The purpose of this study was to investigate the nature of preservice mathematics teachers’ understanding of inequalities. The design of the study was a multi-case study of secondary mathematics education preservice teachers with a focus on their understanding and the lessons that they plan and implement. Data from individual interviews, observations, and documents were collected and analyzed. Five preservice mathematics teachers from a major university in the Southeast United States participated in this study. These preservice teachers were examined during their student teaching field experience semester.

Preservice teachers displayed a robust procedural understanding, which involved an ability to apply the procedures necessary for solving inequalities, during pre-student teaching interviews (Rittle-Johnson & Alibali, 1999). At times, their understanding extended beyond an operational view of inequalities and allowed them to consider the relational nature of inequalities (Knuth et al., 2006). This was especially true while discussing the meaning of a solution of an inequality. Preservice teachers displayed some of the same conceptions noted within prior literature (e.g., treating inequalities as equations, believing that solutions of inequalities must be inequalities). There did not appear to be a relationship between a preservice teacher displaying one conception and any other conception.

Preservice teachers’ understanding of inequalities played a role in their planning and implementation of lessons related to systems of linear inequalities and/or quadratic inequalities. Boundary lines or curves of inequalities seemed to be a recurring focal point where preservice teachers’ understanding influenced their lessons. Preservice teachers often
used and explained the graphing of boundary lines or curves as an extension of graphing corresponding equations; thereby providing opportunities for students to review prior skills and concepts with linear and quadratic equations. Solution strategies were presented in an operational manner which tended to treat boundary lines or curves as an isolated component. This isolation often carried over to descriptions of solutions of inequalities. Preservice teachers described solutions of inequalities as regions and did not mention the inclusion or exclusion of boundary lines or curves. Overall, preservice teachers seemed to view and treat inequalities as something to do rather than something to be interpreted.

Implications of this study indicate a need to consider changes in methods courses. Deficiencies in preservice teachers’ understanding of inequalities need to be addressed within methods courses. Preservice teachers should have opportunities to examine pedagogical decisions that foster a relational view of inequalities (and equations). Additionally, methods courses should help develop preservice teachers’ awareness of how technology can be integrated into lessons without reducing tasks to rote button pushing exercises.
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Examining the Nature of Preservice Teachers’ Understanding of Inequalities Applied in the Practice of Teaching

by
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# TABLE OF CONTENTS

LIST OF TABLES .................................................................................................................. xiii

LIST OF FIGURES ................................................................................................................. xv

CHAPTER 1: INTRODUCTION ................................................................................................. 1

Background of the Study ........................................................................................................ 1

Definition of Terms ................................................................................................................ 4

Significance and Focus of the Study ....................................................................................... 5

Overview of Approach ........................................................................................................... 6

Chapter Summary .................................................................................................................. 7

Organization of the Study ...................................................................................................... 7

CHAPTER 2: LITERATURE REVIEW ...................................................................................... 9

Introduction ............................................................................................................................ 9

Knowledge in Algebra ........................................................................................................... 9

Conceptions surrounding the use of variable ....................................................................... 10

Difficulties with the equal sign and defining an equation .................................................... 13

Knowledge of Inequalities .................................................................................................... 14

Strategies for Solving Inequalities ....................................................................................... 15

  Algebraic manipulation for solving inequalities ................................................................. 15

  Drawing a graph .................................................................................................................. 17

  Using a case approach for solving inequalities ................................................................. 19

Difficulties Encountered With Inequalities ....................................................................... 21

  Difficulties relating inequalities to equations ................................................................. 22

    Solving equations instead of inequalities ..................................................................... 22

    Multiplying or dividing by factors that are not necessarily positive. .......................... 23

  Difficulties with solutions ............................................................................................... 24

    Inequalities will have inequalities as solutions .......................................................... 25
Single number answer tendency .................................................................26
Forming meaningless connections with quadratic roots...............................27
Excluded values ..........................................................................................27
Difficulties with logical connectives .............................................................27
Difficulties with the sign of the factors of products and/or quotients ............29
Difficulties related to the denominators of rational inequalities .....................29
Difficulties in interpreting the meaning of the inequality symbol .................30
Preservice Teachers Understanding of Inequalities .......................................31
Content Knowledge for Teaching Mathematics ............................................33

Lee Shulman and His Colleagues .................................................................34
Deborah Ball and Her Colleagues .................................................................39
Joan Ferrini-Mundy and Her Colleagues ......................................................44
Categories of knowledge ............................................................................47
Overarching categories ................................................................................49
  Bridging ....................................................................................................49
  Decompressing .......................................................................................50
  Trimming. ...............................................................................................51
Conclusion ....................................................................................................54

CHAPTER 3: CONCEPTUAL FRAMEWORK AND METHODOLOGY ..................55

Introduction ..................................................................................................55

Overall Research Design ............................................................................55

Conceptual Framework ..............................................................................56

Pilot Studies .................................................................................................65

Lesson Plan ..................................................................................................65

Task-based Pre-Student Teaching Interview ...............................................68

Defining the Case for the Current Study .....................................................70

Participant Selection ...................................................................................71

Angela ..........................................................................................................73
Christina ......................................................................................................................... 74
Crystal .......................................................................................................................... 74
Heather .......................................................................................................................... 75
Vanessa .......................................................................................................................... 76
Data Collection .............................................................................................................. 77

Pre-Student Teaching Interviews .................................................................................. 81
Lesson Plans ................................................................................................................... 86
Classroom Observations ............................................................................................... 87
Post-Student Teaching Interview ................................................................................ 89
Data Analysis ................................................................................................................ 94

Phase One ...................................................................................................................... 95
Phase Two ...................................................................................................................... 97
Research Validity and Reliability ................................................................................ 105

Subjectivity Statement ................................................................................................. 105

Ethical Issues ................................................................................................................ 107

Chapter Summary ........................................................................................................ 108

CHAPTER 4: UNDERSTANDING REGARDING INEQUALITIES ................................... 109

Aspects of Knowledge Associated with Inequalities ..................................................... 109

Angela ............................................................................................................................ 112

Considering inequalities as equations .......................................................................... 113
Attending to negative values ......................................................................................... 113
Inequalities must have inequalities as solutions ............................................................ 115
Quadratic Inequalities .................................................................................................. 118
Absolute value inequalities .......................................................................................... 124
Systems of inequalities ................................................................................................ 126
Explaining why the sense of an inequality changes ....................................................... 129
Characterizing Angela’s understanding ....................................................................... 130
Quadratic Inequalities ........................................................................................................194
Absolute value inequalities ..........................................................................................199
Systems of inequalities ...............................................................................................201
Explaining why the sense of an inequality changes....................................................203
Characterizing Vanessa’s understanding .....................................................................204
Cross Case Analysis ....................................................................................................204
Strategies for solving inequalities ................................................................................204
  Systems of linear inequalities ..................................................................................205
  Quadratic inequalities ............................................................................................205
Relating inequalities to equations ................................................................................206
  Systems of linear inequalities ..................................................................................206
  Quadratic inequalities ............................................................................................207
Shading as a process .....................................................................................................207
  Systems of linear inequalities ..................................................................................207
  Quadratic inequalities ............................................................................................207
Solutions of inequalities ...............................................................................................208
  Systems of linear inequalities ..................................................................................208
  Quadratic inequalities ............................................................................................208
Possible connections among aspects ..........................................................................209
Chapter Summary ........................................................................................................211

CHAPTER 5: TEACHING SYSTEMS OF LINEAR INEQUALITIES ..................................................213
Descriptions of Classrooms ..........................................................................................214
Lessons Taught Prior to and After Systems of Linear Inequalities ..............................216
Areas of Consideration Prior to the Analysis ...............................................................217
Analysis of Teaching Aspects of Knowledge Associated with Inequalities.................218
Introduction of the aspects .........................................................................................218
Strategies to solve inequalities ....................................................................................219
  Explaining a mathematical idea ..............................................................................220
  View of inequalities ...............................................................................................225
Solving a mathematical problem .................................................. 226
View of inequalities ................................................................. 229
Using technology ....................................................................... 229
View of inequalities ................................................................. 234
Treating inequalities as equations .................................................. 234
Explaining a mathematical idea .................................................. 235
View of inequalities ................................................................. 236
Solving a mathematical problem .................................................. 237
View of inequalities ................................................................. 239
Using technology ....................................................................... 240
View of inequalities ................................................................. 241
The process of shading linear inequalities ..................................... 242
Explaining a mathematical idea .................................................. 242
View of inequalities ................................................................. 248
Solving a mathematical problem .................................................. 249
View of inequalities ................................................................. 254
Using technology ....................................................................... 256
Preservice teachers implemented similar procedures when graphing and shading linear inequalities using graphing calculators .................................................. 258
View of inequalities ................................................................. 258
Solutions to systems of linear inequalities ..................................... 258
Explaining a mathematical idea .................................................. 259
View of inequalities ................................................................. 261
Solving a mathematical problem .................................................. 261
View of inequalities ................................................................. 266
Using technology ....................................................................... 266
View of inequalities ................................................................. 268
Characterizing How Preservice Teachers Approach Systems of Linear Inequalities........268

Chapter Summary ..................................................................................................................270

CHAPTER 6: QUADRATIC INEQUALITIES ..............................................................................271

Descriptions of the Classrooms .............................................................................................272

Areas of Consideration Prior to Analysis .............................................................................273

Analysis of Teaching Aspects of Knowledge Associated with Inequalities .........................275

Introduction of the aspects ......................................................................................................275

Strategies to solve inequalities ..............................................................................................276
  Explaining a mathematical idea ..........................................................................................278
    View of inequalities ..........................................................................................................289
  Solving a mathematical problem .......................................................................................290
    View of inequalities ..........................................................................................................297
  Using technology ................................................................................................................298
    View of inequalities ..........................................................................................................303

Treating inequalities as equations .......................................................................................304
  Explaining a mathematical idea ..........................................................................................304
    View of inequalities ..........................................................................................................305
  Solving a mathematical problem .......................................................................................305
    View of inequalities ..........................................................................................................311
  Using technology ................................................................................................................311
    View of inequalities ..........................................................................................................313

The process of shading quadratic inequalities ....................................................................313
  Explaining a mathematical idea ..........................................................................................313
    View of inequalities ..........................................................................................................316
  Solving a mathematical problem .......................................................................................317
    View of inequalities ..........................................................................................................319
  Using technology ................................................................................................................319
    View of inequalities ..........................................................................................................320

Solutions to quadratic inequalities ......................................................................................321
  Explaining a mathematical idea ..........................................................................................321
Appendix E. Keyword Maps of Classroom Observations. ..........................................................392
LIST OF TABLES

Table 1. Relationship between conceptions of algebra and use of variables.......................... 11
Table 2. Conceptual framework for this study........................................................................ 60
Table 3. Conceptual framework, from a pilot study, that was used to analyze a lesson plan. 66
Table 4. Angela's classroom observation dates and topics covered........................................ 78
Table 5. Christina's classroom observation dates and topics covered. ................................. 78
Table 6. Crystal's classroom observation dates and topics covered......................................... 79
Table 7. Heather's classroom observation dates and topics covered. .................................... 79
Table 8. Vanessa's classroom observation dates and topics covered....................................... 80
Table 9. Timeline outlining data collection specifically relevant to inequalities....................... 80
Table 10. Data sources used to address each research question............................................ 81
Table 11. Link between aspects of knowledge associated with inequalities and questions from pre-student teaching interview................................................................. 83
Table 12. Duration and content areas of the episodes from Angela's post-student teaching interview........................................................................................................................................ 93
Table 13. Duration and content areas of the episodes from Christina's post-student teaching interview........................................................................................................................................ 93
Table 14. Duration and content areas of the episodes from Crystal's post-student teaching interview........................................................................................................................................ 93
Table 15. Duration and content areas of the episodes from Heather's post-student teaching interview........................................................................................................................................ 94
Table 16. Duration and content areas of the episodes from Vanessa's post-student teaching interview. ................................................................. 94

Table 17. Sorting of questions from pre-student teaching interview by commonalities. ...... 96

Table 18. Outline of the procedures conducted in each content area and the data source used for each procedure................................................................. 100
LIST OF FIGURES

Figure 1. Structure of mathematical knowledge for teaching. ................................................. 41

Figure 2. Framework characterizing knowledge for teaching school algebra. ......................... 46

Figure 3. Questions #9 and #10 from the pilot study of the task-based interview. ................. 69

Figure 4. Questions on the post-student teaching interview addressed to the preservice teachers who taught systems of linear inequalities. ......................................................... 90

Figure 5. Questions on the post-student teaching interview addressed to the preservice teachers who taught quadratic inequalities. ................................................................. 91

Figure 6. Coding of Vanessa's classroom observation (February 28, 2011). ...................... 102

Figure 7. Coding of Heather's classroom observation (March 7, 2011). .......................... 103

Figure 8. Problem #1 from the pre-student teaching interview protocol. ......................... 110

Figure 9. Problem #2 from the pre-student teaching interview protocol. ......................... 110

Figure 10. Problem #3 from the pre-student teaching interview protocol. ....................... 111

Figure 11. Problem #4 from the pre-student teaching interview protocol. ....................... 111

Figure 12. Problem #5 from the pre-student teaching interview protocol. ....................... 111

Figure 13. Problem #6 from the pre-student teaching interview protocol. ....................... 111

Figure 14. Problem #7 from the pre-student teaching interview protocol. ....................... 111

Figure 15. Problems #8a and #8b from the pre-student teaching interview protocol. ........ 111

Figure 16. Problem #8c from the pre-student teaching interview protocol. .................... 112

Figure 17. Problem #9 from the pre-student teaching interview protocol. ....................... 112

Figure 18. Problem #10 from the pre-student teaching interview protocol. ..................... 112

Figure 19. Problem #11 from the pre-student teaching interview protocol. ..................... 112
Figure 20. Angela's first attempt to solve #8a. ................................................................. 120

Figure 21. Calculator screen shots as Angela reworks #8a. .............................................. 121

Figure 22. Angela's second attempt to solve #8a. ............................................................... 121

Figure 23. Calculator screen shots from Angela's work on #8b. ..................................... 124

Figure 24. Angela's solution to #8c. ...................................................................................... 125

Figure 25. Angela's attempt to use the elimination method to solve #9. ...................... 126

Figure 26. Angela's algebraic manipulations of #9. ......................................................... 126

Figure 27. Calculator screen shots of Angela's work on #9. ........................................... 127

Figure 28. Angela's handwritten graph for #9. ............................................................... 127

Figure 29. Angela's graphs examining the coordinate (0, 0.5). ................................... 129

Figure 30. The number line Angela used as she explained #11. ...................................... 130

Figure 31. Christina's algebraic manipulation of #4. ....................................................... 133

Figure 32. Christina's algebraic manipulation of #5. ....................................................... 133

Figure 33. Christina's algebraic manipulation of #6. ....................................................... 134

Figure 34. Screen shots of Christina's graph and table of 5x4. .................................... 136

Figure 35. Christina's work for #7. .................................................................................... 137

Figure 36. Screen shots from Christina's work on #8a. .................................................... 138

Figure 37. Christina's solution to #8a. ............................................................................... 139

Figure 38. Christina's work and solution for #8b. ............................................................ 140

Figure 39. Screen shots from Christina's work on #8b. .................................................... 140

Figure 40. Screen shots from Christina's initial attempt to solve #10. ......................... 141

Figure 41. Christina's work for #10. .................................................................................. 142
Figure 42. Screen shots from Christina's revised attempt to solve #10. ...................... 142

Figure 43. Screen shots of the intersection points Christina used to solve #10. ........ 144

Figure 44. Christina's work for #8c. ........................................................................ 145

Figure 45. Christina's attempt to use the elimination method to solve #9. .......... 147

Figure 46. Christina's work for #9. ......................................................................... 147

Figure 47. Screen shots from Christina's work for #9. ........................................... 147

Figure 48. Christina's work for #11. ...................................................................... 150

Figure 49. Crystal's graph of #2f. ......................................................................... 152

Figure 50. Crystal's work for #6. ......................................................................... 153

Figure 51. Crystal's table and graph for #6. ............................................................. 154

Figure 52. Crystal's first attempt to solve #7. ......................................................... 155

Figure 53. Crystal's second attempt to solve #7 and her solution. ....................... 156

Figure 54. Screen shots from Crystal's work on #7. .............................................. 157

Figure 55. Crystal's first attempt to solve #8a using a case strategy. ................... 158

Figure 56. Crystal's second attempt to solve #8a. .................................................... 159

Figure 57. Crystal's work for #8b. ........................................................................ 160

Figure 58. Screen shots from Crystal's work in #10. ............................................. 161

Figure 59. Crystal's work for #8c. ........................................................................ 163

Figure 60. Crystal's work for #9. .......................................................................... 165

Figure 61. Screen shots from Crystal's work for #9. ............................................. 165

Figure 62. Crystal's work for #11. ...................................................................... 167

Figure 63. Heather’s work for #4. ...................................................................... 170
Figure 64. Heather’s work for #5. .............................................................. 171
Figure 65. Heather's work for #6. .............................................................. 171
Figure 66. Heather's work for #3. .............................................................. 173
Figure 67. Screen shots from Heather’s work on #7. .............................. 173
Figure 68. Screen shots from Heather's second attempt for #3. .............. 175
Figure 69. Heather's work for #8a. ........................................................... 177
Figure 70. Screen shots from Heather's work on #8a. ............................ 177
Figure 71. Screen shots from Heather's initial attempt to solve #8b. ....... 178
Figure 72. Heather's initial solution for #8b. ........................................... 178
Figure 73. Heather’s work for #8b. ......................................................... 179
Figure 74. Screen shots from Heather's second attempt to solve #8b. ...... 180
Figure 75. Heather's work for #8c. .......................................................... 181
Figure 76. Heather's work for #9. ............................................................ 183
Figure 77. Screen shots from Heather's work on #9. ............................... 184
Figure 78. Heather's graph for #9. ............................................................ 184
Figure 79. Heather’s work for #11. ......................................................... 186
Figure 80. Vanessa's work for #4. ............................................................. 189
Figure 81. Vanessa's work for #6. ............................................................. 190
Figure 82. Vanessa's work for #3 (Pre-student teaching interview; February 17, 2011). .... 191
Figure 83. Vanessa's work for #7. ............................................................ 192
Figure 84. Screen shots from Vanessa's work on #7. .............................. 193
Figure 85. Screen shots from Vanessa's work on #8a. ............................ 194
Figure 86. Vanessa's work for #8a. ................................................................. 195
Figure 87. Vanessa's work for #8b. .................................................................... 197
Figure 88. Vanessa's work for #10. ..................................................................... 198
Figure 89. Screen shots from Vanessa's work on #10. ......................................... 198
Figure 90. Vanessa's work for #8c. ..................................................................... 199
Figure 91. The graph that Vanessa created for #8c. .............................................. 199
Figure 92. Screen shots from Vanessa's work on #8c. .......................................... 199
Figure 93. Vanessa's representation of her solution for #8c. ............................... 200
Figure 94. Vanessa's work to isolate the variable y for #9. ................................. 201
Figure 95. Screen shots from Vanessa's work on #9. ......................................... 201
Figure 96. Vanessa's graph of her solution for #9. ............................................. 201
Figure 97. Vanessa's work for #11. ................................................................... 203
Figure 98. Vanessa's notes/worksheet outlining the process for solving a system of linear inequalities (Lesson plan; February 28, 2011). ................................................................. 221
Figure 99. Angela's notes outlining the process to graph a system of linear inequalities (Lesson plan; May 2, 2011). ................................................................. 222
Figure 100. Example #3 from Angela's lesson plan (Lesson plan; May 2, 2011) .... 227
Figure 101. Examples #1 and #6 from Vanessa's lesson plan (Lesson plan; February 28, 2011). ........................................................................................................... 227
Figure 102. Example #3a and the graph from Angela's class (Classroom observation; May 2, 2011). ........................................................................................................... 232
Figure 103. Linear Programming problem Crystal worked out in her class (Lesson plan; March 11, 2011) ................................................................. 238

Figure 104. Crystal’s work with the first constraint of a problem (Classroom observation; March 11, 2011) ................................................................................................................................. 239

Figure 105. Crystal switching between an equation and an inequality with the second constraint of a linear programming problem (Classroom observation; March 11, 2011). ... 239

Figure 106. Calculator screenshots from Vanessa class (Classroom observation; February 28, 2011). ................................................................................................................................. 240

Figure 107. Calculator screenshots from Vanessa class (Classroom observation; February 28, 2011). ................................................................................................................................. 240

Figure 108. Calculator screenshots from Vanessa class (Classroom observation; February 28, 2011). ................................................................................................................................. 241

Figure 109. Graph of a system of linear inequalities displayed in Vanessa’s class (Classroom observation; February 28, 2011). ................................................................. 241

Figure 110. Angela’s process for graphing a linear inequality that was projected and discussed (Lesson plan; May 2, 2011) ................................................................. 243

Figure 111. Crystal’s first and second example (Lesson plan; March 10, 2011) ............... 244

Figure 112. Crystal using her hands to illustrate a balance scale (Classroom Observation; March 10, 2011) ................................................................................................................................. 245

Figure 113. Crystal using her hands to illustrate a see-saw (Classroom Observation; March 10, 2011) ................................................................................................................................. 245
Figure 114. Crystal demonstrating the "see-saw" thought process to her students (Classroom observation; March 10, 2011). ............................................................ 246

Figure 115. Crystal’s shading of the "see-saw" thought process (Classroom observation; March 10, 2011). ............................................................ 246

Figure 116. Crystal’s steps to graph a system of linear inequalities (Lesson plan and classroom observation; March 10, 2011). ............................................................ 251

Figure 117. Crystal's student used the test point method (Classroom observation; March 10, 2011). ........................................................................................................................................ 253

Figure 118. Re-creation of the student's mistake in Figure 117. ........................................................................................................................................ 253

Figure 119. Example #1 and the graph from Vanessa's class (Classroom observation; February 28, 2011). ........................................................................................................................................ 257

Figure 120. Calculator screenshots from Vanessa class (Classroom observation; February 28, 2011). ........................................................................................................................................ 257

Figure 121. Calculator screenshots from Vanessa class (Classroom observation; February 28, 2011). ........................................................................................................................................ 258

Figure 122. Vanessa's student handout regarding the solution of a system of linear inequalities (Classroom observation; February 28, 2011). ........................................................................................................................................ 260

Figure 123. Vanessa's first example (Lesson plan; February 28, 2011). ........................................................................................................................................ 263

Figure 124. Vanessa's solution to her first example (Classroom observation; February 28, 2011). ........................................................................................................................................ 263

Figure 125. Vanessa denoting the solution for her first example (Classroom observation; February 28, 2011). ........................................................................................................................................ 264
Figure 126. Example #2 from Vanessa's lesson (Lesson plan; February 28, 2011).......... 265

Figure 127. Students' work for example #2 in Vanessa’s class (Classroom observation; February 28, 2011)................................................................................................................................. 265

Figure 128. Students' work for example #2 in Vanessa’s class (Classroom observation; February 28, 2011)................................................................................................................................. 265

Figure 129. Students' work for example #2 in Vanessa’s class (Classroom observation; February 28, 2011)................................................................................................................................. 265

Figure 130. Calculator screenshots from Vanessa class (Classroom observation; February 28, 2011)................................................................................................................................. 267

Figure 131. Calculator screenshots from Vanessa class (Classroom observation; February 28, 2011)................................................................................................................................. 268

Figure 132. Heather's directions for graphing a quadratic equation with two variables (Lesson plan and Classroom observation; March 1, 2011)............................................................. 279

Figure 133. The "basic rules" from Heather’s lesson on quadratic inequalities with two variables (Classroom observation; March 8, 2011)........................................................................ 280

Figure 134. Slide from Vanessa's lesson with steps for graphing a quadratic inequality (Lesson plan and Classroom observation; April 11, 2011)................................................................. 280

Figure 135. Heather's graph with the vertex and axis of symmetry displayed (Classroom observation; March 7, 2011).................................................................................................................. 282

Figure 136. Heather's graph of the parabola (Classroom observation; March 7, 2011).... 282

Figure 137. Heather's graph of the parabola with shading (Classroom observation; March 7, 2011).................................................................................................................................. 282
Figure 138. Heather's two versions of the solution set (Classroom observation; March 7, 2011). .......................................................... 282

Figure 139. Vanessa identifying the "inside" of the parabola (Classroom observation; April 11, 2011). .......................................................... 284

Figure 140. Vanessa identifying the "outside" of the parabola (Classroom observation; April 11, 2011). .......................................................... 284

Figure 141. Vanessa identifying test point (2, 4) (Classroom observation; April 11, 2011). ........................................................................... 285

Figure 142. Vanessa's work for test point method for example #1 (Classroom observation; April 11, 2011). .......................................................... 285

Figure 143. Vanessa shaded the "outside" of the parabola for example #1 (Classroom observation; April 11, 2011). .................................................. 285

Figure 144. Heather's factored version of her first example of a quadratic inequality with one variable (Classroom observation; March 7, 2011). .......................................................... 287

Figure 145. Heather's "related equation" and solution for first example of a quadratic inequality with one variable (Classroom observation; March 7, 2011). .......................................................... 287

Figure 146. The sign chart from Heather’s work on her first example of a quadratic inequality with one variable (Classroom observation; March 7, 2011). .......................................................... 288

Figure 147. The shaded number line for Heather's first example of a quadratic inequality with one variable (Classroom observation; March 7, 2011). .......................................................... 288

Figure 148. Heather's outline of the graphing strategy for solving quadratic inequalities with one variable (Classroom observation; March 7, 2011). .......................................................... 290
Figure 149. Powerpoint slide Vanessa displayed during her lesson with examples of a quadratic equation with one variable and a quadratic inequality with one variable (Classroom observation; April 11, 2011). ................................................................. 292

Figure 150. Other planned examples of quadratic inequalities with one variable from Vanessa’s lesson (Classroom observation; April 11, 2011). ................................................................. 292

Figure 151. Problems on a review sheet of quadratic inequalities with one variable requested by Vanessa’s students (Classroom observation; April 11, 2011). ................................................................. 292

Figure 152. Heather's first example of a quadratic inequality with one variable (Classroom observation; March 7, 2011). ......................................................................................... 293

Figure 153. Heather's second example of a quadratic inequality with one variable (Classroom observation; March 7, 2011). ......................................................................................... 293

Figure 154. Two examples of quadratic inequalities with one variable from Crystal's lesson (Classroom observation; April 5, 2011). ......................................................................................... 293

Figure 155. Crystal's third example of a quadratic inequality with one variable produced by changing the inequality sign in example #1 (Classroom observation; April 5, 2011). ............. 293

Figure 156. Examples of graphing quadratic equations with two variables from Vanessa’s class (Classroom observation; April 4, 2011). ......................................................................................... 294

Figure 157. Examples of graphing quadratic equations with two variables from Heather’s class (Classroom observation; March 1, 2011). ......................................................................................... 294

Figure 158. Heather's first example of a quadratic inequality with two variables (Classroom observation; March 8, 2011). ......................................................................................... 295
Figure 159. Heather’s second example of a quadratic inequality with two variables
(Classroom observation; March 8, 2011)........................................................................295

Figure 160. Powerpoint slide from Vanessa's lesson with the first example of quadratic
inequality with two variables (Lesson plan; April 11, 2011)........................................296

Figure 161. Vanessa's second example of a quadratic inequality with two variables (Lesson
plan; April 11, 2011)................................................................................................296

Figure 162. Vanessa' third example of a quadratic inequality with two variables (Lesson
plan; April 11, 2011)................................................................................................296

Figure 163. Vanessa's fourth example of a quadratic inequality with two variables (Lesson
plan; April 11, 2011)................................................................................................297

Figure 164. Vanessa entering the quadratic inequality on her TI-84 (Classroom observation;
April 11, 2011)........................................................................................................299

Figure 165. Vanessa selecting the shading for an example on her TI-84 (Classroom
observation; April 11, 2011). ....................................................................................299

Figure 166. Vanessa's graph with shading for an example (Classroom observation; April 11,
2011). .......................................................................................................................299

Figure 167. Vanessa entering a quadratic expression (Classroom observation; April 11,
2011). .......................................................................................................................301

Figure 168. Vanessa selecting an inequality (Classroom observation; April 11, 2011). .... 301

Figure 169. Vanessa's inequality (Classroom observation; April 11, 2011). ............... 301

Figure 170. A graph from Vanessa's work (Classroom observation; April 11, 2011). ....... 301

Figure 171. A graph from Vanessa's work (Classroom observation; April 11, 2011). ....... 302
Figure 172. Vanessa's solution to her first example of a quadratic inequality with one variable (Classroom observation; April 11, 2011) .............................................................. 302

Figure 173. Vanessa's solution and work for an example of a quadratic inequality with one variable (Classroom observation; April 11, 2011) .............................................................. 307

Figure 174. Vanessa's factored version of an example of a quadratic inequality with one variable (Classroom observation; April 11, 2011) .............................................................. 308

Figure 175. For an example of a quadratic inequality with one variable, Vanessa initially set the first factor equal to zero (Classroom observation; April 11, 2011). ........................................ 308

Figure 176. For her first example of a quadratic inequality with one variable, Vanessa set the first factor to be less than zero after a student comment (Classroom observation; April 11, 2011). ............................................................................................................ 309

Figure 177. Vanessa initially wrote x = 7 while solving her first example of a quadratic inequality with one variable (Classroom observation; April 11, 2011). ........................................ 309

Figure 178. Vanessa's displayed work for her first example of a quadratic inequality with one variable (Classroom observation; April 11, 2011) ............................................................................................................ 309

Figure 179. Vanessa's test point method and shading for an example (Classroom observation; April 11, 2011) ............................................................................................................ 312

Figure 180. Vanessa entering an example in to her graphing calculator (Classroom observation; April 11, 2011) ............................................................................................................ 312

Figure 181. Graph of $x^2 + 2x - 3 > 0$ from Heather's lesson plan (Lesson plan; March 7, 2011). ............................................................................................................ 315

Figure 182. Heather’s graph of boundary curve (Classroom observation; March 8, 2011). 318
Figure 183. Heather’s work to evaluate the test point (Classroom observation; March 8, 2011). ................................................................. 318

Figure 184. Heather's shaded graph (Classroom observation; March 8, 2011). ...................... 318

Figure 185. The points plotted on the solution of Vanessa's first example of a quadratic inequality with two variables (Classroom observation; April 11, 2011). ......................... 323

Figure 186. Crystal's work and solution to an example of a quadratic inequality with one variable (Classroom observation; April 5, 2011). ............................................................. 325
CHAPTER 1: INTRODUCTION

Background of the Study

There is evidence of a vicious cycle in which too many prospective teachers enter college with an insufficient understanding of school mathematics, experience little college instruction focused on the mathematics they will teach, and then enter their classrooms inadequately prepared to teach mathematics to the following generations of students.

(Conference Board of the Mathematical Sciences (CBMS), 2001, p. 163)

In 2001, the CBMS report made a series of recommendations that emphasized the importance of mathematical understanding, particularly with regard to preservice teachers. There are many areas underneath the umbrella of mathematics (e.g., algebra, geometry, calculus, statistics, probability, discrete mathematics, etc.) but many argue that a focus on improving teachers’ understanding of algebra is particularly appropriate “because algebra forms the core of the high school mathematics curriculum, improving the teaching and learning of algebra is critical” (Rakes, Valentine, McGatha, & Ronau, 2010, p. 372). In fact, in 2003, the RAND Mathematics Study Panel (2003) purported that “the initial topical choice for focused and coordinated research and development [in Mathematics Education] should be algebra” (p. 43).

The National Council of Teachers of Mathematics (NCTM) declared their view on the importance of algebra by making it one of the five content standards in the Principles and Standards for School Mathematics (NCTM, 2000). Some even go as far as to view Algebra as the new civil right (Moses, 1995). With this “Algebra for all” push, there have been
numerous studies examining the various aspects of algebra (e.g., functions, variables, equations, representations). Along with these recommendations has come much debate surrounding Algebra (e.g., Loveless, 2008; Prevost, 1986; Usiskin, 1987).

If algebra is to be the central focus, then why is there a need for further study? There are a number of studies that have examined preservice, as well as inservice, teachers’ understanding of functions (Even, 1993; Stein, Baxter, & Leinhardt, 1990; Vinner & Dreyfus, 1989; Wilson, 1994). In addition, students’ understanding of functions has been the centerpiece of many studies (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1991; Hollar & Norwood, 1999; O’Callaghan, 1998; Slavit, 1997; Williams, 1998; Yerushalmy, 1991).

Several researchers have turned their attention to how students understand equations (Godfrey & Thomas, 2008; Kieran, 1981; Knuth, Stephens, McNeil, & Alibali, 2006; Koedinger & Nathan, 1998; Nathan & Koedinger, 2000b). Many of these studies focused on middle and high school students’ knowledge and misconceptions of equations and equality. However, the body of research concerning how students and teachers understand relationships that are unequal is much smaller (Bazzini & Tsamir, 2001; Blanco & Garrote, 2007; Lim, 2008; Tsamir & Bazzini, 2002; Vaiyavutjamai & Clements, 2006). The majority of this work has been focused on secondary students (ages 16-17) primarily from Israel and Italy.

Due to the similarity of the structures of equations and inequalities, both share common ground as far as conceptions surrounding the use of variables. This area has been explored in depth (e.g., Booth, 1988; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; MacGregor & Stacey, 1997; McNeil, et al., 2010; Schoenfeld & Arcavi, 1999; Usiskin, 1999;
Those studying conceptions of inequalities (e.g., Blanco & Garrote, 2007; Tsamir & Bazzini, 2002) have noted that there appears to be an overemphasis or an unfounded reliance on the similarities in structure between equations and inequalities that has led to a lack of distinction between the two:

Many students understood the greater than and less than signs to be a nexus between two algebraic expressions. They then carried this nexus through the various steps in solving an inequality without attaching any meaning to it, even to the point of simply substituting an equals sign. (Blanco & Garrote, 2007, p. 224)

This overemphasis or unfounded reliance is characterized by students’ attempts to apply transformational techniques or algebraic manipulations used with equations to solve inequalities (Kieran, 2004).

With consideration to the volumes of studies that examine algebra, there is a deficit in the area of teachers’ knowledge, or for that matter preservice teachers’ knowledge, of inequalities. Many of the authors of the studies that examined students’ understanding of inequalities included statements similar to those made by Tsamir and Bazzini (2002):

…teachers should be familiarized with these intuitive beliefs and with their impact on students’ reasoning when solving inequalities. They should, subsequently, be encouraged to look for ways to promote students’ awareness of various intuitive obstacles and of the need to be ‘on guard’ when solving mathematical problems in conjunction with their knowledge of equations. (p. 809)

Yet, none of the studies talk about whether teachers hold the same ‘intuitive beliefs or obstacles’ (e.g., an equality cannot be a solution to an inequality, inequalities can be solved
in the same manner as equations) as the students who were examined. Additionally, none of the studies examined how teachers develop and implement lessons on inequalities. Before researchers can make recommendations for how teachers should address students’ misconceptions about inequalities, the field needs to know what teachers understand about inequalities. Without an understanding of what teachers know about inequalities, initiatives designed to strengthen preservice teachers’ content knowledge and pedagogical content knowledge within methods courses may rely on false assumptions. This dissertation aims to provide some insight into this issue.

**Definition of Terms**

It is worthwhile to define several of the terms used in this study so that they are used and interpreted consistently.

Inservice teacher – teacher that is employed by a school and licensed in the courses that they teach

Preservice teacher – student in a teacher education program at an institute of higher education pursuing licensure as a teacher

Cooperating teacher – an inservice teacher who has is supervising a preservice teacher during student teaching field experience

Student – the pupils enrolled in a secondary-level class

Lesson – a series of activities designed to explore, practice, and/or explain a specific objective

Equation – a mathematical statement that two or more expressions are equivalent to one another
Inequality – a mathematical statement that two expressions are related to one another in the one of the following manners: less than (<); greater than (>); less than or equal to (≤); greater than or equal to (≥); or not equal to (≠).

Significance and Focus of the Study

Research on teachers’ understanding of inequalities is lacking and has not been a primary focus of the field (Kieran, 1992, 2007; Li 2007). Most studies that have been conducted involved inservice teachers and have scrutinized their understanding of equations. Rarely do studies involve preservice teachers as participants. This study seeks to do its part to fill a void in the knowledge base for mathematics education. By capitalizing on the available information about students’ understanding of solving equations (e.g., Kieran, 1981; Koedinger & Nathan, 1998; MacGregor & Stacey, 1993; Rittle-Johnson & Star, 2009) and solving inequalities (e.g., Tsamir & Bazzini, 2002), this study seeks to add to the limited information that is available about preservice teachers’ understanding of inequalities. In addition, the knowledge gained as a result of this study may potentially help teacher educators as they structure programs, courses, and experiences in an effort to strengthen preservice teachers’ content knowledge.

The purpose of this study was to examine the nature of preservice teachers’ understanding of inequalities and how they use that knowledge in the classroom. The participants of this study were five preservice teachers at a major university in the Southeast United States, who were fulfilling their student teaching field experience requirement.

The research questions guiding this study were the following:
What is the nature of preservice teachers’ understanding of inequalities prior to the student teaching field experience?

How do preservice teachers use their understanding of inequalities while planning and implementing lessons during the student teaching field experience?

Overview of Approach

This study uses a qualitative case study approach as outlined by Yin (1993). The researcher examines the nature of preservice teachers’ understanding of inequalities and how that understanding is applied in the practice of planning and implementing lessons. Five cases, each one of the student teachers, and rich with information, provide the backdrop to answer the first research question of this study. By examining multiple cases, analyses within and among the cases becomes available. For the second research question, there were two cases: systems of linear inequalities and quadratic inequalities. These two cases emerged during data collection when it became clear that there are two different topics in inequalities that are being taught during the student teaching field experience that may draw upon different content knowledge by the preservice teachers.

A series of interviews with the participants were conducted prior to, during, and after the student teaching field experience. The pre-student teaching interview was a task-based interview conducted prior to the preservice teachers’ student teaching field experience. Pre- and post-observation interviews were conducted as necessary. A stimulated recall post-student teaching interview was conducted with the preservice teachers at the end of the student teaching field experience. These interviews were used to explore the preservice
teachers’ understanding of inequalities and identify the manner in which that understanding was used during the planning and implementation of lessons pertaining to inequalities.

The preservice teachers submitted the lesson plans that they created. The lesson plans were examined and used as exploration points during some of the post-student teaching interviews. Additionally, classroom observations of the implementation of the lessons were conducted by the researcher. The classroom observations were video recorded. Relevant aspects of the video recordings were used during the post-student teaching interview to gain insight into the preservice teachers’ decisions and actions.

**Chapter Summary**

The studies that scrutinized inservice teachers seemed to suggest that “teachers lack conceptual knowledge of many topics in the mathematics curriculum” (Attorps, 2005, p. 2). While Bazzini, Tsamir, and their colleagues have conducted studies that have brought to light the strategies, misconceptions, and understanding that students have about inequalities, there is very little in the literature specifically addressing teachers, inservice or preservice. With the realization that there exists a gap in the literature, this study seeks provide valuable information about preservice teachers’ understanding of inequalities and how such understanding is used when planning for and implementing classroom instruction. Informed by interviews, examinations of lesson plans, and classroom observations, a case study analysis is conducted.

**Organization of the Study**

Chapter 2 contains a literature review that reveals how this study is situated within existing works. The conceptual framework and methodology that are used in this study are
discussed in Chapter 3. Chapter 4 contains the results of the data analysis in relation to the preservice teachers’ understanding of inequalities prior to the student teaching field experience. In Chapter 5, the results are presented with regard to how understanding of inequalities was used in the planning and implementation of lessons involving systems of linear inequalities. The results with regard to how understanding of inequalities is used in the planning and implementation of lessons involving quadratic inequalities follow in Chapter 6. Chapter 7 focuses on the overall findings as well as the implications for future research and for mathematics teacher education programs.
CHAPTER 2: LITERATURE REVIEW

Introduction

This chapter lays the foundation on which this study is situated, by synthesizing the available literature. This chapter is separated into three main sections to review the research related to: 1) knowledge in algebra; 2) knowledge of inequalities; and 3) teachers’ content knowledge for teaching mathematics. Within the first two sections, the majority of the reported research focuses on investigating student knowledge. Wherever possible, research pertaining to teacher knowledge is included and synthesized with the work conducted with students.

This literature review draws upon works that include dissertations, research syntheses, journal articles, and conference proceedings. The search is limited in the fact that non-English language journals, dissertations, and proceedings were not considered.

Knowledge in Algebra

Within the subject of algebra, researchers have focused their attention on a variety of topics: functions (e.g., Even, 1993; Lloyd, 1996; Lloyd & Wilson, 1998; Norman, 1992; Wilson, 2004); equal sign (e.g., Kieran, 1981; Knuth, et al., 2006); slope (e.g., Stump, 1999); simplifying algebraic expressions (e.g., Tirosh, Even, & Robinson, 1998); and solving equations (e.g., Nathan & Koedinger, 2000a, 2000b; van Dooren, et al., 2002). There are many areas within algebra that typically cause difficulties for students, as well as teachers. Those areas most pertinent to this current research study on the nature of understanding inequality include, but are not limited to, difficulties with the following: use of variables;
equal sign; and defining an equation. Literature surrounding these areas of difficulties is discussed in upcoming sections.

**Conceptions surrounding the use of variable**

Knuth, Alibali, McNeil, Weinberg, & Stephens (2005) asked if algebra was nothing more than “the study of the 24th letter of the alphabet” (p. 69)? Booth (1984) provided the following comment from an interview with a student.

Julie, aged 14 years, was asked about the expression $3x + 8y + 2x$

I: ‘Do the $x$ and $y$ mean anything there, do they stand for anything?’

J: ‘No, they’re just letters, you have them in algebra.’ (p. 13)

The question posed by Knuth et al. (2006) and Julie’s comment from Booth’s (1984) interview speak to an issue related to use of variables in algebra, even with inequalities. Comments like these should not be surprising since there have been multiple meanings that have been assigned to variable in algebra: unknown, placeholder, symbol, parameter, or argument\(^1\) (Usiskin, 1988). Usiskin (1999) provided an oversimplified summary of the relationships between the different conceptions of algebra and the different uses of variables (see Table 1).

---

\(^1\) This is not an exhaustive list.
Table 1. Relationship between conceptions of algebra and use of variables

<table>
<thead>
<tr>
<th>Conception of Algebra</th>
<th>Use of Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generalized arithmetic</td>
<td>Pattern generalizers (translate, generalize)</td>
</tr>
<tr>
<td>Means to solve certain problems</td>
<td>Unknowns, constants (solve, simplify)</td>
</tr>
<tr>
<td>Study of relationships</td>
<td>Arguments, parameters (relate, graph)</td>
</tr>
<tr>
<td>Structure</td>
<td>Arbitrary marks on paper (manipulate, justify)</td>
</tr>
</tbody>
</table>

Source: Usiskin, 1999, p. 13

At times, teachers do not fully appreciate difficulties that students have with variables. Teachers often assume that students understand that “the same literal symbols stand for the same values throughout [an algebraic] expression” (Olive & Caglayan, 2008, p. 285). In addition, students may haphazardly assign variables without fully defining the variable. Unfortunately, teachers tend to propagate this sloppy notation by writing ‘t = teachers.’ The teacher more than likely said or meant to write ‘t = the number of teachers,’ however there is no guarantee that students have made this connection (Booth, 1988; Stacey & MacGregor, 1997).

Olive & Caglayan (2008) discuss another issue that students have when working with variables. The problems are related to two different kinds of quantity: intensive and extensive. “An extensive quantity can be counted or measured directly, whereas an intensive quantity is derived from the multiplicative combination of two like or unlike quantities” (Olive & Caglayan, 2008, p. 270 – 271).
The issue about use of intensive and extensive quantities is often found in examples that involve coins. All too often, students and/or teachers use ‘\( p = \text{pennies} \)’ without any consideration of whether they are talking about the number of pennies, the value of a penny, or the total value of all the pennies (Olive & Caglayan, 2008). If the assignment of the variable \( p \) is in reference to the number of pennies, then the variable \( p \) would be an extensive quantity. However, if the variable \( p \) is intended to represent the value of the pennies, then the variable \( p \) would be an intensive quantity.

Another issue related to use of variables is the position of the variable in single-variable linear equations or inequalities. A result-unknown problem has a variable that is determined by a series of mathematical operations specified within the problem (e.g., \( x < 7 + 3.2[4 - 1.7] \)). A start-unknown problem has a variable that is needed in order to determine the relationship among the givens (e.g., \( 7.8z - 39 > 39 \)). Stated another way, the unknown in a start-unknown problem is “the start of the process or events described in the problem” (Koedinger & Anderson, 1998, p. 164). Nathan & Koedinger (2000b) noted that start-unknown problems, involving equations, tended to “require algebraic methods or more sophisticated modeling” (p. 170), whereas result-unknown problems, involving equations, could typically be “solved through direct application of arithmetic operations” (p. 170).

For many students, when dealing with equations, start-unknown problems are significantly more difficult than result-unknown problems (Koedinger & Nathan, 2004; Nathan & Koedinger, 2000b). These findings should not be surprising when one considers the amount of exposure to and familiarity with arithmetic problems students gain throughout their education. However, this difficulty in solving start-unknown problems does not simply
disappear with time. Koedinger & Tabachneck (1995) found college-level students working problems with multiple steps and rational numbers still had difficulty solving start-unknown problems (as cited in Nathan & Koedinger, 2000b). There is nothing in the literature to suggest that this would be any different for inequalities.

**Difficulties with the equal sign and defining an equation**

Arcavi (1994), who built on Fey’s (1990) work with number sense, outlines what symbol sense would include. One of the behaviors listed is “an understanding of and an aesthetic feel for the power of symbols” (Arcavi, 1994, p. 31). A symbol of particular interest in algebra is the equal sign. Knuth et al. (2006) examined 177 middle school students’ understanding of the equal sign and their ability to solve equations. The authors found “a strong relation between equal sign understanding and success in solving equations” (Knuth et al., 2006, p. 308). Additionally, Knuth et al. (2006) categorize students’ understanding of the equal sign as operational or relational. Operational responses typically implied a notion of ‘find an answer’ or ‘perform some operations.’ Conversely, relational responses express the equivalency of two sides. The findings of Knuth et al. (2006) are similar to those of other studies (Attorps, 2003; Izsak, Caglayan, & Olive, 2009; van Dooren, Verschaffel, & Onghena, 2002; Wagner & Parker, 1993).

Attorps (2003) pointed out gaps in teachers’ knowledge about the meaning of an equation. The author identified six categories in which the teachers’ concept image of equations did not match the concept definition of equations. Teachers did not consider the following as equations:

2 The study did include an ‘other’ category. It was not discussed in detail in the article. Therefore, I am not going to include it in this paper.
• Identities (e.g., \( \cos \alpha + \cos \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta) \))

• Non-algebraic equations (e.g., \( \int T(x) \, dx = 4x + C \))

• Equations with more than one unknown factor (e.g., \( 7x - 8y = \sqrt{c} \))

• Trivial equations (e.g., \( x = -9 \))

• Functions (e.g., \( f(x) = 11x + 2 \))

(Attorps, 2003, p. 5–6)

In the previous sections, knowledge in algebra was discussed. Specifically, the difficulties surrounding use of variables, the equal sign and defining an equation were reviewed. The next section will focus on available literature pertaining to knowledge of inequalities held by teachers or students.

**Knowledge of Inequalities**

The study of equations and inequalities is a centerpiece of every algebra curriculum. Within the *Principles and Standards for School Mathematics*, the NCTM (2000) outlined an expectation that students should be able to fluidly move between equivalent forms of equations and inequalities and “judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology” (p. 297). However, research on teaching and learning inequalities is limited at best. The majority of studies have been conducted outside of the United States and with secondary students. That does not mean that these studies do not provide relevant and useful information. Due to the similarities between equations and inequalities, some of the findings from studies centered on equations may be used to supplement what is known about inequalities.

In reviewing literature on learning inequalities, several common strategies for approaching inequality statements and problems have been identified. These strategies will
be discussed in the following sections. Within each strategy, potential issues and benefits are examined. In addition, research that identified areas of difficulties for students are explored. Whenever possible, research that utilized teachers as the participants is incorporated and identified.

**Strategies for Solving Inequalities**

Based on their analysis of Israeli high school students, Tsamir and Almog (2001) concluded that students, for the most part, use three strategies to solve inequalities: (1) algebraic manipulation, (2) drawing a graph, and (3) using a case approach. Students were not exclusively implementing a single strategy for all inequalities. The strategy employed by students often depended on the type of inequality (linear, quadratic, multi-variable, etc.) students encountered. The decisions made by students, as to which strategy to utilize, seemed to be affected by those strategies presented in the classroom for that particular type of inequality (Tsamir & Almog, 2001).

Tsamir, Almog, and Tirosh (1998) noted that when it comes to inequalities, “attention is mainly paid to ‘How to solve’ and not to ‘Why to solve it this way?’” (p. 129). This inability to see ‘why’ one solves an inequality with a particular strategy impedes students’ development of “a feeling for when to abandon symbols in favor of other approaches in order to make progress with a problem, or in order to find an easier or more elegant solution or representation” (Arcavi, 1994, p. 31).

**Algebraic manipulation for solving inequalities.** As one would expect, especially in light of the emphasis in traditional school mathematics that is placed on formal algebraic methods, algebraic manipulation is the most prevalent strategy employed by students when
solving inequalities (Tsamir & Almog, 2001; Tsamir et al., 1998). Algebraic manipulations include the following:

- addition or subtraction of the same term to both sides of the inequality;
- multiplication and division of both sides of the inequality by identical factors, with appropriate attention to the orientation of the inequality symbol;
- simplification of an expression through combining like terms or factoring;
- multiplying both sides of the inequality by the square of the denominator;
- “examining quadratic inequalities (i.e., $ax^2 + bx + c > 0$) by first relating to the quadratic roots, or by investigating the sign of ‘a’ and the sign of the determinant” (Tsamir et al., 1998, p. 131); and
- “relating to an inequality of the type $ab > 0$ as a compound system of $\{a > 0 \text{ and } b > 0\}$ or $\{a < 0 \text{ and } b < 0\}$” (Tsamir & Almog, 2001, p. 515).

Even if students are able to apply a formal algebraic method correctly, one may wonder what level of understanding they possess. Steinberg, Sleeman, and Ktorza (1990) found that “many students master procedural details of solving equations without understanding why they are justified” (p. 112). For these students, a formal algebraic method is an instrument that is blindly applied without any thought as to why or how it works. These students lack a relational view of that instrument (Knuth et al., 2006). It can be hypothesized that this lack of a relational view when applying algebraic manipulations is not exclusive to solving equations and may apply to inequalities.

For the most part, students studied in Steinberg et al. (1990) understood how to apply algebraic manipulations (or transformations) to solve equations. A major issue emerged from
the fact that “many students are not sure that an equation that has been derived by a valid transformation has the same solution or are unable to recognize when an equation has been transformed in a way that does not alter the answer” (Steinberg et al., 1990, p. 119). The students possess procedural knowledge of transformations. However, they do not have conceptual knowledge. It is reasonable to assume that this same issue may apply to students’ understanding of transformations applied to inequality statements. For those students without conceptual knowledge of algebraic transformations, an inequality prior to a transformation would be viewed as completely disjoint from the inequality obtained after a transformation.

**Drawing a graph** for solving inequalities. In Tsamir and Almog’s (2001) research, if a student drew a graph of the given function(s) and used it to attain the solution, then they applied the drawing a graph (graphing) strategy. In the Tsamir and Almog study, the students were categorized as applying a graphing strategy if their graphs were on a two dimensional Cartesian coordinate plane. Graphs on a number line or sign chart were not included in this strategy. Sign charts and number lines were tools used by students while invoking an algebraic strategy.

A point of interest is when the students applied a graphing strategy. Tsamir and Almog (2001) noted that students applied a graphing strategy only with rational and quadratic inequalities, not with linear inequalities. Additionally, those students were usually successful, especially when using graphical representations of parabolas for quadratic and rational inequalities.

---

3 The assumed domain is the real number system. This is in line with what is commonly used in school algebra.
One cannot assume that students will learn the same mathematics when solving an inequality graphically as compared to solving using algebraic manipulations (Sackur, 2004). Kieran (2004) noted that graphical representations may help students to form a conceptual understanding of the symbolic form of inequalities. Additionally, use of graphical representations may help students overcome the pitfall of incorrectly applying transformational techniques used with equations while solving inequalities.

Reilly (2010) outlined a discovery activity designed to promote middle school students’ understanding of the solution of systems of linear inequalities by using transparencies. Part of her reasoning for presenting her discovery activity was to provide an alternative to the following “long series of procedural steps that lack meaning” and are typically found in textbooks (Reilly, 2010, p. 56).

1. Graph the equation of the first inequality as if it had an equal sign by finding the x- and y-intercepts or by putting the equation into the \( y = mx + b \) form.

2. Look at the inequality. Determine if the line should be dashed or solid. Erase parts of the line that you already drew if it should have been dashed. (Remind yourself to check the line before you draw it next time.)

3. Do a test case, and see which side of the line should be shaded.

4. Shade it lightly, since more shading will be required.

5. Repeat steps 1 through 4 for the second inequality.

6. Repeat for additional inequalities in your system.

7. Analyze your system graph to determine the overlapped shaded region.

( emphasis in the original; Reilly, 2010, p. 56)
It should be noted that algebraic manipulations may be necessary in order to convert the given inequality into slope-intercept form or to calculate the x and y intercepts (see step 1). Even with the application of these algebraic manipulations, the long series of steps outlined above include the creation of a two-dimensional graph in order to solve the given system of linear inequalities. As a result, Tsamir and Almog (2001) would classify the long series of steps as a graphing strategy.

A graphing strategy assumes an understanding of graphing functions. It provides students with an opportunity for insight into connections between inequalities, equations, variables, and functions. As such, a graphing strategy has “procedural as well as conceptual advantages” (Dreyfus & Eisenberg, 1985, p. 654). One major advantage of a graphing strategy lies in the fact that difficulties are within the steps for solving an inequality by graphing and are not the steps themselves. Reilly’s (2010) first step directs the student to graph a linear equation (e.g., $y = mx + b$). This action may be cumbersome or difficult depending on the values of the slope and y-intercept. With the use of technology, some of the difficulties within this step can be minimized.

**Using a case approach for solving inequalities.** One method to solve inequalities is known as a case approach or the ‘casing to death’ approach (Dreyfus & Eisenberg, 1985). Consider the following inequality: $(3x - 2)(x + 9) < 0$. There are two possibilities to make this product of two factors negative: the first factor must be positive and the second factor must be negative; or vice versa.

Case 1: $+, -$ \quad $3x - 2 > 0$ and $x + 9 < 0$, which is equivalent to $x > \frac{2}{3}$ and $x < -9$. 


These two conditions cannot be satisfied simultaneously, so the solution set for this case is empty.

Case 2: $-, + \ 3x - 2 < 0$ and $x + 9 > 0$, which is equivalent to $x < \frac{2}{3}$ and $x > -9$.

These two conditions are simultaneously satisfied by $-9 < x < \frac{2}{3}$.

Therefore, the solution to the inequality is the union of the two partial solutions sets from the cases: $-9 < x < \frac{2}{3}$. Dreyfus & Eisenberg (1985) noted that “the strengths of the case approach for the mathematically inclined student is exactly its drawbacks for the weaker student; namely the need for careful and systematic treatment of the logical and set-theoretic connectives” (p. 653). The logical and set-theoretic connectives refer to usage of “and” or “or” in problems.

Students in Tsamir and Almog’s (2001) study used a case approach with compound inequalities and with rational inequalities. Some even used both a case approach and a graphing strategy on compound inequalities. This combination of strategies is interesting because of the complementary nature of the two strategies. “[E]ach [strategy] exhibits a view of inequalities which is lacking in the other one” (Dreyfus & Eisenberg, 1985, p. 658).

As students apply a case approach, there is a need to be vigilant and mindful of the possibilities, since in some cases these can be numerous. This requires students to make careful interpretations of the logical connectives of an inequality problem solved using a case approach. On the other hand, a graphing strategy for solving inequalities centers on a visual component. It draws upon previous experiences with graphing equations or functions. Dreyfus and Eisenberg (1985) noted that a graphing approach allows average or below average high school students to find the solution of an inequality easier than when
implementing a case approach. An ability to understand logical connectives and set operations is vital when applying a case approach. Whereas, when students implement a graphing strategy an ability to graph a library of functions is necessary.

Students will use a strategy based on their comfort level, which is affected by what has been shown to them in a classroom. Often they will not consider long-term implications of a dedication to any strategy (Nathan & Koedinger, 2000b). One may say that it does not matter as long as they arrive at the correct solution. Vaiyavutjamai and Clements (2006) noted that low and middle performing participants, who employed a variety of strategies, were unable to describe how the inequality symbols, number lines, and operations all linked together.

**Difficulties Encountered With Inequalities**

In examining available research on inequalities, several researchers point to areas in which students typically experience difficulties as they formulate an understanding of inequalities. These areas of concern included:

- relating inequalities to equations;
- interpretation of solutions;
- understanding logical connectives;
- the sign of the factors of products and/or quotients;
- denominators of rational inequalities; and
- interpreting the meaning of the inequality symbol.
The majority of these studies involve high school level students. However, Üreyen, Mahir & Çetin (2006) discussed how these same difficulties are encountered by students in a college-level Calculus course. The six areas of concern are discussed more fully below.

**Difficulties relating inequalities to equations.** Students encounter difficulties as they try to overlay their prior knowledge and experiences with equations onto inequalities. These difficulties include solving equations instead of inequalities, multiplying or dividing by factors that are not necessarily positive, and forming meaningless connections with quadratic roots.

Vinner, Tall, and others (Tall & Vinner, 1981; Vinner, 1983, 1991; Vinner & Dreyus, 1989) discussed distinctions between a concept definition and one’s concept image. A concept image is “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (Tall & Vinner, 1981, p. 152). The concept definition is the mathematical definition of the concept. The difficulties noted above point to concept images held by students on equations and inequalities that do not match the concept definitions. Students are blurring the lines that separate equations from inequalities, and are thinking of the two in an interchangeable manner.

**Solving equations instead of inequalities.** Students seem to have issues developing appropriate concept images for equations and inequalities. Part of the problem can be traced to teachers’ concept image of equations and inequalities. Attorps (2003) noted that some teachers included inequalities and expressions (e.g., $4x \leq 7x - 23$) in their concept definitions of equations. These findings should raise some level of concern for mathematics
teacher educators. If teachers have difficulties distinguishing between equations and inequalities, then how can we expect students to have a concept image of equations and of inequalities that adheres well to the concept definitions and separates the two (Tall & Vinner, 1981).

Familiarity with equations is part of the cause of the difficulties students, and teachers, encounter with inequalities. Students in a study by Tsamir and Bazzini (2004) “expressed confidence in the correctness of their (incorrect) solution, because all the procedures had been successfully applied ‘a million times’ when solving equations” (p. 809). The structural similarities of equations and inequalities blind students to the subtle, yet important, differences.

**Multiplying or dividing by factors that are not necessarily positive.** In solving rational inequalities, some students violate the stipulation that one is only allowed to multiply both sides of an inequality by a factor which is clearly positive. Tsamir et al. (1998) noted that students will simply multiply both sides of a rational inequality by the denominator without any consideration for non-positive scenarios. While solving an inequality, a student, in a study by Tsamir and Bazzini (2002), commented that “when doing the same operation (i.e., multiplying by \(a\)) on both sides, the equivalency is preserved” (p. 292). A far more frustrating variation of this error occurs when a student multiplies or divides both sides of an inequality by a negative number and does not switch the orientation of the inequality symbol, nor understand why this reversal is needed. Often this incorrect procedure is written off as a careless error, with no consideration as to whether the student has made the error based on connections to solving equations.
**Difficulties with solutions.** A solution to an inequality differs from that of an equation. Generally, solutions to inequalities are sets of numbers which are not discrete. On the other hand, solutions to equations are generally a discrete set of numbers. The difficulties that students have with solutions of inequalities involve an overall understanding of what the solution of an inequality means and an understanding about possible solution(s) to an inequality. Some of these issues point to a lack of symbol sense relative to “the realization of the constant need to check symbol meanings while solving a problem, and to compare and contrast those meanings with one’s own intuitions or with the expected outcome of that problem” (Arcavi, 1994, p. 31). Some students, taught using traditional methods, were able to solve standard inequality problems but were not able to adequately express the meaning of symbols or solutions (Bazzini, Boero, & Garuti, 2001; Bazzini & Tsamir, 2001).

Van Dooren et al. (2002) noted that teachers tend to view algebraic methods as “the only real mathematical problem-solving method” (emphasis in the original, p. 323) and informal methods were not “mathematical method[s] at all” (p. 343). However, Vaiyavutjamai and Clements (2006) pointed out that ninth grade students at low and middle overall academic performance levels had great difficulty describing the answers to inequalities, especially after performing algebraic manipulations. This implies that using a ‘real mathematical problem-solving method’ does not guarantee understanding, let alone a correct solution. Additionally, retention scores lead Vaiyavutjamai and Clements (2006) to conclude that improvements made by many of the low and middle overall performance level students from pre-test to post-test was based on “imperfectly rote-learned knowledge and
skills” (p. 145) and not an instrumental or a relational understanding of inequalities (Skemp, 1976; as cited by Vaiyavutjamai & Clements, 2006).

Inequalities will have inequalities as solutions. Tsamir & Bazzini (2004) analyzed 148 students’ responses to an inequality questionnaire. One of the tasks on the questionnaire asked students to determine the truth value of a statement (i.e., $x = 3^4$ is the only solution to an inequality) and to explain their reasoning. About half of the students provided a correct claim. However, only one in twenty could provide a valid explanation. Of those with the correct claim, two invalid explanations emerged repeatedly. The first invalid explanation indicated that the solution (i.e., $x = 3$) belonged to a system of inequalities. Generally, students used single-variable inequalities whose intersection was the solution. The second invalid explanation pointed to the fact that $x = 3$ belonged to the set of solutions. Even if a student believes or accepts the notion that an equation can be a solution to an inequality, students’ concept images surrounding that notion may be flawed or incomplete.

A second task asked students to find the solution to a given inequality (i.e., $5x^4 \leq 0$). Once again, only half of the students were able to provide the correct answer (i.e., $x = 0$). Many of the students indicated that the solution was an empty set. “Typical explanations were ‘$x^4$ is an expression with an even power and thus it can never be negative’, where students ignored the ‘zero-option’” (Tsamir & Bazzini, 2004, p. 797).

The analysis of the answers to both of these tasks provided some interesting findings. The authors focused on consistencies and inconsistencies in the students’ responses to the two tasks. 20% of all of the students provided inconsistent responses: a correct answer for

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4 This refers to the singleton value not the line defined as $x = 3$. 
task two and an incorrect claim in task one. Even with evidence to the contrary, they still held to the belief that “an inequality can only be the solution of an inequality” (Tsamir & Bazzini, 2004, p. 798).

**Single number answer tendency.** Many students display, what Vaiyavutjamai and Clements (2006) termed, ‘single number answer tendencies’ when solving inequalities. These students are on the opposite side of the spectrum from students in Tsamir & Bazzini (2004). Students with this single number answer tendency believed that the answer to an inequality should be structurally similar to the answer for an equation. In some cases, students were able to perform the correct mathematical operations to solve a given inequality. Their solutions (i.e., $x < -5$) would give an outward appearance that the student had reached the desired end. However, when questioned to explain the meaning of their answer, students responded in a manner similar to the following:

Student: The answer is negative 5.

Interviewer: Your answer is negative 5. Show me your answer on a number line.

Student: [Drew the following, with the $x$ above the $-5$.] The number $x$ is negative 5.

Interviewer: Is there only one answer?

Student: Yes.

(Vaiyavutjamai & Clements, 2006, p. 134)

Students making this mistake appeared to be applying an operational view of inequalities (Knuth et al., 2006).
**Forming meaningless connections with quadratic roots.** As students learn to solve quadratic equations such as $x^2 = 4$ by applying the square-root property, they tend to write solutions as $x = \pm 2$. This shorthand version of the answer is not mathematically incorrect, however students need to understand what this version of the solution really means (i.e., $x = -2$ and $x = 2$). Tsamir and Almog (2001) noted that students attempted to apply the square-root property to inequalities. For example, students applying the square-root property to $x^2 > 81$ would provide the following solution $x > \pm 9$, instead of $x < -9$ or $x > 9$. These same students, when interviewed, could not explain what their answer meant as a solution.

**Excluded values.** With inequalities, two possible areas of excluded values must be considered; non-zero denominators of rational expressions and non-negative values under even-indexed roots. The issues and difficulties that involve excluded values may not be evident in problems in which the excluded value does not affect the final solution. From time to time, students will incorrectly impose restrictions prescribed by an even-indexed radical when restrictions prescribed by a denominator are appropriate, and vice versa (Tsamir & Almog, 2001). In these cases, it is clear that students understand that a restriction is being imposed, however, their lack of symbol sense prevents them from reasoning about the appropriateness of that restriction (Arcavi, 1994).

**Difficulties with logical connectives.** A case approach to solving inequalities hinges on one’s ability to correctly apply logical connectives. One must determine and solve the particular cases that arise as a result of the various restrictions hidden within the problem. Then one must be able to combine those particular case solutions to find the solution to the original problem.
Beyond a case approach, having students address how and why to apply ‘and’ or ‘or’ when working with certain inequalities is a problematic area. Consider a possible solution to the following inequality: \( \frac{x-5}{x+2} < 0 \). The written solution is provided below along with the accompanying interview from Tsamir and Almog (2001).

\[
\begin{align*}
x + 2 &< 0 \\
x &< -2
\end{align*}
\]

\[
\begin{align*}
x - 5 &> 0 \\
x &> 5
\end{align*}
\]

Interviewer: What is one supposed to do with the two results?

Gill: I think that the solution is ‘or’.

Interviewer: How did you get \( x > 5 \)?

Gill: By imposing a positive numerator.

Interviewer: And how did you get \( x < -2 \)?

Gill: By imposing a negative denominator.

Interviewer: So, what should the connection between the two be?

Gill: ‘and’ . . . I think it should be ‘and’ because on the right side we have a negative number . . . to get that, we must have a positive top and also a negative bottom . . . but here, from a positive top and a negative bottom we get no solution . . .

[interviewee pauses to think] Sorry, it should probably be ‘or’. I thought it should be ‘and’ but I got no answer, so it is probably ‘or’ . . .

(p. 518)

From this student’s response, it is clear that he has no reference for which logical connective is appropriate. This may be due to the student’s apparent lack of understanding with regard to why he was “imposing a positive numerator” and a “negative denominator.”
Difficulties with the sign of the factors of products and/or quotients. For product inequalities (e.g., \((x - 4)(2 - x) > 0\)) and quotient inequalities (e.g., \(\frac{x - 4}{2 - x} < 0\)), one possible solution method is to examine the signs of the factors: a case approach. For a product or quotient inequality, the product or quotient is positive if and only if both factors have the same sign. Likewise, for a product or quotient inequality, the product or quotient is negative if and only if the factors have different signs.

Tsamir et al. (1998) found that students did not attend to all of the possibilities when solving inequalities in this manner. For example, if asked to solve \(\frac{x - 4}{2 - x} < 0\), the students may attend to the case where \(x - 4 > 0\) and \(2 - x < 0\). However, they may overlook the case where \(x - 4 < 0\) and \(2 - x > 0\). In a similar manner, students may completely misapply sign rules in favor of preserving the orientation of the inequality symbol for both factors (i.e., \(\frac{x - 4}{2 - x} < 0\) would yield \(x - 4 < 0\) and \(2 - x < 0\)).

Difficulties related to the denominators of rational inequalities. When working with rational inequalities, one technique used to solve them is to find the common denominator of both sides and multiply both sides of the inequality by the square of that common denominator. This technique eliminates the rational expressions on both sides of the inequality and preserves the orientation of the inequality because you are multiplying by a square, which must be non-negative.

For some students, the technique that they apply does not quite fit the above description. Tsamir & Almog (2001) highlighted one student’s solution and interview responses:
\[ \frac{x-5}{x+2} < 0 \Rightarrow \left(\frac{x-5}{x+2}\right)^2 < 0^2 \Rightarrow (x - 5)^2 < 0 \Rightarrow \text{no solution} \]

Interviewer: Can you explain the main ideas of your solution?

Annette: Yes . . . First I had to get rid of the denominator . . . But I could not multiply by x+2 because is not always positive. So, in order to make it positive, I moved from the given expressions to their squares . . . Then, by multiplying both sides by the new denominator, I reached a simple, quadratic inequality.

Interviewer: How did you complete your solution?

Annette: Well, I reached the stage where the inequality was \((x - 5)^2 < 0\). This is an easy inequality . . . because the expression on the right side is always positive, so it cannot be smaller than zero. There is no number to solve the inequality.

(p. 519)

The student’s responses illustrate the confusion encountered when deciding on the appropriate manner to solve rational inequalities. In an effort not to violate the stipulation that one is not allowed to multiply both sides of an inequality by a factor whose sign (positive or negative) is in doubt, the students will simply square everything.

**Difficulties in interpreting the meaning of the inequality symbol.** Part of the development of symbol sense is an awareness of the meaning of a symbol, in this case an inequality symbol (Arcavi, 1994). Lim (2006) explored students’ anticipatory behaviors while working with inequalities. One student in particular, displayed responses that were operational (Knuth et al., 2006). Without hesitation the student “interpreted the inequality as a signal to isolate x and treat it as an ‘equation’” (Lim, 2006, p. 105). On the other hand, another student provided responses of a relational quality (Knuth et al., 2006). This student
reasoned about the relationship by making comparisons between the two expressions of the inequality.

**Preservice Teachers Understanding of Inequalities**

Inequalities are an important concept in mathematics. The reach of inequalities extends beyond the confines of algebra. Inequalities can be found in trigonometry, geometry, operational research (linear programming), probability, real analysis, and other areas. The mathematical concept of inequalities can be found throughout the various stages of mathematical development. The relationship of being “more than” or “less than” is often introduced in early childhood education. As students progress, the objects involved in the relationship become more complicated (transitioning from comparing the number of boys and girls in a classroom to comparing the graphs of quadratic functions on a Cartesian coordinate plane). What follows is a discussion about the understanding that preservice teachers on the cusp of entering their student teaching field experience should possess regarding systems of linear inequalities and quadratic inequalities. This discussion was developed from a review of literature.

Inequalities can complement equations. Often that complementary role is mistakenly identified as a subordinate role and inequalities are classified as equations. Inequalities and equations are separate entities with unique concept definitions. The relationship that is being expressed with equations is one of equivalency or balance. On the other hand, inequalities are relationships that denote an imbalance or a sense of order. Preservice teachers should have separate concept images for equations and inequalities.
Preservice teachers should be familiar with the various solution strategies that exist: algebraic manipulations; graphing; and cases (Tsamir & Almog, 2001). However, the case strategy seems to be a strategy that is rarely encountered. As such, it would not be surprising if preservice teachers were unaware of or unable to employ a case strategy.

Preservice teachers should be able to implement a graphing strategy to solve systems of linear inequalities and quadratic inequalities with one or two variables. Quadratic inequalities with one variable can be solved with either a graphing strategy or an algebraic manipulations strategy. Preservice teachers should be able to implement both strategies when solving quadratic inequalities with one variable. In addition, they should be able to discuss the connection that exists between the symbolic representation of an algebraic manipulations strategy and the visual representation of a graphing strategy. This discussion has the potential to strengthen their students’ understanding of inequalities.

The nuances of the strategies that preservice teachers employ to solve inequalities need not be the exact same. Differences and similarities between strategies are bound to exist. One area in which similarities and differences could occur is in the method advocated by preservice teachers for shading the graph of an inequality. One shading method involves evaluating test points. This is not the only shading method; however, preservice teachers should recognize the test point method due to its prevalence in textbooks.

Preservice teachers should not exhibit difficulties regarding finding the solutions of inequalities that students, who were often at a high school level, displayed in the studies outlined earlier. Preservice teachers’ understanding of solutions of inequalities should include an awareness of the possible solutions that may satisfy a range of inequalities that
they may encounter in secondary mathematics. In addition, preservice teachers should be versed in the meaning of a solution to a system of linear inequalities or quadratic inequalities. This includes being able determine the appropriateness of various representations of the solution: graph or symbolic notation.

The following section outlines existing frameworks related to content knowledge for teaching mathematics. Shulman called into question how the field of education explains the transition from an expert student to a novice teacher: “How does the successful college student transform his or her expertise in the subject matter into a form that high school students can comprehend” (Shulman, 1986, p. 8)? Prestage and Perks (2003) used the terms learner-knowledge and teacher-knowledge when referring to this transition. Learner-knowledge is the “personal subject knowledge that enables one to answer mathematical questions” (Prestage & Perk, 1999b, p. 92). Or more simply put, it is the ability to pass mathematical exams. Teacher-knowledge is the ability to answer questions correctly, explain the ‘why’ questions students will inevitably ask, and utilize “a variety of connections and routes” to navigate students through acquisition of the knowledge (Prestage & Perks, 2003, p. 5). The transition that Shulman alluded to is part of what makes teaching a complex endeavor.

**Content Knowledge for Teaching Mathematics**

In 2007, Skip Fennell, president of the NCTM (2006 – 2008), posed the following question: “How do we ensure that all teachers of mathematics know the mathematics and pedagogy essential for teaching the subject” (p. 3)? This question speaks to a problem that

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5 Learner-knowledge is a subset of teacher-knowledge.
has persistently challenged mathematics education. The problem begins in determining what it is that teachers should know. There seems to be a universal understanding that the subject matter knowledge needed by teachers involves more than a superficial understanding of mathematics. Unfortunately, not much else can be taken as universally agreed upon in defining teacher knowledge.

Researchers in education have presented what they deem to be content knowledge for teaching (Ball, Thames, & Phelps, 2008; Eraut, 1994; Even, 1990; Ma, 1999; Prestage & Perks, 1999a, 1999b, 2003; Shulman, 1986; 1987). While some of these perspectives are based on empirical observations, they move beyond those phenomena by providing models that characterize and generalize. What follows is a discussion of three of those frameworks and how they have been used in studying mathematics teachers.

**Lee Shulman and His Colleagues**

The importance of teachers’ knowledge is commonly agreed upon. However, the question of what is teachers’ knowledge remained largely unanswered until Shulman delivered his Presidential Address at the 1985 American Educational Research Association Conference. Shulman (1986) sought to produce “a more coherent theoretical framework” (p. 9). Based on his work with *Knowledge Growth in Teaching*, he introduced a solution to the ‘missing paradigm’ problem when he included pedagogical content knowledge in his framework for describing knowledge needed by teachers. This inclusion arose as he tried to address the following questions: “How do teachers decide what to teach, how to represent it, how to question students about it and how to deal with problems of misunderstanding” (Shulman, 1986, p. 8)?
Certain categories of teacher knowledge, as defined by Shulman (1987), were already widely accepted. These included the following: “general pedagogical knowledge; knowledge of learners and their characteristics; knowledge of educational contexts; and knowledge of educational ends, purpose, and values and their philosophical and historical grounds” (Shulman, 1987, p. 8). His suggestion was that teachers’ knowledge should be separated into three categories: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. This distinction sparked much debate and research that continues today.

Subject matter content knowledge is what a teacher knows regarding a subject. Teacher’s knowledge about the concepts, procedures, content domains, and the organization of those domains are all part of subject matter content knowledge. This knowledge involves two structures: substantive and syntactic. Grossman, Wilson, and Shulman (1989) defined syntactic structures in the following manner:

The syntactic structures of [mathematics] are the canons of evidence that are used by members of the [mathematics] community to guide inquiry in the field. They are the means by which new knowledge is introduced and accepted. (p. 29)

The substantive structures are “the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts” (Shulman, 1986, p. 9).

Teachers need to be able to address the ‘what’ and the ‘why’ questions that are associated with a mathematical topic or concept. The ‘what’ questions pertain to knowledge that is necessary to understand the topic or concept. These ‘what’ questions are linked to the substantive structures of subject matter content knowledge. The ‘why’ questions are centered
on the syntactic structures and aid the teacher in making instructional decisions. The ‘why’ questions require teachers to be able to deliver and/or evaluate arguments about a topic, concept, procedure, rule, or theorem. In addition, teachers must be able to explain the conditions that would weaken or strengthen those arguments. Included in the ‘why’ questions are the ever popular questions: Why do I need to know this? and Where am I going to use this? In addressing these questions, teachers need to illuminate the connections between the topic or concept at hand and other topics or concepts both within mathematics and external to mathematics. A teacher should pull from both the syntactic and substantive structures when fostering student understanding of a concept. In that, teachers not only need to know what answer to expect but also why a student may give a particular response.

As a category, pedagogical content knowledge was of special interest to Shulman. It is the melding of content knowledge and pedagogical knowledge relative to how material is presented in a classroom. Within this category, one will find “the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction” (Shulman, 1987, p. 8). Pedagogical content knowledge goes beyond this blending to include an awareness of areas of misconceptions and potential pitfalls that a learner may encounter as they interact with a particular topic or concept.

Shulman believed that the field of education was doing a dismal job of addressing the third category of content knowledge, curricular knowledge.

The curriculum is represented by the full range of programs designed for the teaching of particular topics at a given level, the variety of instructional materials available in
relation to those programs, and the set of characteristics that serve as both the indications and contradictions for the use of particular curriculum or program material in particular circumstances. (Shulman, 1986, p. 9)

Here Shulman emphasized the importance of knowledge of alternative curriculum materials as well as lateral curricular knowledge, the teacher’s ability to relate content in their course with that of another course.

Shulman’s work has spurred an avalanche of research (e.g., Baturo & Nason, 1996; Berenson, van Der Valk, Oldham, Runesson, Moreira, & Broekman, 1997; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Carpenter, Fennema, Peterson, & Carey, 1988; Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993; Even, 1990; Ma, 1999; Lowery, 2002; Perressini, Borko, Romagnano, Knuth, & Willis, 2004; Simon, 1995; Simon & Blume, 1994; Sowder, Philipp, Armstrong, & Schappelle, 1998). Some of the research has specifically been conducted with secondary mathematics teachers (e.g., Cooney, Shealy, & Arvold, 1998; Cooney & Wilson, 1995; Even, 1993; Leinhardt, 1989; Leinhardt & Smith, 1985). Some researchers sought to develop courses that fostered the development of teachers’ subject matter knowledge (e.g., Ball, 1988; Lappan & Even, 1989). Other researchers have also recently examined technological pedagogical content knowledge; the knowledge that emerges in the blending of content, pedagogy, and technology (e.g., Lee & Hollebrands, 2008; Mishra & Koehler, 2006). What follows is a brief description of some of the research that builds upon Shulman’s work which helps lay the foundations for this current study.
Krauss et al., (2008) presented “an empirical approach to assessing the knowledge of secondary-level mathematics teachers” (p. 713). The authors developed their assessment instrument in an effort to verify, in an empirical manner, the separation between two of Shulman’s categories: content knowledge and pedagogical content knowledge. The instrument focused on the following content areas: arithmetic, algebra, and geometry. A quasi-experimental approach was employed to determine “the level and connectedness of the two knowledge categories in two groups of teachers with different mathematical expertise” (Krauss et al., 2008, p. 718). Their findings indicated that teachers with more experience displayed higher levels of connectedness between content knowledge and pedagogical content knowledge.

With Shulman’s notion of pedagogical content knowledge in mind, Simon (1995) described the mathematics teaching cycle which included various domains of teacher knowledge, the hypothetical learning trajectory, and the interactions with students. Additionally, Fennema et al. (1996) found that teachers’ knowledge of different strategies that their students use to approach problems was positively correlated with their students’ achievement.

Even and Tirosh (1995) used the phrases “knowing that” and “knowing why” to articulate the subject matter knowledge required by mathematics teachers to adequately address students’ conceptions. These phrases build on the substantive and syntactic structures (Grossman, Wilson, & Shulman, 1989; Shulman, 1986) and address teacher’s knowledge about students. Even and Tirosh (1995) noted that “one cannot assume that teachers' subject

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6 The data for the Krauss et al. (2008) study was obtained from 198 secondary mathematics teachers from Germany.
matter knowledge with respect to the two aspects (‘knowing that’ and ‘knowing why’) are sufficiently comprehensive and articulated for teaching” (p. 18).

Shulman’s work examined teacher’s subject matter knowledge in a broad sense. His framework cut across subjects and was very general. Even and her colleagues modified Shulman’s framework in an attempt to focus on an approach that was topic-specific. Even (1990) outlined a framework that separated one of Shulman’s categories, teacher’s subject matter content knowledge, into seven aspects. Those aspects were the following: essential features; different representations; alternative ways of approaching; the strength of the concept; basic repertoire; knowledge and understanding of a concept; and knowledge of mathematics. The first aspect, essential features, builds on the notions of concept image and concept definition (Vinner, 1981). Even (1990) noted that teachers need a concept image that matches the accepted concept definition. Failure to do so may inhibit not only the teacher’s understanding of the mathematics, but also that of the students.

Ball and her colleagues believed that Shulman’s approach did not fully address the mathematical demand of teaching. “To understand the mathematical work of teaching would require a closer look at practice, with an eye on the mathematical understanding that is needed to carry out the work” (Ball, Lubienski & Mewborn, 2001, p. 449).

Deborah Ball and Her Colleagues

Ball et al. (2008) pointed out the lack of studies that explicitly identify content knowledge that matters for teaching. They acknowledged the numerous studies that use the term pedagogical content knowledge. However, as they noted, “its potential has been only thinly developed” (Ball et al., 2008, p. 389).
In recent years, Ball and her colleagues built on the work of Shulman. The authors sought to improve Shulman’s design in the area of “practical utility” (Ball et al., 2008, p. 390). They operated on the hypothesis that subdividing Shulman’s content knowledge and pedagogical content knowledge is appropriate. This hypothesis built on work in earlier studies (Ball, 1988, 1990, 2000; Ball & Bass, 2000; Delaney, Ball, Hill, Schilling, & Zopf, 2008; Hill, Rowan, & Ball, 2005; Hill, Schilling, & Ball, 2004). The aim was to develop a practice-based theory of mathematical knowledge for teaching (Ball & Bass, 2003). As Hill, Ball, and Schilling (2008) noted, the evidence as to what pedagogical content knowledge entailed was lacking. A few studies had shown that student learning could improve with an emphasis on improving teachers’ pedagogical content knowledge (e.g., Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Franke, Carpenter, & Levi, 2001)

The model of their framework\(^7\) (see Figure 1) separates mathematical knowledge for teaching into two main categories: subject matter knowledge and pedagogical content knowledge. Within subject matter knowledge, Ball and her colleagues constructed three subdivisions. The first subdivision was called common content knowledge (CCK)\(^8\). Ball et al. (2008) defined common content knowledge as “the mathematical knowledge and skill used in settings other than teaching” (p. 399). Specialized content knowledge (SCK), “mathematical knowledge and skill unique to teaching” (Ball et al., 2008, p. 400), was the second subdivision. Tentatively, they included the third subdivision, horizon content

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\(^7\) For earlier versions of the model see Ball, Hill, Bass, & Schilling (2005).

\(^8\) I will not use CCK or any of the other acronyms in this paper. It is too easy for the meaning to get lost in all of the letters.
knowledge. Horizon content knowledge was “an awareness of how mathematical topics are related over the span of mathematics” (Ball et al., 2008, p. 403).

![Figure 1. Structure of mathematical knowledge for teaching. Source: Ball et al., 2008, p. 403](image)

In a similar fashion to subject matter knowledge, Ball and her colleagues constructed three subdivisions within pedagogical content knowledge. The first subdivision was knowledge of content and students (KCS). This subdivision pertained to the “interaction between specific mathematical understanding and familiarity with students and their mathematical thinking” (Ball et al., 2008, p. 401). Knowledge of content and teaching (KCT), examining the “interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning,” was the second subdivision (Ball et al., 2008, p. 401). The third subdivision was knowledge of content and curriculum.
Ball and her colleagues created a representation of the structure of mathematical knowledge for teaching (see Figure 1). In making their conjecture, Ball et al. (2008) were willing to acknowledge that these categories may not be the right ones. They viewed their representation as a step in the refinement and revision process to establish the multidimensional components of content knowledge for teaching.

There was little available in the field that “demonstrate[d] that teachers possess [pedagogical content knowledge] apart from knowledge of the content itself” (emphasis in the original, Hill et al., 2008, p. 373). With this in mind, Hill, Schilling, and Ball (2004) used their framework as a guide to construct an assessment of elementary teachers. This assessment sought to tease out teacher’s pedagogical content knowledge from pedagogical knowledge and content knowledge.

The framework described by Ball and her colleagues played a role in other studies (e.g., Adler & Davis, 2006; Brakoniecki, 2009; Ives, 2009; Kersting, Givvin, Sotelo, & Stigler, 2010; Morris, Hiebert, & Spitzer, 2009; Sleep, 2009; Speer & Wagner, 2009). Speer and Wagner (2009) scrutinized one undergraduate mathematics teacher’s ability to draw on their pedagogical content knowledge and specialized content knowledge to facilitate whole-class discussions. The authors discussed a “practical criteria for distinguishing between the roles of (PCK) [pedagogical content knowledge] and (SCK) [specialized content knowledge] in teaching practice” (Speer & Wagner, 2009, p. 559). In addition, Speer and Wagner (2009) highlighted areas in which a teacher’s ability “to deploy the SCK [specialized content knowledge] necessary to see the potentially productive mathematical ideas embedded in the
students’ solutions” may have allowed the teacher’s students to develop a better mathematical understanding of the topic or concept (p. 557).

Adler and Davis (2006) examined the specialized content knowledge of middle and high school teachers in South Africa. Within their study, Adler and Davis used decompression and compression as terminology that directly related back to the notion of “unpacking” from Ball and Bass (2000). Adler and Davis (2006) found that there was an “absence, rather than presence, of unpacked or elaborated mathematics for teaching in these across-site evaluation tasks, despite their courses being specifically designed for teachers” (p. 291).

Shulman’s framework can and has been applied across subjects in education (Grossman, 1991; Smith & Neale, 1989; Stodolsky & Grossman, 1995; S. Wilson, 1988). The framework presented by Ball and her colleagues can be modified to be applied to different subjects, but it is set-up specifically for mathematics. Others in mathematics education have attempted to consider the specific knowledge needed by teachers for certain domains within school mathematics curricula. For example, Groth (2007) discussed knowledge for teaching statistics while Lee and Hollebrands (2008) examined knowledge for teaching data analysis and probability with technology. Additionally, Swafford, Jones, and Thornton (1997) focused their attention towards geometric content knowledge.

With a focus on the most commonly taught domain in high school, Ferrini-Mundy and her colleagues created a framework that addressed categories of teachers’ knowledge specific to the subject of school algebra. Unlike the other two frameworks, Shulman (1986) and Ball et al. (2008), a centerpiece of the framework proposed by Ferrini-Mundy, Floden,
McCrory, Burrill, & Sandow (2005) were the teaching practices: the actions of teachers where their mathematical knowledge would be applied. The focus on algebra and teaching practices in the framework by Ferrini-Mundy et al. (2005) seemed to align with the goals of this study.

**Joan Ferrini-Mundy and Her Colleagues**

Ferrini-Mundy and her colleagues were influenced by the frameworks of Ball & Bass (2000), Even (1990), and Shulman (1986). They also gave consideration to expectations related to students’ algebraic proficiency, as outlined by the RAND Mathematics Study Panel (2003), which stated students need to have:

- the ability to work flexibly and meaningfully with formulas or algebraic relations—to use them to represent situations, to manipulate them, and to solve the equations they represent;
- a structural understanding of the basic operations of arithmetic and of the notational representations of numbers and mathematical operations (for example, place value, fraction notation, exponentiation);
- a robust understanding of the notion of function, including representing functions (for example, tabular, analytic, and graphical forms); having a good repertoire of the basic functions (linear and quadratic polynomials, and exponential, rational, and trigonometric functions); and using functions to study the change of one quantity in relation to another; and
• knowing how to identify and name significant variables to model quantitative contexts, recognizing patterns, and using symbols, formulas, and functions to represent those contexts.

(p. 44 – 45)

With these frameworks and expectations in mind, Ferrini-Mundy et al. (2005) sought to delineate, describe, and study knowledge for teaching algebra. However, claims regarding the exclusivity of this knowledge to only teachers are not made by the group. Ferrini-Mundy et al. (2005) claim that this knowledge is distinguished by the fact that “it is used in the course of secondary mathematics instruction” (p. 15).

Ferrini-Mundy and her colleagues used a two-dimensional matrix to organize their framework (see Figure 2). The headings of the columns of the matrix are categories of knowledge for algebra teaching: core content knowledge; representation; content trajectories; applications and contexts; language and conventions; and mathematical reasoning and proof. The headings of the rows of the matrix are teaching tasks where a teacher’s mathematical knowledge is applied: analyzing students’ mathematical work and thinking; designing, modifying and selecting mathematical tasks; establishing and revising mathematical goals for students; accessing and using tools and resources for teaching; explaining mathematical ideas and solving mathematical problems; and building and supporting mathematical community and discourse. The final component of the framework is “three overarching categories – decompressing, trimming, bridging – which are mathematical practices infused through all elements of knowledge of algebra for teaching” (Ferrini-Mundy et al., 2005, p. 24).
The headings of the rows of the matrix, teaching tasks, were chosen based on existing works that have delineated the work of teaching (e.g., Ball & Bass, 2000; Chazan, 1999). These tasks provided areas in which mathematics is applied in the classroom. The tasks, ‘particular acts or practices of teaching’, were used as guides to make the knowledge for algebra teaching more visible or accessible to an observer. In the following sections, the categories of knowledge will be expanded and explained.
**Categories of knowledge.** In examining the framework purposed by Ferrini-Mundy and her colleagues (see Figure 2), it is beneficial to illuminate the connections that warrant the inclusion of each category. The first category, core content knowledge, is similar to one of Even’s (1990) seven aspects: essential features. Building on the notion of concept image (Tall & Vinner, 1981; Vinner, 1983, 1991), Even (1990) argues that “teachers need to be able to judge whether an instance belongs to a concept family by using an analytical judgment” that is “based on the concept’s critical attributes” (p. 523).

Representation is one of the process standards outlined in the *Principles and Standards for School Mathematics* (NCTM, 2000). NCTM (2000) considers representation as both a process and a product. That is to say, representations are both “the act of capturing a mathematical concept or relationship in some form” and “the form itself” (p. 66). Ferrini-Mundy et al. (2005) view representations as the ways and means of “recording, organizing, and communicating concepts and procedures” (p. 26).

In Roman mythology, Janus is a god that is depicted with two heads that face in opposite directions. It is said that Janus simultaneously looks into the future and the past. Content trajectories could be called the Janus category. This category is similar to horizon content knowledge (Ball et al., 2008). Teachers need to know what has been and will be done with the concepts that they teach. Content trajectories also contain the notion of alternative approaches as a component. Alternative approaches “includes knowing that the main ideas and concepts in a domain can be organized according to a particular perspective on that content, and with various emphases, and knowing in particular some of the sequences and emphases that instantiate the perspective” (Ferrini-Mundy et al., 2005, p. 28).
As a category, applications and contexts, subsumes “knowledge of problems that arise from situations, contexts, or circumstances outside of algebra, or within a different part of algebra” (Ferrini-Mundy et al., 2005, p. 29). This is an area that is typically associated with ‘word problems,’ the use of which is designed to motivate students to see how mathematics connects with the real-world. Chazan (2000) noted that “a real-world problem is one that someone, perhaps even the student, might encounter outside the school” (p. 40). In addition, he warned that if the students cannot relate to the utility or connections of these real-world problems, then the intent of motivating the students to learn may fail.

Language and conventions “involves understanding aspects of the nature of mathematics, including what is arbitrary, what is based on convention, and what necessary in terms of logical and axiomatic structures” (Ferrini-Mundy et al., 2005, p. 30). School algebra contains definitions and in many cases alternatives to those definitions. As such, one needs to know which definition is most effective to use in various situations. Another area included in this category is notational conventions. Mathematics is froth with symbols. An understanding of distinctions between those symbols is important (Arcavi, 1994). Ferrini-Mundy et al. (2005) pointed out that “knowing conventions overlaps some with the work associated with coordinating mathematical and everyday language” (p. 32). This means that teachers need to know not only the conventions of mathematics, particularly those of school algebra, but also how students’ use of everyday language affects their understanding of those conventions.

The mathematical reasoning and proof category is part of the syntactic structure that was mentioned in Shulman (1986). It includes “knowing the forms of argument and justification that are used on mathematics, the means by which truths are established, and the
level of rigor that is appropriate for the community” (Ferrini-Mundy et al., 2005, p. 32). The inclusion of this category is supported by the NCTM’s (2000) belief that the ability to reason is an essential skill for the development of mathematical understanding.

**Overarching categories.** In addition to the two other components of the framework, Ferrini-Mundy and her colleagues added “three overarching categories – decompressing, trimming, bridging – which are mathematics teaching practices infused through all elements of knowledge of algebra for teaching” (Ferrini-Mundy et al., 2005, p. 24). They allowed these categories to hover over the framework. These overarching categories will be described in more depth.

**Bridging.** The bridging category includes the numerous connections and links that teachers provide, such as: “bridging from students’ understanding to the goals that the teacher is seeking to meet; connecting the ideas of school algebra to those of abstract algebra and real analysis; and linking one area of school mathematics to another” (Ferrini-Mundy et al., 2005, p. 45). The ability of the teacher to make these connections requires a deeper level of understanding of the mathematical content than would be expected of the students. The teacher must align his/her instruction with what has been taught and what will be taught.

Often, in lessons involving single-variable inequalities, students hold the belief that the solution of an inequality must be an inequality (Tsamir & Bazzini, 2004). This deep-seated belief is based on the numerous examples that the student has encountered in which the solution of an inequality was an inequality. However, it is the teacher’s responsibility to provide the bridge, through examples, that will allow the students to understand how the structure of an inequality (i.e., \( x^2 \leq 0 \)) can afford a solution which is an equation (i.e., \( x = \))
In essence, the teacher applies bridging when they find ways to help the students move from their current state of understanding to a desired level of understanding. The desired level of understanding is determined by the learning goals and objectives set forth by the teacher.

**Decompressing.** While Ferrini-Mundy et al. noted that decompressing has its origins in the work of Ball & Bass (2000), decompressing appears to be similar to Brousseau’s didactic transposition. Kang and Kilpatrick (1992) noted that “didactic transposition of knowledge is the transposition from knowledge regarded as a tool to be put to use to knowledge as something to be taught and learned” (p. 2). Decompression refers to one’s ability to “deconstruct one’s own mathematical knowledge into less polished and final form, where elemental components are accessible and visible” (Ball & Bass, 2000, p. 98). Teachers need to be able to take the plastic fairings off of the finely tuned sports bike that is their mathematical knowledge, exposing the frame, the motor, and all the other integral parts that make that machine work the way it does. In doing so, teachers need to realize that they are making the concept more complex for those with a ‘polished and final’ understanding of that concept.

The decompressing by the teacher can serve many purposes. The teacher may decompress a concept to aid in the development of lesson plans. In which case, decompressing a concept, such as graphing a quadratic inequality, may allow the teacher to consider which aspects need to be focal points of the lesson and which aspects can be downplayed. This decompressing may not be evident to the students during the lesson.
Decompressing may also occur in ‘on the fly’ moments in the classroom. Ferrini-Mundy et al. (2005) expected this ‘on the fly’ decompressing to be evident in “direct interactions with students around student misunderstandings and questions, as well as in the design of lessons and choice and sequencing of algebraic tasks” (p. 40). Although at times some of the decompressing completed by the teacher can be difficult to observe, because it is done internally. In these cases, stimulated recall interviews may be beneficial at drawing out the decompressing thought process that was not observable.

**Trimming.** In proceeding through the numerous mathematical courses that are required for licensure in secondary mathematics, teachers have likely accumulated a significant amount of mathematical knowledge beyond that of the secondary curriculum. “Teachers have to *trim* mathematical or contextual content in a way that is mathematically acceptable but leaves intact the content to be learned” (emphasis in the original; Ferrini-Mundy et al., 2005, p. 40 – 41). The notion of trimming is similar to Ma’s (1999) description of Chinese teachers’ ability to present “the simplest form of a certain mathematical idea” (p. 46).

Consider again the finely tuned sports bike analogy from decompressing. Once the teacher has exposed the essential components, a decision must be made. Do you talk about the motor or do you talk about the intake valve, rocker arm, valve cover, engine block, camshaft, crankshaft, piston, spark plug, connecting rod, and rod bearing? The decision about how detailed to get is part of trimming.
From mathematical perspective, a teacher may assign a word problem to his/her students and require the students to write out an inequality and solve it. For example, consider the following problem:

Monique had a great holiday season. She has $900 in her savings account at the beginning of the year. She wants to have at least $200 in the account by the end of May. She withdraws $38 each week from the account. Write an inequality to represent Monique’s situation. How many weeks can Monique withdraw money from her account?

The teacher may decide to provide the students with a list of inequalities and ask the students to determine which one correctly represents the situation (e.g., \(900 - 38w > 200\); \(900 + 38w > 200\); \(900 - 38w \geq 200\); \(900 + 38w \geq 200\); etc.). By providing the students with a list of possible inequalities to choose from, the teacher is ‘trimming’ the problem. The students are not being asked to provide the inequality on their own.

When a teacher trims content, the important mathematical features of that particular content will remain and it will be accessible to students. The degree to which content needs to be trimmed depends on both the content and the students. One needs to take into consideration the future implications of the proposed trimmings, in their own class as well as in other classes. Ferrini-Mundy et al. (2005) refer to the often misused phrase ‘multiplication makes bigger’ to illustrate the consideration that is required. For a teacher working with the natural number system, the statement is correct. However, for the set of integers or rational numbers, the statement falls apart and causes cognitive conflicts for the students (Bell, Swan, & Taylor, 1981).
Trimming also requires an ability to examine a textbook, curriculum, teaching materials, presentation, or student work and determine if trimming has been applied. If trimming has been applied, then one needs to evaluate the appropriateness of the trimming. A similar evaluation occurs for any trimming that the teacher does.

Trimming and decompressing are very similar. Ferrini-Mundy et al. (2005) used the following example to illustrate the differences: Find the solutions of $2x^2 + 4x = 6$ graphically. To accomplish this task, students could graph $f(x) = 2x^2 + 4x$ and $g(x) = 6$. Then the students would need to find the intersection points and make the connection that the $x$-coordinates are the solutions. But, an observant student may notice that both sides of the equation can be divided by 2, yielding $x^2 + 2x = 3$. The teacher, decompressing the problem, may verify the conjecture that this new equation will produce the same solution as the old equation by recalling the following: If $f(x) = g(x)$ and $c$ is an element of the real number system other than zero, then $cf(x) = cg(x)$. With this piece of information, the teacher is then left to determine how much of this reasoning to trim back in order to explain to the students how both sets of functions yield the same solutions. As noted earlier, the level of trimming will be dependent on the students in the class.

Unfortunately, unlike the other frameworks, this framework has not been used often. The authors utilized the framework as part of the development of an assessment, *Knowledge of Algebra for Teaching* (KAT), designed to measure teachers’ knowledge for teaching algebra (Floden & McCrory, 2007). However, follow-up research that reports use of this assessment on a wide scale is not currently available.
Conclusion

Even with the volumes of research that has been conducted surrounding algebra, there are still areas in need of further exploration (Kieran, 1992, 2007). Within this chapter details about the difficulties and issues surrounding inequalities were discussed. The vast majority of what is known comes from studies conducted outside the United States that focused primarily on middle and high school students. In addition, looking at some of the research that exists outside of inequalities (e.g., equations, the equal sign, or variables) has shed light on aspects of inequalities that are not explicitly examined. Even so, little is known about teachers’ understanding of inequalities.

Another area of the research that was reviewed within this chapter pertained to teachers’ content knowledge for teaching mathematics. While there has been more conducted in this area than is discussed within this chapter, the main building blocks of theory that influenced the theoretical framework of this study were included. This framework seeks to examine how teachers take apart their understanding of inequalities and reassemble it in the practice of teaching.

In addressing these research questions, the researcher aims to help mathematics teacher educators to address teachers’ misconceptions surrounding inequalities. Thereby, aiding preservice teachers, who will one day be teachers, to better understand how to help their students to overcome these same misconceptions (Shaughnessy, 1992).

The next chapter will outline the research methodology and conceptual framework of this study.
CHAPTER 3: CONCEPTUAL FRAMEWORK AND METHODOLOGY

Introduction

The purpose of this study was to examine the nature of preservice teachers’ understanding of inequalities. The following research questions guided this study:

- What is the nature of preservice teachers’ understanding of inequalities prior to the student teaching field experience?
- How do preservice teachers use their understanding of inequalities while planning and implementing lessons during the student teaching field experience?

A qualitative case study approach was implemented in an effort to answer the research questions. The five participants of this study were preservice teachers at a major university in the Southeast United States, who were fulfilling their student teaching field experience requirement.

This chapter seeks to present a detailed description of the research methodology and conceptual framework. The description is separated into the following sections: overall research design; conceptual framework; description of a pilot study; definition of the case; description of the participants; identification of the sources of data collected; and explanation of the procedures of the data analysis. Additionally, a discussion about research validity and reliability as well as safeguards against researcher bias and ethical issues of the study are included.

Overall Research Design

This study followed the guidelines of a qualitative case study approach as outlined by Yin (1993). Through qualitative research methods, the researcher was able to examine
“selected issues in great depth with careful attention to detail, context, and nuance” (Patton, 2002, p. 227). The use of a qualitative research methodology was appropriate because this study examined preservice teachers’ understanding of inequalities. The depth of that understanding and the manner in which it is applied in the classroom is difficult to assess via quantitative methods. Additionally, a qualitative research approach with interviews, classroom observations, and written documents (lesson plans) as main sources of data collection allowed the researcher to explore and to ground emerging themes in the data (Strauss & Corbin, 1998).

Since the context cannot be separated from the phenomenon a case study approach was appropriate (Yin, 1993). Within this study, the context was the student teaching semester for preservice teachers. During the student teaching semester, preservice teachers most likely encountered their first opportunity to enact their personal understanding of mathematical topics as they made pedagogical decisions. The preservice teachers’ pedagogical decisions were manifested in the planning and teaching of lessons. The phenomenon was the nature of the preservice teachers’ understanding of inequalities.

**Conceptual Framework**

During this study, the researcher held a radical constructivist perspective. Accordingly, the researcher believed that the formation of knowledge does not occur as part of a transfer from teacher to student, as if one could open the student’s head and pour that knowledge in. The researcher believed that through failure and adaptation one learns of the constraints of the real world in which we live (von Glaserfeld, 1984). The ideas that continue
to hold up are considered to be viable and are maintained; those that do not hold up are discarded.

As a radical constructivist, the researcher believed that the experiences that preservice teachers had with inequalities were unique. In addition, those experiences played a role in the preservice teachers’ understanding of mathematics. This perspective is echoed in the definition of mathematical learning provided by Simon, Tzur, Heinz, Kinzel, and Smith (2000): “Mathematics learning is the process by which humans create and adapt conceptions to organize their experiential worlds” (p. 584).

For this study, the researcher’s intent was to ascertain the ideas, procedures, and understanding that have held up and continue to fit the preservice teacher’s reality regarding the mathematical concept of inequalities (von Glaserfeld, 1984). The researcher was not evaluating preservice teachers’ ability to employ a predetermined best method. Instead, the researcher was evaluating the ideas, procedures, and understanding presented by the preservice teachers against normative mathematical practices as advocated in mathematics education literature and recommendations from organizations such as NCTM (2000), CBMS (2001), and Common Core State Standards Initiative (2010).

Additionally, the researcher believed that preservice teachers relied heavily on their own understanding while developing and implementing lessons during their student teaching field experience. Their reliance on their understanding and repertoire of solution methods, rather than those commonly used by students, was necessitated by their lack of experience in the classroom as a teacher. As such, it was expected that their own understanding of inequalities shaped the explanations, examples, and follow-up activities within their lesson.
In order to explore the research questions of this study, a theoretical lens that would allow the researcher to examine both the preservice teachers’ understanding of inequalities as well as how that understanding was applied during classroom lessons was needed. A conceptual framework designed to meet these needs was developed. The conceptual framework would guide the specific research design, data collection, and analysis for this case study.

The conceptual framework of this study was a matrix design with two overarching views (see Table 2). A brief overview is provided, followed by more detailed descriptions of each component of the framework. The column headings of the framework were aspects of knowledge associated with inequalities. These aspects were identified within a literature review. With regard to this study, there were four primary aspects of knowledge associated with inequalities: 1) Strategies for solving inequalities; 2) Relating inequalities as equations; 3) Shading as a process; and 4) Solutions of inequalities. Within those primary aspects there were other aspects that may be pertinent to more than one primary aspect: domain; use of variable(s); factors of products and/or quotients; logical connectives; and interpretations of the inequality symbol. The row headings were teaching practices: 1) Explaining a mathematical idea; 2) Solving a mathematical problem; and 3) Using technology. These teaching practices were related to some of the tasks of teaching identified within the Ferrini-Mundy et al. (1996) framework (see Figure 2), and were chosen as practices that would likely occur with enough frequency to allow for focused analysis. Additionally, the conceptual framework for this study had two overarching views: operational view of inequalities and relational view of inequalities. These views were adaptations of views of
equations presented by Knuth et al. (2006). An operational view of inequalities involved a belief that an inequality sign was a signal to perform operations or get an answer. A relational view of inequalities involved a belief that an inequality was a statement of mathematical relationship that corresponded to the given inequality symbol. These views were a third dimension of the matrix design. Each cell in the framework represents an event in which a teaching practice (rows) and an aspect of knowledge associated with the mathematical concept of inequalities (columns) occurred. Within each cell of the matrix, displayed data will be classified as an operational view of inequalities or relational view of inequalities.
Table 2. Conceptual framework for this study.

<table>
<thead>
<tr>
<th>Teaching practices</th>
<th>Explaining mathematical ideas</th>
<th>Solving mathematical problems</th>
<th>Using technology</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Strategies for solving inequalities</td>
<td>Relating inequalities as equations</td>
<td>Shading as a process</td>
</tr>
<tr>
<td></td>
<td>Domain</td>
<td>Use of variable(s)</td>
<td>Factors of products and/or quotients</td>
</tr>
</tbody>
</table>

Aspects of Knowledge Associated with Inequalities
The aspect of “strategies employed to solve inequalities” referred to three strategies most often used to solve inequalities: (1) algebraic manipulation, (2) drawing a graph, and (3) using a case approach (Tsamir & Almog, 2001). The case approach strategy⁹ (Dreyfus & Eisenberg, 1985) requires consideration of all of the possibilities that would produce the given inequality. If a graph of the given linear inequalities was drawn and used to attain the solution, then drawing a graph (graphing) strategy was implemented. The strategy of algebraic manipulations included the following:

- addition or subtraction of the same term to both sides of the inequality;
- multiplication and division of both sides of the inequality by identical factors, with appropriate attention to the orientation of the inequality symbol;
- simplification of an expression through combining like terms or factoring;
- multiplying both sides of the inequality by the square of the denominator;
- “examining quadratic inequalities (i.e., \( ax^2 + bx + c > 0 \)) by first relating to the quadratic roots, or by investigating the sign of ‘a’ and the sign of the determinant” (Tsamir et al., 1998, p. 131); and
- “relating to an inequality of the type \( ab > 0 \) as a compound system of \( \{ a > 0 \) and \( b > 0 \} \) or \( \{ a < 0 \) and \( b < 0 \} \)” (Tsamir & Almog, 2001, p. 515).

The aspect of “relating inequalities to equations” was a conception identified within the literature (Attorps, 2003; Tsamir & Bazzini, 2004). This conception was usually related to an incorrect concept image of inequalities and/or equations (Tall & Vinner, 1981). The aspect of “shading as a process” emerged during the researcher’s analysis of pre-student

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⁹ A full description and an example of the case approach strategy can be found in Chapter 2.
teaching interviews. The researcher noticed during the pre-student teaching interviews that the preservice teachers implemented one of the following shading methods: (1) shade above or shade below; or (2) test point. The shade above or shade below method relied on an association between the inequality sign and a relative direction to shade. The test point method involved determining the appropriate region to shade based on the evaluation of a selected coordinate. The preservice teachers’ decision regarding which method to employ seemed to persevere throughout the different types of inequalities encountered during the pre-student teaching interview. Finally, the aspect of “solution(s) of inequalities” referred to the various forms in which the solution of the inequality was represented and/or discussed. Among the representations included in the aspect was shading as an object. Additionally, the issues regarding solutions of inequalities identified during a review of the literature were included in this aspect (Bazzini, Boero, & Garuti, 2001; Bazzini & Tsamir, 2001; Tsamir & Bazzini, 2004; Vaiyavutjamai & Clements, 2006).

The teaching practices of the conceptual framework (see Table 2) were adapted from the framework presented by Ferrini-Mundy et al. (2005). Ferrini-Mundy et al. had categories called tasks of teaching. Two of these categories were adapted to form the teaching practices of the conceptual framework. Explaining mathematical ideas and solving mathematical problems was a teaching task in the Ferrini-Mundy et al. (2005) framework that was separated into two teaching practices for the conceptual framework for this study: 1) Explaining a mathematical idea; and 2) Solving a mathematical problem. Separating these two teaching practice categories in the conceptual framework allowed for a tighter focus. Explaining a mathematical idea was defined to be the act of explaining a mathematical idea
or concept to students (individually, in groups, or as a whole) and/or finding appropriate ways to explain mathematical ideas to students in ways that help them learn. The researcher considered the manner in which the preservice teachers presented strategies, techniques, methods, etc. as belonging to explaining a mathematical idea. Solving mathematical problems was defined as the following: the teacher or the students working through the steps necessary to find the solution to a given problem. In an effort to clarify the coding, the start and end of the problem was dictated by the teacher. The types of problems selected by the preservice teacher to present in the class was considered within the solving mathematical problems teaching practice. The third teaching practice of the conceptual framework for this study, using technology, was adapted from the Ferrini-Mundy et al. (2005) framework. Using technology was defined as occurrences in which a preservice teacher integrated technology, such as graphing calculators, netbooks, web-based applications, etc., into their lessons. The preservice teachers have had extensive attention to the use of technology in their teacher education program and have access to technologies in their field placement classroom during student teaching.

When considering the ways preservice teachers used their knowledge to explain ideas, solve problems, and use technology, it is helpful to consider the ways in which inequalities are treated or talked about. How a preservice teacher treats or considers inequalities when engaged in teaching practices can also provide a lens into their understanding, and the potential understandings they may be promoting for their students. Thus, the overarching lens of an operational or relational view was added to the framework
Rittle-Johnson and Alibali (1999) provided the following definitions for conceptual knowledge and procedural knowledge:

...conceptual knowledge as explicit or implicit understanding of the principles that govern a domain and of the interrelations between pieces of knowledge in a domain. We define procedural knowledge as action sequences for solving problems. These two types of knowledge lie on a continuum and cannot always be separated; however, the two ends of the continuum represent two different types of knowledge. (p. 175)

Building upon these definitions and a study by McNeil and Alibali (2005), Knuth et al. (2006) noted that an operational view of equality entailed a belief that the equal sign was an “announcement of the result of an arithmetic operation” (p. 298). Displaying a relational view of equality meant that one considered the equal sign as “a symbol of mathematical equivalence” (Knuth et al., 2006, p. 298).

This study utilized the definitions of conceptual and procedural knowledge as outlined by Rittle-Johnson and Alibali (1999). In a similar manner as Knuth et al. (2006), an operational view of inequalities involved a belief that an inequality sign was a signal to perform operations or get an answer. In addition, a relational view of inequalities involved a belief that an inequality was a statement of mathematical relationship.

The conceptual framework (see Table 2) afforded the researcher the ability to plan and collect data that would allow for an examination of the interaction between preservice teachers’ understanding of inequalities and their teaching practices. The study was separated

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10 The specifics of that relationship depend on the actual inequality symbol. Since an equation only has one symbol (=), it is easier to talk about and define. On the other hand, an inequality can have one of five symbols which makes it harder to define.
into two phases, as denoted by the research questions. The first phase concentrated on preservice teachers’ understanding of inequalities prior to beginning their student teaching experience. This phase of the study utilized the column headings from the conceptual framework as a guide (see Table 2). The aspects of knowledge associated with inequalities aided in the development and analysis of the pre-student teaching interview (see Appendix C). The aspects were the following: strategies for solving inequalities; domain; relating inequalities to equations; logical connectives; factors of products and/or quotients; use of variable(s); solutions of inequalities; and interpretations of the inequality symbol.

The second phase centered on how understanding of inequalities was utilized while planning and implementing lessons during the student teaching field experience. The conceptual framework (see Table 2) as a whole provided guidance with respect to data collection and analysis in order to address the second research question for each of the content areas: systems of linear inequalities and quadratic inequalities.

**Pilot Studies**

Two separate small scale pilot studies were conducted prior to the commencement of this study. The first pilot study examined a lesson plan submitted by preservice teachers. The study was a pilot of the task-based interview. Based on information gained during these pilot studies, modifications were made to the current study.

**Lesson Plan**

Prior to the start of this study a pair of preservice teachers in one of the methods courses at the same university in which this study was conducted allowed the researcher to examine a lesson plan pertaining to systems of linear inequalities that they created. The
preservice teachers did not wish to allow a classroom observation of the implementation of the lesson. In addition, they did not wish to participate in an interview. However, they were willing to allow the researcher to examine the lesson plan. Therefore, an analysis of the lesson plan as it is written was conducted using a conceptual framework as a guide (see Table 3).

Table 3. Conceptual framework, from a pilot study, that was used to analyze a lesson plan.

<table>
<thead>
<tr>
<th>Aspects of Knowledge Associated with Inequalities</th>
<th>Teaching Practices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategies for solving inequalities</td>
<td>Bridging</td>
</tr>
<tr>
<td>Representations</td>
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<tr>
<td>Relating inequalities to equations</td>
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<tr>
<td>Logical connectives</td>
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<tr>
<td>Factors of products and/or quotients</td>
<td></td>
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<tr>
<td>Denominators of rational inequalities</td>
<td></td>
</tr>
<tr>
<td>Use of variable(s)</td>
<td></td>
</tr>
<tr>
<td>Solutions and interpretations of the inequality symbol</td>
<td></td>
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</tbody>
</table>
The conceptual framework that guided the analysis of this lesson plan (see Table 3) was a modified version of the Ferrini-Mundy et al. (2005) framework. The overarching teaching practices from the Ferrini-Mundy et al. (2005) framework were used as teaching practices. An additional teaching practice, scaffolding, was added after discussions with a committee member. The aspects of knowledge associated with inequalities were created based on a review of literature.

Within the lesson, the preservice teachers had a word problem that they asked the students to solve in small groups. Before the students began the problem, the preservice teachers planned to define two variables in this problem for the students. There was no indication that a discussion about the appropriateness of how the variables were defined occurred. As such, considering only what was written in the lesson plan, this was evidence of trimming as it related to use of variables. In defining the variables, the preservice teachers removed some of the complexity of the problem.

Later in the same problem, the preservice teachers highlighted guiding questions that they asked the students to aid in the formulation of the system of inequalities. These questions were designed to help the students translate from a word problem to a symbolic problem. Based on the framework of this pilot study (see Table 3), this was an example of scaffolding as it related to strategies for solving inequalities. The preservice teachers, while applying scaffolding, took the process of stating the system of inequalities and parsed it into smaller steps. Each step directed the students to write an inequality that later formed the system. The aspect of knowledge associated with inequalities highlighted in this segment of guided questions was strategies for solving inequalities. These particular preservice teachers,
in a similar manner to the teachers in Koedinger and Nathan (2004), seemed to believe that the manner to solve a word problem was to convert it to a symbolic problem. Therefore, the formation of the system of inequalities in symbolic form was part of the solving process.

Issues with the conceptual framework surfaced while conducting the analysis on the lesson plan. While the teaching practices used in this pilot study were defined, it was difficult to describe them in tangible terms. This ambiguity allowed for the possibility of failure to attain reliability. As a result of this pilot study and input from committee members, the conceptual framework was changed to the conceptual framework mentioned earlier (see Table 2).

**Task-based Pre-Student Teaching Interview**

In addition to the document analysis that was conducted on the lesson plan, the researcher conducted a pilot of the task-based interview. The interview was administered to a preservice teacher who had recently completed the student teaching experience. The participant had taught two sections of Algebra I during his internship. The purpose of the pilot interview was to gain a sense of how long the planned interview would take and if any questions needed to be changed to clarify wording, meaning, and/or directions.

In conducting the pilot interview, questions #9 and #10 (see Figure 3) were originally designed to elicit the participant’s knowledge pertaining to graphing and interpreting the solution of inequalities. Based on the responses from the participant, major revisions of these two questions were necessary. The wording of those questions was subsequently changed to focus the participant on the graphing aspects and their own descriptions of the solutions.
During the interview, the participant was asked to graph two linear inequalities. The participant utilized graph paper and produced graphs that were not accurate; the point of intersection was incorrect. The issue manifested itself when the participant drew the lines; he was slightly off in lining up his straight-edge for both lines. The participant knew that the graphs were incorrect, via algebraic manipulations. Yet, the participant stayed with the incorrect graphs to describe the solution and its meaning. Even with a calculator being made available to the participant, he did not use it. With this in mind, the researcher will allow the participants to graph the inequalities in any manner that they deem appropriate.

With respect to the ‘strategies for solving inequalities’ aspect of knowledge associated with inequalities, the participant exhibited at least a procedural understanding (Hiebert & Lefevre, 1986). The dominant strategy for solving inequalities employed by the participant was an algebraic manipulations strategy. Additionally, without any prompting from the researcher, the participant implemented a variation of the case approach when

**Figure 3.** Questions #9 and #10 from the pilot study of the task-based interview.

9. Solve the following inequalities
   a. $-2 \leq 5 - 7x < 19$
   b. $|x - 5| < 13$
   c. $3 \leq \left| \frac{1}{4}x + 1 \right|$ 

10. Given the system: 
    \[
    \begin{align*}
    x + 4y &> 2 \\
    5x - 2y &\leq -1
    \end{align*}
    \]
    Are the following coordinates solutions of this system?
    a. (-1, -2)  b. (-3, 8)  c. (0, 0.5)  d. (5, 2)
solving inequalities that involved absolute values. As noted earlier, the participant was able to solve an inequality using a graphing method. However, unlike the case approach, the researcher had to prompt the participant to utilize this method. As a result, directions for some of the questions were changed in order to ensure that the preservice teachers displayed their understanding of a graphing strategy during the pre-student teaching interview.

**Defining the Case for the Current Study**

A case was defined differently for each of the research questions. Since the first research question addressed understanding of individual preservice teachers, a case was defined to be each of the five preservice teachers who participated in this study. Each case consisted of the understanding displayed by a preservice teacher when solving inequality tasks after completion of all of their methods courses and prior to the student teaching experience. The preservice teacher’s responses to the tasks on the pre-student teaching interview (see Appendix C) were the primary source of data.

There were two cases for the second research question of this study. The cases consisted of the two different topics in inequalities that were being taught by the preservice teachers during the student teaching field experience that may draw upon different content knowledge by the preservice teachers: systems of linear inequalities and quadratic inequalities. Each case consisted of lessons taught by preservice teachers during the student teaching field experience regarding systems of linear inequalities or quadratic inequalities. Case 1 consisted of four preservice teachers (Angela, Christina, Crystal, and Vanessa) taught lessons regarding systems of linear inequalities. Case 2 included lessons pertaining to quadratic inequalities that were taught by three preservice teachers in this study (Crystal,
Heather, and Vanessa). It should be noted that lessons by Crystal and Vanessa were in both cases because they taught both topics during their student teaching field experience.

**Participant Selection**

This study examined preservice teachers at a major university in the Southeast region of the United States during their student teaching field experience semester. The researcher and preservice teachers who potentially could have participated in this study had no control of the preservice teachers’ classroom and subject-level placement to fulfill their student teaching field experience. They could not predetermine the mathematical subjects that would be taught nor could they predetermine the content topics within a subject that they would teach. Therefore, a purposeful and convenient sample was necessary.

Undergraduate preservice teachers were recruited to participate in this study. Subjects who were not undergraduate Mathematics Education majors were not eligible for this study. This excluded graduate level preservice teachers from this study. The researcher wished to ensure that all participants in this study had gone through the same undergraduate Mathematics Education curriculum.

Preservice teachers in this study were selected based on whether they would have the opportunity to teach lessons involving inequalities in an Algebra (Algebra I or Algebra II) course. As a result, preservice teachers who were not assigned to an Algebra I or II course were not included in this study. Also, preservice teachers who were assigned to an Algebra course but would not teach lessons involving inequalities were excluded from this study.

Volunteers for this study were recruited from methods courses that were taught prior to the student teaching field experience. Initially, twenty-five preservice teachers volunteered
to participate in this study. Of the twenty-five, only seven met the requirements that they be undergraduate students who would be teaching lessons involving inequalities in an Algebra course. The researcher made inquiries to the corresponding cooperating teachers as to the specific content of the inequality lessons. One potential participant was teaching linear inequalities, two were teaching systems of linear inequalities, two were teaching quadratic inequalities, and two were teaching systems of linear inequalities and quadratic inequalities.

For purposes of this study, a decision was made to focus on systems of linear inequalities and quadratic inequalities. This decision reduced the potential pool of participants to six. The six remaining potential participants were all white females. A final decision was made as to which five of the six potential participants to include in this study. This decision was based on the topic(s) (system of linear inequalities and/or quadratic inequalities) being taught by the preservice teachers and the location of the schools in which the preservice teachers were placed. Due to the time needed to conduct classroom observations in several classrooms during a short time period (about eight weeks), travel distances needed to be minimized.

The requirements for participation in this study, as outlined in the Informed Consent Form (see Appendix A), was explained to the participants. The participants were guaranteed that pseudonyms would be used. Two copies of the Informed Consent Form were given to the participants. One copy was for their records; the other copy was signed and returned to the researcher. This signed copy was stored on the researcher’s personal computer. As outlined in the Informed Consent Form, the participants were made aware that their involvement in this study was voluntary. They were told that they could terminate their
involvement at any time. None of the participants chose to withdraw from the study prior to its completion. The researcher compensated the participants with a token gift of appreciation. The gift card, which was meant to help them buy things for their classroom, was given to the participants at the end of the post-student teaching interview.

Five white female undergraduate preservice mathematics teachers constituted the purposeful and convenient sample for this study. They were all traditional-aged students (21 – 23). Efforts were made to include minorities and males. However, none of the preservice teachers who volunteered and met the teaching placement requirements of this study fit either of those demographics.

Angela

At the time of this study, Angela was completing the final requirements for Bachelor of Science degrees in Mathematics and Mathematics Education – Mathematics Specialization. She completed her student teaching field experience at a large high school in a small rural school district. Angela was observed while teaching her Algebra I class. The class met five days a week in a 90 minute block format. Angela implemented lessons pertaining to systems of linear equations.

Angela’s 14 students were all freshmen. Approximately half of the students were minorities. The desks were typically arranged in rows facing the front of the classroom. However, from time to time, Angela would rearrange the desks into “stations” around her classroom. Angela’s classroom had a LCD projector connected to a laptop and a wireless scribe tablet. In addition, she had a whiteboard in the front of the classroom. Classroom sets
of netbooks and calculators (TI-83 plus) were readily available within her classroom. Angela’s cooperating teacher noted that none of the students used their own calculators.

**Christina**

At the time of this study, Christina had nearly completed all of the requirements for Bachelor of Science degrees in Mathematics and Mathematics Education – Mathematics Specialization. She completed her student teaching field experience at a large high school in a small rural school district. All of Christina’s classroom observations occurred during her Algebra I class. The class met five days a week in a 90 minute block format. Christina taught systems of linear inequalities to the students in her Algebra I class.

Christina’s cooperating teacher noted that her class included a large number of students who were repeating Algebra I. In addition, the cooperating teacher pointed out that Christina’s class had more inclusion students than any of the Algebra I classes in the recent past. Christina’s class had 18 students, two-thirds of which were minorities. Christina’s classroom had a whiteboard and an overhead projector in the front of the classroom. Classroom sets of netbooks and calculators (TI-83 plus) were readily available within her classroom. Very few of the students used their own calculator during observations. The students’ desks were arranged in rows. The two center rows were facing the whiteboard and projector screen in the front of the classroom. The four other rows, two on each side of the center rows, were facing the center rows.

**Crystal**

At the time of this study, Crystal was completing the last of the requirements for a Bachelor of Science degree in Mathematics Education – Mathematics Specialization. She
completed her student teaching field experience at a large high school in a large suburban school district. All of Crystal’s classroom observations occurred during her Algebra II Honors class. The class met five days a week in a 90 minute block format. She taught both systems of linear inequalities and quadratic inequalities to the students in her Algebra II Honors class.

Crystal’s Algebra II Honors class had 33 students enrolled. Less than one-third of her students were minorities and well over half were classified as sophomores. Crystal and the cooperating teacher mentioned, on separate occasions, that this class had several students who, in their opinion, should not be in an Honors class. The desks were arranged in rows that faced the front whiteboard. Graphing calculators (TI-83 Plus) were available for students; however it appeared that the majority of the students used their own graphing calculators (TI-83 or TI-84 models). A LCD projector attached to a laptop, a whiteboard, and an overhead projector were available for use within Crystal’s classroom.

Heather

Heather had nearly completed all of the requirements for Bachelor of Science degrees in Statistics and Mathematics Education – Statistics Specialization. She completed her student teaching field experience at a large high school in a small rural school district. She taught an Algebra II Honors class that met five days a week in a 90-minute block format. She implemented lessons pertaining to quadratic inequalities. All of Heather’s observations were conducted during this class.

There were 23 students enrolled in Heather’s Algebra II Honors class. Less than a fourth of her students were minorities. The cooperating teacher noted that there were several
students within this particular class who should have been placed into a “regular” Algebra II class. Classroom sets of netbooks and calculators (TI-83 plus) were readily available within Heather’s classroom. About half of the students seemed to use their own calculators (TI-83 or TI-84 models). Students sat in rows that were oriented towards a small whiteboard on the right side of the front of the class. An LCD projector was located on a table in front of the small whiteboard. Connected to the LCD projector were a laptop and a document camera. The laptop and document camera were on a cart to the right side of the class. Heather used the document camera during every observation. She situated the cart so that she was out of the students’ direct line of sight to the projection.

**Vanessa**

At the time of this study, Vanessa was completing the final semester of requirements for a Bachelor of Science degree in Mathematics Education – Mathematics Specialization. She completed her student teaching field experience at a large high school in a large suburban school district. Vanessa’s classroom observations were conducted in her Algebra II class. The class met five days a week in a 90 minute block format. Vanessa taught her Algebra II students systems of linear inequalities and quadratic inequalities.

The enrollment in Vanessa’s Algebra II class was 28 students. Almost half of the students were minorities. The cooperating teacher noted that the number of students classified as sophomores, juniors, or seniors were about the same. The students sat in rows of seats that all faced the front of the classroom. Vanessa had a LCD projector, which was connected to a computer and a document camera, available for use during her lessons. There was a whiteboard at the front of the class. In addition, she had a TI-83 Plus connected to a
television attached to the wall in the front of the classroom next to the whiteboard. A classroom set of graphing calculators (TI-83 Plus) were available for students to use, although about half of the students appeared to use their own graphing calculator during observations. It was difficult to determine the exact models being used by students, but it appeared that students were using either TI-83 or TI-84 graphing calculators.

**Data Collection**

This section provides a description of the data collected within this study. The sources of data for this study were the following: pre-student teaching interviews; lesson plans; video recordings of classroom observations; classroom observation notes; and post-student teaching interviews. The dates of the classroom observations and a brief description of material presented during each classroom observation have been displayed in tables for each of the preservice teachers (see Table 4, Table 5, Table 6, Table 7, and Table 8). A table outlining the data sources that were specifically relevant to inequalities was included (see Table 9). Additionally, a table outlining the data source(s) that were used to address each research question follows (see Table 10).

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11 Lessons involving equations were observed for two reasons: 1) to acclimatize the preservice teacher and students to the video camera and observer; 2) to capture moments that preservice teachers may have potentially referred back to during lessons pertaining to inequalities.
Table 4. Angela's classroom observation dates and topics covered.

<table>
<thead>
<tr>
<th>Date of classroom observation</th>
<th>Topic discussed during the classroom observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 28, 2011</td>
<td>Word problems</td>
</tr>
<tr>
<td>March 1, 2011</td>
<td>Solving systems of linear equations by substitution</td>
</tr>
<tr>
<td>March 2, 2011</td>
<td>Solving systems of linear equations by graphing</td>
</tr>
<tr>
<td>March 3, 2011</td>
<td>Review of solving systems of linear equations</td>
</tr>
<tr>
<td>May 2, 2011</td>
<td>Linear inequalities and Systems of linear inequalities</td>
</tr>
</tbody>
</table>

Table 5. Christina's classroom observation dates and topics covered.

<table>
<thead>
<tr>
<th>Date of classroom observation</th>
<th>Topic discussed during the classroom observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 28, 2011</td>
<td>Solving systems of linear equations by substitution</td>
</tr>
<tr>
<td>March 1, 2011</td>
<td>Solving systems of linear equations by elimination</td>
</tr>
<tr>
<td>March 2, 2011</td>
<td>Solving systems of linear equations by graphing</td>
</tr>
<tr>
<td>March 3, 2011</td>
<td>Review of systems of equations</td>
</tr>
<tr>
<td>March 4, 2011</td>
<td>Solving systems of linear inequalities</td>
</tr>
<tr>
<td>March 7, 2011</td>
<td>Word problems involving systems of equations</td>
</tr>
</tbody>
</table>
Table 6. Crystal's classroom observation dates and topics covered.

<table>
<thead>
<tr>
<th>Date of classroom observation</th>
<th>Topic discussed during the classroom observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>March 10, 2011</td>
<td>Solving systems of linear inequalities</td>
</tr>
<tr>
<td>March 11, 2011</td>
<td>Linear programming</td>
</tr>
<tr>
<td>March 14, 2011</td>
<td>Linear programming</td>
</tr>
<tr>
<td>April 4, 2011</td>
<td>Solving quadratic equations with one variable by factoring</td>
</tr>
<tr>
<td>April 5, 2011</td>
<td>Solving quadratic equations and quadratic inequalities with one variable by factoring</td>
</tr>
<tr>
<td>April 8, 2011</td>
<td>Solving quadratic equations with one variable by completing the square or using the quadratic formula</td>
</tr>
<tr>
<td>April 11, 2011</td>
<td>Review (quadratic inequalities were not discussed)</td>
</tr>
</tbody>
</table>

Table 7. Heather's classroom observation dates and topics covered.

<table>
<thead>
<tr>
<th>Date of classroom observation</th>
<th>Topic discussed during the classroom observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>March 1, 2011</td>
<td>Graphing quadratic functions</td>
</tr>
<tr>
<td>March 2, 2011</td>
<td>Graphing quadratic functions and solving quadratic equations with one variable by factoring</td>
</tr>
<tr>
<td>March 3, 2011</td>
<td>Solving quadratic equations with one variable by completing the square or using the quadratic formula</td>
</tr>
<tr>
<td>March 4, 2011</td>
<td>Discriminant and vertex form of quadratic equations</td>
</tr>
<tr>
<td>March 7, 2011</td>
<td>Solving quadratic inequalities with one variable</td>
</tr>
<tr>
<td>March 8, 2011</td>
<td>Solving quadratic inequalities with two variables</td>
</tr>
</tbody>
</table>
Table 8. Vanessa’s classroom observation dates and topics covered.

<table>
<thead>
<tr>
<th>Date of classroom observation</th>
<th>Topic discussed during the classroom observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 24, 2011</td>
<td>Solving systems of equations</td>
</tr>
<tr>
<td>February 25, 2011</td>
<td>Word problems (systems of equations)</td>
</tr>
<tr>
<td>February 28, 2011</td>
<td>Solving systems of linear inequalities</td>
</tr>
<tr>
<td>April 4, 2011</td>
<td>Solving quadratic equations with two variables by graphing</td>
</tr>
<tr>
<td>April 6, 2011</td>
<td>Solving quadratic equations with one variable by factoring</td>
</tr>
<tr>
<td>April 7, 2011</td>
<td>Solving quadratic equations with one variable using the quadratic formula</td>
</tr>
<tr>
<td>April 8, 2011</td>
<td>Review of solving quadratic equations</td>
</tr>
<tr>
<td>April 11, 2011</td>
<td>Solving quadratic inequalities (two variables and one variable)</td>
</tr>
</tbody>
</table>

Table 9. Timeline outlining data collection specifically relevant to inequalities.

<table>
<thead>
<tr>
<th>Preservice teachers</th>
<th>Angela</th>
<th>Christina</th>
<th>Crystal</th>
<th>Heather</th>
<th>Vanessa</th>
</tr>
</thead>
</table>
Table 10. Data sources used to address each research question.

<table>
<thead>
<tr>
<th>Research Questions</th>
<th>Pre-student teaching interview</th>
<th>Classroom observations (notes and video recordings)</th>
<th>Lesson plans</th>
<th>Post-student teaching interview</th>
</tr>
</thead>
<tbody>
<tr>
<td>Research Question #1</td>
<td>Primary</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Research Question #2 (Systems of Linear Inequalities)</td>
<td>Secondary</td>
<td>Primary</td>
<td>Secondary</td>
<td>Secondary</td>
</tr>
<tr>
<td>Research Question #2 (Quadratic Inequalities)</td>
<td>Secondary</td>
<td>Primary</td>
<td>Secondary</td>
<td>Secondary</td>
</tr>
</tbody>
</table>

In the following sections, the data sources and instruments will be described in the order in which each was collected: pre-student teaching interview; collection of lesson plans; classroom observations; post-student teaching interview.

**Pre-Student Teaching Interviews**

Interviews are commonplace in qualitative case studies. The purpose of interviewing the preservice teachers was to gather detailed, descriptive data that could not be obtained through observation or some other form of data collection (Patton, 2002). Use of the
preservice teachers’ own words allowed the researcher to develop insights about the preservice teachers’ understanding of inequalities (Bogdan & Biklen, 2007).

Two semi-structured interview protocols (see Appendix C and Appendix D) were used for pre-student teaching interviews and post-student teaching interviews. There were two benefits to using a semi-structured interview protocol. The first benefit was preservation of a specific agenda preselected by the researcher. The other benefit was flexibility afforded to the researcher to probe possibly revealing lines of conversation.

The rest of this section focuses on the pre-student teaching interview, which was a data source for both research questions. The pre-student teaching interview (see Appendix C) was conducted prior to the start of the preservice teacher’s student teaching field experience (see Table 9 for the dates of each interview). The preservice teachers were allowed to sign-up for a block of time to conduct the interview. The interviews occurred in a private space at the university. These steps were taken to minimize the intrusiveness of the interviews on the preservice teachers.

The interviews were recorded using an audio recorder and a video recorder. The audio recordings aided in the transcribing of the interviews. The interviews were transcribed verbatim. Through the process of transcribing the interviews, the researcher was immersed in the data. The preservice teachers were offered the opportunity to review a copy of the transcript of the pre-student teaching interview. None of the preservice teachers indicated that they wished to review the transcript.

The pre-student teaching interview followed a task-based interview format. The questions were selected in order to ascertain the preservice teachers’ understanding with
regard to pitfalls and misconceptions discovered in a review of literature. In addition, some of the questions were designed to be consistent with the content areas that the preservice teachers may encounter during their student teaching field experience. These content areas included absolute value inequalities, systems of inequalities, as well as quadratic inequalities.

A table linking the questions used during pre-student teaching interviews to the aspects of knowledge associated with the mathematical concept of inequalities has been included (see Table 11).

Table 11. Link between aspects of knowledge associated with inequalities and questions from pre-student teaching interview.

<table>
<thead>
<tr>
<th>Interview Question</th>
<th>Aspects of Knowledge Associated with Inequalities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1. How would a student define ‘equation’ in a manner that indicated that they have a conceptual understanding of the work?</td>
<td>Relating inequalities to equations; and Use of variable</td>
</tr>
<tr>
<td>Q2. With the previous definition for equation in mind, which of the following are equations:</td>
<td>Relating inequalities to equations; and Use of variable</td>
</tr>
<tr>
<td>a. ( x = x )</td>
<td></td>
</tr>
<tr>
<td>b. ( x = 2 )</td>
<td></td>
</tr>
<tr>
<td>c. ( 0 = 0 )</td>
<td></td>
</tr>
<tr>
<td>d. ( 0 = 1 )</td>
<td></td>
</tr>
<tr>
<td>e. ( V = \frac{4}{3} \pi r^3 )</td>
<td></td>
</tr>
<tr>
<td>f. ( y &gt; 5x - 9 )</td>
<td></td>
</tr>
<tr>
<td>Q3. Consider the set ( S = { x \in \mathbb{R} \mid x=3 } ) and check the validity of the following statement:</td>
<td>Strategies for solving inequalities; Domain; Relating inequalities to equations; Solutions of inequalities; and Interpretation of the inequality symbol</td>
</tr>
<tr>
<td>S can be the solution of both an equation and an inequality. Explain your answer.</td>
<td></td>
</tr>
</tbody>
</table>
Table 11 Continued.

| Q4. Are the following inequalities equivalent: | Strategies for solving inequalities; Domain; Factors of products and/or quotients; Relating inequalities to equations; Use of variable; Solutions of inequalities; and Interpretation of the inequality symbol |
| 7 > ax and \( x < \frac{7}{a} \quad \forall a \in R \) | |

| Q5. Are the following inequalities equivalent statements? | Strategies for solving inequalities; Domain; Relating inequalities to equations; Factors of products and/or quotients; Use of variable(s); Solutions of inequalities; and Interpretation of the inequality symbol |
| \( ax < 5 \) and \( x < \frac{5}{a} \quad \forall a \in R, a \neq 0 \) | |

| Q6. Solve the inequality \((a - 5)x > 2a - 1\), \(x\) being the variable and \(a\) the parameter. | Strategies for solving inequalities; Domain; Relating inequalities to equations; Logical connectives; Factors of products and/or quotients; Use of variable(s); Solutions of inequalities; and Interpretation of the inequality symbol |

| Q7. Indicate the truth set (the solution) of \(5x^4 \leq 0\). | Strategies for solving inequalities; Domain; Relating inequalities to equations; Logical connectives; Solutions of inequalities; and Interpretation of the inequality symbol |

| Q8a. Solve the following inequalities a. \((x - 1)(3x + 7) \geq 0\) | Strategies for solving inequalities; Relating inequalities to equations; Logical connectives; Factors of products and/or quotients; Solutions of inequalities; and Interpretation of the inequality symbol |


Table 11 Continued.

<table>
<thead>
<tr>
<th>Q8b. Solve the following inequalities:</th>
<th>Strategies for solving inequalities; Relating inequalities to equations; Logical connectives; Solutions of inequalities; and Interpretation of the inequality symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>b. $2x^2 + 6x - 17 &lt; x - 13$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Q8c. Solve the following inequalities:</th>
<th>Strategies for solving inequalities; Relating inequalities to equations; Logical connectives; Solutions of inequalities; and Interpretation of the inequality symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>c. $3 \leq \left</td>
<td>\frac{1}{4}x + 1 \right</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Q9. Given the system: $\begin{cases} x + 4y &gt; 2 \ 5x - 2y \leq -1 \end{cases}$</th>
<th>Strategies for solving inequalities; Relating inequalities to equations; Logical connectives; Solutions of inequalities; and Interpretation of the inequality symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>How would you describe the solution of this system? Graph this system and describe the solution of this system. Are the following coordinates solutions of this system? Explain your reasoning:</td>
<td></td>
</tr>
<tr>
<td>$(-1, -2)$   $(-3, 8)$</td>
<td></td>
</tr>
<tr>
<td>$(0, 0.5)$   $(5, 2)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Q10. Solve the following inequality using the graphing technique $x^2 - x + 4 \geq 4x^2$</th>
<th>Strategies for solving inequalities; Relating inequalities to equations; Logical connectives; Solutions of inequalities; and Interpretation of the inequality symbol</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Q11. Why does the inequality sign switch when you multiply or divide both sides of an inequality by a negative number?</th>
<th>Domain; Relating inequalities to equations; Logical connectives; Solutions of inequalities; and Interpretation of the inequality symbol</th>
</tr>
</thead>
</table>
The preservice teachers were asked about certain important features of each question before being asked to solve the questions (Booth, 1984; Newman, 1983). For example, the preservice teachers were asked to describe the set S in their own words for question #3. The researcher reminded the preservice teachers to vocalize their thought process as they completed the questions. The purpose of these pre-student teaching interviews was to determine the preservice teacher’s learner-knowledge regarding inequalities (Prestage & Perk, 1999b). The pre-student teaching interview provided a wealth of knowledge regarding the nature of the preservice teachers’ understanding of inequalities prior to their student teaching field experience.

Lesson Plans

The preservice teachers were asked to submit a copy of the lesson plans that they developed. A lesson plan was submitted by the preservice teachers for every classroom observation. A table was created for each preservice teacher outlining the date and mathematical content of each classroom observation (see Table 4, Table 5, Table 6, Table 7, and Table 8). The content of the lesson plans was inequalities (i.e., systems of linear inequalities, linear programming, and quadratic inequalities) or topics relevant to inequalities (i.e., systems of equations, and quadratic equations). The format of the lesson plan was left to the preservice teacher, cooperating teacher, and university supervisor to determine. As a result, there was variability among the preservice teachers with regard to the level of detail. However, withholding from the preservice teachers what the researcher may be looking for in a lesson plan provided the most natural glimpse of the preservice teachers planning. The
preservice teachers submitted their lesson plans through various methods. Some preservice teachers submitted the plans electronically via email, while others submitted photocopies of the lessons. Unfortunately, not all of the lesson plans were submitted to the researcher prior to classroom observations and those that were submitted prior to the observation were handed to the researcher immediately preceding an observation. As such, a careful review of the lessons plans could not be conducted prior to an observation.

The lesson plans were a source of data used to address the second research question. The lesson plans were used to help determine the extent to which the preservice teacher planned to present material, which included examples used in class, questions they planned to ask, representations they intended to use, etc. In addition, the lesson plans provided a glimpse into what was predicted and planned for, and what was spontaneous in the implementation of a lesson itself.

**Classroom Observations**

The classroom observations and the notes taken during those observations by the researcher were data sources for the second research question. The classroom observations allowed the researcher to take note of evidence as to how the preservice teachers used their understanding of inequalities while implementing a lesson. Detailed classroom observation notes, with the conceptual framework as a guide, were taken during each classroom observation. The classroom observation notes were used to record evidence of what was seen or heard in the classroom rather than an interpretation or evaluation of the methods employed. A benefit of the classroom observation notes was the ability to document events that may occur outside of the view of the camera.
Video recordings of the classroom observations were made, with the files being stored on the researcher’s personal computer. The preservice teachers were not wearing microphones during the observations. Audio was captured by the stationary video camera only. Permission to collect these video recordings was received from the Institutional Review Board (#1816), the local school district, the cooperating teacher, the parents/guardians of the students in the class, and the preservice teacher. During the classroom observations, the video camera was operated by the researcher with the exception of one classroom observation. Vanessa requested that the researcher not attend a class in which she was being observed by her university supervisor (Classroom observation; February 28, 2011). Vanessa agreed to allow the researcher to set-up and record the lesson with a static video camera. As a result, the classroom observation notes for that lesson were based solely on what was seen and heard on the video recording.

It would be naïve to believe that the researcher observed the classes without causing some changes. However, the researcher did not participate in any of the lessons in an effort to minimize the effect of being in the classroom. In addition, the cameras were in the classes a few days before the start of the lessons regarding inequalities. This was done to acclimatize both the preservice teacher and the students to its presence, thereby reducing the Hawthorne effect (Jones, 1992). The video recordings were made to supplement classroom observation notes.

There were a total of 32 classroom observations conducted as part of this study: Angela was observed five times; Christina was observed six times; Crystal was observed seven times; Heather was observed six times; and Vanessa was observed eight times. For
each of the preservice teachers, the classroom observations occurred in the same class. This was done to maintain some consistency and to minimize the preservice teachers’ burden from participating in this study.

After most classroom observations, the researcher was able to ask the preservice teachers questions if necessary. The questions generally pertained to clarification of situations, student-teacher interactions (especially those that occurred as the preservice teacher was walking around the room), and adjustments to future lessons. In depth questions regarding lessons were not asked in an attempt to minimize the potential effects those questions may have on preservice teachers’ future lessons. On a few occasions, the researcher was able to observe a cooperating teacher and a preservice teacher discussing aspects of an observed lesson or future lessons. The researcher made notes about those interactions between the cooperating teacher and the preservice teacher. Those notes were included in classroom observation notes for that day.

**Post-Student Teaching Interview**

The post-student teaching interview (see Appendix D), conducted near the end of the preservice teachers’ student teaching field experience (see Table 9 for dates of each interview), was a data source used to address the second research question. The location of this interview was at the university for most of the preservice teachers. One preservice teacher, Christina, requested that her post-student teaching interview take place at her assigned high school. The site of the post-student teaching interview was selected based on the availability of a private space and convenience for the preservice teachers.
The preservice teachers were asked a set of questions for each content area related to inequalities (i.e., systems of inequalities and/or quadratic inequalities) that they taught during the student teaching field experience (see Figure 4 and Figure 5). These questions were designed to provide the researcher with background information about the lessons and insight into the preservice teachers’ opinions and beliefs regarding systems of linear inequalities and/or quadratic inequalities.

4. With regard to the lessons on inequalities and systems of inequalities, did you collaborate with anyone in the development of those lessons? If so, describe the collaboration to me? How much control do you feel that you had over the development of those lessons? What resources did you use in the development of those lessons (e.g., websites, textbook(s), books)?

5. Considering your lessons on inequalities and systems of inequalities, what were some of the big ideas that you wanted your students to take from those lessons?

6. What challenges were inherent in teaching this mathematical idea? What specific challenges were presented by this class as you taught this mathematical idea? How was your instruction designed to meet these challenges?

Figure 4. Questions on the post-student teaching interview addressed to the preservice teachers who taught systems of linear inequalities.
9. With regard to the lessons on quadratic expressions, equations, and inequalities, did you collaborate with anyone in the development of those lessons? If so, describe the collaboration to me? How much control do you feel that you had over the development of those lessons? What resources did you use in the development of those lessons (e.g., websites, textbook(s), books)?

10. Considering your lessons on quadratic equations and inequalities, what were some of the big ideas that you wanted your students to take from those lessons?

11. What challenges were inherent in teaching this mathematical idea? What specific challenges were presented by this class as you taught this mathematical idea? How was your instruction designed to meet these challenges?

Figure 5. Questions on the post-student teaching interview addressed to the preservice teachers who taught quadratic inequalities.

The rest of the questions on the post-student teaching interview employed stimulated recall (Bloom, 1954). Bloom (1954) was among the first to use stimulated recall. The use of numerous cues or stimuli enabled participants to recall the original moment with greater clarity (Bloom, 1954). Bloom’s (1954) study used audio recordings as the cues for the participants. The stimulated recall method has been employed by other researchers (Dempsey, 2010; O’Brien, 1993) and Schepens, Aelterman, and van Keer (2007) used it when working with preservice teachers.

This study used episodes from the video recordings of the classroom observations as the cues for stimulated recall in post-student teaching interviews. The episodes were selected after an initial analysis of classroom observations. The episodes contained moments that were categorized as explaining a mathematical idea and/or solving a problem based on the
conceptual framework of this study (see Table 2). Some of the episodes captured moments where preservice teachers made decisions that were planned or spontaneous. These episodes were shown to the preservice teachers in an effort to discover their thought process in the moment and rationale behind their decisions. In addition, the preservice teachers were encouraged to elaborate on changes they would make to their explanations with the benefit of hindsight.

In the other episodes, a mistake or issue was exhibited by the preservice teacher or one of their students as they solved an inequality problem. During the viewing of the episode, the mistake or issue was not immediately identified by the researcher. This was done in an effort to determine if the preservice teacher could identify the mistake or issue. In most cases, the preservice teacher commented on the mistake or issue without input from the researcher. This allowed the researcher to ask follow-up questions based on the remarks made by the preservice teacher. On the few occasions when a preservice teacher did not identify the mistake or issue, the researcher asked questions that drew the preservice teacher’s attention to the mistake or issue.

The length of the episodes ranged from thirty seconds to ten minutes. The researcher displayed long episodes in an effort to situate the preservice teacher back into the moment. This was important since great lengths of time had passed between the occurrence of the episodes and viewing of them during the post-student teaching interviews. Additionally, during interviews, the researcher and preservice teachers stopped the episodes to talk about smaller segments within an episode. As a result, longer episodes were not played from start to finish without pauses for discussion. Tables outlining duration and content area (i.e.,
systems of linear inequalities or quadratic inequalities) of the selected episodes for each of
the preservice teachers have been included (see Table 12, Table 13, Table 14, Table 15, and
Table 16).

Table 12. Duration and content areas of the episodes from Angela's post-student teaching
interview.

<table>
<thead>
<tr>
<th>Episode</th>
<th>Duration</th>
<th>Content Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Episode 1</td>
<td>6:23</td>
<td>Systems of linear inequalities</td>
</tr>
<tr>
<td>Episode 2</td>
<td>1:44</td>
<td>Systems of linear inequalities</td>
</tr>
<tr>
<td>Episode 3</td>
<td>4:12</td>
<td>Systems of linear inequalities</td>
</tr>
<tr>
<td>Episode 4</td>
<td>5:48</td>
<td>Systems of linear inequalities</td>
</tr>
<tr>
<td>Episode 5</td>
<td>5:22</td>
<td>Systems of linear inequalities</td>
</tr>
</tbody>
</table>

Table 13. Duration and content areas of the episodes from Christina's post-student teaching
interview.

<table>
<thead>
<tr>
<th>Episode</th>
<th>Duration</th>
<th>Content Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Episode 1</td>
<td>9:58</td>
<td>Systems of linear inequalities</td>
</tr>
<tr>
<td>Episode 2</td>
<td>5:54</td>
<td>Systems of linear inequalities</td>
</tr>
</tbody>
</table>

Table 14. Duration and content areas of the episodes from Crystal's post-student teaching
interview.

<table>
<thead>
<tr>
<th>Episode</th>
<th>Duration</th>
<th>Content Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Episode 1</td>
<td>0:28</td>
<td>Systems of linear inequalities</td>
</tr>
<tr>
<td>Episode 2</td>
<td>2:41</td>
<td>Systems of linear inequalities</td>
</tr>
<tr>
<td>Episode 3</td>
<td>2:23</td>
<td>Systems of linear inequalities</td>
</tr>
<tr>
<td>Episode 4</td>
<td>2:27</td>
<td>Systems of linear inequalities</td>
</tr>
</tbody>
</table>
Table 14 Continued.

<table>
<thead>
<tr>
<th>Episode 5</th>
<th>Duration</th>
<th>Content Area</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>9:51</td>
<td>Quadratic inequalities</td>
</tr>
</tbody>
</table>

Table 15. Duration and content areas of the episodes from Heather's post-student teaching interview.

<table>
<thead>
<tr>
<th>Episode</th>
<th>Duration</th>
<th>Content Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Episode 1</td>
<td>3:08</td>
<td>Quadratic inequalities</td>
</tr>
<tr>
<td>Episode 2a</td>
<td>9:55</td>
<td>Quadratic inequalities</td>
</tr>
<tr>
<td>Episode 2b</td>
<td>9:12</td>
<td>Quadratic inequalities</td>
</tr>
<tr>
<td>Episode 3</td>
<td>8:49</td>
<td>Quadratic inequalities</td>
</tr>
</tbody>
</table>

Table 16. Duration and content areas of the episodes from Vanessa's post-student teaching interview.

<table>
<thead>
<tr>
<th>Episode</th>
<th>Duration</th>
<th>Content Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Episode 1</td>
<td>6:18</td>
<td>Systems of linear inequalities</td>
</tr>
<tr>
<td>Episode 2</td>
<td>3:19</td>
<td>Systems of linear inequalities</td>
</tr>
<tr>
<td>Episode 3</td>
<td>7:19</td>
<td>Quadratic inequalities</td>
</tr>
<tr>
<td>Episode 4</td>
<td>6:05</td>
<td>Quadratic inequalities</td>
</tr>
<tr>
<td>Episode 5</td>
<td>7:30</td>
<td>Quadratic inequalities</td>
</tr>
</tbody>
</table>

Data Analysis

The analysis for this study was conducted in two phases. The phases corresponded to the two research questions of this study. As outlined above, the data sources for this study were the following: pre-student teaching interviews; lesson plans; video recordings of classroom observations; classroom observation notes; and post-student teaching interviews.
The pre-student teaching interview was the source of data for addressing the first research question. The sources of data for the second research question were the pre-student teaching interviews, lesson plans, video recordings of classroom observations, classroom observation notes, and post-student teaching interviews.

**Phase One**

Phase one of the data analysis was conducted in an effort to address the first question of this study: What is the nature of preservice teachers’ understanding of inequalities prior to the student teaching field experience? The analysis in phase one involved scrutinizing the preservice teachers’ responses to tasks on the pre-student teaching interview. The interviews were transcribed verbatim. The transcribed interviews were analyzed using the qualitative software Transana 2.42. As noted earlier, the questions on the pre-student teaching interview were constructed with the row headings of the conceptual framework as a guide (see Table 2). The questions were selected in order to ascertain the researcher’s interpretation of the preservice teachers’ understanding with regard to conceptions discovered in a review of literature.

The researcher started the analysis by sorting the questions based on commonalities. The groups formed by the sorting were not intended to be disjoint, however each question was assigned to a primary group. A table identifying the groups and the questions associated with each group has been included (see Table 17). These commonalities made the analysis easier because the researcher was able to discern patterns in the preservice teachers’ responses. In addition, grouping the questions based on these commonalities allowed the researcher to maintain context in the subsequent coding.
Table 17. Sorting of questions from pre-student teaching interview by commonalities.

<table>
<thead>
<tr>
<th>Groups</th>
<th>Questions from pre-student teaching interview</th>
</tr>
</thead>
<tbody>
<tr>
<td>Considering inequalities as equations</td>
<td>1, 2, other questions based on occurrences</td>
</tr>
<tr>
<td>Attending to negative values</td>
<td>4, 5, 6, other questions based on occurrences</td>
</tr>
<tr>
<td>Inequalities must have inequalities as solutions</td>
<td>3, 7</td>
</tr>
<tr>
<td>Quadratic inequalities</td>
<td>8a, 8b, 10</td>
</tr>
<tr>
<td>Absolute value inequalities</td>
<td>8c</td>
</tr>
<tr>
<td>Systems of inequalities</td>
<td>9</td>
</tr>
<tr>
<td>Explaining why the sense of an inequality changes</td>
<td>11, other questions based on occurrences</td>
</tr>
</tbody>
</table>

Preservice teachers’ pre-student teaching transcribed interviews were uploaded and synced to video recordings of their interviews in Transana 2.42. The researcher created collections\textsuperscript{12}, within Transana 2.42, for each group: considering inequalities as equations; attending to negative values; inequalities must have inequalities as solutions; quadratic inequalities; absolute value inequalities; systems of inequalities; and explaining why the sense of an inequality changes. Within each collection were clips of the preservice teachers’ work on the associated questions from the pre-student teaching interview. Quick clips were created and identified when shorter instances within the clips were relevant to other aspects.

\textsuperscript{12} The terminology collection, clips, and quick clips are used to maintain consistency with the terminology from the Transana software.
Once all of the clips and quick clips were created, the researcher analyzed the clips within each group for each preservice teacher. The analysis was conducted in this fashion because the cases for this phase of the study, addressing the first research question, were defined to be the five preservice teachers. The researcher analyzed all of the aspects for a preservice teacher and characterized the case of that particular preservice teacher before proceeding to the next preservice teacher.

The analyses of the understanding displayed by each preservice teacher in this study were conducted in an effort to address the first question of this study: What is the nature of preservice teachers’ understanding of inequalities prior to the student teaching field experience? The findings of those analyses will be presented in Chapter 4.

**Phase Two**

Based on findings from a literature review and an analysis of the researcher’s interpretation of the understanding of inequalities displayed by the preservice teachers from this study (see Chapter 4), four primary aspects of knowledge associated with the mathematical concept of inequalities emerged as focal points for the analysis of preservice teachers’ lessons in their classrooms relative to inequalities. Those four primary aspects of knowledge associated with inequalities were the following: 1) Strategies for solving inequalities; 2) Relating inequalities as equations; 3) Shading as a process; and 4) Solutions of inequalities. Within those primary aspects there were other aspects that may be pertinent to more than one primary aspect: domain; use of variable(s); factors of products and/or quotients; logical connectives; and interpretations of the inequality symbol.
The teaching practices of the conceptual framework (see Table 2) were adapted from the tasks of teaching from the framework presented by Ferrini-Mundy et al. (2005). Explaining a mathematical idea, solving a mathematical problem, and using technology were the teaching practices. Using these teaching practice categories in the conceptual framework allowed for a tighter focus.

The conceptual framework for this study had two overarching views: operational view of inequalities and relational view of inequalities. These views were a third dimension of the matrix design. Each cell in the framework represents an event in which a teaching practice (rows) and an aspect of knowledge associated with the mathematical concept of inequalities (columns) occurred. Within each cell of the matrix, displayed data will be classified as an operational view of inequalities or relational view of inequalities.

Some of the events within the classroom observations were classified based on the preservice teachers’ intent. If a preservice teacher shaded a half-plane of a graph as part of a graphing solution strategy, then based on the framework that event would be classified as “solving mathematical problems” and “shading as a process.” However, if a preservice teacher shaded a half-plane of a graph in order to illustrate the shade above or shade below method of shading, then that event would be classified as “explaining a mathematical idea” and “shading as a process.”

While observing lessons, events occurred that fell outside of the focus of this final version of the conceptual framework. The purpose of the framework was not to catch and identify everything that occurred. The intent of the framework was to focus on those events that were most relevant to the teaching of inequalities. Additionally, the three teaching
practices (i.e., explaining mathematical ideas, solving mathematical problems, and using technology) were not considered as disjoint categories. The three categories could, and often did, occur in an overlapping fashion with one another. By using the four primary aspects of knowledge associated with inequalities and the three teaching practices as a lens for making sense of the classroom observations, the researcher was able to identify and examine how preservice teachers approached various issues related to systems of linear inequalities and/or quadratic inequalities within their classrooms.

Each cell of the conceptual framework corresponds to the intersection of a teaching practice and an aspect of knowledge associated with inequalities (see Table 2). Thus, considering the first cell, there may be an event in which a preservice teacher worked at ‘explaining mathematical ideas” as it applied to ‘strategies for solve inequalities.’ This cell was indicative of a situation such as when a preservice teacher introduced or explained features related to the various strategies to solve particular inequalities in her classroom.

The omission of a cell, in the planning or implementation of lessons, did not necessarily imply a lack of understanding or an oversight. For example, the aspect “shading as a process” may not be a part of an algebraic manipulations strategy employed when solving quadratic inequalities with one variable. Therefore, a preservice teacher who only implemented an algebraic manipulations strategy would not have any episodes assigned to either cell associated with “shading as a process.” Conversely, a single cell may occur repeatedly in a brief episode or over several lessons.

Phase two of the analysis was conducted in an effort to address the second research question of this study: How do preservice teachers use their understanding of inequalities
while planning and implementing lessons during student teaching? Two cases emerged during data collection when it became clear that there were two different topics in inequalities that were being taught that may draw upon different content knowledge by the preservice teachers: systems of linear inequalities and quadratic inequalities. As a result, the two cases were analyzed separately. The procedures implemented during the analysis of each case were the same (see Table 18).

Table 18. Outline of the procedures conducted in each content area and the data source used for each procedure.

<table>
<thead>
<tr>
<th>Systems of Linear Inequalities</th>
<th>Quadratic Inequalities</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Procedure</strong></td>
<td><strong>Data Source</strong></td>
</tr>
<tr>
<td>Coding using teaching practices</td>
<td>Classroom observations synced with notes</td>
</tr>
<tr>
<td>Coding using aspects of knowledge associated with inequalities</td>
<td>Classroom observations synced with notes</td>
</tr>
<tr>
<td>Examine each aspect of knowledge associated with inequalities individually for commonalities</td>
<td>Clips in Transana from Classroom observations</td>
</tr>
<tr>
<td>Review lesson plans for evidence of planning or forethought</td>
<td>Lesson plans</td>
</tr>
<tr>
<td>Code post-student teaching interview based on content area and aspects</td>
<td>Post-student teaching interview</td>
</tr>
</tbody>
</table>
Table 18 Continued.

<table>
<thead>
<tr>
<th>Examine post-student teaching interview for supporting evidence</th>
<th>Post-student teaching interview</th>
<th>Examine post-student teaching interview for supporting evidence</th>
<th>Post-student teaching interview</th>
</tr>
</thead>
<tbody>
<tr>
<td>Determine alignment or misalignment for each aspect of knowledge associated with inequalities</td>
<td>Pre-student teaching interview</td>
<td>Determine alignment or misalignment for each aspect of knowledge associated with inequalities</td>
<td>Pre-student teaching interview</td>
</tr>
</tbody>
</table>

The analysis of the two cases began by identifying the classroom observations in which systems of linear inequalities or quadratic inequalities were discussed (see Table 9 for the classroom observation dates for each preservice teacher). There were four preservice teachers (Angela, Christina, Crystal, and Vanessa) who taught lessons about systems of linear inequalities. Lessons pertaining to quadratic inequalities were taught by three preservice teachers in this study (Crystal, Heather, and Vanessa)\(^\text{13}\).

The identified classroom observation videos and notes were uploaded and synced in Transana 2.42. The segments of the videos\(^\text{14}\) relevant to systems of linear inequalities or quadratic inequalities were initially coded using the teaching practices from the conceptual framework (see Table 2): explaining a mathematical idea, solving a mathematical problem, or using technology. After a classroom observation involving systems of linear inequalities was coded with the teaching practices, the researcher coded segments of that classroom observation again. This round of coding utilized the primary aspects of knowledge associated

\(^\text{13}\) Vanessa and Crystal taught lessons in both content areas.

\(^\text{14}\) A classroom observation may have included moments outside the scope of this study. These moment outside the scope of the study included dispersing and collecting paperwork, warm-up problems with no connection to inequalities (e.g., applying exponent rules), classroom management incidents, etc. These moments were not coded.
with inequalities from the conceptual framework (see Table 2): strategies for solving inequalities; relating inequalities to equations; shading as a process; and solution(s) of inequalities. This process of coding was repeated for all relevant classroom observations.

Initially the videos of classroom observation were time stamped based on major incidents. However, during the second round of coding the researcher transcribed certain segments of the classroom observations. The segments were transcribed because coding for aspects of knowledge associated with inequalities required a finer grain than coding for teaching practices. The grain of analysis was typically set at the sentence level. Due to the nature of conversation in classroom observations, which involved improper English as well as multiple people talking with instances of cross talk, the grain of analysis was adjusted accordingly. Two examples of the coding of classroom observations have been included (see Figure 6 and Figure 7). The coding of all of the classroom observations has been included (see Appendix E).

![Figure 6. Coding of Vanessa's classroom observation (February 28, 2011).](image-url)
Once the coding of the classroom observations was completed for each of the content areas (systems of linear inequalities and quadratic inequalities), the researcher coded preservice teachers’ post-student teaching interviews. The videos recordings and transcripts of the interviews were uploaded and synced in Transana 2.42. The researcher separated sections of each interview based on the content area (systems of linear inequalities or quadratic inequalities) being discussed. Next, the researcher created collections for each preservice teacher and each content area present in the interview. For example, “Vanessa-Systems” and “Vanessa-Quad” were the names of the two collections associated with Vanessa’s post-student teaching interview. If one of the content areas was not discussed in the post-student teaching interview for one of the preservice teachers, then a collection was not made for that content area with regard to that preservice teacher.

Clips from the post-student teaching interview were created and placed into the appropriate collection. For example, the collection “Vanessa-Systems” contained the following clips: collaboration; use of GC\textsuperscript{15}; big ideas; challenges; addressing challenges; surprises; Episode 1; and Episode 2. These titles were designed to correspond to questions asked during Vanessa’s post-student teaching interview.

\textsuperscript{15} Referring to use of graphing calculator or other technology
Once the classroom observations and post-student teaching interviews were coded, an analysis was conducted. The structure of the analysis was based on the aspects of knowledge associated with inequalities. Within the analysis of each of the aspects, the researcher coordinated evidence from the classroom observations with the corresponding lesson plans. The information from the post-student teaching interviews was used to supplement this analysis. In these interviews, preservice teachers were allowed to elaborate on their thought processes and intent while watching videos of the classroom observations. At the end of the analysis of the teaching practices within each of the primary aspects of knowledge associated with inequalities, attention was given to evaluating whether the classroom actions displayed by the preservice teachers could be classified as an operational view or relational view of inequalities.

The analyses of the two cases required careful integration of multiple data sources. The primary source of data for these analyses was classroom observations, both notes and video recordings. The lesson plans and post-student teaching interviews provided supplementary information. The pre-student teaching interviews were used to determine alignment between classroom actions and displayed understanding. The analyses of the two cases by content areas were conducted in an effort to address the second research question of this study: How do preservice teachers use their understanding of inequalities while planning and implementing lessons during the student teaching field experience? The finding of the analysis of systems of linear inequalities will be presented in Chapter 5. Additionally, the findings of the analysis of quadratic inequalities will be presented in Chapter 6.
Research Validity and Reliability

This study implemented triangulation of sources (Patton, 2002). There were three sources of data: interviews, observations, and document analysis of lesson plans. Through triangulation of data sources, the researcher sought to strengthen the validity and reliability of the study (Merriman, 2002). Patton (2002) noted that triangulation does not imply that the different data sources produce the same result. Triangulation is used as a consistency test.

One of the methods that the researcher used to insure validity was member checks of the interview transcripts. However, the researcher did not provide participants with access to any of the analysis until the publication of this study. Morse (1998) warned against using member checks with analyzed text. She noted that research “reveals nuances, paradoxes, and intricacies that may not be evident to the participants themselves” (Morse, 1998, p. 444).

The researcher created an audit trail that described in detail how data was collected, how analysis codes were created, and how decisions were made (Merriam, 2002). While there is some debate as to the appropriateness of audit trails as a method to ensure research validity and reliability, the researcher erred on the side of taking too many steps (Morse, Barrett, Mayan, Olson, & Spiers, 2002). The audit trail was maintained as an electronic document. As with any log or journal, regular entries were made to prevent gaps in memory on the part of the researcher.

Subjectivity Statement

I am a thirty-seven year old male. I earned my Bachelor of Science degree from Fayetteville State University in Mathematics Education Grades 7 – 12. I have been critical of the preparation that I received from my Alma Mater. My exposure to education specifically
relevant to mathematics was limited to one methods course taught during the six weeks prior to my student teaching experience. The majority of my undergraduate coursework was in mathematics or general education courses, with no attention to teaching methods specific to mathematics.

Upon completing my undergraduate work, I was accepted into North Carolina State University’s mathematics graduate program and received a stipend to be a teaching assistant. I appreciated the deeper understanding of the connective underpinnings of mathematics that I learned in my graduate courses. I found more enjoyment and satisfaction in teaching lower level undergraduate classes than from any of the mathematics courses that I was taking.

After I earned my Master of Science degree from North Carolina State University in Mathematics, I was hired by Louisburg College as a full-time mathematics instructor. Louisburg College had a student population who arrived with some deficiencies in their academic preparation, which warranted the need for remedial coursework. While Louisburg College is not a high school, I would argue that the dynamics of some of my classes were equivalent to high school classes.

Over the seven years I spent at Louisburg College, I was constantly reminded that the teacher has the biggest impact on student learning. However, as a student of direct instruction and a teacher using the standard lecture format, I began to question the effectiveness of this algorithmic approach. I had students that displayed comprehension of topics one semester but showed no comprehension on the same topic the next semester. Were my students getting a deep understanding of mathematics from my lessons or were they just getting by?
As a way to answer these questions, I went back to North Carolina State University to earn a Ph.D. in Mathematics Education. Through my coursework, I have learned that my questions have been the centerpiece of a debate that has gone on for over thirty years. Therefore, I entered this research project as a student who is sorting out his position on the effectiveness of direct instruction. My mathematics coursework leads me to advocate for pedagogical content knowledge in preservice programs. My experiences have shown me the importance of content knowledge. Thus, I believe mathematics preservice programs must present a balance between content and pedagogy.

**Ethical Issues**

Institutional Review Board and local education agency permission were obtained for the protection of the preservice teachers, cooperating teachers, university supervisors, and students in the observed classes. The preservice teachers involved in this study were given an Informed Consent Form (see Appendix A) that outlined their involvement in the study.

The university supervisors were informed that their preservice teacher(s) had agreed to participate in the study. The cooperating teacher and the university supervisor did not have access to video recordings, classroom observation notes, or interview transcripts. This was done to protect the preservice teachers and to foster open dialogue without the fear of reprisal.

Preservice teachers were informed about the recordings from the onset of the study. In addition, this information was included in the Informed Consent Form (see Appendix A) that the participants signed prior to any data collection. As per local/state regulations, forms requesting permission to participate in video recordings were sent to the parents/guardians of
the students within the classes that were studied (see Appendix B). In all cases, the participants were informed that involvement in the study was voluntary and they were free to withdraw from the study at any time.

Chapter Summary

In an effort to address an area in need of further examination, this study seeks to add to the knowledge about preservice mathematics teachers. Specifically, the researcher examined the nature of understanding about inequalities held by preservice teachers and how they apply that understanding in the planning and implementation of lessons. These preservice teachers were all from one major university in the Southeast United States. Five preservice teachers, who had similar exposure to educational methods classes, participated in this study.

This chapter outlined the overall research design, conceptual framework, and data sources used in this study: pre-student teaching interview, classroom observations, classroom observation notes, lesson plans created by the preservice teachers for each classroom observation, and post-student teaching interview. A detailed description of the participants and their classes was included. In addition, the data analysis process and steps that were taken by the researcher to ensure reliability and validity were explained. Finally, the biases held by the researcher and ethical concerns of this study were declared.

The following chapter will divulge the findings of the analyses conducted in order to address the first research question: What is the nature of preservice teachers understanding of inequalities prior to the student teaching field experience?
CHAPTER 4: UNDERSTANDING REGARDING INEQUALITIES

The focus of this chapter is to address the first research question of this study: What is the nature of preservice teachers’ understanding of inequalities prior to the student teaching field experience? For this question, there are five cases; each case was defined to be the individual preservice teachers as the purpose was to examine an individual’s understanding in depth, and to then be able to identify cross-case themes. The source of data for analysis was the preservice teachers’ videos, transcripts, and written responses from the pre-student teaching interviews (Appendix C).

This chapter begins with a description of questions from the pre-student teaching interview and how those questions assess content areas related to understanding inequalities. A discussion of each case relative to the group of aspects of knowledge associated with inequalities will follow. Cross-case themes that emerged during analysis will then be presented. Finally, a summary of the nature of preservice teachers’ understanding of inequalities prior to the student teaching field experience will be stated.

Aspects of Knowledge Associated with Inequalities

Based on findings from a literature review, a series of questions were presented to preservice teachers as part of a task-based interview. This interview was conducted before the preservice teachers began their student teaching field experience on a full-time basis. The researcher started the analysis by sorting the questions based on commonalities. The groups formed by sorting were not intended to be disjoint, however each question was assigned to a primary group. A table identifying the groups and the questions associated with each group has been included (see Table 17). These commonalities made analysis easier because the
The primary focus of questions #1 (see Figure 8) and #2 (see Figure 9) was considering inequalities as equations. Questions #4 (see Figure 11), #5 (see Figure 12), and #6 (see Figure 13) were used to examine preservice teachers’ ability to attend to negative values when working with inequalities. Preservice teachers’ beliefs about inequalities and their solutions were explored through questions #3 (see Figure 10) and #7 (see Figure 14). Preservice teachers’ understanding of quadratic inequalities was investigated using questions #8a, #8b (see Figure 15), and #10 (see Figure 18); absolute value inequalities was the focus of question #8c (see Figure 16); and systems of inequalities was considered in question #9 (see Figure 17). Question #11 (see Figure 19) provided an opportunity to inspect preservice teachers’ understanding of why the sense of an inequality sign changes when the inequality is multiplied or divided by a negative value.

1. How would a student define ‘equation’ in a manner that indicated that they have a conceptual understanding of the work?

*Figure 8. Problem #1 from the pre-student teaching interview protocol.*

2. With the previous definition for equation in mind, which of the following are equations:

- a. \( x = x \)
- b. \( x = 2 \)
- c. \( 0 = 0 \)
- d. \( 0 = 1 \)
- e. \( V = \frac{4}{3}\pi r^3 \)
- f. \( y > 5x - 9 \)

*Figure 9. Problem #2 from the pre-student teaching interview protocol.*
3. Consider the set \( S = \{ x \in \mathbb{R} | x=3 \} \) and check the validity of the following statement:

\[ S \text{ can be the solution of both an equation and an inequality. Explain your answer.} \]

*Figure 10. Problem #3 from the pre-student teaching interview protocol.*

4. Are the following inequalities equivalent:

\[ 7 > ax \text{ and } x < \frac{7}{a} \quad \forall a \in \mathbb{R} \]

*Figure 11. Problem #4 from the pre-student teaching interview protocol.*

5. Are the following inequalities equivalent statements?

\[ ax < 5 \quad \text{and} \quad x < \frac{5}{a} \quad \forall a \in \mathbb{R}, a \neq 0 \]

*Figure 12. Problem #5 from the pre-student teaching interview protocol.*

6. Solve the inequality \((a - 5)x > 2a -1\), \(x\) being the variable and \(a\) the parameter.

*Figure 13. Problem #6 from the pre-student teaching interview protocol.*

7. Indicate the truth set (the solution) of \(5x^4 \leq 0\).

*Figure 14. Problem #7 from the pre-student teaching interview protocol.*

8. Solve the following inequalities

a. \((x - 1)(3x + 7) \geq 0\)

b. \(2x^2 + 6x - 17 < x - 13\)

*Figure 15. Problems #8a and #8b from the pre-student teaching interview protocol.*
8. Solve the following inequalities:
   c. \( 3 \leq \left| \frac{1}{4}x + 1 \right| \)

*Figure 16. Problem #8c from the pre-student teaching interview protocol.*

9. Given the system:
   \[
   \begin{align*}
   x + 4y &> 2 \\
   5x - 2y &\leq -1
   \end{align*}
   \]
   a. How would you describe the solution of this system?
   b. Graph this system and describe the solution of this system.
   c. Are the following coordinates solutions of this system? Explain your reasoning:
      \((-1, -2), (-3, 8), (0, .5), (5, 2)\)

*Figure 17. Problem #9 from the pre-student teaching interview protocol.*

10. Solve the following inequality using the graphing technique
    \(x^2 - x + 4 \geq 4x^2\)

*Figure 18. Problem #10 from the pre-student teaching interview protocol.*

11. Why does the inequality sign switch when you multiply or divide both sides of an inequality by a negative number?

*Figure 19. Problem #11 from the pre-student teaching interview protocol.*

**Angela**

At the time of this study, Angela was a preservice teacher who had earned a 4.0 in all of her coursework. She was in the final semester of completing the requirements for a Bachelors of Science in Mathematics and a Bachelors of Science in Mathematics Education. She believed that her mathematical knowledge was strong and based that belief on the fact that she had “earned at least an ‘A’ in all my math courses.” Several times during the
interview, Angela asked “is that what you are looking for?” In addition, after many of the questions, she asked “is that right?” The lack of feedback from the researcher as to the ‘correctness’ of her responses seemed to frustrate her.

**Considering inequalities as equations**

During her pre-student teaching interview, Angela indicated that she believed that inequalities were equations. When asked to define an equation (see Figure 8), she noted that an equation is “two expressions equal to one another.” Her initial definition seemed to reflect a relational view of equations (Knuth et al., 2006). When presented with a list of six statements (see Figure 9) and asked if the statements were equations or not, she immediately jumped to the inequality statement, #2f. She mentioned that she did not like her definition of an equation. In her opinion, the greater than sign was not a statement of being equal, but she still wanted to include inequalities within equations. In reconciling this conflict, Angela decided to change her definition of equations to “a comparison of two expressions to one another.” While she admitted that she was not fond of this new definition, it allowed her to include inequalities. After working through some of the other statements, Angela was directly asked if inequalities were equations. Angela replied that “they’re a type of equation.” Her statements imply that Angela had included inequalities in her concept image of equations. Based on the changes to her definition, Angela appeared to hold a relational view of inequalities (Knuth et al., 2006).

**Attending to negative values**

A set of three problems were presented to Angela in an effort to determine how well, if at all, she attended to the effects of multiplying or dividing by a negative value on
inequalities. For these problems, the possibility of a negative number was not explicit, unlike a problem similar to $-5x < 20$. Instead a parameter, $a$, was used.

For problem #4 (see Figure 11), Angela was asked if two given inequalities were equivalent, where the parameter $a$ was allowed to be any real number. Angela focused her attention on the fact that if $a$ was equal to zero, then the two inequalities would not be equivalent. At this point in the interview, she had not mentioned anything regarding negative numbers. When given #5 (see Figure 12) which is similar to #4 but excludes the possibility of $a = 0$, she initially said that the two inequalities were equivalent. But when asked if there were any other issues or situations, she noted that negative values of the parameter $a$ would cause you “to flip the sign” which would mean that the two given inequalities were not equivalent. After completing #5, she returned to #4 and said “the zero was just enough for me to say no [these inequalities are not equivalent].” In other words, because she had first attended to zero as a special case, there was not a need for her to continue to consider other cases, such as negative values.

From these statements it appeared that Angela understood the effects of multiplying or dividing by a negative value on inequalities, even if the negative value was not explicit. To solve #6 (see Figure 13), another problem with a parameter $a$, she divided both sides of the inequality by the quantity $(a - 5)$. In doing so, she attended to the possibility of dividing by zero in noting that “$a$ can’t be five, because you can’t divide by zero.” Her statement was correct, but she did not talk about the implications if $a < 5$, which would change the sense of the sign of the inequality.
There was a conflict in interpreting Angela’s ability to negotiate with the possible effects of multiplying or dividing by a negative value. For some of the problems (#4 and #5), Angela was able to show that she was aware of the effects that multiplying or dividing by a negative value has on the sense of an inequality symbol. However, in another problem (#6), she did not attend to the case when $a$ would be less than five.

It was unclear as to why this discrepancy existed, but some possible explanations may lie in the type of problems that Angela was asked. When asked to solve a problem involving an inequality, Angela did not attend to the effects when $a < 5$. This may be related to the inclusion of inequalities in her concept image of equations. When asked to solve, Angela may not differentiate her process for inequalities or equations. On the other hand, when given problems that ask if inequalities are equivalent, Angela may call on knowledge that is specific to inequalities. This specific knowledge may be what allows her to attend to the effects that negative values have on an inequality sign when multiplication or division is involved.

One other possible avenue to consider is the presentation of the parameter itself. In problems #4 and #5, the parameter is by itself in the denominator or being multiplied by the variable. However, in problem #6, the parameter appears in two places and it has values being subtracted from it. These differences may account for inconsistencies in Angela’s ability to deal with the effects of multiplying or dividing an inequality by a negative value.

**Inequalities must have inequalities as solutions**

As Angela examined the statement in #3 (see Figure 10), she believed it to be true for both equations and inequalities. When asked to provide examples of an equation and an
inequality that would make the statement true, she had little difficulty providing an example of an equation: $x + 2 = 5$. As for an inequality, she converted her equation into an inequality (i.e., $x + 2 > 5$) but Angela quickly realized that the solution to this inequality was “$x$ is greater than three, which is not the same thing as $x$ equals three.” Matter of fact, she noted that her solution did not include three. After considering the inequality $x + 2 \geq 5$ and realizing that the solution to that inequality would include values other than three, Angela changed her opinion of the truth value of the statement. She noted that the inequalities cannot have a single value as a solution. She took this belief a step further in the following exchange:

Researcher: Is there a way to do it [referring to creating an inequality] to where I can get it [the solution] to where it’s exclusively three?

Angela: No. It would have to be equal to be exclusive.

Angela seemed to believe that only equations can have a single value as a solution. She easily provided an example to support her belief with respect to equations. Angela understood that her examples of inequalities would not provide a single value solution. Her statements made it clear that she knew that $x \geq 3$ would include values other than three.

Angela’s attempts to create an example for inequalities focused on first degree single variable inequalities. Her choice may point to a connection with first degree single variable equations and the fact that they have one solution. By making the change from an equal sign to an inequality sign, she may have believed that the number of solutions would still hold true.
As Angela started to consider #7 (see Figure 14), she was fixated on the idea that the problem “wants to know when $5x^4$ is negative.” She noted that any value raised to the fourth power will be positive. The researcher asked her to consider zero. Angela correctly stated that zero will satisfy the inequality. It appeared that the researcher jumped ahead of Angela’s thought process because she said “I hadn’t gotten that far yet.” She went back and reaffirmed her earlier statement by noting that $5x^4$ can never be negative. After which, she wrote the correct solution set for the problem.

After writing the solution to #7, Angela said “but I told you that you couldn’t have an inequality with…just one [solution].” As she revisited #3, she was unsure if the value of the solution (i.e., $x = 0$ vs. $x = 3$) played a role in whether the statement in #3, with regard to inequalities, was true. She was willing to accept the truth value of the statement if $x = 0$, pointing to #7 as justification. However, she never produced an inequality with a solution of $x = 3$.

As Angela tried to create an inequality that would satisfy #7, she exhibited behavior discussed within the literature. Tsamir and Almog (2001) noted that some of the students in their study formed meaningless connections to quadratic roots. In trying to find an example of an inequality where $x = 3$ is the only solution, Angela tried $5x^4 \leq 5$, which she simplified to $x^4 \leq 1$. After taking the fourth root of both sides, she said that the answer was $x \leq \pm 1$. However, she quickly noted: “I mean that don’t make sense.” Angela started to exhibit the same flaw as the students in the study conducted by Tsamir and Almog (2001). However, Angela quickly realized that this statement, $x \leq \pm 1$, made no sense and she discarded it.
As Angela discussed how students would answer #7, she made a statement that may be of interest:

Researcher: …what’s the answer?
Angela: Well, this is what they’re going to think the answer is (pointing to $x \leq 0$), by the way that they’re taught to solve inequalities.
Researcher: $x \leq 0$?
Angela: Yes.
Researcher: But what is the answer?
Angela: $x = 0$.

Later as she is revisiting #3, Angela says “I feel like the way that I’ve seen them teach them to solve inequalities, they’ve never have them switch the [inequality sign to an equal sign].” In earlier conversations, Angela never indicated that she observed a class in which inequalities were taught. So it was unclear whether she was basing these statements on experience or beliefs. Tsamir & Bazzini (2004) presented a flaw that was similar to the one raised by Angela: the notion that inequalities will have inequalities as solutions.

Unfortunately, without further examination it is difficult to determine if her statements speak to her beliefs or to her reasoning in the moment or to some experience that she did not mention prior to these questions.

### Quadratic Inequalities

Angela’s confidence seemed to be wavering as the interview progressed. By the time she reached the quadratic inequality problems, she had become more vocal in her doubts.

After completing #5 (see Figure 12), Angela asked: “Can you tell me the answer to this? This
is really bothering me.” Later after completing #7 (see Figure 14) and revisiting #3 (see Figure 10), Angela expressed some frustration.

    Researcher: …do you want to revise your answer to number three?
    Angela:  Yeah (said in a whisper)
    Researcher: You don’t sound confident in that.
    Angela: Well, I didn’t have a very good…I didn’t have a very good explanation for number three (said in a tone of frustration)

In addition, after reading the directions for question #8a (see Figure 15), Angela said “I kind of feel like I don’t know how to now.” These statements lead the researcher to adjust the questioning in an effort to minimize Angela’s level of self-doubt. However, as she encountered mathematical ideas that were causing a perturbation, she often self-initiated a reflection back to prior problems and reconsidered them, though she did not always know how to reconcile the perturbation.

    Angela treated the first quadratic inequality, #8a, in a similar manner as she would an equation (see Figure 20). This is evident from the following statement:

    Angela: …I was going to distribute but then I figured that it would be just how I would end up factoring it. So, if it were a regular equation [emphasis added] set equal to zero then you could take each of the factors and set it equal to zero. So, I took each of them [referring to the factors] and put them greater than or equal to zero.
Figure 20. Angela's first attempt to solve #8a.

It seemed as if she was trying to apply the zero-product property of equations to this particular inequality. She graphed both solutions \((x \geq 1 \text{ and } x \geq -\frac{7}{3})\) on the same number line, noting that they overlapped. She indicated that this was the answer and proceeded to pick \(x = -2\) to substitute into the inequality. She used this value to determine whether the region in between \(-\frac{7}{3}\) and 1 did in fact satisfy the inequality. She determined that the region did not satisfy the inequality. This contradiction to her original solution caused her to doubt her choice of a greater than or equal to sign in the inequality: \(3x + 7 \geq 0\). Instead she decided to make a change to an equation: \(3x + 7 = 0\). This change allowed her to keep the single value of \(x = -\frac{7}{3}\) and the region \(x \geq 1\).

After writing this new solution Angela stopped and plugged the expression on the left side of the original inequality into her calculator and graphed it (see Figure 21). With this graph from her graphing calculator, she realized that the expression on the left side was a quadratic expression and the graph was a parabola. Angela revised her strategy; setting both factors equal to zero and solving (see Figure 22). She called these two solutions the x-intercepts. This may speak to how central the graph was in her thought process and how the graph may have helped her reason with the symbolic representation (Kieran, 2004). From
this point, Angela employed a sign chart method to solve the inequality. She plotted the points \( x = -\frac{7}{3} \) and \( x = 1 \) on a number line as one would expect. In a move that was unexpected, she added \( x = 0 \) to the number line noting that “you should always consider zero.” It was unclear why zero should always be considered, and this was not followed up on by the researcher. The sign chart method as it is used in calculus may be the reason behind her desire to include zero. The inclusion of zero withstanding, Angela correctly implemented the sign chart method; arriving at the correct solution.

![Figure 21. Calculator screen shots as Angela reworks #8a.](image)

![Figure 22. Angela's second attempt to solve #8a.](image)

Angela had some difficulty coordinating between the two dimensional graph of the parabola (see Figure 21) that she used to complete the sign chart and the one dimensional
number line that represented her solution. This difficulty manifested itself as she tried to shade the two dimensional graph in a manner that agreed with her notion of being greater than zero and her solution represented by the shading on the number line (see Figure 22). It seemed that Angela was shading the region of the coordinate plane where the $y$-values were greater than zero. Then she seemed to consider the location of the parabola relative to the shading. Comments, such as “where it is above the $x$-axis” as she is pointing to the parabola (see Figure 21), gave the impression that Angela separated the parabola into two regions, above the $x$-axis and below the $x$-axis, which correspond to $y$-values greater than or less than zero, though she did not make this explicit.

Angela’s final answer flowed from her use of a sign chart method. She was able to successfully navigate her way through the problem once she started to employ a sign chart method. Her decision to use a sign chart method seemed to hinge on her reasoning with the graph (see Figure 21). The graph allowed her to connect the shape, a parabola, with a technique for solving quadratic equations: factoring and the zero product property of equations (Kieran, 2004). Her use of the term “$x$-intercepts” for the values she obtained by setting the factors equal to zero and solving gave the impression that she thought of the problem in terms of the graph. Her success with a sign chart method appeared to be aided by the absence of the variable $y$ within the method. She had difficulty displaying her reasoning; as was evident from the graph she crossed out (see the top of Figure 22).

With more confidence in herself, Angela manipulated the inequality in problem #8b (see Figure 15 for the original inequality), to move all the terms to the left side. After a short attempt at factoring the quadratic expression on the left side of the inequality, Angela
graphed the quadratic expression using a calculator (see Figure 23). Using the calculator’s ability to calculate the zeros of a graph, she found the x-intercepts.

Unlike the previous problem, Angela did not utilize a sign chart to determine the answer. She employed a graphing strategy. She examined the graph on calculator and noted that “it’s negative between these two values.” She then drew a number line with the correct shading. She elaborated that “when x is between those values the inequality holds.” Her number line was correct.

Angela was able to coordinate between the two-dimensional graph that she created on her graphing calculator (see Figure 23) and the single-variable inequality. It seemed that Angela took the structure provided by the sign chart to indicate positive and negative regions and mentally placed that structure on her graph. This move was aided by her use of algebraic manipulations to move the terms from one side leaving zero. It seemed that having zero on one side allowed her to use a tangible object, the x-axis, as a boundary when she examined the graph. It is unclear how Angela would have handled a problem where she was forced to use a constant other than zero or an expression with a variable on one side.
Figure 23. Calculator screen shots from Angela's work on #8b.

Absolute value inequalities

Angela made an error in how she solved the problem that involved an absolute value (see Figure 16 and Figure 24). Her error lies in how she mistakenly set up the problem. She treated the problem as one would if the inequality sign in the original problem was greater than or equal to (≥) instead of less than or equal to (≤). Barring this oversight, the algebraic manipulations approach that she implemented did yield a correct solution.
A point of interest can be seen in the different strategies Angela employed with this problem versus the strategies she used to solve the quadratic inequalities, especially #8a. For the absolute value inequality problem, she did not use her calculator. Matter of fact, she apologized for not using the calculator as she wrote her solution to the problem.

Angela: Sorry I don’t immediately jump to using the calculator like most students would.

Researcher: That’s fine. What are you comfortable with?

Angela: Well I like the calculator, sometimes I just feel like I can’t use it. … because I’m supposed to be doing it by hand.

Another interesting difference revolves around the fact that Angela did not check her solution as she had done with quadratic inequality problems. For #8a, her check involved explicitly picking values to substitute into the inequality. For #8b, her check focused on an examination of the graph on her calculator. In presenting her solution for #8c, she did not graph it on a number line nor did she use her calculator. Based on these factors and her outward appearance, it seemed as if Angela was very certain of her strategy and solution.
Due to time constraints and the social dynamics of the interview, the researcher decided not to delve deeper into this question.

**Systems of inequalities**

For #9 (see Figure 17), Angela initially employed an elimination method (see Figure 25). However, she quickly backed away from using an elimination method, noting “I’m trying to solve it like a system of equations.” Angela performed algebraic manipulations on each inequality in order to isolate the variable y (see Figure 26). She took these steps in order to graph the corresponding equations in her calculator. Angela recreated the calculator generated graph (see Figure 27) on her paper (see Figure 28).

![Figure 25. Angela's attempt to use the elimination method to solve #9.](image1)

![Figure 26. Angela's algebraic manipulations of #9.](image2)
Angela did not use test points to determine the shading for each inequality. Instead, she examined the inequality sign and determined which half-plane to shade.

Angela: When $y$ is greater than this line, $y$ is going to be greater than one half minus [writing one fourth of x]. So $y$ is got to be greater than [shading above the line with negative slope]. $y$ should be greater than or equal to this line [shading above the line with positive slope].

She realized that the graph of one of her lines should have been dashed instead of solid and she made an adjustment to her graph to avoid redrawing the graph. When asked to elaborate...
on the solution to this problem, Angela seemed to try to produce inequalities for $y$ and $x$ that would “define that region [referring to the overlapping shaded region].”

Angela was given a series of coordinates and asked if they were part of the solution or not. She was correct on three out of the four. Matter of fact, when asked about $(-2, 8)$, she noted “that point is in this shaded region [pointing to the overlapping shaded region in Figure 28].” She was unsure if the coordinate $(0, 0.5)$ was part of the solution or not. This point was the intersection of the two boundary lines. Angela correctly identified the point as part of the solution of $5x - 2y \leq -1$, but she did not examine the other inequality while making her determination. At first, the researcher had the opinion that this mistake may be attributable to her graph (see Figure 28). One of her lines should be dashed, a fact that she remembered after drawing the line. However, Angela drew two graphs (see Figure 29) as she tried to decide whether her first impression of the coordinate $(0, 0.5)$ was correct. From these drawings, Angela appeared to have conflicting views of the actual graph of the system. The coordinate $(0, 0.5)$ on the graph on the left is closed whereas on the right the point is open. It seemed that Angela overlooked the fact that a coordinate is part of the solution if it satisfies both of the inequalities. She seemed so transfixed on the shaded region, even trying to define it, that she forgot what the region represents.
Explaining why the sense of an inequality changes

The participants in this study were enrolled in methods courses that preservice teachers typically take immediately prior to their student teaching experience. The professors of these methods courses included discussions about changing the sense of an inequality sign. In one course, the preservice teachers presented several methods to explain why the sense of the inequality sign changes when you multiply or divide both sides of the inequality by a negative number.

When presented with #11 (see Figure 19) Angela mentioned that this was discussed in class a few days before her interview. However, she gave no indication as to whether she presented an explanation in class or not. She started the problem by saying, “I know the answer to this one.” She drew a number line (see Figure 30) and talked about “flipping” or “inverting” the number line. She mentioned that “when you’re doing the negative [multiplying or dividing], you’re essentially flipping the side of the…number line that you’re using.” It seemed as if Angela was referring to a mapping technique explanation that was discussed in the methods course. However, she was unable to clearly articulate a convincing argument for why the sign of the inequality changes.
Angela exhibited an ability to change the sense of the inequality sign if the inequality is multiplied or divided by a negative value. However, her explanation for why the inequality sign “flips” was not complete. Because there were multiple explanations presented in the methods course, Angela may have been confused as to the intricacies of each method.

**Characterizing Angela’s understanding**

Angela was a case of a preservice teacher with high achievement in both mathematics and mathematics education (GPA of 4.0) with an understanding of inequalities that focused on visual cues. She relied on visual cues to aid her as she solved several of the inequalities. These visual cues were primarily graphs, either ones on her graphing calculator or ones she created on her paper. Her graphing calculator was a tool that she used to graph expressions. In addition, she used her graphing calculator when determining shading of inequalities; finding intercepts; and finding intersection points.

With consideration to the framework of this study (see Table 2), Angela’s work will be discussed with respect to whether and when she may have been using a relational or operational approach associated with the mathematical concept of inequalities. She implemented strategies to solve inequalities in a manner that was consistent with both an
operational view and a relational view of inequalities. For example, the strategy that she employed to solve the absolute value inequality problem (see Figure 16) showed that she had a conception of an absolute value as it relates to inequalities. However, she made a mistake in how she set up the problem which yielded an incorrect answer.

Angela appeared to display an operational view of inequalities with regard to some of the aspects. With respect to relating inequalities to equations, she attempted to apply strategies utilized with equations while working with inequalities; an issue highlighted by Blanco & Garrote (2007). In many cases Angela did realize the error of trying to use these strategies, which seems to imply that she was able to reason through such a mistake. Angela displayed other issues which seemed to be coherent with an operational view of inequalities (e.g., identifying boundaries of solution regions, an inability to formally explain why the sense of an inequality changes, and inclusion of inequalities in her concept image of equations). However, Angela did at times display a relational view of inequalities. One example can be seen in her ability to coordinate between two dimensional graphs and single variable inequalities while solving #8b (see Figure 15).

Christina

At the time of this study, Christina was completing the final semester of requirements for a Bachelors of Science degree in Mathematics and a Bachelor of Science degree in Mathematics Education. She was excited about student teaching at a rural high school. She considered her mathematical content knowledge as strong. During the interview Christina laughed and made some jokes. Additionally, she didn’t seem to be tense, fidgety, or nervous; this leads the researcher to believe that she was comfortable during this interview.
Considering inequalities as equations

After pondering #1 (see Figure 8), Christina said the following with regard to her definition of an equation:

Christina: For an equation, the students would know that there’s an equal sign. There’s two sides to it so that normally in an equation they’re looking to solve for something. …The important part is the equal sign.

Christina’s definition seemed to contain elements that point an operational view of equations (Knuth et al., 2006). As Christina worked her way through the statements in #2 (see Figure 9), she initially wrote that #2f was an equation. As she discussed her reasoning for each statement, she made adjustments to her definition of an equation. When she talked about #2f, she called it an inequality. She said “…it’s not an equation… Because there’s not an equal sign.” At this point she changed her answer on the paper to say that #2f was not an equation.

A follow-up question that may have yielded useful information would have been to change the inequality sign, from greater than (>) to greater than or equal to (≥).

Unfortunately, that statement was not presented to Christina in this interview. However, an examination of how she talked about other problems in this interview, that had greater than or equal to (≥) or less than or equal to (≤) signs, did not produce an indication that she considered those problems to be equations.

Christina was not confident in her choice. This was evident from the following exchange:

Researcher: So you don’t think it’s an equation (pointing to #2f)?

Christina: Yeah. But I’m not sure.
This lack of confidence may speak to fluidity in her concept image of an equation. On the other hand, she repeatedly said that an equation must have an equal sign. The centrality of an equal sign in her concept image of an equation leads the researcher to believe she does not include inequalities in her definition.

**Attending to negative values**

Christina was consistent as she responded to #4 (see Figure 11), #5 (see Figure 12), and #6 (see Figure 13). She manipulated the inequalities in all three problems algebraically. She did not consider the possibility that values of the parameter $a$ may result in dividing or multiplying the inequalities by a negative value (see Figure 31, Figure 32, and Figure 33).

![Figure 31. Christina's algebraic manipulation of #4.](image)

![Figure 32. Christina's algebraic manipulation of #5.](image)
Figure 33. Christina's algebraic manipulation of #6.

It seemed as if Christina was concentrating on the process, namely the algebraic manipulations she performed as she “solves for x.” This causes her to overlook the implications of those actions, a pitfall seen in the literature (Steinberg, Sleeman, & Ktorza, 1990). With the problems that pertain to deciding the equivalency of statements, Christina attended to the effects of dividing by zero. However, Christina did not exclude $a = 5$ in #6. After performing algebraic manipulations, she stated: “And that is solved!” She made no mention of excluding values of the variable $a$. This oversight may be attributable to the presentation of the parameter itself. In problems #4 and #5, the parameter was by itself in the denominator or being multiplied by the variable. However, in problem #6, the parameter appears in two places and it has values being subtracted from it. These differences may account for inconsistencies in Christina’s ability to articulate the implications of dividing by zero. Unfortunately, these differences regarding the presentation of the parameter $a$ do not explain why Christina did not consider the effects of multiplying or dividing an inequality by a negative value. One possible explanation may lie in Christina’s thinking about what real numbers represent. It is possible that she did not include negative values in her interpretation of real numbers.
Inequalities must have inequalities as solutions

With little difficulty, Christina was able to produce an example of an equation that satisfied the constraint in problem #3 (see Figure 10 for the original problem). As she considered inequalities, Christina displayed behavior similar to students in a study conducted by Tsamir and Bazzini (2004). She seems to think of solutions to inequalities in terms of other inequalities. This is evident as she talked about whether x = 3 is part of the solution or not.

Christina: …you have x is greater than or equal to 3…but it’s not the solution (x = 3). …Because you have every number greater than 3 in this case.

Later after she explored possibilities with less than or equal to, the researcher summarized what she said.

Researcher: So either it’s [referring to any single value solution such as x = 3] going to be not a solution at all or it’s going to be part of [the solution].

Christina: Yeah.

As Christina started to work on #7 (see Figure 14), she used her graphing calculator to graph “it,” where it was the expression 5x^4 viewed as function of x (see Figure 34). She used the graph “to see when 5x^4 is less than 0.” For Christina, “see” did not mean looking at the graph in Figure 34b, which she quickly bypassed. “See” meant looking at a table of values for the function (see Figure 34c). She changed Δx in the table to “check fractions” (see Figure 34d). Based on a quick scan of the tables, Christina was about to say that 5x^4 could never be less than zero. But she suddenly realized that she had forgotten to consider the “or equal to” part of the inequality sign: “5x^4 is never um, shoot! It can equal 0.”
As Christina examined her solution to #7 (see Figure 35), she noted “that’s not really a solution to the inequality.” This seemed to be more evidence that she did not believe that inequalities have solutions that are not inequalities. After considering what being a solution means, she accepted that $x = 0$ was in fact the solution to $5x^4 \leq 0$.

Researcher: What does it mean to be a solution?

Christina: It makes the statement true.

...  

Researcher: So when I substitute 0 in place of $x$.

Christina: Yes, it makes it true. ... So that’s a solution.
After completing #7, Christina was redirected back to #3 by the researcher. Christina revised her answer based on the solution to #7.

Christina: In the case of #7, $5x^4 \leq 0$, the only solution is $x = 0$. So there could be an inequality such as that.

She accepted the possibility that there exists an inequality such that the solution is $x = 3$. She tried to apply vertical shift transformations to the graph for #7 to create an inequality with a solution of $x = 3$. Her initial attempt was a single variable inequality: $5x^4 + 3 \leq 3$. After realizing that this would yield the same solution as #7, she modified her attempt by changing it to two variable inequality: $5x^4 + 3 \leq y$. She neither graphed nor physically performed algebraic manipulations on either of these attempts. Her attempts were not successful and she quickly moved on.

Prior to working #7, it seemed that Christina believed that inequalities have inequalities as solutions. However, when presented with an example of an inequality that did not have an inequality as it’s solution, she was willing to adjust her concept image of solutions to an inequality. Due to the researcher’s interaction, it is difficult to determine if Christina would have revisited #3 on her own accord.
Quadratic Inequalities

Christina employed a graphing strategy to solve #8a (see Figure 15). She plugged the expression \(3x^2 + 4x - 7\) into her graphing calculator and graphed it (see Figure 36a and Figure 36b). She examined the table of values (see Figure 36c) and said “I’m looking at the table to see based on the graph when y is going to be greater than 0.” She found one of the values, \(x = 1\), on her table. To find the other value, she went back to the graph to get an approximation of the value. Pointing to the approximate value on her graph, she noted “every \(x\) value to the left is going to be greater than [zero] … it’s positive, it’s above the \(x\)-axis which means \(y\) is positive.” The value did not appear on her table, but based on the table she “knew the zero had to be between –2 and –3.” Christina used the “zero” feature on her graphing calculator to find \(x = -2 \frac{1}{3}\) (see Figure 36d).

\[
\text{Figure 36. Screen shots from Christina's work on #8a.}
\]

Christina exhibited some sophisticated coordination between viewing #8a as a function and using the table and graph. She was able to make use of two different representations as she solved the inequality. She used the table to find one critical value of
the solution. To find the other critical value of the solution, she utilized the graph along with the “zero” feature on her graphing calculator. In determining the regions of the solution, she examined her graph. She displayed a robust understanding of a solution through her coordination of graphical and numerical representations.

Tsamir and Almog (2001) discussed some of the difficulties their participants encountered with logical connectives (i.e., ‘and’ and ‘or’). Based on her initial solution to #8a (see Figure 37), Christina appears to have some of the same difficulties. Her solution included ‘and’ as the connector. After verbalizing her solution and noting that “those [referring to her two inequality solutions] actually never intersect,” she changed the connector to ‘or.’ A question in which the inequality solutions overlapped and a logical connective of ‘and’ was appropriate would have been a good follow-up to test the extent of Christina’s understanding of logical connectives.

![Figure 37. Christina's solution to #8a.](image)

Although Christina applied algebraic manipulations to #8b, she did so in order to move the terms to the left-hand side and have zero on the right-hand side (see Figure 38). From this point, she implemented a graphing strategy in much the same manner as she did for #8a. She utilized her graphing calculator to find the solution (see Figure 39). This time
her solution included ‘or’ as the connector and she noted “there’s no way a number can be both.”

Figure 38. Christina's work and solution for #8b.

Figure 39. Screen shots from Christina's work on #8b.

For #8a and #8b, Christina did not check her solutions. She seemed to be confident in her solutions. This confidence may come from visual cues, graphs and tables, provided by her graphing calculator.

For #10 (see Figure 18), Christina was asked to solve the inequality using a graphing strategy. She wanted to perform algebraic manipulations, as she did for #8b, to move all the
terms to one side. However, the researcher asked her to leave the terms as they appear. This was done to explore the depth of Christina’s understanding of the graphing strategy to solve inequalities.

Christina set up two functions, $Y_1$ and $Y_2$, using the left and right expressions from the inequality statement (see Figure 40) into her calculator and utilized the shading feature to get a graph (see Figure 40). After examining the graph, she said “there is a solution set because they overlap [pointing to the areas where the shading overlapped].” Christina created this graph expecting overlapping shaded regions similar to those of systems of inequalities. She admitted “…it’s not really a system.” But using a functions approach, she believed that she “could make it a system.” To this end, she substituted $y$ for each expression and preserved the inequality sign. This created two new inequalities (see Figure 41) which did not match what she had in the calculator.

![Figure 40](image_url). Screen shots from Christina's initial attempt to solve #10.
After Christina revised the shading on her graph (see Figure 42), she was excited that “now the only overlap is in that area.” An area that she said was the solution set and was defined to be “the intersection of these two inequalities [pointing to her inequalities in Figure 41].”

Christina was convinced that a region in two-dimensional space would be the solution to #10. When asked to pick a value that would be part of her solution, she selected the coordinate (0, 3). She plugged this coordinate into the inequalities that comprised her system and showed that (0, 3) does satisfy both inequalities. However, she had difficulty reconciling
her solution and its two-dimensional coordinates with the original single variable inequality.

She was growing frustrated with the problem until she made the following statement:

Christina: I made it into a system. But the original doesn’t have any y’s. So, so if there’s only one variable, the solution set should be on a number line.

When asked by the researcher what the solution would be, she noted “just the x values that are satisfied in the double region (pointing to the overlapping shaded region in Figure 42).”

Christina used her graphing calculator to determine the intersection points of the two parabolas (see Figure 43). With a sense of relief, she used the x-values of the intersection points as she provided the correct solution to #10.

Christina: So our real solution should be $x$ is either less than or equal to 1 or greater than or equal to $-1\frac{1}{3}$. So that should be the real answer.

Her recognition that she needed to use the intersection points of the two functions to determine the solution showed a deep level of understanding of the situation.

Christina wrote the following on her paper as her answer: $-1\frac{1}{3} \leq x \leq 1$. She noted that “$x$ has to be greater than $-1\frac{1}{3}$ for it to be in the double shaded region…” The researcher interrupted her before she could finish her statement. The interruption was an attempt to determine if $-1\frac{1}{3}$ part of her solution, which it was. Unfortunately, the researcher never asked Christina to finish verbalizing the solution.
Christina’s struggles seemed to be centered on her strategy to convert the two expressions into functions. In her defense, the strategy was imposed on her by the researcher. As she introduced the variable $y$, it was reasonable to assume that she was trying to preserve the order of the original inequality. She may have encountered fewer issues while finding the solution(s) if the problem were an equation instead of an inequality.

Christina was able to solve the problem using an alternative to a graphing strategy that she implemented for #8a and #8b. She stumbled in many areas but was able to recover. One area of concern was the fact that she did not talk about what the inequality meant in a manner consistent with a relational view of inequalities. For example, she had difficulty determining the appropriate shading to assign to each function that she created while solving #10 (see Figure 40 and Figure 42). This difficulty may indicate that Christina did not think of the inequality as the object on the left side is greater than or equal to the object on the right side. This lack of a relational view of inequalities appeared as she struggled with her strategy and interpretation of information from her graphing calculator.

**Absolute value inequalities**

Christina correctly separated the absolute value inequality into two inequalities (see Figure 44). As she explained why she created two separate inequalities, Christina wrote a compound inequality. The sense of her inequality signs in this compound inequality were
opposite those of the two inequalities. The incorrect compound inequality did not derail the rest of the problem, because Christina used the two separated inequalities to arrive at her solution.

Christina provided the following as her initial solution: \(-16 \leq x \leq 8\). As she debated whether an ‘or’ or an ‘and’ was the appropriate connector, she realized that her solution was wrong. She noted that it was wrong “because they can’t intersect.” She elaborated that “x can never be both less than or equal to \(-16\) and …greater than or equal to 8.” She scratched out the compound inequality, stating “that’s wrong…it’s or.” She wrote the correct solution:

\[ x \geq 8 \text{ or } x \leq -16. \]

It was interesting that Christina did not employ a function approach with this problem. In a similar manner that she demonstrated in previous problems, she could have set up two functions, \(Y_1\) and \(Y_2\), in her calculator. The two functions could have been defined by using the left and right expressions from the inequality statement. Then she could have utilized the shading feature to get a graph. It may be that the absolute value in the problem
triggered her use of an approach that was different than the function approach she used up to this point. The lack of hesitation as she determined the method she would employ may speak to her confidence and familiarity with this type of inequality and an associated procedure for solving.

Another area of interest lies in how Christina wrote the incorrect and correct solutions. After she isolated the variable $x$ in the two inequalities that she created (Figure 44), instead of using the logical connector ‘and’ she decided to write her initial solution as a compound inequality. The compound inequality that she wrote may provide a glimpse into her concept image of an absolute value inequality. Up to that point, she may have believed that the solution to an absolute value inequality needed to be a region between two values. If she held that belief, she was able to override it as she reasoned about the appropriateness of her initial solution.

Christina’s final solution contains some information that may be intriguing. She determined the set-up of the two inequalities in this problem. She also decided the algebraic manipulations to apply to each inequality in order to isolate the variable $x$. When she finished, the variable $x$ was on the right hand side of both inequalities, where it started. For her final solution, she switched the inequalities to put the variable on the left side. It is possible she did this to satisfy some notion she may hold about standard form for inequalities, or maybe even equations.

**Systems of inequalities**

For #9 (see Figure 17), Christina employed the elimination method (see Figure 45). As soon as she realized that she was not working with a system of linear equations, she
abandoned this approach. In order to “solve both for $y$,” she applied algebraic manipulations to each inequality in order to isolate the variable $y$ (see Figure 46). Her preference to isolate the variable $y$ for both inequalities seemed to be predicated on her desire to utilize her graphing calculator. Once the variable $y$ was isolated in both inequalities, she graphed the inequalities in her calculator (see Figure 47).

\[
\begin{align*}
\begin{cases}
x + 4y & > 2 \\
5x - 2y & \leq -1
\end{cases}
\Rightarrow
\begin{cases}
4y & > 2 - x \\
-2y & \leq -1 - 5x
\end{cases}
\Rightarrow
\begin{cases}
y & > \frac{2 - x}{4} \\
y & \geq \frac{-1 - 5x}{2}
\end{cases}
\]

*Figure 45. Christina's attempt to use the elimination method to solve #9.*

\[
\begin{align*}
\begin{cases}
x + 4y & > 2 \\
5x - 2y & \leq -1
\end{cases}
\Rightarrow
\begin{cases}
4y & > 2 - x \\
-2y & \leq -1 - 5x
\end{cases}
\Rightarrow
\begin{cases}
y & > \frac{2 - x}{4} \\
y & \geq \frac{-1 - 5x}{2}
\end{cases}
\Rightarrow
\begin{cases}
(5, 2) & \text{Not in the solution set} \\
(-1, -2) & \text{No in solution set} \\
(-3, 8) & \text{yes, it is in the solution set} \\
(10, \frac{1}{2}) & \text{Not solution set} \\
0 & \text{Not true}
\end{cases}
\]

*Figure 46. Christina's work for #9.*

*Figure 47. Screen shots from Christina's work for #9.*
Christina pointed at the graph on her calculator and noted “where they’re crossed, that’s the solution set.” Later, she clarified what she meant in saying “the intersection of the shaded regions is the solution set.” Based on these statements, it seems that Christina understands what it means to be a solution to a system of inequalities. Using her graph, she was able to correctly identify whether the coordinates provided to her were solutions or not. In addition, she provided an explanation for why the coordinate (–1, –2) was not a solution. To be considered as part of the solution, Christina noted that the “point (–1, –2) has to satisfy both [inequalities].”

It was interesting that Christina did not talk about whether the boundary line for the inequality $y > \frac{1}{2} - \frac{1}{4}x$ was solid or dotted. She was asked if the point (0, 0.5) was part of the solution. She examined her graph for a moment and then said “for this one, I’d plug it in to check it.” She correctly noted that the point (0, 0.5) would not satisfy her first inequality; therefore it was not a solution. It is unclear if she considered the fact that the point (0, 0.5) lies on the boundary line for $y > \frac{1}{2} - \frac{1}{4}x$, which is not included in the solution.

She seemed to use her graph to determine if the given point was obviously in or out of the overlapping shaded region and for points that were not obvious she employed substitution. A process that she indicated she would ask her students to use in a classroom.

Christina: They can look at the graph for an estimation. And I would kind of jump to the conclusion it wasn’t true [referring to the point (5, 2)]. I would make them look at [pointing to the graph] or substitute the values in to verify their answer.

Christina seemed to have a firm understanding about the solution region of a system of inequalities. She had no problem identifying the region and presenting its meaning. She
implemented two techniques to determine if a given coordinate was part of the solution or not. Her explanations point to an ability to rationalize about specific coordinates. This ability may mask her understanding of the boundary lines as they relate to the solution. When asked to describe the solution, she points to the overlapping shaded region with no mention of the boundary lines. It is unclear if she included the appropriate boundary lines within an object that she called “the overlapping shaded region.”

**Explaining why the sense of an inequality changes**

Recall that a few weeks prior to the interview the preservice teachers had been present during discussions in their methods courses about why the sense of inequality changes when applying multiplication or division of a negative number. To answer #11 (see Figure 19), Christina made use of algebraic manipulations. She created an inequality (see Figure 48) and then she used only addition and subtraction of terms along with division by positive values to arrive at an answer. She took that same inequality and divided by a negative value to produce an answer. Christina said the two inequalities were equivalent because “they’re both eating the same number.” This reference to “eating” was something she said she learned in elementary school.
Christina’s work (see Figure 48) was correct and her explanation seemed to be enough for her to believe she had addressed the question of why. The two examples that she worked through in this interview seemed to be all she needed to provide in order to present a sufficient explanation. Her explanation lacked generalization; this was not a proof by contradiction where a single example to the contrary would have been enough.

**Characterizing Christina’s understanding**

Christina was a case of a preservice teacher, with a double major in mathematics and mathematics education, who seemed comfortable using a functions approach while working with inequalities and using graphing and table features on her graphing calculator as part of her approaches to solving problems. Her graphing calculator was a tool that she used to graph expressions. In addition, she used her graphing calculator when determining shading of
inequalities; finding intercepts; and finding intersection points. For problems that she used her graphing calculator, she did not check her solutions.

Christina’s actions seemed to be coherent with an operational view of inequalities. While applying solution strategies, she appeared to be focused on implementing the steps and producing a solution. She had difficulty with multiplication or division of an inequality by values that are not explicitly stated. There were moments when Christina displayed a relational view of inequalities (e.g., using a graph “to see when $5x^4$ is less than 0”).

**Crystal**

Crystal was a preservice teacher who was in the final semester of completing the requirements for a Bachelors of Science in Mathematics Education. She believed that her mathematical knowledge was strong based on all the math courses she had taken. She seemed relaxed and at ease during the interview.

**Considering inequalities as equations**

As Crystal articulated her definition for an equation, she focused on the word ‘equal.’ She noted that when an expression is set equal to something then it becomes an equation. From what she has mentioned, it is clear that the equal sign is vital component of an equation. Additionally, it appeared that Crystal displayed a relational view of equations (Knuth et al., 2006).

As Crystal considered the statement given in #2f (see Figure 9), she debated whether inequalities were equations. Her initial position was that inequalities were not equations because “there’s multiple answers.” However, after deciding that $x^2 = 4$, a statement given
to her by the researcher, was an equation with multiple answers, she began to change her opinion about #2f.

In her debate about #2f, Crystal seemed to be confused by the proximity of equations and inequalities in her past learning experiences. She mentioned that “you learn the equations and then you just take inequalities.” In addition, she said “you definitely learn inequalities after you learn equations.” This proximity seemed to be what pushed Crystal to change her mind about #2f. She sketched a graph of the inequality (Figure 42) and said “this [referring to #2f] is the equation for this entire area of space [pointing to the shaded region in her graph].” When asked to classify her confidence that #2f and other inequalities provided by the researcher were equations, Crystal said “these are definitely equations.”

![Figure 49. Crystal's graph of #2f.](image)

As Crystal worked through the other problems in the interview, there were moments when she made comments that alluded to an inner conflict as to whether to include inequalities in her concept image of equations. Crystal introduced $3x \geq 9$ while working #3. As she manipulated $3x \geq 9$ algebraically, she called it an inequality. However, later she said
“this equation [laughs] yeah equation is not in the set.” This was not the only instance where Crystal laughed after saying equation while working with an inequality.

Crystal seemed to reluctantly accept her inclusion of inequalities in her concept definition of equations. Her rationale for this inclusion seemed to be primarily focused on the proximity of equations and inequalities in her past educational experiences. Her half-hearted inclusion may have foreshadowed some of the issues that surface in Crystal’s work with inequalities, specifically with attending to negative values.

**Attending to negative values**

Initially, Crystal believed that the two inequalities in #4 (see Figure 11) were equivalent for all real numbers. However upon seeing #5 (see Figure 12), she revised her answer for #4. She noted that “a cannot be equal to zero.” In returning to #5, Crystal called the two given inequalities equivalent. Her focus for both problems is solely on the possibility of dividing by zero. Even as she multiplied one inequality by $a$, she never mentioned the possibility of changing the sense of the inequality sign.

![Figure 50. Crystal’s work for #6.](image)
To solve #6 (see Figure 13), Crystal isolated the variable $x$ in the following manner:

Crystal: I’m gonna divide by the quantity $(a - 5)$, but the only way I can divide by this is assuming that $a - 5$ does not equal zero. So that means $a$ cannot equal 5.

Her process was correct but as was the case with the equivalency problems (#4 and #5) she was solely focused on the possibility of dividing by zero. She made no mention of the implications to the sense of the inequality sign if $a$ were less than five.

As she explained why $a$ could not be equal to five, she mentioned a hole in the graph at $a = 5$. When asked to elaborate on the graph, Crystal drew a table and a graph (see Figure 51). As she created the table, she called $a = 5$ a critical point and said that she would “test” one and six. She used the values for $a$ and $x$, to create two points that she plotted. Crystal connected these two points to form a line and noted that “$x$ is greater than this line,” so she shaded above the line.

*Figure 51. Crystal's table and graph for #6.*
It seemed that Crystal was combining a sign chart method with a table for graphing. She mentioned a hole in the graph at $a = 5$, however she does not represent that “hole” in her graph. It appeared that Crystal evoked a functions approach as she reasoned with #6. Crystal seemed to have an image of inequalities where a graph with shading was a centerpiece.

**Inequalities must have inequalities as solutions**

Crystal did not hesitate to declare that the set $S$ in #3 (see Figure 10) could be a solution to an equation. She provided $3x = 9$ as an example. She did not believe that an inequality could have a solution set with a single value. When asked to talk more about this viewpoint, she wrote $x \geq 3$ and split it into $x = 3$ and $x > 3$. In examining these two components, Crystal noted three was “contained” in $x = 3$ but three was not “contained” in $x > 3$. Her reasoning appeared to be based on a separation of a ‘greater than or equal to’ inequality.

For #7 (see Figure 14), Crystal solved this inequality as if it were an equation (see Figure 53). Once she wrote “$x = 0$” four times, she said “I don’t know if that really makes sense.” She pointed back to “$x \leq 0$” and noted that they were not the same.

*Figure 52. Crystal’s first attempt to solve #7.*
For her second attempt to solve #7, Crystal applied a case-based separation approach. She believed that any number raised to the fourth power would be positive. This indicated that she did not consider zero or mistakenly believed that zero was a positive or negative number.

Researcher: So the only kind of numbers I have are positive or negative?

Crystal: Well, okay. Yes! Yeah, we’re dealing with reals.

She noted that five times any positive number could not be less than zero, so she did not need to consider the “less than” part. With the other component, she wrote that five times a positive number equals zero. Crystal appears to apply the zero-product property of equations as a means to deduce that if $x = 5$ then $y$ must be zero. It was at this point that Crystal realized that she did not consider zero when she wrote positive. She wrote “$x = 0$” as the solution.

![Figure 53](image)

*Figure 53. Crystal's second attempt to solve #7 and her solution.*

Crystal used her graphing calculator after writing her solution as if she was performing a check. She plugged in $5x^4$ for Y1 and 0 for Y2 (see Figure 54a). After entering the values, she utilized the ‘intersection’ function on her graphing calculator (see Figure
54b). With this visual cue, Crystal noted that “this graph touches the x-axis only one time so that why we only have one solution.”

Crystal seemed to display a firm understanding with regard to using a functions approach. This was evident as she entered the two sides of the inequality into her graphing calculator and compared the two graphs. The tangible nature of ‘y = 0’ or the x-axis may have aided her as she formed her conclusions for #7. At this point it is unclear whether her understanding was as firm if one of the functions was not a constant.

Figure 54. Screen shots from Crystal’s work on #7.

Crystal never asked to return to #3. At one point the researcher explicitly asked if the solution to #7, an inequality, was a single value. She said yes and moved on. The researcher decided not to direct her to #3, in the interest of time.

**Quadratic Inequalities**

Crystal employed a case strategy to solve #8a (see Figure 55). She said that these “were the only four choices.” She forgot to include a case, (negative)(negative) > 0. Crystal evaluated the first case, (positive)(positive) > 0. She used those solutions to draw a number line that originally she described as $x \geq \frac{-7}{3}$. Crystal decided with no prompting from the researcher to check her answer by substituting values into the original inequality. When she
checked $x = 0$, she noticed that there was an issue with her solution. She went on to check $x = 1$ and $x = 2$, both of which were correct. Without checking any values less than negative seven thirds, she crossed her first solution out and wrote $[-\frac{7}{3}] \cup [1, \infty)$ as her next solution.

It appeared that Crystal used the inequalities $x - 1 > 0$ and $3x + 7 > 0$ to initially determine the shading on her number line. From there she checked the regions and “critical points” that were shaded to confirm that they should in fact be shaded. This seemed to explain why she did not check any values less than negative seven thirds.

Crystal said “if it’s [referring to any number] not in this interval [pointing at solution] then if we let $x$ be that and we substitute that number in for $x$ then we will get a false statement.” This statement gave the researcher the opinion that she understood what a solution of an inequality meant. So, the researcher asked Crystal to let $x = -5$ and confirm that it was not in the solution, that she defined. As she finished her calculations, Crystal realized that she was wrong.
Crystal: How did I miss that?

Researcher: Miss what?

Crystal: Well, when I broke these apart [pointing to her number line in Figure 55], I thought I was testing all the intervals and obviously not.

Crystal made correct adjustments to her number line and her solution.

The researcher asked Crystal to explain how she would get to this solution as if she was explaining it to a student. After taking a moment to think about the problem, she stated that her mistake lies in her chart. “Instead of having inequality symbols here [pointing to her chart in Figure 55] we should just have equal signs.” With this adjustment to her strategy, Crystal correctly reworked the problem (see Figure 56). After some initial stumbling, she seemed to understand how to apply an algebraic strategy, in this case a sign chart method, to solve this quadratic inequality.

Figure 56. Crystal's second attempt to solve #8a.
Crystal immediately implemented algebraic manipulations on #8b in order to move all the terms to the left side of the inequality (see Figure 57). She said “I would just do the same thing as in the other problem.” She used the quadratic formula to solve the corresponding equation. She called these values her “critical points.” From this point, Crystal employed a sign chart method to solve the inequality. After determining which regions and critical points satisfy her inequality, she correctly identified the solution.

Crystal employed an algebraic manipulations strategy to solve #8a and #8b. For #10 (see Figure 18), she was asked to solve the quadratic inequality using a graphing technique. She entered the two sides of the inequality into her graphing calculator and graphed them (see Figure 58). She used the ‘intersection’ function on her graphing calculator to find the two intersection points. She stated that “they’re equal to each other at these points [pointing
to the two intersection points]." She wanted to employ a sign chart method to solve this inequality but the researcher asked her to try a different strategy. After taking a moment to think and to look at the graph, Crystal indicated that she was unsure of how to solve the inequality without using a sign chart method. Unlike #7, where she was able to apply a functions approach, Crystal did not reach that same level of understanding with #10. Her inability to apply a functions approach with this problem may be due to the fact that the two sides are non-constant expressions, whereas in #7 one side was zero.

Crystal seemed to need a moment to explore possible strategies. It is unclear if she would have realized that her initial solution to #8a was incorrect if the researcher had not asked her to check $x = -5$. She was able to adjust her use of the sign chart method in #8a and arrive at the correct solution. Crystal seemed to be comfortable with a sign chart method because she used it again in #8b and wanted to use it for #10. It is unclear if this is the only method that Crystal could use to solve quadratic inequalities. When she tried to use a graphing strategy to solve #10, she was unsure how she should proceed once she found the
intersection points. It is possible that Crystal may have been able to figure out how to proceed if she were given more time or another problem. Unfortunately, time was an issue at this point in the interview and the researcher decided not to pursue this any further.

**Absolute value inequalities**

Crystal did not like the way #8c (see Figure 16) was presented, so she interchanged the sides correctly. She employed an algebraic manipulations strategy to solve the inequality (see Figure 59). She made a small mistake in her first number line. She plotted and used 16 instead of \(-16\). Taking this mistake and the way that she shaded her number line into consideration, it appeared that she shaded the overlapping region created by the two inequalities. In a manner that was similar to the sign chart method she employed with #8a and #8b, she checked the regions and “critical points” on her number line. There was a moment of confusion when, contrary to her initial graph, the region greater than 16 “checks.” This was unexpected: “…wait, this isn’t right.” Crystal accepted the checks that she performed and wrote \([8, \infty)\) as her solution.
The researcher decided to ask Crystal if she meant to write 16 on her number line or if she wanted \(-16\) as was in her inequality. Upon seeing her mistake, she sounded disappointed. She redrew her number line (see Figure 59). This time she did not shade any parts of the number line. Once again, she employed a sign chart method to determine her solution. This time she produced the correct answer.

The researcher asked Crystal if the sign chart method was necessary to solve the problem.

Researcher: I got \(x \geq 8\) which looks like it’s the same thing as \([8, \infty)\).

Crystal: It is.

Researcher: And then I’ve got \(x \leq -16\) which looks like it’s the exact same thing as \((-\infty, -16]\).
Crystal: I guess technically you might not have to check this number line. …I just want to double check, basically.

She went on to mention that a sign chart method was necessary with the other problems (i.e., #8a and #8b) because “we let it be equal.”

Crystal gave the impression that she relied on the sign chart method to coordinate with her shading of the number line that was based on the two inequalities that she found. When the shading on the number line did not match her interpretation of the sign chart method, she decided to present the answer obtained using a sign chart method. Her decision may be attributable to her comfort level with algebraic techniques.

From her comments, it seemed as if Crystal perceived the sign chart method as a check. However she utilized the method as a solution strategy. It is unclear whether Crystal would use the sign chart method if given another absolute value inequality to solve. Time constraints prevented the researcher from exploring that possibility.

**Systems of inequalities**

For #9 (see Figure 17), Crystal performed algebraic manipulations on each inequality in order to isolate the variable y (see Figure 60). She took these steps in order to graph the corresponding equations in her calculator (see Figure 61a and Figure 61b). After looking at the graph of the equations, Crystal adjusted her calculator setting to have it shade “y is greater than” for both (see Figure 61c). She recreated the graph that was displayed on her graphing calculator (see Figure 61d) on her paper. Using the graph on her paper, she identified the solution. She did not use a dashed line in her graph, but she did correctly note that the solution would not include points on the line with positive slope.
The researcher asked Crystal to determine if a list of coordinates were part of the solution or not. The answers that she provided for each of the coordinates were correct. She used the graph on her graphing calculator as she made her decision about each coordinate. (see Figure 61d). She noted that “to be in the solution set when you plug that ordered pair
into both of these equations; they both have to be true.” This statement seemed to indicate that Crystal did understand what it means to be a solution to a system of inequalities. However, (0, 0.5) was the only coordinate that she substituted into the inequalities to verify the truth value she obtained from the graph.

Crystal appeared to be very comfortable manipulating the inequalities algebraically. She seemed to have a graphing strategy in mind to solve this system of inequalities from the very beginning. She articulated that strategy to the researcher before performing each step.

Crystal seemed confident that she had applied the correct shading for each graph in her graphing calculator. Her comfort level or confidence may explain why she did not use test points to determine the correct shading. It is difficult to determine a reason why she did not use test points. Presenting Crystal with more questions involving systems of inequalities may have yielded answers as to whether she would use test points to determine shading.

**Explaining why the sense of an inequality changes**

Recall that a few weeks prior to the interview the preservice teachers had been present during discussions in their methods courses about why the sense of inequality changes when applying multiplication or division of a negative number. To answer #11 (see Figure 19), Christina made use of algebraic manipulations. She created an inequality (see Figure 62) and then she divided by a negative value to produce an answer. She took that same inequality and used only addition and subtraction of terms to arrive at an answer. Crystal indicated that the answers were the same because of “the reflexive property.”
Crystal’s work (see Figure 62) was correct. Her explanation seemed to be enough for her to believe she had addressed the question of why. It is the researcher’s opinion that her explanation lacked generalization; this was not a proof by contradiction where a single example to the contrary would have been enough. She never addressed whether her method would work for any inequality.

**Characterizing Crystal’s understanding**

Crystal was a case of a preservice teacher who seemed to display an operational view of inequalities. Her inclusion of inequalities in her concept image of equations and her inability to attend to the effects on the sense of the inequality when multiplying or dividing by a negative value appeared consistent with an operational view of inequalities. While Crystal displayed quite a few false starts when applying strategies to solve inequalities, she was able to reason through and find the solution in most of the problems. Her understanding of what an inequality means beyond a specific example seemed to be lacking, as was evident by her inability to finish #10. This seemed coherent with an operational view of inequalities.
Crystal did not apply a single strategy to solve all of the inequality problems. Her choice of strategy seemed to be determined by the type of problem that she encountered. For both #8a and #8b her solution strategy seemed to be more driven by the fact that she was working with quadratics and using algebraic techniques she knows to use with quadratic equations. With the system of linear inequalities problem, she seemed to prefer a graphing strategy.

Crystal’s graphing calculator was a tool that she used to graph, find intersection points, and to perform basic calculations. However, she did not use her graphing calculator to solve #8a or #8b. While it seems that she did not rely on graphs as a primary source of information on many problems, she did seem to rely on charts (sign charts, cases, etc.) as a method to organize and convey her thoughts.

Heather

At the time of this study, Heather was completing the final semester of requirements for a Bachelors of Science degree in Statistics and a Bachelor of Science degree in Mathematics Education. She was excited and nervous about her upcoming student teaching experience. She classified her mathematical content knowledge as strong. Throughout the interview, Heather was able to joke and laugh. This light-hearted nature seemed to be normal for Heather.

Considering inequalities as equations

Heather’s concept image of an equation did not seem to include some inequalities. While considering #2f (see Figure 9), she said the following: “it is an inequality. It is not equal.” She had criteria of “equal” that needed to be met in order for her to consider
something an equation. Her definition of an equation seemed to indicate a relational view of equations (Knuth et al., 2006).

To explore the extent of this notion, the researcher asked Heather to indicate whether \( y \geq 5x - 9 \) was an equation or not. She said that it was an equation “according to my definition.” The inclusion of the “or equal to” in the inequality seemed to be what prompted Heather to identify this statement as an equation. However she was not confident in that answer. Near the end of the interview, Heather was given the opportunity to reconsider her definition of an equation and answers for the statements in #2 (see Figure 9), she did not add or change anything. She said, “I’m good.”

**Attending to negative values**

Heather’s responses throughout the interview indicated that she understood the implications of multiplying or dividing an inequality by a negative value, whether explicitly stated or not. She was able to recognize that given a parameter \( a \), it is possible that operations such as multiplication or division involving \( a \) may affect the sense of the inequality sign. She was able to attend to the effects of multiplying or dividing an inequality by a negative value with equivalency problems (#4 and #5) and with a solving problem (#6).

As Heather considered #4 (see Figure 11), she noted the following:

Heather: …the portion to consider is if \( a \) is negative. Because that would switch. It changes the inequality.

She decided that the two given inequalities were not equivalent. As justification for her statement, she provided a counterexample (see Figure 63). She let \( a = -1 \) and showed that the two inequalities would not be equivalent for that particular value of \( a \). To reinforce the fact
that the two inequalities that she produced using \( a = -1 \) were not equivalent, she graphed both on a number line with different colors. After providing a specific counterexample, Heather generalized her answer as she noted that negative values for \( a \) would prevent the inequalities from being equivalent.

It was interesting that Heather did not consider the implications if \( a = 0 \) for #4. The researcher asked her what types of numbers constituted the real number system. She indicated that it was “all kinds of positive and negative numbers.” It was not until she looked at #5, that she realized that she had forgotten zero. “Man, I didn’t even think about zero. I feel dumb now [said while laughing].” Heather asked if she could go back and add zero to her answer (shown in parentheses in Figure 63).

The researcher would have skipped #5 in the interview if Heather had mentioned zero in #4. Since she did not, #5 was included as a way to determine if she would recognize her oversight in #4. After revising her solution to #4, her work for #5 was minimal (see Figure 64).
To solve #6 (see Figure 13), Heather considered the quantity \((a - 5)\) as a single value or as a “coefficient of \(x\).” She noted that “if you are solving it [referring to the inequality], you would just divide both sides.” Here Heather was outlining the algebraic manipulation strategy that she wanted to use. She mentioned that an issue would occur if \(a = 5\). Up to this point, it is difficult to determine if she was treating the inequality as an equation. However, it became clear that she was thinking in terms of an inequality when she outlined three cases (see Figure 65). She indicated that the value of the parameter \(a\) would determine which answer would be correct.

In examining Heather’s work on these problems, she seems to be able to attend to the effects of multiplying or dividing an inequality by a negative value. The type of problem,

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**Figure 64.** Heather’s work for #5.

**Figure 65.** Heather’s work for #6.
equivalency or solving, did not appear to be of any significance. In addition, the presentation of the parameter $a$ did not seem to be of influence, as she was able to consider a range of possible values for that parameter.

Another interesting habit that Heather displayed while discussing these problems was how she read the inequalities. She did not always read inequalities from left-to-right. Without rewriting the inequality, she would correctly read it from right-to-left. While discussing #4, she wrote $\frac{7}{a} > x$. However, she said the following:

Heather: …$x$ is less than seven over $a$.

Researcher: … you’re actually reading it from right-to-left?

Heather: I mean seven over $a$ is greater than $x$. But that is not the way I read it. She seems to want to read the variable first no matter the orientation of the inequality. In addition, she seemed to read the inequalities in this manner only when the variable was isolated.

**Inequalities must have inequalities as solutions**

Initially, Heather was able to produce an equation such that the solution was the set $S$ (see Figure 66). As she contemplated inequalities, she noted “this would be easier if it was just on integers.” She offered the compound inequality $2 < x < 4$ as an inequality with $S$ as its solution, if $S$ were on the integers instead of the real numbers.

After trying unsuccessfully to create an inequality such that the solution was the set $S$, Heather indicated that she was unsure of how to answer #3. She said that “typically you don’t think of an inequality as having just one possible solution.” She seems to leave the door
open to the possibility that $S$ can be the solution to an inequality but her comments give the impression that she is leaning towards saying it is not possible.

Figure 66. Heather's work for #3.

Almost immediately after reading #7 (see Figure 14), Heather stated her answer: “$x$ can be equal to zero and that’s pretty much it.” She had correctly found the solution set for the inequality $5x^4 \leq 0$. She verbally provided an explanation for her answer that was logical and valid. Without prompting from the researcher, she noted “I can graph it and verify.” After examining her graph (see Figure 67), she said “it hits at (0, 0) and it doesn’t go below the x-axis.”

Figure 67. Screen shots from Heather's work on #7.

After finishing #7, Heather said “dun, dun, dunn, time to tackle number three.” It seems that she realized that her solution to the inequality in #7 was a single value and could
connect to the statement in #3. She entered $x^2 + 3$ into her calculator and examined the graph (see Figure 68a and Figure 68b). After she realized that her initial attempt would not provide the desired solution, she noted the following:

Heather: You could shift it to the right probably. Do $x$ minus three squared.

She wrote a new inequality on her paper (see Figure 66) and graphed the corresponding parabola on her graphing calculator (see Figure 68c and Figure 68d). After examining the graph, she correctly noted that $(x - 3)^2 \leq 0$ is an inequality with $S$ as its solution set.

In reaching her solution, Heather seemed to be using what she understands about even powers like 4 and 2. She seemed to apply that understanding to construct an inequality to give her $x = 3$ as a solution. She knew to use $x - 3$ as an important part of her inequality: “you’re concerned about the $x$-value, you’re not concerned about what is on the right [side of the inequality].” She may have been influenced by thinking through the power of 4 in the prior problem.

Heather’s initial use of her graphing calculator (see Figure 68a and Figure 68b) seemed to be for exploration purposes. Her later use of her graphing calculator (see Figure 68c and Figure 68d) appears to be as a confirmation of how to write the left side of her inequality. Based on the graph for #7, she knew that the parabola needed to touch the $x$-axis in order to satisfy the “equal to zero” constraint.
There was a moment when Heather was revisiting #3 that provided an interesting glimpse into her thinking about inequalities. After writing \((x - 3)(x - 3) \leq 0\) (see Figure 66), Heather explicitly noted that “you are only going to get one solution for that because of the zero-product property.” She may have been applying a property used to solve equations on an inequality. Another possibility may lie in how she reasoned through the problem.

Heather: …you know that it can’t be less than zero because it has to be a positive number.

Researcher: …It doesn’t matter what I have inside here (pointing to the binomial being squared)?

Heather: That’ll be either zero or positive.

Researcher: …the only thing that we are concerned about with this inequality sign is really just the equal to part? Right?

Heather: Uh-huh (nodding).
Heather was not concerned with the “less than” portion of the inequality because she believed that it was not possible. So she turned her focus to the “or equal to” portion of the inequality. This may explain why she utilized an equation solution strategy on an inequality.

**Quadratic Inequalities**

Heather was happy to see “real numbers” when she got to #8. She noted that these problems were “real stuff that I’m used to looking at.” To solve #8a (see Figure 15), she entered the corresponding equation into her graphing calculator and examined the graph (see Figure 70). On graph paper, she drew a two-dimensional graph that approximated what was on her calculator screen (see Figure 69). She made the following comments as she shaded the graph:

Heather: Alright, so I want to know when this parabola is greater than or equal to zero. So, it is greater than or equal to zero when it is above the x-axis. So all this junk over here is going to be above the x-axis (shading two parts of the parabola). So that means your x-values over here (shading part of the x-axis) and your x-values over here (shading part of the x-axis) satisfy the inequality.

Having found the zeros using her graphing calculator, Heather was able to produce the correct answer (see Figure 69).
Heather applied a graphing strategy to solve #8a. She used her graphing calculator as a tool to aid her as she created her own graph. She went so far as to use her graphing calculator to find the minimum (see Figure 70b) because she likes “to have accurate graphs.”

The first sentence in Heather’s statement, “I want to know when this parabola is greater than or equal to zero,” was a solid interpretation of the inequality. Within this statement she appeared to display a relational view of inequalities (Knuth et al., 2006). She articulated what she was looking for while utilizing a functions approach. She displayed an understanding that allowed her to coordinate the two-dimensional parabola with the unidimensional solution. It seemed that Heather would have been able to solve this problem...
with any quadratic on the left side. What was unclear up to this point was her ability to apply her reasoning if the other side was not zero.

Heather determined that she would solve #8b by graphing, as she did with #8a. She entered the corresponding equations into her graphing calculator and graphed them (see Figure 71). She decided to recreate the graph generated by her calculator on her paper (see Figure 72), but she added the shading. While determining where to shade, she said “if you want the parabola is less than the line, you’re…pretty sure you would shade this part.”

![Figure 71. Screen shots from Heather's initial attempt to solve #8b.](image1)

![Figure 72. Heather's initial solution for #8b.](image2)

Once again, Heather articulated a relational view of inequalities (Knuth et al., 2006). She talked about the problem in general terms; “the parabola is less than the line.” Unfortunately, she had difficulties in coordinating her choice to use a two dimensional representation with the single-variable inequality in the problem. After being asked by the researcher to describe her solution, Heather noted that her solutions would be coordinates, (x,
y). It is at this moment that she realized that substituting a coordinate would not be possible because “you don’t have the variable y in there (pointing to the original inequality).” She admitted “I think I might be confused by having the x’s on both sides.”

For her second attempt, Heather decided to apply algebraic manipulations to the original inequality (see Figure 73). She moved all the terms to one side of the inequality. She graphed the corresponding parabola on her graphing calculator and found the zeros (see Figure 74). She recreated the graph on her paper and said “if you want to know that it is less than zero, it is going to be in between those two values (pointing to the zeros).” She shaded the portion of the x-axis between the two zeros and wrote her solution as a compound inequality (see Figure 73). It may have been interesting to find out Heather’s response if she were asked to think about this solution in terms of her prior graph of the parabola and the line.

Figure 73. Heather’s work for #8b.
Heather seemed to prefer a graphing strategy when solving quadratic inequalities with one variable. She had difficulty solving the quadratic inequality with terms, other than zero, on both sides. This difficulty manifested itself in her initial representation of the solution for #8b. She was trying to make two-dimensional region be the solution for a single variable inequality. However this difficulty was only a temporary roadblock in her path to the correct solution. The statements that she made while working these two problems indicated that she had a strong understanding of how to consider two functions in relation to each other.

**Absolute value inequalities**

To solve #8c (see Figure 16), Heather employed a strategy that separated the absolute value inequality into two inequalities (see Figure 75).

Heather: You know that the value inside the absolute value could be…greater than three. Or it could be less than negative three, cause it is the distance on a number line. After applying algebraic manipulations to simplify each inequality, she graphed both inequalities on the same number line (see Figure 75). She checked her solution by selecting...
values in each region to substitute into the original inequality. Once she completed checking all of the regions, she noted “I feel pretty confident that this answer is correct.”

Figure 75. Heather’s work for #8c.

Heather seemed to understand the general situation presented in absolute values problems when she mentioned “finding the distance from zero.” Her strategy which involved creating two inequalities, gave the impression that she understood the nuances that separate absolute value equations from absolute value inequalities.

After being asked by the researcher to articulate the meaning of her correct solution, Heather said the following:

Heather: It means that if you pick a value for $x$ that is eight or greater than that satisfies this inequality (pointing to $8 \leq x$). Let me make sure it does. …I don’t like to talk like I know.

This statement was made after Heather appeared to be finished with the problem. It is difficult to determine if the checks were part of her strategy to solve the problem or made in response to the researcher’s question.

One other event occurred as Heather described her solution. She included the connector ‘and’ the first time she read her solution. However, when asked to describe what
the solution meant, she used ‘or’ in her descriptions of values that “satisfy the [original] inequality.” Later the researcher asked Heather to write the solution with the inequalities. She pointed to the two inequalities that she used to create her number line and solution and indicated those would be her inequalities. She noted that she may add an “or” but she was unsure. She mentioned that “I don’t remember many inequalities.” It is unclear if Heather had a firm understanding of the appropriate logical connective to use when describing her solution. She was hesitant to choose ‘or’ for #8c. However, she was not tentative as she provided her answers for #8a and #8b. One possible explanation for the issues that arose with #8c may be a lack of clarity in the researcher’s questions.

**Systems of inequalities**

As Heather started #9 (see Figure 17), she noted “I wish those were equal signs…because then you could do substitution or elimination or matrices.” She began the problem by performing algebraic manipulations on each inequality in order “to get y on one side by itself so I can pretend that it is just a line.” However, after she isolated $y$ for the first inequality and prior to any graphing (see Figure 76), she made the following comment:

Heather: I would do a dotted line for that…to show that it is not equal to and then shade the part where it is going to be greater than.

So even though she talked about pretending “it is just a line,” her description seemed to indicate that she treated the object as an inequality.
Once she isolated the variable $y$ for each inequality, she employed her graphing calculator as a tool to graph and to help her draw the graphs on paper. She entered the corresponding equations into the graphing calculator one at a time in $Y_1$ and graphed them (see Figure 77a, Figure 77b, Figure 77d, and Figure 77e). Once Heather isolated the variable $y$ in each inequality, she realized that “one half is going to be your $y$-intercept” for both boundary lines. Instead of using the visual cues provided by the graph on her calculator, Heather utilized the ‘table’ feature (see Figure 77c and Figure 77f) to select a second point for each boundary line that she used to create the graph on her paper. Earlier she indicated that she did not like “all the fractions.” The abundance of fractions may have been why Heather utilized the ‘table’ feature on her graphing calculator. She may have employed the calculator as a means to check or she could have done this to avoid the calculations altogether.
As Heather drew the graphs of the inequalities (see Figure 78), she did not use test points to determine the shading of the inequality $y > -\frac{1}{4}x + \frac{1}{2}$. After drawing the dotted line, she said “you want $y$ greater than that, the $y$-values are going to be above that line.”

However, as she was graphing the second inequality $y \geq \frac{5}{2}x + \frac{1}{2}$, she noted “I’m pretty sure you shade this part (pointing to the region “above” the line) but I’m going to pick a point and make sure.” With reassurance from quick mental check, Heather noted that the solution to this system was the “part where both are shaded [pointing to her graph in Figure 78].”
Even though her graph was labeled with equations instead of inequalities, Heather seemed to have a firm understanding of the solution to this system of inequalities. The researcher asked Heather to determine whether specific coordinates were part of the solution or not. She made the correct assignments, solution or not, for each coordinate. She used only her graph to make each determination. Additionally, her reasoning for each assignment was valid and logical. In particular when asked to consider the coordinate (0, 0.5), Heather noted that “although that is a point of intersection, on this one (pointing to the boundary line \( y = -\frac{1}{4}x + \frac{1}{2} \) it’s the dotted line so that can’t be part of [the solution].” This seems to reinforce the notion that Heather did not view the boundary lines as equations.

**Explaining why the sense of an inequality changes**

Recall that a few weeks prior to the interview the preservice teachers had been present during discussions in their methods courses about why the sense of inequality changes when applying multiplication or division of a negative number. To answer #11 (see Figure 19), Heather described a ‘mapping’ approach. She started with a number line and noted that if “you multiply that [pointing to her first number line] by a positive constant” then “the direction stays the same.” However, she said “if you multiply by a negative one or negative two” then the direction will switch (see Figure 79). As Heather was describing “the switch,” she would cross her hands over each other. It seemed that she was trying to provide a visual cue through her hands that served as a signal: “flip the line.”
Heather’s explanation contained a visual element that seemed consistent with her thinking on other problems. She seemed to have a propensity to draw graphs while working problems and to use them as she reasoned. In addition, her explanation to some extent was specific in her use of $-1$. However, as she showed what would happen if the number line was multiplied by $-1$, she mentioned that the orientation of the new number line would change for any negative number. This generalization took her explanation beyond a case-by-case situation to a more universal level.

**Characterizing Heather’s understanding**

Heather was a case of a preservice teacher double majoring in Statistics and Mathematics education whose comments and actions were consistent with a relational view of inequalities. The comments that she made while applying a graphing strategy to solve #8a and #8b seemed to highlight her view of inequalities as a relationship between objects. Even with a relational view of inequalities, issues surfaced during Heather’s interview. However, the issues that she displayed during her interview were minor and did not prevent her from eventually producing the correct solution.
Heather displayed a reflective nature as she answered the questions. From time to time during the interview, she would verbalize her thoughts. It did not seem that her verbalization was meant for the researcher. It appeared that she verbalized as a way to confirm or evaluate her intentions.

**Vanessa**

At the time of this study, Vanessa was completing the final semester of requirements for a Bachelor of Science degree in Mathematics Education with a minor in Accounting. She mentioned that she felt ready to get into the classroom. She classified her mathematical content knowledge as strong and noted that she was “five extra math classes…away from a double major [in Mathematics].”

**Considering inequalities as equations**

As Vanessa outlined components of her definition of an equation, one key component was an equal sign. As she considered #2f (see Figure 9), she noted that it was “not really an equation, it’s an inequality [pointing to the inequality sign].” To find out if changing the inequality sign would affect Vanessa’s opinion, the researcher asked her to consider \( y \geq 5x - 9 \). Without hesitation, she said “no, it’s still an inequality.”

Vanessa’s concept image of equations did not appear to include inequalities. When given the opportunity near the end of the interview to go back and make adjustments to her definition of an equation, she mentioned “an equation is not an inequality.” Her comments gave the impression that she did not consider equations nor inequalities to be a subset of the other. They were separate objects.
Attend to negative values

As Vanessa read #4 (see Figure 11), she had some confusion as to what numbers were included in the real number system. Initially she believed that negative numbers were excluded. However, she noted “that’s natural numbers; negative numbers are included in all real numbers.” Thus, it seemed that her attention to negative numbers came out as she was making sense of the symbolism and as she clarified what was included in the set of real numbers.

Vanessa quickly noted that the two given inequalities were not equivalent (see Figure 80). When asked to explain why, she noted that for the first inequality “to get x isolated by itself you’d have to divide by a on both sides.” She articulated that this new inequality would be true for some cases. However, she clarified “you have to take into account what if a is negative… if a is negative then you have to change the sign of the inequality.”

In addition to recognizing that negative values of a would imply that the two inequalities were not equivalent, Vanessa noted that an issue occurred if a = 0. She focused on the second inequality as she mentioned that “you can’t divide by zero.” She correctly stated that the two inequalities were equivalent if a was a positive number. It is unclear if she meant all positive real numbers are just natural numbers. After her initial confusion about the real number system, she talked about positives, negatives, and zero. The researcher failed to seek clarification as to what numbers Vanessa considered to be “positives.”
Since Vanessa correctly identified all of the values of $a$ that would make the inequalities in #4 equivalent, the researcher decided to skip #5 (see Figure 12). To solve #6 (see Figure 81 for her work), she decided to isolate the variable $x$, by dividing both sides by the quantity $(a - 5)$. She quickly noted that $a$ could not be equal to five, “because then the denominator would be zero.”

Vanessa was aware that certain values of $a$ could be an issue in determining the equivalency of two inequalities, namely zero and negative real numbers. However, her focus on #6 seemed to be on avoiding dividing by zero (i.e., when $a = 5$). She failed to mention the implications if $a$ was less than five.

It is difficult to determine if the presentation of the parameter $a$ caused Vanessa to overlook the effects of multiplying or dividing an inequality by a negative value in #6. In problem #4, the parameter is by itself in the denominator or being multiplied by the variable. However, in problem #6, the parameter appears in two places and it has values being
subtracted from it. These differences may account for inconsistencies in Vanessa’s ability to deal with the effects of multiplying or dividing an inequality by a negative value.

\[
\begin{align*}
\text{if } a = 5 & \quad \Rightarrow x > \frac{2(5)-1}{5-5} = \frac{9}{0} \\
\end{align*}
\]

Figure 81. Vanessa's work for #6.

**Inequalities must have inequalities as solutions**

Vanessa’s initial answer to #3 (see Figure 10) was that the set S could be the solution set for both an equation and inequality. She was quickly able to provide an example of an equation with the solution set S (see Figure 82). The examples of inequalities that she provided contained the set S. This response was similar to those made by students in a study conducted by Tsamir and Bazzini (2004).
The researcher decided to focus Vanessa’s attention on the fact that the set $S$ was part of the solutions for her given examples, not the solution. With a new perspective on what was being asked in the question, she said “then it has to be an equation.” However, she offered a compound inequality, $3 \leq x \leq 3$, as “the only way that it [the set $S$] would be the only solution to an inequality. Like her earlier attempts to provide an inequality, Vanessa’s compound inequality was a response that Tsamir and Bazzini (2004) encountered in their study. This implies that Vanessa’s responses were not unique or atypical. While discussing #8b, Vanessa commented that the answers to “inequalities are solution sets.” When prompted to elaborate on what she meant by a solution set, she gave the impression that a solution set was not a discrete set. This may indicate that she did not believe that certain inequalities have a single-value solution.

Vanessa’s initial attempt to solve #7 (see Figure 83) involved applying algebraic manipulations on the inequality in order to isolate the variable $x$. She drew a number line to represent her solution. The researcher asked her to check $x = -1$, which she indicated was
part of the solution. As she realized that $x = -1$ was not part of the solution, Vanessa said “I’m really confused, because I solved it right.” It seemed that through her use of an algebraic manipulations strategy, she did not consider the effect of raising any number to the $4^{th}$ power (see Figure 83). As a result, she was unable to make sense of this effect in her evaluation of her solution that $x$ is less than or equal to zero.

![Figure 83. Vanessa's work for #7.](image)

In an effort to resolve her confusion, Vanessa decided to utilize her graphing calculator (see Figure 84). She implemented a function approach in order to find the solution and declared “what is shaded (see Figure 84b) should be our solutions.” Vanessa indicated that she did not understand how the calculator and her work both “included $-1$,” but a check of $x = -1$ failed. As she talked about $x = -1$, she was not referring to the vertical line. Instead, she was referring to the point $(-1, 5)$, which was on the parabola $y = 5x^4$. It appeared that Vanessa was having difficulty connecting her two-dimensional representation of the solution with her single-variable inequality. She shaded the area where the $y$-values were less than $5x^4$ (see Figure 84b). However, she did not seem to realize that she was supposed to determine where the graph of the parabola was less than or equal to the line $y = 0$ (the $x$-axis).
Vanessa’s frustration grew after several attempts to reconcile her confusion produced more dead ends. At one point, her argument became cyclical in nature.

Researcher: …if x is less than zero, then $5x^4$ was actually greater than zero?

Vanessa: Yes.

Researcher: But we started by saying that $5x^4$ must be less than or equal to zero.

Vanessa: That’s a problem.

Researcher: Why?

Vanessa: Because you’re saying two different things.

Vanessa admitted that she did not have a correct answer. The researcher decided to move on to the other questions rather than continue probing on #7. In addition, as a result of Vanessa’s inability to determine that the solution to #7 was a single value, the researcher decided not to return to #3 (see Figure 10).

One of the algebraic manipulations that Vanessa performed in the early stages of solving #7 was taking the fourth root of both sides of the inequality. This action seemed to indicate that she was treating the inequality as an equation. She exhibited confidence in her procedures even after examining the graphs produced by her calculator (see Figure 84b):
Vanessa: …I don’t know how to explain that, because I did my math right. I just don’t know why it didn’t work.

Her confidence that she “solved it right” may be connected to her performing these algebraic manipulations on equations many times (Tsamir & Bazzini, 2004).

**Quadratic Inequalities**

For #8a (see Figure 15), Vanessa entered the left side of the inequality into her graphing calculator (see Figure 85a). She noted that the shaded region on the graph (see Figure 85b) was the solution set. As was the case with #7, she seemed to be considering the area where \( y \) was greater than or equal to zero and not where the object was greater than or equal to zero. She set each of the factors on the left side of the inequality to be greater than or equal to zero and solved (see Figure 86). It seemed that she was trying to apply the zero-product property of equations. After graphing each inequality on a number line, Vanessa indicated that the answer was \( x \geq -\frac{7}{3} \) because she used an ‘or’ as the connector.

![Figure 85](image)

*Figure 85. Screen shots from Vanessa’s work on #8a.*
Vanessa indicated that to be a solution, “it [pointing to $x \geq -\frac{7}{3}$ in Figure 86] satisfies the inequality.” Knowing that her solution was incorrect, the researcher asked Vanessa to check $x = 0$. Upon realizing that $x = 0$ was not a solution, she changed her answer to $x \geq 1$. This new information seemed to cause Vanessa to discard her second number line as a solution. With her second number line discarded, she was left with the first number line as a solution.

As a way to confirm that her new solution was correct, Vanessa checked $x = 2$. Finding that it did satisfy the original inequality, she felt somewhat confident in her new solution. The researcher asked her to check $x = -3$. She indicated that “$-3$ isn’t included in the solution set.” However after performing the check, she realized that she was wrong.

Vanessa: …it is working…and that shouldn’t work!
Later, she noted “I really don’t understand why because the algebra is right.” It seemed that she was placing her trust in algebraic manipulations that she performed at the onset of this problem. That trust may be founded on her utilization of the zero-product property for equations, a property that she had no doubt applied numerous times with equations. Unfortunately, she appeared to lose sight of the subtle differences between equations and inequalities.

The researcher believed that Vanessa was becoming more and more frustrated with this problem. That frustration could be heard in her comments: “It’s just not making any sense [exaggerated sigh]!” With a feeling that her frustration was not going to allow her to try any more alternatives, the researcher decided to move on to the next question.

While discussing #7 and #8a, Vanessa made comments similar to the following: “…when inequalities are set equal to zero, weird things happen.” It seemed that she was trying to justify the difficulties that she was encountering. For #8b (see Figure 15), neither side of the inequality started as zero. However, Vanessa decided to “move everything away from this side [pointing to the right side of the inequality],” producing a zero on one side of the inequality.

Vanessa applied the quadratic formula and noted that “those are our solutions if that [pointing to the inequality that she produced in Figure 87] were an equation.” Reaching this point and appearing not to have a clear direction from which to proceed, her frustration surfaced once again. She said “I just don’t know about these.” The researcher decided to proceed to other questions and hoped that Vanessa would return to #8a and #8b.
Unfortunately, when directed back to #8a later in the interview, she said “I have nothing to add.”

Vanessa employed algebraic approaches when working with previous quadratic inequalities. The researcher decided to ask her to solve #10 (see Figure 18) using a graphing technique. It seemed that Vanessa tried to blend a functional approach with a system of inequalities approach to find the solution for #10. The functional approach was implemented to separate the original single-variable inequality into two inequalities with two variables (see Figure 88). From there, Vanessa used her graphing calculator to find the region where the two inequalities overlapped (see Figure 89).

Initially, it appeared that Vanessa was having difficulty expressing the solution to this inequality. She talked about “points” being in the solution set. However, when asked to provide a point from her solution set, she noted “your solution would be the \(x\)-coordinate, it wouldn’t be the \(y\).” She added “you don’t have \(x\) and \(y\) [pointing to the original inequality], so your solutions wouldn’t be in the form \((x, y)\) coordinates. Vanessa, using the ‘trace’ function on her graphing calculator, went on to note that the solution on a number line would
be “about negative three halves over to around one.” She indicated that there was no way to find an exact solution by graphing.

![Figure 88. Vanessa's work for #10.](image)

![Figure 89. Screen shots from Vanessa's work on #10.](image)

While working several of the problems, Vanessa would perform a check. Often these checks produced information that contradicted her answer. In an effort to resolve this perturbation, Vanessa tried to eliminate the part of her solution that contained the contradiction. She did not go back and reevaluate her strategies that produced the erroneous solution(s). She seemed to understand that solutions for these single-variable quadratic inequalities would be uni-dimensional and not two-dimensional. That said she was unable to produce a correct solution for any of these problems. It appeared that she had some ideas on how to proceed towards a solution, but there were gaps in her knowledge that she was unable to work through.
Absolute value inequalities

Vanessa did not like the way #8c (see Figure 16) was presented. She correctly switched the inequality and noted that “it helps me see it better.” Implementing an algebraic manipulations strategy, she separated the absolute value inequality into two inequalities which she solved (see Figure 90). The solution that she produced was correct. She even included the correct logical connector.

\[
\left\{ \begin{array}{l}
3 \leq \frac{1}{4}x + 1 \\
\frac{1}{4}x + 1 \geq 3
\end{array} \right. \quad \text{FOR CALC.}
\]

\[
\left\{ \begin{array}{l}
\frac{1}{4}x + 1 \geq 3 \\
x \geq 2
\end{array} \right.
\quad \text{OR}
\left\{ \begin{array}{l}
\frac{1}{4}x + 1 \leq -3 \\
x \leq -16
\end{array} \right.
\]

Figure 90. Vanessa’s work for #8c.

Figure 91. The graph that Vanessa created for #8c.

Figure 92. Screen shots from Vanessa’s work on #8c.
When asked to describe her solution, Vanessa produced a graph that was based on the solutions of the two inequalities (see Figure 91). It was unclear if she was shading a region or a direction on the $x$-axis. In addition to her graph, Vanessa entered a modified version of the original problem into her graphing calculator and graphed it (see Figure 92). She seemed to be conflicted by the graph she produced and the graph shown on the calculator.

After considering a few values of $x$, she noted that her solution (see Figure 90) was correct. She indicated that the two-dimensional graphs that she and the calculator produced were incorrect. The graph “should just solely be on the number line.” She drew a graph on a number line (see Figure 93) and said “that would be a representation of that [solution].”

![Figure 93. Vanessa's representation of her solution for #8c.](image)

Vanessa’s work and comments seemed to indicate that she displayed a relational view of absolute value inequalities. She was able to explain why she separated the absolute value inequality into two inequalities. Creating the graph or “representation” of the solution was the source of some confusion for Vanessa. However, she was able to produce a number line that represented her solution.
Systems of inequalities

Vanessa was asked to find the solution to a system of linear inequalities (see Figure 17). In order to “solve for \(y\),” she applied algebraic manipulations to each inequality in order to isolate the variable \(y\) (see Figure 94). Once she had the \(y\) isolated for both inequalities, she entered these inequalities into her graphing calculator and graphed them (see Figure 95). She correctly noted that the region where the shading overlapped was the solution.

**Figure 94.** Vanessa's work to isolate the variable \(y\) for #9.

**Figure 95.** Screen shots from Vanessa's work on #9.

**Figure 96.** Vanessa's graph of her solution for #9.
The researcher asked Vanessa to sketch the solution on her paper to clarify what she believed was the solution (see Figure 96). She was aware that the inequality she assigned to Y1, in her graphing calculator, should have been a dotted line. She did not know of a way to get the calculator to show a dotted line and the shading. Unfortunately, as she drew her graph on the paper, she misidentified the lines and made the wrong line dotted (see Figure 96).

The researcher asked Vanessa to indicate whether or not a coordinate was part of the solution. To make that determination, she used the graph displayed on the graphing calculator, not her graph on paper. She moved the cursor on her graphing calculator to an approximate location of the given coordinate and determined if it was in the overlapping shaded region. For the coordinate (0, 0.5), she referred back to the inequalities. She noted that “it’s not included in our solution because it’s greater than [pointing to the inequality labeled 1 in Figure 94].”

While considering the coordinate (0, 0.5), she gave the following reason as to why it was not a solution.

Vanessa: Even though it is included here, it’s not included here [pointing to the shaded regions for each inequality in Figure 95b]. It has to be a solution to both of these.

Vanessa seemed to understand what being a solution to this system of linear inequalities meant. Other than misidentifying which boundary line should have been dotted in her graph, her work and rationale on this problem were correct. It seemed that her operational view of inequalities with regard to this problem was strong.
Explaining why the sense of an inequality changes

Recall that a few weeks prior to the interview the preservice teachers had been present during discussions in their methods courses about why the sense of inequality changes when applying multiplication or division of a negative number. To answer #11 (see Figure 19), Vanessa made use of algebraic manipulations. She created an inequality (see Figure 97) and then she divided by a negative value to produce an answer. She took that same inequality and used only addition and subtraction of terms along with division by positive values to arrive at an answer. She noted that the two answers were the same.

Vanessa’s explanation lacked generalization. Her explanation entailed a specific inequality with specific coefficients. Her explanation seemed to be enough for her to believe she had sufficiently addressed the question of why. She did attempt to provide another possible explanation involving number lines. However, as she started to talk about the explanation involving number lines, she quickly stopped and said “I don’t know how to explain it that way, but it helps me to think about it [referring to #11] like this (pointing to her work in Figure 97).”
Characterizing Vanessa’s understanding

Vanessa was a case of a preservice teacher whose comments and actions were consistent with an operational view of inequalities. It seemed that she did not focus on the subtle differences between inequalities and equations. She gave no indication that she believed that inequalities were equations. However, the strategies that she employed to solve inequalities were similar to those she would use to solve equations (Blanco & Garrote, 2007). She was often trying to implement a functions-based approach using her graphing calculator. However, she did not always correctly consider that she needed to think about each side of the inequality as a function of \( x \) and compare the two. Instead, she treated one side as a function of \( x \) and then employed shading methods as if the inequality had a variable \( y \) in it.

Cross Case Analysis

This section contains a cross case analysis of the preservice teachers’ responses to the tasks in the pre-student teaching interview. This discussion examines the four primary aspects of knowledge associated with the mathematical concept of inequalities as outlined in the framework for this study (see Table 2). Within each aspect, special attention will be given to similarities and differences that surfaced in the individual analyses of systems of linear inequalities and quadratic inequalities. In addition, possible connections among the aspects are discussed.

Strategies for solving inequalities

Preservice teachers seemed to have a preference in their strategy that was related to the type of problem encountered. They tended to stay with that strategy for each of the problems of a specific type. Preservice teachers used their graphing calculators to varying
degrees while implementing a solution strategy. It seemed that the choice of strategy played a major role in how the graphing calculator was employed. Employing an algebraic manipulations strategy seemed to lead to using a graphing calculator only to perform rudimentary arithmetic. On the other hand, while implementing a graphing strategy the graphing calculator was a tool used to graph functions, calculate x-intercepts, calculate values for particular x-values, and calculate the intersection points.

**Systems of linear inequalities.** An examination of participants’ work on the system of linear inequalities problem (see Figure 17) revealed that they employed similar strategies in order to solve the problem. All of the preservice teachers performed algebraic manipulations on each inequality in order to isolate the variable y. The steps taken by preservice teachers to arrive at a solution seemed to match those outlined by Reilly (2010).

**Quadratic inequalities.** Unlike the uniformity of the strategy implemented to solve the system of linear inequalities problem, preservice teachers employed various strategies to solve quadratic inequalities (#8a and #8b; see Figure 15). Christina and Heather implemented a graphing strategy and did not deviate from that strategy for either problem. Crystal, after an unsuccessful attempt at implementing a case strategy for #8a, employed an algebraic manipulations strategy on both #8a and #8b. On the other hand, Angela used an algebraic manipulations strategy that included a sign chart for #8a and a graphing strategy for #8b. Vanessa tried to implement a graphing strategy for #8a. However after encountering difficulty she transitioned to an algebraic manipulations strategy.

While implementing an algebraic manipulations strategy with quadratic inequalities, preservice teachers seemed to prefer to incorporate a sign chart method. An examination of
preservice teachers’ use of a graphing strategy to solve quadratic inequalities revealed some commonalities. Preservice teachers seemed to prefer to have zero on one side of the inequality before graphing. Graphing an inequality with a non-zero term on each side proved to be difficult for preservice teachers. However, in most cases they were able to overcome those difficulties. As a result of the preservice teachers’ desire to have one side of the inequality be zero, what they entered into their graphing calculators was almost identical. Of interest is the fact that in applying a functional approach, the preservice teachers did not enter zero as the second function. They seemed to be drawing on the knowledge that $Y_2=0$ on the calculator is the x-axis. The possible reason for why they may not have included the graph of $y = 0$ is the tangible nature of the x-axis on the graphing calculator.

**Relating inequalities to equations**

At times it was difficult to determine if the occasional mislabeling of inequalities as equations during the course of conversations was more than a slip of the tongue. However, when preservice teachers attempted to implement strategies utilized with equations to solve inequalities, it appeared that they were relating inequalities to equations. Another area in which preservice teachers seemed to relate inequalities to equations was the boundary lines or curves of inequalities.

**Systems of linear inequalities.** Some preservice teachers used their graphing calculators to graph the boundary lines. None of the preservice teachers showed a dotted boundary line on their graphing calculators. It seemed that preservice teachers tracked whether the boundary was solid or dotted without the benefit of a visual display.
**Quadratic inequalities.** While implementing an algebraic manipulations strategy to solve the quadratic inequality problems, preservice teachers converted the inequality that they were working with into a corresponding equation in order to find “key or critical” points. This action seemed to trigger verbal mislabeling of inequalities as equations.

**Shading as a process**

There were two shading methods employed by preservice teachers while graphing inequalities. The first method, denoted as the shade above or shade below method, was an observational connection between direction (i.e., above, below, up, down, left, or right) and an inequality symbol. The second method, denoted as the test point method, utilized test points to determine the appropriate half-plane to shade.

**Systems of linear inequalities.** In making the decision as to which half-plane to shade for each inequality, only one preservice teacher utilized the test point method. As Heather drew her graph, she said “I’m going to pick a point and make sure.” Heather’s use of test points seemed to be a way for her to check her hypothesis about which half-plane to shade. For the others, there was no evidence that they used test points to determine shading. It seemed that the act of isolating the variable $y$ allowed them to utilize a shortcut that involved an association between the inequality sign and the appropriate shading; the shade above or shade below method.

**Quadratic inequalities.** The shade above or shade below method appeared to be the preservice teachers’ preferred way of shading the quadratic inequalities while implementing a graphing strategy. The method seemed to be denoted in a different way: shade inside or shade outside. However, the associative nature of the method was same as that displayed
while shading systems of linear inequalities. None of the preservice teachers indicated a use of test points to determine the appropriate region to shade.

**Solutions of inequalities**

While describing the solution(s) of an inequality, preservice teachers seemed to note that a solution will make the inequality true. The manner in which they represented the solution varied. This variation appeared to be based on the solution strategy that was employed.

**Systems of linear inequalities.** Preservice teachers tended to utilize an overlapping shaded region on a Cartesian coordinate plane to denote the solution of a system of linear inequalities. The manner in which they created the overlapping shaded region varied. Some of the preservice teachers created the graphs on their graphing calculators; others utilized hand drawn graphs. They appeared to use those same graphs in order to identify whether a given coordinate was part of the solution or not.

**Quadratic inequalities.** An issue that many preservice teachers encountered as they solved the quadratic inequality problems was coordinating their introduction of the variable $y$. This issue was a product of their strategy. Preservice teachers, who implemented a graphing strategy, utilized a functional approach. The issue manifested itself in the descriptions of the solution to the quadratic inequalities; many proposed that a two dimensional region was the solution. It was not until they tried to substitute a coordinate with two variables into an inequality with one variable that they realized the existence of the issue.

Another interesting phenomenon related to the issue with coordinating the introduction of the variable $y$ was the format of the inequality. The first quadratic inequality
problem had zero on one side. Christina and Heather graphed the inequality in a two-dimensional plane and examined the graphs relative to the x-axis. Christina was able to modify the second quadratic inequality to get zero on one side and still examine the graphs relative to the x-axis. With #10 she was asked to leave terms other than zero on both sides and find the solution. The issue of coordinating her introduction of the variable y surfaced when she provided a two-dimensional region as her solution. Based on these responses, it appeared that the format of the quadratic inequality may play a role in difficulties faced by the participants in providing a solution.

**Possible connections among aspects**

There were some possible connections that surfaced while examining this particular group of preservice teachers’ understanding of inequalities. One possible connection existed between the preservice teacher’s concept image of equations and their ability to solve inequalities. Vanessa was firm in her belief that inequalities were not equations. However, she had great difficulty solving quadratic inequalities. On the other hand, Angela made an adjustment to her definition of an equation in order to include inequalities. This inclusion in her concept image did not inhibit her ability to solve quadratic inequalities. Thus, with the participants in this study, whether a preservice teacher included inequalities in their concept image of equations or not did not appear to be a predictor of their ability to solve quadratic inequalities.

Another possible connection that warranted examination was the preservice teachers’ choice of strategies and their concept image of equations. When presented with a system of linear inequalities problem, it was interesting that both Angela and Christina began the
problem by employing an elimination method for systems of equations. Both of the participants realized the error of their decision and made comments similar to the following:

Angela: These [pointing to the inequality signs] aren’t the same. So I don’t feel like I can just add those [pointing to the two inequalities on the right side of Figure 25]. Because then what kind of sign do I use?

The false start of treating the system of linear inequalities as if it were a system of linear equations was common to both; however they hold opposing views as to whether inequalities are part of their concept image for equations. Angela included inequalities in her concept image of equations while Christina excluded inequalities from her concept image of equations. Based on their divergent views, it would be difficult to say that their concept image of equations played a role in their decision to use an elimination method for systems of equations. A better explanation probably lies in the similarity of structure between systems of equations and systems of inequalities.

Prior research literature focused on three primary strategies used to solve inequalities: algebraic manipulations of inequalities; graphing; and case method. None of the participants in this study successfully implemented a case strategy to solve any of the inequality problems. Matter of fact, Crystal was the only participant who attempted to use this strategy and she did so for only one problem. Algebraic manipulations of inequalities were the most widely used strategy, a finding that was in line with literature (Tsamir & Almog, 2001; Tsamir et al., 1998). A graphing strategy was implemented a few times; exclusively with quadratic inequality problems (see Figure 15). It seemed that the participants’ concept image of equations did not play a role in the strategy that was employed to solve the inequality
problems. The type of inequality problem the preservice teachers were solving seemed to be a determining factor for which strategy was employed.

**Chapter Summary**

The focus of this chapter was the first research question of this study: What is the nature of preservice teachers’ understanding of inequalities prior to the student teaching field experience? In an effort to address this question, analyses of the five cases were presented; each case was defined to be the individual preservice teachers. The source of data for analyses was pre-student teaching interviews (Appendix C). After individual analyses, a cross case analysis was conducted. The emphasis of the cross case analysis was the four primary aspects of knowledge associated with the mathematical concept of inequalities as outlined in the framework for this study (see Table 2). Within each aspect, special attention was given to similarities and differences that surfaced in the individual analyses of systems of linear inequalities and quadratic inequalities. In addition, possible connections among the aspects were discussed.

The preservice teachers in this study employed a variety of strategies to solve problems that involve inequalities. Within the strategies there were similarities and differences in methods employed by the preservice teachers. Preservice teachers’ choice of strategy was not an indicator of their success in solving inequality problems. Overall, there was varying levels of success in finding correct solutions to some of the problems presented in this interview. This was particularly true for the quadratic inequalities.

With the system of linear inequalities problem, the preservice teachers were fairly uniform in the strategy that they implemented. All of the preservice teachers produced a
solution that appeared to be correct. However, there was some ambiguity as to whether the boundary lines were included in the object that the preservice teacher denoted as the solution. This was apparent as they reasoned with the intersection point of a dotted and solid line boundary lines.

Some of the preservice teachers’ included inequalities in their concept image of equations, others did not. However, this inclusion or lack thereof was not an indicator of preservice teachers’ understanding of solving inequalities. In addition, preservice teachers displayed varying ability to attend to the effect of multiplying or dividing by a negative value. Difficulties did not surface when multiplying or dividing by an explicitly stated negative value. Difficulties emerged while preservice teachers were working with a parameter that was defined to be any real number.

The second research question for this study was the following: How do preservice teachers use their understanding of inequalities while planning and implementing lessons during the student teaching field experience? To examine this question, each of the five preservice teachers was followed into their student teaching semester. After examining all the data collected and the content the preservice teachers were able to teach during their field placement, two cases emerged: the case of teaching systems of inequalities, and the case of teaching quadratic inequalities. The next chapter seeks to address the second research question relative to the case of systems of linear inequalities.
CHAPTER 5: TEACHING SYSTEMS OF LINEAR INEQUALITIES

The second research question for this study was the following: How do preservice teachers use their understanding of inequalities while planning and implementing lessons during the student teaching field experience? To examine this question, five preservice teachers were studied during their student teaching field experience. After scrutinizing the content that the preservice teachers were able to teach during their field placement and the data collected, two cases emerged: 1) teaching systems of linear inequalities; and 2) teaching quadratic inequalities. This chapter will address how understanding of inequalities was used while planning and implementing lessons on systems of linear inequalities. The primary sources of data for analysis were classroom observations and field notes. The secondary sources of data for analysis were pre-student teaching interviews (see Appendix C), post-observation questioning, lesson plans submitted by preservice teachers, and post-student teaching interview (see Appendix D).

Descriptions of each classroom relevant to this case will be provided. An outline of the content that was taught surrounding lessons related to systems of linear inequalities will be presented. In addition, considerations taken into account prior to analysis will be disclosed. The conceptual framework (see Table 2) of this study served as a guide for the analysis. Each cell of the conceptual framework corresponds to the intersection of a teaching practice and an aspect of knowledge associated with inequalities. The manner in which preservice teachers applied the three teaching practices (explaining a mathematical idea, solving a mathematical problem, and using technology) with respect to each of the four primary aspects of knowledge associated with the mathematical concept of inequalities
(strategies for solving inequalities, relating inequalities as equations, shading as a process, and solutions of inequalities) will be discussed. Additionally, the view of inequalities (operational or relational) displayed and/or fostered by preservice teachers while applying each teaching practice with respect to the primary aspects of knowledge will be discussed.

**Descriptions of Classrooms**

Four of the five preservice teachers, who participated in this study, taught lessons pertaining to systems of linear inequalities. Angela and Christina taught Algebra I classes. Crystal taught an Algebra II Honors class and Vanessa taught an Algebra II class. Angela and Christina taught at the same large rural high school. Crystal and Vanessa taught at large suburban high schools.

Christina’s cooperating teacher noted that her class included a large number of students who were repeating Algebra I. In addition, her cooperating teacher pointed out that Christina’s class had more inclusion students than any of the Algebra I classes in the recent past. Christina’s class had 18 students, two-thirds of which were minorities. Christina’s classroom had a whiteboard and an overhead projector in the front of the classroom. Classroom sets of netbooks and calculators (TI-83 plus) were readily available within her classroom. Very few students used their own calculator during observations. Students’ desks were arranged in rows. The two center rows were facing the whiteboard and projector screen in the front of the classroom. The four other rows, two on each side of the center rows, were facing the center rows.

The 14 students in Angela’s class were all freshmen. Approximately half of the students were minorities. The desks were typically arranged in rows facing the front of the
classroom. However, from time to time, Angela would rearrange desks into “stations” around the classroom. Angela’s classroom had a LCD projector connected to a laptop and a wireless scribe tablet. In addition, she had a whiteboard in the front of the classroom. Classroom sets of netbooks and calculators (TI-83 plus) were readily available within her classroom.

Angela’s cooperating teacher noted that none of her students used their own calculators.

Crystal’s Algebra II Honors class had 33 students enrolled. Less than one-third of her students were minorities and well over half were classified as sophomores. Crystal and her cooperating teacher mentioned, on separate occasions, that this class had several students who, in their opinion, should not be in an Honors class. The desks were arranged in rows that faced the front whiteboard. Graphing calculators (TI-83 Plus) were available for students; however it appeared that the majority of the students used their own graphing calculators (TI-83 or TI-84 models). A LCD projector attached to a laptop, a whiteboard, and an overhead projector were available for use within Crystal’s classroom.

The enrollment in Vanessa’s Algebra II class was 28 students. Almost half of the students were minorities. Vanessa’s cooperating teacher noted that the number of students classified as sophomores, juniors, or seniors was about the same. Students sat in rows of seats that all faced the front of the classroom. Vanessa had a LCD projector, which was connected to a computer and a document camera, available for use during her lessons. There was a whiteboard at the front of the class. In addition, she had a TI-84 Plus connected to a TV set attached to the wall in the front of the classroom next to the whiteboard. A classroom set of graphing calculators (TI-83 Plus) were available for students to use, although about half of the students appeared to use their own graphing calculator during observations. It was
difficult to determine the exact models being used by students, but it appeared that students were using either TI-83 or TI-84 graphing calculators.

**Lessons Taught Prior to and After Systems of Linear Inequalities**

Three of the four preservice teachers (Christina, Crystal, and Vanessa) taught lessons pertaining to systems of linear inequalities immediately following lessons about systems of linear equations. For these three preservice teachers, the lessons were taught within the first two weeks of their student teaching field experience. In the days prior to lessons related to systems of linear inequalities, preservice teachers covered manipulating and graphing linear equations in two variables. In addition, they discussed systems of linear equations. As part of those discussions, they taught students to solve systems of linear equations by substitution, graphing, and elimination. Crystal and Vanessa showed their students how to solve systems of linear equations via matrices.

The fourth preservice teacher, Angela, taught her lesson over a month after teaching her students systems of linear equations. The lesson was taught in the final week of her student teaching field experience. Angela’s cooperating teacher made the decision regarding the order in which lessons would be presented. During her post-student teaching interview (May 11, 2011), Angela noted that she would have preferred to teach systems of linear inequalities immediately after systems of linear equations.

For three of the four preservice teachers, the lesson on systems of linear inequalities was the final lesson in a unit. During the two days following systems of linear inequalities, Crystal taught a lesson on linear programming. Portions of her linear programming lesson most directly related to solving systems of linear inequalities were included in the analysis.
However, some parts of that lesson were excluded (i.e., determining the values that satisfy the objective function).

**Areas of Consideration Prior to the Analysis**

The timing of lessons within the student teaching field experience was considered during analysis. As mentioned earlier, three preservice teachers taught these lessons within a week or two of the start of the student teaching field experience. Those three preservice teachers indicated that they started planning their unit, which included systems of linear inequalities, prior to starting their field experience full time. Angela, who taught her lesson near the end of her student teaching experience, believed that the month and half between her lessons on linear equations and her lesson on systems of linear inequalities played a role in her planning.

The influence of cooperating teachers cannot be left out of this discussion. The researcher had a face-to-face discussion with each of the cooperating teachers prior to the beginning of the student teaching field experience. The researcher requested that preservice teachers be the primary source of planning for lessons regarding systems of linear inequalities. All of the cooperating teachers agreed to the request.

When asked in post-student teaching interviews, all of the preservice teachers indicated that they were primarily responsible for lessons that were presented surrounding systems of linear inequalities. The cooperating teachers offered advice and made suggestions as to possible changes. These were actions that cooperating teachers would have performed for any preservice teacher under their supervision. It appeared that lessons regarding systems
of linear inequalities were dictated by the preservice teachers. However, there was evidence that cooperating teachers played a role in some components of the lessons.

**Analysis of Teaching Aspects of Knowledge Associated with Inequalities**

**Introduction of the aspects**

The conceptual framework of this study was a matrix design with two overarching views (see Table 2). The column headings of the framework were aspects of knowledge associated with inequalities. These aspects were identified within a literature review. With regard to this study, there were four primary aspects of knowledge associated with inequalities: 1) Strategies for solving inequalities; 2) Relating inequalities as equations; 3) Shading as a process; and 4) Solutions of inequalities. Within those primary aspects there were other aspects that may be pertinent to more than one primary aspect: domain; use of variable(s); factors of products and/or quotients; logical connectives; and interpretations of the inequality symbol. By using these four primary aspects as an initial lens for making sense of classroom episodes, the researcher was able to identify and examine how preservice teachers implemented the three teaching practices (i.e., explaining a mathematical idea, solving a mathematical problem, and using technology) with respect to each aspect. In addition, the researcher made an assessment as to whether an operational or relational view of inequalities was promoted by the preservice teachers with regard to each teaching practice within each aspect.

The preservice teachers’ decisions regarding how to explain ideas, solve problems, and use technology while teaching lessons related to systems of linear inequalities revealed nuances that will be presented. In this particular case, the aspect of treating inequalities as
equations seemed to occur when preservice teachers or their students were working with the boundary lines within the system of linear inequalities. In addition, how preservice teachers and their students dealt with the process of shading inequalities and solutions of inequalities was studied and will be presented.

**Strategies to solve inequalities**

For purposes of this study, the strategies employed by the preservice teachers were categorized using the descriptions from Tsamir and Almog (2001). Tsamir and Almog (2001) concluded that students, for the most part, use three strategies to solve inequalities: (1) algebraic manipulation, (2) drawing a graph, and (3) using a case approach. A case approach strategy\(^\text{16}\) (Dreyfus & Eisenberg, 1985) requires consideration of all of the possibilities that would produce the given inequality. An algebraic manipulations strategy could include the following:

- addition or subtraction of the same term to both sides of the inequality;
- multiplication and division of both sides of the inequality by identical factors, with appropriate attention to the orientation of the inequality symbol;
- simplification of an expression through combining like terms or factoring;
- multiplying both sides of the inequality by the square of the denominator;
- “examining quadratic inequalities (i.e., \(ax^2 + bx + c > 0\)) by first relating to the quadratic roots, or by investigating the sign of ‘a’ and the sign of the determinant” (Tsamir et al., 1998, p. 131); and
- “relating to an inequality of the type \(ab > 0\) as a compound system of \(\{a > 0 \text{ and } b > \)

\(^{16}\) A full description and an example of the case approach strategy can be found in Chapter 2.
0} or \{a < 0 \text{ and } b < 0\}” (Tsamir & Almog, 2001, p. 515).

If a graph of given linear inequalities was drawn and used to attain the solution, then a drawing a graph (graphing) strategy was implemented. In the Tsamir and Almog (2001) study, students were categorized as applying a graphing strategy if their graphs were on a two dimensional Cartesian coordinate plane. Graphs on a number line or sign chart were not included in this strategy. Sign charts and number lines were tools used by the students while invoking an algebraic manipulations strategy.

There were situations in which the separation between the strategies was unclear. Algebraic manipulations may be performed as part of the implementation of a graphing strategy. An example can be seen when students or preservice teachers isolate the variable $y$ of a linear inequality with two variables in order to graph the boundary line of the inequality. In keeping with the literature and in an effort to maintain clarity, if algebraic manipulations were performed in order to graph an inequality on a two dimensional Cartesian coordinate plane, then a graphing strategy was implemented.

During pre-student teaching interviews, preservice teachers were asked to solve a system of linear inequalities problem (see Figure 17). All four preservice teachers employed a graphing strategy to solve the problem. While teaching, preservice teachers were uniform in advocating a graphing strategy in order to solve systems of linear inequalities. Similarities and differences within the implementation of a graphing strategy existed. Those similarities and differences will be discussed with respect to each of the teaching practices.

**Explaining a mathematical idea.** As one would expect, there were variations in the wording of a graphing strategy as presented by each of the preservice teachers to their
students. Essentially, preservice teachers asked their students to perform the following steps in order to solve a system of linear inequalities using a graphing strategy (see Figure 98 and Figure 99 for specific examples from Vanessa and Angela respectively): 1) graph the boundary line of the first inequality; 2) determine if that boundary line was solid or dashed (dotted) by examining the inequality symbol; 3) shade the appropriate half-plane; 4) repeat steps 1 through 3 for other inequalities; and 5) if possible locate the region of the plane where the shading of each inequality overlaps.

<table>
<thead>
<tr>
<th>Graphing Systems of Linear Inequalities Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. We show the solution to a system of linear inequalities by <strong>graphing</strong>_____ them.</td>
</tr>
<tr>
<td>→ y=mx+b</td>
</tr>
<tr>
<td>2. Graph the line using the <strong>y-intercept</strong>___ and _<em><strong><strong>slope</strong></strong></em>.</td>
</tr>
<tr>
<td>If the inequality is &lt; or &gt;: make the line <strong>dotted/dashed</strong>__ because the solution is <strong>not included</strong>_______ in our solution.</td>
</tr>
<tr>
<td>If the inequality is ≤ or ≥: make the line <strong>solid</strong>__________ because the solution is <strong>included</strong>___________ in our solution.</td>
</tr>
<tr>
<td>3. The solution also includes points not on the line, so you need to <strong>shade</strong>__ the region of the graph:</td>
</tr>
<tr>
<td>shade <strong>up/right</strong>_____ the line for ‘y &gt;’ or ‘y ≥’.</td>
</tr>
<tr>
<td>shade <strong>down/left</strong>_____ the line for ‘y &lt;’ or ‘y ≤’.</td>
</tr>
<tr>
<td>4. The solution to the system of linear inequalities is the <strong>shaded</strong>______ parts of the graph that <strong>overlap</strong>______.</td>
</tr>
</tbody>
</table>

Figure 98. Vanessa’s notes/worksheet outlining the process for solving a system of linear inequalities (Lesson plan; February 28, 2011).
One difference between the preservice teachers surfaced in how they presented the graphing strategy to their students. Vanessa spelled out all of the steps she expected her students to follow to implement a graphing strategy before doing a single example (see Figure 98). Christina jumped right into a system of linear inequalities example and divulged the steps she expected her students to perform as she worked through the example. Crystal and Angela also outlined the procedures they expected their students to use within the context of examples. This decision as to how to present the graphing strategy seemed to be predicated on personal preferences regarding teaching style. However, there were undertones that pointed to assumptions made by the preservice teachers regarding their students’ prior knowledge.

Christina and Vanessa made assumptions as to what their students would be able to do while working with linear inequalities with two variables. During her post-student teaching interview (May 4, 2011), Christina said that “they [her students] had previously been taught how to graph one inequality [referring to a linear inequality with two variables].” She believed that her lesson with systems of linear inequalities would involve recalling that

```
How do we graph them?
1) Graph the line that defines the boundary (y = ax + b).
2) Use a solid line for y ≤ and y ≥
   Use a dashed line for y < and y >.
Why does it matter if we use a solid or dashed line?
What's the difference?
3) Shade above the line for y > and y ≥
   Shade below the line for y < and y ≤
```

Figure 99. Angela's notes outlining the process to graph a system of linear inequalities (Lesson plan; May 2, 2011).
knowledge. As such she decided not to provide step-by-step directions for her version of a graphing strategy prior to her examples as she taught (Lesson plan and classroom observation; March 4, 2011). She noted, in her post-student teaching interview (May 4, 2011), that she wanted her students to “talk [her], piece by piece, through it.” Christina added that her lesson plan allocated more guidance for her students once she discussed the solution of the system of linear inequalities.

Christina and Vanessa seemed to share the same belief that their students would be able to effectively recall prior knowledge related to linear inequalities. Angela and Crystal did not make this assumption. They decided to start their lesson with single variable inequalities on a number line. The reason behind Angela and Crystal’s decision to start their lessons with single variable inequalities did not appear to be the same. In her post-student teaching interview (May 9, 2011), Crystal said “I don’t believe inequalities are covered that much.” This belief seemed to be the reason why she started her lesson with “the first kind of inequalities that we see [pointing to x ≥ 3 on her lesson plan].” Her belief also highlighted her assumptions about the material that was taught to students.

Angela, during her post-student teaching interview (May 11, 2011), noted that a significant amount of time had passed since her students had been exposed to inequalities. In addition, she noted that she had taught her students a unit on exponents and a unit on polynomials since she completed her unit on linear equations. As mentioned earlier, the order of topics covered was dictated by Angela’s cooperating teacher. Angela mentioned that the lesson on inequalities and systems of linear inequalities was “out of place” (Post-student teaching interview; May 11, 2011). She would have preferred to incorporate this lesson into
her linear equations unit. As a result, Angela’s decision to start her lesson with single variable inequalities on a number line seemed to be linked to the timing of the lesson within the academic semester.

Although decisions by Angela and Crystal to start their lesson with single variable inequalities seemed to be based on separate motives, their goal behind this decision seemed to be the same. Based on comments in their post-student teaching interview, Angela and Crystal’s goal appeared to be to establish foundational knowledge of graphing single variable inequalities and build up to implementing a graphing strategy to find the solution of systems of linear inequalities. This became evident upon examination of the progression of examples that both preservice teachers selected for their individual classes. From the very first example, graphing was the focal point in both classrooms. Angela noted that she “wanted them [her students] to make the connection between opened and closed being a dotted or solid line” (Post-student teaching interview; May 2, 2011). Even the manner in which they described a graphing strategy for systems of linear inequalities was a building exercise in that it directly referenced prior examples of linear inequalities.

One possible explanation for these differing assumptions may be that these lessons were taught to students of different levels (e.g., Algebra I, Algebra II, and Algebra II Honors). While that possibility cannot be ignored, it does not seem to fit with the fact that Crystal with her Algebra II Honors students and Angela with her Algebra I students introduced systems of linear inequalities in a similar fashion.

The student teaching experience itself may have affected assumptions that preservice teachers made. During the student teaching field experience for this particular program,
preservice teachers spend most of their time during the first six weeks of a semester in methods block courses. Approximately once a week, the preservice teachers were at their high school observing. During those six weeks, a substantial amount of material was taught to students by the cooperating teacher. Christina noted that her cooperating teacher taught lessons on how to graph inequalities while she was not there.

Christina’s inability to watch her students being taught a specific topic was not unique to her. All of the preservice teachers in this study noted that there were lessons that they did not observe that pertained to material that they taught. For mathematics teachers in general, this is a regular occurrence. Teachers of Algebra II often can only make assumptions as to what was taught to their students in Algebra I. A profound difference may lie in the fact that the cooperating teacher, who taught the prerequisite lesson(s), was observing as preservice teachers called upon that prior knowledge in subsequent lessons. The presence of cooperating teachers, a necessary and fundamental component of the student teaching field experience, may provide some explanation for differences in the manner in which a graphing strategy was presented in the classroom. It is possible that Christina and Vanessa did not start their lessons with single variable inequalities on a number line because they did not want to give the appearance that they needed to re-teach something their cooperating teacher taught earlier.

**View of inequalities.** The manner in which preservice teachers explained to their students how to apply a graphing strategy to solve systems of linear inequalities was consistent with an operational view of inequalities. Preservice teachers’ actions gave the impression that the process was nothing more than a way to obtain the solution. There was no
discussion or explanation about why the steps were appropriate or the interconnections between the steps.

**Solving a mathematical problem.** Based on their responses during post-student teaching interviews, preservice teachers indicated that they selected the examples in the classes in which they were teaching. These selections may have been based on suggestions from cooperating teachers. However, during post-observation questioning, none of the preservice teachers identified a specific system of linear inequalities example that was influenced by their cooperating teacher.

An examination of the examples that preservice teachers selected revealed some interesting commonalities. Of the examples with systems of linear inequalities, the overwhelming majority of the linear inequalities that comprised the systems were ones where the variable $y$ was already isolated. Crystal’s Algebra II Honors class did not encounter a linear inequality where the variable $y$ was not isolated until she transitioned to linear programming (Classroom observation; Day 2: March 11, 2011). Angela’s Algebra I class worked two examples with linear inequalities where the variable $y$ was not isolated (see Figure 100). Likewise, Vanessa’s Algebra II class was given only two examples with linear inequalities where the variable $y$ was not isolated (see Figure 101). Students in Angela’s class utilized two algebraic manipulations to isolate the variable $y$; subtracting the $x$-term from both sides of the inequality and then dividing all of the terms of the inequality by the coefficient of the $y$-term. Whereas, Vanessa’s students only needed to apply one algebraic manipulation to isolate variable $y$; subtracting a constant from both sides of the inequality for the first example and dividing both sides of the inequality by the coefficient of the $y$-term for
the sixth example. It appeared that the level of the course (Algebra I or Algebra II) did not play a role in the number of examples presented with linear inequalities where the variable $y$ was not isolated. Additionally, the level of the course did not seem to be an indicator of the relative difficulty associated with isolating the variable $y$ in the examples selected.

![Example 3: Solve by graphing.](image)

**Figure 100.** Example #3 from Angela's lesson plan (Lesson plan; May 2, 2011).

![Graphing Systems of Linear Inequalities](image)

**Figure 101.** Examples #1 and #6 from Vanessa's lesson plan (Lesson plan; February 28, 2011).

The appearance of the individual linear inequalities was not the only commonality related to examples selected by preservice teachers. All of the linear inequalities selected by the four preservice teachers had an integer value for the $y$-intercept. Matter of fact, two-thirds (32 out of 48) of the linear inequalities in the systems selected by preservice teachers as examples had a $y$-intercept that was a whole number.
The category of number selected to be the slope of the linear inequalities within the systems, as chosen by preservice teachers, was not as uniform as the value of the y-intercept. The slopes of Christina’s linear inequalities were –1, 1, or 2. She did not deviate from those values in any of the eight linear inequalities in her examples. Christina seemed to limit the values of the slope in an effort to address her students’ difficulties with graphing (post-student teaching interview; May 4, 2011). The slopes in Crystal’s examples were exclusively integer values until she transitioned to linear programming (Classroom observation; Day 2: March 11, 2011). Unlike Christina, Crystal did not indicate that her students had major difficulties with graphing. In contrast to Christina and Crystal, Angela and Vanessa presented examples where the slopes of the linear inequalities were a mix of integers and fractions. The variation in the values of slope used by the preservice teachers gave the impression that course level could not be considered as a determining factor.

Preservice teachers made decisions, regarding the manner in which linear inequalities of the systems were presented to their students, which seemed to shape the implementation of the graphing strategy. By presenting the linear inequalities with the variable y isolated, the preservice teachers were able to connect back to graphing linear equations in slope-intercept form (\(y = mx + b\)). This was a topic that most, Angela being the only exception, had covered a few days prior to lessons regarding systems of linear inequalities.

The values of the y-intercept and the slope of the linear inequalities appeared to have been selected to further facilitate the implementation of a graphing strategy. Matter of fact, in restricting every y-intercept to be an integer value, all of the preservice teachers seemed to be structuring their lessons towards graphing the linear inequalities by hand without the
assistance of technology such as graphing calculators. To this end, each of the preservice teachers plotted the y-intercept and used the slope to locate at least one more point. By selecting integer values for the y-intercepts, preservice teachers reduced the difficulties that may have been encountered when graphing without technology. Additionally, restrictions Christina placed on the values of the slope for each linear inequality (i.e., limiting the values to \(-1, 1,\) or \(2\) gave the impression that she wanted to simplify the graphing process even further for her students.

**View of inequalities.** The examples that preservice teachers selected seemed to foster an operational view of inequalities. The linear inequalities in those examples were presented in a preordained format that ushered the implementation of a graphing strategy, as prescribed by the preservice teachers. The purpose of the examples appeared to be the reinforcement of a set of procedures from a graphing strategy.

**Using technology.** The degree to which technology was incorporated into the lesson(s) varied among preservice teachers. Access did not seem to be an issue. All of the preservice teachers had access within their classrooms to a set of graphing calculators. All of the classrooms had a computer or laptop connected to a projector. In addition, each of the students in Angela and Christina’s classes had access to a netbook. The computers, laptops, and netbooks all had wireless access to the internet. With technology readily available, the preservice teachers gravitated to the graphing feature of their graphing calculators.

During post-student teaching interviews, preservice teachers indicated that they did consider other technology. Some of the preservice teachers indicated that they considered “online apps.” Unfortunately, none of the preservice teachers could provide details as to
where those “online apps” were located or features of those “online apps.” In her post-
student teaching interview (May 4, 2011), Christina may have summed up their opinion best
when she said the following: “I may have looked online, but I don't think I seriously
considered using something other than the [graphing] calculator.”

When graphing calculators were employed by preservice teachers, they graphed the
linear equation representing the boundary line of the inequality. This was done in the Y=
screen. In addition, they utilized the shading feature on the Y= screen to represent either
greater than or equal to (≥) or less than or equal to (≤). Preservice teachers were unable to
make the calculator represent greater than (>) or less than (<). In addition, preservice teachers
employed the intersection feature of the graphing calculator to find the coordinate where the
boundary lines crossed.

The preservice teachers of this study typically walked their students step-by-step
through how to use the graphing calculator. The endeavor resembled a button pushing
exercise more than an exploration. The manner in which graphing calculators were utilized
while implementing a graphing strategy could be categorized as “technology as servant;”
meaning that it was used as a “supplementary tool” but not “in creative ways to change the
nature of the [examples]” (Goos et al., 2003, p. 78).

Even with the preservice teachers’ apparent preference towards either TI-83 or TI-84
graphing calculators, their usage of the calculators in the classroom could be considered as
minimal. The technology was not an integrated part of the lessons. The calculators seemed to
be promoted as a checking mechanism. Vanessa told her students that the graphing calculator
was “a great way to check to see if you’re right” (Classroom observation; February 28,
Furthermore, the preservice teachers seemed to employ only certain features of the
calculators. They did not use an app from Texas Instruments called *Inequalz*. The *Inequalz*
app allows users to graph inequalities. When the *Inequalz* app was mentioned to the
preservice teachers in the post-student teaching interview, none of them knew it existed.

The preservice teachers’ decision regarding the use of graphing calculators may have
been heavily influenced by their cooperating teachers. Some of the preservice teachers noted
that the decision to use graphing calculators with this particular unit was widespread. This
was clear from the following comment:

> Vanessa: She [referring to her cooperating teacher] said, as a Math department, we've
decided it's easier to show them this [referring to using the graphing calculator to
solve systems of linear inequalities]. (Post-student teaching interview; May 13, 2011)

Christina provided a glimpse into what appeared to be the driving force behind the push to
employ graphing calculators.

> Researcher: So what was the purpose for the calculators? Why did you use those?
> Christina: Well, a couple of reasons. I guess. One is they can use a calculator on the
[state-level summative assessment]. And that seems to drive a lot in the math
department here. …It's for the kids to be able to pass the [state-level summative
assessment]. (Post-student teaching interview; May 4, 2011)

The cooperating teachers’ desire to incorporate the graphing calculators into lessons
pertaining to systems of inequalities as well as systems of equations seemed to be influenced
by a state-level summative assessment.
Only one of the preservice teachers, Angela, did not utilize graphing calculators in her lesson. The students were provided graphing paper, color pencils, and a straight edge. Angela used a wireless tablet to write on Word documents that she projected onto the screen. She drew her graphs on grid boxes that she placed into the document prior to the lesson (see Figure 102). Angela had difficulty drawing straight lines on the tablet. She reminded her students to use a straight edge when drawing their graphs on graph paper. The tablet allowed Angela to use different colors to graph the boundary lines and to shade the half-planes. Since the students were provided color pencils, they were able to do the same thing on their own graphs.

![Example 3: Solve by graphing](image)

*Figure 102. Example #3a and the graph from Angela's class (Classroom observation; May 2, 2011).*

Angela’s cooperating teacher strongly encouraged her to use graphing calculators in almost every lesson. During the post-student teaching interview (May 11, 2011), Angela
noted that her cooperating teacher “wanted them [the students] to be able to put it [the linear inequalities] into their calculators.” Angela did not show the students how to use their graphing calculators while working with systems of linear inequalities. She said that she was “sick of that being the main thing they learn; being able to put it into their calculators.” It appeared that Angela was cynical with regard to using the calculator. During a classroom observation (May 2, 2011), Angela was graphing a boundary line of an inequality with her students. After graphing the boundary line, Angela asked her students to identify the y-intercept. During her post-student teaching interview (May 11, 2011), as part of a stimulated recall, Angela was shown a video clip that contained that moment of the classroom observation (May 2, 2011). Angela made the following comment in response to her question to her students in the video clip.

Angela: They don’t know. All they have been told is put it in a calculator.

While Angela provided reasons why she did not use the graphing calculators in this situation, it was difficult to determine whether or not she would have used them under different circumstances.

Even though the idea of using a graphing calculator may have been pressed upon the preservice teachers, they were able to articulate potential benefits. Christina seemed to focus on the ability of the graphing calculator to graph linear equations.

Christina: …I think because how much they've [her students] struggled graphing by hand in general if they could see, if they could get it, and put it into calculators. … they're going to … put it [referring to the equation of the boundary line] in the calculator and see the solution and be able to draw it. For my student, I thought that
would satisfy what I was looking for [referring to finding the solution to a system of linear inequalities].

It appeared that she was willing to allow her students to use the graphing calculator as an aid in order to facilitate their understanding of the solution to a system of linear inequalities.

**View of inequalities.** Preservice teachers had technology available to use during their lessons. They decided to use only graphing calculators during their lessons. Students were shown step-by-step how to use the graphing calculator to implement a graphing strategy. As noted earlier, the endeavor resembled a button pushing exercise more than an exploration. Preservice teachers seemed to foster an operational view of inequalities as they used graphing calculators while implementing a graphing strategy. There was very little to no reasoning about the graphs, how they represented the inequality relationships, and how they were related to one another.

**Treating inequalities as equations**

Within the content area of systems of linear inequalities, how preservice teachers treated or thought of inequalities as equations was most obvious in the way they considered the boundary lines. Preservice teachers seemed to prefer having their students isolate the variable y in each inequality in order to graph the boundary lines. The roots of that decision seem to lie in the desire to use the graphing calculator. The process of isolating the variable y produced several moments where the preservice teacher and/or the students treated the inequalities as equations. Those moments were examined and will be discussed.

An examination of lessons was conducted to evaluate the source of treating inequalities as equations. The preservice teacher was considered to be the source if they
provided comments, either written or oral, that treated inequalities as equations. On the other hand, the students were labeled as the source if they made comments designed to treat inequalities as equations. Examples where a preservice teacher and the students were the source will be presented.

**Explaining a mathematical idea.** A perspective to consider may be the level of importance that the preservice teacher placed on separating equations and inequalities. During the observations, there were several moments in which an inequality was called an equation. In the moment, the verbal mislabeling may have been nothing more than a slip of the tongue. There was a moment during Angela’s classroom observation (May 2, 2011) where a student treated inequalities as equations and Angela did not correct her student. Angela outlined her process for graphing a linear inequality with two variables to the class prior to any examples. After providing the steps of her process, Angela asked the class the following question: Why does it matter if we shaded above or below the line? A student offered an answer and Angela commented on the student’s answer.

    Student: If you shaded in a different direction it is not going to be the same equation.
    Angela: It’s not going to be the same equation. Well, it’s not going to change the equation if you shade it wrong.

In this exchange, the student was the source of treating inequalities as equations. It appeared that the student and Angela were not referring to the same object. The student’s use of equation seemed to actually be a reference to the inequality that would be graphed. On the other hand, Angela’s use of equation seemed to be a reference to the boundary line of the inequality that would be graphed. The lack of a specific problem at this point in the lesson
may have contributed to the miscommunication. In the moment, the miscommunication may not have been deemed as critical.

During the stimulated recall portion of Angela’s post-student teaching interview (May 11, 2011), the comments by the student and Angela were part of an episode that she viewed. Without prompting from the researcher, Angela noted that the student was incorrect to say equation. Angela offered the following statement:

Angela: …if I would have been nit-picky about what she actually said about calling it an equation then that would have been just been because you [referring to the researcher] were sitting there. Because I was thinking that we really don't need to get into that right now. (Post-student teaching interview; May 11, 2011)

Angela’s statement seemed to point to a hierarchy. Her “nit-picky” classification that she attached to correcting the student’s misidentification appeared to indicate the low level of importance Angela placed on separating equations and inequalities. This low level may have been assigned because she did not view the misidentification as detrimental to reaching her objective for this lesson.

**View of inequalities.** Angela’s “nit-picky” evaluation may provide a glimpse into how student’s conceptions, surrounding the treatment of inequalities as equations (Tsamir & Bazzini, 2004), develop and persist. Comments made by a teacher, whether intentional or accidental, may be a catalyst for formation of incorrect connections between inequalities and equations. Once those conceptions begin to form and present themselves, a teacher’s decision not to address those misconceptions may serve to reinforce them. Preservice teachers who
actively or passively reinforced incorrect connections between equations and inequalities seemed to foster an operational view of inequalities.

**Solving a mathematical problem.** While solving examples with their students, there were moments when preservice teachers treated inequalities as equations. The majority of the incidents occurred when preservice teachers incorrectly labeled inequalities as equations. These incidents of mislabeling seemed to be accidental and did not appear to be systemic. However, there were a few incidents where preservice teachers seemed to display a conflicted understanding regarding treating inequalities as equations.

In the following example, the preservice teacher acted as the source of relating inequalities to equations. As Crystal graphed a linear inequality with two variables in her class (Classroom observation; March 10, 2011), she told her students to convert the inequality to an equation (i.e., \( y \leq x + 2 \) became \( y = x + 2 \)). She noted that the corresponding equation was the boundary line of the inequality. She said “the line becomes an inequality when I start to shade.” In her post-student teaching interview (May 9, 2011), Crystal said “I liked the way I made it clear that we're not graphing the inequality first, we graph the actual line [referring to equation].” Based on her approach and comments, it appeared that Crystal had blurred the line between equations and inequalities.

Crystal’s blurring of inequalities and equations surfaced in a later lesson as she was showing her students how to solve a linear programming problem (see Figure 103). Once again, she advocated converting the inequalities, the constraints of the linear programming problem, into equations. She applied algebraic manipulations to the inequalities in order to isolate the variable \( y \). As she performed the algebraic manipulations on the first constraint of
the problem, Crystal wrote her transformations out horizontally (see Figure 104). In this horizontal string, she changed the inequality sign to an equal sign. While working with the second constraint in the same linear programming problem, she once again changed the inequality sign to an equal sign (see Figure 105a). However, a few seconds later, she changed the equal sign (see Figure 105a) to an inequality sign (see Figure 105b). She provided no explanation for the change back to an inequality. Her students did not question her about the switch. Even after making the changes to the second constraint (see Figure 105a and Figure 105b), she never went back to the first constraint (see Figure 104) and switched the equal sign back to an inequality sign. It seemed that Crystal was conflicted as to which sign she should select to display; an inequality sign (i.e., \( \leq \)) or an equal sign.

![Linear Programming problem Crystal worked out in her class (Lesson plan; March 11, 2011).](image)

**Figure 103.** Linear Programming problem Crystal worked out in her class (Lesson plan; March 11, 2011).
Figure 104. Crystal's work with the first constraint of a problem (Classroom observation; March 11, 2011).

Figure 105. Crystal switching between an equation and an inequality with the second constraint of a linear programming problem (Classroom observation; March 11, 2011).

**View of inequalities.** Comments made by a preservice teacher, whether intentional or accidental, may be a catalyst for formation of incorrect connections between inequalities and equations. Crystal’s blurring of inequalities and equations during her lesson was consistent with actions she displayed during her pre-student teaching interview (February 17, 2011). Her belief that inequalities were equations seemed to manifest during her lessons as she switched back and forth between an inequality sign and an equal sign (see Figure 105). These actions were consistent with an operational view of inequalities.
Using technology. As mentioned earlier, preservice teachers graphed a linear equation representing the boundary line of an inequality to solve a system of linear inequalities (see Figure 98 and Figure 99 for examples). This step in their graphing strategy was also seen when graphing calculators were employed by preservice teachers to solve systems of linear inequalities. Vanessa’s usage of her graphing calculator, as outlined below, was typical among preservice teachers.

Vanessa: So we know we can graph on our calculator, we can graph linear equations. Does anybody know how we graph inequalities? No. OK, [STUDENT NAME].

Student: We go over to the little line [referring to the line to the left of Y1= on the graph screen; see Figure 106].

Vanessa: Over to this one [moving the cursor to the line to the left of Y1= on the graph screen]? OK, now what do I do?

Student: Change that to a triangle at the bottom.

Vanessa: You hit enter. And look, y'all

Figure 106. Calculator screenshots from Vanessa class (Classroom observation; February 28, 2011).

Figure 107. Calculator screenshots from Vanessa class (Classroom observation; February 28, 2011).
see how that is changing [cycling through the icons]? It changes to all sorts of stuff. OK, I want for the first one, I want it to shade down. So I'm going to make sure that my little triangle is on the bottom (see Figure 107). Everybody see that? It's going to shade down, less than. OK. ... OK, so now we are going to type in our equation, negative x minus one (see Figure 108). Easy enough?

**View of inequalities.** Preservice teachers’ use of graphing calculators while graphing boundary lines of linear inequalities was consistent with an operational view of inequalities. The boundary lines displayed by preservice teachers on their graphing calculators were always solid lines (see Figure 109). They provided little, if any, explanation to their students as to why the boundary line was always solid. This seemed to be linked to how the graphing calculators were employed; as a checking mechanism.

![Figure 108. Calculator screenshots from Vanessa class (Classroom observation; February 28, 2011).](image)

![Figure 109. Graph of a system of linear inequalities displayed in Vanessa's class (Classroom observation; February 28, 2011).](image)
The process of shading linear inequalities

As mentioned earlier, shading can represent two separate entities while working with inequalities; a process and an object. This section examined shading as the process; the manner in which preservice teachers and their students determined which half-plane to shade for an inequality. Two methods of shading emerged from an examination of preservice teachers’ lessons. The first method, denoted as the shade above or shade below method, was an observational connection between direction (i.e., above, below, up, down, left, or right) and an inequality symbol. The second method, denoted as the test point method, utilized test points to determine the appropriate half-plane to shade.

Two trends emerged after an analysis of preservice teachers’ lessons. One of the preservice teachers implemented the test point method and transitioned to the shade above or shade below method, after completing a few examples. The other three preservice teachers preferred to jump straight to the shade above or shade below method.

An examination of the method advocated by preservice teachers in the classroom revealed an unintentional split. Crystal used the test point method after making an on-the-fly decision based on her students’ responses. Angela, Christina, and Vanessa presented only the shade above or shade below method to their students, which followed their lesson plans.

Explaining a mathematical idea. In planning their lessons, it seemed that preservice teachers expected their students to be able to apply the shade above or shade below method. Some preservice teachers did not provide any scaffolding for their students, as Angela

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17 Shading as an object refers to a solution of an inequality. This will be discussed in the next aspect.
provided her students (see Figure 110). In addition, it appeared that some preservice teachers intended this portion of their lesson to be a review. Evidence of this can be seen in Crystal’s wording at the top of her example worksheet: “Remember from Algebra I and First of Year” (see the top of Figure 111).

![How do we graph them?](image)

1) Graph the line that defines the boundary \((y = ax + b)\).
2) Use a solid line for \(y \leq\) and \(y \geq\).
   Use a dashed line for \(y <\) and \(y >\).
   Why does it matter if we use a solid or dashed line?
   What's the difference?
3) Shade above the line for \(y >\) and \(y \geq\)
   Shade below the line for \(y <\) and \(y \leq\)
   Why does it matter if we shaded above or below the line? What's the difference?

*Figure 110. Angela's process for graphing a linear inequality that was projected and discussed (Lesson plan; May 2, 2011).*
Compared to the other preservice teachers, Crystal was more vocal to her students about her own thought processes while shading inequalities during lessons. As Crystal’s students were working examples at their desks (Classroom observation; Day 1: March 10, 2011), she noticed that some of them were having difficulty shading the appropriate half-plane. In an effort to address their difficulty, Crystal shared with her students how she thinks about above and below a boundary line. Her simile drew on the visual imagery of a balance scale and a see-saw.

Crystal: When I look at this green line, I try to think about a scale. So you guys have all seen balance scales? And if you put a lot on this
one side [tilting her body to her left side while placing her hands out as if they were the scales; see Figure 112] the other side is going to go up. Kind of like a see-saw idea. … So when I look at this line, the line is drawn kind of like this [drawing a solid line off to the left side of the graph that is similar to the boundary line for \( y<-2x+1 \)]. So I imagine this right here [pointing to her newly drawn line; see Figure 113] is kind of like my see-saw. And if somebody is on here [pointing to the left side], which way is it going to go? It's going to...this end is going to come down [drawing an arrow pointing down under the left side of the line; see Figure 114], in order for it to be flat [drawing a dotted line horizontally; the pivot is not in the middle of the
line. It is at the right end]. So then if I'm shading for \( y \) is less than this line, then I think of less than as below. So when I think about below the line, I know that I'm going to shade down. So that means for this line, I'm going to shade everything below the line (see Figure 115).

*Figure 114. Crystal demonstrating the "see-saw" thought process to her students (Classroom observation; March 10, 2011).*

*Figure 115. Crystal’s shading of the "see-saw" thought process (Classroom observation; March 10, 2011).*
Crystal told her students that she thinks about a see-saw or a balance scale when trying to determine above or below a line. However, the line that she drew (see Figure 114) as an example pivots at one of the ends, not in the middle. She did not mention if or how vertical lines are considered in her thinking. Also, she never elaborated on the rotational direction that she placed on the line to transform it into a horizontal line (see the dotted line in Figure 114). Even though she shared with her students her own ways of thinking about above or below a line, she did not divulge all of the nuances. There seemed to be two factors that shaped how much she told the students about her way of thinking. The first factor appeared to be the on-the-fly nature of the conversation. There was no evidence in her lesson plans that Crystal intended to present her “see-saw” thought process. The second factor may have been the level of the course. As an Algebra II Honors course, Crystal may have believed that her students would be able to adapt the “see-saw” explanation into their own thought process.

After the class, the researcher was able to observe a conversation between Crystal and her cooperating teacher. The cooperating teacher showed Crystal an alternative way to identify above or below a boundary line. The cooperating teacher’s suggestion involved locating the y-intercept of the boundary line. From the y-intercept, the cooperating teacher indicated that if you go up or above the boundary line then that is greater. In a similar fashion she noted that if you go down or below the boundary line then that is less. The next day (Classroom observation; Day 2: March 11, 2011), Crystal decided to present her cooperating teacher’s alternative suggestion to the students as part of a whole class discussion. The
discussion occurred as Crystal was explaining how to find the solution to the first example of the day.

When asked in her post-student teaching interview (May 9, 2011) which explanation Crystal would use in the future, she noted that she would probably provide both explanations to her students. She mentioned that the see-saw explanation was how she thinks about above or below a boundary line and she said that the see-saw explanation did help some of her students. She offered the following as justification for providing the y-intercept explanation to her students:

Crystal: …a lot of them they need something that's a little more tangible than that [referring to the see-saw explanation]. They can say OK, I put my finger on it [referring to the y-intercept]. I go below. Because they need systematic steps like that [referring to the cooperating teacher’s suggested explanation]. (Post-student teaching interview; May 9, 2011)

The other preservice teachers emphasized an association between direction (above or below) and the sense of the inequality sign while explaining the shade above or shade below method to their students. The wording used by the other preservice teachers to describe the association was similar to what Angela provided in step #3 of her process for graphing a linear inequality (see Figure 110). Angela and Vanessa briefly discussed the association prior to any examples, whereas Christina first mentioned the association as part of her first example.

**View of inequalities.** Preservice teachers tended to provide their students with little, if any, explanation about the shade above or shade below method. Typically, the method was
presented as an association that students needed to memorize. This apparent lack of explanation seemed to be related to preservice teachers’ belief that students had prior understanding of shading inequalities. As a result, their actions, while possibly unintentional, seemed to foster an operational view of inequalities.

With respect to the two shading methods (shade above or shade below; test point), preservice teachers seemed to perceive the test point method as a method consistent with an operational view of inequalities. They appeared to value the test point method only as a fall back explanation. A possible reason for this distinction may lie in the notion that the shade above or shade below method required the students to generalize. Thus, preservice teachers seemed to perceive that the shade above or shade below method was coherent with a relational view of inequalities.

**Solving a mathematical problem.** Preservice teachers utilized a shade above or shade below method while solving systems of linear inequalities with their students. With each inequality, preservice teachers would identify the inequality sign and then determine the direction to shade based on an association (see step #3 in Figure 98). Typically, preservice teachers indicated the shading for each inequality with different colors.

Crystal was the only preservice teacher who used a test point method and a shade above or shade below method to determine the appropriate half-plane to shade. Although a lengthy example, a detailed description of Crystal’s actions illustrate a preservice teacher who drew on her understanding of shading methods to present an alternative shading method (test points) when her students had difficulty utilizing her planned shading method (shade
above or shade below). Crystal provided a glimpse into her impromptu transition\textsuperscript{18} between the shading methods.

During her lesson (Classroom observation; Day 1: March 10, 2011), as Crystal worked through an example (\(y \leq x + 2\), see Figure 111), she asked her students to indicate which half-plane she should shade. Her students were almost evenly split in their choice of the two half-planes.

Crystal: I hear several different things. OK, why don't we test a point, to see first.

Student: (0, 0).

Crystal: (0, 0). OK. So this is the point (0, 0) [labeling it on her graph].

She substituted the values for \(x\) and \(y\) into the inequality and simplified.

Crystal: Is this a true statement? [pointing to \(0 < 2\) on the board]

Students: Yes [choral response]

Crystal: So that means that this point, this point right here [pointing to (0, 0)] on the graph is part of my solution set. So that tells me that I'm going to shade everything kind of under this line.

Crystal indicated, in her post-student teaching interview (May 9, 2011), that the moment when the class was split on which half-plane to shade was when she decided to implement the test point method to determine the appropriate half-plane to shade.

Crystal’s declaration that she made an on-the-fly decision seemed to be supported by an examination of her lesson plan. At no point did she include the use of the test point method as she outlined her steps to graph a system of linear inequalities (see Figure 116).

\textsuperscript{18} Crystal was the only preservice teacher to use both shading methods. As a result, she was the only preservice teacher who transitioned between the shading methods.
Additionally, the first and second examples from her lesson plan did not include any work that would suggest she intended to use the test point method (see Figure 111).

As Crystal continued to her second example (see Figure 111), she started to transition from the test point method to the shade above or shade below method.

Crystal: So for this one it says y is greater than this line. So if we're looking greater than, do you think we are going above the line or below the line?

Students: Above [said by the most vocal group]

Crystal: Above the line. So you're going to be shading up here [shading the graph].

In this instance, Crystal implemented the shade above or shade below method. After which, she utilized a test point in a different manner than the first example.

Crystal: So the same thing we did over here [pointing to the first example], we can test a point. Let’s chose this point [plotting the point (2, 0)]. What is this? This is (2, 0). So x is two and y is zero.
After substituting the values into the inequality, she simplified the inequality by applying arithmetic operations.

Crystal: So we have zero is greater than negative ten thirds. Is this true [pointing at the inequality $0 > -\frac{10}{3}$]? This is a negative number [pointing to $-\frac{10}{3}$]. So is zero greater than any negative number?

Students: Yes [choral response].

Crystal: Right. So I know that my solution is correct.

For this example, Crystal utilized a test point as a confirmation of her earlier decision to shade above the line. She did not tell the students why she selected the point $(2, 0)$ and the students never questioned her on the selection. In addition, she did not discuss what would have happened if the simplified inequality was incorrect. This discussion would have been relevant if a point in the other half-plane had been selected; a plausible scenario since Crystal did not explicitly restrict the location of the test point. However, since the use of test points was an on-the-fly decision, Crystal may not have fully considered all the aspects that needed to be divulged to the students in order to avoid misunderstandings.

During her post-student teaching interview (May 9, 2011), Crystal expressed concerns with the test point method of shading. Her concerns seemed to be centered on a mistake made by a student on the board (Classroom observation; March 10, 2011):

Crystal: The test points is I think prone to a little more error because of the carelessness that can happen just like this [pointing to Figure 117]. (Post-student teaching interview, May 9, 2011)
Figure 117. Crystal's student used the test point method (Classroom observation; March 10, 2011).

Figure 118. Re-creation of the student's mistake in Figure 117.

The student simplified the inequalities and provided the incorrect truth value for the first simplified inequality (see Figure 118). As a result, the student shaded the wrong half-plane. Everything else the student did was correct. The fact that this seemingly minor mistake yielded an incorrect solution appeared to be what Crystal was alluding to in her statement.
Even after expressing her concerns and reservations about the test point method in her post-student teaching interview (May 9, 2011), Crystal noted that she should have considered the test point method implemented by her student.

Crystal: …the best teacher practice should have been to… explain it that way [referring to the shade above or shade below method] but explain it in his way too [referring to the test point method used by one of her students (see Figure 117)]. That's what I should have done.

Crystal acknowledged that showing the students how to determine which half-plane to shade using test points did have benefits. She noted that the test point method may allow students to understand why you shade a particular half-plane.

Crystal planned to show her students how to shade inequalities by using the shade above or shade below method. However, when her students appeared confused, she immediately introduced the test point method. Even though, during post-student teaching interviews preservice teachers indicated that they were familiar with the test point method of shading, none of them planned to utilize the test point method with their students. Preservice teachers implemented a shade above or shade below method while solving systems of linear inequalities.

*View of inequalities*. The association of the inequality symbol and a direction to shade stressed by preservice teachers as part of the shade above or shade below method along with no explanation while solving examples seemed to foster an operational view of inequalities. It appeared that Crystal used the test point method in response to her students’ lack of consensus on an initial example. There was a very good chance that she would not
have implemented the test point method if her students indicated the correct half-plane to shade in her first example. During her post-student teaching interview, Christina was asked if she would change how she explained the process of shading to her students if given another chance.

Christina: I would still ask them if it's greater than where do you think we're going to shade. And if I guess I mean if I got hesitation, then I would try to say, well let's test some points. (Post-student teaching interview; May 4, 2011)

As noted earlier, this seems to indicate that preservice teachers perceived their utilization of the shade above or shade below method as being consistent with a relational view of inequalities.

The curriculum sequence that Crystal followed provided an opportunity to watch her teach lessons involving systems of linear inequalities for three days in a row. This was due to the fact that she covered linear programming immediately after teaching systems of linear inequalities. In the course of the three days the manner in which Crystal indicated the appropriate half-plane for each inequality changed. In the beginning, she was shading the half-plane with colors. As she moved to linear programming, she transitioned to using arrows on the boundary line to “point” to the appropriate half-plane. Those arrows seemed to become smaller and less numerous with each example. Finally, on the third day, Crystal used hand gestures to indicate the appropriate half-plane.

Crystal’s transition to less overt forms of denoting the shaded half-planes may have been in response to her perceptions about the student’s understanding. Crystal seemed to use informal formative evaluations of her students’ work during the classroom observations.
Based on these evaluations, she may have thought that her students had a good understanding of shading. These perceptions may have influenced her decision to change how she indicated the appropriate half-plane; from shading with colors to hand gestures. Additionally, this transition may be shaped by her preference to the shade above or shade below method which for her, and the other preservice teachers, seemed to embody a relational view of inequalities with regard to shading.

**Using technology.** As noted earlier, the preservice teachers’ students seemed to be familiar with shading inequalities in the Y= screen of Texas Instruments calculators (TI-83 and TI-84 models). While using graphing calculators to solve systems of linear inequalities, preservice teachers seemed to utilize that familiarity. Preservice teachers seemed to add another layer to the association with the sense of the inequality sign from the shade above or shade below method when using graphing calculators. It appeared that the inequality sign was linked to direction (up or down) which was linked to a triangle on the left of Y1= on the graph screen. During a classroom observation (February 28, 2011), Vanessa and one of her students outlined how to enter inequalities from an example (see Figure 119) into a graphing calculator.
Student: We go over to the little line [referring to the line to the left of Y1= on the graph screen; see Figure 120].

Vanessa: Over to this one [moving the cursor to the line to the left of Y1= on the graph screen]? OK, now what do I do?

Student: Change that to a triangle at the bottom.
Vanessa: You hit enter. And look, y'all see how that is changing [cycling through the icons]? It changes to all sorts of stuff. OK, I want for the first one, I want it to shade down. So I'm going to make sure that my little triangle is on the bottom (see Figure 121). Everybody see that? It's going to shade down, less than.

Preservice teachers implemented similar procedures when graphing and shading linear inequalities using graphing calculators.

*View of inequalities.* Preservice teachers seemed to be fostering an operational view of inequalities with the way that they taught shading while using graphing calculators. Preservice teachers’ preference towards the shade above or shade below method carried over to graphing calculators. While using graphing calculators the association of the shade above or shade below method was between the inequality sign and a triangle next to Y1 (see Figure 121). There was little discussion about the association during classroom observations; preservice teachers often treated the association as obvious or something that their students should already know.

**Solutions to systems of linear inequalities**

This section focused on references to the solution of a system of linear inequalities made by either the preservice teacher or the students. Unlike the previous section, which considered shading as a process (i.e., test point method or shade above or shade below
This section contemplated shading as an object representing the solution set of an inequality or system of inequalities. There were some commonalities in the preservice teachers’ presentation of solutions of systems of linear inequalities.

**Explaining a mathematical idea.** There seemed to be two focal points that surfaced in the analysis of the preservice teachers’ explanations about the solution of a system of linear inequalities. The first focal point pertained to the preservice teacher’s explanation of the conceptual meaning of the solution of a system of linear inequalities. The second focal point was the emphasis that they placed on the “overlapping” shaded region as the solution. At times, this emphasis did not seem to clearly address whether the boundary lines were a part of the solution.

The analysis of preservice teacher’s explanation of the conceptual meaning of the solution of a system of linear inequalities involved scrutinizing how preservice teachers moved beyond the procedures for finding the solution to a system of linear inequalities. For example, consider how Vanessa (Classroom observation; February 28, 2011) provided her students with a verbal description of the solution of a system of linear inequalities that was typical of the preservice teachers in this study.

Vanessa: After we graph these two [referring to the inequalities of the system], what is our solution going to be?

Student: The shaded parts.

Vanessa: OK, it's the part that is shaded. But is it everything that is shaded? Or is it just one part?

Student: Where they meet.
Vanessa: Where they meet. So it's the shaded parts of the graph that overlap [said as she filled in the blanks on the board; see Figure 122].

![Figure 122. Vanessa's student handout regarding the solution of a system of linear inequalities (Classroom observation; February 28, 2011).](image)

Preservice teachers had a common emphasis on the overlapping shaded regions. They did seem to understand what those regions represented, even though it did not always come out in their lessons with students. When asked in her post-student teaching interview to articulate what students should know about the solution of a system of linear inequalities, Crystal said the following:

Crystal: …for them to understand this is why this region [pointing to the region where the shading overlaps] really is our answer. And also to understand that this region
means more than just a region. It's actually every single little point in the region.
Like this region, the reason it's our answer [is] because this point, this point, like
every single one of these dots you could put in this region, makes the inequalities
ture. (Post-student teaching interview; May 9, 2011)

Crystal’s statement seemed to point to a desire to focus her students’ attention to the fact that
an “overlapping region” is comprised of individual coordinates. At the very end of her
statement, she added a caveat that each and every point will satisfy the inequalities of the
system. The other preservice teachers made comments that were similar to Crystal’s
statement during their post-student teaching interviews about wanting their students to know
about the solutions to systems of linear inequalities.

**View of inequalities.** Preservice teachers displayed actions were consistent with a
relational view of inequalities when they tried to talk about the solution of a system of linear
inequalities as a collection of points that satisfy all of the linear inequalities of the system.
However, preservice teachers’ descriptions that they used with their students of the solution
of a system of linear inequalities as an overlapping shaded region could promote an
operational view of inequalities. Their descriptions gave the appearance that the overlapping
shaded region was a product of a series of steps where consideration of the boundary of the
region was lost.

**Solving a mathematical problem.** The commonality that surfaced among the
preservice teachers was an apparent emphasis placed on the visual perspective of the solution
of a system of linear inequalities. This perspective manifested itself as preservice teachers
graphed the system of linear inequalities with different colors on overhead transparencies or
whiteboards. Consider Christina’s comments made during a classroom observation (March 4, 2011).

Christina: Who can tell me where they think that the solution set is?

Student: In the middle.

Christina: In the middle, yep, where both of our shadings overlap. See how in between the lines there is both blue and green [referring to the two colors she used to shade each inequality] or purple [referring to the color formed by mixing blue and green]? So that is where our solution is. All in here [waving her hand around the region]. … So where both shadings overlap, that's our solution.

Christina’s reliance on the visual blending of colors to demark the solution of the system appeared to be typical among the preservice teachers.

A moment that seemed to illustrate the uncertainty regarding the status of the boundary lines with respect to the solution occurred, during a classroom observation (February 28, 2011), as Vanessa finished explaining her first example (see Figure 123).

Vanessa: What is our solution?

Student: That little part [pointing to the board].

Vanessa: This little part right here [pointing to the “overlapping” shaded region in Figure 124].
After pointing to the solution, in an apparent effort to draw her students’ attention to the appropriate region of her graph, Vanessa denoted the extent of the solution by marking the boundaries (see Figure 125). In marking the boundaries, she drew a solid line over the dotted boundary line (i.e., \( y + 1 < -x \)). It was unclear if Vanessa included portions of the dotted boundary lines in the solution. Since she kept referring to the solution as “the part that
overlaps,” it is plausible that her concept image of the solution did not include any part of the boundary lines of the inequalities. Vanessa’s decision to outline the area representing the solution may have confused her students. She did note during her post-student teaching interview (May 13, 2011) that her students had a very hard time with dotted lines.

Figure 125. Vanessa denoting the solution for her first example (Classroom observation; February 28, 2011).

In Vanessa’s class, the uncertainty regarding the boundary lines and the solution appeared to surface again as she had students work an example on the board (see Figure 126). After allowing the students to work the problem at their desk, Vanessa called on students to provide different components of the graph. The first student was told to graph the boundary line for $x > -2$. The student graphed a dotted line with no indication of shading (see Figure 127). The second student did the same thing for $2y \geq 3x + 6$. This time, the student graphed a solid line with no indication of shading (see Figure 128). Vanessa told a third student to “shade our solution…just the solution” (Classroom observation; February 28,
2011). With some help, the student shaded the appropriate region with a red marker (see Figure 129).

2. \[ x > -2 \]
   \[ 2y \geq 3x + 6 \]

*Figure 126. Example #2 from Vanessa's lesson (Lesson plan; February 28, 2011).*

*Figure 127. Students' work for example #2 in Vanessa’s class (Classroom observation; February 28, 2011).*

*Figure 128. Students' work for an example in Vanessa’s class (Classroom observation; February 28, 2011).*

*Figure 129. Students' work for an example in Vanessa’s class (Classroom observation; February 28, 2011).*
While the three students produced a graph that correctly represented the solution for example #2, the manner in which it was constructed left some questions unanswered. Vanessa’s choice of words created ambiguity regarding the boundary lines and the solution. By telling the last student to “shade our solution,” the students in her class may have inferred that the solution did not take into consideration the boundary lines.

Like Vanessa, preservice teachers may not have believed that a lack of clarity existed as they found the solution of systems of linear inequalities. Each preservice teacher talked about the boundary lines with their students, specifically explaining how to graph the line and whether the line was solid or dotted. During those discussions, preservice teachers noted when the boundary line was included or not included in the solution (see Figure 98 for an example). Preservice teachers seemed to believe that their statements about the inclusion or exclusion of boundary lines were understood by her students and did not need to be reiterated during examples.

**View of inequalities.** Preservice teachers seemed to place an emphasis on the visual perspective of the solution of a system of linear inequalities; overlapping shaded region. The manner in which preservice teachers created the overlapping shaded region while solving systems of linear inequalities during classroom observations was coherent with an operational view of inequalities – the region was a result of a process. The boundaries of the overlapping shaded regions were often neglected by preservice teachers.

**Using technology.** As noted earlier, preservice teachers did not fully address whether the boundary line was solid or dotted while using graphing calculators to solve systems of
linear inequalities during classroom observations. In addition, they stressed an overlapping shaded region as they described to their students the solution of a system of linear inequalities while graphing by hand. This emphasis continued when graphing calculators were utilized by preservice teachers during classroom observations.

Vanessa provided a typical example of how preservice teachers utilized graphing calculators to solve systems of linear inequalities. One of her examples (see Figure 119) contained an inequality with a less than inequality symbol. The way in which Vanessa entered the linear inequalities into her graphing calculator did not account for the dotted boundary line created by the inequality with a less than symbol (see Figure 130). Her displayed solution included a solid boundary line (see Figure 131). Vanessa did not acknowledge the calculator’s limitation.

Figure 130. Calculator screenshots from Vanessa class (Classroom observation; February 28, 2011).
View of inequalities. Preservice teachers’ actions were coherent with an operational view of inequalities with regard to solutions of systems of linear inequalities while using technology, specifically graphing calculators. The limitations of the manner in which they employed graphing calculators to display solutions were not discussed. Additionally, once the shaded region was displayed the example was over and preservice teachers proceeded to the next example; missing opportunities to talk about the meaning of a solution.

Characterizing How Preservice Teachers Approach Systems of Linear Inequalities

Preservice teachers typically have very limited experiences with regard to teaching or observing others teaching system of linear inequalities. Thus, they relied heavily on their own understanding of how to approach systems of linear inequalities when they planned and taught lessons during their student teaching field experience. Their lack of experience led the preservice teachers to make assumptions while planning the lessons that may have differed from those of teachers with experience teaching the prerequisite lessons.

The four preservice teachers, examined in this case, taught their students to implement a graphing strategy in order to find the solution of a system of linear inequalities. The implementation of a graphing strategy was similar in all of the classrooms. This choice
was aligned with their own approaches during an assessment of their understanding of inequalities, analyzed in Chapter 4. Additionally, with the exception of the shading method, the graphing strategy outlined by preservice teachers followed closely the steps presented by Reilly (2010). The manner in which preservice teachers explained, solved problems, and used technology with regard to the solution strategy was consistent with an operational view of inequalities.

For the preservice teachers of this study, slope-intercept form was the primary tool in graphing the boundary lines of linear inequalities (see Figure 98 and Figure 99). Using slope-intercept form allowed them to build on previously taught lessons regarding graphing linear equations. However, their usage of graphing linear equations may have played a role in the incidents in which inequalities were treated as equations. These incidents, which blurred the distinction between inequalities and equations, often occurred while working with the boundary lines of inequalities. The lack of distinction was also seen when graphing calculators were utilized. These occurrences of treating inequalities as equations during classroom observations were in line with an operational view of inequalities.

During their pre-student teaching interview, the preservice teachers seemed to favor the shade above or shade below method of shading linear inequalities. While one of the preservice teachers implemented the test point method in her classroom, the goal of all of the preservice teachers seemed to be to have their students be able to apply the shade above or shade below method. This goal seemed to be shaped by the belief that the shade above or shade below method promoted a relational view of inequalities with regard to shading linear inequalities. However, preservice teachers stressed an association with the sense of the
inequality sign while working with the shade above or shade below method which was coherent with an operational view of inequalities.

The preservice teachers tended to talk about a solution set of a system of linear inequalities as an object. They described a solution set as an overlapping shaded region. This description was part of their understanding (see Chapter 4) and was emphasized to their students in the classroom. Within classrooms, preservice teachers seemed to neglect boundary lines of inequalities when referring to the solution set of the system. This oversight was clearly seen in the preservice teachers’ graphs; specifically their use of colors to denote the solution sets of the linear inequalities. The preservice teachers seemed to place an emphasis on making sure the students were able to identify the appropriate portion of the plane containing the solution. However, that emphasis did not carry over to the boundary lines. As a result, preservice teachers seemed to foster an operational view of inequalities with regard to the solutions of inequalities.

Chapter Summary

This chapter addressed how understanding of inequalities was used while planning and implementing lessons on systems of linear inequalities. The primary sources of data for analysis were classroom observations and field notes. The secondary sources of data for analysis were pre-student teaching interviews (see Appendix C), post-observation questioning, lesson plans submitted by preservice teachers, and post-student teaching interview (see Appendix D). A description of the case of systems of linear inequalities was divulged. The next chapter seeks to address how understanding of inequalities was used while planning and implementing lessons on quadratic inequalities.
CHAPTER 6: QUADRATIC INEQUALITIES

The second research question for this study was the following: How do preservice teachers use their understanding of inequalities while planning and implementing lessons during the student teaching field experience? To examine this question, five preservice teachers were studied during their student teaching field experience. After scrutinizing the content that the preservice teachers were able to teach during their field placement and the data collected, two cases emerged: 1) teaching systems of linear inequalities; and 2) teaching quadratic inequalities. This chapter will address how understanding of inequalities was used while planning and implementing lessons on quadratic inequalities. The primary sources of data for analysis were classroom observations and field notes. The secondary sources of data for analysis were pre-student teaching interviews (see Appendix C), post-observation questioning, lesson plans submitted by preservice teachers, and post-student teaching interview (see Appendix D).

Descriptions of each classroom relevant to this case will be provided. In addition, considerations taken into account prior to analysis will be disclosed. The conceptual framework (see Table 2) of this study served as a guide for the analysis. Each cell of the conceptual framework corresponds to the intersection of a teaching practice and an aspect of knowledge associated with inequalities. The manner in which preservice teachers applied the three teaching practices (explaining a mathematical idea, solving a mathematical problem, and using technology) with respect to each of the four primary aspects of knowledge associated with the mathematical concept of inequalities (strategies for solving inequalities, relating inequalities as equations, shading as a process, and solutions of inequalities) will be
discussed. Additionally, the view of inequalities (operational or relational) displayed and/or fostered by preservice teachers while applying each teaching practice with respect to the primary aspects of knowledge will be discussed.

**Descriptions of the Classrooms**

Three of the five preservice teachers who participated in this study taught lessons pertaining to quadratic inequalities. Crystal and Heather taught Algebra II Honors classes. Vanessa taught an Algebra II class. Heather taught at a large rural high school. Crystal and Vanessa taught at large suburban high schools.

Crystal’s Algebra II Honors class had 33 students enrolled. Less than one-third of her students were minorities and well over half were classified as sophomores. Crystal and her cooperating teacher mentioned, on separate occasions, that this class had several students who, in their opinion, should not be in an Honors class. The desks were arranged in rows that faced the front whiteboard. Graphing calculators (TI-83 Plus) were available for students; however it appeared that a majority of the students used their own graphing calculators (TI-83 or TI-84 models). A LCD projector attached to a laptop, a whiteboard, and an overhead projector were available for use within Crystal’s classroom.

There were 23 students enrolled in Heather’s Algebra II Honors class. Less than a fourth of her students were minorities. Heather’s cooperating teacher noted that there were several students within this particular class who should have been placed into a “regular” Algebra II class. Classroom sets of netbooks and calculators (TI-83 plus) were readily available within Heather’s classroom. About half of the students seemed to use their own calculators (TI-83 or TI-84 models). The students sat in rows that were oriented towards a
small whiteboard on the right side of the front of the class. A LCD projector was located on a
table in front of the small whiteboard. Connected to the LCD projector were a laptop and a
document camera. The laptop and document camera were on a cart to the right side of the
class. Heather used the document camera during every observation. She situated the cart so
that she was out of the students’ direct line of sight to the projection.

The enrollment in Vanessa’s Algebra II class was 28 students. Almost half of
Vanessa’s students were minorities. Her cooperating teacher noted that the number of
students classified as sophomores, juniors, or seniors were about the same. Students sat in
rows of seats that all faced the front of the classroom. Vanessa had a LCD projector, which
was connected to a computer and a document camera, available for use during her lessons.
There was a whiteboard at the front of the class. In addition, she had a TI-83 Plus connected
to a TV set attached to the wall in the front of the classroom next to the whiteboard. A
classroom set of graphing calculators (TI-83 Plus) were available for students to use,
although about half of the students appeared to use their own graphing calculator during
observations. It was difficult to determine the exact models being used by students, but it
appeared that students were using either TI-83 or TI-84 graphing calculators.

**Areas of Consideration Prior to Analysis**

The quadratic inequality lessons taught by Crystal, Heather, and Vanessa were part of
their unit on quadratics which included quadratic expressions and quadratic equations. The
timing of this unit within their student teaching experience varied. Heather taught her unit on
quadratics at the very beginning her student teaching experience. The other two preservice
teachers (Crystal and Vanessa) taught their units five weeks into their student teaching
experience. As a result, the timing of lessons within the student teaching experience was considered during the analysis.

Another area that warranted consideration prior to analysis was the data that was collected. There were two types of quadratic inequalities covered in preservice teachers’ lessons. Crystal taught quadratic inequalities with one variable to her students. The portion of her lesson that pertained to quadratic inequalities with one variable occurred during the latter half of a class period (Classroom observation; April 5, 2011). Heather and Vanessa taught lessons regarding quadratic inequalities with one variable as well as lessons involving quadratic inequalities with two variables. Vanessa’s lesson on one and two variable quadratic inequalities occurred on the same day (Classroom observation; April 11, 2011). Heather covered both types of quadratic inequalities over a two day period (Classroom observation; March 7–8, 2011). Based on the material covered, Heather and Vanessa were the primary sources of data for the analysis presented in this chapter and Crystal was considered as a supporting source of data.

The influence of cooperating teachers cannot be left out of this discussion. The researcher had a face-to-face discussion with each cooperating teacher prior to the beginning of the student teaching experience. The researcher requested that the preservice teachers be the primary source of planning for lessons regarding quadratic inequalities. All of the cooperating teachers agreed to the request.

When asked in post-student teaching interviews, all of the preservice teachers indicated that they were primarily responsible for lessons that were presented surrounding quadratic inequalities. Cooperating teachers offered advice to their preservice teacher and
made suggestions as to possible changes to lessons. These were actions that they would have performed for any preservice teacher under their supervision. It appeared that lessons regarding quadratic inequalities were dictated by the preservice teachers. However, there was evidence that cooperating teacher played a role in some of the lessons.

During her post-student teaching interview (May 9, 2011), Crystal mentioned that her “teacher doesn't really include inequalities” but Crystal included them “because [she] knew [the researcher] wanted to see that.” These influences may have affected Crystal’s decision regarding how much exposure to quadratic inequalities she needed to provide to her students. She may have viewed the inclusion of a few examples of quadratic inequalities as additional practice in factoring for her students. Crystal may have wanted to do more with quadratic inequalities with her students. However, she may have abandoned the idea of doing more in order to be consistent with her cooperating teacher’s pacing. Consequently, the influence of Crystal’s cooperating teacher as well as her involvement in this study seemed to play a role in the structure of Crystal’s quadratic unit.

**Analysis of Teaching Aspects of Knowledge Associated with Inequalities**

**Introduction of the aspects**

The conceptual framework of this study was a matrix design with two overarching views (see Table 2). The column headings of the framework were aspects of knowledge associated with inequalities. These aspects were identified within a literature review. With regard to this study, there were four primary aspects of knowledge associated with inequalities: 1) Strategies for solving inequalities; 2) Relating inequalities as equations; 3) Shading as a process; and 4) Solutions of inequalities. Within those primary aspects there
were other aspects that may be pertinent to more than one primary aspect: domain; use of variable(s); factors of products and/or quotients; logical connectives; and interpretations of the inequality symbol. By using these four primary aspects as an initial lens for making sense of classroom episodes, the researcher was able to identify and examine how preservice teachers implemented the three teaching practices (i.e., explaining a mathematical idea, solving a mathematical problem, and using technology) with respect to each aspect. In addition, the researcher made an assessment as to whether an operational or relational view of inequalities was promoted by the preservice teachers with regard to each teaching practice within each aspect.

The preservice teachers’ decisions regarding how to explain ideas, solve problems, and use technology while teaching lessons related to quadratic inequalities revealed nuances that will be presented. In this particular case, the aspect of treating inequalities as equations seemed to occur when preservice teachers or their students were working with the boundary curve of a quadratic inequality. In addition, how preservice teachers and their students dealt with the process of shading inequalities and solutions of inequalities was studied and will be presented.

**Strategies to solve inequalities**

For purposes of this study, the strategies employed by preservice teachers were categorized using descriptions from Tsamir and Almog (2001). Tsamir and Almog (2001) concluded that students, for the most part, used three strategies to solve inequalities: (1) algebraic manipulation, (2) drawing a graph, and (3) using a case approach. The case
approach strategy\(^\text{19}\) (Dreyfus & Eisenberg, 1985) requires consideration of all of the possibilities that would produce the given inequality. An algebraic manipulations strategy could include the following:

- addition or subtraction of the same term to both sides of the inequality;
- multiplication and division of both sides of the inequality by identical factors, with appropriate attention to the orientation of the inequality symbol;
- simplification of an expression through combining like terms or factoring;
- multiplying both sides of the inequality by the square of the denominator;
- “examining quadratic inequalities (i.e., \(ax^2 + bx + c > 0\)) by first relating to the quadratic roots, or by investigating the sign of ‘a’ and the sign of the determinant” (Tsamir et al., 1998, p. 131); and
- “relating to an inequality of the type \(ab > 0\) as a compound system of \(\{a > 0 \text{ and } b > 0\}\) or \(\{a < 0 \text{ and } b < 0\}\)” (Tsamir & Almog, 2001, p. 515).

Tsamir and Almog (2001) noted that if a graph, on a Cartesian coordinate plane, of a given quadratic inequality was drawn and used to attain a solution, then a graphing strategy was implemented. This additional condition allowed for the exclusion of sign charts and number lines from being considered as part of a graphing strategy. Sign charts and number lines were tools used while invoking an algebraic manipulations strategy to solve quadratic inequalities.

There were situations in which algebraic manipulations were performed as part of the implementation of a graphing strategy. An example can be seen when students or preservice

\(^{19}\) A full description and an example of the case approach strategy can be found in Chapter 2.
teachers isolate the variable \( y \) of a quadratic inequality with two variables in order to graph the boundary curve of an inequality. In keeping with literature and in an effort to maintain clarity, if algebraic manipulations were performed in order to graph an inequality on a Cartesian coordinate plane, then a graphing strategy was implemented.

During pre-student teaching interviews, preservice teachers were asked to solve quadratic inequality problems (see Figure 15 and Figure 18). Preservice teachers employed a graphing strategy or an algebraic manipulation strategy to solve the quadratic inequality problems\(^{20}\). Their success with the strategies varied (see Chapter 4 results).

**Explaining a mathematical idea.** During classroom observations, preservice teachers seemed to implement solution strategies (graphing or algebraic manipulations) based on the type of quadratic inequality presented (quadratic inequalities with one variable or quadratic inequalities with two variables). For quadratic inequalities with two variables a graphing strategy was outlined by preservice teachers. Preservice teachers presented an algebraic manipulations strategy for quadratic inequalities with one variable.

Heather was the only preservice teacher who also employed a graphing strategy to solve quadratic inequalities with one variable. During a classroom observation (March 7, 2011), Heather showed her students how to solve quadratic inequalities with one variable by implementing a graphing strategy as well as an algebraic manipulations strategy for each example. She identified the solution via both strategies before moving on to the next example.

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\(^{20}\) For one of the questions on the pre-student teaching interview, the preservice teachers were asked to implement a graphing strategy to solve the quadratic inequality. Those who had employed a graphing strategy to solve previous quadratic inequality problems were not asked this question.
As preservice teachers outlined a graphing strategy, they recalled procedures taught earlier regarding graphing quadratic equations. For example, the procedure that Heather prescribed to her students for graphing quadratic equations with two variables (see Figure 132) was the starting point for her strategy for graphing quadratic inequalities. Graphing strategies for solving quadratic inequalities with two variables presented by preservice teachers were similar. The nuances of their strategies, as described to their students, contained slight variations. A notable difference involved the incorporation of technology by Vanessa; specifically graphing calculators.

![Figure 132. Heather's directions for graphing a quadratic equation with two variables (Lesson plan and Classroom observation; March 1, 2011).](image)

Preservice teachers did not provide their students with detailed step-by-step directions on how to apply a graphing strategy to solve quadratic inequalities. They started their lesson with a review of an association between the sense of inequality signs and the type of boundary curve; solid or dotted/dashed (see Figure 133 and Figure 134). As preservice teachers introduced their graphing strategy while working an example, it appeared that they expected their students to have some familiarity with graphing quadratic equations. Examples
have been included of Heather’s graphing strategy for quadratic inequalities with one variable\(^{21}\) and Vanessa’s graphing strategy for quadratic inequalities with two variables.

![Figure 133](image1.png)

*Figure 133.* The "basic rules" from Heather’s lesson on quadratic inequalities with two variables (Classroom observation; March 8, 2011).

![Figure 134](image2.png)

*Figure 134.* Slide from Vanessa's lesson with steps for graphing a quadratic inequality (Lesson plan and Classroom observation; April 11, 2011).

The details of Heather’s graphing strategy for quadratic inequalities with one variable emerged as she proceeded through her first example \((x^2 + 2x - 3 > 0); \text{ see Figure 152}.\)

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\(^{21}\) Recall that Heather was the only preservice teacher who employed a graphing strategy with quadratic inequalities with one variable.
During the example, she asked her students questions that guided them through the implementation of the graphing strategy.

Heather: First we need to know what the vertex is. Right? So find the vertex. How do we find the vertex when something is in standard form?

...

Heather: OK, you could complete the square. Or you could use the x-coordinate equals negative b over two a. And then plug that in for y.

After calculating the vertex with input from her students, Heather plotted the vertex and the axis of symmetry on a coordinate plane (see Figure 135).

Heather: What are our methods for graphing?

After some discussion with her students, Heather used an “over 1 up 1a, over 1 up 3a, over 1 up 5a” approach to plot “extra points.” She mentioned to her students that they could find “extra points” by plugging x-values into the “equation.” With points plotted, she drew a parabola (see Figure 136). Heather asked her students to identify the regions of the parabola that were above the x-axis.

Heather: So when is this equation, your parabola [pointing to the parabola; see Figure 136], going to be greater than zero?

She shaded the regions of the parabola that were “greater than zero.” Afterward, she shaded x-values on the x-axis that corresponded to regions where the parabola was above the x-axis (see Figure 137). Finally, Heather wrote two versions of the solution set for the quadratic inequality: compound inequality and set-builder notation (see Figure 138).
Figure 135. Heather's graph with the vertex and axis of symmetry displayed (Classroom observation; March 7, 2011).

Figure 136. Heather's graph of the parabola (Classroom observation; March 7, 2011).

Figure 137. Heather's graph of the parabola with shading (Classroom observation; March 7, 2011).

Figure 138. Heather's two versions of the solution set (Classroom observation; March 7, 2011).
As mentioned earlier, preservice teachers implemented similar graphing strategies to solve quadratic inequalities with two variables. While proceeding through the first example, they seemed to be directing their students through the process of graphing a quadratic inequality with two variables via direct questions and statements. Vanessa provided an illustration of the graphing strategy advocated by preservice teachers (Classroom observation; April 11, 2011).

Vanessa: Are we in standard form or vertex form? … I can look at this [pointing to the quadratic expression on the left side of the inequality in Figure 160] and know that my vertex is (2, 1).

…

Vanessa: Now, am I opening up or down?

…

Vanessa: I can look at my calculator and look at the table [referring to how to get extra ordered pairs to plot in order to graph the parabola].

Vanessa implemented a test point method to determine the appropriate region of the graph to shade.

Vanessa: Now, I'm going to test any point on inside of this parabola [pointing to the "inside" of the parabola; see Figure 139] or the outside [pointing to the "outside" of the parabola; see Figure 140]. Either side. I'm going to go with this point right here [pointing to (2, 4); see Figure 141]. That's going to be two, four. I'm going to test the point (2, 4). … So, I'm just going to substitute in two for \( x \) and four for \( y \).
She substituted the values into the inequality and asked the students questions related to performing the arithmetic. After simplifying, Vanessa had $4 \leq 1$ on the board.

Vanessa: Is four less than or equal to one?

Student: No.

Vanessa: No. It is not [drawing a slash through the less than or equal to sign ($\leq$); see Figure 142]. So the inside portion of this parabola does not satisfy my equation. So, I'm going to shade the outside [shading the "outside" of the parabola; see Figure 143]. Everything outside of this parabola is part of my solution set.

*Figure 139. Vanessa identifying the "inside" of the parabola (Classroom observation; April 11, 2011).*

*Figure 140. Vanessa identifying the "outside" of the parabola (Classroom observation; April 11, 2011).*
The algebraic manipulations strategy employed by preservice teachers was very similar to Heather’s strategy. Some of the terminology differed: Heather talked about “intervals” formed by critical values on a number line, whereas Crystal called those same intervals “chunks” (Classroom observation; April 5, 2011). Another difference could be seen as the solution was presented. Heather always solved quadratic inequalities with one variable
using a graphing strategy and then using an algebraic manipulations strategy. As a result, she never rewrote the solution after completing her algebraic manipulations strategy; instead she scrolled back to the solution that she wrote after completing her graphing strategy earlier in the lesson (see Figure 138).

Heather’s version of an algebraic manipulations strategy required students to begin by writing an inequality in standard form. Her next step involved factoring the quadratic expression. Once the quadratic expression was factored (see Figure 144), Heather told her students to “look at the related equation,” and solve it using “the zero product property” (Classroom observation; March 7, 2011; see Figure 145). The values found after applying the zero product property were plotted on a number line and Heather explained why an “open hole” was used.

Heather: So now we are going to have a number line [drawing a number line]. And then we have an open hole at one and an open hole at negative three.

Student: How do we know?

Heather: Because it's not equal to [pointing to the inequality sign in the original inequality]. It's just greater than. (Classroom observation; March 7, 2011)

It appeared that Heather assumed that her students had a prior understanding of the connection between the inequality sign and the usage of a closed or open plot on a number line. Even after her student’s question, she did not offer much of an explanation.
With the number line separated into three intervals, Heather described the test point method to her students. She asked them to select “easy points in each interval” to test (Classroom observation; March 7, 2011). Testing involved substituting the selected values into standard form of the quadratic inequality with one variable (see Figure 146). The value of each of the test points, positive or negative, was written above its corresponding interval (see Figure 146).

Using a sign chart, Heather explained the connection between the inequality sign and the values of each interval on her number line.
Heather: Since our original inequality [which was in standard form; see Figure 152] was greater than zero, our solution set is going to be where it is positive. So it is going to be these numbers out here [shading the interval $(-\infty, -3)$] and these numbers out here [shading the interval $(1, \infty)$; see Figure 147].

*Figure 146.* The sign chart from Heather’s work on her first example of a quadratic inequality with one variable (Classroom observation; March 7, 2011).

*Figure 147.* The shaded number line for Heather’s first example of a quadratic inequality with one variable (Classroom observation; March 7, 2011).

Vanessa’s algebraic manipulations strategy started with ensuring that the quadratic inequality was in standard form. Once in standard form, she applied either factoring,
completing the square, or the quadratic formula to find values. At no point in her lesson does she define or name these values\textsuperscript{22}. At this point in her algebraic manipulations strategy, Vanessa made use of the logical test feature on her graphing calculator. The way that Vanessa utilized technology will be discussed in more detail under the teaching practice of using technology.

Examples of the three solution strategies employed by preservice teachers while teaching quadratic inequalities to their students were provided. An algebraic manipulations strategy was applied by preservice teachers when teaching quadratic inequalities with one variable. Heather was the only preservice teacher whom presented a graphing strategy as well as an algebraic manipulations strategy with quadratic inequalities with one variable. Preservice teachers outlined a graphing strategy for their students when teaching quadratic inequalities with two variables.

**View of inequalities.** Preservice teachers, in this study, tended to present solution strategies to solve quadratic inequalities that did not promote a relational view of inequalities. While implementing an algebraic manipulations strategy, Heather provided no rationale to her students for considering the “related equation.” Similarly, Crystal never gave justification for converting the inequalities to equations to find the “critical points.” Graphing strategies were tied to steps used to graph quadratic equations with some extra procedures. With either solution strategy, steps were presented to students with the sole purpose of finding the solution which was coherent with an operational view of inequalities.

\textsuperscript{22} Since Vanessa did not name the values, the author will refer to them as [critical] values.
Heather’s graphing strategy for solving quadratic inequalities with one variable seemed to contain a mixture of components that promoted operational and relational views of inequalities. Her statements (see Figure 148) made prior to working an example appeared to be operational in nature. These statements did not provide a connection between zero on one side of the inequality and the x-axis. It appeared that the x-axis was arbitrarily selected when in fact Heather had a clear reason for selecting the x-axis. However, as she progressed through the first example, Heather called on her students to identify the regions of the parabola that were above the x-axis (see Figure 137). In doing so, it appeared that she was trying to foster a relational view of inequalities by highlighting a connection between being greater than zero and being above the x-axis.

**Figure 148.** Heather's outline of the graphing strategy for solving quadratic inequalities with one variable (Classroom observation; March 7, 2011).

**Solving a mathematical problem.** An examination of preservice teachers’ lesson plans revealed two different types of quadratic inequality examples selected by preservice teachers. The first type involved quadratic inequalities with one variable (i.e., \( x^2 + bx + c < \))
The second type involved quadratic inequalities with two variables (i.e., \( y > ax^2 + bx + c \)). Heather and Vanessa presented both types of problems in their lessons. Crystal decided to include only quadratic inequalities with one variable in her lesson.

The order in which preservice teachers decided to present the two types of quadratic inequalities within their lessons was different. Heather opted to start with quadratic inequalities with one variable and then transition, the next day, to quadratic inequalities with two variables. In Vanessa’s lesson, quadratic inequalities with two variables were introduced prior to quadratic inequalities with one variable. Heather may have started with quadratic inequalities with one variable in order to match the structure of her prior lessons pertaining to quadratic equations. She taught her students how to solve quadratic equations with one variable and then moved on to graphing quadratic equations with two variables. On the other hand, Vanessa may have started with quadratic equations with two variables because she viewed it as an extension of graphing quadratic equations with two variables: a lesson she taught prior to the quadratic inequality lesson. The decisions regarding the order were not explored during interviews. However, these decisions seemed to highlight the prior conceptions that each preservice teacher was trying to build upon during their lessons.

While solving quadratic inequalities with one variable, preservice teachers seemed to select quadratic expressions that were factorable (see Figure 149, Figure 150, Figure 151, Figure 152, Figure 153, Figure 154, and Figure 155). At the time of Crystal’s lesson involving quadratic inequalities, Crystal had not taught her students completing the square or the quadratic formula. As a result, Crystal’s quadratic expressions were factorable out of

\[^24\] A less than sign was displayed for illustrative purposes only. Any of the inequality signs may be substituted in place of the less than sign.
necessity. Whereas the other preservice teachers taught their students how to solve quadratic equations with one variable by completing the square or with the quadratic formula in prior lessons. Even though these processes were covered, they only mentioned factoring quadratic expressions.

\[ x^2 + 3x - 18 = 0 \quad x^2 - 6x - 7 < 0 \]

*Figure 149.* Powerpoint slide Vanessa displayed during her lesson with examples of a quadratic equation with one variable and a quadratic inequality with one variable (Classroom observation; April 11, 2011).

\[ x^2 - x > 12 \quad 2x^2 - 5x - 3 \leq 0 \]

*Figure 150.* Other planned examples of quadratic inequalities with one variable from Vanessa’s lesson (Classroom observation; April 11, 2011).

\[
\begin{align*}
#9) & \ x^2 - 5x + 15 < 0 \\
#10) & \ 5x^2 - 27x + 10 \geq 0
\end{align*}
\]

*Figure 151.* Problems on a review sheet of quadratic inequalities with one variable requested by Vanessa’s students (Classroom observation; April 11, 2011).
Figure 152. Heather's first example of a quadratic inequality with one variable (Classroom observation; March 7, 2011).

Figure 153. Heather's second example of a quadratic inequality with one variable (Classroom observation; March 7, 2011).

Figure 154. Two examples of quadratic inequalities with one variable from Crystal's lesson (Classroom observation; April 5, 2011).

Figure 155. Crystal's third example of a quadratic inequality with one variable produced by changing the inequality sign in example #1 (Classroom observation; April 5, 2011).
While teaching quadratic inequalities with two variables, preservice teachers selected examples with the variable $y$ isolated on the left side of the inequality. This decision by preservice teachers to only present examples with the variable $y$ isolated aligned with how they had presented quadratic equations with two variables in prior lessons (see Figure 156 and Figure 157).

<table>
<thead>
<tr>
<th>Solve the following by graphing:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1: $y = x^2$</td>
</tr>
<tr>
<td>Example 2: $y = -x^2$</td>
</tr>
<tr>
<td>Example 3: $y = x^2 + 2$</td>
</tr>
<tr>
<td>Example 4: $y = 2(x + 3)^2 - 5$</td>
</tr>
<tr>
<td>Example 5: $y = x^2 - 4$</td>
</tr>
</tbody>
</table>

*Figure 156. Examples of graphing quadratic equations with two variables from Vanessa’s class (Classroom observation; April 4, 2011).*

<table>
<thead>
<tr>
<th>Graph the following functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1: $y = x^2$</td>
</tr>
<tr>
<td>Example 2: $y = x^2 + 8x + 9$</td>
</tr>
<tr>
<td>Example 3: $y = -x^2 - 4x + 5$</td>
</tr>
</tbody>
</table>

*Figure 157. Examples of graphing quadratic equations with two variables from Heather’s class (Classroom observation; March 1, 2011).*
The quadratic expression on the right side of the quadratic inequalities with two variables was an area of commonalities and differences among preservice teachers. Heather selected a quadratic expression in standard form for her first example (see Figure 158) and a quadratic expression in vertex form for her second example (see Figure 159). Vanessa’s first example contained a quadratic expression in vertex form (see Figure 160). After her first example, all of Vanessa’s quadratic expressions were in standard form (see Figure 161, Figure 162, and Figure 163). Vanessa included quadratic expressions with leading coefficients other than one. On the other hand, the leading coefficient in both of Heather’s examples was one.

Figure 158. Heather's first example of a quadratic inequality with two variables (Classroom observation; March 8, 2011).

Figure 159. Heather's second example of a quadratic inequality with two variables (Classroom observation; March 8, 2011).
Figure 160. Powerpoint slide from Vanessa's lesson with the first example of quadratic inequality with two variables (Lesson plan; April 11, 2011).

Figure 161. Vanessa's second example of a quadratic inequality with two variables (Lesson plan; April 11, 2011).

Figure 162. Vanessa's third example of a quadratic inequality with two variables (Lesson plan; April 11, 2011).
With a limited number of examples presented to students, it appeared that preservice teachers wanted their students to have exposure to quadratic inequalities, but in familiar forms that were parallel to those used to study quadratic equations. This exposure was superficial in nature and did not seem to reveal nuances and possibilities that may have sparked discussion. Opportunities to discuss situations where the solution was either the set of all real numbers, the empty set, or a single value were missed.

*View of inequalities.* The examples used by preservice teachers to provide practice applying the presented solution strategies seemed coherent with an operational view of inequalities. The examples were structured in such a manner as to facilitate the step-by-step nature of the prescribed strategies. Additionally, preservice teachers seemed to incorporate review of previously taught material into the examples of quadratic inequalities that were presented during classroom observations. For example, Vanessa employed factoring, completing the square, and the quadratic formula while working with her students. During a classroom observation (April 11, 2011), Vanessa told her students that parts of her solution strategies were “review from graphing quadratic equations”: a lesson she taught to her
students three days earlier. The prescribed nature of the examples, and the tendency to emphasize reviewing of prior concepts by preservice teachers seemed to foster an operational view of inequalities.

**Using technology.** Preservice teachers were not uniform in terms of the degree to which technology was incorporated into their lessons while implementing solution strategies to solve quadratic inequalities. As noted in the descriptions of the classrooms, availability of technology was apparently not an issue. Heather and Crystal did not integrate any technology into their lessons. Vanessa, on the other hand, incorporated graphing calculators (TI-83s and TI-84s) into her examples, albeit not as an integral part of her examples. The manner in which she utilized graphing calculators while implementing a graphing strategy could be categorized as “technology as servant;” meaning that it was used as a “supplementary tool” but not “in creative ways to change the nature of the [examples]” (Goos et al., 2003, p. 78).

After graphing quadratic inequalities “by hand,” Vanessa showed her students how to use their graphing calculators to graph quadratic inequalities with two variables. For an example of a quadratic inequality with two variables, she provided the following directions to her students:

Vanessa: I'm just going to type in this equation. I have $x$ minus two squared plus one [said as she typed the expression on the right side of the inequality into her calculator; see Figure 164]. Now, since I am less than or equal to, I'm going to go over and I'm going to shade down [cycling through the icons on the right of Y1=; see Figure 165]. Just like I did with linear inequalities [said as the graph was displayed; see Figure 166].
Vanessa seemed to employ only certain features of the graphing calculators (TI-83s or TI-84s). In addition to telling her students to use the “table” feature, she utilized graphing and shading features (see Figure 164, Figure 165, and Figure 166). However, Vanessa’s use of graphing calculators was not limited to moments when she was implementing a graphing
strategy. While implementing an algebraic manipulations strategy, she made use of the “test” feature on her graphing calculator. During her classroom observation (April 11, 2011), Vanessa showed her students how to use the “test” feature while working an example.

Vanessa: Want me to show you something cool in your calculator?

Student: Yeah.

Vanessa: Alright. Go to Y=. I want you to type in x squared minus six x minus seven (see Figure 167). Then hit 2nd button, and do you see the little word that says test? It should be like the math button. 2nd test. I'm going to go to less than [selecting the “<” sign; see Figure 168]. Less than zero. Alright less than zero (see Figure 169). I'm going to hit graph [graph is displayed; see Figure 170]. Let me zoom in just a little [using the zoom feature; see Figure 171]. Do you see there is a nice straight line connected from negative one to seven? See that?

Student: Sure do.

Vanessa used the [critical] values calculated earlier along with the graph created using the test feature on her graphing calculator (see Figure 170) to write the solution of the quadratic inequality (see Figure 172).
Figure 167. Vanessa entering a quadratic expression (Classroom observation; April 11, 2011).

Figure 168. Vanessa selecting an inequality (Classroom observation; April 11, 2011).

Figure 169. Vanessa's inequality (Classroom observation; April 11, 2011).

Figure 170. A graph from Vanessa's work (Classroom observation; April 11, 2011).
During her post-student teaching interview (May 13, 2011), Vanessa revealed that she attended a workshop two weeks prior to teaching her lesson on quadratic inequalities. While at that workshop, she learned about graphing using the “test” feature on Texas Instrument calculators. The researcher asked Vanessa, during her post-student teaching interview (May 13, 2011), about her use of the “test” feature in her class and her students’ understanding of the information provided by the graph.

Researcher: This line that goes from –1 all the way out to 7 … what's the answer that students are getting out of this?

Vanessa: It's all the x values between –1 and 7.

Researcher: Do you think that they will get that with this line sitting up here about y = 1. That's about where, if you look at it [pointing to the line in Figure 170].

Vanessa: Yeah.

Researcher: So, do they get the idea that it's just the x values?
Vanessa: Yeah, I mean I didn't even think about that. Gosh I hope they did. … All I was just talking about [was] your calculator can't just do, just a plain number line. So it has to have the y-axis too.

It seemed that Vanessa did not understand, or at the very least did not tell her students, how the “test” feature operated and why the graph looked the way that it did. She did not appear to know that the “test” feature returns a zero or a one based on the truth value of the given statement. While the line $y = 1$ on the interval $(-1, 7)$ was clearly visible, the line $y = 0$ on the interval $(-\infty, -1) \cup (7, \infty)$ was not visible against the background of the x-axis (see Figure 170).

**View of inequalities.** Vanessa’s use of graphing calculators while presenting her graphing strategy for quadratic inequalities seemed to foster an operational view of inequalities. The fact that she employed the “test” feature may be attributable to her belief that it was “something cool” that she could do on her graphing calculator. It appeared that the “test” feature was just a step in her strategy to solve quadratic inequalities with one variable. There was no attempt on Vanessa’s part to utilize the “test” feature to foster a view of inequalities beyond an operational view, as it was merely a tool to plot a line to represent a solution, a solution she did not explain to her students. Additionally, Vanessa treated the graphing calculator as a checking mechanism while presenting her graphing strategy for quadratic inequalities with two variables. Her incorporation of a graphing calculator appeared to be a button pushing exercise with no connection to the work displayed done “by hand” beyond the final graphs.
Treating inequalities as equations

Within the case of teaching quadratic inequalities, moments occurred in which either the teacher or a student treated inequalities as equations. Some of these moments occurred regardless of the type of quadratic inequality that was being solved. Preservice teachers incorporated, into their solution strategies, connections to methods used while solving equations. These connections could be seen while graphing (see Figure 132) or employing algebraic manipulations (see Figure 145). In addition, examples selected (see Figure 149) seemed to point to a belief held by preservice teachers that inequalities were an extension of equations. This belief was seen in other studies and seemed to inhibit development of a relational view of inequalities (Blanco & Garrote, 2007; Tsamir, Almog, & Tirosh, 1998).

Explaining a mathematical idea. As preservice teachers presented their solution strategies, they tended to build upon procedures taught in prior lessons. The procedures for graphing quadratic equations were the initial steps of preservice teachers graphing strategies. Even though preservice teachers displayed notes about the inclusion or exclusion of the boundary curve (see Figure 133 and Figure 134), this merger of procedures seemed to blur the distinction between equations and inequalities. This blurring on the part of preservice teachers was not limited to graphing strategies. Consider the following comments made by Crystal during a classroom observation regarding how to solve quadratic inequalities with one variable using an algebraic manipulations strategy:

Crystal: Now we are going to start solving quadratic inequalities. So this time instead of these just being equations with equal signs, now they are inequalities. So the same rules apply.
Preservice teachers’ merging of procedures seemed to reinforce one of the issues identified within the literature as a promoting agent for the treatment of inequalities as equations; the proximity of equations and inequalities in lessons (Attorps, 2003; Tsamir and Bazzini, 2004).

**View of inequalities.** Comments made by a preservice teacher, whether intentional or accidental, may be a catalyst for formation of incorrect connections between inequalities and equations. Preservice teachers seemed to blur the distinction between equations and inequalities as they explained their solution strategies. As a result, it appeared as if preservice teachers reinforced incorrect connections between equations and inequalities, which seemed to foster an operational view of inequalities.

**Solving a mathematical problem.** A perspective to consider may be the level of importance that the preservice teacher placed on separating equations and inequalities. During classroom observations, there were several moments in which an inequality was called an equation. In the moment, the verbal mislabeling may have been nothing more than a slip of the tongue. An illustration of such a moment could be seen during Vanessa’s class (Classroom observation; April 11, 2011).

Vanessa: [Having selected the point (0, 0) at a test point] Now I’m going to substitute into my equation [pointing to the original inequality; see Figure 161].

However, preservice teachers’ algebraic manipulations strategy seemed to involve more moments when inequalities were treated as equations. These moments were more than verbal mislabeling. This may be due in part to the fact that within preservice teachers’ algebraic manipulations strategy, each of them had a step that required the students to find “critical” values or points. During her post-student teaching interview (May 12, 2011), Heather was
asked about an example where she wrote “look at related equation” (see Figure 145). Heather noted that she wrote that because “that was what the book said to do.” Her comment seemed to indicate a reliance on the textbook as an authority, which is not surprising or uncommon for preservice teachers.

Although these are lengthy examples, Vanessa’s actions illustrate moments when preservice teachers blur the distinction between inequalities and equations. During her classroom observation (April 11, 2011), Vanessa had sections where her work to solve quadratic inequalities with one variable did not match her final answer. Vanessa asked a student for his solution to the third example of a quadratic inequality with one variable (see Figure 150 for specific example). As she was writing, she would offer phrases of approval (i.e., “good” or “yes”). Within the display of the student’s work (see Figure 173), there was an inequality \( x \leq -\frac{1}{2} \); written in black that contradicted the solution \( -\frac{1}{2} \leq x \leq 3 \); written in red. The issue seemed to be linked to a blurring of inequalities and equations.
An examination of Vanessa’s lesson revealed that issues surrounding her inclusion of contradictory information seemed to be triggered by a comment from a student as she was solving her first example of a quadratic inequality with one variable (i.e., $x^2 - 6x - 7 < 0$; see Figure 149) using an algebraic manipulations strategy.

Vanessa: [transitioning from a solving a quadratic equation with one variable] Let's go through the same process for this [pointing to the example; see Figure 149]. We can solve by factoring, quadratic equation [assuming she meant to say quadratic formula], or completing the square. Which three of those would y'all like to do?

Student: Factoring.

Vanessa factored the quadratic expression on the left side of the inequality [writing $(x - 7)(x + 1) < 0$; see Figure 174].
Vanessa: I'm going to set both of these equal to zero [pointing to the two factors]. So $x - 7$ equals zero [writing $x - 7 = 0$; see Figure 175].

Student 1: Less than.

Vanessa: [erased the equal sign and replaced it with the less than sign, $x - 7 < 0$; see Figure 176] Like that? $x$ equals seven [writing $x = 7$ under $x - 7 < 0$; see Figure 177]. Or $x$ is less than seven [erasing the equal sign and replacing it with the less than sign, $x < 7$; see Figure 178]. And over here, I have $x$ plus one is less than zero [writing $x + 1 < 0$]. So $x$ is less than negative one [writing $x < -1$; see Figure 178]. That look right?

Student 1: Yeah.

*Figure 174.* Vanessa's factored version of an example of a quadratic inequality with one variable (Classroom observation; April 11, 2011).

*Figure 175.* For an example of a quadratic inequality with one variable, Vanessa initially set the first factor equal to zero (Classroom observation; April 11, 2011).
Figure 176. For her first example of a quadratic inequality with one variable, Vanessa set the first factor to be less than zero after a student comment (Classroom observation; April 11, 2011).

Figure 177. Vanessa initially wrote $x = 7$ while solving her first example of a quadratic inequality with one variable (Classroom observation; April 11, 2011).

Figure 178. Vanessa's displayed work for her first example of a quadratic inequality with one variable (Classroom observation; April 11, 2011).

During a stimulated recall video played during her post-student teaching interview (May 13, 2011), Vanessa was shown a video clip of the previously described interaction of her working an example of a quadratic inequality with one variable for her students.
(Classroom observation; April 11, 2011; see Figure 149). Without prompting from the researcher, Vanessa focused on her switching an equal sign to an inequality sign (see Figure 175 and Figure 176).

Vanessa: Oh no.

Researcher: Why are you saying oh no?

Vanessa: Because it's less than its going to be an “and” statement. So x is going to be greater than, no, x is going to be less than 7 and greater than -1.

... 

V: See. See [referring to the comments she made on the video]. That's where it messes up.

T: What messes up?

V: Because x is not less than –1 and less than 7 (see Figure 178) that would just be x is less than 7. It is the region bounded between those two [referring to the critical values]. We talked about less than is "and", so "than" "and." So your x is going to be stuck in the middle and then on a number line where they lie. So the 7 is going to lie over here, –1 is going to lie here [pointing to an imaginary number line on her paper].

Vanessa seemed to realize that she made a mistake in her lesson. The alternative explanation, that she offered, gave the impression that the issue of treating inequalities as equations would not persist. However, it was unclear how Vanessa would enact her alternative explanation in a classroom and if her students would understand this alternative explanation. Additionally, Vanessa’s alternative explanation appeared to rely on an association between an inequality sign and a logical connective (i.e., “and” or “or”). The
manner in which she stressed the “an” in less than seemed to indicate that her association was consistent with an operational view of inequalities.

**View of inequalities.** Preservice teachers’ verbal mislabeling and blurring the distinction between inequalities and equations may provide a glimpse into how student’s conceptions, surrounding the treatment of inequalities as equations (Tsamir & Bazzini, 2004), develop and persist. Comments made by a teacher, whether intentional or accidental, may be a catalyst for formation of incorrect connections between inequalities and equations. Once those conceptions begin to form and present themselves, a teacher’s inability or decision not to address those misconceptions may serve to reinforce them. Preservice teachers who actively or passively reinforced incorrect connections between equations and inequalities seemed to foster an operational view of inequalities.

**Using technology.** Vanessa was the only preservice teachers who used technology during her lessons; specifically graphing calculators. When implementing a graphing strategy to solve quadratic inequalities, Vanessa used her graphing calculator as a checking mechanism. As she graphed quadratic inequalities “by hand,” she was able to display dotted boundary curves (see Figure 179). However, she was unable to create dotted boundary curves while using her graphing calculator (see Figure 180). Vanessa did not provide her students with an explanation as she graphed quadratic inequalities on a graphing calculator; especially with respect to the boundary curve. In a similar manner, when Vanessa utilized the “test” feature on her graphing calculator (see Figure 169), she presented it as “something cool” and never explained to her student the process behind the “test” feature. The lack of explanation
Vanessa provided when utilizing a graphing calculator while solving quadratic inequalities seemed to foster ambiguity between equations and inequalities.

*Figure 179.* Vanessa's test point method and shading for an example (Classroom observation; April 11, 2011).

*Figure 180.* Vanessa entering an example in to her graphing calculator (Classroom observation; April 11, 2011).
**View of inequalities.** When Vanessa used her graphing calculator, she seemed to foster an operational view of inequalities with respect to treating inequalities as equations. There were moments when she blurred the distinction between equations and inequalities. Additionally, she provided little, if any, explanation to her students while using graphing calculators that may have deterred her students from making incorrect connections between equations and inequalities.

**The process of shading quadratic inequalities**

For purposes of this study, shading could represent two separate entities while working with inequalities. This section examined shading as the process within a graphing strategy, not as an object. The focal point of consideration was how preservice teachers and their students determined which half-plane to shade for an inequality. Two methods of shading emerged from an examination of preservice teachers’ lessons. The first method, denoted as the shade inside or shade outside method, was an observational association between direction (i.e., inside, outside, above, below, up, or down) and an inequality symbol. This method was similar to the shade above or shade below method employed with systems of linear inequalities (see Chapter 5). The second method, denoted as the test point method, utilized test points to determine the appropriate region to shade as defined by a boundary curve. This method was identical to test point methods utilized with systems of linear inequalities (see Chapter 5).

**Explaining a mathematical idea.** Preservice teachers provided little explanation prior to working any examples with regard to either shading method (shade above or shade
below or test point). During a classroom observation (April 11, 2011), Vanessa provided an illustration of a verbal explanation of “test a point” and “shading:”

Vanessa: When we graph quadratic inequalities, we are going to test a point on one side of the parabola. If it satisfies that then we shade that part. If it did not satisfy it, then we shade the other part. (Classroom observation; April 11, 2011)

She never utilized a drawing of a parabola while discussing the test point and shading. She stood in front of a blank whiteboard as she referenced “one side of a parabola.” While observing Vanessa in the moment, it seemed as if she expected her students to understand what she meant based solely on a verbal presentation. However, during her post-student teaching interview (May 13, 2011), Vanessa noted that she wanted to “throw it [referring to her graphing strategy] out there” and “we’re going to talk about what it is” in an example that followed. This notion of ‘talk[ing] about” in an example was common among preservice teachers.

Preservice teachers were not consistent with regard to the shading method that they advocated. With either shading method, identifying the “inside” and “outside” of parabolas was a prelude. Vanessa’s preference regarding a shading method seemed to depend on whether she was utilizing technology or not. A test point method was employed when graphing “by hand” and shade above or shade below was used when graphing on calculators. Her use of shade above or shade below method while using graphing calculators was a direct tie back to what she showed her students while solving systems of linear inequalities with a graphing calculator. Heather used a test point method of shading as taught her students to solve quadratic inequalities with two variables. However, while teaching her students how to
solve quadratic inequalities with one variable by graphing, Heather employed a variation of the shade above or shade below method. Since she required students to write the quadratic inequality in standard form, one side of the inequality was always zero. She explained to her students that the quadratic expression on the other side of the inequality had to be positive or negative in order to satisfy the given inequality. With the appropriate value determined, she emphasized to her students that above the x-axis was positive and below the x-axis was negative (see Figure 181 for a representation of what she told her students). In this manner, Heather replaced an evaluation of inside or outside the boundary curve with an evaluation of above or below the x-axis.

Figure 181. Graph of $x^2 + 2x - 3 > 0$ from Heather's lesson plan (Lesson plan; March 7, 2011).

Heather was asked, during her post-student teaching interview (May 12, 2011), if she believed that her students understood the test point method.
Heather: I hope that is what they understood. … I really feel like I failed the first students. Like I wasn't doing a great job. I don't know. I don't think they got it. I think just some people did.

Heather’s comments gave the impression that she had a lot of doubts regarding her students’ understanding of her test point method. Those doubts were not exclusive to the test point method. As she watched a video stimulated recall (Classroom observation; March 7, 2011), Heather commented on a graph that she drew for the students (see Figure 137).

Heather: It's a little confusing.

Researcher: In what way?

Heather: Just all the bold, like what's up with the x-axis and stuff.

Heather’s doubts regarding the shading methods may be linked to the timing of her lessons involving quadratic inequalities within her student teaching field experience.

Heather’s quadratic unit, which included her lessons on quadratic inequalities, was taught during her first two weeks of her student teaching field experience.

**View of inequalities.** Preservice teachers seemed to foster an operational view of inequalities as they outlined a test point method to their students while graphing quadratic inequalities with two variables. They tended to provide their students with little, if any, explanation about their version of a test point shading method. The test point method appeared to be a set of procedures that students were expected to follow. The rationale as to why the method works did not seem to be fully explained to students by preservice teachers. Consider Vanessa’s use of “satisfy that” in her verbal presentation of a test point method, she

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25 Heather taught two Algebra II Honors classes. The observed class was earlier in the day than the other class.
gave the impression that her students would understand her choice of language with no explanation.

While graphing quadratic inequalities with one variable, Heather’s shade above or shade below the x-axis method\textsuperscript{26} seemed to promote a relational view of inequalities. Her shading method emphasized understanding the relationship between the two sides of quadratic inequalities with one variable. One area of concern with her method was the contingency that one side of the quadratic inequality must be zero. Heather could have generalized her shade above or shade below the x-axis method to allow for expressions other than zero. This modification may have fostered a deeper relational view of inequalities.

**Solving a mathematical problem.** As mentioned earlier, preservice teachers employed a test point method of shading while graphing quadratic inequalities with two variables. During a classroom observation (March 8, 2011), Heather provided a typical illustration of how preservice teachers implemented a test point method of shading with their students. With the boundary curve for a quadratic inequality in place (see Figure 182), Heather directed her students to select a “test point” in order to determine “what part to shade” [pointing to the two regions created by the boundary curve]. A student suggested using the origin as the test point. Heather with input from her students substituted the values of the test point into the original inequality and simplified (see Figure 183).

Student 1: That is true [referring to $0 > -7$] so you shade the part with (0, 0) in it.

Student 2: On the outside.

\textsuperscript{26}Heather was the only preservice teacher who implemented a graphing strategy to solve quadratic inequalities with one variable. This shading method was a part of that graphing strategy.
Heather: OK. So we go like this [shading the "outside" of the parabola; see Figure 184].

Student 1: Yeah.

*Figure 182.* Heather’s graph of boundary curve (Classroom observation; March 8, 2011).

*Figure 183.* Heather’s work to evaluate the test point (Classroom observation; March 8, 2011).

*Figure 184.* Heather's shaded graph (Classroom observation; March 8, 2011).
View of inequalities. Preservice teachers’ implementation of a test point method of shading during their lessons appeared to promote an operational view of inequalities. The test point method seemed to be nothing more than another step along the path to finding the solution. However, Heather fostered a relational view of inequalities when she employed her shade above or shade below the x-axis method while solving quadratic inequalities with one variable with her students. Her statement that “the x-axis is zero” displayed a relational view of quadratic inequalities with regard to graphing. This seemed to be aligned with the relational view of quadratic inequalities Heather exhibited during her pre-student teaching interview (see Figure 70).

Using technology. As mentioned earlier, Vanessa encouraged her students to use a test point method as well as the shade inside or shade outside method. She referred to the regions formed by the boundary curve as “inside” and “outside” of the parabola. While graphing quadratic inequalities without technology, she advanced the test point method (see Figure 142). On the other hand, she presented the shade inside or shade outside method when utilizing a graphing calculator (specifically her TI-84; see Figure 165 and Figure 166). The separation of methods that Vanessa followed was not affected by the format of quadratic inequality with two variables. It seemed that the separation of methods was based Vanessa’s desire to promote “conceptual understanding” of quadratic inequalities.

During her post-student teaching interview (May 13, 2011), Vanessa was asked by the researcher to talk about how she would handle a potential conflict between the two shading methods.
Researcher: So, if there is an issue between the graph drawn by hand and what the calculator says, how do they resolve that conflict? Like if they have shaded outside by hand but then on the calculator picked the wrong triangle, the little triangle, picked the wrong one.

Vanessa: You got to check it. You got to test a point. I told them, you can test numerous points. I think one of the girls, she is strong, [and] she comes in to smart lunch just about every day. I told her I said if you don't believe this point pick another one. And you know if you pick the point out here and it did satisfy it and you shade here, test a point here [referring to the other region] and make sure it doesn't satisfy it. But you can test as many points as you want.

Based on her response, Vanessa gave the impression that she endorsed the test point method as the primary method for determining the appropriate region to shade when graphing quadratic inequalities with two variables.

*View of inequalities.* With the graphing calculator, Vanessa seemed to advocate a shade above or shade below method for determining the appropriate region to shade. The association between the sense of the inequality sign and the triangle next to $Y1=$ seemed to bridge back to previously taught lessons regarding linear inequalities. However, Vanessa provided no explanation as the appropriateness or rationale of the association. She treated the association as something her students should memorize. Her actions seemed to be consistent with an operational view of inequalities.
Solutions to quadratic inequalities

This section focused on references to the solution of quadratic inequalities made by either the preservice teacher or their students. Unlike the previous section, which considered shading as a process (i.e., test point method or shade above or shade below method), this section contemplated shading as an object representing the solution set of a quadratic inequality. Some commonalities in preservice teachers’ presentation of solutions of quadratic inequalities will be presented.

Explaining a mathematical idea. The boundary curve of the graph of a quadratic inequality with two variables was an area of interest when examining preservice teachers’ explanation of the meaning of the solution. The inequality sign of quadratic inequalities with two variables dictates whether points on the boundary curve are included in the solution or not. The preservice teachers seemed to place emphasis on this fact during specific moments.

Vanessa provided an example that was typical among the preservice teachers. During her classroom observation (April 11, 2011), Vanessa had the following dialogue with her students while graphing an example of a quadratic inequality with two variables (see Figure 160 for the example):

Vanessa: Now since this is less than or equal to [pointing to the inequality sign; see Figure 160], is my line going to be dotted/dashed or is it going to be a solid line?

Student: Solid [said by two students]

Vanessa: OK, a solid line. Why? STUDENT NAME, why?

Student: Because it is less than or equal to.

[Vanessa drew in a solid parabola through the plotted points; see Figure 139]
Vanessa: Yeah, but why? Why is my line going to be solid for this?

Student: Because the points are included.

Vanessa: Good. They are a part of [it].

At one point, a student offered his understanding for why the parabola should have a solid boundary curve (Classroom observation; April 11, 2011). His reasoning relied on an association between the inequality sign and a solid or dotted boundary curve. Vanessa acknowledged that the student’s association was correct. However, she pushed the students to consider the implications beyond the association. It appeared that preservice teachers wanted to emphasize the relationship between a solid or dotted boundary curve and the inclusion or exclusion of those points in the solution of the quadratic inequality.

Preservice teachers’ emphasis on the boundary curve did not carry over to the end of examples. As an illustration of how preservice teachers did not treat boundary curve near the end of an example, consider Vanessa’s comments after graphing the previously mentioned quadratic inequality (see Figure 160 for the example). While talking about the graph of a solution (see Figure 143), Vanessa noted that “everything outside of this parabola is part of my solution” (Classroom observation; April 11, 2011). Later Vanessa plotted four points on the graph of her solution (see Figure 185). She noted that all of the points were outside the parabola and would satisfy the original inequality (see Figure 160). Vanessa did not mention the boundary curve in this part of her discussion. Preservice teachers may have believed that the emphasis they placed on the boundary curve earlier in examples was enough to ensure students’ understanding. However, their omission of boundary curves in a discussion of the solution may send conflicting signals to their students.
Figure 185. The points plotted on the solution of Vanessa's first example of a quadratic inequality with two variables (Classroom observation; April 11, 2011).

**View of inequalities.** As preservice teachers implemented a graphing strategy for quadratic inequalities with two variables, their explanation of a solution was consistent with a relational view of inequalities. They discussed what it meant to be a solution relative to substituting values into an inequality and determining the truth value. However, preservice teachers did not include a discussion about the points relative to the parabola that formed the boundary curve of the inequality. The lack of a discussion about the boundary curves seemed to foster an operational view of quadratic inequalities.

**Solving a mathematical problem.** Due to the types of quadratic inequalities encountered during the lessons, the solution was not represented in a uniform manner. The solution of quadratic inequalities were represented as a shaded region on a Cartesian coordinate plane, a shaded section on the x-axis of a Cartesian coordinate plane, a shaded section on a number line, a compound inequality, or in interval notation. The determining
factor regarding the representation of a solution seemed to be the type of quadratic inequality being solved.

The sign chart was typically used as part the preservice teachers’ algebraic manipulations strategies while solving quadratic inequalities with one variable. A shaded number line was the product of the sign chart in Crystal and Heather’s lessons. However, none of the preservice teachers represented the solution with a shaded number line. Crystal made the following comment about the solution of a quadratic inequality with one variable during a classroom observation (April 5, 2011).

Crystal: How do I write my answer, to show this region [pointing to the shaded region on her number line; see Figure 186]? Because my answer is not just this number line. The number line shows the answer but when I'm looking for your answer on a test or a quiz or if it's multiple choice you are not going to see a number line shaded. What you are going to see is this shaded part written as an interval notation.

This preference to symbolic representations of the solution could be seen when a graphing strategy was employed to solve quadratic inequalities with one variable. While discussing with her students the solution for an example (see Figure 152 for the example), Heather commented on the appropriate form of the “solution set” (Classroom observation; March 7, 2011).

Heather: A graph [pointing to the graph in Figure 137] is a great thing to look at but a graph is not the solution set for this [pointing to the original inequality; see Figure 152]. You can’t just show a graph and be good for this problem.
The preservice teachers seemed to insist on symbolic notation as the accepted representation of the solution of a quadratic inequality with one variable. The symbolic notation included interval notation as well as compound inequalities. This decision may have been influenced by the preservice teachers’ textbook and/or the cooperating teacher and expectations on standardized tests. Another explanation may lie in the preservice teachers’ perceptions regarding the conventional nature of symbolic notations. It appeared that preservice teachers allowed other authorities to dictate the appropriate representation of the solution.

**View of inequalities.** Preservice teachers’ insistence on symbolic notation for solutions of quadratic inequalities seemed to foster an operational view of inequalities. Preservice teachers did not elaborate on the connections between the various representations of a solution of a quadratic inequality. The manner in which preservice teachers downplayed
graphical representations (e.g., Heather: “A graph is a great thing to look at but…”) appeared to be a missed opportunity to foster a relational view of inequalities.

**Using technology.** As mentioned earlier, graphing calculators were only used by Vanessa, albeit not an integral part. After completing an example “by hand,” Vanessa led her students through a series of steps to produce a graph representing the solution. The way in which Vanessa entered quadratic inequalities into her graphing calculator did not account for a dotted boundary curve created by an inequality with a less than sign or a greater than sign (see Figure 179 and Figure 180). The shaded region depicted on her graphing calculator was compared against her solution created “by hand.” Vanessa gave the impression that this comparison was a formality.

**View of inequalities.** Vanessa’s actions were coherent with an operational view of inequalities with regard to solutions of quadratic inequalities while using technology, specifically graphing calculators. Once the shaded region representing a solution was displayed on her calculator and compared to the solution created “by hand,” an acknowledgement that the solutions matched was made and the example was over. Vanessa seemed to miss opportunities to talk about the meaning of a solution and/or the limitations in the manner in which she employed graphing calculators to display solutions (e.g., exclusion of a boundary curve).

**Characterizing How Preservice Teachers Approach Quadratic Inequalities**

Preservice teachers typically have very limited experiences with regard to teaching or observing others teaching lessons involving quadratic inequalities. Thus, they relied heavily on their own understanding of how to approach quadratic inequalities when they planned and
taught lessons during their student teaching field experience. Their lack of experience led the preservice teachers to make assumptions while planning the lessons that may have differed from those of teachers with experience teaching the prerequisite lessons.

Preservice teachers seemed to implement solution strategies based on the type of quadratic inequalities being taught. Vanessa provided an illustration of the separation between graphing and solving quadratic inequalities as she transitioned from quadratic inequalities with two variables to quadratic inequalities with one variable (Classroom observation: April 11, 2011). It appeared that Vanessa linked “solving by graphing” with quadratic inequalities with two variables. Additionally, she linked “solve by hand” [a reference to the algebraic manipulations strategy] with quadratic inequalities with one variable. On the other hand, Heather referred to the process as solving for both types of quadratic inequalities (see Figure 148 and Figure 133). The separation may not exist for Heather because she used a graphing strategy and an algebraic manipulations strategy to solve quadratic inequalities with one variable. The solution strategies promoted by preservice teachers fostered an operational view of inequalities.

The preservice teachers’ lessons about quadratic inequalities seemed to lack meaningful inclusion of technology. The preservice teachers had various forms of technology available for use within their classroom; not to mention the availability of computer labs at each school. In most instances, technology was not utilized by preservice teachers. The technology that was incorporated into Vanessa’s lessons was a graphing calculator (typically TI-83s and TI-84s). A limited range of features on the graphing calculators were used; these included the logic test, graphing, shading, and table. During pre-student teaching interviews,
Preservice teachers displayed an ability to use their graphing calculators to determine x-intercepts, points of intersection, and values for particular x-values. The manner in which graphing calculators were utilized by Vanessa, the only preservice teacher who incorporated graphing calculators into her lessons, was consistent with an operational view of inequalities.

Preservice teachers’ lessons involving quadratic inequalities seemed to be treated as an opportunity to review other concepts and topics. Heather noted that she structured her quadratic inequality lessons in such a manner to allow her students to get more practice graphing “because they were not good graphers with parabolas. So I wanted to give them more practice” (Post-student teaching interview; May 12, 2011). There were instances when preservice teachers reviewed topics such as completing the square, factoring, and quadratic formula.

In addition to the extra practice, the preservice teachers seemed to want to explicitly connect their lesson(s) on quadratic inequalities to prior topics. Before beginning the first example of a quadratic inequality with one variable (Classroom observation; April 5, 2011), Crystal made the following comment: “So this time instead of these just being equations with equal signs, now they are inequalities, so the same rules apply.” Crystal’s directions to her students seemed to be an attempt to connect the strategy for solving quadratic inequalities with one variable to the strategy she employed to solve quadratic equations with one variable (Classroom observation; April 5, 2011). Preservice teachers seemed to blur the distinction between inequalities and equations as they merged procedures and reviewed prior material. These actions appeared to be consistent with an operational view of inequalities with respect to treating inequalities as equations.
Preservice teachers tended to present a test point method of shading when graphing quadratic inequalities with two variables. They seemed to describe the test point method as a series of steps with no explanation as to the appropriateness of the end result: shading “inside” or “outside” of the parabola. An operational view of inequalities was fostered by preservice teachers as they outlined and implemented a test point method of shading.

The restrictive nature regarding the appropriate representation of a solution of a quadratic inequality displayed by preservice teachers seemed to be coherent with an operational view of inequalities. Symbolic representations, such as interval notation or compound inequalities, were emphasized by preservice teachers as being THE way to represent a solution. Opportunities to establish and/or cultivate connections between the various representations of solutions, including graphical, appeared to be missed.

Chapter Summary

This chapter addressed the ways preservice teachers use their understanding of inequalities while planning and implementing lessons on quadratic inequalities. The primary sources of data for analysis were classroom observations and field notes. The secondary sources of data for analysis were pre-student teaching interviews (see Appendix C), post-observation questioning, lesson plans submitted by preservice teachers, and post-student teaching interview (see Appendix D). A description of the case of quadratic inequalities was divulged. The next and final chapter will discuss the findings as they relate to the research questions; make connections to the literature; note the limitations of this study; and provide implications.
CHAPTER 7: DISCUSSION

Introduction

The purpose of this study was to examine the nature of preservice teachers’ understanding of inequalities and how their understanding is used while planning and implementing lessons regarding inequalities. The participants of this study were five preservice teachers at a major university in the Southeast United States, who were completing their student-teaching field experience.

The research questions of this study were the following:

- What is the nature of preservice teachers’ understanding of inequalities prior to the student teaching field experience?

- How do preservice teachers use their understanding of inequalities while planning and implementing lessons during the student teaching field experience?

These two research questions were addressed through separate case studies. Five cases, representing the individual preservice teachers, were examined for the first research question. The column headings of the conceptual framework (see Table 2) representing different aspects of inequalities served as a lens for analyses of preservice teachers’ understanding about inequalities.

There were two cases for the second research question of this study. The cases consisted of the two different topics in inequalities that were being taught by the preservice teachers during the student teaching field experience that may draw upon different content knowledge by the preservice teachers: systems of linear inequalities and quadratic inequalities. Each case consisted of lessons taught by preservice teachers during the student
teaching field experience regarding systems of linear inequalities or quadratic inequalities. Case 1 consisted of four preservice teachers (Angela, Christina, Crystal, and Vanessa) taught lessons regarding systems of linear inequalities. Case 2 included lessons pertaining to quadratic inequalities that were taught by three preservice teachers in this study (Crystal, Heather, and Vanessa). It should be noted that lessons by Crystal and Vanessa were in both cases because they taught both topics during their student teaching field experience.

The conceptual framework of this study was a matrix design with two overarching views (see Table 2). The column headings of the framework were aspects of knowledge associated with inequalities. These aspects were identified within a literature review. With regard to this study, there were four primary aspects of knowledge associated with inequalities: 1) Strategies for solving inequalities; 2) Relating inequalities as equations; 3) Shading as a process; and 4) Solutions of inequalities. Within those primary aspects there were other aspects that may be pertinent to more than one primary aspect: domain; use of variable(s); factors of products and/or quotients; logical connectives; and interpretations of the inequality symbol. The row headings were teaching practices: 1) Explaining a mathematical idea; 2) Solving a mathematical problem; and 3) Using technology. These teaching practices were related to some of the tasks of teaching identified within the Ferrini-Mundy et al. (1996) framework (see Figure 2), and were chosen as practices that would likely occur with enough frequency to allow for focused analysis. Additionally, the conceptual framework for this study had two overarching views: operational view of inequalities and relational view of inequalities. These views were adaptations of views of equations presented by Knuth et al. (2006). An operational view of inequalities involved a
belief that an inequality sign was a signal to perform operations or get an answer. A relational view of inequalities involved a belief that an inequality was a statement of a mathematical relationship that corresponded to the given inequality symbol. These views were a third dimension of the matrix design. Each cell in the framework represents an event(s) in which a teaching practice (rows) and an aspect of knowledge associated with the mathematical concept of inequalities (columns) occurred. Within each cell of the matrix, events were classified as an operational view of inequalities or relational view of inequalities.

In this chapter, the findings with respect to the two research questions will be presented. In addition, the limitations of the study will be discussed. The chapter will conclude with a description of possible area for future research.

**Summary of Research Questions and Findings**

**Research Question 1**

Findings for the first research question were discussed in Chapter 4.

*Research Question 1: What is the nature of preservice teachers’ understanding of inequalities prior to the student teaching field experience?*

Tsamir and Almog (2001) identified three primary strategies used to solve inequalities: algebraic manipulations of inequalities; graphing; and case method. None of the preservice teachers in this study successfully implemented the case strategy to solve any of the inequality problems during their pre-student teaching interviews. Only one preservice teacher attempted to use a case strategy for one question. She was unsuccessful and switched to an algebraic manipulations strategy. Algebraic manipulation of inequalities was the most widely used solution strategy, a finding that was in line with a review of literature (Tsamir &
Almog, 2001; Tsamir et al., 1998). A graphing strategy was implemented while solving quadratic inequalities (see Figure 15 and Figure 18) and systems of linear inequalities (see Figure 17). Some preservice teachers indicated that they preferred using a graphing strategy. However an absolute value inequality (see Figure 16), which could have been solved graphically, was solved exclusively using algebraic manipulations by all five preservice teachers. The type of inequality being solving seemed to be a determining factor for which strategy was employed by preservice teachers.

Attorps (2003) noted that some teachers incorrectly included inequalities in their concept image of equations. One teacher from Attorps’ study offered the following justification for considering $x + |x - 3| \geq |x - 1| + 2$ as an equation:

Teacher: Yes, it is an equation. It is an inequality. I wonder if it is an equation or not, but I think inequalities are treated in the same section of the textbook as equations, so it is some kind of equation. (Attorps, 2003, p. 6)

The teacher’s justification from Attorps’ study was echoed by Crystal in this study.

Crystal: …you learn the equations and then you just take inequalities. … you definitely learn inequalities after you learn equations.

(Pre-student teaching interview; February 16, 2011)

The results indicate that some preservice teachers included inequalities in their concept image of equations and some did not. The findings of this study seemed to match Attorps (2003) findings. The justification of the proximity of equations and inequalities appeared to be an area of consistency between the studies.
Preservice teachers in this study revealed a tendency to treat inequalities as equations. Tsamir and Bazzini (2004) noted that this tendency manifested itself when the students in their study were solving inequalities. The similarity in the structures of systems of linear equations and systems of linear inequalities seemed to be the catalyst that prompted some preservice teachers to try to use the elimination method for systems of equations to solve the given system of linear inequalities question on their pre-student teaching interview.

Preservice teachers displayed various levels of understanding in accounting for implications of multiplying or dividing an inequality by a negative value. Only one preservice teacher was able to account for the possibility of dividing an inequality by a negative in all three questions presented in this study (see Figure 11, Figure 12, and Figure 13). This finding seemed to indicate that difficulty accounting for implications of dividing or multiplying an inequality by a negative value persisted beyond high school (Tsamir and Bazzini, 2002).

Vaiyavutjamai and Clements (2006) described a tendency of students in their study to provide “single number answers” (p. 134). This tendency was categorized by a belief that the solution to an inequality should be structurally similar to an equation (i.e., a solution of $x > 5$ would be expressed as $x = 5$). None of the preservice teachers in this study displayed this tendency. Since the participants in Vaiyavutjamai and Clements’ study were ninth grade students, this may indicate that the single number answer tendency fades as experience with inequalities increases.

Unlike the single number answer tendency, some other tendencies related to the solutions of inequalities seemed to persist. The high school students in Tsamir and Bazzini’s
(2004) study exhibited a belief that inequalities will have inequalities as solutions. Preservice teachers in this study were given the same questions that Tsamir and Bazzini (2004) presented to their students (see Figure 10 and Figure 14). All of the preservice teachers encountered conflict while considering whether an inequality could have an equation as its solution. Just like students in Tsamir and Bazzini’s (2004) study, preservice teachers offered inequalities where a given solution (i.e., \( x = 3 \)) was part of a solution (e.g., \( x > 2 \)). None of the preservice teachers were able to produce a valid example of an inequality with an equation as a solution until they encountered one (see Figure 14).

During pre-student teaching interviews, preservice teachers were asked to find a solution of a system of linear inequalities (see Figure 17). Preservice teachers seemed to display a robust operational view of inequalities as they produced a solution. Once they produced a solution, they were asked if some coordinates were part of the solution or not. In addressing the question for each coordinate, preservice teachers tended to refer to their graphs. Preservice teachers displayed relational view of inequalities as they explained why the intersection of two corresponding boundary lines was not part of the solution.

**Summary.** Preservice teachers’ actions were consistent with an operational view of inequalities. There were instances of false-starts. But once they were able to recall the “steps,” in most cases preservice teachers were able to produce a correct answer using normative mathematical practices while solving various inequalities (i.e., absolute value inequalities, quadratic inequalities, systems of linear inequalities). Preservice teachers exhibited some of the difficulties noted within prior literature. There did not appear to be a
relationship between the difficulties encountered: displaying one difficulty was not an indicator that another difficulty would accompany it.

Heather was a notable exception among preservice teachers. The difficulties that she exhibited were minor and she was able to produce mathematically correct answers for each question. She consistently displayed actions coherent with a relational view of inequalities. There were moments when Heather talked about inequalities as a relationship between objects.

In the following section, the findings and conclusions related to the second research question will be discussed.

**Research Question 2**

The second research question was separated into two cases based on the concepts regarding inequalities that preservice teachers taught during their student teaching field experience. The two cases were systems of linear inequalities and quadratic inequalities. There were four preservice teachers who taught lessons involving systems of linear inequalities and there were three preservice teachers who taught lessons pertaining to quadratic inequalities. The findings for the second research question with respect to systems of linear inequalities were discussed in Chapter 5. The findings for the second research question with respect to quadratic inequalities were discussed in Chapter 6. A discussion about how preservice teachers engaged in the three teaching practices (explaining mathematical ideas, solving mathematical problems, and using technology) and how they may have been using their understanding while teaching lessons involving inequalities follows.
Research Question 2: How do preservice teachers use their understanding of inequalities while planning and implementing lessons during the student teaching field experience?

Influence of cooperating teacher.

What we see others do, and what we hear them say, inevitably affects what we do and say ourselves. More important still, it reflects upon our thinking.

(von Glaserfeld, 1995, p. 191)

Von Glaserfeld’s words apply to this study in a number of ways. The relationship between a cooperating teacher and a preservice teacher is just one of those ways. The actions of cooperating teachers can be beneficial or detrimental (Bush, 1986), as a result they cannot be ignored. Preservice teachers’ understanding of inequalities played a role in their planning and implementation of lessons during the student teaching field experience. However, that role was mitigated by influences from their cooperating teacher. Within this study, there was evidence that cooperating teachers exerted some influence on their preservice teachers’ lessons. Nowhere was that more clearly evident than use of technology; specifically use of graphing calculators. Cooperating teachers made comments to their preservice teachers about incorporating graphing calculators into lessons. Apparently, the primary reason for incorporating graphing calculators into lessons was to facilitate students passing state-level assessments.

Explaining mathematical ideas. The manner in which preservice teachers engaged in explaining mathematical ideas related to systems of linear inequalities and quadratic inequalities were often very similar. Their explanations illustrate the ways they predominately promoted an operational view of inequalities across the four focal aspects of
knowledge in this study (Strategies for solving inequalities; relating inequalities as equations; shading as a process; and solutions of inequalities).

While planning and teaching lessons related to systems of linear inequalities, all preservice teachers implemented a graphing strategy. Steps that they presented to their students were similar to those noted by Reilly (2010) with the exception of the shading method. Preservice teachers outlined and explained the steps of their graphing strategy either prior to or while working the first example. Preservice teachers were also uniform regarding their decision to outline and explain solution strategies while planning and teaching lessons related to quadratic inequalities. Very little explanation was provided by preservice teachers prior to working examples. The first example was typically used as an opportunity to “show” students the steps of particular strategies. A possible reason for this shift may be preservice teachers’ perceptions of their students’ understanding of inequalities. These perceptions may have been informed by previous lessons taught, in particular lessons related to systems of linear inequalities.

The preservice teachers’ choice of strategy to advocate to their students seemed to be related to the type of inequality being solved and not to any preference on their part. A notable exception to this observation was Heather. Heather seemed to prefer to implement a graphing strategy. Her preference may be related to a visual nature of the strategy. However, within her classroom, she presented both a graphing strategy as well as an algebraic manipulations strategy to her students. She gave the impression that she took into consideration her students’ preferences as well as potential usage in the future.
Strategies for solving systems of linear inequalities and quadratic inequalities that were presented to students typically built upon earlier lessons with systems of linear equations and quadratic equations, respectively. In doing so, preservice teachers implicitly fostered connections between equations and inequalities. At times, connections were explicitly fostered through comments by preservice teachers (e.g., “the same rules apply” Crystal Classroom observation; April 5, 2011). Comments like these dilute the meaning attached to inequality signs (Blanco & Garrote, 2007). The references, connections, and comments seemed to reinforce one of the issues identified within literature as a promoting agent for treating inequalities as equations; the proximity of equations and inequalities in lessons (Attorps, 2003; Tsamir and Bazzini, 2004).

Preservice teachers favored a shade above or shade below method of shading inequalities while solving problems on their own. Their preference carried over into their classrooms. Even though preservice teachers may have believed that their shade above or shade below method promoted a relational view of inequalities, the manner in which they presented it in class often fostered an operational view of inequalities. While this method could have been used to discuss the relationship between the two expressions in an inequality, the preservice teachers only focused on the procedural aspect of the method. In essence, they reduced the shade above or shade below method to an association between the sense of inequality signs and directions. This association may have enabled students to get correct answers, but, as Ball and Forzani (2010) noted, such short cuts and rote associations can also circumvent actual learning. Additionally, this association used by the preservice
teachers retained little or no meaning regarding the relationship implied by inequality signs (Tsamir et al., 1998).

Contrary to the operational view of shading inequalities that was prevalent with other shading methods, Heather’s modified shading method seemed to promote a relational view of inequalities. However, it was specific in nature; needing one side of the inequality to be zero. This lack of generality seemed to align strongly with Heather’s understanding as displayed during her pre-student teaching interview (see Figure 73). This suggests that her understanding was a factor in planning and implementing her quadratic inequality lessons.

Acting as a guide, preservice teachers led their students on a procedural endeavor to find the solution of inequalities (Steinberg, Sleeman, & Ktorza, 1990). While implementing a graphing solution strategy, the steps of the process created a separation between the boundary line or curve and the shading method. This separation seemed to stymie a relational view of inequalities with regard to the meaning of a solution. With systems of linear inequalities, obtaining the “overlapping” shaded region, often represented with different colors, seemed to signal the end of an example or problem. Preservice teachers seemed to place an emphasis on making sure students were able to identify the appropriate portion of the Cartesian coordinate plane containing the solution. However, that emphasis did not carry over to inclusion or exclusion of boundary lines.

Opportunities to discuss the meaning of a solution of inequalities were often missed. This may have been attributable to preservice teachers’ inability to identify comments from their students that could have introduced conversations that fostered a relational view of inequalities (Speer & Wagner, 2009). When those discussions did occur, they were limited in
nature. Preservice teachers made comments about satisfying an inequality. Unfortunately, those comments often neglected to include the boundary line(s) or curve: “everything outside of this parabola is part of my solution [pointing to Figure 143]” (Vanessa; Classroom observation; April 11, 2011). These oversights seemed to be indicative of an inequality relationship losing some of its meaning (Blanco & Garrote, 2007).

Overall, the explanations provided by preservice teachers focused on the steps in procedures, rather than why those steps were connected with the meaning of the relationship signified in an inequality statement. Thus, overall, they did little to explain the relational meaning of inequalities and focused on operational aspects. This was typically consistent with the approaches they took in explaining their own work with inequalities, as discussed in Chapter 4. Most preservice teachers used an operational approach in their own work and in their teaching practices, with occasional statements that indicated they understood the relational aspect of inequality statements. Heather, who displayed more of a relational view of inequalities in her own, was the only preservice teachers that used relational approaches in her practice of explaining the meaning of inequalities.

**Solving mathematical problems.** Preservice teachers engaged in similar teaching practices when solving mathematical problems related to systems of linear inequalities and quadratic inequalities. Their practices promoted an operational view of inequalities across the four focal aspects of knowledge in this study (Strategies for solving inequalities; relating inequalities as equations; shading as a process; and solutions of inequalities).

The type of quadratic inequality being solved seemed to dictate which solution strategy was implemented during lessons (Tsamir & Bazzini, 2004). Quadratic inequalities
with one variable were typically solved using an algebraic manipulation strategy that utilized sign charts. Graphing strategies were employed to solve quadratic inequalities with two variables and systems of linear inequalities.

Inequality examples selected by preservice teachers appeared to serve a dual purpose. First, preservice teachers appeared to want (need) to expose their students to systems of linear inequalities and/or quadratic inequalities, likely based on curriculum requirements. For this purpose, the selection of examples appeared to support reinforcement of a set of procedures from solution strategies dictated by preservice teachers. Based on the structure of some examples, it appeared that preservice teachers favored strategies that emphasized “by hand” over technology.

Second, preservice teachers gave the impression they wanted to incorporate review of concepts taught in prior lessons (i.e., graphing linear equations, factoring, quadratic formula, graphing quadratic equations). At times, preservice teachers’ lessons gave the appearance of being reviews under the guise of learning about inequalities. Reviewed material was so prominent in some cases that it relegated understanding inequalities to a secondary status. These actions by preservice teachers were not haphazard or on-the-fly decisions. Each of the preservice teachers noted a desire to incorporate various “reviews of” or “extra work with” previously taught concepts into their inequality lessons. By diminishing the role of inequalities, preservice teachers seemed to be sacrificing their students’ understanding of inequalities for potential understanding of prior concepts. This second purpose, an opportunity to review, may hint at a belief held by preservice teachers that inequalities are an extension of equations.
While solving problems with inequalities, two methods of shading inequalities were utilized by preservice teachers: 1) test point(s); and 2) shade above or shade below. Preservice teachers had a preference to use a shade above or shade below method when solving inequality problems in the pre-student teaching interviews. In planning and implementing their inequality lessons, preservice teachers used a test point method of shading as a precursor to a shade above or shade below method. Preservice teachers placed an emphasis on utilizing a shade above or shade below method of shading inequalities. Based on responses during post-student teaching interviews, the preservice teachers’ emphasis was a deliberate decision shaped by their beliefs that a shade above or shade below method was more desirable and promoted a relational view of inequalities.

The representation of a solution of an inequality seemed to be dictated by the type of inequality being solved. Preservice teachers used shaded regions on a Cartesian coordinate plane to represent solutions of systems of linear inequalities and quadratic inequalities with two variables. Solutions of quadratic inequalities with one variable were presented as compound inequalities. This consistency gave the impression that preservice teachers’ understanding with regard to the “appropriate” representation of a solution of an inequality affected actions in their lessons. Preservice teachers’ tendency to provide a single “appropriate” representation of a solution, especially when solving quadratic inequalities with one variable, may be related to a notion that only “formal” methods and representations are mathematically acceptable (Van Dooren, et al., 2002). During pre-student teaching interviews, preservice teachers provided these same representations for their solutions of inequalities (see Chapter 4).
Overall, the examples provided by preservice teachers focused on performing predetermined steps in procedures. The connections between the steps often were not elaborated on by preservice teachers as the examples were being solved. The examples appeared to be an exercise in producing a solution that was acceptable to the preservice teachers. Thus, overall, their examples did little to foster the relational meaning of inequalities; instead highlighting the operational aspects.

**Using technology.** The manner in which preservice teachers engaged in using technology related to systems of linear inequalities and quadratic inequalities were similar. Preservice teachers incorporated technology into their solution strategies when teaching systems of linear inequalities more than when teaching quadratic inequalities. Their incorporation of technology, specifically graphing calculators, illustrated the ways they predominately promoted an operational view of inequalities across the four focal aspects of knowledge in this study (Strategies for solving inequalities; relating inequalities as equations; shading as a process; and solutions of inequalities).

With technology readily available, preservice teachers gravitated to the graphing feature of their graphing calculators. They typically walked their students step-by-step through how to use the graphing calculator. The endeavor resembled a button pushing exercise more than an exploration. Additionally, the processes behind features used on graphing calculators, such as the logical test feature, were not described by preservice teachers. The lack of explanation from preservice teachers regarding the use of a graphing calculator while solving inequalities seemed to foster ambiguity between equations and inequalities.
Even with preservice teachers’ apparent preference towards either TI-83 or TI-84 graphing calculators, their usage of graphing calculators in the classroom could be considered as minimal. The technology was not an integrated part of the lessons. The calculators seemed to be promoted as a checking mechanism for solutions obtained “by hand.” At times, this comparison of solutions appeared to be a formality. For most of the preservice teachers, this was consistent with their actions during pre-student teaching interviews. During these interviews, they used their calculators as a “reliable replacement for mental or pen and paper calculations” (Goos et al., 2003, p. 78).

Preservice teachers’ decisions regarding the use of technology seemed to shape the shading method employed. A shade above or shade below method was exclusively used when graphing on calculators. Preservice teachers’ understanding of how to graph inequalities on a graphing calculator appeared to shape their usage of a shade above or shade below method. During pre-student teaching interviews, they used the same method of shading on their graphing calculators that they showed their students in classroom observations. Unfortunately, preservice teachers did not fully address whether the boundary line was solid or dotted while using graphing calculators to solve inequalities during classroom observations. As a result, they did not focus their students’ attention on the boundaries of the shaded regions.

The manner in which graphing calculators were utilized could be categorized as “technology as servant;” meaning that it was used as a “supplementary tool” but not “in creative ways to change the nature of the [examples]” (Goos et al., 2003, p. 78). This was consistent with the manner in which preservice teachers used graphing calculators during
pre-student teaching interview (see Chapter 4). Overall, the usage of technology by preservice teachers focused on the steps necessary to create a shaded region on their graphing calculator. Opportunities to discuss the limitations of their methods and the meaning of those shaded regions were missed. Thus, overall, the way in which technology was used during classroom observations did not foster development of the relational meaning of inequalities. Instead it accentuated the operational aspects.

**Summary.** Preservice teachers’ understanding of inequalities played a role in their planning and implementation of lessons. Their explanation and implementation of solution strategies, during pre-student teaching interviews and in their classroom, tended to build upon strategies associated with equations. This approach seemed to perpetuate a lack of distinction between inequalities and equations in lessons, especially when solving mathematical problems.

Boundary lines or curves of inequalities seemed to be a recurring focal point where preservice teachers’ understanding influenced their lessons. Graphing boundary lines or curves was explained as an extension of graphing corresponding equations thereby providing opportunities to review prior concepts. Solution strategies were employed in an operational manner which tended to treat boundary lines or curves as an isolated component. This isolation was apparent even when graphing calculators were utilized. Additionally, the isolation often carried over to descriptions of solutions of inequalities. When teaching, preservice teachers described solutions of inequalities as regions and did not mention the inclusion or exclusion of boundary lines or curves. This was counter to their displayed understanding on pre-student teaching interviews, where preservice teachers displayed a
relational view of inequalities regarding the inclusion or exclusion of boundary lines or curves in solutions of inequalities.

Ferrini-Mundy and her colleagues identified three overarching teaching practices within their framework (bridging, trimming, and decompressing; see Figure 2). Within this study, there seemed to be moments when preservice teachers applied these teaching practices. During lessons, preservice teachers decided how much to emphasize to their students the exclusion or inclusion of boundary lines or curves. Preservice teachers seemed to apply trimming in a manner that did not leave “intact the content to be learned” (Ferrini-Mundy et al., 2005, p. 40 – 41). Additionally, preservice teachers applied bridging while teaching both systems of linear inequalities and quadratic inequalities. The bridging applied by preservice teachers seemed to reinforce the notion that inequalities are equations.

Limitations

Despite careful consideration in the design and implementation of this study, there were limitations. First and foremost of these limitations was the fact that this study was conducted by a novice researcher. This limitation was minimized through the use of reviews conducted by committee members.

This case study design utilized a limited number of preservice teachers, in a particular part of the country. Therefore, it should be noted that the data collected in this case study was not representative of some larger population. The data corresponded only to teachers in this study and the researcher made no claims of generalization beyond the teachers in this study at the time of this study (2011).
The time constraints were a limitation of this study. The local education agencies in which the preservice teachers were assigned for their student teaching field experience advocated pacing guides for the observed courses. As a result, no matter how much control of the overall development of a unit was afforded to a preservice teacher, the outside influence of the pacing guide potentially limited the time spent on a unit and individual lessons.

Another limitation of the study was the extensive amount of time that passed between the classroom observations and the post-student teaching interview for some of the preservice teachers. The dates of the post-student teaching interview were selected in an effort to minimize the burden placed on the preservice teachers. During some of the post-student teaching interviews, the preservice teachers had difficulty recalling specific information. For example, the preservice teachers were asked about suggestions from their cooperating teachers regarding examples. The preservice teachers could not give remember the exact changes they made based on the suggestions. The use of video-stimulated recall helped to situate the preservice teachers back into the classroom moments. This reduced the issues encountered when discussing events that occurred during a classroom observation.

Data collection was a limitation. In an effort to minimize disruptions in classrooms, the researcher decided to collect audio and video data from a static position. As a result, one-on-one or small group interactions between preservice teachers and students were not recorded. The researcher made comments about some interactions in field notes.

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27 During the last weeks of the student teaching field experience, the student teachers are required to complete a comprehensive project. All of the preservice teachers indicated that they did not want to participate in the post-student teaching interview until their project was submitted.
Another limitation related to data collection involved cooperating teachers. The majority of the information about cooperating teachers came from preservice teachers. Formal interviews with cooperating teachers were not conducted. Cooperating teachers were asked impromptu questions before and after classroom observations. However, those questions were superficial in nature. This study may have benefited from cooperating teachers responses to a post-unit survey. This survey could have addressed feedback provided on lesson plans. In addition, cooperating teachers could have been asked to evaluate implementation of preservice teachers’ lessons, from a content perspective.

The questions that were asked in the pre-student teaching interview were a limitation of this study. In particular, changes needed to be made to the quadratic inequality questions. The preservice teachers should have been asked to implement an algebraic manipulations strategy and/or a graphing strategy based on their responses. Allowing the preservice teachers to select the initial solution strategy provided insight into their preferences. However, their ability to implement the other strategy was left unanswered for some of the preservice teachers.

Implications

This study has shed light on what preservice teachers understand about inequalities after completing methods courses in their mathematics education program as well as how preservice teachers apply their understanding while planning and implementing lessons during their student teaching field experience. Five undergraduate preservice teachers provided a wealth of information about the nature of their understanding with regard to inequalities. Examining these preservice teachers during their student teaching field
experience highlighted the role their understanding played while planning and implementing lessons regarding inequalities. Based on the findings of this study, there are several implications that need to be considered.

The CBMS report (2001) noted that in order for high school mathematics teachers to be “well prepared,” they need an “[u]nderstanding of the ways that basic ideas of number theory and algebraic structures underlie rules for operations on expressions, equations, and inequalities” (p. 40). Additionally, the CBMS report (2001) noted that preservice teachers enter undergraduate studies with “some technical skills” in solving inequalities (p. 124). Preservice teachers will be expected to teach inequalities to their students (Common Core State Standards Initiative, 2010). Mathematics education programs cannot assume that preservice teachers have developed a sufficient understanding of inequalities during their secondary education28 (Even & Tirosh, 1995). The findings of this study would seem to suggest that the emphasis of methods courses needs to be reconsidered.

Within methods courses, preservice teachers should examine pedagogical decisions that foster a relational view of inequalities (and equations). Incorporating cases (Markovits & Smith, 2008; Merseth, 1996) and/or [Japanese] lesson studies (Lewis, Perry, & Murata, 2006; Shimizu, 2002) focused on inequalities (i.e., linear, quadratic, absolute-value) into method courses could provide preservice teachers with valuable experience. Preservice teachers may become better able to recognize pedagogical decisions that limit student views of inequalities to an operational level. Additionally, discussions about alternative pedagogical decisions that foster a relational view of inequalities in students may also nurture preservice teachers’

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28 Inequalities taught in secondary mathematics classrooms are usually not encountered in undergraduate courses.
relational views of inequalities. For example, Heather’s attempt at connecting her solution (see Figure 138) to a quadratic inequality with one variable found by algebraic manipulations (see Figure 144) and by graphing (see Figure 137) would make a good case study for preservice teachers to examine.

Methods courses should help develop preservice teachers’ awareness of how technology can be integrated into lessons without reducing tasks to rote button pushing exercises. As seen in this study, technology served as a secondary component of lessons involving inequalities. Mathematics education programs should ensure that preservice teachers are aware of how to use technology. In addition, methods courses should encourage preservice teachers to consider “the manner in which the subject matter can be changed by the application of technology” (Mirshra & Koehler, 2006, p. 1028). Finally, preservice teachers need to be able to explore techniques that use technology beyond “checking” work done by hand. Preservice teachers could examine Vanessa’s use of the test feature on her graphing calculator while teaching her students how to solve quadratic inequalities with one variable using an algebraic manipulations strategy (see Figure 168) and discuss alternatives.

The manner in which textbooks present and treat inequalities needs to be examined. Bush (1986) noted that “mathematical textbooks played an important role in the planning and teaching of the preservice teachers' lessons” (p. 25). During post-student teaching interviews, preservice teachers made several comments attesting to the influence of their textbook as they planned and implemented their lessons on inequalities. Blurring of the nuances between equations and inequalities is not inhibited by the proximity of equations and inequalities within textbooks. Textbooks may need to make changes in the types of inequality examples
and problems being presented. It is unclear if the examples and problems found in textbooks instill a relational view of inequalities.

The framework of this study (see Table 2) was useful for examining understanding of inequalities. The aspects of knowledge associated with the mathematical concept of inequalities were directly related to findings from literature. These direct connections allowed for foreshadowing, refinement of questions, and comparisons against reported results.

The strength of the framework was its ability to make sense of classroom observations. Using the three teaching practices (explaining mathematical idea, solving mathematical problems, and using technology) from the framework, relevant episodes were easily identified. Additionally, the researcher was able to examine patterns related to when explanations were presented in relation to solving inequalities. Since explaining mathematical idea and solving mathematical problems were adapted from a single category in the framework presented by Ferrini-Mundy et al., (2005), consideration may need to be given to separating their single teaching practice category of explaining mathematical ideas and solving mathematical problems into two parts.

**Suggestions for Future Research**

This study was situated in the student teaching field experience semester. The research questions of this study focused on preservice teachers’ content knowledge and application of their knowledge in a classroom during their student-teaching field experience. Thus suggestions for future research are relevant to mathematics teacher education.
Even and Tirosh (1995) noted that “one cannot assume that teachers' subject-matter knowledge with respect to the two aspects ("knowing that" and "knowing why") are sufficiently comprehensive and articulated for teaching” (p. 18). This comment was affirmed at times in this study. Preservice teachers’ understanding of inequalities contained deficiencies identified within literature. Since subjects in a majority of previous studies were high school students, this seemed to suggest that some deficiencies persist through post-secondary education. A possible area for future research may lie in examining teachers’ understanding of inequalities over an extended period of time. This study was able to take a snapshot of preservice teachers’ understanding. A longitudinal study could chart changes in teachers’ understanding through preservice and inservice years. This information could be used to examine the impact of curricula, classroom field experiences, as well as professional development. Changes to curricula, classroom field experiences, and/or professional development could be informed by knowledge gained.

The second research question of this study focused on how content knowledge was used in planning and implementing lessons during the student teaching field experience. Following inservice teachers for their first five years and examining their lessons regarding inequalities may provide a glimpse into possible evolutions in their content knowledge or how they use that content knowledge to plan and implement lessons regarding inequalities.

The influence of textbooks in planning and implementing lessons was noted by Bush (1986) and by preservice teachers during post-student teaching interviews. A few preservice teachers claimed that they included aspects into lessons that were beyond what was presented in their textbooks. Future research could examine the manner in which preservice teachers
utilized their textbooks. The categories of textbook usage outline by Nicol and Crespo (2006) could be a useful starting point for analyzing preservice teachers’ lesson plans.

Another possible area of future research involves examining preservice (or inservice) teachers’ lessons pertaining to inequalities, particularly quadratic inequalities, with a different framework. Since preservice teachers’ understanding of inequalities is not isolated from their understanding of other mathematical concepts (e.g., equations, functions, graphing, etc.), it may be interesting to determine how the preservice teachers apply the overarching teaching practices from the Ferrini-Mundy et al. (2004) framework: bridging, trimming and decompressing. Scaffolding could be included in the overarching teaching practices as a way to examine how teachers intertwine explanations and examples.

Ball and Forzani (2010) outlined the need to identify and research high-leverage practices; “those practices at the heart of the work of teaching that are most likely to affect student learning” (p. 43). One such practice is being able to provide instructional explanations. Charalambous, Hill, and Ball (2011) noted that providing “instructional explanations lies at the heart of teaching, for it requires transforming the content in mathematically legitimate and pedagogically appropriate ways” (emphasis in the original, p. 443). As seen in this study, preservice teachers encountered difficulties while explaining aspects of inequalities to their students (e.g., providing disjoint solutions strategies, identifying the boundaries of a solution). Thus, while these preservice may have been able to provide rich and meaningful explanations about other topics in their courses, their work with inequalities did not indicate a strong mastery of this instructional practice. Future research
could examine how interventions in methods courses foster preservice teachers’ ability to provide strong explanations about aspects of inequalities to their students.

The inclusion or exclusion of technology seemed to be a recurring theme throughout this study. The blatant exclusion of technology, beyond graphing calculators, with it being so readily available is an area that warrants further study. This could include examining preservice (or inservice) teachers’ technological content knowledge, technological pedagogical knowledge, and technological pedagogical content knowledge (Lee & Hollebrands, 2008; Mishra & Koehler, 2006) as they relate to teaching inequalities and equations.

**Final Thoughts**

The field of mathematics education has been trying to define teacher knowledge ever since Shulman’s (1985) address. The question that constantly arises is: What do secondary mathematics teachers need to understand about mathematics? This question may never be satisfactorily answered for all parties. However, strides have been made and continued to be made towards answering that question. I believe that this study has contributed to answering the question by unraveling the nature of some preservice teachers’ understanding of inequalities.

The findings of this study were mixed. In most areas, preservice teachers displayed an operational view of inequalities that produced mathematically correct solutions (Knuth et al., 2006; Rittle-Johnson & Alibali, 1999). At times, their displayed view of inequalities could be classified as relational; indicative of a deeper or more sophisticated level of understanding. During their lessons, there were instances where actions by preservice teachers promoted an
operational view of inequalities (e.g., disjoint nature of solution strategies). For example, a relational view of inequalities was displayed by preservice teachers while discussing the solution of an inequality while solving on their own. However, this was not true in their classrooms.

Following the lead of Ball, Lubienski, & Mewborn (2001), this study examined “the mathematical understanding that is needed” in the classroom (p. 449). Additionally, this study unveiled the role(s) preservice teachers’ understanding of inequalities played in their lessons. There appeared to be numerous instances where aspects of preservice teachers’ understanding influenced their planning and/or their implementation of lessons regarding inequalities. Preservice teachers seemed to view and treat inequalities as something to do rather than something to be interpreted.
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APPENDICES
Appendix A. Informed Consent Form.

North Carolina State University
INFORMED CONSENT FORM for RESEARCH
Title: Understanding Preservice Teachers’ Knowledge of School Algebra Topics
Faculty Sponsor: Dr Hollylynne Lee

What are some general things you should know about research studies?
You are being asked to take part in a research study. Your participation in this study is voluntary. You have the right to be a part of this study, to choose not to participate or to stop participating at any time without penalty. The purpose of research studies is to gain a better understanding of a certain topic or issue. You are not guaranteed any personal benefits from being in a study. Research studies also may pose risks to those that participate. In this consent form you will find specific details about the research in which you are being asked to participate. If you do not understand something in this form it is your right to ask the researcher for clarification or more information. A copy of this consent form will be provided to you. If at any time you have questions about your participation, do not hesitate to contact the researcher named above.

What is the purpose of this study?
The purpose of this research is to better understand the nature of preservice teachers’ knowledge about several topics in school algebra (e.g., equations, inequalities, rate of change, slope). This study is important because it may help teacher educators design experiences that can foster a deeper level of understanding of topics in school algebra for both the preservice teachers and their future students.

What will happen if you take part in the study?
If you agree to participate in this study, you will be asked to participate in an individual semi-structured interview which may be videotaped. This interview will be conducted prior to your preparation of the lessons being observed. It should take 60 to 75 minutes to complete this interview and will be conducted at NCSU or your assigned high school. You will be asked to submit a series of lessons covering 12 to 20 days of material. You will be asked to allow the researcher to conduct field observations of you implementing the submitted lessons. These field observations may be videotaped, pending approval from the Local Education Agency (LEA). The researcher may ask you to participate in pre- and/or post-observation interviews. These interviews will primarily be used to focus or clarify points of interest in the observation. Finally, you will be asked to participate in an individual semi-structured interview which may be videotaped. This interview will be conducted after you have implemented the lessons. This interview should take 90 to 120 minutes to complete and will be conducted at NCSU or your assigned high school.

Risks
There are no physical or emotional risks associated with participation in this study.
Benefits

The preservice teachers that participate in this study will potentially gain a better understanding about school algebra topics and some of the difficulties students have with these topics. In addition, this study is important because it may help teacher educators design experiences that can foster understanding of algebraic ideas for both the preservice teachers and their future students.

Confidentiality

The information in the study records will be kept confidential to the full extent allowed by law. Data will be stored securely by the principle investigator. Pseudonyms will be used in oral or written reports to avoid linking you to the study.

Compensation

Upon completion of the study, the researcher will provide the participants with a Student E-Membership to the National Council of Teachers of Mathematics (NCTM; $39). If you withdraw from the study prior to its completion, your data will be destroyed and you will not receive any compensation.

What if you are a NCSU student?

Participation in this study is not a course requirement and your participation or lack thereof, will not affect your class standing or grades at NC State.

What if you are a Noyce METS Scholar?

Participation in this study is not a requirement of the Noyce METS program. Your participation or lack thereof, will not affect your status in the program.

What if you have questions about this study?

If you have questions at any time about the study or the procedures, you may contact the researcher, Tyrone Washington, at Poe 502N Campus Box 7801 NCSU Raleigh, NC 27695, or (XXX) XXX-XXXX.

What if you have questions about your rights as a research participant?

If you feel you have not been treated according to the descriptions in this form, or your rights as a participant in research have been violated during the course of this project, you may contact Deb Paxton, Regulatory Compliance Administrator, Box 7514, NCSU Campus (919/515-4514).

Consent to participate

“I have read and understand the above information. I have received a copy of this form. I agree to participate in this study with the understanding that I may choose not to participate or to stop participating at any time without penalty or loss of benefits to which I am otherwise entitled.”
Please check the appropriate box

☐ I agree to participate in all aspects of this study including audio and video recording of interviews and field observations.

☐ I agree to participate in this study including the audio and video recording of interviews, but no video recording of field observations.

☐ I agree to participate in this study including the audio recording of interviews and field observations. However, the interviews and field observations may not be video recorded.

Subject's signature______________________________ Date __________________

Investigator's signature______________________________ Date __________________
Appendix B. Permission to participate in video recording.

FEBRUARY 18, 2011

Dear Parents/Guardians of ________ High School Students:

Your student is currently in a mathematics course that is being taught by a preservice teacher from North Carolina State University. As a part of the requirements for my dissertation, I will be conducting a study that examines preservice teachers’ understanding of algebraic topics. I will use classroom observations to gather information about the preservice teachers’ understanding. Your student is in a class taught by a preservice teacher in this study. These classroom observations will be digitally video-recorded. The videos will focus on the preservice teacher and the board or screen. These videos will be uploaded to a private, password protected server. The videos will be used as part of my analysis. Once the study is completed, the videos will be destroyed.

Please sign and return this form if you do not want your student in the video.

If you decide that you do not want your student in classroom video, we can make arrangements for your student to be outside of the camera’s view while we record the class.

If you have questions or concerns, please email me at htwashin@ncsu.edu.

____________________________________
Tyrone Washington

____________________________________
Hollylynne Stohl Lee, Ph.D.

______________________________

_____ No, I do not give my permission for my student to be in recorded videos.

____________________________________
Parent/Guardian
Student

______/______/2011
Date
Appendix C. Pre-student Teaching Interview Protocol.

I’m Mr. Washington. I really appreciate your taking the time to meet with me today. As you may know, I am interested in learning about how you approach topics in school algebra. I’m going to ask you to work on a few math problems with me. The problems that I will show you today involve topics that are typically covered in an algebra class and so you should have some tools from your past to help you work on them. They are not being graded, and it is okay if you don’t get them right – I just want to see what you do to try to work through them. What you say will remain anonymous.

Because I am interested in how you think about the mathematics, it would really help me if you would talk as much as you can about what you are doing in your head. While you are working and talking, I will probably ask questions to be sure I understand what you are saying. I am asking because I am really interested. I am not asking a question because what you are doing is correct or incorrect, but rather I just want to make sure I understand what you are saying. Chances are I probably just missed something you said or did. Likewise, if I ask a question that does not make sense to you, please tell me. I might take a few notes to help me remember. But, because I don’t want to take too many notes, I would like to tape our conversation. The video camera will be focused on what you are writing or doing.

I have a calculator, a thick marker, some graph paper and some paper available for you to use. I would like for you to write using this marker because it is darker than pencil and will be easier for me to see what you are writing on the video later.

1. How would a student define ‘equation’ in a manner that indicated that they have a conceptual understanding of the word?

2. With the previous definition for equation in mind, which of the following are equations:
   a. \( x = x \)
   b. \( x = 2 \)
   c. \( 0 = 0 \)
   d. \( 0 = 1 \)
   e. \( V = \frac{4}{3} \pi r^3 \)
   f. \( y > 5x – 9 \)
3. Consider the set \( S = \{ x \in \mathbb{R} \mid x = 3 \} \) and check the validity of the following statement:

\( S \) can be the solution of both an equation and an inequality. Explain your answer.

4. Are the following inequalities equivalent:

\[
7 > ax \quad \text{and} \quad x < \frac{7}{a} \quad \forall a \in \mathbb{R}
\]

5. Are the following inequalities equivalent statements?

\[
ax < 5 \quad \text{and} \quad x < \frac{5}{a} \quad \forall a \in \mathbb{R}, a \neq 0
\]

6. Solve the inequality \((a - 5)x > 2a - 1\), \(x\) being the variable and \(a\) the parameter.

7. Indicate the truth set (the solution) of \(5x^4 \leq 0\).

8. Solve the following inequalities
   
   a. \((x - 1)(3x + 7) \geq 0\)

   b. \(2x^2 + 6x - 17 < x - 13\)

   c. \(3 \leq \left| \frac{1}{4}x + 1 \right|\)

9. Find the solution of the given system:

\[
\begin{align*}
    x + 4y & > 2 \\
    5x - 2y & \leq -1
\end{align*}
\]

10. Solve the following inequality using the graphing technique

\[
x^2 - x + 4 \geq 4x^2
\]

11. Why does the inequality sign switch when you multiply or divide both sides of an inequality by a negative number?
Appendix D. Post-student Teaching Interview Protocol.

I’m Mr. Washington. I really appreciate your taking the time to meet with me today. As you may know, I am interested in learning about how you approach topics in school algebra. I’m going to ask you to work on a few math problems with me. I’m going to ask you to look at a video recording from the field observations in your class. After watching the video, I would like to ask you some questions about certain episodes that occurred. You can re-examine the video if you like. As we go through each episode, I want you to try to remember what you were thinking then, not what you think about it now.

Before starting the interview, ask her some biography questions.
   Is she a traditional student?
   Is she from NC?
   Is she a teaching fellow?
   What are her majors?
   How would you categorize your mathematical background?

1. How would a student define ‘equation’ in a manner that indicated that they have a conceptual understanding of the word?

2. With the previous definition for equation in mind, which of the following are equations?
   a. \( x = x \)
   b. \( x = 2 \)
   c. \( 0 = 0 \)
   d. \( 0 = 1 \)
   e. \( V = \frac{4}{3} \pi r^3 \)
   f. \( y > 5x - 9 \)
   g. \( 3x + 7y \leq 21 \)
   h. \( f(x) = ax^2 + bx + c \) \( \text{where } a, b, \text{ and } c \text{ are real numbers} \)

3. With regard to the lessons on systems of equations and systems of inequalities, did you collaborate with anyone in the development of those lessons? If so, describe the collaboration to me? How much control do you feel that you had over the development of those lessons? What resources did you use in the development of those lessons (e.g., websites, textbook(s), books)?

4. Considering your lessons on systems of equations and systems of inequalities, what were some of the big ideas that you wanted your students to take from those lessons?

5. What challenges were inherent in teaching this mathematical idea? What specific challenges were presented by this class as you taught this mathematical idea? How was your instruction designed to meet these challenges?

6. When grading summative assessments, was there anything that the students did regarding inequalities that surprised you?
7. Play Episode 1 (Process)
   Why did you feel it was necessary to lead your lesson with this piece?
   Ask follow-up questions if necessary.

8. Play Episode 2 (First example)
   Ask her what she was thinking as she taught this problem?
   Ask follow-up questions if necessary.

   When showing students how to shade a linear inequality with two-variables you told
   the students to shade above if there is a greater than (>) or greater than or equal to (≥)
   symbol and to shade below if there is a less than (<) or less than or equal to (≤)
   symbol. Do you think that your students fully understood why your statement was
   ‘correct’?

   Why did you want your students to know the ‘above’ and ‘below’ approach to
   shading?

   In the moment, you are often required to make quick decisions as to what to say and
   what to do. You presented the information in a viable manner. I would like to give
   you the luxury of time and ask if there is another way that you would present this idea
   to your students. Can you think of another way to explain this notion? If so, tell me
   about it?

   What about using test points as a method to determine shading?

9. With regard to the lessons on quadratic expressions, equations, and inequalities, did you
   collaborate with anyone in the development of those lessons? If so, describe the
   collaboration to me? How much control do you feel that you had over the
   development of those lessons? What resources did you use in the development of
   those lessons (e.g., websites, textbook(s), books)?

10. Considering your lessons on quadratic equations and inequalities, what were some of the
    big ideas that you wanted your students to take from those lessons?

11. What challenges were inherent in teaching this mathematical idea? What specific
    challenges were presented by this class as you taught this mathematical idea? How
    was your instruction designed to meet these challenges?

12. When grading summative assessments, was there anything that the students did regarding
    inequalities that surprised you?

Transition: After a discussion about a quiz taken the previous day, you start your lesson on
quadratic inequalities.
13. Play Episode 3 (Process: Quadratic inequalities)
   Why did you feel it was necessary to lead your lesson with this piece?
   Ask follow-up questions if necessary.

Transition: This is the third example of the lesson.
14. Play Episode 4 (Quadratic inequalities: \( y < x^2 + 6x + 7 \))
   Talk to me about this problem.
   Ask follow-up questions

Transition: Later in the lesson, you recall how to solve a quadratic equation. Then you give the following example.
15. Play Episode 5 (Quadratic inequality with issue: \( x^2 - 6x - 7 < 0 \))
   Talk to me about this problem.
   Ask follow-up questions

16. How well do you think your students understand what the solution to a system of inequalities is? What about a quadratic inequality?

17. I noticed that you used calculators in the lessons. Could you tell me about how you used the calculators? What purpose did they serve? Would the lessons have been as effective without the calculators? Did you consider using any other technology in these lessons, especially the systems of linear inequalities lesson? Why or why not? Did you consider using any manipulatives (virtual or physical) in these lessons? Why or why not?

18. If you were in charge and were not constrained by standard courses of study, how much time would you allocate to each main idea in these two units?

19. Is there anything else that you would like to add?

Thank you for participating in my study.
Appendix E. Keyword Maps of Classroom Observations.

Systems of Linear Inequalities

Series: Systems of Linear Inequalities
Episode: Angela_Day3_20110302
File: 20110302.wav

Specific Content as it relates to understanding: Shading
Specific Content as it relates to understanding: Solutions
Specific Content as it relates to understanding: Strategies
Specific Content as it relates to understanding: Treating inequalities as equations
Teaching Practices Keywords: Explaining an idea
Teaching Practices Keywords: Solving a problem

Series: Systems of Linear Inequalities
Episode: Christina_Day1_20110304
File: 20110304.wav

Specific Content as it relates to understanding: Shading
Specific Content as it relates to understanding: Solutions
Specific Content as it relates to understanding: Strategies
Teaching Practices Keywords: Explaining an idea
Teaching Practices Keywords: Solving a problem

Series: Systems of Linear Inequalities
Episode: Crystal_Day2_20110311
File: 20110311.wav

Embedded Teaching Practices: Use of other teaching resources
Specific Content as it relates to understanding: Shading
Specific Content as it relates to understanding: Solutions
Specific Content as it relates to understanding: Treating inequalities as equations
Teaching Practices Keywords: Explaining an idea
Teaching Practices Keywords: Solving a problem

Series: Systems of Linear Inequalities
Episode: Vanessa_Day1_20110328
File: 20110328.wav

Teaching Practices Keywords: Solving a problem
Teaching Practices Keywords: Explaining an idea
Specific Content as it relates to understanding: Strategies
Specific Content as it relates to understanding: Treating inequalities as equations
Specific Content as it relates to understanding: Shading
Specific Content as it relates to understanding: Solutions
Embedded Teaching Practices: Use of graphing calculator
Quadratic Inequalities