

ABSTRACT

HUTCHENS, JOHN DAVID. Isomorphy Classes of k -involutions of Algebraic Groups of Type G_2 and F_4 . (Under the direction of Aloysius G. Helminck.)

We continue the full classification of symmetric k -varieties and k -involutions that was initiated by A.G. Helminck and collaborators. Full classifications of k -involutions have been determined for some groups of classical type. Here we begin the classification for groups of exceptional type beginning with groups of type G_2 and F_4 .

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Isomorphy Classes of k -involutions of Algebraic Groups of Type G_2 and F_4

by
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DEDICATION

I would like to dedicate this composition to my parents David and Lestine Hutchens, and thank them for all of their support.

BIOGRAPHY

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Chapter 1

Introduction

I would first like to note that this work fits into a series of papers by A.G. Helminck and collaborators, in particular Ling Wu, Chris Dometrius, Kyle Thompson, Robert Benim and John Absher. Their papers concerning k -involutions and symmetric k -varieties have been valuable. Specifically the series of papers; [1], [2], [19]. I would also like to acknowledge the papers written by A.G. Helminck that initiated the study of symmetric k -varietites and k -involutions; [4], [6], and [5].

1.1 Symmetric spaces

Symmetric spaces have been studied extensively in modern mathematics. They were first studied by Élie Cartan, and later by Berger, Helgason, and others. They have been studied in the realm of Riemannian manifolds and Lie groups. In the field of Riemannian manifolds, symmetric spaces can be defined in terms of geodesics. We will call the shortest path between nearby points, with respect to some Riemannian metric, a *geodesic*. For a more detailed definition see [12]. If V_1 and V_2 are vector spaces with norms N_1 and N_2 , respectively, we call a linear map $s : V_1 \rightarrow V_2$ an *isometry* if $N_2(s(x)) = N_1(x)$ for all $x \in V_1$.

If p is a point on a Riemannian manifold M we define a *geodesic symmetry* to be a map, g , on an open neighborhood of p , fixing p and reversing the geodesics through p . In other words if γ is a geodesic through p , $g(\gamma(t)) = \gamma(-t)$ when $\gamma(0) = p$. We say that M is *locally Riemannian symmetric* if all of its geodesic symmetries are isometries and *globally Riemannian symmetric* if its geodesic symmetries are also defined everywhere on M .

In Lie theory we define a symmetric space in a different way. If we have a Lie group G and a subgroup $H = \{h \in G \mid \theta(h) = h\}$ that is the fixed point group of an involutive automorphism, $\theta : G \rightarrow G$, where $\theta^2 = \text{id}$, then a *symmetric space* is the quotient space G/H . It turns out that a globally Riemannian symmetric space is a special case of the more algebraic definition

given here. Riemannian symmetric spaces occur when the fixed point group of the involution is compact; a lot is known about these spaces over \mathbb{R} and \mathbb{C} . Our goal is to provide some insight into the structure of analogous spaces over local fields and finite fields. To do this we generalize the definition via algebraic group theory.

1.2 Symmetric k -varieties and k -involutions

The problem of identifying all isomorphy classes of symmetric k -varieties is described by Helminck in [6]. There he notes that isomorphy classes of symmetric k -varieties of algebraic groups and isomorphy classes of their k -involutions are in bijection. In the following we provide a classification of isomorphy classes of k -involutions for split types of G_2 and split types of F_4 over certain fields.

A k -involution is an automorphism of order exactly 2, that is defined over the field k . The isomorphy classes of these k -involutions are in bijection with isomorphy classes of the quotient spaces G_k/H_k , where G_k and H_k are the k -rational points of the groups G and $H = G^\theta = \{g \in G \mid \theta(g) = g\}$ respectively, these quotient spaces will be called *symmetric k -varieties*. We can also make the identification $G_k/H_k \cong Q = \{g\theta(g)^{-1} \mid g \in G_k\}$.

1.3 Exceptional groups

Unlike the classical series A_n, B_n, C_n , and D_n , the exceptional groups E_6, E_7, E_8, F_4 , and G_2 do not fit into an infinite family of groups. In this paper we consider k -involutions of exceptional type, specifically of types G_2 and F_4 . Throughout this paper we use the fact that groups of this type can be thought of as automorphisms of certain k -algebras.

Groups of type G_2 over a field k can be thought of as the automorphism group of an octonion algebra over the field k . Octonion algebras are 8 dimensional algebras, whose structure can be thought of as a pair of quaternions with a new product defined in terms of the quaternion product. These algebras are equipped with a map of order 2 that is an analog to complex conjugation. Since, groups of type G_2 are automorphism groups of an octonion algebra, many aspects of their structure can be revealed by the structure of the algebra. So we spend some time in the beginning building up the concept of an octonion algebra, and how subalgebras correspond to the structure of the automorphism group and its k -involutions.

Groups of type F_4 over a field k can be thought of as the automorphism group of an Albert algebra, where an Albert algebra is a 3×3 matrix that is Hermitian up to a diagonal matrix γ , with entries in an octonion algebra over k . In other words $x \in A$, where A is an Albert algebra if and only if,

$$\gamma x \gamma^{-1} = x^*,$$

where $*$ is conjugation with respect to the octonion entries of x along with the transpose of the 3×3 matrix. We, again, spend some time building up the structure of such algebras, and how they relate to other algebras that give us insight into their structure, with a goal of revealing the structure of their automorphism groups. Again, much of the structure of the automorphism group and its k -involutions are given by the structure of the algebra.

In [5], Helminck, lays out a method of classification of k -involutions, which is described in section 4.1. In chapter 4 we follow his classification criteria, and arrive at a full classification of k -involutions of type G_2 and F_4 over certain fields. First we give background information on the algebraic groups in question, thinking of them as automorphisms of certain algebras. We begin by describing these algebras.

Chapter 2

Defining representations

The following treatment of composition algebras and J -algebras is based mostly on the constructions found in [25], [9], and [17]. These sources are cited quite often subsequently along with others when needed. All of the results regarding composition algebras or Albert algebras in the following constructions are well known, and many of the proofs are based on (and sometimes identical to) the source cited.

2.1 Composition algebras

A *composition algebra* C is an algebra over a field k with an identity element e and equipped with a nondegenerate quadratic form N such that for $x, y \in C$, $N(xy) = N(x)N(y)$.

A *quadratic form* on a vector space V over a field k is a mapping $N : V \rightarrow k$ with the following properties:

1. $N(\lambda x) = \lambda^2 N(x)$ with $\lambda \in k$ and $x \in V$.
2. There is an associated bilinear form $\langle , \rangle : V \times V \rightarrow k$ where,
$$\langle x, y \rangle = N(x + y) - N(x) - N(y).$$

Notice that $\langle x, y \rangle = N(x + y) - N(x) - N(y) = N(y + x) - N(y) - N(x) = \langle y, x \rangle$, and so the quadratic form is symmetric. If $\langle x, y \rangle = 0$ we say that $x \perp y$. Two subspaces P and Q are called *orthogonal*, $P \perp Q$, if $\langle x, y \rangle = 0$ for all $x \in P$ and all $y \in Q$. The *orthogonal complement* of P , $P^\perp = \{x \in V | x \perp y \text{ for all } y \in P\}$, where $P \subseteq V$. The bilinear form, \langle , \rangle , is said to be *nondegenerate* if $V^\perp = \{0\}$. When \langle , \rangle is nondegenerate it follows that if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in V$ then $x = y$. A linear subspace W is called *nonsingular* when $\langle , \rangle|_{W \times W}$ is nondegenerate. When W is finite dimensional $\ker\langle u, \cdot \rangle = W$ for $0 \neq u \in W^\perp$ and we have $V = W \oplus W^\perp$, and also W^\perp is nonsingular.

Proposition, [9] 2.1.0.1. *The following are true in a composition algebra;*

$$(a) \ N(e) = 1.$$

$$(b) \ \langle xy_1, xy_2 \rangle = N(x)\langle y_1, y_2 \rangle.$$

$$(c) \ \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle = \langle x_1 y_1, x_2 y_2 \rangle + \langle x_1 y_2, x_2 y_1 \rangle$$

$$(d) \ \langle x, y \rangle \langle e, x \rangle = \langle x, yx \rangle + \langle x^2, y \rangle.$$

Proof. (a) $N(x) = N(ex) = N(e)N(x) \Rightarrow N(e) = 1.$

(b)

$$\begin{aligned} N(x_1y + x_2y) &= \langle x_1y, x_2y \rangle + N(x_1y) + N(x_2y) \\ &= \langle x_1y, x_2y \rangle + N(x_1)N(y) + N(x_2)N(y). \end{aligned}$$

$$N(x_1y + x_2y) = N(x_1 + x_2)N(y) = (\langle x_1, x_2 \rangle + N(x_1) + N(x_2))N(y),$$

so

$$\langle x_1y, x_2y \rangle = \langle x_1, x_2 \rangle N(y).$$

(c) and (d)

$$\begin{aligned} \langle x_1(y_1 + y_2), x_2(y_1 + y_2) \rangle &= \langle x_1, x_2 \rangle N(y_1 + y_2) \\ &= \langle x_1, x_2 \rangle (\langle y_1, y_2 \rangle + N(y_1) + N(y_2)) \end{aligned}$$

and also

$$\begin{aligned} \langle x_1(y_1 + y_2), x_2(y_1 + y_2) \rangle &= \langle x_1 y_1, x_2 y_1 \rangle + \langle x_1 y_1, x_2 y_2 \rangle + \langle x_1 y_2, x_2 y_1 \rangle + \langle x_1 y_2, x_2 y_2 \rangle \\ &= \langle x_1, x_2 \rangle N(y_1) + \langle x_1 y_1, x_2 y_2 \rangle + \langle x_1 y_2, x_2 y_1 \rangle + \langle x_1, x_2 \rangle N(y_2). \end{aligned}$$

and so we have

$$\langle x_1, x_2 \rangle \langle y_1, y_2 \rangle = \langle x_1 y_1, x_2 y_2 \rangle + \langle x_1 y_2, x_2 y_1 \rangle.$$

and

$$\langle x, y \rangle \langle e, x \rangle = \langle x, yx \rangle + \langle x^2, y \rangle.$$

□

Proposition, [25] 2.1.0.2. *Every element x of a composition algebra C satisfies the equation*

$$x^2 - \langle x, e \rangle x + N(x)e = 0.$$

And for $x, y \in C$ we have

$$xy + yx - \langle x, e \rangle y - \langle y, e \rangle x + \langle x, y \rangle e = 0.$$

Proof. Let y be any nonzero element of C , observe

$$\begin{aligned} \langle x^2 - \langle x, e \rangle x + N(x)e, y \rangle &= \langle x^2, y \rangle - \langle x, e \rangle \langle x, y \rangle + N(x) \langle e, y \rangle \\ &= \langle x^2, y \rangle - \langle x, e \rangle \langle x, y \rangle + \langle x, yx \rangle \\ &= 0. \end{aligned}$$

□

Proposition, [25] 2.1.0.3. When $x, y \in e^\perp$ and $x \perp y$, $xy = -yx$.

Proof.

$$\begin{aligned} 0 &= (x + y)^2 - \langle x + y, e \rangle (x + y) + N(x + y)e \\ &= x^2 + xy + yx + y^2 - \langle x, e \rangle x - \langle x, e \rangle y - \langle y, e \rangle x - \langle y, e \rangle y + \langle x, y \rangle e + N(x)e + N(y)e \\ &= xy + yx - \langle x, e \rangle y - \langle y, e \rangle x + \langle x, y \rangle e. \end{aligned}$$

□

Proposition, [25] 2.1.0.4. Let $x \in C$. When the subspace $ke \oplus kx$ is two dimensional and nonsingular, it is a composition algebra.

Proof.

$$\begin{aligned} N(\alpha x + \beta e) &= \langle \alpha x, \beta e \rangle + N(\alpha x) + N(\beta e) \\ &= \alpha \beta \langle x, e \rangle + \alpha^2 N(e) + \beta^2 N(x) \\ &= \alpha^2 + \alpha \beta \langle x, e \rangle + \beta^2 N(x). \end{aligned}$$

The matrix of the linear transformation of multiplication by $\alpha e + \beta x$ is, $\begin{bmatrix} \alpha & -\beta N(x) \\ \beta & \alpha + \beta \langle x, e \rangle \end{bmatrix}$, and

$$\begin{vmatrix} \alpha & -\beta N(x) \\ \beta & \alpha + \beta \langle x, e \rangle \end{vmatrix} = \alpha^2 + \alpha \beta \langle x, e \rangle + \beta^2 N(x).$$

After several lines of straight forward computation we see that

$$N((\alpha_1 e + \beta_1 x)(\alpha_2 e + \beta_2 x)) = N(\alpha_1 e + \beta_1 x)N(\alpha_2 e + \beta_2 x).$$

□

Corollary 2.1.0.5. N is uniquely determined by the algebraic structure of the composition algebra.

Corollary 2.1.0.6. Let C be a composition algebra, then $x \in C$ implies $x^i x^j = x^{i+j}$.

2.1.1 Conjugation and inverses

Define conjugation to be the mapping, $\bar{}: C \rightarrow C$, where $\bar{x} = \langle x, e \rangle e - x = -s_e(x)$.

Proposition: identities I, [9] 2.1.1.1. The following are true in a composition algebra

$$(a) \quad x\bar{x} = x(\langle x, e \rangle e - x) = \langle x, e \rangle x - x^2 = N(x)e = \bar{x}x.$$

$$(b) \quad \bar{x}\bar{y} = \bar{y}\bar{x}$$

$$\begin{aligned} \bar{x}\bar{y} &= (\langle x, e \rangle e - x)(\langle y, e \rangle e - y) \\ &= \langle x, e \rangle \langle y, e \rangle e - \langle x, e \rangle y - \langle y, e \rangle x + xy \\ &= \langle y, e \rangle \langle x, e \rangle e - yx - \langle y, x \rangle e \\ &= \langle yx, e \rangle e + \langle y, x \rangle e - yx - \langle y, x \rangle e \\ &= \langle yx, e \rangle e - yx \\ &= \bar{y}\bar{x}. \end{aligned}$$

$$(c) \quad \bar{\bar{x}} = x$$

$$\begin{aligned} \langle \bar{x}, e \rangle e - \bar{x} &= \langle \langle x, e \rangle e - x, e \rangle e - \langle x, e \rangle e + x \\ &= \langle x, e \rangle \langle e, e \rangle e - \langle x, e \rangle e - \langle x, e \rangle e + x \\ &= 2\langle x, e \rangle e - \langle x, e \rangle e - \langle x, e \rangle e + x = x. \end{aligned}$$

$$(d) \quad \overline{x+y} = \langle x+y, e \rangle e - (x+y) = \langle x, e \rangle e + \langle y, e \rangle e - x - y = \bar{x} + \bar{y}.$$

$$(e) \quad N(\bar{x})e = \bar{\bar{x}}\bar{x} = x\bar{x} = N(x)e.$$

$$\begin{aligned} (f) \quad \langle \bar{x}, \bar{y} \rangle &= N(\bar{x} + \bar{y}) - N(\bar{x}) - N(\bar{y}) = N(\overline{x+y}) - N(x) - N(y) \\ &= N(x+y) - N(x) - N(y) = \langle x, y \rangle. \end{aligned}$$

Proposition: identities II, [9] 2.1.1.2. For $x, y, z \in C$...

$$(a) \quad \langle xy, z \rangle = \langle y, \bar{x}z \rangle$$

$$\begin{aligned} \langle y, \bar{x}z \rangle &= \langle y, (\langle x, e \rangle e - x)z \rangle = \langle y, \langle x, e \rangle z - xz \rangle = \langle x, e \rangle \langle y, z \rangle - \langle y, xz \rangle \\ &= \langle xy, z \rangle + \langle xz, y \rangle - \langle y, xz \rangle = \langle xy, z \rangle \end{aligned}$$

$$(b) \quad \langle xy, z \rangle = \langle x, z\bar{y} \rangle$$

$$(c) \langle xy, \bar{z} \rangle = \langle yz, \bar{x} \rangle.$$

Proposition: identities III, [25] 2.1.1.3. For $x, y, z \in C \dots$

$$(a) x(\bar{y}) = N(x)y$$

let $0 \neq z \in C$ then

$$\langle x(\bar{y}), z \rangle = \langle \bar{y}, \bar{x}z \rangle = N(x)\langle y, z \rangle = \langle N(x)y, z \rangle.$$

$$(b) (\bar{x}y)y = N(y)x$$

$$(c) x(\bar{y}z) + y(\bar{x}z) = \langle x, y \rangle z$$

$$(d) (\bar{x}\bar{y})z + (x\bar{z})y = \langle y, z \rangle x.$$

Proposition: Moufang identities, [25] 2.1.1.4. For $a, x, y \in C \dots$

$$(a) (ax)(ya) = a((xy)a)$$

let $0 \neq z \in C$ then

$$\begin{aligned} \langle (ax)(ya), z \rangle &= \langle ya, (\bar{a}\bar{x})z \rangle = \langle ya, (\bar{x}\bar{a})z \rangle \\ &= \langle y, \bar{x}\bar{a} \rangle \langle a, z \rangle - \langle yz, (\bar{x}\bar{a})a \rangle = \langle xy, \bar{a} \rangle \langle a, z \rangle - N(a)\langle yz, \bar{x} \rangle \end{aligned}$$

$$\begin{aligned} \langle a((xy)a), z \rangle &= \langle (xy)a, \bar{a}z \rangle = \langle xy, \bar{a} \rangle \langle a, z \rangle - \langle (xy)z, a\bar{a} \rangle \\ &= \langle xy, \bar{a} \rangle \langle a, z \rangle - \langle (xy)z, a\bar{a} \rangle = \langle xy, \bar{a} \rangle \langle a, z \rangle - N(a)\langle (xy)z, e \rangle \\ &= \langle xy, \bar{a} \rangle \langle a, z \rangle - N(a)\langle xy, \bar{z} \rangle = \langle xy, \bar{a} \rangle \langle a, z \rangle - N(a)\langle y, \bar{x}\bar{z} \rangle \\ &= \langle xy, \bar{a} \rangle \langle a, z \rangle - N(a)\langle yz, \bar{x} \rangle \end{aligned}$$

$$(b) a(x(ay)) = (a(xa))y$$

$$(c) x(a(ya)) = ((xa)y)a.$$

Proposition: identities IV, [25] 2.1.1.5. For $x, y \in C \dots$

$$(a) (xy)x = x(yx)$$

$$(xy)x = (xy)(ex) = x((ye)x) = x(yx)$$

$$(b) x(xy) = x^2y$$

$$x(\bar{y}x) = (x\bar{x})y$$

$$x(\bar{xy}) = x((\langle x, e \rangle e - x)y) = x(\langle x, e \rangle y - xy) = \langle x, e \rangle xy - x(xy)$$

$$(x\bar{x})y = (x(\langle x, e \rangle e - x))y = (\langle x, e \rangle x - x^2)y = \langle x, e \rangle xy - x^2y.$$

$$(c) (xy)y = xy^2.$$

2.1.2 The associator and composition algebras as alternative algebras

We define the associator, $A(x, y, z) = (xy)z - x(yz)$, and in the same way the commutator measures how commutative an algebra is, the associator measures associativity. It is a trilinear map, and to say an algebra is alternative is to say the associator on the algebra is alternating.

Using 2.1.1.5 we see that when any two entries in a composition algebra are the same the associator vanishes.

Theorem (E. Artin) 2.1.2.1. *The algebra generated by any two elements of an alternative algebra is associative.*

2.1.3 Doubling

Soon we will see, through a theorem of Hurwitz, that the largest composition algebra is eight dimensional, and the only others have dimension 1, 2 or 4. It turns out that composition algebras of dimension 2, 4 and 8 can be defined over any field (including when the characteristic is 2), and that composition algebras of dimension 1 can be defined over any field of characteristic not 2.

We can construct larger composition algebras by a process called doubling, up to a certain dimension. If C is a composition algebra then $C = D \oplus D^\perp$ when D is a nonsingular finite dimensional subspace, and also D^\perp is nonsingular. When D is not all of C , there must exist an $a \in D^\perp$ such that $N(a) \neq 0$.

Proposition, [9] 2.1.3.1. *Let C be a composition algebra and D a finite dimensional composition subalgebra, $D \neq C$. Choose $a \in D^\perp$ with $N(a) \neq 0$, then*

$$D_1 = D \oplus Da$$

is a composition algebra.

Proof. The product, norm and conjugation of D_1 are given by the following...

$$(x + ya)(u + va) = (xu + \lambda\bar{v}y) + (vx + y\bar{u})a$$

$$N(x + ya) = N(x) - \lambda N(y)$$

$$\overline{x + ya} = \bar{x} - ya$$

Where $\lambda = -N(a)$, and dimension of D_1 is twice the dimension of D .

Proposition, [9] 2.1.3.2. *Let C be a composition algebra and D a finite dimensional proper subalgebra, then D is associative.*

First notice that if $a \in D^\perp$, $N(a) \neq 0$, and $x, y \in D$, then $\langle xa, y \rangle = \langle a, \bar{xy} \rangle = 0 \Rightarrow xa \in D^\perp$. Also if $\lambda = -N(a)$,

$$\begin{aligned} N((x + ya)(u + va)) &= N((xu + \lambda\bar{v}y) + (vx + y\bar{u})a) \\ &= \langle xu + \lambda\bar{v}y, (vx + y\bar{u})a \rangle + N(xu + \lambda\bar{v}y) + N(a)N(vx + y\bar{u}) \\ &= \langle xu, (vx + y\bar{u})a \rangle + \langle \lambda\bar{v}y, (vx + y\bar{u})a \rangle + \langle xu, \lambda\bar{v}y \rangle + \\ &\quad N(xu) + N(\lambda\bar{v}y) + N(a)(\langle vx, y\bar{u} \rangle + N(vx) + N(y\bar{u})) \\ &= \langle u, \bar{x}(vx + y\bar{u})a \rangle + \lambda\langle y, v(vx + y\bar{u})a \rangle + \lambda\langle xu, \bar{v}y \rangle + \\ &\quad N(xu) + \lambda^2 N(\bar{v}y) + N(a)(\langle vx, y\bar{u} \rangle + N(vx) + N(y\bar{u})) \\ &= \lambda\langle xu, \bar{v}y \rangle + N(x)N(u) + N(a)N(a)N(v)N(y) - \lambda\langle vx, y\bar{u} \rangle + \\ &\quad N(a)N(v)N(x) + N(a)N(y)N(u). \end{aligned}$$

$$\begin{aligned} N(x + ya)N(u + va) &= (\langle x, ya \rangle + N(x) + N(ya))(\langle u, va \rangle + N(u) + N(va)) \\ &= (N(x) + N(y)N(a))(N(u) + N(v)N(a)) \\ &= N(x)N(u) + N(x)N(v)N(a) + N(y)N(a)N(u) + N(y)N(a)N(v)N(a) \end{aligned}$$

Since $N((x + ya)(u + va)) = N(x + ya)N(u + va)$, we get $\lambda\langle xu, \bar{v}y \rangle - \lambda\langle vx, y\bar{u} \rangle = 0 \Rightarrow \langle xu, \bar{v}y \rangle = \langle vx, y\bar{u} \rangle$.

$$\langle xu, \bar{v}y \rangle = \langle (xu)\bar{y}, \bar{v} \rangle$$

$$\langle (xu)\bar{y}, \bar{v} \rangle = \langle x, \bar{v}(y\bar{u}) \rangle$$

$$\langle (xu)\bar{y}, \bar{v} \rangle = \langle x(\bar{y}\bar{u}), \bar{v} \rangle$$

$$\langle (xu)\bar{y}, \bar{v} \rangle = \langle x(u\bar{y}), \bar{v} \rangle$$

$$\Rightarrow (xu)\bar{y} = x(u\bar{y}).$$

□

Proposition, [25] 2.1.3.3. *A subaglebra $D \oplus Da$, where $a \in D^\perp$ with $N(a) \neq 0$, is associative*

if and only if D is commutative and associative.

Proof.

$$\begin{aligned} ((x + ya)(u + va))(b + ca) &= ((xu + \lambda\bar{v}y) + (vx + y\bar{u})a)(b + ca) \\ &= (xu + \lambda\bar{v}y)b + \lambda\bar{c}(vx + y\bar{u}) + (c(xu + \lambda\bar{v}y) + (vx + y\bar{u})\bar{b})a \end{aligned}$$

$$\begin{aligned} (x + ya)((u + va)(b + ca)) &= (x + ya)((ub + \lambda\bar{c}v) + (cu + v\bar{b})a) \\ &= x(ub + \lambda\bar{c}v) + \lambda\overline{(cu + v\bar{b})y} + ((cu + v\bar{b})x + y(ub + \lambda\bar{c}v))a \\ &= x(ub + \lambda\bar{c}v) + \lambda(\bar{u}\bar{c} + b\bar{v})y + ((cu + v\bar{b})x + y(\bar{b}\bar{u} + \lambda\bar{v}c))a. \end{aligned}$$

$$((x + ya)(u + va))(b + ca) = (x + ya)((u + va)(b + ca))$$

$\Leftrightarrow D$ is commutative and associative.

□

Proposition [9] 2.1.3.4. Let D be a composition algebra and $\lambda \in k^*$. Define on $C = D \oplus D$ a product and quadratic form,

$$(x, y)(u, v) = (xu + \lambda\bar{v}y, vx + y\bar{u})$$

$$N(x, y) = N(x) - \lambda N(y).$$

If D is associative, then C is a composition algebra. C is associative if and only if D is commutative and associative.

Theorem, (A. Hurwitz) 2.1.3.5. Every composition algebra is obtained by repeated doubling starting from ke . Composition algebras have dimension 1, 2, 4, or 8. Composition algebras of dimension 1 or 2 are commutative and associative. Composition algebras of dimension 4 are associative and not commutative. Composition algebras of dimension 8 are neither commutative nor associative.

ke is commutative and associative since k is a field. If $\dim(C) > 1$, we can double to get $ke \oplus ka$, which is commutative and associative, since $x \in ke, \bar{x} = x$ and k is commutative and associative. If $\dim(C) > 2$, then we can double to get $(ke \oplus ka) \oplus (ke \oplus ka)b$, which is associative but not commutative, since $x \neq \bar{x}$ when $x \in ke \oplus ka$. If $\dim(C) > 4$ then we can double and get $((ke \oplus ka) \oplus (ke \oplus ka)b) \oplus ((ke \oplus ka) \oplus (ke \oplus ka)b)c$, which is not associative since $(ke \oplus ka) \oplus (ke \oplus ka)b$ is not commutative. There are no composition algebras larger than the last mentioned since it being not associative implies that $D \oplus Dc, D = (ke \oplus ka) \oplus (ke \oplus ka)b$,

cannot be a proper subalgebra of C by 2.1.3.2 (\oplus is used for emphasis only and is to have the same meaning as \oplus).

Corollary 2.1.3.6. *Any octonion algebra over a field k has an orthogonal basis of the form $e, a, b, ab, c, ac, bc, (ab)c$ with $N(a)N(b)N(c) \neq 0$.*

$$\begin{aligned} & ((ke \oplus ka) \oplus (ke \oplus ka)b) \bigoplus ((ke \oplus ka) \oplus (ke \oplus ka)b)c \\ & = ke \oplus ka \oplus kb \oplus kab \oplus kc \oplus kac \oplus kbc \oplus k(ab)c. \end{aligned}$$

Proposition, [25] 2.1.3.7. *Every element of an octonion algebra is contained in a quaternion subalgebra.*

Proof. Let C be an octonion algebra. If $a \in ke$ then we're done. Let $a \notin ke$ such that $\langle a, e \rangle = 0$ ($a - \frac{1}{2}\langle a, e \rangle e$). If now $N(a) \neq 0$, then $ke \oplus ka$ is nonsingular and is a subalgebra and is contained in the quaternion subalgebra made by doubling. So a is in a quaternion subalgebra.

Now we let $\langle a, e \rangle = 0$ and $N(a) = 0$. Find $b \in e^\perp$ with $\langle a, b \rangle \neq 0$ and $N(b) \neq 0$. $D = ke \oplus kb$ is a subalgebra. Choose $\lambda \in k$ such that $\langle a, b \rangle + \lambda\langle b, b \rangle = 0$.

$$\begin{aligned} N(c) &= N(a + \lambda b) = \lambda\langle a, b \rangle + N(a) + \lambda^2 N(b) \\ &= -\lambda^2\langle b, b \rangle + \lambda^2 N(b) \\ &= -2\lambda^2 N(b) + \lambda^2 N(b) = -\lambda^2 N(b) \neq 0. \end{aligned}$$

This puts $C \in D^\perp$ with $N(c) \neq 0$, and $D \oplus Dc$ is a quaternion algebra with a . \square

2.1.4 The norm determines the composition algebra

A scalar multiple of the norm determines C up to isomorphism.

A σ -similarity t is a surjective σ -linear map from V_1 onto V_2 (vector spaces with nondegenerate bilinear forms), with σ a field isomorphism from k_1 to k_2 such that $N_2(t(x)) = n(t)\sigma(N_1(x))$ for $x \in V_1$ for $n(t) \in k^*$.

$$\begin{aligned} \langle t(x), t(y) \rangle_2 &= N_2(t(x) + t(y)) - N_2(t(x)) - N_2(t(y)) \\ &= N_2(t(x + y)) - N_2(t(x)) - N_2(t(y)) \\ &= n(t)\sigma(N_1(x + y)) - n(t)\sigma(N_1(x)) - n(t)\sigma(N_1(y)) \\ &= n(t)\sigma(N_1(x + y) - N_1(x) - N_1(y)) \\ &= n(t)\sigma(\langle x, y \rangle_1). \end{aligned}$$

Theorem, [25] 2.1.4.1. *If there is a σ -similarity of C onto C' , then the two algebras are σ -isomorphic.*

Proof.

$$N'(t(x)) = n(t)\sigma(N(x)), \text{ so } N'(t(e)) = n(t)\sigma(N(e)) = n(t) \neq 0.$$

Let $T(x) = t(e)^{-1}t(x)$, then

$$N(T(x)) = N(t(e)^{-1}t(x)) = N(t(e)^{-1})N(t(x)) = \sigma(N(x)).$$

So T is a σ -isometry. Also, $T(e) = t(e)^{-1}t(e) = e'$.

Case $\sigma = id$: Our plan is to construct a linear isomorphism, $\varphi : C \rightarrow C'$.

Choose $a \in C$ such that $\langle a, e \rangle = 0$ and $N(a) \neq 0$, where $T(a) = a'$, then $N(a') = N(T(a)) = N(a)$. And $\langle a, e \rangle = \langle T(a), T(e) \rangle = \langle a', e' \rangle$. $D = ke \oplus ka$ and $D' = ke' \oplus ka'$, and so $\varphi(\lambda e + \mu a) = \lambda e' + \mu a'$ is an isomorphism. We find $b \in D^\perp$ such that $N(b) \neq 0$ and $T(b) = b' \in (D')^\perp$. Define $\varphi(x + yb) = \varphi(x) + \varphi(y)b'$, is an isomorphism from $D \oplus Db$ to $D' \oplus D'b'$. Repetition of this process along with Witt's theorem and $\varphi(\bar{x}) = \overline{\varphi(x)}$ gives us the result for composition algebras of all dimension and over all fields of $\text{char} \neq 2$. \square

Corollary 2.1.4.2. *There exists a σ -automorphism of C if and only if there exists a σ -isometry of C onto itself.*

When $N(a) = \lambda \neq 0, N(b) = \mu \neq 0, \langle a, e \rangle = \langle b, e \rangle = \langle a, b \rangle = 0$ we will call a and b a *special (λ, μ) -pair*.

Corollary 2.1.4.3. *There exists a linear automorphism, $\varphi : C \rightarrow C$, of a composition algebra and any two special (λ, μ) -pairs, a, b and a', b' , such that $\varphi(a) = a'$ and $\varphi(b) = b'$.*

2.1.5 Split Composition Algebras

The Cayley-Dickson and Hamilton cases of composition algebras are octonion or quaternion algebras over the real numbers with anisotropic norms, and these are called division composition algebras. If a norm is isotropic the composition algebra is called split. Split composition algebras have zero divisors and division algebras do not. The zero divisors are exactly those elements with zero norm. These elements do not have an inverse, and there is another zero divisor that can be multiplied by each one to get zero. In the division algebra case each nonzero element, x , has an inverse, $N(x)^{-1}\bar{x}$.

Let N be isotropic. Let us find an element $a \in e^\perp$ such that $N(a) = -1$. Let x be such that

$N(x) = 0$. First we consider $x \in e^\perp$. Pick $a' \in e^\perp$ such that $\langle a', x \rangle = 1$. Then

$$\begin{aligned} N(a' - (1 + N(a'))x) &= \langle a', -(1 + N(a'))x \rangle + N(a') + N(-(1 + N(a'))x) \\ &= -\langle a', x \rangle - N(a')\langle a', x \rangle + N(a') + N(1 + N(a'))N(x) \\ &= -1 + 2N(a') + N(x)(\langle 1, N(a') \rangle + N(1) + N(N(a'))) \\ &= -1 - 2 + 0 + 1 + (-1)^2 = -1. \end{aligned}$$

So now if $x \notin e^\perp$, we can write $x = \alpha e + y$ where $y \in e^\perp$. $N(x) = N(\alpha e + y) = \langle \alpha e, y \rangle + \alpha^2 + N(y) = \alpha^2 + N(y) = 0$. So $N(y) = -\alpha^2$. Then letting $a = \alpha^{-1}y$ we get $N(a) = N(\alpha^{-1}y) = \alpha^{-2}N(y) = \alpha^{-2}(-\alpha^2) = -1$. It follows that $ke \oplus ka$ is isotropic and its maximal totally isotropic subspace, $k(e + a)$, has dimension $\frac{1}{2} \dim(k e \oplus k a)$ (the Witt index). By doubling we find that the Witt index of C continues to be $\frac{1}{2} \dim(C)$.

It ends up being the case that all composition algebras over the same field with nondegenerate isotropic norm of the same dimension are isomorphic.

Theorem, [9] 2.1.5.1. *In each dimension 2, 4 and 8 there is exactly one split composition algebra, up to isomorphism, over a given field. These are the only composition algebras that are not division algebras.*

2.1.6 Split Quaternion and Octonion Algebras

For a composition algebra of dimension two we can simply take $N((x, y)) = xy$, for $x, y \in k$.

For a split quaternion algebra we can think of them as 2×2 matrices, $x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ where $x_{ij} \in k$, and have norm $N(x) = x_{11}x_{22} - x_{21}x_{12}$.

$$\begin{aligned} \langle x, y \rangle &= N(x + y) - N(x) - N(y) \\ &= (x_{11} + y_{11})(x_{22} + y_{22}) - (x_{21} + y_{21})(x_{12} + y_{12}) - (x_{11}x_{22} - x_{21}x_{12}) - (y_{11}y_{22} - y_{21}y_{12}) \\ &= x_{11}y_{22} + y_{11}x_{22} - x_{21}y_{12} - y_{21}x_{12} \\ \langle x, e \rangle &= x_{11} + x_{22} \\ &= \text{tr}(x) \\ \bar{x} &= \langle x, e \rangle e - x \\ &= \text{tr}(x)e - x, \end{aligned}$$

$$\text{tr}(x)e - x = \begin{bmatrix} (x_{11} + x_{22}) - x_{11} & -x_{12} \\ -x_{21} & (x_{11} + x_{22}) - x_{22} \end{bmatrix} = \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix}.$$

We can easily obtain a split octonion algebra by doubling. Since all split composition algebras over the same field of the same dimension are isomorphic we can take $\lambda = 1$, and the multiplication will be $(x, y)(u, v) = (xu + \bar{v}y, vx + y\bar{u})$. Our norm will be $N((x, y)) = \det(x) - \det(y)$.

2.1.7 Commuting and Associating Elements of Quaternion and Octonion Algebras

Let C be a quaternion algebra over the field k , and extend to its algebraic closure K . $C_K = K \otimes_k C$ is isotropic (since it's over an algebraically closed field). $C_K \cong M_2(K) \cong K \otimes_k M_2(k)$, whose center is Ke , $C \cong M_2(k)$ whose center is ke .

If z is a central element of an octonion algebra C , then z is contained in a quaternion subalgebra whose center is ke , and so $z \in ke$.

Proposition, [25] 2.1.7.1. *If $(xy)a = x(ya)$ for all $x, y \in C$, an octonion algebra, then $a \in ke$.*

Proof. Let's pick a quaternion subalgebra, D , that contains a . Then $C = D \oplus Db$, for some $b \in D^\perp$. Let $N(b) = -\lambda \neq 0$. Take elements $x = ub, y = b$ for some $u \in D$. $((ub)b)a = (ub)(ba)$.

$$\begin{aligned} ((ub)b)a &= (0 + ub)(0 + eb) \\ &= (\lambda u)a \\ &= \lambda ua \\ ba &= \langle a, e \rangle b - ab \\ &= (\langle a, e \rangle e - a)b \\ &= \bar{a}b \\ (ub)(ba) &= (ub)(\bar{a}b) \\ &= (0 + ub)(0 + \bar{a}b) \\ &= \lambda au \\ &\Rightarrow au = ua \\ &\Rightarrow a \in ke. \end{aligned}$$

□

This concludes what we need to know (and probably a little more than we need to know) about the octonion algebras and their subalgebras in order to continue. Next we will take a look at the defining representations of groups of type F_4 .

2.2 J -algebras

The class of Jordan algebras called J -algebras was first defined by T.A. Springer in [22], and we use the treatment of this type of Jordan algebra from [25] and [22].

2.2.1 Definitions and introduction to basic properties

For now we will let k be a field of characteristic not 2 or 3, and C a composition algebra of dimension eight, as discussed earlier, over k . For any fixed $\gamma_i \in k^*$ and $\gamma = \text{diag}(\gamma_1, \gamma_2, \gamma_3)$, we will define $A = H(C; \gamma)$ be the set of 3×3 , γ -hermitian matrices. Each $x \in A$, where $\xi_i \in k$ and $c_i \in C$, will look like

$$x = h(\xi_1, \xi_2, \xi_3; c_1, c_2, c_3) = \begin{bmatrix} \xi_1 & c_3 & \gamma_1^{-1}\gamma_3\bar{c}_2 \\ \gamma_2^{-1}\gamma_1\bar{c}_3 & \xi_2 & c_1 \\ c_2 & \gamma_3^{-1}\gamma_2\bar{c}_1 & \xi_3 \end{bmatrix},$$

where $\bar{\cdot}$ will denote the following conjugation, $\bar{x} = \langle x, e \rangle e - x$, in C .

We will define a product

$$xy = \frac{1}{2}(x \cdot y + y \cdot x) = \frac{1}{2}((x+y)^2 - x^2 - y^2)$$

with the dot indicating standard matrix multiplication. Notice

$$x(yz) = \frac{1}{2}(x \cdot yz + yz \cdot x) = \frac{1}{4}(x \cdot (y \cdot z + z \cdot y) + (y \cdot z + z \cdot y) \cdot x) \neq (xy)z,$$

however,

$$yx = \frac{1}{2}(y \cdot x + x \cdot y) = \frac{1}{2}(x \cdot y + y \cdot x) = xy.$$

So A is a commutative, nonassociative k -algebra of 3×3 matrices, whose identity element is $e = h(1, 1, 1; 0, 0, 0)$, the usual 3×3 matrix identity.

We will define a quadratic norm, $Q : A \rightarrow k$, with an associated bilinear form

$$\langle x, y \rangle = Q(x + y) - Q(x) - Q(y),$$

and from now on the bilinear form on C will be denoted $N(\cdot, \cdot)$, with

$$Q(x) = \frac{1}{2}(\xi_1^2 + \xi_2^2 + \xi_3^2) + \gamma_3^{-1}\gamma_2 N(c_1) + \gamma_1^{-1}\gamma_3 N(c_2) + \gamma_2^{-1}\gamma_1 N(c_3) = \frac{1}{2}\text{tr}(x^2).$$

Notice the bilinear form is nondegenerate, and $\langle x, y \rangle = \text{tr}(xy)$. With all of this together we will refer to $A \cong H(C; \gamma)$, where C is octonion, as an *Albert algebra*.

Notice that $\langle x, e \rangle = \text{tr}(xe)$, so if $x \in e^\perp$ then $\text{tr}(x) = 0$.

So from this we can look at when $\langle x, e \rangle = 0$ then

$$Q(x^2) = Q(x)^2. \quad (2.1)$$

And then in general

$$\langle xy, z \rangle = \text{tr}(xyz) = \langle x, yz \rangle, \quad (2.2)$$

and also

$$Q(e) = \frac{3}{2}. \quad (2.3)$$

If we consider, alternatively, the class of algebras with a quadratic norm Q , and the identities 2.1-2.3 we get a class of algebras called *J-algebras*. We will refer to a quadratic form that satisfies 2.1-2.3 as a *J-quadratic form*.

A *J-subalgebra* is a linear subspace of A that is nonsingular with respect to Q , contains e and is closed under Jordan multiplication.

An *isomorphism*, $t : A \rightarrow B$, of *J-algebras* over k is a bijective linear transformation, such that $t(xy) = t(x)t(y)$.

Lemma 2.2.1.1. *If $\langle x, e \rangle = \langle y, e \rangle = \langle z, e \rangle = \langle u, e \rangle = 0$, then*

$$2\langle xy, zu \rangle + 2\langle xz, yu \rangle + 2\langle xu, yz \rangle = \langle x, y \rangle \langle z, u \rangle + \langle x, z \rangle \langle y, u \rangle + \langle x, u \rangle \langle y, z \rangle.$$

Proof. If we look at $Q((\lambda x + \mu y + \nu z + \varrho u)^2) = Q(\lambda x + \mu y + \nu z + \varrho u)^2$, using the fact that the degree of the polynomial is 4, and $|k| > 4$, we can say that each side of the equation in the lemma is the coefficient of $\lambda\mu\nu\varrho$ on their respective side of the above equation. \square

Proposition, [22] 2.2.1.2. *If A is a *J-algebra* over k and l is a field extension of k , then $l \otimes_k A$ is a *J-algebra* over l , with the product and quadratic form extended to l .*

Proof. Notice that the equations in the above Lemma and $Q(x^2) = Q(x)^2$ are equivalent, and the equation in the lemma holds for any field extension of k . \square

Proposition, [25] 2.2.1.3. *Every element in a *J-algebra* satisfies the cubic equation*

$$x^3 - \langle x, e \rangle - \left(Q(x) - \frac{1}{2}\langle x, e \rangle^2 \right) x - \det(x)e = 0,$$

where \det is a cubic form on A .

Proof. If we let $x = z = u$ in the lemma above, when $\langle x, e \rangle = \langle y, e \rangle = 0$ we have

$$\begin{aligned} 2\langle xy, xx \rangle + 2\langle xx, yx \rangle + 2\langle xx, yx \rangle &= \langle x, y \rangle \langle x, x \rangle + \langle x, x \rangle \langle y, x \rangle + \langle x, x \rangle \langle y, x \rangle \\ 6\langle x^3, xy \rangle - 6Q(x)\langle x, y \rangle &= 0 \\ \langle x^3 - Q(x)x, y \rangle &= 0. \end{aligned}$$

Since we have assumed $\langle e, y \rangle = 0$ and we know the bilinear form is nondegenerate, so

$$x^3 - Q(x)x = \kappa(x)e,$$

where $\kappa(x)$ is a cubic form with values in k .

We can write any $a \in A$ as $a = x + \frac{1}{3}\langle a, e \rangle$, where $\langle x, e \rangle = 0$. If we insert $a - \frac{1}{3}\langle a, e \rangle$ for x in the above equation we get

$$\begin{aligned} \left(a - \frac{1}{3}\langle a, e \rangle\right)^3 &= a^3 - \frac{1}{3}\langle a, e \rangle a^2 - \frac{2}{3}\langle a, e \rangle a^2 + \frac{2}{9}\langle a, e \rangle^2 a + \frac{1}{9}\langle a, e \rangle^2 a - \frac{1}{27}\langle a, e \rangle^3 \\ &= a^3 - \langle a, e \rangle a^2 + \frac{1}{3}\langle a, e \rangle^2 a - \frac{1}{27}\langle a, e \rangle^3 \end{aligned}$$

and

$$\begin{aligned} Q\left(a - \frac{1}{3}\langle a, e \rangle\right)\left(a - \frac{1}{3}\langle a, e \rangle\right) &= \left(-\frac{1}{3}\langle a, e \rangle^2 + Q(a) + \frac{1}{9}\langle a, e \rangle^2 Q(e)\right)\left(a - \frac{1}{3}\langle a, e \rangle\right) \\ &= \left(Q(a) - \frac{1}{6}\langle a, e \rangle^2\right)\left(a - \frac{1}{3}\langle a, e \rangle\right) \\ &= Q(a)a - \frac{1}{3}Q(a)\langle a, e \rangle - \frac{1}{6}\langle a, e \rangle^2 a + \frac{1}{18}\langle a, e \rangle^3 \end{aligned}$$

therefore

$$a^3 - \langle a, e \rangle a^2 - \left(Q(a) - \frac{1}{2}\langle a, e \rangle^2\right)a - \det(a) = 0, \text{ for all } a \in A,$$

we will call this the *Hamilton-Cayley equation*, where \det is a cubic form with values in k . \square

This give us

$$\begin{aligned} a^3 - \langle a, e \rangle a^2 - \left(Q(a) - \frac{1}{2} \langle a, e \rangle^2 \right) a - \det(a) &= 0 \\ a \left(a^2 - \langle a, e \rangle a - \left(Q(a) - \frac{1}{2} \langle a, e \rangle^2 \right) \right) - \det(a) &= 0 \end{aligned}$$

therefore

$$a^{-1} = \det(a)^{-1} \left(a^2 - \langle a, e \rangle a - \left(Q(a) - \frac{1}{2} \langle a, e \rangle^2 \right) \right) - \det(a).$$

We will call

$$\chi_x(T) = T^3 - \langle x, e \rangle T^2 - \left(Q(x) - \frac{1}{2} \langle x, e \rangle^2 \right) T - \det(x)$$

the *characteristic polynomial* of $x \in A$, and \det the *determinant* of A .

If we take the inner product of both sides of $\chi_x(x)$ with e we get

$$\begin{aligned} 0 &= \langle x^3, e \rangle - \langle x, e \rangle \langle x^2, e \rangle - \left(Q(x) - \frac{1}{2} \langle x, e \rangle^2 \right) \langle x, e \rangle - \det(x) \langle e, e \rangle. \\ &= \langle x^2, x \rangle - \langle x, e \rangle \langle x, x \rangle - Q(x) \langle x, e \rangle + \frac{1}{2} \langle x, e \rangle^3 - \det(x) (Q(e+e) - Q(e) - Q(e)) \\ &= \langle x^2, x \rangle - 2Q(x) \langle x, e \rangle - Q(x) \langle x, e \rangle + \frac{1}{2} \langle x, e \rangle^3 - \det(x) \left(6 - \frac{3}{2} - \frac{3}{2} \right) \\ &= \langle x^2, x \rangle - 3Q(x) \langle x, e \rangle + \frac{1}{2} \langle x, e \rangle^3 - 3 \det(x) \end{aligned}$$

and so we can write

$$\det(x) = \frac{1}{3} \langle x^2, x \rangle - Q(x) \langle x, e \rangle + \frac{1}{6} \langle x, e \rangle^3.$$

Notice

$$\det(e) = \frac{1}{3} \langle (e)^2, e \rangle - Q(e) \langle e, e \rangle + \frac{1}{6} \langle e, e \rangle^3 = 1 - \frac{3}{2}(3) + \frac{1}{6}(27) = 1.$$

So now we can compute $\det(x)$ for any $x = h(\xi_1, \xi_2, \xi_3; c_1, c_2, c_3) \in A = H(C; \gamma_1, \gamma_2, \gamma_3)$,

$$\begin{aligned} \det(x) &= \frac{1}{3} \text{tr}(x^3) - Q(x) \text{tr}(x) + \frac{1}{6} \text{tr}(x)^3 \\ &= \xi_1 \xi_2 \xi_3 - \gamma_3^{-1} \gamma_2 \xi_1 N(c_1) - \gamma_1^{-1} \gamma_3 \xi_2 N(c_2) - \gamma_2^{-1} \gamma_1 \xi_3 N(c_3) + N(c_1 c_2, \bar{c}_3) \end{aligned}$$

The determinant uniquely determines a trilinear form

$$\langle x, x, x \rangle = \det(x).$$

This is given by

$$6\langle x, y, z \rangle = \det(x+y+z) - \det(x+y) - \det(y+z) - \det(x+z) + \det(x) + \det(y) + \det(z), \quad (2.4)$$

which can easily be seen to be true by expanding the trilinear forms on the right side of the equation.

If we replace a with $x+y+z$ in the Hamilton-Cayley equation we get

$$(x+y+z)^3 - \langle x+y+z, e \rangle (x+y+z)^2 - \left(Q(x+y+z) - \frac{1}{2} \langle x+y+z, e \rangle^2 \right) (x+y+z) = \det(x+y+z). \quad (2.5)$$

Any commutative algebra A over a field k , where $\text{char}(k) \neq 2$ such that $x(x^2y) = x^2(xy)$ for all $x, y \in A$, is called a *Jordan algebra*.

Theorem, [22] 2.2.1.4. *All J-algebras are Jordan algebras.*

Proof. We need to show that for any J -algebra A that $x^2(xy) = x(x^2y)$ for all $x, y \in A$. It is enough to show that

$$\langle x^2(xy), z \rangle = \langle x(x^2y), z \rangle$$

for all $z \in A$. This can be rewritten using the commutativity of the multiplication and 2.2 to look like

$$\langle xy, x^2z \rangle = \langle xz, x^2y \rangle.$$

This is already true if y or z are multiples of the identity, by 2.2. So we need to only look at the case when $\langle y, e \rangle = \langle z, e \rangle = 0$. From 2.5 we can collect the terms on each side that are linear in one of x, y or z , and get a long equation that can be found [25] pg. 121. It contains the term $3\langle x, y, z \rangle$, which can be replaced using another lengthy equation until we have

$$\begin{aligned} \langle x^2y, xz \rangle &= -2\langle x(xy), xz \rangle + 2\langle x, e \rangle \langle xy, xz \rangle + \langle x, y \rangle \langle x, xz \rangle \\ &\quad + Q(x) \langle y, xz \rangle - \frac{1}{2} \langle x, e \rangle^2 \langle y, xz \rangle + \langle x, xy \rangle \langle x, z \rangle - \langle x, e \rangle \langle x, y \rangle \langle x, z \rangle. \end{aligned}$$

The right hand side is symmetric in y and z , so we have $\langle x^2y, xz \rangle = \langle x^2z, xy \rangle$. \square

Notice that $k[x]$ the algebra generated by x over the field k for any $x \in A$ is the homomorphic image of an associative algebra, $k[T]/\chi_x(T)$, and so A is power associative. From computations involving the determinant one can show

$$3\langle x, y, z \rangle = \langle xy, z \rangle - \frac{1}{2} \langle x, e \rangle \langle y, z \rangle - \frac{1}{2} \langle y, e \rangle \langle x, z \rangle - \frac{1}{2} \langle z, e \rangle \langle x, y \rangle + \frac{1}{2} \langle x, e \rangle \langle y, e \rangle \langle z, e \rangle. \quad (2.6)$$

2.2.2 Cross product

We will define a product using the symmetric trilinear form from the previous section, $\langle \cdot, \cdot, \cdot \rangle$. This *cross product* will be defined as follows

$$\langle x \times y, z \rangle = 3\langle x, y, z \rangle \text{ for } x, y, z \in A.$$

In this way the cross product is associated with the bilinear form and the determinant.

Lemma, [23] 2.2.2.1. *The following formulas hold in a J-algebra:*

- (a) $x \times y = xy - \frac{1}{2}\langle x, e \rangle y - \frac{1}{2}\langle y, e \rangle x - \frac{1}{2}\langle x, y \rangle e + \frac{1}{2}\langle x, e \rangle \langle y, e \rangle e$
- (b) $x(x \times x) = \det(x)e$
- (c) $(x_1 \times x_2) \times y = \frac{1}{2}(x_1 x_2)y - \frac{1}{2}x_1(x_2 y) - \frac{1}{2}x_2(x_1 y) + \frac{1}{4}\langle x_1, y \rangle x_2 + \frac{1}{4}\langle x_2, y \rangle x_1$
- (d) $(x \times x) \times (x \times x) = \det(x)x$
- (e)
- $4((x \times y) \times (z \times u) + (x \times z) \times (y \times u) + (x \times u) \times (y \times z)) =$
 $3(\langle x, y, z \rangle u + \langle x, y, u \rangle z + \langle x, z, u \rangle y + \langle y, z, u \rangle x)$
- (f) $4x \times (y \times (x \times x)) = \langle x, y \rangle x \times x + \det(x)y$
- (g) $\det(x \times x) = \det(x)^2$

Proof. (a) From equation 2.6 we have

$$\begin{aligned} 3\langle x, y, z \rangle &= \langle xy, z \rangle - \frac{1}{2}\langle x, e \rangle \langle y, z \rangle - \frac{1}{2}\langle y, e \rangle \langle x, z \rangle - \frac{1}{2}\langle z, e \rangle \langle x, y \rangle + \frac{1}{2}\langle x, e \rangle \langle y, e \rangle \langle z, e \rangle \\ &= \langle xy - \frac{1}{2}\langle x, e \rangle y - \frac{1}{2}\langle y, e \rangle x - \frac{1}{2}\langle x, y \rangle e + \frac{1}{2}\langle x, e \rangle \langle y, e \rangle e, z \rangle \\ &= \langle x \times y, z \rangle \end{aligned}$$

(b) Using (a)

$$\begin{aligned} x \times x &= xx - \frac{1}{2}\langle x, e \rangle x - \frac{1}{2}\langle x, e \rangle x - \frac{1}{2}\langle x, x \rangle e + \frac{1}{2}\langle x, e \rangle \langle x, e \rangle e \\ &= x^2 - \langle x, e \rangle x - Q(x) + \frac{1}{2}\langle x, e \rangle^2, \\ x(x \times x) &= x^3 - \langle x, e \rangle x^2 - \left(Q(x) - \frac{1}{2}\langle x, e \rangle^2\right)x \end{aligned}$$

(c) see [25]

(d) Using part (c) and substituting x for x_1 and x_2 and $x \times x$ for y we have

$$\begin{aligned}
& (x \times x) \times (x \times x) \\
&= \frac{1}{2}(xx)(x \times x) - \frac{1}{2}x(x(x \times x)) - \frac{1}{2}x(x(x \times x)) + \frac{1}{4}\langle x, (x \times x) \rangle x + \frac{1}{4}\langle x, (x \times x) \rangle x \\
&= \frac{1}{2}x(\det(x)) - \frac{1}{2}x(\det(x)) - \frac{1}{2}x(\det(x)) + \frac{1}{4}\langle e, x(x \times x) \rangle x + \frac{1}{4}\langle e, x(x \times x) \rangle x \\
&= -\frac{1}{2}\det(x)x + \frac{1}{2}\langle e, e \rangle \det(x)x \\
&= -\frac{1}{2}\det(x)x + \frac{1}{2}(3)\det(x)x \\
&= \det(x)x
\end{aligned}$$

(e),(f),(g) are more similar computations, see [25]. \square

2.2.3 Idempotents of a J -algebra

The idempotents of a J -algebra play an important role. If $u \in A$ and $u^2 = u$ then u is an *idempotent* element.

Lemma, [10] 2.2.3.1. *If $u \in A$ is an idempotent and u is not 0 or e , then $\det(u) = 0$, $Q(u) = \frac{1}{2}$ or 1, $\langle u, e \rangle = 2Q(u)$, $e - u$ is idempotent, $u(e - u) = 0$, $\langle u, e - u \rangle = 0$, and $Q(e - u) = \frac{3}{2} - Q(u)$.*

Proof. We have $\langle u, e \rangle = \text{tr}(ue) = \text{tr}(u) = \text{tr}(u^2) = 2Q(u)$.

This gives us, with idempotency and (1.1),

$$\begin{aligned}
\det(u) &= \frac{1}{3}\langle u^2, u \rangle - Q(u)\langle u, e \rangle + \frac{1}{6}\langle u, e \rangle^3 \\
&= \frac{1}{3}\langle u, e \rangle - Q(u)(2Q(u)) + \frac{1}{6}(2Q(u))^3 \\
&= \frac{2}{3}Q(u) - 2Q(u) + \frac{4}{3}Q(u) \\
&= 0.
\end{aligned}$$

So now we can say that since

$$\begin{aligned}
\det(u) = 0 &= u^3 - \langle u, e \rangle u^2 - (Q(u) - \frac{1}{2} \langle u, e \rangle^2)u \\
&= u - \langle u, e \rangle u - Q(u)u + \frac{1}{2} \langle u, e \rangle^2 u \\
&= \left(1 - \langle u, e \rangle - Q(u) + \frac{1}{2} \langle u, e \rangle^2\right) u \\
&= \left(1 - 2Q(u) - Q(u) + \frac{1}{2} (2Q(u))^2\right) u \\
&= (1 - 3Q(u) + 2Q(u)^2) u.
\end{aligned}$$

This gives us, when $u \neq 0$,

$$Q(u)^2 - \frac{3}{2}Q(u) + \frac{1}{2} = 0.$$

From this we know that $Q(u) = \frac{1}{2}, 1$, when $u \neq 0$. If we look at

$$(e - u)^2 = e^2 - eu - ue + u^2 = e - u - u + u = e - u,$$

we notice that $e - u$ is idempotent. And also that

$$u(e - u) = u - u^2 = 0.$$

And continuing throught the lemma,

$$\langle u, e - u \rangle = \langle u, e \rangle - \langle u, u \rangle = \langle u, e \rangle - \langle u, e \rangle = 0.$$

And finally

$$Q(e - u) = \langle e, -u \rangle + Q(u) + Q(e) = -2Q(u) + Q(u) + \frac{3}{2} = \frac{3}{2} - Q(u),$$

which gives us the fact that if A contains an idempotent not equal to 0 or e then there exists a $u \in A$ that is idempotent such that $Q(u) = \frac{1}{2}$. \square

If $Q(u) = \frac{1}{2}$ then we call u a *primitive idempotent*.

The adjective *primitive* makes sense here since, if $Q(u) = \frac{1}{2}$ we are unable to write u as the sum of two idempotents, since suppose we could, i.e that $u = u_1 + u_2$ and $Q(u) = \frac{1}{2}$. We can see from the following, when $u_1, u_2 \neq 0$,

$$1 = 2Q(u) = \langle u, e \rangle = \langle u_1, e \rangle + \langle u_2, e \rangle = 2Q(u_1) + 2Q(u_2) \geq 2$$

Lemma, [25] 2.2.3.2. Let A be a J -algebra, then $x \in A$ has an inverse $x^{-1} \in A$, such that $xx^{-1} = e$ and $x(x^{-1}y) = x^{-1}(xy)$ for $y \in A$ if and only if $\det(x) \neq 0$, and x^{-1} is unique.

Proof. When $\det(x) \neq 0$ then $x^{-1} = \det(x)^{-1} (x^2 - \langle x, e \rangle x - (Q(x) - \frac{1}{2}\langle x, e \rangle^2))$ is such that $xx^{-1} = e$ by the Hamilton-Cayley equation.

Also

$$\begin{aligned} x(x^{-1}y) &= x \left(\det(x)^{-1} \left(x^2 - \langle x, e \rangle x - \left(Q(x) - \frac{1}{2}\langle x, e \rangle^2 \right) \right) y \right) \\ &= \det(x)^{-1} \left(x(x^2y) - \langle x, e \rangle x(xy) - Q(x)(xy) + \frac{1}{2}\langle x, e \rangle(xy) \right) \\ &= \det(x)^{-1} \left(x^2(xy) - \langle x, e \rangle x(xy) - Q(x)(xy) + \frac{1}{2}\langle x, e \rangle(xy) \right) \\ &= \det(x)^{-1} \left(x^2 - \langle x, e \rangle x - \left(Q(x) - \frac{1}{2}\langle x, e \rangle^2 \right) \right) (xy) \\ &= x^{-1}(xy). \end{aligned}$$

To show uniqueness we will assume for $x \in A$, such that $\det(x) \neq 0$ there exists $z \in A$ such that $xz = e$, then

$$\det(x)z = x^3z - \langle x, e \rangle x^2z - \left(Q(x) - \frac{1}{2}\langle x, e \rangle^2 \right) xz = x^2 - \langle x, e \rangle x - \left(Q(x) - \frac{1}{2}\langle x, e \rangle^2 \right),$$

if we solve for z we get exactly x^{-1} .

Now let us assume there is an $x \in A$ such that $x \neq 0$ and $\det(x) = 0$, then

$$x^2 - \langle x, e \rangle x - \left(Q(x) - \frac{1}{2}\langle x, e \rangle^2 \right) = 0.$$

If we take the inner product of both sides with e we have,

$$\begin{aligned} 0 &= \left\langle x^2 - \langle x, e \rangle x - \left(Q(x) - \frac{1}{2}\langle x, e \rangle^2 \right), e \right\rangle \\ &= \langle x^2, e \rangle - \langle x, e \rangle \langle x, e \rangle - Q(x) \langle e, e \rangle + \frac{1}{2}\langle x, e \rangle^2 \langle e, e \rangle \\ &= 2Q(x) - \langle x, e \rangle^2 - 3Q(x) + \frac{3}{2}\langle x, e \rangle^2 \\ &= -Q(x) + \frac{1}{2}\langle x, e \rangle^2, \end{aligned}$$

which gives us $Q(x) = \frac{1}{2}\langle x, e \rangle^2 \Rightarrow x^2 = \langle x, e \rangle x$. This implies that

$$x = x(x^{-1}x) = x^{-1}(x^2) = x^{-1}\langle x, e \rangle x = \langle x, e \rangle,$$

which implies x is a nonzero scalar multiple of e , but the $\det(e) = 1 \neq 0$ gives us a contradiction. \square

The element x^{-1} will be called the *J-inverse* or sometimes just inverse, but it is not an inverse in the sense of nonassociative algebras since $x(x^{-1}y) = y$ need not be true when $y \notin k[x]$.

2.2.4 Reduced *J*-algebras and their decomposition

We will call a *J*-algebra *reduced* when it contains an idempotent other than 0 or e , by 2.2.3.1 it also contains a primitive idempotent.

If we fix u a primitive idempotent, then from 2.2.3.1 we also see that the inner product (and thus Q) is nondegenerate on the vector space $ke \oplus ku$ since $\langle e, e \rangle = 3$, and $\langle u, e \rangle = \langle u, u \rangle = 1$. Let us define

$$E = (ke \oplus ku)^\perp = \{x \in A \mid \langle x, e \rangle = \langle x, u \rangle = 0\}.$$

Since the quadratic form Q is nondegenerate on E^\perp it must also be nondegenerate on E .

Let us consider $x \in E$ then

$$\langle ux, u \rangle = \langle ux, e \rangle = \langle x, u \rangle = 0,$$

so $ux \in E$.

We can now define the linear transformation

$$t : E \rightarrow E, x \mapsto ux.$$

Lemma, [25] 2.2.4.1. *The linear transformation t is symmetric with respect to $\langle \cdot, \cdot \rangle$ and $t^2 = \frac{1}{2}t$.*

Proof. The symmetry follows from 2.2

$$\langle t(x), y \rangle = \langle ux, y \rangle = \langle x, uy \rangle = \langle x, t(y) \rangle.$$

From 2.2.3.1 and equations (5.13) and (5.14) in [25] we get

$$2u(ux) + ux = 2ux \Rightarrow 2u(ux) = ux \Rightarrow u(ux) = \frac{1}{2}ux \Rightarrow t^2(x) = \frac{1}{2}t(x), \text{ for all } x \in E.$$

This tells us that

$$t^2 = \frac{1}{2}t \Rightarrow t^2 - \frac{1}{2}t = 0 \Rightarrow t(t - \frac{1}{2}) = 0,$$

and that the possible eigenvalues for t are 1 and $\frac{1}{2}$. \square

Corollary 2.2.4.2. *If we let*

$$E_i = \{x \in E \mid t(x) = \frac{1}{2}ix\},$$

then $E_0 \perp E_1$, and $E_0 \oplus E_1 = E$.

Proof. From the symmetry of t with respect to the bilinear form if we let $x \in E_0$ and $y \in E_1$ we get

$$\langle x, y \rangle = \langle x, 2t(y) \rangle = 2\langle x, t(y) \rangle = 2\langle t(x), y \rangle = 2\langle 0, y \rangle = 0.$$

\square

Lemma, [25] 2.2.4.3. *The following are true:*

(a) *For $x, y \in E_0$, $xy = \frac{1}{2}\langle x, y \rangle(e - u)$.*

(b) *For $x, y \in E_1$, $xy = \frac{1}{4}\langle x, y \rangle(e + u) + x \circ y$.*

(c) *If $x \in E_0$ and $y \in E_1$, then $xy \in E_1$.*

Proof. After substituting u for z in equation (5.12) in [25] and simplifying we obtain

$$x(yu) + y(xu) + u(xy) = xy + \frac{1}{2}\langle x, y \rangle u + 3\langle x, y, u \rangle.$$

By applying part (a) of 2.2.4.3 we have

$$x(uy) + y(ux) + u(xy) = xy + \frac{1}{2}\langle x, y \rangle u + \langle ux, y \rangle - \frac{1}{2}\langle x, y \rangle. \quad (2.7)$$

If we let $x, y \in E_0$ then

$$u(xy - \frac{1}{2}\langle x, y \rangle) = xy - \frac{1}{2}\langle x, y \rangle.$$

The only way $z \in A$ is such that $uz = z$ are multiples of u , so

$$xy - \frac{1}{2}\langle x, y \rangle = \kappa u,$$

for some $\kappa \in k$. Then if we look at

$$\begin{aligned}\langle \kappa u, u \rangle &= \langle xy - \frac{1}{2}\langle x, y \rangle, u \rangle \\ \kappa \langle u, e \rangle &= \langle xy, u \rangle - \frac{1}{2}\langle x, y \rangle \langle e, u \rangle \\ \kappa &= -\frac{1}{2}\langle x, y \rangle \\ \Rightarrow xy &= \frac{1}{2}\langle x, y \rangle(e - u).\end{aligned}$$

Now consider $x, y \in E_1$, and define $x \circ y = xy - \frac{1}{4}\langle x, y \rangle(e + u)$, and

$$\begin{aligned}\langle x \circ y, u \rangle &= \langle xy, u \rangle - \frac{1}{4}\langle x, y \rangle \langle e, u \rangle - \frac{1}{4}\langle x, y \rangle \langle u, u \rangle \\ &= \frac{1}{2}\langle x, y \rangle - \frac{1}{4}\langle x, y \rangle - \frac{1}{4}\langle x, y \rangle \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\langle x \circ y, e \rangle &= \langle xy, e \rangle - \frac{1}{4}\langle x, y \rangle \langle e, e \rangle - \frac{1}{4}\langle x, y \rangle \langle u, e \rangle \\ &= \langle x, y \rangle - \frac{3}{4}\langle x, y \rangle - \frac{1}{4}\langle x, y \rangle \\ &= 0,\end{aligned}$$

we have $x \circ y \in E$. From 2.7 we can see that $u(xy) = \frac{1}{2}\langle x, y \rangle u$, and so

$$\begin{aligned}u(x \circ y) &= u\left(xy - \frac{1}{4}\langle x, y \rangle(e + u)\right) \\ &= \frac{1}{2}\langle x, y \rangle u - \frac{1}{4}\langle x, y \rangle u(e + u) \\ &= \frac{1}{2}\langle x, y \rangle u - \frac{1}{4}\langle x, y \rangle(2u) = 0,\end{aligned}$$

and so $x \circ y \in E_0$.

Now we will look at when $x \in E_0$ and $y \in E_1$, then

$$\langle xy, u \rangle = \langle ux, y \rangle = 0,$$

and

$$\langle xy, e \rangle = \langle x, y \rangle = 0,$$

so $xy \in E$. Using 2.7 we see

$$u(xy) + \frac{1}{2}xy = xy \Rightarrow u(xy) = \frac{1}{2}xy,$$

and we have $xy \in E_1$. □

Lemma 2.2.4.4. *The following are true in a J-algebra:*

- (a) $x(xy) = \frac{1}{4}Q(x)y$,
- (b) $x_1(x_2y) + x_2(x_1y) = \frac{1}{4}\langle x_1, x_2 \rangle y$,
- (c) $y \circ xy = \frac{1}{4}Q(y)x$,
- (d) $(y \circ y)y = \frac{1}{4}Q(y)y$,
- (e) $(y_1 \circ y_2)y_3 + (y_2 \circ y_3)y_1 + (y_3 \circ y_1)y_2 = \frac{1}{8}\langle y_1, y_2 \rangle y_2 + \frac{1}{8}\langle y_2, y_3 \rangle y_1 + \frac{1}{8}\langle y_3, y_1 \rangle y_2$,
- (f) $Q(xy) = \frac{1}{4}Q(x)Q(y)$,
- (g) $Q(y \circ y) = \frac{1}{4}Q(y)^2$,

for $x, x_i \in E_0$ and $y, y_i \in E_1$ for $i = 1, 2$.

Proof. Using (5.13) in [25] and the fact that $x, y \in E$ we get

$$2x(xy) + x^2y = Q(x)y + 3\langle x, x, y \rangle.$$

From the initial definition of the trilinear form we have

$$3\langle x, x, y \rangle = \langle x, x \times y \rangle = \left\langle x, xy - \frac{1}{2}(\langle x, e \rangle y + \langle y, e \rangle x + \langle x, y \rangle + \langle x, e \rangle \langle y, e \rangle) \right\rangle = \langle x, xy \rangle.$$

By 2.2.4.4 we have

$$x^2 = \frac{1}{2}\langle x, x \rangle (e - u) = Q(x)(e - u),$$

since $\langle x, x \rangle = \text{tr}(x^2) = 2Q(x)$. And so continuing with the earlier equation

$$\begin{aligned} 2x(xy) + Q(x)(e - u)y &= Q(x)y \\ 2x(xy) + Q(x)y - Q(x)(uy) &= Q(x)y \\ 2x(xy) + Q(x)y - \frac{1}{2}Q(x)y &= Q(x)y \\ 2x(xy) &= \frac{1}{2}Q(x)y \\ x(xy) &= \frac{1}{4}Q(x)y, \end{aligned}$$

and so we have part (a) and part (b) follows by linearization.

First notice

$$y \circ xy = y(xy) - \frac{1}{4}\langle y, xy \rangle(e + u).$$

And then using (5.13) and (5.16) from [25] and 2.2.4.4 we have

$$\begin{aligned} 2y(yx) + y^2x &= Q(y)x + 3\langle y, y, x \rangle \\ xy^2 + 2y(xy) &= Q(y)x + \langle x, y^2 \rangle \end{aligned}$$

Now we can notice that

$$\begin{aligned} y^2 &= \frac{1}{4}\langle y, y, \rangle(e + u) + y \circ y \\ &= \frac{1}{2}Q(y)(e + u) + y \circ y. \end{aligned}$$

Plugging this into our equation above we get

$$x \left(\frac{1}{2}Q(y)(e + u) + y \circ y \right) + 2y(xy) = Q(y)x + \langle x, y^2 \rangle.$$

If we look at

$$\begin{aligned} x(y \circ y) &= \frac{1}{2}\langle x, y \circ y \rangle(e - u) \\ &= \frac{1}{2} \left\langle x, y^2 - \frac{1}{2}Q(y)(e + u) \right\rangle(e - u) \\ &= \frac{1}{2}\langle x, y^2 \rangle(e - u), \end{aligned}$$

and

$$x \left(\frac{1}{2}Q(y)(e + u) \right) = \frac{1}{2}Q(y)x.$$

And now we can see that

$$\begin{aligned} \frac{1}{2}Q(y)x + \frac{1}{2}\langle x, y^2 \rangle(e - u) + 2y(xy) &= Q(y)x + \langle x, y^2 \rangle \\ \frac{1}{2}Q(y)x - \frac{1}{2}\langle x, y^2 \rangle u + 2y(xy) &= Q(y)x + \frac{1}{2}\langle x, y^2 \rangle \\ 2y(xy) - \frac{1}{2}\langle x, y^2 \rangle u &= \frac{1}{2}Q(y)x + \frac{1}{2}\langle x, y^2 \rangle \\ 2y(xy) - \frac{1}{2}\langle y, xy \rangle u &= \frac{1}{2}Q(y)x + \frac{1}{2}\langle y, xy \rangle \\ y(xy) - \frac{1}{4}\langle y, xy \rangle(e + u) &= \frac{1}{4}Q(y)x. \end{aligned}$$

From the Hamilton-Cayley equation and the properties for $y \in E$ we have

$$y^3 - Q(y)y - \det(y) = 0.$$

And if we compute the determinant for $y \in E_1$ we have

$$3\det(y) = \langle y, y \times y \rangle = \langle y, y^2 - \frac{1}{2}Q(y) \rangle = \langle y, y^2 \rangle = \langle y, y \circ y - \frac{1}{2}Q(y)(e + u) \rangle = 0,$$

so we get

$$y^3 = Q(y)y.$$

When we look at

$$\begin{aligned} (y \circ y)y &= (y^2 - \frac{1}{2}Q(y)(e + u))y \\ &= y^3 - \frac{1}{2}Q(y)y - \frac{1}{2}Q(y)(uy) \\ &= Q(y)y - \frac{1}{2}Q(y)y - \frac{1}{4}Q(y)y \\ &= \frac{1}{4}Q(y)y. \end{aligned}$$

This give us (d) and by linearizing we obtain (e).

From 2.2.1.1 we have

$$\begin{aligned} 2\langle xy, xy \rangle + 2\langle xx, yy \rangle + 2\langle xy, xy \rangle &= \langle x, y \rangle \langle x, y \rangle + \langle x, x \rangle \langle y, y \rangle + \langle x, y \rangle \langle x, y \rangle \\ 8Q(xy) + 2\langle x^2, y^2 \rangle &= 4Q(x)Q(y) \\ 4Q(xy) + \langle x^2, y^2 \rangle &= 2Q(x)Q(y). \end{aligned}$$

Seeing that

$$\begin{aligned} \langle x^2, y^2 \rangle &= \langle Q(x)(e - u), \frac{1}{2}Q(y)(e + u) + y \circ y \rangle \\ &= \frac{1}{2}Q(x)Q(y)\langle e - u, e + u \rangle + 0 \\ &= Q(x)Q(y), \end{aligned}$$

since $\langle e - u, e + u \rangle = 2$, substituting this we have (f).

If we use (d) and (f)

$$\frac{1}{4}Q(y \circ y)Q(y) = Q((y \circ y)y) = Q\left(\frac{1}{4}Q(y)y\right) = \frac{1}{16}Q(y)^3.$$

As long as $Q(y) \neq 0$, $Q(y \circ y) = \frac{1}{4}Q(y)^2$, and by Zariski continuity this holds for all $y \in E_1$. \square

Corollary 2.2.4.5. *We will denote the Clifford algebra of the restriction of Q to E_0 by $Cl(Q; E_0)$. The map $\phi : E_0 \rightarrow \text{End}(E_1)$ defined by*

$$\phi(x)(y) = 2xy, \text{ for } x \in E_0, y \in E_1$$

can be extended to a representation of $Cl(Q; E_0)$ in E_1 .

Proof. From the previous Lemma

$$\phi(x)^2(y) = \phi(x)(2xy) = 4x(xy) = Q(x)y,$$

with $x \in E_0$ and $y \in E_1$. So for the extension of ϕ to $T(E_0)$ into $\text{End}(E_1)$ we have the Clifford algebra relation for $Cl(Q; E_0)$. \square

Proposition 2.2.4.6. *Let A be a reduced J-algebra and u a primitive idempotent in A , and let E, E_0, E_1 be as above with respect to u , then*

- (a) $E_0 = \{0\}$ if and only if A is 2-dimensional, and then $A = ku \oplus k(e - u)$, an orthogonal direct sum
- (b) if $E_1 = \{0\}$ then $A = ku \oplus k(e - u) \oplus E_0$, and for $\lambda, \lambda', \mu, \mu' \in k$ and $x, x' \in E_0$ the product and norm are given by

$$(\lambda u + \mu(e - u) + x)(\lambda' u + \mu'(e - u) + x') = \lambda\lambda' u + (\mu\mu' + \frac{1}{2}q(x, x'))(e - u) + \mu x' + \mu' x,$$

and

$$Q(\lambda u + \mu(e - u) + x) = \frac{1}{2}\lambda^2 + \mu^2 + q(x),$$

where q is a nondegenerate quadratic form on E_0 with associative bilinear form $q(,)$. Conversely for any vector space E_0 with a nondegenerate quadratic form q , the above formulas define a J-algebra.

Proof. If $E_0 = \{0\}$ then for $y \in E_1$, $y \circ y = 0$ and so $Q(y) = 0$ by 2.2.4.4, thus we have that $E_1 = \{0\}$, and $A = ku \oplus k(e - u)$.

If $E_1 = \{0\}$ then $A = ku \oplus k(e - u) \oplus E_0$, which are orthogonal vector spaces. The prod-

uct is

$$\begin{aligned}
(\lambda u + \mu(e - u) + x)(\lambda' u + \mu'(e - u) + x') &= \lambda\lambda' + \mu\mu'(e - u) + \mu(e - u)x' + \mu'x(e - u) + xx' \\
&= \lambda\lambda' + \mu\mu'(e - u) + \mu x' + \mu'x + \frac{1}{2}\langle x, x' \rangle(e - u) \\
&= \lambda\lambda' + \left(\mu\mu' + \frac{1}{2}q(x, x') \right) (e - u) + \mu x' + \mu'x,
\end{aligned}$$

where q is the restriction of Q to E_0 . Also, if we have a vector space E_0 with a nondegenerate quadratic form we can have a J -algebra by direct summing $ku \oplus k(e - u)$, and defining the product and quadratic form as above. The dimension of the new algebra will be $\dim(E_0) + 2$.

Now checking 2.1-2.3:

Notice that for 2.1 there is the assumption that $\langle \lambda u + \mu(e - u) + x, e \rangle = 0$, and so

$$\lambda\langle u, e \rangle + \mu\langle e, e \rangle - \mu\langle u, e \rangle = \lambda + 3\mu - \mu = \lambda + 2\mu = 0.$$

So

$$\begin{aligned}
Q((\lambda u + \mu(e - u) + x)^2) &= \frac{1}{2}\lambda^4 + (\mu^2 + \frac{1}{2}q(x, x))^2 + q(2\mu x) \\
&= \frac{1}{2}(-2\mu)^4 + (\mu^2 + q(x))^2 + 4\mu^2q(x) \\
&= 8\mu^4 + \mu^4 + 2q(x)\mu^2 + q(x)^2 + 4q(x)\mu^2 \\
&= 9\mu^4 + 6q(x)\mu^2 + q(x)^2,
\end{aligned}$$

and

$$\begin{aligned}
Q(\lambda u + \mu(e - u) + x)^2 &= \left(\frac{1}{2}\lambda^2 + \mu^2 + q(x) \right)^2 \\
&= \frac{1}{4}\lambda^4 + \frac{1}{2}\lambda^2\mu^2 + \frac{1}{2}\lambda^2q(x) \\
&\quad + \frac{1}{2}\lambda^2\mu^2 + \mu^4 + \mu^2q(x) + \frac{1}{2}\lambda^2q(x) + \mu^2q(x) + q(x)^2 \\
&= \frac{1}{4}\lambda^4 + \lambda^2\mu^2 + \lambda^2q(x) + 2\mu^2q(x) + \mu^4 + q(x)^2 \\
&= \frac{1}{4}(-2\mu)^4 + (-2\mu)^2\mu^2 + (-2\mu)^2q(x) + 2\mu^2q(x) + \mu^4 + q(x)^2 \\
&= 4\mu^4 + 4\mu^4 + 4q(x)\mu^2 + 2q(x)\mu^2 + \mu^4 + q(x)^2 \\
&= 9\mu^4 + 6q(x)\mu^2 + q(x)^2.
\end{aligned}$$

2.2 can be seen to be true by first noticing,

$$\underbrace{(\lambda_1 u + \mu_1(e - u) + x_1)}_X \underbrace{(\lambda_2 u + \mu_2(e - u) + x_2)}_Y = (\lambda_1 \lambda_2)u + \left(\mu_1 \mu_2 + \frac{1}{2}q(x_1, x_2) \right) (e - u) + \mu_1 x_2 + \mu_2 x_2,$$

and then

$$\begin{aligned} \langle XY, Z \rangle &= \left\langle (\lambda_1 \lambda_2)u + \left(\mu_1 \mu_2 + \frac{1}{2}q(x_1, x_2) \right) (e - u) + \mu_1 x_2 + \mu_2 x_2, \underbrace{\lambda_3 u + \mu_3(e - u) + x_3}_Z \right\rangle \\ &= (\lambda_1 \lambda_2) \lambda_3 \langle u, u \rangle + \mu_3 (\mu_1 \mu_2 + \frac{1}{2}q(x, y)) \langle e - u, e - u \rangle + \mu_1 \langle y, z \rangle + \mu_2 \langle x, z \rangle \\ &= (\lambda_1 \lambda_2) \lambda_3 + \mu_3 (\mu_1 \mu_2 + \frac{1}{2}q(x, y))(2) + \mu_1 q(y, z) + \mu_2 q(x, z) \\ &= (\lambda_2 \lambda_3) \lambda_1 + 2\mu_3 \mu_1 \mu_2 + \mu_3 q(x, y) + \mu_1 q(y, z) + \mu_2 q(x, z) \\ &= (\lambda_2 \lambda_3) \lambda_1 + \mu_1 (\mu_2 \mu_3 + \frac{1}{2}q(y, z))(2) + \mu_2 q(z, x) + \mu_3 q(y, x) \\ &= \langle X, YZ \rangle \end{aligned}$$

For 2.3 we consider

$$\lambda u + \mu(e - u) + x = e \Rightarrow \lambda - \mu = 0, \text{ and } \mu = 1, x = 0 \Rightarrow \lambda = 1.$$

so

$$Q(\lambda u + \mu(e - u) + x) = \frac{1}{2}(1)^2 + (1)^2 + q(0) = \frac{3}{2}.$$

□

Notice $k(e - u) \oplus E_0$ is called the Jordan algebra of q and has as an identity element $e - u$, and is a J -algebra if we take the quadratic form to be $\frac{3}{2}Q$.

Lemma, [25] 2.2.4.7. *If $E_1 \neq \{0\}$ then there exists $x_1 \in E_0$ such that $Q(x_1) = \frac{1}{4}$.*

Proof. Notice first that if $E_1 \neq \{0\} \Rightarrow E_0 \neq \{0\}$.

The restriction of Q to E_1 is nondegenerate so there exists $y \in E_1$ such that $Q(y) \neq 0$. If we take $x_1 = Q(y)^{-1}y \circ y$, then

$$Q(x_1) = Q(Q(y)^{-1}y \circ y) = Q(y)^{-2}Q(y \circ y),$$

and by 2.2.4.4

$$Q(y \circ y) = \frac{1}{4}Q(y)^2$$

so

$$Q(x_1) = \frac{1}{4}Q(y)^{-2}Q(y)^2 = \frac{1}{4}.$$

Notice also that

$$x_1y = Q(y)^{-1}(y \circ y)y = Q(y)^{-1}\left(\frac{1}{4}Q(y)\right)y = \frac{1}{4}y.$$

□

We now look at when $e - u$ is the sum of two orthogonal primitive idempotents, when u is a primitive idempotent.

Proposition, [25] 2.2.4.8. *Let A be a reduced J -algebra and u a primitive idempotent of A , then $e - u$ is a sum of two orthogonal primitive idempotents unless*

$$A = ku \oplus k(e - u) \oplus E_0$$

with Q not representing 1 on E_0 . (The second condition is independent of u being chosen such that E_1 is zero.)

Proof. If $E_1 \neq \{0\}$ then there is an element $x_1 \in E_0$ with $Q(x_1) = \frac{1}{4}$ by the previous lemma. Then

$$\left(\frac{1}{2}(e - u) \pm x_1\right)^2 = \left(\frac{1}{4} + \frac{1}{2}q(x_1, x_1)\right)(e - u) \pm x_1 = \left(\frac{1}{4} + \frac{1}{2}2\left(\frac{1}{4}\right)\right)(e - u) \pm x_1 = \frac{1}{2}(e - u) \pm x_1.$$

So $\frac{1}{2}(e - u) + x_1$ and $\frac{1}{2}(e - u) - x_1$ are idempotent elements whose sum is $e - u$, and

$$\begin{aligned} \left\langle \frac{1}{2}(e - u) + x_1, \frac{1}{2}(e - u) - x_1 \right\rangle &= \frac{1}{4}\langle e - u, e - u \rangle - \frac{1}{2}\langle e - u, x_1 \rangle + \frac{1}{2}\langle x_1, e - u \rangle - \langle x_1, x_1 \rangle \\ &= \frac{1}{4}(2) + 0 + 0 - 2\left(\frac{1}{4}\right) \\ &= 0. \end{aligned}$$

If $E_1 = \{0\}$, then $A = ku \oplus k(e - u) \oplus E_0$. We want to consider two elements of A ,

$$a = \lambda u + \mu(e - u) + a_1, \text{ and } b = (e - u) - a.$$

So, we have easily that $a + b = e - u$, and $b = -\lambda u + (1 - \mu)(e - u) - a_1$. To see what will make

these idempotent we look at

$$\begin{aligned} (\lambda u + \mu(e - u) + a_1)^2 &= \lambda^2 u + \left(\mu^2 + \frac{1}{2}q(a_1, a_1) \right) (e - u) + 2\mu a_1 \\ &= \lambda^2 u + (\mu^2 + q(a_1))(e - u) + 2\mu a_1. \end{aligned}$$

So for a to be idempotent we need

$$\lambda = \lambda^2 \Rightarrow \lambda = 0, 1;$$

and

$$2\mu = 1 \Rightarrow \mu = \frac{1}{2};$$

and

$$\mu^2 + q(a_1) = \mu \Rightarrow \frac{1}{4} + q(a_1) = \frac{1}{2} \Rightarrow q(a_1) = \frac{1}{4}.$$

And for b we have

$$\begin{aligned} (-\lambda u + (1 - \mu)(e - u) - a_1)^2 &= \lambda^2 u + \left((1 - \mu)^2 + \frac{1}{2}q(a_1, a_1) \right) (e - u) - 2\mu a_1 \\ &= \lambda^2 u + (1 - 2\mu + \mu^2 + q(a_1))(e - u) - 2\mu a_1. \end{aligned}$$

From this we conclude that

$$\lambda^2 = -\lambda \Rightarrow \lambda = -1, 0;$$

and

$$-2\mu = -1 \Rightarrow \mu = \frac{1}{2};$$

and

$$1 - 2\mu - \mu^2 + q(a_1) = \mu \Rightarrow 1 - 1 + \frac{1}{4} + q(a_1) = \frac{1}{2} \Rightarrow q(a_1) = \frac{1}{4}.$$

So we have that a and b are both idempotent $\Leftrightarrow \lambda = 0, \mu = \frac{1}{2}$, and $Q(a_1) = \frac{1}{4}$. \square

We can now compute $Q(a)$, $Q(b)$, and ab ,

$$Q(a) = Q\left(\frac{1}{2}(e - u) + a_1\right) = \frac{1}{2}(0)^2 + \left(\frac{1}{2}\right)^2 + q(a_1) = \left(\frac{1}{2}\right)^2 + \frac{1}{4} = \frac{1}{2},$$

$$Q(b) = Q\left(\frac{1}{2}(e - u) - a_1\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

and

$$ab = \left(\frac{1}{2}(e - u) + a_1 \right) \left(\frac{1}{2}(e - u) - a_1 \right) = \left(\frac{1}{4} + \frac{1}{2}q(a_1, -a_1) \right) (e - u) + \frac{1}{2}a_1 - \frac{1}{2}a_1 = 0.$$

Let us consider $a \in A$ and its minimal polynomial, m_a . We recall that every element in A satisfies the Cayley-Hamilton equation, χ_a , and so m_a divides χ_a . We also want to consider the algebra generated by $a \in A$ over the field k , $k[a]$, and we will note that $k[a] \cong k[T]/m_a(T)$.

Proposition, [25] 2.2.4.9. *For a common splitting field m_a and χ_a have the same roots, so $m_a = \chi_a$ if χ_a has 3 distinct roots. For $a \notin ke$, the restriction of Q of A to $k[a]$ is nondegenerate if and only if not all roots of χ_a are the same. If χ_a has a root in k , then $k[a]$ contains a primitive idempotent if and only if not all roots are of χ_a are the same.*

Proof. Assume k is a splitting field of χ_a . If $\dim(k[a]) = 1$, then $a = \lambda e$, then $m_a(T) = T - \lambda$. Also,

$$\det(a) = \det(\lambda e) = \frac{\lambda^3}{3}(3) - \frac{3}{2}\lambda^3(3)^2 + \frac{\lambda^2}{6}(27) = \lambda^3,$$

so

$$\begin{aligned} \chi_x(T) &= T^3 - \langle \lambda e, e \rangle T^2 - \left(Q(\lambda e) - \frac{1}{2}\langle \lambda e, e \rangle^2 \right) T - \det(\lambda e) \\ &= T^3 - \lambda(3)T^2 - \left(\lambda^2 \frac{3}{2} - \frac{1}{2}\lambda^2(3)^2 \right) T - \lambda^3 \\ &= T^3 - 3\lambda T^2 + 3\lambda^2 T - \lambda^3 \\ &= (T - \lambda)^3. \end{aligned}$$

Now let us look at the case when $\dim(k[a]) = 2$. If $m_a(T) = (T - \lambda)^2$, we will look at $x = a - \lambda e$.

Notice that

$$x^2 = (a - \lambda e)^2 = 0,$$

and that $x \neq 0 \Rightarrow m_x(T) = T^2$ this implies that $\chi_x(T) = T^3 - \langle x, e \rangle T^2$ since otherwise m_x would not divide χ_x . From this we can conclude that

$$Q(x) - \frac{1}{2}\langle x, e \rangle^2 = 0 \Rightarrow Q(x) = \frac{1}{2}\langle x, e \rangle^2,$$

but also

$$Q(x) = \frac{1}{2}2Q(x) = \frac{1}{2}\langle x, x \rangle = \frac{1}{2}\langle x^2, e \rangle = 0,$$

and so

$$\langle x, e \rangle = 0.$$

From this we have

$$\langle a, e \rangle = \langle x + \lambda e, e \rangle = \langle x, e \rangle + \lambda \langle e, e \rangle = 3\lambda.$$

We can now also compute

$$Q(a) = Q(x + \lambda e) = \langle x, \lambda e \rangle + Q(x) + Q(\lambda e) = \frac{3}{2}\lambda^2.$$

and

$$\begin{aligned} \det(a) &= \frac{1}{3}\langle a^2, a \rangle - Q(x)\langle a, e \rangle + \frac{1}{6}\langle a, e \rangle^3 \\ &= \frac{1}{3}\langle a^2, a \rangle - \frac{3}{2}\lambda^2(3)\lambda + \frac{1}{6}(3\lambda)^3 \\ &= \frac{1}{3}\langle a^2, a \rangle - \frac{9}{2}\lambda^3 + \frac{9}{2}\lambda^3 \\ &= \frac{1}{3}\langle a^3, e \rangle \\ &= \frac{1}{3}\langle x^3 + 3\lambda x^2 + 3\lambda^2 x + \lambda^3, e \rangle \\ &= \lambda^3 \frac{1}{3}\langle e, e \rangle \\ &= \lambda^3. \end{aligned}$$

So

$$\begin{aligned} \chi_a(T) &= T^3 - \langle a, e \rangle T^2 - \left(Q(a) - \frac{1}{2}\langle a, e \rangle^2\right)T - \det(a) \\ &= T^3 - 3\lambda T^2 - \left(\frac{3}{2}\lambda^2 - \frac{9}{2}\lambda^2\right)T - \lambda^3 \\ &= T^3 - 3\lambda T^2 + 3\lambda^2 T - \lambda^3 \\ &= (T - \lambda)^3. \end{aligned}$$

Notice now that if $\lambda = 0$, since $a^2 = 0$ the only idempotent is e in $k[a]$ and that $Q(a) = 0$, so Q is degenerate on $k[a]$.

Now consider $m_a(T) = (T - \lambda)(T - \mu)$ with $\lambda \neq \mu$, then we have

$$k[T]/m_a(T) \cong k \oplus k \cong k[a].$$

So since $k[a]$ is two dimensional we have that it is the direct sum of idempotents u and $e - u$, and we can assume that u is primitive. From this, since $Q(u) = \frac{1}{2}$ and $Q(e - u) = \frac{3}{2} - Q(u)$ we have that Q is nondegenerate on $k[a]$. If we let $a = \alpha u + \beta(e - u)$ then it is straightforward to check that $m_a(T) = (T - \alpha)(T - \beta)$. From this we can conclude that either $\alpha = \lambda$ and $\beta = \mu$ or $\alpha = \mu$

and $\beta = \lambda$. If you compute the characteristic polynomial for a we have $\chi_a(T) = (T - \alpha)(T - \beta)^2$, so we have either $\chi_a(T) = (T - \mu)(T - \lambda)^2$ or $\chi_a(T) = (T - \mu)(T - \lambda)^2$.

The final possibility is that $\dim(k[a]) = 3$. Then we have $m_a(T) = \chi_a(T)$ and if $\chi_a(T)$ has 3 distinct roots $k[a] \cong k \oplus k \oplus k$. From 2.2.4.8 we have that $k[a]$ is spanned by three idempotents u_i for $i = 1, 2, 3$, such that $u_i u_j = 0$ when $i \neq j$. From this we can see that

$$\langle u_i, u_j \rangle = \langle u_i u_j, e \rangle = 0,$$

when $i \neq j$, so each pair are orthogonal. We also see that $\langle u_i, e \rangle = 2Q(u_i)$ so Q is nondegenerate.

If we are in the case where $\chi_a(T) = (T - \lambda)(T - \mu)^2$ with $\lambda \neq \mu$, then we have $k[a] \cong k \oplus k[x]$ with $x \neq 0$ and $x^2 = 0$. We saw previously that the restriction of Q to $k[x]$ is degenerate. Since k and $k[x]$ are ideals generated by orthogonal idempotents and so Q is degenerate on $k[a]$.

It follows from the case $m_a(T) = (T - \lambda)^2$ that Q restricted to $k[a]$ with $\chi_a(T) = (T - \lambda)^3$ is degenerate (Q restricted to $kx \oplus kx^2$ is degenerate and $kx \oplus kx^2$ is orthogonal to ke , in fact the only idempotent in $k[a]$ is e).

Further, if $\chi_a(T)$ does not split over k , but has at least one root in k , then we have either all roots lie in k in which case the above discussion takes care of it. And we have the existence of an idempotent in $k[a]$. Otherwise, the other two distinct roots lie outside of k , but then we have $k[a] \cong k \oplus l$ where l is a quadratic extension of k that contain the roots of $\chi_a(T)$, and $k[a]$ contains an idempotent. \square

Corollary 2.2.4.10. *If k is algebraically closed and $\dim_k(A) > 2$, then $m_x = \chi_x$ for x in a nonempty Zariski open subset of A .*

Proposition 2.2.4.11. *On a J-algebra A over k with $\dim_k(A) > 2$, the J-quadratic form Q and the cubic form \det are determined by the algebra structure of A , i.e. the vector space structure over k along with the product.*

Proof. We only need to show it for k algebraically closed. From the statements just previous there is a nonempty Zariski open subset of A we will call S containing x with χ_x its minimal polynomial. Since \det and Q are determined by coefficients of χ_x , and so the polynomials for \det and Q are determined on S and therefore for all of A . \square

Remark 5.3.11 in [25] has an example of a reduced J-algebra with dimension 2 that does not have a Q and \det determined by its algebra structure.

Corollary 2.2.4.12. *If $\det(x) \neq 0$ then x has a J-algebra inverse x^{-1} and $\det(x^{-1}) = \det(x)^{-1}$.*

Proof. We have already shown the first part. For the second part it is enough to show its truth for the algebraically closed case. Let $\dim_k(A) > 2$. We will have two Zariski open sets one

$$V = \{x \in A \mid \det(x) \neq 0\},$$

and

$$W = \{x \in A \mid m_x = \chi_x\}.$$

Notice that $V \cap W$ would not be empty and that for $x \in V \cap W$,

$$\chi_x(T) = T^3 - \langle x, e \rangle T^2 - \left(Q(x) - \frac{1}{2} \langle x, e \rangle^2 \right) T - \det(x)$$

is the unique cubic polynomial which has x as a root, and also true for x^{-1} and $\chi_{x^{-1}}(T)$ and note also that x and x^{-1} are in the associative subalgebra $k[x]$. We have x^{-1} is a root of $\chi_x(T^{-1})$ and so also of

$$-\det(x)^{-1}T^3\chi_x(T^{-1}) = T^3 + BT^2 + CT - \det(x)^{-1},$$

and so the constant terms must be equal at x^{-1} and so $\det(x^{-1}) = \det(x)^{-1}$ for $x \in V \cap W$, and by Zariski continuity, it holds for all of V . This is also true when $\dim_k(A) = 2$ see [25]. \square

2.2.5 Classification of reduced J-algebras

Fix a primitive idempotent u and assume $E_1 \neq \{0\}$, then fix $x_1 \in E_0$ such that $Q(x_1) = \frac{1}{4}$.

Lemma, [25] 2.2.5.1. *Let s be the linear mapping*

$$s : E_1 \rightarrow E_1 \text{ and } y \mapsto x_1 y,$$

then

- (a) s is symmetric with respect to $\langle \cdot, \cdot \rangle$ and $s^2 = \frac{1}{16}$
- (b) $E_1 = E_+ \oplus E_-$ and $E_+ \perp E_-$ where E_+ and aE_- refer to the eigenspaces of $\frac{1}{4}$ and $-\frac{1}{4}$ respectively
- (c) if $\dim(E_0) > 1$, then both $E_+ \neq \{0\}$ and $E_- \neq \{0\}$
- (d) if $\dim(E_0) = 1$, then $E_1 = E_+$ or $E_1 = E_-$.

Proof. (a) $\langle s(u), v \rangle = \langle x_1 u, v \rangle = \langle u, x_1 v \rangle = \langle u, s(v) \rangle$, for $u, v \in E_1$. Also from a 2.2.4.4, for $x \in E_0$ and $y \in E_1$ we have $x(xy) = \frac{1}{4}Q(x)y$, so $s^2(y) = s(s(y)) = x_1(x_1 y) = \frac{1}{4}(\frac{1}{4})y = \frac{1}{16}y$.

(b) From (a) we can see

$$s^2 = \frac{1}{16} \Rightarrow s^2 - \frac{1}{16} = 0 \Rightarrow s = \pm \frac{1}{4},$$

so s has eigenvalues $\frac{1}{4}$ and $-\frac{1}{4}$, and so $E_1 = E_+ \oplus E_-$.

(c) Since $\dim(E_0) > 1$ we let $x \in E_0$ such that $\langle x, x_1 \rangle = 0$ and $Q(x) \neq 0$. If $E_+ \neq \{0\}$ we can pick $y \in E_+$ with $Q(y) \neq 0$. By 2.2.4.4 we have

$$x_1(x_2y) + x_2(x_1y) = \langle x_1, x_2 \rangle y,$$

for $x_1, x_2 \in E_0$ and $y \in E_1$. In our case $x_1(xy) + x(x_1y) = 0$, so $x_1(xy) = -\frac{1}{4}(xy)$, so $xy \in E_-$. Since we know that $Q(xy) = \frac{1}{4}Q(x)Q(y)$, we have that $Q(xy) \neq 0$ so $xy \neq 0$. This gives us $E_+ \neq \{0\} \Rightarrow E_- \neq \{0\}$, and $E_0 \neq \{0\} \Rightarrow E_1 \neq \{0\}$ happens almost exactly the same way.

(d) Let $E_+ \neq \{0\}$ and $E_- \neq \{0\}$. Let $y \in E_+$ and $z \in E_-$ both with nonzero norms. By the 2.2.4.4

$$4y \circ x_1y = Q(y)x_1 = y \circ y.$$

From 2.2.4.4 we have

$$2(y \circ z)y + (y \circ y)z = \frac{1}{4}Q(y)z + \frac{1}{4}\langle y, z \rangle y.$$

From the fact that $(y \circ y)z = -\frac{1}{4}Q(y)z$ and $\langle y, z \rangle = 0$, we have

$$2(y \circ z)y = \frac{1}{2}Q(y)z,$$

and we can say that $y \circ z \neq 0$. Also,

$$\langle x_1, y \circ z \rangle = \left\langle x_1, yz - \frac{1}{4}\langle y, z \rangle(e + u) \right\rangle = \langle x_1, yz \rangle = -\frac{1}{4}\langle y, z \rangle = 0,$$

which gives us that $\dim(E_0) > 1$ (and also (d) follows immediately). \square

The case we are most interested in is the case when E_+ and E_- are nonzero, where A is a reduced J -algebra with a primitive idempotent u . We want to show that in this case A is isomorphic to an algebra of 3×3 hermitian matrices over composition algebra C . We will take

$$C = x_1^\perp \cap E_0,$$

and give it the structure of a composition algebra.

Lemma 2.2.5.2. *For $y_+, z_+ \in E_+$ and $y_-, z_- \in E_-$ we get,*

$$(a) \quad y_+ \circ z_+ = \frac{1}{2}\langle y_+, z_+ \rangle x_1$$

$$(b) \quad y_- \circ z_- = -\frac{1}{2} \langle y_-, z_- \rangle x_1$$

$$(c) \quad y_+ \circ y_- \in C$$

$$(d) \quad (y_+ \circ y_-)y_+ = \frac{1}{4}Q(y_+)y_-$$

$$(e) \quad (y_+ \circ y_-)y_- = \frac{1}{4}Q(y_-)y_+$$

$$(f) \quad Q(y_+) = Q(y_-) = 0 \text{ if } y_+ \circ y_- = 0 \text{ and } y_+, y_- \neq 0$$

$$(g) \quad Q(y_+ \circ y_-) = \frac{1}{4}Q(y_+)Q(y_-).$$

Proof. (a) and (b) Since from 2.2.4.4 we know $y_+ \circ y_+ = Q(y_+)x_1$ by linearization of this relation we get (a), and similarly for (b).

(c) We already know that $y_+ \circ y_- \in E_0$ all we need to do is compute

$$\langle x_1, y_+ \circ y_- \rangle = \langle x_1, y_+ y_- \rangle = \langle x_1 y_+, y_- \rangle = \frac{1}{4} \langle y_+, y_- \rangle = 0.$$

(d) By 2.2.4.4 we have

$$\begin{aligned} (y_+ \circ y_-)y_+ + (y_- \circ y_+)y_+ + (y_+ \circ y_+)y_- &= \frac{1}{8} (\langle y_+, y_- \rangle y_+ + \langle y_-, y_+ \rangle y_+ + \langle y_+, y_+ \rangle y_-) \\ 2(y_+ \circ y_-)y_+ + (y_+ \circ y_+)y_- &= \frac{1}{4} \langle y_+, y_- \rangle y_+ + \frac{1}{8} \langle y_+, y_+ \rangle y_- \\ 2(y_+ \circ y_-)y_+ + (y_+ \circ y_+)y_- &= \frac{1}{4} Q(y_+)y_- \\ 2(y_+ \circ y_-)y_+ + \frac{1}{2} \langle y_+, y_+ \rangle x_1 y_- &= \frac{1}{4} Q(y_+)y_- \\ 2(y_+ \circ y_-)y_+ - \frac{1}{4} Q(y_+)y_- &= \frac{1}{4} Q(y_+)y_- \\ (y_+ \circ y_-)y_+ &= \frac{1}{4} Q(y_+)y_-. \end{aligned}$$

(e) follows from very nearly the same argument, and (f) follows from (d) and (e).

(g) By 2.2.4.4 we have

$$Q((y_+ \circ y_-)y_+) = \frac{1}{4}Q(y_+ \circ y_-)Q(y_+),$$

and also

$$Q((y_+ \circ y_-)y_+) = Q\left(\frac{1}{4}Q(y_+)y_-\right) = \frac{1}{16}Q(y_+)^2Q(y_-).$$

So when $Q(y_+) \neq 0$ we have

$$\begin{aligned} \frac{1}{4}Q(y_+ \circ y_-)Q(y_+) &= \frac{1}{16}Q(y_+)^2Q(y_-) \\ Q(y_+ \circ y_-) &= \frac{1}{4}Q(y_+)Q(y_-). \end{aligned}$$

This works over an algebraically closed field k and by Zariski continuity the relation holds everywhere. Notice that the set of $Q(y_+) \neq 0$ is open. Now when $Q(y_+) = 0$ we have the argument, also from 2.2.4.4

$$(y_+ \circ y_-) \left((y_+ \circ y_-)y_+ \right) = \frac{1}{4}Q(y_+ \circ y_-)y_+ = (y_+ \circ y_-) \frac{1}{4}Q(y_+)y_- = 0.$$

□

We now want to define the composition algebra structure on C . To do this first we fix $a_+ \in E_+$ and $a_- \in E_-$.

Lemma 2.2.5.3. *The maps*

$$e_+ : C \rightarrow E_+, x \mapsto 4Q(a_-)^{-1}xa_-,$$

$$e_- : C \rightarrow E_-, x \mapsto 4Q(a_+)^{-1}xa_+,$$

are linear isomorphisms. Their inverses are

$$E_+ \rightarrow C, y_+ \mapsto a_- \circ y_+ = a_-y_+,$$

$$E_- \rightarrow C, y_- \mapsto a_+ \circ y_- = a_+y_-.$$

Proof. If I take $x \in C \subset E_0$ then

$$\begin{aligned} x &= 4Q(a_-)^{-1} \frac{1}{4}Q(a_-)x \\ &= 4Q(a_-)^{-1}a_- \circ xa_- \\ &= a \circ (4Q(a_-)^{-1}xa_-) \in Im(e_+^{-1}). \end{aligned}$$

Also,

$$\begin{aligned}
e_+(e_+^{-1}(y_+)) &= e_+(a_- \circ y_+) \\
&= 4Q(a_-)^{-1}(a_- \circ y_+)a_- \\
&= 4Q(a_-)^{-1}\frac{1}{4}Q(a_-)y_+ \\
&= y_+,
\end{aligned}$$

and also the other way, so we have that e_+ is one to one and onto, this is similarly true for e_- . \square

Now we define the composition algebra structure on C . We will use as multiplication \diamond

$$x \diamond y = 16Q(a_+)^{-1}Q(a_-)^{-1}(xa_-)(ya_+) = e_+(x)e_-(y), \text{ for } x, y \in C,$$

and the norm will be

$$N(x) = 4Q(a_+)^{-1}Q(a_-)^{-1}Q(x).$$

Lemma, [25] 2.2.5.4. *The vector space C with \diamond as a product and N as a norm is a composition algebra.*

Proof. First let us see what the identity element should be. We will call it ε .

$$e_+(x)e_-(\varepsilon) = (4Q(a_-)^{-1}xa_-)e_-(\varepsilon),$$

so we need $e_-(\varepsilon) = a_-$, since $(xa_-)a_- = \frac{1}{4}Q(a_-)x$. And also for

$$e_+(\varepsilon)e_-(x) = e_+(\varepsilon)(4Q(a_+)^{-1}xa_+),$$

so we need $e_+(\varepsilon) = a_+$, since $(xa_+)a_+ = \frac{1}{4}Q(a_+)x$. So we need

$$e_+(\varepsilon) = 4Q(a_-)^{-1}\varepsilon a_- = a_+,$$

and

$$e_-(\varepsilon) = 4Q(a_+)^{-1}\varepsilon a_+ = a_-.$$

To this end we can let $\varepsilon = a_+a_-$, since

$$4Q(a_-)^{-1}(a_+a_-)a_- = a_+, \text{ and } 4Q(a_+)^{-1}(a_+a_-)a_+ = a_-.$$

To show the norm permits composition we will compute

$$\begin{aligned}
Q(x \diamond y) &= Q(e_+(x)e_-(y)) \\
&= \frac{1}{4}Q(e_+(x))Q(e_-(y)) \\
&= \frac{1}{4}Q(4Q(a_-)^{-1}xa_-)Q(4Q(a_+)^{-1}ya_+) \\
&= \frac{1}{4}(16)Q(a_-)^{-2}\frac{1}{4}Q(x)Q(a_-)(16)Q(a_+)^{-2}\frac{1}{4}Q(y)Q(a_+) \\
&= 4Q(a_-)Q(a_+)Q(x)Q(y).
\end{aligned}$$

So then

$$\begin{aligned}
N(x \diamond y) &= 4Q(a_+)^{-1}Q(a_-)^{-1}Q(x \diamond y) \\
&= 4Q(a_+)^{-1}Q(a_-)^{-1}4Q(a_-)^{-1}Q(a_+)^{-1}Q(x)Q(y) \\
&= 16Q(a_+)^{-2}Q(a_-)^{-2}Q(x)Q(y),
\end{aligned}$$

and

$$\begin{aligned}
N(x)N(y) &= 4Q(a_+)^{-1}Q(a_-)^{-1}Q(x)4Q(a_+)^{-1}Q(a_-)^{-1}Q(y) \\
&= 16Q(a_+)^{-2}Q(a_-)^{-2}Q(x)Q(y).
\end{aligned}$$

Since the restriction of Q to C is nondegenerate, so is N . \square

We will use $N(\cdot, \cdot)$ to denote the bilinear form associated with N to distinguish from $\langle \cdot, \cdot \rangle$ the bilinear form associated with Q .

Theorem, [25] 2.2.5.5. *A reduced J -algebra A over a field k of characteristic not 2 or 3 with identity element e and quadratic form Q is of the type $A \cong H(C, \gamma)$ if A is not of quadratic type. $H(C, \gamma)$ is an algebra of 3×3 hermitian matrices over composition algebra C over k .*

Proof. We will can assume that there is a primitive idempotent u such that $\dim(E_0) > 1$ and $E_1 \neq 0$. We can fix $x_1 \in E_0$ such that $Q(x_1) = \frac{1}{4}$, and so $E_1 = E_+ \oplus E_-$. Notice that

$$u_1 = u, \quad u_2 = \frac{1}{2}(e - u) + x_1, \quad \text{and} \quad u_3 = \frac{1}{2}(e - u) - x_1$$

are idempotents that sum to e . Using 2.2.5.4 we can write $x \in A$ uniquely as $x = x(\xi_1, \xi_2, \xi_3; c_1, c_2, c_3) =$

$$\xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3 + 2Q(a_+)^{-1}Q(a_-)^{-1}c_1 + 4Q(a_+)^{-1}\bar{c}_2 a_+ + 4Q(a_-)^{-1}\bar{c}_3 a_-,$$

where $\xi_i \in k$ and $c_i \in C$, and $\bar{}$ is conjugation in C . We have seen that $\bar{c}_2 a_+ \in E_-$ and

$\bar{c}_3 a_- \in E_+$. It can be shown (though will not be here) that

$$\begin{aligned} x^2 &= (\xi_1^2 + Q(a_-)N(c_2) + Q(a_+)N(c_3)) u_1 \\ &\quad + (\xi_2^2 + Q(a_+)^{-1}Q(a_-)^{-1}N(c_1) + Q(a_+)N(c_3)) u_2 \\ &\quad + (\xi_3^2 + Q(a_+)^{-1}Q(a_-)^{-1}N(c_1) + Q(a_-)N(c_2)) u_3 \\ &\quad + 2Q(a_+)^{-1}Q(a_-)^{-1} ((\xi_2 + \xi_3)c_1 + Q(a_+)Q(a_-)\bar{c}_3 \diamond \bar{c}_2) \\ &\quad + 4Q(a_+)^{-1}\overline{((\xi_1 + \xi_3)c_2 + Q(a_-)^{-1}\bar{c}_1 \diamond \bar{c}_3)} a_+ \\ &\quad + 4Q(a_-)^{-1}\overline{((\xi_1 + \xi_2)c_3 + Q(a_+)^{-1}\bar{c}_2 \diamond \bar{c}_1)} a_-. \end{aligned}$$

Now if we square an element $h = h(\xi_1, \xi_2, \xi_3; c_1, x_2, c_3) \in H(C; \gamma_1, \gamma_2, \gamma_3)$ we get

$$h^2 = h(\eta_1, \eta_2, \eta_3; d_1, d_2, d_3),$$

where

$$\begin{aligned} \eta_1 &= \xi_1^2 + \gamma_1^{-1}\gamma_3 N(c_2) + \gamma_2^{-1}\gamma_1 N(c_3) \\ \eta_2 &= \xi_2^2 + \gamma_3^{-1}\gamma_2 N(c_1) + \gamma_2^{-1}\gamma_1 N(c_3) \\ \eta_3 &= \xi_3^2 + \gamma_3^{-1}\gamma_2 N(c_1) + \gamma_1^{-1}\gamma_3 N(c_2) \\ d_1 &= (\xi_2 + \xi_3)c_1 + \gamma_2^{-1}\gamma_3 \bar{c}_3 \diamond \bar{c}_2 \\ d_2 &= (\xi_1 + \xi_3)c_2 + \gamma_3^{-1}\gamma_1 \bar{c}_1 \diamond \bar{c}_3 \\ d_3 &= (\xi_1 + \xi_2)c_3 + \gamma_1^{-1}\gamma_2 \bar{c}_2 \diamond \bar{c}_1 \end{aligned}$$

Now the map $\varphi(x(\xi_1, \xi_2, \xi_3; c_1, c_2, c_3)) = h(\xi_1, \xi_2, \xi_3; c_1, x_2, c_3)$ is one to one and onto if we let

$$\gamma_1 = 1, \quad \gamma_2 = Q(a_+)^{-1}, \quad \text{and} \quad \gamma_3 = Q(a_-).$$

And since $\varphi(x)^2 = \varphi(x^2)$ with both algebras being commutative and φ linear we have

$$\varphi((x - y)^2) = \varphi(x)^2 - 2\varphi(xy) + \varphi(y)^2$$

and

$$\varphi(x - y)^2 = \varphi(x)^2 - 2\varphi(x)\varphi(y) + \varphi(y)^2,$$

so φ is an isomorphism. It maps

$$\begin{aligned} u_1 &\mapsto h(1, 0, 0; 0, 0, 0) \\ u_2 &\mapsto h(0, 1, 0; 0, 0, 0) \\ u_3 &\mapsto h(0, 0, 1; 0, 0, 0) \\ x_1 &\mapsto h(0, \frac{1}{2}, -\frac{1}{2}; 0, 0, 0) \\ a_+ &\mapsto h(0, 0, 0; 0, \frac{1}{4}Q(a_+), 0) \\ a_- &\mapsto h(0, 0, 0; 0, 0, \frac{1}{4}Q(a_-)) \end{aligned}$$

Since we already know that $H(C, \gamma)$ satisfies the properties of being a J -algebra we are done. \square

The J -algebras that are isomorphic to $H(C, \gamma)$ will be called *proper J -algebras*, and can be of dimension 6, 9, 15, 27, depending on the dimension of C . The one we are interested in is the proper J -algebra with C an octonion algebra. This algebra will be called an Albert algebra and has dimension 27, and will be shown that its automorphism group is of type F_4 .

Corollary 2.2.5.6. *A reduced J -algebra is proper if and only if the determinant polynomial of A is absolutely reducible.*

Corollary 2.2.5.7. *The isomorphic image of a J -algebra is of the same type (either proper or reduced).*

2.2.6 Properties of reduced J -algebras

We now look for necessary and sufficient conditions for a J -algebra to be reduced.

Theorem, [10] 2.2.6.1. *In a J -algebra A an element $x \in A$ satisfies $x \times x = 0$ if and only if x is a multiple of a primitive idempotent or $x^2 = 0$. So A is reduced if and only if there exists $a \in A$ such that $a \neq 0$ and $a \times a = 0$.*

Proof. Let $x \times x = 0$. If $\langle x, e \rangle = 0$, then from 2.2.2.1 we have that $x^2 - Q(x)e = 0$, but then

$$\langle x^2 - Q(x)e, e \rangle = 0 = \langle x^2, e \rangle - Q(x)\langle e, e \rangle = \langle x, x \rangle - 3Q(x) = 2Q(x) - 3Q(x) = -Q(x),$$

so $x^2 = 0$.

Now if $x^2 = 0$ we have

$$x^3 - \langle x, e \rangle x^2 - \left(Q(x) - \frac{1}{2}\langle x, e \rangle^2 \right) x - \det(x) = 0 \Rightarrow Q(x) = \frac{1}{2}\langle x, e \rangle^2.$$

Since, also,

$$Q(x) = \frac{1}{2}\langle x, x \rangle = \langle x^2, e \rangle = 0,$$

then $\langle x, e \rangle = 0 = Q(x)$, so by 2.2.2.1 $x \times x = 0$.

Now let us consider when $x \times x = 0$ and $\langle x, e \rangle \neq 0$, we shall assume (by rescaling) that $\langle x, e \rangle = 1$. By 2.2.2.1 we have

$$\begin{aligned} x^2 - \frac{1}{2}\langle x, e \rangle x - \frac{1}{2}\langle x, e \rangle x - \frac{1}{2}\langle x, x \rangle e + \frac{1}{2}\langle x, e \rangle \langle x, e \rangle e &= 0 \\ x^2 - \frac{1}{2}(1)x - \frac{1}{2}(1)x - \frac{1}{2}2Q(x)e + \frac{1}{2}(1)(1)e &= 0 \\ x^2 - x - \left(Q(x) - \frac{1}{2}\right)e &= 0. \end{aligned}$$

So then

$$\begin{aligned} \left\langle x^2 - x - \left(Q(x) - \frac{1}{2}\right)e, e \right\rangle &= 0 \\ \langle x^2, e \rangle - \langle x, e \rangle - \left(Q(x) - \frac{1}{2}\right)\langle e, e \rangle &= 0 \\ \langle x, x \rangle - 1 - 3Q(x) - \frac{3}{2} &= 0, \end{aligned}$$

and since $\langle x, e \rangle = 1$ we need $-3Q(x) - \frac{3}{2} = 0 \Rightarrow Q(x) = \frac{1}{2}$. From above this implies that $x^2 = x$, and so is idempotent.

Now let x be a primitive idempotent, then

$$2Q(x) = \langle x, x \rangle = \langle x^2, e \rangle = \langle x, e \rangle = 1,$$

so

$$\begin{aligned} x \times x &= x^2 - \frac{1}{2}\langle x, e \rangle x - \frac{1}{2}\langle x, e \rangle x - \frac{1}{2}\langle x, x \rangle e + \frac{1}{2}\langle x, e \rangle \langle x, e \rangle e \\ &= x - \frac{1}{2}(1)x - \frac{1}{2}(1)x - \frac{1}{2}(1)e + \frac{1}{2}(1)e \\ &= 0. \end{aligned}$$

Now we let $a \neq 0$ such that $a^2 = 0$ and recall that this means $Q(a) = 0$ and $\langle a, e \rangle = 0$. Since Q restricted to e^\perp is nondegenerate there exists $b \in e^\perp$ such that $Q(b) = 0$ and $\langle a, b \rangle = 1$. From

equation (5.13) in [25]

$$\begin{aligned} 2a(ab) + a^2b &= \langle a, b \rangle a + 3\langle a, a, b \rangle \\ 2a(ab) &= a + \langle a \times a, b \rangle \\ 2a(ab) &= a, \end{aligned}$$

also

$$\begin{aligned} 2b(ab) + b^2a &= b + 3\langle a, b, b \rangle \\ 2b(ab) &= -ab^2 + b + \langle a, b \times b \rangle \\ 2b(ab) &= -ab^2 + b + \langle a, b^2 \rangle, \end{aligned}$$

and

$$2a(ab^2) = \langle a, b^2 \rangle a.$$

From equation (5.12) in [25] we can say that

$$a(b(ab)) + b(a(ab)) + (ab)^2 = \frac{3}{2}ab + \frac{1}{2}\langle a, b^2 \rangle a, \quad (2.8)$$

because

$$3\langle a, b, ab \rangle = \langle a \times ab, b \rangle = \langle a(ab) - \frac{1}{2}\langle a, b \rangle e, b \rangle = \langle 0, b \rangle = 0.$$

Let's first notice that

$$\begin{aligned} a(b(ab)) &= a\left(\frac{1}{2}(-ab^2 + b + \langle a, b^2 \rangle e)\right) \\ &= -\frac{1}{2}a(ab^2) + \frac{1}{2}ab + \frac{1}{2}\langle a, b^2 \rangle a \\ &= -\frac{1}{4}\langle a, b^2 \rangle a + \frac{1}{2}ab + \frac{1}{2}\langle a, b^2 \rangle a, \end{aligned}$$

and then that

$$b(a(ab)) = \frac{1}{2}ab.$$

So we can rewrite equation 2.8 as

$$\begin{aligned} -\frac{1}{4}\langle a, b^2 \rangle a + \frac{1}{2}ab + \frac{1}{2}\langle a, b^2 \rangle a + \frac{1}{2}ab + (ab)^2 &= \frac{3}{2}ab + \frac{1}{2}\langle a, b^2 \rangle a \\ (ab)^2 &= \frac{1}{2}ab + \frac{1}{4}\langle a, b^2 \rangle a. \end{aligned}$$

Now we can compute u^2 , where $u = e + \langle a, b^2 \rangle a - 2ab$,

$$\begin{aligned} (e + \langle a, b^2 \rangle a - 2ab)^2 &= e + \langle a, b^2 \rangle a - 2ab \\ &\quad + \langle a, b^2 \rangle a + \langle a, b^2 \rangle^2 a^2 - 2\langle a, b^2 \rangle a(ab) \\ &\quad - 2ab - 2\langle a, b^2 \rangle a(ab) + 4(ab)^2. \end{aligned}$$

If we look at

$$\begin{aligned} \langle a, b^2 \rangle a + \langle a, b^2 \rangle^2 a^2 - 2\langle a, b^2 \rangle a(ab) - 2ab - 2\langle a, b^2 \rangle a(ab) + 4(ab)^2 \\ = \langle a, b^2 \rangle a - \langle a, b^2 \rangle a - 2ab - \langle a, b^2 \rangle a + 4 \left(\frac{1}{2}ab + \frac{1}{4}\langle a, b^2 \rangle a \right) \\ = -2ab - \langle a, b^2 \rangle a + 2ab + \langle a, b^2 \rangle a \\ = 0, \end{aligned}$$

so $u^2 = u$ and so is idempotent. We should also notice that

$$\langle e + \langle a, b^2 \rangle a - 2ab, e \rangle = \langle e, e \rangle - 2\langle a, b \rangle = 3 - 2 = 1,$$

and

$$ua = (e + \langle a, b^2 \rangle a - 2ab)a = a + \langle a, b^2 \rangle a^2 - 2a(ab) = 0.$$

□

Theorem, [10] 2.2.6.2. *A J-algebra A is reduced if and only if the determinant represents zero nontrivially on A.*

Proof. By the previous theorem if A contains a primitive idempotent u, then $u \times u = 0$ which implies $\det(u) = 0$. On the other hand if $x \in A$ such that $\det(x) = 0$, then either $x \times x = 0$ and we are done or $x \times x = y \neq 0$, but then $y \times y = \det(x)x = 0$. □

Proposition, [25] 2.2.6.3. *Let A be a reduced J-algebra and let $u \in A$ be a fixed primitive idempotent, then*

$$(a) t = (Q(y) + 1)^{-1} \left(u + \frac{1}{2}Q(y)(e - u) + y \circ y + y \right) \text{ for } y \in E_1 \text{ and } Q(y) \neq -1,$$

$$(b) t = \frac{1}{2}(e - u) + x + y \text{ with } x \in E_0, Q(x) = \frac{1}{4}, y \in E_1, xy = \frac{1}{4}y, \text{ and } Q(y) = 0$$

are the primitive idempotents in A. In type (b) the condition $Q(y) = 0$ can be replaced with $y \circ y = 0$. The primitive idempotents of type (a) are characterized by $\langle t, u \rangle \neq 0$ and type (b)

are such that $\langle t, u \rangle = 0$. The primitive idempotents of type (a) are such that $\langle t, u \rangle \neq 0$, those of type (b) are such that $\langle t, u \rangle = 0$.

Proof. We first look at $t \in A$ such that $t \times t = 0$, letting $t = \xi e + \eta u + x + y$, where u is a primitive idempotent and $x \in E_0$ and $y \in E_1$,

$$\begin{aligned} t \times t &= t^2 - \frac{1}{2}\langle t, e \rangle t - \frac{1}{2}\langle t, e \rangle t - \frac{1}{2}\langle t, t \rangle e + \frac{1}{2}\langle t, e \rangle \langle t, e \rangle e \\ &= t^2 - \langle t, e \rangle t - \frac{1}{2}\langle t^2, e \rangle e + \frac{1}{2}\langle t, e \rangle^2 e. \end{aligned}$$

So we will start by computing

$$\begin{aligned} t^2 &= (\xi e + \eta u + x + y)^2 \\ &= \xi^2 e + \xi \eta u + \xi x + \xi y + \xi \eta u + \eta^2 u^2 + \eta u x + \eta u y \\ &\quad + \xi x + \eta u x + x^2 + xy + \xi y + \eta u y + xy + y^2 \\ &= \xi^2 e + 2\xi \eta u + 2\xi x + 2\xi y + \eta^2 u + \eta y + x^2 + 2xy + y^2, \end{aligned}$$

and then

$$\begin{aligned} \langle t, e \rangle &= \langle \xi e + \eta u + x + y, e \rangle \\ &= \xi \langle e, e \rangle + \eta \langle u, e \rangle + \langle x, e \rangle + \langle y, e \rangle \\ &= 3\xi + \eta, \end{aligned}$$

so

$$\begin{aligned} \langle t, e \rangle t &= (3\xi + \eta)(\xi e + \eta u + x + y) \\ &= 3\xi^2 e + 3\xi \eta u + 3\xi x + 3\xi y + \xi \eta e + \eta^2 u + \eta x + \eta y, \end{aligned}$$

and

$$\langle t, e \rangle^2 = 9\xi^2 e + 6\xi \eta e + \eta^2 e.$$

We also need to compute

$$\begin{aligned} \langle t^2, e \rangle &= \langle \xi^2 e + 2\xi \eta u + 2\xi x + 2\xi y + \eta^2 u + \eta y + x^2 + 2xy + y^2, e \rangle \\ &= \xi^2 \langle e, e \rangle + (2\xi \eta + \eta^2) \langle u, e \rangle + 2\xi \langle x, e \rangle + (2\xi + \eta) \langle y, e \rangle + \langle x^2, e \rangle + 2 \langle xy, e \rangle + \langle y^2, e \rangle \\ &= 3\xi^2 e + 2\xi \eta e + \eta^2 e + 2Q(x) + 2Q(y). \end{aligned}$$

Now we can say

$$\begin{aligned}
t \times t &= \xi^2 e + 2\xi\eta u + 2\xi x + 2\xi y + \eta^2 u + \eta y + x^2 + 2xy + y^2 \\
&\quad - (3\xi^2 e + 3\xi\eta u + 3\xi x + 3\xi y + \xi\eta e + \eta^2 u + \eta x + \eta y) \\
&\quad - \frac{1}{2}(3\xi^2 e + 2\xi\eta e + \eta^2 e + 2Q(x) + 2Q(y)) \\
&\quad + \frac{1}{2}(9\xi^2 e + 6\xi\eta e + \eta^2 e) \\
&= \xi^2 e + 2\xi\eta u + 2\xi x + 2\xi y + \eta^2 u + \eta y + x^2 + 2xy + y^2 \\
&\quad - 3\xi^2 e - 3\xi\eta u - 3\xi x - 3\xi y - \xi\eta e - \eta^2 u - \eta x - \eta y \\
&\quad - \frac{3}{2}\xi^2 e - \xi\eta e - \frac{1}{2}\eta^2 e - Q(x) - Q(y) \\
&\quad + \frac{9}{2}\xi^2 e + 3\xi\eta e + \frac{1}{2}\eta^2 e \\
&= \xi^2 e - \xi\eta u - \xi y - \xi x + \xi\eta e - \eta x - Q(x)e - Q(y)e + x^2 + 2xy + y^2.
\end{aligned}$$

Now computing

$$x^2 = \frac{1}{2}\langle x, x, \rangle(e - u) = Q(x)(e - u),$$

and

$$y^2 = \frac{1}{4}\langle y, y, \rangle(e + u) + y \circ y = \frac{1}{2}Q(y)(e + u) + y \circ y,$$

while recalling that $y \circ y \in E_0$ and $xy \in E_1$, we arrive at

$$\begin{aligned}
t \times t &= \xi^2 e - \xi\eta u - \xi y - \xi x + \xi\eta e - \eta x \\
&\quad - Q(x)e - Q(y)e + Q(x)(e - u) + 2xy + \frac{1}{2}Q(y)(e + u) + y \circ y \\
&= \left(\xi^2 + \xi\eta - \frac{1}{2}Q(y)\right)e + \left(-\xi\eta - Q(x) + \frac{1}{2}Q(y)\right)u + (y \circ y - (\xi + \eta)x) + (2xy - \xi y).
\end{aligned}$$

So in order for $t \times t = 0$ we need the following equations to hold

$$\begin{aligned}
\xi^2 + \xi\eta - \frac{1}{2}Q(y) &= 0 \\
-\xi\eta - Q(x) + \frac{1}{2}Q(y) &= 0 \\
y \circ y - (\xi + \eta)x &= 0 \\
2xy - \xi y &= 0.
\end{aligned}$$

We will consider three cases $\xi + \eta = 1$, $\xi + \eta = 0$ with $\xi \neq 0$, and $\xi = \eta = 0$.

Case 1($\xi + \eta = 1$): If we solve these equations we get

$$\xi = \frac{1}{2}Q(y), \quad \eta = 1 - \frac{1}{2}Q(y), \text{ and } x = y \circ y.$$

Now our expression for t is,

$$t = u + \frac{1}{2}Q(y)(e - u) + y \circ y + y.$$

If we look at

$$\langle t, e \rangle = \langle u, e \rangle + \frac{1}{2}Q(y)\langle e - u, e \rangle = 1 + Q(y),$$

and so if $Q(y) = -1$ we have $t^2 = 0$ by 2.2.6.1 since t would not be a multiple of a primitive idempotent. When $Q(y) \neq -1$ we have by 2.2.6.1 t is a multiple of a primitive idempotent. In the second case with some computation and substitutions mainly using 2.2.4.3 and 2.2.4.4 we get

$$\begin{aligned} t^2 &= \left(\frac{1}{2}Q(y) + \frac{1}{2}Q(y)^2\right)e + \left(1 + \frac{1}{2}Q(y) - \frac{1}{2}Q(y)^2\right)u + (1 + Q(y))(y \circ y) + (1 + Q(y))y \\ &= (1 + Q(y))u + \frac{1}{2}Q(y)(1 + Q(y))(e - u) + (1 + Q(y))(y \circ y) + (1 + Q(y))y, \end{aligned}$$

so we can see that $(1 + Q(y))^{-1}t$ is an idempotent. Also notice that

$$\langle (1 + Q(y))^{-1}t, e \rangle = (1 + Q(y))^{-1}\langle t, e \rangle = 1 = 2Q((1 + Q(y))^{-1}t),$$

so $Q((1 + Q(y))^{-1}t) = \frac{1}{2}$ and that makes $(1 + Q(y))^{-1}t$ a primitive idempotent.

Case 2($\xi + \eta = 0$, and $\xi \neq 0$): First we can scale ξ and η so that $\xi = \frac{1}{2}$ and we get

$$\xi = \frac{1}{2}, \text{ and } \eta = -\frac{1}{2},$$

so

$$t = \frac{1}{2}(e - u) + x + y.$$

with

$$Q(x) = \frac{1}{4}, \quad Q(y) = 0, \quad y \circ y = 0, \text{ and } xy = \frac{1}{4}y.$$

Notice that if $y \circ y = 0$ and we already know $y \circ y = \frac{1}{4}Q(y)^2$, so $Q(y) = 0$, and also if $Q(y) = 0$ we can let $x = x_1$ in the earlier decomposition of E and have $y \in E_+$ (since $xy = \frac{1}{4}y$ and $x \in E_0$), so by 2.2.4.4 we have $y \circ y = Q(y)x_1 = 0$. The two conditions are equivalent. When

we compute t^2 we can see

$$t^2 = \left(\frac{1}{2}(e - u) + x + y \right)^2 = \frac{1}{4}e - \frac{1}{4}u + \frac{1}{4}e - \frac{1}{4}u + x + y = \frac{1}{2}(e - u) + x + y = t,$$

and also notice

$$\langle t, e \rangle = \left\langle \frac{1}{2}(e - u) + x + y, e \right\rangle = \frac{1}{2}\langle e - u, e \rangle = 1,$$

so t is a primitive idempotent.

Case 3($\xi = \eta = 0$): We have now that

$$t = x + y,$$

so in this case

$$Q(y) = Q(x) = 0, \quad y \circ y = 0, \quad xy = 0.$$

Since

$$t^2 = (x + y)^2 = x^2 + 2xy + y^2 = Q(x)(e - u) + 2xy + \frac{1}{2}Q(y)(e + u) + y \circ y = 0,$$

these elements are nilpotent. \square

Corollary 2.2.6.4. *The elements $t \in A$ with $t^2 = 0$ are a scalar factor of one of the following*

- (a) $t = u - \frac{1}{2}(e - u) + y \circ y + y$ when $y \in E_1$ such that $Q(y) = -1$,
- (b) $t = x + y$ when $x \in E_0$, $y \in E_1$, $Q(x) = Q(y) = 0$, $xy = 0$, and $y \circ y = 0$.

Lemma, [25] 2.2.6.5. *If u and t are primitive idempotents in a J-algebra, then there exists v_0, v_1, \dots, v_n with $n \leq 3$ such that $\langle v_{i-1}, v_i \rangle = 0$ for $1 \leq i \leq n$, where $v_0 = u$ and $v_n = t$.*

Proof. Let $\langle t, e \rangle \neq 0$, then we have

$$t = (Q(b) + 1)^{-1} \left(u + \frac{1}{2}Q(b)(e - u) + b \circ b + b \right),$$

for $b \in E_1$ and $Q(b) \neq -1$. We can compute $\langle t, u \rangle = (Q(b) + 1)^{-1}$. From this we can distinguish two cases

Case 1($\langle t, u \rangle \neq 1$): So in this case we have $Q(b) \neq 0$. So by the previous proposition for $\langle u, v \rangle = 0$, we have

$$v = \frac{1}{2}(e - u) + x + y.$$

Now if we check

$$\begin{aligned}
\langle t, v \rangle &= \left\langle (Q(b) + 1)^{-1} \left(u + \frac{1}{2}Q(b)(e - u) + b \circ b + b \right), \frac{1}{2}(e - u) + x + y \right\rangle \\
&= (Q(b) + 1)^{-1} \left(\frac{1}{4}Q(b)\langle e - u, e - u \rangle + \langle b \circ b, x \rangle + \langle b, y \rangle \right) \\
&= (Q(b) + 1)^{-1} \left(\frac{1}{2}Q(b) + \langle b \circ b, x \rangle + \langle b, y \rangle \right).
\end{aligned}$$

If we let $x = -Q(b)^{-1}b \circ b$ and $y = 0$ we have,

$$\begin{aligned}
\langle t, v \rangle &= (Q(b) + 1)^{-1} \left(\frac{1}{2}Q(b) + \langle b \circ b, x \rangle + \langle b, y \rangle \right) \\
&= (Q(b) + 1)^{-1} \left(\frac{1}{2}Q(b) - Q(b)^{-1}\langle b \circ b, b \circ b \rangle + \langle b, 0 \rangle \right) \\
&= (Q(b) + 1)^{-1} \left(\frac{1}{2}Q(b) - Q(b)^{-1}2Q(b \circ b) \right) \\
&= (Q(b) + 1)^{-1} \left(\frac{1}{2}Q(b) - Q(b)^{-1}2\frac{1}{4}Q(b)^2 \right) \\
&= 0.
\end{aligned}$$

So we have the result holding for $n \leq 2$.

Case 2($\langle t, u \rangle = 1$): In this case we have $Q(b) = 0$. If we take

$$v = \frac{1}{2}(e - u) + x,$$

and we have $\langle t, v \rangle = \langle b \circ b, x \rangle$, if this gives us 1, we can replace x with $-x$ and get $\langle t, v \rangle \neq 1$ and we are in Case 1. \square

2.2.7 The composition algebra is unique

We have seen that all proper reduced J -algebras are isomorphic to an algebra $H(C; \gamma_1, \gamma_2, \gamma_3)$. Next we will see that the composition algebra depends only on A , and not on any choice of x_1, a_+ , or a_- . So it makes sense to call C the composition algebra associated with A .

Theorem, [25] 2.2.7.1. *If A is a proper reduced J -algebra, then the composition algebra C where $A \cong H(C; \gamma_1, \gamma_2, \gamma_3)$ is uniquely determined up to isomorphism.*

Proof. Let u be a fixed primitive idempotent, and t any other idempotent and consider

$$E'_0 = \{x' \in A | \langle x', e \rangle = \langle x', t \rangle = 0, t'x = 0\} = \{x' \in A | \langle x', e \rangle = 0, tx' = 0\}.$$

Now if we fix $x_1 \in E'_0$ such that $Q(x'_1) = \frac{1}{4}$ we can define

$$C' = (x'_1)^\perp \cap E'_0 = \{x' \in E'_0 | \langle x', x'_1 \rangle = 0\}.$$

By Witt's Theorem the restriction of Q to C' is unique to isometry (since C' is the orthogonal complement of x'_1 in E'_0). Since the norm of C' is a multiple of Q ,

$$N(x) = 4Q(a_+)^{-1}Q(a_-)^{-1}Q(x),$$

by an earlier theorem we have that N is unique up to similarity and thus, again by an earlier theorem, the uniqueness of the composition algebra is guaranteed up to isomorphism. All that is left to show is that Q restricted to C' is similar to Q restricted to C' for one choice of x'_1 . By 2.2.6.5 it suffices to show this for t a primitive idempotent with $\langle t, u \rangle = 0$, so t is of type (b) from 2.2.6.3. And so t is of the form

$$t = \frac{1}{2}(e - u) + a + b,$$

with $a \in E_0$, $Q(a) = \frac{1}{4}$, $b \in E_1$, $ab = \frac{1}{4}b$ and $Q(b) = 0$ (also $b \circ b = 0$ can replace $Q(b) = 0$). The space E'_0 consists of the elements

$$x' = \xi e + \eta u + x + y,$$

such that $\langle x', e \rangle = 0$ and $tx' = 0$. If we consider the first equation we arrive at

$$\langle \xi e + \eta u + x + y, e \rangle = 3\xi + \eta = 0,$$

and the second tells

$$\begin{aligned} 0 = tx' &= \left(\frac{1}{2}(e - u) + a + b \right) (\xi e + \eta u + x + y) \\ &= \frac{\xi}{2}(e - u) + \frac{1}{2}(e - u)x + \frac{1}{2}(e - u)y + \xi a + ax + ay + \xi b + \frac{\eta}{2}bu + bx + by \\ &= \left(\frac{\xi}{2} + \frac{1}{2}\langle a, x \rangle + \frac{1}{4}\langle b, y \rangle \right) e + \left(-\frac{\xi}{2} - \frac{1}{2}\langle a, x \rangle + \frac{1}{4}\langle b, y \rangle \right) u \\ &\quad + \left(\frac{1}{2}x + \xi a + b \circ y \right) + \left(\frac{1}{4}y + ay + (\xi + \frac{\eta}{2})b + bx \right). \end{aligned}$$

So now we have the four additional equations

$$\begin{aligned}\xi + \langle a, x \rangle + \frac{1}{2} \langle b, y \rangle &= 0 \\ -\xi - \langle a, x \rangle + \frac{1}{2} \langle b, y \rangle &= 0 \\ \frac{1}{2}x + \xi a + b \circ y &= 0 \\ \frac{1}{4}y + ay + (\xi + \eta)b + bx &= 0.\end{aligned}$$

From these we see that $\eta = -3\xi$, and if we add the first two equations above we see that

$$\frac{1}{2} \langle b, y \rangle = 0, \text{ and } \langle a, x \rangle = -\xi.$$

We also see that

$$x = -2\xi a - 2b \circ y.$$

Further, we see

$$\begin{aligned}ay &= -\frac{1}{4}y - (\xi + \frac{\eta}{2})b - bx \\ &= -\frac{1}{4}y - \xi b - \frac{\eta}{2}b - b(-2\xi a - 2b \circ y) \\ &= -\frac{1}{4}y - \xi b - \frac{\eta}{2}b + 2\xi ab + 2(b \circ y)b \\ &= -\frac{1}{4}y - \xi b + \frac{3\xi}{2}b + \frac{\xi}{2}b + 2(b \circ y)b \\ &= -\frac{1}{4}y + 2\xi b + 2(b \circ y)b.\end{aligned}$$

Using Lemma 1.4.4 we see that

$$\begin{aligned}(b \circ b)y + (b \circ y)b + (y \circ b)b &= \frac{1}{8} \langle b, b \rangle y + \frac{1}{8} \langle b, y \rangle b + \frac{1}{8} \langle y, b \rangle b \\ (b \circ b)y + 2(b \circ y)b &= \frac{1}{4}Q(b)y + \frac{1}{4} \langle b, y \rangle b \\ 2(b \circ y)b &= 0 \\ \Rightarrow (b \circ y)b &= 0,\end{aligned}$$

since $b \circ b = \langle b, y \rangle = Q(b) = 0$, so we have

$$ay = -\frac{1}{4}y + \xi b.$$

Now we have that

$$x = \xi - 3\xi u - 2\xi u - 2b \circ y + y = \xi(e - 3u - 2a) - 2b \circ y + y,$$

and if we let $\xi = \frac{1}{4}$ and $y = \frac{1}{2}b$ we have that

$$ay = \frac{1}{2}ab = \frac{1}{8}b = -\frac{1}{8}b + \frac{1}{4}b = -\frac{1}{4}y + \xi b.$$

We can rewrite $b \circ y = \frac{1}{2}b \circ b = 0$. Notice that

$$x'_1 = \frac{1}{4}(e - 3u - 2a) + \frac{1}{2}b.$$

So if we consider

$$\begin{aligned} \langle x', x'_1 \rangle &= \langle \xi(e - 3u - 2a) - 2b \circ y + y, \frac{1}{4}(e - 3u - 2a) + \frac{1}{2}b \rangle \\ &= 2\xi + \langle x, b \circ y \rangle + \frac{1}{2}\langle b, y \rangle \\ &= 2\xi. \end{aligned}$$

So we need

$$C' = \left\{ x' = -2b \circ y + y \mid y \in E_1, \langle b, y \rangle = 0, ay = -\frac{1}{4}y \right\},$$

and we can pick $x_1 = a$ so we have $b \in E_+$ and $y \in E_-$, so we get $\langle b, y \rangle = 0$. Now we can write

$$C' = \{x' = -2b \circ y + y \mid y \in E_-\},$$

and notice that

$$Q(x') = Q(-2b \circ y + y) = 4Q(b)Q(y) + Q(y) = Q(y),$$

and we have that C' and E_- are isometric and thus similar. Recall e_- is an isomorphism from C to E_- , and

$$\begin{aligned} Q(e_-(x)) &= Q(4Q(a)^{-1}xa_+) \\ &= 16Q(a)^{-2}Q(a_+x) \\ &= 16Q(a)^{-2}\frac{1}{4}Q(a_+)Q(x) \\ &= 4Q(a)^{-1}Q(x), \end{aligned}$$

and we know that

$$\begin{aligned} N(x) &= 4Q(a_+)^{-1}Q(a_-)^{-1}Q(x) \\ &= Q(a_-)^{-1}Q(e_-(x)) \end{aligned}$$

and sure enough

$$Q(a_-)N(x) = Q(e_-(x)),$$

where N is the quadratic form on C , so $n(e_-) = Q(a_-)$ and C and E_- are similar, and by its transitive property we have that C and C' are similar and thus by 1.7.1 in [25] they are isomorphic. \square

Theorem 2.2.7.2. *If A and B are proper J -algebras over k with \det_A and \det_B their respective determinants, then A and B are isomorphic if and only if there exists a linear transformation $t : A \rightarrow B$ such that $\alpha \det_A(x) = \det_B(t(x))$, for some $\alpha \in k^*$.*

Proof. If A and B are isomorphic then so are their associated composition algebras. So we have

$$s : C_A \rightarrow C_B,$$

an isomorphism of composition algebras and we have a linear transformation

$$t : A \rightarrow B,$$

with

$$h(\xi_1, \xi_2, \xi_3; c_1, c_2, c_3) \mapsto h(\lambda_1 \xi_1, \lambda_2 \xi_2, \lambda_3 \xi_3; s(c_1), s(c_2), s(c_3)),$$

and

$$\begin{aligned} \lambda_1 &= (\gamma'_2 \gamma_3)^{-1} \gamma'_3 \gamma_2 \\ \lambda_2 &= (\gamma'_3 \gamma_1)^{-1} \gamma'_1 \gamma_3 \\ \lambda_3 &= (\gamma'_1 \gamma_2)^{-1} \gamma'_2 \gamma_1. \end{aligned}$$

We can then recall that

$$\det_a = \xi_1 \xi_2 \xi_3 - \gamma_3^{-1} \gamma_2 N(c_1) - \gamma_1^{-1} \gamma_3 \xi_2 N(c_2) - \gamma_2^{-1} \gamma_1 N(c_3) + N(c_1 c_2, \bar{c}_3),$$

and that $N(s(c_i)) = N(c_i)$ and $\lambda_1 \lambda_2 \lambda_3 = 1$, and all that is left to check is

$$-(\gamma'_{i+2})^{-1} \gamma'_{i+1} (\gamma'_{i+1} \gamma_{i+2})^{-1} \gamma'_{i+2} \gamma_{i+1} \xi_i = \gamma_{i+2}^{-1} \gamma_{i+1} \xi_i,$$

we see that $\det_A(x) = \det_B(t(x))$.

Now we want to show that if $\alpha \det_A(x) = \det_B(t(x))$ then we have C determined up to isomorphism. By Theorem 1.7.1 in [25] we only need to show that the norms of the composition algebras differ by a nonzero scalar factor. In order to do this we fix $v \in A$ such that $v \neq 0$ and $v \times v = 0$.

We then have two cases v is a primitive idempotent or $v^2 = 0$. Let us consider the quadratic form $F_v(x) = \langle v, x, x \rangle$.

Case 1 (v is a primitive idempotent): We can decompose A with respect to v so that for $x \in A$ we have $x = \xi e + \eta v + a + b$, where $a \in E_0$ and $b \in E_1$. Using equation (5.14) in [25] we have

$$3F_v(x) = \langle vx, x \rangle - Q(x) - \langle x, e \rangle \langle v, x \rangle + \frac{1}{2} \langle x, e \rangle^2.$$

Notice that

$$vx = v(\xi e + \eta v + a + b) = (\xi + \eta)v + \frac{1}{2}b,$$

and so

$$\begin{aligned} \langle vx, x \rangle &= \left\langle (\xi + \eta)v + \frac{1}{2}b, \xi e + \eta v + a + b \right\rangle \\ &= \xi(\xi + \eta)\langle v, e \rangle + \eta(\xi + \eta)\langle v, v \rangle + \frac{1}{2}\langle b, b \rangle \\ &= \xi^2 + 2\xi\eta + \eta^2 + Q(b). \end{aligned}$$

If we go through similar computations using the orthogonality relations we have

$$\begin{aligned} Q(x) &= \frac{1}{2}\langle x, x \rangle = \frac{3}{2}\xi^2 + \xi\eta + \frac{1}{2}\eta^2 + Q(a) + Q(b) \\ \langle v, x \rangle &= \xi + \eta \\ \langle x, e \rangle &= 3\xi + \eta \\ \langle x, e \rangle \langle v, x \rangle &= (3\xi + \eta)(\xi + \eta) = 3\xi^2 + 4\xi\eta + \eta^2 \\ \frac{1}{2}\langle x, e \rangle^2 &= \frac{9}{2}\xi^2 + 3\xi\eta + \frac{1}{2}\eta^2. \end{aligned}$$

So we have that

$$\begin{aligned} 3F_v(x) &= \xi^2 + 2\xi\eta + \eta^2 + Q(b) - \left(\frac{3}{2}\xi^2 + \xi\eta + \frac{1}{2}\eta^2 + Q(a) + Q(b) \right) \\ &\quad - (3\xi^2 + 4\xi\eta + \eta^2) + \frac{9}{2}\xi^2 + 3\xi\eta + \frac{1}{2}\eta^2 \\ &= \xi^2 - Q(a). \end{aligned}$$

So then we have that $R_v = kv \oplus E_1$, where R_v is the radical of F_v . If we look at the quadratic form induced F_v on A/R_v it is the same as restricting F_v to $S_v = ke \oplus E_0$. The restriction to S_v is given by

$$3F_v(\xi e + a) = \xi^2 - Q(a),$$

where $\xi \in k$ and $a \in E_0$. If we let $x_1 \in E_0$ with $Q(x_1) = \frac{1}{4}$ then we can consider S_v as the orthogonal direct sum of $C = x_1^\perp \cap E_0$ and $ke \oplus kx_1$, and use Witt's Theorem to say that F_v determines the restriction of Q to C . This also lets us say that N is a scalar multiple of Q .

Case 2 ($v^2 = 0$): By Theorem 5.5.1 [25] there exists u a primitive idempotent such that $uv = 0$, and so we decompose A with respect to u . By 2.2.6.3 we know that $v \in E_0$ with $Q(v) = 0$. We will take $x = \xi e + \eta u + a_b$ where $\xi, \eta \in k$ and $a \in E_0$ and $b \in E_1$. From equation (5.14) again we get

$$3F_v(x) = \langle v, x^2 \rangle - \langle x, e \rangle \langle v, x \rangle.$$

If we compute, x^2 , we have

$$\begin{aligned} x^2 &= (\xi e + \eta u + a + b)^2 \\ &= \xi^2 e + \xi \eta u + \xi a + \xi b + \xi \eta u + \xi \eta u + \eta^2 u^2 + \eta u a + \eta u b + \xi a + \eta a u + a b + \xi b + \eta b u + b a + b^2 \\ &= \left(\xi^2 + \frac{1}{2} Q(b) \right) e + \left(2\xi \eta + \frac{1}{2} Q(b) + \eta^2 \right) u + (2\xi a + b \circ b) + ((2\xi + \eta)b + 2ab). \end{aligned}$$

So if we can now say that

$$\langle v, x^2 \rangle = 2\xi \langle v, a \rangle + \langle v, b \circ b \rangle,$$

and since

$$\langle x, e \rangle \langle v, x \rangle = (3\xi + \eta) \langle v, a \rangle,$$

we have that

$$3F_v(x) = 2\xi \langle v, a \rangle + \langle v, b \circ b \rangle - (3\xi + \eta) \langle v, a \rangle = -(\xi + \eta) \langle v, a \rangle + \langle v, b \circ b \rangle.$$

So we have

$$R_v = \{ \xi(e - u) + a + b \mid \xi \in k, a \in E_0, b \in E_1, \langle v, a \rangle = 0, vb = 0 \},$$

since v is not in the radical of $\langle \cdot, \cdot \rangle$ restricted to E_0 we have that v is contained in a hyperbolic plane in E_0 . So we have $x_1 \in E_0$ and $c \in x_1^\perp \cap E_0$ such that $Q(x_1) = \frac{1}{4}$ and $Q(c) = -\frac{1}{4}$. We can write v as a nonzero scalar multiple of $x_1 + c$ or take $v = x_1 + c$. We can decompose

$E_1 = E_+ \oplus E_-$ with respect to x_1 , and write $b = b_+ + b_-$. Then

$$vb = (x_1 + c)(b_+ + b_-) = \frac{1}{4}b_+ - \frac{1}{4}b_- + cb_+ + cb_-.$$

In order to have $vb = 0$ we need $cb_+ = \frac{1}{4}b_-$ and $cb_- = -\frac{1}{4}b_+$. If we have $b_- = 4cb_+$, then

$$cb_- = 4c(cb_+) = Q(c)b_+ = -\frac{1}{4}b_+,$$

so if we have that $b_- = 4cb_+$ we have that $vb = 0$. And we can rewrite the radical of F_v as,

$$R_v = \{\xi(e - u) + a + b_+ + 4cb_+ \mid \xi \in k, a \in E_0, \langle v, a \rangle = 0, b_+ \in E_+\}.$$

We have the complementary subspace

$$S_v = \{\xi e + \eta v_1 + b_- \mid \xi, \eta \in k, b_- \in E_-\},$$

and now we fix a $v_1 \in E_0$ such that $\langle v, v_1 \rangle = 1$, $Q(v_1) = 0$. The quadratic form on A/R_v is the same as the restriction of F_v to S_v , which is given by

$$3F_v(\xi e + \eta v_1 + b_-) = -\xi\eta - \frac{1}{2}Q(b_-).$$

We have that S_v is the direct sum of $ke \oplus kv_1$ and E_- , so F_v determines the restriction of Q to E_- , which is a scalar multiple of the norm on C , so F_v determines N up to scalar factor. \square

2.2.8 Norm Classes of Primitive Idempotents

First let us consider the restriction of Q to E_0 . We can write any $x \in E_0$ as $x = \xi x_1 + c$ for $c \in C = x_1^\perp \cap E_0$, and $\xi \in k$. We can compute

$$Q(\xi x_1 + c) = \xi \langle x_1, c \rangle + Q(\xi x_1) + Q(c) = \frac{1}{4}\xi^2 + \frac{1}{4}Q(a_+)Q(a_-)N(c),$$

if we let $\alpha = \frac{1}{4}Q(a_+)Q(a_-)$ we have

$$Q(x) = \frac{1}{4}\xi^2 + \alpha N(c).$$

The isometry class depends on α so it depends on the cosets $\alpha N(C)^*$, where

$$N(C)^* = \{N(c) \mid c \in C, N(c) \neq 0\}.$$

We will call these cosets $\kappa(\alpha) = \alpha N(C)^* \in k^*/N(C)^*$. It turns out that $\kappa(\alpha)$ depends on the

primitive idempotent u , but not on x_1 chosen in E_0 . To see that $\alpha N(C)^*$ is independent of x_1 let us fix u and take another element $x'_1 \in E_0$ such that $Q(x'_1) = \frac{1}{4}$. So $Q(x'_1) = \frac{1}{4}\xi'^2 + \alpha'N'(c')$. From Witt's Theorem we have the existence of a linear transformation $t : C \rightarrow C'$ such that $\alpha N(c) = \alpha'N'(t(c))$. There is a $c \in C$ such that $t(c) \mapsto \varepsilon'$, and if we take that c we have $\alpha N(c) = \alpha'N'(\varepsilon') = \alpha'$. So α and α' are in the same norm class. Therefore α depends on u and not x_1 so we can call $\kappa(\alpha) = \kappa(u)$ the *norm class* of u .

Proposition, [25] 2.2.8.1. *Let A be a proper reduced J-algebra, let u be a primitive idempotent in A and $x_1 \in E_0$ with $Q(x_1) = \frac{1}{4}$. We let*

$$T = (ke \oplus ku \oplus kx_1)^\perp = C \oplus E_1,$$

then the norm classes of the primitive idempotents in A coincides with

$$\{\kappa(Q(t)) \mid t \in T, Q(t) \neq 0\}.$$

Proof. The restriction of Q to T is independent of our choice of u and x_1 , by Witt's Theorem. So we can fix u and x_1 , and proceed with computing $\kappa(v)$ for primitive idempotents.

Case 1 ($\langle v, u \rangle \neq 0$): By 2.2.6.3 we have

$$v = (Q(b) + 1)^{-1} \left(u + \frac{1}{2}Q(b)(e - u) + b \circ b + b \right),$$

with $b \in E_1$ and $Q(b) \neq -1$. First we will compute the zero space, E'_0 of v . We need to solve $vx' = 0$, where $x' = \xi e + \eta u + x + y$. So we need to find the solution to

$$\left(u + \frac{1}{2}Q(b)(e - u) + b \circ b + b \right) (\xi e + \eta u + x + y) = 0.$$

We can look at

$$\begin{aligned} u(\xi e + \eta u + x + y) &= \xi u + \eta u^2 + ux + uy \\ &= (\xi + \eta)u + \frac{1}{2}y, \end{aligned}$$

$$\begin{aligned} \frac{1}{2}Q(b)(e - u)(\xi e + \eta u + x + y) &= \frac{1}{2}Q(b)(\xi(e - u) + \eta(e - u)u + (e - u)x + (e - u)y) \\ &= \frac{1}{2}Q(b)(\xi e - \xi u + x - ux + y - uy) \\ &= \frac{1}{2}Q(b)\xi e - \frac{1}{2}Q(b)\xi u + \frac{1}{2}Q(b)x + \frac{1}{4}Q(b)y, \end{aligned}$$

$$\begin{aligned}
(b \circ b)(\xi e + \eta u + x + y) &= \xi(b \circ b) + \eta u(b \circ b) + x(b \circ b) + y(b \circ b) \\
&= \xi(b \circ b) + \frac{1}{2}\langle b \circ b, x \rangle(e - u) + (b \circ b)y \\
&= \frac{1}{2}\langle b \circ b, x \rangle e - \frac{1}{2}\langle b \circ b, x \rangle u + \xi(b \circ b) + (b \circ b)y,
\end{aligned}$$

$$\begin{aligned}
b(\xi e + \eta u + x + y) &= \xi b + \eta u b + b x + b y \\
&= \frac{1}{4}\langle b, y \rangle e + \frac{1}{4}\langle b, y \rangle u + b \circ y + b x + \left(\xi + \frac{1}{2}\eta\right)b.
\end{aligned}$$

So we obtain the following equations based on the decomposition $A = ke \oplus ku \oplus E_0 \oplus E_1$,

$$\frac{1}{2}Q(b)\xi + \frac{1}{2}\langle b \circ b, x \rangle + \frac{1}{4}\langle b, y \rangle = 0 \quad (2.9)$$

$$-\frac{1}{2}Q(b)\xi - \frac{1}{2}\langle b \circ b, x \rangle + \frac{1}{4}\langle b, y \rangle + \xi + \eta = 0 \quad (2.10)$$

$$\frac{1}{2}Q(b)x + \xi b \circ b + b \circ y = 0 \quad (2.11)$$

$$\left(\frac{1}{4}Q(b) + \frac{1}{2}\right)y + \left(\xi + \frac{1}{2}\eta\right)b + (b \circ b)y + bx = 0. \quad (2.12)$$

We can multiply 2.11 by $2b$ and get

$$Q(b)xb + 2\xi(b \circ b)b + 2(b \circ y)b = 0 \quad (2.13)$$

$$Q(b)bx + \frac{\xi}{2}Q(b)b + \frac{1}{4}\langle b, y \rangle b + \frac{1}{4}Q(b)y - (b \circ b)y = 0. \quad (2.14)$$

Using 2.2.4.4 we can make the substitution

$$\frac{1}{4}\langle b, y \rangle b = \frac{1}{4}Q(b)y - (b \circ b)y - 2(b \circ y)b,$$

and by adding 2.9 and 2.10 we get

$$\frac{1}{2}\langle b, y \rangle = -\xi - \eta. \quad (2.15)$$

In equation 2.12 we can solve for one of the terms and arrive at

$$(b \circ b)y = -\left(\frac{1}{4}Q(b) + \frac{1}{2}\right)y - \left(\xi + \frac{1}{2}\eta\right)b - bx. \quad (2.16)$$

Now we can substitute for $\frac{1}{4}\langle b, y \rangle b$, $\langle b, y \rangle$, and $(b \circ b)y$, in 2.14, and we get

$$2bx + \xi b + y = 0,$$

and so

$$y = -2bx - \xi b. \quad (2.17)$$

If we, conversely, take $y = -2bx - \xi b$ we see that

$$\begin{aligned} \langle b, y \rangle &= \langle b, -2bx - \xi b \rangle \\ &= -2\langle b, bx \rangle - \langle b, \xi b \rangle \\ &= -2\langle b \circ b, x \rangle - 2Q(b). \end{aligned}$$

If we then make the substitution into 2.9 we see that it is satisfied, and for 2.10 we have

$$\begin{aligned} -\frac{1}{2}Q(b)\xi - \frac{1}{2}\langle b \circ b, x \rangle + \frac{1}{4}(-2\langle b \circ b, x \rangle - 2Q(b)) + \xi + \eta &= 0 \\ -Q(b)\xi + \xi + \eta - \langle b \circ b, x \rangle &= 0 \\ (-Q(b) + 1)\xi + \eta - \langle b \circ b, x \rangle &= 0, \end{aligned}$$

so 2.10 is satisfied whenever $(-Q(b) + 1)\xi + \eta - \langle b \circ b, x \rangle = 0$. In 2.11 we have

$$\begin{aligned} \frac{1}{2}Q(b)x + \xi b \circ b + b \circ y &= \frac{1}{2}Q(b)x + \xi b \circ b + b \circ (-2bx - \xi b) \\ &= \frac{1}{2}Q(b)x + \xi b \circ b - 2b \circ xb - \xi b \circ b \\ &= \frac{1}{2}Q(b)x - 2\frac{1}{4}Q(b)x \\ &= 0. \end{aligned}$$

Finally for 2.12 we will compute each term with the new expression for y ,

$$\begin{aligned} \left(\frac{1}{4}Q(b) + \frac{1}{2}\right)y &= \frac{1}{4}Q(b)(-2bx - \xi b) + \frac{1}{2}(-2bx - \xi b) \\ &= -\frac{1}{2}Q(b)xb - \frac{1}{4}\xi Q(b)b - bx - \frac{1}{2}\xi b, \end{aligned}$$

and we will compute

$$\begin{aligned} (b \circ b)y &= (b \circ b)(-2bx - \xi b) \\ &= -2(b \circ b)bx - \xi(b \circ b)b. \end{aligned}$$

With a straight forward application of 2.2.4.4 (e) we have

$$\begin{aligned}(b \circ b)bx + (b \circ xb)b + (b \circ xb)b &= \frac{1}{8}\langle b, b \rangle bx + \frac{1}{8}\langle b, bx \rangle b + \frac{1}{8}\langle b, bx \rangle b \\(b \circ b)bx + \frac{1}{2}Q(b)x &= \frac{1}{4}Q(b)bx + \frac{1}{4}\langle b \circ b, x \rangle b \\(b \circ b)bx &= -\frac{1}{4}Q(b)bx + \frac{1}{4}\langle b \circ b, x \rangle b.\end{aligned}$$

Substituting all of this into 2.12 we get

$$\begin{aligned}\left(\frac{1}{4}Q(b) + \frac{1}{2}\right)y + \left(\xi + \frac{1}{2}\eta\right)b + (b \circ b)y + bx &= 0 \\-\frac{1}{2}Q(b)xb - \frac{1}{4}\xi Q(b)b - bx - \frac{1}{2}\xi b + \xi b + \frac{1}{2}\eta b - 2(b \circ b)bx - \xi(b \circ b)b + bx &= 0 \\-\frac{1}{2}Q(b)xb - \frac{1}{4}\xi Q(b)b + \frac{1}{2}\xi b + \frac{1}{2}\eta b - 2\left(-\frac{1}{4}Q(b)bx + \frac{1}{4}\langle b \circ b, x \rangle b\right) - \frac{1}{4}\xi Q(b)b &= 0 \\-\frac{1}{2}Q(b)xb - \frac{1}{4}\xi Q(b)b + \frac{1}{2}\xi b + \frac{1}{2}\eta b + \frac{1}{2}Q(b)xb - \frac{1}{2}\langle b \circ b, x \rangle b - \frac{1}{4}\xi Q(b)b &= 0 \\-\frac{1}{2}\xi Q(b)b + \frac{1}{2}\xi b + \frac{1}{2}\eta b - \frac{1}{2}\langle b \circ b, x \rangle b &= 0 \\\frac{1}{2}((-Q(b) + 1)\xi + \eta - \langle b \circ b, x \rangle)b &= 0,\end{aligned}$$

but we already need $(-Q(b) + 1)\xi + \eta - \langle b \circ b, x \rangle$ to be zero, so if $y = -2bx - \xi b$ then equations 2.9-2.12 hold if and only if

$$(-Q(b) + 1)\xi + \eta - \langle b \circ b, x \rangle = 0.$$

If we first observe that $\langle x', e \rangle = 0$ implies that

$$\langle \xi e + \eta u + x + y, e \rangle = \xi \langle e, e \rangle + \eta \langle u, e \rangle = 3\xi + \eta = 0.$$

Now we can see that the elements $x' \in E'_0$, are of the form

$$x' = \xi e - 3\xi u + x - (2bx + \xi b), \quad (2.18)$$

with $\xi \in k$ and $x \in E_0$. Now we can rewrite

$$(-Q(b) + 1)\xi + \eta - \langle b \circ b, x \rangle = -(Q(b) + 2)\xi - \langle b \circ b, x \rangle,$$

and so

$$-(Q(b) + 2)\xi - \langle b \circ b, x \rangle = 0. \quad (2.19)$$

Now we can compute $Q(x')$,

$$\begin{aligned} Q(x') &= Q(\xi e - 3\xi u + x - (2bx + \xi b)) \\ &= 3\xi^2 + Q(x) + 2\xi \langle b \circ b, x \rangle + Q(b)Q(x) + \xi^2 Q(b) \\ &= (Q(b) + 3)\xi^2 + (Q(b) + 1)Q(x) + 2\xi \langle b \circ b, x \rangle. \end{aligned}$$

We can consider two cases now $Q(b) = -2$ and $Q(b) \neq -2$.

Case 1: ($Q(b) = -2$ and $\langle u, v \rangle \neq 0$) If this is true we know that $\langle b \circ b, x \rangle = 0$, and we can take $x_1 = \frac{1}{2}b \circ b \in E_0$. Notice that

$$Q(x_1) = Q\left(\frac{1}{2}b \circ b\right) = \frac{1}{4}Q(b \circ b) = \frac{1}{4}\left(\frac{1}{4}\right)(-2)^2 = \frac{1}{4}.$$

and so now we have $x \in x_1^\perp \cap E_0 = C$. Now we can take

$$x'_1 = \frac{1}{2}(e - 3u - b),$$

which is letting $\xi = \frac{1}{2}$ and $x = 0$ in 2.18. If we look at the quadratic form on x'_1 , we get

$$\begin{aligned} Q(x'_1) &= Q\left(\frac{1}{2}(e - 3u - b)\right) \\ &= \frac{1}{4}Q(e - 3u - b) \\ &= \frac{1}{4}(\langle e - 3u, -b \rangle + Q(e - 3u) + Q(-b)) \\ &= \frac{1}{4}(-3\langle u, e \rangle + Q(e) + Q(-3u) - 2) \\ &= \frac{1}{4}\left(-3 + \frac{3}{2} + \frac{9}{2} - 2\right) \\ &= \frac{1}{4}. \end{aligned}$$

We should look at

$$\begin{aligned} \langle x', x'_1 \rangle &= \left\langle \xi e - 3\xi u + x - (2bx + \xi b), \frac{1}{2}(e - 3u - b) \right\rangle \\ &= \frac{1}{2}\xi \langle e, e \rangle - \frac{3}{2}\xi \langle e, u \rangle - \frac{3}{2}\xi \langle e, u \rangle + \frac{9}{2}\xi \langle e, u \rangle + \langle bx, b \rangle + \frac{1}{2}\xi \langle b, b \rangle \\ &= \frac{3}{2}\xi - \frac{3}{2}\xi - \frac{3}{2}\xi + \frac{9}{2}\xi - 2\xi \\ &= \xi, \end{aligned}$$

which is zero only when $\xi = 0$. So we have $C = (x'_1)^\perp \cap E'_0 = \{x' \in E'_0 \mid \xi = 0\}$, and the elements in C' consists of the elements $x - 2bx$ with $x \in C$. If we compute the norm on C we get

$$Q(x') = Q(x - 2bx) = \langle x, -2bx \rangle + Q(x) + Q(-2bx) = Q(x) + Q(b)Q(x) = (Q(b) + 1)Q(x).$$

So, since $\kappa(x) = \kappa(u)$, we have $\kappa(v) = \kappa(Q(b) + 1)\kappa(u)$. Now we can look at another case.

Case 2: ($Q(b) \neq -2, -1$ and $\langle u, v \rangle \neq 0$) Using 2.19 we have that

$$\xi = -(Q(b) + 1)^{-1} \langle b \circ b, x \rangle,$$

and so we can name elements of E'_0 like

$$x'(x) = -(Q(b) + 1)^{-1} \langle b \circ b, x \rangle (e - 3u + b) + x - 2bx,$$

using 2.18. If we now compute the quadratic form on E'_0 ,

$$\begin{aligned} Q(x'(x)) &= Q(-(Q(b) + 2)^{-1} \langle b \circ b, x \rangle (e - 3u + b) + x - 2bx) \\ &= Q(-(Q(b) + 2)^{-1} \langle b \circ b, x \rangle (e - 3u)) + Q(x) + Q((Q(b) + 2)^{-1} \langle b \circ b, x \rangle b - 2bx) \\ &= (Q(b) + 2)^{-2} \langle b \circ b, x \rangle^2 Q(e - 3u) + Q(x) + Q((Q(b) + 2)^{-1} \langle b \circ b, x \rangle b - 2bx) \end{aligned}$$

and so if we compute

$$\begin{aligned} Q(e - 3u) &= -3 \langle e, u \rangle + Q(e) + 9Q(u) \\ &= -3 + \frac{3}{2} + \frac{9}{2} \\ &= 3, \end{aligned}$$

and then

$$\begin{aligned} Q((Q(b) + 2)^{-1} \langle b \circ b, x \rangle b - 2bx) &= \langle (Q(b) + 2)^{-1} \langle b \circ b, x \rangle b, -2bx \rangle + Q((Q(b) + 2)^{-1} \langle b \circ b, x \rangle b) + 4Q(bx) \\ &= -2(Q(b) + 2)^{-1} \langle b \circ b, x \rangle^2 + (Q(b) + 2)^{-2} \langle b \circ b, x \rangle^2 Q(b) + Q(b)Q(x) \\ &= (Q(b) + 2)^{-2} (-2(Q(b) + 2) \langle b \circ b, x \rangle^2 + \langle b \circ b, x \rangle^2 Q(b)) + Q(b)Q(x) \\ &= (Q(b) + 2)^{-2} (-Q(b) - 4) \langle b \circ b, x \rangle^2 + Q(b)Q(x), \end{aligned}$$

so replacing these in the original equation,

$$\begin{aligned}
Q(x'(x)) &= (Q(b) + 2)^{-2} \langle b \circ b, x \rangle^2 Q(e - 3u) + Q(x) + Q((Q(b) + 2)^{-1} \langle b \circ b, x \rangle b - 2bx) \\
&= 3(Q(b) + 2)^{-2} \langle b \circ b, x \rangle^2 + Q(x) + (Q(b) + 2)^{-2} (-Q(b) - 4) \langle b \circ b, x \rangle^2 + Q(b)Q(x) \\
&= Q(b)Q(x) + Q(x) - (Q(b) + 2)^{-2} \langle b \circ b, x \rangle^2 - Q(b)(Q(b) + 2)^{-2} \langle b \circ b, x \rangle^2 \\
&= (Q(b) + 1) (Q(x) - (Q(b) + 2)^{-2} \langle b \circ b, x \rangle^2).
\end{aligned}$$

When can now compute the bilinear form on E'_0 ,

$$\langle x'(x), x'(y) \rangle = Q(x'(x) + x'(y)) - Q(x'(x)) - Q(x'(y))$$

$$\begin{aligned}
Q(x'(x) + x'(y)) &= Q(x'(x + y)) \\
&= (Q(b) + 1) (Q(x + y) - (Q(b) + 2)^{-2} \langle b \circ b, x + y \rangle^2) \\
&= (Q(b) + 1) (Q(x + y) - (Q(b) + 2)^{-2} (\langle b \circ b, x \rangle^2 + 2 \langle b \circ b, x \rangle \langle b \circ b, y \rangle + \langle b \circ b, y \rangle^2)),
\end{aligned}$$

so in the end we have

$$\begin{aligned}
\langle x'(x), x'(y) \rangle &= (Q(b) + 1) (Q(x + y) - (Q(b) + 2)^{-2} (\langle b \circ b, x \rangle^2 + 2 \langle b \circ b, x \rangle \langle b \circ b, y \rangle + \langle b \circ b, y \rangle^2)) \\
&\quad - (Q(b) + 1) (Q(x) - (Q(b) + 2)^{-2} \langle b \circ b, x \rangle^2) \\
&\quad - (Q(b) + 1) (Q(y) - (Q(b) + 2)^{-2} \langle b \circ b, y \rangle^2) \\
&= (Q(b) + 1) (Q(x + y) - Q(x) - Q(y) - 2(Q(b) + 2)^{-2} \langle b \circ b, x \rangle \langle b \circ b, y \rangle) \\
&= (Q(b) + 1) (\langle x, y \rangle - 2(Q(b) + 2)^{-2} \langle b \circ b, x \rangle \langle b \circ b, y \rangle).
\end{aligned}$$

If we now let $Q(b) \neq 0$ we can consider

$$x_1 = Q(b)^{-1}(b \circ b),$$

and

$$x'_1 = \frac{1}{2} Q(b)^{-1} (Q(b) + 1)^{-1} (Q(b) + 2) x'(b \circ b).$$

Notice that

$$Q(x_1) = Q(Q(b)^{-1}(b \circ b)) = Q(b)^{-2} \frac{1}{4} Q(b)^2 = \frac{1}{4},$$

and

$$Q(x'_1) = Q \left(\frac{1}{2} Q(b)^{-1} (Q(b) + 1)^{-1} (Q(b) + 2) x'(b \circ b) \right) = \frac{1}{4} Q(b)^{-2} (Q(b) + 1)^{-2} (Q(b) + 2)^2 Q(x'(b \circ b)),$$

and so we can look at

$$\begin{aligned} Q(x'(b \circ b)) &= (Q(b) + 1) (Q(b \circ b) - (Q(b) + 2)^{-2} \langle b \circ b, b \circ b \rangle^2) \\ &= (Q(b) + 1) \left(\frac{1}{4} Q(b)^2 - (Q(b) + 2)^{-2} (2Q(b \circ b))^2 \right) \\ &= (Q(b) + 1) \left(\frac{1}{4} Q(b)^2 - (Q(b) + 2)^{-2} \left(\frac{1}{2} Q(b)^2 \right)^2 \right) \\ &= (Q(b) + 1) \left(\frac{1}{4} Q(b)^2 - \frac{1}{4} (Q(b) + 2)^{-2} Q(b)^4 \right), \end{aligned}$$

so now

$$\begin{aligned} Q(x'_1) &= \frac{1}{4} Q(b)^{-2} (Q(b) + 1)^{-2} (Q(b) + 2)^2 (Q(b) + 1) \left(\frac{1}{4} Q(b)^2 - \frac{1}{4} (Q(b) + 2)^{-2} Q(b)^4 \right) \\ &= \frac{1}{16} Q(b)^{-2} (Q(b) + 1)^{-2} (Q(b) + 2)^2 (Q(b) + 1) (Q(b)^2 - (Q(b) + 2)^{-2} Q(b)^4) \\ &= \frac{1}{16} (Q(b) + 1)^{-1} ((Q(b) + 2)^2 - Q(b)^2) \\ &= \frac{1}{16} (Q(b) + 1)^{-1} 4(Q(b) + 1) \\ &= \frac{1}{4}. \end{aligned}$$

So the elements of E_0 that are orthogonal to x_1 make up the set $x_1^\perp \cap E_0 = \{x \in E_0 \mid \langle b \circ b, x \rangle = 0\}$, and that

$$C' = \{x'(x) \in E'_0 \mid x \in C\},$$

so as above we get $\kappa(v) = \kappa(Q(b) + 1)\kappa(u)$.

Now we notice that $\kappa(u) = \kappa(Q(a_+)Q(a_-))$ and that our norm classes are of the form

$$Q(a_+)Q(a_-) + Q(a_-)N(c_1) + Q(a_+)N(c_2),$$

for $c_1, c_2 \in C$, which are all values taken by $Q(t) \neq 0$, for some $t \in T$.

Now we must consider the case when v is a primitive idempotent such that $\langle u, v \rangle = 0$. Since we know that the change of u does not affect the possible values of Q on T , then for v it follows from Theorem 1.7.1 that $\kappa(v) = \kappa(Q(a_+))\kappa(u) = \kappa(Q(a_-))$, and so we still have $\kappa(v)$ taking only values of $Q(t)$ such that $t \in T$. \square

Now with the following a theorem and a corollary we have the classification of proper J -algebras over certain fields. We will only be interested in the case that C is an octonion algebra, since we are most interested in the J -algebra structure for the corresponding structure of its automorphism group.

Theorem, [25] 2.2.8.2. *For A and A' , two reduced J -algebras, over isomorphic composition algebras are themselves isomorphic if and only if the quadratic form Q on A and Q' on A' are equivalent. If this is true and u, u' are primitive idempotents in A and A' respectively, then there exists an isomorphism of A onto A' that maps u to u' if and only if $\kappa(u) = \kappa(u')$, i.e. they have the same norm classes.*

Corollary 2.2.8.3. *If C is a split composition algebra there is only one class of Albert algebras over C , and the automorphism group on such an Albert algebra acts transitively on the primitive idempotents.*

2.2.9 Isotopes of Albert algebras

First we will need to refer to two versions of the identity given in 2.2.2.1 (e) by setting $y = x$, $z = x$, and $u = y$ and then by substituting x for y and y for u and z . We end up with the following two identities

$$4(x \times x) \times (x \times y) = \det(x)y + 3\langle x, x, y \rangle x \quad (2.20)$$

$$2(x \times x) \times (y \times y) + 4(x \times y) \times (x \times y) = 3\langle x, x, y \rangle y + 4\langle x, y, y \rangle x. \quad (2.21)$$

We can use these to prove the following lemma.

Lemma, [23] 2.2.9.1. *If $a, x \in A$ and $\det(a) \neq 0$, then $a \times x = 0 \Rightarrow x = 0$.*

Proof. Let us assume $a \times x = 0$, and so

$$\begin{aligned} \langle a, a \times x \rangle &= 0 \\ &= \langle a, a, x \rangle \\ &= 3\langle a, a, x \rangle a, \end{aligned}$$

so using equation 2.20 gives us

$$\begin{aligned} 4(a \times a) \times (a \times x) &= \det(a)x + 3\langle a, a, x \rangle a \\ 0 &= \det(a)x + 0 \\ \det(a)x &= 0, \end{aligned}$$

and from the hypothesis we have that $\det(a)x \neq 0$ so x must be zero. \square

Now we want to define a bilinear form and its corresponding quadratic form and product for the isotope with respect to $a \in A$ with $\det(a) \neq 0$. So let $\det(a) = \lambda \neq 0$, and let us define the relation

$$\langle x, y \rangle_a = -6\lambda^{-1}\langle x, y, a \rangle + 9\lambda^{-2}\langle x, a, a \rangle \langle y, a, a \rangle,$$

for $x, y \in A$. Note first that this is indeed a bilinear form and then that

$$\begin{aligned} \langle x, a \rangle_a &= -6\lambda^{-1}\langle x, a, a \rangle + 9\lambda^{-2}\langle x, a, a \rangle \langle a, a, a \rangle \\ &= -6\lambda^{-1}\langle x, a, a \rangle + 9\lambda^{-1}\langle x, a, a \rangle \\ &= 3\lambda^{-1}\langle x, a, a \rangle. \end{aligned}$$

The form is nondegenerate by the nondegeneracy of the forms defining it. Also we can observe

$$\langle a, a \rangle_a = 3\lambda^{-1}\langle a, a, a \rangle = 3\lambda^{-1}\det(a) = 3.$$

We can rewrite the bilinear form as

$$\langle x, y \rangle_a = \langle x, -2\lambda^{-1}(y \times a) + 3\lambda^{-2}\langle y, a, a \rangle(a \times a) \rangle,$$

and notice that if $\langle x, y \rangle_a = 0$ then

$$\langle x \times a, y \rangle = -\frac{3}{2}\langle y, a, a \rangle \langle x \times a, a \rangle.$$

When $x \neq 0$ we have that $x \times a \neq 0$, so $y = \alpha a$. Now since $\langle x, a \rangle_a \neq 0$ we have that $y = 0$. So we can define a nondegenerate quadratic form that corresponds to \langle , \rangle_a that we will call

$$Q_a(x) = -3\lambda^{-1}\langle x, x, a \rangle + \frac{9}{2}\lambda^{-2}\langle x, a, a \rangle^2. \quad (2.22)$$

By replacing $\langle x, a, a \rangle$ with $\frac{1}{3}\lambda\langle x, a \rangle_a$ we can write

$$Q_a(x) = -3\lambda^{-1}\langle x, x, a \rangle + \frac{1}{2}\langle x, a \rangle_a^2. \quad (2.23)$$

Now we can define the multiplication in the isotope

$$x@y = 4\lambda^{-1}(x \times a) \times (y \times a) + \frac{1}{2}(\langle x, y \rangle_a - \langle x, a \rangle_a \langle y, a \rangle_a)a. \quad (2.24)$$

Proposition 2.2.9.2. *Let $a \in A$ with $\det(a) = \lambda \neq 0$, and define the algebra A_a , which has*

the vector space structure of A , but with norm Q_a and corresponding bilinear form $\langle \cdot, \cdot \rangle_a$, and product $@$. Then A_a is a J -algebra with a as the identity element, and A_a is reduced or proper if and only if A is reduced or proper, respectively.

Proof. First we want to show that $\langle x@y, z \rangle_a$ is symmetric in x, y , and z . So we look at

$$\begin{aligned}\langle x@y, z \rangle_a &= \left\langle 4\lambda^{-1}(x \times a) \times (y \times a) + \frac{1}{2}\langle x, y \rangle_a a - \frac{1}{2}\langle x, a \rangle_a \langle y, a \rangle_a a, z \right\rangle \\ &= 4\lambda^{-1} \underbrace{\langle (x \times a) \times (y \times a), z \rangle_a + \frac{1}{2}\langle x, y \rangle_a \langle z, a \rangle_a - \frac{1}{2}\langle x, a \rangle_a \langle y, a \rangle_a \langle z, a \rangle_a},\end{aligned}$$

where the underlined term is symmetric in x, y , and z . So let us take the first term

$$\begin{aligned}4\lambda^{-1} \langle (x \times a) \times (y \times a), z \rangle_a &= 4\lambda^{-1} \langle (x \times a) \times (y \times a), -2\lambda^{-1}(z \times a) + 3\lambda^{-2}\langle z, a \rangle_a (a \times a) \rangle \\ &= -8\lambda^{-2} \langle (x \times a) \times (y \times a), (z \times a) \rangle + 12\lambda^{-3} \langle (x \times a) \times (y \times a), \langle z, a \rangle_a (a \times a) \rangle \\ &= \underline{-24\lambda^{-2} \langle (x \times a), (y \times a), (z \times a) \rangle + 12\lambda^{-3} \langle (x \times a) \times (y \times a), \langle z, a \rangle_a (a \times a) \rangle},\end{aligned}$$

and notice that the underlined term is symmetric in x, y , and z . What remains from the original expression that has not been shown to be symmetric in all three variables, is

$$\frac{1}{2}\langle x, y \rangle_a \langle z, a \rangle_a + 12\lambda^{-3} \langle (x \times a) \times (y \times a), \langle z, a \rangle_a (a \times a) \rangle.$$

So let us expand

$$\begin{aligned}\frac{1}{2}\langle x, y \rangle_a \langle z, a \rangle_a &= \frac{1}{2} \langle x, -2\lambda^{-1}(y \times a) + 3\lambda^{-2}\langle y, a \rangle_a (a \times a) \rangle \langle z, a \rangle_a \\ &= -\lambda^{-1} \langle x, (y \times a) \rangle \langle z, a \rangle_a + \frac{3}{2}\lambda^{-2} \langle x, \langle y, a \rangle_a (a \times a) \rangle \langle z, a \rangle_a \\ &= \underline{-\lambda^{-1} \langle x, (y \times a) \rangle \langle z, a \rangle_a + \frac{27}{2}\lambda^{-3} \langle x, a \rangle_a \langle y, a \rangle_a \langle z, a \rangle_a},\end{aligned}$$

where the underlined term is symmetric in the three variables, and

$$-\lambda^{-1} \langle x, (y \times a) \rangle \langle z, a \rangle_a = -9\lambda^{-2} \langle x, y, a \rangle \langle z, a, a \rangle.$$

If we now look at

$$\begin{aligned}
12\lambda^{-3} \langle (x \times a) \times (y \times a), \langle z, a, a \rangle (a \times a) \rangle &= 12\lambda^{-3} \langle (x \times a) \times (y \times a), (a \times a) \rangle \langle z, a, a \rangle \\
&= 12\lambda^{-3} \langle (a \times a) \times (a \times x), (y \times a) \rangle \langle z, a, a \rangle \\
&= 3\lambda^{-3} \langle \det(a)x + 3\langle x, a, a \rangle a, (y \times a) \rangle \langle z, a, a \rangle \\
&= 9\lambda^{-2} \langle x, y, a \rangle \langle z, a, a \rangle + \underline{27\lambda^{-3} \langle x, a, a \rangle \langle y, a, a \rangle \langle z, a, a \rangle},
\end{aligned}$$

where the underlined term is symmetric in x, y , and z and so the only terms left not symmetric in our variables are

$$3\lambda^{-2} \langle x, y, a \rangle \langle z, a, a \rangle - 3\lambda^{-2} \langle x, y, a \rangle \langle z, a, a \rangle = 0.$$

Therefore $\langle x @ y, z \rangle$ is symmetric in x, y , and z .

Now we want to work toward computing \det_a , and we will start by defining \times_a . We will use

$$\langle x \times_a y, z \rangle = \langle x, y, z \rangle_a,$$

to define our crossproduct in the isotope A_a , and so we can use 2.2.2.1 to compute

$$x \times_a x = x @ x - \frac{1}{2} \langle x, a \rangle_a x - \frac{1}{2} \langle x, a \rangle_a x - \frac{1}{2} \langle x, x \rangle_a a + \frac{1}{2} \langle x, a \rangle_a \langle x, a \rangle_a a.$$

We can compute

$$x @ x = 4\lambda^{-1} (x \times a) \times (x \times a) + \frac{1}{2} \langle x, x \rangle_a a + \frac{1}{2} \langle x, a \rangle_a \langle x, a \rangle_a a,$$

and when we substitute this in above we have that

$$x \times_a x = 4\lambda^{-1} (x \times a) \times (x \times a) - \langle x, a \rangle_a x.$$

We can then notice that

$$4(x \times a) \times (x \times a) = 3\langle x, x, a \rangle a + 3\langle x, a, a \rangle x - 2(x \times x) \times (a \times a),$$

so we have that

$$x \times_a x = 3\lambda^{-1} \langle x, x, a \rangle a + 3\lambda^{-1} \langle x, a, a \rangle x - 2\lambda^{-1} (x \times x) \times (a \times a) - \langle x, a \rangle_a x$$

and since $3\lambda^{-1}\langle x, a, a \rangle x = \langle x, a \rangle_a x$ and $3\lambda^{-1}\langle x, x, a \rangle = \frac{1}{2}\langle x, a \rangle_a^2 - Q_a(x)$, we have

$$x \times_a x = -2\lambda^{-1}(x \times x) \times (a \times a) - \left(Q_a(x) - \frac{1}{2}\langle x, a \rangle_a^2 \right) a. \quad (2.25)$$

So we can write $3\det_a(x) = \langle x \times_a x, x \rangle_a$, and arrive at

$$\det_a(x) = -2\lambda^{-1}\langle x \times_a x, x, a \rangle + 3\lambda^{-2}\langle x \times_a x, a, a \rangle \langle x, a, a \rangle. \quad (2.26)$$

If we now compute

$$\begin{aligned} \langle x \times_a x, x, a \rangle &= -2\lambda^{-1}\langle (x \times x) \times (a \times a), x, a \rangle - \left(Q_a(x) - \frac{1}{2}\langle x, a \rangle_a^2 \right) \langle x, a, a \rangle \\ &= -2\lambda^{-1}\langle (x \times x) \times (x \times a), x, x \rangle - \left(Q_a(x) - \frac{1}{2}\langle x, a \rangle_a^2 \right) \langle x, a, a \rangle, \end{aligned}$$

and if we then notice

$$(x \times x) \times (x \times a) = \frac{1}{4}\det(x)a + \frac{3}{4}\langle x, x, a \rangle x,$$

we have

$$\begin{aligned} \langle x \times_a x, x, a \rangle &= -\frac{1}{2}\lambda^{-1}\det(x)\langle a, a, a \rangle - \frac{3}{4}\lambda^{-1}\langle x, x, a \rangle \langle x, a, a \rangle - \left(Q_a(x) - \frac{1}{2}\langle x, a \rangle_a^2 \right) \langle x, a, a \rangle \\ &= -\frac{1}{2}\det(x) - \left(\frac{1}{2}Q_a(x) - \frac{1}{4}\langle x, a \rangle_a^2 \right) \langle x, a, a \rangle \\ &= -\frac{1}{2}\det(x) - \frac{1}{6}\lambda \left(Q_a(x) - \frac{1}{2}\langle x, a \rangle_a^2 \right) \langle x, a \rangle_a. \end{aligned}$$

Now if we compute

$$\begin{aligned} \langle x \times_a x, a, a \rangle &= -2\lambda^{-1}\langle (x \times x) \times (a \times a), a, a \rangle - \left(Q_a(x) - \frac{1}{2}\langle x, a \rangle_a^2 \right) \langle a, a, a \rangle \\ &= -2\lambda^{-1}\langle (x \times x) \times (a \times a), a, a \rangle - \lambda \left(Q_a(x) - \frac{1}{2}\langle x, a \rangle_a^2 \right), \end{aligned}$$

we can then see that

$$\begin{aligned}
2(x \times x) \times (a \times a) &= 3\langle x, x, a \rangle a + 3\langle x, a, a \rangle x - 4(x \times a) \times (x \times a) \\
&= -3\lambda^{-1}\langle x, x, a \rangle \langle a, a, a \rangle - 3\lambda^{-1}\langle x, a, a \rangle^2 + 4\lambda^{-1}\langle (x \times a) \times (x \times a), a, a \rangle \\
&= -3\langle x, x, a \rangle - 3\lambda^{-1}\langle x, a, a \rangle^2 + \langle x, x, a \rangle + 3\lambda^{-1}\langle x, a, a \rangle^2 \\
&= -2\langle x, x, a \rangle \\
&= \frac{2}{3}\lambda Q_a(x) - \frac{1}{3}\lambda\langle x, a \rangle_a^2,
\end{aligned}$$

and so

$$\begin{aligned}
\langle x \times_a x, a, a \rangle &= \frac{2}{3}\lambda Q_a(x) - \frac{1}{3}\lambda\langle x, a \rangle_a^2 - \lambda Q_a(x) + \frac{1}{2}\lambda\langle x, a \rangle_a^2 \\
&= -\frac{1}{3}\lambda \left(Q_a(x) - \frac{1}{2}\langle x, a \rangle_a^2 \right).
\end{aligned}$$

We now substitute these expressions into the equation for $\det_a(x)$, and get

$$\begin{aligned}
\det_a(x) &= -2\lambda^{-1} \left(-\frac{1}{2}\det(x) - \frac{1}{6}\lambda \left(Q_a(x) - \frac{1}{2}\langle x, a \rangle_a^2 \right) \langle x, a \rangle_a \right) \\
&\quad + 3\lambda^{-2} \left(-\frac{1}{3}\lambda \left(Q_a(x) - \frac{1}{2}\langle x, a \rangle_a^2 \right) \right) \langle x, a, a \rangle \\
&= \lambda^{-1}\det(x) + \frac{1}{3} \left(Q_a(x) - \frac{1}{2}\langle x, a \rangle_a^2 \right) - \frac{1}{3} \left(Q_a(x) - \frac{1}{2}\langle x, a \rangle_a^2 \right) (3\lambda^{-1}\langle x, a, a \rangle) \\
&= \lambda^{-1}\det(x).
\end{aligned}$$

In the case where $a = e$ we get $\det(e) = 1 = \lambda$, so $\det_e(x) = \det(x)$ for $x \in A$.

If we look at

$$\begin{aligned}
\langle x, y \rangle_e &= -6\langle x, y, e \rangle + 9\langle x, e, e \rangle \langle y, e, e \rangle \\
&= -2\langle x \times e, y \rangle + \langle x, e \times e \rangle \langle y, e \times e \rangle,
\end{aligned}$$

so if we compute

$$\begin{aligned}
x \times e &= x - \frac{1}{2}\langle x, e \rangle e - \frac{1}{2}\langle e, e \rangle x - \frac{1}{2}\langle x, e \rangle e + \frac{1}{2}\langle x, e \rangle \langle e, e \rangle e \\
&= -\frac{1}{2}x - \langle x, e \rangle e + \frac{3}{2}\langle x, e \rangle e \\
&= -\frac{1}{2}x + \frac{1}{2}\langle x, e \rangle e,
\end{aligned}$$

so we end up with

$$\begin{aligned}
\langle x, y \rangle_e &= -2 \left\langle -\frac{1}{2}x + \frac{1}{2}\langle x, e \rangle e, \right\rangle + \langle x, e \rangle \langle y, e \rangle \\
&= \langle x, y \rangle - \langle x, e \rangle \langle y, e \rangle + \langle x, e \rangle \langle y, e \rangle \\
&= \langle x, y \rangle.
\end{aligned}$$

The last thing to see that $x@y = xy$ so we look at

$$x@y = 4(x \times e) \times (y \times e) + \frac{1}{2}(\langle x, y \rangle - \langle x, e \rangle \langle y, e \rangle)e,$$

and we start by computing

$$\begin{aligned}
4(x \times e) \times (y \times e) &= 4 \left(-\frac{1}{2}x + \frac{1}{2}\langle x, e \rangle e \right) \times \left(-\frac{1}{2}y + \frac{1}{2}\langle y, e \rangle e \right) \\
&= (x \times y) - \langle y, e \rangle (x \times e) - \langle x, e \rangle (y \times e) + \langle x, e \rangle \langle y, e \rangle e \\
&= x \times y + \frac{1}{2}\langle x, e \rangle y + \frac{1}{2}\langle y, e \rangle x - \langle x, e \rangle \langle y, e \rangle e + \langle x, e \rangle \langle y, e \rangle e \\
&= x \times y + \frac{1}{2}\langle x, e \rangle y + \frac{1}{2}\langle y, e \rangle x,
\end{aligned}$$

which leaves us with

$$\begin{aligned}
x@y &= (x \times y) + \frac{1}{2}\langle x, e \rangle y + \frac{1}{2}\langle y, e \rangle x + \frac{1}{2}\langle x, y \rangle e - \frac{1}{2}\langle x, e \rangle \langle y, e \rangle e \\
&= xy.
\end{aligned}$$

We now have that $A_e = A$, and A_a and A have the same determinant up to nonzero factor, so by 2.2.5.7, 2.2.5.6, 2.2.7.2 and 2.2.6.2 we have that A_a is reduced or proper J -algebra is and only if A is, where the isomorphism is the identity map. \square

Proposition, [10] 2.2.9.3. *Let A be a proper J -algebra, and let $a, b \in A$, with $\det(a) \det(b) \neq 0$, then the following are equivalent:*

(a) $A_a \cong A_b$,

(b) there exists a linear transformation $t : A \rightarrow A$ such that $t(a) = b$ and

$$\langle t(x), t(y), t(z) \rangle = \det(a)^{-1} \det(b) \langle x, y, z \rangle,$$

(c) if A is reduced $\det(a)^{-1} \langle x, y, a \rangle$ and $\det(b)^{-1} \langle x, y, b \rangle$ are equivalent.

Proposition, [25] 2.2.9.4. *If A is a J -algebra of dimension greater than 2, then a linear transformation t of A is an automorphism if and only if $t(e) = e$ and $\det(t(x)) = \det(x)$ for $x \in A$.*

Proof. Using 2.2.4.11 we get the \Leftarrow direction. So let t be a linear transformation that leaves e invariant and leaves \det invariant. Since $A_e = A$ we can write

$$\langle x, y \rangle = -6\langle x, y, e \rangle + 9\langle x, e, e \rangle \langle y, e, e \rangle,$$

and by 2.4 if t leaves \det invariant it leaves $\langle \cdot, \cdot, \cdot \rangle$ invariant, and so we then have that $\langle \cdot, \cdot \rangle$ is left invariant by t . If we then notice

$$\begin{aligned} \langle t(x \times y), t(z) \rangle &= \langle x \times y, z \rangle \\ &= \langle x, y, z \rangle \\ &= \langle t(x), t(y), t(z) \rangle \\ &= \langle t(x) \times t(y), t(z) \rangle, \end{aligned}$$

we see that $t(x \times y) = t(x) \times t(y)$ since the form is nondegenerate. We can compute

$$\begin{aligned} t(x \times y) &= t\left(xy - \frac{1}{2}\langle x, e \rangle y - \frac{1}{2}\langle y, e \rangle x - \frac{1}{2}\langle x, y \rangle e + \frac{1}{2}\langle x, e \rangle \langle y, e \rangle e\right) \\ &= t(xy) - \frac{1}{2}\langle x, e \rangle t(y) - \frac{1}{2}\langle y, e \rangle t(x) - \frac{1}{2}\langle x, y \rangle t(e) + \frac{1}{2}\langle x, e \rangle \langle y, e \rangle t(e) \\ &= t(xy) - \frac{1}{2}\langle x, e \rangle t(y) - \frac{1}{2}\langle y, e \rangle t(x) - \frac{1}{2}\langle x, y \rangle e + \frac{1}{2}\langle x, e \rangle \langle y, e \rangle e, \end{aligned}$$

and then

$$\begin{aligned} t(x) \times t(y) &= t(x)t(y) - \frac{1}{2}\langle t(x), e \rangle t(y) - \frac{1}{2}\langle t(y), e \rangle t(x) - \frac{1}{2}\langle t(x), t(y) \rangle e + \frac{1}{2}\langle t(x), e \rangle \langle t(y), e \rangle e \\ &= t(x)t(y) - \frac{1}{2}\langle t(x), t(e) \rangle t(y) - \frac{1}{2}\langle t(y), t(e) \rangle t(x) \\ &\quad - \frac{1}{2}\langle t(x), t(y) \rangle e + \frac{1}{2}\langle t(x), t(e) \rangle \langle t(y), t(e) \rangle e \\ &= t(x)t(y) - \frac{1}{2}\langle x, e \rangle t(y) - \frac{1}{2}\langle y, e \rangle t(x) - \frac{1}{2}\langle x, y \rangle e + \frac{1}{2}\langle x, e \rangle \langle y, e \rangle e, \end{aligned}$$

which implies that $t(xy) = t(x)t(y)$. □

2.2.10 Automorphisms of an Albert Algebra

We want to consider automorphisms of an Albert algebra A , which we will denote $\text{Aut}(A)$. To do this we first consider the automorphisms that leave a fixed primitive unipotent element in

A invariant. We will call this group $\text{Aut}(A)_u$. Recalling that $A = ku \oplus k(e - u) \oplus E_0 \oplus E_1$ where E_0 and E_1 are the eigenspaces corresponding to the eigenvalues 0 and $\frac{1}{2}$ respectively. An automorphism $s \in \text{Aut}(A)_u$ must fix these two spaces as well

$$us(x) = s(u)s(x) = s(ux) = s(0) = 0, \quad us(y) = s(u)s(y) = s(uy) = s\left(\frac{1}{2}y\right) = \frac{1}{2}s(y),$$

for $x \in E_0$ and $y \in E_1$. So we have that s is orthogonal. Since s is an automorphism we know it induces orthogonal transformations t on E_0 and v on E_1 so that for $x \in E_0$ and $y \in E_0$,

$$v(xy) = t(x)v(y).$$

Proposition 2.2.10.1. *Let A be a proper reduced J -algebra and u a primitive idempotent and E_0 and E_1 the zero and half space of u respectively, then for every rotation t of E_0 there is a similarity v of E_1 such that*

$$v(xy) = t(x)v(y),$$

where $x \in E_0$ and $y \in E_1$. In fact when $t = s_{a_1} \cdots s_{a_{2h}}$ then we can take $v(y) = a_1(\cdots(a_{2h}(y))\cdots)$ for $y \in E_0$.

Proof. So let us look at

$$\begin{aligned} -s_a(x)ay &= -(x - Q(a)^{-1}\langle a, x \rangle a)(ay) \\ &= (-x + Q(a)^{-1}\langle a, x \rangle a)(ay) \\ &= -x(ay) + \frac{1}{4}(4Q(a)^{-1}\langle a, x \rangle a(ay)), \end{aligned}$$

and we know that $y = 4Q(a)^{-1}a(ay)$, so

$$\begin{aligned} -s_a(x)ay &= -x(ay) + \frac{1}{4}\langle a, x \rangle y \\ &= a(xy), \end{aligned}$$

when $x \in E_0$ and $y \in E_1$. If t is a rotation then t is the product of an even number of reflections. Notice in general that if t_1, v_1 and t_2, v_2 satisfy

$$v_1(xy) = t_1(x)v_1(y) \quad v_2(xy) = t_2(x)v_2(y),$$

then

$$v_1v_2(xy) = v_1(t_2(x)v_2(y)) = t_1(t_2(x))v_1(v_2(y)).$$

So for t any product of an even number of reflections we get

$$v(xy) = a_1(\cdots(a_{2h}(xy))\cdots) = s_{a_1} \cdots s_{a_{2h}}(x)a_1(\cdots(a_{2h}(y))\cdots) = t(x)v(y).$$

Multiplication by $a \in E_0$ is a similarity since

$$Q(ay) = \frac{1}{4}Q(a)Q(y),$$

it has multiplier $\frac{1}{4}Q(a)$. It is also easy to show that the spinor norm of t is the same as the square class of the multiplier of v . \square

Proposition, [25] 2.2.10.2. *If A is an Albert algebra then for any rotation t of E_0 the similarity v of E_1 such that the previous proposition is satisfied is unique up to multiplication.*

Proof. In order to prove this it suffices to show that for the $t = \text{id}$ case $v = \lambda \text{id}$.

In Corollary 1.4.5 we found a representation of the Clifford algebra $\text{Cl}(Q; E_0)$ in E_0 where $\varphi(x)(y) = 2xy$. The $\dim(E_0) = 9$, since $E_0 = C \oplus kx_1$. The $\dim(E_1) = 16$, since $A = ku \oplus k(e - u) \oplus E_0 \oplus E_1$ has dimension 27. So the $\dim(\text{Cl}(Q; E_0)) = 2^9 = 2^8 + 2^8$. When taken over the algebraic closure of k this is the sum of two full matrix algebras each of dimension $2^8 = 16^2$, so the irreducible representations are of dimension 16. This tells us that φ is absolutely irreducible since the $\dim(E_1) = 16$. Now if we notice

$$\varphi(v(xy)) = \varphi(x)(v(y)) = 2xv(y) = v(2xy) = v(\varphi(x)(y)),$$

then φ and v commute so $v = \lambda \text{id}$ by Schur's lemma. \square

Proposition, [25] 2.2.10.3. *If t is an orthogonal transformation of E_0 that is not a rotation, then there is not a similarity v satisfying 2.2.10.1.*

Proof. Assume that t is an orthogonal transformation of E_0 and is not a rotation. So we can write $t = s_a t_1$ where s_a is a reflection and t_1 is a rotation, but that there exists v such that 2.2.10.1 holds. We know that t_1^{-1} is a rotation so there must exist a similarity of E_1 such that 2.2.10.1 holds, we will call it v_1 . We will let v_a be the similarity that exists for $-s_a$ per 2.2.10.1. Now we can define $w = vv_1v_a$,

$$w(xy) = vv_1v_a(xy) = -s_a t_1 t_1^{-1} s_a(x) w(y) = -xw(y),$$

for $x \in E_0$ and $y \in E_1$. Recall again the representation of $\text{Cl}(Q; E_0)$ in E_1 , φ , and notice

$$w(xy) = w\left(\frac{1}{2}\varphi(x)(y)\right) = \frac{1}{2}w(\varphi(x)(y)) = -xw(y) = -\frac{1}{2}\varphi(x)(w(y)),$$

so we see that

$$w\varphi(x) = -\varphi(x)w.$$

So w^2 must commute with φ so using Schur's lemma again we have $w^2 = \lambda \text{id}$ which says we can write $E_1 = W_+ \oplus W_-$, the sum of the eigenspaces of w with eigenvalues $\pm\lambda$. We have seen above that for $x \in E_0$, $\varphi(x)$ exchanges these spaces so they must have equal dimension. The Clifford algebra $\text{Cl}^+(Q; E_0)$ leaves the eigenspaces invariant and for the Albert algebra $\dim(\text{Cl}^+(Q; E_0)) = 2^8$ a full matrix algebra over the algebraic closure of k so it has irreducible representations of dimension 16, but the restriction of φ to the even Clifford algebra has invariant subspaces W_+ and W_- of dimension 8, a contradiction. \square

Lemma, [25] 2.2.10.4. *In a proper reduced J-algebra we have for any rotation t of E_0 and similarity v of E_1 satisfying equation $v(xy) = t(x)v(y)$,*

$$v(y) \circ v(y) = n(v)t(y \circ y).$$

Proof. If we replace x in $v(xy) = t(x)v(y)$ with $y \circ y$ we see that

$$v((y \circ y)y) = v\left(\frac{1}{4}Q(y)y\right) = \frac{1}{4}Q(y)v(y),$$

and on the other hand

$$v((y \circ y)y) = t(y \circ y)v(y),$$

so

$$t(y \circ y)v(y) = \frac{1}{4}Q(y)v(y).$$

Now we can take $x_1 = Q(y)^{-1}t(y \circ y) \in E_0$ when $Q(y) \neq 0$, and we get

$$Q(x_1) = Q(Q(y)^{-1}t(y \circ y)) = Q(y)^{-2}Q(t(y \circ y)) = Q(y)^{-2}Q(y \circ y) = Q(y)^{-2}Q(y)^2 \frac{1}{4} = \frac{1}{4}.$$

Also notice that

$$x_1v(y) = Q(y)^{-1}t(y \circ y)v(y) = \frac{1}{4}v(y),$$

which puts $v(y) \in E_+$ if we decompose E_1 with x_1 , and by 2.2.5.2 (a) we get that

$$v(y) \circ v(y) = \frac{1}{2}\langle v(y), v(y) \rangle x_1 = n(v)Q(y)Q(y)^{-1}t(y \circ y) = n(v)t(y \circ y).$$

Over the algebraic closure of k this holds for all of E_1 by Zariski continuity. \square

Theorem, [25] 2.2.10.5. *Let A be a reduced Albert algebra over k and u a primitive idempotent in A with eigenspaces E_0 and E_1 , corresponding to eigenvalues 0 and $\frac{1}{2}$ respectively, then the*

restriction map

$$Res_{E_0} : s \mapsto s|_{E_0},$$

is a homomorphism of $\text{Aut}(A)_u$ onto the reduced orthogonal group $O'(Q; E_0)$ with a kernel of order 2.

Proof. Let $s \in \text{Aut}(A)_u$, then s leaves E_0 and E_1 invariant and induces orthogonal transformations t and v on E_0 and E_1 respectively. Since s is an automorphism it satisfies $s(xy) = t(x)v(y)$ for $x \in E_0$ and $y \in E_1$. And we have already observed that t must be a rotation. Also, notice that since v is orthogonal $n(v) = 1$, which must be the same as the spinor norm of t , so $\sigma(t) = 1$. Then $s|_{E_0}$ is in the kernel of the spinor norm map, which means that $s|_{E_0} \in O'(Q; E_0)$ and is a homomorphism by assumption.

To prove surjectivity let us consider $t \in O'(Q; E_0)$ so $\sigma(t) = 1 = \nu(v)$ there exists a similarity v to be such that $t(x)v(y) = v(xy)$ and so $n(v) = 1$, so here $O'(Q; E_0)$ must contain only reflections and v must be orthogonal. We can define the linear transformations $s : A \rightarrow A$, so that

$$s(e) = e, \quad s(u) = u, \quad s|_{E_0} = t, \quad s|_{E_1} = v.$$

In order to show that this is an automorphism we need only to show that $s(z^2) = s(z)^2$ for $z \in A$, since A is a commutative algebra. So for $z = ae + bu + x + y \in A$

$$\begin{aligned} s(z^2) &= s(a^2e + 2abu + 2ax + b^2u + by + 2ay + x^2 + 2xy + y^2) \\ &= a^2e + 2abu + b^2u + 2as(x) + bs(y) + 2as(y) + s(x^2) + 2s(xy) + s(y^2) \\ &= a^2e + 2abu + b^2u + 2at(x) + bv(y) + 2av(y) + 2v(xy) + Q(x)(e - u) \\ &\quad + \frac{1}{2}Q(y)(e + u) + v(y) \circ v(y), \end{aligned}$$

since

$$s(x^2) = \frac{1}{2}\langle x, x \rangle s(e - u) = Q(x)(e - u),$$

and

$$\begin{aligned} s(y^2) &= \frac{1}{4}\langle y, y \rangle s(e + u) + t(y \circ y) \\ &= \frac{1}{2}Q(y)(e + u) + n(v)^{-1}v(y) \circ v(y) \\ &= \frac{1}{2}Q(y)(e + u) + v(y) \circ v(y). \end{aligned}$$

Now if we compute

$$\begin{aligned}
s(z)^2 &= (ae + bu + s(x) + s(y))^2 \\
&= a^2e + 2abu + b^2u + 2at(x) + bv(y) + 2av(y) + 2t(x)v(y) + t(x)^2 + v(y)^2 \\
&= a^2e + 2abu + b^2u + 2at(x) + bv(y) + 2av(y) + 2v(xy) + Q(x)(e - u) \\
&\quad + \frac{1}{2}Q(y)(e + u) + v(y) \circ v(y),
\end{aligned}$$

since

$$t(x)^2 = \frac{1}{2}\langle t(x), t(x) \rangle(e - u) = Q(x)(e - u),$$

and

$$v(y)^2 = \frac{1}{4}\langle v(y), v(y) \rangle(e + u) + v(y) \circ v(y) = \frac{1}{2}Q(y)(e + u) + v(y) \circ v(y).$$

So we have that $s \in \text{Aut}(A)_u$ and $\text{Res} : \text{Aut}(A)_u \rightarrow O'(Q, E_0)$ is surjective.

For the kernel of Res we look at $t = \text{id}$. From before we know that $v = \lambda \text{id}$ and since $n(v) = 1$ $v = \pm \lambda \text{id}$. So the kernel consists of two elements. \square

Now we want to show that the square classes of the multipliers in E_1 coincide with the group of spinor norms of rotations in E_0 .

Lemma, [25] 2.2.10.6. *Let A be a proper reduced J -algebra and $y, z \in E_1$ where $Q(y)Q(z) \neq 0$, then there exist elements $a_1, a_2, \dots, a_l \in E_0$ such that*

$$z = a_1(a_2(\cdots(a_ly)\cdots)),$$

and we can always do so such that l is even.

Proof. In our first case we will take $y \circ y = \lambda z \circ z$ for some $\lambda \in k^*$, so

$$Q(y \circ y) = \frac{1}{4}Q(y)^2 = \frac{1}{4}\lambda^2Q(z)^2 = \lambda^2Q(z \circ z),$$

and we have that

$$Q(y)^2 = \lambda^2Q(z)^2.$$

This tells us that $Q(y) = \pm \lambda Q(z)$. Then we can take $x_1 = Q(z)^{-1}z \circ z$, which gives us

$$Q(x_1) = Q(Q(z)^{-1}z \circ z) = Q(z)^{-2}Q(z \circ z) = \frac{1}{4}.$$

We also notice that

$$x_1z = Q(z)^{-1}(z \circ z)z = Q(z)^{-1}\frac{1}{4}Q(z)z = \frac{1}{4}z,$$

so $z \in E_+$. We know that $y \in E_\pm$. If $y \in E_-$ then by 2.2.5.3 there exists a $c \in C$ such that $z = cy$. If $y \in E_+$ we take any $c \in C$ with $Q(c) \neq 0$ then we have $y' = cy \in E_-$ and so there exists a $c' \in C$ such that $z = c'y' = c'cy$.

For our other case let us assume $y \circ y$ and $z \circ z$ are linearly independent, and we will obtain all other cases by linearity. We have

$$Q(y \circ y) = \frac{1}{4}Q(y)^2 = (Q(y)Q(z)^{-1})^2 \frac{1}{4}Q(z)^2 = (Q(y)Q(z)^{-1})^2 Q(z \circ z),$$

so let us call $Q(y)Q(z)^{-1} = \alpha$, then

$$Q(y \circ y) = \alpha^2 Q(z \circ z) = Q(\alpha z \circ z).$$

Now by Witt's Theorem there is an orthogonal transformation t of E_0 where $t(y \circ y) = \alpha z \circ z$. Since $\dim(E_0) > 1$ we can assume that t is a rotation. By 2.2.10.1 $t = s_{a_1} \cdots s_{a_l}$ where l is even means that there exists v a similarity of E_1 such that

$$v(w) = a_1(\cdots (a_l w) \cdots),$$

by 2.2.10.4

$$v(y) \circ v(y) = n(v)t(y \circ y) = n(v)\alpha z \circ z,$$

and we can take $\lambda = \alpha$ and by the first case $v(y)$ can be transformed to z by one or two multiplications by an element of E_0 .

If we end up with an odd number of multiplications we can multiply by $4x_1$, since

$$4x_1 z = 4Q(z)^{-1}(z \circ z)z = 4Q(z)^{-1}\frac{1}{4}Q(z)z = z.$$

□

We can now show that for every similarity v of E_1 there exists a rotation t of E_0 such that $\nu(v) = \sigma(t)$, and then using 2.2.10.1 we can prove the following proposition.

Proposition, [10] 2.2.10.7. *Let A be a proper reduced J -algebra and u a primitive idempotent of A with E_0 and E_1 its 0 and $\frac{1}{2}$ spaces, then the group of spinor norms of rotations of E_0 with respect to the restriction of Q to E_0 coincides with the group of square classes of multipliers of similarities of E_1 with respect to Q restricted to E_1 .*

Proof. With 2.2.10.1 in mind we need only to show that for every similarity v of E_1 there is a rotation t of E_0 such that $\nu(v) = \sigma(t)$. So let us take $y \in E_1$ such that $Q(y) \neq 0$. From the previous Lemma we have

$$z = a_1(\cdots (a_l y) \cdots) = v(y),$$

where l is even. Then $n(v) = Q(a_1) \cdots Q(a_l)$, so for $t = s_{a_1} \cdots s_{a_l}$ a rotation we have

$$\sigma(t) = Q(a_1) \cdots Q(a_l),$$

and so $\sigma(t) = \nu(v)$. \square

In the next proof we want to consider the algebraic groups in the setting when $k = K$ an algebraically closed field. We can get the Albert algebra over k to be defined over K by $A_K = K \otimes_k A$, and we can think of A as a subset of A_K . We will denote the algebraic group $\text{Aut}(A_K)_u$. Recall the $\text{Spin}(Q; E_0)$ is the algebraic group that is a subset of the even Clifford group $\Gamma^+(Q; E_0)$ with $a \in \Gamma^+(Q; E_0)$ being also in $\text{Spin}(Q; E_0)$ if and only if $Q(a) = 1$, where $Q(a_1 \circ a_2 \circ \cdots \circ a_{2h}) = Q(a_1)Q(a_2) \cdots Q(a_{2h})$, with \circ the Clifford algebra product.

Proposition 2.2.10.8. $\text{Aut}(A_K)_u$ is isomorphic to $\text{Spin}(Q; E_0)$.

Proof. Let $\psi : \text{Spin}(Q; E_0) \rightarrow \text{Aut}(A_K)_u$ that takes $s \mapsto \psi(s)$, where $s = a_1 \circ a_2 \circ \cdots \circ a_{2h}$ and $\psi(s)$ is the linear map of A_K that leaves e and u invariant and

$$\psi(s)|_{E_0} = s_{a_1} s_{a_2} \cdots s_{a_{2h}} = t, \text{ and } \psi(s)|_{E_1}(y) = a_1(a_2(\cdots(a_{2h}y)\cdots)) = v(y).$$

We want to show that this is a homomorphism of algebraic groups and then that it is bijective homomorphism of algebraic groups, and use results from the theory of algebraic groups to conclude that ψ is an isomorphism (as it also must preserve the variety structure).

It is clearly a homomorphism. By Theorem 1.10.5 and we have that ψ is bijective, with both $\text{Spin}(Q; E_0)$ and $\text{Aut}(A_K)_u$ being double covers of $\text{O}'(Q; E_0)$, which coincides with $\text{SO}(Q; E_0)$ over an algebraically closed field. Let the map $\pi : \text{Spin}(Q; E_0) \rightarrow \text{SO}(Q; E_0)$ be the canonical map taking

$$a_1 \circ \cdots \circ a_{2h} \mapsto s_{a_1} \cdots s_{a_{2h}},$$

then $\pi(s) = \text{Res}_{E_0}(\psi(s))$. Since $\text{char}(k) \neq 2$ we have that π is a separable homomorphism, and so by [24], 4.3.7 we have that $d\pi$ is bijective on the Lie algebra. This tells us that $d\psi$ is bijective and so by [24] 5.3.3 we have that ψ is a isomorphism. \square

Corollary 2.2.10.9. The Lie algebra of $\text{Aut}(A_K)_u$, denoted $L = L(\text{Aut}(A_K)_u)$, is the space of derivations of A_K with $du = 0$.

Proof. From [7], p. 77 we know that L is contained by the space S of derivations of A_K such that $du = 0$.

If $d \in S$ then $de = du = 0$, and $d(E_i) \subset E_i$ for $i = 0, 1$. So we can write $d_i = d|_{E_i}$. From

Lemma 1.4.3 we get for $x, y \in E_0$

$$\begin{aligned} d(xy) &= d\left(\frac{1}{2}\langle x, y \rangle(e - u)\right) \\ &= \frac{1}{2}\langle x, y \rangle d(e - u) \\ &= 0, \end{aligned}$$

on one hand, and then $d(xy) = d(x)y + xd(y)$, so if we compute

$$d(x)y = \frac{1}{2}\langle d(x), y \rangle(e - u),$$

and

$$xd(y) = \frac{1}{2}\langle x, d(y) \rangle(e - u),$$

and putting it together we notice

$$0 = \frac{1}{2}(\langle d(x), y \rangle + \langle x, d(y) \rangle)(e - u),$$

so

$$\langle d_0(x), y \rangle + \langle x, d_0(y) \rangle = 0.$$

Through a similar computation the same is true for d_1 . This gives us that d_0 is skew symmetric with respect to $Q|_{E_0}$. From [24] 7.4.7 we get that the map $d \mapsto d_0$ maps S to $\mathrm{SO}(Q; E_0)$ linearly. If $d_0(x) = 0$ for $x \in E_0$ then $d_1(xy) = d_0(x)y + xd_1(y) = xd_1(y)$. This gives us, as we saw in 2.2.10.5, that $d_1 = \lambda \text{id}$. Notice that

$$\langle d_1(y), y \rangle + \langle y, d_1(y) \rangle = 0 \Rightarrow \langle y, d_1(y) \rangle = 0,$$

since the bilinear form is symmetric and the characteristic of the field is not 2. So then

$$\lambda \langle y, y \rangle = \langle y, d_1(y) \rangle = 0,$$

which gives us that $\lambda = 0$ so the map $d \mapsto d_0$ is injective. By a dimension argument

$$\dim(L) \leq \dim(S) \leq \dim(L(\mathrm{SO}(Q; E_0))) = \dim(\mathrm{SO}(Q; E_0)) = \dim(\mathrm{Aut}(A_K)_u) = \dim(L),$$

so S must be L . \square

First Tits Construction, [14] 2.2.10.10. Let n be the cubic norm form of a degree 3 asso-

ciative algebra A over R , and let $\mu \in R^*$. We define a module

$$J(A, \mu) = A_0 \oplus A_1 \oplus A_2,$$

to be the direct sum of three copies of A , and define c , N , T , and $\#$ by

$$c = \text{id} \oplus 0 \oplus 0 \tag{2.27}$$

$$N(a) = n(a_0) + \mu n(a_1) + \mu^2 n(a_2) - t(a_0 a_1 a_2) \tag{2.28}$$

$$T(a) = t(a_0) \tag{2.29}$$

$$T(a, b) = t(a_0, b_0) + t(a_1, b_2) + t(a_2, b_1) \tag{2.30}$$

$$a^\# = (a_0^\# - \mu a_1 a_2) \oplus (\mu a_2^\# - a_0 a_1) \oplus (a_1^\# - a_2 a_0) \tag{2.31}$$

$$(ab)^\# = b^\# a^\#, \tag{2.32}$$

where n and t are the norm and trace on A for $x = (a_0, a_1, a_2)$ and $y = (b_0, b_1, b_2)$. Then $(N, \#, c)$ is a sharpened cubic form and $J(N, \#, c)$ is a Jordan algebra.

We want to construct a split Albert algebra over a field k . To this end we pick our associative algebra $A = M_3(k)$ with $n = \det$, $t = \text{tr}$, and $c = \text{id}$, and $\mu = 1$.

Proposition 2.2.10.11. *The algebra $J(M_3(k), 1)$ is a split Albert algebra.*

Proof. If we consider the Albert algebra coming from the first Tits construction with $A = J(M_3(k), 1)$. We can show that $(M_3(k))^+ \cong M_3(k) \oplus 0 \oplus 0$. So by, [17] 11.2, A is not a division algebra so must be reduced, so $A \cong H(C, \gamma)$. And C contains a copy of $k \oplus k$, so A is split, [17]. \square

This concludes the material about J -algebras that we will need to discuss k -involution of algebraic groups of type F_4 .

Chapter 3

Algebraic groups of type G_2 and F_4

3.1 The automorphism group of an octonion algebra is of type G_2

We want to talk about the group $\text{Aut}(C)$, the linear automorphism group of composition algebras, specifically octonion algebras. First we will consider automorphisms of octonion algebras that leave quaternion subalgebras invariant.

Consider $x + ya \in D \oplus Da$, where $a \in D^\perp$, and $t \in \text{Aut}(C)$ such that $t(D) = D$. Since $t(C) = C$, then $t(D^\perp) = D^\perp$.

$$t(x + ya) = t(x) + t(y)t(a)$$

Which we can write using $u, v : D \rightarrow D$,

$$t(x + ya) = u(x) + v(y)a$$

$$\begin{aligned} (v(x)a)u(w) &= (0 + v(x)a)(u(w) + 0a) \\ &= (v(x)\overline{u(w)})a \\ t((xa)w) &= t(xa)t(w) = (t(x)t(a))t(w) \\ &= (v(x)a)u(w) \\ &= v(x)u(\bar{w})a \end{aligned}$$

$$\begin{aligned}
(xa)w &= (0 + xa)(w + 0a) = (x\bar{w})a \\
t((x\bar{w})a) &= t(x\bar{w})t(a) \\
&= v(x\bar{w})a \\
\overline{u(w)} &= u(\bar{w}) \\
v(x\bar{w}) &= v(x)u(\bar{w}) \\
\Rightarrow v(xy) &= v(x)u(y)
\end{aligned}$$

If we let $x = e$ and $v(x) = p$, for $p \in D$ and since v is an isometry $N(v(e)) = N(p) = N(e) = 1$. So we get $v(y) = pu(y)$.

So,

$$t(x + ya) = u(x) + (pu(y))a$$

And with use of the use of the Skolem-Noether Theorem, since D is associative and central simple $\text{Aut}(D)$ consists of only inner-automorphisms.

$$t(x + ya) = cxc^{-1} + (pcyc^{-1})a$$

and $N(c) \neq 0$.

3.1.1 Dimension of G

We will let $G = \text{Aut}(C_K)$, where K is the algebraic closure of some field k .

So we have a quaternion subalgebra $D \subset C$. D_K is the quaternion subalgebra over the algebraic closure of k , the field that D is over. Let $D_K(1)$ be the algebraic group of $p \in D_K$ such that $N(p) = 1$, and $D(1)$ the analog over k .

D_K is isomorphic to $\text{GL}_2(K)$ with determinant as norm, which makes $D_K(1)$ isomorphic to $\text{SL}_2(K)$. So $D_K(1)$ is three dimensional and connected.

Proposition, [9] 3.1.1.1. *There is an isomorphism between the the algebraic groups of K -automorphisms of C_K that fix D_K elementwise, we'll call it G_K^D , and $D_K(1)$.*

Proof. Define a map $\varphi : G_K^D \rightarrow D_K(1)$, by $t(x + ya) = x + (\varphi(t)y)a = u(x) + (pu(y))a$ where $u = id$. If $t_1, t_2 \in G_K^D$ are such that $t_1 = t_2$ then $x + (\varphi(t_1)y)a = x + (\varphi(t_2)y)a \Leftrightarrow \varphi(t_1) = \varphi(t_2)$.

Take $p \in D_K(1)$ then $N(p) = 1$ and is mapped from the $t_p \in G_K^D$ such that $t_p(x + ya) = x + (py)a$. Which means $G_K^D \cong D_K(1) \cong \text{SL}_2(K)$, so G_K^D is connected and of dimension three. \square

Proposition, [25] 3.1.1.2. $\rho : \text{SF} \rightarrow \text{SO}(N_1)$, where $t \in \text{SF}, t \mapsto t \mid_{e^\perp}$ is an isomorphism.

G is contained in the stablizer of e in $\text{O}(N)$, the orthogonal group of the quadratic form assoicated with N . Let F be the stabilizer of e in $\text{O}(N)$, then $\text{SF} = F \cap \text{SO}(N)$

Proof. Define for $u \in \mathrm{SO}(N \mid e^\perp)$ its extension $\varepsilon(u)(e) = e$. $\varepsilon = \rho^{-1}$ is a homomorphism and so ρ is an isomorphism. \square

Proposition, [9] 3.1.1.3. $G = \mathrm{Aut}(C_K)$ is a connected algebraic group with dimension 14.

Proof. Consider C_K the octonion algebra over K the algebraic closure of k . With K being algebraically closed we must have special pairs (as solutions to the equations in the definition). By Witt's theorem we have a linear isometry t (also a rotation) that maps special pairs to special pairs leaving e fixed. SF acts transitively on the set of special pairs. Let H in SF be the stabilizer of a special pair a, b , then H is isomorphic to the special orthogonal group of a five dimensional quadratic form N_5 . The restriction to $(Ke \oplus Ka \oplus Kb \oplus)^{\perp}$ yields the isomorphism as in 4.1.2. So $X = SF / H$ is a homogenous space and can be identified with the set of special pairs.

$$\begin{aligned} \dim(\mathrm{SO}(N \mid e^\perp)) &= \binom{7}{2} = 21 \\ \dim(\mathrm{SO}(N_5)) &= \binom{5}{2} = 10 \\ \dim(X) &= 21 - 10 = 11, \end{aligned}$$

and G operates transitively on the set of special pairs X . Since $D_K = Ke \oplus Ka \oplus Kb \oplus Kab$, G_K^D is the stabilizer of a, b , giving us $X \cong G/G_K^D$. And we notice

$$\dim(G) = \dim(X) + \dim(G_K^D) = 11 + 3 = 14.$$

So then we see that X is an irreducible algebraic variety, G_K^D is connected, and X is connected, so G is connected. \square

3.1.2 Other observations

The automorphism group G of C_K acts transitively on the set of quaternion subalgebras.

The group G is a subgroup of $\mathrm{SO}(N)$, since it is connected and contains the identity and is already a subgroup of $\mathrm{O}(N)$.

3.1.3 Derivations and the Lie algebra of the automorphism group

A derivation of C will be a linear map of C such that for $x, y \in C$, $d(xy) = xd(y) + d(x)y$.

$$\begin{aligned}
d(ee) &= ed(e) + d(e)e \\
d(e) &= 2d(e) \\
\Rightarrow d(e) &= 0
\end{aligned}$$

Also if d and f are derivations of C ...

$$\begin{aligned}
[d, f](xy) &= (df - fd)(xy) \\
&= df(xy) - fd(xy) \\
&= d(xf(y) + f(x)y) - f(xd(y) + d(x)y) \\
&= d(xf(y)) + d(f(x)y) - f(xd(y)) - f(d(x)y) \\
&= xdf(y) + d(x)f(y) + f(x)d(y) + df(x)y - xfd(y) - f(x)d(y) - d(x)f(y) - fd(x)y \\
&= xdf(y) - xfd(y) + df(x)y - fd(x)y \\
&= x(df - fd)(y) + (df - fd)(x)y \\
&= x[d, f](y) + [d, f](x)y
\end{aligned}$$

Let $d \in \text{Der}(C)$ then there are linear maps d_0 and d_1 such that

$$d(x) = d_0(x) + d_1(x)a$$

for $x \in D$, where $C = D \oplus Da$.

$$\begin{aligned}
d(xy) &= xd(y) + d(x)y = x(d_0(y) + d_1(y)a) + (d_0(x) + d_1(x)a)y \\
&= xd_0(y) + d_0(x)y + (d_1(y)x + d_1(x)\bar{y})a \\
&= d_0(xy) + d_1(xy)a \\
\Rightarrow d_0(xy) &= xd_0(y) + d_0(x)y, \text{ is in } \text{Der}(C) \\
d_1(xy) &= d_1(x)\bar{y} + d_1(y)x.
\end{aligned}$$

$$\begin{aligned}
d_1(e) &= d_1(e)e + d_1(e)e \\
&= 2d_1(e) \\
\Rightarrow d_1(e) &= 0, \text{ since char } k \neq 2.
\end{aligned}$$

Lemma, [25] 3.1.3.1. *Let S denote the space of linear maps like d_1 above. If $\dim(D) = 2$ then $\dim(S) \leq 2$, and if $\dim(D) = 4$ then $\dim(S) \leq 8$.*

Proof. Since $d_1(e) = 0$, if D is 2 dimensional S is the set of linear maps from a 2 dimensional vector space into a 2 dimensional vector space determined by the image of one vector so $\dim(S) \leq 2$. If D is 4 dimensional and since we have a rule for what happens to $d_1(ab)$ and $d_1(e) = 0$ the maps in S are completely determined by the images of two vectors into a 4 dimensional vector space, so $\dim(S) \leq 8$. \square

If we consider $\text{Der}_0(C)$ the space of derivations whose restriction to D is zero we get the following result.

Lemma, [25] 3.1.3.2. $\text{Der}_0(C) \leq 1$ if C is 4 dimensional and $\dim(\text{Der}_0(C)) \leq 3$ if C is 4 dimensional.

Proof. Let $d \in \text{Der}_0$ then since $C = D \oplus Da$, d must be determined by da . Recall $a^2 = \lambda$ and $ax = \bar{x}a$ for $x \in D$. If $\dim(D) = 2$ then D is commutative and $bx + \bar{x}b = 0 \Rightarrow (x + \bar{x})b = 0$ for all $x \in D \Rightarrow b = 0$, since this would say that there was no zero in the base field, $\Rightarrow \dim(D) = 0$.

If $\dim(D) = 4$ we will use the fact that $bx = -\bar{x}b$ to say

$$b(xy) = -(\bar{y}\bar{x})b = -\bar{y}(\bar{x}b) = \bar{y}(bx) = (\bar{y}b)x = -(by)x = -b(yx)$$

$\Rightarrow b(xy + yx) = 0$ for all $x, y \in D$. Let us consider the case where $k = K$ is algebraically closed, and it will be resolved for $k \neq K$ shortly. If K is algebraically closed then $D \cong M_2(K)$ so we can take $x = e_{12}$ and $y = e_{21}$, which gives us $xy + yx = e_{11} + e_{22} = I_{2 \times 2} \Rightarrow b = 0$. So we have that $da = ca$ and $(ca)a + a(ca) = 0 \Rightarrow c(aa) + (aa)\bar{c} = (c + \bar{c})\lambda = 0 \Rightarrow c + \bar{c} = 0 \Rightarrow c \in e^\perp \subset D$. \square

Lemma, [25] 3.1.3.3. If C is 2 dimensional then $\text{Der}(C) = 0$. If C is 4 dimensional then $\dim(\text{Der}(C)) \leq 3$. If C is 8 dimensional then $\dim(\text{Der}(C)) \leq 14$.

Proof. If $\dim(C) = 2$, then $C = k[a]$ with the relation $x^2 - \langle x, e \rangle x + N(x)e = 0$ for all $x \in C$. And we get the following

$$d(a^2 - \langle a, e \rangle a + N(a)e) = d(a^2) - \langle a, e \rangle d(a) = 2a(da) - \langle a, e \rangle d(a) = 0$$

Since we are assuming $\text{char}(k) \neq 0$ we can take $\langle a, e \rangle = 0 \Rightarrow 2a(da) = 0 \Rightarrow da = 0 \Rightarrow d = 0$.

Let us consider $\dim(C) > 2$, then we will look at the exact sequence

$$0 \rightarrow \text{Der}_0(C) \rightarrow \text{Der}(C) \rightarrow \text{Der}(D) \oplus S \rightarrow 0.$$

So we get an upper bound on $\dim(\text{Der}(C))$ by

$$\dim(\text{Der}(C)) \leq \dim(\text{Der}_0(C)) + \dim(\text{Der}(D)) + \dim(S).$$

□

From the previous results we get the bounds for the other two cases.

Proposition, [25] 3.1.3.4. *Let C be an octonion algebra over a field k , then $\text{Der}_k(C)$ is the Lie algebra of $\text{Aut}(C)$.*

Proof. Since we know $\text{Der}(C_K) = K \otimes_K \text{Der}_k(C)$, all we need to show is that $\text{Der}_K(C)$ is the Lie algebra of $\text{Aut}(C)_K$.

If we consider the map $\mu : C_K \otimes C_K \rightarrow C_K$, which is defined by $\mu(x \otimes y) = xy$, then \mathbf{G}_K is the stabilizer of μ in the algebraic group $\text{GL}(C_K)$. And a result from [7] gives us that the elements of the Lie algebra of the algebraic groups of automorphisms of a K -algebra are the derivations. □

Corollary 3.1.3.5. *If C is defined over k then so is $\text{Aut}(C)$.*

3.2 k -split maximal torus of G_2

To place our results as an extension of [5] and to fit into Helminck's invariant approach, the next step is to identify k -split maximal tori contained in our automorphism groups.

3.2.1 Maximal Torus of G_2

We will consider a torus T_K of $\text{Aut}(C_K)$ with elements like $t_{\lambda,\mu}(x + ya) = c_\lambda xc_\lambda^{-1} + (c_\mu yc_\lambda^{-1})a$, where $x, y \in D_K$, $a \in D_K^\perp$, and $C_K = D_K \oplus D_Ka$.

$$t_{\lambda,\mu} = \text{diag}(1, \lambda^2, \lambda^{-2}, 1, \lambda^{-1}\mu, \lambda\mu, \lambda^{-1}\mu^{-1}, \lambda\mu^{-1}),$$

where C_K has a basis $(e_{ij}, 0), (0, e_{ij})$, with $i, j = 1, 2$.

If there is a reparameterization $\beta = \lambda\mu$ and $\gamma = \lambda\mu^{-1}$, then

$$t_{\lambda,\mu} = \text{diag}(1, \beta\gamma, \beta^{-1}\gamma^{-1}, 1, \gamma^{-1}, \beta, \beta^{-1}, \gamma)$$

and there is an isomorphism $t_{\beta,\gamma} \mapsto (\beta, \gamma) \in (K^*, K^*)$.

The eigenspaces of $t_{\lambda,\mu}$ are the spans of $(e_{11}, 0) + (e_{22}, 0), (e_{12}, 0), (e_{21}, 0)$, and all of $(0, e_{ij})$ with $i, j = 1, 2$. Which leaves the eigenspaces of $t_{\lambda,\mu}$ invariant, which in turn implies that t leaves D_K invariant. And as we have seen before we can write t in the form

$$t : x + ya \mapsto cxc^{-1} + (pcyc^{-1})a$$

where $\det(c) = \det(p) = 1$. From this we concluded that certain restrictions of t happen thusly

$$t(x) = cxc^{-1}$$

$$t(ya) = (pcyc^{-1})a$$

Since each of these restrictions must also leave the eigenspaces invariant

$$ce_{12}c^{-1} = a_1e_{12} \Rightarrow ce_{12} = a_1e_{12}c$$

$$ce_{21}c^{-1} = a_2e_{21} \Rightarrow ce_{21} = a_2e_{21}c$$

$$\begin{aligned} &\Rightarrow \begin{bmatrix} 0 & c_{11} \\ 0 & c_{21} \end{bmatrix} = a_1 \begin{bmatrix} c_{21} & c_{22} \\ 0 & 0 \end{bmatrix} \\ \text{and } &\begin{bmatrix} c_{12} & 0 \\ c_{22} & 0 \end{bmatrix} = a_2 \begin{bmatrix} 0 & 0 \\ c_{11} & c_{12} \end{bmatrix} \\ &\Rightarrow c = \begin{bmatrix} \iota & 0 \\ 0 & \iota^{-1} \end{bmatrix} \end{aligned}$$

And similarly for $p = \begin{bmatrix} \kappa & 0 \\ 0 & \kappa^{-1} \end{bmatrix}$, and so $t \in T_K$, and T_K is a maximal torus.

Proposition 3.2.1.1. *The center of $\text{Aut}(C_K)$ consists of only the identity.*

Proof. A central element must commute with all of $\text{Aut}(C_K)$ including T_K , so T_K must contain the center. So it must have the form $z_{\lambda,\mu} : x + ya \mapsto c_\lambda xc_\lambda^{-1} + (c_\mu yc_\lambda^{-1})a$. And so it must leave D_K invariant and thus have the form $z : x + ya \mapsto cxc^{-1} + (pcyc^{-1})a$, and $N(p) = 1$.

So $z|_{D_K} = \text{diag}(1, \lambda^2, \lambda^{-2}, 1)$ must commute with all inner automorphisms, which would consist of all of $M_2(K)$. This is only true when $\lambda^2 = 1$, which makes $z|_{D_K} = id$. \square

This holds for all quaternion subalgebras of the octonion algebra, and since every element in the octonion algebra is contained in a quaternion subalgebra $z = id$.

Proposition, [25] 3.2.1.2. *The only invariant subspaces of C_K are Ke and e^\perp .*

Proof. Let V be an invariant subspace of C_K . We are going to use conjugation to show the span of V . In the 1 and 2 dimensional case it is trivial since they are commutative.

So let C_K be four dimensional, so $C_K = M_2(K)$. By Skolem-Noether every automorphism is inner. So let $t \in M_2(K)$ be a diagonal matrix with distinct eigenvalues, then the map $x \mapsto txt^{-1}$ has the eigenspaces $Ke_{11} + Ke_{22}, Ke_{12}, Ke_{21}$. Therefore V is spanned by vectors of the form

$\alpha e_{11} + \beta e_{22}, \gamma e_{12}, \delta e_{21}$. In the first case let $e_{12} \in V$, then

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -\lambda^2 & \lambda \end{bmatrix}$$

So e_{21} and $e_{11} - e_{22}$ must be in V as well, as it is an invariant subspace.

Now let $e_{21} \in V$, then

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & -\lambda^2 \\ 1 & -\lambda \end{bmatrix}$$

So e_{12} and $e_{11} - e_{22}$ are in V as well.

Let $\alpha e_{11} + \beta e_{22} \in V$.

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ -\beta\lambda + \alpha\lambda & \beta \end{bmatrix}$$

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta\lambda + \alpha\lambda \\ 0 & \beta \end{bmatrix}$$

With $\alpha \neq \beta$ V also includes e_{12} and e_{21} . So $V \supset e^\perp$.

If $\alpha = \beta$ then $\alpha(e_{11} + e_{22})$ commutes with all $M_2(K)$ and so $V = Ke$. If e_{12} and e_{21} are not in V , then $V = Ke$. \square

We will build the 8 dimensional case out of the 4 dimensional one. Let V be an invariant subspace of C_K , a composition algebra, where $\dim(C_K) = 8$. Let D_1 be an quaternion subalgebra, and $V_1 = V \cap D_1$. Recall that every automorphism of D_1 can be extended to an automorphism of C_K . This means that V_1 is invariant under $\text{Aut}(D_1)$, so $V_1 = \{0\}$ or Ke or $e^\perp \cap D_1$ or D_1 . Since there must exist an automorphism φ of C_K that takes D_1 to D_2 , another quaternion subalgebra, $\varphi(V_1) = V_2$. With every element of C_K being contained in a quaternion subalgebra $V = \{0\}$ or Ke or e^\perp or C_K .

Corollary 3.2.1.3. $\text{Aut}(C_K)$ has a faithful irreducible representation.

The 7 dimensional representation in e^\perp .

Theorem, [25] 3.2.1.4. $\text{Aut}(C_K)$ and $\text{Aut}(C)$ are of type G_2 .

Proof. We know that $\text{Aut}(C_K)$ is a connected linear algebraic group of dimension 14 and rank 2, and that $\text{Aut}(C_K)$ has a faithful representation. So $\text{Aut}(C_K)$ is reductive. The center of $\text{Aut}(C)$ is trivial, so $\text{Aut}(C)$ is semisimple. From the dimension, rank, and the fact that $\dim(\text{Aut}(C_K)) =$

rank + $|R|$, where R is the root system, we know that the root system has 12 elements and is 2 dimensional and irreducible (since the only reducible root system of dimension 2 has 4 elements), and so $\text{Aut}(C)$ is of type G_2 . \square

3.3 The automorphism groups of an Albert algebra is of type F_4

3.3.1 $\text{Aut}(A_K)$ is of type F_4

Lemma, [25] 3.3.1.1. $\text{Aut}(A_K)$ is a connected linear algebraic group of dimension 52.

Proof. Consider the action of $\text{Aut}(A_K)$ on the variety V of primitive idempotent elements of A_K . By 2.2.8.3 we have that $\text{Aut}(A_K)$ acts transitively on V . We want to show that V is an irreducible variety of dimension 16. To do this we consider the primitive idempotent elements

$$u_1 = u, \quad u_2 = \frac{1}{2}(e - u) + x_1, \quad u_3 = \frac{1}{2}(e - u) - x_1,$$

and define the subset of V ,

$$V_i = \{t \in V \mid \langle t, u_i \rangle \neq 0\},$$

which by inspection is a Zariski open subset of V_i . So the elements in V_1 are of type (a) of 2.2.6.3, which are the elements,

$$(Q(y) + 1)^{-1} \left(u + \frac{1}{2}Q(y)(e - u) + y \circ y + y \right),$$

for $y \in E_1$ with $Q(y) \neq -1$. Since the $y \in E_1$ such that $Q(y) \neq -1$ form a Zariski open subset in E_1 , and thus is dense in E_1 , the dimension of V_1 must be the dimension of E_1 , which is 16. And this is true of V_2 and V_3 since they are written in terms of u_1 . If we have $t \in V$ and $t \notin V_1$, then it is of type (b) from 2.2.6.3 so

$$t = \frac{1}{2}(e - u) + x + y,$$

for $x \in E_0$, $Q(x) = \frac{1}{4}$, $y \in E_1$, and $Q(y) = 0$. When we compute

$$\begin{aligned} \left\langle \frac{1}{2}(e - u) + x + y, \frac{1}{2}(e - u) + x_1 \right\rangle &= \frac{1}{4}(\langle (e - u), (e - u) \rangle + \langle x, x_1 \rangle) \\ &= \frac{1}{2} + \langle x, x_1 \rangle, \end{aligned}$$

so

$$\langle t, u_2 \rangle = \frac{1}{2} - \langle x, x_1 \rangle,$$

so $\langle t, u_1 \rangle$ and $\langle t, u_2 \rangle$ cannot be zero at the same time so t has to be in V_2 or V_3 , which gives us that

$$V = V_1 \cup V_2 \cup V_3.$$

Also, we notice that $V_2 \cap V_3 \neq \emptyset$, since $\langle t, u_2 \rangle$ and $\langle t, u_3 \rangle$ can be nonzero at the same time. The same is true for $V_1 \cap V_2$ and $V_1 \cap V_3$, and so each $V_i \cap V_j$ is a dense open set of V_i . We also know that $V_1 \cap V_2 \cap V_3$ is nonempty and so must be a dense open subset of the V_i . The Zariski closure of $V_1 \cap V_2 \cap V_3$ must be all of V , which is now the closure of an irreducible subset, and so is irreducible with dimension 16.

We saw in 2.2.10.8 that $\text{Aut}(A_K)_u$ is isomorphic to $\text{Spin}(Q; E_0)$ which is a quasi-simple algebraic group of type B_4 and so has dimension 36. From [24] 5.5.9 (with $\text{Aut}(A_K)_u$ the connected component containing the identity and V the closed subgroup) we get that $\text{Aut}(A_K)$ is connected and of dimension 52. \square

3.4 k -split maximal torus of type F_4

Here we construct a k -split maximal torus of type F_4 using an action of two copies of SL_3 on an Albert algebra.

3.4.1 Action of $\text{SL}_3(k) \times \text{SL}_3(k)$ on $A = J(M_3(k), 1)$

Proposition 3.4.1.1. *The map*

$$f_{uv} : A \rightarrow A, \text{ where } (a_0, a_1, a_2) \mapsto (ua_0u^{-1}, ua_1v^{-1}, va_2u^{-1}),$$

is an automorphism of A if and only if $(u, v) \in \text{SL}_3(k) \times \text{SL}_3(k)$.

Proof. In order for f_{uv} to be an automorphism of $A = J(M_3(k), 1)$ we need f_{uv} to preserve the base point, the sharp map and the norm. A map of the form f_{uv} will clearly fix the basepoint $c = (\text{id}, 0, 0)$ for any $u \in \text{GL}_3(k)$. Next we check that f_{uv} preserves the norm. So we look at

$$\begin{aligned} N(f_{uv}(a)) &= N((ua_0u^{-1}, ua_1v^{-1}, va_2u^{-1})) \\ &= n(ua_0u^{-1}) + n(ua_1v^{-1}) + n(va_2u^{-1}) - t(ua_0u^{-1}ua_1v^{-1}va_2u^{-1}) \\ &= n(u)n(a_0)n(u^{-1}) + n(u)n(a_1)n(v^{-1}) + n(v)n(a_2)n(u^{-1}) - t(ua_0a_1a_2u^{-1}) \\ &= n(a_0) + n(a_1) + n(a_2) - t(a_0a_1a_2) \\ &= N(a). \end{aligned}$$

Let us look at the sharp map

$$a^\# = \left(a_0^\# - a_1 a_2, a_2^\# - a_0 a_1, a_1^\# - a_2 a_0 \right).$$

then

$$\begin{aligned} f_{uv}(a^\#) &= f_{uv}\left(a_0^\# - a_1 a_2, a_2^\# - a_0 a_1, a_1^\# - a_2 a_0\right) \\ &= \left(u(a_0^\# - a_1 a_2)u^{-1}, u(a_2^\# - a_0 a_1)v^{-1}, v(a_1^\# - a_2 a_0)u^{-1}\right) \\ &= \left(ua_0^\#u^{-1} - ua_1 a_2 u^{-1}, ua_2^\#v^{-1} - ua_0 a_1 v^{-1}, va_1^\#u^{-1} - va_2 a_0 u^{-1}\right) \\ &= \left(ua_0^\#u^{-1} - ua_1 v^{-1}va_2 u^{-1}, ua_2^\#v^{-1} - ua_0 u^{-1}ua_1 v^{-1}, va_1^\#u^{-1} - va_2 u^{-1}ua_0 u^{-1}\right). \end{aligned}$$

Since $a_i^\# = a_i^2 - t(a_i)a_i + s(a_i)c$ and $x^{-1} = x^\#$ if and only if $x \in \mathrm{SL}_3(k)$,

$$\begin{aligned} x(a_i^\#)y^{-1} &= (x^{-1})(a_i^\#)(y)^\# \\ &= (x^{-1})(ya_i)^\# \\ &= (ya_i x^{-1})^\#. \end{aligned}$$

For a_0 let $x = y = u$, a_1 let $x = u$ and $y = v$, a_2 let $x = v$ and $y = u$. \square

Now if we consider a k -split maximal torus in $\mathrm{SL}_3(k)$,

$$T = \left\{ \begin{bmatrix} u_1 & & \\ & u_1^{-1}u_2 & \\ & & u_2^{-1} \end{bmatrix} \mid u_1, u_2 \in k \right\} \subset \mathrm{SL}_3(k),$$

we see that it is of rank 2 and so a k -split maximal torus $T \times T \subset \mathrm{SL}_3(k) \times \mathrm{SL}_3(k) \subset F_4$ must have rank 4 and we have that $\mathcal{T} = T \times T \subset F_4$ is a k -split maximal torus. In accordance with the action of f_{uv} on A , our k -split maximal torus takes the following form, for $t \in \mathcal{T}$,

$$\begin{aligned} t = \mathrm{diag}(1, u_1^2 u_2^{-1}, u_1 u_2 \mid u_1^{-2} u_2, 1, u_1^{-1} u_2^2 \mid u_1^{-1} u_2^{-1}, u_1 u_2^{-2}, 1 \mid \\ u_1 v_1^{-1}, u_1 v_1 v_2^{-1}, u_1 v_2 \mid u_1^{-1} u_2 v_1^{-1}, u_1^{-1} u_2 v_1 v_2^{-1}, u_1^{-1} u_2 v_2 \mid u_2^{-1} v_1^{-1}, u_2^{-1} v_1 v_2^{-1}, u_2^{-1} v_2 \mid \\ u_1^{-1} v_1, u_1 u_2^{-1} v_1, u_2 v_1 \mid u_1^{-1} v_1^{-1} v_2, u_1 u_2^{-1} v_1^{-1} v_2, u_2 v_1^{-1} v_2 \mid u_1^{-1} v_2^{-1}, u_1 u_2^{-1} v_2^{-1}, u_2 v_2^{-1}), \end{aligned}$$

and if we make the identification $a = u_1 u_2$, $b = u_1 v_1^{-1}$, $c = u_1 v_2$, and $d = u_2 v_1^{-1} v_2$ we have an isomorphism to (k^*, k^*, k^*, k^*) .

Chapter 4

k -involutions

First we introduce the three invariants from [5] shown to give a full classification. Then give a full classification of k -involutions for algebraic groups of type G_2 and F_4 within the context of the classification criteria set up by Helminck in [5].

4.1 Invariants

In the following we introduce a characterization of k -involutions of an algebraic group of a specific type given by Helminck. First we need to define some terms. We say a torus A is θ -split, for an involution θ , if for all $a \in A$, $\theta(a) = a^{-1}$. A torus is (θ, k) -split if it is both θ -split and k -split. The group of characters and root spaces associated with a torus T are denoted by $X^*(T)$ and $\Phi(T)$ respectively. We will also denote by

$$A_\theta^- = \{a \in A \mid \theta(a) = a^{-1}\}^\circ,$$

the connected component of the set of elements inverted by θ , and by

$$I_k(A_\theta^-) = \{a \in A_\theta^- \mid \theta \circ \text{Inn}(a) \text{ is a } k\text{-involution}\},$$

the elements of $I_k(A_\theta^-)$ will be called k -inner elements. Let A be a maximal k -split torus and A_θ^- a maximal (θ, k) -split torus. Let S be a maximal k -split torus. For $T \supset S$ a maximal k -torus, an involution

$$\theta \in \text{Aut}(X^*(T), X^*(S), \Phi(T), \Phi(S))$$

is *admissible* if there exists a k -involution $\tilde{\theta} \in \text{Aut}(G, T, S)$ such that $T_{\tilde{\theta}}^-$ is a maximal $\tilde{\theta}$ -split torus, $S_{\tilde{\theta}}^-$ is a maximal $(\tilde{\theta}, k)$ -split torus, and $\tilde{\theta}|_{(X^*(T), X^*(S), \Phi(T), \Phi(S))} = \theta$. The full classification can be completed with the classification of the following three invariants, [5],

- (1) classification of admissible involutions of $(X^*(T), X^*(A), \Phi(T), \Phi(A))$, where T is a maximal torus in G , A is a maximal k -split torus contained in T
- (2) classification of the G_k -isomorphy classes of k -involutions of the k -anisotropic kernel of G
- (3) classification of the G_k -isomorphy classes of k -inner elements $a \in I_k(A_\theta^-)$.

When considering split algebraic groups, we only need to consider invariants of type (1) and (2) from above.

Now that we have seen the importance of k -involutions in the study of symmetric k -varieties we continue with the aim of classifying the k -involutions of split groups of type G_2 and F_4 over certain fields, and in some cases provide their fixed point groups. The next two chapter will provide background on the structure of groups of type G_2 and F_4 needed to understand their k -involutions.

4.2 k -involutions of G_2

4.2.1 Involutions from a Maximal Torus

Recall that there is a maximal torus, T , of the form $t = \text{diag}(1, \zeta\eta, \zeta^{-1}\eta^{-1}, 1, \eta^{-1}, \zeta, \zeta^{-1}, \eta)$ with respect to the basis $\{(e_{ij}, 0), (0, e_{ij})\}_{1 \leq i, j \leq 2}$, where e_{ij} are the 2×2 matrices with a one in the ij position and zeros everywhere else.

An involution will be a map, in this case an inner automorphism \mathcal{I}_g , such that $\mathcal{I}_g \neq \text{id}$ and $\mathcal{I}_g^2 = \text{id}$. Let's consider \mathcal{I}_t for $t \in T$. Then

$$\mathcal{I}_t(x) = txt^{-1}$$

and so

$$\mathcal{I}_t^2(x) = t^2xt^{-2} \text{ and we want } t^2xt^{-2} = x$$

or in other words $t^2x = xt^2$, which puts $t^2 \in Z(G) = \{\text{id}\}$. As long as $t \notin Z(G)$, \mathcal{I}_t is an involution.

Remark 4.2.1.1. If $\theta \in \text{Inn}(G)$ and $\theta = \mathcal{I}_t$ is a k -involution if and only if $t^2 \in Z(G)$.

Since groups of type G_2 have a trivial center, the problem of classifying k -involutions for $\text{Aut}(C)$, where C is a split octonion algebra, is the same as classifying the conjugacy classes of elements of order 2 in $\text{Aut}(C)$ that preserve the k -structure of $\text{Aut}(C)$.

Using the above statement and the following result of Jacobson we can show that certain k -involutions given by conjugation by elements coming from the maximal k -split torus T are isomorphic.

Proposition, [9] 4.2.1.2. *Let C be an octonion algebra over k , then the conjugacy class of quadratic elements, $t \in G = \text{Aut}(C)$ such that $t^2 = \text{id}$ are in bijection with the isomorphism classes of quaternion subalgebras of C .*

In particular if $t \in \text{Aut}(C)$ has order 2, then it leaves some quaternion subalgebra D elementwise fixed giving us the 1-eigenspace. Then D^\perp is the (-1) -eigenspace. If $gtg^{-1} = s$ for some $g \in G$, then s has order 2 and $g(D) = D'$, D' a quaternion subalgebra elementwise fixed by s , and $D \cong D'$. This is shown in more generality below.

Corollary 4.2.1.3. *Let C be an octonion algebra over k and D and D' quaternion subalgebras of C . If $s, t \in G = \text{Aut}(C)$ are elements of order 2 and s, t fix D, D' elementwise respectively, then $s \cong t$ if and only if $D \cong D'$ over k .*

Corollary 4.2.1.4. *For $s, t \in \text{Aut}(C)$, $\mathcal{I}_t \cong \mathcal{I}_s$ if and only if s and t leave isomorphic quaternion subalgebras invariant.*

Lemma 4.2.1.5. $\mathcal{I}_g \mathcal{I}_\varepsilon \mathcal{I}_g^{-1} = \mathcal{I}_{g\varepsilon g^{-1}}$

Proof. We can apply the left hand side to an element $y \in G$,

$$\begin{aligned} \mathcal{I}_g \mathcal{I}_\varepsilon \mathcal{I}_g^{-1}(y) &= g(\varepsilon(g^{-1}yg)\varepsilon^{-1})g^{-1} \\ &= (g\varepsilon g^{-1})y(g\varepsilon g^{-1})^{-1} \\ &= \mathcal{I}_{(g\varepsilon g^{-1})}(y). \end{aligned}$$

□

Proposition 4.2.1.6. *If $\varepsilon_1^2 = \varepsilon_2^2 = \text{id}$ and $\varepsilon_1, \varepsilon_2 \in T$ a maximal torus of G when $Z(G) = \{\text{id}\}$, then $\mathcal{I}_{\varepsilon_1} \cong \mathcal{I}_{\varepsilon_2}$ if and only if $\varepsilon_1 = n\varepsilon_2 n^{-1}$ for some $n \in N_G(T)$.*

Proof. If $n\varepsilon_2 n^{-1} = \varepsilon_1$ for $n \in N_G(T)$ then $\mathcal{I}_{\varepsilon_1} \cong \mathcal{I}_{\varepsilon_2}$ via the isomorphism $\text{Inn}(\mathcal{I}_n)$.

Now we let $\mathcal{I}_{\varepsilon_1} \cong \mathcal{I}_{\varepsilon_2}$, and so there exists a $g \in G$ such that $\mathcal{I}_{\varepsilon_1} = \mathcal{I}_g(\mathcal{I}_{\varepsilon_2})\mathcal{I}_g^{-1}$, then we have by 4.2.1.5,

$$(g\varepsilon_2 g^{-1})^{-1} \varepsilon_1 y = y(g\varepsilon_2 g^{-1})^{-1} \varepsilon_1,$$

for all $y \in G$ and so $(g\varepsilon_2 g^{-1})^{-1} \varepsilon_1 \in Z(G) = \{\text{id}\}$. Thus we have that $\varepsilon_1^{-1} = (g\varepsilon_2 g^{-1})^{-1}$, so $\varepsilon_1 = g\varepsilon_2 g^{-1}$. So now notice $S = gTg^{-1}$ is a maximal torus containing ε_1 . The group $Z_G(\varepsilon_1)$ contains S and T , so there exists an $x \in Z_G(\varepsilon_1)$ such that $xSx^{-1} = T$. We know that $S = gTg^{-1}$ so

$$xgTg^{-1}x^{-1} = xgT(xg)^{-1} = T,$$

which has $xg \in N_G(T)$. We notice that

$$\begin{aligned}\mathcal{I}_{xg}\mathcal{I}_{\varepsilon_2}\mathcal{I}_{xg}^{-1} &= \mathcal{I}_{xg\varepsilon_2(xg)^{-1}} \\ &= \mathcal{I}_{xg\varepsilon_2g^{-1}x^{-1}} \\ &= \mathcal{I}_{x\varepsilon_1x^{-1}} \\ &= \mathcal{I}_{\varepsilon_1},\end{aligned}$$

which from the previous argument we have $(xg)\varepsilon_2(xg)^{-1} = \varepsilon_1$.

□

Using the previous proposition it is possible to find elements $n, m \in N_G(T)$ such that

$$t_{(-1,-1)} = n(t_{(-1,1)})n^{-1} = m(t_{(1,-1)})m^{-1}.$$

It is also possible to show, and perhaps more illustrative, that they leave isomorphic quaternion subalgebras invariant, and thus by 4.2.1.4 provide us with isomorphic k -involutions.

Proposition 4.2.1.7. $t_{(-1,-1)} \cong t_{(-1,1)} \cong t_{(1,-1)}$.

Proof. Let $G = \text{Aut}(C) \supset T = \{\text{diag}(1, \beta\gamma, \beta^{-1}\gamma^{-1}, 1, \gamma^{-1}, \beta, \beta^{-1}, \gamma) \mid \beta, \gamma \in k^*\}$, then G is an algebraic group over the field k , $\text{char}(k) \neq 2$. The automorphism $t_{(-1,-1)}$ leaves the split quaternion subalgebra $(M_2(k), 0)$ elementwise fixed, and $(M_2(k), 0)$ is a split subalgebra.

The element of order 2,

$$t_{(1,-1)} = \text{diag}(1, -1, -1, 1, -1, 1, 1, -1),$$

leaves the quaternion subalgebra,

$$k \underbrace{\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, 0 \right)}_e \bigoplus k \underbrace{\left(0, \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \right)}_a \bigoplus k \underbrace{\left(0, \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \right)}_b \bigoplus k \underbrace{\left(\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, 0 \right)}_{ab},$$

elementwise fixed. Notice that $(b-a)(e+ab) = (0, 0)$, and so the quaternion subalgebra is split.

And

$$t_{(-1,1)} = \text{diag}(1, -1, -1, 1, 1, -1, -1, 1),$$

leaves the quaternion algebra

$$k \underbrace{\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, 0 \right)}_e \bigoplus k \underbrace{\left(\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, 0 \right)}_a \bigoplus k \underbrace{\left(0, \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \right)}_b \bigoplus k \underbrace{\left(0, \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \right)}_{ab},$$

elementwise fixed. Notice that $(ab + b)(e + a) = (0, 0)$, and so the quaternion subalgebra is split. Since over a given field k every split quaternion subalgebra is isomorphic, we have that $t_{(-1,-1)} \cong t_{(-1,1)} \cong t_{(1,-1)}$. \square

Corollary 4.2.1.8. $\mathcal{I}_{t_{(-1,-1)}} \cong \mathcal{I}_{t_{(-1,1)}} \cong \mathcal{I}_{t_{(1,-1)}}$.

So from now on we refer to a representative of the isomorphic class containing $\mathcal{I}_{t_{(-1,-1)}}$, $\mathcal{I}_{t_{(-1,1)}}$, and $\mathcal{I}_{t_{(1,-1)}}$ as \mathcal{I}_t , when there is no ambiguity.

Lemma 4.2.1.9. *There is only one isomorphy class of k -involutions when k is a finite field of order greater than 2, complex numbers, or when k is a complete, totally imaginary algebraic number field.*

Proof. In these cases only split quaternion algebras exist, [15], [25]. \square

In [26] Yokota talks about the maps γ , γ_C , and γ_H and shows are they isomorphic, and that they are also isomorphic to any composition of maps between them. In his paper he defines a conjugation coming from complexification. In particular we can look at γ_H , which is the complexification conjugation on the quaternion level of an octonion algebra over \mathbb{R} . If we take $u + vc \in H \oplus Hc$ where $u, v \in H$ and $c \in H^\perp$ his map is

$$\gamma_H(u + vc) = u - vc,$$

which in our presentation of the octonion algebra would look like,

$$\gamma_H \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \right) = \left(\begin{bmatrix} u_{11} & -u_{12} \\ -u_{21} & u_{22} \end{bmatrix}, \begin{bmatrix} v_{11} & -v_{12} \\ -v_{21} & v_{22} \end{bmatrix} \right),$$

and corresponds to our map $\mathcal{I}_{t_{(-1,1)}}$.

4.2.2 Maximal θ -split torus

Rather than trying to find a maximal θ -split torus, where $\theta \cong \mathcal{I}_t$, and then computing its maximal k -split subtorus, we find a k -involution θ that splits our maximal k -split torus of the form

$$T = \{\text{diag}(1, \beta\gamma, \beta^{-1}\gamma^{-1}, 1, \gamma^{-1}, \beta, \beta^{-1}, \gamma) \mid \beta, \gamma \in k^*\}.$$

It is straight forward to check that

$$s = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \bigoplus \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix},$$

is an element of $\text{Aut}(C)$, where C is the split octonion algebra described above over a field k , $\text{char}(k) \neq 2$. It is immediate that T is a \mathcal{I}_s -split torus.

Proposition 4.2.2.1. *T is a maximal (\mathcal{I}_s, k) -split torus.*

Proof. Notice first that if $t \in T$ that $\mathcal{I}_s(t) = t^{-1}$, and next that T is k -split and is a maximal torus. \square

Proposition 4.2.2.2. $\mathcal{I}_s \cong \mathcal{I}_t$

Proof. The element s is an automorphism of order 2 of C , our split octonion algebra described above, that leaves the following quaternion algebra fixed elementwise,

$$k \left(\underbrace{\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, 0}_{e} \right) \oplus k \left(\underbrace{\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, 0}_{a} \right) \oplus k \left(0, \underbrace{\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}}_{b} \right) \oplus k \left(0, \underbrace{\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}}_{ab} \right).$$

Notice that $(b + ab)(e + a + b + ab) = 0$, and so the quaternion subalgebra is split. \square

4.2.3 Another isomorphy class of k -involutions over certain fields

We have seen that our maximal torus $T = T_{\mathcal{I}_s}^-$, and so we can look at elements of T for k -inner elements of \mathcal{I}_s that will give us new conjugacy classes over fields for which quaternion division algebras can exist. The fields we are interested in include the real numbers, p -adics, and rationals.

Lemma 4.2.3.1. *For C a split octonion algebra over a field $k = \mathbb{R}, \mathbb{Q}_2, \mathbb{Q}$,*

$$s \cdot t_{(1,-1)} \in \text{Aut}(C),$$

leaves a quaternion division subalgebra elementwise fixed.

Proof. The element $s \cdot t_{(1,-1)} \in \text{Aut}(C)$ leaves the following quaternion subalgebra elementwise fixed,

$$k \left(\underbrace{\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, 0}_{e} \right) \oplus k \left(0, \underbrace{\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}}_a \right) \oplus k \left(0, \underbrace{\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}}_b \right) \oplus k \left(\underbrace{\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, 0}_{ab} \right).$$

All basis elements are such that $x\bar{x} = 1$, and so have a norm isomorphic to the 2-Pfister form $\left(\frac{-1, -1}{k}\right)$, where $k = \mathbb{R}, \mathbb{Q}_2, \mathbb{Q}$, which corresponds to a quaternion division algebra over each respective field. Moreover, over $k = \mathbb{R}$ or \mathbb{Q}_2 there is only one quaternion division algebra up to isomorphism. \square

Lemma 4.2.3.2. For C a split octonion algebra over a field $k = \mathbb{Q}_p$ and $p > 2$ $s \cdot t_{(-N_p, -pN_p^{-1})}$ leaves a division quaternion algebra elementwise fixed.

Proof. The element $s \cdot t_{(-N_p, -pN_p^{-1})}$ leaves the following quaternion subalgebra elementwise fixed,

$$k \underbrace{\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, 0 \right)}_e \oplus k \underbrace{\left(0, \begin{bmatrix} & -N_p \\ 1 & \end{bmatrix} \right)}_a \oplus k \underbrace{\left(\begin{bmatrix} & p \\ 1 & \end{bmatrix}, 0 \right)}_b \oplus k \underbrace{\left(0, \begin{bmatrix} N_p & \\ & -p \end{bmatrix} \right)}_{ab},$$

with $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 = \{1, p, N_p, pN_p\}$. This algebra is isomorphic to $\left(\frac{p, N_p}{\mathbb{Q}_p}\right)$, which is a representative of the unique isomorphy class of quaternion division algebras for a given p . \square

Theorem 4.2.3.3. Let $\theta = \mathcal{I}_s$ and $G = \text{Aut}(C)$ where C is a split octonion algebra over a field k , then

1. when $k = \mathbb{R}, \mathbb{Q}, \mathbb{Q}_2$; θ and $\theta \circ \mathcal{I}_{t_{(1, -1)}}$ are representatives of 2 isomorphy classes of k -involutions of G . In the cases $k = \mathbb{R}$ or \mathbb{Q}_2 these are the only cases, but this is not true for $k = \mathbb{Q}$.
2. when $k = \mathbb{Q}_p$ and $p > 2$, we have two isomorphy classes of k -involutions.
3. when $k = \mathbb{C}$ and \mathbb{F}_p with $p > 2$; there is only one isomorphy class of k -involutions of G .

Proof. For part (1) we only need to notice that over the fields \mathbb{R} , \mathbb{Q} and \mathbb{Q}_2 that θ and $\theta \circ \mathcal{I}_{t_{(1, -1)}}$ leave nonisomorphic subalgebras elementwise fixed and so by 4.2.1.2, 4.2.1.8, and 4.2.2.2 they are not isomorphic. And by 2.2.6.2 and 4.2.3.1 these are the only 2 quaternion subalgebras up to isomorphism. There are no other possible isomorphy classes.

For part (2) by 2.2.6.2, 4.2.3.2, [16].

For part (3) by 2.2.6.2, 4.2.1.7, and 4.2.1.8 we have the result. \square

4.3 Fixed point groups

In order to compute the fixed point groups of each k -involution we first look at how such elements of $\text{Aut}(C)$ act on C .

Lemma 4.3.0.4. Let $t \in \text{Aut}(C) = G$ such that $t^2 = \text{id}$ and $D \subset C$ the quaternion algebra elementwise fixed by t then $f \in G^{\mathcal{I}_t} = \{g \in G \mid \mathcal{I}_t(g) = g\}$ if and only if f leaves D invariant.

Proof. (\Rightarrow): Let $D \subset C$ be fixed elementwise by t and $f \in G^t$, then $\mathcal{I}_t(f) = f$. Now let $c \in C$ be any element of the octonion algebra containing D , then we can write $C = D \oplus D^\perp$ and

$c = a + b$ where $a \in D$ and $b \in D^\perp$. Since t is a k -involution it has only ± 1 as eigenvalues and $t(a + b) = a - b$. Furthermore,

$$\begin{aligned}\mathcal{I}_t(f)(c) &= tft^{-1}(c) \\ &= tft(c) \\ &= tft(a + b) \\ &= tft(a) + tft(b) \\ &= tf(a) + f(b),\end{aligned}$$

and since $\mathcal{I}_t(f)(c) = f(c) = f(a + b)$ for all $c \in C$,

$$f(a) + f(b) = tf(a) + f(b).$$

From this we can conclude that $tf(a) = f(a)$ so $f(a) \in D$.

(\Leftarrow): If we assume, conversely, that $D \subset C$ is the subalgebra fixed elementwise by t and $f(D) = D$ then $f(D^\perp) = D^\perp$, and

$$\begin{aligned}tft^{-1}(c) &= tft(a + b) \\ &= tft(a) + tft(b) \\ &= tf(a) - tf(b) \\ &= f(a) + f(b) \\ &= f(c),\end{aligned}$$

for all $c \in C$ and we have the result. \square

Every involution in $\text{Aut}(C)$ leaves some quaternion algebra, D , elementwise fixed. Now we need only to see what automorphisms of C leave D invariant. In every case we will consider if we have a fixed quaternion algebra $D \subset C$, then $C = D \oplus D^\perp$ with respect to N . The automorphisms of C that leave D invariant, denoted $\text{Aut}(C, D)$ are of the form,

$$s(x + ya) = s(x) + s(y)s(a),$$

where $s \in \text{Aut}(C, D)$; $x, y \in D$ and $a \in D^\perp$ such that $N(a) \neq 0$. Since s leaves D invariant we can further see that

$$s(y) \in D, \text{ and } s(a) \in D^\perp \text{ such that } N(s(a)) \neq 0,$$

and we can write

$$s(x + ya) = s_{dp}(x + ya) = dxd^{-1} + (pdyd^{-1})a,$$

where $d, p \in D$ and $N(d) \neq 0, N(p) = 1$.

First let us consider the case where D is a split quaternion algebra. In this case $D \cong M_2(k)$ and $d \in \mathrm{GL}_2(k)$ and $p \in \mathrm{SL}_2(k)$.

Proposition 4.3.0.5. *When $t \in \mathrm{Aut}(C)$ is an involution and leaves $D \subset C$, a split quaternion subalgebra, elementwise fixed, then*

$$G^{\mathcal{I}_t} \cong \mathrm{PGL}_2(k) \times \mathrm{SL}_2(k).$$

Proof. If we consider the map

$$\psi : \mathrm{GL}_2(k) \times \mathrm{SL}_2(k) \rightarrow \mathrm{Aut}(C, D), \text{ where } (d, p) \mapsto s_{dp},$$

is surjective. The kernel is given by $\ker(\psi) = \{(\alpha \cdot e, e) \mid \alpha \in k^*\}$. \square

In the case where the involution t leaves a quaternion division algebra D invariant, we have the same initial set up, i.e., $s \in \mathrm{Aut}(C, D)$ then

$$s(x + ya) = s_{dp}(x + ya) = dxd^{-1} + (pdyd^{-1})a,$$

where $N(d) \neq 0$ and $N(p) = 1$, only $D \not\cong M_2(k)$ it is isomorphic to Hamilton's quaternions over k . In this case $N(d) \neq 0$ only tells us that $d \neq 0 \in D$.

Proposition 4.3.0.6. *When $t \in \mathrm{Aut}(C)$ is an involution and leaves $D \subset C$, a quaternion division algebra, elementwise fixed, then $G^{\mathcal{I}_t} \cong \mathrm{SO}(D_0, N) \times \mathrm{Sp}(1)$ where $D_0 = ke^\perp$.*

Proof. Then $N(p) = p\bar{p} = 1$ tells us that $p \in \mathrm{Sp}(1)$ is the group of 1×1 symplectic matrices over D . If we consider the surjective homomorphism

$$\psi : D^* \times \mathrm{Sp}(1) \rightarrow \mathrm{Aut}(C, D) \text{ where } (d, p) \mapsto s_{dp},$$

and $D^* = D - \{0\}$ is the group consisting of the elements of D having inverses, its kernel, $\ker(\psi) = \{(\alpha \cdot e, e) \mid \alpha \in k^*\}$. So we have

$$D^*/Z(D^*) \times \mathrm{Sp}(1) \cong \mathrm{Inn}(D^*) \times \mathrm{Sp}(1).$$

Jacobson tells us, [9], that all automorphisms of D^* are inner and leave the identity fixed. Also included in [9] is that $\mathrm{Inn}(D^*) \cong \mathrm{SO}(D_0, N)$, where $D_0 \subset D$ such that $D_0 = e^\perp$, the three dimensional subspace, and $\mathrm{SO}(D_0, N)$ is the group of rotations of D_0 with respect to $N|_{D_0}$. \square

4.3.1 k -involutions when $k = \mathbb{Q}$

Isomorphy classes of quaternion algebras over k are given by equivalence classes of 2-Pfister forms $\left(\frac{\alpha, \beta}{k}\right)$ over k corresponding to the quadratic form, [11],

$$N(x) = x_0^2 + (-\alpha)x_1^2 + (-\beta)x_2^2 + (\alpha\beta)x_3^2,$$

while octonion algebras depend on 3-Pfister forms, $\left(\frac{\alpha, \beta, \gamma}{k}\right)$ with quadratic form,

$$N(x) = x_0^2 + (-\alpha)x_1^2 + (-\beta)x_2^2 + (\alpha\beta)x_3^2 + (-\gamma)x_4^2 + (\alpha\gamma)x_5^2 + (\beta\gamma)x_6^2 + (-\alpha\beta\gamma)x_7^2.$$

It is not difficult to show that for a prime p , such that $p \equiv 3 \pmod{4}$,

$$\left(\frac{-1, p}{\mathbb{Q}}\right)$$

is a division algebra. Further, for p and q distinct primes both equivalent to $3 \pmod{4}$

$$\left(\frac{-1, p}{\mathbb{Q}}\right) \not\cong \left(\frac{-1, q}{\mathbb{Q}}\right),$$

see [18] Exercise 1.7.4.

Example 4.3.1.1. Let $C = (M_2(\mathbb{Q}), M_2(\mathbb{Q}))$. We can find an involution of $\text{Aut}(C)$ leaving $D \cong \left(\frac{-1, p}{\mathbb{Q}}\right)$ elementwise fixed by first constructing a basis for D . If we pick

$$a = \left(0, \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\right) \text{ and } b = \left(\begin{bmatrix} & p \\ 1 & \end{bmatrix}, 0\right),$$

then $a^2 = -1$ and $b^2 = p$. Then the other basis elements of D are $e = (\text{id}, 0)$ and

$$ab = \left(0, \begin{bmatrix} -1 & \\ & -p \end{bmatrix}\right).$$

Let $p \equiv 3 \pmod{4}$, then D is a division algebra and

$$s_p = \begin{bmatrix} & & 1 \\ & p & \\ p^{-1} & & \end{bmatrix} \bigoplus \begin{bmatrix} & & p^{-1} \\ & 1 & \\ p & & \end{bmatrix} \in \text{Aut}(C),$$

leaves D elementwise fixed. Notice $s_p = s \cdot t_{(p,1)}$. There is a \mathbb{Q} -involution for each $p \equiv 3 \pmod{4}$

of which there are an infinite number.

4.3.2 k is an algebraic number field

When the quaternion or octonion algebra is taken over an algebraic number field we have that 2 quadratic forms are equivalent if and only if they are equivalent over all local fields k_ν , where ν varies over all places of k , which is a result of Hasse's Theorem [20]. Hasse's Theorem also tells us the number of possible anisotropic quadratic forms corresponding to nonisomorphic octonion algebras is given an upper bound of 2^r where r is the number of real places of k [25].

This concludes our discussion and classification of k -involutions of type G_2 .

4.4 k -involutions of F_4

Here we recall some facts outlined earlier that will help us with the classification of k -involutions of type F_4 , and then we proceed to the classification.

4.4.1 Split Albert algebra

We will let k be a field over characteristic not 2, and C a split composition algebra of dimension eight, as discussed earlier, over k . For any fixed $\gamma_i \in k^*$, we will define $A = H(C, \gamma)$ be the set of 3×3 γ -hermitian matrices. Each $x \in A$, where $\xi_i \in k$ and $c_i \in C$, will look like

$$x = h(\xi_1, \xi_2, \xi_3; c_1, c_2, c_3) = \begin{bmatrix} \xi_1 & c_3 & \gamma_1^{-1}\gamma_3\bar{c}_2 \\ \gamma_2^{-1}\gamma_1\bar{c}_3 & \xi_2 & c_1 \\ c_2 & \gamma_3^{-1}\gamma_2\bar{c}_1 & \xi_3 \end{bmatrix},$$

where - will denote the following conjugation, $\bar{x} = \langle x, e \rangle e - x$, in C .

We will define a product

$$xy = \frac{1}{2}(x \cdot y + y \cdot x) = \frac{1}{2}((x+y)^2 - x^2 - y^2)$$

with the dot indicating standard matrix multiplication. Notice

$$x(yz) = \frac{1}{2}(x \cdot yz + yz \cdot x) = \frac{1}{4}(x \cdot (y \cdot z + z \cdot y) + (y \cdot z + z \cdot y) \cdot x) \neq (xy)z,$$

however,

$$yx = \frac{1}{2}(y \cdot x + x \cdot y) = \frac{1}{2}(x \cdot y + y \cdot x) = xy.$$

So A is a commutative, nonassociative k -algebra of 3×3 matrices, whose identity element is $e = h(1, 1, 1; 0, 0, 0)$, the usual 3×3 matrix identity.

We will define a quadratic norm, $Q : A \rightarrow k$, with an associated bilinear form

$$\langle x, y \rangle = Q(x + y) - Q(x) - Q(y),$$

and from now on the bilinear form on C will be denoted $N(\cdot, \cdot)$, with

$$Q(x) = \frac{1}{2} (\xi_1^2 + \xi_2^2 + \xi_3^2) + \gamma_3^{-1} \gamma_2 N(c_1) + \gamma_1^{-1} \gamma_3 N(c_2) + \gamma_2^{-1} \gamma_1 N(c_3) = \frac{1}{2} \text{tr}(x^2).$$

Notice the bilinear form is nondegenerate, and $\langle x, y \rangle = \text{tr}(xy)$. With all of this together we will refer to $A = H(C; \gamma)$, where C is octonion and γ is a diagonal matrix with entries in k^* , as an *Albert algebra*.

4.4.2 Decomposition by a primitive idempotent (Peirce decomposition)

The idempotent elements of a J -algebra play an important role. If $u \in A$ and $u^2 = u$ then u is an *idempotent* element.

Lemma 4.4.2.1. *If $u \in A$ is an idempotent element and u is not 0 or e , then $\det(u) = 0$, $Q(u) = \frac{1}{2}$ or 1, $\langle u, e \rangle = 2Q(u)$, $e - u$ is idempotent, $u(e - u) = 0$, $\langle u, e - u \rangle = 0$, and $Q(e - u) = \frac{3}{2} - Q(u)$.*

Proof. We have $\langle u, e \rangle = \text{tr}(ue) = \text{tr}(u) = \text{tr}(u^2) = 2Q(u)$.

This gives us, with idempotency and (1.1),

$$\begin{aligned} \det(u) &= \frac{1}{3} \langle u^2, u \rangle - Q(u) \langle u, e \rangle + \frac{1}{6} \langle u, e \rangle^3 \\ &= \frac{1}{3} \langle u, e \rangle - Q(u)(2Q(u)) + \frac{1}{6} (2Q(u))^3 \\ &= \frac{2}{3} Q(u) - 2Q(u) + \frac{4}{3} Q(u) \\ &= 0. \end{aligned}$$

So now we can say that since

$$\begin{aligned}
\det(u) = 0 &= u^3 - \langle u, e \rangle u^2 - (Q(u) - \frac{1}{2} \langle u, e \rangle^2)u \\
&= u - \langle u, e \rangle u - Q(u)u + \frac{1}{2} \langle u, e \rangle^2 u \\
&= \left(1 - \langle u, e \rangle - Q(u) + \frac{1}{2} \langle u, e \rangle^2\right) u \\
&= \left(1 - 2Q(u) - Q(u) + \frac{1}{2} (2Q(u))^2\right) u \\
&= (1 - 3Q(u) + 2Q(u)^2) u.
\end{aligned}$$

This gives us, when $u \neq 0$,

$$Q(u)^2 - \frac{3}{2}Q(u) + \frac{1}{2} = 0.$$

From this we know that $Q(u) = \frac{1}{2}, 1$, when $u \neq 0$. If we look at

$$(e - u)^2 = e^2 - eu - ue + u^2 = e - u - u + u = e - u,$$

we notice that $e - u$ is idempotent. And also that

$$u(e - u) = u - u^2 = 0.$$

And continuing throught the lemma,

$$\langle u, e - u \rangle = \langle u, e \rangle - \langle u, u \rangle = \langle u, e \rangle - \langle u, e \rangle = 0.$$

And finally

$$Q(e - u) = \langle e, -u \rangle + Q(u) + Q(e) = -2Q(u) + Q(u) + \frac{3}{2} = \frac{3}{2} - Q(u),$$

which gives us the fact that if A contains an idempotent not equal to 0 or e then there exists an idempotent $u \in A$ such that $Q(u) = \frac{1}{2}$. \square

If $Q(u) = \frac{1}{2}$ then we call u a *primitive idempotent*.

Theorem, [25] 4.4.2.2. *If A is a proper reduced J-algebra, then the composition algebra C where $A \cong H(C, \gamma)$ is uniquely determined up to isomorphism.*

If we fix a primitive idempotent $u \in A$, and let

$$E_0 = \{a \in e^\perp \subset A \mid ua = 0\}, \text{ and let } E_1 = \{a \in e^\perp \subset A \mid ua = \frac{1}{2}a\},$$

then an Albert algebra has a decomposition

$$A = ku \oplus k(e - u) \oplus E_0 \oplus E_1,$$

called the *Peirce decomposition*. For example if we consider $A = H(C, \gamma)$, and let

$$u = \begin{bmatrix} 1 & & \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix},$$

then

$$e - u = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix},$$

and elements E_0 and E_1 are of the form,

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \xi & x_1 \\ \cdot & \overline{x_1} & -\xi \end{bmatrix}, \begin{bmatrix} \cdot & x_3 & \overline{x_2} \\ \overline{x_3} & \cdot & \cdot \\ x_2 & \cdot & \cdot \end{bmatrix},$$

respectively.

4.4.3 First Tits construction

We first need to define a sharped cubic form, $(N, \#, c)$ on X , a module over a ring of scalars. First we will choose a basepoint c , such that $N(c) = 1$. Another component we define is the $\#$ quadratic map, for $x \in X$, such that

$$T(x^\#, y) = N(x; y) \tag{4.1}$$

$$x^{\#\#} = N(x)x \tag{4.2}$$

$$c\#x = T(y)c - y. \tag{4.3}$$

And further we define the following forms,

$$T(x) = N(c; x) \quad (4.4)$$

$$S(x) = N(x; c) \quad (4.5)$$

$$T(x, y) = T(x)T(y) - N(c, x, y) \quad (4.6)$$

$$S(x, y) = T(x)T(y) - T(x, y) \quad (4.7)$$

$$T(x) = T(x, c) \quad (4.8)$$

$$T(c) = 3 = S(c) \quad (4.9)$$

$$x\#y = (x + y)^\# - x^\# - y^\#. \quad (4.10)$$

A space X with $(N, \#, c)$ such that all the above are satisfied is called a *sharped cubic form* on X . From a sharped cubic form $(N, \#, c)$ we can construct a unital Jordan algebra, $J(N, \#, c)$, which has unit element c , and a U operator defined by

$$U_x y = T(x, y)x - x^\# \# y.$$

We can define other products using the U operator as follows

$$x^2 = U_x c \quad (4.11)$$

$$\{x, y\} = U_{x,y} c \quad (4.12)$$

$$xy = \frac{1}{2}U_{x,y}c. \quad (4.13)$$

Proposition 4.4.3.1. *Any sharped cubic form $(N, \#, c)$ gives a unital Jordan algebra $J(N, \#, c)$ with unit, c and product*

$$xy = \frac{1}{2}(x\#y + T(x)y + T(y)x - S(x, y)c).$$

Finally, we relate the sharp product and sharp map to the above products in the following way,

$$x^\# = x^2 - T(x)x + S(x)c \quad (4.14)$$

$$x\#y = \{x, y\} - T(x)y - T(y)x + S(x, y)c \quad (4.15)$$

$$0 = x^3 - T(x)x^2 + S(x)x - N(x)c. \quad (4.16)$$

From here we define the Tits first construction for making Jordan algebras of cubic form type of degree 3 out of associative algebras of degree 3. Let X be an associative algebra of

degree 3 over a unital commutative ring R with a cubic norm form n satisfying the following axioms beginning with

$$x^3 - t(x)x^2 + s(x)x - n(x)1 = 0, \quad (4.17)$$

with $t(x) = n(1; x)$, $s(x) = n(x; 1)$, $n(1) = 1$. Notice that also if $x^\# = x^2 - t(x)x + s(x)1$ we can write

$$xx^\# = x^\#x = n(x)1. \quad (4.18)$$

The second axiom is

$$n(x; y) = t(x^\#, y), \quad (4.19)$$

and the last is

$$t(x, y) = t(xy). \quad (4.20)$$

Since we can also think of $t(x, y) = t(x)t(y) - n(1, x, y)$, and $n(\ , \ , \)$ is symmetric,

$$s(x, y) = n(x, y, 1) = n(1, x, y) = t(x)t(y) - t(x, y) = t(x)t(y) - t(xy).$$

Proposition 4.4.3.2. *Let X be a unital associative algebra with a basepoint such that $n(c) = 1$, $x^\# = x^2 - t(x)x + s(x)1$, and $x\#y = (x + y)^\# - x^\# - y^\#$. Then any cubic form on X satisfies*

$$t(1) = s(1) = 3 \quad (4.21)$$

$$1^\# = 1 \quad (4.22)$$

$$s(x, 1) = 2t(x) \quad (4.23)$$

$$1\#x = t(x)1 - x \quad (4.24)$$

$$s(x) = t(x^\#) \quad (4.25)$$

$$2s(x) = t(x)^2 - t(x^2). \quad (4.26)$$

For the remainder of the construction we follow [17].

First Tits Construction 4.4.3.3. *Let n be the cubic norm form of a degree 3 associative algebra X over R , and let $\mu \in R^*$. We define a module*

$$J(X, \mu) = X_0 \oplus X_1 \oplus X_2,$$

to be the direct sum of three copies of X , and define c , N , T , and $\#$ by

$$c = \text{id} \oplus 0 \oplus 0 \quad (4.27)$$

$$N(x) = n(x_0) + \mu n(x_1) + \mu^2 n(x_2) - t(x_0 x_1 x_2) \quad (4.28)$$

$$T(x) = t(x_0) \quad (4.29)$$

$$T(x, y) = t(x_0, y_0) + t(x_1, y_2) + t(x_2, y_1) \quad (4.30)$$

$$x^\# = (x_0^\# - \mu x_1 x_2) \oplus (\mu x_2^\# - x_0 x_1) \oplus (x_1^\# - x_2 x_0) \quad (4.31)$$

$$(xy)^\# = y^\# x^\#, \quad (4.32)$$

where n and t are the norm and trace on X for $x = (x_0, x_1, x_2)$ and $y = (y_0, y_1, y_2)$. Then $(N, \#, c)$ is a sharped cubic form and $J(N, \#, c)$ is a Jordan algebra.

Proof. See [14]. \square

We want to construct a split Albert algebra over a field k . To this end we pick our associative algebra $X = M_3(k)$ with $n = \det$, $t = \text{tr}$, and $c = \text{id}$, and $\mu = 1$.

Proposition 4.4.3.4. *The algebra $J(M_3(k), 1)$ is a split Albert algebra.*

Proof. This is constructed in [3], [17]. \square

Here we beginning looking at examples of k -involutions of type F_4 with our goal being a complete classification of isomorphy classes of k -involutions over certain fields.

4.4.4 Involutions of G_2 as involutions of F_4

Theorem 4.4.4.1. *G_2 is a subgroup of F_4 and all involutions in G_2 are involutions of F_4 .*

Proof. For the split case every Albert algebra A is isomorphic over the given field, k , so we can take $\gamma_i = 1$, for $i = 1, 2, 3$. We can think of elements of A as having the form

$$\begin{bmatrix} \eta_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \eta_2 & x_1 \\ x_2 & \overline{x_1} & \eta_3 \end{bmatrix} \leftrightarrow \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

where $\eta_i \in k$ and $x_i \in C$, an eight dimensional split octonion algebra. The product of two

elements $x, y \in A$ would look like

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2\eta_1\omega_1 + 2\operatorname{Re}(\bar{x}_2y_2) + 2\operatorname{Re}(x_3\bar{y}_3) \\ 2\eta_2\omega_2 + 2\operatorname{Re}(\bar{x}_3y_3) + 2\operatorname{Re}(x_1\bar{y}_1) \\ 2\eta_3\omega_3 + 2\operatorname{Re}(\bar{x}_1y_1) + 2\operatorname{Re}(x_2\bar{y}_2) \\ (\eta_2 + \eta_3)y_1 + (\omega_2 + \omega_3)x_1 + \bar{y}_2x_3 + \bar{x}_2y_3 \\ (\eta_1 + \eta_3)y_2 + (\omega_1 + \omega_3)x_2 + \bar{y}_3x_1 + \bar{x}_3y_1 \\ (\eta_1 + \eta_2)y_3 + (\omega_1 + \omega_2)x_3 + \bar{y}_1x_2 + \bar{x}_1y_2 \end{bmatrix}.$$

We can let $t \in \operatorname{Aut}(C)_k$ act on A as follows

$$t \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ t(x_1) \\ t(x_2) \\ t(x_3) \end{pmatrix}.$$

To see that this is an automorphism of A we look at

$$t(xy) = t \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{1}{2} t \begin{pmatrix} 2\eta_1\omega_1 + 2\operatorname{Re}(\bar{x}_2y_2) + 2\operatorname{Re}(x_3\bar{y}_3) \\ 2\eta_2\omega_2 + 2\operatorname{Re}(\bar{x}_3y_3) + 2\operatorname{Re}(x_1\bar{y}_1) \\ 2\eta_3\omega_3 + 2\operatorname{Re}(\bar{x}_1y_1) + 2\operatorname{Re}(x_2\bar{y}_2) \\ (\eta_2 + \eta_3)y_1 + (\omega_2 + \omega_3)x_1 + \bar{y}_2x_3 + \bar{x}_2y_3 \\ (\eta_1 + \eta_3)y_2 + (\omega_1 + \omega_3)x_2 + \bar{y}_3x_1 + \bar{x}_3y_1 \\ (\eta_1 + \eta_2)y_3 + (\omega_1 + \omega_2)x_3 + \bar{y}_1x_2 + \bar{x}_1y_2 \end{pmatrix}.$$

Now if we look at the action of t on the fourth component of xy ,

$$t((\eta_2 + \eta_3)y_1 + (\omega_2 + \omega_3)x_1 + \bar{y}_2x_3 + \bar{x}_2y_3) = (\eta_2 + \eta_3)t(y_1) + (\omega_2 + \omega_3)t(x_1) + t(\bar{y}_2x_3) + t(\bar{x}_2y_3),$$

and we can notice that

$$t(\bar{x}) = t(\langle x, e \rangle e - x) = \langle t(x), t(e) \rangle e - t(x) = \langle t(x), e \rangle e - t(x) = \overline{t(x)},$$

since $t \in \text{Aut}(C) \subset \text{SO}(N)$, we have that

$$\begin{aligned} t((\eta_2 + \eta_3)y_1 + (\omega_2 + \omega_3)x_1 + \overline{y_2x_3} + \overline{x_2y_3}) \\ = (\eta_2 + \eta_3)t(y_1) + (\omega_2 + \omega_3)t(x_1) + \overline{t(y_2x_3)} + \overline{t(x_2y_3)} \\ = (\eta_2 + \eta_3)t(y_1) + (\omega_2 + \omega_3)t(x_1) + \overline{t(y_2)t(x_3)} + \overline{t(x_2)t(y_3)}, \end{aligned}$$

and this is also true for the fifth and sixth component of xy as they have the same form as the fourth component. If we look at the first component of xy we see that

$$\text{Re}(t(x_3)t(\overline{y_3})) = \text{Re}(t(x_3\overline{y_3})) = \text{Re}(x_3\overline{y_3}),$$

since t leaves the identity component of C invariant. So we can say that

$$t(xy) = \frac{1}{2} \begin{bmatrix} 2\eta_1\omega_1 + 2\text{Re}(\overline{t(x_2)}t(y_2)) + 2\text{Re}(t(x_3)\overline{t(y_3)}) \\ 2\eta_2\omega_2 + 2\text{Re}(\overline{t(x_3)}t(y_3)) + 2\text{Re}(t(x_1)\overline{t(y_1)}) \\ 2\eta_3\omega_3 + 2\text{Re}(\overline{t(x_1)}t(y_1)) + 2\text{Re}(t(x_2)\overline{t(y_2)}) \\ (\eta_2 + \eta_3)t(y_1) + (\omega_2 + \omega_3)t(x_1) + \overline{t(y_2)t(x_3)} + \overline{t(x_2)t(y_3)} \\ (\eta_1 + \eta_3)t(y_2) + (\omega_1 + \omega_3)t(x_2) + \overline{t(y_3)t(x_1)} + \overline{t(x_3)t(y_1)} \\ (\eta_1 + \eta_2)t(y_3) + (\omega_1 + \omega_2)t(x_3) + \overline{t(y_1)t(x_2)} + \overline{t(x_1)t(y_2)} \end{bmatrix} = t(x)t(y).$$

□

We can also note at this point that this is true for any $A = H(C; \gamma_1, \gamma_2, \gamma_3)$, since t is linear.

4.4.5 An involution not from G_2

Theorem 4.4.5.1. *There is a series of k -involutions not from G_2 in F_4 when the $\gamma_i = 1$ in A , the Albert algebra for which F_4 is the automorphism group. They are of the form $s_{ij} : A \rightarrow A$, where η_i and η_j switch positions, each of the octonion elements are conjugated, and switch the i^{th} and j^{th} position of the octonion elements. The one we are most concerned with is*

$$s_{23} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_3 \\ \eta_2 \\ \overline{x_1} \\ \overline{x_3} \\ \overline{x_2} \end{pmatrix}.$$

Proof. There is an involution induced when we switch η_i and η_j , $i \neq j$. We will demonstrate

this now using η_1 and η_2 . So if $s_{12} : A \rightarrow A$ is the map

$$\begin{aligned}s_{12} : \eta_1 &\leftrightarrow \eta_2 \\x_1 &\mapsto \overline{x_2} \\x_2 &\mapsto \overline{x_1} \\x_3 &\mapsto \overline{x_3},\end{aligned}$$

we have

$$s_{12} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \eta_1 \\ \eta_3 \\ \overline{x_2} \\ \overline{x_1} \\ \overline{x_3} \end{pmatrix},$$

and this is an involution, since

$$s_{12} \begin{pmatrix} \eta_2 \\ \eta_1 \\ \eta_3 \\ \overline{x_2} \\ \overline{x_1} \\ \overline{x_3} \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \overline{\overline{x_1}} \\ \overline{\overline{x_2}} \\ \overline{\overline{x_3}} \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Now let us show this is a automorphism, it is clearly a bijection,

$$\begin{aligned}s_{12}(xy) &= s_{12} \begin{pmatrix} 2\eta_1\omega_1 + 2\operatorname{Re}(\overline{x_2}y_2) + 2\operatorname{Re}(x_3\overline{y_3}) \\ 2\eta_2\omega_2 + 2\operatorname{Re}(\overline{x_3}y_3) + 2\operatorname{Re}(x_1\overline{y_1}) \\ 2\eta_3\omega_3 + 2\operatorname{Re}(\overline{x_1}y_1) + 2\operatorname{Re}(x_2\overline{y_2}) \\ (\eta_2 + \eta_3)y_1 + (\omega_2 + \omega_3)x_1 + \overline{y_2x_3} + \overline{x_2y_3} \\ (\eta_1 + \eta_3)y_2 + (\omega_1 + \omega_3)x_2 + \overline{y_3x_1} + \overline{x_3y_1} \\ (\eta_1 + \eta_2)y_3 + (\omega_1 + \omega_2)x_3 + \overline{y_1x_2} + \overline{x_1y_2} \end{pmatrix} \\&= \frac{1}{2} \begin{pmatrix} 2\eta_2\omega_2 + 2\operatorname{Re}(\overline{x_3}y_3) + 2\operatorname{Re}(x_1\overline{y_1}) \\ 2\eta_1\omega_1 + 2\operatorname{Re}(\overline{x_2}y_2) + 2\operatorname{Re}(x_3\overline{y_3}) \\ 2\eta_3\omega_3 + 2\operatorname{Re}(\overline{x_1}y_1) + 2\operatorname{Re}(x_2\overline{y_2}) \\ \overline{(\eta_1 + \eta_3)y_2 + (\omega_1 + \omega_3)x_2 + \overline{y_3x_1} + \overline{x_3y_1}} \\ \overline{(\eta_2 + \eta_3)y_1 + (\omega_2 + \omega_3)x_1 + \overline{y_2x_3} + \overline{x_2y_3}} \\ \overline{(\eta_1 + \eta_2)y_3 + (\omega_1 + \omega_2)x_3 + \overline{y_1x_2} + \overline{x_1y_2}} \end{pmatrix}\end{aligned}$$

$$= \frac{1}{2} \begin{bmatrix} 2\eta_2\omega_2 + 2\operatorname{Re}(\bar{x}_3y_3) + 2\operatorname{Re}(x_1\bar{y}_1) \\ 2\eta_1\omega_1 + 2\operatorname{Re}(\bar{x}_2y_2) + 2\operatorname{Re}(x_3\bar{y}_3) \\ 2\eta_3\omega_3 + 2\operatorname{Re}(\bar{x}_1y_1) + 2\operatorname{Re}(x_2\bar{y}_2) \\ (\eta_1 + \eta_3)\bar{y}_2 + (\omega_1 + \omega_3)\bar{x}_2 + y_3x_1 + x_3y_1 \\ (\eta_2 + \eta_3)\bar{y}_1 + (\omega_2 + \omega_3)\bar{x}_1 + y_2x_3 + x_2y_3 \\ (\eta_1 + \eta_2)\bar{y}_3 + (\omega_1 + \omega_2)\bar{x}_3 + y_1x_2 + x_1y_2 \end{bmatrix}.$$

Now if we look at

$$\begin{aligned} s_{12}(x)s_{12}(y) &= s_{12} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} s_{12} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \eta_1 \\ \eta_3 \\ \bar{x}_2 \\ \bar{x}_1 \\ \bar{x}_3 \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \\ \omega_3 \\ \bar{y}_2 \\ \bar{y}_1 \\ \bar{y}_3 \end{pmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 2\eta_2\omega_2 + 2\operatorname{Re}(x_1\bar{y}_1) + 2\operatorname{Re}(\bar{x}_3y_3) \\ 2\eta_1\omega_1 + 2\operatorname{Re}(x_3\bar{y}_3) + 2\operatorname{Re}(\bar{x}_2y_2) \\ 2\eta_3\omega_3 + 2\operatorname{Re}(x_2\bar{y}_2) + 2\operatorname{Re}(\bar{x}_1y_1) \\ (\eta_1 + \eta_3)\bar{y}_2 + (\omega_1 + \omega_3)\bar{x}_2 + \bar{y}_1\bar{x}_3 + \bar{x}_1\bar{y}_3 \\ (\eta_2 + \eta_3)\bar{y}_1 + (\omega_2 + \omega_3)\bar{x}_1 + \bar{y}_3\bar{x}_2 + \bar{x}_3\bar{y}_2 \\ (\eta_2 + \eta_1)\bar{y}_3 + (\omega_2 + \omega_1)\bar{x}_3 + \bar{y}_2\bar{x}_1 + \bar{x}_2\bar{y}_1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 2\eta_2\omega_2 + 2\operatorname{Re}(x_1\bar{y}_1) + 2\operatorname{Re}(\bar{x}_3y_3) \\ 2\eta_1\omega_1 + 2\operatorname{Re}(x_3\bar{y}_3) + 2\operatorname{Re}(\bar{x}_2y_2) \\ 2\eta_3\omega_3 + 2\operatorname{Re}(x_2\bar{y}_2) + 2\operatorname{Re}(\bar{x}_1y_1) \\ (\eta_1 + \eta_3)\bar{y}_2 + (\omega_1 + \omega_3)\bar{x}_2 + x_3y_1 + y_3x_1 \\ (\eta_2 + \eta_3)\bar{y}_1 + (\omega_2 + \omega_3)\bar{x}_1 + x_2y_3 + y_2x_3 \\ (\eta_2 + \eta_1)\bar{y}_3 + (\omega_2 + \omega_1)\bar{x}_3 + x_1y_2 + y_1x_2 \end{bmatrix}, \end{aligned}$$

so we have that $s_{12}(xy) = s_{12}(x)s_{12}(y)$, i.e. $s_{12} \in \operatorname{Aut}(A)$. We get three distinct elements $s_{ij} \in \operatorname{Aut}(A)$ such that $i < j$, and $s_{ij}^2 \in Z(\operatorname{Aut}(A)) = \{\text{id}\}$. \square

On the other hand if we use the

$$A = ke \oplus k(e - u) \oplus \underbrace{kz_1 \oplus C}_{E_0} \oplus E_1,$$

we can fix a primitive idempotent element and see the maps s_{ij} differently. So let

$$u = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & \ddots \end{bmatrix},$$

then through simple computations we have, in the case where the $\gamma_i = 1$,

$$e - u = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}, \quad z_1 = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot \\ \cdot & \cdot & -\frac{1}{2} \end{bmatrix}$$

and the elements of C and E_1 look like

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & x_1 \\ \cdot & \overline{x_1} & \cdot \end{bmatrix} \text{ and } \begin{bmatrix} \cdot & x_3 & \overline{x_2} \\ \overline{x_3} & \cdot & \cdot \\ x_2 & \cdot & \cdot \end{bmatrix},$$

respectively, where x_i are elements of a split octonion algebra over k . The relevant map that leaves u invariant is s_{23} . Under this new basis an element of A would look like,

$$\begin{bmatrix} f & x_3 & \overline{x_2} \\ \overline{x_3} & g + h & x_1 \\ x_2 & \overline{x_1} & g - h \end{bmatrix} \leftrightarrow \begin{bmatrix} f \\ g \\ h \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Now the map s_{23} , when using this u decomposition basis, would look like

$$s_{23} \begin{pmatrix} \begin{bmatrix} f \\ g \\ h \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{pmatrix} = \begin{bmatrix} f \\ g \\ -h \\ \overline{x_1} \\ \overline{x_3} \\ \overline{x_2} \end{bmatrix},$$

and we can see that this fixes u and $(e - u)$, while leaving the subspaces, E_0 and E_1 , invariant.

4.4.6 The subspaces E_0 and E_1 for $A = J(M_3(k), 1)$

We can fix a primitive idempotent element in the split Albert algebra with underlying vector space, $M_3(k) \oplus M_3(k) \oplus M_3(k)$, as described above,

$$u = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \oplus 0 \oplus 0.$$

In this case we get the subspaces

$$E_0 = z_1 + \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & * \\ \cdot & * & \cdot \end{bmatrix} \oplus \begin{bmatrix} * & * & * \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \oplus \begin{bmatrix} * & \cdot & \cdot \\ * & \cdot & \cdot \\ * & \cdot & \cdot \end{bmatrix},$$

where

$$z_1 = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot \\ \cdot & \cdot & -\frac{1}{2} \end{bmatrix} \oplus 0 \oplus 0,$$

and

$$E_1 = \begin{bmatrix} \cdot & * & * \\ * & \cdot & \cdot \\ * & \cdot & \cdot \end{bmatrix} \oplus \begin{bmatrix} \cdot & \cdot & \cdot \\ * & * & * \\ * & * & * \end{bmatrix} \oplus \begin{bmatrix} \cdot & * & * \\ \cdot & * & * \\ \cdot & * & * \end{bmatrix}.$$

Recall that $\text{Aut}(A)$ acts transitively on primitive idempotents when A is of split type Corollary 2.2.8.3. If we fix a primitive idempotent element u then clearly for $t \in \text{Aut}(A)$, $t(u)$ is a primitive idempotent. If u, u_1, u_2 are primitive idempotents such that $\langle u, u_1 \rangle = \langle u, u_2 \rangle = \langle u_1, u_2 \rangle = 0$ and $u + u_1 + u_2 = e$, then

$$\begin{aligned} t(u + u_1 + u_2) &= t(e) \\ t(u) + t(u_1) + t(u_2) &= e. \end{aligned}$$

We, also, notice that $t \in \text{Aut}(A)$ leaves Q and thus $\langle \cdot, \cdot \rangle$ invariant. So $\langle t(u), t(u_1) \rangle = \langle t(u), t(u_2) \rangle = \langle t(u_1), t(u_2) \rangle = 0$. Let us denote the automorphisms of order 2 in $\text{Aut}(A)$ by $\text{Aut}_2(A)$.

If $s \in \text{Aut}_2(A)$ and $\langle u, s(u) \rangle \neq 0$, then by 2.2.6.3,

$$s(u) = (Q(y) + 1)^{-1} \left(u + \frac{1}{2} Q(y)(e - u) + y \circ y + y \right),$$

where $y \in E_1$ and $Q(y) \neq -1$. So when we look at $s^2(u) = u$ and let $\alpha = (Q(y) + 1)$ we have,

$$\begin{aligned} s^2(u) &= u = \alpha \left(s(u) + \frac{1}{2}Q(y)(s(e) - s(u)) + s(y) \circ s(y) + s(y) \right). \\ &= \alpha \left[\alpha \left(u + \frac{1}{2}Q(y)(e - u) + y \circ y + y \right) \right. \\ &\quad \left. + \frac{1}{2}Q(y) \left(e - \alpha \left(u + \frac{1}{2}Q(y)(e - u) + y \circ y + y \right) \right) + s(y) \circ s(y) + s(y) \right] \\ &= \alpha \left[\left(\alpha - \frac{\alpha}{4}Q(y) + \frac{\alpha}{4}Q(y)^2 \right) u + \left(\frac{\alpha+1}{2}Q(y) - \frac{\alpha}{4}Q(y)^2 \right) e \right. \\ &\quad \left. + \left(\alpha - \frac{\alpha}{2}Q(y) \right) y \circ y + s(y) \circ s(y) + \left(\alpha - \frac{\alpha}{2}Q(y) \right) y + s(y) \right]. \end{aligned}$$

The following results on elements of order 2 in $\text{Aut}(A)$, where A is an Albert algebra, can be found in [10], along with further results on elements of order 2 in $\text{Aut}(A)$ not needed presently.

Theorem, [10] 4.4.6.1. *Let A be a finite dimensional exceptional central simple Jordan algebra. Then A is reduced if and only if $\text{Aut}(A)$ contains elements of period two. If the condition holds then any $s \in \text{Aut}(A)$ having period two is either;*

- (I) *a reflection in a sixteen dimensional central simple subalgebra of degree three,*
- (II) *the center element $r_u \neq 1$ in a subgroup $\text{Aut}(A)_u$, u is a primitive idempotent.*

Corollary, [10] 4.4.6.2. *An involution of type (I) leaves a subalgebra $H_K \subset A_K \cong H(C, \gamma)$ where K is algebraically closed and $H_K \cong H(D, \gamma)$ where $D \subset C$ is a quaternion subalgebra over K .*

Proposition, [10] 4.4.6.3. *If A is split then any two automorphisms of A of period two and type (II) are conjugate in $\text{Aut}(A)$, also when k is either finite or algebraically closed, then any two automorphisms of A of period two and type (I) are conjugate in $\text{Aut}(A)$.*

If we fix u as above r_u will be defined by

$$r_u \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ x_1 \\ -x_2 \\ -x_3 \end{pmatrix}$$

Let us see that r_u is an automorphism, since it clearly has order two. First let us compute

$$\begin{aligned}
r_u(xy) &= r_u \left(\frac{1}{2} \begin{bmatrix} 2\eta_1\omega_1 + 2\operatorname{Re}(\bar{x}_2y_2) + 2\operatorname{Re}(x_3\bar{y}_3) \\ 2\eta_2\omega_2 + 2\operatorname{Re}(\bar{x}_3y_3) + 2\operatorname{Re}(x_1\bar{y}_1) \\ 2\eta_3\omega_3 + 2\operatorname{Re}(\bar{x}_1y_1) + 2\operatorname{Re}(x_2\bar{y}_2) \\ (\eta_2 + \eta_3)y_1 + (\omega_2 + \omega_3)x_1 + \bar{y}_2x_3 + \bar{x}_2y_3 \\ (\eta_1 + \eta_3)y_2 + (\omega_1 + \omega_3)x_2 + \bar{y}_3x_1 + \bar{x}_3y_1 \\ (\eta_1 + \eta_2)y_3 + (\omega_1 + \omega_2)x_3 + \bar{y}_1x_2 + \bar{x}_1y_2 \end{bmatrix} \right) \\
&= \frac{1}{2} \begin{bmatrix} 2\eta_1\omega_1 + 2\operatorname{Re}(\bar{x}_2y_2) + 2\operatorname{Re}(x_3\bar{y}_3) \\ 2\eta_2\omega_2 + 2\operatorname{Re}(\bar{x}_3y_3) + 2\operatorname{Re}(x_1\bar{y}_1) \\ 2\eta_3\omega_3 + 2\operatorname{Re}(\bar{x}_1y_1) + 2\operatorname{Re}(x_2\bar{y}_2) \\ (\eta_2 + \eta_3)y_1 + (\omega_2 + \omega_3)x_1 + \bar{y}_2x_3 + \bar{x}_2y_3 \\ -((\eta_1 + \eta_3)y_2 + (\omega_1 + \omega_3)x_2 + \bar{y}_3x_1 + \bar{x}_3y_1) \\ -((\eta_1 + \eta_2)y_3 + (\omega_1 + \omega_2)x_3 + \bar{y}_1x_2 + \bar{x}_1y_2) \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 2\eta_1\omega_1 + 2\operatorname{Re}(\bar{(-x}_2)(-y_2)) + 2\operatorname{Re}((-x_3)\bar{(-y}_3)) \\ 2\eta_2\omega_2 + 2\operatorname{Re}(\bar{(-x}_3)(-y_3)) + 2\operatorname{Re}(x_1\bar{y}_1) \\ 2\eta_3\omega_3 + 2\operatorname{Re}(\bar{x}_1y_1) + 2\operatorname{Re}((-x_2)\bar{(-y}_2)) \\ (\eta_2 + \eta_3)y_1 + (\omega_2 + \omega_3)x_1 + \bar{(-y}_2)(-x_3) + \bar{(-x}_2)(-y_3) \\ (\eta_1 + \eta_3)(-y_2) + (\omega_1 + \omega_3)(-x_2) + \bar{(-y}_3)x_1 + \bar{(-x}_3)y_1 \\ (\eta_1 + \eta_2)(-y_3) + (\omega_1 + \omega_2)(-x_3) + \bar{y}_1\bar{(-x}_2) + \bar{x}_1\bar{(-y}_2) \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ x_1 \\ -x_2 \\ -x_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ y_1 \\ -y_2 \\ -y_3 \end{bmatrix} \\
&= r_u \left(\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) r_u \left(\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} \right).
\end{aligned}$$

From 4.4.6.3 we then see that $s_u \cong r_u$ for any k where $\operatorname{char}(k) \notin \{2, 3\}$.

Lemma 4.4.6.4. *The k -involutions of the form $\theta \circ t_{(\pm 1, \pm 1)}$ or $t_{(\pm 1, \pm 1)} \in \text{Aut}(C) \subset \text{Aut}(A)$ are type (I). The k -involutions of the form $r_u \in \text{Aut}(A)$ are of type (II).*

Lemma 4.4.6.5. *Let $A \cong H(C, \gamma)$ be a split Albert algebra. If $t \in \text{Aut}(A)$ is of type (I) then t is isomorphic to an involution in $\text{Aut}(\tilde{C})$ for some $\tilde{C} \cong C$.*

Proof. If $t \in \text{Aut}(A)$ is of type (a) then t has a subalgebra $H = H(\tilde{D}, \gamma) \cong H(D, \gamma) \subset A \cong H(C, \gamma)$ as its fixed point algebra. If this is the case then $t|_{H^\perp} = -1$, and $H^\perp \cong H(D, \gamma)^\perp$, which are the elements of the form $H^\perp = H(\tilde{D}^\perp, \gamma)$ that have zero entries along their diagonal. Since t leaves \tilde{C} invariant t must be an automorphism of C . Then if we restrict t to the composition algebra \tilde{C} and consider an element $a + b \in \tilde{C} = \tilde{D} \oplus \tilde{D}^\perp$, then $t(a + b) = a - b$ and t must be the involution of \tilde{C} that leaves \tilde{D} fixed. \square

4.4.7 $\mathcal{T} = T_\theta^-$

An involution θ such that for all $t \in \mathcal{T}$, $\theta(t) = t^{-1}$, is constructed as follows; let

$$p = \begin{bmatrix} 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 \end{bmatrix}.$$

Then let

$$g = \begin{bmatrix} p & \cdot & \cdot \\ \cdot & \cdot & p \\ \cdot & p & \cdot \end{bmatrix},$$

and $\theta = \mathcal{I}_g$ is an involution such that $\mathcal{I}_g(t) = t^{-1}$ for all $t \in \mathcal{T}$. We need only to verify that $g \in \text{Aut}(A)$. First notice that

$$\theta(a) = g(a_0, a_1, a_2) = (a_0^T, a_2^T, a_1^T),$$

where x^T is the transpose of $x \in M_3(k)$. Now we need only to check that g preserves the base point, sharp map, and the norm. First notice that the basepoint $c = (\text{id}, 0, 0)$ is fixed by g . If we recall that

$$a^\# = (a_0^\# - a_1 a_2, a_2^\# - a_0 a_1, a_1^\# - a_2 a_0).$$

We begin with $g(a^\#) = \left((a_0^\# - a_1 a_2)^T, (a_1^\# - a_2 a_0)^T, (a_2^\# - a_0 a_1)^T \right)$. Let us look at

$$\begin{aligned} (a_0^\# - a_1 a_2)^T &= \left(a_0^\# \right)^T - (a_1 a_2)^T \\ &= (a_0^2 - t(a_0) a_0 + s(a_0) c)^T - a_2^T a_1^T \\ &= (a_0^T)^2 - t(a_0) a_0^T + s(a_0) c - a_2^T a_1^T \\ &= (a_0^T)^\# - a_2^T a_1^T, \end{aligned}$$

noticing that $(a^2)^T = (a^T)^2$. Recall that in $M_3(k)$, $s(a) = \frac{1}{2}(t(a)^2 - t(a^2))$, and $t(a_0^T) = t(a_0)$. From this we can say that

$$(a_0^\# - a_1 a_2)^T = (a_0^T)^2 - t(a_0^T) a_0^T + s(a_0^T) c - a_2^T a_1^T.$$

Now considering the second and third copies of $M_3(k) \in A$ we look at

$$\begin{aligned} \left((a_1^\# - a_2 a_0)^T, (a_2^\# - a_0 a_1)^T \right) &= \left((a_1^\# - a_2 a_0)^T, (a_2^\# - a_0 a_1)^T \right) \\ &= \left((a_1^T)^\# - a_0^T a_2^T, (a_2^T)^\# - a_1^T a_0^T \right). \end{aligned}$$

So all together we have

$$\begin{aligned} g(a^\#) &= \left((a_0^T)^\# - a_2^T a_1^T, (a_1^T)^\# - a_0^T a_2^T, (a_2^T)^\# - a_1^T a_0^T \right) \\ &= g(a)^\#. \end{aligned}$$

Now we only have left to check that $N(g(a)) = N(a)$. So to that end we recall

$$N(a) = n(a_0) + n(a_1) + n(a_2) - t(a_0 a_1 a_2).$$

In our case n is the determinant of $a_i \in M_3(k)$ and t is the trace,

$$\begin{aligned} N(g(a)) &= n(a_0^T) + n(a_2^T) + n(a_1^T) - t(a_0^T a_2^T a_1^T) \\ &= n(a_0) + n(a_1) + n(a_2) - t(a_0^T (a_1 a_2)^T) \\ &= n(a_0) + n(a_1) + n(a_2) - t((a_1 a_2 a_0)^T) \\ &= n(a_0) + n(a_1) + n(a_2) - t(a_1 a_2 a_0) \\ &= n(a_0) + n(a_1) + n(a_2) - t(a_0 a_1 a_2) \\ &= N(a), \end{aligned}$$

and so we have that $\theta \in \text{Aut}(A)$.

4.4.8 k -involutions of type F_4

We also note that $t \in \text{Aut}(C) \subset \text{Aut}(A)$, t leaves a quaternion subalgebra of C , the octonion algebra that A is taken to be over. We can again take these elements θ and $\theta \circ \mathcal{I}_{t(1,-1)}$, as above. In this case t leaves the space $H(D, \gamma)$ fixed, where D is the quaternion subalgebra left fixed by $t|_C$.

The element s_u will have a matrix representation of the form

$$s_{23} = \text{diag}(1, 1, -1) \oplus J \oplus \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix},$$

where

$$J = \begin{bmatrix} & & 1 \\ & -1 & \\ & & -1 \\ 1 & & \end{bmatrix} \bigoplus \text{diag}(-1, -1, -1, -1).$$

This map has an eigenspace of dimension 11 for the 1-eigenspace, and of dimension 16 for the (-1) -eigenspace. If we think of an element of A as being of the column vector form above, and of each octonion entry as being in $M_2(k)$, then an element in the Albert algebra A has the form

$$\begin{bmatrix} f \\ g \\ h \\ (a, b) \\ (u, v) \\ (x, y) \end{bmatrix}.$$

The 1-eigenspace has an arbitrary element of the taking the form

$$\begin{bmatrix} f \\ g \\ 0 \\ \left(\begin{bmatrix} a_{11} \\ u_{11} & -u_{12} \\ -u_{21} & u_{22} \end{bmatrix}, \begin{bmatrix} 0 \\ -v_{11} & -v_{12} \\ -v_{21} & -v_{22} \end{bmatrix} \right) \\ \left(\begin{bmatrix} u_{22} & u_{12} \\ u_{21} & u_{11} \end{bmatrix}, \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \right) \end{bmatrix}.$$

The eigenspace corresponding to the eigenvalue -1 has elements of the form,

$$\begin{bmatrix} 0 \\ 0 \\ h \\ \left(\begin{bmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) \\ \left(\begin{bmatrix} -u_{11} & u_{12} \\ u_{21} & -u_{22} \end{bmatrix}, \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \right) \\ \left(\begin{bmatrix} u_{22} & u_{12} \\ u_{21} & u_{11} \end{bmatrix}, \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \right) \end{bmatrix}.$$

4.4.9 Involution that inverts entire k -split maximal torus

Let us call $\mathcal{I}_g = \theta$ as above, and consider $\theta \circ \mathcal{I}_t$ where $t \in \mathcal{T}$. The k -involution, $\theta(\mathcal{I}_t) = \mathcal{I}_{\theta_t}$,

$$\mathcal{I}_{\theta_t} = \begin{bmatrix} \mathcal{I}_{\theta_t,0} & & & \\ & \ddots & & \cdot \\ & \cdot & \ddots & \mathcal{I}_{\theta_t,2} \\ \cdot & \mathcal{I}_{\theta_t,1} & & \cdot \end{bmatrix},$$

where $\mathcal{I}_{\theta_t,0} =$

$$\begin{bmatrix} 1 & \cdot \\ \cdot & \cdot & \cdot & u_1^{-2}u_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_1^{-1}u_2^{-1} & \cdot & \cdot \\ \cdot & u_1^2u_2^{-1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & u_1u_2^{-2} & \cdot \\ \cdot & \cdot & u_1u_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & u_1^{-1}u_2^2 & \cdot & \cdot & \cdot \\ \cdot & 1 \end{bmatrix}$$

$\mathcal{I}_{\theta_t,2} =$

$$\begin{bmatrix} u_1^{-1}v_1 & \cdot \\ \cdot & \cdot & \cdot & u_1^{-1}v_1^{-1}v_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_1^{-1}v_2^{-1} & \cdot & \cdot \\ \cdot & u_1u_2^{-1}v_1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & u_1u_2^{-1}v_1^{-1}v_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & u_1u_2^{-1}v_2^{-1} & \cdot \\ \cdot & \cdot & u_2v_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & u_2v_1^{-1}v_2 & \cdot & \cdot & \cdot \\ \cdot & u_2v_2^{-1} & \end{bmatrix}$$

$\mathcal{I}_{\theta_t,1} =$

$$\begin{bmatrix} u_1v_1^{-1} & \cdot \\ \cdot & \cdot & \cdot & u_1^{-1}u_2v_1^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_2^{-1}v_1^{-1} & \cdot & \cdot \\ \cdot & u_1v_1v_2^{-1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & u_1^{-1}u_2v_1v_2^{-1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & u_2^{-1}v_1v_2^{-1} & \cdot \\ \cdot & \cdot & u_1v_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & u_1^{-1}u_2v_2 & \cdot & \cdot & \cdot \\ \cdot & u_2^{-1}v_2 & \end{bmatrix}$$

4.4.10 Another involution

According to [5] there should be an involution of the root system of F_4 that lifts to the group over certain fields, which leaves a three dimensional k -split torus fixed and sends a one dimensional k -split torus to its inverse. In order to see this lifted involution we first consider the involution on the root space. The involution is represented by the following diagram in [5] with $\Delta = \{\alpha_i\}_1^4$.

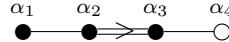


Figure 4.1: Diagram for involution fixing rank 3 split torus

If we consider an alternate set of simple roots,

$$\begin{aligned}\beta_1 &= \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 \\ \beta_2 &= \alpha_1 + \alpha_2 \\ \beta_3 &= -\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4 \\ \beta_4 &= \alpha_2 + \alpha_3.\end{aligned}$$

Notice that $\langle \beta_i, \alpha_4 \rangle = 2 \frac{(\beta_i, \alpha_4)}{(\alpha_4, \alpha_4)} = 0$ for $1 \leq i \leq 3$, and so the reflection s_{α_4} should be the projection of the involution we need on the root space $\Delta' = \{\beta_i\}_1^4$. If we make the identification,

$$\begin{aligned}\alpha_1 &\sim \left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} c & & & \\ & c^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) \\ \alpha_2 &\sim \left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & d & & \\ & & d^{-1} & \\ & & & 1 \end{bmatrix} \right) \\ \alpha_3 &\sim \left(\begin{bmatrix} a & & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) \\ \alpha_4 &\sim \left(\begin{bmatrix} 1 & & & \\ & b & & \\ & & b^{-1} & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right),\end{aligned}$$

then

$$\begin{aligned}\beta_1 &\sim \left(\begin{bmatrix} a^4 & & \\ & a^{-2} & \\ & & a^{-2} \end{bmatrix}, \begin{bmatrix} a & & \\ & a & \\ & & a^{-2} \end{bmatrix} \right) \\ \beta_2 &\sim \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} b & & \\ & 1 & \\ & & b^{-1} \end{bmatrix} \right) \\ \beta_3 &\sim \left(\begin{bmatrix} c^{-2} & & \\ & c & \\ & & c \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \right) \\ \beta_4 &\sim \left(\begin{bmatrix} d & & \\ & d^{-1} & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & d & \\ & & d^{-1} \end{bmatrix} \right).\end{aligned}$$

If we consider the action $(u, v) \in \mathrm{SL}_3(k) \times \mathrm{SL}_3(k)$, then $\tilde{s}_{\alpha_4}(u, v) = (sus, v)$, where

$$s = \begin{bmatrix} -1 & & \\ & & -1 \\ & -1 & \end{bmatrix}.$$

The involution \tilde{s}_v acts on $(a_0, a_1, a_2) \in A$ by switching the second and third column of a_1 , the second and third row of a_2 , and negates the third and second components, i.e.

$$\begin{aligned}\tilde{s}_u(a_0, a_1, a_2) &= (s_{u_0}(a_0), s_{u_1}(a_1), s_{u_2}(a_2)) \\ &= \left(\begin{bmatrix} a_{11}^0 & a_{13}^0 & a_{12}^0 \\ a_{21}^0 & a_{33}^0 & a_{32}^0 \\ a_{31}^0 & a_{23}^0 & a_{22}^0 \end{bmatrix}, - \begin{bmatrix} a_{11}^1 & a_{13}^1 & a_{12}^1 \\ a_{21}^1 & a_{23}^1 & a_{22}^1 \\ a_{31}^1 & a_{33}^1 & a_{32}^1 \end{bmatrix}, - \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 \\ a_{31}^2 & a_{32}^2 & a_{33}^2 \\ a_{21}^2 & a_{22}^2 & a_{23}^2 \end{bmatrix} \right).\end{aligned}$$

It is clear that \tilde{s}_u leaves $(\mathrm{id}, 0, 0)$ fixed, and that

$$N(\tilde{s}_u(a)) = \det(s_{u_0}(a_0)) + \det(s_{u_1}(a_1)) + \det(s_{u_2}(a_2)) - \mathrm{tr}(s_{U_0}(a_0)s_{v_1}(a_1)s_{v_2}(a_2)) = N(a)$$

It is computationally tedious, but not difficult to check, that $\tilde{s}_u(a)^\# = \tilde{s}_u(a^\#)$. In Appendix A there are the details of this calculation via Maple™. Notice that it leaves E_0 and E_1 invariant.

We can also consider s acting on the first component of $\mathrm{SL}_3(k) \times \mathrm{SL}_3(k)$ by $s_{\alpha_4}(u, v) = (sus, v)$. To do this we can just reverse the components of the tori of $\mathrm{SL}_3(k) \times \mathrm{SL}_3(k)$ corresponding to the root spaces labeled above as α_i and β_j with $1 \leq i, j \leq 4$.

This gives the element of $\text{Aut}(A)$,

$$\tilde{s}_u = \underbrace{\begin{bmatrix} 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 \\ \cdot & 1 & \cdot \\ \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix}}_{s_{u_0}} \oplus$$

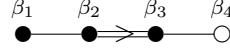
$$\underbrace{\begin{bmatrix} -1 & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}}_{s_{u_1}} \oplus$$

$$\underbrace{\begin{bmatrix} -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & -1 \end{bmatrix}}_{s_{u_2}}$$

It turns out that \tilde{s}_u is exactly the map s_{23} as described in (section 1.14). So you can refer to the proof above that $\tilde{s}_u \in \text{Aut}(A)$.

4.4.11 Involution leaving a rank 3 k -split torus fixed

Let us take $\sigma = \mathcal{I}_{\tilde{s}_u}$ to be the representative of the involution corresponding to the involution diagram,



and so in the k -split case there should be a one dimensional torus that is inverted by σ , which will give us a maximal (σ, k) -split torus S_σ^- . Since $\sigma|_{X^*(\mathcal{T})} = s_{\alpha_4}$ that maximal torus inverted by σ is the action of the element in the maximal torus of $\mathrm{SL}_3(k)$ that corresponds to α_4 . Using the $f_{(u,v)}$ action from $\mathrm{SL}_3(k) \times \mathrm{SL}_3(k)$, S_σ^- comes from the action of the form $f_{(\tilde{\alpha}_4, \mathrm{id})}$, where

$$\tilde{\alpha}_4 = \begin{bmatrix} 1 & & \\ & d & \\ & & d^{-1} \end{bmatrix}.$$

So our maximal (σ, k) -split torus is of the form

$$\begin{aligned} S_\sigma^- = \{ & \mathrm{diag}(1, d^{-1}, d, d, 1, d^2, d^{-1}, d^{-2}, 1) \\ & \oplus \mathrm{diag}(1, 1, 1, d, d, d, d^{-1}, d^{-1}, d^{-1}) \\ & \oplus \mathrm{diag}(1, d^{-1}, d, 1, d^{-1}, d, 1, d^{-1}, d) \mid d \in k \}. \end{aligned}$$

Notice that

$$\sigma \circ \mathcal{I}_t(x) = \tilde{s}_v t x t^{-1} \tilde{s}_v = (\tilde{s}_v t) x (\tilde{s}_v t)^{-1} = \mathcal{I}_{(\tilde{s}_v t)}(x),$$

and if $\mathcal{I}_{(\tilde{s}_v t)}$ is a k -involution then $(\tilde{s}_v t)^2 \in Z(\mathrm{Aut}(A)) = \{\mathrm{id}\}$.

The k -inner elements will come from S_σ^- by [5]. There could be non-isomorphic k -involutions that differ by a k -inner element of the form,

$$\sigma \circ \mathcal{I}_d = \underbrace{\begin{bmatrix} 1 & \cdot \\ \cdot & \cdot & d & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & d^{-1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d^{-1} & \cdot & \cdot \\ \cdot & 1 \\ \cdot & d^{-2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & d & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & d^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}}_{s_{u_0}(d)} \oplus$$

$$\underbrace{\begin{bmatrix} -1 & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -d^{-1} & \cdot & \cdot \\ \cdot & -d^{-1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & -d & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -d & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -d & \cdot & \cdot & \cdot \end{bmatrix}}_{s_{u_1}(d)} \oplus$$

$$\underbrace{\begin{bmatrix} -1 & \cdot \\ \cdot & \cdot & -d & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -d^{-1} & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -d^{-1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & -d & \cdot \\ \cdot & -d^{-1} & \cdot \end{bmatrix}}_{s_{u_2}(d)},$$

but 4.4.6.3 tells us they are all in the same isomorphic class.

4.4.12 Isomorphy classes of k -involutions of $\text{Aut}(A)$

For our two presentations of the split Albert algebra over a given field, we will consider two different ways to assure linear maps are automorphisms. For $H(C, \gamma)$ we can just check the map in question is a bijection and respects the multiplication. When we consider the Albert algebra in the first Tits construction form, we can check that the map is a bijection and respects the norm and adjoint maps, and leaves the base point fixed, [17].

4.4.13 k -involutions

From the combinatorial classification of invariants of type (I) given by [5] we know we should have two involutions corresponding to the following diagrams

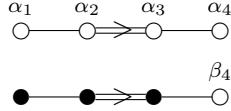


Figure 4.2: Diagrams of type θ and σ

The first corresponds to a k -involution $\theta : G \rightarrow G$ that splits an entire k -split maximal torus, and the second diagram corresponds to a k -involution $\sigma : G \rightarrow G$ that splits a rank 1 k -split torus and fixes a k -split torus of rank 3.

4.4.14 k -inner elements

We have two isomorphy classes of invariants of type (I) as set out by [5]. Once we describe the G_k -isomorphy classes of k -inner elements for our representatives of admissible k -involutions of type (I), our task will be complete. We will follow [10] and call automorphisms in the isomorphy class of k -involutions of the form $\theta \circ \mathcal{I}_t$, $t \in I_k(\mathcal{T}_\theta^-)$ of type (I), and those of the form $\sigma \circ \mathcal{I}_s$, $s \in I_k(\mathcal{T}_\sigma^-)$ of type (II).

By 4.4.6.3 all elements of order 2 in $\text{Aut}(A)$ of type (II) are conjugate for any k where $\text{char}(k) \neq 2$. So we have the following theorem.

Theorem 4.4.14.1. *All k -involutions of the form $\sigma \circ \mathcal{I}_s$, $s \in I_k(\mathcal{T}_\sigma^-)$ are isomorphic.*

The isomorphy classes of k -involutions of the form $\theta \circ \mathcal{I}_t$, $t \in I_k(\mathcal{T}_\theta^-)$ are said to be of type

(I) and leave a 15 dimensional subalgebra isomorphic to an algebra of the form $H(D, \gamma)$ where $D \subset C$ is a quaternion subalgebra of a split octonion algebra.

We can use a generalization of a result from the section on groups of type G_2 .

Lemma 4.4.14.2. *Let A be a k -algebra and D and D' subalgebras of A . If $t, t' \in G = \text{Aut}(A)$ are elements of order 2 and t, t' fix D, D' elementwise respectively, then $t \cong t'$ if and only if $D \cong D'$ over k .*

Proof. Let $D, D' \subset A$ such that $t(a) = a$ and $t'(a') = a'$ for all $a \in D$ and $a' \in D'$, and let D and D' be the largest such subalgebras with respect to t and t' . First we will show sufficiency. Let $D \cong D'$, and let $g \in \text{Aut}(A)$ be such that $g(D') = D$. For $c \in A$ $c = a + b$ where $a \in D$ and $b \in A - D$. Since t, t' are of order 2 we have D (resp. D') is the 1-eigenspace and $A - D$ (resp. $A - D'$) is the (-1) -eigenspace of t (resp. t'). Then we have,

$$\begin{aligned} gt'g^{-1}(a + b) &= gt'(a' + b') \\ &= g(a' - b') \\ &= a - b \\ &= t(a + b), \end{aligned}$$

since $g(A - D') = A - D$. To show necessity we start by assuming there exists a $g \in \text{Aut}(A)$ such that $gt'g^{-1} = t$, which implies that $t'g^{-1} = g^{-1}t$, and from this we see that

$$\begin{aligned} t'g^{-1}(a + b) &= g^{-1}t(a + b) \\ t'(g^{-1}(a) + g^{-1}(b)) &= g^{-1}(a - b) \\ t'(g^{-1}(a) + g^{-1}(b)) &= g^{-1}(a) - g^{-1}(b). \end{aligned}$$

This shows us that $g^{-1}(a) \in D'$ for all $a \in D$, and so $D \cong D'$. \square

So we know that the isomorphy classes of k -involutions are in bijection with isomorphy classes of subalgebras fixed by elements of order 2 in $\text{Aut}(A)$. For the k -involutions of the form $\theta \circ \mathcal{I}_r$ we know that there is only one isomorphy class when k is algebraically closed or when k is finite. Over \mathbb{R} or \mathbb{Q}_p the isomorphy classes of $H(D, \gamma)$ are based on whether D is split or not. If D is split there is only one isomorphy class. If D is a division algebra there can be more than one parametrized by equivalence classes of elements of k not represented by $N|_D$, 2.2.8.2.

The norm class of \mathbb{R} is $\{1, -1\}$ and N represents every element of \mathbb{Q}_p , [25], so when D is a division algebra we get two isomorphy classes of $H(D, \gamma)$ over \mathbb{R} and one isomorphy class over \mathbb{Q}_p .

Theorem 4.4.14.3. *For the algebraic group $\text{Aut}(A)$, where A is a split Albert algebra the k -involutions of the form $\theta \circ \mathcal{I}_r$ where $r \in I_k(S_\theta^-)$*

- (1) *for k algebraically closed, and \mathbb{F}_p , there is only one isomorphy class with representative taken to be θ ,*
- (2) *for $k = \mathbb{R}$ there are three distinct isomorphy classes, corresponding to a quadratic form on $H(D, \gamma)$ being split (D is split), positive definite (D is division and $\gamma = \text{id}$), or indeterminate (D is division $\gamma = \text{diag}(-1, 1, 1)$),*
- (3) *for $k = \mathbb{Q}_p$ there are 2 isomorphy classes one fixing $H(D, \text{id})$ when D is a split quaternion algebras, and one fixing $H(D, \text{id})$ when D is a division algebra over k .*
- (4) *for $k = \mathbb{Q}$ there are an infinite number of isomorphy classes.*

Proof. (1) This is immediate from 4.4.6.3.

- (2) Over \mathbb{R} there are only two isomorphy classes of quaternion algebras, and the norm class of the primitive idempotent is $\kappa(\alpha) = \{-1, 1\}$. All of the fifteen dimensional J -algebras over a split quaternion algebra are isomorphic. We just need to look at when D is a division quaternion algebra. The values for γ can be taken from $\kappa(\alpha)$, and so we can have γ as plus or minus the identity matrix, or a matrix having one or two negative ones and the rest positive ones. Now by 2.2.7.2 and 2.2.8.1 we see there are only two isomorphy classes of $H(D, \gamma)$ when D is a division algebra, any isomorphy class of $H(D, \gamma)$ has the case when $\gamma = \text{id}$ or $\gamma = (-1, 1, 1)$ as a representative.
- (3) Over \mathbb{Q}_p we have two isomorphy classes of D , unlike the real case N_D takes on all values in \mathbb{Q}_p , so $\kappa(\alpha) = \{1\}$, [25] p 22, [16]. So the only unique value γ is $\gamma = \text{id}$, and we have only two isomorphy classes for $H(D, \gamma)$, when D is split, or when D is a division algebra.
- (4) In the case where D is split, we have only one isomorphy class of $H(D, \gamma)$. We already have an infinite number of isomorphy classes of division quaternion algebras when $k = \mathbb{Q}$, and each of these will be parameterized by values of γ not represented by N_D .

□

4.5 Fixed point groups

We can again refer to 4.3.0.4, and compute the fixed point groups of each k -involution of the form $\theta \circ \mathcal{I}_r$ by the subgroup of $\text{Aut}(A)$ that leaves certain subalgebras $H(D, \gamma) \subset H(C, \gamma) \cong A$ invariant.

Proposition 4.5.0.4. *If $\theta_* = \mathcal{I}_t$ is a k -involution of $\text{Aut}(A)$ such that*

$$t_*(\eta_1, \eta_2, \eta_3, x_1, x_2, x_3) = (\eta_1, \eta_2, \eta_3, t(x_1), t(x_2), t(x_3)),$$

where $t \in \text{Aut}(C)$ such that $t^2 = \text{id}$, fixes a division algebra D , then θ_ is a k -involution of $\text{Aut}(A)$ fixing $H(D, \text{id})$.*

Proof. By 4.4.6.2 and 4.4.6.1 elements in $\text{Aut}(A)$ of order 2 that either leave a subalgebra of the Albert algebra of dimension 11 or 15. An element of the form

$$t_*(\eta_1, \eta_2, \eta_3, x_1, x_2, x_3) = (\eta_1, \eta_2, \eta_3, t(x_1), t(x_2), t(x_3)),$$

leave a 15 dimensional subalgebra of the form $H(D, \text{id}) \subset A$ fixed, and therefore is of type (I). \square

Using a representative of the above type, we can decompose the Albert algebra $A \cong H(C, \text{id}) = H(D, \text{id}) \oplus K \cdot a$, where

$$K \cdot a = \left\{ \begin{bmatrix} 0 & x_{31}a & -x_{21}a \\ -x_{31}a & 0 & x_{11}a \\ x_{21}a & -x_{11}a & 0 \end{bmatrix} \mid x_{i1} \in D, a \in D^\perp \right\}$$

So we can consider elements of $H(C, \text{id})$ of the form $x_0 + x_1 \cdot a$ where $x_0 \in H(D, \text{id})$, $x_1 \in K$, and $a = \text{diag}(a, a, a)$. An automorphism of the Albert algebra that leaves $H(D, \text{id})$ invariant is of the form, $\tilde{t} \in \text{Aut}(H(C, \text{id}))$ such that

$$\tilde{t}(x_0 + x_1 \cdot a) = \tilde{t}(x_0) + \tilde{t}(x_1 \cdot a),$$

and since $\tilde{t}(H(D, \text{id})) = H(D, \text{id})$, we must have that $\tilde{t}(K \cdot a) = K \cdot a$. This allows us to consider the automorphism of \tilde{t} as made up of two maps $r \in \text{Aut}(H(D, \text{id}))$ and s a linear transformation of 3×3 skew symmetric quaternion matrices. The linear transformation s must also interact with r to respect the multiplication of $H(C, \text{id})$. To see how this works let us consider $x_0 + x_1 \cdot a, y_0 + y_1 \cdot a \in H(C, \text{id})$,

$$(x_0 + x_1 \cdot a)(y_0 + y_1 \cdot a) = x_0 y_0 + (x_1 \cdot a)(y_1 \cdot a) + (x_1 \cdot a)y_0 + x_0(y_1 \cdot a),$$

where $x_0 y_0, (x_1 \cdot a)(y_1 \cdot a) \in H(D, \text{id})$ and $(x_1 \cdot a)y_0, x_0(y_1 \cdot a) \in K \cdot a$. To see this notice that

$$x_0(y_1 \cdot a) = \frac{1}{2} (x_0 \cdot (y_1 \cdot a) + (y_1 \cdot a) \cdot x_0) = (y_1 \cdot \bar{x} - (y_1 \cdot \bar{x})^T) \cdot a,$$

we will define $(y_1 \cdot \bar{x} - (y_1 \cdot \bar{x})^T) \equiv x_0 * y_1$. Notice that $y_1 \cdot \bar{x} - (y_1 \cdot \bar{x})^T$ is a skew-symmetric matrix with entries in D , and since the J -algebra multiplicaiton is commutative we can see that $(x_1 \cdot a) y_0 \in K \cdot a$ also. The $*$ map gives us a representation of $H(D, \text{id})$ on $K \cdot a$.

Now to see that $(x_1 \cdot a) (y_1 \cdot a)$ is in $H(D, \text{id})$, we work out the multiplication,

$$(x_1 \cdot a) (y_1 \cdot a) = \frac{1}{2} ((x_1 \cdot a) \cdot (y_1 \cdot a) + (y_1 \cdot a) \cdot (x_1 \cdot a)) = \frac{\alpha}{2} (\bar{x}_1 \cdot y_1 + \bar{y}_1 \cdot x_1)^T,$$

where $\alpha = -N(a)$. It is straight forward to check that $\bar{x}_1 \cdot y_1 + \bar{y}_1 \cdot x_1$ is in $H(D, \text{id})$. We will denote $\frac{\alpha}{2} (\bar{x}_1 \cdot y_1 + \bar{y}_1 \cdot x_1)^T \equiv x_1 \bullet y_1$. So let's rewrite the multiplication under this decomposition in the following way,

$$\begin{aligned} (x_0 + x_1 \cdot a) (y_0 + y_1 \cdot a) &= x_0 y_0 + (x_1 \cdot a) (y_1 \cdot a) + (x_1 \cdot a) y_0 + x_0 (y_1 \cdot a) \\ &= (x_0 y_0 + x_1 \bullet y_1) + ((x_0 * y_1) + (y_0 * x_1)) \cdot a. \end{aligned}$$

In order to determine the structure of a map $\tilde{t} \in \text{Aut}(A, H(D, \gamma))$ that is an automorphism of the Albert algebra leaving a subalgebra of the from $H(D, \gamma)$ invariant we would need

$$\begin{aligned} \tilde{t}((x_0 + x_1 \cdot a) (y_0 + y_1 \cdot a)) &= \tilde{t}(x_0 + x_1 \cdot a) \tilde{t}(y_0 + y_1 \cdot a) \\ &= (t(x_0) + t(x_1 \cdot a)) (t(y_0) + t(y_1 \cdot a)) \\ &= (r(x_0) + s(x_1) \cdot a) (r(y_0) + s(y_1) \cdot a) \\ &= r(x_0)r(y_0) + s(x_1) \bullet s(y_1) + ((r(x_0) * s(y_1) + r(y_0) * s(x_1)) \cdot a), \end{aligned}$$

and if we multiply first we have

$$\begin{aligned} \tilde{t}((x_0 + x_1 \cdot a) (y_0 + y_1 \cdot a)) &= \tilde{t}(x_0 y_0 + (x_1 \cdot a) (y_1 \cdot a) + (x_1 \cdot a) y_0 + x_0 (y_1 \cdot a)) \\ &= \tilde{t}((x_0 y_0 + x_1 \bullet y_1) + ((x_0 * y_1) + (y_0 * x_1)) \cdot a) \\ &= r(x_0 y_0 + x_1 \bullet y_1) + s((x_0 * y_1) + (y_0 * x_1)) \cdot a \\ &= r(x_0 y_0) + r(x_1 \bullet y_1) + (s(x_0 * y_1) + s(y_0 * x_1)) \cdot a. \end{aligned}$$

So we have the equations,

$$r(x_1 \bullet y_1) = s(x_1) \bullet s(y_1) \tag{4.33}$$

$$s(x_0 * y_1) = r(x_0) * s(y_1). \tag{4.34}$$

Writing out these two equations using their respective definitions we get

$$r(x_1 \bullet y_1) = r\left(\frac{\alpha}{2}(\bar{x}_1 \cdot y_1 + \bar{y}_1 \cdot x_1)^T\right) = \frac{\alpha}{2}\left(\overline{s(x_1)} \cdot s(y_1) + \overline{s(y_1)} \cdot s(x_1)\right)^T \quad (4.35)$$

$$s(x_0 * y_1) = s\left(\frac{1}{2}(y_1 \cdot \bar{x}_0 - (y_1 \cdot \bar{x}_0)^T)\right) = \frac{1}{2}\left(s(y_1) \cdot \overline{r(x_0)} - (s(y_1) \cdot \overline{r(x_0)})^T\right). \quad (4.36)$$

From here will need a theorem by Martindale that relates associative algebras with involution to Jordan algebras. First we need some definitions. Here we follow [13], and define a *central simple algebra* to be a vector space over k closed under some bilinear multiplication, which is simple and whose center consists of ke only, where e is the identity element of the vector space. We can associate with these algebras certain anti-automorphisms called CSA-involutions. A *CSA-involution* of a central simple algebra M over a field k is a map $\phi : M \rightarrow M$ such that if $x, y \in M$ then

- (a) $\phi(x + y) = \phi(x) + \phi(y)$
- (b) $\phi(xy) = \phi(y)\phi(x)$
- (c) $\phi^2(x) = x.$

If we restrict the CSA-involution to $ke \subset M$ we get an automorphism, and the CSA-involution must restrict to the identity on ke if it is not of order 2. If a CSA-involution leaves ke fixed pointwise it is called a *CSA-involution of the first kind*, and if it is of order 2 on ke we say it is a *CSA-involution of the second kind*.

An *associative specialization* is a homomorphism of a Jordan algebra J into an algebra M^+ , where M is an associative algebra and M^+ is the Jordan algebra with Jordan product constructed from the associative product in M . An associative specialization $\phi : J \rightarrow M^+$ has the following property

$$\phi(ab) = \frac{1}{2}(\phi(a)\phi(b) + \phi(b)\phi(a)),$$

with $a, b \in J$.

An associative algebra M with CSA-involution is called *perfect* if and only if any associative specialization of $\text{Her}(M, \gamma)$ has a unique extension to a homomorphism of M .

Theorem, [8] 4.5.0.5. *Let M be a central simple associative algebra with CSA-involution $\phi : M \rightarrow M$ and identity element, $e = \sum_1^3 u_i$ where u_i , are orthogonal idempotents such that $u_i \in H$, where H are the elements that are ϕ -hermitian and $Mu_iM = M$ for $1 \leq i \leq 3$. Then M is perfect.*

Corollary 4.5.0.6. *Let D be an central simple associative algebra with CSA-involution and identity element, and let D be an associative algebra with conjugation ${}^- : D \rightarrow D$, with the CSA-involution on 3×3 matrices over D , ϕ , constructed from conjugation in D by, $\phi(x) = \gamma \bar{x}^T \gamma^{-1}$, then D is perfect.*

Since a quaternion algebra over a field k , is central simple with an involution given by quaternion conjugation, we have that homomorphisms of $H_3(D, \gamma)$ extend uniquely to homomorphism of $(M_3(D), \phi)$, where ϕ is the CSA-involution constructed from the quaternion conjugation on D . From this we can say that our map r extends uniquely to an automorphism of $(M_3(D), \phi)$, and r has norm 1. So there exists an element $R \in \mathrm{Sp}(3, k)$ such that $r(m) = Rm\phi(R)$, and the only elements that commute with ϕ and multiplication are $\pm \mathrm{id}$. This is a group of type $\mathrm{PSp}(3, k)$, [13]. The fixed point group is of the form,

$$\tilde{t}(x_0 + x_1 \cdot a) = r(x_0) + s(x_1) \cdot a = R(x_0)\phi(R) + (sR(x_1)\phi(R)) \cdot a.$$

This contains the fixed point group for t , the restriction of \tilde{t} to one of the copies of the octonion algebra as above. Recall that t is of the form

$$t(x + ya) = t(x) + t(y)a = dxd^{-1} + (pdyd^{-1})a,$$

where $d, p \in D$, the quaternion algebra fixed by an element of order 2 in $\mathrm{Aut}(C)$. We saw earlier that $d \in \mathrm{Aut}(D)$ and $p \in \mathrm{Sp}(1) = \{x \in D \mid x\bar{x} = 1\}$. So

$$\mathrm{Aut}(D) \times \mathrm{Sp}(1) \subset \mathrm{Aut}(M_3(D), \phi) \times S,$$

but $\mathrm{Aut}(D) \subset \mathrm{Aut}(M_3(D), \phi) = \mathrm{PSp}(3)$ and is of rank 3 leaving S being a group of rank 1 with elements of norm 1, and so $S \cong \mathrm{Sp}(1)$. So we have indicated the following proposition.

Proposition 4.5.0.7. *For a k -involution of type F_4 of the form $\theta \circ \mathcal{I}_t$, where $t \in \mathcal{T}_\theta^-$, the fixed point group is of the form*

$$\mathrm{Aut}(M_3(D), \phi) \times \mathrm{Sp}(1, k).$$

For involutions of type σ we have seen that $\mathrm{Spin}(Q, E_0)$ acts irreducibly on $\mathrm{Aut}(A)_u$, and so leaves invariant the subalgebra of dimension 11 that is fixed by this involution. So this must be the fixed point group.

Proposition 4.5.0.8. *For a k -involution of type F_4 of the form $\sigma \circ \mathcal{I}_t$, where $t \in \mathcal{T}_\sigma^-$ is $\mathrm{Spin}(Q, E_0)$.*

We now conclude the classification of isomorphy classes of k -involutions of algebraic groups of type G_2 and F_4 over the fields \mathbb{F}_q when $\mathrm{char}(\mathbb{F}_q) \neq 2, \mathbb{Q}_p$, and \mathbb{C} and \mathbb{R} which were already known. We have also put these results into the context of the theory laid out by Helminck in [5].

4.6 Galois cohomology

The classification of isomorphy classes of k -involutions is related to a certain Galois cohomology group. We now discuss this correspondence in detail. First we will recall a few notions from the theory of Galois cohomology that can be found in [21]. For the sake of our discussion we refer to a ground field k , and K is the algebraic closure of k .

Let us first look at $H^0(Gal_k, \text{Aut}(X, K))$, where $\text{Aut}(X, K)$ is the automorphism group of X defined over k , and Gal_k is the absolute Galois group of k , and so $H^0(Gal_k, \text{Aut}(X, K)) = \text{Aut}(X, k)$, where we are thinking of $\text{Aut}(X,)$ as a functor from the algebraic separable extensions of k with values in the category of groups. If we consider $H^1(Gal_k, \text{Aut}(X, K))$ we are considering the K/k -forms of X , where a K/k -form of X will be any object Y defined over k that becomes isomorphic to X when the ground field of X and Y are extended to K .

In a broad sense this paper is concerned with $\text{Aut}(C)$, where C is a composition algebra of dimension 8 defined over k . And so the isomorphy classes of $\text{Aut}(C)$ over a given field k correspond to $H^1(Gal_k, \text{Aut}(C, K))$.

In particular we are concerned with isomorphy classes of k -involutions, which we have found to correspond to certain isomorphy classes of fixed point groups of $\text{Aut}(C)$ defined over k . Further, these fixed point groups always leave invariant a quaternion subalgebra D , and so over a given k our classes of k -involutions correspond to the K/k -forms of D .

From this we see that $H^1(Gal_k, \text{Aut}(C, D, K))$ is in bijective correspondence with the isomorphy classes of k -involutions, where we are thinking of $\text{Aut}(C, D, K) \subset \text{Aut}(C, K)$ as the subgroup that leaves a quaternion subalgebra, D , invariant. If we let $\text{Aut}(C, K) = G$, then the fixed point group G^θ is the same as $Z_G(\theta)$, the centralizer of θ in G .

Proposition 4.6.0.9. *There is a bijection between $H^1(Gal_k, \text{Aut}(C, D, K))$ and $H^1(Gal_k, Z_G(\theta))$.*

Proof. This follows from 4.2.1.4 and 4.3.0.4. □

We now have a bijection between $H^1(Gal_k, \text{Aut}(C, D, K))$ and $H^1(Gal_k, Z_G(\theta))$. Both of these are in bijective correspondence to the isomorphy classes of k -involutions, \mathcal{C}_k . We can think of $H^1(Gal_k, Z_G(\theta))$ as the classes of K/k -forms of $\text{Aut}(C, D, K)$, and we have the following corollary.

Corollary 4.6.0.10. *The map,*

$$\begin{aligned} Z_G : \mathcal{C}_k &\rightarrow H^1(Gal_k, Z_G(\theta)) \\ [\theta] &\mapsto [G^\theta], \end{aligned}$$

is bijective.

Proof. This follows from the fact that isomororphy classes of fixed point groups of k -involutions are in bijection with isomorphy classes of their fixed point groups, which is described in [5,6]. \square

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APPENDIX

Appendix A

Computations

The following were computation done in a MapleTM to show that the two bijective maps of order 2 described in section 4.4 are in fact automorphisms of an Albert algebra. It is easy to check that each map leaves the basepoint invariant. The following computations show that each map leaves the norm invariant and respects the # map.

```

> with(LinearAlgebra);
[&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm,
BilinearForm, CharacteristicMatrix, CharacteristicPolynomial, Column,
ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix,
ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation,
CrossProduct, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix,
Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues, Eigenvectors,
Equal, ForwardSubstitute, FrobeniusForm, GaussianElimination, GenerateEquations,
GenerateMatrix, Generic, GetResultDataType, GetResultShape, GivensRotationMatrix,
GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm,
HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite,
IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, KroneckerProduct,
LA_Main, LUDecomposition, LeastSquares, LinearSolve, LyapunovSolve, Map, Map2,
MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse, MatrixMatrixMultiply,
MatrixNorm, MatrixPower, MatrixScalarMultiply, MatrixVectorMultiply,
MinimalPolynomial, Minor, Modular, Multiply, NoUserValue, Norm, Normalize, NullSpace,
OuterProductMatrix, Permanent, Pivot, PopovForm, QRDecomposition, RandomMatrix,
RandomVector, Rank, RationalCanonicalForm, ReducedRowEchelonForm, Row,
RowDimension, RowOperation, RowSpace, ScalarMatrix, ScalarMultiply, ScalarVector,
SchurForm, SingularValues, SmithForm, StronglyConnectedBlocks, SubMatrix, SubVector,
SumBasis, SylvesterMatrix, SylvesterSolve, ToeplitzMatrix, Trace, Transpose,
TridiagonalForm, UnitVector, VandermondeMatrix, VectorAdd, VectorAngle,
VectorMatrixMultiply, VectorNorm, VectorScalarMultiply, ZeroMatrix, ZeroVector, Zip]

```

=

```

> a0 := 
$$\begin{bmatrix} a_{011} & a_{012} & a_{013} \\ a_{021} & a_{022} & a_{023} \\ a_{031} & a_{032} & a_{033} \end{bmatrix};$$


$$a0 := \begin{bmatrix} a_{011} & a_{012} & a_{013} \\ a_{021} & a_{022} & a_{023} \\ a_{031} & a_{032} & a_{033} \end{bmatrix}$$
 (2)

```

=

```

> a1 := 
$$\begin{bmatrix} a_{111} & a_{112} & a_{113} \\ a_{121} & a_{122} & a_{123} \\ a_{131} & a_{132} & a_{133} \end{bmatrix};$$


$$a1 := \begin{bmatrix} a_{111} & a_{112} & a_{113} \\ a_{121} & a_{122} & a_{123} \\ a_{131} & a_{132} & a_{133} \end{bmatrix}$$
 (3)

```

$$\begin{aligned}
 > a2 := & \begin{bmatrix} a211 & a212 & a213 \\ a221 & a222 & a223 \\ a231 & a232 & a233 \end{bmatrix}; \\
 & a2 := \begin{bmatrix} a211 & a212 & a213 \\ a221 & a222 & a223 \\ a231 & a232 & a233 \end{bmatrix} \tag{4}
 \end{aligned}$$

$$\begin{aligned}
 > A1 := - & \begin{bmatrix} a111 & a113 & a112 \\ a121 & a123 & a122 \\ a131 & a133 & a132 \end{bmatrix}; \\
 & A1 := \begin{bmatrix} -a111 & -a113 & -a112 \\ -a121 & -a123 & -a122 \\ -a131 & -a133 & -a132 \end{bmatrix} \tag{5}
 \end{aligned}$$

$$\begin{aligned}
 > A2 := - & \begin{bmatrix} a211 & a212 & a213 \\ a231 & a232 & a233 \\ a221 & a222 & a223 \end{bmatrix}; \\
 & A2 := \begin{bmatrix} -a211 & -a212 & -a213 \\ -a231 & -a232 & -a233 \\ -a221 & -a222 & -a223 \end{bmatrix} \tag{6}
 \end{aligned}$$

$$\begin{aligned}
 > \\
 > B0 := - & \begin{bmatrix} a011 & a013 & a012 \\ a031 & a033 & a032 \\ a021 & a023 & a022 \end{bmatrix}; \\
 & B0 := \begin{bmatrix} -a011 & -a013 & -a012 \\ -a031 & -a033 & -a032 \\ -a021 & -a023 & -a022 \end{bmatrix} \tag{7}
 \end{aligned}$$

$$\begin{aligned}
 > BI := - & \begin{bmatrix} -a111 & -a112 & -a113 \\ -a131 & -a132 & -a133 \\ -a121 & -a122 & -a123 \end{bmatrix}; \\
 & BI := \begin{bmatrix} a111 & a112 & a113 \\ a131 & a132 & a133 \\ a121 & a122 & a123 \end{bmatrix} \tag{8}
 \end{aligned}$$

$$\begin{aligned}
> B2 := - \begin{bmatrix} -a_{211} & -a_{213} & -a_{212} \\ -a_{221} & -a_{223} & -a_{222} \\ -a_{231} & -a_{233} & -a_{232} \end{bmatrix}; \\
B2 := \begin{bmatrix} a_{211} & a_{213} & a_{212} \\ a_{221} & a_{223} & a_{222} \\ a_{231} & a_{233} & a_{232} \end{bmatrix}
\end{aligned} \tag{9}$$

$$\begin{aligned}
> D2 := expand(Determinant(B0) + Determinant(B1) + Determinant(B2) - Trace(B0 \\
&\quad .B1.B2));
D2 := -a_{111}a_{222}a_{233} - a_{111}a_{133}a_{122} + a_{221}a_{233}a_{212} + a_{211}a_{223}a_{232} \\
&\quad - a_{221}a_{213}a_{232} - a_{131}a_{112}a_{123} + a_{231}a_{213}a_{222} - a_{021}a_{013}a_{032} \\
&\quad + a_{031}a_{013}a_{022} + a_{111}a_{132}a_{123} - a_{031}a_{023}a_{012} + a_{121}a_{112}a_{133} \\
&\quad + a_{011}a_{032}a_{023} + a_{131}a_{122}a_{113} + a_{021}a_{033}a_{012} - a_{011}a_{033}a_{022} \\
&\quad - a_{121}a_{132}a_{113} - a_{231}a_{223}a_{212} + a_{211}a_{011}a_{111} + a_{211}a_{013}a_{131} \\
&\quad + a_{211}a_{012}a_{121} + a_{221}a_{011}a_{112} + a_{221}a_{013}a_{132} + a_{221}a_{012}a_{122} \\
&\quad + a_{231}a_{011}a_{113} + a_{231}a_{013}a_{133} + a_{231}a_{012}a_{123} + a_{213}a_{031}a_{111} \\
&\quad + a_{213}a_{033}a_{131} + a_{213}a_{032}a_{121} + a_{223}a_{031}a_{112} + a_{223}a_{033}a_{132} \\
&\quad + a_{223}a_{032}a_{122} + a_{233}a_{031}a_{113} + a_{233}a_{033}a_{133} + a_{233}a_{032}a_{123} \\
&\quad + a_{212}a_{021}a_{111} + a_{212}a_{023}a_{131} + a_{212}a_{022}a_{121} + a_{222}a_{021}a_{112} \\
&\quad + a_{222}a_{023}a_{132} + a_{222}a_{022}a_{122} + a_{232}a_{021}a_{113} + a_{232}a_{023}a_{133} \\
&\quad + a_{232}a_{022}a_{123}
\end{aligned} \tag{10}$$

$$\begin{aligned}
> dI - D2, \\
dI + a_{211}a_{222}a_{233} + a_{111}a_{133}a_{122} - a_{221}a_{233}a_{212} - a_{211}a_{223}a_{232} \\
&\quad + a_{221}a_{213}a_{232} + a_{131}a_{112}a_{123} - a_{231}a_{213}a_{222} + a_{021}a_{013}a_{032} \\
&\quad - a_{031}a_{013}a_{022} - a_{111}a_{132}a_{123} + a_{031}a_{023}a_{012} - a_{121}a_{112}a_{133} \\
&\quad - a_{011}a_{032}a_{023} - a_{131}a_{122}a_{113} - a_{021}a_{033}a_{012} + a_{011}a_{033}a_{022} \\
&\quad + a_{121}a_{132}a_{113} + a_{231}a_{223}a_{212} - a_{211}a_{011}a_{111} - a_{211}a_{013}a_{131} \\
&\quad - a_{211}a_{012}a_{121} - a_{221}a_{011}a_{112} - a_{221}a_{013}a_{132} - a_{221}a_{012}a_{122} \\
&\quad - a_{231}a_{011}a_{113} - a_{231}a_{013}a_{133} - a_{231}a_{012}a_{123} - a_{213}a_{031}a_{111} \\
&\quad - a_{213}a_{033}a_{131} - a_{213}a_{032}a_{121} - a_{223}a_{031}a_{112} - a_{223}a_{033}a_{132} \\
&\quad - a_{223}a_{032}a_{122} - a_{233}a_{031}a_{113} - a_{233}a_{033}a_{133} - a_{233}a_{032}a_{123} \\
&\quad - a_{212}a_{021}a_{111} - a_{212}a_{023}a_{131} - a_{212}a_{022}a_{121} - a_{222}a_{021}a_{112} \\
&\quad - a_{222}a_{023}a_{132} - a_{222}a_{022}a_{122} - a_{232}a_{021}a_{113} - a_{232}a_{023}a_{133} \\
&\quad - a_{232}a_{022}a_{123}
\end{aligned} \tag{11}$$

$$\begin{aligned}
> a0sharp := a0.a0 - ScalarMultiply(a0, Trace(a0)) + ScalarMultiply\left(IdentityMatrix(3), \\
&\quad \frac{1}{2} \cdot (Trace(a0)^2 - Trace(a0.a0))\right);
\end{aligned} \tag{12}$$

$$\begin{aligned}
a0sharp := & \left[\left[\frac{1}{2} a011^2 - (a011 + a022 + a033) a011 + \frac{1}{2} (a011 + a022 + a033)^2 \right. \right. \\
& - \frac{1}{2} a022^2 - a032 a023 - \frac{1}{2} a033^2, a011 a012 + a012 a022 + a013 a032 - (a011 \\
& + a022 + a033) a012, a011 a013 + a023 a012 + a013 a033 - (a011 + a022 \\
& + a033) a013], \\
& \left[a021 a011 + a022 a021 + a023 a031 - (a011 + a022 + a033) a021, \frac{1}{2} a022^2 \right. \\
& - (a011 + a022 + a033) a022 + \frac{1}{2} (a011 + a022 + a033)^2 - \frac{1}{2} a011^2 \\
& - a013 a031 - \frac{1}{2} a033^2, a021 a013 + a022 a023 + a023 a033 - (a011 + a022 \\
& + a033) a023 \Big], \\
& \left[a031 a011 + a032 a021 + a033 a031 - (a011 + a022 + a033) a031, a031 a012 \right. \\
& + a032 a022 + a033 a032 - (a011 + a022 + a033) a032, \frac{1}{2} a033^2 - (a011 + a022 \\
& + a033) a033 + \frac{1}{2} (a011 + a022 + a033)^2 - \frac{1}{2} a011^2 - a012 a021 - \frac{1}{2} a022^2 \Big] \Big]
\end{aligned} \tag{12}$$

> asharp0 := a0sharp - a1.a2;

$$\begin{aligned}
asharp0 := & \left[\left[\frac{1}{2} a011^2 - (a011 + a022 + a033) a011 + \frac{1}{2} (a011 + a022 + a033)^2 \right. \right. \\
& - \frac{1}{2} a022^2 - a032 a023 - \frac{1}{2} a033^2 - a111 a211 - a112 a221 - a113 a231, \\
& a011 a012 + a012 a022 + a013 a032 - (a011 + a022 + a033) a012 - a111 a212 \\
& - a112 a222 - a113 a232, a011 a013 + a023 a012 + a013 a033 - (a011 + a022 \\
& + a033) a013 - a111 a213 - a112 a223 - a113 a233], \\
& \left[a021 a011 + a022 a021 + a023 a031 - (a011 + a022 + a033) a021 - a121 a211 \right. \\
& - a122 a221 - a123 a231, \frac{1}{2} a022^2 - (a011 + a022 + a033) a022 + \frac{1}{2} (a011 \\
& + a022 + a033)^2 - \frac{1}{2} a011^2 - a013 a031 - \frac{1}{2} a033^2 - a121 a212 - a122 a222 \\
& - a123 a232, a021 a013 + a022 a023 + a023 a033 - (a011 + a022 + a033) a023 \\
& - a121 a213 - a122 a223 - a123 a233 \Big], \\
& \left[a031 a011 + a032 a021 + a033 a031 - (a011 + a022 + a033) a031 - a131 a211 \right]
\end{aligned} \tag{13}$$

```


$$\begin{aligned}
& - a_{132}a_{221} - a_{133}a_{231}, a_{031}a_{012} + a_{032}a_{022} + a_{033}a_{032} - (a_{011} + a_{022} \\
& + a_{033})a_{032} - a_{131}a_{212} - a_{132}a_{222} - a_{133}a_{232}, \frac{1}{2}a_{033}^2 - (a_{011} + a_{022} \\
& + a_{033})a_{033} + \frac{1}{2}(a_{011} + a_{022} + a_{033})^2 - \frac{1}{2}a_{011}^2 - a_{012}a_{021} - \frac{1}{2}a_{022}^2 \\
& - a_{131}a_{213} - a_{132}a_{223} - a_{133}a_{233} \Big] \\
> asharp011 := expand \left( \frac{1}{2}a_{011}^2 - (a_{011} + a_{022} + a_{033})a_{011} + \frac{1}{2}(a_{011} + a_{022} \\
& + a_{033})^2 - \frac{1}{2}a_{022}^2 - a_{023}a_{032} - \frac{1}{2}a_{033}^2 - a_{111}a_{211} - a_{112}a_{221} \\
& - a_{113}a_{231} \right); \\
asharp011 := a_{033}a_{022} - a_{032}a_{023} - a_{111}a_{211} - a_{112}a_{221} - a_{113}a_{231} \tag{14}
\end{aligned}$$


$$\begin{aligned}
> asharp012 := expand (a_{011}a_{012} + a_{022}a_{012} + a_{032}a_{013} - (a_{011} + a_{022} \\
& + a_{033})a_{012} - a_{111}a_{212} - a_{112}a_{222} - a_{113}a_{232}); \\
asharp012 := a_{013}a_{032} - a_{033}a_{012} - a_{111}a_{212} - a_{112}a_{222} - a_{113}a_{232} \tag{15}
\end{aligned}$$


$$\begin{aligned}
> asharp013 := expand (a_{011}a_{013} + a_{012}a_{023} + a_{013}a_{033} - (a_{011} + a_{022} \\
& + a_{033})a_{013} - a_{111}a_{213} - a_{112}a_{223} - a_{113}a_{233}); \\
asharp013 := a_{023}a_{012} - a_{013}a_{022} - a_{111}a_{213} - a_{112}a_{223} - a_{113}a_{233} \tag{16}
\end{aligned}$$


$$\begin{aligned}
> asharp021 := expand (a_{021}a_{011} + a_{022}a_{021} + a_{023}a_{031} - (a_{011} + a_{022} \\
& + a_{033})a_{021} - a_{121}a_{211} - a_{122}a_{221} - a_{123}a_{231}); \\
asharp021 := a_{023}a_{031} - a_{021}a_{033} - a_{121}a_{211} - a_{122}a_{221} - a_{123}a_{231} \tag{17}
\end{aligned}$$


$$\begin{aligned}
> asharp022 := expand \left( \frac{1}{2}a_{022}^2 - (a_{011} + a_{022} + a_{033})a_{022} + \frac{1}{2}(a_{011} + a_{022} \\
& + a_{033})^2 - \frac{1}{2}a_{011}^2 - a_{013}a_{031} - \frac{1}{2}a_{033}^2 - a_{121}a_{212} - a_{122}a_{222} \\
& - a_{123}a_{232} \right); \\
asharp022 := a_{011}a_{033} - a_{013}a_{031} - a_{121}a_{212} - a_{122}a_{222} - a_{123}a_{232} \tag{18}
\end{aligned}$$


$$\begin{aligned}
> asharp023 := expand (a_{021}a_{013} + a_{022}a_{023} + a_{023}a_{033} - (a_{011} + a_{022} \\
& + a_{033})a_{023} - a_{121}a_{213} - a_{122}a_{223} - a_{123}a_{233}); \\
asharp023 := a_{021}a_{013} - a_{023}a_{011} - a_{121}a_{213} - a_{122}a_{223} - a_{123}a_{233} \tag{19}
\end{aligned}$$


$$\begin{aligned}
> asharp031 := expand (a_{031}a_{011} + a_{032}a_{021} + a_{033}a_{031} - (a_{011} + a_{022} \\
& + a_{033})a_{031} - a_{131}a_{211} - a_{132}a_{221} - a_{133}a_{231}); \\
asharp031 := a_{032}a_{021} - a_{031}a_{022} - a_{131}a_{211} - a_{132}a_{221} - a_{133}a_{231} \tag{20}
\end{aligned}$$


$$\begin{aligned}
> asharp032 := expand (a_{031}a_{012} + a_{032}a_{022} + a_{033}a_{032} - (a_{011} + a_{022} \\
& + a_{033})a_{032} - a_{131}a_{212} - a_{132}a_{222} - a_{133}a_{232}); \\
asharp032 := a_{031}a_{012} - a_{032}a_{011} - a_{131}a_{212} - a_{132}a_{222} - a_{133}a_{232} \tag{21}
\end{aligned}$$


```

```

> asharp033 := expand  $\left( \frac{1}{2} a033^2 - (a011 + a022 + a033) a033 + \frac{1}{2} (a011 + a022 + a033)^2 - \frac{1}{2} a011^2 - a012 a021 - \frac{1}{2} a022^2 - a131 a213 - a132 a223 - a133 a233 \right);$ 
asharp033 := a011 a022 - a012 a021 - a131 a213 - a132 a223 - a133 a233 (22)
=> Sasharp0 := Matrix([ [asharp011, asharp013, asharp012], [asharp031, asharp033, asharp032], [asharp021, asharp023, asharp022]]);
Sasharp0 := [[a033 a022 - a032 a023 - a111 a211 - a112 a221 - a113 a231, a023 a012 - a013 a022 - a111 a213 - a112 a223 - a113 a233, a013 a032 - a033 a012 - a111 a212 - a112 a222 - a113 a232],
[a032 a021 - a031 a022 - a131 a211 - a132 a221 - a133 a231, a011 a022 - a012 a021 - a131 a213 - a132 a223 - a133 a233, a031 a012 - a032 a011 - a131 a212 - a132 a222 - a133 a232],
[a023 a031 - a021 a033 - a121 a211 - a122 a221 - a123 a231, a021 a013 - a023 a011 - a121 a213 - a122 a223 - a123 a233, a011 a033 - a013 a031 - a121 a212 - a122 a222 - a123 a232]]]
> B0sharp := B0.B0 - ScalarMultiply(B0, Trace(B0))
+ ScalarMultiply  $\left( \text{IdentityMatrix}(3), \frac{1}{2} \cdot (\text{Trace}(B0)^2 - \text{Trace}(B0 \cdot B0)) \right);$ 
B0sharp := [[ $\frac{1}{2} a011^2 + (-a011 - a033 - a022) a011 + \frac{1}{2} (-a011 - a033 - a022)^2 - \frac{1}{2} a022^2 - a032 a023 - \frac{1}{2} a033^2, a011 a013 + a023 a012 + a013 a033 + (-a011 - a033 - a022) a013, a011 a012 + a012 a022 + a013 a032 + (-a011 - a033 - a022) a012,$ 
 $a031 a011 + a032 a021 + a033 a031 + (-a011 - a033 - a022) a031, \frac{1}{2} a033^2 + (-a011 - a033 - a022) a033 + \frac{1}{2} (-a011 - a033 - a022)^2 - \frac{1}{2} a011^2 - a012 a021 - \frac{1}{2} a022^2, a031 a012 + a032 a022 + a033 a032 + (-a011 - a033 - a022) a032,$ 
 $a021 a011 + a022 a021 + a023 a031 + (-a011 - a033 - a022) a021, a021 a013 + a022 a023 + a023 a033 + (-a011 - a033 - a022) a023, \frac{1}{2} a022^2 + (-a011 - a033 - a022) a022 + \frac{1}{2} (-a011 - a033 - a022)^2 - \frac{1}{2} a011^2 - a013 a031]$ ]] (24)

```

$$\begin{aligned}
& \left[\left[-\frac{1}{2} a033^2 \right] \right] \\
> & \text{Bsharp0} := \text{B0sharp} - \text{B1.B2}, \\
\text{Bsharp0} := & \left[\left[\frac{1}{2} a011^2 + (-a011 - a033 - a022) a011 + \frac{1}{2} (-a011 - a033 - a022)^2 \right. \right. \quad (25) \\
& - \frac{1}{2} a022^2 - a032 a023 - \frac{1}{2} a033^2 - a111 a211 - a112 a221 - a113 a231, \\
& a011 a013 + a023 a012 + a013 a033 + (-a011 - a033 - a022) a013 - a111 a213 \\
& - a112 a223 - a113 a233, a011 a012 + a012 a022 + a013 a032 + (-a011 - a033 \\
& - a022) a012 - a111 a212 - a112 a222 - a113 a232], \\
& \left. \left[a031 a011 + a032 a021 + a033 a031 + (-a011 - a033 - a022) a031 - a131 a211 \right. \right. \\
& - a132 a221 - a133 a231, \frac{1}{2} a033^2 + (-a011 - a033 - a022) a033 + \frac{1}{2} (\\
& - a011 - a033 - a022)^2 - \frac{1}{2} a011^2 - a012 a021 - \frac{1}{2} a022^2 - a131 a213 \\
& - a132 a223 - a133 a233, a031 a012 + a032 a022 + a033 a032 + (-a011 - a033 \\
& - a022) a032 - a131 a212 - a132 a222 - a133 a232] \Bigg], \\
& \left[a021 a011 + a022 a021 + a023 a031 + (-a011 - a033 - a022) a021 - a121 a211 \right. \\
& - a122 a221 - a123 a231, a021 a013 + a022 a023 + a023 a033 + (-a011 - a033 \\
& - a022) a023 - a121 a213 - a122 a223 - a123 a233, \frac{1}{2} a022^2 + (-a011 - a033 \\
& - a022) a022 + \frac{1}{2} (-a011 - a033 - a022)^2 - \frac{1}{2} a011^2 - a013 a031 - \frac{1}{2} a033^2 \\
& - a121 a212 - a122 a222 - a123 a232] \Bigg] \\
> & \text{Bsharp011} := \text{expand} \left(\frac{1}{2} a011^2 - (a011 + a022 + a033) a011 + \frac{1}{2} (a011 + a022 \right. \\
& + a033)^2 - \frac{1}{2} a022^2 - a023 a032 - \frac{1}{2} a033^2 - a111 a211 - a112 a221 \\
& \left. - a113 a231 \right); \\
\text{Bsharp011} = & a033 a022 - a032 a023 - a111 a211 - a112 a221 - a113 a231 \quad (26) \\
> & \text{Bsharp012} := \text{expand} (a011 a013 + a012 a023 + a013 a033 - (a011 + a022 + a033) a013 \\
& - a111 a213 - a112 a223 - a113 a233); \\
\text{Bsharp012} = & a023 a012 - a013 a022 - a111 a213 - a112 a223 - a113 a233 \quad (27)
\end{aligned}$$

```

> Bsharp013 := expand (a011 a012 + a022 a012 + a032 a013 - (a011 + a022
+ a033) a012 - a111 a212 - a112 a222 - a113 a232);
Bsharp013 := a013 a032 - a033 a012 - a111 a212 - a112 a222 - a113 a232      (28)

>
> Bsharp021 := expand (a031 a011 + a032 a021 + a033 a031 - (a011 + a022
+ a033) a031 - a131 a211 - a132 a221 - a133 a231);
Bsharp021 := a032 a021 - a031 a022 - a131 a211 - a132 a221 - a133 a231      (29)

> Bsharp022 := expand  $\left( \frac{1}{2} a033^2 - (a011 + a022 + a033) a033 + \frac{1}{2} (a011 + a022 + a033)^2 - \frac{1}{2} a011^2 - a012 a021 - \frac{1}{2} a022^2 - a131 a213 - a132 a223 - a133 a233 \right)$ 
Bsharp022 := a011 a022 - a012 a021 - a131 a213 - a132 a223 - a133 a233      (30)

> Bsharp023 := expand (a031 a012 + a032 a022 + a033 a032 - (a011 + a022
+ a033) a032 - a131 a212 - a132 a222 - a133 a232);
Bsharp023 := a031 a012 - a032 a011 - a131 a212 - a132 a222 - a133 a232      (31)

>
> Bsharp031 := expand (a021 a011 + a022 a021 + a023 a031 - (a011 + a022
+ a033) a021 - a121 a211 - a122 a221 - a123 a231);
Bsharp031 := a023 a031 - a021 a033 - a121 a211 - a122 a221 - a123 a231      (32)

> Bsharp032 := expand (a021 a013 + a022 a023 + a023 a033 - (a011 + a022
+ a033) a023 - a121 a213 - a122 a223 - a123 a233);
Bsharp032 := a021 a013 - a023 a011 - a121 a213 - a122 a223 - a123 a233      (33)

> Bsharp033 := expand  $\left( \frac{1}{2} a022^2 - (a011 + a022 + a033) a022 + \frac{1}{2} (a011 + a022 + a033)^2 - \frac{1}{2} a011^2 - a013 a031 - \frac{1}{2} a033^2 - a121 a212 - a122 a222 - a123 a232 \right)$ 
Bsharp033 := a011 a033 - a013 a031 - a121 a212 - a122 a222 - a123 a232      (34)

> Bsharp0expand := Matrix ([[Bsharp011, Bsharp012, Bsharp013], [Bsharp021, Bsharp022,
Bsharp023], [Bsharp031, Bsharp032, Bsharp033]]));
Bsharp0expand := [[a033 a022 - a032 a023 - a111 a211 - a112 a221 - a113 a231,
a023 a012 - a013 a022 - a111 a213 - a112 a223 - a113 a233, a013 a032 - a033 a012
- a111 a212 - a112 a222 - a113 a232],
[a032 a021 - a031 a022 - a131 a211 - a132 a221 - a133 a231, a011 a022
- a012 a021 - a131 a213 - a132 a223 - a133 a233, a031 a012 - a032 a011
- a131 a212 - a132 a222 - a133 a232],
[a023 a031 - a021 a033 - a121 a211 - a122 a221 - a123 a231, a021 a013
- a023 a011 - a121 a213 - a122 a223 - a123 a233, a011 a033 - a013 a031
- a121 a212 - a122 a222 - a123 a232]]]

```

$$\begin{aligned}
 > & \text{MatrixAdd}(\text{Sasharp0}, -\text{Bsharp0expand}); \\
 & \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \\
 \boxed{>} & (\text{check}, ?, ?)
 \end{aligned} \tag{36}$$

$$\begin{aligned}
 > & a2; \\
 & \left[\begin{array}{ccc} a_{211} & a_{212} & a_{213} \\ a_{221} & a_{222} & a_{223} \\ a_{231} & a_{232} & a_{233} \end{array} \right]
 \end{aligned} \tag{37}$$

$$\begin{aligned}
 > & a2sharp := a2.a2 - \text{ScalarMultiply}(a2, \text{Trace}(a2)) + \text{ScalarMultiply}\left(\text{IdentityMatrix}(3), \frac{1}{2} \cdot (\text{Trace}(a2)^2 - \text{Trace}(a2.a2))\right); \\
 a2sharp := & \left[\left[\begin{array}{c} \frac{1}{2} a_{211}^2 - (a_{211} + a_{222} + a_{233}) a_{211} + \frac{1}{2} (a_{211} + a_{222} + a_{233})^2 \\ - \frac{1}{2} a_{222}^2 - a_{223} a_{232} - \frac{1}{2} a_{233}^2, a_{211} a_{212} + a_{212} a_{222} + a_{213} a_{232} - (a_{211} + a_{222} + a_{233}) a_{212}, a_{211} a_{213} + a_{223} a_{212} + a_{213} a_{233} - (a_{211} + a_{222} + a_{233}) a_{213} \end{array} \right], \right. \\
 & \left. \begin{array}{c} a_{221} a_{211} + a_{222} a_{221} + a_{223} a_{231} - (a_{211} + a_{222} + a_{233}) a_{221}, \frac{1}{2} a_{222}^2 - (a_{211} + a_{222} + a_{233}) a_{222} + \frac{1}{2} (a_{211} + a_{222} + a_{233})^2 - \frac{1}{2} a_{211}^2 - a_{213} a_{231} - \frac{1}{2} a_{233}^2, a_{221} a_{213} + a_{222} a_{223} + a_{223} a_{233} - (a_{211} + a_{222} + a_{233}) a_{223} \end{array} \right], \\
 & \left[\left[\begin{array}{c} a_{231} a_{211} + a_{232} a_{221} + a_{233} a_{231} - (a_{211} + a_{222} + a_{233}) a_{231}, a_{231} a_{212} + a_{232} a_{222} + a_{233} a_{232} - (a_{211} + a_{222} + a_{233}) a_{232}, \frac{1}{2} a_{233}^2 - (a_{211} + a_{222} + a_{233}) a_{233} + \frac{1}{2} (a_{211} + a_{222} + a_{233})^2 - \frac{1}{2} a_{211}^2 - a_{212} a_{221} - \frac{1}{2} a_{222}^2 \end{array} \right] \right]
 \end{aligned} \tag{38}$$

$$\begin{aligned}
 > & asharp1 := \text{MatrixAdd}(a2sharp, -a0.a1); \\
 asharp1 := & \left[\left[\begin{array}{c} \frac{1}{2} a_{211}^2 - (a_{211} + a_{222} + a_{233}) a_{211} + \frac{1}{2} (a_{211} + a_{222} + a_{233})^2 \end{array} \right] \right]
 \end{aligned} \tag{39}$$

$$\begin{aligned}
& - \frac{1}{2} a_{222}^2 - a_{223} a_{232} - \frac{1}{2} a_{233}^2 - a_{011} a_{111} - a_{013} a_{131} - a_{012} a_{121}, \\
& a_{211} a_{212} + a_{212} a_{222} + a_{213} a_{232} - (a_{211} + a_{222} + a_{233}) a_{212} - a_{011} a_{112} \\
& - a_{013} a_{132} - a_{012} a_{122}, a_{211} a_{213} + a_{223} a_{212} + a_{213} a_{233} - (a_{211} + a_{222} \\
& + a_{233}) a_{213} - a_{011} a_{113} - a_{013} a_{133} - a_{012} a_{123}], \\
& \left[a_{221} a_{211} + a_{222} a_{221} + a_{223} a_{231} - (a_{211} + a_{222} + a_{233}) a_{221} - a_{021} a_{111} \right. \\
& - a_{023} a_{131} - a_{022} a_{121}, \frac{1}{2} a_{222}^2 - (a_{211} + a_{222} + a_{233}) a_{222} + \frac{1}{2} (a_{211} \\
& + a_{222} + a_{233})^2 - \frac{1}{2} a_{211}^2 - a_{213} a_{231} - \frac{1}{2} a_{233}^2 - a_{021} a_{112} - a_{023} a_{132} \\
& - a_{022} a_{122}, a_{221} a_{213} + a_{222} a_{223} + a_{223} a_{233} - (a_{211} + a_{222} + a_{233}) a_{223} \\
& - a_{021} a_{113} - a_{023} a_{133} - a_{022} a_{123}], \\
& \left[a_{231} a_{211} + a_{232} a_{221} + a_{233} a_{231} - (a_{211} + a_{222} + a_{233}) a_{231} - a_{031} a_{111} \right. \\
& - a_{033} a_{131} - a_{032} a_{121}, a_{231} a_{212} + a_{232} a_{222} + a_{233} a_{232} - (a_{211} + a_{222} \\
& + a_{233}) a_{232} - a_{031} a_{112} - a_{033} a_{132} - a_{032} a_{122}, \frac{1}{2} a_{233}^2 - (a_{211} + a_{222} \\
& + a_{233}) a_{233} + \frac{1}{2} (a_{211} + a_{222} + a_{233})^2 - \frac{1}{2} a_{211}^2 - a_{212} a_{221} - \frac{1}{2} a_{222}^2 \\
& - a_{031} a_{113} - a_{033} a_{133} - a_{032} a_{123}] \Big] \\
> & \text{asharp111 := expand}\left(\frac{1}{2} a_{211}^2 - (a_{211} + a_{222} + a_{233}) a_{211} + \frac{1}{2} (a_{211} + a_{222} \\
& + a_{233})^2 - \frac{1}{2} a_{222}^2 - a_{223} a_{232} - \frac{1}{2} a_{233}^2 - a_{011} a_{111} - a_{013} a_{131} \\
& - a_{012} a_{121}\right); \quad (40) \\
> & \text{asharp112 := expand}(a_{211} a_{212} + a_{212} a_{222} + a_{213} a_{232} - (a_{211} + a_{222} \\
& + a_{233}) a_{212} - a_{011} a_{112} - a_{013} a_{132} - a_{012} a_{122}); \quad (41) \\
> & \text{asharp113 := expand}(a_{211} a_{213} + a_{223} a_{212} + a_{213} a_{233} - (a_{211} + a_{222} \\
& + a_{233}) a_{213} - a_{011} a_{113} - a_{013} a_{133} - a_{012} a_{123}); \quad (42) \\
> & \text{asharp121 := expand}(a_{221} a_{211} + a_{222} a_{221} + a_{223} a_{231} - (a_{211} + a_{222} \\
& + a_{233}) a_{221} - a_{021} a_{111} - a_{023} a_{131} - a_{022} a_{121})
\end{aligned}$$

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+ a233) a221 - a021 a111 - a023 a131 - a022 a121);
asharp121 := a223 a231 - a221 a233 - a021 a111 - a023 a131 - a022 a121      (43)

> asharp122 := expand  $\left( \frac{1}{2} a222^2 - (a211 + a222 + a233) a222 + \frac{1}{2} (a211 + a222 + a233)^2 - \frac{1}{2} a211^2 - a213 a231 - \frac{1}{2} a233^2 - a021 a112 - a023 a132 - a022 a122 \right);$ 
asharp122 := a211 a233 - a213 a231 - a021 a112 - a023 a132 - a022 a122      (44)

> asharp123 := expand  $(a221 a213 + a222 a223 + a223 a233 - (a211 + a222 + a233) a223 - a021 a113 - a023 a133 - a022 a123);$ 
asharp123 := a221 a213 - a223 a211 - a021 a113 - a023 a133 - a022 a123      (45)

>
> asharp131 := expand  $(a231 a211 + a232 a221 + a233 a231 - (a211 + a222 + a233) a231 - a031 a111 - a033 a131 - a032 a121);$ 
asharp131 := a232 a221 - a231 a222 - a031 a111 - a033 a131 - a032 a121      (46)

> asharp132 := expand  $(a231 a212 + a232 a222 + a233 a232 - (a211 + a222 + a233) a232 - a031 a112 - a033 a132 - a032 a122);$ 
asharp132 := a231 a212 - a232 a211 - a031 a112 - a033 a132 - a032 a122      (47)

> asharp133 := expand  $\left( \frac{1}{2} a233^2 - (a211 + a222 + a233) a233 + \frac{1}{2} (a211 + a222 + a233)^2 - \frac{1}{2} a211^2 - a212 a221 - \frac{1}{2} a222^2 - a031 a113 - a033 a133 - a032 a123 \right);$ 
asharp133 := a211 a222 - a212 a221 - a031 a113 - a033 a133 - a032 a123      (48)

> Sasharp1 := Matrix ( -[[asharp111, asharp112, asharp113], [asharp131, asharp132, asharp133], [asharp121, asharp122, asharp123]] );
Sasharp1 := [[-a222 a233 + a223 a232 + a011 a111 + a013 a131 + a012 a121,
-a213 a232 + a233 a212 + a011 a112 + a013 a132 + a012 a122, -a223 a212
+ a213 a222 + a011 a113 + a013 a133 + a012 a123],
[-a232 a221 + a231 a222 + a031 a111 + a033 a131 + a032 a121, -a231 a212
+ a232 a211 + a031 a112 + a033 a132 + a032 a122, -a211 a222 + a212 a221
+ a031 a113 + a033 a133 + a032 a123],
[-a223 a231 + a221 a233 + a021 a111 + a023 a131 + a022 a121, -a211 a233
+ a213 a231 + a021 a112 + a023 a132 + a022 a122, -a221 a213 + a223 a211
+ a021 a113 + a023 a133 + a022 a123]]      (49)

>
> B2sharp := B2.B2 - ScalarMultiply(B2, Trace(B2))
+ ScalarMultiply  $\left( \text{IdentityMatrix}(3), \frac{1}{2} \cdot (\text{Trace}(B2)^2 - \text{Trace}(B2 \cdot B2)) \right);$ 

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(50)

$$\begin{aligned}
B2sharp := & \left[\left[\frac{1}{2} a_{211}^2 - (a_{211} + a_{223} + a_{232}) a_{211} + \frac{1}{2} (a_{211} + a_{223} + a_{232})^2 \right. \right. \\
& - \frac{1}{2} a_{223}^2 - a_{222} a_{233} - \frac{1}{2} a_{232}^2, a_{211} a_{213} + a_{213} a_{223} + a_{233} a_{212} - (a_{211} \\
& + a_{223} + a_{232}) a_{213}, a_{211} a_{212} + a_{213} a_{222} + a_{212} a_{232} - (a_{211} + a_{223} \\
& + a_{232}) a_{212}], \\
& \left[a_{221} a_{211} + a_{223} a_{221} + a_{231} a_{222} - (a_{211} + a_{223} + a_{232}) a_{221}, \frac{1}{2} a_{223}^2 \right. \\
& - (a_{211} + a_{223} + a_{232}) a_{223} + \frac{1}{2} (a_{211} + a_{223} + a_{232})^2 - \frac{1}{2} a_{211}^2 \\
& - a_{231} a_{212} - \frac{1}{2} a_{232}^2, a_{212} a_{221} + a_{222} a_{223} + a_{232} a_{222} - (a_{211} + a_{223} \\
& + a_{232}) a_{222} \Big], \\
& \left[a_{231} a_{211} + a_{221} a_{233} + a_{232} a_{231} - (a_{211} + a_{223} + a_{232}) a_{231}, a_{213} a_{231} \right. \\
& + a_{223} a_{233} + a_{233} a_{232} - (a_{211} + a_{223} + a_{232}) a_{233}, \frac{1}{2} a_{232}^2 - (a_{211} + a_{223} \\
& + a_{232}) a_{232} + \frac{1}{2} (a_{211} + a_{223} + a_{232})^2 - \frac{1}{2} a_{211}^2 - a_{221} a_{213} - \frac{1}{2} a_{223}^2 \Big] \Big]
\end{aligned} \tag{50}$$

> $Bsharp1 := B2sharp - B0.BI;$

$$\begin{aligned}
Bsharp1 := & \left[\left[\frac{1}{2} a_{211}^2 - (a_{211} + a_{223} + a_{232}) a_{211} + \frac{1}{2} (a_{211} + a_{223} + a_{232})^2 \right. \right. \\
& - \frac{1}{2} a_{223}^2 - a_{222} a_{233} - \frac{1}{2} a_{232}^2 + a_{011} a_{111} + a_{012} a_{121} + a_{013} a_{131}, \\
& a_{211} a_{213} + a_{213} a_{223} + a_{233} a_{212} - (a_{211} + a_{223} + a_{232}) a_{213} + a_{011} a_{112} \\
& + a_{012} a_{122} + a_{013} a_{132}, a_{211} a_{212} + a_{213} a_{222} + a_{212} a_{232} - (a_{211} + a_{223} \\
& + a_{232}) a_{212} + a_{011} a_{113} + a_{012} a_{123} + a_{013} a_{133}], \\
& \left[a_{221} a_{211} + a_{223} a_{221} + a_{231} a_{222} - (a_{211} + a_{223} + a_{232}) a_{221} + a_{031} a_{111} \right. \\
& + a_{032} a_{121} + a_{033} a_{131}, \frac{1}{2} a_{223}^2 - (a_{211} + a_{223} + a_{232}) a_{223} + \frac{1}{2} (a_{211} \\
& + a_{223} + a_{232})^2 - \frac{1}{2} a_{211}^2 - a_{231} a_{212} - \frac{1}{2} a_{232}^2 + a_{031} a_{112} + a_{032} a_{122} \\
& + a_{033} a_{132}, a_{212} a_{221} + a_{222} a_{223} + a_{232} a_{222} - (a_{211} + a_{223} + a_{232}) a_{222} \\
& + a_{031} a_{113} + a_{032} a_{123} + a_{033} a_{133} \Big], \\
& \left[a_{231} a_{211} + a_{221} a_{233} + a_{232} a_{231} - (a_{211} + a_{223} + a_{232}) a_{231} + a_{021} a_{111} \right]
\end{aligned} \tag{51}$$

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+ a022 a121 + a023 a131, a213 a231 + a223 a233 + a233 a232 - (a211 + a223
+ a232) a233 + a021 a112 + a022 a122 + a023 a132,  $\frac{1}{2}$  a2322 - (a211 + a223
+ a232) a232 +  $\frac{1}{2}$  (a211 + a223 + a232)2 -  $\frac{1}{2}$  a2112 - a221 a213 -  $\frac{1}{2}$  a2232
+ a021 a113 + a022 a123 + a023 a133] ]

```

= > $Bsharp111 := expand \left(\frac{1}{2} a211^2 + (-a211 - a223 - a232) a211 + \frac{1}{2} (-a211 - a223 - a232)^2 - \frac{1}{2} a223^2 - a222 a233 - \frac{1}{2} a232^2 + a011 a111 + a012 a121 + a013 a131 \right);$

(52)

= > $Bsharp112 := expand (a211 a213 + a213 a223 + a233 a212 + (-a211 - a223 - a232) a213 + a011 a112 + a012 a122 + a013 a132);$

$Bsharp112 := -a213 a232 + a233 a212 + a011 a112 + a013 a132 + a012 a122$ (53)

= > $Bsharp113 := expand (a211 a212 + a213 a222 + a212 a232 + (-a211 - a223 - a232) a212 + a011 a113 + a012 a123 + a013 a133);$

$Bsharp113 := -a223 a212 + a213 a222 + a011 a113 + a013 a133 + a012 a123$ (54)

= >

= > $Bsharp121 := expand (a221 a211 + a223 a221 + a231 a222 + (-a211 - a223 - a232) a221 + a031 a111 + a032 a121 + a033 a131);$

$Bsharp121 := -a232 a221 + a231 a222 + a031 a111 + a033 a131 + a032 a121$ (55)

= > $Bsharp122 := expand \left(\frac{1}{2} a223^2 + (-a211 - a223 - a232) a223 + \frac{1}{2} (-a211 - a223 - a232)^2 - \frac{1}{2} a211^2 - a231 a212 - \frac{1}{2} a232^2 + a031 a112 + a032 a122 + a033 a132 \right);$

$Bsharp122 := -a231 a212 + a232 a211 + a031 a112 + a033 a132 + a032 a122$ (56)

= > $Bsharp123 := expand (a212 a221 + a222 a223 + a232 a222 + (-a211 - a223 - a232) a222 + a031 a113 + a032 a123 + a033 a133);$

$Bsharp123 := -a211 a222 + a212 a221 + a031 a113 + a033 a133 + a032 a123$ (57)

= >

= > $Bsharp131 := expand (a231 a211 + a221 a233 + a232 a231 + (-a211 - a223 - a232) a231 + a021 a111 + a022 a121 + a023 a131);$

$Bsharp131 := -a223 a231 + a221 a233 + a021 a111 + a023 a131 + a022 a121$ (58)

= > $Bsharp132 := expand (a213 a231 + a223 a233 + a233 a232 + (-a211 - a223 - a232) a233 + a021 a112 + a022 a122 + a023 a132);$

$Bsharp132 := -a211 a233 + a213 a231 + a021 a112 + a023 a132 + a022 a122$ (59)

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> Bsharp133 := expand  $\left( \frac{1}{2} a232^2 + (-a211 - a223 - a232) a232 + \frac{1}{2} (-a211 - a223 - a232)^2 - \frac{1}{2} a211^2 - a221 a213 - \frac{1}{2} a223^2 + a021 a113 + a022 a123 + a023 a133 \right);$ 
Bsharp133 :=  $-a221 a213 + a223 a211 + a021 a113 + a023 a133 + a022 a123$  (60)
> Bsharp1expand := Matrix([ [ Bsharp111, Bsharp112, Bsharp113 ], [ Bsharp121, Bsharp122, Bsharp123 ], [ Bsharp131, Bsharp132, Bsharp133 ] ]);  

Bsharp1expand := [[ -a222 a233 + a223 a232 + a011 a111 + a013 a131 + a012 a121, -a213 a232 + a233 a212 + a011 a112 + a013 a132 + a012 a122, -a223 a212 + a213 a222 + a011 a113 + a013 a133 + a012 a123 ],  

[ -a232 a221 + a231 a222 + a031 a111 + a033 a131 + a032 a121, -a231 a212 + a232 a211 + a031 a112 + a033 a132 + a032 a122, -a211 a222 + a212 a221 + a031 a113 + a033 a133 + a032 a123 ],  

[ -a223 a231 + a221 a233 + a021 a111 + a023 a131 + a022 a121, -a211 a233 + a213 a231 + a021 a112 + a023 a132 + a022 a122, -a221 a213 + a223 a211 + a021 a113 + a023 a133 + a022 a123 ]]
> MatrixAdd (Sasharp1, - Bsharp1expand );

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
 (62)
>
> a1sharp := a1. a1 - ScalarMultiply(a1, Trace(a1)) + ScalarMultiply(IdentityMatrix(3),  


$$\frac{1}{2} \cdot (Trace(a1)^2 - Trace(a1.a1))$$
);
a1sharp :=  $\left[ \left[ \frac{1}{2} a111^2 - (a111 + a122 + a133) a111 + \frac{1}{2} (a111 + a122 + a133)^2 - \frac{1}{2} a122^2 - a123 a132 - \frac{1}{2} a133^2, a111 a112 + a112 a122 + a113 a132 - (a111 + a122 + a133) a112, a111 a113 + a112 a123 + a113 a133 - (a111 + a122 + a133) a113 \right],$   


$$\left[ a121 a111 + a122 a121 + a123 a131 - (a111 + a122 + a133) a121, \frac{1}{2} a122^2 - (a111 + a122 + a133) a122 + \frac{1}{2} (a111 + a122 + a133)^2 - \frac{1}{2} a111^2 - a113 a131 - \frac{1}{2} a133^2, a121 a113 + a122 a123 + a123 a133 - (a111 + a122 + a133) a123 \right]$$
 (63)

```

$$\begin{aligned}
& + a133) a123 \Big], \\
& \left[a131 a111 + a132 a121 + a133 a131 - (a111 + a122 + a133) a131, a131 a112 \right. \\
& + a132 a122 + a133 a132 - (a111 + a122 + a133) a132, \frac{1}{2} a133^2 - (a111 + a122 \\
& \left. + a133) a133 + \frac{1}{2} (a111 + a122 + a133)^2 - \frac{1}{2} a111^2 - a112 a121 - \frac{1}{2} a122^2 \right] \Big] \\
> \quad & asharp2 := MatrixAdd (a1sharp, -a2.a0); \\
asharp2 := & \left[\left[\frac{1}{2} a111^2 - (a111 + a122 + a133) a111 + \frac{1}{2} (a111 + a122 + a133)^2 \right. \right. \\
& - \frac{1}{2} a122^2 - a123 a132 - \frac{1}{2} a133^2 - a211 a011 - a212 a021 - a213 a031, \\
& a111 a112 + a112 a122 + a113 a132 - (a111 + a122 + a133) a112 - a211 a012 \\
& - a212 a022 - a213 a032, a111 a113 + a112 a123 + a113 a133 - (a111 + a122 \\
& + a133) a113 - a211 a013 - a212 a023 - a213 a033], \\
& \left[a121 a111 + a122 a121 + a123 a131 - (a111 + a122 + a133) a121 - a221 a011 \right. \\
& - a222 a021 - a223 a031, \frac{1}{2} a122^2 - (a111 + a122 + a133) a122 + \frac{1}{2} (a111 \\
& + a122 + a133)^2 - \frac{1}{2} a111^2 - a113 a131 - \frac{1}{2} a133^2 - a221 a012 - a222 a022 \\
& - a223 a032, a121 a113 + a122 a123 + a123 a133 - (a111 + a122 + a133) a123 \\
& \left. - a221 a013 - a222 a023 - a223 a033 \right], \\
& \left[a131 a111 + a132 a121 + a133 a131 - (a111 + a122 + a133) a131 - a231 a011 \right. \\
& - a232 a021 - a233 a031, a131 a112 + a132 a122 + a133 a132 - (a111 + a122 \\
& + a133) a132 - a231 a012 - a232 a022 - a233 a032, \frac{1}{2} a133^2 - (a111 + a122 \\
& + a133) a133 + \frac{1}{2} (a111 + a122 + a133)^2 - \frac{1}{2} a111^2 - a112 a121 - \frac{1}{2} a122^2 \\
& \left. - a231 a013 - a232 a023 - a233 a033 \right] \Big] \\
> \quad & asharp211 := expand \left(\frac{1}{2} a111^2 - (a111 + a122 + a133) a111 + \frac{1}{2} (a111 + a122 \right. \\
& \left. + a133)^2 - \frac{1}{2} a122^2 - a123 a132 - \frac{1}{2} a133^2 - a211 a011 - a212 a021
\end{aligned}$$

```


$$- a213 a031);$$


$$\text{asharp211 := } a122 a133 - a123 a132 - a211 a011 - a212 a021 - a213 a031 \quad (65)$$


$$> \text{asharp212 := expand (} a111 a112 + a112 a122 + a113 a132 - (a111 + a122$$


$$+ a133) a112 - a211 a012 - a212 a022 - a213 a032);$$


$$\text{asharp212 := } a113 a132 - a112 a133 - a211 a012 - a212 a022 - a213 a032 \quad (66)$$


$$> \text{asharp213 := expand (} a111 a113 + a112 a123 + a113 a133 - (a111 + a122$$


$$+ a133) a113 - a211 a013 - a212 a023 - a213 a033);$$


$$\text{asharp213 := } a112 a123 - a113 a122 - a211 a013 - a212 a023 - a213 a033 \quad (67)$$


$$>$$


$$> \text{asharp221 := expand (} a121 a111 + a122 a121 + a123 a131 - (a111 + a122$$


$$+ a133) a121 - a221 a011 - a222 a021 - a223 a031);$$


$$\text{asharp221 := } a123 a131 - a121 a133 - a221 a011 - a222 a021 - a223 a031 \quad (68)$$


$$> \text{asharp222 := expand (} \frac{1}{2} a122^2 - (a111 + a122 + a133) a122 + \frac{1}{2} (a111 + a122$$


$$+ a133)^2 - \frac{1}{2} a111^2 - a113 a131 - \frac{1}{2} a133^2 - a221 a012 - a222 a022$$


$$- a223 a032);$$


$$\text{asharp222 := } a111 a133 - a113 a131 - a221 a012 - a222 a022 - a223 a032 \quad (69)$$


$$> \text{asharp223 := expand (} a121 a113 + a122 a123 + a123 a133 - (a111 + a122$$


$$+ a133) a123 - a221 a013 - a222 a023 - a223 a033);$$


$$\text{asharp223 := } a121 a113 - a123 a111 - a221 a013 - a222 a023 - a223 a033 \quad (70)$$


$$>$$


$$> \text{asharp231 := expand (} a131 a111 + a132 a121 + a133 a131 - (a111 + a122$$


$$+ a133) a131 - a231 a011 - a232 a021 - a233 a031);$$


$$\text{asharp231 := } a132 a121 - a131 a122 - a231 a011 - a232 a021 - a233 a031 \quad (71)$$


$$> \text{asharp232 := expand (} a131 a112 + a132 a122 + a133 a132 - (a111 + a122$$


$$+ a133) a132 - a231 a012 - a232 a022 - a233 a032);$$


$$\text{asharp232 := } a131 a112 - a132 a111 - a231 a012 - a232 a022 - a233 a032 \quad (72)$$


$$> \text{asharp233 := expand (} \frac{1}{2} a133^2 - (a111 + a122 + a133) a133 + \frac{1}{2} (a111 + a122$$


$$+ a133)^2 - \frac{1}{2} a111^2 - a112 a121 - \frac{1}{2} a122^2 - a231 a013 - a232 a023$$


$$- a233 a033);$$


$$\text{asharp233 := } a111 a122 - a112 a121 - a231 a013 - a232 a023 - a233 a033 \quad (73)$$


$$>$$


$$> \text{Sasharp2 := Matrix (} -[[\text{asharp211, asharp213, asharp212}], [\text{asharp221, asharp223,}$$


$$\text{asharp222}], [\text{asharp231, asharp233, asharp232}]]);$$


$$\text{Sasharp2 := } [[-a122 a133 + a123 a132 + a211 a011 + a212 a021 + a213 a031,}$$


$$-a112 a123 + a113 a122 + a211 a013 + a212 a023 + a213 a033, -a113 a132 \quad (74)$$


```

```

+ a112 a133 + a211 a012 + a212 a022 + a213 a032],
[-a123 a131 + a121 a133 + a221 a011 + a222 a021 + a223 a031, -a121 a113
+ a123 a111 + a221 a013 + a222 a023 + a223 a033, -a111 a133 + a113 a131
+ a221 a012 + a222 a022 + a223 a032],
[-a132 a121 + a131 a122 + a231 a011 + a232 a021 + a233 a031, -a111 a122
+ a112 a121 + a231 a013 + a232 a023 + a233 a033, -a131 a112 + a132 a111
+ a231 a012 + a232 a022 + a233 a032]]
```

>

```

> BIsharp := BI.BI - ScalarMultiply(BI, Trace(BI))
    + ScalarMultiply(IdentityMatrix(3),  $\frac{1}{2} \cdot (\text{Trace}(BI)^2 - \text{Trace}(BI \cdot BI))$ );
BIsharp :=  $\left[ \left[ \frac{1}{2} a111^2 - (a111 + a132 + a123) a111 + \frac{1}{2} (a111 + a132 + a123)^2 \right.$  (75)
 $- \frac{1}{2} a132^2 - a122 a133 - \frac{1}{2} a123^2, a111 a112 + a112 a132 + a113 a122 - (a111
+ a132 + a123) a112, a111 a113 + a112 a133 + a113 a123 - (a111 + a132
+ a123) a113],$ 
 $\left[ a131 a111 + a132 a131 + a121 a133 - (a111 + a132 + a123) a131, \frac{1}{2} a132^2 \right.$ 
 $- (a111 + a132 + a123) a132 + \frac{1}{2} (a111 + a132 + a123)^2 - \frac{1}{2} a111^2
- a121 a113 - \frac{1}{2} a123^2, a113 a131 + a133 a132 + a123 a133 - (a111 + a132
+ a123) a133],$ 
 $\left[ a121 a111 + a131 a122 + a123 a121 - (a111 + a132 + a123) a121, a112 a121
+ a132 a122 + a122 a123 - (a111 + a132 + a123) a122, \frac{1}{2} a123^2 - (a111 + a132
+ a123) a123 + \frac{1}{2} (a111 + a132 + a123)^2 - \frac{1}{2} a111^2 - a131 a112 - \frac{1}{2} a132^2 \right]$ 
```

> Bsharp2 := MatrixAdd(BIsharp, -B2.B0);

```

Bsharp2 :=  $\left[ \left[ \frac{1}{2} a111^2 - (a111 + a132 + a123) a111 + \frac{1}{2} (a111 + a132 + a123)^2 \right.$  (76)
 $- \frac{1}{2} a132^2 - a122 a133 - \frac{1}{2} a123^2 + a211 a011 + a212 a021 + a213 a031,
a111 a112 + a112 a132 + a113 a122 - (a111 + a132 + a123) a112 + a211 a013
+ a212 a023 + a213 a033, a111 a113 + a112 a133 + a113 a123 - (a111 + a132
+ a123) a113 + a211 a012 + a212 a022 + a213 a032],$ 
```

$$\begin{aligned} & \left[a_{131}a_{111} + a_{132}a_{131} + a_{121}a_{133} - (a_{111} + a_{132} + a_{123})a_{131} + a_{221}a_{011} \right. \\ & + a_{222}a_{021} + a_{223}a_{031}, \frac{1}{2}a_{132}^2 - (a_{111} + a_{132} + a_{123})a_{132} + \frac{1}{2}(a_{111} \right. \\ & + a_{132} + a_{123})^2 - \frac{1}{2}a_{111}^2 - a_{121}a_{113} - \frac{1}{2}a_{123}^2 + a_{221}a_{013} + a_{222}a_{023} \\ & + a_{223}a_{033}, a_{113}a_{131} + a_{133}a_{132} + a_{123}a_{133} - (a_{111} + a_{132} + a_{123})a_{133} \end{aligned}$$

$$+ a_{221}a_{012} + a_{222}a_{022} + a_{223}a_{032}],$$

$$\begin{aligned} & \left[a_{121}a_{111} + a_{131}a_{122} + a_{123}a_{121} - (a_{111} + a_{132} + a_{123})a_{121} + a_{231}a_{011} \right. \\ & + a_{232}a_{021} + a_{233}a_{031}, a_{112}a_{121} + a_{132}a_{122} + a_{122}a_{123} - (a_{111} + a_{132} \\ & + a_{123})a_{122} + a_{231}a_{013} + a_{232}a_{023} + a_{233}a_{033}, \frac{1}{2}a_{123}^2 - (a_{111} + a_{132} \\ & + a_{123})a_{123} + \frac{1}{2}(a_{111} + a_{132} + a_{123})^2 - \frac{1}{2}a_{111}^2 - a_{131}a_{112} - \frac{1}{2}a_{132}^2 \\ & \left. + a_{231}a_{012} + a_{232}a_{022} + a_{233}a_{032} \right] \end{aligned}$$

$$\begin{aligned} > Bsharp211 := & expand\left(\frac{1}{2}a_{111}^2 + (-a_{111} - a_{132} - a_{123})a_{111} + \frac{1}{2}(-a_{111} - a_{132} \right. \\ & - a_{123})^2 - \frac{1}{2}a_{132}^2 - a_{122}a_{133} - \frac{1}{2}a_{123}^2 + a_{211}a_{011} + a_{213}a_{031} \\ & \left. + a_{212}a_{021}\right); \end{aligned}$$

$$Bsharp211 := -a_{122}a_{133} + a_{123}a_{132} + a_{211}a_{011} + a_{212}a_{021} + a_{213}a_{031} \quad (77)$$

$$\begin{aligned} > Bsharp212 := & expand(a_{111}a_{112} + a_{112}a_{132} + a_{113}a_{122} + (-a_{111} - a_{132} \\ & - a_{123})a_{112} + a_{211}a_{013} + a_{213}a_{033} + a_{212}a_{023}); \\ & Bsharp212 := -a_{112}a_{123} + a_{113}a_{122} + a_{211}a_{013} + a_{212}a_{023} + a_{213}a_{033} \end{aligned} \quad (78)$$

$$\begin{aligned} > Bsharp213 := & expand(a_{111}a_{113} + a_{112}a_{133} + a_{113}a_{123} + (-a_{111} - a_{132} \\ & - a_{123})a_{113} + a_{211}a_{012} + a_{213}a_{032} + a_{212}a_{022}); \\ & Bsharp213 := -a_{113}a_{132} + a_{112}a_{133} + a_{211}a_{012} + a_{212}a_{022} + a_{213}a_{032} \end{aligned} \quad (79)$$

$$\begin{aligned} > > Bsharp221 := & expand(a_{131}a_{111} + a_{132}a_{131} + a_{121}a_{133} + (-a_{111} - a_{132} \\ & - a_{123})a_{131} + a_{221}a_{011} + a_{223}a_{031} + a_{222}a_{021}); \\ & Bsharp221 := -a_{123}a_{131} + a_{121}a_{133} + a_{221}a_{011} + a_{222}a_{021} + a_{223}a_{031} \end{aligned} \quad (80)$$

$$\begin{aligned} > Bsharp222 := & expand\left(\frac{1}{2}a_{132}^2 + (-a_{111} - a_{132} - a_{123})a_{132} + \frac{1}{2}(-a_{111} - a_{132} \right. \\ & - a_{123})^2 - \frac{1}{2}a_{111}^2 - a_{121}a_{113} - \frac{1}{2}a_{123}^2 + a_{221}a_{013} + a_{223}a_{033} \end{aligned}$$

```

+ a222 a023);
Bsharp222 := -a121 a113 + a123 a111 + a221 a013 + a222 a023 + a223 a033      (81)
> Bsharp223 := expand (a113 a131 + a133 a132 + a123 a133 + (-a111 - a132
- a123) a133 + a221 a012 + a223 a032 + a222 a022);
Bsharp223 := -a111 a133 + a113 a131 + a221 a012 + a222 a022 + a223 a032      (82)
>
> Bsharp231 := expand (a121 a111 + a131 a122 + a123 a121 + (-a111 - a132
- a123) a121 + a231 a011 + a233 a031 + a232 a021);
Bsharp231 := -a132 a121 + a131 a122 + a231 a011 + a232 a021 + a233 a031      (83)
> Bsharp232 := expand (a112 a121 + a132 a122 + a122 a123 + (-a111 - a132
- a123) a122 + a231 a013 + a233 a033 + a232 a023);
Bsharp232 := -a111 a122 + a112 a121 + a231 a013 + a232 a023 + a233 a033      (84)
> Bsharp233 := expand ( $\frac{1}{2} a123^2 + (-a111 - a132 - a123) a123 + \frac{1}{2} (-a111 - a132$ 
 $- a123)^2 - \frac{1}{2} a111^2 - a131 a112 - \frac{1}{2} a132^2 + a231 a012 + a233 a032$ 
 $+ a232 a022$ );
Bsharp233 := -a131 a112 + a132 a111 + a231 a012 + a232 a022 + a233 a032      (85)
> Bsharp2expand := Matrix ([[Bsharp211, Bsharp212, Bsharp213], [Bsharp221, Bsharp222,
Bsharp223], [Bsharp231, Bsharp232, Bsharp233]]));
Bsharp2expand := [[-a122 a133 + a123 a132 + a211 a011 + a212 a021 + a213 a031,
-a112 a123 + a113 a122 + a211 a013 + a212 a023 + a213 a033, -a113 a132
+ a112 a133 + a211 a012 + a212 a022 + a213 a032],
[-a123 a131 + a121 a133 + a221 a011 + a222 a021 + a223 a031, -a121 a113
+ a123 a111 + a221 a013 + a222 a023 + a223 a033, -a111 a133 + a113 a131
+ a221 a012 + a222 a022 + a223 a032],
[-a132 a121 + a131 a122 + a231 a011 + a232 a021 + a233 a031, -a111 a122
+ a112 a121 + a231 a013 + a232 a023 + a233 a033, -a131 a112 + a132 a111
+ a231 a012 + a232 a022 + a233 a032]]]
> MatrixAdd (Sasharp2, -Bsharp2expand );
[ 0  0  0 ]
[ 0  0  0 ]
[ 0  0  0 ]      (87)

```

```

> with(LinearAlgebra);
[&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm,
BilinearForm, CharacteristicMatrix, CharacteristicPolynomial, Column,
ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix,
ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation,
CrossProduct, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix,
Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues, Eigenvectors,
Equal, ForwardSubstitute, FrobeniusForm, GaussianElimination, GenerateEquations,
GenerateMatrix, Generic, GetResultDataType, GetResultShape, GivensRotationMatrix,
GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm,
HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite,
IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, KroneckerProduct,
LA_Main, LUDecomposition, LeastSquares, LinearSolve, LyapunovSolve, Map, Map2,
MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse, MatrixMatrixMultiply,
MatrixNorm, MatrixPower, MatrixScalarMultiply, MatrixVectorMultiply,
MinimalPolynomial, Minor, Modular, Multiply, NoUserValue, Norm, Normalize, NullSpace,
OuterProductMatrix, Permanent, Pivot, PopovForm, QRDecomposition, RandomMatrix,
RandomVector, Rank, RationalCanonicalForm, ReducedRowEchelonForm, Row,
RowDimension, RowOperation, RowSpace, ScalarMatrix, ScalarMultiply, ScalarVector,
SchurForm, SingularValues, SmithForm, StronglyConnectedBlocks, SubMatrix, SubVector,
SumBasis, SylvesterMatrix, SylvesterSolve, ToeplitzMatrix, Trace, Transpose,
TridiagonalForm, UnitVector, VandermondeMatrix, VectorAdd, VectorAngle,
VectorMatrixMultiply, VectorNorm, VectorScalarMultiply, ZeroMatrix, ZeroVector, Zip]

```

```

> a0 := 
$$\begin{bmatrix} a_{011} & a_{012} & a_{013} \\ a_{021} & a_{022} & a_{023} \\ a_{031} & a_{032} & a_{033} \end{bmatrix};$$


```

$$a0 := \begin{bmatrix} a_{011} & a_{012} & a_{013} \\ a_{021} & a_{022} & a_{023} \\ a_{031} & a_{032} & a_{033} \end{bmatrix} \quad (2)$$

```

> a1 := 
$$\begin{bmatrix} a_{111} & a_{112} & a_{113} \\ a_{121} & a_{122} & a_{123} \\ a_{131} & a_{132} & a_{133} \end{bmatrix};$$


```

$$a1 := \begin{bmatrix} a_{111} & a_{112} & a_{113} \\ a_{121} & a_{122} & a_{123} \\ a_{131} & a_{132} & a_{133} \end{bmatrix} \quad (3)$$

$$\begin{aligned}
 > a2 := & \begin{bmatrix} a211 & a212 & a213 \\ a221 & a222 & a223 \\ a231 & a232 & a233 \end{bmatrix}; \\
 & a2 := \begin{bmatrix} a211 & a212 & a213 \\ a221 & a222 & a223 \\ a231 & a232 & a233 \end{bmatrix} \tag{4}
 \end{aligned}$$

$$\begin{aligned}
 > A1 := - & \begin{bmatrix} a111 & a113 & a112 \\ a121 & a123 & a122 \\ a131 & a133 & a132 \end{bmatrix}; \\
 & A1 := \begin{bmatrix} -a111 & -a113 & -a112 \\ -a121 & -a123 & -a122 \\ -a131 & -a133 & -a132 \end{bmatrix} \tag{5}
 \end{aligned}$$

$$\begin{aligned}
 > A2 := - & \begin{bmatrix} a211 & a212 & a213 \\ a231 & a232 & a233 \\ a221 & a222 & a223 \end{bmatrix}; \\
 & A2 := \begin{bmatrix} -a211 & -a212 & -a213 \\ -a231 & -a232 & -a233 \\ -a221 & -a222 & -a223 \end{bmatrix} \tag{6}
 \end{aligned}$$

$$\begin{aligned}
 > d1 := & \text{expand}(\text{Determinant}(a0) + \text{Determinant}(a1) + \text{Determinant}(a2) - \text{Trace}(a0.a1 \\
 & .a2)); \\
 d1 := & a021 a032 a013 - a211 a223 a232 - a131 a122 a113 - a121 a112 a133 \\
 & - a021 a012 a033 + a111 a122 a133 - a031 a022 a013 - a231 a222 a213 \\
 & - a111 a123 a132 - a221 a212 a233 + a231 a212 a223 + a031 a012 a023 \\
 & + a211 a222 a233 + a131 a112 a123 - a011 a023 a032 + a011 a022 a033 \\
 & + a121 a132 a113 - a211 a011 a111 - a211 a012 a121 - a211 a013 a131 \\
 & - a221 a011 a112 - a221 a012 a122 - a221 a013 a132 - a231 a011 a113 \\
 & - a231 a012 a123 - a231 a013 a133 - a212 a021 a111 - a212 a022 a121 \\
 & - a212 a023 a131 - a222 a021 a112 - a222 a022 a122 - a222 a023 a132 \\
 & - a232 a021 a113 - a232 a022 a123 - a232 a023 a133 - a213 a031 a111 \\
 & - a213 a032 a121 - a213 a033 a131 - a223 a031 a112 - a223 a032 a122 \\
 & - a223 a033 a132 - a233 a031 a113 - a233 a032 a123 - a233 a033 a133 \\
 & + a221 a232 a213 \\
 > D1 := & \text{expand}(\text{Determinant}(a0) + \text{Determinant}(A1) + \text{Determinant}(A2) - \text{Trace}(a0.A1 \\
 & .A2)); \\
 D1 := & a021 a032 a013 - a211 a223 a232 - a131 a122 a113 - a121 a112 a133 \tag{7}
 \end{aligned}$$

(8)

$$\begin{aligned}
& - a_{021} a_{012} a_{033} + a_{111} a_{122} a_{133} - a_{031} a_{022} a_{013} - a_{231} a_{222} a_{213} \\
& - a_{111} a_{123} a_{132} - a_{221} a_{212} a_{233} + a_{231} a_{212} a_{223} + a_{031} a_{012} a_{023} \\
& + a_{211} a_{222} a_{233} + a_{131} a_{112} a_{123} - a_{011} a_{023} a_{032} + a_{011} a_{022} a_{033} \\
& + a_{121} a_{132} a_{113} - a_{211} a_{011} a_{111} - a_{211} a_{012} a_{121} - a_{211} a_{013} a_{131} \\
& - a_{221} a_{011} a_{112} - a_{221} a_{012} a_{122} - a_{221} a_{013} a_{132} - a_{231} a_{011} a_{113} \\
& - a_{231} a_{012} a_{123} - a_{231} a_{013} a_{133} - a_{212} a_{021} a_{111} - a_{212} a_{022} a_{121} \\
& - a_{212} a_{023} a_{131} - a_{222} a_{021} a_{112} - a_{222} a_{022} a_{122} - a_{222} a_{023} a_{132} \\
& - a_{232} a_{021} a_{113} - a_{232} a_{022} a_{123} - a_{232} a_{023} a_{133} - a_{213} a_{031} a_{111} \\
& - a_{213} a_{032} a_{121} - a_{213} a_{033} a_{131} - a_{223} a_{031} a_{112} - a_{223} a_{032} a_{122} \\
& - a_{223} a_{033} a_{132} - a_{233} a_{031} a_{113} - a_{233} a_{032} a_{123} - a_{233} a_{033} a_{133} \\
& + a_{221} a_{232} a_{213}
\end{aligned}$$

$$> dI - Dl; \quad 0 \quad (9)$$

$$\begin{aligned}
> a0sharp := a0.a0 - \text{ScalarMultiply}(a0, \text{Trace}(a0)) + \text{ScalarMultiply}\left(\text{IdentityMatrix}(3), \right. \\
& \left. \frac{1}{2} \cdot (\text{Trace}(a0)^2 - \text{Trace}(a0.a0))\right); \\
a0sharp := & \left[\left[\frac{1}{2} a_{011}^2 - (a_{011} + a_{022} + a_{033}) a_{011} + \frac{1}{2} (a_{011} + a_{022} + a_{033})^2 \right. \right. \quad (10) \\
& - \frac{1}{2} a_{022}^2 - a_{023} a_{032} - \frac{1}{2} a_{033}^2, a_{011} a_{012} + a_{012} a_{022} + a_{032} a_{013} - (a_{011} \\
& + a_{022} + a_{033}) a_{012}, a_{011} a_{013} + a_{012} a_{023} + a_{013} a_{033} - (a_{011} + a_{022} \\
& + a_{033}) a_{013}], \\
& \left[a_{021} a_{011} + a_{022} a_{021} + a_{023} a_{031} - (a_{011} + a_{022} + a_{033}) a_{021}, \frac{1}{2} a_{022}^2 \right. \\
& - (a_{011} + a_{022} + a_{033}) a_{022} + \frac{1}{2} (a_{011} + a_{022} + a_{033})^2 - \frac{1}{2} a_{011}^2 \\
& - a_{013} a_{031} - \frac{1}{2} a_{033}^2, a_{021} a_{013} + a_{022} a_{023} + a_{023} a_{033} - (a_{011} + a_{022} \\
& + a_{033}) a_{023}], \\
& \left[a_{031} a_{011} + a_{032} a_{021} + a_{033} a_{031} - (a_{011} + a_{022} + a_{033}) a_{031}, a_{031} a_{012} \right. \\
& + a_{032} a_{022} + a_{033} a_{032} - (a_{011} + a_{022} + a_{033}) a_{032}, \frac{1}{2} a_{033}^2 - (a_{011} + a_{022} \\
& + a_{033}) a_{033} + \frac{1}{2} (a_{011} + a_{022} + a_{033})^2 - \frac{1}{2} a_{011}^2 - a_{012} a_{021} - \frac{1}{2} a_{022}^2 \left. \right] \left. \right]
\end{aligned}$$

$$> \text{MatrixAdd}(a0sharp, -A1.A2); \quad \left[\left[\frac{1}{2} a_{011}^2 - (a_{011} + a_{022} + a_{033}) a_{011} + \frac{1}{2} (a_{011} + a_{022} + a_{033})^2 - \frac{1}{2} a_{022}^2 \right. \right. \quad (11)$$

$$\begin{aligned}
& - a_{023} a_{032} - \frac{1}{2} a_{033}^2 - a_{111} a_{211} - a_{113} a_{231} - a_{112} a_{221}, a_{011} a_{012} \\
& + a_{012} a_{022} + a_{032} a_{013} - (a_{011} + a_{022} + a_{033}) a_{012} - a_{111} a_{212} - a_{113} a_{232} \\
& - a_{112} a_{222}, a_{011} a_{013} + a_{012} a_{023} + a_{013} a_{033} - (a_{011} + a_{022} + a_{033}) a_{013} \\
& - a_{111} a_{213} - a_{113} a_{233} - a_{112} a_{223}], \\
& \left[a_{021} a_{011} + a_{022} a_{021} + a_{023} a_{031} - (a_{011} + a_{022} + a_{033}) a_{021} - a_{121} a_{211} \right. \\
& - a_{123} a_{231} - a_{122} a_{221}, \frac{1}{2} a_{022}^2 - (a_{011} + a_{022} + a_{033}) a_{022} + \frac{1}{2} (a_{011} \\
& + a_{022} + a_{033})^2 - \frac{1}{2} a_{011}^2 - a_{013} a_{031} - \frac{1}{2} a_{033}^2 - a_{121} a_{212} - a_{123} a_{232} \\
& - a_{122} a_{222}, a_{021} a_{013} + a_{022} a_{023} + a_{023} a_{033} - (a_{011} + a_{022} + a_{033}) a_{023} \\
& - a_{121} a_{213} - a_{123} a_{233} - a_{122} a_{223}], \\
& \left[a_{031} a_{011} + a_{032} a_{021} + a_{033} a_{031} - (a_{011} + a_{022} + a_{033}) a_{031} - a_{131} a_{211} \right. \\
& - a_{133} a_{231} - a_{132} a_{221}, a_{031} a_{012} + a_{032} a_{022} + a_{033} a_{032} - (a_{011} + a_{022} \\
& + a_{033}) a_{032} - a_{131} a_{212} - a_{133} a_{232} - a_{132} a_{222}, \frac{1}{2} a_{033}^2 - (a_{011} + a_{022} \\
& + a_{033}) a_{033} + \frac{1}{2} (a_{011} + a_{022} + a_{033})^2 - \frac{1}{2} a_{011}^2 - a_{012} a_{021} - \frac{1}{2} a_{022}^2 \\
& - a_{131} a_{213} - a_{133} a_{233} - a_{132} a_{223}] \Big] \\
> & \text{Sa0sharp11 := expand} \left(\frac{1}{2} a_{011}^2 - (a_{011} + a_{022} + a_{033}) a_{011} + \frac{1}{2} (a_{011} + a_{022} \right. \\
& + a_{033})^2 - \frac{1}{2} a_{022}^2 - a_{023} a_{032} - \frac{1}{2} a_{033}^2 - a_{111} a_{211} - a_{113} a_{231} \\
& \left. - a_{112} a_{221} \right); \\
& \text{Sa0sharp11 := } a_{022} a_{033} - a_{023} a_{032} - a_{111} a_{211} - a_{113} a_{231} - a_{112} a_{221} \quad (12) \\
> & \text{Sa0sharp12 := expand} (a_{011} a_{012} + a_{012} a_{022} + a_{032} a_{013} - (a_{011} + a_{022} \\
& + a_{033}) a_{012} - a_{111} a_{212} - a_{113} a_{232} - a_{112} a_{222}); \\
& \text{Sa0sharp12 := } a_{032} a_{013} - a_{012} a_{033} - a_{111} a_{212} - a_{113} a_{232} - a_{112} a_{222} \quad (13) \\
> & \text{Sa0sharp13 := expand} (a_{011} a_{013} + a_{012} a_{023} + a_{013} a_{033} - (a_{011} + a_{022} \\
& + a_{033}) a_{013} - a_{111} a_{213} - a_{113} a_{233} - a_{112} a_{223}); \\
& \text{Sa0sharp13 := } a_{012} a_{023} - a_{022} a_{013} - a_{111} a_{213} - a_{113} a_{233} - a_{112} a_{223} \quad (14) \\
> & \text{Sa0sharp21 := expand} (a_{021} a_{011} + a_{022} a_{021} + a_{023} a_{031} - (a_{011} + a_{022} \\
& + a_{033}) a_{021} - a_{121} a_{211} - a_{123} a_{231} - a_{122} a_{221});
\end{aligned}$$

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$$+ a033) a021 - a121 a211 - a123 a231 - a122 a221);$$


$$Sa0sharp21 := a023 a031 - a021 a033 - a121 a211 - a123 a231 - a122 a221 \quad (15)$$


> Sa0sharp22 := expand  $\left( \frac{1}{2} a022^2 - (a011 + a022 + a033) a022 + \frac{1}{2} (a011 + a022 + a033)^2 - \frac{1}{2} a011^2 - a013 a031 - \frac{1}{2} a033^2 - a121 a212 - a123 a232 - a122 a222 \right);$ 

$$Sa0sharp22 := a011 a033 - a013 a031 - a121 a212 - a123 a232 - a122 a222 \quad (16)$$


> Sa0sharp23 := expand  $(a021 a013 + a022 a023 + a023 a033 - (a011 + a022 + a033) a023 - a121 a213 - a123 a233 - a122 a223);$ 

$$Sa0sharp23 := a021 a013 - a023 a011 - a121 a213 - a123 a233 - a122 a223 \quad (17)$$


>
> Sa0sharp31 := expand  $(a031 a011 + a032 a021 + a033 a031 - (a011 + a022 + a033) a031 - a131 a211 - a133 a231 - a132 a221);$ 

$$Sa0sharp31 := a032 a021 - a031 a022 - a131 a211 - a133 a231 - a132 a221 \quad (18)$$


> Sa0sharp32 := expand  $(a031 a012 + a032 a022 + a033 a032 - (a011 + a022 + a033) a032 - a131 a212 - a133 a232 - a132 a222);$ 

$$Sa0sharp32 := a031 a012 - a032 a011 - a131 a212 - a133 a232 - a132 a222 \quad (19)$$


> Sa0sharp33 := expand  $\left( \frac{1}{2} a033^2 - (a011 + a022 + a033) a033 + \frac{1}{2} (a011 + a022 + a033)^2 - \frac{1}{2} a011^2 - a012 a021 - \frac{1}{2} a022^2 - a131 a213 - a133 a233 - a132 a223 \right);$ 

$$Sa0sharp33 := a011 a022 - a012 a021 - a131 a213 - a133 a233 - a132 a223 \quad (20)$$


> Sa0sharp := Matrix ([[Sa0sharp11, Sa0sharp12, Sa0sharp13], [Sa0sharp21, Sa0sharp22, Sa0sharp23], [Sa0sharp31, Sa0sharp32, Sa0sharp33]]));

$$Sa0sharp := [[a022 a033 - a023 a032 - a111 a211 - a113 a231 - a112 a221, a032 a013 - a012 a033 - a111 a212 - a113 a232 - a112 a222, a012 a023 - a022 a013 - a111 a213 - a113 a233 - a112 a223], \quad (21)
[a023 a031 - a021 a033 - a121 a211 - a123 a231 - a122 a221, a011 a033 - a013 a031 - a121 a212 - a123 a232 - a122 a222, a021 a013 - a023 a011 - a121 a213 - a123 a233 - a122 a223], \quad
[a032 a021 - a031 a022 - a131 a211 - a133 a231 - a132 a221, a031 a012 - a032 a011 - a131 a212 - a133 a232 - a132 a222, a011 a022 - a012 a021 - a131 a213 - a133 a233 - a132 a223]]]$$


>
> Sasharp0 := MatrixAdd (a0sharp, -a1.a2);

$$Sasharp0 := \left[ \left[ \frac{1}{2} a011^2 - (a011 + a022 + a033) a011 + \frac{1}{2} (a011 + a022 + a033)^2 \right] \right] \quad (22)$$


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$$\begin{aligned}
& - \frac{1}{2} a_{022}^2 - a_{023} a_{032} - \frac{1}{2} a_{033}^2 - a_{111} a_{211} - a_{113} a_{231} - a_{112} a_{221}, \\
& a_{011} a_{012} + a_{012} a_{022} + a_{032} a_{013} - (a_{011} + a_{022} + a_{033}) a_{012} - a_{111} a_{212} \\
& - a_{113} a_{232} - a_{112} a_{222}, a_{011} a_{013} + a_{012} a_{023} + a_{013} a_{033} - (a_{011} + a_{022} \\
& + a_{033}) a_{013} - a_{111} a_{213} - a_{113} a_{233} - a_{112} a_{223}], \\
& \left[a_{021} a_{011} + a_{022} a_{021} + a_{023} a_{031} - (a_{011} + a_{022} + a_{033}) a_{021} - a_{121} a_{211} \right. \\
& - a_{123} a_{231} - a_{122} a_{221}, \frac{1}{2} a_{022}^2 - (a_{011} + a_{022} + a_{033}) a_{022} + \frac{1}{2} (a_{011} \\
& + a_{022} + a_{033})^2 - \frac{1}{2} a_{011}^2 - a_{013} a_{031} - \frac{1}{2} a_{033}^2 - a_{121} a_{212} - a_{123} a_{232} \\
& - a_{122} a_{222}, a_{021} a_{013} + a_{022} a_{023} + a_{023} a_{033} - (a_{011} + a_{022} + a_{033}) a_{023} \\
& - a_{121} a_{213} - a_{123} a_{233} - a_{122} a_{223}], \\
& \left[a_{031} a_{011} + a_{032} a_{021} + a_{033} a_{031} - (a_{011} + a_{022} + a_{033}) a_{031} - a_{131} a_{211} \right. \\
& - a_{133} a_{231} - a_{132} a_{221}, a_{031} a_{012} + a_{032} a_{022} + a_{033} a_{032} - (a_{011} + a_{022} \\
& + a_{033}) a_{032} - a_{131} a_{212} - a_{133} a_{232} - a_{132} a_{222}, \frac{1}{2} a_{033}^2 - (a_{011} + a_{022} \\
& + a_{033}) a_{033} + \frac{1}{2} (a_{011} + a_{022} + a_{033})^2 - \frac{1}{2} a_{011}^2 - a_{012} a_{021} - \frac{1}{2} a_{022}^2 \\
& - a_{131} a_{213} - a_{133} a_{233} - a_{132} a_{223}] \\
& > Sasharp011 := expand \left(\frac{1}{2} a_{011}^2 - (a_{011} + a_{022} + a_{033}) a_{011} + \frac{1}{2} (a_{011} + a_{022} \\
& + a_{033})^2 - \frac{1}{2} a_{022}^2 - a_{023} a_{032} - \frac{1}{2} a_{033}^2 - a_{111} a_{211} - a_{113} a_{231} \\
& - a_{112} a_{221} \right); \\
& Sasharp011 := a_{022} a_{033} - a_{023} a_{032} - a_{111} a_{211} - a_{113} a_{231} - a_{112} a_{221} \quad (23) \\
& > Sasharp012 := expand (a_{011} a_{012} + a_{012} a_{022} + a_{032} a_{013} - (a_{011} + a_{022} \\
& + a_{033}) a_{012} - a_{111} a_{212} - a_{113} a_{232} - a_{112} a_{222}); \\
& Sasharp012 := a_{032} a_{013} - a_{012} a_{033} - a_{111} a_{212} - a_{113} a_{232} - a_{112} a_{222} \quad (24) \\
& > Sasharp013 := expand (a_{011} a_{013} + a_{012} a_{023} + a_{013} a_{033} - (a_{011} + a_{022} \\
& + a_{033}) a_{013} - a_{111} a_{213} - a_{113} a_{233} - a_{112} a_{223}); \\
& Sasharp013 := a_{012} a_{023} - a_{022} a_{013} - a_{111} a_{213} - a_{113} a_{233} - a_{112} a_{223} \quad (25) \\
& > \\
& > Sasharp021 := expand (a_{021} a_{011} + a_{022} a_{021} + a_{023} a_{031} - (a_{011} + a_{022} \\
& + a_{033}) a_{021} - a_{121} a_{211} - a_{123} a_{231} - a_{122} a_{221});
\end{aligned}$$

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Sasharp021 := a023 a031 - a021 a033 - a121 a211 - a123 a231 - a122 a221      (26)
> Sasharp022 := expand  $\left( \frac{1}{2} a022^2 - (a011 + a022 + a033) a022 + \frac{1}{2} (a011 + a022 + a033)^2 - \frac{1}{2} a011^2 - a013 a031 - \frac{1}{2} a033^2 - a121 a212 - a123 a232 - a122 a222 \right);$ 
Sasharp022 := a011 a033 - a013 a031 - a121 a212 - a123 a232 - a122 a222      (27)
> Sasharp023 := expand  $(a021 a013 + a022 a023 + a023 a033 - (a011 + a022 + a033) a023 - a121 a213 - a123 a233 - a122 a223);$ 
Sasharp023 := a021 a013 - a023 a011 - a121 a213 - a123 a233 - a122 a223      (28)
>
> Sasharp031 := expand  $(a031 a011 + a032 a021 + a033 a031 - (a011 + a022 + a033) a031 - a131 a211 - a133 a231 - a132 a221);$ 
Sasharp031 := a032 a021 - a031 a022 - a131 a211 - a133 a231 - a132 a221      (29)
> Sasharp032 := expand  $(a031 a012 + a032 a022 + a033 a032 - (a011 + a022 + a033) a032 - a131 a212 - a133 a232 - a132 a222);$ 
Sasharp032 := a031 a012 - a032 a011 - a131 a212 - a133 a232 - a132 a222      (30)
> Sasharp033 := expand  $\left( \frac{1}{2} a033^2 - (a011 + a022 + a033) a033 + \frac{1}{2} (a011 + a022 + a033)^2 - \frac{1}{2} a011^2 - a012 a021 - \frac{1}{2} a022^2 - a131 a213 - a133 a233 - a132 a223 \right);$ 
Sasharp033 := a011 a022 - a012 a021 - a131 a213 - a133 a233 - a132 a223      (31)
> Sasharp0expand := Matrix ([[Sasharp011, Sasharp012, Sasharp013], [Sasharp021, Sasharp022, Sasharp023], [Sasharp031, Sasharp032, Sasharp033]]);
Sasharp0expand := [[a022 a033 - a023 a032 - a111 a211 - a113 a231 - a112 a221, a032 a013 - a012 a033 - a111 a212 - a113 a232 - a112 a222, a012 a023 - a022 a013 - a111 a213 - a113 a233 - a112 a223],
[a023 a031 - a021 a033 - a121 a211 - a123 a231 - a122 a221, a011 a033 - a013 a031 - a121 a212 - a123 a232 - a122 a222, a021 a013 - a023 a011 - a121 a213 - a123 a233 - a122 a223],
[a032 a021 - a031 a022 - a131 a211 - a133 a231 - a132 a221, a031 a012 - a032 a011 - a131 a212 - a133 a232 - a132 a222, a011 a022 - a012 a021 - a131 a213 - a133 a233 - a132 a223]]]
>
> MatrixAdd (Sa0sharp, -Sasharp0expand);

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (33)$$


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> A2sharp := A2.A2 - ScalarMultiply(A2, Trace(A2) )
    + ScalarMultiply(IdentityMatrix(3),  $\frac{1}{2} \cdot (\text{Trace}(A2)^2 - \text{Trace}(A2.A2))$ );
A2sharp :=  $\left[ \left[ \begin{array}{l}
\left[ \frac{1}{2} a211^2 + (-a211 - a232 - a223) a211 + \frac{1}{2} (-a211 - a232 - a223)^2 \right. \\
- \frac{1}{2} a232^2 - a222 a233 - \frac{1}{2} a223^2, a211 a212 + a212 a232 + a222 a213 + (-a211 \\
- a232 - a223) a212, a211 a213 + a212 a233 + a213 a223 + (-a211 - a232 \\
- a223) a213], \\
\left[ a231 a211 + a232 a231 + a233 a221 + (-a211 - a232 - a223) a231, \frac{1}{2} a232^2 + ( \right. \\
- a211 - a232 - a223) a232 + \frac{1}{2} (-a211 - a232 - a223)^2 - \frac{1}{2} a211^2 - a213 a221 \\
- \frac{1}{2} a223^2, a231 a213 + a232 a233 + a233 a223 + (-a211 - a232 - a223) a233 \Big], \\
\left[ a221 a211 + a222 a231 + a223 a221 + (-a211 - a232 - a223) a221, a221 a212 \right. \\
+ a222 a232 + a223 a222 + (-a211 - a232 - a223) a222, \frac{1}{2} a223^2 + (-a211 \\
- a232 - a223) a223 + \frac{1}{2} (-a211 - a232 - a223)^2 - \frac{1}{2} a211^2 - a212 a231 \\
- \frac{1}{2} a232^2 \Big] \right] \right]$ 
> MatrixAdd(A2sharp, -a0.A1);
 $\left[ \left[ \frac{1}{2} a211^2 + (-a211 - a232 - a223) a211 + \frac{1}{2} (-a211 - a232 - a223)^2 - \frac{1}{2} a232^2 \right. \right.$  (35)
 $- a222 a233 - \frac{1}{2} a223^2 + a011 a111 + a012 a121 + a013 a131, a211 a212 \\
+ a212 a232 + a222 a213 + (-a211 - a232 - a223) a212 + a011 a113 + a012 a123 \\
+ a013 a133, a211 a213 + a212 a233 + a213 a223 + (-a211 - a232 - a223) a213 \\
+ a011 a112 + a012 a122 + a013 a132], \\
 $\left[ a231 a211 + a232 a231 + a233 a221 + (-a211 - a232 - a223) a231 + a021 a111 \right. \\
+ a022 a121 + a023 a131, \frac{1}{2} a232^2 + (-a211 - a232 - a223) a232 + \frac{1}{2} ( \\
- a211 - a232 - a223)^2 - \frac{1}{2} a211^2 - a213 a221 - \frac{1}{2} a223^2 + a021 a113 \\
+ a022 a123 + a023 a133, a231 a213 + a232 a233 + a233 a223 + (-a211 - a232$$ 
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$$\begin{aligned}
& - a_{223}) a_{233} + a_{021} a_{112} + a_{022} a_{122} + a_{023} a_{132} \Big], \\
& \left[ a_{221} a_{211} + a_{222} a_{231} + a_{223} a_{221} + (-a_{211} - a_{232} - a_{223}) a_{221} + a_{031} a_{111} \right. \\
& + a_{032} a_{121} + a_{033} a_{131}, a_{221} a_{212} + a_{222} a_{232} + a_{223} a_{222} + (-a_{211} - a_{232} \\
& - a_{223}) a_{222} + a_{031} a_{113} + a_{032} a_{123} + a_{033} a_{133}, \frac{1}{2} a_{223}^2 + (-a_{211} - a_{232} \\
& - a_{223}) a_{223} + \frac{1}{2} (-a_{211} - a_{232} - a_{223})^2 - \frac{1}{2} a_{211}^2 - a_{212} a_{231} - \frac{1}{2} a_{232}^2 \\
& \left. + a_{031} a_{112} + a_{032} a_{122} + a_{033} a_{132} \right] \Big]
\end{aligned}$$


>



$Sasharp111 := expand \left( \frac{1}{2} a_{211}^2 + (-a_{211} - a_{232} - a_{223}) a_{211} + \frac{1}{2} (-a_{211} - a_{232} - a_{223})^2 - \frac{1}{2} a_{232}^2 - a_{233} a_{222} - \frac{1}{2} a_{223}^2 + a_{011} a_{111} + a_{012} a_{121} + a_{013} a_{131} \right);$  (36)



$Sasharp112 := expand (a_{211} a_{212} + a_{212} a_{232} + a_{213} a_{222} + (-a_{211} - a_{232} - a_{223}) a_{212} + a_{011} a_{113} + a_{012} a_{123} + a_{013} a_{133});$  (37)



$Sasharp113 := expand (a_{211} a_{213} + a_{212} a_{233} + a_{213} a_{223} + (-a_{211} - a_{232} - a_{223}) a_{213} + a_{011} a_{112} + a_{012} a_{122} + a_{013} a_{132});$  (38)



$Sasharp121 := expand (a_{231} a_{211} + a_{232} a_{231} + a_{233} a_{221} + (-a_{211} - a_{232} - a_{223}) a_{231} + a_{021} a_{111} + a_{022} a_{121} + a_{023} a_{131});$  (39)



$Sasharp122 := expand \left( \frac{1}{2} a_{232}^2 + (-a_{211} - a_{232} - a_{223}) a_{232} + \frac{1}{2} (-a_{211} - a_{232} - a_{223})^2 - \frac{1}{2} a_{211}^2 - a_{213} a_{221} - \frac{1}{2} a_{223}^2 + a_{021} a_{113} + a_{022} a_{123} + a_{023} a_{133} \right);$  (40)



$Sasharp123 := expand (a_{231} a_{213} + a_{232} a_{233} + a_{233} a_{223} + (-a_{211} - a_{232} - a_{223}) a_{233} + a_{021} a_{112} + a_{022} a_{122} + a_{023} a_{132});$  (41)



>



$Sasharp131 := expand (a_{221} a_{211} + a_{222} a_{231} + a_{223} a_{221} + (-a_{211} - a_{232} - a_{223}) a_{221} + a_{031} a_{111} + a_{032} a_{121} + a_{033} a_{131});$


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$$- a223) a221 + a031 a111 + a032 a121 + a033 a131);$$


$$\text{Sasharp131} := a222 a231 - a221 a232 + a031 a111 + a032 a121 + a033 a131 \quad (42)$$

> Sasharp132 := expand(a221 a212 + a222 a232 + a223 a222 + (-a211 - a232

$$- a223) a222 + a031 a113 + a032 a123 + a033 a133);$$


$$\text{Sasharp132} := a221 a212 - a222 a211 + a031 a113 + a032 a123 + a033 a133 \quad (43)$$

> Sasharp133 := expand( $\left( \frac{1}{2} a223^2 + (-a211 - a232 - a223) a223 + \frac{1}{2} (-a211 - a232\right.$ 

$$- a223)^2 -  $\frac{1}{2} a211^2 - a212 a231 - \frac{1}{2} a232^2 + a031 a112 + a032 a122$ 

$$+ a033 a132 \right);$$


$$\text{Sasharp133} := a211 a232 - a212 a231 + a031 a112 + a032 a122 + a033 a132 \quad (44)$$

> Sasharp1 := Matrix([[Sasharp111, Sasharp112, Sasharp113], [Sasharp121, Sasharp122,

$$\text{Sasharp123}], [Sasharp131, Sasharp132, Sasharp133]]);$$


$$\text{Sasharp1} := [[a223 a232 - a222 a233 + a011 a111 + a012 a121 + a013 a131, a222 a213$$


$$- a212 a223 + a011 a113 + a012 a123 + a013 a133, a212 a233 - a232 a213$$


$$+ a011 a112 + a012 a122 + a013 a132],$$


$$[a233 a221 - a231 a223 + a021 a111 + a022 a121 + a023 a131, a211 a223$$


$$- a213 a221 + a021 a113 + a022 a123 + a023 a133, a231 a213 - a233 a211$$


$$+ a021 a112 + a022 a122 + a023 a132],$$


$$[a222 a231 - a221 a232 + a031 a111 + a032 a121 + a033 a131, a221 a212$$


$$- a222 a211 + a031 a113 + a032 a123 + a033 a133, a211 a232 - a212 a231$$


$$+ a031 a112 + a032 a122 + a033 a132]]$$

>
> a2sharp := a2.a2 - ScalarMultiply(a2, Trace(a2)) + ScalarMultiply(IdentityMatrix(3),

$$\frac{1}{2} \cdot (Trace(a2)^2 - Trace(a2.a2));$$


$$\text{a2sharp} := \left[ \left[ \frac{1}{2} a211^2 - (a211 + a222 + a233) a211 + \frac{1}{2} (a211 + a222 + a233)^2\right.\right.$$


$$-  $\frac{1}{2} a222^2 - a223 a232 - \frac{1}{2} a233^2, a211 a212 + a212 a222 + a232 a213 - (a211$ 

$$+ a222 + a233) a212, a211 a213 + a212 a223 + a213 a233 - (a211 + a222$$


$$+ a233) a213],$$


$$\left. \left[ a221 a211 + a222 a221 + a231 a223 - (a211 + a222 + a233) a221, \frac{1}{2} a222^2\right.\right.$$


$$- (a211 + a222 + a233) a222 +  $\frac{1}{2} (a211 + a222 + a233)^2 - \frac{1}{2} a211^2$ 

$$- a231 a213 -  $\frac{1}{2} a233^2, a213 a221 + a223 a222 + a233 a223 - (a211 + a222$ 

$$+ a233) a213 \right] \right] \quad (46)$$$$$$$$$$

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+ a233) a223],  

[[ a231 a211 + a221 a232 + a233 a231 - (a211 + a222 + a233) a231, a212 a231  

+ a222 a232 + a232 a233 - (a211 + a222 + a233) a232,  $\frac{1}{2}$  a2332 - (a211 + a222  

+ a233) a233 +  $\frac{1}{2}$  (a211 + a222 + a233)2 -  $\frac{1}{2}$  a2112 - a221 a212 -  $\frac{1}{2}$  a2222 ]]]  

> MatrixAdd (a2sharp, -a0.a1);  

[[  $\frac{1}{2}$  a2112 - (a211 + a222 + a233) a211 +  $\frac{1}{2}$  (a211 + a222 + a233)2 -  $\frac{1}{2}$  a2222 (47)  

- a223 a232 -  $\frac{1}{2}$  a2332 - a011 a111 - a012 a121 - a013 a131, a211 a212  

+ a212 a222 + a232 a213 - (a211 + a222 + a233) a212 - a011 a112 - a012 a122  

- a013 a132, a211 a213 + a212 a223 + a213 a233 - (a211 + a222 + a233) a213  

- a011 a113 - a012 a123 - a013 a133],  

[ a221 a211 + a222 a221 + a231 a223 - (a211 + a222 + a233) a221 - a021 a111  

- a022 a121 - a023 a131,  $\frac{1}{2}$  a2222 - (a211 + a222 + a233) a222 +  $\frac{1}{2}$  (a211  

+ a222 + a233)2 -  $\frac{1}{2}$  a2112 - a231 a213 -  $\frac{1}{2}$  a2332 - a021 a112 - a022 a122  

- a023 a132, a213 a221 + a223 a222 + a233 a223 - (a211 + a222 + a233) a223  

- a021 a113 - a022 a123 - a023 a133],  

[ a231 a211 + a221 a232 + a233 a231 - (a211 + a222 + a233) a231 - a031 a111  

- a032 a121 - a033 a131, a212 a231 + a222 a232 + a232 a233 - (a211 + a222  

+ a233) a232 - a031 a112 - a032 a122 - a033 a132,  $\frac{1}{2}$  a2332 - (a211 + a222  

+ a233) a233 +  $\frac{1}{2}$  (a211 + a222 + a233)2 -  $\frac{1}{2}$  a2112 - a221 a212 -  $\frac{1}{2}$  a2222  

- a031 a113 - a032 a123 - a033 a133]]]  

> asharp111 := expand ( $\frac{1}{2}$  a2112 - (a211 + a222 + a233) a211 +  $\frac{1}{2}$  (a211 + a222  

+ a233)2 -  $\frac{1}{2}$  a2222 - a232 a223 -  $\frac{1}{2}$  a2332 - a011 a111 - a012 a121

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$$- a013 a131);$$


$$\text{asharp111} := a222 a233 - a223 a232 - a011 a111 - a012 a121 - a013 a131 \quad (48)$$


$$> \text{asharp112} := \text{expand}(a211 a212 + a212 a222 + a213 a232 - (a211 + a222 + a233) a212 - a011 a112 - a012 a122 - a013 a132);$$


$$\text{asharp112} := a232 a213 - a212 a233 - a011 a112 - a012 a122 - a013 a132 \quad (49)$$


$$> \text{asharp113} := \text{expand}(a211 a213 + a212 a223 + a213 a233 - (a211 + a222 + a233) a213 - a011 a113 - a012 a123 - a013 a133);$$


$$\text{asharp113} := a212 a223 - a222 a213 - a011 a113 - a012 a123 - a013 a133 \quad (50)$$


$$>$$


$$> \text{asharp121} := \text{expand}(a221 a211 + a222 a221 + a231 a223 - (a211 + a222 + a233) a221 - a021 a111 - a022 a121 - a023 a131);$$


$$\text{asharp121} := a231 a223 - a233 a221 - a021 a111 - a022 a121 - a023 a131 \quad (51)$$


$$> \text{asharp122} := \text{expand}\left(\frac{1}{2} a222^2 - (a211 + a222 + a233) a222 + \frac{1}{2} (a211 + a222 + a233)^2 - \frac{1}{2} a211^2 - a231 a213 - \frac{1}{2} a233^2 - a021 a112 - a022 a122 - a023 a132\right);$$


$$\text{asharp122} := a233 a211 - a231 a213 - a021 a112 - a022 a122 - a023 a132 \quad (52)$$


$$> \text{asharp123} := \text{expand}(a213 a221 + a223 a222 + a233 a223 - (a211 + a222 + a233) a223 - a021 a113 - a022 a123 - a023 a133);$$


$$\text{asharp123} := a213 a221 - a211 a223 - a021 a113 - a022 a123 - a023 a133 \quad (53)$$


$$>$$


$$> \text{asharp131} := \text{expand}(a231 a211 + a221 a232 + a233 a231 - (a211 + a222 + a233) a231 - a031 a111 - a032 a121 - a033 a131);$$


$$\text{asharp131} := a221 a232 - a222 a231 - a031 a111 - a032 a121 - a033 a131 \quad (54)$$


$$> \text{asharp132} := \text{expand}(a212 a231 + a222 a232 + a232 a233 - (a211 + a222 + a233) a232 - a031 a112 - a032 a122 - a033 a132);$$


$$\text{asharp132} := a212 a231 - a211 a232 - a031 a112 - a032 a122 - a033 a132 \quad (55)$$


$$> \text{asharp133} := \text{expand}\left(\frac{1}{2} a233^2 - (a211 + a222 + a233) a233 + \frac{1}{2} (a211 + a222 + a233)^2 - \frac{1}{2} a211^2 - a221 a212 - \frac{1}{2} a222^2 - a031 a113 - a032 a123 - a033 a133\right);$$


$$\text{asharp133} := a222 a211 - a221 a212 - a031 a113 - a032 a123 - a033 a133 \quad (56)$$


$$> \text{Salsharp} := \text{Matrix}(-[[\text{asharp111}, \text{asharp113}, \text{asharp112}], [\text{asharp121}, \text{asharp123}, \text{asharp122}], [\text{asharp131}, \text{asharp133}, \text{asharp132}]]);$$


$$\text{Salsharp} := [[a223 a232 - a222 a233 + a011 a111 + a012 a121 + a013 a131, a222 a213 - a212 a223 + a011 a113 + a012 a123 + a013 a133, a212 a233 - a232 a213 + a011 a112 + a012 a122 + a013 a132], \quad (57)$$


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[ a233 a221 - a231 a223 + a021 a111 + a022 a121 + a023 a131, a211 a223
- a213 a221 + a021 a113 + a022 a123 + a023 a133, a231 a213 - a233 a211
+ a021 a112 + a022 a122 + a023 a132],
[ a222 a231 - a221 a232 + a031 a111 + a032 a121 + a033 a131, a221 a212
- a222 a211 + a031 a113 + a032 a123 + a033 a133, a211 a232 - a212 a231
+ a031 a112 + a032 a122 + a033 a132]]
> MatrixAdd (SaIsharp, -SasharpI);

```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (58)$$

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>
> A1sharp := A1.A1 - ScalarMultiply (A1, Trace (A1)) + ScalarMultiply (IdentityMatrix (3),

$$\frac{1}{2} \cdot (Trace(A1)^2 - Trace(A1.A1)) \Big);$$

A1sharp := [[ [  $\frac{1}{2} a111^2 + (-a111 - a123 - a132) a111 + \frac{1}{2} (-a111 - a123 - a132)^2$  ] ] (59)

$$- \frac{1}{2} a123^2 - a122 a133 - \frac{1}{2} a132^2, a111 a113 + a113 a123 + a112 a133 + (-a111 - a123 - a132) a113, a111 a112 + a113 a122 + a112 a132 + (-a111 - a123 - a132) a112 ],$$


$$[ a121 a111 + a123 a121 + a122 a131 + (-a111 - a123 - a132) a121, 
$$\frac{1}{2} a123^2 + (-a111 - a123 - a132) a123 + \frac{1}{2} (-a111 - a123 - a132)^2 - \frac{1}{2} a111^2 - a112 a131$$
 ] ] (59)

$$- \frac{1}{2} a132^2, a121 a112 + a123 a122 + a122 a132 + (-a111 - a123 - a132) a122 ],$$


$$[ a131 a111 + a133 a121 + a132 a131 + (-a111 - a123 - a132) a131, a131 a113 + a133 a123 + a132 a133 + (-a111 - a123 - a132) a133, 
$$\frac{1}{2} a132^2 + (-a111 - a123 - a132) a132 + \frac{1}{2} (-a111 - a123 - a132)^2 - \frac{1}{2} a111^2 - a113 a121$$
 ] ] (59)

$$- \frac{1}{2} a123^2 ] ]$$

> MatrixAdd (A1sharp, -A2.a0);$$$$

```

$$\left[\left[\frac{1}{2} a111^2 + (-a111 - a123 - a132) a111 + \frac{1}{2} (-a111 - a123 - a132)^2 - \frac{1}{2} a123^2 \right] \right] \quad (60)$$

$$\begin{aligned}
& -a_{122}a_{133} - \frac{1}{2}a_{132}^2 + a_{211}a_{011} + a_{212}a_{021} + a_{213}a_{031}, a_{111}a_{113} \\
& + a_{113}a_{123} + a_{112}a_{133} + (-a_{111} - a_{123} - a_{132})a_{113} + a_{211}a_{012} + a_{212}a_{022} \\
& + a_{213}a_{032}, a_{111}a_{112} + a_{113}a_{122} + a_{112}a_{132} + (-a_{111} - a_{123} - a_{132})a_{112} \\
& + a_{211}a_{013} + a_{212}a_{023} + a_{213}a_{033}],
\end{aligned}$$

$$\begin{aligned}
& \left[a_{121}a_{111} + a_{123}a_{121} + a_{122}a_{131} + (-a_{111} - a_{123} - a_{132})a_{121} + a_{231}a_{011} \right. \\
& + a_{232}a_{021} + a_{233}a_{031}, \frac{1}{2}a_{123}^2 + (-a_{111} - a_{123} - a_{132})a_{123} + \frac{1}{2}(\\
& -a_{111} - a_{123} - a_{132})^2 - \frac{1}{2}a_{111}^2 - a_{112}a_{131} - \frac{1}{2}a_{132}^2 + a_{231}a_{012} \\
& + a_{232}a_{022} + a_{233}a_{032}, a_{121}a_{112} + a_{123}a_{122} + a_{122}a_{132} + (-a_{111} - a_{123} \\
& - a_{132})a_{122} + a_{231}a_{013} + a_{232}a_{023} + a_{233}a_{033}], \\
& \left[a_{131}a_{111} + a_{133}a_{121} + a_{132}a_{131} + (-a_{111} - a_{123} - a_{132})a_{131} + a_{221}a_{011} \right. \\
& + a_{222}a_{021} + a_{223}a_{031}, a_{131}a_{113} + a_{133}a_{123} + a_{132}a_{133} + (-a_{111} - a_{123} \\
& - a_{132})a_{133} + a_{221}a_{012} + a_{222}a_{022} + a_{223}a_{032}, \frac{1}{2}a_{132}^2 + (-a_{111} - a_{123} \\
& - a_{132})a_{132} + \frac{1}{2}(-a_{111} - a_{123} - a_{132})^2 - \frac{1}{2}a_{111}^2 - a_{113}a_{121} - \frac{1}{2}a_{123}^2 \\
& \left. + a_{221}a_{013} + a_{222}a_{023} + a_{223}a_{033} \right]
\end{aligned}$$

$$\begin{aligned}
> Sasharp211 := & \text{expand} \left(\frac{1}{2}a_{111}^2 + (-a_{111} - a_{123} - a_{132})a_{111} + \frac{1}{2}(-a_{111} - a_{123} \right. \\
& - a_{132})^2 - \frac{1}{2}a_{123}^2 - a_{122}a_{133} - \frac{1}{2}a_{132}^2 + a_{211}a_{011} + a_{212}a_{021} \\
& \left. + a_{213}a_{031} \right);
\end{aligned}$$

(61)

$$\begin{aligned}
> Sasharp212 := & \text{expand} (a_{111}a_{113} + a_{113}a_{123} + a_{112}a_{133} + (-a_{111} - a_{123} \\
& - a_{132})a_{113} + a_{211}a_{012} + a_{212}a_{022} + a_{213}a_{032}); \\
Sasharp212 := & a_{112}a_{133} - a_{113}a_{132} + a_{211}a_{012} + a_{212}a_{022} + a_{213}a_{032}
\end{aligned}$$

(62)

$$\begin{aligned}
> Sasharp213 := & \text{expand} (a_{111}a_{112} + a_{113}a_{122} + a_{112}a_{132} + (-a_{111} - a_{123} \\
& - a_{132})a_{112} + a_{211}a_{013} + a_{212}a_{023} + a_{213}a_{033}); \\
Sasharp213 := & a_{113}a_{122} - a_{112}a_{123} + a_{211}a_{013} + a_{212}a_{023} + a_{213}a_{033}
\end{aligned}$$

(63)

$$\begin{aligned}
> Sasharp221 := & \text{expand} (a_{121}a_{111} + a_{123}a_{121} + a_{122}a_{131} + (-a_{111} - a_{123} \\
& - a_{132})a_{121} + a_{231}a_{011} + a_{232}a_{021} + a_{233}a_{031});
\end{aligned}$$

...

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Sasharp221 := a122 a131 - a121 a132 + a231 a011 + a232 a021 + a233 a031      (64)
> Sasharp222 := expand  $\left( \frac{1}{2} a123^2 + (-a111 - a123 - a132) a123 + \frac{1}{2} (-a111 - a123 - a132)^2 - \frac{1}{2} a111^2 - a112 a131 - \frac{1}{2} a132^2 + a231 a012 + a232 a022 + a233 a032 \right);$ 
Sasharp222 := a111 a132 - a112 a131 + a231 a012 + a232 a022 + a233 a032      (65)
> Sasharp223 := expand  $(a121 a112 + a123 a122 + a122 a132 + (-a111 - a123 - a132) a122 + a231 a013 + a232 a023 + a233 a033);$ 
Sasharp223 := a121 a112 - a122 a111 + a231 a013 + a232 a023 + a233 a033      (66)
>
> Sasharp231 := expand  $(a131 a111 + a133 a121 + a132 a131 + (-a111 - a123 - a132) a131 + a221 a011 + a222 a021 + a223 a031);$ 
Sasharp231 := a133 a121 - a131 a123 + a221 a011 + a222 a021 + a223 a031      (67)
> Sasharp232 := expand  $(a131 a113 + a133 a123 + a132 a133 + (-a111 - a123 - a132) a133 + a221 a012 + a222 a022 + a223 a032);$ 
Sasharp232 := a131 a113 - a133 a111 + a221 a012 + a222 a022 + a223 a032      (68)
> Sasharp233 := expand  $\left( \frac{1}{2} a132^2 + (-a111 - a123 - a132) a132 + \frac{1}{2} (-a111 - a123 - a132)^2 - \frac{1}{2} a111^2 - a113 a121 - \frac{1}{2} a123^2 + a221 a013 + a222 a023 + a223 a033 \right);$ 
Sasharp233 := a111 a123 - a113 a121 + a221 a013 + a222 a023 + a223 a033      (69)
> Sasharp2 := Matrix ([[Sasharp211, Sasharp212, Sasharp213], [Sasharp221, Sasharp222, Sasharp223], [Sasharp231, Sasharp232, Sasharp233]]);
Sasharp2 := [[a123 a132 - a122 a133 + a211 a011 + a212 a021 + a213 a031, a112 a133 - a113 a132 + a211 a012 + a212 a022 + a213 a032, a113 a122 - a112 a123 + a211 a013 + a212 a023 + a213 a033],
[a122 a131 - a121 a132 + a231 a011 + a232 a021 + a233 a031, a111 a132 - a112 a131 + a231 a012 + a232 a022 + a233 a032, a121 a112 - a122 a111 + a231 a013 + a232 a023 + a233 a033],
[a133 a121 - a131 a123 + a221 a011 + a222 a021 + a223 a031, a131 a113 - a133 a111 + a221 a012 + a222 a022 + a223 a032, a111 a123 - a113 a121 + a221 a013 + a222 a023 + a223 a033]]]
>
> a1sharp := a1. a1 - ScalarMultiply(a1, Trace(a1)) + ScalarMultiply  $\left( \text{IdentityMatrix}(3), \frac{1}{2} \cdot (Trace(a1)^2 - Trace(a1.a1)) \right);$ 

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(71)

$$\begin{aligned}
a1sharp := & \left[\left[\frac{1}{2} a111^2 - (a111 + a122 + a133) a111 + \frac{1}{2} (a111 + a122 + a133)^2 \right. \right. \\
& - \frac{1}{2} a122^2 - a123 a132 - \frac{1}{2} a133^2, a111 a112 + a112 a122 + a113 a132 - (a111 \\
& + a122 + a133) a112, a111 a113 + a112 a123 + a113 a133 - (a111 + a122 \\
& + a133) a113], \\
& \left[a121 a111 + a122 a121 + a131 a123 - (a111 + a122 + a133) a121, \frac{1}{2} a122^2 \right. \\
& - (a111 + a122 + a133) a122 + \frac{1}{2} (a111 + a122 + a133)^2 - \frac{1}{2} a111^2 \\
& - a131 a113 - \frac{1}{2} a133^2, a113 a121 + a123 a122 + a133 a123 - (a111 + a122 \\
& + a133) a123 \Big], \\
& \left[a131 a111 + a121 a132 + a133 a131 - (a111 + a122 + a133) a131, a112 a131 \right. \\
& + a122 a132 + a132 a133 - (a111 + a122 + a133) a132, \frac{1}{2} a133^2 - (a111 + a122 \\
& + a133) a133 + \frac{1}{2} (a111 + a122 + a133)^2 - \frac{1}{2} a111^2 - a121 a112 - \frac{1}{2} a122^2 \Big] \Big]
\end{aligned} \tag{71}$$

> MatrixAdd(a1sharp, -a2.a0);

$$\begin{aligned}
& \left[\left[\frac{1}{2} a111^2 - (a111 + a122 + a133) a111 + \frac{1}{2} (a111 + a122 + a133)^2 - \frac{1}{2} a122^2 \right. \right. \\
& - a123 a132 - \frac{1}{2} a133^2 - a211 a011 - a212 a021 - a213 a031, a111 a112 \\
& + a112 a122 + a113 a132 - (a111 + a122 + a133) a112 - a211 a012 - a212 a022 \\
& - a213 a032, a111 a113 + a112 a123 + a113 a133 - (a111 + a122 + a133) a113 \\
& - a211 a013 - a212 a023 - a213 a033], \\
& \left[a121 a111 + a122 a121 + a131 a123 - (a111 + a122 + a133) a121 - a221 a011 \right. \\
& - a222 a021 - a223 a031, \frac{1}{2} a122^2 - (a111 + a122 + a133) a122 + \frac{1}{2} (a111 \\
& + a122 + a133)^2 - \frac{1}{2} a111^2 - a131 a113 - \frac{1}{2} a133^2 - a221 a012 - a222 a022 \\
& - a223 a032, a113 a121 + a123 a122 + a133 a123 - (a111 + a122 + a133) a123 \\
& - a221 a013 - a222 a023 - a223 a033 \Big], \\
& \left[a131 a111 + a121 a132 + a133 a131 - (a111 + a122 + a133) a131 - a231 a011 \right.
\end{aligned} \tag{72}$$

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$$\begin{aligned}
& - a_{232}a_{021} - a_{233}a_{031}, a_{112}a_{131} + a_{122}a_{132} + a_{132}a_{133} - (a_{111} + a_{122} \\
& + a_{133})a_{132} - a_{231}a_{012} - a_{232}a_{022} - a_{233}a_{032}, \frac{1}{2}a_{133}^2 - (a_{111} + a_{122} \\
& + a_{133})a_{133} + \frac{1}{2}(a_{111} + a_{122} + a_{133})^2 - \frac{1}{2}a_{111}^2 - a_{121}a_{112} - \frac{1}{2}a_{122}^2 \\
& - a_{231}a_{013} - a_{232}a_{023} - a_{233}a_{033} \Big] \\
> a2sharp11 := expand \left( \frac{1}{2}a_{111}^2 - (a_{111} + a_{122} + a_{133})a_{111} + \frac{1}{2}(a_{111} + a_{122} \\
& + a_{133})^2 - \frac{1}{2}a_{122}^2 - a_{123}a_{132} - \frac{1}{2}a_{133}^2 - a_{211}a_{011} - a_{212}a_{021} \\
& - a_{213}a_{031} \right); \\
& a2sharp11 := a_{122}a_{133} - a_{123}a_{132} - a_{211}a_{011} - a_{212}a_{021} - a_{213}a_{031} \quad (73) \\
> a2sharp12 := expand (a_{111}a_{112} + a_{112}a_{122} + a_{113}a_{132} - (a_{111} + a_{122} \\
& + a_{133})a_{112} - a_{211}a_{012} - a_{212}a_{022} - a_{213}a_{032}); \\
& a2sharp12 := a_{113}a_{132} - a_{112}a_{133} - a_{211}a_{012} - a_{212}a_{022} - a_{213}a_{032} \quad (74) \\
> a2sharp13 := expand (a_{111}a_{113} + a_{112}a_{123} + a_{113}a_{133} - (a_{111} + a_{122} \\
& + a_{133})a_{113} - a_{211}a_{013} - a_{212}a_{023} - a_{213}a_{033}); \\
& a2sharp13 := a_{112}a_{123} - a_{113}a_{122} - a_{211}a_{013} - a_{212}a_{023} - a_{213}a_{033} \quad (75) \\
> \\
> a2sharp21 := expand (a_{121}a_{111} + a_{122}a_{121} + a_{131}a_{123} - (a_{111} + a_{122} \\
& + a_{133})a_{121} - a_{221}a_{011} - a_{222}a_{021} - a_{223}a_{031}); \\
& a2sharp21 := a_{131}a_{123} - a_{133}a_{121} - a_{221}a_{011} - a_{222}a_{021} - a_{223}a_{031} \quad (76) \\
> a2sharp22 := expand \left( \frac{1}{2}a_{122}^2 - (a_{111} + a_{122} + a_{133})a_{122} + \frac{1}{2}(a_{111} + a_{122} \\
& + a_{133})^2 - \frac{1}{2}a_{111}^2 - a_{131}a_{113} - \frac{1}{2}a_{133}^2 - a_{221}a_{012} - a_{222}a_{022} \\
& - a_{223}a_{032} \right); \\
& a2sharp22 := a_{133}a_{111} - a_{131}a_{113} - a_{221}a_{012} - a_{222}a_{022} - a_{223}a_{032} \quad (77) \\
> a2sharp23 := expand (a_{113}a_{121} + a_{123}a_{122} + a_{133}a_{123} - (a_{111} + a_{122} \\
& + a_{133})a_{123} - a_{221}a_{013} - a_{222}a_{023} - a_{223}a_{033}); \\
& a2sharp23 := a_{113}a_{121} - a_{111}a_{123} - a_{221}a_{013} - a_{222}a_{023} - a_{223}a_{033} \quad (78) \\
> \\
> a2sharp31 := expand (a_{131}a_{111} + a_{121}a_{132} + a_{133}a_{131} - (a_{111} + a_{122} \\
& + a_{133})a_{131} - a_{231}a_{011} - a_{232}a_{021} - a_{233}a_{031}); \\
& a2sharp31 := a_{121}a_{132} - a_{122}a_{131} - a_{231}a_{011} - a_{232}a_{021} - a_{233}a_{031} \quad (79) \\
> a2sharp32 := expand (a_{112}a_{131} + a_{122}a_{132} + a_{132}a_{133} - (a_{111} + a_{122} \\
& + a_{133})a_{132} - a_{231}a_{012} - a_{232}a_{022} - a_{233}a_{032}); \\
& a2sharp32 := a_{112}a_{131} - a_{111}a_{132} - a_{231}a_{012} - a_{232}a_{022} - a_{233}a_{032} \quad (80)
\end{aligned}$$


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> a2sharp33 := expand  $\left( \frac{1}{2} a133^2 - (a111 + a122 + a133) a133 + \frac{1}{2} (a111 + a122 + a133)^2 - \frac{1}{2} a111^2 - a121 a112 - \frac{1}{2} a122^2 - a231 a013 - a232 a023 - a233 a033 \right);$ 
a2sharp33 := a122 a111 - a121 a112 - a231 a013 - a232 a023 - a233 a033 (81)
> Sa2sharp := Matrix ( -[[a2sharp11, a2sharp12, a2sharp13], [a2sharp31, a2sharp32, a2sharp33], [a2sharp21, a2sharp22, a2sharp23]] );
Sa2sharp := [[a123 a132 - a122 a133 + a211 a011 + a212 a021 + a213 a031, a112 a133 - a113 a132 + a211 a012 + a212 a022 + a213 a032, a113 a122 - a112 a123 + a211 a013 + a212 a023 + a213 a033],
[a122 a131 - a121 a132 + a231 a011 + a232 a021 + a233 a031, a111 a132 - a112 a131 + a231 a012 + a232 a022 + a233 a032, a121 a112 - a122 a111 + a231 a013 + a232 a023 + a233 a033],
[a133 a121 - a131 a123 + a221 a011 + a222 a021 + a223 a031, a131 a113 - a133 a111 + a221 a012 + a222 a022 + a223 a032, a111 a123 - a113 a121 + a221 a013 + a222 a023 + a223 a033]]
> MatrixAdd (Sa2sharp, -Sasharp2);

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
 (83)

```