ABSTRACT

MOHAMED, SAAD A. AHMED. A Lie Theoretic Approach to the Full Transformation Semigroup. (Under the direction of Mohan Putcha.)

The full transformation semigroup is the semigroup analogue of the symmetric group. Any semigroup is isomorphic to a semigroup of transformations . This semigroup arises naturally in automata theory, a branch of theoretical computer science .

The purpose of this thesis is to study the full transformation semigroup in a new way. We accomplish this by realizing that the full transformation semigroup is a subsemigroup of the monoid of all $n \times n$ matrices.This allows us to transfer Lie theoretic concepts to the full transformation semigroup. In particular we find analogues of Borel and parabolic subgroups, root elements and Chevalley's big cell.

A Lie Theoretic Approach to the Full Transformation Semigroup

by
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A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh, North Carolina

2012

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DEDICATION

To my parents.

BIOGRAPHY

Saad A. A. Mohamed was born in a small town called Aryiayna December 15, 1968 in Libya and grew up in the same city and still considered it home. He has attended schools in Libya until he received his master's degree in mathematics in 2000. Saad stared his work as a high school math teacher before he got his master's degree, then he became a member faculty of the mathematics department at University Of Aljable Algarbi . In 2007, Saad came to US to complete his study. He attended some language courses to improve his English at University of Pittsburgh for about one year. In 2008, he moved to Raleigh,NC for gradute school at North Carolina State University, where he received another master's degree in applied mathematics in 2010 and he completed his Ph.D in 2012.

ACKNOWLEDGEMENTS

First of all,I would like to thank Allah(God)for every thing. I would like to thank my advisor,Dr. Mohan Putcha, for his advice to me in my research and study,I am appreciative of his patience,support and guidance .I would like to thank my whole family (parents,wife, brothers and sisters)for their supplication and many years of support for me.

Also, thanks are extended to the members of my committee: Dr. Kailash Misra , Dr. Ernest Stitzinger , Dr. Thomas Lada and Dr. Jonathan Horowitz for my both Preliminary and Final exam.I would like to thank each faculty and staff member of Mathematics Department.

I do not forget to thank my Country Libya for giving me a chance to complete my study, and I wish a good life for all people in Libya.

Finally, a special thanks to my wife and kids(yes kids) for their patience for all my study period.

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Introduction

The full transformation semigroup $\mathcal{T}_n$ has been much studied, see for example [1],[2],[8],[9], [13] and[14]. In this thesis we find a new way to study the full transformation semigroup.

Since $\mathcal{T}_n$ is a submonoid of the monoid $M_n(F)$ of all $n \times n$ matrices over a field F , we can apply Lie theoretic aspects of the unit group $GL_n(F)$, see[4],[5]and[6],to $M_n(F)$, and then to $\mathcal{T}_n$.

We obtain the analogues $\mathcal{T}_n^+$ and $\mathcal{T}_n^-$ of Borel and opposite Borel subgroups and introduce the concept of root elements of $\mathcal{T}_n$.

We prove the following results:

1. We prove that $\mathcal{T}_n = S_n \mathcal{T}_n^+$, where S_n is the symmetric group.(Theorem 3.2.1)

2. We show that $\mathcal{T}_n^+$ is generated by the positive root elements and that this gives a presentation for $\mathcal{T}_n^+$.(Proposition 3.1.8 and Theorem 3.1.10)

3. We study the conjugates $\pi \mathcal{T}_n^+ \pi$ and their intersections,we find a precise formula for $|\mathcal{T}_n^+ \cap \pi \mathcal{T}_n^+ \pi^{-1}|$ (Theorem 3.3.4),and notice that this is not determined by $\ell(\pi)$,like it is for $GL_n(F)$.

4. By Schur's theorem any element of $M_n(F)$ is similar to an upper triangular matrix. The analogous result is not true for $\mathcal{T}_n$. We prove $\sigma \in \mathcal{T}_n$ is conjugate to an element of $\mathcal{T}_n^+$ if and only if $\sigma^n = \sigma^{n+1}$. (Theorem 3.4.2)

5. We study the analogue $\mathcal{T}_n^- \mathcal{T}_n^+$ of Chevalley's big cell $B^- B$ of $GL_n(F)$, we find that all idempotents are in $\mathcal{T}_n^- \mathcal{T}_n^+$ (Theorem 3.5.1), we also find necessary conditions for $\sigma \in \mathcal{T}_n$ to be in $\mathcal{T}_n^- \mathcal{T}_n^+$.(Theorem 3.5.3)

However the general problem of characterizing the elements of $\mathcal{T}_n^- \mathcal{T}_n^+$ remains open.

6. We define analogues P and P^- in $\mathcal{T}_n$ of opposite parabolic subgroups in $GL_n(F)$. We determine $|P|$ and $|P^-|$ (Theorem 3.6.1,Theorem 3.6.2) and find that unlike in $GL_n(F)$, P and P^- are usually not of the same size. We also determine the root element of P and P^-. (Theorem 3.6.5)

Chapter 1

Semigroups

We will start with some basic information about semigroups, ideals and how to classify them by Green's relations, after that we will concentrate on the full linear transformation semigroups . We know that a semigroup is a group with out the inverse and the identity conditions. Likewise we also know that a monoid is almost a group ,or a group missing only the inverse requirement. So the monoid is a semigroup with the identity, and the group is a monoid has inverse for each element.

1.1 Semigroups

A non-empty set S with associative binary operation (.) is called a Semigroup and denoted by $(S,.)$. If S is finite, then $|S|$ will denote to the number of elements of S . If S has an identity element 1 such that $x.1 = 1.x = x\ \forall x\ \in S$, in this case we call S a monoid . If S does not have an identity, always we can adjoin an element works as 1 , so $S^1 = S \cup \{1\}$ will be a monoid.

Also, if S does not have a zero element 0 such that $0.x = x.0 = 0\ \forall x\ \in S$, we can adjoin an element 0 to S to get $S^0 = S\cup\{0\}$ as a semigroup having zero.An element $s \in S$ is said to have an inverse if there exists an element $x \in S$ such that $s = sxs$ and $x = xsx$,if every element in S has a unique inverse,denoted x^{-1} ,then S is called an inverse semigroup.

The element $e \in S$ is called idempotent, if $e^2 = e$, will denote to the set of all idempotents elements of the semigroup S by $E(S)$,we will give the idempotent elements more attention, note that $E(S)$ is a subsemigruop of S .

If $H \subseteq S$, $H \neq \phi$, then H is a subsemigroup of S , if H is closed under the same binary operation.

If $A \subset S$, $A\emptyset\emptyset$, then the subsemigroup generated by A , is the smallest subsemigroup of S containing A , and denoted by $\langle A\rangle$, which consisting of all finite products $a_1 a_2 \ldots a_n$ of the elements of A , we call $\langle A\rangle$ cyclic subsemigroup if A has just one element.In general, we call S a cyclic semigroup, if it can generate by a single element. If A and B are non-empty subsets of a semigroup S , then we define AB as

$$\{ab|\ a \in A, b \in B\}\ .$$

An element $s \in S$ is said to be regular if there exist $x \in S$ such that $s = sxs$ and $x = xsx$. S is called a regular semigroup if every element of S is regular[1]and[2]. We note that if $sxs = s$ then $e = sx$ is an idempotent element of S such that $es = e$,

$$
\begin{aligned}
e^2 &= (sx)(sx) \\
&= (sxs)x \\
&= sx \\
&= e\ , \\
es &= sxs \\
&= s.
\end{aligned}
$$

Example 1.1.1. Let $S = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ be the set of the even integers numbers, then S is a semigroup with usual multiplication, which has zero 0 ,and does not have identity. So $S^1 = S \cup \{1\}$,and $E(S) = \{0\}$.

Example 1.1.2. Let $S = \{1, 2, 3, \ldots\}$, then S is an infinite semigroup with the usual multiplication , which has an identity but does not have a zero, so $(S, .)$ is a monoid. therefore $S^1 = S$, $S^0 = S \cup \{0\}$ and $E(S) = \{1\}$.

Example 1.1.3. For any set $S \neq \emptyset$ define the two semigroups $L(s) = (S, .)$ where $(.)$ is defined as $x.y = x\ \forall x, y \in S$,and $R(S) = (S, .)$ where $(.)$ is defined as $x.y = y\ \forall\ x, y \in S$, as long as S has more than one element, $L(S)$ and $R(S)$ are non-commutative semigroups without identity. $E(L(S)) = L(S)$ and $E(R(S)) = R(S)$.

Example 1.1.4. Let X be any set,and let 2^X be the set of all subsets of X , then $S_1 = (2^X, \cup)$ and $S_2 = (2^X, \cap)$ are semigroups with identities and zeros.For S_1 the identity is the empty subset and the zero is the set X , and S_2 has the set X as identity and the empty set as zero, $E(S_1) = S_1$ and $E(S_2) = S_2$.

Example 1.1.5. Let X be a set. Then $S = (2^{X \times X}, .)$ is the semigroup of all relations on X, where $(.)$ defined by $R_1.R_2 = \{(x, y) : \text{ for some } z \in X, (z, x) \in R_1, (z, y) \in R_2\}$ .If $R \in S$ let $R^{-1} = \{(y, x) : (x, y) \in R\}$, then $(R.T)^{-1} = T^{-1}.R^{-1}$ for all $R, T \in S$. The identity here is the relation $I \in S$, where $I = \{(x, x) : x \in X\}$ .So $S^1 = S$,hence S is a monoid.

Example 1.1.6. $Z^{(2 \times 2)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z \right\}$ is a semigroup with , the identity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the zero $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and many idempotents as $\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$.

1.2 Ideals, Homomorphisms And Green's Relations

A non-empty subset I of a semigroup S is called left ideal, if for all $i \in I$, $s \in S$, then $s.i \in I$,that is $SI \subseteq I$,also called right ideal, if for all $i \in I$, $s \in S$, then $i.s \in I$,that is $IS \subseteq I$ and I is called an ideal, if satisfies both conditions. Ideals,right and left ideals are subsemigroups.An ideal I of a semigroup S is called principal ideal if generated by one element $s \in S$, and will denoted by $I = \langle s \rangle$.

An ideal I of S is called a proper ideal,if it does not contain ideals of S other than 0 , and it does not contained in ideals of S other than S it self. A semigroup S is simple if it has no proper ideals.

If $I_1, I_2, \ldots, I_n$ are all the ideals of S , then $I_1 \ldots I_n = I_1 \cap \cdots \cap I_n$, this ideal is the unique minimal ideal of S ,it is called the kernel of S , and denoted by $K(S)$ .If a semigroup S has a zero, then $K(S) = 0$.

For any semigroup S the sets of all ideals, right ideals, and left ideals are closed under the operations of union and non-empty intersection.

Definition 1.2.1. Let S_1, S_2 be semigroups, then

$$\varphi : S_1 \to S_2 \text{ is a homomorphism iff } \varphi(x_1 x_2) = \varphi(x_1)\varphi(x_2) \; \forall x_1, x_2 \in S_1$$

We call φ monomorphism or $(1-1)$ homomorphism if φ is $(1-1)$ and we call φ epimorphism or onto homomorphism if φ is onto and here we call S_2 a homomorphic image of S_1 , also φ is isomorphism if it is $(1-1)$ and onto, in this case we say S_1 and S_2 are isomorphic.

Example 1.2.2. Let S_1, S_2 as defined in example 1.1.4 , and let $\varphi : S_1 \to S_2$ defined by $\varphi(A) = A^c$, where A^c is the complement of A .Then φ is an isomorphism, so S_1, S_2 are isomorphic.

Example 1.2.3. Let I be an ideal of a semigroup S , and let S/I be the quotient semigroup,which is defined to be $((S - I) \cup \{0\}, .)$, where $0 \notin (S - I)$ and

$$s_1 . s_2 = \begin{cases} s_1 s_2, & \text{if } s_1 s_2 \in (S - I) \\ 0 & \text{otherwise.} \end{cases}$$

The natural epimorphism $\eta : S \longrightarrow S/I$ is given by

$$\eta(s) = \begin{cases} s, & \text{if } s \in S - I \\ 0, & \text{if } s \in I. \end{cases}$$

then we note that $S/S = \{0\}$ and $S/\emptyset = S^0$.

Now we introduce some of the most commonly used ideas in semigroup theory, which are Green's relations. Green's relations were first studied by J.A.Green in [8],it defined by some relations as we will see in the following definitions.

Definition 1.2.4. Let S be any semigroup , and $s \in S$, then the principal ideals generated by s and Green's relations on S are defined as following :

1. $L(s) = S^1 s$ is the principal left ideal generated by s and

$$s_1 \mathcal{L} s_2 \iff L(s_1) = L(s_2).$$

2. $R(s) = sS^1$ is the principal right ideal generated by s and

$$s_1 \mathcal{R} s_2 \iff R(s_1) = R(s_2).$$

3. $J(s) = S^1 s S^1$ is the principal ideal generated by s and

$$s_1 \mathcal{J} s_2 \iff J(s_1) = J(s_2) .$$

4. $s_1 \mathcal{H} s_2 \iff s_1 \mathcal{L} s_2$ and $s_1 \mathcal{R} s_2$.

5. $s_1 \mathcal{D} s_2 \iff \exists s \in S$ such that $s_1 \mathcal{L} s$ and $s \mathcal{R} s_2$ or , equivalently, $\iff \exists t \in S$ such that $s_1 \mathcal{R} t$ and $t \mathcal{L} s_2$.

$\mathcal{L}$,$\mathcal{R}$,$\mathcal{J}$,$\mathcal{H}$, and $\mathcal{D}$ are equivalence relations on S , then we can talk about the equivalence classes for these relations. For example, let L_s be the $\mathcal{L}$ -class containing s ,that is the set of all elements of S which are $\mathcal{L}$ -equivalent to s , so

$$L_s = \{ t \in S : s \mathcal{L} t \} .$$

The same for the other relations .

Example 1.2.5. let $(S, .)$ be a semigroup, where $S = \{a, b, c\}$ and $(.)$ defined as in the following table: $E(S) = \{a, c\}$ and S has identity which a ,so $S = S^1$,then

.	a	b	c
a	a	b	c
b	b	a	c
c	c	b	c

$S^1a = \{a, b, c\}, S^1b = \{a, b\}$ and $S^1c = \{c\}$.

Therefore, $L_a = \{a\}$, $L_b = \{b\}$ and $L_c = \{c\}$.

Also, $aS^1 = \{a, b, c\}$, $bS^1 = \{a, b, c\}$ and $cS^1 = \{b, c\}$, then $R_a = \{a, b\} = R_b$ and $R_c = \{c\}$.

$J(a) = S^1aS^1 = S^1$ and $J(b) = S^1bS^1 = S^1$, so $J(a) = J(b) \Rightarrow a\mathcal{J}b.$

$J(c) = S^1cS^1 = \{b, c\}$. $H(a) = \{a\}$, $H(b) = \{b\}$ and $H(c) = \{c\}$.

From[4] we have the following theorem to know more about idempotents :

Theorem 1.2.6. *The following three conditions on a semigroup S are equivalent :*

(i) S is regular , and any two idempotent elements of S commute with other;

(ii) Every principal right ideal and every principal left ideal of S has a unique idempotent generator ;

(iii) S is an inverse semigroup (i.e., every element of S has a unique inverse in S).

The following properties can be found in [4]. For a semigroup S , the $\mathcal{J}, \mathcal{L}$ and $\mathcal{R}$ relations can be ordered by the following orderings;

1. $J_a \leq J_b$ iff $J(a) \subseteq J(b)$.

2. $R_a \leq R_b$ iff $R(a) \subseteq R(b)$.

3. $L_a \leq L_b$ iff $L(a) \subseteq L(b)$.

These orderings are reflexive , antisymmetric, and transitive. Also, we can note that:

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1. $\mathcal{R}$-classes and $\mathcal{L}$-classes are disjoint unions of $\mathcal{H}$ classes.

2. $\mathcal{J}$-classes are disjoint union of $\mathcal{L}$-classes and , also is disjoint union of $\mathcal{R}$-classes.Hence $\mathcal{J}$-classes are disjoint union of $\mathcal{H}$-classes.

3. Every $\mathcal{H}$-class is the intersection of an $\mathcal{L}$-class and $\mathcal{R}$-class.

4. The intersection of an $\mathcal{L}$-class and $\mathcal{R}$-class is either empty or is an $\mathcal{H}$-class.

It is obvious that $\mathcal{R} \subseteq \mathcal{J}$, $\mathcal{L} \subseteq \mathcal{J}$ and $\mathcal{D} \subseteq \mathcal{J}$, so the Figure(1.1) shows the relation between these classes on the semigroup S .

$$\mathcal{H} \quad \nearrow \quad \mathcal{L} \quad \searrow \quad \mathcal{D} \rightarrow \mathcal{J}$$
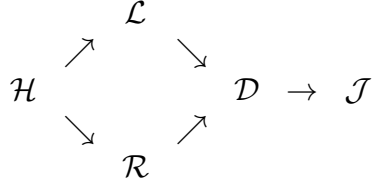$$\searrow \quad \mathcal{R} \quad \nearrow$$

Figure 1.1: Green's classes in the semigroup S

Note: For all finite semigroups S , $\mathcal{J} = \mathcal{D}$.

1.3 Full Transformation Semigroups.

For any non-empty set X . Let T_X be the set of all functions (or linear transformations from X to itself $f : X \rightarrow X$, then T_X with the operation of composition of functions is called full transformation semigroup.

For each $\alpha \in T_X$ we associate two things : the range of α which denoted X_α or $\alpha(X)$ and the equivalence relation on X denoted by π_α and defined by $x\pi_\alpha y$ if $\alpha(x) = \alpha(y)$. The equivalence classes X_{π_α} of X under this relation has the same number of elements as the range of α , $|X_{\pi_\alpha}| = |X_\alpha|$ and this number called the rank of α .

In this thesis we will focus on the semigroup of all functions from the set $\{1, 2, 3, \ldots, n\}$ to itself, which denoted by $\mathcal{T}_n$, as example of the full transformation semigroup. (i.e. $\mathcal{T}_n = \{\sigma : \sigma : \{1, 2, 3, \ldots, n\} \rightarrow \{1, 2, 3, \ldots, n\}, \sigma$ function or linear transformation),and in this case we have a few notations will be used to represent the elements in $\mathcal{T}_n$ as appropriate:

1. Two line notation, if $\sigma \in \mathcal{T}_n$,then we can write σ as

$$
\begin{pmatrix}
1 & 2 & \ldots & n \\
\sigma(1) & \sigma(2) & \ldots & \sigma(n)
\end{pmatrix}
$$

For example , $\sigma \in \mathcal{T}_5, \sigma(1) = 2, \sigma(2) = 2, \sigma(3) = 1, \sigma(4) = 3$ and $\sigma(5) = 2$, then

$$
\sigma =
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 2 & 1 & 3 & 2
\end{pmatrix}
$$

2. One line notation,since the first line in the two line notation is the same, so we can omit it and just write the second line to represent σ as

$$\sigma = \begin{pmatrix} \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$$

For example, we can write $\sigma \in \mathcal{T}_5$ which defined above as

$$\sigma = \begin{pmatrix} 2 & 2 & 1 & 3 & 2 \end{pmatrix}$$

3. Matrix notation , let $\sigma \in \mathcal{T}_n$, we can indicate $\sigma(j) = i$ by place a 1 in the $(i, j)-$entry of $n \times n$ matrix. For example, again we can write the $\sigma \in \mathcal{T}_5$ which defined above as

$$\sigma = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note: $|\mathcal{T}_n| = n^n$.

Example 1.3.1. Let S be the semigroup of all functions from the set $X = \{1, 2, 3\}$ to it self ,so $S = \{\sigma : \sigma : \{1, 2, 3\} \rightarrow \{1, 2, 3\}, \sigma$ function or linear transformation $\}$
$|S| = 3^3 = 27$ So

$$S = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \Big\}.$$

$$E(S) = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\}.$$

From [8]we have the following property with it's proof, to get the number of idempotents in the semigroup $\mathcal{T}_n$:

Property 1.3.2. The number e_n of the idempotents in the semigroup $\mathcal{T}_n$ equals to

$$e_n = \sum_{k=1}^{n} \binom{n}{k} k^{n-k}. \tag{1.1}$$

Proof. To define an idempotent α of rank k we have to choose a $k-$element set $im(\alpha)$ (this can be done in $\binom{n}{k}$ different ways), and then we have to define a mapping from $\{1, 2, \ldots n\} \setminus im(\alpha)$ to $im(\alpha)$ in an arbitrary way (this can be done in k^{n-k} different ways). Hence $\mathcal{T}_n$ contains exactly $\binom{n}{k} k^{n-k}$ idempotents of rank k . The statement is now obtained applying the sum rule. $\qquad\square$

Example 1.3.3. The number of the idempotents elements in $\mathcal{T}_3$ is

$$
e_3 = \sum_{k=1}^{3} \binom{3}{k} k^{3-k} = \binom{3}{1} + \binom{3}{2} + \binom{3}{3}
$$
$$
= 3 + 6 + 1
$$
$$
= 10.
$$

and the number of the idempotents elements in $\mathcal{T}_4$ is

$$
e_4 = \sum_{k=1}^{4} \binom{4}{k} k^{4-k} = \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}
$$
$$
= 4 + 24 + 12 + 1
$$
$$
= 41.
$$

From [8] we have the following theorems and corollaries (also the proofs can be found in[8]) which show some properties of the Green's relations on the full transformation semigroup $\mathcal{T}_n$:

Theorem 1.3.4. *Let* $S = \mathcal{T}_n$ *,for each* $\alpha \in \mathcal{T}_n$ *the right principal ideal generated by* α *has the following form:*

$$
\alpha S = \{\beta \in S : im(\beta) \subseteq im(\alpha)\} \ .
$$

Corollary 1.3.5. *1. Let* $S = \mathcal{T}_n$ *,and* $\alpha \in S$ *such that* $rank(\alpha) = k$ *, then*

$$
|\alpha S| = k^n \ .
$$

2. the semigroup $\mathcal{T}_n$ has exactly $2^n - 1$ different principal right ideals.

Theorem 1.3.6. *Let $S = \mathcal{T}_n$,for each $\alpha \in \mathcal{T}_n$ the left principal ideal generated by α has the following form:*

$$S\alpha = \{\beta \in S : dom(\alpha) \subseteq dom(\alpha) \quad and \quad \pi_\beta \subseteq \pi_\alpha\} \ .$$

where $(x\pi_\alpha y \Leftrightarrow \alpha(x) = \alpha(y))$.

Corollary 1.3.7.   *1. Let $S = \mathcal{T}_n$,and $\alpha \in S$ such that $rank(\alpha) = k$, then*

$$|S\alpha| = n^k \ .$$

2. the semigroup $\mathcal{T}_n$ has exactly

$$B_n = \sum_{k=1}^{n} S(n,k)$$

different principal left ideals (B_n is the number of unordered partitions of the set $\{1, 2, \ldots, n\}$ into disjoint unions of nonempty k blocks) .

Theorem 1.3.8. *Let $S = \mathcal{T}_n$,for each $\alpha \in \mathcal{T}_n$ the principal ideal generated by α has the following form:*

$$S\alpha S = \{\beta \in S : \ rank(\beta) \leq \ rank(\alpha)\} \ .$$

Corollary 1.3.9.   *1. Let $S = \mathcal{T}_n$,and $\alpha \in S$ such that rank(α)= k , then*

$$|S\alpha S| = \sum_{i=1}^{k} S(n,i)\frac{n!}{(n-i)!}$$

.

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2. the semigroup $\mathcal{T}_n$ contains n different principal ideals.

Lemma 1.3.10. If α, $\beta \in \mathcal{T}_x$,then:

1. $\alpha \mathcal{R} \beta \iff X_\alpha = X_\beta$.(α , β have the same range).

2. $\alpha \mathcal{L} \beta \iff X_{\pi_\alpha} = X_{\pi_\beta}$.(α , β have the same partition).

3. α and β are $\mathcal{D}-$equivalent $\iff$ They have the same rank.

Theorem 1.3.11. *In the finite full transformation semi group $\mathcal{T}_x$ we have:*

1. $\mathcal{D} = \mathcal{J}$.

2. *There is a one-to-one correspondence between the set of all principal ideals of $\mathcal{T}_x$ and the set of all cardinal numbers $r \leq |x|$ such that the principal ideal corresponding to r consists of all elements of $\mathcal{T}_x$ of rank $\leq r$.*

3. *There is a one-to-one correspondence between the set of all $\mathcal{D}-$classes of $\mathcal{T}_x$ and the set of all cardinal numbers $\leq r$ such that the $\mathcal{D}-$class $\mathcal{D}_r$ corresponding to r consists of all elements of $\mathcal{T}_x$ of rank r .*

4. *Let r be a cardinal number $\leq |X|$, there is a one-to-one correspondence between the set of all $\mathcal{L}-$classes in $\mathcal{D}_r$ and the set of all subsets Y of X of cardinal r such that the $\mathcal{L}-$class corresponding to Y consists of all elements of $\mathcal{T}_x$ having range Y .*

5. *Let r be a cardinal number $\leq |X|$, there is a one-to-one correspondence between the set of all $\mathcal{R}-$classes in $\mathcal{D}_r$ and the set of all partitions X_{π_α} of X for which $|X_{\pi_\alpha}| = r$ such that the $\mathcal{R}-$class corresponding to X_{π_α} consists of all elements of $\mathcal{T}_x$ having partition X_{π_α}.*

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6. *Let r be a cardinal number $\leq |X|$, there is a one-to-one correspondence between the set of all $\mathcal{H}-$classes in $\mathcal{D}_r$ and the set of all pairs (X_{π_α}, Y) , where X_{π_α} is a partition of X and Y is a subset of X such that $|X_{\pi_\alpha}| = |Y| = r$,such that the $\mathcal{H}-$class corresponding to (X_{π_α}, Y) consists of all elements of $\mathcal{T}_x$ having partition X_{π_α} and range Y.*

In other words, for $\mathcal{T}_n$, let $\sigma, \beta \in \mathcal{T}_n$, then: $\alpha \mathcal{R} \beta \iff$ they have the same range and $\alpha \mathcal{L} \beta \iff$ they have the same fibers (a fiber of a map $f : X \to Y$ is the set $f^{-1}(y) = \{x \in X : f(x) = y\}$). $\mathcal{R}-$ classes are in $(1-1)$ correspondence with subsets of the set $\{1, 2, \ldots, n\}$ and $\mathcal{L}$ -classes are in $(1-1)$ correspondence with partitions of the set $\{1, 2, \ldots, n\}$.

Let J be a $\mathcal{J}-$class of a semigroup S . Let $R_1, \ldots, R_m$ be the $\mathcal{R}-$classes in J , and let $L_1, \ldots, L_n$ be the $\mathcal{L}-$classes in J . Then, the $\mathcal{H}-$classes in J are exactly $\{\mathcal{H} = R_i \cap L_j : i = 1, \ldots, m; j = 1, \ldots, n\}$. So the following table shows a picture of $J-$classes , each row an $\mathcal{R}-$class , each column an $\mathcal{L}-$class, and the intersection of each of row and column an $\mathcal{H}-$class.

Table 1.1:   Eggbox of $\mathcal{R}, \mathcal{L}, and\ \mathcal{H} - Classes\ in\mathcal{T}_n$

R_1	H_{11}					
R_i		H_{ij}				
R_m					H_{mn}	
	L_1	L_j			L_n	

Lets show these relations on $\mathcal{T}_3$, we will use first line notation form $(a_1a_2a_3)$ as defined before in the beginning of section (1.3), for the mapping $1 \rightarrow a_1, 2 \rightarrow a_2$,and $3 \rightarrow a_3$,i.e, $\begin{pmatrix} 1 & 2 & 3 \\ a_1 & a_2 & a_3 \end{pmatrix}$, $a_i \in \{1, 2, 3\}$, $i = 1, 2, 3$ just to save space in the tables.

Example 1.3.12. Let $X = \{1, 2, 3\}$, then $\mathcal{T}_X = \mathcal{T}_3 = \{(123), (132), (213), (231), (312),$ $(321), (111), (112), (113), (221), (222), (223), (331), (332), (333), (122), (133), (211), (233),$ $(311), (322), (121), (131), (212), (232), (313), (323)\}$.

Next table shows $\mathcal{R}, \mathcal{L}, \mathcal{D} = \mathcal{J}\ and\ \mathcal{H} - Classes$ for $\mathcal{T}_3$.

Table 1.2: $\mathcal{R}, \mathcal{L}, and \ \mathcal{H}$-Classes of $\mathcal{T}_3$

X/α \ X_α	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
{1,2,3}	(111)*	(222)*	(333)*				
{1} , {2,3}				(122)* (211)	(133)* (311)	(233) (322)	
{2}, {1,3}				(121)* (212)	(131) (313)	(232) (323)*	
{3}, {1,2}				(112) (221)	(113)* (331)	(223)* (332)	
{1}, {2}, {3}							(123)*(312) (132)(231) (213)(321)

Each row except the first row in the table(Range of α X_α) is $\mathcal{R}$ $-$class , each column except the first column(Partition of X/α) is $\mathcal{L}$ $-$class and the intersections are $\mathcal{H}-$classes. The $\mathcal{D}_r = \mathcal{J}_r - classes$, $(r = 1, 2, 3)$ where $\mathcal{J}_r$ is the set of all elements in $\mathcal{T}_3$ of rank r , which : $\mathcal{J}_1 = \{(111), (222), (333)\}$,

$\mathcal{J}_2 = \{(122), (211), (133), (311), (233), (322), (121), (212), (131), (313), (232), (323), (112),$
$(221), (113), (331), (223), (332)\}$ and $\mathcal{J}_3 = \{(123), (312), (132), (231), (213), (321)\}$.

Thus , $\mathcal{T}_3$ has the following eggbox structure :

Table 1.3: Eggbox Diagram for $\mathcal{T}_3$

1				
1				
1				
1				
	2	2	2	
	2	2	2	
	2	2	2	
				6

From the eggbox , we can see that $|\mathcal{T}_3| = 3(1) + 9(2) + 6(1) = 27$.

Now ,let see all of those properties in $\mathcal{T}_4$, for that we will use the following example from[1].

Example 1.3.13. Let $X = \{1, 2, 3, 4\}$ then $\mathcal{T}_X = \mathcal{T}_4$ and $|\mathcal{T}_4| = 4^4 = 256$ elements, here we have the unit group S_4 which we denoted by $J_4(\mathcal{R}_4 = \mathcal{L}_4 = \mathcal{H}_4)$, also we have the maps of rank 3 which we denoted by J_3 . Then we have the maps of rank 2 which formed $\mathcal{J}-$class J_2 , finely we have the set of functions of rank 1 , $\mathcal{J}-$ class J_1 , and these $\mathcal{J}-$classes are ordered linearly as shown in the following diagram:

$$J_1$$

$$|$$

$$J_2$$

$$|$$

$$J_3$$

$$|$$

$$J_4$$

Figure 1.2: $\mathcal{J}$-classes for $\mathcal{T}_4$

The next tables show that what each $\mathcal{J}-$class looks like. As the previous example, here , there are four $\mathcal{J}-$classes $J_r(r = 1, 2, 3, 4)$, where J_r is the set of all functions of rank r. The headings for the rows are partitions of the set X, and for the columns are the ranges of the functions. Starred elements are idempotents.

Table 1.4: J_1-class in $\mathcal{T}_4$

J_1	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
$\{1, 2, 3, 4\}$	$(1111)^*$	$(2222)^*$	$(3333)^*$	$(4444)^*$

Table 1.5: J_2 Class for $\mathcal{T}_4$

J_2	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
$\{1\},\{2,3,4\}$	(1222)*	(1333)*	(1444)*	(2333)	(2444)	(3444)
	(2111)	(3111)	(4111)	(3222)	(4222)	(4333)
$\{2\},\{1,3,4\}$	(1211)*	(1311)	(1411)	(2322)	(2422)	(3433)
	(2122)	(3133)	(4144)	(3233)*	(4244)*	(4344)
$\{3\},\{1,2,4\}$	(1121)*	(1131)*	(1141)	(2232)*	(2242)	(3343)
	(2212)	(3313)	(4414)	(3323)	(2424)	(4434)*
$\{4\},\{1,2,3\}$	(1112)	(1113)	(1114)*	(2223)	(2224)*	(3334)*
	(2221)	(3331)	(4441)	(3332)	(4442)	(4443)
$\{1,2\},\{3,4\}$	(1122)	(1133)*	(1144)*	(2233)*	(2244)*	(3344)
	(2211)	(3311)	(4411)	(3322)	(4422)	(4433)
$\{1,3\},\{2,4\}$	(1212)*	(1313)	(1414)*	(2233)	(2424)	(3434)*
	(2121)	(3131)	(4141)	(3232)*	(4242)	(4343)
$\{2,3\},\{1,4\}$	(1221)*	(1331)*	(1441)	(2332)	(2442)	(3443)
	(2112)	(3113)	(4114)	(3223)	(4224)*	(4334)*

Table 1.6: J_3 Class for $\mathcal{T}_4$

J_3	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$
$\{1\},\{2\},\{3,4\}$	$(1233)^*(2133)$ $(2311)(3211)$ $(3122)(1322)$	$(1244)^*(2144)$ $(2411)(4211)$ $(4122)(1422)$	$(1344)(3144)$ $(3411)(4311)$ $(4133)(1433)$	$(2344)(3244)$ $(3422)(4322)$ $(4233)(2433)$
$\{1\},\{3\},\{24\}$	$(1232)^*(2131)$ $(2313)(2312)$ $(3121)(1323)$	$(1242)(2141)$ $(2414)(4212)$ $(4121)(1424)$	$(1343)(3141)$ $(3414)(4313)$ $(4131)(1434)^*$	$(2343)(3242)$ $(3424)(4323)$ $(4232)(2434)$
$\{1\},\{4\},\{2,3\}$	$(1223)(2113)$ $(2331)(3221)$ $(3112)(1332)$	$(1224)^*(2114)$ $(2441)(4221)$ $(4112)(1442)$	$(1334)^*(3114)$ $(3441)(4331)$ $(4331)(1443)$	$(2334)(3224)$ $(3442)(4332)$ $(4223)(2443)$
$\{2\},\{3\},\{1,4\}$	$(1231)^*(2132)$ $(2312)(3213)$ $(3123)(1321)$	$(1241)(2142)$ $(2412)(4214)$ $(4124)(1421)$	$(1341)(3143)$ $(3413)(4314)$ $(4134)(1431)$	$(2342)(3243)$ $(3423)(4324)$ $(4234)^*(2432)$
$\{2\},\{4\},\{1,3\}$	$(1213)(2123)$ $(2321)(3231)$ $(3132)(1312)$	$(1214)^*(2124)$ $(2421)(4241)$ $(4142)(1412)$	$(1314)(3134)$ $(3431)(4341)$ $(4143)(1413)$	$(2324)(3234)^*$ $(3432)(4342)$ $(4243)(2423)$
$\{3\},\{4\},\{1,2\}$	$(1123)(2213)$ $(2231)(3321)$ $(3312)(1132)$	$(1124)(2214)$ $(2241)(4421)$ $(4412)(1142)$	$(1134)^*(3314)$ $(3341)(4431)$ $(4413)(1143)$	$(2234)^*(3324)$ $(3342)(4432)$ $(4423)(2243)$

Table 1.7: Eggbox Diagram for $\mathcal{T}_4$

J_4 $\boxed{24}$ $1.1.24 = 24$

6	6	6	6
6	6	6	6
6	6	6	6
6	6	6	6
6	6	6	6
6	6	6	6

J_3 $6.4.6 = 144$

2	2	2	2	2	2	2
2	2	2	2	2	2	2
2	2	2	2	2	2	2
2	2	2	2	2	2	2
2	2	2	2	2	2	2
2	2	2	2	2	2	2

J_2 $7.6.2 = 84$

J_1 $\boxed{1\ |\ 1\ |\ 1\ |\ 1}$ $1.4.1 = 4$

                     $Total$ $4^4 = 256$

.

1.4 Connection with Automata Theory

In automata theory [3] and [12], full transformation semigroups arise naturally. One consider a finite set of states Q and a finite input alphabet X, and a next state function $\delta : X \times Q \longrightarrow Q$. Thus each $x \in X$ gives rise to a function $\delta_x : Q \longrightarrow Q$, given by $\delta_x(q) = \delta(x, q)$. Given a string of inputs $x_1, x_2, \ldots, x_m$ (not necessarily distinct), the effect of this string of inputs is given by $\delta_{x_n} \circ \delta_{x_{n-1}} \circ \cdots \circ \delta_{x_1}$. Thus if $Q = \{1, 2, \ldots, n\}$ then each $\delta_x \in \mathcal{T}_n$ and studying strings of inputs is the same as studying the subsemigroup of $\mathcal{T}_n$ generated by $\delta_x, x \in X$.

Example 1.4.1. For example let $X = \{x, y\}$ *and* $Q = \{1, 2, 3\}$ .If the next state function $\delta : X \times Q \longrightarrow Q$ is given by

$$(x, 1) \longrightarrow 2$$
$$(x, 2) \longrightarrow 2$$
$$(x, 3) \longrightarrow 3$$
$$(y, 1) \longrightarrow 3$$
$$(y, 2) \longrightarrow 3$$
$$(y, 3) \longrightarrow 1$$

Then $\delta_x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}, \delta_y = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix} \in \mathcal{T}_3$.

1.5 $M_n(F)$ Monoid of $n \times n$ Matrices

If F is a field, let $M_n(F)$ denote multiplication monoid of all $n \times n$ matrices over F , also called a linear semigroup. $M_n(F)$ contains many idempotent elements such as $e = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where I_r is the $r \times r$ identity matrix. The subset of $M_n(F)$ of all

invertible $n \times n$ matrices under the matrix multiplication is a group, called the general linear group,which denoted by $GL_n(F)$ .Also, the subset

$$SL_n(F) = \{A \in GL_n(F) \mid det(A) = 1\} \ .$$

with the matrix multiplication,called the special linear group $SL_n(F)$.

If F is algebraically closed field , then

$$dim M_n(F) = n^2 \ .$$

If $F = \mathbb{F}_q$,the finite field with q elements, then

$$|M_n(F)| = q^{n^2} \ .$$

It will be important for us to realize $\mathcal{T}_n$ as a semigroup of matrices . If $\sigma \in \mathbf{T}_n$, let $A(\sigma)$ be the matrix with 1 in $(\sigma(i), i)^{th}$ position and 0 elsewhere. So each column of $A(\sigma)$ has exact one non-zero entry that is 1 , and this will be the same as the matrix notation which defined in section (1.3).

Example 1.5.1. Let $\sigma = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \in \mathcal{T}_2$, then σ can represents by 2×2 matrix $A(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

If $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix} \in \mathcal{T}_3$, so σ can represents by 3×3 matrix as $A(\sigma) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Also, if $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 4 \end{pmatrix} \in \mathcal{T}_5$, then $A(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$

In the next example will list all $\mathcal{T}_3$ elements in matrix form, after we listed them in two lines form(example 1.3.1).

Example 1.5.2. *For $n = 3, \mathcal{T}_3 = \{\sigma : \sigma : \{1,2,3\} \to \{1,2,3\}, \sigma$ function or linear transformation $\}, |\mathcal{T}_3| = 3^3 = 27$ So*

$$\mathcal{T}_3 = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\left. \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \right.$$

$$\left.\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\right\}$$

$$E(\mathcal{T}_3) = \left\{\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\1&1&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\1&1&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&1\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&0\\0&1&1\end{pmatrix}, \begin{pmatrix}1&0&1\\0&1&0\\0&0&0\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}0&0&0\\0&1&0\\1&0&1\end{pmatrix}, \begin{pmatrix}1&1&0\\0&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0\\1&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}.$$

Recalling some linear algebra and matrix theory, we see that :

For the semigroup $M_n(F)$:

$A\mathcal{R}B \iff A$ and B are column equivalent .

$A\mathcal{L}B \iff A$ and B are row equivalent .

$A\mathcal{J}B \iff rank(A) = rank(B)$.

For the semigroup $\mathcal{T}_n$ we have :

$\alpha\mathcal{R}\beta \iff \alpha$ and β have the same fiber (preimages) that is ($R-classes$ correspond to the set of all maps with the same fibers) .

$\alpha\mathcal{L}\beta \iff \alpha$ and β have the same range ($L-classes$ correspond to the set of all maps with the same range).

Lastly, $\alpha\mathcal{J}\beta \iff \alpha$ and β have the same rank , that is ($J-classes$ correspond to the set of all maps with the same rank).

Chapter 2

Lie Theory of $GL_n(F)$ and $M_n(F)$

If F is algebraic closed field, $G = GL_n(F)$ is the unit group of the monoid $M_n(F)$, then $dimGl_n(F) = n^2$, and if $F = \mathbb{F}_q$, then:

$$|Gl_n(F)| = (q^n - 1)(q^n - q) \ldots (q^n - q^{n-1}) .$$

We review some basic concepts about algebraic group theory, which can be found in [4], [5] and [6] as they pertain to $Gl_n(F)$. The maximal tours T is the group of diagonal matrices.

$W = N_G(T)/T$ is called the Weyl group of G.

2.1 Weyl Group and The Rook Monoid

For $GL_n(F)$, Weyl group $W = S_n$ generated by simple reflections $(i\ i+1), 1 \leq i \leq n$. $R_n = \overline{N_G(T)}/T$ is called the Rook monoid. It consists of all $0-1$ matrices with at most one non-zero entry in each row and column.

Example 2.1.1. Let $G = GL_2(\mathbb{R})$ the group of all invertible matrices,and

$$T = \left\{ \begin{pmatrix} a & o \\ o & b \end{pmatrix} : ab \neq 0 \right\}$$

$$\text{then} \quad N_G(T) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} : ab \neq 0 \right\}$$

$$, \quad W = N_G(T)/T = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \cong S_2$$

$$\text{and} \quad \overline{N_G(T)} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \right\} ,so$$

$\overline{N_G(T)}/T =$

$$R_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Definition 2.1.2. If $\sigma \in S_n$, then the length $\ell(\sigma)$ is the smallest number of simple reflections that σ is a product of. See[7] .It is also given by

$$\ell(\sigma) = |\{(i,j) : i < j, \ \sigma(i) > \sigma(j)\}| .$$

Example 2.1.3. If $\sigma(123) \in S_n$, then $\ell(\sigma) = 2$. If $\sigma = (23) \in S_n$, then $\ell(\sigma) = 1$.

2.2 Borel Subgroup

Let $G = GL_n(F)$,the group B of all invertible upper triangular $n \times n$ matrices is a Borel subgroup of G ,this means that B is a maximal invertible connected solvable

subgroup of G .Also B^- , the set of all lower triangular matrices is called the opposite Borel subgroup of G .

The maximal tours $T = B \cap B^-$ is the group of diagonal matrices .More details about Borel subgroup can be found in [4],[5], and [6].

2.3   Schur's Theorem

If F is an algebraically closed field, Schur's theorem states that every matrix in $M_n(F)$ is similar to an upper triangular matrix.[16]and [17]

So, that means for any matrix $A \in M_n(F)$, $A = xUx^{-1}$ for some $x \in M_n(F)$ and some upper triangular matrix U .

In other words, that is gives :

$$GL_n(F) = \bigcup_{x \in G} xBx^{-1}.$$

where B is the group of upper triangular invertible matrices ,and

$$M_n(F) = \bigcup_{x \in G} x\overline{B}\, x^{-1}.$$

where $\overline{B}$ is the monoid of all upper triangular matrices.

(Schur's theorem is not valid in $\mathcal{T}_n$ as we will see later).

2.4   Bruhat Decomposition

In $Gl_n(F)$, the Bruhat decomposition says that given $G = Gl_n(F)$, the Borel subgroup B of upper triangular invertible matrices , and the Weyl group W of permutation matrices, then

$$G = \bigsqcup_{w \in S_n} BwB$$

So, that means we can write any element of G as the product of an upper triangular invertible matrix , a permutation matrix, and an upper triangular invertible matrix.

Example 2.4.1. If $n = 2$, $GL_n(F) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, ac \neq 0 \right\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \neq 0, ac \neq bd \right\}$

Renner decomposition[15] extends the Bruhat decomposition of $GL_n(F)$ to $M_n(F)$

$$M_n(F) = \bigsqcup_{\sigma \in R_n} B\sigma B \ .$$

Where B is $n \times n$ invertible upper triangular matrix. This allows us to express elements of $M_n(F)$ as a product of an upper triangular invertible matrix ,some element in R_n and an upper triangular invertible matrix.

Example 2.4.2. $M_n(F) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, ac \neq 0 \right\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \neq 0, ad \neq bc \right\} \cup$

$\left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \neq 0 \right\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \neq 0, ad = bc \right\} \cup$

$\left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} : d \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} .$

2.5 Chevalley's Big Cell

Let B denote to the group of invertible upper triangular matrices and B^- denote the group of invertible lower triangular . Then B^-B is called the Chevalley's big cell. It is important in the structure theory of algebraic groups[4],[5] and[6]. By the Bruhat decomposition

$$G = \bigsqcup_{x \in S_n} B^- x B$$

Now $x^{-1} B^- x \subseteq B^- B$. So

$$G = \bigcup_{x \in S_n} x B^- B \ .$$

This means that any invertible matrix A can be written as :

$$A = \sigma L U$$

where $\sigma \in S_n, L$ is a lower triangular matrix and U is upper triangular matrix. This is called the LU decomposition in the linear algebra [16] and [17] (useful for solving system equations). Moreover A has LU decomposition (that is it's $B^- B$) if and only if all the principal minor of A are non-zero.

Example 2.5.1. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{pmatrix}$, then A can be expressed as a product of

lower triangular matrix $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{pmatrix}$ and upper triangular matrix

$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so

$$A = LU$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

LU does not always exist(even if A non-singular),for example, if $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$

is non-singular and can not factored as $A = LU$ .However interchanging 2nd and 3rd

rows leads to an LU decomposition . So A can be written as $A = \sigma LU$ where

$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

2.6 Roots

For $GL_n(F)$ the root system in $1-1$ corresponding with

$$\Phi = \{(i,j)|i,j = 1,2,\ldots,n, i \neq j\} .$$

For $\alpha = (i,j) \in \Phi$, we can think of α as a homomorphism , $\begin{pmatrix} a_1 & & \\ O & \ddots & O \\ & & a_n \end{pmatrix}$

$\longrightarrow$ a_i/a_j from T to F .This is the algebraic group analogue of the map in Lie

algebras, $\begin{pmatrix} a_1 & & \\ O & \ddots & O \\ & & a_n \end{pmatrix}$ $\longrightarrow$ $a_i - a_j$. When $i < j$, the root α is positive , and when

$i > j$,the root α is said to be negative .

Let Φ^+ denote to set of all positive roots and Φ^- the set of all negative roots . So

$$\Phi = \Phi^+ \bigcup \Phi^- .$$

For $\alpha \in \Phi$, the root subgroup X_α consists of all elements matrices

$$I + \alpha E_{ij} .$$

We note that $X_\alpha \cong (F, +)$. If F is algebraically closed, then $dim X_\alpha = 1$. If $F = \mathbb{F}_q$

,then $|X_\alpha| = q$. Now $U = \displaystyle\prod_{\alpha \in \Phi^+} X_\alpha$ where the product is in any order .See [4],[5] and[6]

for details.

Example 2.6.1. For $n = 3$, the positive root subgroups are:

$$X_{12} = \left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, X_{13} = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \; and, X_{23} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

The negative root subgroups are:

$$X_{21} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, X_{31} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \right\} \; and, X_{32} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} \right\}.$$

Note that:

$$X_{12}X_{23}X_{13} = X_{23}X_{12}X_{13} = X_{12}X_{13}X_{13} = X_{23}X_{13}X_{12} = X_{13}X_{12}X_{23} = X_{13}X_{23}X_{12}.$$

The group U will be the set of all products of $X_{12}, X_{23} \; and \; X_{13}$ in any order, where

$$U = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in F \right\}$$

similarly the group

$$U^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} : a, b, c \in F \right\}.$$

is a product of X_{21}, X_{31} and X_{32} in any order.

2.7   Borel Subgroups and Intersections

The Borel subgroup B decompose as $B = UT$, where U is consists of upper triangular matrices with 1's on the diagonal and T consists of diagonal matrices. So

$$B = \begin{pmatrix} 1 & * & \ldots & * \\ 0 & 1 & \ldots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \ldots & 1 \end{pmatrix} \begin{pmatrix} * & 0 & \ldots & 0 \\ 0 & * & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & * \end{pmatrix}$$

If $\pi \in S_n$, then

$$B \cap \pi B \pi^{-1} = [U \cap \pi U \pi^{-1}]T \ .$$

If F is algebraically closed, then

$$dim[U \cap \pi U \pi^{-1}] = \tfrac{n^2 - n}{2} - \ell(\pi) \ .$$

If $F = \mathbb{F}_q$,

$$|U \cap \pi U \pi^{-1}| = q^{\frac{n^2 - n}{2} - \ell(\pi)} \ .$$

where

$$\ell(\pi) = |(i, j) : i < j, \pi(i) > \pi(j)|. \ (\text{definition } (2.1.2) \ )$$

Thus in either case , the size of $B \cap \pi B \pi^{-1}$ depends only on the length $\ell(\pi)$ of π .(More details can found in [4]). In chapter 3 we will see that, this is not valid in $\mathcal{T}_n$.

2.8 Parabolic Subgroups

If $n = n_1 + n_2 + \cdots + n_t$, then the block upper triangular matrices of the form

$$\begin{pmatrix} A_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & A_t \end{pmatrix}$$

where A_i , is $n_i \times n_i$ matrix, $1 \le i \le t$ (Stars in the matrix , it could be any numbers),

form a subgroup group P called a parabolic subgroup of $Gl_n(F)$ and denoted by P .

The subgroup of block lower triangular matrices of the form

$$\begin{pmatrix} A_1 & 0 & 0 \\ * & \ddots & 0 \\ * & * & A_t \end{pmatrix}$$

is called the opposite parabolic subgroup and denoted by P^- .If F is algebraically

closed , then

$$dim P = dim P^-$$

If $F = \mathbb{F}_q$, then

$$|P| = |P^-| .$$

So opposite parabolic subgroups have the same size .

Example 2.8.1. If $n = 2 + 3$. In this case, the elements of the parabolic subgroup P

of $GL_n(F)$ have the form

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}$$

and the elements of the opposite parabolic subgroup P^- have the form

$$\begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

where the stars in both matrices can be any elements from F, and the 2×2 and 3×3 diagonal blocks are invertible.

Chapter 3

Lie Theory and The Full Transformation Semigroup

The full transformation semigroup $\mathcal{T}_n$ plays a central role in semigroup theory similar to the role that the symmetric group plays in group theory. The full transformation semigroup $\mathcal{T}_n$ and some special subsemigroups of $\mathcal{T}_n$ have been much studied over the last fifty years. Because $\mathcal{T}_n \subseteq M_n(F)$, we can define in $\mathcal{T}_n$ upper triangular transformations $\mathcal{T}_n^+$ and lower triangular transformation $\mathcal{T}_n^-$ as:

$$\mathcal{T}_n^+ = \{\ \sigma \in \mathcal{T}_n : \sigma(i) \leq i \ , \forall i\ \}\ .$$

and

$$\mathcal{T}_n^- = \{\ \sigma \in \mathcal{T}_n : \sigma(i) \geq i \ , \forall i\ \}\ .$$

These are analogues of the Borel subgroup B and the opposite Borel subgroup B^- of $GL_n(F)$.

In this chapter we will concentrate on the semigroups $\mathcal{T}_n, \mathcal{T}_n^+$,and $\mathcal{T}_n^-$, and will try

to connect our results with Lie theory.

Note that $\sigma(i) = i$ (the Identity function)is in both $\mathcal{T}_n^+$ and $\mathcal{T}_n^-$.

3.1 Borel Subsemigroups

We begin with the analogue of the Borel subgroups B and B^-, which are $\mathcal{T}_n^+$ and $\mathcal{T}_n^-$ in $\mathcal{T}_n$. We will see, where they are act the same and where they act differently. Let us start with $\mathcal{T}_3^+$ and $\mathcal{T}_3^-$, which we will see again and again :

Example 3.1.1. *For* $n = 3$,

$$\mathcal{T}_3^+ = \{ \text{ linear transformation } \sigma \text{ where}$$

$$\sigma : \{1,2,3\} \to \{1,2,3\} \text{ ,and } \sigma(i) \leq i \ \forall i \in \{1,2,3\}\} \Rightarrow$$

$$\mathcal{T}_3^+ = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

and

$$\mathcal{T}_3^- = \{ \text{ linear transformation } \sigma \text{ where}$$

$$\sigma : \{1,2,3\} \to \{1,2,3\} \text{ ,and } \sigma(i) \geq i \ \forall i \in \{1,2,3\}\} \Rightarrow$$

$$\mathcal{T}_3^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right\}.$$

Theorem 3.1.2. $|\mathcal{T}_n^+| = |\mathcal{T}_n^-| = n!$.

Proof. for any $\sigma \in \mathcal{T}_n^+$, $\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \dots & \sigma(n) \end{pmatrix}$ and $\sigma(i) \leq i \ \forall i \in$ $\{1,2,3\dots,n\}$

there is only one possibility for $\sigma(1)$ which is 1, there are two possibilities for $\sigma(2)$ which are $1, 2$ and so on ...

There are n possibilities for $\sigma(n)$ which $1, 2, 3 \ldots n$

Therefore, $|\mathcal{T}_n^+| = n!$. Similarly for $|\mathcal{T}_n^-| = n!$. $\qquad\qquad\square$

Example 3.1.3. For $n = 3, \mathcal{T}_3 = \{\sigma| \ \sigma : \ \{1, 2, 3\} \rightarrow \{1, 2, 3\}, \ \sigma$ function or linear transformation $\}$, $|\mathcal{T}_3| = 3^3 = 27$ where $|\mathcal{T}_3^+| = 3! = 6 \Rightarrow$

$$\mathcal{T}_3^+ = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

and $|\mathcal{T}_3^-| = 3! = 6 \Rightarrow$

$$\mathcal{T}_3^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right\}.$$

The rest of $\mathcal{T}_3$ is:

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

$$E(\mathcal{T}_3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

In chapter 2 ,section 6, we saw how the root system for $GL_n(F)$ was defined. The root subgroups were defined as: $X_{ij} = \{I + \alpha E_{ij}\}, i \neq j$, $\alpha \in F$ and called positive if $i < j$ and Negative if $i > j$.

That means the positive root subgroups will be as: $\left\{ \begin{pmatrix} 1 & & \\ 0 & \ddots & \alpha \\ & & 1 \end{pmatrix} \right\}$ and the negative

root subgroups will be as: $\left\{ \begin{pmatrix} 1 & & \\ \alpha & \ddots & 0 \\ & & 1 \end{pmatrix} \right\}$. The analogous elements of $\mathcal{T}_n$ are the root

elements e_{ij} which we now define.

Definition 3.1.4. $e_{ij} = I - E_{jj} + E_{ij}, i \neq j$ where I the identity matrix,E_{jj} is the Zero matrix with 1 in $(j\ j)$ entry , and E_{ij} is the Zero with 1 in $(i\ j)$ entry. (i.e $e_{ij}(j) = i$ and $e_{ij}(k) = k$, for all $k \neq j$,$k \in \{1, 2, 3, \cdots, n\}$) . e_{ij} is called Positive Root element whenever $i < j$ and Negative Root element when $i > j$.

Example 3.1.5. $\left\{ e_{14} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_{13} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, e_{12} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$

$$e_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, e_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Big\}$$

are the positive root elements in $\mathcal{T}_4$ and

$$\Big\{ e_{41} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, e_{42} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_{43} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$e_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, e_{32} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, e_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\}$$

are the negative root elements in $\mathcal{T}_4$.

Proposition 3.1.6. Every $\sigma \in \mathcal{T}_n^+$ can be written as a product of finite number of $e_{ij}, i \leq j$ where $e_{ij}(j) = i$ and $e_{ij}(k) = k \ \forall k \neq j$.

Proof. let $\sigma \in \mathcal{T}_n^+$, suppose j_1 is the largest number $\leq n$ such that $\sigma(j_1) \neq j_1$, say $\sigma(j_1) = i_1$ (i.e $\sigma(k) = k \ \forall j_1 < k \leq n$) then $\sigma = e_{i_1 j_1} \sigma_1$ where $\sigma_1(j_1) = j_1$, and $\sigma_1(k) = \sigma(k) \forall k \neq j_1$. Now $\sigma_1(k) = k \ \forall j_1 \leq k \leq n$ again, suppose j_2 is the largest number $\leq j_1$ such that $\sigma_1(j_2) \neq j_2$, let say $\sigma_1(j_2) = i_2$, then $\sigma_1 = e i_2 j_2 \sigma_2$ where $\sigma_2(j_2) = j_2$ and $\sigma_2(k) = k \forall j_2 \leq k \leq n$ so $\sigma = e_{i_1 j_1} e i_2 j_2 \sigma_2$, but since n is finite,if we continue with this way , we will reach the identity. Therefore, $\sigma = e_{i_1 j_1} e_{i_2 j_2} \dots e_{i_m j_m} \dot{I}$ where $m \leq n$. □

Let define a standard form for the elements of $\mathcal{T}_n$ which will be used as needed later.

Definition 3.1.7. Define the standard form for $\sigma \in \mathcal{T}_n^+$ as:

$$\sigma = e_{i_1 n} e_{i_2 (n-1)} \ldots e_{i_t 1}.$$

Example 3.1.8. If $\sigma \in \mathcal{T}_5^+$, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 2 & 5 \end{pmatrix}$ then

$$\sigma = e_{24}\sigma_1$$

$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 4 & 5 \end{pmatrix} = e_{24}e_{23}\sigma_2$, where $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$.

So $\sigma = e_{24}e_{23}e_{22}e_{11}$. which is in the standard form as desired.

Now we know that S_n can be generated by simple reflections $(i\ i+1), 1 \le i \le n$,but this is not true in case of $\mathcal{T}_n^+$.

For example, $\mathcal{T}_3^+ \neq \langle e_{12},\ e_{23} \rangle$ (e_{13} not there). So the question here, what elements in $\mathcal{T}_n^+$ can generate $\mathcal{T}_n^+$?

The following theorem can answer this question :

Theorem 3.1.9. $\mathcal{T}_n^+$ *can be presented as:*

$$\langle e_{ij}, i \le j \le n | e_{ij}{}^2 = e_{ij}, e_{ij}e_{jl} = e_{il}e_{ij}, e_{ij}e_{kj} = e_{kj}\ \text{and}\ e_{kj}e_{ij} = e_{ij} \rangle \ .$$

Proof. We know that every $\sigma \in \mathcal{T}_n^+$ can be written as a product of $e_{ij}, i \le j$ and $e_{ij}(m) = m\ \forall m \neq j, e_{ij}(j) = i$. (Proposition(3.1.6))

So $(e_{ij})^2(m) = e_{ij}(m) = m\ \forall m \neq j$, and $(e_{ij})^2(j) = (e_{ij})(i) = i$,then $(e_{ij})^2 = e_{ij}$.

Also, $e_{ij}e_{jl} = e_{ij}(m) = m\ \forall m \neq l, m \neq j$, $eije_{jl}(l) = e_{ij}(j) = i, e_{ij}e_{jl}(j) = e_{ij}(j) = i$.

Then $e_{ij}e_{jl} = e_{il}e_{ij} = e_{ij}e_{il}$,and $e_{ij}e_{kj}(m) = e_{ij}(m) = m\ \forall m \neq j$

$e_{ij}e_{kj}(j) = e_{ij}(k) = k\ \forall k \neq j$. Then $e_{ij}e_{kj} = e_{kj}$. Similarly, $e_{kj}e_{ij}(m) = m\ \forall m \neq j$

$e_{kj}e_{ij}(j) = e_{kj}(i) = i$ then $e_{kj}e_{ij} = e_{ij}$. So $\mathcal{T}_n^+$ satisfied these conditions .

Now, if there is any other semigroup S satisfies the same conditions

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(i.e $\mathcal{S} = \langle a_{ij}, i \leq j \leq n | a_{ij}{}^2 = a_{ij}, a_{ij}a_{jl} = a_{il}a_{ij}, a_{ij}a_{kj} = a_{kj}$ and $a_{kj}a_{ij} = a_{ij}\rangle$).

We need to show that $|\mathcal{S}| = n! = |\mathcal{T}_n^+|$.

If $X = \{$ all upper transformation which in the standard form $\}$,and $\sigma \in X$,so

$\sigma = a_{i_1 j_1} a_{i_2 j_2} \ldots a i_t j_t$ where $j_1 > j_2 \cdots > j_t$, and $i_k \leq j_k \forall 1 \leq k \leq t$, $a_{ij}.\sigma = a_{ij}(a_{i_1 j_1} \ldots a_{i_t j_t})$. Consider the following cases :

(i) $i = i_1, j = j_1 \Rightarrow a_{ij}\sigma = \sigma$, since $a_{ij}{}^2 = a_{ij}$.

(ii) $i \leq i_1$, or $i \geq i_1, j = j_1 \Rightarrow a_{ij}\sigma = a_{ij}a_{i_1 j_1} \ldots a_{i_t j_t} = a_{i_1 j_1}a_{ij}a_{i_2 j_2} \ldots a_{i_t j_t}$, since

$a_{i_1 j_1}a_{ij} = a_{ij} = a_{i_1 j_1}$, and that is true $\forall k$ such that $j < j_k$ (i.e $a_{ij}a_{i_k j_k} = a_{i_k j_k}aij$).

Therefore, $a_{ij}\sigma \in X \Rightarrow a_{ij}X \Rightarrow XX \subset X, a_{ij} \in X \Rightarrow X = S, |S| \leq n!$. $\qquad \square$

Example 3.1.10. $\mathcal{T}_3^+$ generated by 1 , e_{12}, e_{13} and e_{23}. $\mathcal{T}_3^+$ can be presented as

$\mathcal{T}_3^+ = \langle \{1, a, b, c : a^2 = a, b^2 = b = cb, c^2 = cb = c, ab = ba = ac\}\rangle$.

3.2   A Decomposition of $\mathcal{T}_n$

Since S_n is the unit group of the semigroup $\mathcal{T}_n$ we look at the product $S_n \mathcal{T}_n^+$.

Theorem 3.2.1. $\mathcal{T}_n = S_n \mathcal{T}_n^+$. *(i.e any $\sigma \in \mathcal{T}_n$, can be expressed as a product of some element $\pi \in S_n$ and $\sigma^+ \in \mathcal{T}_n^+$.)*

Proof. Let $\sigma \in \mathcal{T}_n$, so every row of σ is made up of one's and zero's, we look to the 1st element of the 1st row which is zero, say ($1j$ position) and we look to the column which contains this zero, it will contains 1, say in i row (ij position), then we change the 2nd row by i row. Then $\sigma_1 = (2\ i)\sigma$.

Now, again we look to the 2nd row in σ_1 and do the same thing which we did with 1st row in σ, that is we look for the 1st zero after the 1st one in the 2nd row , say in $2k$ position , and go down below this position till we find the 1st one, say in the lk position , then

we change the 3rd row by the lth row , to get $\sigma_2 = (3\ l)(2\ i)\sigma$ and we continue this process till we reach the last row and will get $\sigma = (2\ i)^{-1}(3\ l)^{-1}\ldots(m\ t)^{-1}\sigma_m$ where $(2\ i)^{-1}(3\ l)^{-1}\ldots(m\ t)^{-1} \in S_n$ and $\sigma_m \in \mathcal{T}_n$ and $m \leq n$. Therefore, $\mathcal{T}_n = S_n\mathcal{T}_n^+$. $\qquad\square$

Example 3.2.2. *Let* $\sigma = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}_4$, then $\sigma = \pi\sigma^+$,where

$$\pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in S_4 \text{ and } \sigma^+ = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}_4^+.$$

3.3 Intersections

In this section we study intersection of conjugacy of $\mathcal{T}_n^+$.

To know more about $|\mathcal{T}_n^+ \cap \pi\mathcal{T}_n^+\pi^{-1}|$,and $\mathcal{T}_n^+ \cap \pi\mathcal{T}_n^+\pi^{-1}$.

Let take the following example for $\mathcal{T}_4^+$.

Example 3.3.1. Consider $\mathcal{T}_4^+$, which as follows :

$$\mathcal{T}_4^+ = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$

$$\left(\begin{array}{cccc}1&1&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&1&0&1\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&1&0&0\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&1\end{array}\right),$$

$$\left(\begin{array}{cccc}1&1&0&0\\0&0&0&1\\0&0&1&0\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&1&0&0\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{array}\right),\ \left(\begin{array}{cccc}1&0&1&1\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&0&0&1\\0&1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right),$$

$$\left(\begin{array}{cccc}1&0&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&0&0&0\\0&1&1&1\\0&0&0&0\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&0&0&0\\0&1&1&0\\0&0&0&1\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&0&0&0\\0&1&1&0\\0&0&0&0\\0&0&0&1\end{array}\right),\ \left(\begin{array}{cccc}1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&0\end{array}\right),$$

$$\left(\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&0&1&0\\0&1&0&1\\0&0&0&0\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&0&1&0\\0&1&0&0\\0&0&0&1\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&0&1&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{array}\right)\right\}.$$

we know that $|S_4| = 24$. If $\pi \in S_4$, so there are two questions can be asked here:

What is the $\mathcal{T}_4^+ \cap \pi\mathcal{T}_4^+\pi^{-1}$?

And what is the $|\mathcal{T}_4^+ \cap \pi\mathcal{T}_4^+\pi^{-1}|$?

The answer is :

$$\mathcal{T}_4^+ \cap \pi\mathcal{T}_4^+\pi^{-1} = \left\{\left(\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right),\ \left(\begin{array}{cccc}1&0&1&1\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&0&0&1\\0&1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right),\ \left(\begin{array}{cccc}1&0&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right),\right.$$

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Bigg\}.
$$

and $|\mathcal{T}_4^+ \cap \pi\mathcal{T}_4^+\pi^{-1}| = 12$.

Also, if $\pi = (23)$ then:

$$
\mathcal{T}_4^+ \cap \pi\mathcal{T}_4^+\pi^{-1} = \Bigg\{
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$\left. \left\{ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \right. ,$$

and $|\mathcal{T}_4^+ \cap \pi \mathcal{T}_4^+ \pi^{-1}| = 16$.

If $\pi = (34) = \pi^{-1}$, $then |\mathcal{T}_4^+ \cap \pi \mathcal{T}_4^+ \pi^{-1}| = 18$,

if $\pi = (13) = \pi^{-1}$, then $|\mathcal{T}_4^+ \cap \pi \mathcal{T}_4^+ \pi^{-1}| = 4$,

if $\pi = (14) = \pi^{-1}$, then $|\mathcal{T}_4^+ \cap \pi \mathcal{T}_4^+ \pi^{-1}| = 2$,

for $\pi = (24) = \pi^{-1}$, then $|\mathcal{T}_4^+ \cap \pi \mathcal{T}_4^+ \pi^{-1}| = 8$,

if $\pi = (123)$ then $\pi^{-1} = (213)$ and in this case $|\mathcal{T}_4^+ \cap \pi \mathcal{T}_4^+ \pi^{-1}| = 8$.

Now if $\sigma \in \mathcal{T}_n^+$, what are the conditions must σ has to $(12)\sigma(12)$ be in $\mathcal{T}_n^+$. The answer is just

$$\sigma(2) \neq 1 ,$$

for $(23)\sigma(23)$ to be in $\mathcal{T}_n^+$ the condition is

$$\sigma(3) \neq 2 ,$$

and for $(14)\sigma(14)$ to be in $\mathcal{T}_n^+$, σ must satisfies the following conditions :

$$\sigma(4) = 4, \sigma(2) \neq 1 \quad and \ \sigma(3) \neq 1.$$

In general,

$$(i \ i+1)\sigma(i \ i+1) \in \mathcal{T}_n^+ \implies \sigma(i+1) \neq i.$$

And more generally,we can conclude that if $\sigma \in \mathcal{T}_n^+$,and $(i \ j) \in S_n, i \leq j$ then $(ij)\sigma(ij)$ to be in $\mathcal{T}_n^+, \sigma$ must has the following conditions :

$$\sigma(j) = j \ or \ \sigma(j) < i \ and \ \sigma(k) \neq i \forall i < k < j .$$

Then

$$|\mathcal{T}_n^+ \cap (i\ i+1)\mathcal{T}_n^+(i\ i+1)| = \tfrac{i}{i+1}n!\ .$$

Therefore, we can conclude that

$$|\mathcal{T}_n^+ \cap (ij)\mathcal{T}_n^+(ij)| = \tfrac{i}{j}\Pi_{i<k<j}\ \tfrac{k-1}{k}n! = \tfrac{i^2}{j(j-1)}n!\ .$$

Proposition 3.3.2. If $\pi = (i\ i+1)$ then $|\mathcal{T}_n^+ \cap \pi\mathcal{T}_n^+\pi^{-1}| = \tfrac{i}{i+1}n!$ and in general if $\pi = (i\ j)$ then $|\mathcal{T}_n^+ \cap \pi\mathcal{T}_n^+\pi^{-1}| = \tfrac{i^2}{j(j-1)}n!.$

In chapter 2, for the $GL_n(F)$ we found that, the size of $B \cap \pi B \pi^{-1}$ depends on the length of π ,but this not true for $\mathcal{T}_n$, the following example shows that:

Example 3.3.3. Let $\pi = (12)$, then

$$\mathcal{T}_3^+ \cap \pi\mathcal{T}_3^+\pi^{-1} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Therefore $|\mathcal{T}_3^+ \cap \pi\mathcal{T}_3^+\pi^{-1}| = 3$, and for $\pi = (23)$, then

$$\mathcal{T}_3^+ \cap \pi\mathcal{T}_3^+\pi^{-1} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Therefore $|\mathcal{T}_3^+ \cap \pi\mathcal{T}_3^+\pi^{-1}| = 4.$

So (12) and (23) have the same length, but $|\mathcal{T}_3^+\cap(12)\mathcal{T}_3^+(12)^{-1}| \neq |\mathcal{T}_3^+\cap(23)\mathcal{T}_3^+(23)^{-1}|\ .$

The following theorem gives the rule to get$|\mathcal{T}_n^+ \cap \pi^{-1}\mathcal{T}_n^+\pi|\ .$

Theorem 3.3.4. *If* $\pi \in S_n$, *then* $|\mathcal{T}_n^+ \cap \pi^{-1}\mathcal{T}_n^+\pi| = \alpha_1 \cdot \alpha_2 \ldots \alpha_n$ *where*

$$\alpha_i = |\{j : j \leq i, \pi(j) \leq \pi(i)\}|.$$

Proof. If $\sigma \in \mathcal{T}_n^+ \cap \pi^{-1}\mathcal{T}_n^+\pi$, then $\sigma \in \mathcal{T}_n^+$ and $\sigma \in \pi^{-1}\mathcal{T}_n^+\pi^{-1}$,

so $\sigma(i) \leq i \forall 1 \leq i \leq n \rightarrow ..(1)$ and $\sigma = \pi^{-1}\theta\pi$ for some

$$\theta \in \mathcal{T}_n^+ \quad \Rightarrow \quad \sigma = \pi^{-1}\theta\pi$$
$$\Rightarrow \quad \pi\sigma\pi^{-1} \in \mathcal{T}_n^+$$
$$\Rightarrow \quad \pi\sigma\pi^{-1}(i) \leq i \forall 1 \leq i \leq n$$
$$\Rightarrow \quad \pi\sigma(i) \leq \pi(i) \rightarrow ..(2)$$

so , if

$$\sigma(i) = j \quad \Rightarrow \quad j \leq i \text{ and } \pi(j) \leq \pi(i)$$
$$\Rightarrow \quad \sigma(i) \in \{j : j \leq i, \pi(j) \leq \pi(i)\}$$
$$\Rightarrow \quad \alpha_i = |\{j; j \leq i, \pi(j) \leq pi(i)\}|$$

which the number of possibilities of $\pi(i)$. Therefore $|\mathcal{T}_n^+ \cap \pi^{-1}\mathcal{T}_n^+\pi| = \alpha_1 \cdot \alpha_2 \cdots \alpha_n$. $\square$

Example 3.3.5. If $\pi = (123)$, then $|\mathcal{T}_3^+ \cap \pi\mathcal{T}_3^+\pi^{-1}| = \alpha_1 \cdot \alpha_2 \cdot \alpha_3$ where:

$\alpha_1 = |\{j : j \leq 1, \pi(j) \leq \pi(1) \}| = 1,$

$\alpha_2 = | \{j : j \leq 2, \pi(j) \leq \pi(2) = 3 \}| = 2$

and $\alpha_3 = |\{j : j \leq 3, \pi(j) \leq \pi(3) = 1 \}| = 1.$

Therefore,

$$
\begin{aligned}
|\mathcal{T}_3^+ \cap \pi \mathcal{T}_3^+ \pi^{-1}| \; &= \; \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \\
&= \; 1 \cdot 2 \cdot 1 \\
&= \; 2.
\end{aligned}
$$

3.4 Unions

We have seen that in Schur's theorem, every element of $GL_n(\mathbb{C})$ is a similar to a triangular matrix. This is not true of $\mathcal{T}_n$, to figure out what $X = \bigcup_{\pi \in S_n} (\pi \mathcal{T}_n^+ \pi^{-1})$ is? Let start with $n = 3$ as an example:

Example 3.4.1. $X = \bigcup_{\pi \in S_3} (\pi \mathcal{T}_3^+ \pi^{-1}) =$

$$
\left\{ I, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \right.
$$

$$
\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
$$

$$
\left. \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\}.
$$

Proposition 3.4.2. If $\sigma \in \mathcal{T}_n^+$, then $\sigma^N = \sigma^{N+1}$ for some $N \leq n$.

Proof. If $\sigma \in \mathcal{T}_n^+$,by using the definition of $\mathcal{T}_3^+$ and since n is finite, then we have $\sigma(i) \leq i \ \forall \ 1 \leq i \leq n$, so $i \geq \sigma(i) \geq \sigma^2(i) \geq \sigma^3(i) \cdots \geq \sigma^N = \sigma^{N+1}$ for some $N \leq n$. $\qquad\square$

As a result of this proposition ,if $\pi \in \mathcal{S}_n, \sigma \in \mathcal{T}_n^+$ and $\sigma_1 = \pi\sigma\pi^{-1}$, then $\sigma_1^n = \sigma_1^{n+1}$. So for all $\sigma \in X, \sigma^n = \sigma^{n+1}$.

The following theorem, gives us the answer of ,what $X = \cup_{\pi \in S_n}(\pi\mathcal{T}_n^+\pi^{-1})$ is? When is $\sigma \in \mathcal{T}_n$ conjugate to an element of $\mathcal{T}_n^+$? Since it gives the conditions on σ to be in X .

Theorem 3.4.3. If $X = \cup_{\pi \in S_n}(\pi\mathcal{T}_n^+\pi^{-1})$, *then* $\sigma \in X \Leftrightarrow \sigma^n = \sigma^{n+1}$.

Proof. ($\Longrightarrow$)

$$If \sigma \in X \ \Rightarrow \ \sigma = \pi^{-1}\theta\pi \ , \theta \in \mathcal{T}_n^+$$
$$\Rightarrow \ \pi\sigma\pi^{-1} = \theta \in \mathcal{T}_n^+$$
$$\Rightarrow \ \pi\sigma\pi^{-1}(i) \leq i \ \forall \ i = 1, 2, \ldots, n$$
$$\Rightarrow \ (\pi\sigma\pi^{-1})^2(i) \leq \pi\sigma\pi^{-1}(i) \leq i \forall i = 1, 2, \ldots, n$$
$$\Rightarrow \ (\pi\sigma\pi^{-1})^k(i) \leq (\pi\sigma\pi^{-1})^{k-1}(i) \leq \cdots \leq \pi\sigma\pi^{-1} \leq i \forall \ i = 1, 2, \ldots, n$$

So there is some N_i such that
$(\pi\sigma\pi^{-1})^N_{(i}+1)(i) = (\pi\sigma\pi^{-1})_i^N(i) \Rightarrow \pi\sigma^{N_i+1}\pi^{-1}(i) = \pi\sigma^{N_i}\pi^{-1} = \pi\sigma_i^N\pi^{-1}$, therefore take $M = maxN_i$ to get $\sigma^M(i) = \sigma^{M+1}(i)\forall i = 1, 2, \ldots n$.

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($\Longleftarrow$) Now if $\exists\, N$ such that $\sigma^N = \sigma^{N+1}$, we need to show that $\sigma \in X$. Define the relation $\prec$ on $\{1, 2, \ldots n\}$ as : $i \prec j \ \exists\, t \geq 0$ such that $\sigma^t(i) = j$,and prove that $\prec$ is partially ordered(reflexive, antisymmetric and transitive).

(i) $\prec$ reflexive : $\sigma^0(i) = 1(i) = i \ \forall i \in 1, 2, \ldots n$ .So $i \prec i \ \forall i = 1, 2, \ldots n$.

(ii)$\prec$ antisymmetric : if $i \prec j, j \prec i$,suppose that

$$\sigma^t(i) = j, t > 0, \sigma^s(j) = i, s \geq 0 \ \Rightarrow \ \sigma^{(t+s)}(i) = \sigma^s(j) = i$$
$$\Rightarrow \ \sigma^{(t+s)k}(i) = i \forall k$$
$$\Rightarrow \ (t+s)k > N \text{ for some } k.$$

i.e $\exists M > N$ such that $\sigma^N(i) = \sigma^M(i) = i, M > N \Rightarrow \sigma^N(i) = i$.

But $j = \sigma^t(i) = \sigma^t(\sigma^N(i)) = \sigma^{(t+N)}(i) = \sigma^N(i) = i$. Therefore, $\prec$ is symmetric .

(iii)$\prec$ is transitive: If

$$i \prec j, j \prec k \ \Rightarrow \ \sigma^t(i) = j, \sigma^s(j) = k \text{ for some t,s} \geq 0$$
$$\Rightarrow \ k = \sigma^s(j) = \sigma^s(\sigma^t(i)) = \sigma^{(s+t)}(i)$$
$$\Rightarrow \ i \prec k.$$

Then $\prec$ is partially ordered, so $\prec$ can be extended to a linear ordered, i.e $1, 2, \ldots n = i_1, i_2, \ldots i_n$ that means $\exists\, \pi \in \mathcal{S}_n$ such that $i_K = \pi(k)$, $i_k \prec i_l \Rightarrow k \leq l$, so

$$\pi(k) = \pi(l) \ \Rightarrow \ k \leq l$$
$$\Rightarrow \ \sigma^t(\pi(k)) = \pi(l)$$
$$\Rightarrow \ \pi^{-1}\sigma^t(k) \leq l.$$

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$\square$

3.5   Big Cell

We have seen that the big cell B^-B is of much importance in group theory . So we study the analogues product $\mathcal{T}_n^- \mathcal{T}_n^+$ of the subsemigroups $\mathcal{T}_n^-$ and $\mathcal{T}_n^+$ to know more about it,and connect our results with what we already know about the Borel subgroups B and B^- . For that we will introduce two theorems,we begin with:

Theorem 3.5.1. *Every idempotent element* $e \in \mathcal{T}_n$ *is in* $\mathcal{T}_n^- \mathcal{T}_n^+$ *, that means* e *can be written as a product of positive root elements and negative root elements*

(i.e $e \in \mathcal{T}_n$ *,$e^2 = e$* $\Rightarrow$ *$e = e^- e^+ \in \mathcal{T}_n^- \mathcal{T}_n^+$ for some elements* $e^- \in \mathcal{T}_n^-$ *and* $e^+ \in \mathcal{T}_n^+$ *).*

Proof. Let $e \in \mathcal{T}_n$,be an idempotent element, so $e^2 = e$ $\Rightarrow$ $e^2(i) = e(i) = e(e(i))$.
Now if $e \in \mathcal{T}_n^+$,by the definition of $\mathcal{T}_n^+$,

$$e \in \mathcal{T}_n^+ \Rightarrow e(i) \leq i$$
$$\Rightarrow e^2(i) = e(e(i)) = e(\leq i) \text{ which } \leq i \text{ (since } e \in \mathcal{T}_n^+ \text{).}$$

$$Also, if \ e \in \mathcal{T}_n^- \Rightarrow e(i) \geq i$$
$$\Rightarrow e^2(i) = e(e(i)) = e(\geq i) \text{ which } \geq i \text{ (since } e \in \mathcal{T}_n^- \text{).}$$

Therefore, in both cases $e \in \mathcal{T}_n^- \mathcal{T}_n^+$ (since $\mathcal{T}_n^-, \mathcal{T}_n^+ \subset \mathcal{T}_n^- \mathcal{T}_n^+$).

Now if $e \in \mathcal{T}_n - \{\mathcal{T}_n^+ \cup \mathcal{T}_n^-\} \Rightarrow e(i) \leq i$ for some $i's$ and $e(i) \geq i$ for others, then we can write e as a product of $e^- \in \mathcal{T}_n^-$ and $e^+ \in \mathcal{T}_n^+$,

$$\text{where } e^-(i) = \begin{cases} e(i) & \text{if } e(i) \geq i \\ i & , \quad otherwise \end{cases}.$$

$$\text{and } e^+(i) = \begin{cases} e(i) & \text{if } e(i) \leq i \\ i & , \quad otherwise \end{cases}.$$

Therefore $e = e^- e^+ \in \mathcal{T}_n^- \mathcal{T}_n^+$.

If $e(i) \geq i \Rightarrow e(i) = e^-(i)$ and $e^+(i) = i$. If $e(i) \leq i \Rightarrow e(i) = e^+(i)$ and $e^-(i) = i$. So in both cases $e = e^- e^+ \in \mathcal{T}_n^- \mathcal{T}_n^+$.

$\square$

Example 3.5.2. Let $e = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ be an idempotent element $(e^2 = e)$, in $\mathcal{T}_5$

then $e = e^- e^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $(e = e^- e^+ \in \mathcal{T}_5^- \mathcal{T}_5^+)$.

Also for $e = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}_7, \ e = e^- e^+ \in \mathcal{T}_7^- \mathcal{T}_7^+ \ $ where

$$e^- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad e^+ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We now find some necessary conditions for σ to be in $\mathcal{T}_n^- \mathcal{T}_n^+$:

Theorem 3.5.3. *If $\sigma \in \mathcal{T}_n^- \mathcal{T}_n^+$, then σ is satisfies the following two conditions:*

1. *If σ is $(1-1)$ on $\{1, 2, \ldots, m\}$, then $\sigma(i) \geq i \ \forall i = 1, 2, \ldots, m$.*

2. *If $\{1, 2, \ldots, m\} \subseteq Rang(\sigma)$,then $\sigma(i) \leq i \ \forall i = 1, 2, \ldots, m$.*

Proof. Suppose $\sigma \in \mathcal{T}_n^- \mathcal{T}_n^+$, so $\sigma = \sigma^- \sigma^+$ where $\sigma^- \in \mathcal{T}_n^-$ and $\sigma^+ \in \mathcal{T}_n^+$.

1. If σ is $(1-1)$ on $\{1, 2, \ldots, m\}$, then σ^+ will be $(1-1)$, because if

$$\sigma^+(i) = \sigma^+(j), \; i, \; j \in \{1, 2, \ldots, m\} \; \Rightarrow \; \sigma^-\sigma^+(i) = \sigma^-\sigma^+(j)$$
$$\Rightarrow \; \sigma(i) = \sigma(j)$$
$$\Rightarrow \; i = j \; since \; \sigma \; is \; (1-1).$$

And we can show (by induction)that σ^+ is the Identity map on $\{1, 2, \ldots, m\}$, at $i = 1$, $\sigma^+(1) = 1$ (from σ^+ definition)

$$\sigma^+(2) \leq 2 \Rightarrow \sigma^+(2) = 1 \; or \; 2 \;,$$

but σ^+ is $(1-1)$, so $\sigma^+(2) \neq 1 = \sigma(1)$,then $\sigma^+(2) = 2$ and so on $\ldots$
Suppose $\sigma^+(j) = j$ for $j < i$, since $\sigma^+(i) \leq i$, then if

$$\sigma^+(i) = j, \; j < i \; \Rightarrow \sigma^+(i) = \sigma^+(j) = j.$$

But since σ^+ is $(1-1)$ on $\{1, 2, \ldots, m\}$, that means $i = j$ which a contradiction.
So $\sigma^+(i) = i$, therefore σ^+ is the identity map on $\{1, 2, \ldots, m\}$, that means

$$\sigma(i) = \sigma^-\sigma^+(i) = \sigma^-(i) \geq i \; \forall i = 1, 2, \ldots, m \quad (from \; \sigma \; definition \;).$$

2. IF $\{1, 2, \ldots, m\} \subseteq Rang \; \sigma$, then $\sigma(i) \leq i \; \forall i = 1, 2, \ldots, m$.
If $\sigma = \sigma^-\sigma^+$, $\{1, 2, \ldots, m\} \subseteq Rang \; \sigma \subseteq Rang \; \sigma^-$. First, will try to prove that σ^- is the Identity map on $\{1, 2, \ldots, m\}$ and we will do that by induction,since that

$$1 \in Rang \; \sigma^- \; \Rightarrow \; 1 = \sigma^-(j) for \; some \; j$$
$$\Rightarrow \; 1 = \sigma^-(j) \geq j$$
$$\Rightarrow \; j = 1, \; so \; \sigma^-(1) = 1.$$

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$$2 \in Rang \, \sigma^- \quad \Rightarrow \quad 2 = \sigma^-(k) \; for \; some \; k$$

$$\Rightarrow \quad 2 = \sigma^-(k) \geq k \; , so \; k = 1 \; or \; k = 2.$$

If $k = 1 \Rightarrow 2 = \sigma^-(1) = 1$ which a contradiction, so $k = 2 \Rightarrow \sigma^-(2) = 2$, and so on ...

Suppose that $\sigma^-(j) = j \; \forall j < i$, $i, j \in \{1, 2, \ldots, m\}$, want to show that $\sigma^-(i) = i$.

Since $2 \in Rang\sigma \Rightarrow 2 = \sigma^-(k) \geq k$, $k = 1$ or $k = 2$, then $2 = \sigma^-(2)$. Same argument for

$$i \in Rang \, \sigma^- \quad \Rightarrow \quad i = \sigma^-(k) \geq K \; so \; i > k$$

$$\Rightarrow \quad i = \sigma^-(k) = k, \; then \; i = k \; , \sigma^-(i) = i.$$

$\square$

Example 3.5.4. For $n = 3$, we know that $|\mathcal{T}_3| = 3^3 = 27$ where $|\mathcal{T}_3^+| = 3! = 6$.

$\Rightarrow \mathcal{T}_3^+ =$

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

and $|\mathcal{T}_3^-| = 3! = 6$.

$\Rightarrow \mathcal{T}_3^- =$

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right\}.$$

$$\mathcal{T}_3^- \mathcal{T}_3^+ = \left\{ \mathcal{T}_3^-, \mathcal{T}_3^+, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\}$$

And

$$\mathcal{T}_3^+ \mathcal{T}_3^- = \left\{ \mathcal{T}_3^-, \mathcal{T}_3^+, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

So $|\mathcal{T}_3^- \mathcal{T}_3^+| = |\mathcal{T}_3^+ \mathcal{T}_3^-| = 16$.

By theorem(3.5.1) we know that each element of:

$$E(\mathcal{T}_3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \right.$$

$$\left.\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right\}\right. \quad \text{is in} \quad \mathcal{T}_3^- \mathcal{T}_3^+ .$$

Now let see which $\sigma \in \mathcal{T}_3$ and which $\sigma \notin \mathcal{T}_3^- \mathcal{T}_3^+$?

$$\left\{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right\}$$

All of these elements, they do not satisfies the condition(ii) of theorem (3.5.3),and just

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{are satisfy condition (i) .}$$

In $GL_n(F)$, $A \in B^- B$ if and only if all the minors of A are non-zero.The analogous problem in $\mathcal{T}_n$ seems much more difficult.

Problem Let $\sigma \in \mathcal{T}_n$ .Find necessary and sufficient condition for σ to be in $\mathcal{T}_n^- \mathcal{T}_n^+$.

For $GL_n(F)$ and any $\pi \in W = S_n$ we know that

$$\pi B \pi^{-1} \subseteq B^- B .$$

One can ask, is this true for the full transformation semigroup $\mathcal{T}_n$?

(i.e is $\pi \mathcal{T} \pi^{-1} \subseteq \mathcal{T}_n^- \mathcal{T}_n^+$?)

The answer is NO! , and the following example shows that.

Example 3.5.5. If $\pi = (12)$,and $\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ then $\pi\sigma\pi^{-1} =$

$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which not in $\mathcal{T}_3^- \mathcal{T}_3^+$ (see example(3.5.4)).

3.6   Parabolic Semigroups

We now define the analogues of parabolic subgroups of $GL_n(F)$.

Let $\sigma \in \mathcal{T}_n$ where $n = n_1 + n_2 + \cdots + n_t$ satisfies the following conditions

:

$\sigma(i) \leq n_1 \; \forall i \leq n_1$

$\sigma(i) \leq n_1 + n_2 \; \forall n_1 < i \leq n_1 + n_2$

$\vdots$

$\sigma(i) \leq n \; \forall n_1 + n_2 + \cdots + n_{t-1} < i \leq n$

So σ will be as:

$$\sigma =$$

$$
\begin{array}{cccc}
n_1 & n_2 & \cdots & n_t
\end{array}
$$

$$
\left(
\begin{array}{ccccccccc}
* & * & \cdots* & * & * & \cdots* & & * & * & \cdots* \\
\vdots & \ddots & \cdots* & * & * & \cdots* & & * & * & \cdots* \\
* & * & \cdots* & * & * & \cdots* & & * & * & \cdots* \\
& & & & & \cdots & & & & \\
& & & \vdots & \vdots & \cdots & & * & * & \cdots* \\
& & & * & * & \cdots* & & \vdots & \vdots & \cdots \\
& & & & & & & * & * & \cdots*
\end{array}
\right)
$$

Let denote by P_n^+ for the set of all σs in this form .

Theorem 3.6.1. *P_n^+ is a subsemigroup of $\mathcal{T}_n$ and*

$$|P_{n_1+n_2+\cdots+n_t}^+| = n_1^{n_1}(n_1+n_2)^{n_2}(n_1+n_2+n_3)^{n_3}\ldots(n_1+n_2+\cdots+n_t)^{n_t} .$$

Let P_n^- denote set of all element of σ of the form:

$$\sigma =$$

$$
\begin{array}{ccc}
n_1 & n_2 & \cdots & n_t
\end{array}
$$

$$
\left(
\begin{array}{cccccccccccc}
* & * & \cdots & * & & & & & & & & \\
* & * & \cdots & * & * & * & \cdots & * & & & & \\
* & * & \cdots & * & * & * & \cdots & * & & & & \\
& & & & & & & & \cdots & & & \\
* & * & \cdots & * & * & * & \cdots & * & & * & * & \cdots & * \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & & & \vdots & \ddots & \cdots & * \\
* & * & \cdots & * & * & * & \cdots & * & & * & * & \cdots & *
\end{array}
\right)
$$

We call P_n^- opposite parabolic subsemigroup.

Theorem 3.6.2. *P_n^- is a subsemigroup of $\mathcal{T}_n$ and*

$$|P_n^-| = (n_1 + \cdots + n_t)^{n_1}(n_2 + \cdots + n_t)^{n_2} \ldots n_t^{n_t} .$$

Example 3.6.3. If $n_1 = 1$ and $n_2 = 1$,so $n = 1 + 1 = 2$.

Then $\sigma(i) \leq n_1 = 1 \ \forall \ i \leq 1 \Rightarrow \sigma(1) = 1$ and $\sigma(i) \leq n_1 + n_2 = 1 + 1 = 2 \ \forall \ 1 < i \leq 2$,so $\sigma(2) = 1 \ or \ 2$,then we have just two σ's satisfies these conditions :

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ and \ \sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$So \ P_{1+1}^+ = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} .$$

Also, P_{1+1}^- has two elements: $\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$.

Example 3.6.4. If $n_1 = 1$ and $n_2 = 2$,so $n = 1 + 2 = 3$.

Then

$$\sigma(i) \leq n_1 = 1 \ \forall \ i \leq 1 \Rightarrow \sigma(1) = 1$$

and

$$\sigma(i) \leq n_1 + n_2 = 1 + 2 = 3 \ \forall \ 1 < i \leq 3 \ \Rightarrow \ \sigma(2) = 1, 2 \ or \ 3 \ , and$$

$\sigma(3) = 1, 2 \ or \ 3$ then $|P_{1+2}^+| = 1 \times 3^2 = 9$ and $|P_{1+2}^-| = (1+2)^1 2^2 = 12$

as shown below satisfies these conditions:

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\sigma_4 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_5 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \sigma_6 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\sigma_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \sigma_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, and \ \sigma_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Also,

$$P_{1+2}^- = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$\left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Thus we see that the opposite parabolic subsemigroups P_{1+2}^+, P_{1+2}^- have different sizes .

So we see that unlike for $GL_n(F)$ (see section (2.8)),

$P_{n_1+n_2+\cdots+n_t}^+$ and $P_{n_1+n_2+\cdots+n_t}^-$ can have different sizes.

Now we can define the root elements e_{ij} as before. $e_{ij}(j) = i$ and, $e_{ij}(k) = k \; \forall k \neq j$. Let $P_{n_1+n_2+\cdots+n_t}$ be the set of all σ's in this form, so the following theorem gives the rule to find the number of all root elements in $P_{n_1+n_2+\cdots+n_t}$.

Theorem 3.6.5. *Let* R_e *be the number of root elements in* P_n *, then*

$$R_e = (n_1^2 - n_1) + (n_1 + n_2).n_2 - n_2 + (n_1 + n_2 + n_3).n_3 - n_3 + \cdots + n.n_t - n_t \ .(i.e.$$

$$R_e = n_1^2 + n_2^2 + \cdots + n_t^2 - n_1 - n_1 n_2 - n_1 n_3 - n_2 n_3 \cdots - n_1 n_t - n_2 n_t - \cdots n_{t-1} n_t \ ).$$

Proof. If e_{ij} is a root element in P_n , the e_{ij} satisfies all conditions,which have given in the definition of P_n ,

then we have $n_1^2 - n_1$ root elements satisfies the condition $i, j \leq n_1, i \neq j$,we have $(n_1 + n_2)n_2 - n_2$ root elements satisfies the condition $n_1 < j \leq n_1 + n_2$, $i \leq n_1 + n_2$,we have $(n_1 + n_2 + n_3)n_3 - n_2$ elements satisfies the condition $n_1 + n_2 < j \leq n_1 + n_2 + n_3$, $i \leq n_1 + n_2 + n_3$, . . . and we have $n.n_t - n_t$ root elements satisfies the last condition. Therefore, the number R_e of root elements in P_n is

$$R_e = (n_1^2 - n_1) + (n_1 + n_2).n_2 - n_2 + (n_1 + n_2 + n_3).n_3 - n_3 + \cdots + n.n_t - n_t \ .$$
$$(\text{i.e } R_e = n_1^2 + n_2^2 + \cdots + n_t^2 - n_1 - n_1 n_2 - n_1 n_3 - \ldots n_1 n_t - \ldots n_{t-1} n_t \ )$$

$\square$

Example 3.6.6. Let P_n be a set as defined above, where

$n = n_1 + n_2$, $n_1 = 1$ and $n_2 = 2$,then P_{1+2} contains nine elements.

The number of root elements in P^+_{1+2} is :

$$(n_1^2 - n_1) + (n_1 + n_2).n_2 - n_2 = (1^2 - 1) + (1 + 2).2 - 2$$

$$= 0 + 6 - 2$$

$$= 4$$

$\Rightarrow$ which are

$$e_{12}, e_{23}, e_{13} \ and \ e_{23} \ .$$

P^-_{1+2} in this case will contain the same number of root elements which 4 ,

they are :

$$e_{12}, e_{13}, e_{32} \ and \ e_{23} \quad .$$

For more explanation , let $n = 2+3$, then P^+_{2+3} and P^-_{2+3} will contains

each σ in the following forms :

$$
\begin{pmatrix}
* & * & \bigm| & * & * & * \\
* & * & \bigm| & * & * & * \\
\hline
0 & 0 & \bigm| & * & * & * \\
0 & 0 & \bigm| & * & * & * \\
0 & 0 & \bigm| & * & * & *
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
* & * & \bigm| & 0 & 0 & 0 \\
* & * & \bigm| & 0 & 0 & 0 \\
\hline
* & * & \bigm| & * & * & * \\
* & * & \bigm| & * & * & * \\
* & * & \bigm| & * & * & *
\end{pmatrix}
$$

In this case the number of root elements in P^{+}_{2+3} is :

$$
(n_1^2 - n_1) + (n_1 + n_2).n_2 - n_2 \;=\; (2^2 - 2) + (2 + 3).3 - 3
$$

$$
=\; 14
$$

Which , they are :

e_{21}, e_{12}, e_{43}, e_{53}, e_{54}, e_{13}, e_{23}, e_{14}, e_{24}, e_{34}, e_{15}, e_{25}, e_{35}, e_{45} .

The same thing ,if we consider $n = n_1 + n_2 + n_3$, where

$n_1 = 2, n_2 = 3, n_3 = 2$ .Then $P_n^+ = P_{2+3+2}^+$ will contains each σ as:

$$\begin{pmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \end{pmatrix}$$

and in this case the number of root elements in P_{2+3+2}^+ will be:

$$(2^2 - 2) + (2+3).3 - 3 + (2+3+2).2 - 2 \;\; = \;\; 26$$

e_{21}, e_{12}, e_{13}, e_{23}, e_{43}, e_{53}, e_{14}, e_{24}, e_{34}, e_{54}, e_{15}, e_{25}, e_{35}, e_{45}, e_{16},

e_{26}, e_{36}, e_{46}, e_{56}, e_{76}, e_{17}, e_{27}, e_{37}, e_{47}, e_{57} , e_{67} ,

and P_{2+3+2}^- will contains each σ as:

$$\begin{pmatrix} * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix}$$

P_{2+3+2}^{-} will contains the same number of root elements as p_{2+3+2}^{+} ,which 26 as following:

e_{21}, e_{12}, e_{31}, e_{32}, e_{41}, e_{42}, e_{51}, e_{52}, e_{61}, e_{62}, e_{71}, e_{72}, e_{43}, e_{34}, e_{53}, e_{35},

e_{54}, e_{45}, e_{63}, e_{64}, e_{65}, e_{73}, e_{74}, e_{75}, e_{76}, e_{67} .

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