II})$		F_4		A_1	A_1
$F_4^{1,4}(\text{II}, \text{I})$		BC_1		BC_1	A_1
$F_4^{4,4}(\text{I}, \epsilon_i), i = 1, 4$		F_4		F_4	F_4

Table 4.2 – Continued

$G_2^{2,2}(I, \epsilon_1)$		G_2		G_2	G_2
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Chapter 5

On Maximal \mathbb{R} -split tori

5.1 Associated Pairs

With the \mathfrak{h} -conjugacy classes of the (θ, σ) -stable maximal quasi \mathbb{R} -split tori determined we can now discuss the requirements to determine the $\mathfrak{h}_{\mathbb{R}}$ -conjugacy classes of (θ, σ) -stable maximal \mathbb{R} -split tori. In between, we will find the \mathfrak{h} -conjugacy classes of (θ, σ) -stable maximal quasi \mathbb{R} -split tori which are \mathfrak{h} -conjugate to a maximal \mathbb{R} -split torus. One tool we will use in this investigation is the associated pair. We consider the commuting involutions (θ, σ) and the associated pair is $(\theta, \sigma\theta)$. Recall, $\sigma\theta = \theta\sigma$.

The following diagram from [10] and [17] helps to explain the relationship between associated pairs and original pairs.

$$\begin{array}{ccccc}
 (\mathfrak{g}, \mathfrak{h}) & \xleftarrow{\text{associated}} & (\mathfrak{g}, \mathfrak{h}^a) & \xleftarrow{\text{dual}} & (\mathfrak{g}^{ad}, \mathfrak{h}^d) \\
 (\theta, \sigma) & & (\theta, \sigma\theta) & & (\sigma\theta, \theta) \\
 \uparrow & & & & \uparrow \\
 \text{dual} & & & & \text{associated} \\
 \downarrow & & & & \downarrow \\
 (\mathfrak{g}^d, \mathfrak{h}^d) & \xleftarrow{\text{associated}} & (\mathfrak{g}^d, \mathfrak{h}^a) & \xleftarrow{\text{dual}} & (\mathfrak{g}^{ad}, \mathfrak{h}) \\
 (\sigma, \theta) & & (\sigma, \sigma\theta) & & (\sigma\theta, \sigma)
 \end{array}$$

Figure 5.1: Associated and Dual Pairs

So for each of the 171 pairs, we will find the associate pair. The reason behind using the associate pair is based on the fact that we need to retain the "R-splitness". In the quasi R-split classification, the "flipping" pieces need not flip the torus into an R-split piece. The associated pair $(\theta, \sigma\theta)$ can give us information about the maximal θ -split (R-split) torus in the fixed point group of σ . We can consider the projection to the (-1)-eigenspaces of θ , σ , and $\sigma\theta$. We always require θ -split.

Previously we looked at the action of σ on Φ_θ to determine the σ -split portion inside the θ -split torus. Similarly, we can look at the action of $\sigma\theta$ on Φ_θ to find the $\sigma\theta$ -split portion inside the θ -split torus. It is important to note that the torus from the associated pair is different than that of the original pair.

We will call the maximal R-split torus for (θ, σ) , \mathfrak{a} (as usual), and the maximal R-split torus for $(\theta, \sigma\theta)$, \mathfrak{s} . So $\mathfrak{s}_{\sigma\theta}^-$ is maximal $\sigma\theta$ -split. However, we know that the torus is already θ -split which implies that $\mathfrak{s}_{\sigma\theta}^-$ maximal $\sigma\theta$ -split is equivalent to \mathfrak{s}_σ^+ is a maximal in the fixed point group.

Essentially we have just built a structure where we known the top (maximal σ -split inside a θ -split) and the bottom (maximal σ -fixed portion inside a θ -split). Once we know the rank of the "top" torus and the "bottom" torus, we know the how many levels live in between. We will do the same analysis as we did for the original pair on the associated pair. We can gather information on the distance and the levels between the top and the bottom tori by considering the rank.

Lemma 5.1.1 ([10, Lemma 9.18]) *Let \mathfrak{a} and \mathfrak{s} be as above. Call the singular rank the difference in rank of the (θ, σ) -stable maximal (σ, \mathbb{R}) -split torus and the (θ, σ) -stable maximal (\mathbb{R}) -split , σ -fixed torus Then we have the following.*

$$\text{singular rank} = \dim(\mathfrak{a}_\sigma^-) + \dim(\mathfrak{s}_{\sigma\theta}^-) - \dim(\mathfrak{a})$$

Essentially the rank is determined by restricting to the $\dim(\mathfrak{a})$ and finding how large both the (+1) and (-1)-eigenspaces of σ are in \mathfrak{a} . This way everything stays \mathbb{R} -split (ie. θ -split). Last, we are working off of the fact that commuting involutions are in a 1-1 correspondence with real semisimple spaces.

5.2 Standard Tori and (θ, σ) -singular Involutions

The same process used for the quasi \mathbb{R} -split tori can be used for the \mathbb{R} -split tori. Recall, $\mathfrak{A}_k^{(\theta, \sigma)}$ is the set of all (θ, σ) -stable maximal k -split tori. Similar to Chapter 3 any $\mathfrak{a}_1 \in \mathfrak{A}_k^{(\theta, \sigma)}$ can be put into standard position with either \mathfrak{a} or \mathfrak{s} (the maximal σ -split and σ -fixed tori) and we can discuss a_i -standard involutions in $W(\mathfrak{a})$. As with σ -singular involutions we have corresponding theory in the \mathbb{R} -split case. Here we will finally talk about (θ, σ) -singular involutions.

Definition 5.2.1 *Let $\mathfrak{a} \in \mathfrak{A}_k^{(\theta, \sigma)}$ and $w \in W(\mathfrak{a})$. Then w is (θ, σ) -singular if*

1. w^2
2. $\sigma w = w\sigma$
3. the involutions $\sigma|[\mathfrak{g}_w, \mathfrak{g}_w]$ and $\sigma\theta|[\mathfrak{g}_w, \mathfrak{g}_w]$ are k -split.

A root $\alpha \in \Phi(a)$ is called (θ, σ) -singular if the reflection $s_\alpha \in W(\mathfrak{a})$ is (θ, σ) -singular.

Proposition 5.2.2 ([10, Proposition 8.9]) *Let $\mathfrak{a} \in \mathfrak{A}_k^{(\theta, \sigma)}$ with \mathfrak{a}_σ^- maximal. Then there is a one to one correspondence between the \mathfrak{h}_k^+ -conjugacy classes of $\mathfrak{A}_k^{(\theta, \sigma)}$ and the $W(\mathfrak{a}, \mathfrak{h}_k^+)$ -conjugacy classes of (θ, σ) -singular roots of $W(\mathfrak{a})$*

In order to classify the $\mathfrak{h}_{\mathbb{R}}^+$ conjugacy classes of tori, we need once again only consider the conjugacy classes of roots in the Weyl group. However, the classification is quite

difficult compared to the quasi \mathbb{R} -split case. First, we need to consider the conjugacy in $W(\mathfrak{a}, \mathfrak{h}_{\mathbb{R}}^+)$ which means we must determine $W(\mathfrak{a}, \mathfrak{h}_{\mathbb{R}}^+)$ from $\Phi(\mathfrak{a}, \mathfrak{h}_{\mathbb{R}}^+)$. Second, we must actually find the (θ, σ) -singular roots. Luckily, we have simplified the process a bit because of the quasi \mathbb{R} -split case. Recall the σ -singular involutions. These involutions satisfied many of the qualifications for (θ, σ) -singular roots except one:

$$\sigma\theta|[\mathfrak{g}_w, \mathfrak{g}_w] \text{ is } \mathbb{R}\text{-split}$$

We will have to consider this case in particular. But in the search for (θ, σ) -singular roots we can already restrict to $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ because each root must also be a σ -singular roots and we have already showed all σ -singular roots are contained in $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-)$.

As stated previously, the (θ, σ) -singular involutions are elements in the Weyl group of $W(\mathfrak{a})$ where \mathfrak{a} is an (θ, σ) -stable maximal (θ, σ) -split torus. Based on the requirements all potential (θ, σ) -singular roots, α (s_α is a (θ, σ) -singular involution) must be roots of the maximal (σ, \mathbb{R}) -split ((θ, σ) -split) torus and be roots of the original θ -split torus.

We call the root system of a maximal θ -split torus Φ_θ . Using the diagrams discussed in Chapter 2, we can determine the type of Φ_θ and consider the action of σ on the root system. This step will determine the dimension of the maximal σ -split torus in the maximal θ -split or \mathbb{R} -split torus. In the Appendix A, we represent this step in the box $\sigma|\Phi_\theta$.

Also, we can determine the root system and type of the maximal (σ, \mathbb{R}) -split torus. We will call the root system $\Phi_{\theta, \sigma}$. We repeat this process with the pair $(\theta, \sigma\theta)$ is a similar manner. We are interested in potential (θ, σ) -singular (($\theta, \sigma\theta$)-singular) roots which will be found in $\Phi_\theta \cap \Phi_{\theta, \sigma}$ ($\Phi_\theta \cap \Phi_{\theta, \sigma\theta}$).

We will begin by describing the conjugacy classes of involution in $W(\Phi_\theta \cap \Phi_{\theta, \sigma})$ and then pass to the (θ, σ) -singular involutions.

So the W -conjugacy classes of involutions in W are determined by the type of $\Phi(w)$. However, we want to focus on the involutions in $W(\Phi_\theta \cap \Phi_{(\theta,\sigma)})$ and their conjugacy classes in W . First we must find the σ -maximal involutions in $\Phi_\theta \cap \Phi_{(\theta,\sigma)}$. So now with the full classification of the σ -maximal involutions in $\Phi_\theta \cap \Phi_{(\theta,\sigma)}$, we return to our question of (θ, σ) -singular involutions and their $W(\mathfrak{a})$ -conjugacy classes as discussed before. We can use the previous process and purely focus on the maximal (θ, σ) -singular involutions. The rank of $\Phi(w_m)$ is called the singular rank and is the rank described in Lemma 5.1.1. We will use the following lemmas from [10, Chapter 9].

Lemma 5.2.3 *Let w_m be a maximal (θ, σ) -singular involution in $W(\Phi_\theta \cap \Phi_{\theta,\sigma})$. Every (θ, σ) -involution in $W(\Phi_\theta \cap \Phi_{\theta,\sigma})$ is conjugate under $W(\mathfrak{a}, \mathfrak{h}_\mathbb{R}^+)$ with a (θ, σ) -singular involutions w satisfying $\mathfrak{a}_w^- \subset \mathfrak{a}_{w_m}^-$.*

Lemma 5.2.4 *Let w_m be a maximal (θ, σ) -singular in $W(\Phi_\theta \cap \Phi_{(\theta,\sigma)})$ and w a (θ, σ) -singular involution in $W(\Phi_\theta \cap \Phi_{(\theta,\sigma)})$ with $\mathfrak{a}_w^- \subset \mathfrak{a}_{w_m}^-$. Then $w \cdot w_m$ is a (θ, σ) -singular involution in $W(\Phi_\theta \cap \Phi_{\theta,\sigma})$.*

We still need to determine which involutions are (θ, σ) -singular involutions in $W(\mathfrak{a}, \mathfrak{a}_\sigma^-)$. Let \mathfrak{s} be a (θ, σ) -stable maximal \mathbb{R} -split torus of \mathfrak{g} with \mathfrak{s}_σ^+ maximal which is standard with respect to \mathfrak{a} . Then $\mathfrak{s}_\sigma^- \subset \mathfrak{a}_\sigma^-$. Let $\Pi_1 = \{\alpha \in \Phi | \sigma(\alpha) = -\alpha\}$. Let $\Delta(\mathfrak{s})$ be the basis of $\Phi(\mathfrak{s})$ with respect to a $(-\sigma)$ -order on $\Phi(\mathfrak{s})$. This induces an order on Π_1 . Let $w_{\Pi_1}^0$ be the longest root of $W(\Pi_1)$ with respect to the ordering. Then $w_{\Pi_1}^0(\Phi^+(\mathfrak{s})) = \sigma(\Phi^+(\mathfrak{s}))$. We can identify Π_1 with a subset Π of $\Phi(\mathfrak{a})$ where $w_{\Pi_1}^0$ maps to w_Π^0 . If $w(\sigma)$ is a σ -maximal involution of $W(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ such that $\mathfrak{a}_{w_{\Pi_1}^0}^- \subset \mathfrak{a}_{w(\sigma)}^-$, then $w_m = w(\sigma) \cdot w_\Pi^0$ is maximal (θ, σ) -singular involution. So to classify these involutions we need to determine Π_1 and a list of σ -maximal involutions. Deriving the former involves employing our associate pair of involutions. Previous we looked at the diagram of $\sigma|\Phi(\mathfrak{a})$ for the restricted root

system. In this case, we look at the diagrams of $\sigma\theta|\Phi(\mathfrak{s})$ since $\mathfrak{s}_\sigma^+ = \mathfrak{s}_{\sigma\theta}^-$ is maximal with respect to $\sigma\theta$.

So, our first task is to determine the type of the maximal involution possible in $\Phi_\theta \cap \Phi_{\theta,\sigma}$. In the appendix tables, we determine the maximal involution in the intersection of our root spaces for both the pair (θ, σ) and $(\theta, \sigma\theta)$. There are essentially four cases that could occur.

- Case 1: There are no (θ, σ) - or no $(\theta, \theta\sigma)$ - singular roots. An empty intersection means there are no candidates to flip a one dimensional piece from the σ -split to the portion fixed by σ . If there are no (θ, σ) -singular roots then the singular rank is 0, we know there is only one $\mathfrak{h}_{\mathbb{R}}$ -conjugacy class of σ -stable maximal \mathbb{R} -split tori.
- Case 2: If singular rank is r and w_m is of type $r \cdot A_1$, then all (θ, σ) -stable involutions w with $\mathfrak{a}_w^- \subset \mathfrak{a}_{w_m}^-$ are (θ, σ) -singular. Therefore at each level we will achieve a torus of proper dimension from the top of the diagram to the bottom. We consider both $\Phi_\theta \cap \Phi_{\theta,\sigma}$ and $\Phi_\theta \cap \Phi_{\theta,\sigma\theta}$. There is more work to be done to see if any of the levels split when considered over $W(\mathfrak{a}, \mathfrak{h}_{\mathbb{R}}^+)$.
- Case 3: If the singular rank is r and the w_m is of type B_r, C_r, D_r, E_r ($r = 6, 7$ or 8), F_r ($r = 4$), or G_r ($r = 2$), then all involutions in the diagram are (θ, σ) -singular. Again, we need to consider the splitting of these involutions in $W(\mathfrak{a}, \mathfrak{h}_{\mathbb{R}}^+)$.
- Case 4: w_m is of type $r \cdot A_1 \cdot X_\ell$ where $X_\ell = B_\ell, C_\ell, D_\ell, E_\ell$ ($\ell = 6, 7$ or 8), F_ℓ ($\ell = 4$), or G_ℓ ($\ell = 2$) and singular rank is $r + \ell$. In order to determine the conjugacy classes of singular involutions we must consider w_m in pieces. We can determine $(\theta, \sigma)|_{\mathfrak{g}_{r \cdot A_1}}$ from the classification of pairs in [7]. From here we can also look at the associated pair $(\theta, \sigma), (\theta, \sigma\theta)|_{\mathfrak{g}_{r \cdot A_1}}$. We can do a similar procedure for G_{X_ℓ} . We cannot make

the same claims as before if w is a (θ, σ) -singular involution and is of type $r \cdot A_1 + X_\ell$. Not all the involutions of this type are not necessarily (θ, σ) -singular. However, as discussed previous, we don't believe that this type occurs as a maximal involutions.

5.3 $W(\mathfrak{a}, \mathfrak{h}_k^+)$ -conjugacy classes

Once we determine the (θ, σ) -singular involutions, if we consider their conjugacy classes in $W(\mathfrak{a})$ then we have a classification of $\mathfrak{A}_0^{(\theta, \sigma)}$ ((θ, σ) -stable maximal quasi k -split tori \mathfrak{h} -conjugate to a maximal k -split torus). Once the involutions are determine, then this classification falls out in the exact manner as the quasi \mathbb{R} -split case. However, then we consider these conjugacy classes over $W(\mathfrak{a}, \mathfrak{h}_\mathbb{R}^+)$ then two involutions of the same type in $W(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ (which were conjugate under $W(\mathfrak{a})$) will not be conjugate under $W(\mathfrak{a}, \mathfrak{h}_\mathbb{R}^+)$.

Example Consider the following case for $\ell = 7, p = 2, q = 4, i = 1$.

Type (θ, σ)	Type $(\theta, \theta\sigma)$	singular rank
$A_\ell^{p,q}(\text{III}_a, \text{III}_a, \epsilon_i)$ $p < q - p + 2i \leq \frac{1}{2}(\ell + 1)$ $1 \leq p < q \leq \frac{1}{2}(\ell + 1)$ $0 \leq i \leq p$	$A_\ell^{p,q-p+2i}(\text{III}_a, \text{III}_a, \epsilon_{p-i})$	p
$A_7^{2,4}(\text{III}_a, \text{III}_a, \epsilon_1)$	$A_7^{2,4}(\text{III}_a, \text{III}_a, \epsilon_1)$	2

We can reference Table 4.2 and determine that $\Phi(\mathfrak{a}) = BC_2$ and $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-) = BC_2$. The rank of the maximal σ -split torus is 2 and the rank of the maximal σ -fixed torus is also 2, but the rank of the maximal \mathbb{R} -split (ie. θ -split) torus is also 2. This says that at the "top" the maximal \mathbb{R} -split torus is a σ -split torus and at the "bottom" the torus is

a σ -fixed torus. We need to determine what is happening with the the tori in between. Since we go from top to bottom, we will consider the tori that are standard to \mathfrak{a} where $\dim((\mathfrak{a}_i)^-_\sigma) = 2, 1$, and 0.

Through direction computation on the six roots in $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ (ie. $e_1 \pm e_2, e_1, 2e_1, e_2, 2e_2$) we can determine that only two roots are (θ, σ) -singular. These roots are the unique short roots, usually denoted e_1 and e_2 .

In $W(\mathfrak{a})$ these roots of type A_1 are conjugate. So we can view the conjugacy classes of (θ, σ) -singular roots in $W(\mathfrak{a})$ as the blackened dots in the diagram below. Recall, this classifies the quasi \mathbb{R} -split tori that are \mathfrak{h} -conjugacy to a maximal \mathbb{R} -split torus.

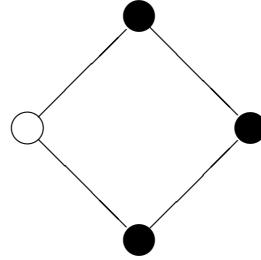


Figure 5.2: $\mathfrak{A}_0^{(\theta, \sigma)} / \mathfrak{h}$ in Example 5.3

We see there is one conjugacy class at each level. So all tori $\mathfrak{a}_i \in \mathfrak{a}_0^{\theta, \sigma}$ such that $\dim((\mathfrak{a}_i)^-_\sigma) = 2$ are conjugate. Similarly those with dimension 1 and 0. However, if we consider the conjugacy classes of these singular involutions in $W(\mathfrak{a}, \mathfrak{h}_{\mathbb{R}}^+) = BC_1 + BC_1$ then e_1 and e_2 while both type A_1 are no longer conjugate. So the one dimension piece will split and there will two $\mathfrak{h}_{\mathbb{R}}^+$ -conjugacy classes of tori where the rank of the σ -split portion is of rank 1.

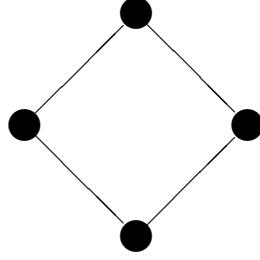


Figure 5.3: $\mathfrak{A}_{\mathbb{R}}^{(\theta,\sigma)}/\mathfrak{h}_{\mathbb{R}}^+$ in Example 5.3

In this small example, it was not difficult to calculate the (θ, σ) -singular roots or $W(\mathfrak{a}, \mathfrak{h}_{\mathbb{R}}^+)$; however, that will not always be the case. Let us consider some methods from [10] to make the calculations in general more convenient.

Definition 5.3.1 *Let H be k -open subgroup of G_σ . Then we say that H is pseudo-connected if*

$$H_k = Z_{H_k^+}(A)H_k^0$$

Proposition 5.3.2 *Let H be an open k -subgroup of G_σ . Then the following are equivalent.*

1. $W(A_\sigma^-, G_{\sigma\theta}) \cong W(A_\sigma^-, H_k^+)$
2. H_k is pseudo-connected.

Since we are working on the Lie algebra in order to lift into the group, we need to restrict to the case when H_k is pseudo-connected. So instead of looking at the conjugacy classes in $W(\mathfrak{a}, \mathfrak{h}_k^+)$ it will suffice to consider the conjugacy classes in $W(\mathfrak{a}, \mathfrak{g}_{\sigma\theta})$.

Proposition 5.3.3 *Let w_1, w_2 be (θ, σ) -singular involutions. Then w_1 and w_2 are conjugate under $W(\mathfrak{a}, \mathfrak{h}_k^+)$ if and only if w_1 and w_2 are conjugate under $W(\mathfrak{a}_\sigma^-, \mathfrak{h}_k^+)$.*

Corollary 5.3.4 *Let w_1 and w_2 be (θ, σ) -singular involutions. Then the following are equivalent.*

1. w_1 and w_2 are conjugate under $W(\mathfrak{a}, \mathfrak{h}_k^+)$.
2. w_1 and w_2 are conjugate under $W(\mathfrak{a}_\sigma^-, \mathfrak{h}_k^+)$.
3. w_1 and w_2 are conjugate under $W(\mathfrak{a}, \mathfrak{g}_{\sigma\theta})$.
4. w_1 and w_2 are conjugate under $W(\mathfrak{a}_\sigma^-, \mathfrak{g}_{\theta\sigma})$.

So now we have a further description of where the (θ, σ) -singular involutions live.

While we could previously restrict ourselves to $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ we can now only look in $\Phi(\mathfrak{a}, \mathfrak{g}_{\sigma\theta})$ or any of the equivalent spaces above. We have chosen to get more insight into $\Phi(\mathfrak{a}, \mathfrak{g}_{\sigma\theta})$.

To start, we must determine $\mathfrak{g}_{\sigma\theta}(\mathbb{R})$ and the roots of the maximal torus in $\mathfrak{g}_{\sigma\theta}$ in order to discuss the intersection of the rootsystems. Using the information from the associated pair, we actually have type of involution of $\sigma\theta$ and can determine the fixed point group when paired with the Cartan involution θ .

Table 5.1 has a complete list of $\mathfrak{g}_{\sigma\theta}$ for our pairs (θ, σ) . We should recall that we may have the action of a quadratic element, in which case we will still write $\mathfrak{g}_{\sigma\theta}$ with the understanding that we are considering $\mathfrak{g}_{\sigma\theta \text{Int}(\epsilon)}$ when applicable. Last, $\Phi(\mathfrak{t})$ is the type of the maximal torus in $\mathfrak{g}_{\sigma\theta}$.

To complete the classification, we must determine $\Phi(\mathfrak{a}, \mathfrak{g}_{\sigma\theta})$ which can be done by looking that the multiplicities of the roots and we must determine which roots, λ , have the property that $\sigma\theta|[\mathfrak{g}_{s_\lambda}, \mathfrak{g}_{s_\lambda}]$ is \mathbb{R} -split. The (θ, σ) -singular involutions have been determined in some cases already. The following are conjectures as to how the classification will proceed.

Theorem 5.3.5 [7] A root $\lambda \in \Phi(\mathfrak{a})$ is in $\Phi(\mathfrak{a}, \mathfrak{g}_{\sigma\theta})$ (resp. $\Phi(\mathfrak{a}, \mathfrak{g}_{\sigma\theta \text{Int}(\epsilon)})$) if and only if $m^+(\lambda, \sigma\theta) > 0$ (resp. $m^+(\lambda, \sigma\theta \text{Int}(\epsilon)) > 0$).

Conjecture The (θ, σ) -singular roots in $\Phi(\mathfrak{a}, \mathfrak{g}_{\sigma\theta})$ are those roots where both $m^+(\lambda, \sigma\theta) > 0$ and $m^-(\lambda, \sigma\theta) < 0$ (resp. $m^+(\lambda, \sigma\theta \text{Int}(\epsilon)) > 0$ and $m^-(\lambda, \sigma\theta \text{Int}(\epsilon)) > 0$).

Recall, the four cases discussed previously in the previous section. Here we summarize them into three with extra information derived from observations so far on (θ, σ) -singular involutions.

- Case 1: There are no (θ, σ) - or no $(\theta, \theta\sigma)$ - singular roots. We see this in the case when $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ is empty or when $\Phi(\mathfrak{s}, \mathfrak{s}_{\sigma\theta}^-)$ is empty. An empty intersection means there are no candidates to flip a one dimensional piece from the σ -split to the portion fixed by σ . Also, If there are no (θ, σ) -singular roots then the singular rank is 0, we can conclude there is only one \mathfrak{h}_k -conjugacy class of σ -stable maximal \mathbb{R} -split tori in \mathfrak{g} . This occurs in 18 of the 171 cases.
- Case 2: [10] If singular rank is r and $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ or $\Phi(\mathfrak{s}, \mathfrak{s}_{\sigma\theta}^-)$ is of type $r \cdot A_1$, then all (θ, σ) -stable involutions w with $\mathfrak{a}_w^- \subset \mathfrak{a}_{w_m}^-$ are (θ, σ) -singular. We still must verify that w_m is of type $r \cdot A_1$ is (θ, σ) -singular. In most cases, $\Phi(\mathfrak{t})$ will be type X_ℓ for B_r, C_r, D_r, E_r ($r = 6, 7$, or 8), F_r ($r = 4$), or G_r ($r = 2$). Then our conjugacy classes will not split.
- Case 3: If the singular rank is r and the $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ or $\Phi(\mathfrak{s}, \mathfrak{s}_{\sigma\theta}^-)$ is of type B_r, C_r, D_r, E_r ($r = 6, 7$, or 8), F_r ($r = 4$), or G_r ($r = 2$), then all involutions in the diagram are (θ, σ) -singular if w_m is of type B_r, C_r, D_r, E_r ($r = 6, 7$, or 8), F_r ($r = 4$), or G_r ($r = 2$). Again, we need to consider the splitting of these involutions in $W(\mathfrak{a}, \mathfrak{h}_\mathbb{R}^+)$. In fact, in some classes $\Phi(\mathfrak{t})$ agrees well and has the same rank; however, we also have

$\Phi(\mathfrak{t}) = X_i + X_j$, where $X = A, B, C, BC, D$ or E . This will cause the roots to split.

In this case, often the singular rank is less than the maximal rank in $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ or $\Phi(\mathfrak{s}, \mathfrak{s}_{\sigma\theta}^-)$ which complicates matters.

Table 5.1: Associated Pairs and Fixed Point Groups

Type (θ, σ)	Type $(\theta, \theta\sigma)$	singular rank	$\mathfrak{g}_{\theta\sigma Int(\epsilon_i)}$	$\Phi(\mathfrak{t})$
$A_{2\ell+1}^{2\ell+1,\ell}(\text{I}, \text{II})$	$A_{2\ell+1}^{2\ell+1,\ell+1}(\text{I}, \text{III}_b, \epsilon_0)$	0	$sl(\ell+1, \mathbb{C}) + so(2)$	A_ℓ
$A_{4\ell-1}^{2\ell-1,4\ell-1}(\text{II}, \text{I})$	$A_{4\ell-1}^{2\ell-1,2\ell}(\text{II}, \text{III}_b, \epsilon_\ell)$	ℓ	$sl(2\ell, \mathbb{C}) + so(2)$	$A_{2\ell-1}$
$A_{4\ell+1}^{2\ell,4\ell+1}(\text{II}, \text{I})$	$A_{4\ell+1}^{2\ell,2\ell+1}(\text{II}, \text{III}_b)$	ℓ	$sl(2\ell, \mathbb{C} + so(2)$	$A_{2\ell}$
$A_{2\ell-1}^{2\ell-1,\ell}(\text{I}, \text{III}_b, \epsilon_0)$	$A_{2\ell-1}^{2\ell-1,\ell-1}(\text{I}, \text{II})$	0	$sp(\ell, \mathbb{R})$	C_ℓ
$A_{4\ell-1}^{2\ell,4\ell-1}(\text{III}_b, \text{I}, \epsilon_0)$	$A_{4\ell-1}^{2\ell,2\ell-1}(\text{III}_b, \text{II}, \epsilon_\ell)$	ℓ	$sp(2\ell, \mathbb{R})$	$C_{2\ell}$
$A_{4\ell+1}^{2\ell+1,4\ell+1}(\text{III}_b, \text{I}, \epsilon_0)$	$A_{4\ell+1}^{2\ell+1,2\ell}(\text{III}_b, \text{II})$	ℓ	$sp(2\ell+1, \mathbb{R})$	$C_{2\ell+1}$
$A_{4\ell-1}^{2\ell-1,2\ell}(\text{II}, \text{III}_b, \epsilon_\ell)$	$A_{4\ell-1}^{2\ell-1,4\ell-1}(\text{II}, \text{I})$	ℓ	$so^*(4\ell)$	C_ℓ
$A_{4\ell+1}^{2\ell,2\ell+1}(\text{II}, \text{III}_b)$	$A_{4\ell+1}^{2\ell,4\ell+1}(\text{II}, \text{I})$	ℓ	$so^*(4\ell+2)$	BC_ℓ
$A_{4\ell-1}^{2\ell,2\ell-1}(\text{III}_b, \text{II}, \epsilon_\ell)$	$A_{4\ell-1}^{2\ell,2\ell-1}(\text{III}_b, \text{I}, \epsilon_0)$	ℓ	$so^*(4\ell)$	C_ℓ
$A_{4\ell+1}^{2\ell+1,2\ell}(\text{III}_b, \text{II})$	$A_{4\ell+1}^{2\ell+1,4\ell+1}(\text{III}_b, \text{I}, \epsilon_0)$	ℓ	$so^*(4\ell+2)$	BC_ℓ
$A_{2\ell-1}^{2\ell-1,\ell}(\text{I}, \text{III}_b, \epsilon_\ell)$	$A_{2\ell-1}^{2\ell-1}(\text{I}, \epsilon_\ell)$	ℓ	$so(\ell, \ell)$	B_ℓ
$A_{2\ell-1}^{\ell,2\ell-1}(\text{III}_b, \text{I}, \epsilon_\ell)$	$A_{2\ell-1}^{\ell,2\ell-1}(\text{III}_b, \text{I}, \epsilon_\ell)$	ℓ	$so(\ell, \ell)$	B_ℓ
$A_{2\ell-1}^{2\ell-1,2\ell-1}(\text{I}, \epsilon_\ell)$	$A_{2\ell-1}^{2\ell-1,\ell}(\text{I}, \text{III}_b, \epsilon_\ell)$	ℓ	$sl(\ell, \mathbb{R}) + sl(\ell, \mathbb{R}) + \mathbb{R}$	$\begin{matrix} A_{\ell-1} \\ A_{\ell-1} + \mathbb{R} \end{matrix}$
$A_{4\ell-1}^{2\ell-1,2\ell}(\text{II}, \text{III}_b, \epsilon_0)$	$A_{4\ell-1}^{2\ell-1}(\text{II}, \epsilon_\ell)$	ℓ	$sp(\ell, \ell)$	C_ℓ

Table 5.1 – Continued

$A_{4\ell-1}^{2\ell,2\ell-1}(\text{III}_b, \text{II}, \epsilon_0)$	$A_{4\ell-1}^{2\ell,2\ell-1}(\text{III}_b, \text{II}, \epsilon_0)$	0	$sp(\ell, \ell)$	C_ℓ
$A_{4\ell-1}^{2\ell-1}(\text{II}, \epsilon_\ell)$	$A_{4\ell-1}^{2\ell-1,2\ell}(\text{II}, \text{III}_b, \epsilon_0)$	ℓ	$su^*(2\ell) + su^*(2\ell) + \mathbb{R}$	$\begin{matrix} A_{\ell-1} \\ A_{\ell-1} + \mathbb{R} \end{matrix}$
$A_\ell^{\ell,p}(\text{I}, \text{III}_a)$ $1 \leq 2p \leq \ell$	$A_\ell^{\ell,\ell}(\text{I}, \epsilon_p)$	p	$so(\ell+1-p, p)$	B_p
$A_\ell^{p,\ell}(\text{III}_a, \text{I})$ $1 \leq 2p \leq \ell$	$A_\ell^{p,\ell}(\text{III}_a, \text{I})$	p	$so(p, \ell-p+1)$	B_p
$A_\ell^\ell(\text{I}, \epsilon_p)$ $1 \leq 2p \leq \ell$	$A_\ell^{\ell,p}(\text{I}, \text{III}_a)$	p	$sl(p, \mathbb{R}) + sl(\ell-p+1, \mathbb{R}) + \mathbb{R}$	$\begin{matrix} A_{p-1} \\ A_{\ell-p} + \mathbb{R} \end{matrix}$
$A_{4\ell-1}^{2p,2\ell-1}(\text{III}_a, \text{II})$ $1 \leq 4p \leq 4\ell$	$A_{4\ell-1}^{2p,2\ell-1}(\text{III}_a, \text{II})$	0	$sp(2\ell-p, p)$	BC_p
$A_{4\ell+1}^{2p,2\ell}(\text{III}_a, \text{II})$ $1 \leq 4p \leq 4\ell$	$A_{4\ell+1}^{2p,2\ell}(\text{III}_a, \text{II})$	0	$sp(2\ell-p+1, p)$	BC_p
$A_{4\ell-1}^{2\ell-1,2p}(\text{II}, \text{III}_a)$ $1 \leq 4p \leq 4\ell+2$	$A_{4\ell-1}^{2\ell-1,2\ell-1}(\text{II}, \epsilon_p)$	p	$sp(2\ell-p, p)$	BC_p
$A_{4\ell+1}^{2\ell,2p}(\text{II}, \text{III}_a)$ $1 \leq 4p \leq 4\ell+2$	$A_{4\ell+1}^{2\ell,2\ell}(\text{II}, \epsilon_p)$	p	$sp(2\ell+1-p, p)$	BC_p
$A_{2\ell+1}^{\ell,\ell}(\text{II}, \epsilon_p)$ $1 \leq 2p \leq \ell+1$	$A_{2\ell+1}^{\ell,2p}(\text{II}, \text{III}_a)$	p	$su^*(2p) + su^*(2\ell+1-2p) + \mathbb{R}$	$A_p + A_{\ell-p}$
$A_{2\ell-1}^{\ell,\ell}(\text{III}_b, \epsilon_\ell)$	$A_{2\ell-1}^{\ell,\ell}(\text{III}_b, \epsilon_\ell)$	ℓ	$sl(\ell, \mathbb{C}) + \mathbb{R}$	$A_{\ell-1} + \mathbb{R}$

Table 5.1 – Continued

$A_{\ell}^{p,p}(III_a, \epsilon_p)$ $p < \ell + 1 - 2p$	$A_{\ell}^{p,\ell+1-2p}(III_a, \epsilon_p)$	p	$su(\ell+1-2p, 0) + su(p, p)$	BC_p
$A_{2\ell-1}^{\ell,p}(III_b, III_a, \epsilon_i)$ $\ell - p + 2i \leq p$ $1 \leq p < \ell$ $0 \leq i \leq p - 1$	$A_{2\ell-1}^{\ell,\ell-p+2i}(III_b, III_a, \epsilon_i)$	$2i$	$su(i, \ell - p + i) + su(\ell - i, p - i) + so(2)$	$BC_i + BC_{p-i}$
$A_{2\ell-1}^{\ell,p}(III_b, III_a, \epsilon_i)$ $p < \ell - p + 2i \leq \ell$ $1 \leq p < \ell$ $0 \leq i \leq p - 1$	$A_{2\ell-1}^{\ell,\ell-p+2i}(III_b, III_a, \epsilon_{\ell-p+i})$	$2i$	$su(\ell - p + i, i) + su(p - i, \ell - i) + so(2)$	$BC_i + BC_{p-i}$
$A_{2\ell-1}^{\ell,p}(III_b, III_a, \epsilon_i)$ $p \leq \ell + p - 2i < \ell$ $1 \leq p < \ell$ $0 \leq i \leq p - 1$	$A_{2\ell-1}^{\ell,\ell+p-2i}(III_b, III_a, \epsilon_{\ell-i})$	$2p - 2i$	$su(i, \ell - p + i) + su(\ell - i, p - i) + so(2)$	$BC_i + BC_{p-i}$
$A_{2\ell-1}^{\ell,p}(III_b, III_a, \epsilon_i)$ $\ell + p - 2i < p$ $1 \leq p < \ell$ $0 \leq i \leq p - 1$	$A_{2\ell-1}^{\ell,\ell-p+2i}(III_b, III_a, \epsilon_{\ell-i})$	$2p - 2i$	$su(i, \ell - p + i) + su(\ell - i, p - i) + so(2)$	$BC_i + BC_{p-i}$
$A_{2\ell-1}^{p,\ell}(III_a, III_b, \epsilon_i)$ $\ell - p + 2i \leq p$ $1 \leq p < \ell$ $0 \leq i \leq p - 1$	$A_{2\ell-1}^{p,\ell-p+2i}(III_b, III_a, \epsilon_{\ell-p+i})$	$\ell - p + 2i$	$su(i, \ell - p + i) + su(\ell - i, p - i) + so(2)$	$BC_i + BC_{p-i}$

Table 5.1 – Continued

$A_{2\ell-1}^{p,\ell}(\text{III}_a, \text{III}_b, \epsilon_i)$	$A_{2\ell-1}^{p,\ell-p+2i}(\text{III}_b, \text{III}_a, \epsilon_{p-i})$	p	$su(i, \ell - p + i) + su(\ell - i, p - i) + so(2)$	$BC_i + BC_{p-i}$
$p < \ell - p + 2i \leq \ell$				
$1 \leq p < \ell$				
$0 \leq i \leq p - 1$				
$A_{2\ell-1}^{p,\ell}(\text{III}_a, \text{III}_b, \epsilon_i)$	$A_{2\ell-1}^{p,\ell+p-2i}(\text{III}_b, \text{III}_a, \epsilon_i)$	p	$su(i, \ell - p + i) + su(\ell - i, p - i) + so(2)$	$BC_i + BC_{p-i}$
$p \leq \ell + p - 2i < \ell$				
$1 \leq p < \ell$				
$0 \leq i \leq p - 1$				
$A_{2\ell-1}^{p,\ell}(\text{III}_a, \text{III}_b, \epsilon_i)$	$A_{2\ell-1}^{p,\ell+p-2i}(\text{III}_b, \text{III}_a, \epsilon_{\ell-i})$	$\ell + p - 2i$	$su(i, \ell - p + i) + su(\ell - i, p - i) + so(2)$	$BC_i + BC_{p-i}$
$\ell + p - 2i < p$				
$1 \leq p < \ell$				
$0 \leq i \leq p - 1$				
$A_\ell^{q,p}(\text{III}_a, \text{III}_a, \epsilon_i)$	$A_\ell^{q,q-p+2i}(\text{III}_a, \text{III}_a, \epsilon_i)$	$2i$	$su(\ell + 1 - q - i, p - i) + su(q - p + i, i) + so(2)$	$BC_i + BC_{p-i}$
$q - p + 2i \leq p < q$				
$1 \leq p < q \leq \frac{1}{2}(\ell + 1)$				
$0 \leq i \leq p$				
$A_\ell^{q,p}(\text{III}_a, \text{III}_a, \epsilon_i)$	$A_\ell^{q,q-p+2i}(\text{III}_a, \text{III}_a, \epsilon_{p-i})$	p	$su(\ell + 1 - q - i, p - i) + su(q - p + i, i) + so(2)$	$BC_i + BC_{p-i}$
$p < q < q - p + 2i \leq \frac{1}{2}(\ell + 1)$				
$1 \leq p < q \leq \frac{1}{2}(\ell + 1)$				
$0 \leq i \leq p$				

Table 5.1 – Continued

$A_\ell^{q,p}(\text{III}_a, \text{III}_a, \epsilon_i)$					
$q \leq \ell + 1 - q + p - 2i$					
$\ell + 1 - q + p - 2i < \frac{1}{2}(\ell + 1)$	$A_\ell^{q,\ell+1-q+p-2i}(\text{III}_a, \text{III}_a, \epsilon_{q-p+i})$	p	$su(\ell + 1 - q - i, p - i) +$	$BC_i +$	
$1 \leq p < q \leq \frac{1}{2}(\ell + 1)$			$su(q - p + i, i) + so(2)$	BC_{p-i}	
$0 \leq i \leq p$					
$A_\ell^{q,p}(\text{III}_a, \text{III}_a, \epsilon_i)$					
$p \leq \ell + 1 - q + p - 2i < q$	$A_\ell^{q,\ell+1-q+p-2i}(\text{III}_a, \text{III}_a, \epsilon_{\ell+1-q-i})$	$\ell + 1 - 2q + 2p - 2i$	$su(\ell + 1 - q - i, p - i) +$	$BC_i +$	
$1 \leq p < q \leq \frac{1}{2}(\ell + 1)$			$su(q - p + i, i) + so(2)$	BC_{p-i}	
$0 \leq i \leq p$					
$A_\ell^{q,p}(\text{III}_a, \text{III}_a, \epsilon_i)$					
$\ell + 1 - q + p - 2i < q$	$A_\ell^{q,\ell+1-q+p-2i}(\text{III}_a, \text{III}_a, \epsilon_{\ell-q-i})$	$\ell + 1 - 2q + 2p - 2i$	$su(\ell + 1 - q - i, p - i) +$	$BC_i +$	
$1 \leq p < q \leq \frac{1}{2}(\ell + 1)$			$su(q - p + i, i) + so(2)$	BC_{p-i}	
$0 \leq i \leq p$					
$A_\ell^{p,q}(\text{III}_a, \text{III}_a, \epsilon_i)$					
$q - p + 2i \leq p$					
$1 \leq p < q \leq \frac{1}{2}(\ell + 1)$	$A_\ell^{p,q-p+2i}(\text{III}_a, \text{III}_a, \epsilon_{q-p+i})$	$2i$	$su(\ell + 1 - q - i, p - i) +$	$BC_i +$	
$0 \leq i \leq p$			$su(q - p + i, i) + so(2)$	BC_{p-i}	
$A_\ell^{p,q}(\text{III}_a, \text{III}_a, \epsilon_i)$					
$p < q - p + 2i \leq \frac{1}{2}(\ell + 1)$	$A_\ell^{p,q-p+2i}(\text{III}_a, \text{III}_a, \epsilon_{p-i})$	p	$su(\ell + 1 - q - i, p - i) +$	$BC_i +$	
$1 \leq p < q \leq \frac{1}{2}(\ell + 1)$			$su(q - p + i, i) + so(2)$	BC_{p-i}	
$0 \leq i \leq p$					

Table 5.1 – Continued

$A_\ell^{p,q}(\text{III}_a, \text{III}_a, \epsilon_i)$				
$p < q \leq \ell + 1 - q + p - 2i$				
$\ell + 1 - q + p - 2i < \frac{1}{2}(\ell + 1)$	$A_\ell^{p,\ell+1-q+p-2i}(\text{III}_a, \text{III}_a, \epsilon_i)$	p	$su(\ell + 1 - q - i, p - i) +$	$BC_i +$
$1 \leq p < q \leq \frac{1}{2}(\ell + 1)$			$su(q - p + i, i) + so(2)$	BC_{p-i}
$0 \leq i \leq p$				
$A_\ell^{p,q}(\text{III}_a, \text{III}_a, \epsilon_i)$				
$p \leq \ell + 1 - q + p - 2i < q$	$A_\ell^{p,\ell+1-q+p-2i}(\text{III}_a, \text{III}_a, \epsilon_i)$	p	$su(\ell + 1 - q - i, p - i) +$	$BC_i +$
$1 \leq p < q \leq \frac{1}{2}(\ell + 1)$			$su(q - p + i, i) + so(2)$	BC_{p-i}
$0 \leq i \leq p$				
$A_\ell^{p,q}(\text{III}_a, \text{III}_a, \epsilon_i)$				
$\ell + 1 - q + p - 2i < p < q$	$A_\ell^{p,\ell+1-q+p-2i}(\text{III}_a, \text{III}_a, \epsilon_{\ell+1-q-i})$	$\ell + 1 - 2q + 2p - 2i$	$su(\ell + 1 - q - i, p - i) +$	$BC_i +$
$1 \leq p < q \leq \frac{1}{2}(\ell + 1)$			$su(q - p + i, i) + so(2)$	BC_{p-i}
$0 \leq i \leq p$				
$B_\ell^{p,p}(\text{I}_a, \epsilon_i)$				
$p < \ell - p + 2i$	$B_\ell^{p,p}(\text{I}_a, \epsilon_i)$	p	$so(p, p) + so(2\ell + 1 - p - i, 0)$	B_p
$1 \leq p < \ell$				
$(0 \leq i \leq p)$				
$B_\ell^{q,p}(\text{I}_a, \text{I}_a, \epsilon_i)$				
$q - p + 2i \leq p < q$	$B_\ell^{q,q-p+2i}(\text{I}_a, \text{I}_a, \epsilon_i)$	$2i$	$so(q - p + i, i) +$	$B_i + B_{p-i}$
$1 \leq p < q \leq \ell$			$so(2\ell + 1 - q - i, p - i)$	
$0 \leq i \leq p$				

Table 5.1 – Continued

$B_\ell^{q,p}(I_a, I_a, \epsilon_i)$				
$q < q - p + 2i \leq \ell$	$B_\ell^{q,q-p+2i}(I_a, I_a, \epsilon_{p-i})$	p	$so(q - p + i, i) +$ $so(2\ell + 1 - q - i, p - i)$	$B_i + B_{p-i}$
$1 \leq p < q \leq \ell$				
$0 \leq i \leq p$				
$B_\ell^{q,p}(I_a, I_a, \epsilon_i)$				
$q \leq 2\ell + 1 - q + p - 2i < \ell$	$B_\ell^{q,2\ell+1-q+p-2i}(I_a, I_a, \epsilon_{q-p+i})$	p	$so(q - p + i) + so(2\ell +$ $1 - q - i, p - i)$	$B_i + B_{p-i}$
$1 \leq p < q \leq \ell$				
$0 \leq i \leq p$				
$B_\ell^{q,p}(I_a, I_a, \epsilon_i)$				
$p \leq 2\ell + 1 - q + p - 2i < q$	$B_\ell^{q,2\ell+1-q+p-2i}(I_a, I_a, \epsilon_{2\ell+1-q-i})$	$2\ell + 1 - 2q + 2p - 2i$	$so(q - p + i, i) +$ $so(2\ell + 1 - q - i, p - i)$	$B_i + B_{p-i}$
$1 \leq p < q \leq \ell$				
$0 \leq i \leq p$				
$B_\ell^{q,p}(I_a, I_a, \epsilon_i)$				
$2\ell + 1 - q + p - 2i < p < q$	$B_\ell^{q,2\ell+1-q+p-2i}(I_a, I_a, \epsilon_{2\ell+1-q-i})$	$2\ell + 1 - 2q + 2p - 2i$	$so(q - p + i, i) +$ $so(2\ell + 1 - q - i, p - i)$	$B_i + B_{p-i}$
$1 \leq p < q \leq \ell$				
$0 \leq i \leq p$				
$B_\ell^{p,q}(I_a, I_a, \epsilon_i)$				
$q - p + 2i \leq p < q$	$B_\ell^{p,q-p+2i}(I_a, I_a, \epsilon_{q-p+i})$	$q - p + 2i$	$so(q - p + i, i) +$ $so(2\ell + 1 - q - i, p - i)$	$B_i + B_{p-i}$
$1 \leq p < q \leq \ell$				
$0 \leq i \leq p$				

Table 5.1 – Continued

$B_\ell^{p,q}(I_a, I_a, \epsilon_i)$				
$p < q < q - p + 2i \leq \ell$	$B_\ell^{q,q-p+2i}(I_a, I_a, \epsilon_{p-i})$	p	$so(q-p+i, i) +$ $so(2\ell+1-q-i, p-i)$	$B_i + B_{p-i}$
$1 \leq p < q \leq \ell$				
$0 \leq i \leq p$				
$B_\ell^{p,q}(I_a, I_a, \epsilon_i)$				
$p < q \leq 2\ell+1-q+p-2i < \ell$	$B_\ell^{p,2\ell+1-q+p-2i}(I_a, I_a, \epsilon_i)$	p	$so(q-p+i, i) +$ $so(2\ell+1-q-i, p-i)$	$B_i + B_{p-i}$
$1 \leq p < q \leq \ell$				
$0 \leq i \leq p$				
$B_\ell^{p,q}(I_a, I_a, \epsilon_i)$				
$p \leq 2\ell+1-q+p-2i < q$	$B_\ell^{p,2\ell+1-q+p-2i}(I_a, I_a, \epsilon_i)$	p	$so(q-p+i, i) +$ $so(2\ell+1-q-i, p-i)$	$B_i + B_{p-i}$
$1 \leq p < q \leq \ell$				
$0 \leq i \leq p$				
$B_\ell^{p,q}(I_a, I_a, \epsilon_i)$				
$2\ell+1-q+p-2i < p < q$	$B_\ell^{q,2\ell+1-q+p-2i}(I_a, I_a, \epsilon_{2\ell-q-i})$	$2\ell+1-q+p-2i$	$so(q-p+i, i) +$ $so(2\ell+1-q-i, p-i)$	$B_i + B_{p-i}$
$1 \leq p < q \leq \ell$				
$0 \leq i \leq p$				
$C_\ell^{\ell,p}(I, II_a)$ ($2p \leq \ell$)	$C_\ell^{\ell,\ell}(I, \epsilon_p)$	p	$su(\ell-p, p) + so(2)$	BC_p
$C_\ell^{p,\ell}(II_a, I)$ ($2p \leq \ell$)	$C_\ell^{p,\ell}(II_a, I)$	p	$su(\ell-p, p)$	BC_p
$C_\ell^{\ell,\ell}(I, \epsilon_p)$ ($2p \leq \ell$)	$C_\ell^{\ell,p}(I, II_a)$	p	$sp(p, \mathbb{R}) + sp(\ell-p, \mathbb{R})$	$C_p + C_{\ell-p}$

Table 5.1 – Continued

$C_{2\ell}^{2\ell,\ell}(I, II_b, \epsilon_0)$	$C_{2\ell}^{2\ell,\ell}(I, II_b, \epsilon_0)$	0	$sp(\ell, \mathbb{C})$	C_ℓ
$C_{2\ell}^{\ell,2\ell}(II_b, I, \epsilon_0)$	$C_{2\ell}^\ell(II_b, \epsilon_\ell)$	ℓ	$sp(\ell, \mathbb{C})$	C_ℓ
$C_{2\ell}^{\ell,\ell}(II_b, \epsilon_\ell)$	$C_{2\ell}^{\ell,2\ell}(II_b, I, \epsilon_0)$	ℓ	$su^*(2\ell) + \mathbb{R}$	$A_{\ell-1} + \mathbb{R}$
$C_{2\ell}^{2\ell,\ell}(I, II_b, \epsilon_\ell)$	$C_{2\ell}^{2\ell,2\ell}(I, \epsilon_\ell)$	ℓ	$su(\ell, \ell) + so(2)$	C_ℓ
$C_{2\ell}^{\ell,2\ell}(II_b, I, \epsilon_\ell)$	$C_{2\ell}^{\ell,2\ell}(II_b, I, \epsilon_\ell)$	ℓ	$su(\ell, \ell) + so(2)$	C_ℓ
$C_{2\ell}^{2\ell,2\ell}(I, \epsilon_\ell)$	$C_{2\ell}^{2\ell,2\ell}(I, \epsilon_\ell)$	ℓ	$sp(\ell, \mathbb{R}) + sp(\ell, \mathbb{R})$	$C_\ell + C_\ell$
$C_\ell^{p,p}(II_a, \epsilon_p)$	$C_\ell^{p,\ell+1-2p}(II_a, \epsilon_p)$	p	$sp(\ell - 2p, 0) + sp(p, p)$	BC_p
$C_\ell^{q,p}(II_a, II_a, \epsilon_i)$ $q - p + 2i \leq p$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$	$C_\ell^{q,q-p+2i}(II_a, II_a, \epsilon_i)$	$2i$	$sp(\ell - q - i, p - i) +$ $sp(q - p + i)$	$BC_i + BC_{p-i}$
$C_\ell^{q,p}(II_a, II_a, \epsilon_i)$ $p < q < q - p + 2i \leq \frac{1}{2}\ell$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$	$C_\ell^{q,q-p+2i}(II_a, II_a, \epsilon_{p-i})$	p	$sp(\ell - q - i, p - i) +$ $sp(q - p + i)$	$BC_i + BC_{p-i}$
$C_\ell^{q,p}(II_a, II_a, \epsilon_i)$ $q \leq l - q + p - 2i < \frac{1}{2}\ell$ $(0 \leq i \leq p)$	$C_\ell^{q,\ell-q+p-2i}(II_a, II_a, \epsilon_{q-p+i})$	p	$sp(\ell - q - i, p - i) +$ $sp(q - p + i)$	$BC_i + BC_{p-i}$

Table 5.1 – Continued

$C_\ell^{q,p}(\Pi_a, \Pi_a, \epsilon_i)$				
$p \leq \ell q + p - 2i < q$	$C_\ell^{q,\ell-q+p-2i}(\Pi_a, \Pi_a, \epsilon_{\ell-q-i})$	$\ell - 2q + 2p - 2i$	$sp(\ell - q - i, p - i) +$ $sp(q - p + i)$	$BC_i +$ BC_{p-i}
$1 \leq p < q \leq \frac{1}{2}\ell$				
$(0 \leq i \leq p)$				
$C_\ell^{q,p}(\Pi_a, \Pi_a, \epsilon_i)$				
$q - p + 2i \leq p$	$C_\ell^{q,\ell-q+p-2i}(\Pi_a, \Pi_a, \epsilon_{\ell-q-i})$	$\ell - 2q + 2p - 2i$	$sp(\ell - q - i, p - i) +$ $sp(q - p + i)$	$BC_i +$ BC_{p-i}
$1 \leq p < q \leq \frac{1}{2}\ell$				
$(0 \leq i \leq p)$				
$C_\ell^{p,q}(\Pi_a, \Pi_a, \epsilon_i)$				
$q - p + 2i \leq p$	$C_\ell^{p,q-p+2i}(\Pi_a, \Pi_a, \epsilon_{q-p+i})$	$q - p + 2i$	$sp(q - p + i, i) + sp(\ell -$ $i, p - i)$	$BC_i +$ BC_{p-i}
$1 \leq p < q \leq \frac{1}{2}\ell$				
$(0 \leq i \leq p)$				
$C_\ell^{p,q}(\Pi_a, \Pi_a, \epsilon_i)$				
$p < q - p + 2i \leq \frac{1}{2}\ell$	$C_\ell^{p,q-p+2i}(\Pi_a, \Pi_a, \epsilon_{p-i})$	p	$sp(q - p + i, i) + sp(\ell -$ $i, p - i)$	$BC_i +$ BC_{p-i}
$1 \leq p < q \leq \frac{1}{2}\ell$				
$(0 \leq i \leq p)$				
$C_\ell^{p,q}(\Pi_a, \Pi_a, \epsilon_i)$				
$q \leq \ell - q + p - 2i < \frac{1}{2}\ell$	$C_\ell^{p,\ell-q+p-2i}(\Pi_a, \Pi_a, \epsilon_i)$	p	$sp(q - p + i, i) + sp(\ell -$ $i, p - i)$	$BC_i +$ BC_{p-i}
$1 \leq p < q \leq \frac{1}{2}\ell$				
$(0 \leq i \leq p)$				

Table 5.1 – Continued

$C_\ell^{p,q}(\Pi_a, \Pi_a, \epsilon_i)$	$C_\ell^{p,\ell-q+p-2i}(\Pi_a, \Pi_a, \epsilon_i)$	p	$sp(q-p+i, i) + sp(\ell-i, p-i)$	$BC_i + BC_{p-i}$
$p \leq \ell - q + p - 2i < q$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$				
$C_\ell^{p,q}(\Pi_a, \Pi_a, \epsilon_i)$	$C_\ell^{p,\ell-q+p-2i}(\Pi_a, \Pi_a, \epsilon_{\ell-q-i})$	$\ell - q + p - 2i$	$sp(q-p+i, i) + sp(\ell-i, p-i)$	$BC_i + BC_{p-i}$
$\ell - q + p - 2i < p$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$				
$C_{2\ell}^{\ell,p}(\Pi_b, \Pi_a, \epsilon_i)$	$C_{2\ell}^{\ell,\ell-p+2i}(\Pi_b, \Pi_a, \epsilon_i)$	$2i$	$sp(\ell-p+i, i) + sp(\ell-i, p-i)$	$BC_i + BC_{p-i}$
$\ell - p + 2i \leq p$ $0 \leq i \leq p - 1$ $1 \leq p \leq \ell$				
$C_{2\ell}^{\ell,p}(\Pi_b, \Pi_a, \epsilon_i)$	$C_{2\ell}^{\ell,\ell-p+2i}(\Pi_b, \Pi_a, \epsilon_{p-i})$	$2i$	$sp(\ell-p+i, i) + sp(\ell-i, p-i)$	$BC_i + BC_{p-i}$
$p < \ell - p + 2i \leq \ell$ $0 \leq i \leq p - 1$ $1 \leq p \leq \ell$				
$C_{2\ell}^{\ell,p}(\Pi_b, \Pi_a, \epsilon_i)$	$C_{2\ell}^{\ell,\ell+p-2i}(\Pi_b, \Pi_a, \epsilon_{\ell-i})$	$2p - 2i$	$sp(\ell-p+i, i) + sp(\ell-i, p-i)$	$BC_i + BC_{p-i}$
$p \leq l + p - 2i < l$ $0 \leq i \leq p - 1$ $1 \leq p \leq \ell$				

Table 5.1 – Continued

$C_{2\ell}^{\ell,p}(\Pi_b, \Pi_a, \epsilon_i)$	$C_{2\ell}^{\ell,\ell+p-2i}(\Pi_b, \Pi_a, \epsilon_{\ell-i})$	$2p - 2i$	$sp(\ell - p + i, i) + sp(\ell - i, p - i)$	$BC_i + BC_{p-i}$
$\ell + p - 2i < p$				
$0 \leq i \leq p - 1$				
$1 \leq p \leq \ell$				
$C_{2\ell}^{p,\ell}(\Pi_a, \Pi_b, \epsilon_i)$	$C_{2\ell}^{p,\ell-p+2i}(\Pi_a, \Pi_a, \epsilon_{\ell-p+i})$	$\ell - p + 2i$	$sp(\ell - p + i, i) + sp(\ell - i, p - i)$	$BC_i + BC_{p-i}$
$\ell - p + 2i \leq p$				
$0 \leq i \leq p - 1$				
$1 \leq p \leq \ell$				
$C_{2\ell}^{p,\ell}(\Pi_a, \Pi_b, \epsilon_i)$	$C_{2\ell}^{p,\ell-p+2i}(\Pi_a, \Pi_a, \epsilon_{p-i})$	p	$sp(\ell - p + i, i) + sp(\ell - i, p - i)$	$BC_i + BC_{p-i}$
$p < \ell - p + 2i \leq \ell$				
$0 \leq i \leq p - 1$				
$1 \leq p \leq \ell$				
$C_{2\ell}^{p,\ell}(\Pi_a, \Pi_b, \epsilon_i)$	$C_{2\ell}^{p,\ell+p-2i}(\Pi_a, \Pi_a, \epsilon_i)$	p	$sp(\ell - p + i, i) + sp(\ell - i, p - i)$	$BC_i + BC_{p-i}$
$p \leq \ell + p - 2i < \ell$				
$0 \leq i \leq p - 1$				
$1 \leq p \leq \ell$				
$C_{2\ell}^{p,\ell}(\Pi_a, \Pi_b, \epsilon_i)$	$C_{2\ell}^{p,\ell+p-2i}(\Pi_a, \Pi_a, \epsilon_{\ell-i})$	$\ell + p - 2i$	$sp(\ell - p + i, i) + sp(\ell - i, p - i)$	$BC_i + BC_{p-i}$
$\ell + p - 2i < p$				
$0 \leq i \leq p - 1$				
$1 \leq p \leq \ell$				

Table 5.1 – Continued

$D_\ell^{q,p}(I_a, I_a, \epsilon_i)$				
$q - p + 2i \leq p$				
$1 \leq p < q \leq \ell - 1$	$D_\ell^{q,q-p+2i}(I_a, I_a, \epsilon_i)$	$2i$	$so(q - p + i, i) +$	$B_i + B_{p-i}$
$(0 \leq i \leq p - 1)$			$so(2\ell - q - i, p - i)$	
<hr/>				
$D_\ell^{q,p}(I_a, I_a, \epsilon_i)$				
$p < q < q - p + 2i \leq \ell$				
$1 \leq p < q \leq \ell - 1$	$D_\ell^{q,q-p+2i}(I_a, I_a, \epsilon_{p-i})$	p	$so(q - p + i, i) +$	$B_i + B_{p-i}$
$(0 \leq i \leq p - 1)$			$so(2\ell - q - i, p - i)$	
<hr/>				
$D_\ell^{q,p}(I_a, I_a, \epsilon_i)$				
$p < q \leq 2\ell - q + p - 2i < \ell$				
$1 \leq p < q \leq \ell - 1$	$D_\ell^{q,2\ell-q+p-2i}(I_a, I_a, \epsilon_{q-p+ii})$	p	$so(q - p + i, i) +$	$B_i + B_{p-i}$
$(0 \leq i \leq p - 1)$			$so(2\ell - q - i, p - i)$	
<hr/>				
$D_\ell^{q,p}(I_a, I_a, \epsilon_i)$				
$p \leq 2\ell - q + p - 2i < q$				
$1 \leq p < q \leq \ell - 1$	$D_\ell^{q,2\ell-q+p-2i}(I_a, I_a, \epsilon_{2\ell-q-i})$	$2\ell - 2q + 2p - 2i$	$so(q - p + i, i) +$	$B_i + B_{p-i}$
$(0 \leq i \leq p - 1)$			$so(2\ell - q - i, p - i)$	
<hr/>				
$D_\ell^{q,p}(I_a, I_a, \epsilon_i)$				
$2\ell - q + p - 2i < p$				
$1 \leq p < q \leq \ell - 1$	$D_\ell^{q,2\ell-q+p-2i}(I_a, I_a, \epsilon_{2\ell-q-i})$	$2\ell - 2q + 2p - 2i$	$so(q - p + i, i) +$	$B_i + B_{p-i}$
$(0 \leq i \leq p - 1)$			$so(2\ell - q - i, p - i)$	

Table 5.1 – Continued

$D_\ell^{p,q}(I_a, I_a, \epsilon_i)$				
$q - p + 2i \leq p$				
$1 \leq p < q \leq \ell - 1$	$D_\ell^{q,q-p+2i}(I_a, I_a, \epsilon_{q-p+i})$	$q - p + 2i$	$so(q - p + i, i) +$	$B_i + B_{p-i}$
$0 \leq i \leq p - 1$			$so(2\ell - q - i, p - i)$	
<hr/>				
$D_\ell^{p,q}(I_a, I_a, \epsilon_i)$				
$p < q < q - p + 2i \leq \ell$				
$1 \leq p < q \leq \ell - 1$	$D_\ell^{q,q-p+2i}(I_a, I_a, \epsilon_{p-i})$	p	$so(q - p + i, i) +$	$B_i + B_{p-i}$
$0 \leq i \leq p - 1$			$so(2\ell - q - i, p - i)$	
<hr/>				
$D_\ell^{p,q}(I_a, I_a, \epsilon_i)$				
$p < q \leq 2\ell - q + p - 2i < \ell$				
$1 \leq p < q \leq \ell - 1$	$D_\ell^{p,2\ell-q+p-2i}(I_a, I_a, \epsilon_i)$	p	$so(q - p + i, i) +$	$B_i + B_{p-i}$
$0 \leq i \leq p - 1$			$so(2\ell - q - i, p - i)$	
<hr/>				
$D_\ell^{p,q}(I_a, I_a, \epsilon_i)$				
$p \leq 2\ell - q + p - 2i < q$				
$1 \leq p < q \leq \ell - 1$	$D_\ell^{p,2\ell-q+p-2i}(I_a, I_a, \epsilon_i)$	p	$so(q - p + i, i) +$	$B_i + B_{p-i}$
$0 \leq i \leq p - 1$			$so(2\ell - q - i, p - i)$	
<hr/>				
$D_\ell^{p,q}(I_a, I_a, \epsilon_i)$				
$2\ell - q + p - 2i < p < q$				
$1 \leq p < q \leq \ell - 1$	$D_\ell^{q,2\ell-q+p-2i}(I_a, I_a, \epsilon_{2\ell-q-i})$	$2\ell - q + p - 2i$	$so(q - p + i, i) +$	$B_i + B_{p-i}$
$0 \leq i \leq p - 1$			$so(2\ell - q - i, p - i)$	

Table 5.1 – Continued

$D_\ell^{\ell,p}(I_b, I_a, \epsilon_i)$	$D_\ell^{\ell,\ell-p+2i}(I_b, I_a, \epsilon_i)$	$2i$	$so(i, \ell - p + i) + so(\ell - i, p - i)$	$B_i + B_{p-i}$
$\ell - p + 2i \leq p$				
$0 \leq i \leq p - 1$				
$1 \leq p < \ell$				
$D_\ell^{\ell,p}(I_b, I_a, \epsilon_i)$	$D_\ell^{\ell,\ell-p+2i}(I_b, I_a, \epsilon_{p-i})$	$2i$	$so(i, \ell - p + i) + so(\ell - i, p - i)$	$B_i + B_{p-i}$
$p < \ell - p + 2i \leq \ell$				
$0 \leq i \leq p - 1$				
$1 \leq p < \ell$				
$D_\ell^{\ell,p}(I_b, I_a, \epsilon_i)$	$D_\ell^{\ell,\ell+p-2i}(I_b, I_a, \epsilon_{\ell-i})$	$2p - 2i$	$so(i, \ell - p + i) + so(\ell - i, p - i)$	$B_i + B_{p-i}$
$p \leq \ell + p - 2i < \ell$				
$0 \leq i \leq p - 1$				
$1 \leq p < \ell$				
$D_\ell^{\ell,p}(I_b, I_a, \epsilon_i)$	$D_\ell^{\ell,\ell+p-2i}(I_b, I_a, \epsilon_{\ell-i})$	$2p - 2i$	$so(i, \ell - p + i) + so(\ell - i, p - i)$	$B_i + B_{p-i}$
$\ell + p - 2i < p$				
$0 \leq i \leq p - 1$				
$1 \leq p < \ell$				
$D_\ell^{p,\ell}(I_a, I_b, \epsilon_i)$	$D_\ell^{p,\ell-p+2i}(I_b, I_a, \epsilon_{\ell-p+i})$	$\ell - p + 2i$	$so(i, \ell - p + i) + so(\ell - i, p - i)$	$B_i + B_{p-i}$
$\ell - p + 2i \leq p$				
$1 \leq p < \ell$				
$0 \leq i \leq p - 1$				

Table 5.1 – Continued

$D_\ell^{p,\ell}(I_a, I_b, \epsilon_i)$	$D_\ell^{p,\ell-p+2i}(I_b, I_a, \epsilon_{p-i})$	p	$so(i, \ell - p + i) + so(\ell - i, p - i)$	$B_i + B_{p-i}$
$p \leq \ell - p + 2i \leq \ell$				
$1 \leq p < \ell$				
$0 \leq i \leq p - 1$				
$D_\ell^{p,\ell}(I_a, I_b, \epsilon_i)$	$D_\ell^{p,\ell+p-2i}(I_b, I_a, \epsilon_i)$	p	$so(i, \ell - p + i) + so(\ell - i, p - i)$	$B_i + B_{p-i}$
$p < \ell + p - 2i \leq \ell$				
$1 \leq p < \ell$				
$0 \leq i \leq p - 1$				
$D_\ell^{p,\ell}(I_a, I_b, \epsilon_i)$	$D_\ell^{p,\ell+p-2i}(I_b, I_a, \epsilon_{\ell-i})$	$\ell + p - 2i$	$so(i, \ell - p + i) + so(\ell - i, p - i)$	$B_i + B_{p-i}$
$\ell + p - 2i < p$				
$1 \leq p < \ell$				
$0 \leq i \leq p - 1$				
$D_{2\ell}^{\ell,4p}(III_a, I_a)$ $(1 \leq 2p \leq \ell - 1)$	$D_{2\ell}^{\ell,\ell}(III_a, \epsilon_p)$	$2p$	$su(2\ell - 2p, 2p)$	BC_{2p}
$D_{2\ell}^{2p,\ell}(I_a, III_a)$ $(1 \leq 2p \leq 2\ell - 1)$	$D_{2\ell}^{2p,\ell}(I_a, III_a)$	0	$su(2\ell - p, p)$	BC_p
$D_{2\ell}^{\ell,\ell}(III_a, \epsilon_p)$ $(1 \leq 2p \leq \ell - 1)$	$D_{2\ell}^{\ell,4p}(III_a, I_a)$	$2p$	$so^*(4p) + so^*(4\ell - 4p)$	$C_p + C_{\ell-p}$
$D_{2\ell+1}^{\ell,4p}(III_b, I_a)$ $(1 \leq 2p \leq \ell)$	$D_{2\ell+1}^{\ell,\ell}(III_b, \epsilon_p)$	$2p$	$su(2\ell + 1 - 2p, 2p)$	BC_{2p}
$D_{2\ell+1}^{2p,\ell}(I_a, III_b)$ $(1 \leq 2p \leq 2\ell)$	$D_{2\ell+1}^{2p,\ell}(I_a, III_b)$	0	$su(2\ell - p + 1, p) + so(2)$	BC_p

Table 5.1 – Continued

$D_{2\ell+1}^{\ell,\ell}(\text{III}_b, \epsilon_p)$ ($1 \leq 2p \leq \ell$)	$D_{2\ell+1}^{\ell,4p}(\text{III}_b, \text{I}_a)$	$2p$	$so^*(4p) + so^*(4\ell - 4p + 2)$	$C_p +$ $BC_{\ell-p}$
$D_{4\ell}^{2\ell,4\ell}(\text{III}_a, \text{I}_b, \epsilon_0)$	$D_{4\ell}^{2\ell,2\ell}(\text{III}_a, \epsilon_\ell)$	2ℓ	$su(2\ell, 2\ell)$	$C_{2\ell}$
$D_{2\ell}^{2\ell,\ell}(\text{I}_b, \text{III}_a, \epsilon_0)$	$D_{2\ell}^{2\ell,\ell}(\text{I}_b, \text{III}_a, \epsilon_0)$	ℓ	$su(\ell, \ell) + so(2)$	C_ℓ
$D_{4\ell}^{2\ell,2\ell}(\text{III}_a, \epsilon_\ell)$	$D_{4\ell}^{2\ell,4\ell}(\text{III}_a, \text{I}_b, \epsilon_0)$	ℓ	$so^*(4\ell) + so^*(4\ell)$	$C_\ell + C_\ell$
$D_{2\ell}^{\ell,2\ell}(\text{III}_a, \text{I}_b, \epsilon_\ell)$	$D_{2\ell}^{\ell,2\ell}(\text{III}_a, \text{I}_b, \epsilon_\ell)$	ℓ	$so(2\ell, \mathbb{C})$	D_ℓ
$D_{2\ell}^{2\ell,\ell}(\text{I}_b, \text{III}_a, \epsilon_\ell)$	$D_{2\ell}^{2\ell,2\ell}(\text{I}_b, \epsilon_\ell)$	ℓ	$so(2\ell, \mathbb{C})$	D_ℓ
$D_{2\ell}^{2\ell,2\ell}(\text{I}_b, \epsilon_\ell)$	$D_{2\ell}^{2\ell,\ell}(\text{I}_b, \text{III}_a, \epsilon_\ell)$	ℓ	$sl(2\ell, \mathbb{R}) + \mathbb{R}$	$A_{2\ell-1} + \mathbb{R}$
$D_{2\ell+1}^{\ell,2\ell+1}(\text{III}_b, \text{I}_b)$	$D_{2\ell+1}^{\ell,2\ell+1}(\text{III}_b, \text{I}_b)$	0	$so(2\ell + 1, \mathbb{C})$	B_ℓ
$D_{2\ell+1}^{2\ell+1,\ell}(\text{I}_b, \text{III}_b)$	$D_{2\ell+1}^{2\ell+1,2\ell+1}(\text{I}_b, \epsilon_\ell)$	ℓ	$so(2\ell + 1, \mathbb{C})$	B_ℓ
$D_{2\ell+1}^{2\ell+1,2\ell+1}(\text{I}_b, \epsilon_\ell)$	$D_{2\ell+1}^{2\ell+1,\ell}(\text{I}_b, \text{III}_b)$	ℓ	$sl(2\ell + 1, \mathbb{R}) + \mathbb{R}$	$A_{2\ell} + \mathbb{R}$
$D_\ell^{p,p}(I_a, \epsilon_p)$	$D_\ell^{p,\ell-2p}(I_a, \epsilon_p)$	p	$sp(p, p) + sp(\ell - 2p, 0)$	BC_p
$E_6^{6,4}(\text{I}, \text{II}, \epsilon_0)$	$E_6^{6,2}(\text{I}, \text{IV})$	0	$f_{4(4)}$	F_4
$E_6^{4,6}(\text{II}, \text{I}, \epsilon_0)$	$E_6^{4,2}(\text{II}, \text{IV})$	1	$f_{4(4)}$	F_4
$E_6^{6,2}(\text{I}, \text{IV})$	$E_6^{6,4}(\text{I}, \text{II}, \epsilon_0)$	0	$su^*(6) + su(2)$	A_2
$E_6^{2,6}(\text{IV}, \text{I})$	$E_6^{2,4}(\text{IV}, \text{II})$	1	$su^*(6) + su(2)$	A_2
$E_6^{4,2}(\text{II}, \text{IV})$	$E_6^{4,6}(\text{II}, \text{I}, \epsilon_0)$	1	$sp(3, 1)$	BC_1

Table 5.1 – Continued

$E_6^{2,4}(\text{IV}, \text{II})$	$E_6^{2,6}(\text{IV}, \text{I})$	1	$sp(3, 1)$	BC_1
$E_6^{6,4}(\text{I}, \text{II}, \epsilon_1)$	$E_6^{6,6}(\text{I}, \epsilon_2)$	4	$sp(4, \mathbb{R})$	C_4
$E_6^{4,6}(\text{II}, \text{I}, \epsilon_1)$	$E_6^{4,6}(\text{II}, \text{I}, \epsilon_1)$	4	$sp(4, \mathbb{R})$	C_4
$E_6^{6,6}(\text{I}, \epsilon_2)$	$E_6^{6,4}(\text{I}, \text{II}, \epsilon_1)$	4	$sl(6, \mathbb{R}) + sl(2, \mathbb{R})$	$A_5 + A_1$
$E_6^{6,2}(\text{I}, \text{III})$	$E_6^{6,6}(\text{I}, \epsilon_1)$	2	$sp(2, 2)$	C_2
$E_6^{2,6}(\text{III}, \text{I})$	$E_6^{2,6}(\text{III}, \text{I})$	2	$sp(2, 2)$	C_2
$E_6^{6,6}(\text{I}, \epsilon_1)$	$E_6^{6,2}(\text{I}, \text{III})$	2	$so(5, 5) + \mathbb{R}$	$B_5 + \mathbb{R}$
$E_6^{4,2}(\text{II}, \text{III}, \epsilon_0)$	$E_6^{4,2}(\text{II}, \text{III}, \epsilon_0)$	0	$so^*(10) + so(2)$	BC_2
$E_6^{2,4}(\text{III}, \text{II}, \epsilon_0)$	$E_6^{2,2}(\text{III}, \epsilon_1)$	2	$so^*(10) + so(2)$	BC_2
$E_6^{2,2}(\text{III}, \epsilon_1)$	$E_6^{2,4}(\text{III}, \text{II}, \epsilon_0)$	2	$su(5, 1) + sl(2, \mathbb{R})$	$BC_1 + A_1$
$E_6^{2,2}(\text{III}, \epsilon_2)$	$E_6^{2,2}(\text{III}, \epsilon_2)$	2	$su(5, 1) + sl(2, \mathbb{R})$	$BC_1 + A_1$
$E_6^{4,2}(\text{II}, \text{III}, \epsilon_1)$	$E_6^{4,4}(\text{II}, \epsilon_4)$	2	$so(8, 2) + so(2)$	B_2
$E_6^{2,4}(\text{III}, \text{II}, \epsilon_1)$	$E_6^{2,4}(\text{III}, \text{II}, \epsilon_1)$	0	$su(4, 2) + su(2)$	BC_2
$E_6^{4,4}(\text{II}, \epsilon_4)$	$E_6^{4,2}(\text{II}, \text{III}, \epsilon_1)$	2	$so(6, 2) + so(2)$	B_4
$E_6^{4,4}(\text{II}, \epsilon_1)$	$E_6^{4,4}(\text{II}, \epsilon_1)$	4	$su(3, 3) + sl(2, \mathbb{R})$	$C_3 + A_1$
$E_6^{2,2}(\text{III}, \text{IV})$	$E_6^{2,2}(\text{III}, \text{IV})$	0	$f_{4(-20)}$	BC_1
$E_6^{2,2}(\text{IV}, \text{III})$	$E_6^{2,2}(\text{IV}, \epsilon_1)$	1	$f_{4(-20)}$	BC_1

Table 5.1 – Continued

$E_6^{2,2}(\text{IV}, \epsilon_1)$	$E_6^{2,2}(\text{IV}, \text{III})$	1	$so(9, 1) + \mathbb{R}$	$B_1 + \mathbb{R}$
$E_7^{7,4}(\text{V}, \text{VI}, \epsilon_0)$	$E_7^{7,3}(\text{V}, \text{VII})$	0	$e_{6(2)} + so(2)$	F_4
$E_7^{4,7}(\text{VI}, \text{V}, \epsilon_0)$	$E_7^{4,3}(\text{VI}, \text{VII}, \epsilon_1)$	2	$e_{6(2)} + so(2)$	F_4
$E_7^{7,3}(\text{V}, \text{VII})$	$E_7^{7,4}(\text{V}, \text{VI}, \epsilon_0)$	0	$so^*(12) + su(2)$	C_3
$E_7^{3,7}(\text{VII}, \text{V})$	$E_7^{3,4}(\text{VII}, \text{VI}, \epsilon_1)$	2	$so^*(12) + su(2)$	C_3
$E_7^{4,3}(\text{VI}, \text{VII}, \epsilon_1)$	$E_7^{4,7}(\text{VI}, \text{V}, \epsilon_0)$	2	$su(6, 2)$	BC_2
$E_7^{3,4}(\text{VII}, \text{VI}, \epsilon_1)$	$E_7^{3,7}(\text{VII}, \text{V})$	2	$su(6, 2)$	BC_2
$E_7^{7,4}(\text{V}, \text{VI}, \epsilon_1)$	$E_7^{7,t}(\text{V}, \epsilon_1)$	4	$su(4, 4)$	C_4
$E_7^{4,4}(\text{VI}, \epsilon_1)$	$E_7^{4,4}(\text{VI}, \epsilon_1)$	4	$so^*(12) + sl(2, \mathbb{R})$	$C_3 + A_1$
$E_7^{4,4}(\text{VI}, \epsilon_4)$	$E_7^{4,4}(\text{VI}, \epsilon_4)$	4	$so(8, 4) + su(2)$	B_4
$E_7^{4,7}(\text{VI}, \text{V}, \epsilon_1)$	$E_7^{4,7}(\text{VI}, \text{V}, \epsilon_1)$	4	$su(4, 4)$	C_4
$E_7^{7,7}(\text{V}, \epsilon_1)$	$E_7^{7,4}(\text{V}, \text{VI}, \epsilon_1)$	4	$so(6, 6) + sl(2, \mathbb{R})$	$D_6 + A_1$
$E_7^{7,3}(\text{V}, \text{VII}, \epsilon_1)$	$E_7^{7,7}(\text{V}, \epsilon_7)$	3	$su^*(8)$	A_3
$E_7^{3,7}(\text{VII}, \text{V}, \epsilon_1)$	$E_7^{3,7}(\text{VII}, \text{V}, \epsilon_1)$	3	$su^*(8)$	A_3
$E_7^{7,7}(\text{V}, \epsilon_7)$	$E_7^{7,3}(\text{V}, \text{VII}, \epsilon_1)$	3	$e_{6(6)} + \mathbb{R}$	$E_6 + \mathbb{R}$
$E_7^{7,7}(\text{V}, \epsilon_2)$	$E_7^{7,7}(\text{V}, \epsilon_2)$	7	$sl(8, \mathbb{R})$	A_7
$E_7^{3,4}(\text{VII}, \text{VI})$	$E_7^{3,3}(\text{VII}, \epsilon_3)$	2	$e_{6(-14)} + so(2)$	BC_2

Table 5.1 – Continued

$E_7^{4,3}(\text{VI}, \text{VII})$	$E_7^{4,3}(\text{VI}, \text{VII})$	2	$e_{6(-14)} + so(2)$	BC_2
$E_7^{3,3}(\text{VII}, \epsilon_3)$	$E_7^{3,4}(\text{VII}, \text{VI})$	2	$so(10, 2) + sl(2, \mathbb{R})$	$B_2 + A_1$
$E_7^{3,3}(\text{VII}, \epsilon_1)$	$E_7^{3,3}(\text{VII}, \epsilon_1)$	3	$e_{6(-26)} + sl(2, \mathbb{R})$	$A_2 + A_1$
$E_8^{8,4}(\text{VIII}, \text{IX}, \epsilon_0)$	$E_8^{8,4}(\text{VIII}, \text{IX}, \epsilon_0)$	0	$e_{7(-5)} + so(2)$	F_4
$E_8^{4,8}(\text{IX}, \text{VIII}, \epsilon_0)$	$E_8^{4,4}(\text{IX}, \epsilon_4)$	4	$e_{7(-5)} + so(2)$	F_4
$E_8^{4,4}(\text{IX}, \epsilon_4)$	$E_8^{4,8}(\text{IX}, \text{VIII}, \epsilon_0)$	4	$so(12, 4)$	B_4
$E_8^{4,4}(\text{IX}, \epsilon_1)$	$E_8^{4,4}(\text{IX}, \epsilon_1)$	4	$e_{7(-25)} + sl(2, \mathbb{R})$	$C_3 + A_1$
$E_8^{8,4}(\text{VIII}, \text{IX}, \epsilon_1)$	$E_8^{8,8}(\text{VIII}, \epsilon_8)$	4	$so^*(16)$	C_4
$E_8^{4,8}(\text{IX}, \text{VIII}, \epsilon_1)$	$E_8^{4,8}(\text{IX}, \text{VIII}, \epsilon_1)$	0	$so^*(16)$	C_4
$E_8^{8,8}(\text{VIII}, \epsilon_8)$	$E_8^{8,4}(\text{VIII}, \text{IX}, \epsilon_1)$	4	$e_{7(7)} + sl(2, \mathbb{R})$	$E_7 + A_1$
$F_4^{4,1}(\text{I}, \text{II})$	$F_4^{4,4}(\text{I}, \epsilon_4)$	1	$sp(2, 1) + su(2)$	BC_1
$F_4^{1,4}(\text{II}, \text{I})$	$F_4^{1,4}(\text{II}, \text{I})$	1	$sp(2, 1) + su(2)$	BC_1
$F_4^{4,4}(\text{I}, \epsilon_4)$	$F_4^{4,1}(\text{I}, \text{II})$	1	$so(5, 4)$	B_4
$F_4^{4,4}(\text{I}, \epsilon_1)$	$F_4^{4,4}(\text{I}, \epsilon_1)$	4	$sp(3, \mathbb{R}) + sl(2, \mathbb{R})$	$C_3 + A_1$
$G_2^{2,2}(\text{I}, \epsilon_1)$	$G_2^{2,2}(\text{I}, \epsilon_1)$	2	$sl(2, \mathbb{R}) + sl(2, \mathbb{R})$	$A_1 + A_1$

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APPENDIX

Appendix A

Complete Tables for Classification

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell+1}^{2\ell+1,\ell}(\text{I}, \text{II})$		$A_{2\ell+1}^{2\ell+1,\ell+1}(\text{I}, \text{III}_b, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		\emptyset	$\ell+1$		$(\ell+1) \cdot A_1$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
\emptyset	$A_{2\ell+1}$	id	$(\ell+1) \cdot A_1$	0	- - - -
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sl(\ell+1, \mathbb{C}) + so(2)$			$AI \times AI + \mathbb{C}$		A_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{4\ell-1}^{2\ell-1, 4\ell-1}(\text{II}, \text{I})$		$A_{4\ell-1}^{2\ell-1, 2\ell}(\text{II}, \text{III}_b, \epsilon_\ell)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
$2\ell-1$		$A_{2\ell-1}$	ℓ		$\ell \cdot A_1$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$A_{2\ell-1}$	$A(\ell)$	$\ell \cdot A_1$	ℓ	2 2 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sl(2\ell, \mathbb{C}) + so(2)$				$AI \times AI + \mathbb{C}$	$A_{2\ell-1}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{4\ell+1}^{2\ell, 4\ell+1}(\text{II}, \text{I})$		$A_{4\ell+1}^{2\ell, 2\ell+1}(\text{II}, \text{III}_b)$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2ℓ		$A_{2\ell}$	ℓ		$\ell \cdot A_1$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$A_{2\ell}$	$A(\ell)$	$\ell \cdot A_1$	ℓ	2 2 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sl(2\ell+1, \mathbb{C}) + so(2)$				$AI \times AI + \mathbb{C}$	$A_{2\ell}$

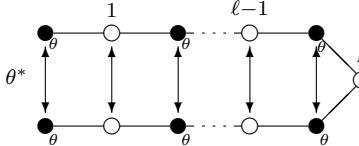
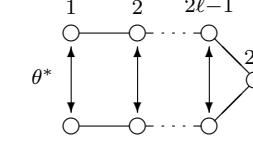
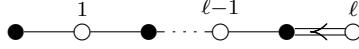
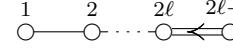
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell-1}^{2\ell-1,\ell}(\text{I, III}_b, \epsilon_0)$		$A_{2\ell-1}^{2\ell-1,\ell-1}(\text{I, II})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		$\ell \cdot A_1$	$\ell - 1$		\emptyset
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$A_{2\ell-1}$	$A(\ell)$	\emptyset	0	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell, \mathbb{R})$				CI	C_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
$A_{4\ell-1}^{2\ell, 4\ell-1}(\text{III}_b, \text{I}, \epsilon_0)$		$A_{4\ell-1}^{2\ell, 2\ell-1}(\text{III}_b, \text{II}, \epsilon_\ell)$			
2ℓ		$C_{2\ell}$	ℓ		$\ell \cdot A_1$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$C_{2\ell}$	$C_{2\ell}$	$C(2\ell)$	$\ell \cdot A_1$	ℓ	1 $\frac{1}{\lambda_\ell : 0}$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(2\ell, \mathbb{R})$				CI	$C_{2\ell}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{4\ell+1}^{2\ell+1, 4\ell+1}(\text{III}_b, \text{I}, \epsilon_0)$		$A_{4\ell+1}^{2\ell+1, 2\ell}(\text{III}_b, \text{II})$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
$2\ell + 1$		$C_{2\ell+1}$	ℓ		$\ell \cdot A_1$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$C_{2\ell+1}$	$C_{2\ell+1}$	$C(2\ell + 1)$	$\ell \cdot A_1$	ℓ	1 $\frac{1}{\lambda_\ell : 0}$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(2\ell + 1, \mathbb{R})$				CI	$C_{2\ell+1}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{4\ell-1}^{2\ell-1,2\ell}(\text{II}, \text{III}_b, \epsilon_\ell)$		$A_{4\ell-1}^{2\ell-1,4\ell-1}(\text{II}, \text{I})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		$\ell \cdot A_1$	$2\ell - 1$		$A_{2\ell-1}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$A_{2\ell-1}$	$A(\ell)$	$\ell \cdot A_1$	ℓ	1 1 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so^*(4\ell)$				$DIII_a$	C_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{4\ell+1}^{2\ell, 2\ell+1}(\text{II}, \text{III}_b)$		$A_{4\ell+1}^{2\ell, 4\ell+1}(\text{II}, \text{I})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		$\ell \cdot A_1$	2ℓ		$A_{2\ell}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$A_{2\ell}$	$A(\ell)$	$\ell \cdot A_1$	ℓ	1 1 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so^*(4\ell + 2)$				$DIII_b$	BC_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{4\ell-1}^{2\ell, 2\ell-1}(\text{III}_b, \text{II}, \epsilon_\ell)$		$A_{4\ell-1}^{2\ell, 2\ell-1}(\text{III}_b, \text{I}, \epsilon_0)$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		$\ell \cdot A_1$	2ℓ		$C_{2\ell}$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$C_{2\ell}$	$A(\ell)$	$C_{2\ell}$	ℓ	1 1 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so^*(4\ell)$				$DIII_a$	C_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{4\ell+1}^{2\ell+1, 2\ell}(\text{III}_b, \text{II})$		$A_{4\ell+1}^{2\ell+1, 4\ell+1}(\text{III}_b, \text{I}, \epsilon_0)$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		$\ell \cdot A_1$	$2\ell+1$		$C_{2\ell+1}$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$C_{2\ell+1}$	$A(\ell)$	$C_{2\ell+1}$	ℓ	1 1 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so^*(4\ell+2)$			$DIII_b$		BC_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell-1}^{2\ell-1,\ell}(\text{I}, \text{III}_b, \epsilon_\ell)$		$A_{2\ell-1}^{2\ell-1}(\text{I}, \epsilon_\ell)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		$\ell \cdot A_1$	$2\ell-1$		$A_{2\ell-1}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$A_{2\ell-1}$	$A(\ell)$	$\ell \cdot A_1$	ℓ	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(\ell, \ell)$				BI	B_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell-1}^{\ell, 2\ell-1}(\text{III}_b, \text{I}, \epsilon_\ell)$		$A_{2\ell-1}^{\ell, 2\ell-1}(\text{III}_b, \text{I}, \epsilon_\ell)$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		C_ℓ	ℓ		C_ℓ
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_ℓ	C_ℓ	$B(\ell)$	C_ℓ	ℓ	1 $\begin{matrix} 1 \\ \lambda_\ell : 0 \end{matrix}$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(\ell, \ell)$				BI	B_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell-1}^{2\ell-1,2\ell-1}(\text{I}, \epsilon_\ell)$		$A_{2\ell-1}^{2\ell-1,\ell}(\text{I}, \text{III}_b, \epsilon_\ell)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
$2\ell-1$		$A_{2\ell-1}$	ℓ		$\ell \cdot A_1$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$A_{2\ell-1}$	$A(\ell)$	$\ell \cdot A_1$	ℓ	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sl(\ell, \mathbb{R}) + sl(\ell, \mathbb{R}) + \mathbb{R}$				$AI + AI + \mathbb{C}$	$A_{\ell-1} + A_{\ell-1} + \mathbb{R}$

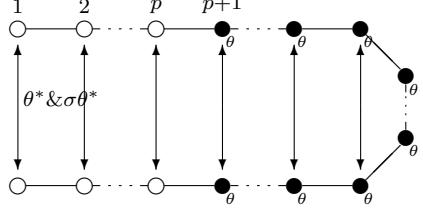
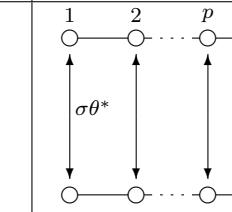
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{4\ell-1}^{2\ell-1, 2\ell}(\text{II}, \text{III}_b, \epsilon_0)$		$A_{4\ell-1}^{2\ell-1}(\text{II}, \epsilon_\ell)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		$\ell \cdot A_1$	$2\ell - 1$		$A_{2\ell-1}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$A_{2\ell-1}$	$A(\ell)$	$\ell \cdot A_1$	ℓ	1 1 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell, \ell)$				CII_b	C_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{4\ell-1}^{2\ell, 2\ell-1}(\text{III}_b, \text{II}, \epsilon_0)$		$A_{4\ell-1}^{2\ell, 2\ell-1}(\text{III}_b, \text{II}, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		$\ell \cdot A_1$	ℓ		$\ell \cdot A_1$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$C_{2\ell}$	$A(\ell)$	$\ell \cdot A_1$	0	1 1 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell, \ell)$				CII_b	C_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{4\ell-1}^{2\ell-1}(\text{II}, \epsilon_\ell)$		$A_{4\ell-1}^{2\ell-1, 2\ell}(\text{II}, \text{III}_b, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
$2\ell - 1$		$A_{2\ell-1}$	ℓ		$\ell \cdot A_1$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$A_{2\ell-1}$	$A(\ell)$	$\ell \cdot A_1$	ℓ	2 2 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su^*(2\ell) + su^*(2\ell) + \mathbb{R}$			$AI + AI + \mathbb{C}$		$A_{\ell-1} + A_{\ell-1} + \mathbb{R}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_\ell^{\ell, p}(\text{I}, \text{III}_a)$ $1 \leq 2p \leq \ell$		$A_\ell^{\ell, \ell}(\text{I}, \epsilon_p)$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		$p \cdot A_1$	ℓ		A_ℓ
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$p \cdot A_1$	A_ℓ	$A(p)$	$\ell \cdot A_1$	p	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(\ell + 1 - p, p)$				BI, ℓ even, DI_a , ℓ odd	B_p

Type (θ, σ)		diagram (θ, σ)		Type $(\theta, \sigma\theta)$		diagram $(\theta, \sigma\theta)$		
$A_\ell^{p,\ell}(\text{III}_a, \text{I})$ $1 \leq 2p \leq \ell$				$A_\ell^{p,\ell}(\text{III}_a, \text{I})$				
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$		$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$		$\Phi_{\sigma\theta} \cap \Phi_\theta$	
p			BC_p	p			BC_p	
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$	$m^-(\lambda)$	$m^+(2\lambda)$	$m^-(2\lambda)$
B_p	BC_p	$B(p)$	B_p	p	1 $\lambda_p : 4$	1 4	0 $2\lambda_p : 1$	0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$		
$so(\ell + 1 - p, p)$				BI, ℓ even, DI_a, ℓ odd		B_p		

Type (θ, σ)		diagram (θ, σ)		Type $(\theta, \sigma\theta)$		diagram $(\theta, \sigma\theta)$	
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$		
$A_\ell^\ell(\mathrm{I}, \epsilon_p)$ $1 \leq 2p \leq \ell$			$A_\ell^{\ell,p}(\mathrm{I}, \mathrm{III}_a)$				
ℓ		A_ℓ	p		$p \cdot A_1$		
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$	$m^-(\lambda)$	$m^+(2\lambda)$
$\ell \cdot A_1$	A_ℓ	$A(\lfloor \frac{\ell+1}{2} \rfloor)$	$p \cdot A_1$	p	1	0	0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$		
$sl(p, \mathbb{R}) + sl(\ell - p + 1, \mathbb{R}) + \mathbb{R}$				$AI + AI + \mathbb{C}$	$A_{p-1} + A_{\ell-p} + \mathbb{R}$		

Type (θ, σ)		diagram (θ, σ)	Type $(\theta, \sigma\theta)$		diagram $(\theta, \sigma\theta)$
$A_{4\ell-1}^{2p, 2\ell-1}(\text{III}_a, \text{II})$ $1 \leq 4p \leq 4\ell$			$A_{4\ell-1}^{2p, 2\ell-1}(\text{III}_a, \text{II})$		
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		\emptyset	p		\emptyset
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
\emptyset	BC_{2p}	id	\emptyset	0	- - - -
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(2\ell - p, p)$				CII_a	BC_p

Type (θ, σ)		diagram (θ, σ)	Type $(\theta, \sigma\theta)$		diagram $(\theta, \sigma\theta)$
$A_{4\ell+1}^{2p,2\ell}(\text{III}_a, \text{II})$ $1 \leq 4p \leq 4\ell$			$A_{4\ell+1}^{2p,2\ell}(\text{III}_a, \text{II})$		
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		\emptyset	p		\emptyset
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
\emptyset	BC_{2p}	id	\emptyset	0	- - - -
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(2\ell - p + 1, p)$				CII_a	BC_p

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{4\ell-1}^{2\ell-1, 2p}(\text{II}, \text{III}_a)$ $1 \leq 4p \leq 4\ell + 2$		$A_{4\ell-1}^{2\ell-1, 2\ell-1}(\text{II}, \epsilon_p)$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		$p \cdot A_1$	$2\ell - 1$		$A_{2\ell-1}$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$p \cdot A_1$	$A_{2\ell-1}$	$A(p)$	$\ell \cdot A_1$	p	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(2\ell - p, p)$			CII_a		BC_p

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{4\ell+1}^{2\ell, 2p}(\text{II}, \text{III}_a)$ $1 \leq 4p \leq 4\ell + 2$		$A_{4\ell+1}^{2\ell, 2\ell}(\text{II}, \epsilon_p)$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		$p \cdot A_1$	2ℓ		$A_{2\ell}$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$p \cdot A_1$	$A_{2\ell}$	$A(p)$	$\ell \cdot A_1$	p	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(2\ell - p + 1, p)$			CII_a		BC_p

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell+1}^{\ell,\ell}(\text{II}, \epsilon_p)$ $1 \leq 2p \leq \ell + 1$		$A_{2\ell+1}^{\ell,2p}(\text{II}, \text{III}_a)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		A_ℓ	p		$p \cdot A_1$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\lfloor \frac{\ell+1}{2} \rfloor \cdot A_1$	A_ℓ	$A(\lceil \frac{\ell+1}{2} \rceil)$	$p \cdot A_1$	p	2 2 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su^*(2p) + su^*(2\ell - 2p + 1) + \mathbb{R}$			$AI + AI + \mathbb{C}$		$A_p + A_{\ell-p-1} + \mathbb{R}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell-1}^{\ell,\ell}(\text{III}_b, \epsilon_\ell)$		$A_{2\ell-1}^{\ell,\ell}(\text{III}_b, \epsilon_\ell)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		C_ℓ	ℓ		C_ℓ
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_ℓ	C_ℓ	$B(\ell)$	C_ℓ	ℓ	1 $\begin{matrix} 1 \\ \lambda_\ell : 0 \end{matrix}$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sl(\ell, \mathbb{C}) + \mathbb{R}$				$AI \times AI + \mathbb{C}$	$A_{\ell-1} + \mathbb{R}$

Type (θ, σ)		diagram (θ, σ)	Type $(\theta, \sigma\theta)$		diagram $(\theta, \sigma\theta)$
$A_\ell^{p,p}(\text{III}_a, \epsilon_p)$ $p < \ell + 1 - 2p$			$A_\ell^{p, \ell+1-2p}(\text{III}_a, \epsilon_p)$		
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	p		BC_p
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	BC_p	p	1 $\lambda_p : \ell - 2p + 1$ 0 1
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell + 1 - 2p, 0) + su(p, p)$				$AIII_a$	BC_p

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell-1}^{\ell, p}(\text{III}_b, \text{III}_a, \epsilon_i)$ $\ell - p + 2i \leq p$ $0 \leq i \leq p - 1$		$A_{2\ell-1}^{\ell, \ell-p+2i}(\text{III}_b, \text{III}_a, \epsilon_i)$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		C_p	$\ell - p + 2i$		$C_{\ell-p+2i}$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_p	C_ℓ	$C(p)$	$C_{\ell-p+2i}$	$2i$	$\begin{matrix} 2 \\ \lambda_p : 1 \end{matrix}$ 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell - p + i, i) + su(\ell - i, p - i) + so(2)$			$AIII_b + AIII_b + \mathbb{C}$		$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell-1}^{\ell, p}(\text{III}_b, \text{III}_a, \epsilon_i)$ $p < \ell - p + 2i \leq \ell$ $0 \leq i \leq p - 1$		$A_{2\ell-1}^{\ell, \ell-p+2i}(\text{III}_b, \text{III}_a, \epsilon_{\ell-p+i})$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		C_p	$\ell - p + 2i$		$C_{\ell-p+2i}$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_p	C_ℓ	$C(p)$	$C_{\ell-p+2i}$	$2i$	$\begin{matrix} 2 \\ \lambda_p : 1 \end{matrix}$ 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell - p + i, i) + su(\ell - i, p - i) + so(2)$				$AIII_b + AIII_b + \mathbb{C}$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell-1}^{\ell, p}(\text{III}_b, \text{III}_a, \epsilon_i)$ $p \leq \ell + p - 2i < \ell$ $0 \leq i \leq p - 1$		$A_{2\ell-1}^{\ell, \ell+p-2i}(\text{III}_b, \text{III}_a, \epsilon_{\ell-i})$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		C_p	$\ell + p - 2i$		$C_{\ell+p-2i}$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_p	C_ℓ	$C(p)$	$C_{\ell+p-2i}$	$2p - 2i$	$\begin{matrix} 2 \\ \lambda_p : 1 \end{matrix}$ 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell - p + i, i) + su(\ell - i, p - i) + so(2)$			$AIII_b + AIII_b + \mathbb{C}$		$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell-1}^{\ell, p}(\text{III}_b, \text{III}_a, \epsilon_i)$ $\ell + p - 2i < p$ $0 \leq i \leq p - 1$		$A_{2\ell-1}^{\ell, \ell+p-2i}(\text{III}_b, \text{III}_a, \epsilon_{\ell-i})$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		C_p	$\ell + p - 2i$		$C_{\ell+p-2i}$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_p	C_ℓ	$C(p)$	$C_{\ell+p-2i}$	$2p - 2i$	$\begin{matrix} 2 \\ \lambda_p : 1 \end{matrix}$ 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell - p + i, i) + su(\ell - i, p - i) + so(2)$			$AIII_b + AIII_b + \mathbb{C}$		$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell-1}^{p,\ell}(\text{III}_a, \text{III}_b, \epsilon_i)$ $\ell - p + 2i \leq p$ $0 \leq i \leq p-1$		$A_{2\ell-1}^{p,\ell-p+2i}(\text{III}_a, \text{III}_a, \epsilon_{\ell-p+i})$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	$\ell - p + 2i$		$BC_{\ell-p+2i}$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_ℓ	$B(p)$	$BC_{\ell-p+2i}$	$\ell - p + 2i$	$\begin{matrix} 2 \\ \lambda_p : 2(\ell-q) \end{matrix} \quad \begin{matrix} 0 \\ 2(q-p) \end{matrix} \quad \begin{matrix} 0 \\ 2\lambda_p : 1 \end{matrix} \quad \begin{matrix} 0 \\ \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell - p + i, i) + su(\ell - i, p - i) + so(2)$				$AIII_b + AIII_b + \mathbb{C}$	$BC_i + BC_{p-i}$

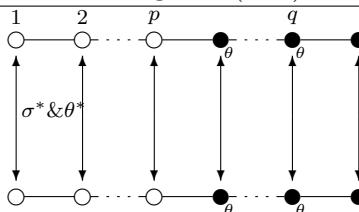
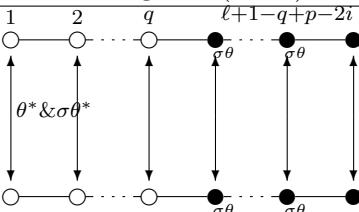
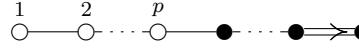
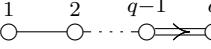
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell-1}^{p,\ell}(\text{III}_a, \text{III}_b, \epsilon_i)$ $p < l - p + 2i \leq l$ $0 \leq i \leq p-1$		$A_{2\ell-1}^{p,\ell-p+2i}(\text{III}_a, \text{III}_a, \epsilon_{p-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	p		BC_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	BC_p	p	$\begin{matrix} 2 \\ \lambda_p : 2(\ell-q) \end{matrix} \quad \begin{matrix} 0 \\ 2(q-p) \end{matrix} \quad \begin{matrix} 0 \\ 2\lambda_p : 1 \end{matrix} \quad \begin{matrix} 0 \\ \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell-p+i, i) + su(\ell-i, p-i) + so(2)$				$AIII_b + AIII_b + \mathbb{C}$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell-1}^{p,\ell}(\text{III}_a, \text{III}_b, \epsilon_i)$ $p \leq \ell + p - 2i < \ell$ $0 \leq i \leq p - 1$		$A_{2\ell-1}^{p,\ell+p-2i}(\text{III}_a, \text{III}_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	p		BC_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	BC_p	p	$\begin{matrix} 2 \\ \lambda_p : 2(\ell-q) \end{matrix} \quad \begin{matrix} 0 \\ 2(q-p) \end{matrix} \quad \begin{matrix} 0 \\ 2\lambda_p : 1 \end{matrix} \quad \begin{matrix} 0 \\ \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell-p+i, i) + su(\ell-i, p-i) + so(2)$				$AIII_b + AIII_b + \mathbb{C}$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_{2\ell-1}^{p,\ell}(\text{III}_a, \text{III}_b, \epsilon_i)$ $\ell + p - 2i < p$ $0 \leq i \leq p - 1$		$A_{2\ell-1}^{p,\ell+p-2i}(\text{III}_a, \text{III}_a, \epsilon_{\ell-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	$\ell + p - 2i$		$BC_{\ell+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	$BC_{\ell+p-2i}$	$\ell + p - 2i$	$\begin{matrix} 2 \\ \lambda_p : 2(\ell - q) \end{matrix} \quad \begin{matrix} 0 \\ 2(q - p) \end{matrix} \quad \begin{matrix} 0 \\ 2\lambda_p : 1 \end{matrix} \quad 0$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell - p + i, i) + su(\ell - i, p - i) + so(2)$				$AIII_b + AIII_b + \mathbb{C}$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_\ell^{q,p}(\text{III}_a, \text{III}_a, \epsilon_i)$ $q - p + 2i \leq p < q$ $1 \leq p < q \leq \frac{1}{2}(\ell + 1)$ $(0 \leq i \leq p)$		$A_\ell^{q, q-p+2i}(\text{III}_a, \text{III}_a, \epsilon_i)$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	$q - p + 2i$		BC_{q-p+2i}
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_q	$B(p)$	BC_{q-p+2i}	$2i$	$\begin{array}{cccc} 2 & 0 & 0 & 0 \\ \lambda_p : 2(\ell - q) & 2(q - p) & 2\lambda_p : 1 & 0 \end{array}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell + 1 - q - i, p - i) + su(q - p + i, i) + so(2)$				$AIII_b + AIII_b + \mathbb{C}$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_\ell^{q,p}(\text{III}_a, \text{III}_a, \epsilon_i)$ $p < q < q - p + 2i$ $q - p + 2i \leq \frac{1}{2}(l+1)$ $1 \leq p < q \leq \frac{1}{2}(l+1)$ $(0 \leq i < p)$		$A_\ell^{q,q-p+2i}(\text{III}_a, \text{III}_a, \epsilon_{p-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	q		BC_q
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_q	$B(p)$	BC_q	p	$\begin{matrix} 2 & 0 & 0 & 0 \\ \lambda_p : 2(\ell-q) & 2(q-p) & 2\lambda_p : 1 & \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell+1-q-i, p-i) + su(q-p+i, i) + so(2)$				$AIII_b + AIII_b + \mathbb{C}$	$BC_i + BC_{p-i}$

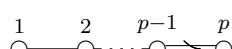
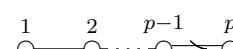
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_\ell^{q,p}(\text{III}_a, \text{III}_a, \epsilon_i)$ $q \leq \ell + 1 - q + p - 2i$ $\ell + 1 - q + p - 2i < \frac{1}{2}(\ell + 1)$ $1 \leq p < q \leq \frac{1}{2}(\ell + 1)$ $(0 \leq i \leq p)$		$A_\ell^{q, \ell+1-q+p-2i}(\text{III}_a, \text{III}_a, \epsilon_{q-p+i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	q		BC_q
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_q	$B(p)$	BC_q	p	$\begin{array}{cccc} 2 & 0 & 0 & 0 \\ \lambda_p : 2(\ell - q) & 2(q - p) & 2\lambda_p : 1 & 0 \end{array}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell + 1 - q - i, p - i) + su(q - p + i, i) + so(2)$				$AIII_b + AIII_b + \mathbb{C}$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_\ell^{q,p}(\text{III}_a, \text{III}_a, \epsilon_i)$ $p \leq \ell + 1 - q + p - 2i < q$ $1 \leq p < q \leq \frac{1}{2}(\ell + 1)$ $(0 \leq i \leq p)$		$A_\ell^{q, \ell+1-q+p-2i}(\text{III}_a, \text{III}_a, \epsilon_{\ell+1-q-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	$\ell + 1 - q + p - 2i$		$BC_{\ell+1-q+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_q	$B(p)$	$BC_{\ell+1-q+p-2i}$	$\ell + 1 - 2q + 2p - 2i$	$\begin{array}{cccc} 2 & 0 & 0 & 0 \\ \lambda_p : 2(\ell - q) & 2(q - p) & 2\lambda_p : 1 & \end{array}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell + 1 - q - i, p - i) + su(q - p + i, i) + so(2)$				$AIII_b + AIII_b + \mathbb{C}$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_\ell^{q,p}(\text{III}_a, \text{III}_a, \epsilon_i)$ $\ell + 1 - q + p - 2i < q$ $1 \leq p < q \leq \frac{1}{2}(\ell + 1)$ $(0 \leq i \leq p)$		$A_\ell^{q, \ell+1-q+p-2i}(\text{III}_a, \text{III}_a, \epsilon_{\ell+1-q-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	$\ell + 1 - q + p - 2i$		$BC_{\ell+1-q+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_q	$B(p)$	$BC_{\ell+1-q+p-2i}$	$\ell + 1 - 2q + 2p - 2i$	$\begin{array}{cccc} 2 & 0 & 0 & 0 \\ \lambda_p : 2(\ell - q) & 2(q - p) & 2\lambda_p : 1 & \end{array}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell + 1 - q - i, p - i) + su(q - p + i, i) + so(2)$				$AIII_b + AIII_b + \mathbb{C}$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_\ell^{p,q}(\text{III}_a, \text{III}_a, \epsilon_i)$ $q - p + 2i \leq p$ $1 \leq p < q \leq \frac{1}{2}(\ell + 1)$ $(0 \leq i \leq p)$		$A_\ell^{p, q-p+2i}(\text{III}_a, \text{III}_a, \epsilon_{q-p+i})$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	$q - p + 2i$		BC_{q-p+2i}
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	BC_{q-p+2i}	$2i$	$\begin{array}{cccc} 2 & 0 & 0 & 0 \\ \lambda_p : 2(\ell - q) & 2(q - p) & 2\lambda_p : 1 & \end{array}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell + 1 - q - i, p - i) + su(q - p + i, i) + so(2)$			$AIII_b + AIII_b + \mathbb{C}$		$BC_i + BC_{p-i}$

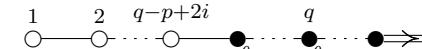
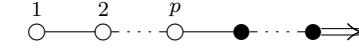
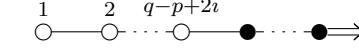
Type (θ, σ)		diagram (θ, σ)	Type $(\theta, \sigma\theta)$		diagram $(\theta, \sigma\theta)$
$A_\ell^{p,q}(\text{III}_a, \text{III}_a, \epsilon_i)$ $p < q - p + 2i \leq \frac{1}{2}(\ell + 1)$ $1 \leq p < q \leq \frac{1}{2}(\ell + 1)$ $(0 \leq i \leq p)$			$A_\ell^{p,q-p+2i}(\text{III}_a, \text{III}_a, \epsilon_{p-i})$		
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	p		BC_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	BC_p	p	$\begin{matrix} 2 \\ \lambda_p : 2(\ell - q) \end{matrix}$ $\begin{matrix} 0 \\ 2(q - p) \end{matrix}$ $\begin{matrix} 0 \\ 2\lambda_p : 1 \end{matrix}$ 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell + 1 - q - i, p - i) + su(q - p + i, i) + so(2)$				$AIII_b + AIII_b + \mathbb{C}$	$BC_i + BC_{p-i}$

Type (θ, σ)		diagram (θ, σ)	Type $(\theta, \sigma\theta)$		diagram $(\theta, \sigma\theta)$
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	BC_p		BC_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	BC_p	p	$\begin{array}{cccc} 2 & 0 & 0 & 0 \\ \lambda_p : 2(\ell - q) & 2(q - p) & 2\lambda_p : 1 & \end{array}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell + 1 - q - i, p - i) + su(q - p + i, i) + so(2)$			$AIII_b + AIII_b + \mathbb{C}$		$BC_i + BC_{p-i}$

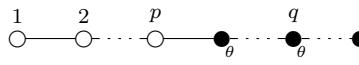
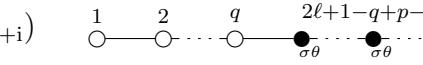
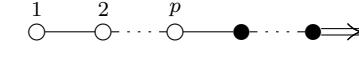
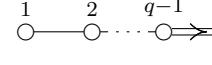
Type (θ, σ)		diagram (θ, σ)	Type $(\theta, \sigma\theta)$		diagram $(\theta, \sigma\theta)$
$A_\ell^{p,q}(\text{III}_a, \text{III}_a, \epsilon_i)$ $p \leq \ell + 1 - q + p - 2i < q$ $1 \leq p < q \leq \frac{1}{2}(\ell + 1)$ $(0 \leq i \leq p)$			$A_\ell^{p, \ell+1-q+p-2i}(\text{III}_a, \text{III}_a, \epsilon_i)$		
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	p		BC_p
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	BC_p	p	$\begin{array}{cccc} 2 & 0 & 0 & 0 \\ \lambda_p : 2(\ell - q) & 2(q - p) & 2\lambda_p : 1 & 0 \end{array}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell + 1 - q - i, p - i) + su(q - p + i, i) + so(2)$				$AIII_b + AIII_b + \mathbb{C}$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$A_\ell^{p,q}(\text{III}_a, \text{III}_a, \epsilon_i)$ $\ell + 1 - q + p - 2i < p < q$ $1 \leq p < q \leq \frac{1}{2}(\ell + 1)$ $(0 \leq i \leq p)$		$A_\ell^{p, \ell+1-q+p-2i}(\text{III}_a, \text{III}_a, \epsilon_{\ell+1-q-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	$\ell + 1 - q + p - 2i$		$BC_{\ell+1-q+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	$BC_{\ell+1-q+p-2i}$	$\ell + 1 - 2q + 2p - 2i$	$\begin{array}{cccc} 2 & 0 & 0 & 0 \\ \lambda_p : 2(\ell - q) & 2(q - p) & 2\lambda_p : 1 & 0 \end{array}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell + 1 - q - i, p - i) + su(q - p + i, i) + so(2)$				$AIII_b + AIII_b + \mathbb{C}$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$B_\ell^{p,p}(I_a, \epsilon_i)$ $p < \ell - p + 2i$ $1 \leq p < \ell$ $(0 \leq i \leq p)$		$B_\ell^{p,p}(I_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	p		B_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_p	$B(p)$	B_{pi}	p	$\lambda_p : \begin{matrix} 1 & 0 \\ \ell - q + 1 & \ell - q \end{matrix} \quad 0 \quad 0$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(p, p) + so(2\ell + 1 - p - i, 0)$				BI	B_p

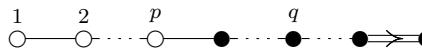
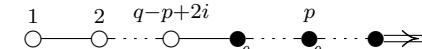
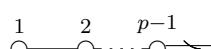
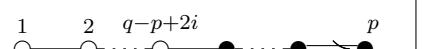
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$B_\ell^{q,p}(I_a, I_a, \epsilon_i)$ $q - p + 2i \leq p < q$ $1 \leq p < q \leq \ell$ $(0 \leq i \leq p)$		$B_\ell^{q,q-p+2i}(I_a, I_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	$q - p + 2i$		B_{q-p+2i}
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_q	$B(p)$	B_{q-p+2i}	$2i$	$\lambda_p : \begin{matrix} 1 & 0 \\ \ell-q+1 & \ell-q \end{matrix} \quad 0 \quad 0$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell + 1 - q - i, p - i) + so(q - p + i, i)$				$BI + BI$	$B_i + B_{p-i}$

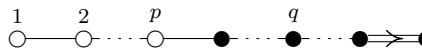
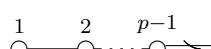
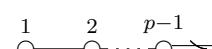
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$B_\ell^{q,p}(I_a, I_a, \epsilon_i)$ $q < q < q - p + 2i \leq \ell$ $1 \leq p < q \leq \ell$ $(0 \leq i \leq p)$		$B_\ell^{q,q-p+2i}(I_a, I_a, \epsilon_{p-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	q		B_q
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_q	$B(p)$	B_q	p	$\lambda_p : \begin{matrix} 1 & 0 \\ \ell-q+1 & \ell-q \end{matrix} \quad 0 \quad 0$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell + 1 - q - i, p - i) + so(q - p + i, i)$				$BI + BI$	$B_i + B_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$B_\ell^{q,p}(I_a, I_a, \epsilon_i)$ $q \leq 2\ell + 1 - q + p - 2i < l$ $1 \leq p < q \leq \ell$ $(0 \leq i \leq p)$		$B_\ell^{q,2\ell+1-q+p-2i}(I_a, I_a, \epsilon_{q-p+i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	q		B_q
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_q	$B(p)$	B_q	p	$\lambda_p : \begin{matrix} 1 & 0 \\ \ell - q + 1 & \ell - q \end{matrix} \quad 0 \quad 0$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell + 1 - q - i, p - i) + so(q - p + i, i)$				$BI + BI$	$B_i + B_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$B_\ell^{q,p}(I_a, I_a, \epsilon_i)$ $p \leq 2\ell + 1 - q + p - 2i < q$ $1 \leq p < q \leq \ell$ $(0 \leq i \leq p)$		$B_\ell^{q,2\ell+1-q+p-2i}(I_a, I_a, \epsilon_{2\ell+1-q-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	$2\ell + 1 - q + p - 2i$		$B_{2\ell+1-q+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_q	$B(p)$	$B_{2\ell+1-q+p-2i}$	$2\ell + 1 - 2q + 2p - 2i$	$\begin{matrix} 1 & 0 \\ \lambda_p : \ell - q + 1 & \ell - q \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell + 1 - q - i, p - i) + so(q - p + i, i)$				$BI + BI$	$B_i + B_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$B_\ell^{q,p}(I_a, I_a, \epsilon_i)$ $2\ell + 1 - q + p - 2i < p$ $1 \leq p < q \leq \ell$ $(0 \leq i \leq p)$		$B_\ell^{q,2\ell+1-q+p-2i}(I_a, I_a, \epsilon_{2\ell+1-q-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	$2\ell + 1 - q + p - 2i$		$B_{2\ell+1-q+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_q	$B(p)$	$B_{2\ell+1-q+p-2i}$	$2\ell + 1 - 2q + 2p - 2i$	$\begin{matrix} 1 & 0 \\ \lambda_p : \ell - q + 1 & \ell - q \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell + 1 - q - i, p - i) + so(q - p + i, i)$				$BI + BI$	$B_i + B_{p-i}$

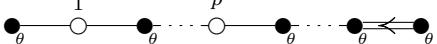
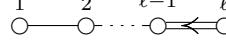
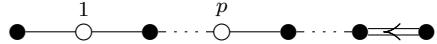
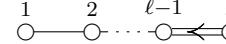
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$B_\ell^{p,q}(I_a, I_a, \epsilon_i)$ $q - p + 2i < p < q$ $1 \leq p < q \leq \ell$ $(0 \leq i \leq p)$		$B_\ell^{p,q-p+2i}(I_a, I_a, \epsilon_{q-p+i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	$q - p + 2i$		B_{q-p+2i}
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_p	$B(p)$	B_{q-p+2i}	$q - p + 2i$	$\begin{matrix} 1 \\ \lambda_p : (2\ell - q \\ -p + 1) \end{matrix}$ $\begin{matrix} 0 \\ q - p \end{matrix}$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell + 1 - q - i, p - i) + so(q - p + i, i)$				$BI + BI$	$B_i + B_{p-i}$

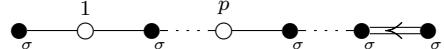
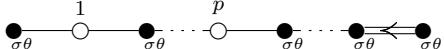
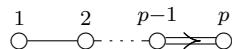
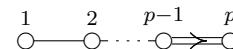
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$B_\ell^{p,q}(I_a, I_a, \epsilon_i)$ $p < q - p + 2i \leq \ell$ $1 \leq p < q \leq \ell$ $(0 \leq i \leq p)$		$B_\ell^{p,q-p+2i}(I_a, I_a, \epsilon_{p-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	p		B_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_p	$B(p)$	B_p	p	$\begin{matrix} 1 \\ \lambda_p : (2\ell - q \\ -p + 1) \end{matrix}$ $\begin{matrix} 0 \\ q - p \end{matrix}$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell + 1 - q - i, p - i) + so(q - p + i, i)$				$BI + BI$	$B_i + B_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$B_\ell^{p,q}(I_a, I_a, \epsilon_i)$ $q \leq 2\ell + 1 - q + p - 2i < \ell$ $1 \leq p < q \leq \ell$ $(0 \leq i \leq p)$		$B_\ell^{p,2\ell+1-q+p-2i}(I_a, I_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	p		B_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_p	$B(p)$	B_p	p	$\begin{matrix} 1 \\ \lambda_p : (2\ell - q \\ -p + 1) \end{matrix}$ $\begin{matrix} 0 \\ q - p \end{matrix}$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell + 1 - q - i, p - i) + so(q - p + i, i)$				$BI + BI$	$B_i + B_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$B_\ell^{p,q}(I_a, I_a, \epsilon_i)$ $p \leq 2\ell + 1 - q + p - 2i < q$ $1 \leq p < q \leq \ell$ $(0 \leq i \leq p)$		$B_\ell^{p,2\ell+1-q+p-2i}(I_a, I_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	p		B_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_p	$B(p)$	B_p	p	$\begin{matrix} 1 \\ \lambda_p : (2\ell - q \\ -p + 1) \end{matrix}$ $\begin{matrix} 0 \\ q - p \end{matrix}$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell + 1 - q - i, p - i) + so(q - p + i, i)$				$BI + BI$	$B_i + B_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$B_\ell^{p,q}(I_a, I_a, \epsilon_i)$ $2\ell + 1 - q + p - 2i < p$ $1 \leq p < q \leq l$ $(0 \leq i \leq p)$		$B_\ell^{p,2\ell+1-q+p-2i}(I_a, I_a, \epsilon_{2\ell+1-q-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	$2\ell + 1 - q + p - 2i$		$B_{2\ell+1-q+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_p	$B(p)$	$B_{2\ell+1-q+p-2i}$	$2\ell + 1 - q + p - 2i$	$\lambda_p : (2\ell - q \atop -p + 1) \quad \begin{matrix} 1 \\ 0 \\ q - p \end{matrix} \quad 0 \quad 0$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell + 1 - q - i, p - i) + so(q - p + i, i)$				$BI + BI$	$B_i + B_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_\ell^{\ell,p}(\text{I}, \text{II}_a)$ $(2p \leq \ell)$		$C_\ell^{\ell,\ell}(\text{I}, \epsilon_p)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		$p \cdot A_1$	ℓ		C_l
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$p \cdot A_1$	C_ℓ	$A(p)$	C_ℓ	p	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell - p, p) + so(2)$				$AIII_a + \mathbb{C}$	BC_p

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_\ell^{p,\ell}(\text{II}_a, \text{I})$ $(2p \leq \ell)$		$C_\ell^{p,\ell}(\text{II}_a, \text{I})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	p		BC_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	BC_p	p	$\begin{array}{cccc} 2 & 2 & & 0 \\ \lambda_p : 2(\ell - 2p) & 2(\ell - 2p) & 2\lambda_p : 1 & 2 \end{array}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell - p, p) + so(2)$				$AIII_a + \mathbb{C}$	BC_p

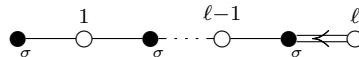
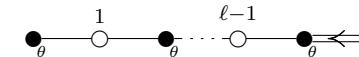
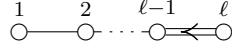
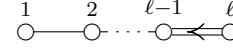
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_\ell^{\ell,\ell}(\mathrm{I}, \epsilon_p)$ $(2p \leq \ell)$		$C_\ell^{\ell,p}(\mathrm{I}, \mathrm{II}_a)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		C_ℓ	p		$p \cdot A_1$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_ℓ	C_ℓ	$B(\ell)$	$p \cdot A_1$	p	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(p, \mathbb{R}) + sp(\ell - p, \mathbb{R})$				$CI + CI$	$C_p + C_{\ell-p}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_{2\ell}^{2\ell,\ell}(\text{I}, \text{II}_b, \epsilon_0)$		$C_{2\ell}^{2\ell,\ell}(\text{I}, \text{II}_b, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		$\ell \cdot A_1$	ℓ		$\ell \cdot A_1$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	C_ℓ	$A(\ell)$	$\ell \cdot A_1$	0	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell, \mathbb{C})$				$CI \times CI$	C_ℓ

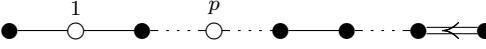
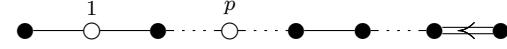
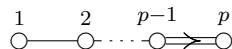
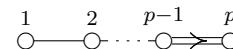
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_{2\ell}^{\ell,2\ell}(\text{II}_b, I, \epsilon_0)$		$C_{2\ell}^\ell(\text{II}_b, \epsilon_\ell)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		C_ℓ	ℓ		C_ℓ
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_ℓ	C_ℓ	$B(\ell)$	C_ℓ	ℓ	2 2 $\lambda_\ell : 1$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell, \mathbb{C})$				$CI \times CI$	C_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_{2\ell}^{\ell,\ell}(\text{II}_b, \epsilon_\ell)$		$C_{2\ell}^{\ell,2\ell}(\text{II}_b, \text{I}, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		C_ℓ	ℓ		C_ℓ
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_ℓ	C_ℓ	$B(\ell)$	C_ℓ	ℓ	2 $\lambda_\ell : 1$ 2 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su^*(2\ell) + \mathbb{R}$				$AII + \mathbb{C}$	$A_{\ell-1} + \mathbb{R}$

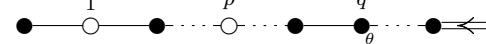
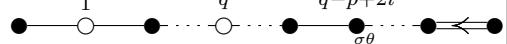
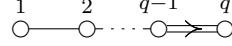
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_{2\ell}^{2\ell,\ell}(I, II_b, \epsilon_\ell)$		$C_{2\ell}^{2\ell,2\ell}(I, \epsilon_\ell)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		$\ell \cdot A_1$	2ℓ		$C_{2\ell}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$C_{2\ell}$	$A(\ell)$	$C_{2\ell}$	ℓ	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell, \ell) + so(2)$				$III_b + \mathbb{C}$	C_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_{2\ell}^{\ell,2\ell}(\Pi_b, I, \epsilon_\ell)$		$C_{2\ell}^{\ell,2\ell}(\Pi_b, I, \epsilon_\ell)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		C_ℓ	ℓ		C_ℓ
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_ℓ	C_ℓ	$B(\ell)$	C_ℓ	ℓ	$\begin{matrix} 2 \\ \lambda_\ell : 1 \end{matrix}$ 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell, \ell) + so(2)$				$AIII_b + \mathbb{C}$	C_ℓ

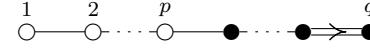
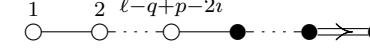
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_{2\ell}^{2\ell,2\ell}(\text{I}, \epsilon_\ell)$		$C_{2\ell}^{2\ell,\ell}(\text{I}, \text{II}_b, \epsilon_\ell)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2ℓ		$C_{2\ell}$	ℓ		C_ℓ
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$C_{2\ell}$	$C_{2\ell}$	$B(2\ell)$	C_ℓ	ℓ	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell, \mathbb{R}) + sp(\ell, \mathbb{R})$				$CI + CI$	$C_\ell + C_\ell$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_\ell^{p,p}(\text{II}_a, \epsilon_i)$ $p < \ell - 2p + 1$ $1 \leq p < \frac{1}{2}\ell$ $(0 \leq i \leq p)$		$C_\ell^{p,\ell-2p+1}(\text{II}_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	rank $\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	p		BC_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda) \quad m^-(\lambda) \quad m^+(2\lambda) \quad m^-(2\lambda)$ $\begin{matrix} 4 \\ \lambda_p : \\ 4(\ell - q - p) \end{matrix} \quad \begin{matrix} 0 \\ 4(q - p) \end{matrix} \quad \begin{matrix} 0 \\ 2\lambda_p : 3 \end{matrix} \quad \begin{matrix} 0 \\ \end{matrix}$
BC_p	BC_p	$B(p)$	BC_p	p	
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - 2p, 0) + sp(p, p)$				CII_a	BC_p

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_\ell^{q,p}(\Pi_a, \Pi_a, \epsilon_i)$ $q - p + 2i \leq p$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$		$C_\ell^{q,q-p+2i}(\Pi_a, \Pi_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	$q - p + 2i$		BC_{q-p+2i}
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_q	$B(p)$	BC_{q-p+2i}	$2i$	$\begin{matrix} 4 \\ \lambda_p : \\ 4(\ell - q - p) \end{matrix}$ $\begin{matrix} 0 \\ 4(q - p) \end{matrix}$ $\begin{matrix} 0 \\ 2\lambda_p : 3 \end{matrix}$ 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - q - i, p - i) + sp(q - p + i, i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_\ell^{q,p}(\Pi_a, \Pi_a, \epsilon_i)$ $p < q < q - p + 2i \leq \frac{1}{2}\ell$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$		$C_\ell^{q,q-p+2i}(\Pi_a, \Pi_a, \epsilon_{p-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	q		BC_q
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_q	$B(p)$	BC_q	p	$\begin{matrix} 4 \\ \lambda_p : \\ 4(\ell - q - p) \end{matrix}$ $\begin{matrix} 0 \\ 4(q - p) \\ 2\lambda_p : 3 \end{matrix}$ 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - q - i, p - i) + sp(q - p + i, i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_\ell^{q,p}(\Pi_a, \Pi_a, \epsilon_i)$ $q \leq \ell - q + p - 2i < \frac{1}{2}\ell$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$		$C_\ell^{q, \ell-q+p-2i}(\Pi_a, \Pi_a, \epsilon_{q-p+i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	q		BC_q
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_q	$B(p)$	B_q	p	$\begin{array}{ccccc} 4 & 0 & 0 & 0 \\ \lambda_p : & 4(\ell - q - p) & 4(q - p) & 2\lambda_p : 3 & 0 \end{array}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - q - i, p - i) + sp(q - p + i, i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_\ell^{q,p}(\Pi_a, \Pi_a, \epsilon_i)$ $p \leq q + p - 2i < q$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$		$C_\ell^{q, \ell-q+p-2i}(\Pi_a, \Pi_a, \epsilon_{\ell-q-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	$\ell - q + p - 2i$		$BC_{\ell-q+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_q	$B(p)$	$BC_{\ell-q+p-2i}$	$\ell - 2q + 2p - 2i$	$\begin{array}{cccc} 4 & 0 & 0 & 0 \\ \lambda_p : & 4(q-p) & 2\lambda_p : 3 & \\ 4(\ell-q-p) & & & 0 \end{array}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - q - i, p - i) + sp(q - p + i, i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

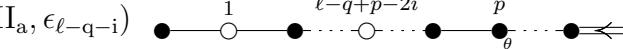
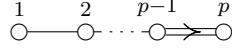
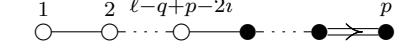
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$					
$C_\ell^{q,p}(\Pi_a, \Pi_a, \epsilon_i)$ $\ell - q + p - 2i < p$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$		$C_\ell^{q, \ell-q+p-2i}(\Pi_a, \Pi_a, \epsilon_{\ell-q-i})$						
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$			
p		BC_p	$\ell - q + p - 2i$		$BC_{\ell-q+p-2i}$			
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $\lambda_p :$ $4(ell-q-p)$	$m^-(\lambda)$ 0 $4(q-p)$	$m^+(2\lambda)$ 0 $2\lambda_p : 3$	$m^-(2\lambda)$ 0
BC_p	BC_q	$B(p)$	$BC_{\ell-q+p-2i}$	$\ell - 2q + 2p - 2i$	$\begin{matrix} 4 \\ \lambda_p : \\ 4(ell-q-p) \end{matrix}$	$\begin{matrix} 0 \\ 4(q-p) \end{matrix}$	$\begin{matrix} 0 \\ 2\lambda_p : 3 \end{matrix}$	0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$		
$sp(\ell - q - i, p - i) + sp(q - p + i, i)$				$CII_a + CII_a$		$BC_i + BC_{p-i}$		

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_\ell^{p,q}(\Pi_a, \Pi_a, \epsilon_i)$ $q - p + 2i \leq p$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$		$C_\ell^{p,q-p+2i}(\Pi_a, \Pi_a, \epsilon_{q-p+i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	$q - p + 2i$		BC_{q-p+2i}
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	BC_{q-p+2i}	$q - p + 2i$	$\begin{matrix} 4 \\ \lambda_p : \\ 4(\ell - q - p) \end{matrix}$ $\begin{matrix} 0 \\ 4(q - p) \end{matrix}$ $\begin{matrix} 0 \\ 2\lambda_p : 3 \end{matrix}$ 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - q - i, p - i) + sp(q - p + i, i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_\ell^{p,q}(\Pi_a, \Pi_a, \epsilon_i)$ $p < q - p + 2i \leq \frac{1}{2}\ell$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$		$C_\ell^{p,q-p+2i}(\Pi_a, \Pi_a, \epsilon_{p-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	p		BC_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	BC_p	p	$\begin{matrix} 4 \\ \lambda_p : \\ 4(\ell - q - p) \end{matrix}$ $\begin{matrix} 0 \\ 4(q - p) \\ 2\lambda_p : 3 \end{matrix}$ 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - q - i, p - i) + sp(q - p + i, i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_\ell^{p,q}(\Pi_a, \Pi_a, \epsilon_i)$ $q \leq \ell - q + p - 2i < \frac{1}{2}\ell$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$		$C_\ell^{p, \ell-q+p-2i}(\Pi_a, \Pi_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	p		B_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	BC_p	p	$\begin{matrix} 4 \\ \lambda_p : \\ 4(\ell - q - p) \end{matrix}$ $\begin{matrix} 0 \\ 4(q - p) \end{matrix}$ $\begin{matrix} 0 \\ 2\lambda_p : 3 \end{matrix}$ 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - q - i, p - i) + sp(q - p + i, i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_\ell^{p,q}(\Pi_a, \Pi_a, \epsilon_i)$ $p \leq \ell - q + p - 2i < q$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$		$C_\ell^{p, \ell-q+p-2i}(\Pi_a, \Pi_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	p		B_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	BC_p	p	$\begin{matrix} 4 \\ \lambda_p : \\ 4(\ell - q - p) \end{matrix}$ $\begin{matrix} 0 \\ 4(q - p) \\ 2\lambda_p : 3 \end{matrix}$ 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - q - i, p - i) + sp(q - p + i, i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_\ell^{p,q}(\Pi_a, \Pi_a, \epsilon_i)$ $\ell - q + p - 2i < p$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$		$C_\ell^{p, \ell-q+p-2i}(\Pi_a, \Pi_a, \epsilon_{\ell-q-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	$\ell - q + p - 2i$		$BC_{\ell-q+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	$BC_{\ell-q+p-2i}$	$\ell - q + p - 2i$	$\begin{matrix} 4 \\ \lambda_p : \\ 4(\ell - q - p) \end{matrix}$ $\begin{matrix} 0 \\ 4(q - p) \end{matrix}$ $\begin{matrix} 0 \\ 2\lambda_p : 3 \end{matrix}$ 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - q - i, p - i) + sp(q - p + i, i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_{2\ell}^{\ell,p}(\text{II}_b, \text{II}_a, \epsilon_i)$ $\ell - p + 2i \leq p$ $0 \leq i \leq p - 1$ $1 \leq p \leq \ell$		$C_{2\ell}^{\ell, \ell-p+2i}(\text{II}_b, \text{II}_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		C_p	$\ell - p + 2i$		$C_{\ell-p+2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $\lambda_p : 3$ 4 0 0 0
C_p	C_ℓ	$B(p)$	$C_{\ell-p+2i}$	$2i$	
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - p + i, i) + sp(\ell - i, p - i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_{2\ell}^{\ell,p}(\Pi_b, \Pi_a, \epsilon_i)$ $p < \ell - p + 2i \leq \ell$ $0 \leq i \leq p-1$ $1 \leq p \leq \ell$		$C_{2\ell}^{\ell, \ell-p+2i}(\Pi_b, \Pi_a, \epsilon_{\ell-p+i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		C_p	$\ell - p + 2i$		$C_{\ell-p+2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_p	C_ℓ	$B(p)$	$C_{\ell-p+2i}$	$2i$	$\frac{4}{\lambda_p : 3}$ 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - p + i, i) + sp(\ell - i, p - i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

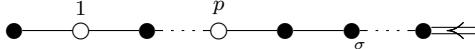
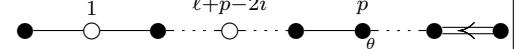
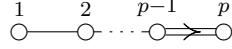
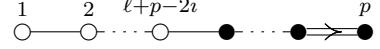
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_{2\ell}^{\ell,p}(\Pi_b, \Pi_a, \epsilon_i)$ $p \leq \ell + p - 2i < \ell$ $0 \leq i \leq p - 1$ $1 \leq p \leq \ell$		$C_{2\ell}^{\ell, \ell+p-2i}(\Pi_b, \Pi_a, \epsilon_{\ell-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		C_p	$\ell + p - 2i$		$C_{\ell+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_p	C_ℓ	$B(p)$	$C_{\ell+p-2i}$	$2p - 2i$	$\lambda_p : 3$ 4 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - p + i, i) + sp(\ell - i, p - i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_{2\ell}^{\ell,p}(\Pi_b, \Pi_a, \epsilon_i)$ $\ell + p - 2i < p$ $0 \leq i \leq p-1$ $1 \leq p \leq \ell$		$C_{2\ell}^{\ell, \ell+p-2i}(\Pi_b, \Pi_a, \epsilon_{\ell-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		C_p	$\ell + p - 2i$		$C_{\ell+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_p	C_ℓ	$B(p)$	$C_{\ell+p-2i}$	$2p - 2i$	$\lambda_p : 3$ 4 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - p + i, i) + sp(\ell - i, p - i)$			$CII_a + CII_a$		$BC_i + BC_{p-i}$

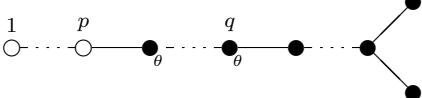
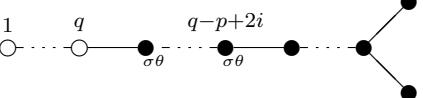
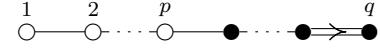
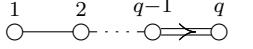
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_{2\ell}^{p,\ell}(\Pi_a, \Pi_b, \epsilon_i)$ $\ell - p + 2i \leq p$ $0 \leq i \leq p - 1$ $1 \leq p \leq \ell$		$C_{2\ell}^{p,\ell-p+2i}(\Pi_a, \Pi_a, \epsilon_{\ell-p+i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	$\ell - p + 2i$		$BC_{\ell-p+2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	$BC_{\ell-p+2i}$	$\ell - p + 2i$	$\begin{matrix} 4 & 0 & 0 & 0 \\ \lambda_p : 4(\ell - p) & 4(\ell - p) & 2\lambda_p : 3 & 0 \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - p + i, i) + sp(\ell - i, p - i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_{2\ell}^{p,\ell}(\Pi_a, \Pi_b, \epsilon_i)$ $p < \ell - p + 2i \leq l$ $0 \leq i \leq p-1$ $1 \leq p \leq \ell$		$C_{2\ell}^{p,\ell-p+2i}(\Pi_a, \Pi_a, \epsilon_{p-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	p		BC_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	BC_p	p	$\begin{matrix} 4 & 0 & 0 & 0 \\ \lambda_p : 4(\ell-p) & 4(\ell-p) & 2\lambda_p : 3 & \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell-p+i, i) + sp(\ell-i, p-i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_{2\ell}^{p,\ell}(\text{II}_a, \text{II}_b, \epsilon_i)$ $p \leq \ell + p - 2i < l$ $0 \leq i \leq p - 1$ $1 \leq p \leq \ell$		$C_{2\ell}^{p,\ell+p-2i}(\text{II}_a, \text{II}_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	p		BC_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	BC_p	p	$\begin{matrix} 4 & 0 & 0 & 0 \\ \lambda_p : 4(\ell-p) & 4(\ell-p) & 2\lambda_p : 3 & \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - p + i, i) + sp(\ell - i, p - i)$			$CII_a + CII_a$		$BC_i + BC_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$C_{2\ell}^{p,\ell}(\Pi_a, \Pi_b, \epsilon_i)$ $\ell + p - 2i < p$ $0 \leq i \leq p-1$ $1 \leq p \leq \ell$		$C_{2\ell}^{p,\ell+p-2i}(\Pi_a, \Pi_a, \epsilon_{\ell-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	$\ell + p - 2i$		$BC_{\ell+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_p	BC_p	$B(p)$	$BC_{\ell+p-2i}$	$\ell + p - 2i$	$\begin{array}{cccc} 4 & 0 & 0 & 0 \\ \lambda_p : 4(\ell-p) & 4(\ell-p) & 2\lambda_p : 3 & 0 \end{array}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(\ell - p + i, i) + sp(\ell - i, p - i)$				$CII_a + CII_a$	$BC_i + BC_{p-i}$

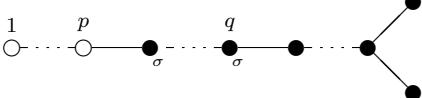
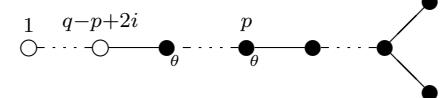
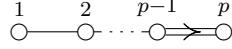
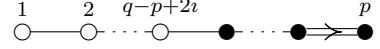
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{q,p}(\mathrm{I}_a, \mathrm{I}_a, \epsilon_i)$ $q - p + 2i \leq p$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p - 1)$		$D_\ell^{q,q-p+2i}(\mathrm{I}_a, \mathrm{I}_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	$q - p + 2i$		B_{q-p+2i}
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_q	$B(p)$	B_{q-p+2i}	$2i$	$\lambda_p : 4(\ell - q) \quad 4(q - p) \quad 0 \quad 0$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell - q - i, p - i) + so(q - p + i, i)$				$DI_a + DI_a$	$B_i + B_{p-i}$

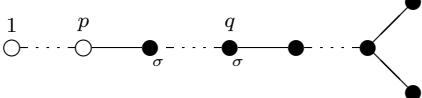
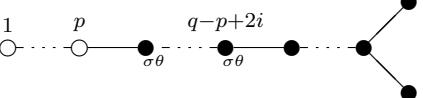
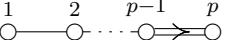
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{q,p}(\mathbf{I}_a, \mathbf{I}_a, \epsilon_i)$ $p < q < q - p + 2i \leq l$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p - 1)$		$D_\ell^{q,q-p+2i}(\mathbf{I}_a, \mathbf{I}_a, \epsilon_{p-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	rank $\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	q		B_q
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_q	$B(p)$	B_q	p	$\lambda_p : 4(\ell - q) \quad 4(q - p) \quad 0 \quad 0$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell - q - i, p - i) + so(q - p + i, i)$				$DI_a + DI_a$	$B_i + B_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{q,p}(I_a, I_a, \epsilon_i)$ $q \leq 2\ell - q + p - 2i < l$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p - 1)$		$D_\ell^{q,2\ell-q+p-2i}(I_a, I_a, \epsilon_{q-p+i})$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	q		B_q
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_q	$B(p)$	B_q	p	$\begin{array}{cccc} 1 & 0 & 0 & 0 \\ \lambda_p : 4(\ell - q) & 4(q - p) & 2\lambda_p : 1 & 0 \end{array}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell - q - i, p - i) + so(q - p + i, i)$				$DI_a + DI_a$	$B_i + B_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{q,p}(\mathbf{I}_a, \mathbf{I}_a, \epsilon_i)$ $p \leq 2\ell - q + p - 2i < q$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p - 1)$		$D_\ell^{q,2\ell-q+p-2i}(\mathbf{I}_a, \mathbf{I}_a, \epsilon_{2\ell-q-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	$2\ell - q + p - 2i$		$B_{2\ell-q+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_q	$B(p)$	$B_{2\ell-q+p-2i}$	$2\ell - 2q + 2p - 2i$	$\begin{matrix} 1 & 0 \\ \lambda_p : 4(\ell - q) & 4(q - p) \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell - q - i, p - i) + so(q - p + i, i)$				$DI_a + DI_a$	$B_i + B_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{q,p}(\mathbf{I}_a, \mathbf{I}_a, \epsilon_i)$ $2\ell - q + p - 2i < p$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p - 1)$		$D_\ell^{q,2\ell-q+p-2i}(\mathbf{I}_a, \mathbf{I}_a, \epsilon_{2\ell-q-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	$2\ell - q + p - 2i$		$B_{2\ell-q+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_q	$B(p)$	$B_{2\ell-q+p-2i}$	$2\ell - 2q + 2p - 2i$	$\begin{matrix} 1 & 0 \\ \lambda_p : 4(\ell - q) & 4(q - p) \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell - q - i, p - i) + so(q - p + i, i)$				$DI_a + DI_a$	$B_i + B_{p-i}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{p,q}(I_a, I_a, \epsilon_i)$ $q - p + 2i \leq p$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p - 1)$		$D_\ell^{p,q-p+2i}(I_a, I_a, \epsilon_{q-p+i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	$q - p + 2i$		B_{q-p+2i}
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_p	$B(p)$	B_{q-p+2i}	$q - p + 2i$	$\begin{matrix} 1 & 0 \\ \lambda_p : 4(\ell - q) & 4(q - p) \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell - q - i, p - i) + so(q - p + i, i)$				$DI_a + DI_a$	$B_i + B_{p-i}$

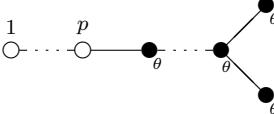
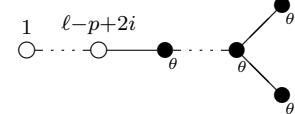
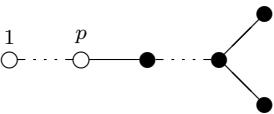
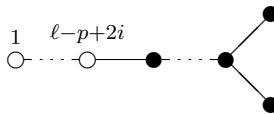
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{p,q}(\mathbf{I}_a, \mathbf{I}_a, \epsilon_i)$ $p < q - p + 2i \leq l$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p - 1)$		$D_\ell^{p,q-p+2i}(\mathbf{I}_a, \mathbf{I}_a, \epsilon_{p-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	p		B_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_p	$B(p)$	B_p	p	$\lambda_p : 4(\ell - q) \quad 4(q - p) \quad 0 \quad 0$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell - q - i, p - i) + so(q - p + i, i)$				$DI_a + DI_a$	$B_i + B_{p-i}$

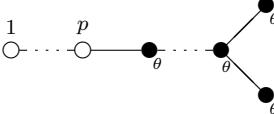
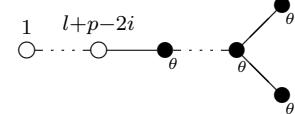
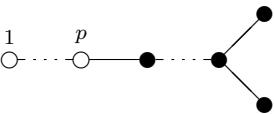
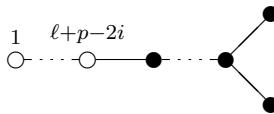
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{p,q}(I_a, I_a, \epsilon_i)$ $q \leq 2\ell - q + p - 2i < l$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p - 1)$		$D_\ell^{p,2\ell-q+p-2i}(I_a, I_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	p		B_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_p	$B(p)$	B_p	p	$\lambda_p : 4(\ell - q) \quad 4(q - p) \quad 0 \quad 0$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell - q - i, p - i) + so(q - p + i, i)$				$DI_a + DI_a$	$B_i + B_{p-i}$

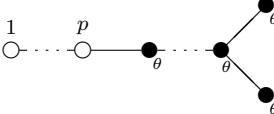
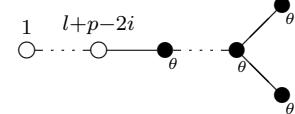
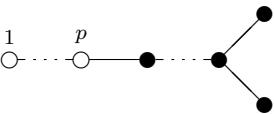
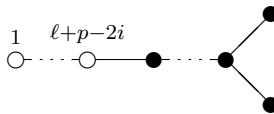
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{p,q}(I_a, I_a, \epsilon_i)$ $p \leq 2\ell - q + p - 2i < q$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p - 1)$		$D_\ell^{p,2\ell-q+p-2i}(I_a, I_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	p		B_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_p	$B(p)$	B_p	p	$\lambda_p : 4(\ell - q) \quad 4(q - p) \quad 0 \quad 0$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell - q - i, p - i) + so(q - p + i, i)$				$DI_a + DI_a$	$B_i + B_{p-i}$

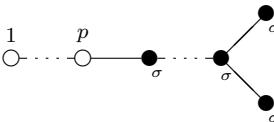
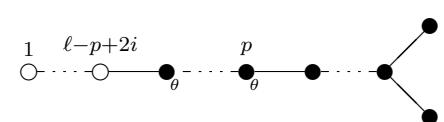
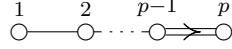
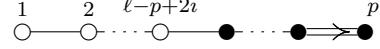
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{p,q}(\mathbf{I}_a, \mathbf{I}_a, \epsilon_i)$ $2\ell - q + p - 2i < p$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p - 1)$		$D_\ell^{p,2\ell-q+p-2i}(\mathbf{I}_a, \mathbf{I}_a, \epsilon_{2\ell-q-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	$2\ell - q + p - 2i$		$B_{2\ell-q+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_p	$B(p)$	$B_{2\ell-q+p-2i}$	$2\ell - q + p - 2i$	$\lambda_p : 4(\ell - q) \quad 4(q - p) \quad 0 \quad 0$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell - q - i, p - i) + so(q - p + i, i)$				$DI_a + DI_a$	$B_i + B_{p-i}$

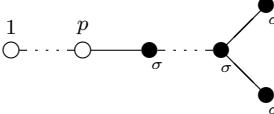
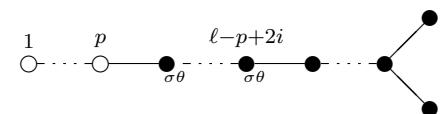
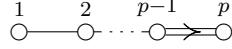
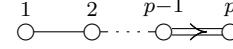
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{\ell,p}(I_b, I_a, \epsilon_i)$ $\ell - p + 2i \leq p$ $0 \leq i \leq p - 1$ $1 \leq p < \ell$		$D_\ell^{\ell, \ell-p+2i}(I_b, I_a, \epsilon_i)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		D_p	$\ell - p + 2i$		$D_{\ell-p+2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
D_p	D_ℓ	$D(p)$	$D_{\ell-p+2i}$	$2i$	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(\ell - p + i, i) + so(\ell - i, p - i)$				$DI_a + DI_a$	$B_i + B_p - i$

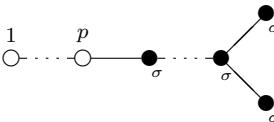
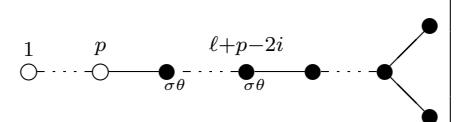
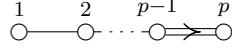
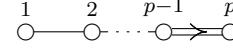
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{\ell, p}(\mathrm{I}_b, \mathrm{I}_a, \epsilon_i)$ $p < \ell - p + 2i \leq \ell$ $0 \leq i \leq p - 1$ $1 \leq p < \ell$		$D_\ell^{\ell, \ell-p+2i}(\mathrm{I}_b, \mathrm{I}_a, \epsilon_{p-i})$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		D_p	$\ell - p + 2i$		$D_{\ell-p+2i}$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
D_p	D_ℓ	$D(p)$	$D_{\ell-p+2i}$	$2i$	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(\ell - p + i, i) + so(\ell - i, p - i)$				$DI_a + DI_a$	$B_i + B_p - i$

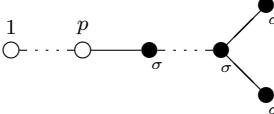
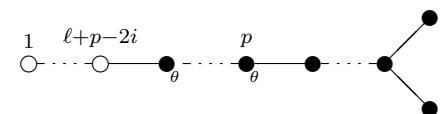
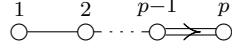
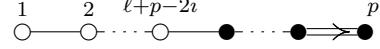
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{\ell, p}(\mathrm{I}_b, \mathrm{I}_a, \epsilon_i)$ $p \leq \ell + p - 2i < \ell$ $0 \leq i \leq p - 1$ $1 \leq p < \ell$		$D_\ell^{\ell, \ell+p-2i}(\mathrm{I}_b, \mathrm{I}_a, \epsilon_{\ell-i})$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		D_p	$\ell + p - 2i$		$D_{\ell+p-2i}$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
D_p	D_ℓ	$D(p)$	$D_{\ell+p-2i}$	$2p - 2i$	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(\ell - p + i, i) + so(\ell - i, p - i)$				$DI_a + DI_a$	$B_i + B_p - i$

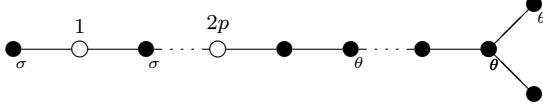
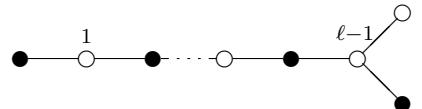
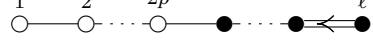
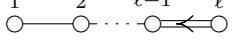
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{\ell,p}(I_b, I_a, \epsilon_i)$ $\ell + p - 2i < p$ $0 \leq i \leq p - 1$ $1 \leq p < \ell$		$D_\ell^{\ell, \ell+p-2i}(I_b, I_a, \epsilon_{\ell-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		D_p	$\ell + p - 2i$		$D_{\ell+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
D_p	D_ℓ	$D(p)$	$D_{\ell+p-2i}$	$2p - 2i$	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(\ell - p + i, i) + so(\ell - i, p - i)$				$DI_a + DI_a$	$B_i + B_p - i$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$					
$D_\ell^{p,\ell}(I_a, I_b, \epsilon_i)$ $\ell - p + 2i \leq p$ $0 \leq i \leq p - 1$ $0 \leq p < \ell$		$D_\ell^{p,\ell-p+2i}(I_a, I_a, \epsilon_{\ell-p+i})$						
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$			
p		B_p	$\ell - p + 2i$		$B_{\ell-p+2i}$			
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $\lambda_p : \ell - p$	$m^-(\lambda)$ $\ell - p$	$m^+(2\lambda)$ 0	$m^-(2\lambda)$ 0
B_p	B_p	$B(p)$	$B_{\ell-p+2i}$	$\ell - p + 2i$	$\begin{matrix} 1 \\ \lambda_p : \ell - p \end{matrix}$	$\begin{matrix} 0 \\ \ell - p \end{matrix}$	0	0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$			
$so(\ell - p + i, i) + so(\ell - i, p - i)$				$DI_a + DI_a$	$B_i + B_p - i$			

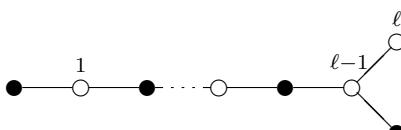
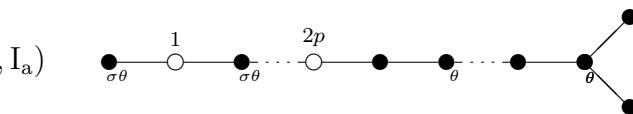
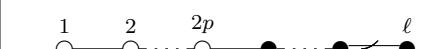
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{p,\ell}(I_a, I_b, \epsilon_i)$ $p < \ell - p + 2i \leq \ell$ $0 \leq i \leq p - 1$ $0 \leq p < \ell$		$D_\ell^{p,\ell-p+2i}(I_a, I_a, \epsilon_{p-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	p		B_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_p	$B(p)$	B_p	p	$\lambda_p : \ell - p$ $\ell - p$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(\ell - p + i, i) + so(\ell - i, p - i)$				$DI_a + DI_a$	$B_i + B_p - i$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$					
$D_\ell^{p,\ell}(I_a, I_b, \epsilon_i)$ $p \leq \ell + p - 2i < \ell$ $0 \leq i \leq p - 1$ $0 \leq p < \ell$		$D_\ell^{p,\ell+p-2i}(I_a, I_a, \epsilon_i)$						
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$			
p		B_p	p		B_p			
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $\lambda_p : \ell - p$	$m^-(\lambda)$ $\ell - p$	$m^+(2\lambda)$ 0	$m^-(2\lambda)$ 0
B_p	B_p	$B(p)$	B_p	p	1	0	0	0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$			
$so(\ell - p + i, i) + so(\ell - i, p - i)$			$DI_a + DI_a$		$B_i + B_p - i$			

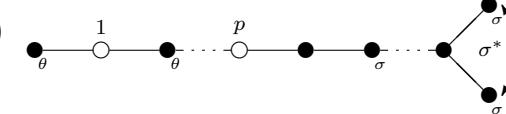
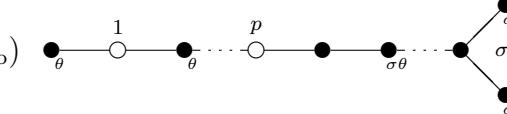
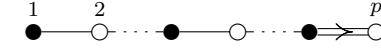
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{p,\ell}(I_a, I_b, \epsilon_i)$ $\ell + p - 2i < p$ $0 \leq i \leq p - 1$ $0 \leq p < \ell$		$D_\ell^{p,\ell+p-2i}(I_a, I_a, \epsilon_{l-i})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	$\ell + p - 2i$		$B_{\ell+p-2i}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_p	$B(p)$	$B_{\ell+p-2i}$	$\ell + p - 2i$	$\lambda_p : \begin{matrix} 1 & 0 \\ \ell-p & \ell-p \end{matrix} \quad \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(\ell - p + i, i) + so(\ell - i, p - i)$				$DI_a + DI_a$	$B_i + B_p - i$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_{2\ell}^{\ell, 4p}(\text{III}_a, \text{I}_a)$ $(1 \leq 2p \leq \ell - 1)$		$D_{2\ell}^{\ell, \ell}(\text{III}_a, \epsilon_p)$			
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
$2p$		C_{2p}	ℓ		C_ℓ
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_{2p}	C_ℓ	$B(2p)$	C_ℓ	$2p$	2 $\lambda_p : 1$ 2 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(2\ell - 2p, 2p)$				$AIII_a$	BC_{2p}

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_{2\ell}^{2p,\ell}(\text{I}_a, \text{III}_a)$ $(1 \leq 2p \leq 2\ell - 1)$		$D_{2\ell}^{2p,\ell}(\text{I}_a, \text{III}_a)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		$p \cdot A_1$	p		$p \cdot A_1$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$p \cdot A_1$	B_{2p}	$A(p)$	$p \cdot A_1$	0	$\begin{matrix} 2 \\ \lambda_p : 1 \end{matrix}$ 2 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(2\ell - p, p) + so(2)$				$AIII_a$	BC_p

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_{2\ell}^{\ell,\ell}(\text{III}_a, \epsilon_p)$ $(1 \leq 2p \leq \ell - 1)$		$D_{2\ell}^{\ell,4p}(\text{III}_a, \text{I}_a)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		C_ℓ	$2p$		C_{2p}
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_ℓ	C_ℓ	$B(\ell)$	C_{2p}	$2p$	2 $\lambda_p : 1$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so^*(4p) + so^*(4\ell - 4p)$				$DIII_a + DIII_a$	$C_p + C_{\ell-p}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$					
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$			
$D_{2\ell+1}^{\ell,2p}(\text{III}_b, \text{I}_a)$ $(1 \leq 2p \leq \ell)$		$D_{2\ell+1}^{\ell,\ell}(\text{III}_b, \epsilon_p)$						
$2p$		BC_{2p}	ℓ		BC_ℓ			
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $\lambda_p : 2(2\ell - 2p + 1)$	$m^-(\lambda)$ $2(2\ell - 2p + 1)$	$m^+(2\lambda)$ $2\lambda_p : 1$	$m^-(2\lambda)$ 0
BC_{2p}	BC_ℓ	$B(2p)$	BC_ℓ	$2p$	2 $\lambda_p : 2(2\ell - 2p + 1)$	2 $2(2\ell - 2p + 1)$	0	0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$			
$su(2\ell - 2p + 1, 2p)$				$AIII_a$	BC_{2p}			

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_{2\ell+1}^{2p,\ell}(\text{I}_a, \text{III}_b)$ $(1 \leq 2p \leq 2\ell)$		$D_{2\ell+1}^{2p,\ell}(\text{I}_a, \text{III}_b)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		$p \cdot A_1$	p		$p \cdot A_1$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$p \cdot A_1$	B_{2p}	$A(p)$	$p \cdot A_1$	0	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(2\ell - p + 1, p) + so(2)$				$AIII_a$	BC_p

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_{2\ell+1}^{\ell,\ell}(\text{III}_b, \epsilon_p)$ $(1 \leq 2p \leq 2\ell)$		$D_{2\ell+1}^{\ell,4p}(\text{III}_b, \text{I}_a)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		BC_ℓ	$2p$		BC_{2p}
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_ℓ	BC_ℓ	$B(\ell)$	BC_{2p}	$2p$	2 2 0 $\lambda_\ell : 1$ 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so^*(4p) + so^*(4\ell - 4p + 2)$			$DIII_a + DIII_b$		$C_p + BC_{\ell-p}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_{4\ell}^{2\ell,4\ell}(\text{III}_a, \text{I}_b, \epsilon_0)$		$D_{4\ell}^{2\ell,2\ell}(\text{III}_a, \epsilon_\ell)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2ℓ		$C_{2\ell}$	2ℓ		$C_{2\ell}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$C_{2\ell}$	$C_{2\ell}$	$B(2\ell)$	$C_{2\ell}$	2ℓ	2 2 0 $\lambda_\ell : 1$ 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(2\ell, 2\ell)$				$AIII_b$	$C_{2\ell}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_{2\ell}^{2\ell,\ell}(\text{I}_b, \text{III}_a, \epsilon_0)$		$D_{2\ell}^{2\ell,\ell}(\text{I}_b, \text{III}_a, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		$\ell \cdot A_1$	ℓ		$\ell \cdot A_1$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$D_{2\ell}$	$A(\ell)$	$\ell \cdot A_1$	ℓ	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(\ell, \ell) + so(2)$				$AI\!II\!I_b + \mathbb{C}$	C_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_{4\ell}^{2\ell,2\ell}(\text{III}_a, \epsilon_\ell)$		$D_{4\ell}^{2\ell,4\ell}(\text{III}_a, \text{I}_b, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2ℓ		$C_{2\ell}$	ℓ		C_ℓ
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$C_{2\ell}$	$C_{2\ell}$	$B(2\ell)$	C_ℓ	ℓ	$\begin{matrix} 2 & 2 \\ \lambda_\ell : 1 & 0 \\ & 0 \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so^*(4\ell) + so^*(4\ell)$				$D\text{III}_a + D\text{III}_a$	$C_\ell + C_\ell$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_{2\ell}^{\ell,2\ell}(\text{III}_a, \text{I}_b, \epsilon_\ell)$		$D_{2\ell}^{\ell,2\ell}(\text{III}_a, \text{I}_b, \epsilon_\ell)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		C_ℓ	ℓ		C_ℓ
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_ℓ	C_ℓ	$B(\ell)$	C_ℓ	ℓ	$\begin{matrix} 2 & 2 \\ \lambda_\ell : 1 & 0 \\ & 0 \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell, \mathbb{C})$				$D_\ell \times D_\ell$	D_ℓ

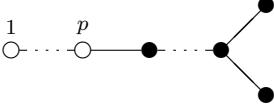
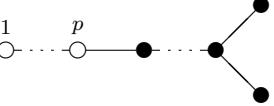
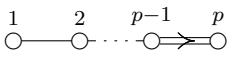
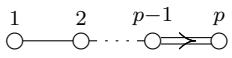
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_{2\ell}^{2\ell,\ell}(\text{I}_b, \text{III}_a, \epsilon_\ell)$		$D_{2\ell}^{2\ell,2\ell}(\text{I}_b, \epsilon_\ell)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
ℓ		$\ell \cdot A_1$	2ℓ		$D_{2\ell}$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$D_{2\ell}$	$A(\ell)$	$D_{2\ell}$	ℓ	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(\ell, \mathbb{C})$				$D_\ell \times D_\ell$	D_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
$D_{2\ell}^{2\ell}(\text{I}_b, \epsilon_\ell)$		$D_{2\ell}^{2\ell,\ell}(\text{I}_b, \text{III}_a, \epsilon_\ell)$			
2ℓ		$D_{2\ell}$	ℓ		$\ell \cdot A_1$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$D_{2\ell}$	$D_{2\ell}$	$D(\ell)$	$\ell \cdot A_1$	ℓ	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sl(2\ell, \mathbb{R}) + \mathbb{R}$				$AI + \mathbb{C}$	$A_{2\ell-1} + \mathbb{R}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
$D_{2\ell+1}^{\ell,2\ell+1}(\text{III}_b, \text{I}_b)$		$D_{2\ell+1}^{\ell,2\ell+1}(\text{III}_b, \text{I}_b)$			
ℓ		BC_ℓ	ℓ		BC_ℓ
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_ℓ	BC_ℓ	$B(\ell)$	BC_ℓ	0	2 2 0 $\lambda_\ell : 1$ 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell+1, \mathbb{C})$				$BI \times BI$	B_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
$D_{2\ell+1}^{2\ell+1, \ell}(\text{I}_b, \text{III}_b)$		$D_{2\ell+1}^{2\ell+1, 2\ell+1}(\text{I}_b, \epsilon_\ell)$			
ℓ		$\ell \cdot A_1$	$2\ell + 1$		$D_{2\ell+1}$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$\ell \cdot A_1$	$D_{2\ell+1}$	$A(\ell)$	$D_{2\ell+1}$	ℓ	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(2\ell + 1, \mathbb{C})$				$BI \times BI$	B_ℓ

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
rank $\Phi_{\sigma, \theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma, \theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta, \theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
$D_{2\ell+1}^{2\ell+1, 2\ell+1}(\text{I}_b, \epsilon_\ell)$		$D_{2\ell+1}^{2\ell+1, \ell}(\text{I}_b, \text{III}_b)$			
$2\ell + 1$		$D_{2\ell+1}$	ℓ		$\ell \cdot A_1$
max.involution $\Phi_{\sigma, \theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta, \theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$D_{2\ell+1}$	$D_{2\ell+1}$	$D(2\ell + 1)$	$\ell \cdot A_1$	ℓ	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sl(2\ell + 1, \mathbb{R}) + \mathbb{R}$				$AI + \mathbb{C}$	$A_{2\ell} + \mathbb{R}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$D_\ell^{p,p}(I_a, \epsilon_p)$ $p < \ell - 2p$		$D_\ell^{p,\ell-2p}(I_a, \epsilon_p)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		B_p	p		B_p
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_p	B_p	$B(p)$	B_p	p	$\frac{1}{\lambda_p : 4(\ell - p)}$ 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(p, p) + sp(\ell - 2p, 0)$				$CII_a + CII_a$	BC_p

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{6,4}(\text{I}, \text{II}, \epsilon_0)$		$E_6^{6,2}(\text{I}, \text{IV})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		D_4	2		\emptyset
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
D_4	E_6	$A(4)$	\emptyset	0	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$f_{4(4)}$				FI	F_4

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
$E_6^{4,6}(\text{II}, \text{I}, \epsilon_0)$		$E_6^{4,2}(\text{II}, \text{IV})$			
4		F_4	1		A_1
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
F_4	F_4	$F(4)$	A_1	1	1 $\lambda_1, \lambda_2 : 0$ $\lambda_3, \lambda_4 : 1$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$f_{4(4)}$				FI	F_4

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{6,2}(\text{I}, \text{IV})$		$E_6^{6,4}(\text{I}, \text{II}, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		\emptyset	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
\emptyset	E_6	id	F_4	0	- - - -
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su^*(6) + su(2)$			$AII + \mathbb{C}$		A_2

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{2,6}(\text{IV}, \text{I})$		$E_6^{2,4}(\text{IV}, \text{II})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		A_2	1		A_1
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
A_1	A_2	$A(1)$	A_1	1	4 4 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su^*(6) + su(2)$				$AII + \mathbb{C}$	A_2

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{4,2}(\text{II}, \text{IV})$		$E_6^{4,6}(\text{II}, \text{I}, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
1		A_1	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
A_1	F_4	$A(1)$	F_4	1	5 3 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(3, 1)$				CII_a	BC_1

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{2,4}(\text{IV}, \text{II})$		$E_6^{2,6}(\text{IV}, \text{I})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
1		A_1	2		A_2
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
A_1	A_2	$A(1)$	A_1	1	3 5 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(3, 1)$				CII_a	BC_1

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{6,4}(\text{I}, \text{II}, \epsilon_1)$		$E_6^{6,6}(\text{I}, \epsilon_2)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		D_4	6		E_6
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
D_4	E_6	$A(4)$	E_6	4	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(4, \mathbb{R})$				CI	C_4

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{4,6}(\text{II}, \text{I}, \epsilon_1)$		$E_6^{4,6}(\text{II}, \text{I}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		F_4	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
F_4	F_4	$F(4)$	F_4	4	1 $\lambda_1, \lambda_2 : 0$ $\lambda_3, \lambda_4 : 1$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(4, \mathbb{R})$				CI	C_4

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{6,6}(\text{I}, \epsilon_2)$		$E_6^{6,4}(\text{I}, \text{II}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
6		E_6	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
D_4	E_6	$A(4)$	F_4	4	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sl(6, \mathbb{R}) + sl(2, \mathbb{R})$				$AI + AI$	$A_5 + A_1$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{6,2}(\text{I}, \text{III})$		$E_6^{6,6}(\text{I}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		$2 \cdot A_1$	6		E_6
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$2 \cdot A_1$	E_6	$A(2)$	E_6	2	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(2, 2)$				$CIIf$	C_2

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{2,6}(\text{III}, \text{I})$		$E_6^{2,6}(\text{III}, \text{I})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		BC_2	2		BC_2
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_2	BC_2	$B(2)$	BC_2	2	$\lambda_1 : 3$ 3 $2\lambda_1 : 0$ $\lambda_2 : 4$ 4 $2\lambda_2 : 1$ 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(2, 2)$				CII_b	C_2

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{6,6}(\mathrm{I}, \epsilon_1)$		$E_6^{6,2}(\mathrm{I}, \mathrm{III})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
6		E_6	2		$2 \cdot A_1$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
E_6	E_6	$A(4)$	$2 \cdot A_1$	2	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(5, 5) + \mathbb{R}$				$DI_b + \mathbb{C}$	$B_5 + \mathbb{R}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{4,2}(\text{II}, \text{III}, \epsilon_0)$		$E_6^{4,2}(\text{II}, \text{III}, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		B_2	2		B_2
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_2	F_4	$B(2)$	B_2	0	1 $\lambda_1, \lambda_2 : 0$ $\lambda_3, \lambda_4 : 1$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so^*(10) + so(2)$				$DIII_b + \mathbb{C}$	BC_2

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{2,4}(\text{III}, \text{II}, \epsilon_0)$		$E_6^{2,2}(\text{III}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		BC_2	2		BC_2
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_2	BC_2	$B(2)$	BC_2	2	$4 \quad \begin{matrix} \lambda_1 : 2 \\ \lambda_2 : 4 \end{matrix} \quad \begin{matrix} 0 \\ 1 \end{matrix} \quad \begin{matrix} 0 \\ 0 \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so^*(10) + so(2)$				$DIII_b + \mathbb{C}$	BC_2

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{2,2}(\text{III}, \epsilon_1)$		$E_6^{2,4}(\text{III}, \text{II}, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		BC_2	2		BC_2
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_2	BC_2	$B(2)$	BC_2	2	$4 \quad \begin{matrix} \lambda_1 : 2 \\ \lambda_2 : 4 \end{matrix} \quad \begin{matrix} 0 \\ 1 \end{matrix} \quad \begin{matrix} 0 \\ 0 \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(5, 1) + sl(2, \mathbb{R})$				$AIII_a + AI$	$BC_1 + A_1$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
$E_6^{2,2}(\text{III}, \epsilon_2)$		$E_6^{2,4}(\text{III}, \text{II}, \epsilon_0)$			
2		BC_2	2		BC_2
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_2	BC_2	$B(2)$	BC_2	2	$4 \quad \begin{matrix} \lambda_1 : 2 \\ \lambda_2 : 4 \end{matrix} \quad \begin{matrix} 0 \\ 1 \end{matrix} \quad \begin{matrix} 0 \\ 0 \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(5, 1) + sl(2, \mathbb{R})$				$AIII_a + AI$	$BC_1 + A_1$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{4,2}(\text{II}, \text{III}, \epsilon_1)$		$E_6^{4,4}(\text{II}, \epsilon_4)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		B_2	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_2	F_4	$B(2)$	F_4	2	1 $\lambda_1, \lambda_2 : 0$ $\lambda_3, \lambda_4 : 1$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(8, 2) + so(2)$				$DI_a + \mathbb{C}$	B_2

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{2,4}(\text{III}, \text{II}, \epsilon_1)$		$E_6^{2,4}(\text{III}, \text{II}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		BC_2	2		BC_2
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_2	BC_2	$B(2)$	BC_2	0	$4 \quad \begin{matrix} \lambda_1 : 2 \\ \lambda_2 : 4 \end{matrix} \quad \begin{matrix} 0 \\ 1 \end{matrix} \quad \begin{matrix} 0 \\ 0 \end{matrix}$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(4, 2) + su(2)$				$AIII_a + \mathbb{C}$	BC_2

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
$E_6^{4,4}(\text{II}, \epsilon_4)$		$E_6^{4,2}(\text{II}, \text{III}, \epsilon_1)$			
4		F_4	2		B_2
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
F_4	F_4	$F(4)$	B_2	2	1 $\lambda_1, \lambda_2 : 0$ $\lambda_3, \lambda_4 : 1$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(6, 4) + so(2)$				$DI_a + \mathbb{C}$	B_4

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
$E_6^{4,4}(\text{II}, \epsilon_1)$		$E_6^{4,4}(\text{II}, \epsilon_1)$			
4		F_4	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
F_4	F_4	$F(4)$	F_4	4	1 $\lambda_1, \lambda_2 : 0$ $\lambda_3, \lambda_4 : 1$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(3,3) + sl(2, \mathbb{R})$				$AIII_b + AI$	$C_3 + A_1$

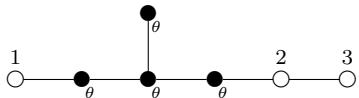
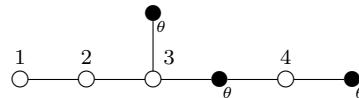
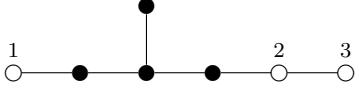
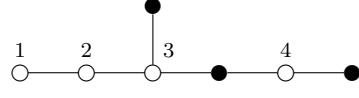
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{2,2}(\text{III}, \text{IV})$		$E_6^{2,2}(\text{III}, \text{IV})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
1		A_1	1		A_1
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
A_1	BC_2	$A(1)$	A_1	0	8 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$f_{4(-20)}$				FII	BC_1

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{2,2}(\text{IV}, \text{III})$		$E_6^{2,2}(\text{IV}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
1		A_1	2		A_2
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
A_1	A_2	$A(1)$	A_1	1	8 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$f_{4(-20)}$				FII	BC_1

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_6^{2,2}(\text{IV}, \epsilon_1)$		$E_6^{2,2}(\text{IV}, \text{III})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		A_2	1		A_1
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
A_1	A_2	$A(1)$	A_1	1	8 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(9, 1) + \mathbb{R}$				$DI_a + \mathbb{C}$	$B_1 + \mathbb{R}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{7,4}(V, VI, \epsilon_0)$		$E_7^{7,3}(V, VII)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		D_4	3		C_3
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
D_4	E_7	$D(4)$	C_3	0	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$e_{6(2)} + so(2)$				$EII + \mathbb{C}$	F_4

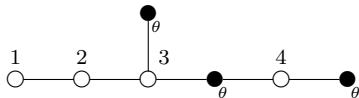
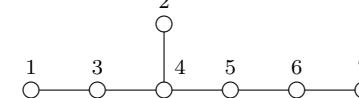
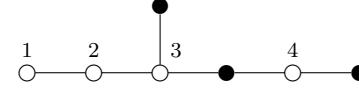
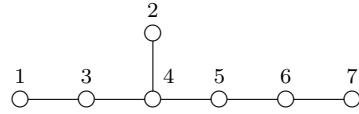
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{4,7}(\text{VI}, \text{V}, \epsilon_0)$		$E_7^{4,3}(\text{VI}, \text{VII}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		F_4	2		C_2
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
F_4	F_4	$F(4)$	C_2	2	1 $\lambda_1, \lambda_2 : 0$ $\lambda_3, \lambda_4 : 1$ 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$e_{6(2)} + so(2)$				$EII + \mathbb{C}$	F_4

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{7,3}(\text{V}, \text{VII})$		$E_7^{7,4}(\text{V}, \text{VI}, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
3		$3 \cdot A_1$	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$3 \cdot A_1$	E_7	$A(3)$	F_4	0	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so^*(12) + su(2)$				$DIII_a + \mathbb{C}$	C_3

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{3,7}(\text{VII}, \text{V})$		$E_7^{3,4}(\text{VII}, \text{VI}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
3		C_3	2		B_2
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_3	C_3	$C(3)$	B_2	2	$\lambda_1 : 1$ 0 0 0 $\lambda_2, \lambda_3 : 4$ 4 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so^*(12) + su(2)$				$DIII_a + \mathbb{C}$	C_3

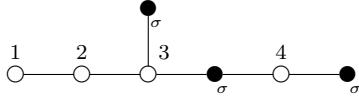
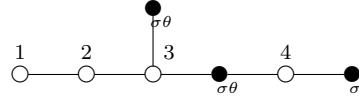
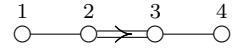
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{4,3}(\text{VI}, \text{VII}, \epsilon_1)$		$E_7^{4,7}(\text{VI}, \text{V}, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		B_2	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_2	F_4	$B(2)$	F_4	2	$\lambda_1 : 2$ 2 0 0 $\lambda_2 : 1$ 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(6, 2)$				$AIII_a$	BC_2

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{3,4}(\text{VII}, \text{VI}, \epsilon_1)$		$E_7^{3,7}(\text{VII}, \text{V})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		C_2	3		C_3
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_2	C_3	$C(2)$	C_3	2	$\lambda_1 : 6$ 2 0 0 $\lambda_2 : 1$ 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(6, 2)$				$AIII_a$	BC_2

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{7,4}(V, VI, \epsilon_1)$		$E_7^{7,7}(V, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		D_4	7		E_7
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
D_4	E_7	$D(4)$	E_7	4	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(4, 4)$				$AIII_b$	C_4

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{4,4}(\text{VI}, \epsilon_1)$		$E_7^{4,4}(\text{VI}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		F_4	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
F_4	F_4	$F(4)$	F_4	4	$\lambda_1, \lambda_2 : 1$ 0 $\lambda_3, \lambda_4 : 2$ 2 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so^*(12) + sl(2, \mathbb{R})$				$DIII_a + AI$	$C_3 + A_1$

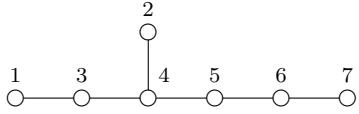
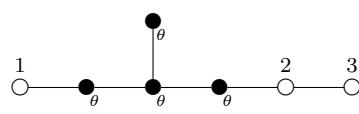
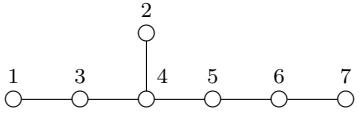
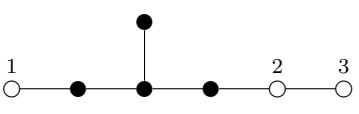
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{4,4}(\text{VI}, \epsilon_4)$		$E_7^{4,4}(\text{VI}, \epsilon_4)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		F_4	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
F_4	F_4	$F(4)$	F_4	4	$\lambda_1, \lambda_2 : 1$ 0 $\lambda_3, \lambda_4 : 2$ 2 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(8, 4) + su(2)$				$DI_a + \mathbb{C}$	B_4

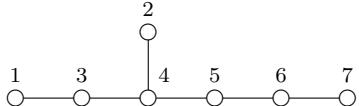
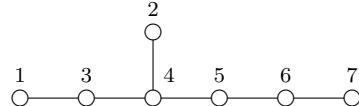
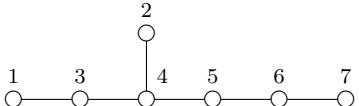
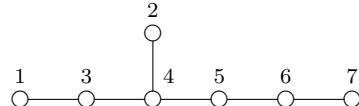
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{4,7}(\text{VI}, \text{V}, \epsilon_1)$		$E_7^{4,7}(\text{VI}, \text{V}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		F_4	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
F_4	F_4	$F(4)$	F_4	4	$\lambda_1, \lambda_2 : 1$ 0 $\lambda_3, \lambda_4 : 2$ 2 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su(4, 4)$				$AIII_b$	C_4

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{7,7}(V, \epsilon_1)$		$E_7^{7,4}(V, VI, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
7		E_7	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
E_7	E_7	$E(7)$	F_4	4	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(6, 6) + sl(2, \mathbb{R})$				$DI_b + AI$	$D_6 + A_1$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{7,3}(V, VII, \epsilon_1)$		$E_7^{7,7}(V, \epsilon_7)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
3		$3 \cdot A_1$	7		E_7
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
$3 \cdot A_1$	E_7	$A(3)$	E_7	3	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su^*(8)$				AII	A_3

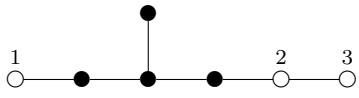
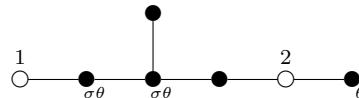
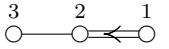
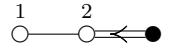
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{3,7}(\text{VII}, \text{V}, \epsilon_1)$		$E_7^{3,7}(\text{VII}, \text{V}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
3		C_3	3		C_3
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_3	C_3	$C(3)$	C_3	3	$\lambda_1 : 1 \quad 0$ $\lambda_2, \lambda_3 : 4 \quad 4 \quad 0 \quad 0$
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t}) \text{ or } \Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$su^*(8)$				AII	A_3

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{7,7}(V, \epsilon_7)$		$E_7^{7,3}(V, VII, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
7		E_7	3		$3 \cdot A_1$
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
E_7	E_7	$E(7)$	$3 \cdot A_1$	3	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$e_{6(6)} + \mathbb{R}$				$EI + \mathbb{C}$	$E_6 + \mathbb{R}$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{7,7}(V, \epsilon_2)$		$E_7^{7,7}(V, \epsilon_2)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
7		E_7	7		E_7
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
E_7	E_7	$E(7)$	E_7	7	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sl(8, \mathbb{R})$				AI	A_7

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{3,4}(\text{VII}, \text{VI})$		$E_7^{3,3}(\text{VII}, \epsilon_3)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		C_2	3		C_3
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
B_2	C_3	$B(2)$	C_3	2	$\lambda_1 : 6$ 2 0 0 $\lambda_2 : 1$ 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$e_{6(-14)}$				$EIII + \mathbb{C}$	BC_2

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{4,3}(\text{VI}, \text{VII})$		$E_7^{4,3}(\text{VI}, \text{VII})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		C_2	2		C_2
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_2	F_4	$B(2)$	C_2	2	$\lambda_1 : 6$ 2 0 0 $\lambda_2 : 1$ 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$e_{6(-14)}$				$EIII + \mathbb{C}$	BC_2

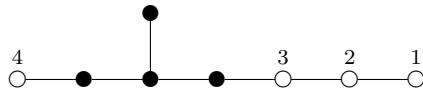
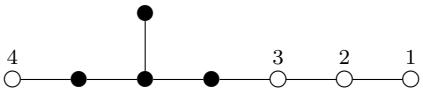
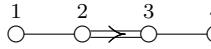
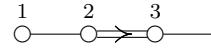
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{3,3}(\text{VII}, \epsilon_3)$		$E_7^{3,4}(\text{VII}, \text{VI})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
3		C_3	2		C_2
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_3	C_3	$B(3)$	C_2	2	$\lambda_1 : 6$ 2 0 0 $\lambda_2 : 1$ 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(10, 2) + sl(2, \mathbb{R})$				$DI_a + AI$	$B_2 + A_1$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_7^{3,3}(\text{VII}, \epsilon_1)$		$E_7^{3,3}(\text{VII}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
3		C_3	3		C_3
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
C_3	C_3	$B(3)$	C_3	3	$\lambda_1 : 6$ 2 0 0 $\lambda_2 : 1$ 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$e_{6(-26)} + sl(2, \mathbb{R})$				$EIV + AI$	$A_2 + A_1$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_8^{8,4}(\text{VIII}, \text{IX}, \epsilon_0)$		$E_8^{8,4}(\text{VIII}, \text{IX}, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		D_4	4		D_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
D_4	E_8	$D(4)$	D_4	0	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$e_{7(-5)} + so(2)$				$EVI + \mathbb{C}$	F_4

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_8^{4,8}(\text{IX}, \text{VIII}, \epsilon_0)$		$E_8^4(\text{IX}, \epsilon_4)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		F_4	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
F_4	F_4	$F(4)$	F_4	4	$\lambda_1, \lambda_2 : 1$ 0 $\lambda_3, \lambda_4 : 4$ 4 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$e_{7(-5)} + so(2)$				$EVI + \mathbb{C}$	F_4

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_8^{4,4}(\text{IX}, \epsilon_4)$		$E_8^{4,8}(\text{IX}, \text{VIII}, \epsilon_0)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		F_4	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
F_4	F_4	$F(4)$	F_4	4	$\lambda_1, \lambda_2 : 1$ 0 $\lambda_3, \lambda_4 : 4$ 4 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(12, 4)$				DI_a	B_4

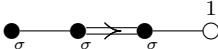
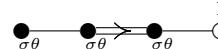
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_8^{4,4}(\text{IX}, \epsilon_1)$		$E_8^{4,4}(\text{IX}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		F_4	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
F_4	F_4	$F(4)$	F_4	4	$\lambda_1, \lambda_2 : 1$ 0 $\lambda_3, \lambda_4 : 4$ 4 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$e_{7(-25)} + sl(2, \mathbb{R})$				$EVII + AI$	$C_3 + A_1$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_8^{8,4}(\text{VIII}, \text{IX}, \epsilon_1)$		$E_8^8(\text{VIII}, \epsilon_8)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		D_4	8		E_8
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
D_4	E_8	$D(4)$	E_8	4	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so^*(16)$			$DIII_a$		C_4

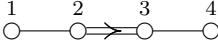
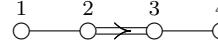
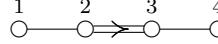
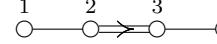
Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_8^{4,8}(\text{IX}, \text{VIII}, \epsilon_1)$		$E_8^{4,8}(\text{IX}, \text{VIII}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		F_4	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
F_4	F_4	$F(4)$	F_4	0	$\lambda_1, \lambda_2 : 1$ 0 $\lambda_3, \lambda_4 : 4$ 4 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so^*(16)$				$DIII_a$	C_4

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$E_8^{8,8}(\text{VIII}, \epsilon_8)$		$E_8^{8,4}(\text{VIII}, \text{IX}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
8		E_8	4		D_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
E_8	E_8	$E(8)$	D_4	4	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$			$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$		$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$e_{7(7)} + sl(2, \mathbb{R})$			$EV + AI$		$E_7 + A_1$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$F_4^{4,1}(\text{I}, \text{II})$		$F_4^4(\text{I}, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
1		A_1	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
A_1	F_4	$A(1)$	F_4	1	4 3 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(2, 1) + su(2)$				$CII_a + \mathbb{C}$	BC_1

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$F_4^{1,4}(\text{II}, \text{I})$		$F_4^{1,4}(\text{II}, \text{I})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
1	○	BC_1	1	○	BC_1
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
BC_1	BC_1	$B(1)$	BC_1	1	4 4 3 4
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(2, 1) + su(2)$				$CII_a + \mathbb{C}$	BC_1

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$F_4^{4,4}(\text{I}, \epsilon_4)$		$F_4^{4,1}(\text{I}, \text{II})$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		F_4	1		A_1
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
F_4	F_4	$F(4)$	A_1	1	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$so(5, 4)$				BI	B_4

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$F_4^{4,4}(I, \epsilon_1)$		$F_4^{4,4}(I, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
4		F_4	4		F_4
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
F_4	F_4	$F(4)$	F_4	4	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sp(3, \mathbb{R}) + sl(2, \mathbb{R})$				$CI + AI$	$C_3 + A_1$

Type (θ, σ)	diagram (θ, σ)	Type $(\theta, \sigma\theta)$	diagram $(\theta, \sigma\theta)$		
$G_2^{2,2}(I, \epsilon_1)$		$G_2^{2,2}(I, \epsilon_1)$			
rank $\Phi_{\sigma,\theta}$	$\sigma \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
2		G_2	2		G_2
max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W -conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank	$m^+(\lambda)$ $m^-(\lambda)$ $m^+(2\lambda)$ $m^-(2\lambda)$
G_2	G_2	$B(2)$	G_2	2	1 0 0 0
$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$				$(\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}, \bar{\theta})$	$\Phi(\mathbf{t})$ or $\Phi(\mathbf{t}_1) + \Phi(\mathbf{t}_2)$
$sl(2, \mathbb{R}) + sl(2, \mathbb{R})$				$AI + AI$	$A_1 + A_1$