ABSTRACT

AHMED, NEVEEN. Portfolio Choice: An Empirical Investigation. (Under the direction of Denis Pelletier.)

In this dissertation we study the optimal portfolio selection problem. In this respect we develop an estimation technique to compute single and multi-period portfolio weights of an infinitely-lived investor who invests in \( N \) risky assets and one risk-free asset using the first-order condition “Euler equation” from the investor utility maximization problem.

The dissertation is composed of three chapters. The first chapter analyses and computes the single-period optimal portfolio choice of an infinitely lived investor. In the second chapter we extend our analysis for the multi-period optimal portfolio choice. Finally, the third chapter we empirically introduce consumption growth as a source of long-term risk and hence a source of influence on the optimal portfolio choice.

The investor is assumed to have one of two sets of preference representations: Epstein-Zin (EZ) recursive utility function or habit formation (HF) utility function. We investigate the portfolio weights generated from these utility functions for different sets of preferences parameters including the risk-aversion parameter and the intertemporal elasticity of substitution parameter. We find that the optimal portfolio weights differ greatly across time and across utility functions. Our results show that more risk averse investors tend to hold fewer stocks than less risk-averse ones. Moreover, we found that the introduction of consumption growth in our GARCH-in-Mean specification has an impact on the composition of the investor’s optimal portfolio choice.

**Keywords:** Portfolio Choice, Epstein-Zin, Stock markets.

**JEL Classification:** G0.G11.G17
Portfolio Choice: An Empirical Investigation

by

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DEDICATION

To my late Dad, Mum, Farah and Prof Karlyn.
BIOGRAPHY

Neveen was born in Cairo, Egypt. She had her undergraduate study at the Economics department, at the Faculty of Economics and Political Science, Cairo University. Neveen was awarded Ford scholarship to pursue her masters degree, meanwhile she was awarded a governmental scholarship to pursue her PhD degree. Neveen decided to accept the governmental scholarship and joined North Carolina State University (NCSU) in 2006. She had her masters degree from North Carolina State University on May 2010, and her Ph.D. degree from NCSU on May, 2012 under the supervision of Prof. Denis Pelletier. Neveen’s research interests concentrate on portfolio analysis.
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Introduction

Individuals differ in their investment motivations: some invest in order to finance higher consumption in the short run, some make longer term investments to secure higher income at retirement. Choosing the optimal combination of stocks given the enormous number of stocks available is a critical decision. The investor’s choice of optimal portfolio weights has been studied in the financial economics literature since the seminal paper of Markowitz (1952). Recently there has been growing attention directed to this fundamental financial economic decision. These recent developments have put optimal portfolio choice on the forefront of financial research. This chapter is concerned with finding an estimation technique to compute the single period optimal portfolio weights for an infinitely lived investor.

Markowitz (1952) introduced the theory of optimal portfolio selection by introducing mean-variance (M-V) analysis to compute the optimal portfolio weights. One interesting feature of Markowitz’s formulation is that it accounts for the tradeoff between risk (measured by the variance) and expected return. However, the Markowitz formulation can tackle the investor optimization problem only under very restrictive assumptions about the investor’s preferences or the distribution of returns (see Campbell and Viceira (2002)). Several critiques were directed at this method as being a myopic optimization that only involves single period portfolio optimization. Critiques mainly focused on the inability of M-V analysis to account for changes in the investment opportunity set (changes in expected returns or covariances, see Brennan et al. (1997)) which is described by the state variables.

The optimal portfolio choice, the weight for each asset available to the investor that maximizes his expected utility, is a financial decision that entails the use of information
affecting future returns (state variables), the extent of the investor’s risk aversion, and how the asset’s volatility evolves over time. It also requires finding a technique that links all relevant information to find the optimal portfolio weights.

In order to model changes in investment opportunity researchers incorporate state variables that could determine the evolution of asset returns involved in the portfolio optimization. They start from a value function for an investor who maximize his utility $U(C_t)$ subject to a budget constraint,

$$W_{t+1} = (W_t - C_t)(w_t'r_{t+1} + (1 - i'w_t)r_f),$$

where $W_t$ is the wealth of the investor at time $t$, $C_t$ is consumption, $r_{t+1}$ is the $N \times 1$ vector of returns for the risky assets, $r_f$ is the risk-free rate of return, $i$ is an $N \times 1$ vector of ones and $w_t$ is the $N \times 1$ vector of portfolio weights.

Thus the investor’s problem is characterized by a sequence of optimization problems, the investor’s wealth $W_t$ and a set of state variables $Z_t$ that could help predict future returns (e.g., dividend price ratios, default risks, ...). For an investor wanting to choose his optimal portfolio for the next $\tau$ periods,

$$V_t(W_t, Z_t) = \max_{\{w_t, C_t\}_{t=0}^{t+\tau}} E_t[U(W_{t+\tau})],$$

which gives the following Bellman equation,

$$V_t(\tau, W_t, Z_t) = \max_{w_t, C_t} E_t[V_{t+1}(\tau - 1, W_{t+1}, Z_{t+1})].$$

Bellman’s principle of optimality states that “An optimal policy has the property that, whatever the initial state and decision are, the remaining decisions must constitute an
optimal policy with regard to the state resulting from the first decision, see Miranda and Fackler (2002).

A vast literature covers techniques for finding the solution to the previous Bellman equation. Mostly numerical techniques have been used to solve the value function using discretization of the state space or projection methods. The latter approach requires the use of extensive sophisticated numerical analysis with various assumptions about the elasticity of substitution in some cases or on the value at which linearization is done in other cases. Our proposed methodology takes into consideration the change in investment opportunity set, and uses the Generalized Method of Moment (GMM) to estimate the optimal portfolio weights.

The thesis focuses on finding the optimal portfolio weights both for the single-period and the multi-period portfolio problem of an infinitely-lived investor. We study this problem for two preferences representations: Epstein-Zin (EZ) and habit formation (HF) utility functions. Our model uses Euler equations as moment conditions in the GMM estimation as illustrated below in the model’s section.

We adopt the EZ utility because unlike other utility functions such as logarithmic or power utility, it can disentangle the investor’s risk aversion and his elasticity of intertemporal substitution (EIS). With additive utility, the EIS is just the reciprocal of the coefficient of risk aversion which lead to several empirical inconsistencies. Moreover, other preference forms lead to intertemporal inconsistencies and the first order conditions of optimization (the Euler equations) are applicable only for consumers that myopically ignore the fact that plans constructed at any given time will not be carried out in the future, while EZ results in recursive structures that are intertemporally consistent (see Epstein and Zin (1991)).

Habit formation is the second form of utility that we are going to utilize in computing
optimal portfolio weights. Preferences exhibiting habit formation stems from a class of non-separable time preferences. Revived interest in habit formation utility function is an attempt to solve an empirical anomaly in which there is a strong dependence of observed and recommended asset allocations on the investment horizon. The horizon puzzle is tackled by introducing preferences in which utility is defined with respect to a non-zero lower-bound on wealth or consumption [see Ferson and Constantinides (1991)]. Moreover, habit formation takes into consideration the effect of past consumption on the instantaneous utility level; an increase in past consumption decreases the utility and lead to temporal non-separable utility.

The thesis is organized as follows. The first chapter presents the single-period optimal portfolio choice. The second chapter presents the Multi-period optimal portfolio choice. The third chapter analyses the multi-period portfolio choice with consumption growth and returns volatility, finally, we conclude.
Chapter 1

Portfolio Choice: Single-Period

1.1 Summary

This chapter analyzes the single-period optimal portfolio choice. Our main objective is to find the optimal portfolio weights for an infinitely lived investor who maximizes his utility subject to a budget constraint. In this chapter we assume that the investor make his portfolio choice period after period. We perform an empirical analysis assuming that the investor enjoys either Epstein-Zin utility function or habit-formation utility function. The rest of the paper is organized as follows. Section 2 critically analyzes the literature on portfolio selection and asset pricing, Section 3 introduces the model. Section 4 presents the methodology used to solve the model. Section 5 analyzes the data statistics and in Section 6 we estimate the model’s parameters. Finally, Section 7 discusses the empirical results, and Section 8 concludes.
1.2 Literature Review

The literature on portfolio choice dates back to Markowitz (1952) employing Mean-Variance analysis which was the start of the theoretical background in the portfolio selection literature. Markowitz (1952) demonstrated that mean-variance preferences could be reconciled with the Von Neumann and Morgenstern (1944) theory of choice by assuming quadratic utility or the multivariate normal distribution of returns. However, Markowitz’s model was subject to several criticisms; as an example it is only applicable for quadratic utility which restricts the relation between the elasticity of substitution and risk aversion, and is not monotonically increasing in wealth. Moreover, the Markowitz formulation does not account for time varying return distributions and the optimal portfolio choice represents a single myopic optimization. A myopic portfolio is one that depends only upon current wealth and the distribution of returns currently available, as defined in Ingersoll (1987).

Sharpe (1964) and Lintner (1965) built on Markowitz’s work and formulated a model of portfolio choice, the Capital Asset Pricing Model (CAPM). The CAPM predicted a positive relationship between average returns and covariation with market returns. This result was challenged by Fama and French (1992) who discovered a significant, negative relationship. Moreover, the original derivation of the CAPM was based on the assumption that investors’ preferences can be defined over the mean and variance of a portfolio’s distribution of returns. Due to the restrictive nature of the assumptions underlying the CAPM, several attempts have been made to apply it to a more general context.

Academic finance has been drifting towards a purely empirical (statistical) approach to modeling asset prices, e.g. Fama and French (1996), Brennan et al. (1997), Campbell and Viceria (1996), Brandt (1999). Recently, several developments in the literature
comprises estimation techniques that try to account for the variation in the returns distribution across time and its influence on portfolio choice for long-horizon investors. Moreover, recent literature on portfolio selection moved away from using quadratic and log utility functions into other forms of preferences, as for example Epstein-Zin (EZ) utility and habit formation. The failure of time separable models to explain the equity premium puzzle led to the evolution of habit formation utility as an attempt to solve this puzzle, see Sundaresan (1989). For example the inconsistency associated with log utility from the restrictive relation between risk aversion and elasticity of intertemporal substitution (EIS) led to the use of EZ utility which unlike other utility functions allows for the separation between investor’s risk aversion and elasticity of intertemporal substitution.

The literature review is organized as follows. First, I present Markowitz mean-variance formulation. Then I analyze the CAPM of Sharpe (1964) and Lintner (1965). Finally, I critically analyze some of the "post CAPM" literature which incorporates recent developments in the portfolio selection literature.

1.2.1 Mean Variance analysis

Markowitz (1952) formulated the mean-variance analysis for the portfolio selection problem. The Markowitz model assumes that investors are risk averse and when they choose their portfolio they care only about the mean and the variance of a single period portfolio return. An investor will minimize the variance of the portfolio by choosing optimal portfolio weights for a given expected return. Thus Markowitz illustrated the idea that higher expected return can only be obtained by bearing more risk. The mean variance problem can also be formulated where the investor is maximizing expected return for a given amount of risk that she can tolerate, or as a minimization problem of portfolio
return subject to predetermined target of expected return as follows:

$$\min_{w} Var[r_p] = w' \Sigma w \quad \text{subject to} \quad w' \mu + (1 - w'i)r_f = \bar{\mu}$$  \hspace{1cm} (1.1)$$

where $w$ are portfolio weights, $\mu$ is the vector of expected returns for the risky assets, $r_f$ is the risk-free return, $r_p$ is the portfolio’s return, $\Sigma$ is the variance-covariance matrix of the returns on the risky assets, and the predetermined target expected return $\bar{\mu}$. The minimization problem gives rise to a solution of the portfolio weights ($w^*$) that is function of $\Sigma$, $\mu$ and $\bar{\mu}$.

Markowitz work led to a flow of research in the area of portfolio choice, yet his work reflects a myopic investment strategy where the investor cares only about the distribution of his wealth next period. Recently with growing evidence on predictability of the distribution of stock returns, researchers started to account for the time varying investment opportunity set by examining the dynamic portfolio choice.

### 1.2.2 Capital Asset Pricing Model

Sharpe (1964) builds on the Markowitz model of portfolio selection by casting Markowitz’s micro-model of choice into an equilibrium framework. Sharpe tried to identify an efficient portfolio and added two assumptions to the Markowitz model to identify the optimal portfolio. The first assumption is homogeneity of investor expectations; investors agree on a distribution for returns which is assumed to be the true one from period $t$ to $t+1$. The second assumption is a common rate of interest; the borrowing and lending occurs at the risk-free rate regardless of how much is borrowed or lent.

Sharpe showed that the portfolio of risky assets held by investors in equilibrium must coincide with the market portfolio. Thus the risk premium on any asset is a linear function
of the asset’s contribution to the risk of the market portfolio: the asset’s β:

\[ E(R_i) = r_f + \beta_i [E(R_m) - r_f] \]  \hspace{1cm} (1.2)

where \( R_i \) is the return on asset \( i \), \( r_f \) is the risk free rate of interest, \( R_m \) is the return on the market portfolio and \( \beta_i \) is asset \( i \)’s beta, which reflects the sensitivity of the expected excess asset returns to the expected excess market returns. Betas exceeding one indicate more than average riskiness, i.e. these assets contribution to overall risk is above average and hence a higher expected return is required to compensate for higher risk. Betas below one indicate a lower than average risk contribution.

Lintner (1965) examined three scenarios for portfolio choice. Starting with a choice of holding cash and a single common stock, to borrow and invest in a single common stock or to hold saving deposits, and finally the choice between several stocks along with saving deposits. Lintner found that stocks’ values vary directly with the intercept and the correlation with the market and inversely with residual variance of the regression on market index. Moreover, he concluded that the best portfolio is the one with the best combination of risk and return, which is not the Markowitz efficient portfolio with lowest risk even for ”a risk averse” investor.

The author found that the gain from diversification happens when the stocks are negatively correlated with the market and when the residual variances of the assets are not zero. Moreover the gain of diversification when stocks are positively correlated with the market happens if the risk coming from assets returns is smaller than the ”index component” returns and the risk of the general index.

Finally Lintner claimed that the best portfolio is the one with the highest \( \theta \) where \( \theta \)
is defined by
\[ \theta = \frac{\bar{r} - r_f}{\sigma_r} \]  
(1.3)

where \( \bar{r} \) is the the rate of return expected on the portfolio, \( \sigma_r \) is the standard deviation of this return and \( r_f \) is the risk-free rate.

### 1.2.3 Post CAPM

Merton (1969) was one of the pioneer researchers who drew attention to the need to account for the long-term when choosing a portfolio. He examined the optimal portfolio problem in a continuous time framework. Merton examined the case of an investor with constant relative risk aversion who chooses between two assets and found an explicit solution for the optimal portfolio weights as a function of expected returns and variances. Merton confirmed Samuelson (1969)'s results that for constant relative risk aversion utility the portfolio decision is independent from the consumption one. He found that for an investor with small relative risk aversion the substitution effect dominates the wealth effect, i.e for an increase in the expected mean returns, an investor tends to save more so as to invest in the risky asset and hence consume less.

Campbell and Viceria (1996) solve for the optimal portfolio and consumption choice of an infinitely lived investor. The investor chooses between a risk-less asset with a constant return and one risky asset with a mean reverting expected return. They solved for the optimal portfolio choice using the method of undetermined coefficients. They start by approximating the Euler equation using a second order Taylor approximation and approximating the budget constraint which becomes linear in log consumption and quadratic in the portfolio weight on the risky asset. The portfolio weight solution is linear in the state variable. The authors guess a form for the optimal portfolio and consumption
policies; they show that these policies satisfy the approximate Euler equation and budget constraint, and they show that the parameters of the policies can be identified from the parameters of the model.

The optimal portfolio weights are decomposed into two components: a myopic one, which is positively correlated with expected excess return and negatively correlated with investors risk aversion, and a hedging component. The hedging component in turn consists of two parts; one that relates unexpected asset returns with state variables (the state variable used in the empirical analysis is the log of the dividend-price ratio) and is influenced by the risk aversion parameter. The other part consists of the covariance between unexpected asset returns and the conditional variance of log consumption that is influenced by both the risk aversion and elasticity of intertemporal substitution.

In their empirical analysis they find that variation in the coefficient of relative risk aversion has more influence on portfolio choice than the coefficient of intertemporal elasticity of substitution because the optimal portfolio weights depends on intertemporal elasticity of substitution only indirectly through the log-linearization parameter of the budget constraint. They examined the portfolio and consumption choice for different values of risk aversion and elasticity of intertemporal substitution parameters.

They find that for a relative risk aversion above one (logarithmic investor) the hedging demand is positive. On the other hand, when the investor has a relative risk aversion coefficient lower than one, the hedging demand is negative. This result comes from the idea that the covariance between the unexpected stock returns and revisions in expected future stock return is negative. This means that stocks tend to have higher returns when the expected future return falls (when the investment opportunity set worsens). An investor with low risk aversion likes to hold assets that deliver wealth in a good investment opportunity setting. While a high risk aversion investor likes to hold assets
that deliver wealth in the bad state, thus he will hold a positive hedging demand. However, they pointed to the fact that risk-aversion limits the investors exposure to the risky asset in all states of the world. Also their result showed that the optimal portfolio weights are very responsive to changes in expected excess returns \(x_t\). They find that the optimal portfolio rule for myopic investor (logarithmic investor, \(\gamma = 1\)) is

\[
\alpha_t = 1/2\gamma + 1/(\gamma \sigma^2)x_t
\]

while the mean hedging demand is given by the following equation:

\[
\alpha_{t, hedging} = \alpha_{t, total} - \alpha_{t, myopic} = \alpha_{t, total}(\mu; \gamma; \psi) - (1/\gamma)\alpha_{t, total}(\mu; 1; \psi).
\]

They find that mean hedging demand is positive for risk aversion above one and accounts for 20 – 50 percent of stock demand, suggesting that the intertemporal hedging demand motive is strong for risk-averse investors.

Brandt (1999) presents a nonparametric approach to solving the optimal portfolio problem. The author estimated single and multi-period portfolio and consumption rules for an investor with constant relative risk aversion and a one month to 20 years horizon. The investor chooses between two securities, the value-weighted NYSE index and a 30-day Treasury bill from January 1947 to December 1996 where nominal returns are deflated by the rate of change in the Consumer Price Index. Four variables describe the investment opportunity set that forecast time-varying risk premium and volatility. These variables are dividend yield, default premium, term premium and lagged excess return.

The main idea is that since the investment opportunity set may be time varying, \(i.e\) the distribution of returns may vary over time (what is called a stochastic invest-
ment opportunity set) returns become correlated with observed forecasting variables (for example dividend-price ratio). In order to take into consideration this time-varying investment opportunity set the author used conditional method of moments to estimate the optimal portfolio weights and consumption. This method accounts for the change in the investment opportunity set through a weighting function that includes state variables. The author contrasts the conditional method of moment’s result with one that uses a linear regression of a single period portfolio choice of an investor with CRRA utility with risk aversion parameter $\gamma$ equal to 5. The returns are generated using a regression of $\ln(1 + R_{t+1}^e)$ on $\ln(z_t)$ where $z$ are the forecasting variables, then solving for the optimal portfolio weights given the implied conditional distribution of returns.

The author estimated a single period and multi-period portfolio problem. The difference between the two portfolios is what is called the hedging demand, which arises from the investor’s attempt to hedge against predictable changes in the investment opportunity set thus trying to smooth the effects of predictable changes in the investment opportunity set. Brandt found that the portfolio choice depends on the forecasting variables, investor’s horizon and rebalancing period in the multi period case. He found that the portfolio choice varies significantly with the dividend yield, default premium, term premium and lagged excess return. Moreover, decisions are less nonlinear in dividend yield and excess returns than they are in default and term premiums. Also an investor in a multi-period problem tends to hold a greater fraction of savings in equity than does a single period investor. Moreover, the author finds that after an initial decline the holding of stock increases with the horizon.

Barberis (2000) investigated empirically the effect of investors horizon on portfolio allocation. He analyzes the case of an investor with power utility in a discrete time framework. The investor chooses his portfolio between two assets (treasury bill and stock
Barberis examined the effect of both horizon and parameter uncertainty on the investor's portfolio choice. Barberis argues that since returns are predictable, the investor's horizon plays an important role on portfolio weights, thus variation in expected returns over time can lead to horizon effects. Barberis explicitly accounted for parameter uncertainty, which he called estimation risk in the portfolio choice. Returns are shown to be predictable with a VAR model for state variables. Dividend yield is the state variable that governs expected returns. Parameter uncertainty is captured using a Bayesian approach by a posterior distribution of the parameter given the data and he compares these results with one that fixes the parameter.

The author analyzes two portfolio problems: a static buy-hold problem and a dynamic one with rebalancing. For the static case, he finds a strong horizon effect for long-term investors because of predictability in asset returns. When returns are predictable in such a way as to induce a mean reversion it reduces the variance of returns and make stocks appear less risky for long horizon investors.

In the dynamic rebalancing case he found a horizon effect too but in this case the horizon effect is due to differences in the relative risk aversion coefficient: a more risk averse investor will invest more in stocks than a shorter horizon investor, what they referred to as hedging demand. This is because when there is a negative correlation between shocks to realized and expected returns, the investor will hedge against such shocks by increasing the holding of stocks.

Moreover, the author finds that when he incorporates uncertainty in the parameters, the result is affected. In some cases the horizon effect still exists but becomes weaker and in other cases the result is completely reversed. According to the author, parameter uncertainty increases the variance of the distribution for cumulative returns especially at the long horizon and that is why stocks become less attractive. In the dynamic case,
he finds that investors wait to update their posterior distribution or learn about the parameters. This can lead to holding less stocks in comparison with a short horizon investor. Finally, he finds that parameter uncertainty lowers the sensitivity of portfolio choice to the state variable, which may lead to a gradual shift in portfolio composition over time.

Brennan et al. (1997) estimate portfolio choice for three types of investment strategies. The first is for a myopic investor who has one-month horizon, the second is a dynamic portfolio choice for a long term investor with a 20-year horizon. Finally, the authors analyze an investment strategy for what they called a 1992 strategy in which the horizon is calculated as the number of months remaining to January 1992. The dynamic problem reflects the changing investment opportunity set which is described by the following variables: short-term interest rate, dividend yield on common stock, yield on the long-term bond. They assumed that the investor can choose his portfolio between cash, stock, and a long-term bond. They used monthly data from January 1972 to December 1991. Stock returns are taken as the CRSP value weighted market index.

The authors used an optimal control method to solve for the optimal portfolio choice, and presented a numerical solution by discretizing the state space. They found that under the assumption of a constant 20-year horizon, cash has a higher proportion under the first strategy because it is considered a risk-less asset with a one month horizon but not for longer horizon, this is because of reinvestment rate uncertainty. The results for the last strategy are between the first and second strategy. The stocks holding is more for the 20-year horizon than the one month horizon. The authors explained that the greater investment in stock for the long horizon is because of the influence of mean reversion in stock prices on lowering asset volatility and hence it becomes less risky over the long horizon. Moreover they estimated the certainty equivalence under each investment
strategy and found that the one month myopic strategy has the most volatile pattern as the strategy fails to hedge against shifts in the investment opportunity set compared with the 1992 strategy.

Garlappi and Skoulakis (2008) developed a numerical method for the solution of a large class of discrete time dynamic portfolio choices which they name State Variable Decomposition (SVD). They worked with the certainty equivalence instead of working with the value function itself. They analyzed both a static and dynamic portfolio problem. First they start by solving the recursion characterizing the dynamic problem, working with the certainty equivalence which is a monotonic transformation of the value function. Then they decomposed each state variable into a sum of its conditional mean and the corresponding zero mean shock, and then separate choice variables from shocks in a multiplicative way and computed the conditional expectation.

They approximate the certainty equivalence by a Taylor series centered at the conditional mean of the state variable. In this way the state variable decomposition is able to reduce the original problem into an approximate one in which conditional expectations are functions only of shocks to the state variable and not the choice variable. They used backward recursion to solve for the value function, starting with solution to the problem in terminal period T and proceed backward to time zero. In the static portfolio choice problem they examined constant relative risk aversion (CRRA) utility function and normally distributed excess returns and solved numerically for optimal portfolio weights. In the dynamic portfolio choice problem they evaluated the SVD and then calibrated the model utilizing three assets, a nominal treasury bill, stocks and long-term treasury bond. They model the investment opportunity set by a VAR that includes return on three assets (short-term ex-post real interest rate, excess stock returns and excess bond returns). Also, they tested their model using EZ utility and found that the SVD method
is successful in producing asset weights that are 1% apart from the results obtained with a quadrature method.

Constantinides (1990) proposed the habit formation utility function as an attempt to solve the equity premium puzzle. He proposed a utility function where the utility level depends on current consumption and a weighted average of past consumption, what is called internal or "intrinsic habit formation" as opposed to external habit formation where the utility depends on aggregate consumption.

Ferson and Constantinides (1991) examined a habit formation utility function trying to find whether the durability or the persistence effect dominates and hence whether the parameter of habit formation in the utility function is positive or negative respectively. They utilized GMM to estimate the Euler equation that is derived from the utility maximization problem to predict future returns of common stocks and bond portfolios and future growth rates of consumption. Asset returns are measured in excess of the 3-month treasury bill returns. Several instruments are examined. They start by introducing instruments such as lagged values of consumption and asset returns but they found that financial ratios do a better job as instruments than the lagged consumption and returns because the monthly data are subject to measurement error and time aggregation in contrast with financial ratios. Finally, they found that using quarterly and yearly data, the time separable coefficient is negative which means that habit persistence dominates the effect of consumption durability.

Abel (1990) introduced a utility function that incorporates three classes of utility functions: a time-separable utility, a habit formation and "catching up with the Jones" utility function (the relative consumption model). Basically Abel attempted to introduce habit formation utility to try to explain the equity premium puzzle. He solved for an explicit pricing formula using an iid assumption on consumption growth. For the
time-separable preferences and relative consumption model he was able to derive closed-form solutions for the unconditional expected returns and he calculated numerically the unconditional expected returns in the habit formation model. Abel found that for the time-separable utility the equity premium does not converge to the historical average (600 basis point). For the relative consumption model, the result is better than other models, the unconditional rates of return on stocks and bonds in this model are closer to the historical averages: the equity premium is 463 basis points yet the conditional expected rates of return vary too much. Finally for the habit formation, Abel found that the result is sensitive to the choice of the risk aversion parameter.

1.3 Model

We present two discrete-time models that solve a portfolio choice problem for an infinitely lived investor who is maximizing his utility by choosing the level of consumption and choosing between investing in $N$ risky assets and a risk-free one. The investor is subject to a lifetime budget constraint. The two models correspond to two sets of utility functions. First, Epstein-Zin (EZ) utility function and then habit formation (HF) utility function. We assume no transaction costs and no labor income in our model.

1.3.1 Epstein-Zin utility

We start by assuming that a representative investor has an Epstein-Zin (EZ) utility representation. EZ utility solves the problem associated with power utility in which the elasticity of intertemporal substitution (EIS) is the inverse of relative risk aversion. Epstein and Zin (1991) introduced a generalization of power utility based on recursive utility that can disentangle EIS from risk aversion. The risk aversion coefficient describes the
consumer’s reluctance to substitute consumption across states of the world, while the EIS reflects consumer’s willingness to substitute consumption over time (see Campbell and Viceira (2002)). Thus, it is not necessary that an investor have a low EIS if his risk aversion is high.

The representative investor optimizes by choosing the level of consumption and the portfolio weights that maximize his lifetime utility subject to the intertemporal budget constraint where wealth next period equals the portfolio return times the reinvested wealth. This optimization results in Euler equations which are orthogonality conditions that must be satisfied. The Euler equations depend in a nonlinear way on current and future consumption, the assets’ return, and on parameters which characterize the preferences.

The representative investor maximizes the following recursive structure of the utility function:

\[
U_t = \left( C_t^\rho + \beta E_t[U_{t+1}^{1-\gamma}]^{\frac{\rho}{1-\gamma}} \right) \frac{1}{\beta} 
\]

subject to the following budget constraint

\[
a_{t+1} = (a_t - C_t)R_{m,t+1}. 
\]

The time discount factor is represented by \(\beta\). The risk aversion parameter is \(\gamma\), which measures the reluctance to trade consumption today for a fair gamble. The EIS is equal to \(\frac{1}{1-\rho}\). It reflects the investor’s willingness to transfer consumption over time in response to changes in interest rates. The wealth at period \(t + 1\) is \(a_{t+1}\) and \(C_t\) is the consumption at time \(t\). The investor can influence the future flow of consumption by trading in the
risky financial assets. The return $R_{m,t+1}$ on the investor’s portfolio is equal to

$$R_{m,t+1} = w_t'(i + r_{t+1}) + (1 - w_t'i)(1 + r_f).$$  \hspace{1cm} (1.6)

Where $i$ is vector of ones, $w_t$ are the weights for the risky assets, $r_f$ is the risk-free rate, and $r_{t+1}$ is the risky stock returns at $t + 1$.

This utility maximization problem can be represented as a dynamic optimization through the following Bellman equation:

$$J(a_t, I_t) = \max \left\{ C_t, w_t \right\} \left[ C_t^{\rho} + (\beta(E_tJ(a_{t+1}, I_{t+1})^{1-\gamma})^{\frac{\rho}{\rho - 1}}\right]^{\frac{1}{\rho}}$$ \hspace{1cm} (1.7)

The first order condition of the previous utility maximization problem with respect to consumption and portfolio weights leads to the following set of Euler equations (see Appendix A for details):

$$E_t \left[ \beta^{\frac{1-\gamma}{\rho}} \left( \frac{C_{t+1}}{C_t} \right)^{(1-\gamma)\frac{\rho - 1}{\rho}} (R_{m,t+1})^{\frac{1-\gamma}{\rho} - 1} (1 + r_{i,t+1}) - 1 \right] = 0 \quad \forall i = 1, \ldots, N. \hspace{1cm} (1.8)$$

### 1.3.2 Habit formation utility

Preferences exhibiting habit formation stem from a class of time non-separable preferences. Time-separable utility functions assume implicitly that the satisfaction an agent gets from a bundle of consumption goods will be the same regardless of his past consumption experience. Time-separable utility implies that the marginal rates of substitution between two dates depend on the consumption level in these two periods. Moreover, habit formation resembles EZ as it drives a wedge between the relative risk aversion of the representative agent and the elasticity of intertemporal substitution in consumption, i.e
it does not impose the restrictive relation that the EIS is the reciprocal of the coefficient of relative risk aversion.

The distinguishing feature of these models is that current utility depends not only on current consumption, but also on a habit stock formed from past consumption. The larger the habit, the less pleasure is received from a given amount of consumption, and the larger must be new purchases to gain the same benefit. The agent tries to maximize the following utility function, as in Ferson and Constantinides (1991)

\[
E_t \left[ \sum_{i=0}^{\infty} \beta^i \frac{(C_{t+i} + b_1 C_{t-1+i})^{1-\gamma}}{1-\gamma} \right] \quad (1.9)
\]

subject to the same budget constraint in equation (1.5). The maximization problem is presented using dynamic optimization in the following Bellman equation:

\[
V(a_t, C_{t-1}) = \max_{C_t, w_t} \left[ u(C_t, C_{t-1}) + \beta E_t(V(a_{t+1}, C_t)) \right] \quad (1.10)
\]

We take the FOC with respect to consumption and the optimization results in the following Euler condition. We follow Ferson and Constantinides (1991) in denoting \( b_0 = 1 \) and \( b_2 = 0 \):

\[
E_t \left[ \sum_{i=1}^{2} \beta^i \left( \frac{\tilde{C}_{t+i}}{C_t} \right)^{-\gamma} (b_{i-1} R_{m,t+1} - b_i) - 1 \right] = 0 \quad (1.11)
\]

with

\[
\tilde{C}_{t+1}^{-\gamma} = (C_{t+1} + b_1 C_t)^{-\gamma},
\]
\[
\tilde{C}_{t+2}^{-\gamma} = (C_{t+2} + b_1 C_{t+1})^{-\gamma},
\]
\[
\tilde{C}_t^{-\gamma} = (C_t + b_1 C_{t-1})^{-\gamma}.
\]
The parameter $b_1$ captures the strength of habit formation. If $b_1 = 0$, the utility is time-separable with a constant relative risk-aversion utility function. When $b_1$ is not equal to zero the model is said to exhibit habit persistence. In this model an individual’s habit level depends on his or her own level of past consumption. A negative $b_1$ means that there is a habit persistence effect, i.e., past consumption has a negative effect on current utility because you need to exceed this level of consumption in order to have an increase in the utility. While a positive $b_1$ represents a durability effect where your past consumption has a durable effect and hence increase current utility. Since we use data on consumption of non-durables and services we will restrict ourselves to the case of negative $b_1$.

Then we take the FOC of equation (1.10) with respect to portfolio weights and the optimization results in the following Euler conditions.

$$E_t[\{\tilde{C}_{t+1} - \gamma_{t+1} + b_1 \beta \tilde{C}_{t+2}\} (r_{t+1} - ir_f)] = 0 \quad (1.12)$$

Then we premultiply equation (1.12) by $w_t$, rearranging and making use of $R_{m,t}$ (see appendix 2 for details), we get:

$$E_t\left[\left(\tilde{C}_{t+1} - \gamma_{t+1} + b_1 \beta \tilde{C}_{t+2}\right) [R_{m,t+1} - (1 + r_f)]\right] = 0 \quad (1.13)$$

We combine equation (1.11) and equation (1.13) and get the following set of Euler equations, which we use to compute optimal portfolio weights.

$$E_t\left[\left(\tilde{C}_{t+1} - \gamma_{t+1} + b_1 \beta \tilde{C}_{t+2}\right) \{(i + r_{t+1}) - iR_{m,t+1}\}\right] = 0 \quad (1.14)$$
1.3.3 Comparison of EZ and HF utility functions

We move now to a comparison of the Euler equations for the two utility functions discussed above. The Euler equation for the EZ utility is:

$$E_t \left[ \beta^{\frac{1-\gamma}{\rho}} \left( \frac{C_{t+1}}{C_t} \right)^{(1-\gamma)\frac{\rho-1}{\rho}} (1 + R_{m,t+1})^{(1-\gamma)\frac{1}{\rho} - 1} (1 + r_{i,t+1}) - 1 \right] = 0$$

while the Euler equation for Habit formation is:

$$E_t \left[ \tilde{C}_{t+1}^{(1-\gamma)} + b \beta C_{t+2}^{(1-\gamma)} \right] \left\{ (i + r_{t+1}) - iR_{m,t+1} \right\} = 0$$

Comparing the two Euler equations, we notice they have the following terms in common. Both include consumption growth, portfolio returns, relative risk aversion coefficient and they are discounted using the time discount factor $\beta$. They differ in the format by which the parameters enter into the Euler equations. As an example, they differ in the form by which risk aversion parameter enter into the Euler equation. Moreover, HF utility has an additional parameter $b$, the habit persistent parameter, to capture the impact of past consumption on current level of utility. Specifically the HF Euler equation accounts for the impact of consumption growth from the past two periods on utility, while the EZ accounts only for the impact of current consumption on utility.

1.4 Computing portfolio weights from FOC

From the model section, we can observe that the portfolio weights are implicitly defined in the market return $R_m$ defined in equation (1.6). To compute the optimal portfolio weights we need to carry out several steps. The first step is to draw inferences about the
data generating process $f(y_{t+1}|I_t)$, where $y_{t+1}$ consist of future assets returns and future consumption growth. In this respect, we assume that the returns and consumption are conditionally Gaussian, then we model the evolution of the mean and variance-covariance matrix. First, in order to model the conditional mean, we implement a Vector Autoregression (VAR). The VAR framework captures the dependence of expected returns on past values of returns and other state variables (that we are going to include in our model in the second chapter) and the evolution of these state variables (see Campbell and Viceira (2002)). Following the empirical evidence about the predictability of stock returns we assumed the following VAR(1):

$$
\begin{bmatrix}
  r_{t+1} \\
gc_{t+1}
\end{bmatrix}
= C + \Phi
\begin{bmatrix}
  r_t \\
gc_t
\end{bmatrix} + \epsilon_{t+1}
$$

(1.15)

where $gc_t$ refers to the growth rate of consumption at time $t$. The vector $C$ and matrix $\Phi$ is a diagonal matrix that represents the VAR parameters.

Second, we employ a Constant Conditional Correlation AutoRegressive Conditional Heteroskedasticity (CCC-ARCH) model to estimate the variance-covariance matrix of the next period error term for the risky assets and consumption ($\epsilon_{t+1}$). This entails the following steps. The conditional variance of element $i$ of $\epsilon_{t+1}$ is given by the following ARCH model:

$$
\sigma^2_{i,t+1} = \omega_i + \alpha_i \epsilon_{i,t}^2
$$

(1.16)

where $\frac{\omega}{(1-\alpha_i)}$ is the unconditional variance for the innovation of series $i$. Second, I compute the next period variance-covariance matrix by employing a CCC model defined by the following equation :

$$
\Sigma_{t+1} = D_{t+1} \Gamma D_{t+1}
$$
where $D_{t+1}$ is the next period standard deviation computed from the ARCH equation (3.9), $\Sigma_{t+1}$ is the variance-covariance matrix, and $\Gamma$ is the correlation matrix. The correlation matrix can be simply estimated using the sample variance-covariance of the standardized error terms:

$$
\hat{Q} = \frac{1}{T} \sum_{t=1}^{T} \left( \hat{D}_{t}^{-1} \hat{\epsilon}_{t} \right) \left( \hat{D}_{t}^{-1} \hat{\epsilon}_{t} \right)'
$$

$$
\hat{\Gamma} = \hat{R}_{t}^{-1} \hat{Q} \hat{R}_{t}^{-1}
$$

where $R_t$ is a diagonal matrix composed of the square-root of the elements on the diagonal of $\hat{Q}$.

Third, after specifying the conditional distribution of returns and consumption, we use Monte Carlo simulation to simulate $M$ realizations for next period returns and consumption under a Gaussian distribution with mean equal to $\hat{C} + \hat{\Phi} \begin{bmatrix} r_t \\ gc_t \end{bmatrix}$ and variance covariance matrix $\Sigma_{t+1}$. Then we re-estimate $\Phi$ and $\Sigma$ each time we move one step forward in time.

Finally, in order to compute the optimal portfolio weights we evaluate the conditional expectation of the first order condition of the utility maximization (the Euler equation). We use Generalized Method of Moments (GMM) to solve for the optimal portfolio weights. The Euler equations are utilized as moment conditions, where the unknown parameters to be estimated are the portfolio weights, given the simulated returns and consumption from the first step.

The Euler equation defines an error term $u_{i,t}$ for each asset $i = 1, \ldots, N$, such that $E_t[u_{i,t+1}] = 0$, where $E_t[.]$ stands for conditional expectation given information at time $t$, and $u_{t+1}$ is a vector of $N$ errors. If the conditional expectation of the Euler equation
is zero then the unconditional expectation will be equal to zero by the law of iterated expectations, and we can use the Euler equations as moment conditions. For each asset’s return \((r_{i,t})\) we have one orthogonality condition. In our empirical investigation we have \(N\) assets and \(N\) orthogonality conditions that are functions of the \(N\) portfolio weights, thus the GMM is exactly identified.

For the single-period portfolio problem we use equation (1.8) for EZ utility and equation (1.14) for habit formation utility as moment conditions for each asset, where \(i\) runs from 1 to \(N\). This gives us \(N\) parameters (portfolio weights, \(w_{i,t}\)) to estimate and \(N\) moment conditions at each point of time \(t\), thus GMM is exactly identified. For a given simulated vector of returns \((r_{j,t+1})\) and consumption \((C_{t+1})\), the error term for each asset \(i\) for the EZ utility is

\[
u_{it}(w_t) = \beta^{\frac{1-\gamma}{\rho}} \left( \frac{C_{t+1}}{C_t} \right)^{\frac{1-\gamma}{\rho}} \left( 1 + R_{m,t+1}^{(j)} \right)^{-1} \left( 1 + r_{i,t+1}^{(j)} \right) - 1. \tag{1.19}
\]

The optimal portfolio is then the one that solves these Euler equations. And our objective is to find \(\hat{w}\) (the optimal portfolio weights, we do not impose short-sale constraint on \(w_{i,t}\) or on the risk-free asset, \(i.e\) the optimal weights can take negative values, and can be greater than one). We use GMM as presented in the following optimization to compute \(\hat{w}\):

\[
\hat{w} = \arg \min_{w \in \mathbb{R}^N} \left( \frac{1}{M} \sum_{j=1}^{M} u_t^{(j)}(w) \right)' \left( \frac{1}{M} \sum_{t=1}^{M} u_t^{(j)}(w) \right)
\]

where \(u_t^{(j)}(w) = (u_{1t}^{(j)}(w), u_{2t}^{(j)}(w), \ldots, u_{Nt}^{(j)}(w))'\) and \(M\) is the number of simulated observations or trajectories of returns and consumption. For the habit formation utility function we proceed in a similar way to compute the optimal portfolio weights. We use the Euler equation (1.14) to define our error term \(u_{it}^{(j)}(w_t)\) that define the moment conditions.
1.5 Data

We use monthly data from Federal Reserve Economic Data (FRED)(data on the risk-free asset, we take the return on three-months treasury bills as the risk-free rate), Yahoo finance (data on assets’ returns) and the Center for Research in Security Prices of the University of Chicago (CRSP)(data on S&P 500). We cover the period from May 1986 until October 2009, which constitutes 279 observations. We exclude the last 14 observations to avoid the crash period of 2009, which leaves us with 265 observations. The data are categorized into two subcategories: the in-sample period (observations 1 to 250) and the out-of-sample period (observations 250-265). Consumption data for consumer non-durable goods and services, are obtained from FRED. To compute consumption growth we take the first difference of the logarithm of consumption data. We examine five stocks to represent firms that are well established in the stock market. These five risky stocks are: Citi Group Corporation (CITI), IBM, General electric (GE), Microsoft (MS) and Bank of America Corporation (BAA). The returns on these assets are computed by taking the first difference of the logarithm price of these assets.

Summary statistics of the data are presented in the following tables. In Table 1.1, we report the sample mean and standard deviation for the asset returns and consumption growth. The mean return ranges from 0.0021 (0.21%) for MS to 0.01 (1%) for GE, while the mean growth of consumption is 0.0043. As expected the mean return is positive for all assets in our sample. The mean return for the five assets collectively is 0.0064 (0.64%) with standard deviation of 0.006. Moreover, though the whole dissertation we take the sample average of the three-month treasury bill (0.002) as the risk-free rate.

In addition, Table 1.1 shows sample autocorrelation for lag of order 1. The autocorrelations decay toward zero at longer lags for all the variables. The skewness for the
different series are reported in Table 1.1. We can see that for consumption and all the assets except Microsoft, the skewness is negative indicating that their distribution has a long left tail with distribution clustering to the right of the mean. Some authors try to examine the impact of higher moments like the skewness and kurtosis on portfolio choice, see for example Ameida and Garcia (2008).

The correlation between the return on the various assets and consumption growth is presented in Table 1.2. The highest correlation is between Bank of America Corporation and Citi Group Corporation and this is expected as they are both financial services firms. The lowest correlation between returns is between Bank of America and Microsoft. The correlation between consumption growth with asset return is highest for Microsoft and lowest for Bank of America, the sign is positive for all assets’ correlation with each other and with consumption, except for Bank of America and consumption growth.

The p-value of the correlation coefficient is significant for all assets return but it is not significant for the correlation between assets’ return and consumption. Table 1.3 and 1.4 present the results of fitting VAR-ARCH model to the data, and the standard errors are reported in brackets. Table 1.4 presents the ARCH parameters. We observe that MS has the most volatility (computed as $\omega/(1 - \alpha)$), while BAA has the most persistence in the volatility (as indicated by $\alpha$).
Table 1.2: Correlation between risky assets and consumption growth

<table>
<thead>
<tr>
<th>Variable</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
<th>Cons</th>
</tr>
</thead>
<tbody>
<tr>
<td>CITI</td>
<td>1</td>
<td>0.651</td>
<td>0.296</td>
<td>0.314</td>
<td>0.772</td>
<td>0.0342</td>
</tr>
<tr>
<td>IBM</td>
<td>0.651</td>
<td>1</td>
<td>0.376</td>
<td>0.462</td>
<td>0.598</td>
<td>0.062</td>
</tr>
<tr>
<td>GE</td>
<td>0.296</td>
<td>0.376</td>
<td>1</td>
<td>0.436</td>
<td>0.218</td>
<td>0.007</td>
</tr>
<tr>
<td>MS</td>
<td>0.314</td>
<td>0.462</td>
<td>0.436</td>
<td>1</td>
<td>0.259</td>
<td>0.072</td>
</tr>
<tr>
<td>BAA</td>
<td>0.772</td>
<td>0.598</td>
<td>0.218</td>
<td>0.259</td>
<td>1</td>
<td>-0.004</td>
</tr>
<tr>
<td>Cons</td>
<td>0.034</td>
<td>0.062</td>
<td>0.007</td>
<td>0.072</td>
<td>-0.004</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1.3: Vector AutoRegression parameters for assets ’ returns and consumption growth

<table>
<thead>
<tr>
<th>Variable</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
<th>Cons</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi_0$</td>
<td>0.01444</td>
<td>0.0122</td>
<td>0.0062</td>
<td>0.0237</td>
<td>0.0143</td>
<td>0.0051</td>
</tr>
<tr>
<td>Standard errors</td>
<td>(0.0058)</td>
<td>(0.0040)</td>
<td>(0.0050)</td>
<td>(0.0064)</td>
<td>(0.004)</td>
<td>(0.0005)</td>
</tr>
<tr>
<td>$\Phi_1$</td>
<td>-0.0288</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Standard errors</td>
<td>(0.0698)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Phi_2$</td>
<td>0</td>
<td>-0.02338</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Standard errors</td>
<td>(0.0699)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Phi_3$</td>
<td>0</td>
<td>0</td>
<td>-0.0238</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Standard errors</td>
<td>(0.0720)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Phi_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-0.0675</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Standard errors</td>
<td>(0.0675)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Phi_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-0.0584</td>
<td>0</td>
</tr>
<tr>
<td>Standard errors</td>
<td>(0.0724)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Phi_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.2018</td>
</tr>
<tr>
<td>Standard errors</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.0576)</td>
</tr>
</tbody>
</table>
Table 1.4: ARCH-CCC parameters

<table>
<thead>
<tr>
<th>Variable</th>
<th>$\omega$</th>
<th>$\alpha_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CITI</td>
<td>0.00740</td>
<td>0.235892</td>
</tr>
<tr>
<td></td>
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<td>0.0948</td>
</tr>
<tr>
<td>IBM</td>
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<tr>
<td></td>
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<td>0.0794</td>
</tr>
<tr>
<td>GE</td>
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<td>0.1291</td>
</tr>
<tr>
<td></td>
<td>0.0005</td>
<td>0.0749</td>
</tr>
<tr>
<td>MS</td>
<td>0.0087</td>
<td>0.2392</td>
</tr>
<tr>
<td></td>
<td>0.0008</td>
<td>0.0904</td>
</tr>
<tr>
<td>BAA</td>
<td>0.0046</td>
<td>0.5530</td>
</tr>
<tr>
<td></td>
<td>0.0003</td>
<td>0.0947</td>
</tr>
<tr>
<td>Cons</td>
<td>0.000</td>
<td>0.195</td>
</tr>
<tr>
<td></td>
<td>0.2133</td>
<td>0.0802</td>
</tr>
</tbody>
</table>

1.6 Estimating the Model’s Parameters

Our model makes use of two sets of preference representations: Epstein-Zin (EZ) utility function and habit formation (HF) utility function. Each of these utility functions comprise three preference parameters. We need to calibrate the models for the risk aversion ($\gamma$), elasticity of intertemporal substitution ($\frac{1}{1-\rho}$), and the time discount factor ($\beta$) parameters in the Epstein-Zin utility function. We also need to calibrate the HF utility parameters: the habit persistence parameter ($b_1$), concavity parameter ($\gamma$) and discount factor parameter ($\beta$).

There is no clear consensus in the literature on the values of these parameters. We use Generalized Method of Moments to estimate the model parameters. We utilize historical data for monthly consumption and assets’ returns over the period May 1986 until October 2009, which constitutes 279 observations. We exclude the last 14 observations to avoid the crash period of 2009, which leaves us with 265 observations, and we use the return on
the S&P index as a proxy for the market rate of return. We examine different scenarios of our model using different values of the parameters which are taken from the literature.

1.6.1 Epstein-Zin Utility

The model’s parameters for the EZ utility are: the risk aversion parameter $\gamma$, the elasticity of intertemporal substitution ($\frac{1}{1-\rho}$) and the time discount factor $\beta$. The following table summarizes the results. We find that the estimated discount factor ($\beta$) is sensitive to the choice of initial values, while risk-aversion ($\gamma$) parameters equals 17, and the EIS ($\frac{1}{1-\rho}$) equals $-0.151$. The second row in the following table presents the standard errors for the estimated parameters. There is no general consensus about the values of these parameters in the literature, for example authors tend to use values that range from 5-20 for $\gamma$, and an EIS of 1.5, or 2 (see for example Brandt (1999), Bansal et al. (2007a)).
1.6.2 Habit Formation utility

The models’ parameters for the HF utility are: $\gamma$ which is defined as a concavity parameter that is approximately equal to the relative risk aversion and the habit persistence parameter $b_1$. Both have a wide range of values that are documented in the literature and there is no consensus about their values. Initially, we use a two-stage GMM to estimate the habit, and risk aversion parameters. We employ the Euler equations as the moment conditions and we use lagged assets’ returns (we use one lag) as instruments. Our estimate of $b_1$ is -0.038 and our estimate of $\gamma$ is about 5.

1.7 Empirical Results

In this section we present our results for the single-period optimal portfolio weights for EZ utility function and HF utility function using different sets of values for the preference parameters. We calibrate our models using parameters estimates taken from the literature. Our target is to compute the optimal portfolio weights for two scenarios, a high risk-aversion and a low risk-aversion investor. Hence, we used the literature values for $\gamma = 10$ since it is representation of high risk-aversion investor, which may be also be represented with $\gamma = 17$ but to avoid duplication of work we used the literature estimate.
In addition, we choose another value from the literature that represents the low-risk aversion investor.

1.7.1 Epstein-Zin Portfolio

In order to compute the optimal portfolio weights for a representative investor, we evaluate the conditional distribution of the Euler equation presented in equation (1.14). Using the method presented in section 1.4 we calibrate the model with different values of the preference parameters; \( \gamma \) and \( \rho \). We examine the model with the following set of values for the risk aversion \( \gamma \in \{5,10\} \), (used by several authors for example, Brandt (1999), and Bansal et al. (2007a)) and EIS equal to 1.5, (Bansal et al. (2007a). Moreover, we use the sample mean (0.002) for the value of the risk-less asset’s return. We first present figures of our results then we analyze the results tabulated below.

Figure 1.1: Optimal portfolio weights for EZ utility with \( \beta = 0.996 \), \( \gamma = 5 \), \( \rho = 0.5 \)
Table 1.6: Optimal portfolio weights for EZ utility: $\beta = 0.996, \gamma = 5, \rho = 0.5$

<table>
<thead>
<tr>
<th></th>
<th>July 09</th>
<th>August 09</th>
<th>September 09</th>
<th>October 09</th>
<th>November 09</th>
<th>December 09</th>
</tr>
</thead>
<tbody>
<tr>
<td>CITI</td>
<td>-0.224</td>
<td>-0.586</td>
<td>-0.346</td>
<td>-0.060</td>
<td>-0.311</td>
<td>-0.240</td>
</tr>
<tr>
<td>IBM</td>
<td>0.488</td>
<td>0.495</td>
<td>0.545</td>
<td>0.537</td>
<td>0.454</td>
<td>0.674</td>
</tr>
<tr>
<td>GE</td>
<td>1.174</td>
<td>1.429</td>
<td>0.962</td>
<td>0.726</td>
<td>1.201</td>
<td>1.064</td>
</tr>
<tr>
<td>MS</td>
<td>0.292</td>
<td>0.204</td>
<td>0.298</td>
<td>0.299</td>
<td>0.307</td>
<td>0.386</td>
</tr>
<tr>
<td>BAA</td>
<td>0.517</td>
<td>0.749</td>
<td>0.737</td>
<td>0.663</td>
<td>0.605</td>
<td>0.117</td>
</tr>
</tbody>
</table>

Table 1.6 summarizes the results of the last six periods of Figure 1.1. It reports the single-period optimal portfolio choice for an investor who updates his choice each period taking into consideration new information about expected returns and the variance-covariance matrix of returns. We find that an investor with $\gamma = 5$ and $\rho = 0.5$ will short sell CITI between 6% to 58% of his savings, allocate more than 100% of his savings in GE. The investor will also direct between 45% to 67% of his savings to IBM, between 20% to 38% to MS, and between 11.7% to 74.9% to BAA. Whenever the sum of the optimal weights of the assets is greater than one, this means that we are borrowing at the risk-free rate and hence we are leveraged. A general observation is that in the USA the law has a cap on borrowing (no more than 50% of investor’s investment). However, as we didn’t impose constraint on borrowing this limitation on leverage is not observed in our empirical findings.

The Figures 1.2 and 1.3 present the estimated variances and means of the five assets which we believe drive the changes in optimal portfolio weights across time. Specifically sharp changes in portfolio weights can be attributed to changes in the assets’ variances over time. For example as shown below we can observe that the sharp decline in the optimal portfolio weights of MS before the 10th period accompany a sharp increase in the variance of MS. Also, we observe that the sharp changes in portfolio weights which can
be also attributed to changes in the assets’ variances over time. For example as shown below we can observe that the sharp decline in the optimal portfolio weights of GE before the 15th period accompanies a sharp increase in the variance of GE.

Figure 1.2: Estimated variances of the assets

Table 1.7: Optimal portfolio weights for the EZ utility: $\beta = 0.996$, $\gamma = 10$, $\rho = 0.5$.

<table>
<thead>
<tr>
<th>weight</th>
<th>July 09</th>
<th>August 09</th>
<th>September 09</th>
<th>October 09</th>
<th>November 09</th>
<th>December 09</th>
</tr>
</thead>
<tbody>
<tr>
<td>CITI</td>
<td>-0.099</td>
<td>-0.246</td>
<td>-0.206</td>
<td>-0.041</td>
<td>-0.170</td>
<td>-0.143</td>
</tr>
<tr>
<td>GE</td>
<td>0.610</td>
<td>0.702</td>
<td>0.515</td>
<td>0.401</td>
<td>0.579</td>
<td>0.544</td>
</tr>
<tr>
<td>IBM</td>
<td>0.246</td>
<td>0.230</td>
<td>0.290</td>
<td>0.250</td>
<td>0.237</td>
<td>0.337</td>
</tr>
<tr>
<td>MS</td>
<td>0.152</td>
<td>0.104</td>
<td>0.150</td>
<td>0.162</td>
<td>0.158</td>
<td>0.215</td>
</tr>
<tr>
<td>BA</td>
<td>0.228</td>
<td>0.338</td>
<td>0.374</td>
<td>0.302</td>
<td>0.306</td>
<td>0.068</td>
</tr>
</tbody>
</table>

Table 1.7 summarizes the results of the optimal portfolio weights of an investor who
is more risk averse than in the first case. As expected higher risk aversion \((\gamma = 10)\) leads to a lower portfolio of equity, represented in lower holding of four out of the five assets. The investor’s holding of IBM dropped from over 100\% (for a low risk averse investor) to a range of 40\% to 70\% (the investor with higher risk aversion), from a maximum of 38\% for a low risk aversion investor to a range between 10\% to 15.8\% for MS, and from a maximum of 74\% for a low risk averse investor to a range between 6.8\% to 37.4\% for BA . Finally, for an investor with a risk aversion equal to 10 we find that he will short sale CITI from 9\% to 20\%, GE from 18 \% to 19\%.

1.7.2 Habit Formation utility

This section presents the single-period optimal portfolio choice for an infinitely-lived investor when his preferences are represented by a habit formation utility function. We examine different scenarios of the model by changing the values of the calibrated pa-
Figure 1.4: Optimal portfolio weights for the EZ utility with $\beta = 0.99$, $\gamma = 10$, $\rho = 0.5$ parameters that are taken from the literature. Specifically, we investigate the impact of changing the values of the habit persistence parameter ($b_1$) and the concavity parameter ($\gamma$) on the results of the model.

Ferson and Constantinides (1991) estimated $b_1$, and $\gamma$ using different instrumental variables (for example they used instrumental variables such as lagged returns and consumption versus lagged financial variables such as dividend yield on CRSP index, the return on one and three months Treasury-bill). The values of these parameters change with the choice of the instrumental variables, for example when they used one lag for the financial variables as the instruments they found that the there is an evidence of time-nonseparable model with a negative $b_1$, with $"b_1 = -0.717, \gamma = 8"$. And using two lags of financial variables as instruments, they found the parameters to be; $"b_1 = -0.717, \gamma = 8"$.

The next figures present our results for an investor with habit persistence parameter
$b_1 = -0.36$, and relative risk aversion $\gamma = 1.918$ and then for "$b_1 = -0.717, \gamma = 8$".

The graphs show that the optimal portfolio weights change over time as we update our estimate of future expected returns and variance-covariance of the assets. Table 1.8 presents the first case "$b_1 = -0.36, \gamma = 1.918$" and Table 1.9 presents results for the second case "$b_1 = -0.717, \gamma = 8$" for the optimal portfolio weights for an investor with these parameter values. The optimal portfolio weights are presented in Figures 1.5 and 1.6. Table 1.8 and 1.9 report the last six periods of our estimation for the optimal portfolio weights using habit formation utility. The optimal portfolio weights in the first scenario ($b_1 = -0.36, \gamma = 1.918$) vary from a holding of 85% of wealth in equities to 97% of wealth. The composition of the portfolio varies both across time and across scenarios. In the first scenario we find that (CITI) varies from 14% to 20%, the second asset (IBM) varies from 14.7% to 17.9%, the thirds asset’s (GE) weight varies from 16% to 20% , the
Figure 1.6: Optimal portfolio weights habit formation with $b_1 = -0.717, \gamma = 8, \beta = 0.996$

Table 1.8: Optimal portfolio weights habit formation: $b_1 = -0.36, \gamma = 1.918, \beta = 0.996$

<table>
<thead>
<tr>
<th></th>
<th>July 09</th>
<th>August 09</th>
<th>September 09</th>
<th>October 09</th>
<th>November 09</th>
<th>December 09</th>
</tr>
</thead>
<tbody>
<tr>
<td>CITI</td>
<td>0.150</td>
<td>0.179</td>
<td>0.204</td>
<td>0.190</td>
<td>0.141</td>
<td>0.147</td>
</tr>
<tr>
<td>IBM</td>
<td>0.176</td>
<td>0.175</td>
<td>0.172</td>
<td>0.147</td>
<td>0.179</td>
<td>0.166</td>
</tr>
<tr>
<td>GE</td>
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<td>0.202</td>
<td>0.176</td>
<td>0.181</td>
<td>0.186</td>
</tr>
<tr>
<td>MS</td>
<td>0.207</td>
<td>0.207</td>
<td>0.201</td>
<td>0.152</td>
<td>0.197</td>
<td>0.194</td>
</tr>
<tr>
<td>BAA</td>
<td>0.170</td>
<td>0.215</td>
<td>0.201</td>
<td>0.182</td>
<td>0.167</td>
<td>0.159</td>
</tr>
</tbody>
</table>

Table 1.9: Optimal portfolio weights habit formation: $b_1 = -0.717, \gamma = 8, \beta = 0.996$

<table>
<thead>
<tr>
<th></th>
<th>July 09</th>
<th>August 09</th>
<th>September 09</th>
<th>October 09</th>
<th>November 09</th>
<th>December 09</th>
</tr>
</thead>
<tbody>
<tr>
<td>CITI</td>
<td>0.147</td>
<td>0.175</td>
<td>0.201</td>
<td>0.194</td>
<td>0.139</td>
<td>0.144</td>
</tr>
<tr>
<td>IBM</td>
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<td>0.148</td>
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<td>GE</td>
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<td>0.194</td>
<td>0.205</td>
<td>0.176</td>
<td>0.185</td>
<td>0.193</td>
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<tr>
<td>MS</td>
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<td>0.195</td>
<td>0.152</td>
<td>0.191</td>
<td>0.195</td>
</tr>
<tr>
<td>BAA</td>
<td>0.177</td>
<td>0.217</td>
<td>0.204</td>
<td>0.182</td>
<td>0.170</td>
<td>0.172</td>
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</tbody>
</table>
fourth asset (MS) varies from 15% to 20%, finally the fifth asset (BAA) varies from 15.9% to 21.5%.

In the second scenario \( (b_1 = -0.717, A = 8) \) optimal portfolio weights vary from about 84% of investor wealth to 98%. Table 1.9 presents the results for the second scenario where the habit persistence parameter is equal to \((-0.717)\) and the concavity parameter \(A\) is equal 8. The investor in this case holds between 14% to 20% of his savings in CITI, between 14.8% and 17.4% in IBM, and between 16.9% and 20.5% in GE. Finally the investor directs between 15.2% to 20.7% to MS, and between 17% to 21.4% to BAA.

### 1.8 Conclusion

We were able to derive the optimal portfolio weights from the conditional Euler equation using GMM after specifying data generating processes for returns and consumption growth and using a Monte Carlo simulations to draw consumption and returns values so as to evaluate the conditional expected values in equation (1.8) and (1.11).

For the single-period optimal portfolio choice of the Epstein-Zin (EZ) utility, results show a tendency to have a higher portfolio of equities for lower risk aversion investors. The differences across periods are affected mainly from changes in the estimated variances.

In addition, the results of the single-period portfolio choice for the habit formation (HF) utility function showed less variation across periods. Moreover the HF results showed less sensitivity of the results to the choice of the calibrated parameters.

Finally, we observe that the resulted optimal portfolio weights of the EZ and the HF models are very different. For example the EZ weights shows variations across time and across calibrated parameters, while that of HF do not show much variation. Moreover, we find that the optimal portfolio weights in the EZ model is influenced by variations in
the variances, while the HF weights are not.
Chapter 2

Multip-Period Portfolio Choice

2.1 Summary

Individuals differ in their investment motivations, some invest in order to finance higher consumption in the short-run, some make longer term investment to secure higher income at retirement, and others invest for the goal of paying for kids tuition in the future. Choosing the optimal combination of stocks given the enormous number of stocks available becomes a critical decision. Usually the investors care about their portfolios over several periods of time, computing the optimal portfolio weights for multiple periods is an empirical issue that has received revived interest in the past few years, and which we address here.

There has been growing attention directed to the multi-period portfolio choice (see Barberis (2000), Brennan et al. (1997), Garlappi and Skoulakis (2008)). This interest stems from the fact that most investment decisions are done on a multi-period basis. This requires the investor to account for changes in the investment opportunity set. This investment opportunity set is described by state variables, such as lagged returns, price-
dividend ratio, default premiums and risk premiums. In this chapter we are going to use lagged returns as our state variable.

In this chapter we compute multi-period optimal portfolio weights for an infinitely lived investor who invest in $N$ risky assets. The optimal multi-period ahead portfolio choice is the computation of portfolio weights not only for the next period given current period’s information but we find the optimal portfolio weights for the next $\tau$ periods ahead ($\tau = 6$ in the empirical application) given information available till the current period. We will use two standard models of preferences, Epstein-Zin and habit formation.

The model used in this chapter stems from the one in the first chapter. For the EZ utility the investor will maximize his utility as described in the first chapter’s equation (1.4) subject to the budget constraint in equation (1.5), which are repeated here for the convenience of the reader:

$$U_t = \left( C_t^\rho + \beta E_t[U_{t+1}^{1-\gamma}]^{\frac{\rho}{1-\gamma}} \right)^{\frac{1}{\rho}}$$  \hspace{1cm} (2.1)

subject to the following budget constraint

$$a_{t+1} = (a_t - C_t)R_{m,t+1}.$$  \hspace{1cm} (2.2)

The main difference from the first chapter is that we use here the Euler equation to find the optimal portfolio choice for several periods ahead ($\tau$ periods ahead). The Euler equations for $\tau$ periods ahead is:

$$E_{t+\tau} \left[ \beta^{\frac{1-\gamma}{\rho}} \left( \frac{C_{t+\tau}}{C_{t+\tau-1}} \right)^{(1-\gamma)\frac{\rho-1}{\rho}} \left( 1 + R_{m,t+\tau} \right)^{(1-\gamma)\frac{\rho-1}{\rho}} (1 + r_{i,t+\tau} - 1) \right] = 0$$  \hspace{1cm} (2.3)
For $\tau = 1, \ldots, 6$ periods ahead.

The investor also maximizes his utility in the case of habit formation utility described in the first chapter and result in the following first order conditions:

The FOC with respect to consumption:

\[
E_t \left[ \sum_{i=1}^{2} \beta^i \left( \frac{\tilde{C}_{t+i}}{C_t} \right)^{-\gamma} (b_i - b_{i-1} R_{m,t+1} - b_t) - 1 \right] = 0 \quad (2.4)
\]

with

\[
\tilde{C}_{t+1}^{-\gamma} = (C_{t+1} + b_1 C_t)^{-\gamma},
\]

\[
\tilde{C}_{t+2}^{-\gamma} = (C_{t+2} + b_1 C_{t+1})^{-\gamma},
\]

\[
\tilde{C}_{t}^{-\gamma} = (C_{t} + b_1 C_{t-1})^{-\gamma}.
\]

The FOC with respect to portfolio weights ($w_t$):

\[
E_t[ [C_{t+1}^{-\gamma} + b_1 \beta [C_{t+2}^{-\gamma}]] [R_{m,t+1} - (1 + r_f)]] = 0 \quad (2.5)
\]

We combine equation (2.4) and equation (2.5) and get the following set of Euler equations, which we use to compute optimal portfolio weights.

For the multi-period portfolio choice we also use the following set of Euler equations:

\[
E_t \left[ Y_{t+\tau} \{ (i + r_{t+\tau}) - i R_{m,t+\tau} \} \right] = 0 \quad (2.6)
\]

where $Y_{t+\tau} = [\tilde{C}_{t+\tau}^{-\gamma} + b_1 \beta [\tilde{C}_{t+\tau+1}^{-\gamma}]]$
with

\[
\tilde{C}_{t+\tau}^{-\gamma} = (C_{t+\tau} + b_t C_{t+\tau-1})^{-\gamma},
\]

\[
\tilde{C}_{t+\tau+1}^{-\gamma} = (C_{t+\tau+1} + b_t C_{t+\tau})^{-\gamma},
\]

Again we are using the resulting Euler equation to find the optimal portfolio weights for \( \tau \) periods ahead.

This chapter is structured as follows: we discuss the methodology in Section 2, then we present empirical estimates of model’s parameters in Section 3. Finally, we conclude in Section 4.

### 2.2 Optimal Portfolio Choice: Multi-period

To compute the optimal multi-period portfolio choice, we need to estimate the conditional expectation presented in the Euler equation for EZ utility (equation (2.3)) and habit formation utility (equation (2.6)). We follow the same steps outlined in the first chapter with some changes to account for the multi-periods aspect of our optimization as discussed in the next section. We dynamically estimate the \( \tau \) periods ahead conditional distribution of returns and consumption given information until the current period.

At each time period \( t \) we estimate the mean of the next \( \tau \) periods returns using a vector AutoRegressive (VAR) model. Then we estimate the \( \tau \) periods variance-covariance matrix using Constant Conditional Correlation AutoRegressive Conditional Heteroskedasticity (CCC-ARCH). At each \( \tau \) period we simulate M observations of returns and consump-
tion. If \( E_{t+\tau} = \hat{C} + \hat{\Phi} \begin{bmatrix} \tau_{t,\tau} \\ g_{c,t,\tau} \end{bmatrix} \) then from the law of iterated expectations \( E_t = \hat{C} + \hat{\Phi} \begin{bmatrix} \tau_{t,\tau} \\ g_{c,t,\tau} \end{bmatrix} \). From \( I_t \), we simulate paths for returns and consumption to have \( C_{t+\tau+1}^{(j)} \) and \( \tau_{t+\tau+1}^{(j)} \). In order to simulate the paths of returns and consumption growth we estimate the VAR and ARCH parameters once every time period then we draw paths of simulated returns and consumption growth each \( \tau \) period, and update this simulated returns as we move \( \tau \) period in the future by updating the data generating function \( f(y_{t+1}|I_t) \).

And then we use GMM to compute optimal portfolio weights. We compute the optimal portfolio weights by estimating the conditional expectation of the first order condition of the utility maximization (the Euler equations), where these Euler equations are used to define the error terms that are used in the GMM estimation.

Thus for the multiple-period portfolio problem at time \( t \) we are interested in computing the optimal portfolio at time \( t + \tau \), we start by simulating multiple trajectories \((\tau_{t+1}^{(j)}, g_{c,t+1}^{(j)}), (\tau_{t+2}^{(j)}, g_{c,t+2}^{(j)}), \ldots, (\tau_{t+\tau+1}^{(j)}, g_{c,t+\tau+1}^{(j)})\). For a given simulated trajectory, we compute the following error term for asset \( i \):

\[
\begin{align*}
  u_{it}^{(j)}(w_{t+\tau}) &= \beta \frac{1-\gamma}{\rho} \left( \frac{C_{t+\tau+1}^{(j)}}{C_{t+\tau}^{(j)}} \right) \left( 1 + \rho \left( 1 + \frac{1-\gamma}{\rho} \right) \right) - 1. 
\end{align*}
\]

Then again the optimal portfolio is the one that solves these Euler equations (equation (2.3) for EZ utility and (2.6) for HF utility). For the habit formation utility function we proceed in a similar way to compute the optimal portfolio weights. We use the Euler equation 2.3 to define our error term \( u_{it}^{(j)}(w_i) \) that define the moment conditions.
2.3 Empirical application: multi-period ahead portfolio

We compute the optimal portfolio weights for the out-of-sample observations (data from 150 to 265), which are 116 months, at each month the objective is to find the optimal portfolio weights for six periods ahead. For example starting from period 150 (which corresponds to September 1998) we estimate paths of consumption growth and returns.

We start by estimating the conditional distribution for next period ($\tau=1$) (simulated observation 151, which corresponds to October 1998) given information until period 150, then we estimate the conditional distribution for the next period ($\tau=2$) (simulated observation 152, which corresponds to November 1998) given information at time $t$ but we make use of the simulated error and simulated returns and consumption at $\tau=1$ (151) in estimating the mean and the variance-covariance matrix of the $\tau=2$ period. Thus at each month we simulate paths of $\tau$ periods ahead of returns and consumption growth, by estimating the VAR and GARCh parameters once at the beginning of $\tau=1$ but by updating the available our means and variance-covariance matrix of returns and consumption growth. We keep repeating the same process till $\tau=6$ period (observation 156, which corresponds to March 1999), every time we make use of simulated returns and consumption of the previous period in estimating the mean and variance-covariance matrix of the next period till the sixth period ($\tau=6$).

Then we use the Euler equation as moment condition and employ GMM to compute optimal portfolio weights. Once we compute the optimal portfolio choice for six periods, we restart the estimation process but now we are conditioning on observed $t+1$ (observation 151) information instead of information at time $t$ (observation 150). Subsequently, we carry out the same procedures described above for another six periods.
We continue our estimation till the last available observations in the out-of sample observations (t=265), and we compute the optimal portfolio weights for another six-periods using same procedures outlined above.

For the empirical findings we started by taking the in-sample to be from observation 1-250 and the out of sample to be observations from 251 to observation 265. The initial optimal portfolio weights indicated a tendency for the weights to decline towards zero. The optimal portfolio weights are driven by either changes in means or changes in the variances. We explore changes in mean and it showed no unusual changes for the five assets and consumption growth. While the variances showed a major change when we change the start date of the out-of-sample period to be period 150. The data show that the variances of the five assets are larger when we base our in sample till the 150th observation, and hence the optimization always takes into consideration a reverting variance and hence weights decline toward zero. When we allow the in-sample to include observation 1 to observation 150 the optimal portfolio no longer tends to move to zero.

The main difference between the EZ and HF is that for the EZ we simulate paths of returns and consumption (for 6 periods ahead) and compute the optimal portfolio weights after we update the expected mean and variance-covariance matrix which derive information from the simulated path of return and consumption growth as we proceed further in horizon. While in HF at each point we simulated paths of consumption for two steps ahead which entre the Euler equation in addition to the simulated returns.

2.3.1 Epstein-Zin Utility

The following tables and figures present the results of the multi-period portfolio choice under several scenarios. Each scenario refers to different set of calibrated parameters.
Each curve in the following figures represents six periods of optimal portfolio weights conditional on current month information. Thus, for each asset we have 26 curves, and each curve represents six periods ahead of optimal portfolio weights. We present two tables for each curve: the first table summarizes the $\tau$ period results of the optimal portfolio choice for the period before the last of the out-of-sample observation and the second table demonstrate the results from the six period portfolio choice for the last out-of-sample observation.

![Figure 2.1: Weight-Epstein-Zin Multi-period: (\(\beta = 0.996, \gamma = 5, \rho = 0.5\))](image)

Table 2.1 presents optimal portfolio weights for six periods ahead given the information available till the period before the last of the out-of the sample observations. To be specific, this summarizes the results of the curve before the last in each graph of Figure 2.1. We find that an investor with $\gamma = 5$ and $\rho = 0.5$ varies his holding of CITI from $-34\%$ to $-5.2\%$ where the composition of CITI across time change with less shot-selling
Table 2.1: Optimal portfolio weights six periods ahead for the next to last period (September, 2009) EZ utility function: $\beta = 0.996$, $\gamma = 5$, $\rho = 0.5$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.3442</td>
<td>0.6954</td>
<td>0.5167</td>
<td>0.1853</td>
<td>0.6860</td>
</tr>
<tr>
<td>2</td>
<td>-0.1652</td>
<td>0.5865</td>
<td>0.4156</td>
<td>0.1125</td>
<td>0.4143</td>
</tr>
<tr>
<td>3</td>
<td>-0.0960</td>
<td>0.5704</td>
<td>0.3660</td>
<td>0.0859</td>
<td>0.2665</td>
</tr>
<tr>
<td>4</td>
<td>-0.0974</td>
<td>0.5091</td>
<td>0.3252</td>
<td>0.1039</td>
<td>0.2388</td>
</tr>
<tr>
<td>5</td>
<td>-0.0894</td>
<td>0.4449</td>
<td>0.3179</td>
<td>0.0692</td>
<td>0.2517</td>
</tr>
<tr>
<td>6</td>
<td>-0.0521</td>
<td>0.4112</td>
<td>0.2849</td>
<td>0.0603</td>
<td>0.2128</td>
</tr>
</tbody>
</table>

as move in the future. The investor holds between 41.1% to 69.5% of his savings in IBM, where the investor start with 70% of his wealth in IBM and end with 28%. In addition the investor holds between 28.5% to 51.6% in GE. Moreover, the investors holds between 6% to 18.5% in MS, and between 21% to 68.8% of his savings in BAA.
Table 2.2: Optimal portfolio weights six periods ahead for the last period, (October, 2009) EZ utility function: $\beta = 0.996$, $\gamma = 5$, $\rho = 0.5$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.2972</td>
<td>0.7163</td>
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<td>0.2892</td>
<td>0.2274</td>
</tr>
<tr>
<td>2</td>
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<td>0.6166</td>
<td>0.4666</td>
<td>0.1360</td>
<td>0.1082</td>
</tr>
<tr>
<td>3</td>
<td>-0.0187</td>
<td>0.5027</td>
<td>0.3934</td>
<td>0.0991</td>
<td>0.1423</td>
</tr>
<tr>
<td>4</td>
<td>-0.0397</td>
<td>0.4940</td>
<td>0.3455</td>
<td>0.0940</td>
<td>0.1338</td>
</tr>
<tr>
<td>5</td>
<td>-0.0616</td>
<td>0.4445</td>
<td>0.3254</td>
<td>0.0826</td>
<td>0.1461</td>
</tr>
<tr>
<td>6</td>
<td>-0.0606</td>
<td>0.4313</td>
<td>0.3395</td>
<td>0.0736</td>
<td>0.1092</td>
</tr>
</tbody>
</table>

The above table presents the six-periods optimal portfolio choices for an investor with $\gamma = 5$ and $\rho = 0.5$ given the information available till the last observation in our out-of-sample observations. This summarizes the last curve in each graph of Figure 2.1. We find that an investor with $\gamma = 5$ and $\rho = 0.5$ will sell CITI short from about 1.9% to 29.7%. The investor will hold from 43% to 71.6% of his savings in IBM, from 32% to 66% of GE and hold from 7.4% to 28.9% of MS. Finally, the investor will hold from 10.8% to 22.7% of his portfolio in BAA.

Table 2.3: Optimal portfolio weights six periods ahead for the next to the last period, (September, 2009) EZ utility: $\beta = 0.996$, $\gamma = 10$, $\rho = 0.5$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.2493</td>
<td>0.4913</td>
<td>0.3560</td>
<td>0.1194</td>
<td>0.4811</td>
</tr>
<tr>
<td>2</td>
<td>-0.1115</td>
<td>0.3940</td>
<td>0.2810</td>
<td>0.0888</td>
<td>0.2731</td>
</tr>
<tr>
<td>3</td>
<td>0.0607</td>
<td>-0.2599</td>
<td>-0.2280</td>
<td>-0.0006</td>
<td>-0.1747</td>
</tr>
<tr>
<td>4</td>
<td>-0.0822</td>
<td>0.3302</td>
<td>0.2512</td>
<td>0.0544</td>
<td>0.1701</td>
</tr>
<tr>
<td>5</td>
<td>-0.0569</td>
<td>0.2925</td>
<td>0.2222</td>
<td>0.0570</td>
<td>0.1574</td>
</tr>
<tr>
<td>6</td>
<td>-0.0114</td>
<td>0.1933</td>
<td>0.2305</td>
<td>0.0843</td>
<td>0.1145</td>
</tr>
</tbody>
</table>
Table 2.3 presents the results of the optimal portfolios of an investor who is more risk averse ($\gamma = 10$) and has an Elasticity of Intertemporal substitution (EIS) of 2. It describes the six-period optimal portfolio choice for the investor given the information available till the last observation in our out-of-sample observation. Apparently the value of $\gamma$ has a major impact on composition of the investor’s portfolio as opposed when $\gamma$ is equal to 5.

This table summarizes last curve in each graph of Figure 2.2. We find that an investor will short-sell CITI from 1.1% to 25% in all periods except the third period. The investor will have a positive holdings of IBM in all periods except the third period, which ranges between 19% and 49%. The investor will have a positive holding of GE in all periods except the third one, his positive holdings range between 22% and 35%. The investor will also hold between 5.4% and 11.9% of his wealth in MS except in the third period where he has negative holding if MS. Finally, the investor holds a portfolio that ranges between

Figure 2.2: Portfolio Weights Epstein-Zin Multi-period: $\beta = 0.996$, $\gamma = 10$, $\rho = 0.5$
-17% to 48% in BAA.
Table 2.4: Optimal portfolio weights six periods ahead for the last period, (October, 2009) EZ utility: $\beta = 0.996$, $\gamma = 10$, $\rho = 0.5$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.1548</td>
<td>0.4845</td>
<td>0.4421</td>
<td>0.1608</td>
<td>0.1312</td>
</tr>
<tr>
<td>2</td>
<td>0.0301</td>
<td>-0.3057</td>
<td>-0.2691</td>
<td>-0.0200</td>
<td>-0.0716</td>
</tr>
<tr>
<td>3</td>
<td>0.0276</td>
<td>-0.2737</td>
<td>-0.2341</td>
<td>-0.0136</td>
<td>-0.0690</td>
</tr>
<tr>
<td>4</td>
<td>0.0338</td>
<td>-0.2350</td>
<td>-0.2164</td>
<td>-0.0105</td>
<td>-0.0810</td>
</tr>
<tr>
<td>5</td>
<td>-0.0748</td>
<td>-0.0576</td>
<td>-0.0151</td>
<td>-0.1424</td>
<td>-0.0913</td>
</tr>
<tr>
<td>6</td>
<td>0.0231</td>
<td>-0.1923</td>
<td>-0.1872</td>
<td>-0.0140</td>
<td>-0.0746</td>
</tr>
</tbody>
</table>

Table 2.4 we find that an investor with high risk aversion ($\gamma = 10$) and $\rho$ equal 0.5, he will allocate between -15% to 3.3% of his wealth to CITI across the six time periods. The investor will choose to hold IBM between -30.5% and 48%, while allocating between -26% and 44.2% of his saving to GE. He will hold between -14% and 16% of his wealth to MS, and between -9% to 13% in BAA.
Figure 2.3: portfolio Weights Epstein-Zin Multi-period: $\beta = 0.996$, $\gamma = 5$, $\rho = -0.5$
Table 2.5: Optimal portfolio weights six periods ahead for the next to the last period (September, 2009), E-Z utility: $\beta = 0.996$, $\gamma = 5$, $\rho = -0.5$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.1910</td>
<td>0.9107</td>
<td>0.9968</td>
<td>-0.0172</td>
<td>0.6414</td>
</tr>
<tr>
<td>2</td>
<td>-0.2276</td>
<td>0.8716</td>
<td>0.7108</td>
<td>-0.0002</td>
<td>0.5520</td>
</tr>
<tr>
<td>3</td>
<td>-0.1751</td>
<td>0.7601</td>
<td>0.5895</td>
<td>0.0208</td>
<td>0.4487</td>
</tr>
<tr>
<td>4</td>
<td>-0.1439</td>
<td>0.6947</td>
<td>0.5392</td>
<td>0.0123</td>
<td>0.3573</td>
</tr>
<tr>
<td>5</td>
<td>-0.1210</td>
<td>0.6327</td>
<td>0.4910</td>
<td>0.0204</td>
<td>0.3310</td>
</tr>
<tr>
<td>6</td>
<td>-0.1116</td>
<td>0.6014</td>
<td>0.4761</td>
<td>0.0088</td>
<td>0.2790</td>
</tr>
</tbody>
</table>

Table 2.5 presents optimal portfolio weights for six-period ahead of optimal portfolio choice for a less risk-averse investor than the previous one ($\gamma = 5$) and EIS that is less than 1, given the information available till the period before the last of the out-of-the-sample observations. This table summarizes the results of the curve before the last in each graph of figure 2.5. We find that the investor will short sell CITI from about 11% to 22% of his savings. The investor holds from 60% to 91% of his wealth in IBM. The investor will be holding from 47% to 99% of GE and hold from -1.7% to 2% of MS. The investor will hold from 27% to 64% of BAA.
Table 2.6: Optimal portfolio weights six periods ahead for the last period (October, 2009), EZ utility: $\beta = 0.996$, $\gamma = 5$, $\rho = -0.5$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1922</td>
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</tr>
<tr>
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<td>0.1660</td>
</tr>
<tr>
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<td>0.1593</td>
</tr>
<tr>
<td>4</td>
<td>-0.0598</td>
<td>0.6938</td>
<td>0.5532</td>
<td>0.0265</td>
<td>0.1682</td>
</tr>
<tr>
<td>5</td>
<td>-0.0594</td>
<td>0.6469</td>
<td>0.5107</td>
<td>0.0228</td>
<td>0.1647</td>
</tr>
<tr>
<td>6</td>
<td>-0.0633</td>
<td>0.5940</td>
<td>0.4870</td>
<td>0.0326</td>
<td>0.1526</td>
</tr>
</tbody>
</table>

Table 2.6 presents the six-period optimal portfolio choices for the more risk-averse investor with an EIS of less than 1, given the information available till the last observation in our out-of-sample observations. This summarizes the last curve in each graph of Figure 3. We find that the investor is short selling CITI from 5.9% to 6.6% in all periods except in the first period where she holds a positive amount of her savings in CITI. The investor holds from 59% to 94% of her wealth in IBM. The investor will be holding from 48% to 88% of GE and hold from 2.9% to 4.7% of MS. The investor will short 4.9% of BAA in the first period and have a positive holdings of BAA in all other periods ranging from 15.2% to 16.6%.
The previous demonstrated results show that the investor choose substantially different portfolio over time and over horizon. We observe several interesting results for the optimal portfolio weights over time, over the different values for the risk aversion and finally over the 6 time periods. For four out of the five assets (IBM, General Electric (GE), Microsoft (MS) and Bank of America corporation (BAA)), we observe that the optimal portfolio weights become more positive or less negative with low values of relative risk aversion $\gamma = 5$ in comparison when we allow $\gamma$ to increase to 10.

Our results for the four assets (IBM, GE, MS and BAA) change slightly when we change the value of $\rho$ that reflects the Elasticity of Intertemporal Substitution (EIS) given $\gamma$. We observe a considerable change in the results of the fourth asset MS when we change the value of $\rho$. An investor with a high value of $\gamma = 10$ refers to a highly conservative investor who would be expected to hold few risky assets as he will not be willing to buy risky assets for its risk premium, however for the first asset CITI we observe that his holdings in the second till the fourth period become positive. Thus, the investor here is holding more risky assets than expected, because he is hedging against changes in investment opportunity set. To be specific, the investor hedges against changes in the investment opportunity set meaning that "for a highly conservative investor, he holds the risky asset only if it covaries with declines in interest rates, compensating the portfolio for the reduction in income that occurs when the interest rate falls" (Campbell and Viceira (2002)).
2.3.2 Habit-Formation utility

In this section we present the multi-period optimal portfolio choice of an infinitely-lived investor with a habit formation (HF) utility function. We follow the same steps that we used to estimate the optimal portfolio choice for the EZ utility function, the only difference is that we evaluate the Euler equation defined in equation (2.6) and use it to define our error term that is used in GMM.

We examine two different scenarios that reflect different values of the calibrated parameters, \((b_1)\) the habit persistence parameter and the concavity parameter \((\gamma)\) which are taken from the literature, Ferson and Constantinides (1991).

Figure 2.4: Optimal Weights HF utility Multi-period: \(\beta = 0.996\), \(\gamma = 1.98\), \(b = -0.361\)
Figure 2.4 above represents the multi-period optimal portfolio choice for an infinitely lived investor with HF utility, knowing information until August 2009 and finding the optimal portfolio weights from September 2009 and for six periods ahead. The optimal portfolio allocation changes with the habit persistence parameter $b_1$ and the concavity parameter $\gamma$. A negative $b_1$ indicates that the investor receives less pleasure from a given amount of consumption, and the larger must be the new purchases to gain the same benefit.

Table 2.7: Optimal portfolio weights six periods ahead for the next to last period (September, 2009), HF: $\beta = 0.996, \gamma = 1.98, b = -0.361$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</tr>
<tr>
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</tbody>
</table>

We find that for the period before last in the previous figure, when $\beta = 0.996, \gamma = 1.98, b_1 = -0.361$ the investor is holding between 14.1% and 17.8% of his wealth in CITI. He will choose to hold between 17.2% and 18.2% of his wealth in IBM, and between 16.4% and 17.6% in GE. Finally, the investor will hold between 18.6% and 22.4% of his saving in MS and 16% to 17.8% in BAA.
Table 2.8: Optimal portfolio weights six periods ahead for the last period (October, 2009), HF: $\beta = 0.996$, $\gamma = 1.98$, $b = -0.361$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1472</td>
<td>0.1621</td>
<td>0.1788</td>
<td>0.1850</td>
<td>0.1560</td>
</tr>
<tr>
<td>2</td>
<td>0.1640</td>
<td>0.1809</td>
<td>0.1636</td>
<td>0.2178</td>
<td>0.1729</td>
</tr>
<tr>
<td>3</td>
<td>0.1753</td>
<td>0.1796</td>
<td>0.1671</td>
<td>0.2224</td>
<td>0.1767</td>
</tr>
<tr>
<td>4</td>
<td>0.1720</td>
<td>0.1775</td>
<td>0.1589</td>
<td>0.2167</td>
<td>0.1681</td>
</tr>
<tr>
<td>5</td>
<td>0.1698</td>
<td>0.1749</td>
<td>0.1614</td>
<td>0.2175</td>
<td>0.1691</td>
</tr>
<tr>
<td>6</td>
<td>0.1658</td>
<td>0.1756</td>
<td>0.1605</td>
<td>0.2151</td>
<td>0.1669</td>
</tr>
</tbody>
</table>

In table 2.8 we find that for the last period in the previous figure, when $\beta = 0.996$, $\gamma = 1.98$, $b_1 = -0.361$, the investor is holding between 14.7% and 17.5% of his wealth in CITI. He will choose to hold between 16% and 18.1% of his wealth in IBM, and between 16% and 17.8% in GE. Finally, the investor will hold between 18.5% and 22.2% of his saving in MS and 15.6% to 16.9% in BAA.
Figure 2.5: Optimal Weights HF utility Multi-period: $b_1 = -0.717$, $\gamma = 8$
Table 2.9: Optimal portfolio weights six periods ahead for the next to last period (September, 2009), HF: $b_1 = -0.717, \gamma = 8$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1439</td>
<td>0.1701</td>
<td>0.1693</td>
<td>0.1821</td>
<td>0.1581</td>
</tr>
<tr>
<td>2</td>
<td>0.1724</td>
<td>0.1801</td>
<td>0.1604</td>
<td>0.2188</td>
<td>0.1727</td>
</tr>
<tr>
<td>3</td>
<td>0.1753</td>
<td>0.1844</td>
<td>0.1500</td>
<td>0.2121</td>
<td>0.1747</td>
</tr>
<tr>
<td>4</td>
<td>0.1755</td>
<td>0.1807</td>
<td>0.1469</td>
<td>0.2095</td>
<td>0.1767</td>
</tr>
<tr>
<td>5</td>
<td>0.1772</td>
<td>0.1803</td>
<td>0.1438</td>
<td>0.2055</td>
<td>0.1742</td>
</tr>
<tr>
<td>6</td>
<td>0.1705</td>
<td>0.1806</td>
<td>0.1447</td>
<td>0.2053</td>
<td>0.1752</td>
</tr>
</tbody>
</table>

The investor in the next to last period of Table 2.9 and for six periods ahead, where $b_1 = -0.717, \gamma = 8$ is holding between 14.3% and 17.7% of his wealth in CITI. He will choose to hold between 17% and 18.4% of his wealth in IBM, and between 14.4% and 16.9% in GE. Finally, the investor will hold between 18.2% and 21.8% of his saving in MS and 15.8% to 17.7% in BAA.

Table 2.10: Optimal portfolio weights six periods ahead for the last period (October, 2009), HF: $b_1 = -0.717, \gamma = 8, \beta = 0.996$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1541</td>
<td>0.1618</td>
<td>0.1650</td>
<td>0.1703</td>
<td>0.1539</td>
</tr>
<tr>
<td>2</td>
<td>0.1605</td>
<td>0.1703</td>
<td>0.1487</td>
<td>0.1857</td>
<td>0.1573</td>
</tr>
<tr>
<td>3</td>
<td>0.1701</td>
<td>0.1785</td>
<td>0.1439</td>
<td>0.2025</td>
<td>0.1705</td>
</tr>
<tr>
<td>4</td>
<td>0.1684</td>
<td>0.1782</td>
<td>0.1454</td>
<td>0.2013</td>
<td>0.1655</td>
</tr>
<tr>
<td>5</td>
<td>0.1701</td>
<td>0.1782</td>
<td>0.1464</td>
<td>0.2039</td>
<td>0.1660</td>
</tr>
<tr>
<td>6</td>
<td>0.1727</td>
<td>0.1778</td>
<td>0.1434</td>
<td>0.2022</td>
<td>0.1696</td>
</tr>
</tbody>
</table>

The investor in the last period of Table 2.9 and for six periods ahead, where $b_1 =
$\gamma = 8$ is holding between 15.4% and 17.2% of his wealth in CITI. He will choose to hold between 16% and 17.8% of his wealth in IBM, and between 14.3% and 16.5% in GE. Finally, the investor will hold between 17% and 20% of his saving in MS and 15.3% to 16.9% in BAA.

In general we find that the weights of the EZ utility function shows considerable variation across time and cross different values of the calibrated parameters when compared with HF utility function that its results show a stable trend and less variability. For example the optimal portfolio weights for EZ utility ranges between -29% to 73% in the case of $\gamma = 5$ and $\rho = 0.5$, and between -25% and 48% for the case of $\gamma = 10$ and $\rho = 0.5$, with while the HF utility’s optimal portfolio weights over the different values of $\gamma$ and $b_1$ range from 14% and 24%.
2.4 Conclusion

We were able to compute the multi-period optimal portfolio choice for a representative investor with Epstein-Zin or habit formation preferences taking into account changes in the investment opportunity set described by the state variable (lagged returns). We used Monte Carlo simulation to draw future returns and consumption growth after specifying the data generating process using Vector Auto regression and CCC-ARCH models.

The Epstein-Zin results shows variation in the optimal portfolio weights over time and over different values of the calibrated parameters, relative risk aversion ($\gamma$) and the Elasticity of Intertemporal Substitution ($\frac{1}{1-\rho}$). There is a tendency for portfolio weights to decrease with an increase in the risk aversion represented by $\gamma$.

The multi-period optimal portfolio choice for an investor with habit formation utility shows less variation over time and over different values of the calibrated parameters: the habit persistence parameter ($b_1$) and the approximate relative risk aversion ($\gamma$) when compared with the behavior of the EZ utility when we change its calibrated parameters.
Chapter 3

Portfolio Choice with Consumption Growth: Multi-period

3.1 Summary

A recent development in asset pricing and portfolio selection is the Bansal and Yaron (2004) Long-Run Risk (LRR) model. This model was developed to explain several stylized facts and puzzles in the finance literature. The model is centered around time varying consumption volatility, which is considered a channel for economic uncertainty. Bansal and Yaron (2004) includes time varying consumption volatility and growth in the Stochastic Discount Factor (SDF) and shows their impact on the risk premium demanded by investors. Bansal et al. (2007a), Bansal et al. (2009), and Guofu Zhou (2009) extend Bansal and Yaron (2004). Beeler and Campbell (2009), Malloy et al. (2009) assess the ability of the LRR model to explain anomalous financial market facts.

In this chapter we extend our work in the first two chapters by using the Bansal and Yaron (2004) model. We incorporate consumption growth in our model, and examine its
impact on optimal portfolio selection. In addition we include a heteroskedasticity and a moving average term to our specification of that data generating process of the future consumption growth and expected returns. Our model takes into account the findings of Bansal et al. (2007a), and Bansal et al. (2009) that a rise in consumption growth increases expectations about future cash flows and increases asset valuation. In our model we expect to see an impact of changing consumption growth on portfolio choice. We use a GARCH-In-Mean model, which incorporates time variation in consumption growth.

The rest of the paper is organized as follow. Section 2 presents a review of the Long-Run Risk model. Section 3 presents our model and methodology. Section 4 analyzes the empirical findings, and Section 5 concludes.

3.2 Literature

Recent literature attempts to explain changes in asset prices through changes in consumption volatility and expected growth. Specifically economic uncertainty is described by the conditional volatility of consumption, which influences stock valuation. This relation has been studied by Bansal and Yaron (2004), Bansal et al. (2009), and Beeler and Campbell (2009). The LRR model suggests that a rise in economic uncertainty leads to a fall in asset prices, and that asset prices help predict future economic uncertainty.

The literature models variation in \((p_t - y_t)\) (the price-dividend ratio used by Campbell and Shiller (1988)) as a function of variation in expected cash flow growth \((g_{yt+j})\) and/or variation in expected asset returns \((r_{t+j})\).

\[
p_t - y_t = k_0 + E_t \sum_{j=1}^{\infty} (k_{1j})[g_{yt+j} - r_{t+j}] \quad \text{and}
\]

\[
\text{var}(p_t - y_t) = (k_{1j})[\text{cov}(g_{yt+j}, p_T - y_t) - \text{cov}(r_{t+j}, p_t - y_t)]
\]

\[
k_{1j}\frac{\text{cov}(g_{yt+j}, p_T - y_t)}{\text{var}(p_T - y_t)}
\]

is the fraction of the variation in \((p_t - y_t)\) that is related to fluctu-
ating expected growth, and \( \frac{k^2 \text{cov}(r_{t+j}, p_t - y_t)}{\text{var}(p_t - y_t)} \) is the fraction that is related to variation in expected returns.

Previously the literature analyzed variation in \((p_t - y_t)\) by fixing the growth of cash flow. This treatment caused expected returns to vary with risk aversion (e.g., Campbell and Viceria (1999), and Bansal et al. (2002)). The more recent literature examines the impact of changes in economic uncertainty on asset valuation through the introduction of a time-varying consumption volatility channel. For example, Bansal and Yaron (2004) stressed the importance of cash-flow news, which works through persistent variation in the growth rate and volatility of consumption. For example consumption growth increases investors’ expectation of future cash flows and hence increases stock prices. The following section presents the literature on the LRR model, first we present Bansal et al. (2004, 2007) contributions to the literature, then we present part of the literature that assesses the LRR model.

Bansal et al. (2002), and Bansal and Yaron (2004) developed the Long-Run Risk model (LRR) which explains stock price variation as a result of a persistent fluctuations in the mean and volatility of aggregate consumption. They examine the impact of economic uncertainty and changes in expected growth rates of consumption and dividends on asset valuation. Their model includes a small persistent component in consumption growth and consumption volatility that can affect stock prices. They model stock returns as having two long-run risk channels: the first emerges from long-run fluctuations in expected growth, and the second emerge from long-run fluctuations in consumption volatility. In their model consumption and dividend growth rates contain a small persistent expected growth rate component. Consumption growth drives up stock prices by increasing investors’ expectations of future cash flows, while fluctuating consumption volatility captures time-varying economic uncertainty. They showed that their model is
able to account for asset market features, such as a low risk free rate, and equity premium puzzles.

The main idea behind their LRR model is that the degree of persistence in the expected growth rate news affects the volatility of the price-dividend ratio and also determines the risk premium on the asset. Specifically, they find that fluctuating economic uncertainty (conditional volatility of consumption) directly affects price-dividend ratios, and a rise in economic uncertainty leads to a fall in asset prices.

Bansal and Yaron (2004) use an Epstein-Zin (EZ) utility representation to show that shocks to expected growth are long-lasting and hence affect expected growth and the volatility computation for long-horizon investors. They model the dynamics of consumption and dividends as follows:

\begin{align*}
g_{c,t+1} & = \mu_c + x_t + \sigma_t \eta_{t+1} \quad (3.1) \\
x_{t+1} & = \rho x_t + \psi \sigma_t e_{t+1} \quad (3.2) \\
\sigma_{t+1}^2 & = \bar{\sigma}^2 + \nu (\sigma_t^2 - \bar{\sigma}^2) + \sigma_\omega \omega_{t+1} \quad (3.3) \\
g_{d,t+1} & = \mu_d + \phi x_t + \pi \sigma_t \eta_{t+1} + \psi \sigma_t \mu_{d,t+1} \quad (3.4)
\end{align*}

where $g_{c,t+1}$ and $g_{d,t+1}$ are the growth rate in consumption and dividends respectively. $x_t$ is the small predictable component of the long-risk in consumption growth. $e_{t+1} \quad w_{t+1}, \quad \eta_{t+1}$ are i.i.d. shocks. The parameter $\rho$ determines the persistence in the conditional mean of consumption growth. $\sigma_t$ is a common time-varying volatility in consumption.

The equation for the equity premium has two sources of systematic risk. The first relates to fluctuations in expected consumption growth and the second relates to fluctuations in consumption volatility. The equity premium in the presence of time-varying economic uncertainty is given by:
\[ E_t(r_{m,t+1} - r_{f,t}) = \beta_{m,e} \lambda_{m,e} \sigma_t^2 + \beta_{m,w} \lambda_{m,w} \sigma_w^2 - 0.5 \text{var}_t(r_{m,t+1}) \]

The market compensation for bearing stochastic volatility risk in consumption is determined by \( \lambda_{m,w} \), and the time-varying risk premium on the market portfolio \( \sigma_t \), fluctuates. \( \text{var}_t(r_{m,t+1}) = \beta_{m,e}^2 \sigma_t^2 + \psi_\sigma^2 \sigma_t^2 + \beta_{m,w}^2 \sigma_w^2 \). The exposure of market return to expected growth rate news is \( \beta_{m,e} \), and the price of expected growth risk depends on \( \lambda_{m,e} \).

Asset prices in the LRR model are determined by expectations of future growth and volatility, where changes in the expectation lead to movement in current price-dividend ratios. Bansal and Yaron (2004) claim that an increase in economic uncertainty raises risk premia and lowers the market price-dividend ratio. They calibrate the model at the monthly frequency. They estimated the elasticity of intertemporal substitution (EIS) and find it to be larger than one. They point out that an IES larger than 1 is critical for capturing the negative correlation between price-dividend ratios and consumption volatility.

For the empirical application they set the risk aversion parameter to 10 and EIS to 1.5. An EIS larger than one implies that the substitution effect dominates the income effect, which means that increases in future consumption growth increase future dividends which raise stock prices. The substitution effect dominates the effect of an increase in the interest rate that decreases stock prices. Moreover, as indicated in the LRR model, with an IES bigger than \( 1/\gamma \), the inverse of relative risk-aversion, an asset that pays off when there is an upward revision in expected consumption growth is risky and commands a premium. Bansal and Yaron (2004) suggest that future prices increase with expected future consumption growth and fall with consumption volatility. Thus, they found a negative relation between consumption volatility and current price-dividend ratios.

Bansal et al. (2007b) evaluate the LRR model and develop a method to estimate the
model. They use a model of a representative agent having Epstein-Zin preferences and use the Generalized Method of Moments in the estimation. They use the same dynamics for consumption and dividends as Bansal and Yaron (2004). The main difference between the models of Bansal et al. (2007b) and Bansal and Yaron (2004) is that the former allows for correlation between consumption and dividend growth, thus allowing an i.i.d. shock \((\eta)\) to influence the dividend process and, thus, be an additional source of risk beside the short-run and the long-run risk of consumption and the risk from consumption volatility. Bansal et al. (2007b) show that the LRR model is able to price the time series and cross section of returns.

The risk premium is given by:

\[
E_t[(r_{j,t+1} - r_{f,t}) + 0.5\sigma_{t,r}^2] = \sum_{i=\{\eta,e,w\}} \beta_{i,j}^t \lambda_i \sigma_{i,t}^2
\]

where \(\beta_{i,j}^t\) is the beta with respect to the three sources of risk: the short-run risk \((\sigma_{t,\eta_{t+1}})\), the long-run risk \((\sigma_{t,e_{t+1}})\), and the volatility risk \((\sigma_{w,w_{t+1}})\).

For the empirical application Bansal et al. (2007b) calibrated the model using monthly data. They derive time-aggregated annual growth rates of consumption and dividends to match key aspects of annual aggregate consumption and dividend data. They use six asset returns: the risk-free rate, the market return, and the returns on value, growth, large-firm and small-firm portfolios. They find that the long-run risk is persistent and is predicted by the price-dividend ratio, which accounts for most of the risk premia (about 50%), while the short-run and volatility risks account for the rest. Moreover they find that assets with large mean returns react more to innovation in the long-run risk and economic uncertainty (volatility of consumption).

Bansal et al. (2007a) use the Bansal and Yaron (2004) model of long-run risk and the improved model solution of Bansal et al. (2007b) to evaluate the plausibility of predicted returns and cash flows, by comparing data with LRR model predictions. Bansal et al.
(2007a) assess the predictability of cash flow growth and hence its impact on asset prices. They also test whether the LRR model is able to generate finite sample properties that are consistent with empirical findings about returns, cash flows and consumption growth.

The findings of Bansal et al. (2007a) center on the existence of return predictability that emerges as a result of time-varying risk premia and time-varying volatility of consumption and cash flows. They find that consumption growth is driven by a small persistence component "x_t". They argue that the LRR model explains and matches the low level of predictability of returns and consumption growth.

For the empirical application Bansal et al. (2007a) use annual data on consumption, and returns from the NYSE, AMEX, and NASDAQ from 1930 to 2006. Their model assumes a complete market and a representative agent having Epstein-Zin preferences. They take the consumption and dividends dynamics from Bansal and Yaron (2004). They use improved approximate analytical solutions for the LRR model. They conjecture the price to consumption ratio to be: $z_t = A_0 + A_1 x_t + A_2 \sigma_t^2$. They solve for $z_t$, which they assume to be endogenous, and then derive the innovation to the return to wealth and the innovation to the IMRS. Then they compute the risk premia for the different assets. They specify the risk premium on the market portfolio to be:

$$E_t[r_{m,t+1}r_{f,t} + 0.5 \sigma_{t,r_m}^2] = \beta_{\eta,m} \lambda_\eta \sigma_t^2 + \beta_{e,m} \lambda_e \sigma_t^2 + \beta_{w,m} \lambda_w \sigma_w^2$$

where $\beta_{\eta,m}$ is the beta of the market return with respect to the short-run risk ($\eta$), $\beta_{e,m}$ is the beta of market return with respect to the long-run risk innovation ($e_t$), and $\beta_{w,m}$ is the beta of the market return with respect to economic uncertainty risk ($w_t$). They find that their model is able to match data characteristics about consumption, dividends and asset returns. Moreover, they find that their model is able to generate a low risk-free rate observed in the data and reasonable volatility in the price-dividend ratio.

Bansal et al. (2007a) find that there is a weak evidence of predictability of consum-
tion growth when they use the log of the price-dividend ratio. The relation gets stronger when they add the real interest rate as one of the regressors. However, they find a very weak predictability of asset returns using the price-dividend ratio as the regressor, which may limit the use of price-dividend ratio to predict future returns.

Bansal et al. (2009) analyze their previous LRR model and contrast their results to that of the external habit model of Campell and Cochrane (1999). They calibrated the LRR model of Bansal and Yaron (2004) using the approximate analytical method of Bansal, Kiku, and Yaron (2007b). Their model has the same structure as the model of Bansal and Yaron (2004). They assume all shocks are i.i.d normal and are orthogonal to each other. They specify the same dynamics for consumption and dividend growth, but add one more dynamic by allowing the i.i.d consumption shock $\eta_{t+1}$ to be correlated with the dividend process and thus adding a new channel of risk. They use an approximate analytical solution to solve for asset prices’ dynamics. Their results suggest that there is a small long-run predictable component in consumption growth and that consumption volatility is time-varying.

Bansal et al. (2009) used annual data that runs from 1929 to 2008. The assets analyzed are NYSE, AMEX, and NASDAQ from CRSP, and they used the 3-month Treasury bills adjusted for inflation as the risk-free asset. They show empirically that consumption growth is predictable. They used a vector-autoregression that is based on consumption growth, the price-dividend ratio, and the real risk-free rate. The long-run risk premium is realized when investors have a preference for early resolution of uncertainty, that is when $\gamma$ is larger than the reciprocal of IES. They find that 75% of the risk premium arises from the long-run growth and volatility risks. To be specific they found that the long-run growth risk affects mostly the level of risk premiums whereas the volatility contributes more to the variability of asset prices.
Finally, Bansal et al. (2009) compare their results for the LRR model with results for a habit formation utility function. Specifically the Campbell and Viceria (1999) backward-looking model. Bansal et al. (2009) find that the forward-looking LRR model accounts for the key dynamic properties of asset-market data. The basic difference between the two models is that in the LRR model current price-dividend ratios are determined by time-varying expected growth and consumption volatility, hence a change in the current price-dividend ratio reduce future expected growth and increases future volatility, while in the habit formation model the shock of habit is driven by lagged consumption growth, and a reduction in growth rates raises risk-aversion, the equity premium and the discount rate reducing the price-dividend ratio. The authors argue that the data better support their forward-looking model in which the price-dividend ratio can predict future volatility in consumption.

Beeler and Campbell (2009) also evaluate the LRR model of Bansal and Yaron (2004). They compare implications of Bansal and Yaron (2004) (the BY model), and Bansal et al. (2007b) (the BKY model) using quarterly data, annual data with monthly frequency and time-aggregated data. They use the same data that Bansal and Yaron (2004) use for 1930-2006. Beeler and Campbell (2009) find that the LRR model does match the consumption growth, dividend growth and stock returns but it understates the volatility of the log price-dividend ratio. They run several other tests of the BY and BKY model to test whether they match data moments. They test for the autocorrelation of consumption and dividend growth and find that the LRR model implies higher correlation than the data suggest.

Beeler and Campbell (2009) also show that asset-pricing properties of the LRR model change with the preference parameters of the representative agent. In particular they find that the Elasticity of Inter temporal Substitution (EIS) has the biggest effect on the
behavior of asset prices. Bansal et al. (2009) find that the LRR model has problems in calculating the real equilibrium interest rate when the EIS is low and the real interest rate is negative as the model includes precautionary savings which cause negative and infinitely-lived consumption or dividend claims with infinite price.

Beeler and Campbell (2009) also evaluate the relative importance of consumption and volatility shocks in the BY and BKY models. They find that the BKY model, which assumes a big role for the volatility of consumption produces a higher risk premium even when shocks to consumption growth are i.i.d, an outcome stemming from the BKY model’s assumption about the correlation between dividend and consumption growth. Thus in the BKY model most of the equity premium and variability in stock prices come from time-varying volatility.

Beeler and Campbell (2009) evaluate the LRR model by testing the ability of the log price-dividend ratio to predict long-run consumption and dividend growth. They regress the excess stock return, consumption growth and dividend growth on the log price-dividend ratio at the start of the measurement period. They find that asset prices are more responsive to lagged consumption growth than future consumption growth produced by the long-run model. They also find the BKY model requires extreme volatility to match the data.

Beeler and Campbell (2009) find that the BY model is consistent with the data in that the log-price dividend ratio predicts consumption volatility with a negative sign. They find weaker evidence that the BY model is consistent with the data in that the log price-dividend ratio can predict dividend or return volatility. Finally, they find that the BKY model matches the data in its ability to predict consumption and dividend volatility using asset prices.
3.3 Model and Methodology

Given the empirical evidence about the time varying consumption volatility and of the impact of consumption growth on asset prices (see Bansal et al. (2002), Campbell (2003)) and motivated by the work of Bansal and Yaron (2004) we extend our model to compute the optimal portfolio choice of an infinitely lived investor taking into consideration changes in the growth rate and volatility of consumption. Prior work finds that the fluctuation in expected growth and volatility of consumption are related to observable macroeconomic fluctuations and could help in interpreting market movements. Specifically, prior work suggests that consumption growth and volatility are sources of systematic risk and shocks to them carry separate risk premiums and, hence, should affect portfolio choice. In our model we include these fluctuations and examine their effects on optimal portfolio choice. In this context we examine the effect of consumption growth on portfolio selection.

We build our model based on a GARCH-In-Mean specification. We use the Euler equation derived from the optimal portfolio choice of an investor with EZ utility preference:

\[
E_{t+\tau} \left[ \beta^{1-\gamma} \left( \frac{C_{t+\tau}}{C_{t-1}} \right)^{(1-\gamma)\rho-1} \rho \left( 1 + R_{m,t+\tau} \right)^{(1-\gamma)\rho-1} \left( 1 + r_{i,t+\tau} \right) - 1 \right] = 0 \tag{3.5}
\]

where \( \tau \) is the number of periods ahead (1 to 6).

We also use the Euler equation for an investor with habit formation utility:

\[
E_{t+\tau} [ Y_{t+\tau} \left( (i + r_{t+\tau}) - iR_{m,t+\tau} \right) ] = 0 \tag{3.6}
\]
where \( Y_{t+\tau} = \left[ \tilde{C}_{t+\tau}^{-\gamma} + b_1 \beta [\tilde{C}_{t+\tau+1}] \right] \)

with

\[
\begin{align*}
\tilde{C}_{t+\tau}^{-\gamma} & = (C_{t+\tau} + b_1 C_{t+\tau-1})^{-\gamma}, \\
\tilde{C}_{t+\tau+1}^{-\gamma} & = (C_{t+\tau+1} + b_1 C_{t+\tau})^{-\gamma}
\end{align*}
\]

Again we are using the resulting Euler equation to find the optimal portfolio weights for \( \tau \) periods ahead.

We use GMM to estimate the Euler equations as in the previous chapters. The difference between this chapter and the previous chapters is in the specification of the data generating process \( f(y_{t+1}|I_t) \), and in the error terms. We model the returns \( r_t \) and consumption growth \( gc_t \) as follows:

\[
\begin{align*}
\begin{align*}
 r_{t+1} & = H_1 + \Phi_r r_t + \theta_r gc_t + v_r \sqrt{\sigma^2_{r,t}} + a_r \epsilon_{r,t} + \epsilon_{r,t+1} \\
 gc_{t+1} & = H_2 + \Phi_c gc_t + v_c \sqrt{\sigma^2_{c,t}} + a_c \epsilon_{c,t} + \epsilon_{c,t+1}
\end{align*}
\end{align*}
\]

\( H_1, H_2 \) and \( \Phi_r \), and \( \Phi_c \) are the VAR parameters, where \( \Phi_r \) is a diagonal matrix. In this model we add \( gc_t \), the growth rate of consumption, as an explanatory variable, since shocks to consumption growth are expected to affect the risk premium on an asset and, hence, the optimal portfolio weights. \( \theta_r \) captures the effect of consumption growth on returns, we expect to see a positive values for the vector \( \theta_r \). In addition we add a heteroskedasticity term and a moving average in equation 7 and 8.

We then employ a Constant Conditional Correlation Generalized AutoRegressive

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Conditional Heteroskedasticity (CCC-GARCH-in-Mean) model to estimate the variance-covariance matrix of the next period error term for the risky assets and consumption ($\epsilon_t$). The difference between the previous chapters and this one mainly pertains to the modeling of the error term here is based on the GARCH-In-Mean specification.

We then use the error term to obtain the GARCH parameters using Maximum Likelihood estimation. Then we employ these parameters to estimate the next period conditional variance of $\epsilon_{t+1}$, which we specify as:

$$
\sigma_{i,t+1}^2 = \omega_i + \alpha_i \epsilon_{i,t}^2 + \beta_i \sigma_t^2 \tag{3.9}
$$

where $\frac{\omega}{(1-\alpha_i-\beta_i)}$ is the unconditional variance for the innovation for series $i$, $\alpha_i$, $\beta_i$ are GARCH parameters. Then we compute the next-period variance-covariance matrix by employing a CCC model defined by the following equation:

$$
\Sigma_{t+1} = D_{t+1} \Gamma D_{t+1}
$$

where $D_{t+1}$ is the next-period standard deviation computed from the GARCH equation (3.9), $\Sigma_{t+1}$ is the variance-covariance matrix, and $\Gamma$ is the correlation matrix. The correlation matrix can be simply estimated using the sample variance-covariance of the standardized error terms:

$$
\hat{Q} = \frac{1}{T} \sum_{t=1}^{T} \left( \hat{D}_t^{-1} \hat{\epsilon}_t \right) \left( \hat{D}_t^{-1} \hat{\epsilon}_t \right)'
$$

$$
\hat{\Gamma} = R_t^{-1} \hat{Q} R_t^{-1} \tag{3.11}
$$

where $R_t$ is a diagonal matrix composed of the square-root of the elements on the diagonal.
of \( \dot{Q} \).

We then follow a methodology similar to the one outlined in the second chapter to compute the optimal portfolio weights. We also estimate the conditional expectation presented in the Euler equation for EZ utility and Habit formation utility using GMM (refer to equations (5) and (6)).

### 3.4 Empirical Results

Before we can estimate our models we must first calibrate the utility functions. The Epstein-Zin requires values for the risk aversion parameter \( \gamma \), the time discount factor \( \beta \), and the EIS \( \left( \frac{1}{1-\rho} \right) \). There is no consensus in the literature on the values of \( \gamma \) and EIS. In the first chapter we estimated these parameter using Generalized Method of Moments. We found that the \( (\gamma) \) equals 17, and that \( \rho \) equals -0.151.

The habit formation function requires values for a concavity parameter which is approximately equal to the relative risk aversion \( (\gamma) \), a habit persistence parameter \( (b_1) \), and a time discount factor \( (\beta) \). Again there is no consensus about the value of \( \gamma \) and \( b_1 \). In the first chapter we estimated these parameters using GMM. Our estimates of \( \gamma \) and \( b_1 \) were 5 and -0.038 respectively. In the empirical section we examine different scenarios for our model using different values of the parameters which are taken from the literature.

Table 3.1 presents the parameter estimates of the GARCH-in-Mean model. We emphasize three important parameters: the impact of consumption growth on asset returns \( (\theta_r) \), the impact of volatility of consumption growth on consumption growth \( (\nu_c) \) and the impact of volatility of asset returns on asset returns. In a consumption-based asset pricing model the aggregate stock market is treated as a claim on aggregate consumption. Expected future consumption growth increases both future dividends and real interest
rates, effects which pull stock prices in opposite directions. In cases where the EIS is
greater than one future consumption growth increases stock prices because the positive
impact on future dividends dominates the negative effect from an increase in real interest
rates. Thus we expect a positive effect of consumption growth on asset returns and a
positive values for $\theta_r$. Table 3.1 shows positive values of $\theta_r$ for all assets. We also expect
the consumption volatility parameter $\upsilon_c$ to be negative as increases in volatility has a
negative impact on future consumption growth. This expectation is born out by the $\upsilon_c$
estimates in Table 3.1 which is negative. While the literature on the impact of return
volatility is not settled yet, we observe a positive impact of return volatility on assets
return for all assets except for IBM which has a negative value for $\upsilon$.

Table 3.1: Estimates of the Model Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
<th>gc</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_r$</td>
<td>0.3474</td>
<td>0.0400</td>
<td>0.6969</td>
<td>1.2419</td>
<td>0.0971</td>
<td>0</td>
</tr>
<tr>
<td>$\upsilon$</td>
<td>0.0308</td>
<td>-0.3325</td>
<td>0.1513</td>
<td>0.1088</td>
<td>0.0126</td>
<td>-0.5864</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>0.8089</td>
<td>-0.7931</td>
<td>0.4729</td>
<td>-0.4937</td>
<td>0.8912</td>
<td>0.1003</td>
</tr>
<tr>
<td>$\upsilon_c$</td>
<td>-0.7902</td>
<td>0.7790</td>
<td>-0.4839</td>
<td>0.5121</td>
<td>-1.0000</td>
<td>-0.3007</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.0003</td>
<td>0.0001</td>
<td>0.0008</td>
<td>0.0003</td>
<td>0.0006</td>
<td>4.37E-05</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.1580</td>
<td>0.1626</td>
<td>0.1240</td>
<td>0.1196</td>
<td>0.4418</td>
<td>0.4235</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.8420</td>
<td>0.8366</td>
<td>0.7699</td>
<td>0.8516</td>
<td>0.5582</td>
<td>1.00E-10</td>
</tr>
<tr>
<td>$H$</td>
<td>-0.0029</td>
<td>0.0374</td>
<td>-0.0108</td>
<td>0.0115</td>
<td>-0.0002</td>
<td>0.0084</td>
</tr>
</tbody>
</table>
In the following section we demonstrate the empirical results of our model. We use monthly data for the five risky assets (IBM, GE, BAA, MS and CITI) chosen in chapters one and two. We use the three-month return on US Treasury bills as the risk-free rate. First we present and analyze EZ results then we present the habit formation results.

3.4.1 Epstein Zin

In this chapter we divide the data just as we did in the first two chapters. Specifically we divide the data into in-sample observations (observation 1 to 249 corresponding to May, 1986 till December, 2006) and out-of-sample observations (observation from 250 to 265, corresponding to January, 2006 till October, 2009). We examine different scenarios using different values for the calibrated parameters. Specifically we use values of the risk aversion measure $\gamma$ equal to 5 and 10, and use values of EIS greater than one. Figure 3.1 presents the optimal portfolio weights for the five risky assets over six periods from $t=250$ till $t=265$. In each period the risk-free weight is the residual of 1 minus the sum of the risky assets’ weights ($1 - \sum_{i=1}^{5} w_i$).

Table 3.2: Optimal portfolio weights six periods ahead of the next to last period (September, 2009): $\gamma = 5, \rho = 0.5, \beta = 0.996$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.3861</td>
<td>0.4966</td>
<td>-0.5589</td>
<td>-0.3535</td>
<td>-0.1801</td>
</tr>
<tr>
<td>2</td>
<td>-0.1999</td>
<td>-0.0585</td>
<td>-0.3127</td>
<td>-0.0604</td>
<td>-0.1750</td>
</tr>
<tr>
<td>3</td>
<td>-0.4123</td>
<td>0.3371</td>
<td>0.9398</td>
<td>0.2898</td>
<td>0.3349</td>
</tr>
<tr>
<td>4</td>
<td>-0.2085</td>
<td>0.5384</td>
<td>0.8682</td>
<td>0.4946</td>
<td>0.4061</td>
</tr>
<tr>
<td>5</td>
<td>-0.2753</td>
<td>0.2539</td>
<td>0.6216</td>
<td>0.1851</td>
<td>0.2727</td>
</tr>
<tr>
<td>6</td>
<td>-0.1166</td>
<td>0.6959</td>
<td>0.5913</td>
<td>0.4404</td>
<td>0.3004</td>
</tr>
</tbody>
</table>
Figure 3.1: Portfolio weights Epstein-Zin multi-period: $\gamma = 5$, $\rho = 0.5$, $\beta = 0.996$

Table 3.2 presents the optimal portfolio weights of the five risky assets knowing information until period 264 and computing the optimal portfolio weights for six periods ahead (i.e, from period 265 till period 270). For CITI we find that the investor short sale this asset from 38% in the first period to 11% in the sixth period. For IBM the optimal portfolio weights range from negative 5% in the second period to 53% in fourth period. Optimal weights range from negative 31% in the first period to 93% in the second period for GE. While for MS optimal weights range from negative 35% in the first period to 49% in the fourth period. Finally, the investor holds from -18% of BAA in the first period to 30% in the sixth period.

Table 3.3 presents the optimal portfolio weights for the five risky assets computed for periods 265 through 271 knowing information until period 265. An investor who has low risk-aversion ($\gamma = 5$) holds from -16.6% of second period savings in CITI to -1.4% of fifth period saving in CITI. The same investor holds from -21% to 64.7% of IBM in the second
Table 3.3: Optimal portfolio weights six periods ahead of the last period (October, 2009): $\gamma = 5, \rho = 0.5, \beta = 0.996$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.1494</td>
<td>0.1562</td>
<td>-0.2301</td>
<td>0.9140</td>
<td>-0.4672</td>
</tr>
<tr>
<td>2</td>
<td>-0.1669</td>
<td>-0.2142</td>
<td>-0.956</td>
<td>-0.1052</td>
<td>0.2005</td>
</tr>
<tr>
<td>3</td>
<td>-0.0362</td>
<td>0.6187</td>
<td>0.1362</td>
<td>0.5186</td>
<td>-0.0794</td>
</tr>
<tr>
<td>4</td>
<td>-0.3245</td>
<td>0.6475</td>
<td>0.2061</td>
<td>0.7103</td>
<td>0.4714</td>
</tr>
<tr>
<td>5</td>
<td>-0.0145</td>
<td>0.5838</td>
<td>0.3740</td>
<td>0.3929</td>
<td>0.0182</td>
</tr>
<tr>
<td>6</td>
<td>-0.1810</td>
<td>1.2212</td>
<td>0.3919</td>
<td>0.4675</td>
<td>0.3405</td>
</tr>
</tbody>
</table>

and fourth periods respectively. The same investor holds between -10.5% and 71% of his savings in GE, and between -10.5% and 91% in MS. Finally, the investor holds between -46% and 47% of his saving in BAA.
Figure 3.2: Portfolio weights Epstein-Zin Multi-period: $\gamma = 10$, $\rho = 0.5$, $\beta = 0.996$
Table 3.4: Optimal portfolio weights six periods ahead of the next to last period (September, 2009): $\gamma = 10$, $\rho = 0.5$, $\beta = 0.996$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.2503</td>
<td>0.4175</td>
<td>-0.0405</td>
<td>-0.2810</td>
<td>-0.4607</td>
</tr>
<tr>
<td>2</td>
<td>-0.4094</td>
<td>0.5738</td>
<td>0.9652</td>
<td>0.2729</td>
<td>0.5357</td>
</tr>
<tr>
<td>3</td>
<td>-0.2837</td>
<td>0.9628</td>
<td>0.5319</td>
<td>0.1388</td>
<td>0.1814</td>
</tr>
<tr>
<td>4</td>
<td>-0.1275</td>
<td>0.3374</td>
<td>0.5047</td>
<td>0.3012</td>
<td>0.2482</td>
</tr>
<tr>
<td>5</td>
<td>-0.1392</td>
<td>0.7514</td>
<td>0.3564</td>
<td>0.1021</td>
<td>0.1380</td>
</tr>
<tr>
<td>6</td>
<td>-0.0642</td>
<td>0.3860</td>
<td>0.3705</td>
<td>0.2623</td>
<td>0.1857</td>
</tr>
</tbody>
</table>

When we change the investor’s risk aversion measure from 5 to 10 we find that the investor invests less savings in the risky assets. Table 3.4 shows that the optimal portfolio weights for the six periods ahead (period 265 through 270) knowing information through period 264. Table 3.4 shows a general decline in the weights compared with table 2. For example the investor now holds from -40% to -6% of his savings in CITI, between 33% to 96% in IBM, and between -4% to 96% in GE, and between -28% and 30% in MS. The weights in BAA ranges between -40% and 53%.

Table 3.5: Optimal portfolio weights six periods ahead of the last period (October, 2009): $\gamma = 10$, $\rho = 0.5$, $\beta = 0.996$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.0804</td>
<td>0.6449</td>
<td>-0.1620</td>
<td>0.5350</td>
<td>-0.2405</td>
</tr>
<tr>
<td>2</td>
<td>-0.0910</td>
<td>0.0238</td>
<td>-1.2920</td>
<td>-0.1157</td>
<td>0.0533</td>
</tr>
<tr>
<td>3</td>
<td>0.2221</td>
<td>-0.7252</td>
<td>-0.7395</td>
<td>-0.2084</td>
<td>-0.4380</td>
</tr>
<tr>
<td>4</td>
<td>-0.1651</td>
<td>0.8945</td>
<td>0.1508</td>
<td>0.4431</td>
<td>0.3166</td>
</tr>
<tr>
<td>5</td>
<td>-0.0097</td>
<td>0.3548</td>
<td>0.2023</td>
<td>0.2023</td>
<td>0.0039</td>
</tr>
<tr>
<td>6</td>
<td>-0.1087</td>
<td>0.7505</td>
<td>0.1981</td>
<td>0.3265</td>
<td>0.2033</td>
</tr>
</tbody>
</table>
Raising the investors measure of risk aversion from 5 to 10 also reduces the savings in risky assets reported in Table 3.5 although the effect is less pronounce. Table 3.5 shows the optimal portfolio weights computed for periods 266 through 271 knowing information up to period 265. Compared with Table 3, Table 5 shows a general decline in weights for MS, and BAA. Specifically, the investors chooses to allocate between -16% and 22% of his savings to CITI, and between -72% to 89% for IBM. Investment in BAA is little affected with portfolio weights ranging between -24% of his wealth to 20%.

3.4.2 Habit Formation utility

In this section we present the multi-period optimal portfolio choice of an infinitely-lived investor with a Habit Formation (HF) utility function. We examine two different scenarios that reflect different values of the calibrated parameters: the habit persistence parameter $b$, and the concavity parameter $\gamma$. We examine the case where $\beta = 0.996$, $b_1 = -0.717$, $\gamma = 8$, and the case where $\beta = 0.996$, $\gamma = 1.98$, $b = -0.361$ (Ferson and Constantinides (1991)). The results of the optimal portfolio weights are presented in the following figures and tables, each figure consists of portfolio weights for six periods ahead for each of the five assets.

Table 3.6 presents the optimal portfolio choice of an investor with habit formation utility and with the following calibrated parameters: a time discount factor equal to 0.996, an approximate relative risk-aversion measure equal to 8, and a habit persistence parameter equal to $-0.717$. The investor optimally chooses the weights for the five assets given information up till period 264 and for 6 periods ahead. The investor chooses optimally to allocate between 14.6% and 20.7% of his savings to CITI, and between 16.2% and 20.5% to IBM. The investor directs 15.8% to 17.9% of his savings to GE and between 14.4%
Table 3.6: Optimal portfolio weights six periods ahead of the period next to last HF: \( \beta = 0.996, b_1 = -0.717, \gamma = 8 \)

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2021</td>
<td>0.1943</td>
<td>0.1791</td>
<td>0.1442</td>
<td>0.1850</td>
</tr>
<tr>
<td>2</td>
<td>0.1572</td>
<td>0.1629</td>
<td>0.1592</td>
<td>0.1593</td>
<td>0.1619</td>
</tr>
<tr>
<td>3</td>
<td>0.2082</td>
<td>0.2035</td>
<td>0.1705</td>
<td>0.1591</td>
<td>0.1911</td>
</tr>
<tr>
<td>4</td>
<td>0.1491</td>
<td>0.1732</td>
<td>0.1586</td>
<td>0.1596</td>
<td>0.1687</td>
</tr>
<tr>
<td>5</td>
<td>0.2079</td>
<td>0.2050</td>
<td>0.1645</td>
<td>0.1650</td>
<td>0.1915</td>
</tr>
<tr>
<td>6</td>
<td>0.1463</td>
<td>0.1785</td>
<td>0.1594</td>
<td>0.1606</td>
<td>0.1721</td>
</tr>
</tbody>
</table>

and 16.5% to MS, and between 16% and 19% to BAA.

Table 3.7: Optimal portfolio weights six periods ahead of the last period HF: \( \beta = 0.996, b_1 = -0.717, \gamma = 8 \)

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1983</td>
<td>0.1521</td>
<td>0.1524</td>
<td>0.1701</td>
<td>0.1849</td>
</tr>
<tr>
<td>2</td>
<td>0.1722</td>
<td>0.1612</td>
<td>0.1592</td>
<td>0.1603</td>
<td>0.1524</td>
</tr>
<tr>
<td>3</td>
<td>0.1932</td>
<td>0.1527</td>
<td>0.1566</td>
<td>0.1570</td>
<td>0.1783</td>
</tr>
<tr>
<td>4</td>
<td>0.1813</td>
<td>0.1614</td>
<td>0.1597</td>
<td>0.1649</td>
<td>0.1482</td>
</tr>
<tr>
<td>5</td>
<td>0.1914</td>
<td>0.1535</td>
<td>0.1581</td>
<td>0.1546</td>
<td>0.1735</td>
</tr>
<tr>
<td>6</td>
<td>0.1869</td>
<td>0.1610</td>
<td>0.1603</td>
<td>0.1674</td>
<td>0.1477</td>
</tr>
</tbody>
</table>

Table 3.7 presents the optimal portfolio choice of an investor with habit formation utility with the same calibration as the previous table, but knowing information until period 265 the investor chooses the optimal portfolio weights for the next six periods. The investor allocates between 17% and 19.8% of his savings to CITI, between 15.2% and 16% to IBM, and between 15.2% and 16% to GE. In addition, the investor allocates 15.4% of his wealth in the fifth period and 17% of his savings in the first period to MS.
while allocating between 14.7% and 18.4% to BAA.

Figure 3.3: Weight- Habit Formation Multi-period HF: $\beta = 0.996$, $\gamma = 1.98$, $b = -0.361$
Table 3.8: Optimal portfolio weights six periods ahead of the period next to last HF: \( \beta = 0.996, \gamma = 1.98, b = -0.361 \)

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2025</td>
<td>0.1941</td>
<td>0.1791</td>
<td>0.1443</td>
<td>0.1848</td>
</tr>
<tr>
<td>2</td>
<td>0.1572</td>
<td>0.1630</td>
<td>0.1591</td>
<td>0.1593</td>
<td>0.1619</td>
</tr>
<tr>
<td>3</td>
<td>0.2067</td>
<td>0.2047</td>
<td>0.1707</td>
<td>0.1589</td>
<td>0.1920</td>
</tr>
<tr>
<td>4</td>
<td>0.1493</td>
<td>0.1731</td>
<td>0.1586</td>
<td>0.1596</td>
<td>0.1685</td>
</tr>
<tr>
<td>5</td>
<td>0.2070</td>
<td>0.2056</td>
<td>0.1648</td>
<td>0.1647</td>
<td>0.1922</td>
</tr>
<tr>
<td>6</td>
<td>0.1474</td>
<td>0.1747</td>
<td>0.1594</td>
<td>0.1604</td>
<td>0.1700</td>
</tr>
</tbody>
</table>

Table 3.8 presents the optimal portfolio choice of an investor with habit formation utility with the following calibration: time discount factor equal to 0.996, an approximate relative risk-aversion measure equal to 1.98, and a habit persistence parameter equal to -0.361. The investor chooses to invest 14.7% of his wealth in the sixth period to 20.7% in the fifth period in CITI, and between 16.3% to 20.5% in IBM. Moreover, he directs between -8% to 24.6% of his savings in GE, between 15.9% and 17.9% in MS. Finally, the investor optimally allocates between 16.19% in the second period and 19.22% in the fifth period in BAA.
Table 3.9: Optimal portfolio weights six periods ahead of the last period HF: $\beta = 0.996$, $\gamma = 1.98$, $b = -0.361$

<table>
<thead>
<tr>
<th>Period ahead</th>
<th>CITI</th>
<th>IBM</th>
<th>GE</th>
<th>MS</th>
<th>BAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1985</td>
<td>0.1520</td>
<td>0.1525</td>
<td>0.1701</td>
<td>0.1850</td>
</tr>
<tr>
<td>2</td>
<td>0.1719</td>
<td>0.1612</td>
<td>0.1592</td>
<td>0.1603</td>
<td>0.1524</td>
</tr>
<tr>
<td>3</td>
<td>0.1938</td>
<td>0.1528</td>
<td>0.1566</td>
<td>0.1572</td>
<td>0.1788</td>
</tr>
<tr>
<td>4</td>
<td>0.1821</td>
<td>0.1616</td>
<td>0.1598</td>
<td>0.1649</td>
<td>0.1481</td>
</tr>
<tr>
<td>5</td>
<td>0.1905</td>
<td>0.1531</td>
<td>0.1582</td>
<td>0.1545</td>
<td>0.1737</td>
</tr>
<tr>
<td>6</td>
<td>0.1875</td>
<td>0.1611</td>
<td>0.1601</td>
<td>0.1673</td>
<td>0.1479</td>
</tr>
</tbody>
</table>

Table 3.9 presents the optimal portfolio choice of an investor with habit formation utility with the same calibrated parameters as in the previous table, but knowing information until period 265 the investor chooses the optimal portfolio weights for six periods ahead. The investor chooses to allocate between 17.2% and 19.8% of his wealth in CITI, between 15.3% and 16.2% in IBM, and between 15.2% and 15.9% in GE. In addition the investor directs between 15.4% and 17% of his savings to MS, and between 15.2% and 18.5% to BAA.
3.5 Conclusion

We computed the optimal portfolio weights of an investor who has either Epstein-Zin or Habit Formation preferences. We made use of Bansal and Yaron (2004) results which explain stock prices as a result of fluctuations in the growth rate and volatility of consumption. We incorporated consumption growth and volatility of consumption and returns and examined their effects on optimal portfolio weights. In addition we added a heteroskedastic and a moving average term to that describe the data generating process of the future consumption growth and the expected returns.

We found that consumption growth and volatility of returns affect expected return positively. We computed the portfolio weights for six-periods ahead knowing information up till the current period, where we made use of Monte-Carlo simulation to compute expected returns and consumption growth. The optimal portfolio weights for the Epstein-Zin preference show considerable variation over time and over different calibrated values of the relative risk aversion parameter.

In addition, we computed the multi-period optimal portfolio choice for a representative investor who has a Habit Formation preferences. The optimal portfolio weights show less variability across different time horizons and across different calibrated habit persistent parameters and the approximation of relative risk aversion when compared with the EZ results.

Finally, we compared the portfolio returns of the risky assets that resulted from the optimization processes in the second and third chapters. We find that when an investor takes into consideration the changes in the volatility of returns and consumption growth when choosing the optimal portfolio weights, the return on the resulting portfolio return is generally higher than when he ignores return volatility and consumption growth when
choosing his portfolio.
Conclusion

We computed the optimal portfolio weights of an investor who has either Epstein-Zin or Habit Formation preferences. In the first and second essay, we examined the optimal portfolio choices of an infinitely lived investor who invest in N risky assets and one risk free asset and optimally chooses the portfolio weights for one period and multi-periods. We were able to compute the optimal portfolio choices for this investor. The optimal portfolio choices over the single period and the multi-period showed considerable variations in the optimal portfolio weights across time and across different parameterization of the model.

Finally, in the third essay we made use of Bansal and Yaron (2004) results which explain stock prices as a result of fluctuations in the growth rate and volatility of consumption. We incorporated consumption growth and volatility of consumption and returns and examined their effects on optimal portfolio weights. In addition we added a heteroskedastic and a moving average term to that describe the data generating process of the future consumption growth and the expected returns. And we found an impact of the introduction of these variables on the investor’s optimal portfolio choices. The optimal portfolio weights show variability across time and across different parametrization of the mode. However, for the habit format utility less variability were shown across different time horizons and across different calibrated habit persistent parameters and the approximation of relative risk aversion when compared with the EZ results.
REFERENCES


Appendix A

EZ optimization

As noted in section 3.1, the EZ problem can be represented as follows. The investor wants to maximize the following recursive structure of the utility function:

$$U_t = [C_t^\rho + \beta E_t(U_{t+1}^{1-\gamma})^{\frac{\rho}{1-\gamma}}]^{\frac{1}{\rho}}$$ \hspace{1cm} (A.1)

subject to the following budget constraint

$$a_{t+1} = (a_t - C_t)R_{m,t+1}$$ \hspace{1cm} (A.2)

where $\beta$ is time discount factor, $\frac{1}{1-\rho}$ is the coefficient of elasticity of intertemporal substitution (EIS). The wealth at period $t+1$ is $a_{t+1}$, and $C_t$ is the consumption at time $t$. $R_{m,t+1}$ the portfolio return and $R_{m,t+1}$ is equal to

$$R_{m,t+1} = w_t'(i + r_{t+1}) + (1 - w_t'i)(1 + r_f)$$ \hspace{1cm} (A.3)
where \( r_f \) is return on risk free asset, \( w' \) is a vector of portfolio weights for each asset, \( r_{t+1} \) are returns on risky assets, and \( i \) is an \((N \times 1)\) vector of ones.

The above maximization problem can be represented using dynamic programming by the following Bellman equation:

\[
J(a_t, I_t) = \max_{\{C_t, w_t\}} \left[ C_t^\rho + \beta(E_tJ(a_{t+1}, I_{t+1}))^{1-\gamma} \right]^{\frac{1}{1-\gamma}}
\]

(A.4)

Since the certainty equivalent is homogenous with respect to wealth we can infer that \( J_t(a_t) = A_t a_t \). Thus,

\[
A_t a_t = \max_{\{C_t, w_t\}} \left[ C_t^\rho + \beta(a_t - C_t)^\rho \mu_t \left( A_{t+1}(w'_t(i + r_{t+1}) + (1 - w'_t i)(1 + r_f)) \right) \right]^{\frac{1}{\gamma}}
\]

(A.5)

where the certainty equivalent is \( \mu_t(v_{t+1}) = E_t[v_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}} \).

A first intermediate step is to show that at each point of time we have \( \frac{C_t}{a_t} = A_t^{\frac{\mu}{\mu-1}} \). To show this result start by taking the F.O.C of equation (A.5) with respect to \( C_t \): assuming we have the optimal portfolio \( w^*_C \) (giving the optimal certainty equivalent \( \mu^*_t \))

\[
\rho C_t^{\rho-1} - \beta \rho (a_t - C_t)^{\rho-1} \mu^\rho = 0
\]

(A.6)

Thus,

\[
C_t^{\rho-1} = \beta (a_t - C_t)^{\rho-1} \mu^\rho
\]

(A.7)

plugging equation (A.7) into (A.5), the value function is now equal to

\[
A_t a_t = \left[ C_t^\rho + \beta (a_t - C_t)^\rho \mu^\rho \right]^{\frac{1}{\gamma}}
\]

(A.8)

\[
= C_t^{\rho-1}(a_t - C_t)
\]

99
Therefore,

\[ A_t^\rho a_t^\rho = C_t^\rho + C_t^{\rho-1} a_t - C_t^\rho \]  

(A.9)

Thus,

\[ \frac{C_t}{a_t} = \frac{A_t^\rho}{A_t^{\rho-1}} \]  

(A.10)

Knowing this result, the certainty equivalent can then be expressed as:

\[ \mu^* = E_t[A_{t+1}^{1-\gamma}a_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}} \]  

(A.11)

\[ = E_t[A_{t+1}^{1-\gamma}(a_t-C_t)^{1-\gamma} \left\{ w'_t(i + r_{t+1}) + (1 - w'_t)i(1 + r_f)^{1-\gamma} \right\}]^{\frac{1}{1-\gamma}} \]  

(A.12)

\[ = R_m^* \]

Substituting equation (A.10) in equation (A.11), we get

\[ \mu^*_t = E_t \left[ \left( \frac{C_t^{\rho-1}}{[(a_t - C_t)(1 + R_{m,t+1}^*)]^{\rho-1}} \right)^{1-\gamma} (a_t - C_t)^{1-\gamma}(1 + R_{m,t+1}^*)^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \]  

(A.13)

Plugging \( \mu^* \) in equation (A.7) we have:

\[ C_t^{\rho-1} = \beta(a_t-C_t)^{\rho-1} E_t \left[ \left( \frac{C_t^{\rho-1}}{[(a_t - C_t)(1 + R_{m,t+1}^*)]^{\rho-1}} \right)^{1-\gamma} (a_t - C_t)^{1-\gamma}(1 + R_{m,t+1}^*)^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \]  

(A.14)

and,

\[ C_t^{\rho-1} = \beta(a_t-C_t)^{\rho-1} E_t \left[ \left( \frac{C_t^{\rho-1}}{[(1 + R_{m,t+1})]^{\rho-1}} \right)^{1-\gamma} (1 + R_{m,t+1}^*)^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \]  

(A.15)
Thus,

\[ 1 = \beta E_t \left[ \left( \frac{C_{t+1}}{C_t} \right)^{\frac{\rho - 1}{ho} (1 - \gamma)} (1 + R_{m,t+1})^{\frac{1 - \gamma}{\rho}} \right]^{1 - \gamma} \]  \hspace{1cm} (A.16)

Second, we maximize with respect to portfolio weights:

\[ \max_{w_t} \mu_t \left( A_{t+1} [w_t'(i + r_{t+1}) + (1 - w_t'i)(1 + r_f t + 1)] \right) \]  \hspace{1cm} (A.17)

\[ = \max_{w_t} E_t \left[ A_{t+1}^{1 - \gamma} [w_t'(i + r_{t+1}) + (1 - w_t'i)(1 + r_f t + 1)]^{1 - \gamma} \right]^{\frac{1}{1 - \gamma}} \]  \hspace{1cm} (A.18)

F.O.C:

\[ E_t \left[ A_{t+1}^{1 - \gamma} (1 - \gamma) [w_t'(i + r_{t+1}) + (1 - w_t'i)(1 + r_f t + 1)] \right] (i + r_{t+1}) - i(1 + r_f t + 1) = 0 \]  \hspace{1cm} (A.19)

\[ = 1 + R_{m,t+1}^* \]

\[ E_t \left[ A_{t+1}^{1 - \gamma} (1 + R_{m,t+1}^*)^{-\gamma} (r_{t+1} - ir f t + 1) \right] = 0 \]  \hspace{1cm} (A.20)

Since, \( A_{t+1} = \left( \frac{C_{t+1}}{a_{t+1}} \right)^{\frac{\rho - 1}{\rho}} \)

\[ = \left( \frac{C_{t+1}}{a_{t+1} C_t} \right) (1 + R_{m,t+1}^*)^{\frac{\rho - 1}{\rho}} \]

Thus,

\[ E_t \left[ C_{t+1}^{\frac{\rho - 1}{\rho} (1 - \gamma)} (1 + R_{m,t+1}^*)^{\frac{1 - \gamma}{\rho} - \gamma} (r_{t+1} - ir f t + 1) \right] = 0 \]  \hspace{1cm} (A.21)

\[ E_t \left[ C_{t+1}^{\frac{\rho - 1}{\rho} (1 - \gamma)} (1 + R_{m,t+1}^*)^{\frac{1 - \gamma}{\rho} - 1} (r_{t+1} - ir f t + 1) \right] = 0 \]  \hspace{1cm} (A.22)

\[ E_t \left[ C_{t+1}^{\frac{\rho - 1}{\rho} (1 - \gamma)} (1 + R_{m,t+1}^*)^{\frac{1 - \gamma}{\rho} - 1} (r_{t+1} - ir f t + 1) \right] = 0 \]  \hspace{1cm} (A.23)
Thus,

$$E_t \left[ \frac{C_{t+1}}{C_t} \frac{\bar{w} + 1}{\rho} (1 - \gamma) \left( 1 + R_{m,t}^* \right)^{\frac{1 - \gamma}{\rho} - 1} \left( (i + r_{t+1}) - (1 + r_{f,t+1}) \right) \right] = 0 \quad (A.24)$$

Multiply by $w'_t$

$$E_t \left[ \left( \frac{C_{t+1}}{C_t} \frac{\bar{w} + 1}{\rho} (1 - \gamma) \left( 1 + R_{m,t}^* \right)^{\frac{1 - \gamma}{\rho} - 1} (w'_t (i + r_{t+1}) - w'_t (1 + r_{f,t+1})) \right) = 0 \quad (A.25)$$

$$E_t \left[ \left( \frac{C_{t+1}}{C_t} \frac{\bar{w} + 1}{\rho} (1 - \gamma) \left( 1 + R_{m,t}^* \right)^{\frac{1 - \gamma}{\rho} - 1} \right) \right] \left( 1 - \gamma \rho \right) = E_t \left[ \left( \frac{C_{t+1}}{C_t} \frac{\bar{w} + 1}{\rho} (1 - \gamma) \left( 1 + R_{m,t}^* \right)^{\frac{1 - \gamma}{\rho} - 1} \right) \right] \left( 1 + r_{f,t+1} \right)$$

$$E_t \left[ \left( \frac{C_{t+1}}{C_t} \frac{\bar{w} + 1}{\rho} (1 - \gamma) \left( 1 + R_{m,t}^* \right)^{\frac{1 - \gamma}{\rho} - 1} \right) \right] \left( 1 - \gamma \rho \right) = \beta^{\frac{1 - \gamma}{\rho}} \left( 1 - \gamma \rho \right)$$

Which gives us,

$$\beta^{\frac{1 - \gamma}{\rho}} E_t \left[ \left( \frac{C_{t+1}}{C_t} \frac{\bar{w} + 1}{\rho} (1 - \gamma) \left( 1 + R_{m,t}^* \right)^{\frac{1 - \gamma}{\rho} - 1} \right) \right] = 1 \quad (A.27)$$

which leads us to the Euler equation no.8

$$E_t \left[ \beta^{\frac{1 - \gamma}{\rho}} \left( \frac{C_{t+1}}{C_t} \frac{\bar{w} + 1}{\rho} (1 - \gamma) \left( 1 + R_{m,t}^* \right)^{\frac{1 - \gamma}{\rho} - 1} \right) \left( 1 + r_{i,t+1} \right) - 1 \right] = 0$$
Appendix B

Habit Formation Optimization

The investor preference is represented by the following utility function:

\[(1 - \rho)^{-1} \sum_{t=0}^{\infty} \beta^t \frac{C_{t}^{1-\gamma}}{1-\gamma} \]  \hspace{1cm} (B.1)

where \(C_{t} = \sum_{\tau=0}^{\infty} b_{\tau} C_{t-\tau}\). Following Ferson and Constantinides (1991), we assume that \(b_0 = 1, b_1 \in \mathbb{R}, b_{\tau} = 0\) for \(\tau \geq 2\).

The investor maximizes the following utility function,

\[E_t \left[ \sum_{i=0}^{\infty} \beta^{i} \frac{(C_{t+i} + b_{1} C_{t-1+i})^{1-\gamma}}{1-\rho} \right],\]

subject to the following budget constraint,

\[a_{t+1} = (a_t - C_t) R_{m,t+1},\]
Which can be represented in the following Bellman equation:

\[ V(a_t, C_{t-1}) = \max_{c_t}(u(C_t, C_{t-1}) + \beta E_t(V(a_{t+1}, C_t))) \]  

(B.2)

The F.O.C w.r.t \( c_t \):

\[
u_1(C_t, C_{t-1}) + \beta E_t[V_1(a_{t+1}, C_t) \left( \frac{\partial a_{t+1}}{\partial c_t} \right) + V_2(a_{t+1}, C_t)] = 0 \]  

(B.3)

\[ = -R_{m,t+1} \]

where \( V_1 \) and \( V_2 \) are the partial derivatives of \( V \) with respect to the first argument of the function, and the second argument respectively, Equation B.3 implies an optimal \( c_t^* \).

For the optimal \( c_t \) we have

\[ V(a_t, C_{t-1}) = u(C_t, C_{t-1}) + \beta E_t[V(a_{t+1}, C_t)] \]  

(B.4)

We take the partial derivative of (B.2) with respect to \( a_t \)

\[ V_1(a_t, C_{t-1}) = u_1(C_t, C_{t-1}) \left( \frac{\partial c_t}{\partial a_t} \right) + \beta E_t[V_1(a_{t+1}, C_t) \left( \frac{\partial a_{t+1}}{\partial c_t} \right) \left( \frac{\partial c_t}{\partial a_t} \right) + V_2(a_{t+1}, C_t) \left( \frac{\partial a_{t+1}}{\partial a_t} \right) + \\
V_2(a_{t+1}, C_t) \left( \frac{\partial c_t}{\partial a_t} \right) \left( \frac{\partial c_t}{\partial a_t} \right) + \beta E_t[V_1(a_{t+1}, C_t)(R_{m,t+1})] \] \]

\[ = u_1(C_t, C_{t-1}) \left( \frac{\partial c_t}{\partial a_t} \right) + \beta E_t[V_1(a_{t+1}, C_t)(R_{m,t+1})] \]

(B.5)

We also take the partial derivative of B.5 with respect to \( C_{t-1} \)
\[ V_2(a_t, C_{t-1}) = u_1(C_t, C_{t-1}) \frac{\partial c_t}{\partial C_{t-1}} + u_2(C_t, C_{t-1}) + \beta E_t[V_1(a_{t+1}, C_t)(-R_{m,t+1}) \frac{\partial C_t}{\partial C_{t-1}} + V_2(a_{t+1}, C_t) \frac{\partial C_t}{\partial C_{t-1}}] \]
\[
= u_1(C_t, C_{t-1}) + \beta E_t[V_1(a_{t+1}, C_t)(-R_{m,t+1}) + V_2(a_{t+1}, C_t)] \frac{\partial c_t}{\partial C_{t-1}} + u_2(C_t, C_{t-1})
\]
\[
= 0 \text{ from (B.3)}
\]

\[ \implies V_2(a_t, C_{t-1}) = u_2(C_t, C_{t-1}) \quad (B.6) \]

We plug (B.5) and (B.6) into (B.3)

\[ u_1(C_t, C_{t-1}) - V_1(a_t, C_{t-1}) + \beta E_t[u_2(C_{t+1}, C_t)] = 0 \]

So we get

\[ V_1(a_t, C_{t-1}) = u_1(C_t, C_{t-1}) + \beta E_t[u_2(C_{t+1}, C_t)] \quad (B.7) \]

Using (B.7), we can replace \( V_1 \) in (B.3) with a combination of \( u_1 \) and \( u_2 \):

\[ u_1(C_t, C_{t-1}) + \beta E_t \left\{ u_1(C_{t+1}, C_t) + \beta E_{t+1}[u_2(C_{t+2}, C_{t+1})](-R_{m,t+1}) + u_2(C_{t+1}, C_t) \right\} = 0. \]

Substituting for the value of \( u_1(C_t, C_{t-1}), u_1(C_{t+1}, C_t), u_2(C_{t+2}, C_{t+1}), u_2(C_{t+1}, C_t) \)
gives us:

\[ (C_t + b_1 C_{t-1})^{-\gamma} = \beta E_t \left[ R_{m,t+1} \left( (C_{t+1} + b_1 C_t)^{-\gamma} + \beta E_{t+1}b_1 (C_{t+2} + b_1 C_{t+1})^{-\gamma} \right) - b_1 (C_{t+1} + b_1 C_t)^{-\gamma} \right] \]
\[
= C_t^{-\gamma} = C_{t+1}^{-\gamma} = C_{t+2}^{-\gamma} =
\]
\[ C_{t+1} \]
\[ C_t^{-\gamma} - \beta E_t \left[ (C_{t+1}^{-\gamma} + b_1 \beta E_{t+1}[C_{t+2}^{-\gamma}]) R_{m,t+1} \right] + b_1 \beta E_0[C_{t+1}]^{-\gamma} = 0 \quad (B.9) \]
where $\tilde{C}_t = C_t + b_1 C_{t-1}$, which gives:

$$
\tilde{C}_t^{-\gamma} = \beta E_t[\tilde{C}_{t+1}^{-\gamma}(R_{m,t+1} - b_1)] + \beta^2 E_t[b_1 R_{m,t+1} \tilde{C}_{t+2}^{-\gamma}]
$$

(B.10)

and can be rearranged to give the following equation

$$
E_t \left[ \sum_{i=1}^{2} \beta^i \left( \frac{\tilde{C}_{t+i}}{\tilde{C}_t} \right)^{-\gamma} (b_{i-1} R_{m,t+1} - b_i) - 1 \right] = 0
$$

(B.11)

with

$$
\begin{align*}
\tilde{C}_{t+1}^{-\gamma} &= (C_{t+1} + b_1 C_t)^{-\gamma}, \\
\tilde{C}_{t+2}^{-\gamma} &= (C_{t+2} + b_1 C_{t+1})^{-\gamma}, \\
\tilde{C}_t^{-\gamma} &= (C_t + b_1 C_{t-1})^{-\gamma}.
\end{align*}
$$

Next we take the F.O.C of equation (B.2) w.r.t. $w_t$

$$
\beta E_t[V_1(a_{t+1}, C_t) \frac{\partial a_{t+1}}{\partial w_t}] = 0
$$

(B.12)

$$
\beta E_t[V_1(a_{t+1}, C_t)(a_t - C_t) \frac{\partial R_{m,t+1}}{\partial w_t}] = 0
$$

(B.13)

$$
E_t[V_1(a_{t+1}, C_t) \frac{\partial R_{m,t+1}}{\partial w_t}] = 0
$$

(B.14)

$$
\beta E_t[V_1(a_{t+1}, C_t)(r_{t+1} - i r_f)] = 0
$$

(B.15)
Using equation (B.7), we get

\[ E_t \left[ \{ u_1(C_{t+1}, C_t) + \beta E_{t+1}(u_2(C_{t+2}, C_{t+1}) \} (r_{t+1} - ir_f) \} = 0 \]  \hspace{1cm} (B.16) 

\[ E_t[\{\tilde{C}_{t+1}^{-\gamma} + b_1 \beta E_{t+1}[\tilde{C}_{t+2}^{-\gamma}]\} (r_{t+1} - ir_f)]] = 0 \]  \hspace{1cm} (B.17)

We have \( R_{m,t+1} = w'_t(i + r_{t+1}) + (1 - w'_t) (1 + r_f) \)

Which we can rewrite \( R_{m,t+1} \) as follows:

\[
R_{m,t+1} = w'_t i + w'_t r_{t+1} + (1 - w'_t)i + (1 - w'_t) r_f \\
= w'_t i + w'_t r_{t+1} - w'_t r_f + (1 - w'_t) r_f \\
= w'_t r_{t+1} + (1 - w'_t) r_f + 1
\]

which gives:

\[ R_{m,t+1} = w'_t (r_{t+1} - ir_f) + (r_f + 1) \]  \hspace{1cm} (B.18)

We premultiply equation (B.17) by \( w'_t \)

\[ E_t[\{\tilde{C}_{t+1}^{-\gamma} + b_1 \beta [\tilde{C}_{t+2}^{-\gamma}]\} w'_t (r_{t+1} - ir_f)] = 0 \]  \hspace{1cm} (B.19)

Using equation (B.18) we can write equation (B.19) as follows:

\[ E_t[\{\tilde{C}_{t+1}^{-\gamma} + b_1 \beta [\tilde{C}_{t+2}^{-\gamma}]\} (R_{m,t+1} - (1 + r_f))] = 0 \]  \hspace{1cm} (B.20)

Let \( E_t \left[ \tilde{C}_{t+1}^{-\gamma} + b_1 \beta [\tilde{C}_{t+2}^{-\gamma}] \right] = Y_{t+1} \)

Therefore, we can write equation (B.20) as follows:

\[ E_t[Y_{t+1} R_{m,t+1} - Y_{t+1}(1 + r_f)] = 0 \]  \hspace{1cm} (B.21)
Thus we have,

\[ E_t[Y_{t+1}R_{m,t+1}] = E_t[Y_{t+1}](1 + r_f) \]  

(B.22)

We can rewrite equation (B.17) as follows:

\[ E_t[Y_{t+1}(r_{t+1} - ir_f)] = 0 \]  

(B.23)

\[ E_t[Y_{t+1}[(i + (r_{t+1} - [1 + r_f]i)] = 0 \]  

(B.24)

\[ E_t[Y_{t+1}(i + r_{t+1})] = E_t[Y_{t+1}](1 + r_f) i \]  

(B.25)

from equation (B.22)

\[ E_t[Y_{t+1}((i + r_{t+1}) - iR_{m,t+1})] = 0 \]  

(B.26)