ABSTRACT

TOLLEY, MELISSA MARIE. The Connections Between A_∞ and L_∞ Algebras. (Under the direction of Dr. Thomas Lada.)

In the work of Kajiura and Stasheff, we are given the definition of A_∞ strong homotopy derivations. By proving an alternate, but equivalent, definition for these derivations, we are able to take this idea and develop a corresponding definition for L_∞ strong homotopy derivations. From here we show this definition is not only consistent with the ideas behind our alternate A_∞ strong homotopy derivation definition, but also consistent with the symmetrization of A_∞ algebras to L_∞ algebras, thus showing this is the correct definition to use. We then define strong homotopy inner derivations for these algebras, resulting in examples of A_∞ and L_∞ strong homotopy derivations.

One of our goals here is to find connections between A_∞ and L_∞ algebras. We show that there are two ways to start with a lower level A_∞ algebra structure and lift to an L_∞ algebra structure on the corresponding coalgebra, both resulting in exactly the same L_∞ algebra. We show that skew-symmetrizing then lifting maps is equivalent to lifting then symmetrizing the maps of the lower level A_∞ algebra.

To show these connections throughout the paper, we start with the work from Michael Allocca, where an explicit example of an A_∞ algebra is given. By using definitions of Stasheff and Lada, we are then able to construct a corresponding L_∞ algebra, then lift these two examples on coalgebras, resulting in four explicitly stated A_∞ and L_∞ algebras which we use throughout the paper. To complete our concrete examples, we find explicit strong homotopy derivations for the lifted A_∞ and L_∞ algebras, giving two concrete examples of algebras and corresponding homotopy derivations.

The Connections Between A_∞ and L_∞ Algebras

by
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DEDICATION

To all those who helped along the way, those looking from above and those by my side, those in faith and those in humanity.

BIOGRAPHY

Melissa Tolley was born in Asheville, NC to parents Roland and Carolyn Tolley, and grew up in Black Mountain, NC until the age of 14 when she moved to Swannanoa, NC. From the age of five she was helping (or hindering) her father with trig problems at their living room coffee table. Throughout primary and elementary school, Melissa was involved with the Duke TIP program and the Academically Gifted program offered through her school. When she went to middle school, she became involved with MathCounts where she (finally) met other people who enjoyed doing math in their free time and her interest in math grew from there.

Throughout high school, Melissa dreamed of being a anesthesiologist while still competing in math events. As a junior in high school, Melissa spent a summer at the University of North Carolina at Wilmington through Summer Ventures where she learned that she wouldn't have to give up math while pursuing her medical school dream by double majoring. In high school she was awarded "The Math Award" during her senior year, setting the path for college.

During her first semester at Agnes Scott College, Melissa found out that a major in Biology was not for her, so she quickly dropped the medical school dream, changed her major to only Mathematics, and picked up an Economics minor. By tutoring privately and in the Agnes Scott tutorial center, Melissa realized she wanted to teach in a college setting and wished to pursue a Ph.D in mathematics.

In January of 2008, Dr. Ernest Stitzinger called her with the news that North Carolina State University was inviting her to their graduate program. She moved to Raleigh in August of 2008 and loved everything the city had to offer for five years while working hard towards a Ph.D in mathematics under the direction of Dr. Tom Lada.

At the time of this writing, she will defend in March of 2013 and start her job as Assistant Professor at Wingate College in August, a title she still can't believe is hers.

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If you've ever given me any encouragement, directly or indirectly, your name belongs here, because you deserve some credit. But since acknowledgements can't be longer than the actual dissertation, I'll cut it short.

To the two people responsible for giving me life, Mama and Daddy, you taught me to go forth in this world without ever letting anything/anyone hold me back. You taught me to be independent without regard. You taught me to reach for the stars because anything is attainable. And you did it with love. You can't put a price on that. Hey, I'm (finally) graduating!

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To my NC State people, CBach, George, Kaska, and Christophe: These last 5 years would have been unbearable with you. Christina, we've been together since before grad school even started! You're a great friend, and more importantly, an amazing person. I miss you (and Oph) everyday and I'm still plotting a way to get you back as my roommate. Believe me when I say, your friendship is invaluable. George and Kaska, you are my crew, and a pillar of structural

support in my life. Christophe, you deserve recognition for always being someone I could talk to. I'll always owe you for that.

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There are plenty of others whose names need to be listed. And not just people, social media helped too: Facebook, Twitter, Jersey Shore, Rob & Big, and Pandora for knowing exactly how to provide a soundtrack to my life. You can't do math ALL the time. And let's not forget Cash Money Records, taking over for the 99-2000. That was a big year.

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Definitions

From Stasheff, we obtain the definition of an A_∞ algebra [9]:

Definition 1 (A_∞ Algebra). Let V be a graded vector space. An A_∞ structure on V is a collection of linear maps $m_k : V^{\otimes k} \to V$ of degree $2 - k$ that satisfy the identity

$$\sum_{\lambda=}^{n-1} \sum_{k=1}^{n-\lambda} \alpha m_{n-k+1}(x_1 \otimes \cdots \otimes x_\lambda \otimes m_k(x_{\lambda+1} \otimes \cdots \otimes x_{\lambda+k}) \otimes x_{\lambda+k+1} \otimes \cdots \otimes x_n) = 0 \qquad (1.1)$$

where $\alpha = (-1)^{k+\lambda+k\lambda+kn+k(|x_1|+\cdots+|x_\lambda|)}$, for all $n \geq 1$.

From Lada and Stasheff, we have the definition of an L_∞ Algebra [8]:

Definition 2 (L_∞ Algebra). An L_∞ algebra structure on a graded vector space V is a collection of skew symmetric linear maps $l_n : V^{\otimes n} \to V$ of degree $2 - n$ that satisfy the relation

$$\sum_{i_j=n+1} \sum_{\sigma} (-1)^\sigma (-1)^{e(\sigma)} (-1)^{i(j-1)} l_j(l_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0 \qquad (1.2)$$

where $(-1)^\sigma$ is the sign of the permutation, $e(\sigma)$ is the product of the degrees of the permuted elements, and σ is taken over all $(i, n - i)$ unshuffles.

There are equivalent descriptions of A_∞ and L_∞ algebras given by degree one coderivations on the coalgebras $T^C(\downarrow V)$ and $S^C(\downarrow V)$, respectively, with $D^2 = 0$. From Kajiura and Stasheff [5], we obtain the following definitions:

Definition 3 (A_∞ Algebra). Let A be a $\mathbb{Z}$-graded vector space $A = \bigoplus_{r \in \mathbb{Z}} A^r$ and suppose that there exists a collection of degree one multi-linear maps

$$m := \{m_k : A^{\otimes k} \to A\}_{k \geq 1}$$

(A, m) is called an A_∞ algebra when the multi-linear maps m_k satisfy the following relation

$$\sum_{k+l=n+1} \sum_{i=1}^{k} (-1)^{o_1+\cdots+o_{i-1}} m_k(o_1, \ldots, o_{i-1}, m_l(o_i, \ldots, o_{i_l-1}), o_{i+l}, \ldots, o_n) = 0 \qquad (1.3)$$

for $n \geq 1$, where o_j on (-1) denotes the degree of o_j.

Definition 4 (L_∞ Algebra). Let L be a graded vector space and suppose that a collection of degree one graded symmetric linear maps $l := \{l_k : L^{\otimes k} \to L\}_{k \geq 1}$ is given. (L, l) is called an L_∞ algebra if and only if the maps satisfy the following relation

$$\sum_{\sigma \in S_{k+l=n}} (-1)^{\epsilon(\sigma)} l_{1+l}(l_k(c_{\sigma(1)}, \ldots, c_{\sigma(k)}), c_{\sigma(k+1)}, \ldots, c_{\sigma(n)}) = 0 \qquad (1.4)$$

for $n \geq 1$, where $(-1)^{\epsilon(\sigma)}$ is the Kozsul sign of the permutation.

Theorem 5. *[6] If $\{m_n : V^{\otimes n} \to V\}$ is an A_∞ structure, then $l_n = \displaystyle\sum_{\sigma \in S_n} (-1)^\tau m_n \circ \sigma$ where τ is the multiplication of the sign of σ and the Koszul sign, gives an L_∞ structure.*

We will use the definitions from Lada and Stasheff for beginning work, and the alternate definitions once we move to the coalgebras in later work.

Before we go any further, we discuss permutations versus unshuffles. If we consider the element (x, y, z), the permutations are

$$(x, y, z)$$
$$(x, z, y)$$
$$(y, x, z)$$
$$(y, z, x)$$
$$(z, x, y)$$
$$(z, y, x)$$

However, the unshuffles are:

$$(x, y, z)$$
$$(x, y, (z))$$
$$(x, z, (y))$$
$$(y, z, (x))$$
$$(x, (y, z))$$
$$(y, (x, z))$$
$$(z, (x, y))$$

The difference here is that permutations do not keep order, whereas unshuffles do. For the unshuffles, we look at ways to break up the number of elements, so in our example we can break up 3 by $(3, 0)$, $(2, 1)$, and $(1, 2)$. Note that $(3, 0)$ and $(0, 3)$ are the same. We use unshuffles in the definitions of L_∞ algebras and permutations in the above theorem and in the definition of A_∞ algebra.

Definition 6. (Strong homotopy derivation for A_∞ Algebras) A *strong homotopy derivation* of degree one of an A_∞-algebra (A, m) consists of a collection of multi-linear maps of degree one

$$\theta := \{\theta_q | A^{\otimes q} \to A\}_{q \geq 1}$$

satisfying the following relations:

$$
\begin{aligned}
0 = \sum_{r+s=q+1} \sum_{i=0}^{r-1} & (-1)^{\beta(s,i)} \theta_r(o_1, \ldots, o_i, m_s(o_{i+1}, \ldots, o_{i+s}), \ldots, o_q) \\
& + (-1)^{\beta(s,i)} m_r(o_1, \ldots, o_i, \theta_s(o_{i+1}, \ldots, o_{i+s}), \ldots, o_q)
\end{aligned}
\tag{1.5}
$$

Here the sign $\beta(s, i) = o_1 + \cdots + o_i$ results from moving m_s, respectively θ_s, past $(o_1, \ldots, o_i)$.

Definition 7. (Strong Homotopy Derivation for L_∞ Algebras) A *strong homotopy derivation* of degree one of an L_∞ algebra consists of a collection of symmetric, multi-linear maps of degree one

$$\theta := \{\theta_q | L^{\otimes q} \to L\}_{q \geq 1}$$

satisfying relations:

$$
\sum_{\substack{j=1 \\ \sigma \in U(j,n-j)}}^{j=n} (-1)^{\epsilon(\sigma)} \theta_{n-j+1}(l_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)})
$$
$$
+ (-1)^{\epsilon(\sigma)} l_{n-j+1}(\theta_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)}) = 0
$$

(1.6)

where $(-1)^{\epsilon(\sigma)}$ is the sign of the unshuffle.

Chapter 2

Finite Examples

In Chapter 1 we gave definitions for our two algebras, in this chapter we present (and justify) two A_∞ and two L_∞, one at each level, that we will reference throughout this paper.

2.1 Finite A_∞ Example

Allocca and Lada used our first definition to find a small finite dimensional example [1]:

Example 8. Let V denote a graded vector space given by $V = \oplus V_n$ where V_0 has basis $< v_1, v_2 >$, V_1 has basis $< w >$, and $V_n = 0$ for $n \neq 0, 1$. The structure on V is defined by the linear maps $m_n : V^{\otimes n} \to V$:

$$m_1(v_1) = m_1(v_2) = w$$

$$\text{For } n \geq 2 : m_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes(n-2)-k}) = (-1)^k s_n v_1, \ 0 \leq k \leq n-2$$

$$m_n(v_1 \otimes w^{\otimes(n-2)} \otimes v_2) = s_{n+1} v_1$$

$$m_n(v_1 \otimes w^{\otimes(n-1)}) = s_{n+1} w$$

where $s_n = (-1)^{\frac{(n+1)(n+2)}{2}}$, and $m_n = 0$ when evaluated on any element of $V^{\otimes n}$ that is not listed above.

Our goal here was to show an L_∞ algebra could arise from this finite A_∞ algebra example. To do our work, we used the following theorem that creates a relationship between the two algebras (at the lower level). [6]

Theorem 9. *If* $\{m_n : V^{\otimes n} \to V\}$ *is an* A_∞ *structure, then* $l_n = \displaystyle\sum_{\sigma \in S_n} (-1)^\tau m_n \circ \sigma$ *where* τ *is the multiplication of the sign of* σ *and the Koszul sign, gives an* L_∞ *structure.*

For the Koszul sign and sign of σ, this comes from the degree of the permuted elements along with the number of transpositions done. For example, on the element (x, y, z), the Koszul sign for $l_2(l_1(y), x, z)$ would be $(-1)^1(-1)^{|x||y|}$ because x and y have been switched, so we have one transposition. These signs will play an important part in our work.

2.2 Finite L_∞ Example

From this finite example and using the above theorem, we are able to construct a finite L_∞ algebra:

Example 10. Consider the graded vector space $V = V_0 \oplus V_1$ where V_0 has basis $< v_1, v_2 >$ and V_1 has basis $< w >$. We show that this space has an L_∞ structure given by:

$$l_1(v_1) = l_1(v_2) = w$$
$$\text{For } n \geq 2, l_n(v_1 \otimes w^{\otimes(n-1)}) = (n-1)!s_{n+1}w$$
$$l_n(v_1 \otimes w^{\otimes(n-2)} \otimes v_2) = (n-2)!s_{n+1}v_1$$

where $s_n = (-1)^{\frac{(n+1)(n+2)}{2}}$ and $l_n = 0$ when evaluated on any element of $V^{\otimes n}$ that is not listed.

Here, the tensor product, $\otimes$, is the skew-symmetric tensor, i.e. the wedge product. Throughout this paper will will use $\otimes$ instead of $\wedge$ for L_∞ algebra to keep notation consistent, but keep in mind at the lower level of L_∞ algebras, we have that $\otimes$ is the skew-symmetric tensor product and at the higher level, $\otimes$ is the symmetric tensor.

First, note that from the Theorem and the maps m_n (and using that l_n is skew-symmetric), the only nonzero terms in the sum $\sum_{\sigma \in S_n} (-1)^\tau m_n \circ \sigma$, will be those acting on the following elements of $V^{\otimes n}$: v_1, v_2, $v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{(n-2)-k}$ for $0 \leq k \leq n-2$, $v_1 \otimes w^{\otimes(n-2)} \otimes v_2$, and $v_1 \otimes w^{\otimes(n-1)}$, for $n \geq 2$.

Now look at l_1. Since the only permutation of one element is the identity, we have that

$$l_1(v_1) = m_1(v_1) = w$$

and

$$l_1(v_2) = m_1(v_2) = w$$

6

Next, we look at l_2 before we look at a generic n, to get a feel for how these permutations work. From our list above, the only terms that will give nonzero entries are $v_1 \otimes w$ and $v_1 \otimes v_2$. We look at each of these individually.

We have that

$$
\begin{aligned}
l_2(v_1 \otimes v_2) &= m_2(v_1 \otimes v_2) - m_2(v_2 \otimes v_1) \\
&= s_3 v_1 - 0 \\
&= s_3 v_1
\end{aligned}
$$

$$
\begin{aligned}
l_2(v_1 \otimes w) &= m_2(v_1 \otimes w) - m_2(w \otimes v_1) \\
&= s_3 w - 0 \\
&= s_3 w
\end{aligned}
$$

Now, let $n \geq 3$. We first look at $l_n(v_1 \otimes w^{\otimes(n-1)})$. When we look at the sum $\sum_{\sigma \in S_n} (-1)^\tau m_n \circ \sigma$ acting on this element, the only non-zero terms will be $m_n(v_1 \otimes w^{\otimes(n-1)})$ for each σ permuting the $w's$. Any other term will have $m_n(w \otimes \cdots)$, which is zero, as A_∞-algebra mappings are neither symmetric nor skew-symmetric. Now we consider the Koszul sign of each of these terms. Since $w \in V_1$, when we permute any two $w's$ we get a coefficient sign of $+1$. That is, we get (-1) for a transposition of two $w's$ and a $(-1)(-1)^{1 \cdot 1}$ as the Koszul sign since these terms are in V_1, giving a positive sign for each term. The number of nonzero terms is the number of ways we can permute the $n-1$ $w's$, which is $(n-1)!$. Now we have $(n-1)!$ terms, each one is

$$
m_n(v_1 \otimes w^{\otimes(n-1)}) = s_{n+1} w
$$

therefore

$$
\begin{aligned}
l_n(v_1 \otimes w^{\otimes(n-1)}) &= \Big(\sum_{\sigma \in S_n} (-1)^\tau m_n \circ \sigma \Big)(v_1 \otimes w^{\otimes(n-1)}) \\
&= (n-1)! s_{n+1} w
\end{aligned}
$$

Next, look at $l_n(v_1 \otimes w^{\otimes(n-2)} \otimes v_2)$. When we expand this in the summation $\sum_{\sigma \in S_n} (-1)^\tau m_n \circ \sigma$, we see that the only nonzero terms will be of the form $m_n(v_1 \otimes w^{\otimes(n-2)} \otimes v_2)$, when we permute; this is because the m_n maps involving v_1 and v_2 for $n \geq 3$ are only defined when v_1 is the first

term and v_2 is the last term.

In a similar fashion as before, when we permute $w's$ we get a positive Koszul number. The number of terms in the sum will be the number of ways we can permute the $w^{\otimes(n-2)}$, which is $(n-2)!$. Since each term is positive, we get:

$$
\begin{aligned}
l_n(v_1 \otimes w^{\otimes(n-2)} \otimes v_2) &= (n-2)! m_n(v_1 \otimes w^{\otimes(n-2)} \otimes v_2) \\
&= (n-2)! s_{n+1} v_1
\end{aligned}
$$

The last nonzero element to consider is $v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes(n-2)-k}$. We will show for $n \geq 2$, $l_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes(n-2)-k}) = 0$. For explanation purposes, we will distinguish the two v_1 as v_{1_1} and v_{1_2}, so we are looking for

$$
l_n(v_{1_1} \otimes w^{\otimes k} \otimes v_{1_2} \otimes w^{\otimes(n-2)-k})
$$

When we expand the summation, the only nonzero terms will be of the form

$$
m_n(v_{1_1} \otimes w^{\otimes k} \otimes v_{1_2} \otimes w^{\otimes(n-2)-k})
$$

and

$$
m_n(v_{1_2} \otimes w^{\otimes k} \otimes v_{1_1} \otimes w^{\otimes(n-2)-k})
$$

for some permutations on $w's$.

Note that there are $n-1$ terms of the form $m_n(v_{1_1} \otimes w^{\otimes k} \otimes v_{1_2} \otimes w^{\otimes(n-2)-k})$, these are:

$$
v_{1_1} \otimes v_{1_2} \otimes w^{\otimes(n-2)}
$$

$$
v_{1_1} \otimes w \otimes v_{1_2} \otimes w^{\otimes(n-3)}
$$

$$
\vdots
$$

$$
v_{1_1} \otimes w^{\otimes(n-2)} \otimes v_{1_2}
$$

Similarly, there are $n-1$ terms of the form $m_n(v_{1_2} \otimes w^{\otimes k} \otimes v_{1_1} \otimes w^{\otimes(n-2)-k})$, each one corresponding to switching v_{1_1} and v_{1_2} from above. These are the only nonzero terms since a w in the first coordinate gives a zero for m_n, that is $m_n(w, \ldots) = 0$.

We look at the correspondence of the sign of $m_n(v_{1_1} \otimes w^{\otimes k} \otimes v_{1_2} \otimes w^{\otimes(n-2)-k})$ and the sign of $m_n(v_{1_2} \otimes w^{\otimes k} \otimes v_{1_1} \otimes w^{\otimes(n-2)-k})$. Say the sign of $m_n(v_{1_1} \otimes w^{\otimes k} \otimes v_{1_2} \otimes w^{\otimes(n-2)-k})$ is $+1$.

Then the sign of

$$m_n(v_{1_1} \otimes v_{1_2} \otimes w^{\otimes k} \otimes w^{\otimes(n-2)-k})$$

is $(+1)(-1)^k$ since we have done k transpositions, with each transposition between two elements of degree $+1$. Continuing, the sign of

$$m_n(v_{1_2} \otimes v_{1_1} \otimes w^{\otimes k} \otimes w^{\otimes(n-2)-k})$$

is $(+1)(-1)^k(-1)$ since we've transposed two elements, each of degree 0.

Moving those k $w's$ back to the right, gives the sign of

$$m_n(v_{1_2} \otimes w^{\otimes k} \otimes v_{1_1} \otimes w^{\otimes(n-2)-k})$$

as $(+1)(-1)^k(-1)(-1)^k$ since we've done k transpositions, each with two elements of degree 1 and 0, so a sign of $+1$. Simplifying this sign gives:

$$\begin{aligned} (+1)(-1)^k(-1)(-1)^k &= (-1)(-1)^{2k} \\ &= -1 \end{aligned}$$

Hence, when the sign of $m_n(v_{1_1} \otimes w^{\otimes k} \otimes v_{1_2} \otimes w^{\otimes(n-2)-k})$ is $+1$, the sign of $m_n(v_{1_2} \otimes w^{\otimes k} \otimes v_{1_1} \otimes w^{\otimes(n-2)-k})$ is -1. The $n-1$ of the first type then cancel out with the $n-1$ of the second type, giving us 0 in the summation.

Therefore,

$$l_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes(n-2)-k}) = 0$$

for $n \geq 2$.

Because we used the theorem presented before, we know this is an example of an L_∞ algebra, but we explicitly prove this is an L_∞ algebra in the next section. Although these calculations are unnecessary (as the theorem provides our proof), this is a way to show how the mappings work together in addition to proving the accuracy of our calculations.

2.2.1   Showing Sum Relation

Next, to verify that this finite example is, in fact, an L_∞ algebra, we show the relation

$$\sum_{i_j=n+1} \sum_\sigma (-1)^\sigma (-1)^{e(\sigma)} (-1)^{i(j-1)} l_j(l_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0$$

holds on the maps for each n. It is important to note that we are still using the definitions for A_∞ and L_∞ algebras that have not been desuspended, that is, we are using the first two

definitions.

For $n = 1$, we have that

$$
\begin{aligned}
l_1(l_1(v_1)) &= l_1(w) \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
l_1(l_1(v_2)) &= l_1(w) \\
&= 0
\end{aligned}
$$

For $n = 2$, the only elements of V that are nonzero when maps are applied are $v_1 \otimes w$ and $v_1 \otimes v_2$, so we show this sum is zero on each element:

For $v_1 \otimes w$ we have:

$$
\begin{aligned}
&\sum_{i_j=n+1} \sum_{\sigma} (-1)^{\sigma}(-1)^{e(\sigma)}(-1)^{i(j-1)} l_j(l_i(\sigma(v_1, w))) \\
&= (-1)^0(-1)^0(-1)^{2(1-1)} l_1(l_2(v_1 \otimes w)) + (-1)^0(-1)^0(-1)^{1(2-1)} l_2(l_1(v_1), w) \\
&\quad + (-1)^1(-1)^{0 \cdot 1}(-1)^{1(2-1)} l_2(l_1(w), v_1) \\
&= l_1(l_2(v_1 \otimes w)) - l_2(l_1(v_1), w) + l_2(l_1(w), v_1) \\
&= l_1(s_3 w) - l_2(w, w) + l_2(0, v_1) \\
&= s_3 \cdot 0 - 0 + 0 \\
&= 0
\end{aligned}
$$

For $v_1 \otimes v_2$ we have:

$$\sum_{i_j=n+1}\sum_{\sigma}(-1)^{\sigma}(-1)^{e(\sigma)}(-1)^{i(j-1)}l_j(l_i(\sigma(v_1,v_2)))$$

$$= \quad (-1)^0(-1)^0(-1)^{2(1-1)}l_1(l_2(v_1\otimes v_2)) + (-1)^0(-1)^0(-1)^{1(2-1)}l_2(l_1(v_1),v_2)$$

$$+(-1)^1(-1)^{0\cdot 0}(-1)^{1(2-1)}l_2(l_1(v_2),v_1)$$

$$= \quad l_1(l_2(v_1,v_2)) - l_2(l_1(v_1),v_2) + l_2(l_1(v_2),v_1)$$

$$= \quad l_1(s_3v_1) - l_2(w,v_2) + l_2(w,v_1)$$

$$= \quad s_3l_1(v_1) - 0 - l_2(v_1,w)$$

$$= \quad s_3w - s_3w$$

$$= \quad 0$$

Next, we move to $n \geq 3$. As a precursor to the generalized result, we will show the relations hold on the two elements that give nonzero maps, $v_1 \otimes w \otimes w$ and $v_1 \otimes w \otimes v_2$. Two comments are of importance here, we use w_1 and w_2 to keep track of order. These do not denote two different elements in V_1, as both are w. Secondly, in terms where multiple transpositions occur, we multiply by more than one Koszul sign, one for each transposition.

For $v_1 \otimes w \otimes w$ we have:

$$\sum_{i_j=n+1}\sum_{\sigma}(-1)^{\sigma}(-1)^{e(\sigma)}(-1)^{i(j-1)}l_j(l_i(\sigma(v_1,w,w)))$$

$$= \quad (-1)^0(-1)^0(-1)^{1(3-1)}l_3(l_1(v_1),w_1,w_2)$$

$$+(-1)^{0\cdot 1}(-1)^1(-1)^{1(3-1)}l_3(l_1(w_1),v_1,w_2)$$

$$+(-1)^{0\cdot 1}(-1)^{1\cdot 1}(-1)^2(-1)^{1(3-1)}l_3(l_1(w_2),v_1,w_1)$$

$$+(-1)^0(-1)^0(-1)^{2(2-1)}l_2(l_2(v_1,w_1),w_2)$$

$$+(-1)^{1\cdot 1}(-1)^1(-1)^{2(2-1)}l_2(l_2(v_1,w_2),w_1)$$

$$+(-1)^{0\cdot 1}(-1)^{0\cdot 1}(-1)^2(-1)^{2(2-1)}l_2(l_2(w_1,w_2),v_1)$$

$$+(-1)^0(-1)^0(-1)^{3(1-1)}l_1(l_3(v_1,w_1,w_2))$$

$$= \quad l_3(l_1(v_1),w_1,w_2) - l_3(l_1(w_1),v_1,w_2) - l_3(l_1(w_2),v_1,w_1) + l_2(l_2(v_1,w_1),w_2)$$

$$+l_2(l_2(v_1,w_2),w_1) + l_2(l_2(w_1,w_2),v_1) + l_1(l_3(v_1,w_1,w_2))$$

$$= \quad l_3(w,w,w) - l_3(0,v_1,w_2) - l_3(0,v_1,w_1) + l_2(s_3w,w) + l_2(s_3w,w)$$

$$+l_2(0,v_1) + l_1(2!s_4w)$$

$$= \quad 0$$

For $v_1 \otimes w \otimes v_2$ we have:

$$\sum_{i_j=n+1} \sum_\sigma (-1)^\sigma (-1)^{e(\sigma)} (-1)^{i(j-1)} l_j(l_i(\sigma(v_1, w, v_2)))$$

$$
\begin{aligned}
&= (-1)^0 (-1)^0 (-1)^{1(3-1)} l_3(l_1(v_1), w, v_2) \\
&\quad + (-1)^{0\cdot1} (-1)^1 (-1)^{1(3-1)} l_3(l_1(w), v_1, v_2) \\
&\quad + (-1)^{0\cdot0} (-1)^{0\cdot1} (-1)^2 (-1)^{1(3-1)} l_3(l_1(v_2), v_1, w) \\
&\quad + (-1)^0 (-1)^0 (-1)^{2(2-1)} l_2(l_2(v_1, w), v_2) \\
&\quad + (-1)^{0\cdot1} (-1)^1 (-1)^{2(2-1)} l_2(l_2(v_1, v_2), w) \\
&\quad + (-1)^{0\cdot0} (-1)^{0\cdot1} (-1)^2 (-1)^{2(2-1)} l_2(l_2(w, v_2), v_1) \\
&\quad + (-1)^0 (-1)^0 (-1)^{3(1-1)} l_1(l_3(v_1, w, v_2)) \\
&= l_3(l_1(v_1), w, v_2) - l_3(l_1(w), v_1, v_2) + l_3(l_1(v_2), v_1, w) + l_2(l_2(v_1, w), v_2) \\
&\quad - l_2(l_2(v_1, v_2), w) + l_2(l_2(w, v_2), v_1) + l_1(l_3(v_1, w, v_2)) \\
&= l_3(w, w, v_2) - l_3(0, v_1, v_2) + l_3(w, v_1, w) + l_2(s_3 w, v_2) \\
&\quad - l_2(s_3 v_1, w) + l_2(0, v_1) + l_1(s_4 v_1) \\
&= -2 s_4 w - s_3 s_3 w + s_4 w \\
&= 2w - w - w \\
&= 0
\end{aligned}
$$

Now we move to the generalized case of $n \geq 3$. The two elements to consider here are $v_1 \otimes w^{\otimes n-1}$ and $v_1 \otimes w^{\otimes n-2} \otimes v_2$.

For $v_1 \otimes w^{\otimes n-1}$ each entry in the sum is of the form:

$$l_j(l_i(v_1, w^{\otimes n-j-2}), w^{\otimes j-1}) \text{ or } l_j(l_i(w^{\otimes i}), v_1, w^{\otimes n-i-1})$$

But simplifying these gives:

$$l_j((i-1)! s_{i+1} w), w^{\otimes j-1}) \text{ or } l_j(0, v_1, w^{\otimes n-i-1})$$

In either case, the term is zero. Hence the sum is zero and the relation holds.

Our last case to consider is the sum acting on $v_1 \otimes w^{\otimes n-2} \otimes v_2$. First, we note that the only way we are able to get nonzero terms is when we have elements from our initial list where the maps were defined. All other terms in the sum will be zero. Hence, the only nonzero terms are:

$$l_n(l_1(v_2), v_1, w^{\otimes(n-2)}),\ l_{n-1}(l_2(v_1, v_2), w^{\otimes(n-2)}),\ l_1(l_n(v_1, w^{\otimes(n-2), v_2})),$$

$$\text{and } l_{n-i+1}(l_i(v_1,, w^{\otimes(i-2)}, v_2), w^{\otimes n-i}))\ \text{where } n > i > 2$$

Also note that for each i where $2 < i < n$, we will have $\binom{n-2}{n-i}$ terms because we can choose any $n - i$ of the w-elements. Without considering the coefficients, our sum is:

$$\pm l_1(l_n(v_1, w^{\otimes n-2}, v_2)) \pm l_{n-1}(l_2(v_2, v_2), w^{\otimes n-2}) \pm l_n(l_1(v_2), v_1, w^{\otimes n-2}),$$

$$\text{and } \pm \sum_{2<i<n} \binom{n-2}{n-i} l_{n-i+1}(l_i(v_1, w^{\otimes i-2}, v_2), w^{\otimes n-i})$$

Now we find the coefficients. For our last term, we first permute the w-elements, then move the $i - 2$ w-elements past the v_2 element. Permuting elements from V_1 results in a positive sign (a -1 for the permutation multiplied by a $(-1)^{1 \cdot 1}$ for the Koszul sign), so we leave these positive one multiplications out. The only sign that is left is the $(-1)^{0 \cdot 1}(-1)^{n-i} = (-1)^{n-i}$, which comes from moving v_2 past $n - i$ w-elements. Now our sum becomes:

$$
\begin{aligned}
=\ & (-1)^{n \cdot 0} l_1(l_n(v_1, w^{\otimes n-2}, v_2)) + (-1)^{2(n-2)}(-1)^{n-2} l_{n-1}(l_2(v_2, v_2), w^{\otimes n-2}) \\
& (-1)^{n-1}(-1)^{n-1} l_n(l_1(v_2), v_1, w^{\otimes n-2}) \\
& + \sum_{2<i<n} (-1)^{n-i}(-1)^{i(n-i)} \binom{n-2}{n-i} l_{n-i+1}(l_i(v_1, w^{\otimes i-2}, v_2), w^{\otimes n-i}) \\
=\ & (-1)^{n \cdot 0} l_1((n-2)! s_{n+1} v_1) + (-1)^{2(n-2)}(-1)^{n-2} l_{n-1}(s_3 v_1, w^{\otimes n-2}) \\
& (-1)^{n-1}(-1)^{n-1} l_n(w, v_1, w^{\otimes n-2}) \\
& + \sum_{2<i<n} (-1)^{n-i}(-1)^{i(n-i)} \binom{n-2}{n-i} l_{n-i+1}((i-2)! s_{i+1} v_1, w^{\otimes n-i}) \\
=\ & (n-2)! s_{n+1} w + (-1)^{n-2} s_n w - (n-1)! s_{n+1} w \\
& + \sum_{2<i<n} \frac{(n-2)!}{(n-i)!(i-2)!}(-1)^{n-i}(-1)^{i(n-i)}(i-2)! s_{i+1}(n-i)! s_{n-i+2} w
\end{aligned}
$$

To show this sum is zero, it is equivalent to show the coefficients of w add to 0. And since each has a factor of $(n-2)!$, we can divide by $(n-2)!$ and simplify exponents to get:

$$
\begin{aligned}
& (-1)^{\frac{(n+2)(n+2)}{2}} + (-1)^{n-2}(-1)^{\frac{(n+1)(n+2)}{2}} - (n-1)(-1)^{\frac{(n+2)(n+3)}{2}} \\
& + \sum_{2<i<n} (-1)^{n-i}(-1)^{i(n-i)}(-1)^{\frac{(i+2)(i+3)}{2}}(-1)^{\frac{(n-i+3)(n-i+4)}{2}}
\end{aligned}
$$

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Also note that $(-1)^{n+2} = (-1)^{n-2}$, so we can simplify the second term to be $(-1)^{\frac{(n+2)(n+3)}{2}}$ and so our sum becomes:

$$(3-n)(-1)^{\frac{(n+2)(n+3)}{2}} + \sum_{2<i<n}(-1)^{n-i}(-1)^{i(n-i)}(-1)^{\frac{(i+2)(i+3)}{2}}(-1)^{\frac{(n-i+3)(n-i+4)}{2}}$$

We use a computer programming language (Maple) to simplify the sum. Also note, that our sum only depends on exponents being even or odd. And since these exponents are being divided by 2, we can take every term modulo 4 in the numerator of the exponents. So we have:

$$
\begin{aligned}
&(3-n)(-1)^{\frac{(n+2)(n+3)}{2}} - (n+\tfrac{1}{2})(-1)^{\frac{n(n+5)}{2}} + \tfrac{7}{2}(-1)^{\frac{n(n+9)}{2}} \quad \text{from Maple} \\
=\ &(3-n)(-1)^{\frac{(n+2)(n+3)}{2}} + (n+\tfrac{1}{2})(-1)^{\frac{n^2+5n-2}{2}} - \tfrac{7}{2}(-1)^{\frac{9n^2+n-2}{2}} \\
\equiv\ &(3-n)(-1)^{\frac{(n+2)(n+3)}{2}} + (n+\tfrac{1}{2})(-1)^{\frac{n^2+n-2}{2}} - \tfrac{7}{2}(-1)^{\frac{n^2+n-2}{2}} \quad \text{mod } 4 \\
\equiv\ &(3-n)(-1)^{\frac{(n+2)(n+3)}{2}} + (n+\tfrac{1}{2}-\tfrac{7}{2})(-1)^{\frac{n^2+n-2}{2}} \quad \text{mod } 4 \\
\equiv\ &(3-n)(-1)^{\frac{(n+2)(n+3)}{2}} + (n-3)(-1)^{\frac{n^2+n+6}{2}} \quad \text{mod } 4 \\
=\ &(3-n)(-1)^{\frac{(n+2)(n+3)}{2}} + (n-3)(-1)^{\frac{(n+2)(n+3)}{2}} \\
=\ &0
\end{aligned}
$$

Now we have shown that for any n,

$$\sum_{i_j=n+1}\sum_{\sigma}(-1)^{\sigma}(-1)^{e(\sigma)}(-1)^{i(j-1)}l_j(l_i(v_{\sigma(1)},\ldots,v_{\sigma(i)}),v_{\sigma(i+1)},\ldots,v_{\sigma(n)})=0$$

thus proving that our finite example is, in fact, an L_∞ algebra.

2.3 Finite Desuspended A_∞ Example

On a higher level (where all maps have degree one and we use our second definition), we use work from Michael Allocca. From Allocca's paper [1], he desuspended our previous A_∞ algebra and proved the following gives an A_∞ algebra structure:

Example 11. Let $V = V_{-1} + V_0$ be given by $V_{-1} = \langle x_1, x_2 \rangle$ and $V_0 = \langle y \rangle$. The following maps describe an A_∞ structure on V:

$$\hat{m}_1(x_1) = \hat{m}_1(x_2) = y$$
$$\text{For } n \geq 2, \ \hat{m}_n(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{n-2-k}) = x_1 \text{ for } 0 \leq k \leq n - 2$$
$$\hat{m}_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = x_1$$
$$\hat{m}_n(x_1 \otimes y^{\otimes n-1}) = y$$

2.4 Finite Desuspended L_∞ Example

From our previous example of an L_∞ Algebra, we need to desuspend this algebra to find a finite example on a desuspended coalgebra. To do this, we look at Lada's paper [6] where he shows how to obtain these desuspended maps, we let $W = W_{-1} + W_0$ where $W_{-1} = \langle x_1, x_2 \rangle$ and $W_0 = \langle y \rangle$ such that the desuspension operator, $\downarrow$, is given by: $\downarrow v_1 = x_1$, $\downarrow v_2 = x_2$, and $\downarrow w = y$. Then the collection of degree one symmetric linear maps $\hat{l}_n : W^{\otimes n} \to W$ given by $\hat{l}_n = (-1)^{\frac{n(n-1)}{2}} \downarrow \circ l_n \circ \uparrow^{\otimes n}$ gives an L_∞ algebra structure, given by our second definition of L_∞ algebra.

To find these $\hat{l}_n$ maps, we apply the above map to x_1, x_2, $x_1 \otimes y^{\otimes n-1}$, and $x_1 \otimes y^{\otimes n-2} \otimes x_2$, as these come from the maps of l_n. As a note about signs, each time we apply a desuspension, we need to consider how many terms this operator has moved past, as an example

$$l_3 \circ \uparrow^{\otimes 3} (x_1, y, y) = (-1)^{|x_1| + |y| + |x_1|} l_3(v_1, w, w)$$

since one operator has moved past x_1 and y, one has moved past x_1, and one hasn't moved past anything. Then we have four calculations to perform:

(i) On x_1:

$$
\begin{aligned}
\hat{l}_1(x_1) &= (-1)^{\frac{1(1-1)}{2}} \downarrow \circ l_1 \circ \uparrow^{\otimes 1} (x_1) \\
&= (-1)^{\frac{1(1-1)}{2}} \downarrow \circ l_1(v_1) \\
&= (-1)^{\frac{1(1-1)}{2}} \downarrow (-1)^{\frac{(1+2)(1+3)}{2}} w \\
&= y
\end{aligned}
$$

(ii) On x_2:

$$
\begin{aligned}
\hat{l}_1(x_2) &= (-1)^{\frac{1(1-1)}{2}} \downarrow \circ l_1 \circ \uparrow^{\otimes 1} (x_2) \\
&= (-1)^{\frac{1(1-1)}{2}} \downarrow \circ l_1 (v_2) \\
&= (-1)^{\frac{1(1-1)}{2}} \downarrow (-1)^{\frac{(1+2)(1+3)}{2}} w \\
&= y
\end{aligned}
$$

(iii) On the element $x_1 \otimes y^{\otimes n-1}$, we have:

$$
\begin{aligned}
\hat{l}_n(x_1 \otimes y^{\otimes n-1}) &= (-1)^{\frac{n(n-1)}{2}} \downarrow \circ l_n \circ \uparrow^{\otimes n} (x_1 \otimes y^{\otimes n-1}) \\
&= (-1)^{\frac{n(n-1)}{2}} (-1)^{n-1} \downarrow \circ l_n (v_1 \otimes w^{\otimes n-1}) \\
&= (-1)^{\frac{n(n-1)}{2}} (-1)^{n-1} \downarrow (-1)^{\frac{(n+2)(n+3)}{2}} (n-1)! w \\
&= (-1)^{\frac{n(n-1)+2n-2+(n+2)(n+3)}{2}} (n-1)! y \\
&= (n-1)! y
\end{aligned}
$$

The reason we can simplify this last exponent so much is due to the fact that everything is modulo 2 in the exponent of -1. Since we have that $\frac{n(n-1)+2n-2+(n+2)(n+3)}{2} = (n+1)(n+2)$, where either $n+1$ or $n+2$ is even, the exponent of -1 is even, so

$$
(-1)^{\frac{n(n-1)+2n-2+(n+2)(n+3)}{2}} = +1
$$

(iv) Lastly, we look at $x_1 \otimes y^{\otimes n-2} \otimes x_2$:

$$
\begin{aligned}
\hat{l}_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) &= (-1)^{\frac{n(n-1)}{2}} \downarrow \circ l_n \circ \uparrow^{\otimes n} (x_1 \otimes y^{\otimes n-2} \otimes x_2) \\
&= (-1)^{\frac{n(n-1)}{2}} (-1)^{n-1} \downarrow \circ l_n (v_1 \otimes w^{\otimes n-2} \otimes v_2) \\
&= (-1)^{\frac{n(n-1)+2n-2}{2}} (-1)^{\frac{(n+2)(n+3)}{2}} \downarrow (n-2)! v_1 \\
&= (-1)^{\frac{n(n-1)}{2}} (-1)^{\frac{(n+2)(n+3)}{2}} (n-2)! x_1 \\
&= (n-2)! x_1
\end{aligned}
$$

Again, we can simplify this exponent because everything is modulo 2 in the power of -1.

This work gives us the following:

Example 12. Let $W = W_{-1} + W_0$ be given by $W_{-1} = \langle x_1, x_2 \rangle$ and $W_0 = \langle y \rangle$, which has been desuspended from our previous finite L_∞ algebra given by V. The maps given by $\hat{l}_n : W^{\otimes n} \to W$

where

$$\hat{l}_1(x_1) = \hat{l}_1(x_2) = y$$
$$\hat{l}_n(x_1 \otimes y^{\otimes n-1}) = (n-1)!y$$
$$\hat{l}_n(x_1 \otimes y^{n-2} \otimes x_2) = (n-2)!x_1$$

give an L_∞ structure, as defined in the second definition using a coalgebra.

2.4.1 Showing Sum Relation

By the work of Lada [6], we know our desuspended algebra is an example of an L_∞ algebra, as he proves that by setting $\hat{l}_n = \downarrow \circ l_n \circ \uparrow^{\otimes n}$, the result is an L_∞ algebra. To verify the accuracy of our work, and to show how the mappings work inside the double sum, we show (by definition) that this is, in fact, an L_∞ algebra. To prove this, we look to our sum in the second definition of L_∞ (1.4) algebra and show that

$$\sum_{\sigma \in S_{k+l=n}} (-1)^{\epsilon(\sigma)} l_{1+l}(l_k(c_{\sigma(1)}, \ldots, c_{\sigma(k)}), c_{\sigma(k+1)}, \ldots, c_{\sigma(n)}) = 0$$

We show this double sum is zero on each of our four inputs as follows

(i) We have that $\hat{l}_1 \circ \hat{l}_1(x_1) = \hat{l}_1(y) = 0$, so the definition holds.

(ii) We also have that $\hat{l}_1 \circ \hat{l}_1(x_2) = \hat{l}_1(y) = 0$, so again the definition holds.

(iii) When we look at this double sum on $x_1 \otimes y^{\otimes n-1}$, the only terms we need to consider are those where x_1 is in the first position. Here we have:

$$\pm \hat{l}_{n-j+1}(\hat{l}_j(x_1 \otimes y^{\otimes j-1}), y^{\otimes n-j}) = \pm \hat{l}_{n-j+1}((j-1)!y, y^{\otimes n-j})$$
$$= 0$$

or we have the term

$$\hat{l}_n(\hat{l}_1(x_1), y^{\otimes n-1}) = \hat{l}_n(y, y^{\otimes n-1})$$
$$= 0$$

Therefore each term in the double sum is zero and hence the definition holds.

(iv) Lastly, we look at the double sum on the element $x_1 \otimes y^{\otimes n-2} \otimes x_2$. Inside the double sum there are two types of elements we need to consider, as all others will be zero. These nonzero terms are

(I) $\pm \hat{l}_n(\hat{l}_1(x_2), x_1 \otimes y^{\otimes n-2})$

(II) $\pm \hat{l}_i(\hat{l}_j(x_1 \otimes y^{\otimes j-2} \otimes x_2), y^{\otimes n-j})$ for $j = 2, \ldots n-1$.

We go through these and look at each element, then add them to get zero.

(I) Since we have switched x_1 and x_2, both of degree -1, we have that

$$\begin{aligned}
-\hat{l}_n(\hat{l}_1(x_2), x_1, y^{\otimes n-2}) &= -\hat{l}_n(y, x_1, y^{\otimes n-2}) \\
&= -(n-1)!y
\end{aligned}$$

(II) Lastly, we have that,

$$\begin{aligned}
\pm \hat{l}_i(\hat{l}_j(x_1 \otimes y^{\otimes j-2} \otimes x_2), y^{\otimes n-j}) &= \hat{l}_i((j-2)!x_1, y^{\otimes n-j}) \\
&= (n-j)!(j-2)!y
\end{aligned}$$

Now note that there are $\binom{n-2}{j-2}$ elements of this form for each $j = 2, \ldots n..$ Since there are $\binom{n-2}{j-2}$ terms, when we add them all up we get:

$$\begin{aligned}
&\sum_{j=2}^{n} \binom{n-2}{j-2}(n-j)!(j-2)!y \\
={}& \sum_{j=2}^{n} \frac{(n-2)!}{(n-j)!(j-2)!}(n-j)!(j-2)!y \\
={}& \sum_{j=2}^{n} (n-2)!y \\
={}& [n(n-2)! - (n-2)!]y
\end{aligned}$$

Now, we have these two types of terms, only one each of type (I), and we've already added up the $\binom{n-2}{j-2}$ terms of type (II) for $j = 2, \ldots, n$. We add all of these to get:

$$\begin{aligned}
[n(n-2)! - (n-2)!]y + -(n-1)!y &= [-(n-1)(n-2)! + (n-1)(n-2)!]y \\
&= (n-2)![-(n-1) + (n-1)]y \\
&= 0
\end{aligned}$$

Therefore, the double sum from our definition of L_∞ algebra holds on all types of elements and our maps on the desuspended algebra do, in fact, give an L_∞ algebra from the second definition given.

2.5 Desuspended Connection

From our work, we've seen that we can get an example of an L_∞ algebra from an A_∞ algebra at the lower level by setting $l_n = \sum_{\sigma \in S_n} (-1)^\tau m_n \circ \sigma$, where $(-1)^\tau$ is the product of the sign of the permutation with the degrees of the permuted elements. Then, we can lift each of these algebras to achieve A_∞ and L_∞ algebras that have been desuspended. The question we then asked, is are these desuspended algebras related in the same was that the lower level algebras are? That is to say, for our example, does the following diagram commute:
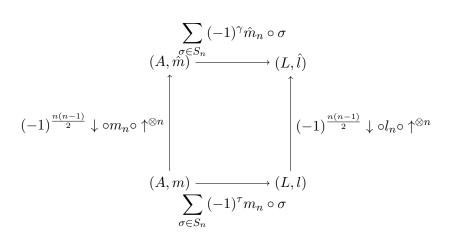
$$(A, \hat{m}) \xrightarrow{\sum_{\sigma \in S_n} (-1)^\gamma \hat{m}_n \circ \sigma} (L, \hat{l})$$

$$(-1)^{\frac{n(n-1)}{2}} \downarrow \circ m_n \circ \uparrow^{\otimes n} \qquad\qquad (-1)^{\frac{n(n-1)}{2}} \downarrow \circ l_n \circ \uparrow^{\otimes n}$$

$$(A, m) \xrightarrow[\sum_{\sigma \in S_n} (-1)^\tau m_n \circ \sigma]{} (L, l)$$

Figure 2.1: Ways to Lift Our Example

Where $(-1)^\gamma$ comes from the degrees of the permuted elements. That is to say, if we permuted x and y within our sum, we would have a coefficient of $(-1)^{|x||y|}$. Note that at the upper level, because the maps are symmetric, we do not need to account for the degree of the permutation inside the summation, only the degree of the permuted elements.

For our example, this answer is yes, as we show by acting $\sum_{\sigma \in S_n} (-1)^\gamma \hat{m}_n \circ \sigma$ on the elements, $x_1, x_2, x_1 \otimes y^{\otimes n-1}, x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes n-2-k}$, and $x_1 \otimes y^{\otimes n-2} \otimes x_2$, and show this gives our L_∞ algebra example.

Look at $\hat{l}_1$. Since the only permutation of one element is the identity, we have that

$$\hat{l}_1(x_1) = \hat{m}_1(x_1) = y$$

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and

$$\hat{l}_1(x_2) = \hat{m}_1(x_2) = y$$

Now, let $n \geq 2$. We first look at $\hat{l}_n(x_1 \otimes y^{\otimes(n-1)})$. When we look at the sum $\sum_{\sigma \in S_n} (-1)^\gamma \hat{m}_n \circ \sigma$ acting on this element, the only non-zero terms will be $\hat{m}_n(x_1 \otimes y^{\otimes(n-1)})$ for each σ permuting the $y's$. Any other term will have $\hat{m}_n(y \otimes \cdots)$, which is zero. Now we consider the degree sign of each of these terms. Since $y \in W_0$, when we permute any two $y's$ we get a coefficient sign of $1^{0\cdot0} = 1$, giving a positive sign for each term. The number of nonzero terms is the number of ways we can permute the $n-1$ $y's$, which is $(n-1)!$. Now we have $(n-1)!$ terms, each one is

$$\hat{m}_n(x_1 \otimes y^{\otimes(n-1)}) = y$$

therefore

$$\begin{aligned}
\hat{l}_n(x_1 \otimes y^{\otimes(n-1)}) &= (\sum_{\sigma \in S_n} (-1)^\gamma \hat{m}_n \circ \sigma)(x_1 \otimes y^{\otimes(n-1)}) \\
&= (n-1)!y
\end{aligned}$$

Next, look at $\hat{l}_n(x_1 \otimes y^{\otimes(n-2)} \otimes x_2)$. When we expand this in the summation $\sum_{\sigma \in S_n} (-1)^\gamma \hat{m}_n \circ \sigma$, we see that the only nonzero terms will be of the form $\hat{m}_n(x_1 \otimes y^{\otimes(n-2)} \otimes x_2)$, when we permute; this is because the $\hat{m}_n$ maps involving x_1 and x_2 for $n \geq 2$ are only defined when x_1 is the first term and x_2 is the last term.

In a similar fashion as before, when we permute $y's$ we get a positive degree number. The number of terms in the sum will be the number of ways we can permute the $y^{\otimes(n-2)}$, which is $(n-2)!$. Since each term is positive, we get:

$$\begin{aligned}
\hat{l}_n(x_1 \otimes y^{\otimes(n-2)} \otimes x_2) &= (n-2)!\hat{m}_n(x_1 \otimes y^{\otimes(n-2)}) \\
&= (n-2)!x_1
\end{aligned}$$

The last nonzero element to consider is $x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes(n-2)-k}$. We will show for $n \geq 2$, $\hat{l}_n(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes(n-2)-k}) = 0$. For explanation purposes, we will distinguish the two x_1 as x_{1_1} and x_{1_2}, so we are looking for

$$\hat{l}_n(x_{1_1} \otimes y^{\otimes k} \otimes x_{1_2} \otimes y^{\otimes(n-2)-k})$$

When we expand the summation, the only nonzero terms will be of the form

$$\hat{m}_n(x_{1_1} \otimes y^{\otimes k} \otimes x_{1_2} \otimes y^{\otimes(n-2)-k})$$

and

$$\hat{m}_n(x_{1_2} \otimes y^{\otimes k} \otimes x_{1_1} \otimes y^{\otimes(n-2)-k})$$

for some permutations on $y's$.

Note that there are $n-1$ terms of the form $\hat{m}_n(x_{1_1} \otimes y^{\otimes k} \otimes x_{1_2} \otimes y^{\otimes(n-2)-k})$, these are:

$$x_{1_1} \otimes x_{1_2} \otimes y^{\otimes(n-2)}$$

$$x_{1_1} \otimes y \otimes x_{1_2} \otimes y^{\otimes(n-3)}$$

$$\vdots$$

$$x_{1_1} \otimes y^{\otimes(n-2)} \otimes x_{1_2}$$

Similarly, there are $n-1$ terms of the form $\hat{m}_n(x_{1_2} \otimes y^{\otimes k} \otimes x_{1_1} \otimes y^{\otimes(n-2)-k})$, each one corresponding to switching x_{1_1} and x_{1_2} from above. These are the only nonzero terms since a y in the first coordinate gives a zero for $\hat{m}_n$.

We look at the correspondence of the sign of $\hat{m}_n(x_{1_1} \otimes y^{\otimes k} \otimes x_{1_2} \otimes y^{\otimes(n-2)-k})$ and the sign of $\hat{m}_n(x_{1_2} \otimes y^{\otimes k} \otimes x_{1_1} \otimes y^{\otimes(n-2)-k})$. Say the sign of $\hat{m}_n(x_{1_1} \otimes y^{\otimes k} \otimes x_{1_2} \otimes y^{\otimes(n-2)-k})$ is $+1$. Then the sign of

$$\hat{m}_n(x_{1_1} \otimes x_{1_2} \otimes y^{\otimes k} \otimes y^{\otimes(n-2)-k})$$

is $(+1)$ since time we permute any y and x_1, the degrees of which are 0 and -1, respectively, we get a corresponding $(-1)^{0 \cdot -1} = +1$. Continuing, the sign of

$$\hat{m}_n(x_{1_2} \otimes x_{1_1} \otimes y^{\otimes k} \otimes y^{\otimes(n-2)-k})$$

is $(+1)(-1)$ since we've transposed two elements, each of degree -1.

Moving those k $w's$ back to the right, gives the sign of

$$\hat{m}_n(x_{1_2} \otimes y^{\otimes k} \otimes x_{1_1} \otimes y^{\otimes(n-2)-k})$$

is $(+1)(-1)(+1)$, for the same reasoning as above. Hence, when the sign of $\hat{m}_n(x_{1_1} \otimes y^{\otimes k} \otimes x_{1_2} \otimes y^{\otimes(n-2)-k})$ is $+1$, the sign of $\hat{m}_n(x_{1_2} \otimes y^{\otimes k} \otimes x_{1_1} \otimes y^{\otimes(n-2)-k})$ is -1. The $n-1$ of the first type then cancel out with the $n-1$ of the second type, giving us 0 in the summations.

Therefore,
$$\hat{l}_n(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes(n-2)-k}) = 0$$

for $n \geq 2$. Another way to look as this is that from properties of $\hat{l}_n$, we have:

$$\hat{l}_n(x_{1_2} \otimes y^{\otimes k} \otimes x_{1_1} \otimes y^{\otimes(n-2)-k}) = \hat{l}_n(x_1 \otimes x_1 \otimes y^{\otimes(n-2)})$$
$$= (-1)^{-1\cdot-1}\hat{l}_n(x_1 \otimes x_1 \otimes y^{\otimes(n-2)}) \text{ from the degrees of } x_1$$

And since
$$\hat{l}_n(x_1 \otimes x_1 \otimes y^{\otimes(n-2)}) = (-1)\hat{l}_n(x_1 \otimes x_1 \otimes y^{\otimes(n-2)})$$

we must have that $\hat{l}_n(x_{1_2} \otimes y^{\otimes k} \otimes x_{1_1} \otimes y^{\otimes(n-2)-k}) = 0$.

From this work, we can see that these $\hat{l}$ are precisely those we found by lifting our finite L_∞ algebra, and therefore our diagram:
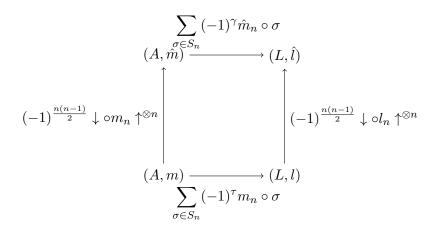

Figure 2.2: Commuting Diagram to Lift in Our Example

does, in fact, commute, which gives rise to the idea that you can symmetrize and then lift to go from a lower level A_∞ algebra to a desuspend L_∞ algebra, or you can lift and then symmetrize. We will look at this in more detail in chapter 7.

2.6 Our Four Examples

For simplicity, our four concrete examples we will use through this paper are:

Example 13 (A_∞)**.** Let V denote a graded vector space given by $V = \oplus V_n$ where V_0 has basis $< v_1, v_2 >$, V_1 has basis $< w >$, and $V_n = 0$ for $n \neq 0, 1$. The structure on V is defined by the linear maps $m_n : V^{\otimes n} \to V$:

$$m_1(v_1) = m_1(v_2) = w$$

$$\text{For } n \geq 2 : m_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes (n-2)-k}) = (-1)^k s_n v_1, \ 0 \leq k \leq n - 2$$

$$m_n(v_1 \otimes w^{\otimes (n-2)} \otimes v_2) = s_{n+1} v_1$$

$$m_n(v_1 \otimes w^{\otimes (n-1)}) = s_{n+1} w$$

where $s_n = (-1)^{\frac{(n+1)(n+2)}{2}}$, and $m_n = 0$ when evaluated on any element of $V^{\otimes n}$ that is not listed above.

Example 14 (Desuspended A_∞)**.** Let $W = W_{-1} + W_0$ be given by $W_{-1} = \langle x_1, x_2 \rangle$ and $W_0 = \langle y \rangle$. The following maps describe an A_∞ structure on V:

$$\hat{m}_1(x_1) = \hat{m}_1(x_2) = w$$
$$\text{For } n \geq 2, \ \hat{m}_n(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{n-2-k}) = x_1 \text{ for } 0 \leq k \leq n - 2$$
$$\hat{m}_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = y_1$$
$$\hat{m}_n(x_1 \otimes y^{\otimes n-1}) = y$$

Example 15 (L_∞)**.** Consider the graded vector space $V = V_0 \oplus V_1$ where V_0 has basis $< v_1, v_2 >$ and V_1 has basis $< w >$. We show that this space has an L_∞ structure given by:

$$l_1(v_1) = l_1(v_2) = w$$
$$\text{For } n \geq 2 \ l_n(v_1 \otimes w^{\otimes (n-1)}) = (n-1)! s_{n+1} w$$
$$l_n(v_1 \otimes w^{\otimes (n-2)} \otimes v_2) = (n-2)! s_{n+1} v_1$$

where $s_n = (-1)^{\frac{(n+1)(n+2)}{2}}$ and $l_n = 0$ when evaluated on any element of $V^{\otimes n}$ that is not listed.

Example 16 (Desuspended L_∞)**.** Let $W = W_{-1} + W_0$ be given by $W_{-1} = \langle x_1, x_2 \rangle$ and $W_0 = \langle y \rangle$, which has been desuspended from our previous finite L_∞ algebra given by V. The

maps given by $\hat{l}_n : W^{\otimes n} \to W$ where

$$
\begin{aligned}
\hat{l}_1(x_1) = \hat{l}_1(x_2) &= y \\
\hat{l}_n(x_1 \otimes y^{\otimes n-1}) &= (n-1)!y \\
\hat{l}_n(x_1 \otimes y^{n-2} \otimes x_2) &= (n-2)!x_1
\end{aligned}
$$

give an L_∞ structure, as defined in the second definition using a coalgebra.

Chapter 3

Alternate A_∞ Strong Homotopy Derivation Definition

It is important to note that for the remainder of this paper, we will use the definitions of A_∞ and L_∞ algebras on a desuspended algebra.

We next look into the work of Hiroshige Kajiura and Jim Stasheff on homotopy algebras inspired by classical open-closed string field theory [5]. Here, Kajiura and Stasheff give the following definition:

Definition 17. (Strong homotopy derivation) A *strong homotopy derivation* of degree one of an A_∞-algebra (A, m) consists of a collection of multi-linear maps of degree one

$$\theta := \{\theta_q | A^{\otimes q} \to A\}_{q \geq 1}$$

satisfying the following relations:

$$
\begin{aligned}
0 = & \sum_{r+s=q+1} \sum_{i=0}^{r-1} (-1)^{\beta(s,i)} \theta_r(o_1, \dots, o_i, m_s(o_{i+1}, \dots, o_{i+s}), \dots, o_q) \\
& + (-1)^{\beta(s,i)} m_r(o_1, \dots, o_i, \theta_s(o_{i+1}, \dots, o_{i+s}), \dots, o_q)
\end{aligned}
\tag{3.1}
$$

Here the sign $\beta(s, i) = o_1 + \cdots + o_i$ results from moving m_s, respectively θ_s, past $(o_1, \dots, o_i)$.

In their paper, Kajiura and Stasheff go on to say that this sum is equivalent to seeing θ as a coderivation of $T^c A$ with no constant term and such that $[m, \theta] = 0$. Keep in mind, these maps have been lifted, so we are now using our second definition of A_∞ algebra.

First we note that for lifted m and θ, these maps have degree one. So when we apply these maps to elements, we don't need to worry about multiplying by the degree of the map, also the

commutator bracket is given by

$$m \circ \theta - (-1)^{|m||\theta|}\theta \circ m$$

and since both degrees are one, that is $|m| = |\theta| = 1$, we have the commutator bracket is reduced to $m \circ \theta + \theta \circ m$. We show this is equivalent to Kajiura and Stasheff's definition of a strong homotopy derivation.

We look at this bracket on one element:

$$
\begin{aligned}
[m, \theta](x) &= m\theta(x) + \theta m(x) \\
&= m_1\theta_1(x) + \theta_1 m_1(x)
\end{aligned}
$$

This is equivalent to the sum given by (3.1), and we later use that $m_1\theta_1 = -\theta_1 m_1$, since the sum (and hence the bracket) are set to zero by definition.

Now we work on two elements:

$$
\begin{aligned}
[m, \theta](x, y) &= m(\theta_2(x, y) + \theta_1(x) \otimes y + (-1)^{|x|}x \otimes \theta_1(y)) \\
&\quad + \theta(m_2(x, y) + m_1(x) \otimes y + (-1)^{|x|}x \otimes m_1(y)) \\
&= m_1\theta_2(x, y) + m_2(\theta_1(x), y) + m_1\theta_1(x) \otimes y + (-1)^{|\theta_1(x)|}\theta_1(x) \otimes m_1(y) \\
&\quad + (-1)^{|x|}m_2(x, \theta_1(y)) + (-1)^{|x|}m_1(x) \otimes \theta_1(y) + (-1)^{|x|+|x|}x \otimes m_1\theta_1(y) \\
&\quad + \theta_1 m_2(x, y) + \theta_2(m_1(x), y) + \theta_1 m_1(x) \otimes y + (-1)^{|m_1(x)|}m_1(x) \otimes \theta_1(y) \\
&\quad + (-1)^{|x|}\theta_2(x, m_1(y)) + (-1)^{|x|}\theta_1(x) \otimes m_1(y)
\end{aligned}
$$

These signs come from the elements that m_i or θ_i has moved past. Technically, the coefficient for, say, $x \otimes m_1(y)$ is $(-1)^{|x||m_1|}$, but as we said before $|m_1| = 1$, so we don't write the degrees of the maps. Now we look at terms that cancel:

$$(-1)^{|\theta_1(x)|}\theta_1(x) \otimes m_1(y) + (-1)^{|x|}\theta_1(x) \otimes m_1(y) = 0$$

since $|\theta_1(x)| = 1 + |x|$, so $(-1)^{|x|}\theta_1(x) \otimes m_1(y) = -(-1)^{|\theta_1(x)|}\theta_1(x) \otimes m_1(y)$.

Similarly,

$$(-1)^{|x|}m_1(x) \otimes \theta(y) + (-1)^{|m_1(x)|}m_1(x) \otimes \theta_1(y) = 0$$

Now we use the fact that $m_1\theta_1 = -\theta_1 m_1$ to get that

$$m_1\theta_1(x) \otimes y + \theta_1 m_1(x) \otimes y = 0$$

and

$$(-1)^{|x|+|x|} x \otimes m_1\theta_1(y) + (-1)^{|x|+|x|} x \otimes \theta_1 m_1(y) = 0$$

This leaves us with

$$
\begin{aligned}
[m,\theta](x,y) &= m_1\theta_2(x,y) + m_2(\theta_1(x),y) + (-1)^{|x|}m_2(x,\theta_1(y)) \\
&+ \theta_1 m_2(x,y) + \theta_2(m_1(x),y) + (-1)^{|x|}\theta_2(x,m_1(y))
\end{aligned}
$$

which is equivalent to (3.1) on two inputs.

Before we generalize this on n inputs, we show the process in more detail with three elements:

$$[m,\theta](x,y,z) \;=\; m_1\theta_3(x,y,z) + m_2(\theta_2(x,y),z)$$
$$+\, m_1\theta_2(x,y)\otimes z + (-1)^{|\theta_2(x,y)|}\theta_2(x,y)\otimes m_1(z)$$
$$+\,(-1)^{|x|}m_2(x_1,\theta_2(y,z)) + (-1)^{|x|}m_1(x)\otimes\theta_2(y,z)$$
$$+\,(-1)^{|x|+|x|}x\otimes m_1\theta_2(y,z) + m_3(\theta_1(x),y,z)m_2(\theta_1(x),y)\otimes z$$
$$+\,(-1)^{|\theta_1(x)|}\theta_1(x)\otimes m_2(y,z) + m_1\theta_1(x)\otimes y\otimes z$$
$$+\,(-1)^{|\theta_1(x)|}\theta_1(x)\otimes m_1(y)\otimes z + (-1)^{|\theta_1(x)|+|y|}\theta_1(x)\otimes y\otimes m_1(z)$$
$$+\,(-1)^{|x|}m_3(x,\theta_1(y),z) + (-1)^{|y|}m_2(x,\theta_1(y))\otimes z$$
$$+\,(-1)^{|x|+|y|}x\otimes m_2(\theta_1(y),z) + (-1)^{|x|}m_1(x)\otimes\theta_1(y)\otimes z$$
$$=\; (-1)^{|x|+|x|}x\otimes m_1\theta_1(y)\otimes z + (-1)^{|x|+|\theta_1(y)|+|x|}x\otimes\theta_1(y)\otimes m_1(z)$$
$$+\,(-1)^{|x|+|y|}m_3(x,y,\theta_1(z)) + (-1)^{|x|+|y|}m_2(x,y)\otimes\theta_1(z)$$
$$+\,(-1)^{|x|+|y|+|x|}x\otimes m_2(y,\theta_1(z)) + (-1)^{|x|+|y|}m_1(x)\otimes y\otimes\theta_1(z)$$
$$+\,(-1)^{|x|+|\theta_1(y)|+|x|}x\otimes m_1(y)\otimes\theta_1(z) + (-1)^{|x|+|y|+|x|+|y|}x\otimes y\otimes m_1\theta_1(z)$$
$$+\,\theta_1 m_3(x,y,z) + \theta_2(m_2(x,y),z) + \theta_1 m_2(x,y)\otimes z$$
$$+\,(-1)^{|m_2(x,y)|}m_2(x,y)\otimes\theta_1(z) + (-1)^{|x|}\theta_2(x_1,m_2(y,z))$$
$$+\,(-1)^{|x|}\theta_1(x)\otimes m_2(y,z) + (-1)^{|x|+|x|}x\otimes\theta_1 m_2(y,z)$$
$$+\,\theta_3(m_1(x),y,z) + \theta_2(m_1(x),y)\otimes z + (-1)^{|m_1(x)|}m_1(x)\otimes\theta_2(y,z)$$
$$+\,\theta_1 m_1(x)\otimes y\otimes z + (-1)^{|m_1(x)|}m_1(x)\otimes\theta_1(y)\otimes z$$
$$+\,(-1)^{|m_1(x)|+|y|}m_1(x)\otimes y\otimes\theta_1(z) + (-1)^{|x|}\theta_3(x,m_1(y),z)$$
$$+\,(-1)^{|y|}\theta_2(x,m_1(y))\otimes z + (-1)^{|x|+|y|}x\otimes\theta_2(m_1(y),z)$$
$$+\,(-1)^{|x|}\theta_1(x)\otimes m_1(y)\otimes z$$
$$=\; (-1)^{|x|+|x|}x\otimes\theta_1 m_1(y)\otimes z + (-1)^{|x|+|m_1(y)|+|x|}x\otimes m_1(y)\otimes\theta_1(z)$$
$$+\,(-1)^{|x|+|y|}\theta_3(x,y,m_1(z)) + (-1)^{|x|+|y|}\theta_2(x,y)\otimes m_1(z)$$
$$+\,(-1)^{|x|+|y|+|x|}x\otimes\theta_2(y,m_1(z)) + (-1)^{|x|+|y|}\theta_1(x)\otimes y\otimes m_1(z)$$
$$+\,(-1)^{|x|+|m_1(y)|+|x|}x\otimes\theta_1(y)\otimes m_1(z) + (-1)^{|x|+|y|+|x|+|y|}x\otimes y\otimes\theta_1 m_1(z)$$

As a remark on the signs, consider the last element, $(-1)^{|x|+|y|+|x|+|y|}x\otimes y\otimes\theta_1 m_1(z)$. This came from θ acting on the element $(-1)^{|x|+|y|}x\otimes y\otimes m_1(z)$. The sign of $(-1)^{|x|+|y|}x\otimes y\otimes m_1(z)$ came from moving m_1 past x and y (we aren't permuting elements, so we only include signs obtained by moving the map past elements), and the fact that m_1 has degree one. The reason for the extra $(-1)^{|x|+|y|}$ in the term $(-1)^{|x|+|y|+|x|+|y|}x\otimes y\otimes\theta_1 m_1(z)$ came from moving the θ_1 past x and y, just like the m_1. Again, since the degree of θ is one, we don't bother to multiply

the exponent by the degree of the map.

Now we show that every term involving a tensor cancels out. First we look at those terms involving a $m_1\theta_1(x_i)$. Those that cancel are:

$$
\begin{aligned}
m_1\theta_1(x) \otimes y \otimes z \quad &\text{with} \quad \theta_1 m_1(x) \otimes y \otimes z \\
(-1)^{|x|+|x|} x \otimes \theta_1 m_1(y) \otimes z \quad &\text{with} \quad (-1)^{|x|+|x|} x \otimes m_1\theta_1(y) \otimes z \\
(-1)^{|x|+|y|+|x|+|y|} x \otimes y \otimes \theta_1 m_1(z) \quad &\text{with} \quad (-1)^{|x|+|y|+|x|+|y|} x \otimes y \otimes m_1\theta_1(z)
\end{aligned}
$$

Note that the coefficients are the same for these terms, which is no coincidence and we prove later.

Next, we look at repeated terms and show they add to zero. For example,

$$
\begin{aligned}
(-1)^{|\theta_2(x,y)|}\theta_2(x,y) \otimes m_1(z) + (-1)^{|x|+|y|}\theta_2(x,y) \otimes m_1(z) &= (-1)^{|x|+|y|+1}\theta_2(x,y) \otimes m_1(z) \\
&\quad +(-1)^{|x|+|y|}\theta_2(x,y) \otimes m_1(z) \\
&= -(-1)^{|x|+|y|}\theta_2(x,y) \otimes m_1(z) \\
&\quad +(-1)^{|x|+|y|}\theta_2(x,y) \otimes m_1(z) \\
&= 0
\end{aligned}
$$

All others of this form are:

$$
\begin{aligned}
(-1)^{|x|}m_1(x) \otimes \theta_2(y,z) \ &+\ (-1)^{|m_1(x)|}m_1(x) \otimes \theta_2(y,z) = 0 \\
(-1)^{|\theta_1(x)|}\theta_1(x) \otimes m_2(y,z) \ &+\ (-1)^{|x|}\theta_1(x) \otimes m_2(y,z) = 0 \\
(-1)^{|\theta_1(x)|}\theta_1(x) \otimes m_1(y) \otimes z \ &+\ (-1)^{|x|}\theta_1(x) \otimes m_1(y) \otimes z = 0 \\
(-1)^{|\theta_1(x)|+|y|}\theta_1(x) \otimes y \otimes m_1(z) \ &+\ (-1)^{|x|+|y|}\theta_1(x) \otimes y \otimes m_1(z) = 0 \\
(-1)^{|x|}m_1(x) \otimes \theta_1(y \otimes z) \ &+\ (-1)^{|m_1(x)|}m_1(x) \otimes \theta_1(y \otimes z) = 0 \\
(-1)^{|x|+|\theta_1(y)|+|x|}x \otimes \theta_1(y) \otimes m_1(z) \ &+\ (-1)^{|x|+|y|+|x|}x \otimes \theta_1(y) \otimes m_1(z) = 0 \\
(-1)^{|x|+|y|}m_2(x,y) \otimes \theta_1(z) \ &+\ (-1)^{|m_2(x,y)|}m_2(x,y) \otimes \theta_1(z) = 0 \\
(-1)^{|x|+|y|}m_1(x) \otimes y \otimes \theta_1(z) \ &+\ (-1)^{|m_1(x)|+|y|}m_1(x) \otimes y \otimes \theta_1(z) = 0 \\
(-1)^{|x|+|\theta_1(y)|+|x|}x \otimes m_1(y) \otimes \theta_1(z) \ &+\ (-1)^{|x|+|y|+|x|}x \otimes m_1(y) \otimes \theta_1(z) = 0
\end{aligned}
$$

Now, we consider those tensor terms that are left. These are of the form $x \otimes \cdots$ and $\cdots \otimes z$. Simplifying these gives:

$$m_1\theta_2(x,y) \otimes z + m_2(\theta_1(x),y) \otimes z + (-1)^{|x|}m_2(x,\theta_1(y)) \otimes z + \theta_1 m_2(x,y) \otimes z$$
$$+\theta_2(m_1(x),y) \otimes z + (-1)^{|x|}\theta_2(x,m_1(y)) \otimes z$$
$$= \quad [m_1\theta_2(x,y) + m_2(\theta_1(x),y) + (-1)^{|x|}m_2(x,\theta_1(y)) + \theta_1 m_2(x,y)$$
$$+\theta_2(m_1(x),y) + (-1)^{|x|}\theta_2(x,m_1(y))] \otimes z$$
$$= \quad 0 \otimes z \text{ from the relationship on two elements previously}$$
$$= \quad 0$$

In the same way, we have,

$$(-1)^{|x|+|x|}x \otimes m_1\theta_2(y,z) + (-1)^{|x|+|x|}x \otimes m_2(\theta_1(y),z) + (-1)^{|x|+|y|+|x|}x \otimes m_2(y,\theta_1(z))$$
$$+(-1)^{|x|+|x|}x \otimes \theta_1 m_2(y,z) + (-1)^{|x|+|x|}x \otimes \theta_2(m_1(y),z) + (-1)^{|x|+|y|+|x|}x \otimes \theta_2(y,m_1(z))$$
$$= \quad x \otimes [m_1\theta_2(y,z) + m_2(\theta_1(y),z) + (-1)^{|y|}m_2(y,\theta_1(z))$$
$$+\theta_1 m_2(y,z) + \theta_2(m_1(y),z) + (-1)^{|y|}\theta_2(y,m_1(z))]$$
$$= \quad x \otimes 0 \text{ from the relation on two elements and } (-1)^{2|x|} = 1$$
$$= \quad 0$$

After all these cancellations, we are left with

$$\begin{aligned}
[m,\theta](x,y,z) &= m_1\theta_3(x,y,z) + m_2(\theta_2(x,y),z) + (-1)^{|x|}m_2(x_1,\theta_2(y,z)) \\
&\quad + m_3(\theta_1(x),y,z) + (-1)^{|x|}m_3(x,\theta_1(y),z) + (-1)^{|x|+|y|}m_3(x,y,\theta_1(z)) \\
&\quad + \theta_1 m_3(x,y,z) + \theta_2(m_2(x,y),z) + (-1)^{|x|}\theta_2(x_1,m_2(y,z)) \\
&\quad + \theta_3(m_1(x),y,z) + (-1)^{|x|}\theta_3(x,m_1(y),z) + (-1)^{|x|+|y|}\theta_3(x,y,m_1(z))
\end{aligned}$$

which is precisely the sum given by Kajiura and Stasheff.

Now we prove that this bracket is equivalent to the (3.1) on a generic number of inputs. We have that

$$\begin{aligned}
[m,\theta](x_1,x_2,\ldots,,x_n) &= m\theta(x_1,x_2,\ldots,x_n) + \theta m(x_1,x_2,\ldots,x_n)\\[4pt]
&= m\Big(\sum_{j=1}^{n}(-1)^{\beta(i)}x_1\otimes\cdots\otimes x_i\otimes\theta_j(x_{i+1},\ldots,x_{i+j})\otimes\cdots\otimes x_n\Big)\\[4pt]
&\quad +\theta\Big(\sum_{j=1}^{n}(-1)^{\beta(i)}x_1\otimes\cdots\otimes x_i\otimes m_j(x_{i+1},\ldots,x_{i+j})\otimes\cdots\otimes x_n\Big)\\[4pt]
&= \sum_{p=1}^{n}\sum_{j=1}^{n}(-1)^{\alpha(s)}(-1)^{\beta(i)}x_1\otimes x_s\otimes m_p(x_{s+1},\ldots,x_{s+p})\otimes x_i\otimes\cdots\\[4pt]
&\quad \cdots\theta_j(x_{i+1},\ldots,x_{i+j})\otimes\cdots\otimes x_n\\[4pt]
&\quad +\sum_{p=1}^{n}\sum_{j=1}^{n}(-1)^{\alpha(s)}(-1)^{\beta(i)}x_1\otimes x_s\otimes\theta_p(x_{s+1},\ldots,x_{s+p})\otimes x_i\otimes\cdots\\[4pt]
&\quad \cdots m_j(x_{i+1},\ldots,x_{i+j})\otimes\cdots\otimes x_n
\end{aligned}$$

where $\beta(q)=\alpha(q)=|x_1|+\cdots+|x_q|$. We only need to show that any term with a tensor product cancels in the above sum to show this is equivalent to (3.1). We do this in the same way as with three elements.

Note that there are three types of tensor terms:

(i) $x_1\otimes\cdots\otimes\theta_1 m_1(x_j)\otimes\cdots\otimes x_n$ (or $m_1\theta_1$)

(ii) $x_1\otimes\cdots\otimes m_i(x_j,\ldots,x_{j+i})\otimes\cdots\otimes\theta_q(x_l,\ldots,x_{l+q})\otimes\cdots\otimes x_n$ (or m_i and θ_q are switched)

(iii) $x_1\otimes\cdots\otimes x_i\otimes m_j(x_{i+1},\ldots,\theta_q(x_s,\ldots,x_{s+q}),\ldots)\otimes\cdots\otimes x_n$ (or m_j and θ_q are switched)

Note that we've shown the bracket is equivalent to (3.1) for two and three inputs, as we will be using induction to show these equations are equivalent. Let two elements be our base case and assume that $[m,\theta](x_1,\ldots,x_{n-1})=0$. We prove $[m,\theta](x_1,\ldots,x_n)=0$ (or is equivalent to (3.1)) by induction. Consider term (i). This comes from θ acting on $x_1\otimes\cdots\otimes m_1(x_j)\otimes\cdots\otimes x_n$. Firstly, the sign for $x_1\otimes\cdots\otimes m_1(x_j)\otimes\cdots\otimes x_n$ is $(-1)^{|x_1|+\cdots+|x_{j-1}|}$ since we have moved m_1 past the first $j-1$ terms. When we apply θ, we move θ_1 past the first $j-1$ terms again, giving (with the coefficient) the term:

$$(-1)^{2(|x_1|+\cdots+|x_{j-1}|)}x_1\otimes\cdots\otimes\theta_1 m_1(x_j)\otimes\cdots\otimes x_n$$

So, this term has a coefficient of $+1$. Note that there is another term from the second half of the sum, again with a coefficient of $+1$ (for the same reason as above) of the form

$$(-1)^{2(|x_1|+\cdots+|x_{j-1}|)}x_1\otimes\cdots\otimes m_1\theta_1(x_j)\otimes\cdots\otimes x_n$$

And since $m_1\theta_1 = -\theta_1 m_1$ (from before), we have that

$$x_1 \otimes \cdots \otimes \theta_1 m_1(x_j) \otimes \cdots \otimes x_n + x_1 \otimes \cdots \otimes m_1\theta_1(x_j) \otimes \cdots \otimes x_n = 0$$

So all terms of form (i) sum to 0.

Next, we move to terms of form (ii). The term

$$x_1 \otimes \cdots \otimes m_i(x_j, \ldots, x_{j+i}) \otimes \cdots \otimes \theta_q(x_l, \ldots, x_{l+q}) \otimes \cdots \otimes x_n$$

comes from applying θ_m to the term $(-1)^{|x_1|+\cdots|x_{j-1}|}x_1 \otimes \cdots \otimes x_{j-1} \otimes m_i(x_j, \ldots, x_{j+1}) \otimes \cdots \otimes x_n$, where m_i has moved past the first $j-1$ terms. Once we apply θ_m, we have the term:

$$(-1)^{|x_1|+\cdots+|x_{j-1}|+|x_1+\cdots+|m_i(x_j,\ldots,x_{j+i})|+|x_{j+i+1}|+\cdots+|x_{l-1}|}x_1 \otimes$$
$$\cdots \otimes m_i(x_j, \ldots, x_{j+i}) \otimes \cdots \otimes \theta_q(x_l, \ldots, x_{l+q}) \otimes \cdots \otimes x_n$$

Now we have another term in the second half of the sum by applying m_i to

$$(-1)^{|x_1|+\cdots+|x_{l-1}|}x_1 \otimes \cdots \otimes \theta_q(x_l, \ldots, x_{l+q}) \otimes \cdots \otimes x_n$$

which came from moving θ_q past the first $q-1$ terms. This gives the term:

$$(-1)^{|x_1|+\cdots+|x_{l-1}|+|x_1|+\cdots+|x_{j-1}|}x_1 \otimes \cdots \otimes m_i(x_j, \ldots, x_{j+i}) \otimes \cdots \otimes \theta_q(x_l, \ldots, x_{l+q}) \otimes \cdots \otimes x_n$$

And note that

$$
\begin{aligned}
&(-1)^{|x_1|+\cdots+|x_{j-1}|+|x_1|+\cdots+|m_i(x_j,\ldots,x_{j+i})|+|x_{j+i+1}|+\cdots+|x_{l-1}|} \\
=\ &(-1)^{|x_1|+\cdots+|x_{j-1}|+|x_1|+\cdots+|x_j|\cdots|x_{j+i}|+1+|x_{j+i+1}|+\cdots+|x_{l-1}|} \\
=\ &(-1)^{|x_1|+\cdots+|x_{l-1}|+|x_1|+\cdots+|x_{j-1}|+1} \\
=\ &-(-1)^{|x_1|+\cdots+|x_{l-1}|+|x_1|+\cdots+|x_{j-1}|}
\end{aligned}
$$

Hence,

$$(-1)^{|x_1|+\cdots+|x_{j-1}|+|x_1+\cdots+|m_i(x_j,\ldots,x_{j+i})|+|x_{j+i+1}|+\cdots+|x_{l-1}|}x_1 \otimes \cdots \otimes$$

$$\otimes m_i(x_j,\ldots,x_{j+i}) \otimes \cdots \otimes \theta_q(x_l,\ldots,x_{l+q}) \otimes \cdots \otimes x_n$$

$$+(-1)^{|x_1|+\cdots+|x_{l-1}|+|x_1|+\cdots+|x_{j-1}|}x_1 \otimes \cdots \otimes \cdots \otimes m_i(x_j,\ldots,x_{j+i}) \otimes$$

$$\otimes \cdots \otimes \theta_q(x_l,\ldots,x_{l+q}) \otimes \cdots \otimes x_n$$

$$= \quad -(-1)^{|x_1|+\cdots+|x_{l-1}|+|x_1|+\cdots+|x_{j-1}|}x_1 \otimes \cdots \otimes m_i(x_j,\ldots,x_{j+i}) \otimes \cdots \otimes \theta_q(x_l,\ldots,x_{l+q}) \otimes$$

$$\otimes \cdots \otimes x_n$$

$$+(-1)^{|x_1|+\cdots+|x_{l-1}|+|x_1|+\cdots+|x_{j-1}|}x_1 \otimes \cdots \otimes m_i(x_j,\ldots,x_{j+i}) \otimes \cdots \otimes \theta_q(x_l,\ldots,x_{l+q}) \otimes$$

$$\otimes \cdots \otimes x_n$$

$$= \quad 0$$

So these terms of type (ii) also sum to 0.

Lastly, we look to those of type (iii). Consider all the terms left (after canceling those of form (i) and (ii)) of the form

$$x_1 \otimes \cdots \otimes x_i \otimes m_j(x_{i+1},\ldots,\theta_q(x_s,\ldots,x_{s+q}),\ldots) \otimes x_b \otimes \cdots \otimes x_n$$

where i is the largest subscript on the left and b is the smallest subscript on the right of this form. Then factoring out on the left and right gives:

$$x_1 \otimes \cdots \otimes x_i \otimes [m,\theta](x_{i+1},\ldots,x_{b-1}) \otimes x_b \otimes \cdots \otimes x_n$$

Note that by induction,
$$[m,\theta](x_{i+1},\ldots,x_{b-1}) = 0$$

so these terms combine to zero. Also, we can work our way outward, meaning after performing this cancelation for i and b, we look to x_j where $j < i$ and x_a where $a > b$, and perform induction again. Hence, all terms of form (iii) combine to zero.

Therefore, our three types of terms add to zero, leaving only those of the form

$$(-1)^{\beta(i)}\theta_r(x_1,\ldots,x_i,m_s(x_{i+1},\ldots,x_{i+s}),\ldots,x_n)$$

and

$$(-1)^{\beta(i)}m_r(x_1,\ldots,x_i,\theta_s(x_{i+1},\ldots,x_{i+s}),\ldots,x_n)$$

which is precisely (3.1).

From this work, our double sum can be thought of as a commutator bracket on m and θ. This gives an alternate definition for a strong homotopy derivation on an A_∞ algebra and helps us develop a corresponding definition for an L_∞ strong homotopy derivation in our later work.

Chapter 4

L_∞ Strong Homotopy Derivation Definition

4.1 Strong Homotopy Derivations on L_∞ Algebras

Given the definition of strong homotopy derivations of A_∞ algebras [5], we knew there should be a corresponding definition for L_∞. As we looked previously, a strong homotopy derivation for A_∞ consists of a collection of maps satisfying (3.1), but an equivalent definition is a collection of degree one maps, θ, where $[m, \theta] = 0$. Using this same idea, we worked backwards by saying if (L, l) is an L_∞ algebra and θ a strong homotopy derivation, then $[l, \theta] = 0$. (We give these details later.) From this relation, we get the definition:

Definition 18 ((Strong Homotopy Derivation for L_∞ Algebras)). A *strong homotopy derivation* of degree one of an L_∞ algebra consists of a collection of symmetric, multi-linear maps of degree one

$$\theta := \{\theta_q | L^{\otimes q} \to L\}_{q \geq 1}$$

satisfying relations:

$$
\begin{aligned}
\sum_{\substack{j=1 \\ \sigma \in U(j, n-j)}}^{j=n} & (-1)^{\epsilon(\sigma)} \theta_{n-j+1}(l_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)}) \\
& + (-1)^{\epsilon(\sigma)} l_{n-j+1}(\theta_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)}) = 0
\end{aligned}
\tag{4.1}
$$

where $(-1)^{\epsilon(\sigma)}$ is the sign of the unshuffle.

4.2 Developing this Definition

We now show this is consistent with $[\theta, l] = 0$. First, note that as with the A_∞ case, we don't bother multiplying exponents by the degree of the maps when we carry through the l or θ, as these both have degree one. The difference here is we have to consider the signs of unshuffles as we carry through the maps, and for the same reason

$$[\theta, l] = \theta \circ l - (-1)^{|\theta||l|} l \circ \theta = \theta \circ l + l \circ \theta$$

Consider this bracket on one element:

$$\begin{aligned}
[\theta, l](x) &= (\theta \circ l)(x) + (l \circ \theta)(x) \\
&= \theta_1 l_1(x) + l_1 \theta_1(x)
\end{aligned}$$

which is consistent to (4.1), since there are no unshuffles to consider. Now we look to two inputs. Note the signs of the unshuffles as we apply θ and l.

$$\begin{aligned}
[\theta, l](x, y) &= \theta(l_2(x, y) + l_1(x) \otimes y + (-1)^{|x||y|} l_1(y) \otimes x) \\
&\quad + l(\theta_2(x, y) + \theta_1(x) \otimes y + (-1)^{|x||y|} \theta_1(y) \otimes x) \\
&= \theta_1 l_2(x, y) + \theta_2(l_1(x), y) + \theta_1 l_1(x) \otimes y + (-1)^{|l_1(x)||y|} \theta_1(y) \otimes l_1(x) \\
&\quad + (-1)^{|x||y|} \theta_2(l_1(y), x) + (-1)^{|x||y|} \theta_1 l_1(y) \otimes x \\
&\quad + (-1)^{|x||y| + |l_1(y)||x|} \theta_1(x) \otimes l_1(y) + l_1 \theta_2(x, y) + l_2(\theta_1(x), y) \\
&\quad + l_1 \theta_1(x) \otimes y + (-1)^{|\theta_1(x)||y|} l_1(y) \otimes \theta_1(x) \\
&\quad + (-1)^{|x||y|} l_2(\theta_1(y), x) + (-1)^{|x||y|} l_1 \theta_1(y) \otimes x \\
&\quad + (-1)^{|x||y| + |\theta_1(y)||x|} l_1(x) \otimes \theta_1(y)
\end{aligned}$$

Now we use the property that $x \otimes y = (-1)^{|x||y|} y \otimes x$. So,

$$(-1)^{|l_1(x)||y|} \theta_1(y) \otimes l_1(x) + (-1)^{|x||y|+|\theta_1(y)||x|} l_1(x) \otimes \theta_1(y)$$
$$= (-1)^{(|x|+1)|y|} \theta_1(y) \otimes l_1(x) + (-1)^{|x||y|+(|y|+1)|x|} l_1(x) \otimes \theta_1(y)$$
$$= (-1)^{(|x|+1)|y|+|l_1(y)||\theta_1(x)|} l_1(x) \otimes l_1(y) + (-1)^{|x||y|+(|y|+1)|x|} l_1(x) \otimes \theta_1(y)$$
$$= (-1)^{(|x|+1)|y|+(|y|+1)(|x|+1)} l_1(x) \otimes l_1(y) + (-1)^{|x||y|+(|y|+1)|x|} l_1(x) \otimes \theta_1(y)$$
$$= (-1)^{|x||y|+|y|+|x||y|+|x|+|y|+1} l_1(x) \otimes l_1(y) + (-1)^{|x||y|+|x||y|+|x|} l_1(x) \otimes \theta_1(y)$$
$$= (-1)^{|x|+1} l_1(x) \otimes l_1(y) + (-1)^{|x|} l_1(x) \otimes \theta_1(y) \text{ since } (-1)^{2m} = 1 \text{ for all } m$$
$$= -(-1)^{|x|} l_1(x) \otimes l_1(y) + (-1)^{|x|} l_1(x) \otimes \theta_1(y)$$
$$= 0$$

Similarly,

$$(-1)^{|x||y|+|l_1(y)||x|} \theta_1(x) \otimes l_1(y) + (-1)^{|\theta_1(x)||y|} l_1(y) \otimes \theta_1(x) = 0$$

Now look at our other tensor terms. We use the fact that $\theta_1 \circ l_1 = -l_1 \circ \theta$, from before, to say:

$$(-1)^{|x||y|} \theta_1 l_1(y) \otimes x + (-1)^{|x||y|} l_1 \theta_1(y) \otimes x = [(-1)^{|x||y|} \theta_1 l_1(y) + (-1)^{|x||y|} l_1 \theta_1(y)] \otimes x$$
$$= [(-1)^{|x||y|} \theta_1 l_1(y) - (-1)^{|x||y|} \theta_1 l_1(y)] \otimes x$$
$$= 0 \otimes x$$
$$= 0$$

Similarly,
$$l_1 \theta_1(x) \otimes y + \theta_1 l_1(x) \otimes y = 0$$

Now, we have reduced the bracket to:

$$[\theta, l](x, y) = \theta_1 l_2(x, y) + \theta_2(l_1(x), y) + (-1)^{|x||y|} \theta_2(l_1(y), x)$$
$$+ l_1 \theta_2(x, y) + l_2(\theta_1(x), y) + (-1)^{|x||y|} l_2(\theta_1(y), x)$$

Which is consistent with (4.1).

We next show that (4.1) is consistent with our bracket on n inputs. Much like the A_∞ case, we show this by induction (since we have proved our base case of $n = 2$), so assume the bracket

definition for strong homotopy derivation is consistent with (4.1) for any number of inputs less than n. We look at

$$[\theta, l](x_1, \ldots, x_n) \quad = \quad \theta \circ l(x_1, \ldots, x_n) + l \circ \theta(x_1, \ldots, x_n)$$

Since we only consider unshuffles and don't actually move θ and l through the term

$$(x_1, \ldots, x_n)$$

every term begins with $\theta(x_i, \ldots)$ or $l(x_i, \ldots)$. So, to show this is consistent with (4.1), we only need to show that all terms of the form

$$l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes \cdots$$

and

$$\theta_q(l_j(x_{i_1}, \ldots, x_{i_j}), x_{i_{j+1}}, \ldots, x_{q+j-1}) \otimes \cdots$$

cancel with some other term(s) in the sum. Note that in no instance will we have a term of the form:

$$l_p(x_{q_1}, \ldots, x_{q_p}) \otimes x_t \otimes \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes \cdots$$

because in an unshuffle we always keep order, meaning after applying θ, the term involving θ is now the first term when we apply l, so this has to remain either in the first part of the unshuffle (this would result in the second form from above) or the second part of the unshuffle (resulting in the first form from above).

First we consider term one,

$$l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes \cdots$$

and note there is a corresponding term of the form

$$\theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \cdots$$

Our primary goal is to find the coefficients of these two terms, then add the two terms, resulting in zero.

Consider

$$l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes \cdots$$

This comes from applying l to $\theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes \cdots$. For this term, we need to figure the sign of the unshuffle. Each time we move x_{j_α} past another x_β, we get a factor of -1 from a transposition, but we want to be careful not to double count transpositions. Here, we get the coefficient of

$$(-1)^{\displaystyle |x_{j_1}| \sum_{\substack{i<j_1 \\ i \neq j_\alpha}} |x_i| + |x_{j_2}| \sum_{\substack{i<j_2 \\ i \neq j_\alpha}} |x_i| + \cdots + |x_{j_m}| \sum_{\substack{i<j_m \\ i \neq j_\alpha}} |x_i|}$$

Now, once we apply l we get the term $l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes \cdots$ with a coefficient of -1 to the following exponent:

$$\gamma = |x_{j_1}| \sum_{\substack{i<j_1 \\ i \neq j_\alpha}} |x_i| + |x_{j_2}| \sum_{\substack{i<j_2 \\ i \neq j_\alpha}} |x_i| + \cdots + |x_{j_m}| \sum_{\substack{i<j_m \\ i \neq j_\alpha}} |x_i| + |x_{q_1}| \sum_{\substack{i<q_1 \\ i \neq j_\alpha \\ i \neq q_\beta}} |x_i| + \cdots$$

$$+ |x_{q_p}| \sum_{\substack{i<q_p \\ i \neq j_\alpha \\ i \neq q_\beta}} |x_i| + \sum_{i=1}^{p} |x_{q_i}| |\theta_m(x_{j_1}, \ldots, x_{j_m})|$$

This gives the term

$$(-1)^\gamma l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes \cdots$$

and using properties of the skew-symmetric tensor product, we know that:

$$
\begin{aligned}
(-1)^\gamma l_p(x_{q_1}, \ldots, x_{q_p}) \quad \otimes \quad & \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes \cdots = \\
= \quad & (-1)^\gamma (-1)^{|l_p(x_{q_1}, \ldots, x_{q_p})||\theta_m(x_{j_1}, \ldots, x_{j_m})|} \theta_m(x_{j_1}, \ldots, x_{j_m}) \\
& \otimes l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \cdots \\
= \quad & (-1)^{\gamma + (|x_{q_1}| + \cdots + |x_{q_p}| + 1)(|x_{j_1}| + \cdots + |x_{j_m}| + 1)} \theta_m(x_{j_1}, \ldots, x_{j_m}) \\
& \otimes l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \cdots \\
= \quad & (-1)^\Gamma \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \cdots
\end{aligned}
$$

Hence, we have rewritten our first term as

$$(-1)^\Gamma \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \cdots \qquad (4.2)$$

Now, consider (as we said previously) the corresponding term:

$$\theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \cdots$$

Doing the same process on this term gives the coefficient of -1 with exponent:

$$\begin{aligned}
\delta \; = \; & |x_{q_1}| \sum_{\substack{i<q_1 \\ i\neq q_\alpha}} |x_i| + \cdots + |x_{q_p}| \sum_{\substack{i<q_p \\ i\neq q_\alpha}} |x_i| + |x_{j_1}| \sum_{\substack{i<j_1 \\ i\neq j_\alpha \\ i\neq q_\beta}} |x_i| + \cdots + \\
& + |x_{j_m}| \sum_{\substack{i<j_m \\ i\neq q_\beta \\ i\neq j_\alpha}} |x_i| + \sum_{i=1}^{m} |x_{j_i}| |l_p(x_{q_1}, \ldots, x_{q_p})|
\end{aligned}$$

Giving us the term the finalized term:

$$(-1)^\delta \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \cdots \tag{4.3}$$

If we can show that $(4.2) + (4.3) = 0$, then we have shown that all terms of the form $\theta_p(x_{q_1}, \ldots, x_{q_p}) \otimes l_m(x_{j_1}, \ldots, x_{j_m}) \otimes \cdots$ sum to 0. We do this by showing that $-(-1)^\Gamma = (-1)^\delta$, which is equivalent to showing that $\Gamma + 1 = \delta$.

We first expand out Γ and δ slightly:

$$\begin{aligned}
\Gamma \; = \; & |x_{j_1}| \sum_{\substack{i<j_1 \\ i\neq j_\alpha}} |x_i| + |x_{j_2}| \sum_{\substack{i<j_2 \\ i\neq j_\alpha}} |x_i| + \cdots + |x_{j_m}| \sum_{\substack{i<j_m \\ i\neq j_\alpha}} |x_i| + |x_{q_1}| \sum_{\substack{i<q_1 \\ i\neq j_\alpha \\ i\neq q_\beta}} |x_i| + \cdots \\
& + |x_{q_p}| \sum_{\substack{i<q_p \\ i\neq j_\alpha \\ i\neq q_\beta}} |x_i| + |x_{q_1}| + \cdots + |x_{q_p}| + |x_{q_1}||x_{j_1}| + |x_{q_1}||x_{j_2}| + \cdots + \\
& + |x_{q_2}||x_{j_1}| + \cdots + |x_{q_2}||x_{j_m}| + \cdots + |x_{q_p}||x_{j_1}| + \cdots + |x_{q_p}||x_{j_m}| + 1 + \\
& + |x_{q_1}| + \cdots + |x_{q_p}| + |x_{q_1}||x_{j_1}| + |x_{q_1}||x_{j_2}| + \cdots + |x_{q_1}||x_{j_m}| + |x_{q_2}||x_{j_1}| + \cdots \\
& + |x_{q_2}||x_{j_m}| + \cdots + |x_{q_p}||x_{j_1}| + \cdots |x_{q_p}||x_{j_m}| + |x_{j_1}| + \cdots + |x_{j_m}|
\end{aligned}$$

Note that $x_{q_1} + \cdots + x_{q_p}$ appears twice in this sum, so we can make $\hat{\Gamma}$ where:

$$\hat{\Gamma} \;=\; |x_{j_1}| \sum_{\substack{i<j_1 \\ i\neq j_\alpha}} |x_i| + |x_{j_2}| \sum_{\substack{i<j_2 \\ i\neq j_\alpha}} |x_i| + \cdots + |x_{j_m}| \sum_{\substack{i<j_m \\ i\neq j_\alpha}} |x_i| + |x_{q_1}| \sum_{\substack{i<q_1 \\ i\neq j_\alpha \\ i\neq q_\beta}} |x_i| + \cdots$$

$$+ |x_{q_p}| \sum_{\substack{i<q_p \\ i\neq j_\alpha \\ i\neq q_\beta}} |x_i| + |x_{q_1}||x_{j_1}| + |x_{q_1}||x_{j_2}| + \cdots + |x_{q_2}||x_{j_1}| + \cdots +$$

$$+ |x_{q_2}||x_{j_m}| + \cdots + |x_{q_p}||x_{j_1}| + \cdots + |x_{q_p}||x_{j_m}| + 1 + |x_{q_1}||x_{j_1}| +$$

$$+ |x_{q_1}||x_{j_2}| + \cdots + |x_{q_1}||x_{j_m}| + |x_{q_2}||x_{j_1}| + \cdots + |x_{q_2}||x_{j_m}| + \cdots + |x_{q_p}||x_{j_1}| +$$

$$+ \cdots + |x_{q_p}||x_{j_m}| + |x_{j_1}| + \cdots + |x_{j_m}|$$

And we have that

$$\delta \;=\; |x_{q_1}| \sum_{\substack{i<q_1 \\ i\neq q_\alpha}} |x_i| + \cdots + |x_{q_p}| \sum_{\substack{i<q_p \\ i\neq q_\alpha}} |x_i| + |x_{j_1}| \sum_{\substack{i<j_1 \\ i\neq j_\alpha \\ i\neq q_\beta}} |x_i| + \cdots +$$

$$+ |x_{j_m}| \sum_{\substack{i<j_m \\ i\neq q_\beta \\ i\neq j_\alpha}} |x_i| + |x_{q_p}| \sum_{\substack{i<q_p \\ i\neq j_\alpha \\ i\neq q_\beta}} |x_i| + |x_{j_1}| + \cdots + |x_{j_m}| + |x_{j_1}||x_{q_1}|$$

$$+ |x_{j_1}||x_{q_2}| + \cdots + |x_{j_1}||x_{q_p}| + |x_{j_2}||x_{q_1}| + \cdots + |x_{j_2}||x_{q_p}|$$

$$+ \cdots + |x_{j_m}||x_{q_1}| + \cdots |x_{j_m}||x_{q_p}|$$

Also note that both $\hat{\Gamma}$ and δ have the terms

$$|x_{j_1}||x_{q_1}| + |x_{j_1}||x_{q_2}| + \cdots \quad + \quad |x_{j_1}||x_{q_p}| + |x_{j_2}||x_{q_1}| +$$

$$+ \cdots + |x_{j_2}||x_{q_p}| + \cdots \quad + \quad |x_{j_m}||x_{q_1}| + \cdots |x_{j_m}||x_{q_p}|$$

and

$$|x_{j_1}| + \cdots + |x_{j_m}|$$

so we can reduce these terms to say showing $\Gamma + 1 = \delta$ is equivalent to showing $\hat{\Gamma} + 1 = \delta$, which is equivalent to showing $\tilde{\Gamma} + 1 = \tilde{\delta}$ where

$$\tilde{\Gamma} \;=\; |x_{j_1}| \sum_{\substack{i<j_1 \\ i\neq j_\alpha}} |x_i| + |x_{j_2}| \sum_{\substack{i<j_2 \\ i\neq j_\alpha}} |x_i| + \cdots + |x_{j_m}| \sum_{\substack{i<j_m \\ i\neq j_\alpha}} |x_i| + |x_{q_1}| \sum_{\substack{i<q_1 \\ i\neq j_\alpha \\ i\neq q_\beta}} |x_i| + \cdots$$

$$+ |x_{q_p}| \sum_{\substack{i<q_p \\ i\neq j_\alpha \\ i\neq q_\beta}} |x_i| + |x_{q_1}||x_{j_1}| + |x_{q_1}||x_{j_2}| + \cdots + |x_{q_1}||x_{j_m}| + |x_{q_2}||x_{j_1}| + \cdots$$

$$+ |x_{q_2}||x_{j_m}| + \cdots + |x_{q_p}||x_{j_1}| + \cdots |x_{q_p}||x_{j_m}| + 1$$

and

$$\tilde{\delta} \;=\; |x_{q_1}| \sum_{\substack{i<q_1 \\ i\neq q_\alpha}} |x_i| + \cdots + |x_{q_p}| \sum_{\substack{i<q_p \\ i\neq q_\alpha}} |x_i| + |x_{j_1}| \sum_{\substack{i<j_1 \\ i\neq j_\alpha \\ i\neq q_\beta}} |x_i| + \cdots +$$

$$+ |x_{j_m}| \sum_{\substack{i<j_m \\ i\neq q_\beta \\ i\neq j_\alpha}} |x_i| + |x_{q_p}| \sum_{\substack{i<q_p \\ i\neq j_\alpha \\ i\neq q_\beta}} |x_i|$$

We show this by showing each term appears the same number of times in $\tilde{\Gamma}$ and in $\tilde{\delta}$, with the exception of the $+1$ appearing in $\tilde{\Gamma}$.

(I) Consider the term $|x_{q_k}||x_i|$. We have two cases:

    (a) if $i \neq j_\alpha$ for any α, then we don't need to worry about repeating this element, as it appears exactly once on each side.

    (b) if $i = j_\alpha$ for some α, then we again have two possibilities.

        (i) Say $q_k < j_\alpha$. Then this term appears twice in $\tilde{\Gamma}$, once in $x_{j_\alpha} \sum_{\substack{i<j_\alpha \\ i\neq j_\beta}} |x_i|$ and once in the non-summation terms. For $\tilde{\delta}$, since $q_k < j_\alpha$, this term does not appear in $\tilde{\delta}$. Appearing twice in $\tilde{\Gamma}$ is equivalent to appearing zero times in $\tilde{\delta}$ since $(-1)^2 = 1$.

        (ii) Now say $q_k > j_\alpha$. This term will appear once as a non-summation term in $\tilde{\Gamma}$, and once in $\tilde{\delta}$ in the sum $|x_{q_k}| \sum_{\substack{i<q_k \\ i\neq q_\beta}} |x_i|$. So this term appears the same number of times in each exponent.

(II) Now consider the term (the only other type), $|x_{j_k}||x_i|$. We have two cases:

(a) If $i \neq q_\alpha$ for any α, then we don't need to worry about repeating this element, as it appears exactly once on each side.

(b) If $i = q_\alpha$ for some α, then we again have two possibilities.

    (i) Say $j_k < q_\alpha$. This appears only as a non-summation term in $\tilde{\Gamma}$, so only one term. For $\tilde{\delta}$, this appears again only once in the sum $|x_{q_\alpha}| \sum\limits_{i < q_\alpha, i \neq q_\beta} |x_i|$, appearing the same number of times in each exponent.

    (ii) Now say $j_k > q_\alpha$. Then this term appears twice in $\tilde{\Gamma}$, once in $x_{j_k} \sum\limits_{\substack{i < j_k \\ i \neq j_\beta}} |x_i|$ and once in the non-summation terms. For $\tilde{\delta}$, since $j_k < q_\alpha$, this term does not appear in $\tilde{\delta}$. Again, appearing twice in $\tilde{\Gamma}$ is equivalent to appearing zero times in $\tilde{\delta}$.

Since these are the only types of terms and they appear an equal (or equivalent) number of times in both $\tilde{\Gamma}$ and $\tilde{\delta}$, we have that $\tilde{\Gamma} + 1 = \tilde{\delta}$ and so $\Gamma + 1 = \delta$. Therefore,

$$
\begin{aligned}
(-1)^\gamma l_p(x_{q_1}, \ldots, x_{q_p}) \quad &\otimes \quad \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes \cdots \\
&+ \quad (-1)^\delta \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \cdots \\
&= \quad (-1)^\Gamma \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \cdots \\
&\quad + (-1)^\delta \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \cdots \\
&= \quad -(-1)^\delta \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \cdots \\
&\quad + (-1)^\delta \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \cdots \\
&= \quad 0
\end{aligned}
$$

So all terms of the form $\theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \cdots$ sum to 0, keeping consistent with our definition of an L_∞ algebra strong homotopy derivation (4.1).

Lastly, we consider elements of the form

$$
\theta_q(l_j(x_{i_1}, \ldots, x_{i_j}), x_{i_{j+1}}, \ldots, x_{q+j-1}) \otimes \cdots
$$

Recall that (by our induction argument), (4.1) is equivalent to the bracket structure on $n-1$ elements. Start with $n-1$ elements in the otimes, so as an example, all elements of the form

$$
\theta_1 l_1(x_{\sigma(1)}) \otimes x_{\sigma(2)} \otimes \cdots
$$

If we take all the terms (of which there are only two), we now can use induction and the

properties of skew-symmetric tensor products to say this is $0 \otimes x_{\sigma(2)} \otimes \cdots$. We continue this by collecting terms with $n-2$ terms in the otimes. Then by induction, we again will get their sum to be zero. Hence, all of these terms add to zero.

Since both types of terms

$$l_p(x_{q_1}, \ldots, x_{q_p}) \otimes \theta_m(x_{j_1}, \ldots, x_{j_m}) \otimes \cdots$$

and

$$\theta_q(l_j(x_{i_1}, \ldots, x_{i_j}), x_{i_{j+1}}, \ldots, x_{q+j-1}) \otimes \cdots$$

add to zero, we are only left with terms such as

$$\theta_{n-j+1}(l_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)})$$

and

$$l_{n-j+1}(\theta_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)})$$

which is exactly what we see in (4.1). Hence, this definition for L_∞ strong homotopy derivation is consistent with the bracket. So our definition works in the same way the definition of A_∞ strong homotopy derivation does.

4.3 Relating Strong Homotopy Derivations

After finding a definition for L_∞-algebra strong homotopy derivation, we then asked the question, if an L_∞-algebra is produced from symmetrizing an A_∞-algebra, does an L_∞-strong homotopy derivation result from symmetrizing an A_∞-strong homotopy derivation?

Our result here is, yes, the two are connected the same way A_∞ and L_∞-algebras are. To prove this result, we look at Lada's work in *Commutators of A_∞ Structures* [6]. In this paper, we use the notation that $\Lambda^* V$ is the cofree commutative coalgebra on V, and $T^* V$ the cofree coalgebra on the graded vector space V. Here the projections are given by $\pi_n : T^* V \to T^n V$ and $p_n : \Lambda^* V \to \Lambda^n V$. We have a correspondence between the two coalgebras via a coalgebra injective map

$$\chi(v_1 \otimes \cdots \otimes v_n) = \sum_{\sigma \in S_n} (-1)^{e(\sigma)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

where $(-1)^{e(\sigma)}$ is the sign of the permutation.

We now reference Lada's Proposition 5 [6]:

Proposition. *(Proposition* 5*) Suppose that* $f : T^*V \to V$ *is a linear map which extends to the coderivation* $\hat{f} : T^*V \to T^*V$. *Then the diagram*

$$
\begin{array}{ccc}
\Lambda^*V & \xrightarrow{\ \chi\ } & T^*V \\
\Big\uparrow f \,\hat{\circ}\, \chi & & \Big\uparrow \hat{f} \qquad \searrow \pi_1 \\
\Lambda^*V & \xrightarrow[\ \chi\ ]{} & T^*V \xrightarrow[\ f\ ]{} V
\end{array}
$$

Figure 4.1: Lada's Proposition 5

commutes. Here, $f \,\hat{\circ}\, \chi$ *is the extension of the map* $f \circ \chi : \Lambda^*V \to V$ *to the coderivation* $f \,\hat{\circ}\, \chi : \Lambda^*V \to \Lambda^*V$.

Now, let (V, m) be an A_∞ structure and extend this to an L_∞ structure given by (V, l), where l is found by skew-symmetrizing m, as we did before in Theorem 3. This gives us the diagram:

$$
\begin{array}{ccc}
\Lambda^c V & \xrightarrow{\ \chi\ } & T^c V \\
\Big\uparrow \hat{l} = m \,\hat{\circ}\, \chi & & \Big\uparrow \hat{m} \qquad \searrow \pi_1 \\
\Lambda^c V & \xrightarrow[\ \chi\ ]{} & T^c V \xrightarrow[\ m\ ]{} V \\
& l = m \circ \chi &
\end{array}
$$

Figure 4.2: Proposition 5 With A_∞ & L_∞ Maps

We let (V, θ) give a strong homotopy derivation structure and we define θ' to be the symmetriczation of θ, again using Theorem 3. This gives the picture:

Figure 4.3: Proposition 5 With A_∞ & L_∞ SH Derivation Maps

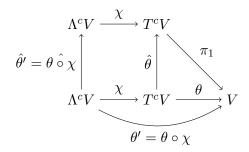Our goal here is to show that θ' is an L_∞ strong homotopy derivation. Using our definition and work with L_∞ strong homotopy derivations, we know the definition holds if and only if $[\hat{l}, \hat{\theta}'] = 0$. We have shown this is equivalent to the definition in previous work. To prove this, we use that θ gives an A_∞ strong homotopy derivation on V, so $[\hat{m}, \hat{\theta}] = 0$. Now we apply χ to get:

$$
\begin{aligned}
\chi[\hat{l}, \hat{\theta}'] &= \chi(\hat{l}\hat{\theta}' + \hat{\theta}'\hat{l}) \\
&= \chi\hat{l}\hat{\theta}' + \chi\hat{\theta}'\hat{l} \\
&= \hat{m}\chi\hat{\theta}' + \hat{\theta}\chi\hat{l} \\
&= \hat{m}\hat{\theta}\chi + \hat{\theta}\hat{m}\chi \\
&= [\hat{m}, \hat{\theta}]\chi \\
&= 0
\end{aligned}
$$

This comes from the fact that the diagrams commute, so $\chi \circ \hat{l} = \hat{m} \circ \chi$, and the fact that $[\hat{m}, \hat{\theta}] = 0$ since θ is a strong homotopy derivation.

So we've shown that $\chi[\hat{l}, \hat{\theta}'] = 0$, and since χ is injective, this means that $[\hat{l}, \hat{\theta}] = 0$. Hence, when we symmetrize a strong homotopy derivation for an A_∞ algebra, we do, in fact, get a strong homotopy derivation for an L_∞ algebra. Another point of importance here is that by showing that strong homotopy derivations are connected in the same way that A_∞ and L_∞ algebras are connected shows that our definition of L_∞ strong homotopy derivations is the definition we should be using.

Chapter 5

Example of A_∞ and L_∞ SH-Derivations

After giving the definition of strong homotopy derivations for both A_∞ and L_∞ algebras, we next look to finding a canonical example for these derivations. To do this, we look back to basic Lie algebras from [4] and notice how he defines an *inner derivation* to fix an element a, then $D_a(x) = xa - ax$.

5.1   On A_∞ Algebras

Our goal was to use this to define θ_1 for an A_∞ algebra using m_2 as the multiplication, so we set $\theta_1(x) = m_2(x, a) - m_2(a, x)$, where a is a fixed element in the vector space. This definition worked with the double sum from (3.1), but when we defined something similar for θ_2, we noticed the problems with negatives. So, we went back and redefined $\theta_1 = m_2(x, a) + m_2(a, x)$. We first show this is consistant with (3.1), i.e., does given a strong homotopy derivation.

First, let $\theta_1 = m_2(x, a) + m_2(a, x)$. From the double sum, (3.1), we have that

$$\theta_1 m_1(x) + m_1 \theta_1(x) = 0$$

should hold. Expanding this out and using our basic relation on A_∞ algebra maps (using our second definition of A_∞ algebra since maps have been lifted), we know that

$$m_1 m_2(x, y) + m_2(m_1(x), y) + (-1)^{|x|} m_2(x, m_1(y)) = 0$$

So we have,

$$\begin{aligned}
\theta_1 m_1(x) + m_1\theta_1(x) &= m_2(m_1(x), a) + m_2(a, m_1(x)) + m_1 m_2(x, a_+m_1 m_2(a, x) \\
&= m_1 m_2(x, a) + m_2(m_1(x), a) + (-1)^{|x|} m_2(x, m_1(a)) \\
&\quad + m_1 m_2(a, x) + (-1)^{|a|} m_2(a, m_1(x)) + m_2(m_1(a), y) \\
&= 0
\end{aligned}$$

This last line comes from setting restrictions on a. We set $m_1(a) = 0$ and $|a| = 2k$ for some $k \in \mathbb{Z}$. Thus, our definition for θ_1 is consistent with (3.1) and can be used to define a strong homotopy derivation for A_∞ algebras.

Next, we define $\theta_2(x, y) = m_3(x, y, a) + m_3(x, a, y) + m_3(a, x, y)$ and show this is consistent with the definition of strong homotopy derivation, (3.1). For this, we look back to the definition of A_∞ algebra to get the relationship among the m_i's. From this we have:

$$m_1 m_3(x, y, z) + m_2(m_2(x, y), z) + (-1)^{|x|} m_2(x, m_2(y, z)) + m_3(m_1(x), y, z)$$

$$+ (-1)^{|x|} m_3(x, m_1(y), z) + (-1)^{|x|+|y|} m_3(x, y, m_1(z)) = 0$$

To show θ_2 is consistent with the definition of a strong homotopy derivation, we plug into the double sum, (3.1), to get:

$$\begin{aligned}
&\theta_1 m_2(x, y) + \theta_2(m_1(x), y) + (-1)^{|x|} \theta_2(x, m_1(y)) \\
&+ m_1 \theta_2(x, y) + m_2(\theta_1(x), y) + (-1)^{|x|} m_2(x, \theta_1(y)) \\
&= m_2(m_2(x, y), a) + m_2(a, m_2(x, y)) + m_3(m_1(x), y, a) \\
&\quad + m_3(m_1(x), a, y) + m_3(a, m_1(x), y) + (-1)^{|x|} m_3(x, m_1(y), a) \\
&\quad + (-1)^{|x|} m_3(x, a, m_1(y)) + (-1)^{|x|} m_3(a, x, m_1(y)) + m_1 m_3(x, y, a) \\
&\quad + m_1 m_3(x, a, y) + m_1 m_3(a, x, y) + m_2(m_2(x, a), y) \\
&\quad + m_3(m_2(a, x), y) + (-1)^{|x|} m_2(x, m_2(y, a)) + (-1)^{|x|} m_2(x, m_2(a, y))
\end{aligned}$$

And since $m_1(a) = 0$ and $|a| = 2k$ for some $k \in \mathbb{Z}$, we can alter this sum to the following way

$$
\begin{aligned}
= \quad & m_2(m_2(x,y),a) + m_3(m_1(x),y,a) + (-1)^{|x|}m_3(x,m_1(y),a) \\
& + m_1 m_3(x,y,a) + (-1)^{|x|}m_2(x,m_2(y,a)) + (-1)^{|x|+|y|}m_3(x,y,m_1(a)) \\
& + m_2(a,m_2(x,y)) + (-1)^{|a|}m_3(a,m_1(x),y) + (-1)^{|x|}m_3(a,x,m_1(y)) \\
& + m_1 m_3(a,x,y) + m_2(m_2(a,x),y) + m_3(m_1(a),x,y) + m_3(m_1(x),a,y) \\
& + (-1)^{|x|+|a|}m_3(x,a,m_1(y)) + m_1 m_3(x,a,y) + m_2(m_2(x,a),y) \\
& + (-1)^{|x|}m_2(x,m_2(a,y)) + (-1)^{|x|}m_3(x,m_1(a),y) \\
= \quad & 0 + 0 + 0 \\
= \quad & 0
\end{aligned}
$$

Hence, the way in which we defined θ_2 is consistent with the definition of strong homotopy derivation on an A_∞ algebra.

Now, we define θ_n for a generic n and show this works with (3.1). Define

$$
\theta_n(x_1, \ldots, x_n) = m_{n+1}(x_1, \ldots, x_n, a) + m_{n+1}(x_1, \ldots, x_{n-1}, a, x_n)
$$

$$
+ \cdots + m_{n+1}(x, a, x_2, \ldots, x_n) + m_{n+1}(a, x_1, \ldots, x_n)
$$

Let's look at the double sum, (3.1) on n elements:

$$
\sum_{r+s=n+1} \sum_{i=1}^{r-1} (-1)^{\beta(i)} \theta_r(x_1, \ldots, x_i, m_s(x_{i+1}, \ldots, x_{i+s}), \ldots, x_n)
$$

$$
+ \sum_{r+s=n+1} \sum_{i=1}^{r-1} (-1)^{\beta(i)} m_r(x_1, \ldots, x_i, \theta_s(x_{i+1}, \ldots, x_{i+s}), \ldots, x_n)
$$

where $\beta(i) = |x_1| + |x_2| + \cdots + |x_i|$ and we know from the definition of A_∞ algebra that

$$
\sum_{k+l=n+1} \sum_{i=1}^{k} (-1)^{\beta(i)} m_k(x_1, \ldots, x_{i-1}, m_l(x_i, \ldots, x_{i+l-1}), \ldots, x_n) = 0
$$

Using how we've defined θ_i, the double sum now becomes:

$$\sum_{r+s=n+1} \sum_{i=1}^{r-1} (-1)^{\beta(i)} \theta_r(x_1, \ldots, x_i, m_s(x_{i+1}, \ldots, x_{i+s}), \ldots, x_n)$$

$$+ \sum_{r+s=n+1} \sum_{i=1}^{r-1} (-1)^{\beta(i)} m_r(x_1, \ldots, x_i, \theta_s(x_{i+1}, \ldots, x_{i+s}), \ldots, x_n)$$

$$= \sum_{r+s=n+1} \sum_{i=1}^{r-1} [(-1)^{\beta(i)} m_{r+1}(x_1, \ldots, x_i, m_s(x_{i+1}, \ldots, x_{i+s}), \ldots, x_n, a)$$

$$+ (-1)^{\beta(i)} m_{r+1}(x_1, \ldots, x_i, m_s(x_{i+1}, \ldots, x_{i+s}), \ldots, x_{n-1}, a, x_n) + \cdots +$$

$$+ m_{r+1}(x_1, a, x_2, \ldots, x_i, m_s(x_{i+1}, \ldots, x_{i+s}), \ldots, x_n)$$

$$+ m_{r+1}(a, x_1, \ldots, x_i, m_s(x_{i+1}, \ldots, x_{i+s}), \ldots, x_n)]$$

$$+ \sum_{r+s=n+1} \sum_{i=1}^{r-1} [(-1)^{\beta(i)} m_r(x_1, \ldots, x_i, m_{s+1}(x_{i+1}, \ldots, x_{i+s}, a), \ldots, x_n)$$

$$+ (-1)^{\beta(i)} m_r(x_1, \ldots, x_i, m_{s+1}(x_{i+1}, \ldots, x_{i+s-1}, a, x_{i+s}), \ldots, x_n) + \cdots +$$

$$+ (-1)^{\beta(i)} m_r(x_1, \ldots, x_i, m_{s+1}(x_{i+1}, a, x_{i+2}, \ldots, x_{i+s}), \ldots, x_n)$$

$$+ (-1)^{\beta(i)} m_r(a, x_1, \ldots, x_i, m_{s+1}(x_{i+1}, \ldots, x_{i+s}), \ldots, x_n)]$$

And since $|a| = 2k$ for some $k \in \mathbb{Z}$ and $m_1(a) = 0$, we can make this sum as follows:

$$= \sum_{r+s=n+1} \sum_{i=1}^{r-1} [(-1)^{\beta(i)} m_{r+1}(x_1, \ldots, x_i, m_s(x_{i+1}, \ldots, x_{i+s}), \ldots, x_n, a)$$

$$+ (-1)^{\beta(i)} m_{r+1}(x_1, \ldots, x_i, m_s(x_{i+1}, \ldots, x_{i+s}), \ldots, x_{n-1}, a, x_n) + \cdots +$$

$$+ (-1)^{\beta(i)+|a|} m_{r+1}(x_1, a, x_2, \ldots, x_i, m_s(x_{i+1}, \ldots, x_{i+s}), \ldots, x_n)$$

$$+ (-1)^{\beta(i)+|a|} m_{r+1}(a, x_1, \ldots, x_i, m_s(x_{i+1}, \ldots, x_{i+s}), \ldots, x_n)]$$

$$+ \sum_{r+s=n+1} \sum_{i=1}^{r-1} [(-1)^{\beta(i)} m_r(x_1, \ldots, x_i, m_{s+1}(x_{i+1}, \ldots, x_{i+s}, a), \ldots, x_n)$$

$$+ (-1)^{\beta(i)} m_r(x_1, \ldots, x_i, m_{s+1}(x_{i+1}, \ldots, x_{i+s-1}, a, x_{i+s}), \ldots, x_n) + \cdots +$$

$$+ (-1)^{\beta(i)} m_r(x_1, \ldots, x_i, m_{s+1}(x_{i+1}, a, x_{i+2}, \ldots, x_{i+s}), \ldots, x_n)$$

$$+ (-1)^{\beta(i)} m_r(a, x_1, \ldots, x_i, m_{s+1}(x_{i+1}, \ldots, x_{i+s}), \ldots, x_n)]$$

$$+ [(-1)^{\beta(n)} m_{n+1}(x_1, \ldots, x_n, m_1(a)) + (-1)^{\beta(n-1)} m_{n+1}(x_1, \ldots, x_{n-1}, m_1(a), x_n) +$$

$$+ \cdots + (-1)^1 m_{n+1}(x_1, m_1(a), x_2, \ldots, x_n) + m_{n+1}(m_1(a), x_1, \ldots, x_n)]$$

Note that when we originally expanded out the sum with only m_i's, at no point would

$m_1(a)$ appear because we only have m_i acting on $(x_1, \ldots, x_n)$, so if m_i acts on an element, it must contain an x_j. This is why we add the last set of terms. Additionally, in the first set of terms, we need to point out the added $(-1)^{|a|}$ because a has been moved past m_i, and to keep signs consistent we need this extra coefficient. Note that we can keep the equality here because $|a| = 2k$ for some $k \in \mathbb{Z}$.

Now we rewrite the double sum yet again:

$$= \sum_{k+l=n+1} \sum_{i=1}^{k} (-1)^{\beta(i)} m_{k+1}(x_1, \ldots, x_{i-1}, m_l(x_i, \ldots, x_{i+l-1}), \ldots, x_n, a)$$

$$+ \sum_{k+l=n+1} \sum_{i=1}^{k} (-1)^{\beta(i)} m_{k+1}(x_1, \ldots, x_{i-1}, m_l(x_i, \ldots, x_{i+l-1}), \ldots, x_{n-1}, a, x_n) + \cdots +$$

$$+ \sum_{k+l=n+1} \sum_{i=1}^{k} (-1)^{\beta(i)+|a|} m_{k+1}(x_1, a, x_2, \ldots, x_{i-1}, m_l(x_i, \ldots, x_{i+l-1}), \ldots, x_n)$$

$$+ \sum_{k+l=n+1} \sum_{i=1}^{k} (-1)^{\beta(i)+|a|} m_{k+1}(a, x_1, x_2, \ldots, x_{i-1}, m_l(x_i, \ldots, x_{i+l-1}), \ldots, x_n)$$

Each of these is 0 as a direct result of the definition of A_∞ algebra maps on $n+1$ elements, or we could think of this at $n+1$ copies of the sum definition of A_∞ algebra. Hence, our definition for θ_n gives a strong homotopy for an A_∞ algebra because the double sum, (3.1), holds.

5.2 On L_∞ Algebras

Just as we did before with finding a canonical example of an A_∞ algebra strong homotopy derivation, we will do the same for a L_∞ algebra strong homotopy derivation.

For this section, we again use our second definition of L_∞ algebra (1.4) along with our definition of an L_∞ algebra strong homotopy derivation (4.1). Now, let (L, l) be an L_∞ algebra and a be a fixed element in our algebra such that $l_1(a) = 0$ and a has even degree, i.e., $|a| = 2k$ for some $k \in \mathbb{Z}$. We will show that by setting

$$\theta_n(x_1, \ldots, x_n) = l_{n+1}(x_1, \ldots, x_n, a)$$

we obtain a strong homotopy derivation. Before we prove that this works with our definition, we look at the case where $n = 1$ first.

Define $\theta_1(x) = l_2(x, a)$. To prove this is consistent with our definition of strong homotopy derivation, we should get 0 when we plug in one element to our double sum (4.1). Additionally,

we use our relationship from the definition of L_∞ algebra (1.4) to say that $l_1(l_1(x)) = 0$. Here we get:

$$\theta_1(l_1(x)) + l_1(\theta_1'(x)) \quad = \quad l_2(l_1(x), a) + l_1(l_2(x, a))$$

And if we make our a such that $l_1(a) = 0$, then this is equal to:

$$= \quad l_2(l_1(x), a) + l_1(l_2(x, a)) + (-1)^{|x||a|+1} l_2(l_1(a), x)$$
$$= \quad 0$$

Because this comes directly from our definition of L_∞ algebra on (x, a).

Now let $\theta_1(x_1, \ldots, x_n) := l_{n+1}(x_1, \ldots, x_n, a)$. We show that by defining θ_n in this way, we have an L_∞ strong homotopy derivation, i.e., that the sum from (4.1) is 0.

Acting the sum on n-inputs gives:

$$\sum_{\substack{j=1 \\ \sigma \in U(j, n-j)}}^{j=n} (-1)^{\epsilon(\sigma)} \theta_{n-j+1}(l_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)})$$
$$+ (-1)^{\epsilon(\sigma)} l_{n-j+1}(\theta_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)})$$

$$= \sum_{\substack{j=1 \\ \sigma \in U(j, n-j)}}^{j=n} (-1)^{\epsilon(\sigma)} l_{n-j+2}(l_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)}, a)$$
$$+ (-1)^{\epsilon(\sigma)} l_{n-j+1}(l_{j+1}(x_{\sigma(1)}, \ldots, x_{\sigma(j)}, a), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)})$$

$$= \sum_{\substack{j=1 \\ \sigma \in U(j, n-j)}}^{j=n} (-1)^{\epsilon(\sigma)} l_{n-j+2}(l_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)}, a)$$
$$+ (-1)^{\epsilon(\sigma)+|a||x_{\sigma(j+1)}|+\cdots+|a||x_{\sigma(n)}|} l_{n-j+1}(l_{j+1}(x_{\sigma(1)}, \ldots, x_{\sigma(j)}, a), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)})$$
$$+ (-1)^{|x_1||a|+\cdots+|x_n||a|} l_{n+1}(l_1(a), x_1, \ldots, x_n)$$

$$= \quad 0$$

because this is precisely the relation between l_i's in the definition of L_∞ algebra on $(x_1, \ldots, x_n, a)$. Since we get that (4.1) is 0, defining θ_n in this way does give a strong homotopy derivation.

By finding these two ways to find A_∞ and L_∞ strong homotopy derivations, we can now write out explicit examples in the next chapter.

Chapter 6

Concrete Examples

Now that we have a way to construct these strong homotopy derivations, we go back to our concrete examples and explicitly define a strong homotopy derivation.

6.1 Concrete A_∞ Example

Recall from Allocca's paper [1]:

Example 19 (A finite A_∞ Algebra)**.** Let $W = W_{-1} + W_0$ be given by $W_{-1} = \langle x_1, x_2 \rangle$ and $W_0 = \langle y \rangle$. The following maps describe an A_∞ structure on W:

$$
\begin{aligned}
\hat{m}_1(x_1) = \hat{m}_1(x_2) &= y \\
\text{For } n \geq 2, \ \hat{m}_n(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{n-2-k}) &= x_1 \text{ for } 0 \leq k \leq n-2 \\
\hat{m}_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) &= x_1 \\
\hat{m}_n(x_1 \otimes y^{\otimes n-1}) &= y
\end{aligned}
$$

From this definition, we can see that the degree of y is even and $\hat{m}_1(y) = 0$. Now we define

$$
\theta_n(x_1, \ldots, x_n) := \hat{m}_{n+1}(x_1, \ldots, x_n, y) + \cdots + \hat{m}_{n+1}(y, x_1, \ldots, x_n)
$$

as we did before, but replacing a with y. Now we go through and find explicitly what these θ are. Note that the only terms we need to check are:

(i) x_1

(ii) $x_1 \otimes x_2$

(iii) $x_2 \otimes x_1$

(iv) $x_1 \otimes x_1$

(v) $x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes n-2-k}$

(vi) $x_1 \otimes y^{\otimes n-2} \otimes x_2$

(vii) $x_1 \otimes y^{\otimes k} \otimes x_2 \otimes y^{\otimes n-2-k}$

(viii) $x_1 \otimes y^{\otimes n-1}$

We go through each of these, apply θ_n, then find a more simplified form.

(i) For x_1, we get

$$\begin{aligned} \theta_1(x_1) &= \hat{m}_2(x_1, y) + \hat{m}_2(y, x_1) \\ &= y \end{aligned}$$

(ii) For $x_1 \otimes x_2$, we have that

$$\begin{aligned} \theta_2(x_1, x_2) &= \hat{m}_3(x_1, x_2, y) + \hat{m}_3(x_1, y, x_2) + \hat{m}_3(y, x_1, x_2) \\ &= x_1 \end{aligned}$$

(iii) For $x_2 \otimes x_1$, we have that $\theta_2(x_2, x_1) = 0$ because $\hat{m}_3$ is 0 whenever x_2 is our first element or y is our first element.

(iv) For $x_1 \otimes x_1$, we have

$$\begin{aligned} \theta_2(x_1, x_1) &= \hat{m}_3(x_1, x_1, y) + \hat{m}_3(x_1, y, x_1) + \hat{m}_3(y, x_1, x_1) \\ &= 2x_1 \end{aligned}$$

(v) For $x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes n-2-k}$, we have

$$\begin{aligned} \theta_n(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes n-2-k}) &= \hat{m}_{n+1}(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes n-1-k}) + \cdots \\ &\quad + \hat{m}_{n+1}(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{n-1-k}) + \\ &\quad + \hat{m}_{n+1}(x_1 \otimes y^{\otimes k+1} \otimes x_1 \otimes y^{\otimes n-2-k}) + \cdots \\ &\quad + \hat{m}_{n+1}(x_1 \otimes y^{\otimes k+1} \otimes x_1 \otimes y^{\otimes n-2}) + \\ &\quad + \hat{m}_{n+1}(y \otimes x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes n-2-k}) \end{aligned}$$

Note that there are $n - 1 - k$ terms of the form $\hat{m}_{n+1}(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes n-1-k})$ and

$k + 1$ terms of the form $\hat{m}_{n+1}(x_1 \otimes y^{\otimes k+1} \otimes x_1 \otimes y^{\otimes n-2})$, so if we add these together we get:

$$\begin{aligned} \theta_n(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes n-2-k}) &= (n-1-k)x_1 + (k+1)x_1 \\ &= nx_1 \end{aligned}$$

(vi) For $x_1 \otimes y^{\otimes n-1} \otimes x_2$, we have:

$$\begin{aligned} \theta_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) &= \hat{m}_{n+1}(x_1 \otimes y^{\otimes n-2} \otimes x_2 \otimes y) + \hat{m}_{n+1}(x_1 \otimes y^{\otimes n-1} \otimes x_2) + \\ &\quad + \cdots + \hat{m}_{n+1}(x_1 \otimes y^{\otimes n-1} \otimes x_2) + \hat{m}_{n+1}(y, x_1 \otimes y^{\otimes n-1} \otimes x_2) \end{aligned}$$

Note that there are $n-1$ terms of the form $\hat{m}_{n+1}(x_1 \otimes y^{\otimes n-1} \otimes x_2)$, so we get:

$$\begin{aligned} \theta_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) &= x_1 + (n-1)x_1 \\ &= nx_1 \end{aligned}$$

(vii) For $x_1 \otimes y^{\otimes k} \otimes x_2 \otimes y^{\otimes n-2-k}$, we have that $\hat{m}_{n+1}$ is nonzero only when x_2 is the last element to be acted on. Since this won't happen when we distribute the extra y throughout, then $\theta_n(x_1 \otimes y^{\otimes k} \otimes x_2 \otimes y^{\otimes n-2-k}) = 0$

(viii) Lastly, for $x_1 \otimes y^{\otimes n-1}$, we have that:

$$\theta_n(x_1 \otimes y^{\otimes n-1}) = \hat{m}_{n+1}(x_1 \otimes y^{\otimes n}) + \cdots + \hat{m}_{n+1}(x_1 \otimes y^{\otimes n}) + \hat{m}_{n+1}(y \otimes x_1 \otimes y^{\otimes n-1})$$

And since there are n terms of the form $\hat{m}_{n+1}(x_1 \otimes y^{\otimes n})$, we get that:

$$\theta_n(x_1 \otimes y^{\otimes n-1}) = ny$$

Then if we write out the explicitly defined strong homotopy derivation we have:

Example 20. Let $W = W_{-1} + W_0$ be given by $W_{-1} = \langle x_1, x_2 \rangle$ and $W_0 = \langle y \rangle$, where an A_∞ algebra structure has been given by

$$\begin{aligned} \hat{m}_1(x_1) = \hat{m}_1(x_2) &= y \\ \text{For } n \geq 2, \ \hat{m}_n(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{n-2-k}) &= x_1 \text{ for } 0 \leq k \leq n-2 \\ \hat{m}_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) &= x_1 \\ \hat{m}_n(x_1 \otimes y^{\otimes n-1}) &= y \end{aligned}$$

Then the following gives a strong homotopy derivation on this coalgebra:

$$\theta_1(x_1) = y$$
$$\text{For } n \geq 2,\ \theta_n(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes n-2-k}) = nx_1 \text{ where } 0 \leq k \leq n-2$$
$$\theta_n(x_1 \otimes y^{\otimes n-1}) = ny$$
$$\theta_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = nx_1$$

6.1.1 Verifying the Definition

To reiterate that this is consistent with our definition of strong homotopy derivation, we show the double sum (3.1) from our definition works on $x_1 \otimes x_1$ and $x_1 \otimes y \otimes y$, just to show how cancellation works and to double check ourselves.

From the definition of A_∞ strong homotopy derixation, we have:

$$\theta_1(\hat{m}_2(x_1, x_1)) + \theta_2(\hat{m}_1(x_1), x_1) + (-1^{|x_1|})\theta_2(x_1, \hat{m}_1(x_1)) + \hat{m}_1(\theta_2(x_1, x_1)) +$$
$$+\hat{m}_2(\theta_1(x_1), x_1) + (-1)^{|x_1|}\hat{m}_2(x_1, \theta_1(x_1))$$
$$= \theta_1(x_1) + \theta_2(y, x_1) + (-1)^{|x_1|}\theta_2(x_1, y) + \hat{m}_1(x_1) + \hat{m}_2(y, x_1) + (-1)^{|x_1|}\hat{m}_2(x_1, y)$$
$$= y + (-1)^{|x_1|}y + y + (-1)^{|x_1|}y$$
$$= y - y + y - y$$
$$= 0$$

And now on $x_1 \otimes y \otimes y$ we have:

$$\theta_1(\hat{m}_3(x_1, y, y)) + \theta_2(\hat{m}_2(x_1, y), y) + (-1)^{|x_1|}\theta_2(x_1, \hat{m}_2(y, y)) + \theta_3(\hat{m}_1(x_1), y, y)$$
$$+(-1)^{|x_1|}\theta_3(x_1, \hat{m}_1(y), y) + (-1)^{|x_1|+|y|}\theta_3(x_1, y, \hat{m}_1(y)) + \hat{m}_1(\theta_3(x_1, y, y)) +$$
$$+\hat{m}_2(\theta_2(x_1, y), y) + (-1)^{|x_1|}\hat{m}_2(x_1, \theta_2(y, y)) + \hat{m}_3(\theta_1(x_1), y, y) +$$
$$+(-1)^{|x_1|}m_3(x_1, \hat{m}_1(y), y) + (-1)^{|x_1|+|y|}\hat{m}_3(x_1, y, \theta_1(y))$$
$$= \theta_1(y) + \theta_2(y, y) + (-1)^{|x_1|}\theta_2(x_1, 0) + \theta_3(y, y, y) +$$
$$+(-1)^{|x_1|}(x_1, 0, y) + (-1)^{|x_1|+|y|}\theta_3(x_1, y, 0) + \hat{m}_1(2y) + \hat{m}_2(y, y) +$$
$$+(-1)^{|x_1|}\hat{m}_2(x_1, 0) + \hat{m}_3(y, y, y) + (-1)^{|x_1|}m_3(x_1, 0, y) +$$
$$+(-1)^{|x_1||y|}\hat{m}_3(x_1, y, 0)$$
$$= 0$$

These are two examples to show that our example of a strong homotopy derivation on an A_∞ algebra is consistent with our definition. We know the technique for that θ_n works, as we

proved this earlier. This is just a way to double check our work.

Next we more to a concrete example of an L_∞ algebra. As a reminder, here was our finite L_∞ algebra from before:

6.2 Concrete L_∞ Example

Our definition of strong homotopy derivation is only defined on the desupsended coalgebras, so here, we use the work from chapter 2 to use the lifted L_∞ algebra example:

Example 21 (Desuspended L_∞). Let $W = W_{-1} + W_0$ be given by $W_{-1} = \langle x_1, x_2 \rangle$ and $W_0 = \langle y \rangle$, which has been desuspended from our previous finite L_∞ algebra given by W. The maps given by $\hat{l}_n : W^{\otimes n} \to W$ where

$$
\begin{aligned}
\hat{l}_1(x_1) = \hat{l}_1(x_2) &= y \\
\hat{l}_n(x_1 \otimes y^{\otimes n-1}) &= (n-1)!y \\
\hat{l}_n(x_1 \otimes y^{n-2} \otimes x_2) &= (n-2)!x_1
\end{aligned}
$$

give an L_∞ structure, as defined in the second definition using a coalgebra.

From here we can now give an explicit example of a strong homotopy derivation on our L_∞ algebra.

From our work before, we know that setting $\hat{\theta}_n(x_1, \ldots, x_n) = \hat{l}_{n+1}(x_1, \ldots, x_n, a)$ where $|a| = 2k$ for some $k \in \mathbb{Z}$ and $\hat{l}_1(a) = 0$ gives a strong homotopy derivation structure on our L_∞ algebra. In our example, y has the properties that $\hat{l}_1(y) = 0$ and $|y| = 0$. So we set

$$
\hat{\theta}_n(x_1, \ldots, x_n) = \hat{l}_{n+1}(x_1, \ldots, x_n, y)
$$

and find what these $\hat{\theta}_n$ actually are. For this, we only need to plug in x_1, $x_1 \otimes y^{\otimes n-1}$, and $x_1 \otimes y^{\otimes n-2} \otimes x_2$. The reason we don't worry about x_2 is that $\hat{l}_2(x_2, y) = 0$.

Now we plug in our three terms:

(i) For x_1, we have,

$$
\begin{aligned}
\hat{\theta}_1(x_1) &= \hat{l}_2(x_1, y) \\
&= y
\end{aligned}
$$

(ii) Next, we evaluation on $x_1 \otimes y^{\otimes n-1}$, to get

$$\hat{\theta}_n(x_1 \otimes y^{\otimes n-1}) = \hat{l}_{n+1}(x_1 \otimes y^n)$$
$$= n!y$$

(iii) Lastly, we plug in $x_1 \otimes y^{\otimes n-2} \otimes x_2$,

$$\hat{\theta}_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = \hat{l}_{n+1}(x_1 \otimes y^{\otimes n-1} \otimes x_2)$$
$$= (n-1)!x_1$$

What we wish to show is that by setting

$$\hat{\theta}_1(x_1) = y$$
$$\hat{\theta}_n(x_1 \otimes y^{\otimes n-1}) = n!y$$
$$\hat{\theta}_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = (n-1)!x_1$$

then we get a strong homotopy derivation structure on our L_∞ algebra. Before we explicitly state this example, we prove, using the definition, that this is an L_∞ strong homotopy derivation structure by showing

$$\sum_{\substack{j=1 \\ \sigma \in U(j,n-j)}}^{j=n} (-1)^{\epsilon(\sigma)} \theta_{n-j+1}(l_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)})$$
$$+ (-1)^{\epsilon(\sigma)} l_{n-j+1}(\theta_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)}) = 0$$
(6.1)

where $(-1)^{|\sigma|}$ is the sign of the unshuffle. We show this double sum is zero on the elements x_1, $x_1 \otimes y^{\otimes n-1}$, and $x_1 \otimes y^{\otimes n-2} \otimes x_2$. Although we have already showed this $\hat{\theta}$ structure should work, we do it again now that $\hat{\theta}$ has been defined explicitly.

(i) For x_1, we have that $\hat{\theta}_1(\hat{l}_1(x_1)) + \hat{l}_1(\hat{\theta}_1(x_1)) = 0$, so the double sum definition holds.

(ii) For $x_1 \otimes y^{\otimes n-1}$, the only terms of importance are:

$$\pm \hat{l}_i(\hat{\theta}_j(x_1 \otimes y^{\otimes j-1}), y^{\otimes n-j}) \pm \hat{\theta}_i(\hat{l}_j(x_1 \otimes y^{\otimes j-1}), y^{\otimes n-j})$$
$$= \hat{l}_i((j-1)!y, y^{\otimes n-j}) \pm \hat{\theta}_i((j-1)!y, y^{\otimes n-j})$$
$$= 0$$

So each term in the double sum is zero, hence the definition holds.

(iii) Lastly we have the term $x_1 \otimes y^{\otimes n-2} \otimes x_2$. The terms that will give us nonzero elements are:

(I) $\pm \hat{l}_{n-j+1}(\hat{\theta}_j(x_1 \otimes y^{\otimes j-2} \otimes x_2), y^{\otimes n-j})$ for $j = 2, \ldots, n$.

(II) $\pm \hat{\theta}_n(\hat{l}_1(x_2), x_1, y^{\otimes n-2})$

(III) $\hat{\theta}_{n-j+1}(\hat{l}_j(x_1 \otimes y^{\otimes j-2} \otimes x_2), y^{\otimes n-j})$ for $j = 2, \ldots, n$.

For the second type, we have that

$$
\begin{aligned}
\pm \hat{\theta}_n(\hat{l}_1(x_2), x_1, y^{\otimes n-2}) &= -\hat{\theta}_n(y, x_1, y^{\otimes n-2}) \\
&= -n!y
\end{aligned}
$$

For those of of type (I), note that there are $\binom{n-2}{j-2}$ of these for each $j = 2, \ldots, n$. So when we add these terms up, we get:

$$
\begin{aligned}
&\sum_{j=2}^{n} \sum_{\sigma} \hat{l}_{n-j+1}(\hat{\theta}_j(x_1 \otimes y^{\otimes j-2} \otimes x_2), y^{\otimes n-j}) \\
= &\sum_{j=2}^{n} \frac{(n-2)!}{(j-2)!(n-j)!} \hat{l}_{n-j+1}((j-1)!x_1, y^{\otimes n-j}) \\
= &\sum_{j=1}^{n} \frac{(n-2)!}{(j-2)!(n-j)!} (j-1)!(n-j)!y \\
= &\sum_{j=2}^{n} (j-1)(n-2)!y \\
= &\sum_{j=2}^{n} j(n-2)!y - \sum_{j=1}^{n} (n-2)!y \\
= &(n-2)!\frac{n(n+1)}{2}y - (n-2)!y - n(n-2)!y + (n-2)!y \\
= &(n-2)!\frac{n(n+1)}{2}y - n(n-2)!y
\end{aligned}
$$

Lastly, for those of type (III), note that there are $\binom{n-2}{j-2}$ of these for each $j = 2, \ldots, n$.

Adding these together gives:

$$\sum_{j=2}^{n} \binom{n-2}{j-2} \hat{\theta}_{n-j+1}(\hat{l}_j(x_1 \otimes y^{\otimes j-2} \otimes x_2), y^{\otimes n-j})$$

$$= \sum_{j=2}^{n} \frac{(n-2)!}{(n-j)!(j-2)!} \hat{\theta}_{n-j+1}((j-2)!x_1, y^{\otimes n-j})$$

$$= \sum_{j=2}^{n} \frac{(n-2)!}{(n-j)!}(n-j+1)!y$$

$$= (n-2)!(-1)^{n^2}(n-j+1)y$$

$$= \sum_{j=2}^{n}(-1)^{n^2}n(n-2)!y - \sum_{j=1}^{n}(n-2)!jy + \sum_{j=1}^{n}(n-2))!y$$

$$= n^2(n-2)!y - n(n-2)!y - (n-2)!\frac{n(n+1)}{2}y +$$
$$+(n-2)!y + n(n-2)!y - (n-2)!y$$

$$= n^2(n-2)!y - (n-2)!\frac{n(n+1)}{2}y$$

Now that we have added up the three different types of terms, we add the results together which will give us the double sum on the element $x_1 \otimes y^{\otimes n-2} \otimes x_2$. Adding these together gives:

$$n^2(n-2)!y - (n-2)!\frac{n(n+1)}{2}y + (n-2)!\frac{n(n+1)}{2}y$$
$$-n(n-2)!y - n!y$$

$$= n^2(n-2)!y - n(n-2)!y - n!y$$

$$= (n-2)!y[n^2 - n - n(n-1)]$$

$$= 0$$

Therefore, this example is consistent with the definition of a strong homotopy derivation for L_∞ algebras, so formally we state this example as:

Example 22. Let $W = W_{-1} + W_0$ be given by $W_{-1} = \langle x_1, x_2 \rangle$ and $W_0 = \langle y \rangle$ with maps given by $\hat{l}_n : W^{\otimes n} \to W$ where

$$\hat{l}_1(x_1) = \hat{l}_1(x_2) = y$$
$$\hat{l}_n(x_1 \otimes y^{\otimes n-1}) = (n-1)!y$$
$$\hat{l}_n(x_1 \otimes y^{n-2} \otimes x_2) = (n-2)!x_1$$

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as an L_∞ structure. Then a strong homotopy derivation on W is given by the following symmetric maps $\hat{\theta} : W^{\otimes n} \to W$:

$$
\begin{aligned}
\hat{\theta}_1(x_1) &= y \\
\hat{\theta}_n(x_1 \otimes y^{\otimes n-1}) &= n!y \\
\hat{\theta}_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) &= (n-1)!x_1
\end{aligned}
$$

Where $\hat{\theta}_n$ is zero on elements where no permutation is listed.

Now we have concrete examples for an A_∞ strong homotopy derivation and a L_∞ strong homotopy derivation to go along with our previous A_∞ and L_∞ algebras.

Chapter 7

Two Ways to Lift

In chapter 2, we saw that for our examples of A_∞ and L_∞ algebras, the following diagram commutes:

$$
\begin{array}{ccc}
& \displaystyle\sum_{\sigma\in S_n}(-1)^\gamma \hat{m}_n \circ \sigma & \\
(A,\hat{m}) & \xrightarrow{\hspace{3cm}} & (L,\hat{l}) \\
\Big\uparrow{\scriptstyle (-1)^{\frac{n(n-1)}{2}}\,\downarrow\,\circ m_n\circ\,\uparrow^{\otimes n}} & & \Big\uparrow{\scriptstyle (-1)^{\frac{n(n-1)}{2}}\,\downarrow\,\circ l_n\circ\,\uparrow^{\otimes n}} \\
(A,m) & \xrightarrow{\hspace{3cm}} & (L,l) \\
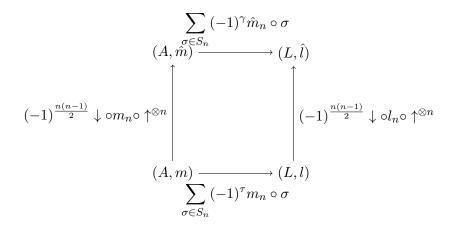& \displaystyle\sum_{\sigma\in S_n}(-1)^\tau m_n \circ \sigma &
\end{array}
$$

Figure 7.1: Two Ways to Lift

Where $(-1)^\gamma$ comes from the degrees of the permuted elements and $\tau = \gamma \cdot \epsilon(\sigma)$, where $\epsilon(\sigma)$ gives the degree of the permutation.

In this chapter we prove that this diagram, in general, commutes and thus show that there are two ways to go from a lower level A_∞ algebra to an upper level L_∞ algebra.

Before we start our work, we briefly clarify the desuspension operator, $\uparrow^{\otimes n}$. When we apply this map, much like the maps for an A_∞ algebra map, any time we move the operator past an

entry, we have to account for the degree of that element. Looking at

$$\uparrow^{\otimes n} (x_1, \ldots, x_n)$$

To desuspsned x_1 we haven't moved an operator past any entries. To desuspend x_2, we have moved $\uparrow$ past x_1. To desuspend x_3, we have moved $\uparrow$ past x_1 and x_2. To desuspend x_4, we have moved $\uparrow$ past x_1, x_2, and x_3. This gives the coefficient of

$$(-1)^{|x_1|} \cdot (-1)^{|x_1|+|x_2|} \cdot (-1)^{|x_1|+|x_2|+|x_3|} \cdots (-1)^{|x_1|+|x_2|+\cdots+|x_{n-1}|}$$

Combining these exponents, we see that there are $n-1$ of $|x_1|$, $n-2$ of $|x_2|$, etc. Therefore, the sign that comes from $\uparrow^{\otimes n}$ is

$$(-1)^{\sum\limits_{i=1}^{n}(n-i)|x_i|}$$

To show this diagram commutes, we start with $\hat{l}_n(x_1, \ldots, x_n)$ at the upper level, and work backwards to show we achieve the same results. There are a few things to note here, when we desuspend, we will let $\uparrow x_i = v_i$, and denote γ_x and γ_v as the signs that come from permuting x_i's and v_i's, respectively.

Working backwards along the left side of this diagram, we have that

$$
\begin{aligned}
\hat{l}_n(x_1, \ldots, x_n) &= \sum_{\sigma \in S_n} (-1)^{\gamma_x} \hat{m}_n \circ \sigma(x_1, \ldots, x_n) \\
&= \sum_{\sigma \in S_n} (-1)^{\gamma_x} (-1)^{\frac{n(n-1)}{2}} \downarrow \circ m_n \circ \uparrow^{\otimes n} \circ \sigma(x_1, \ldots, x_n)
\end{aligned}
$$

Along the right side, we have

$$
\begin{aligned}
\hat{l}_n(x_1, \ldots, x_n) &= (-1)^{\frac{n(n-1)}{2}} \downarrow \circ l_n \circ \uparrow^{\otimes n} (x_1, \ldots, x_n) \\
&= (-1)^{\frac{n(n-1)}{2}} \downarrow \circ \sum_{\sigma \in S_n} (-1)^{\tau} m_n \circ \sigma \circ \uparrow^{\otimes n} (x_1, \ldots, x_n) \\
&= (-1)^{\frac{n(n-1)}{2}} \downarrow \circ \sum_{\sigma \in S_n} (-1)^{\gamma_v \cdot \epsilon(\sigma)} m_n \circ \sigma \circ \uparrow^{\otimes n} (x_1, \ldots, x_n)
\end{aligned}
$$

Our goal is to show these two sums are equal. To do this, we look at terms. Once we apply the desuspension operator, all terms will be of the form $(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$. If we can show that all coefficients for each σ are equivalent, the we will have show these two sums are equal.

Instead of looking at a general permutation, we look at a general transposition and show the coefficients are equal. Once we show this equality, we will use the fact that any permutation can be written as a product of transpositions, so coefficients of a permutation will be a product of coefficients of transpositions.

Let σ be a transposition that transposes v_j and v_{j-1}, or x_j and x_{j-1} at the higher level. We now find the coefficient along the two sides of the diagram.

Along the left side, $(-1)^{\gamma_x} = (-1)^{|x_j||x_{j-1}|}$ since we have only transposed these two elements. We will still have $(-1)^{\frac{n(n-1)}{2}}$, and by applying the operator $\uparrow^{\otimes n}$, we have a coefficient of

$$(-1)^{(n-i)|x_{j-1}|+(n-i-1)|x_j|+\sum_{i=1}^{j-2}(n-i-1)|x_i| + \sum_{i=j+1}^{n}(n-i)|x_i|}$$

. These two terms $(n-i)|x_{j-1}|$ and $(n-i-1)|x_j|$ in the exponent come from the operator $\uparrow$ having to move past one less entry to desuspend x_j and moving past one extra to desuspend x_{j-1}.

Along the right side of the diagram, we have $(-1)^{\frac{n(n-1)}{2}}$ from the lifting, $(-1)^{\epsilon(\sigma)} = (-1)^1$, and $(-1)^{\gamma_v} = (-1)^{|v_j||v_{j-1}|}$ from one transposition switching v_j and v_{j-1}. Since we applied the desuspsension operator first, this was applied to all x_i, hence we get a coefficient of

$$(-1)^{\sum_{i=1}^{n}(n-i)|x_i|}.$$

Therefore our two coefficients for $(v_1, v_2, \ldots, v_j, v_{j-1}, v_{j+1}, \ldots, v_n)$ are:

$$(-1)^{\frac{n(n-1)}{2}+|x_j||x_{j-1}|+(n-i)|x_{j-1}|+(n-i-1)|x_j|+\sum_{i=1}^{j-2}(n-i-1)|x_i| + \sum_{i=j+1}^{n}(n-i)|x_i|} \tag{7.1}$$

and

$$(-1)^{\frac{n(n-1)}{2}+1+|v_j||v_{j-1}|+\sum_{i=1}^{n}(n-i)|x_i|} \tag{7.2}$$

Showing these are equivalent, reduces to showing the two exponents of (-1) are equivalent modulo 2, so we working backwards starting with the left side (7.1)

$$\frac{n(n-1)}{2} + |x_j||x_{j-1}| + (n-i)|x_{j-1}| + (n-i-1)|x_j| +$$

$$+ \sum_{i=1}^{j-2}(n-i)|x_i| + \sum_{i=j+1}^{n}(n-i)|x_i|$$

$$= \frac{n(n-1)}{2} + |x_j||x_{j-1}| + \sum_{i=j+1}^{n}(n-i)|x_i| +$$

$$+(n-i)|x_{j-1}| + \sum_{i=1}^{j-2}(n-i)|x_i| + (n-i-1)|x_j|$$

And the right side (7.2) can be seen as:

$$\frac{n(n-1)}{2} + 1 + |v_j||v_{j-1}| + \sum_{i=1}^{n}(n-i)|x_i|$$

$$= \frac{n(n-1)}{2} + 1 + (|x_j|+1)(|x_{j-1}|+1) + \sum_{i=1}^{n}(n-i)|x_i|$$

$$= \frac{n(n-1)}{2} + |x_j||x_{j-1}| + |x_j| + |x_{j-1}| + \sum_{i=1}^{n}(n-i)|x_i| \bmod 2$$

Note that both terms have $\frac{n(n-1)}{2}$ as well as $|x_j||x_{j-1}|$, so we can cancel these. Also, note that

$$\sum_{i=1}^{n}(n-i)|x_i| - (\sum_{i=1}^{j-2}(n-i)|x_i| + \sum_{i=j+1}^{n}(n-i)|x_i|)$$

$$= (n-j+1)|x_{j-1}| + (n-j)|x_j|$$

Once we cancel these last terms from our reduced forms of (7.1) and (7.2), we are left with showing that

$$(n-j)|x_{j-1}| + (n-j-1)|x_j| = |x_j| + |x_{j-1}| + (n-j+1)|x_{j-1}| + (n-j)|x_j|$$

modulo 2. This is true, as

$$(n-j)|x_{j-1}| + (n-j-1)|x_j| = n|x_{j-1}| - j|x_{j-1}| + n|x_j| - j|x_j| - |x_j|$$

and

$$
\begin{aligned}
|x_j| + |x_{j-1}| + (n-j+1)|x_{j-1}| + (n-j)|x_j| \;=\;& |x_{j-1}| + |x_{j-1}| + |x_j| \\
& + n|x_{j-1}| - j|x_{j-1}| + n|x_j| - j|x_j| \\
\cong\;& |x_j| + n|x_{j-1}| - j|x_{j-1}| + n|x_j| - j|x_j|
\end{aligned}
$$

Thus, the two coefficients for $(v_1, v_2, \ldots, v_j, v_{j-1}, v_{j+1}, \ldots, v_n)$ are equivalent, and any element that comes from one transposition has equivalent coefficients. Therefore, when working backwards in both ways to look at $\hat{l}_n(x_1, \ldots, x_n)$ and expanding the sum, any transposition has the same coefficient.

Consider the following theorem from Garrett [3],

Theorem 23. *The permutation group S_n on n things $\{1, 2, \ldots, n\}$ is generated by adjacent transpositions s_i.*

Instead of looking at a general permutation, we have looked at a general adjacent transposition and shown the coefficients are equal. Since any permutation can be written as a product of transpositions, the coefficients of a permutation will be a product of coefficients of transpositions. Therefore, the coefficients of each of the permutations on $(v_1, \ldots, v_n)$ are equivalent, resulting in the two sums being equal, and so our diagram
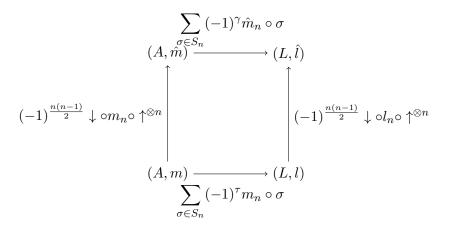

$$
\begin{array}{ccc}
& \displaystyle\sum_{\sigma \in S_n} (-1)^{\gamma}\hat{m}_n \circ \sigma & \\
(A, \hat{m}) & \xrightarrow{\hspace{3cm}} & (L, \hat{l}) \\[2em]
\Big\uparrow {\scriptstyle (-1)^{\frac{n(n-1)}{2}}\; \downarrow\, \circ m_n \circ\, \uparrow^{\otimes n}} & & \Big\uparrow {\scriptstyle (-1)^{\frac{n(n-1)}{2}}\; \downarrow\, \circ l_n \circ\, \uparrow^{\otimes n}} \\[2em]
(A, m) & \xrightarrow{\hspace{3cm}} & (L, l) \\
& \displaystyle\sum_{\sigma \in S_n} (-1)^{\tau}m_n \circ \sigma &
\end{array}
$$

Figure 7.2: Commuting Diagram for Two Ways to Lift

does, in fact, commute.

The result of this chapter is that if we start with a lower level A_∞ algebra, we can either skew-symmetrize then lift, or lift and then symmetrize to result in exactly the same desuspended L_∞ algebra.

Chapter 8

An Extra L_∞ Example

By overlooking a degree of a mapping, I was able to find an additional example of an L_∞ algebra and corresponding strong homotopy derivation:

This work gives us the following:

Example 24. Let $W = W_{-1} + W_0$ be given by $W_{-1} = \langle x_1, x_2 \rangle$ and $W_0 = \langle y \rangle$, which has been desuspended from our previous finite L_∞ algebra given by V. The maps given by $\hat{l}_n : W^{\otimes n} \to W$ where

$$
\begin{aligned}
\hat{l}_1(x_1) = \hat{l}_1(x_2) &= y \\
\hat{l}_n(x_1 \otimes y^{\otimes n-1}) &= (-1)^{n^2+1}(n-1)!y \\
\hat{l}_n(x_1 \otimes y^{n-2} \otimes x_2) &= (-1)^{n^2+1}(n-2)!x_1
\end{aligned}
$$

give an L_∞ structure, as defined in the second definition using a coalgebra.

We first show that this is, in fact, an L_∞ algebra. To prove this, we look to our sum in the second definition of L_∞ algebra and show that

$$
\sum_{\sigma \in S_{k+l=n}} (-1)^{\epsilon(\sigma)} l_{1+l}(l_k(c_{\sigma(1)}, \ldots, c_{\sigma(k)}), c_{\sigma(k+1)}, \ldots, c_{\sigma(n)}) = 0 \tag{8.1}
$$

We show this double sum is zero on each of our four inputs as follows

(i) We have that $\hat{l}_1 \circ \hat{l}_1(x_1) = \hat{l}_1(y) = 0$, so the definition holds.

(ii) We also have that $\hat{l}_1 \circ \hat{l}_1(x_2) = \hat{l}_1(y) = 0$, so again the definition holds.

(iii) When we look at this double sum on $x_1 \otimes y^{\otimes n-1}$, the only terms we need to consider are

those where x_1 is in the first position. Here we have:

$$\pm \hat{l}_{n-j+1}(\hat{l}_j(x_1 \otimes y^{\otimes j-1}), y^{\otimes n-j}) \;=\; \pm \hat{l}_{n-j+1}((-1)^{j^2+1}(j-1)!y, y^{\otimes n-j})$$
$$=\; 0$$

Therefore each term in the double sum is zero and hence the definition holds.

(iv) Lastly, we look at the double sum on the element $x_1 \otimes y^{\otimes n-2} \otimes x_2$. Inside the double sum there are four types of elements we need to consider, as all others will be zero. These nonzero terms are

(I) $\pm \hat{l}_n(\hat{l}_1(x_2), x_1 \otimes y^{\otimes n-2})$

(II) $\pm \hat{l}_i(\hat{l}_j(x_1 \otimes y^{\otimes j-2} \otimes x_2), y^{\otimes n-j})$ for $j = 2, \ldots n-1$.

We go through these and look at each element, then add them to get zero.

(I) Since we have switch x_1 and x_2, both of degree -1, we have that

$$-\hat{l}_n(\hat{l}_1(x_2), x_1, y^{\otimes n-2}) \;=\; -\hat{l}_n(y, x_1, y^{\otimes n-2})$$
$$=\; (-1)^{n^2+1}(n-1)!y$$
$$=\; (-1)^{n^2}(n-1)!y$$

(II) Lastly, we have that,

$$\pm \hat{l}_i(\hat{l}_j(x_1 \otimes y^{\otimes j-2} \otimes x_2), y^{\otimes n-j}) \;=\; \hat{l}_{n-j+1}((-1)^{j^2+1}(j-2)!x_1, y^{\otimes n-j})$$
$$=\; (-1)^{j^2+1}(-1)^{(n-j+1)^2+1}(n-j)!(j-2)!y$$

Now note that there are $\binom{n-2}{j-2}$ elements of this form for each $j = 2, \ldots, n$.. Since there are $\binom{n-2}{j-2}$ terms, when we add them all up we get:

$$\sum_{j=2}^{n}(-1)^{j^2+1}(-1)^{(n-j+1)^2+1}\binom{n-2}{j-2}(n-j)!(j-2)!y$$
$$=\; \sum_{j=2}^{n}(-1)^{j^2+1}(-1)^{(n-j+1)^2+1}\frac{(n-2)!}{(n-j)!(j-2)!}(n-j)!(j-2)!y$$
$$=\; \sum_{j=2}^{n}(-1)^{n^2+1}(n-2)!y$$
$$=\; (-1)^{n^2+1}n(n-2)!y - (-1)^{n^2+1}(n-2)!y$$

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For the exponent of -1, we have that

$$j^2 + 1 + (n - j + 1)^2 + 1 \quad \equiv \quad j^2 + n^2 - 2nj + 2n + j^2 - 2j + 1$$

$$\equiv \quad n^2 + 1$$

Hence, the exponent simplifies when we look modulo 2.

Now, we have these two types of term, only one each of type (I), and we've already added up the $\binom{n-2}{j-2}$ terms of type (II) for $j = 2, \ldots, n$. We add all of these to get:

$$
\begin{aligned}
& (-1)^{n^2}(n-1)!y + (-1)^{n^2+1}n(n-2)!y - (-1)^{n^2+1}(n-2)!y \\
= \ & ((-1)^{n^2}(n-1) + (-1)^{n^2+1}n - (-1)^{n^2+1})(n-2)!y \\
= \ & ((-1)^{n^2}n - (-1)^{n^2} - (-1)^{n^2}n + (-1)^{n^2})(n-2)!y \\
= \ & 0(n-2)!y \\
= \ & 0
\end{aligned}
$$

Therefore, the double sum from our definition of L_∞ algebra holds on all types of elements and our desuspended algebra is, in fact, an L_∞ algebra. From here we can now give an explicit example of a strong homotopy derivation on our L_∞ algebra.

From our work before, we know that setting $\hat{\theta}_n(x_1, \ldots, x_n) = \hat{l}_{n+1}(x_1, \ldots, x_n, a)$ where $|a| = 2k$ for some $k \in \mathbb{Z}$ and $\hat{l}_1(a) = 0$ gives a strong homotopy derivation structure on our L_∞ algebra. In our example, y has the properties that $\hat{l}_1(y) = 0$ and $|y| = 0$. So we set

$$\hat{\theta}_n(x_1, \ldots, x_n) = \hat{l}_{n+1}(x_1, \ldots, x_n, y)$$

and find what these $\hat{\theta}_n$ actually are. For this, we only need to plug in x_1, $x_1 \otimes y^{\otimes n-1}$, and $x_1 \otimes y^{\otimes n-2} \otimes x_2$. The reason we don't worry about x_2 is that $\hat{l}_2(x_2, y) = 0$.

Now we plug in our three terms:

(i) For x_1, we have,

$$
\begin{aligned}
\hat{\theta}_1(x_1) & = \hat{l}_2(x_1, y) \\
& = -y
\end{aligned}
$$

(ii) Next, we try $x_1 \otimes y^{\otimes n-1}$, to get

$$
\begin{aligned}
\hat{\theta}_n(x_1 \otimes y^{\otimes n-1}) &= \hat{l}_{n+1}(x_1 \otimes y^n) \\
&= (-1)^{(n+1)^2+1} n! y \\
&= (-1)^{n^2} n! y
\end{aligned}
$$

(iii) Lastly, we plug in $x_1 \otimes y^{\otimes n-2} \otimes x_2$,

$$
\begin{aligned}
\hat{\theta}_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) &= \hat{l}_{n+1}(x_1 \otimes y^{\otimes n-1} \otimes x_2) \\
&= (-1)^{n^2} (n-1)! x_1
\end{aligned}
$$

What we wish to show is that by setting

$$
\begin{aligned}
\hat{\theta}_1(x_1) &= y \\
\hat{\theta}_n(x_1 \otimes y^{\otimes n-1}) &= (-1)^{n^2} n! y \\
\hat{\theta}_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) &= (-1)^{n^2} (n-1)! x_1
\end{aligned}
$$

then we get a strong homotopy derivation structure on our L_∞ algebra. Before we explicitly state this example, we prove, using the definition, that this is an L_∞ strong homotopy derivation structure by showing

$$
\begin{aligned}
\sum_{\substack{j=1 \\ \sigma \in U(j,n-j)}}^{j=n} &(-1)^{\epsilon(\sigma)} \theta_{n-j+1}(l_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)}) \\
&+ (-1)^{\epsilon(\sigma)} l_{n-j+1}(\theta_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)}) = 0
\end{aligned}
\tag{8.2}
$$

where $(-1)^{|\sigma|}$ is the sign of the unshuffle. We show this double sum is zero on the elements x_1, $x_1 \otimes y^{\otimes n-1}$, and $x_1 \otimes y^{\otimes n-2} \otimes x_2$. Although we have already showed this $\hat{\theta}$ structure should work, we do it again now that $\hat{\theta}$ has been defined explicitly.

(i) For x_1, we have that $\hat{\theta}_1(\hat{l}_1(x_1)) + \hat{l}_1(\hat{\theta}_1(x_1)) = 0$, so the double sum definition holds.

(ii) For $x_1 \otimes y^{\otimes n-1}$, the only terms of importance are:

$$
\begin{aligned}
&\pm \hat{l}_i(\hat{\theta}_j(x_1 \otimes y^{\otimes j-1}), y^{\otimes n-j}) \pm \hat{\theta}_i(\hat{l}_j(x_1 \otimes y^{\otimes j-1}), y^{\otimes n-j}) \\
&= \hat{l}_i(\alpha y, y^{\otimes n-j}) \pm \hat{\theta}_i(\alpha y, y^{\otimes n-j}) \\
&= 0
\end{aligned}
$$

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So each term in the double sum is zero, hence the definition holds.

(iii) Lastly we have the term $x_1 \otimes y^{\otimes n-2} \otimes x_2$. The terms that will give us nonzero elements are:

$\quad$ (I) $\pm \hat{l}(n-j+1(\hat{\theta}_j(x_1 \otimes y^{\otimes j-2} \otimes x_2), y^{\otimes n-j})$ for $j = 2, \ldots, n$.

$\quad$ (II) $\pm \hat{\theta}_n(\hat{l}_1(x_2), x_1, y^{\otimes n-2})$

$\quad$ (III) $\hat{\theta}_{n-j+1}(\hat{l}_j(x_1 \otimes y^{\otimes j-2} \otimes x_2), y^{\otimes n-j})$ for $j = 2, \ldots, n$.

For the second type, we have that

$$\begin{aligned}
\pm \hat{\theta}_n(\hat{l}_1(x_2), x_1, y^{\otimes n-2}) &= -\hat{\theta}_n(y, x_1, y^{\otimes n-2}) \\
&= -(-1)^{n^2} n! y
\end{aligned}$$

For those of of type (I), note that there are $\binom{n-2}{j-2}$ of these for each $j = 2, \ldots, n$. So when we add these terms up, we get:

$$\begin{aligned}
& \sum_{j=2}^{n} \sum_{\sigma} \hat{l}_{n-j+1}(\hat{\theta}_j(x_1 \otimes y^{\otimes j-2} \otimes x_2), y^{\otimes n-j}) \\
=& \sum_{j=2}^{n} \frac{(n-2)!}{(j-2)!(n-j)!} \hat{l}_{n-j+1}((-1)^{j^2}(j-1)! x_1, y^{\otimes n-j}) \\
=& \sum_{j=1}^{n} \frac{(n-2)!}{(j-2)!(n-j)!} (-1)^{j^2}(j-1)!(-1)^{(n-j+1)^2+1}(n-j)! y \\
=& \sum_{j=2}^{n} (-1)^{n^2}(j-1)(n-2)! y \\
=& \sum_{j=2}^{n} (-1)^{n^2} j (n-2)! y - \sum_{j=1}^{n} (-1)^{n^2}(n-2)! y \\
=& (-1)^{n^2}(n-2)! \frac{n(n+1)}{2} y - (-1)^{n^2}(n-2)! y - (-1)^{n^2} n(n-2)! y + (-1)^{n^2}(n-2)! y \\
=& (-1)^{n^2}(n-2)! \frac{n(n+1)}{2} y - (-1)^{n^2} n(n-2)! y
\end{aligned}$$

Lastly, for those of type (III), note that there are $\binom{n-2}{j-2}$ of these for each $j = 2, \ldots, n$.Adding

these together gives:

$$\sum_{j=2}^{n} \binom{n-2}{j-2} \hat{\theta}_{n-j+1}(\hat{l}_j(x_1 \otimes y^{\otimes j-2} \otimes x_2), y^{\otimes n-j})$$

$$= \sum_{j=2}^{n} \frac{(n-2)!}{(n-j)!(j-2)!} \hat{\theta}_{n-j+1}((-1)^{j^2+1}(j-2)! x_1, y^{\otimes n-j})$$

$$= \sum_{j=2}^{n} \frac{(n-2)!}{(n-j)!}(-1)^{j^2+1}(-1)^{(n-j+1)^2}(n-j+1)! y$$

$$= (n-2)!(-1)^{n^2}(n-j+1)y$$

$$= \sum_{j=2}^{n}(-1)^{n^2} n(n-2)! y - \sum_{j=1}^{n}(-1)^{n^2}(n-2)! jy + \sum_{j=1}^{n}(-1)^{n^2}(n-2))! y$$

$$= (-1)^{n^2} n^2 (n-2)! y - (-1)^{n^2} n(n-2)! y - (-1)^{n^2}(n-2)! \frac{n(n+1)}{2} y +$$

$$\quad + (-1)^{n^2}(n-2)! y + (-1)^{n^2} n(n-2)! y - (-1)^{n^2}(n-2)! y$$

$$= (-1)^{n^2} n^2 (n-2)! y - (-1)^{n^2}(n-2)! \frac{n(n+1)}{2} y$$

Now that we have added up the three different types of terms, we add the results together which will give us the double sum on the element $x_1 \otimes y^{\otimes n-2} \otimes x_2$. Adding these together gives:

$$(-1)^{n^2} n^2 (n-2)! y - (-1)^{n^2}(n-2)! \frac{n(n+1)}{2} y + (-1)^{n^2}(n-2)! \frac{n(n+1)}{2} y$$

$$\quad -(-1)^{n^2} n(n-2)! y - (-1)^{n^2} n! y$$

$$= (-1)^{n^2} n^2 (n-2)! y - (-1)^{n^2} n(n-2)! y - (-1)^{n^2} n! y$$

$$= (-1)^{n^2}(n-2)! y[n^2 - n - n(n-1)]$$

$$= 0$$

Therefore, this example is consistent with the definition of a strong homotopy derivation for L_∞ algebras, so formally we state this example as:

Example 25. Let $W = W_{-1} + W_0$ be given by $W_{-1} = \langle x_1, x_2 \rangle$ and $W_0 = \langle y \rangle$ with maps given by $\hat{l}_n : W^{\otimes n} \to W$ where

$$\hat{l}_1(x_1) = \hat{l}_1(x_2 = y$$

$$\hat{l}_n(x_1 \otimes y^{\otimes n-1}) = (-1)^{n^2+1}(n-1)! y$$

$$\hat{l}_n(x_1 \otimes y^{n-2} \otimes x_2) = (-1)^{n^2+1}(n-2)! x_1$$

as an L_∞ structure. Then a strong homotopy derivation on W is given by the following symmetric maps $\hat{\theta} : W^{\otimes n} \to W$:

$$
\begin{aligned}
\hat{\theta}_1(x_1) &= y \\
\hat{\theta}_n(x_1 \otimes y^{\otimes n-1}) &= (-1)^{n^2} n! y \\
\hat{\theta}_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) &= (-1)^{n^2}(n-1)! x_1
\end{aligned}
$$

Where $\hat{\theta}_n$ is zero on elements where no permutation is listed.

This is another example of an L_∞ algebra and strong homotopy derivation. Again, this resulted by overlooking a corresponding sign from a previous L_∞ algebra example, but in the end, we have another concrete example of an L_∞ algebra and corresponding strong homotopy derivation.

REFERENCES

[1] Allocca, Michael P., and Tom Lada, 2010: A Finite Dimensional A_∞ Algebra Example. *Georgian Math Journal*, **Vol. 12, No. 1**, 1-12.

[2] Daily, Marilyn and Tom Lada, 2005: A Finite Dimensional L_∞ Algebra Example in Gauge Theory. *Homology, Homotopy, and Applications*, **Vol. 7, No. 2**, 87-93.

[3] Garrett, Paul B., 2008: *Abstract Algebra*, CRC Press, 451pp.

[4] Jacobson, Nathan, 1966: *Lie Algebras*, John Wiley & Sons, Inc., 73-83.

[5] Kajiura, Hiroshige, and Jim Stasheff, 2006: Homotopy Algebras Inspired by Classical Open-Closed String Field Theory. *Communications Mathematical Phsyics*, **Vol. 263**, 553-581.

[6] Lada, Tom, 1999: Commutators of A_∞ Structures. *Contemporary Mathematics*, **Vol. 227**, 227-233.

[7] Lada, T. and M. Markl, 1994: Strongly Homotopy Lie Algebras. *Communications in Algebra*, **Vol. 23, No. 6**, 2147-2161.

[8] Lada, T. and J. Stasheff, 1993: Introduction to SH Lie Algebras for Physicists. *International Journal of Theoretical Physics*, **Vol. 32**, 1087-1103.

[9] J. Stasheff, 1963: Homotopy Associativity of H-Spaces II. *Transactions of the American Mathematical Society*, **Vol. 108, No. 2**, 293-312.