ABSTRACT

MARONCELLI, DANIEL MICHAEL. Existence of Solutions to Nonlinear Boundary Value Problems at Higher Dimensional Resonance. (Under the direction of Jesús Rodríguez.)

The focus of this paper is the study of nonlinear boundary value problems. We investigate the existence of solutions to both impulsive differential equations and discrete-time difference equations. We concentrate on the case of resonance; that is, the case where the dimension of the solution space to the associated linear homogeneous problem is nontrivial. In particular, we focus on the case where the dimension of the solution space is strictly greater than one.

We begin by considering nonlinear impulsive boundary value problems of the form

$$x'(t) = A(t)x(t) + f(t, x(t)), \quad t \in [0,1] \setminus \{t_1, t_2, \cdots, t_k\}$$

$$x(t_i^+) - x(t_i^-) = J_i(x(t_i^-)), \quad i = 1, \cdots, k$$

subject to boundary conditions

$$Bx(0) + Dx(1) = 0,$$

where the points $t_i, i = 1, \cdots, k$, are fixed with $0 < t_1 < t_2 < \cdots < t_k < 1$. Criteria for the solvability the nonlinear boundary value problem are established using topological degree theory in combination with the Lyapunov-Schmidt procedure.

Next we focus on the solvability of weakly nonlinear problems of the form

$$x'(t) = A(t)x(t) + g(t) + \varepsilon f(t, x(t)), \quad t \in [0,1] \setminus \{t_1, t_2, \cdots, t_k\}$$

$$x(t_i^+) - x(t_i^-) = w_i, \quad i = 1, ..., k$$

subject to boundary conditions

$$Bx(0) + Dx(1) = 0.$$

Our analysis uses an implicit function theorem argument along with the Lyapunov-Schmidt procedure to prove the existence of solutions for small ε.

We then analyze nonlinear, discrete, multipoint boundary value problems of the form

$$x(t+1) = A(t)x(t) + f(t, x(t)),$$

subject to

$$\sum_{i=0}^{m} B_i x(i) = 0.$$

The analysis here is similar to that of the impulsive boundary value problem. Again, our focus is the case of resonance.

Lastly, we study least squares solutions to linear boundary value problems of the form

$$x'(t) = A(t)x(t) + h(t), \quad t \neq t_1 < t_2 < ... < t_k \in [0, 1]$$

$$x(t_i^+) - x(t_i^-) = v_i, \quad i = 1, ..., k$$

subject to

$$Bx(0) + Dx(1) = 0.$$

We obtain a complete characterization of the least squares solution with minimal $L^2([0, 1], \mathbb{R}^n)$ norm.

Existence of Solutions to Nonlinear Boundary Value Problems
at Higher Dimensional Resonance

by
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DEDICATION

To my my wonderful family: Donovan, Isabella, and Hailey.

BIOGRAPHY

Daniel Michael Maroncelli was born on February 17, 1984 in Billings, Montana to his wonderful parents Janet and Mike Maroncelli. Dan has one sister, Jenny Maroncelli. In 2007, Dan graduated from Montana State University with his bachelor's degree in Civil Engineering. It was while at Montana State that Dan met his bride to be, Hailey Maroncelli. Dan and Hailey were blessed with their first child, Donovan Michael, on April 5, 2008. In May of 2009, Dan completed his master's degree in mathematics at Montana State, upon which he decided to pursue his doctorate in mathematics at North Carolina State University. On June 3, 2011 Dan and Hailey were blessed with their second child, Isabella Marie. At the time of this writing Dan is looking forward to the new challenges that await him as an academic professional.

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to make it without you. I am truly blessed to have shared this experience with such a wonderful person like you.

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Chapter 1

Introduction

This paper is devoted to the study of nonlinear boundary value problems at resonance. In particular, we focus on the case where the solution space to an associated linear homogeneous boundary value problem has dimension greater than one. In each chapter, the properties of the nonlinearities and their interaction with the solution space of the linear homogeneous problem play a crucial role.

In chapter 2 we analyze the solvability of impulsive differential equations of the form

$$x'(t) = A(t)x(t) + f(t, x(t)), \quad t \in [0,1] \setminus \{t_1, t_2, \cdots, t_k\}$$

$$x(t_i^+) - x(t_i^-) = J_i(x(t_i^-)), \quad i = 1, \cdots, k$$

subject to boundary conditions

$$Bx(0) + Dx(1) = 0,$$

We will assume that f, each J_i, and A are continuous. $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$, $J_i\colon \mathbb{R}^n \to \mathbb{R}^n$, and for each $t \in [0,1]$, $A(t)$ is an $n \times n$ matrix. The matrices B and D are $n \times n$ and, in order to avoid redundancies, we will assume that the augmented matrix $[B|D]$ has full row rank. Our approach is topological, using degree theory and the Lyapunov-Schmidt procedure.

In chapter 3 we focus on the solvability of weakly nonlinear impulsive systems of the form

$$x'(t) = A(t)x(t) + g(t) + \varepsilon f(t, x(t)), \quad t \in [0, 1] \setminus \{t_1, t_2, \cdots, t_k\}$$

$$x(t_i^+) - x(t_i^-) = w_i, \quad i = 1, ..., k$$

subject to boundary conditions

$$Bx(0) + Dx(1) = 0.$$

The points $t_i, i = 1, \cdots, k$, are fixed with $0 < t_1 < t_2 < \cdots < t_k < 1$. We assume f and A are continuous, $f \colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ and for each t, $A(t)$ is a $n \times n$ matrix. The $w_i, i = 1, \cdots, k$, are elements of $\mathbb{R}^n$, B and D are $n \times n$ matrices, and in order to have a linearly independent system of boundary conditions, we assume the augmented matrix $[B|D]$ has full row rank. ε is a "small" real parameter, and $x(t_i^+)$ and $x(t_i^-)$ denote the left and right-hand limits for x at the points t_i respectively, and the function g is piecewise continuous. Our approach will utilize an implicit function theorem argument in combination with the Lyapunov-Schmidt procedure.

In chapter 4 we look at multipoint boundary value problems for discrete systems. Again, our focus is on the case of resonance. In particular, we analyze problems of the form

$$x(t + 1) = A(t)x(t) + f(t, x(t)),$$

subject to

$$\sum_{i=0}^{m} B_i x(i) = 0.$$

We will assume that for each $t \in \mathbb{N} = \{0, 1, 2, \cdots\}$, $A(t)$ is an $n \times n$ invertible matrix. We assume $f \colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous, m is a fixed integer greater than two, each $B_i, i = 0, \cdots, m$, is $n \times n$ matrix and the augmented matrix $[B_1|B_2| \cdots |B_m]$ has full row rank.

Lastly, in chapter 5, our goal is to characterize least squares solutions to boundary value problems of the following form

$$x'(t) = A(t)x(t) + h(t), \quad a.e. \, [0,1] \tag{1.1}$$

$$x(t_i^+) - x(t_i^-) = v_i, \quad i = 1, ..., k \tag{1.2}$$

subject to

$$Bx(0) + Dx(1) = 0. \tag{1.3}$$

The points $t_i, i = 1, \cdots, k$, are fixed with $0 < t_1 < t_2 < \cdots < t_k < 1$. For each $t \in [0,1]$, $A(t)$ is an $n \times n$ matrix. The map $t \to A(t)$ is assumed to have components in $L^2([0,1], \mathbb{R})$ and h is assumed to be in $L^2([0,1], \mathbb{R}^n)$. The v_i, $i = 1, \cdots, k$, are elements of $\mathbb{R}^n$, and B and D are $n \times n$ matrices with the augmented matrix $[B|D]$ having full row rank. As a consequence of our analysis, we completely characterize the least squares solution of minimal $L^2[0,1]$ norm.

Remark 1.0.1. We would like to the remark that each chapter is self-contained and thus may be read in any order that the reader sees fit.

Chapter 2

On the solvability of nonlinear impulsive boundary value problems

In this chapter we provide criteria for the solvability of nonlinear, impulsive, two-point boundary value problems. We consider problems of the form

$$x'(t) = A(t)x(t) + f(t, x(t)), \quad t \in [0, 1] \setminus \{t_1, t_2, \cdots, t_k\} \tag{2.1}$$

$$x(t_i^+) - x(t_i^-) = J_i(x(t_i^-)), \quad i = 1, \cdots, k \tag{2.2}$$

subject to boundary conditions

$$Bx(0) + Dx(1) = 0, \tag{2.3}$$

where the points $t_i, i = 1, \cdots, k$, are fixed with $0 < t_1 < t_2 < \cdots < t_k < 1$.

Throughout the discussion we will assume that f, each J_i, and A are continuous. $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$, $J_i\colon \mathbb{R}^n \to \mathbb{R}^n$, and for each $t \in [0, 1]$, $A(t)$ is an $n \times n$ matrix. The matrices B and D are $n \times n$ and, in order to avoid redundancies, we will assume that the augmented matrix $[B|D]$ has full row rank.

The main objective of this paper is the study of nonlinear, impulsive boundary value problems at resonance; that is, systems where the associated linear homogeneous problem has nontrivial solutions. Our approach is based on the use of topological degree theory in conjunction with the Lyapunov-Schmidt procedure. The results we obtain depend on properties of the nonlinearities, as well as the solution space of the associated linear homogeneous problem.

There is an extensive literature regarding degree theory, the Lyapunov-Schmidt procedure, and projection schemes in nonlinear analysis. General theoretical results and applications to boundary value problems in differential equations can be found in [6, 8, 15, 27, 28, 30, 33, 34, 35, 38]. The solvability of discrete systems is considered in [7, 25, 29]. Those interested in the theory and application of impulsive systems may consult [13, 23, 24, 26, 36].

2.1 Preliminaries

We will formulate the nonlinear boundary value problem (2.1)-(2.3) as an operator equation. In order to do so, we introduce appropriate spaces and operators. $PC_{\{t_i\}}[0,1]$ will represent the set of $\mathbb{R}^n$-valued continuous functions on $[0,1] \setminus \{t_1, \cdots, t_k\}$ which have right and left-hand limits at each t_i, $i = 1, \cdots, k$. On $PC_{\{t_i\}}[0,1]$ we will use the supremum norm; that is,

$$\|\phi\| = \sup_{t \in [0,1] \setminus \{t_1, \cdots, t_k\}} |\phi(t)|,$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^n$. It is well known that when endowed with this norm, $PC_{\{t_i\}}[0,1]$ is a Banach space. The subset of $PC_{\{t_i\}}[0,1]$ consisting of continuously differentiable functions $\phi : [0,1] \setminus \{t_1, \cdots, t_k\} \to \mathbb{R}^n$ such that $\phi^{'}$ has finite right and left-hand limits at each t_i, $i = 1, \cdots, k$, will be denoted by $PC^1_{\{t_i\}}[0,1]$. Finally, we define

$$X = \{\phi \in PC_{\{t_i\}}[0,1] \mid B\phi(0) + D\phi(1) = 0\}.$$

The norms on $PC^1_{\{t_i\}}[0,1]$ and X will be the same as on $PC_{\{t_i\}}[0,1]$.

We now introduce mappings $\mathcal{L}$ and $\mathcal{F}$. The domain of $\mathcal{L}$, written $dom(\mathcal{L})$, is given by

$dom(\mathcal{L}) = PC^1_{\{t_i\}}[0,1] \cap X.$

The mapping $\mathcal{L}\colon \mathrm{dom}(\mathcal{L}) \subset X \to PC_{\{t_i\}}[0,1] \times \mathbb{R}^{nk}$ is defined by

$$\mathcal{L}x = \begin{bmatrix} x'(\cdot) - A(\cdot)x(\cdot) \\ x(t_1^+) - x(t_1^-) \\ \vdots \\ x(t_k^+) - x(t_k^-) \end{bmatrix}.$$

The nonlinear operator $\mathcal{F}\colon PC_{\{t_i\}}[0,1] \to PC_{\{t_i\}}[0,1] \times \mathbb{R}^{nk}$ is given by

$$\mathcal{F}x = \begin{bmatrix} f(\cdot, x(\cdot)) \\ J_1(x(t_1^-)) \\ \vdots \\ J_k(x(t_k^-)) \end{bmatrix}.$$

We make $PC_{\{t_i\}}[0,1] \times \mathbb{R}^{nk}$ a Banach space by introducing the following norm:

$$\left\| \begin{bmatrix} h(\cdot) \\ v_1 \\ \vdots \\ v_k \end{bmatrix} \right\| = \max\{\|h\|, |v_1|, |v_2|, \cdots, |v_k|\}.$$

Remark 2.1.1. With the definitions as above, it is clear that solving the nonlinear boundary value problem (2.1)-(2.3) is equivalent to solving $\mathcal{L}x = \mathcal{F}x$.

Before focusing on the nonlinear boundary problem (2.1)-(2.3) , we analyze the linear homogeneous problem

$$x'(t) = A(t)x(t), \quad t \in [0,1] \setminus \{t_1, t_2, \cdots, t_k\} \tag{2.4}$$

subject to boundary conditions (2.3), as well as the linear nonhomogeneous problem

$$
\begin{aligned}
&x'(t) = A(t)x(t) + h(t), \quad t \in [0,1] \setminus \{t_1, t_2, \cdots, t_k\} \\
&x(t_i^+) - x(t_i^-) = v_i, \quad i = 1, ..., k
\end{aligned}
\tag{2.5}
$$

subject to the same boundary conditions. Here we assume $h \in PC_{\{t_i\}}[0,1]$ and each v_i, $i = 1, \cdots, k$, is an element of $\mathbb{R}^n$.

It is clear that a function x is a solution to the linear nonhomogeneous problem (2.5) subject to boundary conditions (2.3) if and only if $\mathcal{L}x = \begin{bmatrix} h \\ v \end{bmatrix}$, where $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}$. Taking $\begin{bmatrix} h \\ v \end{bmatrix} = 0$, we see that the solution space of the linear homogeneous problem (2.4) subject to the boundary conditions (2.3) is given by the $Ker(\mathcal{L})$. We now characterize $Ker(\mathcal{L})$ and $Im(\mathcal{L})$.

Proposition 2.1.2. *A function x is a solution to the linear homogeneous problem (2.4) subject to the boundary conditions (2.3) if and only if $x(t) = \Phi(t)c$ for some $c \in Ker(B + D\Phi(1))$. Here $\Phi(\cdot)$ is the principal fundamental matrix solution to $x' = A(\cdot)x$.*

Proof.

$$
\begin{aligned}
\mathcal{L}x = 0 \quad &\Longleftrightarrow \quad x' = A(\cdot)x \text{ and } Bx(0) + Dx(1) = 0 \\
&\Longleftrightarrow \quad x = \Phi(\cdot)x(0) \text{ and } Bx(0) + Dx(1) = 0 \\
&\Longleftrightarrow \quad \text{there exists } c \in \mathbb{R}^n, \text{ such that } x = \Phi(\cdot)c \text{ and } Bc + D\Phi(1)c = 0.
\end{aligned}
$$

□

Corollary 2.1.3. *The solution space of the linear homogeneous problem (2.4) subject to the boundary conditions (2.3) has the same dimension as the $Ker(B + D\Phi(1))$.*

We now choose vectors $b_1, \cdots, b_p$, where $p \leq n$, from $\mathbb{R}^n$ which form a basis for $Ker(B + D\Phi(1))$ and make the following definition:

Definition 2.1.4. We define $S(t)$ to be the $n \times p$ matrix whose ith column is $S_i(t) := \Phi(t)b_i$.

Corollary 2.1.5. *A function x is a solution to the linear homogeneous problem (2.4) with boundary conditions (2.3) if and only if $x(\cdot) = S(\cdot)\alpha$ for some $\alpha \in \mathbb{R}^p$.*

Proposition 2.1.6. *Let $\{c_1, \cdots, c_p\}$ be a basis for $Ker\left((B + D\Phi(1))^T\right)$. Then the linear non-homogeneous problem (2.5) subject to the boundary conditions (2.3) has a solution if and only if for each $i = 1, \cdots, p$, we have*

$$\left\langle c_i, D\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i\right)\right\rangle = 0.$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$.

Proof. It is well documented, see [13, 36], that $\mathcal{L}x = \begin{bmatrix} h \\ v \end{bmatrix}$ if and only if x is given by the variation of parameters formula

$$x(t) = \Phi(t)\left(x(0) + \int_0^t \Phi^{-1}(s)h(s)ds + \sum_{t_i < t} \Phi^{-1}(t_i)v_i\right)$$

and x satisfies the boundary conditions (2.3).

Imposing the boundary conditions, we have $\begin{bmatrix} h \\ v \end{bmatrix} \in Im(\mathcal{L})$ if and only if there exists

$w \in \mathbb{R}^n$ such that

$$
\begin{aligned}
&Bw + D\left(\Phi(1)\left(w + \int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i\right)\right) \\
&\iff [B + D\Phi(1)]w = -D\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i\right) \\
&\iff D\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i\right) \in Im\left(B + D\Phi(1)\right).
\end{aligned}
$$

Using the fact that $Im(B + D\Phi(1))$ is the orthogonal complement of $Ker\left((B + D\Phi(1))^{\mathrm{T}}\right)$, we have that

$$
\begin{bmatrix} h \\ v \end{bmatrix} \in Im(\mathcal{L})
$$

if and only if

$$
\left\langle c, D\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i\right)\right\rangle = 0
$$

for all $c \in Ker\left((B + D\Phi(1))^{\mathrm{T}}\right)$. □

If we now define $W := [c_1, ..., c_p]$ and $\Psi(t)^T := W^T D\Phi(1)\Phi^{-1}(t)$, we get the following corollary:

Corollary 2.1.7. *The linear nonhomogeneous problem (2.5) with boundary conditions (2.3) has a solution if and only if* $\int_0^1 \Psi^T(s)h(s)ds + \sum_{i=1}^{k} \Psi^T(t_i)v_i = 0.$

Remark 2.1.8. It is now clear that the linear nonhomogeneous boundary value problem (2.5) subject to the boundary conditions (2.3) has a unique solution if and only if $B + D\Phi(1)$ is invertible. If this is the case, $\mathcal{L}$ is a bijection. We then have, for each element $\begin{bmatrix} h \\ v \end{bmatrix} \in PC_{\{t_i\}}[0, 1]$, that the unique solution to $\mathcal{L}x = \begin{bmatrix} h \\ v \end{bmatrix}$ is given by

$$x(t) = \mathcal{L}^{-1} \begin{bmatrix} h \\ v \end{bmatrix} (t) = \Phi(t) \left(-[B + D\Phi(1)]^{-1} D\Phi(1) \left(\int_0^1 \Phi^{-1}(s)h(s)ds \right.\right.$$
$$\left.\left. + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i \right) \right)$$
$$+ \Phi(t) \left(\int_0^t \Phi^{-1}(s)h(s)ds + \sum_{t_i < t} \Phi^{-1}(t_i)v_i \right).$$

2.2 Main Results

In this section we focus on the nonlinear boundary value problem

$$x'(t) = A(t)x(t) + f(t, x(t)), \quad t \in [0,1] \setminus \{t_1, t_2, \cdots, t_k\}$$

$$x(t_i^+) - x(t_i^-) = J_i(x(t_i^-)), \quad i = 1, \cdots, k$$

with boundary conditions

$$Bx(0) + Dx(1) = 0.$$

We are mainly interested in systems at resonance and our principle result in this regard is theorem 2.2.7. In this theorem we establish conditions for the existence of solutions which are based on the interplay between the nonlinearities $f, J_1, \cdots, J_k$ and the solution space of the linear homogeneous problem (2.4) subject to the boundary conditions (2.3).

In theorem 2.2.1 we present criteria for the solvability of (2.1)-(2.3) in the nonresonant case. The analysis in this case is simpler and the results obtained here are based on the growth rate of the nonlinearities.

Theorem 2.2.1. *Suppose that the only solution to the linear homogeneous problem (2.4) subject to the boundary conditions (2.3) is the trivial solution. If there exist real numbers M_1, M_2, and α, with $0 \le \alpha < 1$, such that for all $t \in [0,1]$ and $y \in \mathbb{R}^n$, $|f(t,y)| \le M_1|y|^\alpha + M_2$ and $|J_i(y)| \le M_1|y|^\alpha + M_2$, then the nonlinear boundary value problem (2.1)-(2.3) has a solution.*

Proof. Define $H : PC_{\{t_i\}}[0,1] \to PC_{\{t_i\}}[0,1]$ by

$$\begin{aligned}[H(x)](t) =& \Phi(t)\left(-[B + D\Phi(1)]^{-1}D\Phi(1)\left(\int_0^1 \Phi^{-1}(s)f(s,x(s))ds \right.\right.\\ &\left.\left. + \sum_{i=1}^{k} \Phi^{-1}(t_i)J_i(x(t_i^-))\right)\right)\\ &+ \Phi(t)\left(\int_0^t \Phi^{-1}(s)f(s,x(s))ds + \sum_{t_i<t} \Phi^{-1}(t_i)J_i(x(t_i^-))\right).\end{aligned}$$

From remark 2.1.8, it is clear that the solutions of (2.1)-(2.3) are precisely the fixed points of H.

Using the fact that for all $t \in [0,1]$ and $y \in \mathbb{R}^n$

$$|f(t,y)| \le M_1|y|^\alpha + M_2$$

and

$$|J_i(y)| \le M_1|y|^\alpha + M_2,$$

it follows that there exist B_1, B_2 such that

$$\|H(x)\| \le B_1\|x\|^\alpha + B_2.$$

Since $\alpha < 1$, we may choose r sufficiently large such that $B_1r^\alpha + B_2 \le r$. With this in mind, we define

$$\mathcal{B} = \{x \in PC_{\{t_i\}}[0,1] : \|x\| \le r\}.$$

It is clear that $H(\mathcal{B}) \subset \mathcal{B}$. From basic properties of integral operators, it is evident that H is compact. The existence of a fixed point for H is now a consequence of Schauder's theorem. $\square$

We now turn our attention to the case in which the linear homogeneous problem (2.4) subject to the boundary conditions (2.3) has a nontrivial solution space. In this case we analyze (2.1)-(2.3) using a projection scheme known as the Lyapunov-Schmidt procedure. To do so we construct projections onto the $Ker(\mathcal{L})$ and $Im(\mathcal{L})$.

Let $V \colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto $Ker(B + D\Phi(1))$.

Definition 2.2.2. Define $P \colon X \to X$ by

$$[Px](t) = \Phi(t)Vx(0).$$

Proposition 2.2.3. *P is a projection onto $Ker(\mathcal{L})$.*

Proof. $[P^2x](t) = \Phi(t)V^2x(0) = \Phi(t)Vx(0) = [Px](t)$, thus P is a projection. From the characterization of $Ker(\mathcal{L})$, it follows that $Im(P) = Ker(\mathcal{L})$. $\square$

Let $T \colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto $Ker(W^T D\Phi(1))$. It follows from corollary 2.1.7 that

$$\begin{bmatrix} h \\ v \end{bmatrix} \in Im(\mathcal{L}) \text{ if and only if } [I - T]\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i\right) = 0.$$

Definition 2.2.4. Define $E : PC_{\{t_i\}}[0,1] \times \mathbb{R}^{nk} \to PC_{\{t_i\}}[0,1] \times \mathbb{R}^{nk}$ by

$$E\begin{bmatrix} h \\ v \end{bmatrix} = \begin{bmatrix} h(\cdot) - \Phi(\cdot)[I - T]\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\right) \\ v \end{bmatrix}.$$

Proposition 2.2.5. *E is a projection onto $Im(\mathcal{L})$.*

Proof.

$$[I-T]\left(\int_0^1 \Phi^{-1}(s)\left[h(s)-\Phi(s)\left(I-T\right)\left(\int_0^1 \Phi^{-1}(u)h(u)du \right.\right.\right.$$
$$\left.\left.\left. + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\right)\right] + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\right)$$

$$= [I-T]\left(\int_0^1 \Phi^{-1}(s)h(s) + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\right) - [I-T]^2\left(\int_0^1 \Phi^{-1}(u)h(u)du \right.$$
$$\left. + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\right) = 0.$$

It follows that $E^2 = E$ and that $Im(E) \subset Im(\mathcal{L})$.

To see that $Im(\mathcal{L}) \subset Im(E)$ note that if $\begin{bmatrix} h \\ v \end{bmatrix} \in Im(\mathcal{L})$, then

$$[I-T]\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\right) = 0.$$

We then have

$$\Phi(\cdot)[I-T]\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\right) = 0,$$

from which it follows that $E\begin{bmatrix} h \\ v \end{bmatrix} = \begin{bmatrix} h \\ v \end{bmatrix}$. □

For the sake of completeness, we now give a self-contained description of the Lyapunov-Schmidt projection procedure.

Proposition 2.2.6. *Solving $\mathcal{L}x = \mathcal{F}x$ is equivalent to solving the system*

$$\begin{cases} x = Px + M_p E\mathcal{F}x \\ \qquad \textit{and} \\ (I-E)\mathcal{F}x = 0 \end{cases}$$

where M_p *is* $\mathcal{L}^{-1}_{|Ker(P)\cap dom(\mathcal{L})}$.

Proof. We have

$$\mathcal{L}x = \mathcal{F}x \Longleftrightarrow \begin{cases} E[\mathcal{L}x - \mathcal{F}x] = 0 \\ \text{and} \\ (I-E)[\mathcal{L}x - \mathcal{F}x] = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \mathcal{L}x - E\mathcal{F}x = 0 \\ \text{and} \\ (I-E)\mathcal{F}x = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} M_p\mathcal{L}x - M_pE\mathcal{F}x = 0 \\ \text{and} \\ (I-E)\mathcal{F}x = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} (I-P)x - M_pE\mathcal{F}x = 0 \\ \text{and} \\ (I-E)\mathcal{F}x = 0 \end{cases} .$$

□

We now come to our main result concerning the nonlinear boundary value problem (2.1)-(2.3). Before stating the result, we make some introductory assumptions and definitions.

In the following it will be assumed that for sufficiently large r, the map

$$(t,x) \to \begin{bmatrix} f(t,x) \\ J_1(x) \\ \vdots \\ J_k(x) \end{bmatrix}$$

is Lipschitz, in x, on the complement of $B(0,r)$. Here we use the standard convention of denoting, for any normed space Y, $\{y \in Y : \|y\| < r\}$ by $B(0,r)$. More specifically, we assume there exist real numbers R_0 and L, such that for all $t \in [0,1]$ and any x and $y \in \mathbb{R}^n$ with $|x| > R_0$ and $|y| > R_0$, we have

$$\left\| \begin{bmatrix} f(t,x) - f(t,y) \\ J_1(x) - J_1(y) \\ \vdots \\ J_k(x) - J_k(y) \end{bmatrix} \right\| \leq L|x-y|.$$

We let, for $r \geq R_0$, $L(r)$ denote the smallest Lipschitz constant on the complement of $B(0,r)$.

The following observation will be used in what follows. Since the map taking $(t,\alpha) \to S(t)\alpha$ is a continuous mapping, it attains its minimum on the compact set

$$\mathcal{O} := [0,1] \times \{\alpha \in \mathbb{R}^p : \, |\alpha| = 1\}.$$

For each $\alpha \neq 0$, $S(\cdot)\alpha$ is a nonzero solution to (2.4) and so $\eta := \inf_{(t,\alpha)\in\mathcal{O}} |S(t)\alpha| > 0$.

Theorem 2.2.7. *Suppose the following conditions hold:*

C1. The functions $f, J_1, \cdots, J_k$, are bounded, say by b.

C2. There exist real numbers R, $d > 0$, and β such that for all $\alpha \in \mathbb{R}^p$ with $|\alpha| > R$,

$$\left| \int_0^1 \Psi^T(t) f(t, S(t)\alpha) dt + \sum_{i=1}^k \Psi^T(t_i) J_i(S(t_i)\alpha) \right| \geq d$$

and

$$\left\langle \alpha, \int_0^1 \Psi^T(t) f(t, S(t)\alpha) dt + \sum_{i=1}^k \Psi^T(t_i) J_i(S(t_i)\alpha) \right\rangle \geq \beta > -d^2.$$

C3. $\lim_{r\to\infty} L(r)(k+1) \|M_p E\| \left\|\Psi^T(\cdot)\right\| b < \min\left\{ \frac{\sqrt{d^2+\beta}}{\sqrt{2}}, d \right\}.$

(Here $\left\|\Psi^T(\cdot)\right\| = \sup_{t\in[0,1]} \left\|\Psi^T(t)\right\|$*).*

Then there exists a solution to the nonlinear boundary value problem (2.1)-(2.3).

Proof. Since the functions $f, J_1, \cdots, J_k$ are bounded, we may choose a common bound. As above, let b denote this common bound. Clearly,

$$\|\mathcal{F}x\| \leq b$$

for each x in $PC_{\{t_i\}}[0,1]$. For convenience, we assume $\{b_1, b_2, \cdots, b_p\}$ (definition 2.1.4) and $\{c_1, c_2, \cdots, c_p\}$ (proposition 2.1.6) have been chosen such that

$$\|S(\cdot)\| \leq 1$$

and

$$\left\|\Psi^T(\cdot)\right\| \leq 1.$$

From C1., C2., and C3., there exists a positive real number, which we also denote by R, such that for all $\alpha \in \mathbb{R}^p$ with $|\alpha| \geq R$ and each real number $r \geq R$, we have the following:

1. $\left| \int_0^1 \Psi^T(t) f(t, S(t)\alpha) dt + \sum_{i=1}^k \Psi^T(t_i) J_i(S(t_i)\alpha) \right| \geq d.$

2. $\left\langle \alpha, \int_0^1 \Psi^T(t) f(t, S(t)\alpha) dt + \sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha) \right\rangle \geq \beta > -d^2.$

3. $L(r)(k+1)\left\|M_p E\right\| b < \min\left\{ \dfrac{\sqrt{d^2+\beta}}{\sqrt{2}}, d \right\}.$

Here $\|M_pE\|$ denotes the operator norm of M_pE.

We will establish the existence of a solution to (2.1)-(2.3) by showing the existence of a fixed point for an operator H.

We define the operator $H : \mathbb{R}^p \times Im(I-P) \to \mathbb{R}^p \times Im(I-P)$ by

$$H(\alpha, x) = \begin{bmatrix} \alpha - \displaystyle\int_0^1 \Psi^T(t) f(t, S(t)\alpha + x(t)) dt - \sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha + x(t_i^-)) \\ \\ M_p E F(S(\cdot)\alpha + x) \end{bmatrix}.$$

We use the max norm on the space $\mathbb{R}^p \times Im(I-P)$; that is, $\|(\alpha, x)\| = \max\{|\alpha|, \|x\|\}$.

For $h \in PC_{\{t_i\}}[0,1]$ and $v \in \mathbb{R}^{nk}$ define

$$\begin{aligned} N_{h,v}(t) =& \Phi(t) \left(-M_{BD} D\Phi(1) \left(\int_0^1 \Phi^{-1}(s) h(s) ds + \sum_{i=1}^{k} \Phi^{-1}(t_i) v_i \right) \right) \\ &+ \Phi(t) \left(\int_0^t \Phi^{-1}(s) h(s) ds + \sum_{t_i < t} \Phi^{-1}(t_i) v_i \right), \end{aligned}$$

where M_{BD} denotes the right inverse of $B + D\Phi(1)$ when restricted to orthogonal complement of $Ker(B + B\Phi(1))$. Since

$$N_{h,v}(0) = -M_{BD} D\Phi(1) \left(\int_0^1 \Phi^{-1}(s) h(s) ds + \sum_{i=1}^{k} \Phi^{-1}(t_i) v_i \right),$$

we have $VN_{h,v}(0) = 0$ and thus $P(N_{h,v}) = 0$. Further, from the characterization of the $Im(L)$, it follows that $L(N_{h,v}) = \begin{bmatrix} h \\ v \end{bmatrix}$. Since $M_p\left(\begin{bmatrix} h \\ v \end{bmatrix}\right)$ is the unique element with these two properties, it follows that

$$M_p\left(\begin{bmatrix} h \\ v \end{bmatrix}\right)(t) = \Phi(t)\left(-M_{BD}D\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\right)\right)$$
$$+ \Phi(t)\left(\int_0^t \Phi^{-1}(s)h(s)ds + \sum_{t_i<t}\Phi^{-1}(t_i)v_i\right).$$

From basic properties of integral operators, we have that M_p is compact. Combining the compactness of M_p and the boundedness of $\mathcal{F}$, we see that H is a compact operator. Further, from proposition 2.2.6, having a solution to (2.1)-(2.3) is equivalent to H having a fixed point.

We choose $R^* > \max\left\{(k+1)b, \dfrac{R + \|M_pE\|\, b}{\eta}\right\}$ and define

$\Omega := B(0, R^*) \times B(0, \|M_pE\|\, b)$.

We will show that $deg(I - H, \Omega, 0) \neq 0$. To this end, define

$Q : [0,1] \times \overline{\Omega} \to \mathbb{R}^p \times Im(I - P)$

by

$Q(\lambda, (\alpha, x)) =$

$$\begin{bmatrix} (1-\lambda)\alpha + \lambda\left(\int_0^1 \Psi^T(t)f(t, S(t)\alpha + x(t))dt + \sum_{i=1}^{k}\Psi^T(t_i)J_i(S(t_i)\alpha + x(t_i^-)\right) \\ \\ x - \lambda M_pEF(S(\cdot)\alpha + x) \end{bmatrix}$$

Using the fact that

$deg(Q(0,\cdot,\cdot), \Omega, 0) = deg(I, \Omega, 0) = 1,$

and that Q is clearly a homotopy between I and $I - H$, the result will follow once we show

$0 \notin Q(\lambda, \partial(\Omega))$ for each $\lambda \in (0, 1)$.

Now, it is clear that $(\alpha, x) \in \partial(\Omega)$ if and only if

$$|\alpha| = R^* \text{ and } \|x\| \leq \|M_p E\| \, b,$$

or

$$|\alpha| \leq R^* \text{ and } \|x\| = \|M_p E\| \, b.$$

With this in mind, let (α, x) be in $\partial(\Omega)$ and assume $|\alpha| \leq R^*$ with $\|x\| = \|M_p E\| \, b$. It follows that

$$\begin{aligned} \|x - \lambda M_p E F(S(\cdot)\alpha + x)\| &\geq |\|x\| - \lambda \|M_p E F(S(\cdot)\alpha + x)\|| \\ &\geq \|M_p E\| \, b - \lambda \|M_p E\| \, b > 0. \end{aligned}$$

Thus, $Q(\lambda, (\alpha, x)) \neq 0$.

Now suppose (α, x) is in $\partial(\Omega)$ and assume $|\alpha| = R^*$ with $\|x\| \leq \|M_p E\| \, b$. We then have

$$\left| (1 - \lambda)\alpha + \lambda \left(\int_0^1 \Psi^T(t) f(t, S(t)\alpha + x(t)) dt + \sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha + x(t_i^-)) \right) \right|$$

$$\begin{aligned} \geq & \left| (1 - \lambda)\alpha + \lambda \left(\int_0^1 \Psi^T(t) f(t, S(t)\alpha) dt + \sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha) \right) \right| \\ - & \left| \lambda \left(\int_0^1 \Psi^T(t) f(t, S(t)\alpha) dt + \sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha) \right) \right. \\ & \quad \left. - \lambda \left(\int_0^1 \Psi^T(t) f(t, S(t)\alpha + x(t)) dt + \sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha + x(t_i^-)) \right) \right| . \end{aligned}$$

Since $|\alpha| = R^*$, it follows that

$$\begin{aligned}
&\left|(1-\lambda)\alpha + \lambda\left(\int_0^1 \Psi^T(t)f(t,S(t)\alpha)dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\right)\right|^2 \\
&= (1-\lambda)^2|\alpha|^2 + \lambda^2\left|\left(\int_0^1 \Psi^T(t)f(t,S(t)\alpha)dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\right)\right|^2 \\
&+2(1-\lambda)\lambda\left\langle \alpha, \int_0^1 \Psi^T(t)f(t,S(t)\alpha)dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\right\rangle \\
&\geq \left((1-\lambda)^2+\lambda^2\right)\left|\int_0^1 \Psi^T(t)f(t,S(t)\alpha)dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\right|^2 \\
&+2(1-\lambda)\lambda\left\langle \alpha, \int_0^1 \Psi^T(t)f(t,S(t)\alpha)dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\right\rangle \\
&\geq \left((1-\lambda)^2+\lambda^2\right)d^2 + 2(1-\lambda)\lambda\beta.
\end{aligned}$$

For $\lambda \in [0,1]$, the function $\lambda \to ((1-\lambda)^2+\lambda^2)d^2 + 2(1-\lambda)\lambda\beta$ has a minimum of either $\dfrac{d^2+\beta}{2}$ or d^2. Thus,

$$\left|(1-\lambda)\alpha + \lambda\left(\int_0^1 \Psi^T(t)f(t,S(t)\alpha)dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\right)\right| \geq \min\left\{\frac{\sqrt{d^2+\beta}}{\sqrt{2}}, d\right\}.$$

Using the fact that

$$|\alpha| \geq \frac{R + \|M_pE\|\, b}{\eta},$$

we get

$$\inf_{t\in[0,1]} |S(t)\alpha| \geq \eta\left(\frac{R}{\eta}\right) = R,$$

and

$$\inf_{t\in[0,1]} |S(t)\alpha + x(t)| \geq \eta\left(\frac{R + \|M_pE\|\, b}{\eta}\right) - \|M_pE\|\, b = R.$$

It follows that

$$\left|(1-\lambda)\alpha+\lambda\left(\int_0^1 \Psi^T(t)f(t,S(t)\alpha+x(t))dt+\sum_{i=1}^{k}\Psi^T(t_i)J_i(S(t_i)\alpha+x(t_i^-))\right)\right|$$

$$\begin{aligned}
&\geq \min\left\{\frac{\sqrt{d^2+\beta}}{\sqrt{2}},d\right\}-\left\|\Psi^T(\cdot)\right\| L(R)(k+1)\left\|x\right\| \\
&\geq \min\left\{\frac{\sqrt{d^2+\beta}}{\sqrt{2}},d\right\}-L(R)(k+1)\left\|M_pE\right\| b \\
&> 0.
\end{aligned}$$

□

Remark 2.2.8. Theorem 2.2.7 is a considerable extension of the ideas appearing in [25, 33, 34] in many ways. First, it allows for continuous systems with impulses. Most importantly, it places no restriction on the dimension of the solution space of the linear homogeneous problem (2.4) with boundary conditions (2.3).

2.3 Examples

The following examples illustrate ways in which the hypothesis of the main result can be satisfied.

In our first example we analyze the solvability of

$$x'(t)=f(x(t)),\quad t\in[0,1]\setminus\left\{\frac{1}{4}\right\}$$

$$x\left(\frac{1}{4}^+\right)-x\left(\frac{1}{4}^-\right)=J\left(x\left(\frac{1}{4}^-\right)\right)$$

subject to

$$Bx(0) + Dx(1),$$

where

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $A = 0$, it follows that $\Phi(t) = I$ for all $t \in [0, 1]$, and therefore

$$B + D\Phi(1) = B + D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We choose

$$W^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } S(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It follows that

$$\Psi^T(t) = W^T.$$

We now take

$$f(x_1,x_2,x_3) = \begin{bmatrix} \dfrac{x_2+\sin(x_2+x_3)}{1+\sqrt{x_1^2+x_2^2+x_3^2}} \\ \dfrac{x_3}{1+\sqrt{x_1^2+x_2^2+x_3^2}} \\ f_3(x_1,x_2,x_3) \end{bmatrix}$$

and

$$J(x_1,x_2,x_3) = \begin{bmatrix} \dfrac{-x_3}{1+\sqrt{x_1^2+x_2^2+x_3^2}} \\ \dfrac{\cos(x_1+x_2)+x_2}{1+\sqrt{x_1^2+x_2^2+x_3^2}} \\ J_3(x_1,x_2,x_3) \end{bmatrix},$$

where f_3 and J_3 are bounded continuous functions. We then have

$$\int_0^1 \Psi^T(t) f(t, S(t)\alpha)dt + \sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha) = \begin{bmatrix} \dfrac{\alpha_1-\alpha_2+\sin(\alpha_1+\alpha_2)}{1+\sqrt{\alpha_1^2+\alpha_2^2}} \\ \dfrac{\alpha_1+\alpha_2+\cos(\alpha_1)}{1+\sqrt{\alpha_1^2+\alpha_2^2}} \end{bmatrix}.$$

Now,

$$\left| \begin{bmatrix} \dfrac{\alpha_1-\alpha_2+\sin(\alpha_1+\alpha_2)}{1+\sqrt{\alpha_1^2+\alpha_2^2}} \\ \dfrac{\alpha_1+\alpha_2+\cos(\alpha_1)}{1+\sqrt{\alpha_1^2+\alpha_2^2}} \end{bmatrix} \right|^2$$

$$= \frac{2|\alpha|^2 + 2(\alpha_1-\alpha_2)\sin(\alpha_1+\alpha_2) + 2(\alpha_1+\alpha_2)\cos(\alpha_1)}{(1+|\alpha|)^2}$$

$$+ \frac{\sin^2(\alpha_1+\alpha_2) + \cos^2(\alpha_1)}{(1+|\alpha|)^2}$$

and

$$\left\langle \begin{bmatrix} \alpha_1 \\ \\ \alpha_2 \end{bmatrix}, \begin{bmatrix} \dfrac{\alpha_1 - \alpha_2 + \sin(\alpha_1 + \alpha_2)}{1 + \sqrt{\alpha_1^2 + \alpha_2^2}} \\ \\ \dfrac{\alpha_1 + \alpha_2 + \cos(\alpha_1)}{1 + \sqrt{\alpha_1^2 + \alpha_2^2}} \end{bmatrix} \right\rangle = \frac{|\alpha|^2 + \alpha_1 \sin(\alpha_1 + \alpha_2) + \alpha_2 \cos(\alpha_1)}{1 + |\alpha|}.$$

Thus, we may choose a real number R such that for each $\alpha \in \mathbb{R}^p$ with $|\alpha| \geq R$,

$$\left\| \begin{bmatrix} \dfrac{\alpha_1 - \alpha_2 + \sin(\alpha_1 + \alpha_2)}{1 + \sqrt{\alpha_1^2 + \alpha_2^2}} \\ \\ \dfrac{\alpha_1 + \alpha_2 + \cos(\alpha_1)}{1 + \sqrt{\alpha_1^2 + \alpha_2^2}} \end{bmatrix} \right\| > 1$$

and

$$\left\langle \begin{bmatrix} \alpha_1 \\ \\ \alpha_2 \end{bmatrix}, \begin{bmatrix} \dfrac{\alpha_1 - \alpha_2 + \sin(\alpha_1 + \alpha_2)}{1 + \sqrt{\alpha_1^2 + \alpha_2^2}} \\ \\ \dfrac{\alpha_1 + \alpha_2 + \cos(\alpha_1)}{1 + \sqrt{\alpha_1^2 + \alpha_2^2}} \end{bmatrix} \right\rangle > 0.$$

We now assume that, for $i = 1, 2, 3$, $\dfrac{\partial f_3}{\partial x_i}$ and $\dfrac{\partial J_3}{\partial x_i}$ exist and that

$$\lim_{r \to \infty} \sup_{|x| > r} \frac{\partial f_3}{\partial x_i}(x) < \infty$$

and

$$\lim_{r \to \infty} \sup_{|x| > r} \frac{\partial J_3}{\partial x_i}(x) < \infty.$$

An easy calculation shows

$$Df(y_1, y_2, y_3) =$$

$$
c(y)\begin{bmatrix} -y_1y_2 - y_1\sin(y_2+y_3) & -y_1y_3 & \dfrac{\partial f_3}{\partial x_1}(y_1,y_2,y_3) \\ \\ d(y)(1+\cos(y_2+y_3)) - y_2^2 - y_2\sin(y_2+y_3) & -y_2y_3 & \dfrac{\partial f_3}{\partial x_2}(y_1,y_2,y_3) \\ \\ d(y)(\cos(y_2+y_3)) - y_2y_3 - y_3\sin(y_2+y_3) & d(y)-y_3^2 & \dfrac{\partial f_3}{\partial x_3}(y_1,y_2,y_3) \end{bmatrix}^T,
$$

where $c(y) = \dfrac{1}{|y|(1+|y|)^2}$ and $d(y) = |y|(1+|y|)$.

It is then clear that

$$L_0^*(r) := \sup_{|x|>r} \|Df(x)\|$$

satisfies $\lim_{r\to\infty} L_0^*(r) = 0$. A simialr calculation shows the same is true for

$$L_i^*(r) := \sup_{|x|>r} \|DJ_i(x)\|.$$

An application of the integral mean value theorem then shows that $C3$. is satisfied. Thus, by theorem 2.2.7, the nonlinear boundary value problem has a solution.

Remark 2.3.1. We have chosen the matrix A to be 0 in order to convey the essential ideas of theorem 2.2.7; that is, the relationship between the behavior of the nonlinearities and the solution space of the associated linear homogeneous boundary value problem. It should be clear that a similar analysis can be carried out when the matrix A is nonzero.

For our second example we focus on the solvability of

$$x'(t) = A(t)x(t) + f(t, x(t)), \quad t \in [0,1] \setminus \{t_1, t_2, \cdots, t_k\}$$

$$x(t_i^+) - x(t_i^-) = J_i(x(t_i^-)), \quad i = 1, \cdots, k$$

subject to

$$Bx(0) + Dx(1) = 0,$$

when, for large α, $\sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha)$ is bounded away from 0. That is, we assume that there exists positive real numbers R_1 and d, such that for all $\alpha \in \mathbb{R}^p$ with $|\alpha| > R_1$,

$$\left| \sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha) \right| > d.$$

If we assume the following:

1. There exists a real number R_2 such that for all $\alpha \in \mathbb{R}^p$ with $|\alpha| > R_2$,

$$\left\langle \alpha, \int_0^1 \Psi^T(t) f(t, S(t)\alpha) dt + \sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha) \right\rangle \geq 0.$$

2. $\lim_{r \to \infty} L(r) = 0,$

then theorem 2.2.7 guarantees that the nonlinear boundary value problem has a solution provided, for large α,

$$\left| \int_0^1 \Psi^T(t) f(t, S(t)\alpha) dt \right| < d.$$

We would like to point out the relative simplicity of computing

$$\left| \sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha) \right|.$$

Chapter 3

Weakly nonlinear boundary value problems with impulses

In the following chapter we will be analyzing weakly nonlinear, impulsive, boundary value problems of the form

$$x'(t) = A(t)x(t) + g(t) + \varepsilon f(t, x(t)), \quad t \in [0,1] \setminus \{t_1, t_2, \cdots, t_k\} \tag{3.1}$$

$$x(t_i^+) - x(t_i^-) = w_i, \quad i = 1, ..., k \tag{3.2}$$

subject to boundary conditions

$$Bx(0) + Dx(1) = 0. \tag{3.3}$$

The points $t_i, i = 1, \cdots, k$, are fixed with $0 < t_1 < t_2 < \cdots < t_k < 1$. We assume f and A are continuous, $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ and for each t, $A(t)$ is a $n \times n$ matrix. The $w_i, i = 1, \cdots, k$, are elements of $\mathbb{R}^n$, B and D are $n \times n$ matrices, and in order to have a linearly independent system of boundary conditions, we assume the augmented matrix $[B|D]$ has full row rank. ε is a "small" real parameter, $x(t_i^+)$ and $x(t_i^-)$ denote the right and left-hand limits for x at the points t_i respectively, and the function g belongs to $PC_{\{t_i\}}[0,1]$ which will be defined below.

We present a qualitative analysis of the dependence of solutions on the "small" parameter ε. This analysis allows us to establish a connection between the nonlinear boundary value problem and the associated linear homogeneous boundary value problem

$$x'(t) = A(t)x(t), \tag{3.4}$$

subject to boundary conditions (3.3), as well as the linear nonhomogeneous problem

$$x'(t) = A(t)x(t) + h(t), \quad t \in [0,1] \setminus \{t_1, t_2, \cdots, t_k\} \tag{3.5}$$

$$x(t_i^+) - x(t_i^-) = v_i, \quad i = 1, ..., k$$

with the same boundary conditions.

Our emphasis will be on the resonant case; that is, the case in which the linear homogeneous problem (3.4) subject to the boundary conditions (3.3) has a nontrivial solution space. In this case we analyze (3.1)-(3.3) using a projection scheme, often referred to as the Lyapunov-Schmidt procedure, in combination with the implicit function theorem. Our work is self-contained, but for those readers interested in seeing further applications of Lyapunov-Schmidt reduction and its generalizations, we suggest [8, 17, 33, 37].

For completeness, we include an analysis of the nonresonant case. In this case we establish the existence of solutions by direct applications of the implicit function theorem and the contraction mapping theorem.

3.1 Preliminaries

$PC_{\{t_i\}}[0,1]$ will represent the set of $\mathbb{R}^n$-valued continuous functions on $[0,1]\setminus\{t_1,\cdots,t_k\}$ which have right and left-hand limits at each t_i, $i=1,\cdots,k$. The norm used on $PC_{\{t_i\}}[0,1]$ will be the supremum norm; that is,

$$\|\phi\| = \sup_{t\in[0,1]\setminus\{t_1,\cdots,t_k\}} |\phi(t)|,$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^n$. With this norm, $PC_{\{t_i\}}[0,1]$ becomes a Banach space. The subset of $PC_{\{t_i\}}[0,1]$ consisting of those functions, ϕ, for which $\phi' \in PC_{\{t_i\}}[0,1]$ will be denoted by $PC^1_{\{t_i\}}[0,1]$. Finally, we define

$$X = \{\phi \in PC_{\{t_i\}}[0,1] \mid B\phi(0) + D\phi(1) = 0\}.$$

The supremum norm will be used on both $PC^1_{\{t_i\}}[0,1]$ and X.

We wish to formulate the nonlinear boundary value problem as an operator problem. To do so we define the following operators.

The operator $\mathcal{L}\colon \mathrm{dom}(\mathcal{L}) \subset X \to PC_{\{t_i\}}[0,1]\times\mathbb{R}^{nk}$ is defined by

$$\mathcal{L}x = \begin{bmatrix} x'(\cdot) - A(\cdot)x(\cdot) \\ x(t_1^+) - x(t_1^-) \\ \vdots \\ x(t_k^+) - x(t_k^-) \end{bmatrix},$$

where

$$dom(\mathcal{L}) = PC^1_{\{t_i\}}[0,1] \cap X.$$

The nonlinear operator $\mathcal{F}\colon PC_{\{t_i\}}[0,1] \to PC_{\{t_i\}}[0,1]\times\mathbb{R}^{nk}$ is defined by

$$\mathcal{F}(x) = \begin{bmatrix} f(\cdot, x(\cdot)) \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We use the max norm on $PC_{\{t_i\}}[0,1] \times \mathbb{R}^{nk}$; that is,

$$\left\| \begin{bmatrix} h(\cdot) \\ v_1 \\ \vdots \\ v_k \end{bmatrix} \right\| = \max\{\|h\|, |v_1|, |v_2|, \cdots, |v_k|\}.$$

Remark 3.1.1. Let

$$w = \begin{bmatrix} g \\ w_1 \\ \vdots \\ w_k \end{bmatrix},$$

then solving the nonlinear boundary value problem (3.1)-(3.3) is equivalent to solving $\mathcal{L}x = \varepsilon \mathcal{F}x + w$.

To establish a connection between the nonlinear boundary value problem and the associated linear homogeneous and nonhomogeneous boundary problems, we now characterize the $Im(\mathcal{L})$. We choose $\{c_1, \cdots, c_p\}$ as a basis for $Ker\left((B + D\Phi(1))^T\right)$, and define the following:

$$W = [c_1, ..., c_p]$$

and

$$\Psi(t)^T = W^T D\Phi(1)\Phi^{-1}(t).$$

Proposition 3.1.2. $\mathcal{L}x = \begin{bmatrix} h \\ v \end{bmatrix}$ *if and only if* $\int_0^1 \Psi^T(s)h(s)ds + \sum_{i=1}^{k} \Psi^T(t_i)v_i = 0.$

Proof. It is well documented, see [13, 36], that $\mathcal{L}x = \begin{bmatrix} h \\ v \end{bmatrix}$ if and only if x is given by the variation of parameters formula, and satisfies the boundary conditions (3.3); that is, if and only if

$$x(t) = \Phi(t)\left(x(0) + \int_0^t \Phi^{-1}(s)h(s)ds + \sum_{t_i<t} \Phi^{-1}(t_i)v_i\right)$$

and x satisfies (3.3).

Imposing the boundary conditions, we have

$\begin{bmatrix} h \\ v \end{bmatrix} \in Im(\mathcal{L})$ if and only if there exists $w \in \mathbb{R}^n$ such that

$$\begin{aligned} &Bw + D\left(\Phi(1)(w + \int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i\right) = 0 \\ &\iff [B + D\Phi(1)]w = -D\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i\right) \\ &\iff D\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i\right) \in Im\left(B + D\Phi(1)\right). \end{aligned}$$

The result now follows from the fact that $Ker\left((B + D\Phi(1))^{\mathrm{T}}\right)$ is the orthogonal complement of $Im(B + D\Phi(1))$. □

Corollary 3.1.3. *The linear nonhomogeneous boundary value problem (3.5) with boundary conditions (3.3) has a unique solution if and only if $B + D\Phi(1)$ is invertible.*

Corollary 3.1.4. *The solution space of the linear homogeneous problem (3.4) subject to the boundary conditions (3.3) has the same dimension as the*
$Ker(B + D\Phi(1))$.

Choosing vectors $b_1, \cdots, b_p$, where $p \leq n$, from $\mathbb{R}^n$ which form a basis for $Ker(B + D\Phi(1))$ we make the following definition:

Definition 3.1.5. We define $S(t)$ to be the $n \times p$ matrix whose ith column is $S_i(t) := \Phi(t)b_i$.

It is now easily seen that a function x is a solution to linear homogeneous problem (3.4) subject to the boundary conditions (3.3) if and only if $x(\cdot) = S(\cdot)\alpha$ for some $\alpha \in \mathbb{R}^p$.

In order to use arguments involving the implicit function theorem, we now establish the continuous differentiability of $\mathcal{F}$ under appropriate conditions on f. For those readers interested in calculus in Banach spaces, we suggest [9, 14].

Proposition 3.1.6. *Suppose f has a continuous partial derivative with respect to x, then $\mathcal{F}$ is continuously differentiable. Further,*

$$D\mathcal{F}(x_0)h = \begin{bmatrix} \frac{\partial f}{\partial x}(\cdot, x_0(\cdot))h(\cdot) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Proof. Let $R\colon PC_{\{t_i\}}[0,1] \to PC_{\{t_i\}}[0,1]$ be defined by

$$[R(x)](t) = f(t, x(t)).$$

For $x_0 \in PC_{\{t_i\}}[0,1]$ we define

$$K(x_0) : PC_{\{t_i\}}[0,1] \to PC_{\{t_i\}}[0,1]$$

by

$$[K(x_0)h](t) = \frac{\partial f}{\partial x}(t, x_0(t))h(t).$$

It follows that

$$\begin{aligned}
&\|R(x_0+h) - R(x_0) - K(x_0)h\| \\
&\quad = \sup_{t\in[0,1]\setminus\{t_1,\cdots,t_k\}} \left| f(t,x_0(t)+h(t)) - f(t,x_0(t)) - \frac{\partial f}{\partial x}(t,x_0(t))h(t) \right| \\
&\quad \leq \sup_{t\in[0,1]\setminus\{t_1,\cdots,t_k\}} \sup_{w\in[x_0(t),x_0(t)+h(t)]} \left\| \frac{\partial f}{\partial x}(t,w) - \frac{\partial f}{\partial x}(t,x_0(t)) \right\| |h(t)|,
\end{aligned}$$

where, for s, $r \in \mathbb{R}^n$, $[s,r]$ denotes the line segment between s and r. It follows that R is differentiable.

To see the continuity of DR notice that

$$\begin{aligned}
&\|DR(x) - DR(x_0)\| \\
&\quad = \sup_{\|h\|=1} \|DR(x)h - DR(x_0)h\| \\
&\quad = \sup_{\|h\|=1} \sup_{t\in[0,1]\setminus\{t_1,\cdots,t_k\}} \left| \frac{\partial f}{\partial x}(t,x(t))h(t) - \frac{\partial f}{\partial x}(t,x_0(t))h(t) \right| \\
&\quad \leq \sup_{t\in[0,1]\setminus\{t_1,\cdots,t_k\}} \left| \frac{\partial f}{\partial x}(t,x(t)) - \frac{\partial f}{\partial x}(t,x_0(t)) \right|.
\end{aligned}$$

The continuity of DR is thus a consequence of the continuity of $\dfrac{\partial f}{\partial x}$.

□

3.2 Main Results

We now come to our first result regarding the nonlinear boundary value problem (3.1)-(3.3). For the moment we focus our attention to when $B + D\Phi(1)$ is invertible and prove the existence of solutions in two cases. In the first case, we assume that f has a continuous partial derivative with respect to x and prove the existence of solutions using an implicit function argument. In the second case, we assume f is Lipschitz with respect to x and obtain the existence of solutions using an application of the contraction mapping theorem.

Theorem 3.2.1. *Suppose that the only solution to (3.4) subject to the boundary conditions (3.3) is the trivial solution. If f has a continuous partial derivative with respect to x, then for each "small" ε there is a solution to the nonlinear boundary value problem (3.1)-(3.3).*

Proof. We have seen that $\mathcal{L}$ invertible if and only if the only solution to (3.4) subject to the boundary conditions (3.3) is the trivial solution. With this in mind we define

$$H : \mathbb{R} \times X \to X$$

by

$$H(\varepsilon, x) = x - \varepsilon \mathcal{L}^{-1} \mathcal{F}(x) - \mathcal{L}^{-1} w.$$

It follows that x is a solution to (3.1)-(3.3) if and only if $H(\varepsilon, x) = 0$. Clearly, $H(0, \mathcal{L}^{-1}w) = 0$ and $\dfrac{\partial H}{\partial x}(0, \mathcal{L}^{-1}w) = I$, where I denotes the identity map on X. Therefore, by the implicit function theorem, (3.1)-(3.3) has a solution for each small ε. □

Theorem 3.2.2. *Suppose that the only solution to (3.4) subject to the boundary conditions (3.3) is the trivial solution. If f is Lipschitz with respect to x; that is, if for all $t \in [0,1]$ there exists an M such that $|f(t,x) - f(t,y)| \leq M|x - y|$, then for each $\varepsilon < \dfrac{1}{\|\mathcal{L}^{-1}\|\, M}$ there is a unique solution to the nonlinear boundary value problem (3.1)-(3.3).*

Proof. Define, for $\varepsilon < \dfrac{1}{\|\mathcal{L}^{-1}\|\, M}$,

$$H_\varepsilon : X \to X$$

by

$$H_\varepsilon(x) = \varepsilon \mathcal{L}^{-1} \mathcal{F}(x) + \mathcal{L}^{-1} w.$$

The solutions to (3.1)-(3.3) are the fixed points of $H_\varepsilon(x)$.

Using the fact that f is Lipschitz with respect to x, we get

$$\begin{aligned}
\|H_\varepsilon(x) - H_\varepsilon(y)\| &= \left\|\varepsilon\mathcal{L}^{-1}(\mathcal{F}(x) - \mathcal{F}(y))\right\| \\
&\leq \varepsilon \left\|\mathcal{L}^{-1}\right\| \|\mathcal{F}(x) - \mathcal{F}(y)\| \\
&\leq \varepsilon \left\|\mathcal{L}^{-1}\right\| M \|x - y\|.
\end{aligned}$$

Since $\varepsilon < \dfrac{1}{\|\mathcal{L}^{-1}\| M}$, we have that H_ε is a contraction. By the contraction mapping theorem, H_ε has a unique fixed point. □

We now turn our attention to the focus of this paper, the resonant case. So that we may analyze (3.1)-(3.3) using a Lyapunov-Schmidt reduction, we construct projections onto the $Ker(L)$ and $Im(L)$. The construction of the projections P and E that follow appears in [22]. We include the details for the readers convenience.

Let $V\colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto $Ker(B + D\Phi(1))$.

Definition 3.2.3. Define $P\colon X \to X$ by

$$[Px](t) = \Phi(t)Vx(0)$$

Proposition 3.2.4. *P is a projection onto $Ker(L)$.*

Proof. Combine corollary 3.1.4 with the fact that V is a projection. □

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto $Ker(W^T D\Phi(1))$. It follows, from Proposition 3.1.2, that

$$\begin{bmatrix} h \\ v \end{bmatrix} \in ImL \text{ iff } [I - T]\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i\right) = 0.$$

Definition 3.2.5. Define $E : PC_{\{t_i\}}[0,1] \times \mathbb{R}^{nk} \to PC_{\{t_i\}}[0,1] \times \mathbb{R}^{nk}$ by

$$E\begin{bmatrix} h \\ v \end{bmatrix} = \begin{bmatrix} h(\cdot) - \Phi(\cdot)[I-T]\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\right) \\ \\ v \end{bmatrix}$$

Proposition 3.2.6. *E is a projection onto $Im(L)$.*

Proof.

$$\begin{aligned}[I-T]\Bigg(\int_0^1 \Phi^{-1}(s)\Big[h(s) - \Phi(s)\,(I-T)\Big(\int_0^1 \Phi^{-1}(u)h(u)du \\ + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\Big)\Big] + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\Bigg)\end{aligned}$$

$$\begin{aligned}= [I-T]\left(\int_0^1 \Phi^{-1}(s)h(s) + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\right) - [I-T]^2\Bigg(\int_0^1 \Phi^{-1}(u)h(u)du \\ + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\Bigg) = 0.\end{aligned}$$

The assertion is now clear. □

For the sake of completeness we now give a self-contained description of the Lyapunov-Schmidt projection procedure. For further details, and a vast number of applications and generalizations of this method, the reader may consult [1, 3, 6, 11, 27, 30] and the references therein.

Proposition 3.2.7. *Solving $\mathcal{L}x = \varepsilon\mathcal{F} + w$ is equivalent to solving the system*

$$\begin{cases} \varepsilon(I-E)\mathcal{F}(x) + (I-E)w = 0 \\ \\ \qquad\qquad \textit{and} \\ \\ (I-P)x - \varepsilon M_p E\mathcal{F}(x) - M_p E w = 0 \end{cases},$$

where M_p *is* $\mathcal{L}^{-1}_{|(Ker(P)\cap dom(L)}$.

Proof. We have

$$\mathcal{L}x = \varepsilon\mathcal{F}x + w \iff \begin{cases} (I-E)(\mathcal{L}x - \varepsilon\mathcal{F}(x) - w) = 0 \\ \text{and} \\ E(\mathcal{L}x - \varepsilon\mathcal{F}(x) - w) = 0 \end{cases}$$

$$\iff \begin{cases} (I-E)(\varepsilon\mathcal{F}(x) + w) = 0 \\ \text{and} \\ \mathcal{L}x - E\left(\varepsilon\mathcal{F}(x) + w\right) = 0 \end{cases}$$

$$\iff \begin{cases} (I-E)(\varepsilon\mathcal{F}(x) + w) = 0 \\ \text{and} \\ M_p\mathcal{L}x - M_pE(\varepsilon\mathcal{F}(x) + w) = 0 \end{cases}$$

$$\iff \begin{cases} (I-E)(\varepsilon\mathcal{F}(x) + w) = 0 \\ \text{and} \\ (I-P)x - M_pE(\varepsilon\mathcal{F}(x) + w) = 0 \end{cases}.$$

□

The following proposition will play a significant role in the proof of the main result.

Proposition 3.2.8. *The operator* $\mathcal{L}$ *is a Fredholm mapping of Index 0.*

Proof. The $Ker(\mathcal{L})$ is finite dimensional. In fact, from corollary 3.1.4, we have $dim(Ker(\mathcal{L})) = dim(Ker\,(B + D\Phi(1))$.

Further, $Im(\mathcal{L})$ is closed since E is a continuous projection with $Im(E) = Im(\mathcal{L})$.

Finally,

$$\begin{aligned}
dim\left((PC_{\{t_i\}}[0,1]\times\mathbb{R}^{nk})/Im(\mathcal{L})\right) &= dim\,(Im(I-E))\\
&= dim\,(Im(I-T))\\
&= dim\left(Ker(W^T D\Phi(1))^{\perp}\right)\\
&= dim\left(Im(\Phi(1)^T D^T W)\right).
\end{aligned}$$

If $D^T W$ is 1-1, then

$$\begin{aligned}
dim\left(Im(\Phi(1)^T D^T W)\right) &= dim(Im(W))\\
&= dim(Ker(\mathcal{L})).
\end{aligned}$$

Thus, the result will follow once we show $D^T W$ is 1-1. To this end, suppose $D^T Wx = 0$ for some nonzero x. It would follow that $Wx \in Ker(D^T)$. Combining this with the fact that $Wx \in Ker((B + D\Phi(1))^T)$, we would conclude $Wx \in Ker(B^T)$ and thus $[B|D]$ would not have full row rank. The result now follows. □

We now come to our main result regarding the nonlinear boundary value problem (3.1)-(3.3). In what follows we assume that the linear nonhomogeneous problem

$$x'(t) = A(t)x(t) + g(t), \quad t \in [0,1] \setminus \{t_1, t_2, \cdots, t_k\}$$

$$x(t_i^+) - x(t_i^-) = w_i, \quad i = 1, ..., k$$

subject to the boundary conditions (3.3) has a solution. From Proposition 3.1.2, this happens if and only if

$$\int_0^1 \Psi^T(t)g(t)dt + \sum_{i=1}^{k} \Psi^T(t_i)w_i = 0. \tag{3.6}$$

We define

$H \colon \mathbb{R} \times X \to Im(I-E) \times Im(I-P)$

by

$$H(\varepsilon, x) = \left(\begin{array}{c} (I-E)\mathcal{F}(x) \\ \\ (I-P)x - \varepsilon M_p E(\mathcal{F}(x)) - M_p E w \end{array} \right).$$

Combining Proposition 3.2.7 and assumption (3.6), we have that for nonzero ε, solving (3.1)-(3.3) is equivalent to solving $H(\varepsilon, x) = 0$.

A characterization of M_p will be helpful in proving the main result. Let M_{BD} denote the right inverse of $B + D\Phi(1)$ when restricted to the orthogonal complement of $Ker(B + B\Phi(1))$. For $h \in PC_{\{t_i\}}[0,1]$ and $v \in \mathbb{R}^{nk}$, notice

$$VM_{BD}D\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i\right) = 0.$$

It now follows from the characterization of the $Im(L)$ that

$$\begin{aligned} M_p\left(\begin{bmatrix} h \\ v \end{bmatrix}\right)(t) =& \Phi(t)\left(-M_{BD}D\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\right)\right. \\ & \left. + \int_0^t \Phi^{-1}(s)h(s)ds + \sum_{t_i<t}\Phi^{-1}(t_i)v_i\right). \end{aligned}$$

Theorem 3.2.9. *Suppose f is continuously differentiable with respect to x. If there exsists an $\alpha \in \mathbb{R}^p$ with*

$$\int_0^1 \Psi^T(t) f\left(t, S(t)\alpha + M_p\left(w\right)(t)\right) dt = 0$$

and

$$\int_0^1 \Psi^T(t) \frac{\partial f}{\partial x}\left(u, S(t)\alpha + M_p\left(w\right)(t)\right) S(t) dt$$

invertible, then for each "small" ε there is a solution, x_ε, to the boundary value problem (3.1)-

(3.3). Further, $\lim_{\varepsilon \to 0} \|x_\varepsilon - \hat{x}\| = 0$*, where*

$$\hat{x}(\cdot) = S(t)\alpha + M_p\left(w\right)(t).$$

Proof. Define

$$\hat{x}(\cdot) = S(t)\alpha + M_p\left(w\right)(t)$$

as above. Combining (3.6) with the fact that

$$\int_0^1 \Psi^T(t) f\left(t, S(t)\alpha + M_p\left(w\right)(t)\right) du = 0,$$

we conclude $H(0, \hat{x}) = 0$.

From the definition of H it follows that

$DH(\varepsilon, x)(\alpha, w) =$

$$\begin{pmatrix} (I-E)D\mathcal{F}(x)(w) \\ \\ (I-P)w - \varepsilon M_p ED\mathcal{F}(x)(w) - \alpha M_p E\mathcal{F}(x) \end{pmatrix}.$$

Thus,

$$\frac{\partial H}{\partial x}(0, \hat{x})(h) = \begin{pmatrix} (I-E)D\mathcal{F}(\hat{x})(h) \\ \\ (I-P)h \end{pmatrix}.$$

If $\dfrac{\partial H}{\partial x}(0, \hat{x})(h) = 0$, it follows that

$$(I-P)h = 0$$

and

$$(I-E)D\mathcal{F}(\hat{x})(Ph) = 0.$$

Using the fact that Ph is in the $Ker(\mathcal{L})$, there exists an α^* in $\mathbb{R}^p$ such that $Ph = S(\cdot)\alpha^*$, thus

$$(I - E)D\mathcal{F}(\hat{x})(S(\cdot)\alpha^*) = 0.$$

From Proposition 3.1.6 , this happens if and only if

$$\int_0^1 \Psi^T(t)\frac{\partial f}{\partial x}(t, \hat{x}(t))S(t)\alpha^* dt = 0.$$

Since

$$\int_0^1 \Psi^T(t)\frac{\partial f}{\partial x}(t, \hat{x}(t))S(t)dt$$

is invertible, it follows that $\alpha^* = 0$ and therefore $Ph = 0$. We therefore conclude, $\frac{\partial H}{\partial x}(0, \hat{x})(\cdot)$ is 1-1.

We now show that $\frac{\partial H}{\partial x}(0, \hat{x})(\cdot)$ is a bijection. From Proposition 3.2.8, and the fact that

$$\int_0^1 \Psi^T(u)\frac{\partial f}{\partial x}(u,\hat{x}(u))S(u)du$$

is invertible, it follows that $(I - E)D\mathcal{F}(\hat{x})_{|_{Im(P)}}$ is a bijection. Let $\begin{bmatrix} p \\ q \end{bmatrix} \in Im(I - E) \times Im(I - P)$. We then have that there is an $r \in Im(P)$ with

$$(I - E)D\mathcal{F}(\hat{x})(r) = p - (I - E)D\mathcal{F}(\hat{x})q.$$

If we define $h = r + q$, then it follows that

$$\begin{aligned}
\frac{\partial H}{\partial x}(0,\hat{x})(h) &= \begin{bmatrix} (I-E)D\mathcal{F}(\hat{x})(r+q) \\ (I-P)(r+q) \end{bmatrix} \\
&= \begin{bmatrix} (I-E)D\mathcal{F}(\hat{x})(r) + (I-E)D\mathcal{F}(\hat{x})(q) \\ (I-P)(r) + (I-P)(q) \end{bmatrix} \\
&= \begin{bmatrix} p - (I-E)D\mathcal{F}(\hat{x})(q) + (I-E)D\mathcal{F}(\hat{x})(q) \\ q \end{bmatrix} \\
&= \begin{bmatrix} p \\ q \end{bmatrix}.
\end{aligned}$$

Thus, $\dfrac{\partial H}{\partial x}(0,\hat{x})(\cdot)$ is a bijection. The result now follows from the implicit function theorem. $\square$

Remark 3.2.10. When applied to the case of impulsive systems subject to periodic boundary conditions, Theorem 3.2.9 represents a natural extension of results obtained by Lewis, [16], for classical ordinary differential equations. For more details on this topic the reader is referred to [11, 16]

3.3 Example

In this example we illustrate the use of Theorem 3.2.9. We analyze the solvability of

$$x'(t) = Ax(t) + \varepsilon f(t, x(t)),\ t \neq \frac{1}{2}$$

$$x\left(\frac{1}{2}^+\right) - x\left(\frac{1}{2}^-\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

subject to

$$Bx(0) + Dx(1) = 0.$$

Here

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & e \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It follows that

$$\Phi(t) = e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \text{ and } B + D\Phi(1) = \begin{bmatrix} 1 & e \\ 1 & e \end{bmatrix}.$$

Choosing a basis we get

$$W^T = [1, -1], S(t) = \begin{bmatrix} e^t(e-t) \\ -e^t \end{bmatrix}, \Psi^T(t) = [0, e^{1-t}]$$

and

$$\Psi^T\left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [0, e^{\frac{1}{2}}]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$$

Thus, (3.6) is satisfied. Further, $M_{BD}D\Phi(1)\sum_{i=1}^{k}\Phi^{-1}(t_i)v_i = 0$.

For $\alpha \in \mathbb{R}^p$ define

$$x_\alpha(t) = S(t)\alpha + \Phi(t)\sum_{t_i<t}\Phi^{-1}(t_i)v_i$$

$$= \begin{cases} e^t\begin{pmatrix} \alpha(e-t) \\ \\ -\alpha \end{pmatrix} & \text{for} \quad 0 \le t < 1/2, \\ \\ e^t\begin{pmatrix} \alpha(e-t) + \dfrac{1}{\sqrt{e}} \\ \\ -\alpha \end{pmatrix} & \text{for} \quad 1/2 < t \le 1 \end{cases}.$$

Define

$\phi : \mathbb{R} \to \mathbb{R}$

by

$\phi(\alpha) = \int_0^1 \Psi^T(t) f(t, x_\alpha(t))dt.$

Since f has a continuous partial dervivative with respect to x, an easy calculation shows that ϕ is differentiable with

$$\phi'(\alpha) = \int_0^1 \Psi^T(t)\frac{\partial f}{\partial x}(t, x_\alpha(t))S(t)dt.$$

Thus, if there exists an α_0 such that $\phi(\alpha_0) = 0$ and $\phi'(\alpha_0) \neq 0$, then by Theorem 3.2.9 the nonlinear boundary value problem has a solution for "small" ε.

For a specific example, take

$f(t, x_1, x_2) = \begin{bmatrix} f_1(t, x_1, x_2) \\ t + x_1 - {x_2}^3 \end{bmatrix}.$

It follows that

$$\begin{aligned}
\int_0^1 \Psi^T(t) \begin{bmatrix} f_1(t, x_\alpha(t)) \\ f_2(t, x_\alpha(t)) \end{bmatrix} dt &= \int_0^1 [0, e^{1-t}] \begin{bmatrix} f_1(t, x_\alpha(t)) \\ f_2(t, x_\alpha(t)) \end{bmatrix} dt \\
&= \int_0^1 e^{1-t} f_2(t, x_\alpha(t)) dt \\
&= c_1 \alpha^3 + c_2 \alpha + c_3,
\end{aligned}$$

where $c_1 = \displaystyle\int_0^1 e^{2t+1} dt$, $c_2 = \dfrac{e^2 - e}{2}$ and $c_3 = \displaystyle\int_0^1 te^{1-t} dt + \frac{1}{2} e^{\frac{1}{2}}$.

Thus, there exists a $\alpha_0 \in \mathbb{R}$ such that $\phi(\alpha_0) = c_1 {\alpha_0}^3 + c_2 \alpha_0 + c_3 = 0$ and $\phi'(\alpha_0) = 3c_1 {\alpha_0}^2 + c_2 > 0$.

Chapter 4

On the solvability of multipoint boundary value problems for discrete systems at resonance

In this paper we analyze nonlinear, discrete, multipoint boundary value problems of the form

$$x(t+1) = A(t)x(t) + f(t, x(t)), \tag{4.1}$$

subject to

$$\sum_{i=0}^{m} B_i x(i) = 0. \tag{4.2}$$

Throughout our discussion, we assume that for each $t \in \mathbb{N} = \{0, 1, 2, \cdots\}$, $A(t)$ is an $n \times n$ invertible matrix. We assume $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous, m is a fixed integer greater than two, each $B_i, i = 0, \cdots, m$, is $n \times n$ matrix and the augmented matrix $[B_1|B_2|\cdots|B_m]$ has full row rank.

The dimension of the solution space for linear homogeneous problem

$$x(t+1) = A(t)x(t), \tag{4.3}$$

subject to the boundary conditions (4.2), will play a critical role in our analysis. Our results depend intimately on the interaction between the nonlinearity f and the solution space of the linear homogeneous problem. We will primarily be concerned with the case of resonance; that is, the case when the solution space to (4.3), (4.2) is nontrivial. In particular, we will focus on the case in which the solution space has dimension greater than one. In this regard, our results constitute a significant generalization of the ideas in [25, 31], where the solution space is assumed to be less than two. Our approach uses a projection scheme, the Lyapunov-Schmidt procedure, in combination with topological degree theory. For readers interested in the solvability of discrete systems we suggest [7, 29, 27, 28] and the references therein. Those interested in the use of similar ideas in differential equations should consult [2, 17, 33, 38].

4.1 Preliminaries

We will formulate the nonlinear boundary value problem (4.1)-(4.2) as an operator problem. To do so we introduce appropriate spaces and operators. We define

$$X = \left\{ \phi : \{0, 1, 2, \cdots, m\} \to \mathbb{R}^n \mid \sum_{i=0}^{m} B_i \phi(i) = 0 \right\},$$

and

$$Z = \left\{ \phi : \{0, 1, 2, \cdots, m-1\} \to \mathbb{R}^n \right\}.$$

We use the sup norm on both X and Z; that is, for $\phi \in X$

$$\|\phi\| = \max_{t \in \{0,1,2,\cdots,m\}} |\phi(t)|,$$

and for $\psi \in Z$

$$\|\psi\| = \max_{t \in \{0,1,2,\cdots,m-1\}} |\psi(t)|.$$

Here $|\cdot|$ represents the standard Euclidean norm on $\mathbb{R}^n$. It is clear that X and Z are finite-dimensional Banach spaces under these norms. We define a linear operator

$\mathcal{L}\colon X \to Z$ by

$[\mathcal{L}x](t) = x(t+1) - A(t)x(t)$, and we introduce a nonlinear operator

$\mathcal{F} : X \to Z$

defined by

$[\mathcal{F}x](t) = f(t, x(t))$.

Remark 4.1.1. With the definitions above, it is clear that solving the nonlinear boundary value problem (4.1)-(4.2) is equivalent to solving $\mathcal{L}x = \mathcal{F}x$. It is equally clear that the solution space of the linear homogeneous problem (4.3), (4.2) is given by the $Ker(\mathcal{L})$.

Let

$$\Phi(t) = \begin{cases} I & \text{if } t = 0 \\ A(t-1)A(t-2)\cdots A(0) & \text{if } t = 1, 2, \cdots \end{cases}$$

It is well known, see [12], that Φ is the principal fundamental matrix solution to (4.3).

While analyzing the nonlinear boundary value problem (4.1)-(4.2), it will be useful to have a characterization of the $Im(\mathcal{L})$.

Proposition 4.1.2. *An element $h \in Z$ is contained in the $Im(\mathcal{L})$ if and only if*

$$B_1\Phi(1)\Phi^{-1}(1)h(0) + \cdots + B_m\Phi(m)\sum_{i=0}^{m-1}\Phi^{-1}(i+1)h(i) \in Ker\left(\left(\sum_{i=0}^{m} B_i\Phi(i)\right)^T\right)^{\perp}.$$

Proof. From the variation of parameters formula, see [12], we have

$\mathcal{L}x = h$ if and only if there exists an element $x \in X$ such that

$$x(t) = \Phi(t)x(0) + \Phi(t)\sum_{i=0}^{t-1}\Phi^{-1}(i+1)h(i).$$

Here $\sum_{i=0}^{-1}\Phi^{-1}(i+1)h(i)$ is taken to be 0.

Imposing the boundary conditions, we have $h \in Im(\mathcal{L})$ if and only if there exists $w \in \mathbb{R}^n$ such that

$$\sum_{i=0}^{m} B_i\Phi(i)w + \left(B_1\Phi(1)\Phi^{-1}(1)h(0) + \cdots + B_m\Phi(m)\sum_{i=0}^{m-1}\Phi^{-1}(i+1)h(i)\right) = 0,$$

which is clearly equivalent to

$$B_1\Phi(1)\Phi^{-1}(1)h(0) + \cdots + B_m\Phi(m)\sum_{i=0}^{m-1}\Phi^{-1}(i+1)h(i) \in Im\left(\sum_{i=0}^{m} B_i\Phi(i)\right).$$

Using the fact that $Im\left(\sum_{i=0}^{m} B_i\Phi(i)\right) = Ker\left(\left(\sum_{i=0}^{m} B_i\Phi(i)\right)^T\right)^{\perp}$, the result follows. $\square$

Remark 4.1.3. It follows from Proposition 4.1.2 that $\mathcal{L}$ is invertible if and only if $\sum_{i=0}^{m} B_i\Phi(i)$ is invertible. To see this note that if $\mathcal{L}$ is invertible, then from the proof of Proposition 4.1.2, $\sum_{i=0}^{m} B_i\Phi(i)$ is one-to-one. Since $\sum_{i=0}^{m} B_i\Phi(i)$ is an $n \times n$ matrix, $\sum_{i=0}^{m} B_i\Phi(i)$ is also onto. The invertibility of $\sum_{i=0}^{m} B_i\Phi(i)$ implying the invertibility of $\mathcal{L}$ follows since in this case we have that the unique solution to $\mathcal{L}x = h$ is given by

$$\begin{aligned} x(t) =& \Phi(t) \left(- \left[\sum_{i=0}^{m} B_i \Phi(i) \right]^{-1} \left(\sum_{j=1}^{m} \sum_{i=0}^{j-1} B_j \Phi(j) \Phi^{-1}(i+1) h(i) \right) \right. \\ & \left. + \sum_{i=0}^{t-1} \Phi^{-1}(i+1) h(i) \right). \end{aligned}$$

We now introduce some notation to simplify our characterization of the $Im(\mathcal{L})$. We let $c_1, c_2, \cdots, c_p$ denote a basis for the $Ker\left(\left(\sum_{i=0}^{m} B_i \Phi(i) \right)^T \right)$. If we define $W = [c_1, ..., c_p]$ and $\Psi^T : \{0, 1, 2, \cdots, m-1\} \to \mathbb{R}^n$ by

$$\Psi^T(t) = \sum_{i=t+1}^{m} W^T B_i \Phi(i) \Phi^{-1}(t+1),$$

then by simply rearranging $\sum_{j=1}^{m} \sum_{i=0}^{j-1} B_j \Phi(j) \Phi^{-1}(i+1) h(i)$, we get the following corollary:

Corollary 4.1.4. *An element $h \in Z$ is contained in the $Im(\mathcal{L})$ if and only if $\sum_{i=0}^{m-1} \Psi^T(i) h(i) = 0$.*

From Remark 4.1.3, we have that the linear homogeneous problem (4.3), (4.2) has a nontrivial solution space whenever $\sum_{i=0}^{m} B_i \Phi(i)$ is singular. It will be useful to have a description of the solution space in this case. From Remark 4.1.1, this is equivalent to finding a description of the $Ker(\mathcal{L})$.

Proposition 4.1.5. *The solution space of the linear homogenous problem (4.3), (4.2) and the $Ker\left(\sum_{i=0}^{m} B_i \Phi(i) \right)$ have the same dimension.*

Proof. Taking $h = 0$ in the variation of parameters formula, we have

$$\begin{aligned}\mathcal{L}x = 0 \quad &\Longleftrightarrow \quad x(t) = \Phi(t)x(0) \text{ for all } t \in \{0, 1, \cdots, m\} \text{ and } \sum_{i=0}^{m} B_i x(i) = 0 \\ &\Longleftrightarrow \quad \exists c \in \mathbb{R}^n \text{ such that } x(\cdot) = \Phi(\cdot)c \text{ and } \sum_{i=0}^{m} B_i \Phi(i) c = 0.\end{aligned}$$

It follows that the map $c \to \Phi(\cdot)c$ is an isomorphism from $Ker\left(\sum_{i=0}^{m} B_i \Phi(i)\right)$ to $Ker(\mathcal{L})$. □

To simplify notation, we choose vectors $b_1, \cdots, b_p$, where $p \leq n$, from $\mathbb{R}^n$ which form a basis for $Ker\left(\sum_{i=0}^{m} B_i \Phi(i)\right)$ and make the following definition:

Definition 4.1.6. We define $S(t)$ to be the $n \times p$ matrix whose ith column is $S_i(t) := \Phi(t)b_i$.

We now get the following characterization of the $Ker(\mathcal{L})$: a function $x \in Ker(\mathcal{L})$ if and only if $x(\cdot) = S(\cdot)\alpha$ for some $\alpha \in \mathbb{R}^p$.

Remark 4.1.7. For each $\alpha \neq 0$, $S(\cdot)\alpha$ is a nonzero solution to (4.3); it follows that

$$\min_{t=0,1,2,\cdots,m} |S(t)\alpha| > \ 0.$$

4.2 Main Results

We now turn our attention to analyzing the solvability of the nonlinear boundary value problem (4.1)-(4.2). Recall this problem has the following form:

$$x(t+1) = A(t)x(t) + f(t, x(t)),$$

subject to

$$\sum_{i=0}^{m} B_i x(i) = 0.$$

Our primary concern is the case of resonance and our principal result in this regard is Theorem 4.2.6. The results we obtain in Theorem 4.2.6 depend largely on the relationship between the nonlinearity f and the solution space of the linear homogenous problem (4.3),(4.2). Our approach will be topological, utilizing topological degree theory in conjunction with the Lyapunov-Schmidt procedure.

For the sake of completeness we include an analysis of the nonresonant case; this is the content of Theorem 4.2.1. The analysis here is simpler and will depend, for the most part, on the growth of the nonlinearity f. It should be noted that by placing fewer growth restrictions on the nonlinearity f, Theorem 4.2.1 is an extension of the results for the nonresonant case found in [31].

Theorem 4.2.1. *Suppose that the only solution to (4.3), (4.2) is the trivial solution. Suppose further that there exists a function* $g : \mathbb{R}^n \to \mathbb{R}^+$ *and a real number* M *such that for all* $t \in \{0, 1, 2, \cdots, m\}$ *and* $s \in \mathbb{R}^n$, $|f(t, s)| \le M|s| + g(s)$. *If* $g(s) \le g(w)$ *when* $|s| \le |w|$ *and* $\lim_{|s|\to\infty} \dfrac{g(s)}{|s|} = 0$, *then the nonlinear boundary value problem (4.1), (4.2) has a solution provided* M *is "sufficiently" small.*

Proof. Define $H : X \to X$ by

$$[H(x)](t) =\Phi(t) \left(-[\sum_{i=0}^{m} B_i\Phi(i)]^{-1} \left(\sum_{j=1}^{m}\sum_{i=0}^{j-1} B_j\Phi(j)\Phi^{-1}(i+1)f(i, x(i)) \right) + \sum_{i=0}^{t-1} \Phi^{-1}(i+1)f(i, x(i)) \right).$$

From Remark 4.1.3 we have that $H = \mathcal{L}^{-1}\mathcal{F}$ and thus the solutions to the nonlinear boundary value problem (4.1), (4.2) are precisely the fixed points of H.

Using the fact that for all $t \in \{0, 1, 2, \cdots, m\}$ and $s \in \mathbb{R}^n$

$$|f(t, s)| \le M|s| + g(s),$$

it follows that there exists real numbers B_1 and B_2 such that

$$\|H(x)\| \leq MB_1 \|x\| + B_2 g(x(\beta_x)).$$

Here β_x is any point with $x(\beta_x) = \|x\|$. If $MB_1 < 1$, we may choose r sufficiently large such that for all s with $|s| \leq r$,

$$B_2 g(s) < (1 - MB_1)r.$$

We define

$$\mathcal{B} = \{x \in X : \|x\| \leq r\}.$$

It is clear that $H(\mathcal{B}) \subset \mathcal{B}$. The existence of a solution is now a consequence of Brouwer's fixed point theorem. □

Remark 4.2.2. By taking $g : \mathbb{R}^n \to \mathbb{R}^+$ as $g(x) = M_1|x|^\alpha + M_2$, $0 \leq \alpha < 1$ and M_1 and $M_2 > 0$, we see that sublinear growth is special case of Theorem 4.2.1.

We now focus our attention on the resonant case. From Remark 4.1.3 we know this is equivalent to the matrix $\sum_{i=0}^{m} B_i\Phi(i)$ being singular. We will analyze (4.1), (4.2) using a projection scheme known as the Lyapunov-Schmidt procedure. To do so we need projections onto the $Ker(\mathcal{L})$ and $Im(\mathcal{L})$. In this regard, we choose to follow [32]. For those readers interested in learning more about the Lyapunov-Schmidt procedure and similar Alternative Methods, as well as their applications to differential and difference equations, we suggest [1, 4, 5, 6, 10, 11, 27, 28].

Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto $Ker\left(\sum_{i=0}^{m} B_i\Phi(i)\right)$.

Definition 4.2.3. Define $P : X \to X$ by

$$[Px](t) = \Phi(t)Vx(0).$$

Definition 4.2.4. Define $E : Z \to Z$ by

$$[Eh](t)] = h(t) - \Psi(t)\left(\sum_{j=0}^{m-1}\Psi^T(j)\Psi(j)\right)^{-1}\sum_{i=0}^{m-1}\Psi^T(i)h(i).$$

The proofs that E and P are projections, as well as many other properties of these maps may be found in [32].

The following proposition is the result of the Lyapunov-Schmidt reduction. For the interested reader, the proof of the result may be found in [33].

Proposition 4.2.5. *Solving $\mathcal{L}x = \mathcal{F}x$ is equivalent to solving the system*

$$\begin{cases} x = Px + M_p E\mathcal{F}x \\ \qquad and \\ (I - E)\mathcal{F}x = 0 \end{cases}$$

where M_p is $\mathcal{L}^{-1}_{|(Ker(P)\cap dom(\mathcal{L})}$.

We now come to our main result concerning the nonlinear boundary value problem (4.1)-(4.2). In the following it will be assumed that f is bounded, say by b. It will also be assumed that there exists a real number R_0 such that for all $r \geq R_0$ there exist a set, U_r, for which the following properties hold:

1. f is Lipschitz in x on U_{R_0}.

2. For all $t \in \{0, 1, \cdots, m\}$, for all $\alpha \in \mathbb{R}^p$ with $|\alpha| \geq r$, and for all $x \in \mathbb{R}^n$ with $|x| \leq \|M_pE\|\, b$, we have $S(t)\alpha + x \subset U_r$. Here $\|M_pE\|$ represents the operator norm of M_pE.

By intersecting if needed, we may assume the sets U_r are decreasing. With this in mind we let, for $r \geq R_0$, $L(r)$ denote the smallest Lipschitz constant for f on U_r. Note that $L(r)$ is decreasing in r, so that $\lim_{r\to\infty} L(r)$ exists.

The following observation will be used in what follows. Since the map taking $(t, \alpha) \to S(t)\alpha$ is a continuous mapping, it attains its minimum on the compact set

$$\mathcal{O} := \{0, \cdots, m\} \times \{\alpha \in \mathbb{R}^p : |\alpha| = 1\}.$$

From Remark 4.1.7, we have that $\eta = \inf\limits_{(t,\alpha) \in \mathcal{O}} |S(t)\alpha| > 0$.

Theorem 4.2.6. *Suppose the following conditions hold:*

C1. There exist real numbers R, $k > 0$, and γ such that for all $\alpha \in \mathbb{R}^p$ with $|\alpha| > R$,

$$\left| \sum_{i=0}^{m-1} \Psi^T(i) f(i, S(i)\alpha) \right| \geq k.$$

and

$$\left\langle \alpha, \sum_{i=0}^{m-1} \Psi^T(i) f(i, S(i)\alpha) \right\rangle \geq \gamma > -k^2.$$

C2. $\lim\limits_{r \to \infty} L(r) m \left\| M_p E \right\| \left\| \Psi^T(\cdot) \right\| b < \min \left\{ \dfrac{\sqrt{k^2 + \gamma}}{\sqrt{2}}, k \right\}$

Here $\left\| \Psi^T(\cdot) \right\|$ *denotes* $\sup\limits_{t \in \{0,1,\cdots,m-1\}} \left\| \Psi^T(t) \right\|$,

then there exists a solution to the nonlinear boundary value problem (4.1)-(4.2).

Proof. As above, we let b denote a bound for f. We may assume, without loss of generality, that $\|S(\cdot)\| \leq 1$.

We assume, by renaming if needed, that R is such that for all $\alpha \in \mathbb{R}^p$ and every $r > 0$ with $|\alpha| \geq R$ and $r \geq R$, the following hold:

1. $\left| \sum\limits_{i=0}^{m-1} \Psi^T(i) f(i, S(i)\alpha) \right| \geq k.$

2. $\left\langle \alpha, \sum_{i=0}^{m-1} \Psi^T(i) f(i, S(i)\alpha) \right\rangle \geq \gamma > -k^2$.

3. $L(r)m \|M_p E\| \left\|\Psi^T(\cdot)\right\| b < \min\left\{ \frac{\sqrt{k^2+\gamma}}{\sqrt{2}}, k \right\}$.

We define an operator $H : \mathbb{R}^p \times Im(I-P) \to \mathbb{R}^p \times Im(I-P)$ by

$$H(\alpha, x) = \begin{bmatrix} \displaystyle\sum_{j=0}^{m-1} \Psi^T(j) f(j, S(j)\alpha + x(j)) \\ \\ x - M_p EF(S(\cdot)\alpha + x) \end{bmatrix}.$$

From Proposition 4.2.5, we have the solutions to the nonlinear boundary value problem (4.1)-(4.2) are precisely the zeros of H. We will show that for a suitable choice of Ω, we have $deg(H, \Omega, 0) \neq 0$. To this end, choose

$$R^* > \max\left\{ m \left\|\Psi^T(\cdot)\right\| b, R \right\}$$

and define

$\Omega := B(0, R^*) \times B(0, \|M_p E\| b)$.

Further, define

$Q : [0,1] \times \overline{\Omega} \to \mathbb{R}^p \times Im(I-P)$

by

$$Q(\lambda, (\alpha, x)) = \begin{bmatrix} (1-\lambda)\alpha + \lambda \displaystyle\sum_{j=0}^{m-1} \Psi^T(j) f(j, S(j)\alpha + x(j)) \\ \\ x - \lambda M_p EF(S(\cdot)\alpha + x) \end{bmatrix}.$$

Since Q is clearly a homotopy between I and H, the proof will be complete provided we show

$0 \notin Q(\lambda, \partial(\Omega))$ for each $\lambda \in (0,1)$.

Now, it is clear that $(\alpha, x) \in \partial(\Omega)$ if and only if

$$|\alpha| = R^* \text{ and } \|x\| \leq \|M_p E\| \, b,$$

or

$$|\alpha| \leq R^* \text{ and } \|x\| = \|M_p E\| \, b.$$

If $(\alpha, x) \in \partial(\Omega)$ with $|\alpha| \leq R^*$ and $\|x\| = \|M_p E\| \, b$, then

$$\begin{aligned} \|x - \lambda M_p EF(S(\cdot)\alpha + x)\| &\geq \left| \|x\| - \lambda \|M_p EF(S(\cdot)\alpha + x)\| \right| \\ &\geq \|M_p E\| \, b - \lambda \|M_p E\| \, b > 0, \end{aligned}$$

and it follows that $Q(\lambda, (\alpha, x)) \neq 0$.

Now, if $(\alpha, x) \in \partial(\Omega)$ with $|\alpha| = R^*$ and $\|x\| \leq \|M_p E\| \, b$, then for all $t \in \{0, 1, \cdots, m\}$

$$S(t)\alpha + x(t) \in U_{R^*}.$$

Since $R^* > R$ and the sets U_r are decreasing in r, we have $S(t)\alpha + x(t) \in U_R$. Using the fact that f is Lipschitz on U_R, we have

$$\begin{aligned} \left| \sum_{j=0}^{m-1} \Psi^T(j) \Big[f(j, S(j)\alpha) - f(j, S(j)\alpha + x(j)) \Big] \right| &\leq L(R) m \left\| \Psi^T(\cdot) \right\| \|x\| \\ &\leq L(R) m \left\| \Psi^T(\cdot) \right\| \|M_p E\| \, b. \end{aligned}$$

Now, since $\left\langle \alpha, \displaystyle\sum_{i=0}^{m-1} \Psi^T(i) f(i, S(i)\alpha) \right\rangle \geq \gamma > -k^2$, we have, by writing the norm in terms of inner products,

$$\left|(1-\lambda)\alpha+\lambda\sum_{j=0}^{m-1}\Psi^T(j)f(j,S(j)\alpha+x(j))\right|^2 \geq$$

$$\left((1-\lambda)^2+\lambda^2\right)k^2+2(1-\lambda)\lambda\gamma.$$

A simple calculation shows that for $\lambda \in [0,1]$,

$$\left((1-\lambda)^2+\lambda^2\right)k^2+2(1-\lambda)\lambda\gamma \geq \min\left\{\frac{k^2+\gamma}{2}, k^2\right\}.$$

Using the fact that

$$\left|(1-\lambda)\alpha+\lambda\sum_{j=0}^{m-1}\Psi^T(j)f(j,S(j)\alpha+x(j))\right| \geq$$

$$\left|(1-\lambda)\alpha+\lambda\sum_{j=0}^{m-1}\Psi^T(j)f(j,S(j)\alpha)\right|$$

$$-\lambda\left|\sum_{j=0}^{m-1}\Psi^T(j)\Big[f(j,S(j)\alpha)-f(j,S(j)\alpha+x(j))\Big]\right|,$$

and that $L(R)m\left\|\Psi^T(\cdot)\right\|\|M_pE\|b < \min\left\{\frac{\sqrt{k^2+\gamma}}{\sqrt{2}}, k\right\}$, it follows that

$$\left|(1-\lambda)\alpha+\lambda\sum_{j=0}^{m-1}\Psi^T(j)f(j,S(j)\alpha+x(j))\right| > 0.$$

Thus, $Q(\lambda,(\alpha,x)) \neq 0$. The proof is now complete by the invariance of degree under homotopy.

$\square$

4.3 Example

Consider

$$x(t+1) = A(t)x(t) + f(t, x(t))$$

subject to

$$\sum_{i=0}^{m} B_i x(i) = 0,$$

where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}, B_m = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -2(m-1) \end{bmatrix},$$

and $B_i = 0$ for $i \neq 1, 2, m$.

Since A is constant, we have that $\Phi(t) = A^t = \begin{bmatrix} 1 & 0 & 2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

It follows that

$$\sum_{i=0}^{m} B_i \Phi(i) = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}.$$

Thus, the solution space to the linear homogenous problem has dimension 2.

We choose $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ as a basis for $Ker\left(\sum_{i=0}^{m} B_i\Phi(i)\right)$. It follows that

$$S(t) = \begin{bmatrix} 0 & 2(t-1) \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Define $M = \begin{bmatrix} (m-1) & 0 & -(m-1)m \\ 1 & 0 & -2 \end{bmatrix}$.

For $m > 2$, we have that $M|_{Ker(M)^\perp}$ is invertible. We denote the inverse simply by M^{-1}.

We now define

$$\begin{aligned} f(x_1, x_2, x_3) &= \frac{1}{(1 + x_2^2 + x_3^2)^{1.5}} M^{-1} \begin{bmatrix} cx_2^3 + \ln(1 + x_2^2 + x_3^2) \\ cx_3^3 + \tan^{-1}(x_2 + x_3) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ f_2(x_1, x_2, x_3) \\ 0 \end{bmatrix}, \end{aligned}$$

where f_2 is a bounded continuous function. Here c is a positive constant.

If we choose $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ as a basis for $Ker\left(\left(\sum_{i=0}^{m} B_i\Phi(i)\right)^T\right)$, then

$$\sum_{i=0}^{m-1} \Psi^T(i) f(i, S(i)\alpha) = \begin{bmatrix} (m-1) & 0 & -(m-1)m \\ 1 & 0 & -2 \end{bmatrix} f(0, \alpha_1, \alpha_2)$$

$$= Mf(0, \alpha_1, \alpha_2)$$

$$= \frac{1}{(1+|\alpha|^2)^{1.5}} \begin{bmatrix} c\alpha_1^3 + \ln(1+|\alpha|^2) \\ c\alpha_2^3 + \tan^{-1}(\alpha_1 + \alpha_2) \end{bmatrix}.$$

Now,

$$\left| \frac{1}{(1+|\alpha|^2)^{1.5}} \begin{bmatrix} c\alpha_1^3 + \ln(1+|\alpha|^2) \\ c\alpha_2^3 + \tan^{-1}(\alpha_1 + \alpha_2) \end{bmatrix} \right|^2$$

$$= \frac{1}{(1+|\alpha|^2)^3} \left(c^2\alpha_1^6 + c^2\alpha_2^6 + 2c\alpha_1^3 \ln(1+|\alpha|^2) + 2c\alpha_2^3 \tan^{-1}(\alpha_1+\alpha_2) \right.$$
$$\left. + \ln^2(1+|\alpha|^2) + (\tan^{-1}(\alpha_1+\alpha_2))^2 \right).$$

We also have that,

$$\left\langle \alpha, \Psi^T(i) f(i, S(i)\alpha) \right\rangle$$

$$= \frac{1}{(1+|\alpha|^2)^{1.5}} \left(c\alpha_1^4 + c\alpha_2^4 + \alpha_1 \ln(1+|\alpha|^2) + \alpha_2 \tan^{-1}(\alpha_1+\alpha_2) \right).$$

Taking $|\alpha|$ to be large, we see C2. is satisfied.

We now choose

$$U_r = \left\{ x \in \mathbb{R}^3 \mid \sqrt{x_2^2 + x_3^2} \geq r - d \right\},$$

where d is a fixed real number greater than $\|M_p E\|\, b$. It is clear that U_r contains $S(t)\alpha + x$ for all $\alpha \in \mathbb{R}^2$ with $|\alpha| \geq r$ and all $x \in \mathbb{R}^3$ with $|x| \leq \|M_p E\|\, b$. Now an easy calculation shows the following:

$$\frac{\partial f}{\partial x_1}(y_1, y_2, y_3) = \begin{bmatrix} 0 \\ \frac{\partial f_2}{\partial x_1} \\ 0 \end{bmatrix},$$

$$\frac{\partial f}{\partial x_2}(y_1, y_2, y_3) = K(y) \begin{bmatrix} 3y_2^2 d(y) + 2y_2 - 3y_2(cy_2^3 + \ln(1 + y_2^2 + y_3^2)) \\ \dfrac{d(y)}{(1 + (y_2 + y_3)^2)} - 3y_2({y_2}^3 + \tan^{-1}(y_2 + y_3)) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial f_2}{\partial x_2} \\ 0 \end{bmatrix},$$

$$\frac{\partial f}{\partial x_3}(y_1, y_2, y_3) = K(y) \begin{bmatrix} 2y_3 - 3y_3(cy_2^3 + \ln(1 + y_2^2 + y_3^2)) \\ \dfrac{d(y)}{(1 + (y_2 + y_3)^2)} - 3y_3({y_2}^3 + \tan^{-1}(y_2 + y_3)) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial f_2}{\partial x_3} \\ 0 \end{bmatrix},$$

where $K(y) = \dfrac{1}{(1 + y_2^2 + y_3^2)^{2.5}} M^{-1}$ and $d(y) = 1 + y_2^2 + y_3^2$.

If we assume, for $i = 1, 2, 3$, that $\lim_{r \to \infty} \sup_{|y| > r} \dfrac{\partial f_2}{\partial x_i}(y_1, y_2, y_3) = 0$, then clearly $\lim_{r \to \infty} \sup_{|y| > r} \|Df(y)\| = 0$. An application of the integral mean value theorem shows $\lim_{r \to \infty} L(r) = 0$. Therefore, by Theorem 4.2.6, the nonlinear boundary value problem has a solution.

Chapter 5

A least squares solution to linear boundary value problems with impulses

In the following we will be concerned with finding least squares solutions to

$$x'(t) = A(t)x(t) + h(t), \quad a.e. \, [0, 1] \tag{5.1}$$

$$x(t_i^+) - x(t_i^-) = v_i, \quad i = 1, ..., k \tag{5.2}$$

subject to

$$Bx(0) + Dx(1) = 0. \tag{5.3}$$

The points $t_i, i = 1, \cdots, k$, are fixed with $0 < t_1 < t_2 < \cdots < t_k < 1$. For each $t \in [0, 1]$, $A(t)$ is an $n \times n$ matrix. The components of $A(\cdot)$ are assumed to be in $L^2([0, 1], \mathbb{R})$ and the function h is assumed to be in $L^2([0, 1], \mathbb{R}^n)$. The v_i, $i = 1, \cdots, k$, are elements of $\mathbb{R}^n$, and B and D are $n \times n$ matrices with the augmented matrix $[B|D]$ having full row rank.

In our analysis we obtain a complete description for the least squares solution of minimal $L^2([0,1],\mathbb{R}^n)$ norm. Our analysis is intimately related to the idea of generalized inverses. For those readers interested in the method of least squares, as well as ideas regarding generalized inverses and generalized Green's functions as they apply to differential equations, we suggest [18, 19, 20, 21, 33, 35].

5.1 Preliminaries

The linear boundary value problem will be viewed as an operator equation. To formulate the problem, we introduce the following. $PAC_{\{t_i\}}[0,1]$ will represent the subset of $L^2([0,1],\mathbb{R}^n)$ consisting of functions which are absolutely continuous on every compact subinterval of $[0,1]\setminus\{t_1,\cdots,t_k\}$. We define

$$dom(\mathcal{L}) = \{\phi \in PAC_{\{t_i\}}[0,1] \mid \phi' \in L^2([0,1],\mathbb{R}^n) \text{ and } B\phi(0)+D\phi(1)=0\}.$$

We define an inner-product on $L^2([0,1],\mathbb{R}^n)\times\mathbb{R}^{nk}$ by

$$\left\langle \begin{bmatrix} h_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} h_2 \\ v_2 \end{bmatrix} \right\rangle = \int_0^1 {h_1}^T(s)h_2(s)ds + \sum_{i=1}^{k} {v_{1,i}}^T v_{2,i},$$

where for $j=1,2$,

$$v_j = \begin{bmatrix} v_{j,1} \\ \vdots \\ v_{j,k} \end{bmatrix}.$$

It is clear that $L^2([0,1],\mathbb{R}^n)\times\mathbb{R}^{nk}$ becomes a Hilbert space under the above inner-product.

We define an operator$\mathcal{L}: dom(\mathcal{L}) \to L^2([0,1],\mathbb{R}^n)\times\mathbb{R}^{nk}$ by

$$\mathcal{L}x = \begin{bmatrix} x'(\cdot) - A(\cdot)x(\cdot) \\ x(t_1^+) - x(t_1^-) \\ \vdots \\ x(t_k^+) - x(t_k^-) \end{bmatrix}.$$

Remark 5.1.1. It is clear, from the previous defintions, that finding a least squares solution to (5.1)-(5.3) is equivalent to finding a least squares solution to the operator equation $\mathcal{L}x = \begin{bmatrix} h \\ v \end{bmatrix}$.

To obtain descriptions of our least squares solutions, we will construct projections onto the $Ker(\mathcal{L})$ and $Im(\mathcal{L})$. To aid in the construction of these projections, we now completely characterize both the kernel and image of $\mathcal{L}$.

Proposition 5.1.2. *A function $x \in Ker(\mathcal{L})$ if and only if $x(t) = \Phi(t)c$ for some $c \in Ker(B + D\Phi(1))$. Here $\Phi(\cdot)$ is the principal fundamental matrix solution to $x' = A(t)x$.*

Proof. $\mathcal{L}x = 0$ if and only if

$$x' = A(t)x \; a.e. \, [0,1] \text{ and } Bx(0) + Dx(1) = 0,$$

which happens if and only if

$$x = \Phi(\cdot)x(0) \text{ and the boundary conditions hold,}$$

which is equivalent to

$$\exists c \in \mathbb{R}^n \text{ such that } x = \Phi(\cdot)c \text{ and } Bc + D\Phi(1)c = 0.$$

□

We now turn to a characterization of the $Im(\mathcal{L})$. To do so, we introduce the following notation. We let $\{c_1, \cdots, c_p\}$ be a basis for $Ker\left((B + D\Phi(1))^T\right)$. We define

$$W = [c_1, ..., c_p]$$

and

$$\Psi(t)^T = W^T D\Phi(1)\Phi^{-1}(t).$$

Lastly, we define $S = span\left\{\begin{bmatrix} \Psi_j(\cdot) \\ \vec{\Psi_j} \end{bmatrix}, j = 1, ..., p\right\}$,

where

$$\begin{bmatrix} \Psi_j(\cdot) \\ \vec{\Psi_j} \end{bmatrix} = \begin{bmatrix} \Psi_j(\cdot) \\ \Psi_j(t_1), \\ \vdots \\ \Psi_j(t_k) \end{bmatrix}.$$

Here $\Psi_j(\cdot)$ denotes the jth column of $\Psi(\cdot)$.

Proposition 5.1.3. *$\mathcal{L}x = \begin{bmatrix} h \\ v \end{bmatrix}$ if and only if $\int_0^1 \Psi^T(s)h(s)ds + \sum_{i=1}^{k} \Psi^T(t_i)v_i = 0$; that is, if and only if $\left\langle \begin{bmatrix} \Psi_j(\cdot) \\ \vec{\Psi_j} \end{bmatrix}, \begin{bmatrix} h \\ v \end{bmatrix} \right\rangle = 0$ for each $j = 1, \cdots, p$.*

Proof. It is well documented that $\mathcal{L}x = \begin{bmatrix} h \\ v \end{bmatrix}$ if and only if

$$x(t) = \Phi(t)\left(x(0) + \int_0^t \Phi^{-1}(s)h(s)ds + \sum_{t_i < t} \Phi^{-1}(t_i)v_i\right).$$

Imposing the boundary conditions, we have

$\begin{bmatrix} h \\ v \end{bmatrix} \in Im(\mathcal{L})$ if and only if there exists $w \in \mathbb{R}^n$ such that

$$Bw + D\left(\Phi(1)(w + \int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i\right).$$

This is clearly equivalent to there existing a $w \in \mathbb{R}^n$ such that

$$[B + D\Phi(1)]w = -D\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i\right),$$

which is equivalent to

$$D\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i\right) \in Im\left(B + D\Phi(1)\right).$$

Since $Im(B + D\Phi(1)) = Ker\left((B + D\Phi(1))^T\right)^{\perp}$, the result follows. $\square$

Corollary 5.1.4. *The image of $\mathcal{L}$ is equal to $S^{\perp}$.*

5.2 Least squares solution with minimal norm

In this section we characterize the least squares solution with minimal norm for the linear boundary value problem

$$x'(t) = A(t)x(t) + h(t), \quad a.e.\,[0,1]$$

$$x(t_i^+) - x(t_i^-) = v_i, \quad i = 1, ..., k$$

subject to

$$Bx(0) + Dx(1) = 0.$$

From Proposition 5.1.2, it follows that there exist a basis, $\alpha_1, \cdots, \alpha_p$, for $Ker(B + D\Phi(1))$ such that

$$\{\Phi(\cdot)\alpha_1, ..., \Phi(\cdot)\alpha_p\}$$

is an orthonormal basis for the $Ker(\mathcal{L})$.

We define

$$P : L^2([0,1], \mathbb{R}^n) \to L^2([0,1], \mathbb{R}^n)$$

by

$$Px = \sum_{j=1}^{p} \langle \Phi(\cdot)\alpha_j, x \rangle \, \Phi(\cdot)\alpha_j$$

and

$$Q : L^2([0,1], \mathbb{R}^n) \times \mathbb{R}^{nk} \to L^2([0,1], \mathbb{R}^n) \times \mathbb{R}^{nk}$$

by

$$Q \begin{bmatrix} h \\ v \end{bmatrix} = \sum_{j=1}^{p} \left\langle \begin{bmatrix} \psi_j(\cdot) \\ \vec{\psi}_j \end{bmatrix}, \begin{bmatrix} h \\ v \end{bmatrix} \right\rangle \begin{bmatrix} \psi_j(\cdot) \\ \vec{\psi}_j \end{bmatrix}.$$

It is clear that P and $I - Q$ are the orthogonal projections onto $Ker(\mathcal{L})$ and $Im(\mathcal{L})$, respectively.

Proposition 5.2.1. *The least squares solution to (5.1)-(5.3) with minimal $L^2([0,1], \mathbb{R}^n)$ norm is given by $M_p(I - Q) \begin{bmatrix} h \\ v \end{bmatrix}$, where $M_p = \mathcal{L}^{-1}_{|Ker(P) \cap dom(\mathcal{L})}$.*

Proof. It is clear that any least squares solution, x, satisfies $\mathcal{L}x = (I - Q) \begin{bmatrix} h \\ v \end{bmatrix}$.

Since

$$\begin{aligned}
\|x\|^2 &= \|Px + (I-P)x\|^2 \\
&= \left\| Px + M_p(I-Q)\begin{bmatrix} h \\ v \end{bmatrix} \right\|^2 \\
&= \|Px\|^2 + \left\| M_p(I-Q)\begin{bmatrix} h \\ v \end{bmatrix} \right\|^2,
\end{aligned}$$

we see that $\|x\|$ is a minimum precisely when $Px = 0$. The result now follows. $\square$

Theorem 5.2.2. *The least squares solution to (5.1)-(5.3) with minimal $L^2([0,1], \mathbb{R}^n)$ norm is given by*

$$\begin{aligned}
x(t) =& \Phi(t)\left((Ec+\beta) + \int_0^t \Phi^{-1}(s)\left[h(s) - \sum_{j=1}^{p}\left[\int_0^1 \psi_j^T(u)h(u)du + \right.\right.\right. \\
& \left.\left. \sum_{i=1}^{k} \psi_j^T(t_i)v_i \right] \Psi_j(s) \right] ds + \sum_{t_i<t} \Phi^{-1}(t_i)\left(v_i - \sum_{j=1}^{p}\left[\int_0^1 \psi_j^T(u)h(u)du \right.\right. \\
& \left.\left.\left. + \sum_{i=1}^{k} \psi_j^T(t_i)v_i \right] \Psi_j(t_i) \right)\right).
\end{aligned}$$

Here $E = [\alpha_1, ..., \alpha_p]$, and $c \in \mathbb{R}^p$ and $\beta \in Ker(B + D\Phi(1))^{\perp}$ are the unique elements satisfying

$$\begin{aligned}
c_i = & -\int_0^1 \alpha_i^T \Phi^T(s)\Phi(s)\beta - \int_0^1 \alpha_i^T \Phi^T(s)\Phi(s)\left(\int_0^s \Phi^{-1}(u)\left[h(u) \right.\right. \\
& \left. - \sum_{j=1}^{p}\left[\int_0^1 \psi_j^T(y)h(y)dy + \sum_{i=1}^{k} \psi_j^T(t_i)v_i \right] \Psi_j(u) \right] du + \sum_{t_i<s} \Phi^{-1}(t_i)\left(v_i \right. \\
& \left.\left. - \sum_{j=1}^{p}\left[\int_0^1 \psi_j^T(y)h(y)dy + \sum_{i=1}^{k} \psi_j^T(t_i)v_i \right] \Psi_j(t_i) \right) ds \right).
\end{aligned}$$

and

$$\beta = -TD\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\right),$$

where

$$T = [B + D\Phi(1)]^{-1}_{|Ker(B+D\Phi(1))^{\perp}}.$$

Remark 5.2.3. We would like to point out, as will be evident from the proof below, that when $A(\cdot)$ and h are continuous the the least squares solution will actually satisfy

$$x'(t) = A(t)x(t) + h(t) \quad \text{for all } t \in [0,1] \setminus \{t_1, \cdots, t_k\}.$$

Proof. With Proposition 5.2.1 in mind, we search for a description of M_p. Now, for $\begin{bmatrix} g \\ u \end{bmatrix} \in Im(\mathcal{L})$, $M_p \begin{bmatrix} g \\ u \end{bmatrix}$ is the unique element in $dom(\mathcal{L})$ satisfying the following:

1. $\mathcal{L}M_p \begin{bmatrix} g \\ u \end{bmatrix} = \begin{bmatrix} g \\ u \end{bmatrix}.$

2. $PM_p \begin{bmatrix} g \\ u \end{bmatrix} = 0.$

We now show that

$$\begin{aligned} M_p\left(\begin{bmatrix} g \\ u \end{bmatrix}\right)(t) =& \Phi(t)\left(Ec^* + \beta\right) \\ &+ \Phi(t)\left(\int_0^t \Phi^{-1}(s)g(s)ds + \sum_{t_i<t}\Phi^{-1}(t_i)u_i\right), \end{aligned}$$

for all $\begin{bmatrix} g \\ u \end{bmatrix} \in Im(\mathcal{L})$, where

$$c_i^* = -\int_0^1 \alpha_i^T \Phi^T(s)\Phi(s)\left(\beta + \int_0^s \Phi^{-1}(u)g(u)du + \sum_{t_i<s} \Phi^{-1}(t_i)u_i\right) ds.$$

From Proposition 5.1.3, it is clear that

$$\mathcal{L}\left(\Phi(t)\left(Ec^* + \beta\right) + \Phi(t)\left(\int_0^t \Phi^{-1}(s)g(s)ds + \sum_{t_i<t} \Phi^{-1}(t_i)u_i\right)\right) = \begin{bmatrix} g \\ u \end{bmatrix}.$$

Now,

$$\int_0^1 \alpha_i^T \Phi(s)^T \left[\Phi(s)\left(Ec^* + \beta + \int_0^s \Phi^{-1}(u)g(u)du + \sum_{t_i<s} \Phi^{-1}(t_i)u_i\right)\right] ds =$$

$$\int_0^1 \alpha_i^T \Phi^T(s)\Phi(s)\left(c_i^*\alpha_i + \beta + \int_0^s \Phi^{-1}(u)g(u)du + \sum_{t_i<s} \Phi^{-1}(t_i)u_i\right) ds =$$

$$c_i^* + \int_0^1 \alpha_i^T \Phi^T(s)\Phi(s)\left(\beta + \int_0^s \Phi^{-1}(u)g(u)du + \sum_{t_i<s} \Phi^{-1}(t_i)u_i\right) ds = 0,$$

Since $Px = 0$ if and only if for each i, $i = 1, \cdots, p$, we have $\langle \Phi(\cdot)\alpha_i, x\rangle = 0$, it follows that

$$P\left(\Phi(t)\left(Ec^* + \beta\right) + \Phi(t)\left(\int_0^t \Phi^{-1}(s)g(s)ds + \sum_{t_i<t} \Phi^{-1}(t_i)u_i\right)\right) = 0.$$

Thus,

$$\begin{aligned} M_p\left(\begin{bmatrix} g \\ u \end{bmatrix}\right)(t) =& \Phi(t)\left(Ec^* + \beta\right) \\ &+ \Phi(t)\left(\int_0^t \Phi^{-1}(s)g(s)ds + \sum_{t_i<t} \Phi^{-1}(t_i)u_i\right). \end{aligned}$$

The result now follows for an arbitrary $\begin{bmatrix} h \\ v \end{bmatrix} \in L^2\left([0,1],\mathbb{R}^n\right) \times \mathbb{R}^{nk}$ by replacing $\begin{bmatrix} g \\ u \end{bmatrix}$ in the description of M_p with $(I-Q)\begin{bmatrix} h \\ v \end{bmatrix}$. $\square$

REFERENCES

[1] S. Bancroft, J.K. Hale, and D. Sweet. Alternative problems for nonlinear functional equations. *J. Differ. Equ.*, 4:40–56, 1968.

[2] A. Boucherif. Nonlinear three-point boundary value problems. *J. Math. Anal. Appl.*, 77:577–600, 1980.

[3] A. Boucherif. Nonlinear multipoint boundary value problems. *Nonlinear Anal.*, 10, No. 9:957–964, 1986.

[4] L. Cesari. Functional analysis and periodic solutions of nonlinear differential equations. *Contr. Differential Equations*, 1:149–187, 1963.

[5] L. Cesari. Functional analysis and galerkin's method. *Michigan Math. J*, 11:385–414, 1964.

[6] S. Chow and J. K. Hale. *Methods of Bifurcation Theory.* Springer, Berlin, 1982.

[7] Debra L. Etheridge and Jesús Rodríguez. Periodic solutions of nonlinear discrete-time systems. *App. Anal.*, 62:119–137, 1996.

[8] R. E. Gaines and J. L. Mawhin. *Coincidence Degree and Nonlinear Differential Equations (Lecture Notes in Mathematics).* Springer-Verlag, 1977.

[9] C. Gerhardt. *Analysis II.* International Press, Somerville, MA, 2006.

[10] J. K. Hale. *Applications of alternative problems.* Lecture Notes, Brown University, Providence, RI, 1971.

[11] J. K. Hale. *Ordinary Differential Equations.* Robert E. Kreiger Publishing Company, Malabar FL, 1980.

[12] W. G. Kelley and A.C. Peterson. *Difference Equations.* Academic Press, New York, 1978.

[13] I. V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov. *Theory of Impulsive Differential Equations.* World Scientific, Singapore, 1989.

[14] Serge Lang. *Real Analysis, second ed.* Addison-Wesley Publishing Company, Reading, MA, 1983.

[15] A. C. Lazer and D. F. Leach. Bounded perturbations of forced harmonic oscillators at resonance. *Ann. Mat. Pura Appl.*, 82:49–68, 1969.

[16] D. C. Lewis. On the role of first integrals in the perturbation of periodic solutions. *Ann. Math.*, 63:535–548, 1956.

[17] B. Liu and J.S. Yu. Solvability of multi-point boundary value problem at resonance (iii). *Appl. Math. Comput.*, 129:119–143, 2002.

[18] J. Locker. The method of least squares for boundary value problems. *Trans. Amer. Math. Soc.*, 154:57–68, 1971.

[19] J. Locker. On constructing least squares solutions to two-point boundary value problems. *Trans. Amer. Math. Soc.*, 203:175–183, 1975.

[20] J. Locker. The generalized green's function for an nth order linear differential operator. *Trans. Amer. Math. Soc.*, 228:243–268, 1977.

[21] J. Locker. *Functional analysis and two-point differential operators.* Longman Scientific & Technical, 1986.

[22] D. Maroncelli and J. Rodríguez. On the solvability of nonlinear impulsive boundary value problems. *submitted*, 2012.

[23] J. J. Nieto. Basic theory of nonresonance impulsive periodic problems of first order. *J. Math. Anal. Appl.*, 205:423–433, 1997.

[24] J. J. Nieto. Periodic boundary value problems for first-order impulsive ordinary differential equations. *Nonlinear Anal.*, 51:1223–1232, 2002.

[25] David Pollack and Padraic Taylor. Multipoint boundary value problems for discrete nonlinear systems at resonance. *Int. J. Pure Appl. Math.*, 63:311–326, 2010.

[26] I. Rachunkova and M. Tvrdy. Existence results for impulsive second-order periodic problems. *Nonlinear Anal.*, 59:133–146, 2004.

[27] Jesús Rodríguez. An alternative method for boundary vallue problems with large nolinearities. *J. Differ. Equ.*, 43:157–167, 1982.

[28] Jesús Rodríguez. Galerkin's method for ordinary differential equations subject to generalizer nonlinear boundary conditions. *J. Differential Equations*, 97:112–126, 1992.

[29] Jesús Rodríguez and Debra L. Etheridge. Periodic solutions of nonlinear second-order difference equations. *Adv. Difference Eqn.*, pages 173–192, 2005, no. 2.

[30] Jesús Rodríguez and D. Sweet. Projection methods for nonlinear boundary value problems. *J. Differ. Equ.*, 58:282–293, 1985.

[31] Jesús Rodríguez and Padraic Taylor. Scalar discrete nonlinear multipoint boundary value problems. *J. Math. Anal. Appl.*, 330:876–890, 2007.

[32] Jesús Rodríguez and Padraic Taylor. Weakly nonlinear discrete multipoint boundary value problems. *J. Math. Anal. Appl.*, 329:77–91, 2007.

[33] Jesús Rodríguez and Padraic Taylor. Multipoint boundary value problems for nonlinear ordinary differential equations. *Nonlinear Anal.*, 68:3465–3474, 2008.

[34] Jesús F. Rodríguez. Existence theory for nonlinear eigenvalue problems. *Appl. Anal.*, 87:293–301, 2008, no. 3.

[35] N. Rouche and J. Mawhin. *Ordinary differential equations.* Pitman, London, 1980.

[36] A. M. Samoilenko and N. A. Perestyuk. *Impulsive Differential Equations.* World Scientific, Singapore, 1995.

[37] Wayne Spealman and Daniel Sweet. The alternative method for solutions in the kernel of a bounded linear functional. *J. Differ Equ.*, 37:297–302, 1980.

[38] Z. Zhao and J. Liang. Existence of solutions to functional boundary value problems of second-order nonlinear differential equations. *J. Math. Anal. Appl.*, 373:614–634, 2011.