ABSTRACT

WRIGHT, JUSTIN PAUL. Periodic Dynamical Systems of Population Models. (Under the direction of Dr. John E. Franke.)

We show that P. Cull's concept of enveloping functions can be applied to periodic systems of population models to ensure the existence of globally asymptotically stable attractors for such systems without ever considering the compositions of the maps involved. We give conditions to ensure that a period-n system of population models sharing a fixed point and enveloping function has a globally asymptotically stable trivial geometric cycle. We also give conditions on two-periodic systems of population models that ensure the existence of a globally attracting geometric 2-cycle as well as providing bounds for the location of the attractor when the population models do not share a fixed point. Perturbation techniques are applied to show that a periodic dynamical system that is a perturbation of an enveloped population model has a globally attracting geometric cycle.

Periodic Dynamical Systems of Population Models

by
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BIOGRAPHY

Justin Wright was born to Cindy and David Wright on January 30, 1984 in Williamsport, Pennsylvania. He grew up in Williamsport with no intent of obtaining a college education let alone a Ph.D. His attempts to enlist in both the Air Force and Navy after high school failed due to a minor medical condition. Justin then spent a year working in a women's underwear factory. While suffering through severe boredom and financial difficulties, he decided to attend Pennsylvania College of Technology in hopes of eventually pursuing an engineering degree at Penn State University. Due to severe boredom, he decided to transfer to the mathematics program at Bloomsburg University of Pennsylvania where he excelled both academically and as a Wal-Mart employee. After attending an REU program at North Carolina State University in the summer of 2007, Justin decided to pursue a graduate degree in applied mathematics.

Justin began his graduate career at North Carolina State University in the summer of 2008 and received his Master's degree in 2011. He conducted the research for his Ph.D. dissertation under the guidance of John Franke in discrete dynamical systems.

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Introduction

Over the past 50 years there has been a continuing trend of using mathematics to analyze both biological and ecological systems (see [6], [12], [24], [35], [36]). Specifically, biologists and ecologists often use mathematical methods to study the growth and decline of a population or to study the interaction of several populations. Mathematical models allow biologists and ecologists to study both how a population changes in size as well as what causes the population to change.

Biologists and ecologists often employ difference equations to study populations. Such equations are of the form

$$x_{t+1} = f(x_t), \tag{1.1}$$

where f is a continuous function whose input is the state or size of a population, x_t, at time t and whose output is the state or size of the population at time $t+1$. Models like (1.1) are discrete in time, unlike the natural process of population growth which is continuous. However, populations are usually measured at discrete time intervals and some populations, such as plant populations, are annual, so difference equations may offer a better method for studying data collected from a given population. Furthermore, difference equations only require an understanding of functions rather than more complicated concepts like differential equations or stochastic methods ([16]). Difference equations are primarily used when the long-term behavior of the population is of interest.

While biologists and ecologists are often most interested in finding a model that accurately predicts the behavior of a population, mathematicians are often interested in determining what behaviors a given model is capable of displaying. Despite the deterministic and relatively simple nature of models based on difference equations like (1.1), their dynamics can be surprisingly complicated. In fact, since Robert May's seminal paper "Some simple models with complex dynamics" ([27]), the models used to predict the behavior of populations have become their own area of mathematical interest. Such models may predict unbounded growth, the decline of an overcrowded population, multiple attractors, the decline of a population with too few individuals, or even chaotic behavior.

A common model for predicting population growth is the Beverton-Holt model,

$$x_{t+1} = \frac{rKx_t}{K + (r-1)x_t},$$

where K gives the carrying capacity and r is a growth factor. The traditional Beverton-Holt model is well known and predicts that small populations will grow monotonically until they reach the carrying capacity and large populations will monotonically decrease until they reach the carrying capacity. In an effort to develop a more accurate model for the growth of a population, Cushing and Henson considered a Beverton-Holt model with a periodically forced carrying capacity in [14] and [15]. That is, the carrying capacity of the model was allowed to change in accordance with seasonal fluctuations. Many biological and ecological systems exhibit periodic variations in intrinsic and extrinsic parameters such as fluctuations in season and climate affecting growth parameters, interaction coefficients, and carrying capacities ([7], [12], [13], [23]). In [15], Cushing and Henson gave conditions that ensured the existence of a globally attracting 2-cycle in a 2-periodic environment. The authors also established that the periodic environment was detrimental to the population by showing that the average over the 2-cycle in the periodic environment is smaller than the globally attracting fixed point of a non-periodic Beverton-Holt model whose carrying capacity was the average of the carrying capacities in the periodic case. The authors conjectured in [14] that these results continue to hold if an arbitrary period is allowed with the 2-cycle being replaced by an arbitrary cycle.

The traditional study of difference equations does not allow the model in question to change in time. As such, Cushing and Henson were forced to apply ad hoc methods in their approach to the period-2 case in [15]. Multiple frameworks have recently been introduced to allow for the study of periodic time-dependent difference equations. In [21], the concept of a cylinder map was used by Franke and Selgrade to represent a periodic dynamical system using an autonomous (time-independent) system. The authors also established theorems concerning the long term behavior of such systems. In [17], Elaydi and Sacker introduced skew-product dynamical systems and geometric cycles to study Cushing and Henson's conjectures. The authors of [17] were in fact able to validate the Cushing and Henson conjectures which appeared in [14].

Our primary concern in this work is Cushing and Henson's first conjecture concerning the existence of an attractor for the periodic Beverton-Holt model. Elaydi and Sacker were able to confirm the conjecture for the Beverton-Holt model and several other models, but their methods rely heavily on the function that gives the model. Therefore, their methods do not generalize to the general study of periodic dynamical systems. Within the study of time-independent difference equations there are many techniques for establishing the existence of an attractor, both locally and globally. However, little work has been done at this time to offer conditions on a periodic dynamical system to ensure the existence of an attractor for such a system. Therefore, a major goal of this work is to establish conditions that will ensure the existence of a globally attracting periodic solution for a periodic dynamical system.

Inspired by the first Cushing and Henson conjecture, we focus our efforts on a general class of functions that contains the Beverton-Holt model known as population models. Population models represent a specific class of difference equations that exhibit properties that make them especially useful in the prediction of the behavior of a population. Population models assume the population has a carrying capacity in its environment and that the population will grow if it is below that carrying capacity and will decline if it is above that carrying capacity. However, population models do not allow for the consideration of multiple species and they do not allow for the extinction of a population.

Population models have been studied extensively since at least the 1980's, and have the interesting property that local stability often implies global stability. This fact has been established using Liapunov functions in [20] and [22] as well as techniques by Singer ([34]), Rosenkranz ([31]), and Cull ([8],[9],[10]). Many of these techniques suffer from being difficult to apply in many settings. Cull provided a simple technique utilizing enveloping functions in [11] to give conditions under which local stability implied global stability for population models. Enveloping has the benefit of being simple to understand and apply even for those without a strong mathematical background.

Within this work, we first establish that Cull's concept of enveloping can be extended to periodic dynamical systems of population models if the carrying capacity does not change over time. This result allows the population models within the periodic dynamical system to take any form and allows for any period. However, this result does not directly apply to the Cushing and Henson conjectures where it was assumed that the carrying capacity was fluctuating.

The second major result is a series of theorems that can be applied to period-2 dynamical systems of population models to establish the existence of a global attractor. While these theorems restrict the period of the system to period-2, they allow the maps to have different carrying capacities and algebraic expressions. Furthermore, they provide a simple set of conditions that can be checked to ensure the existence of a globally attracting 2-cycle. A method is then outlined that allows these theorems to be applied to general n-periodic dynamical systems.

We conclude with several results concerning the perturbation of periodic dynamical systems. The first result ensures that the perturbation of an enveloped population model may be enveloped by a translation of the original enveloping function. The second set of results gives conditions that ensure that a geometric cycle persists under the perturbation of the periodic dynamical system. The final result ensures that if a population model is enveloped then a periodic dynamical system of population models "close" to the original function will have a globally attracting geometric cycle.

CHAPTER 2

Background and Motivation

In this chapter, we introduce the necessary background for the rest of this work. It is hoped that the topics discussed in this chapter will lead the reader to understand the general context and motivation for this work.

2.1 Preliminary Notation

Throughout this work we will have need to refer to a metric space, generally denoted as X, with metric d. In most cases, X is the space real valued vectors of length n, denoted $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, we find it convenient to begin indexing at 0. That is,

$$x = (x_0, x_1, \ldots, x_{n-1}) \in \mathbb{R}^n.$$

This indexing is convenient since we will often be working modulo an integer n. For $x, y \in \mathbb{R}^n$ and $i = 0, 1, \ldots, n-1$ we use

$$d(x, y) = ||x - y||_\infty = \max_i |x_i - y_i|.$$

We will denote the open ball of $x \in X$ by

$$\mathcal{B}(x, r) = \{y \in X : d(x, y) < r\}$$

and the closed ball by

$$\bar{\mathcal{B}}(x, r) = \{y \in X : d(x, y) \leq r\}$$

where $r > 0$.

For $\mathcal{I} \subset \mathbb{R}^n$ we will denote the space of k continuously differentiable functions from $\mathcal{I}$ to $\mathcal{I}$ by $\mathcal{C}^k(\mathcal{I})$. In some instances we will have need to refer to the space of continuous mappings from a topological space Y to a topological space Z which we will denote using $C(Y, Z)$.

On occassion we will use $\mathbb{R}^+$ to denote $\{x \in \mathbb{R} : x > 0\}$ and $\mathbb{Z}^+$ to denote $\{x \in \mathbb{Z} : x > 0\}$.

2.2 Autonomous Difference Equations

Here we present some of the basic concepts and terminology associated with difference equations. For $f \in \mathcal{C}^0(\mathcal{I})$ and $x \in \mathcal{I}$ we define the first order autonomous difference equation

$$x_{t+1} = f(x_t), \quad t \in \mathbb{Z}^+. \tag{2.1}$$

Equation (2.1) is called *autonomous* because the function f does not depend on time. We define the forward orbit of an initial condition $x_0 \in \mathcal{I}$ under (2.1) by the ordered set

$$\mathcal{O}^+(x_0) = \big\{x_0, x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), \ldots, x_n = f^n(x_0), \ldots\big\}$$

where

$$f^n(x) = \underbrace{(f \circ \ldots \circ f)}_{n \text{ times}}(x).$$

Since the orbit is determined by iteration of f, we often think of the study of (2.1) as studying the iteration of a map.

Our primary concern in the study of (2.1) is the asymptotic behavior of an orbit. Of particular interest are *periodic points*, that is $x^* \in \mathcal{I}$ for which there exists $k \in \mathbb{Z}^+$ such that

$$f^k(x^*) = x^*.$$

In this instance, we refer to x^* as a periodic point with period k. If there is no r, where $1 \leq r < k$, such that $f^r(x^*) = x^*$ then we say that x^* is a point of minimal period k. If a point is said to have period k then it should be assumed that the point has minimal period k unless otherwise stated. In the event that $f(x^*) = x^*$, we call x^* a fixed point of (2.1) or synonymously a fixed point of f.

A period k point, x^*, of (2.1) is said to be *attracting* or *locally attracting* if there exists $\eta > 0$ such that for all $x \in \mathcal{I}$, if $d(x^*, x) < \eta$ then $\lim_{n\to\infty} f^{nk}(x) = x^*$. If $\lim_{n\to\infty} f^{nk}(x) = x^*$ for all $x \in \mathcal{I}$ then we call x^* *globally attracting.*

We refer to a period k point of (2.1) as *stable* if for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathcal{I}$, if $d(x^*, x) < \delta$ then $d(f^{nk}(x), x^*) < \epsilon$ for all $n \in \mathbb{Z}^+$. If x^* is not stable we refer to it as *unstable*. If a periodic point is both attracting and stable then it is referred to as *asymptotically stable*. It is known (see [32]) that for a continuous map on the real line, a globally attracting fixed point must be stable.

The following lemma is a well known result that we use throughout this work without reference.

Lemma 2.2.1. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{n\to\infty} f^n(x) = L$ then $f(L) = L$.*

Proof. Suppose $\lim_{n\to\infty} f^n(x) = L$. Then

$$\begin{aligned} L &= \lim_{n\to\infty} f(f^{n-1}(x)) \\ &= f(\lim_{n\to\infty} f^{n-1}(x)) \\ &= f(L). \end{aligned}$$

□

A fixed point x^* of a map $f \in \mathcal{C}^1(\mathbb{R})$ is said to be *hyperbolic* if $|f'(x^*)| \neq 1$ and *nonhyperbolic* if $|f'(x^*)| = 1$. Hyperbolicity can be used to determine the nature of a fixed point as per the following theorem which appears with proof in [16].

Theorem 2.2.1. *Let x^* be a hyperbolic fixed point of a map f, where f is continuously differentiable at x^*. Then the following statements hold true:*

1. *If $|f'(x^*)| < 1$, then x^* is attracting.*
2. *If $|f'(x^*)| > 1$, then x^* is unstable.*

While we do not give the full proof here, we mention that given the assumptions in the theorem there exists $r > 0$ and $\mathcal{B}(x^*, r)$ such that $|f'(x)| \leq \lambda < 1$ for all $x \in \mathcal{B}(x^*, r)$. Then, using induction and the Mean Value Theorem, it can be shown that for $x_0 \in \mathcal{B}(x^*, r)$

$$|f^n(x_0) - x^*| \leq \lambda^n |x_0 - x^*|. \tag{2.2}$$

Then for $r_1 \in (0, r)$, (2.2) gives

$$f(\bar{\mathcal{B}}(x^*, r_1)) \subset \mathcal{B}(x^*, r_1).$$

Furthermore, (2.2) shows that f is a contraction on $\mathcal{B}(x^*, r_1)$.

Trapping regions play an important role in understanding the asymptotic dynamics of a discrete dynamical system. The following definition is adapted from [30].

Definition 2.2.1. An open set $U \subset X$ is a trapping region for f if $f(\bar{U}) \subset U$.

It should be clear that the presence of an attracting hyperbolic fixed point, x^*, for a map, f, ensures that there is an open neighborhood of x^* that is a trapping region for f.

2.3 Periodically Forced Difference Equations and the Cushing and Henson Conjectures

A clear limitation of first order autonomous difference equations is that the model is not allowed to change over time. However, biological and ecological systems often have parameters that depend on time either intrinsically or extrinsically. In [15], Cushing and Henson considered the well known Beverton-Holt model

$$y_{t+1} = \frac{ry_t}{1 + (r-1)(y_t/K)} \tag{2.3}$$

where r is an intrinsic growth rate and K is the carrying capacity of the species. It is well established that if $r \leq 1$ then the fixed point 0 is globally attracting on $(0, \infty)$ and if $r > 1$ then K is globally attracting.

To explore the dynamics of a population in a periodic environment, K was replaced by the periodic sequence $K = K(t)$. For simplicity, the sequence was restricted to period-2 and represented as $K = K_{av}(1 + \alpha(-1)^t)$ where K_{av} is the average of K over time and $\alpha \in [0, 1)$. By rescaling $x_t = y_t/K$, the authors arrived at the periodically forced Beverton-Holt model

$$x_{t+1} = \frac{rx_t}{1 + (r-1)(x_t/(1 + \alpha(-1)^t))}. \tag{2.4}$$

Cushing and Henson went on to show that (2.4) has a globally attracting two-cycle $\{c_0, c_1\}$. That is, the orbit of c_0 under (2.4) is

$$\mathcal{O}^+(c_0) = \{c_0, c_1, c_0, c_1, \ldots\}.$$

Furthermore, it was shown that $0 < c_0 < c_1$ and $\frac{1}{2}(c_0 + c_1) < K_{av}$. In other words, the average of the terms in the globally attracting cycle for (2.4) is less than the global attractor, K_{av}, for (2.3). This lead the authors to call the globally attracting two-cycle of (2.4) *attenuate*. In fact, Cushing and Henson established this result in the period-2 case for a general class of models that are, like the Beverton-Holt model, monotone increasing and concave down.

Results similar to those of Cushing and Henson had previously been established for logistic differential equations in [3], [4], [26], and [29]. Later results showed the deleterious effects of a periodic environment to be model dependent and may even be advantageous ([5]). More recent results in [28] have shown that a given model may have resonant cycles (advantageous) or attenuate cycles (deleterious) based on the parameters of the model.

Cushing and Henson's work lead them to the Cushing and Henson conjectures concerning an n-periodic Beverton-Holt model of the form

$$x_{t+1} = \frac{rK_t}{K_t + (r-1)x_t} x_t \tag{2.5}$$

where K_t is an n-periodic sequence. The Cushing and Henson conjectures as stated in [14] are

(a) Equation (2.5) has a positive n-periodic solution $y_t > 0$, and it is globally attracting for $x_0 > 0$.

(b) If $n > 2$, the strict inequality $av(y_t) < av(K_t)$ holds. Here av denotes the average of a periodic cycle, i.e.

$$av(y_t) = \frac{1}{p} \sum_{i=1}^{p-1} y_t.$$

Clearly, Cushing and Henson proved these results for $n = 2$ in [15]. While periodically forced ecological models have been of interest since the 1970's, the Cushing and Henson conjectures seem to have launched a great deal of interest in the area of periodic dynamical systems. However, Cushing and Henson's work in [15] was purely based on (2.4). That is, their approach was purely ad hoc and difficult to generalize even to a period-3 example. In the next two sections, we present efforts made to add structure to the study of periodic dynamical systems.

The concept described in the first Cushing and Henson conjecture is the topic of primary concern throughout this work. In short, we desire to establish conditions that ensure that a period-n dynamical system will have a global attractor. However, whereas Cushing and Henson only allowed the parameters of a model to change over time, we will allow the model to change over time. It should be noted that we will not prove the Cushing and Henson conjectures in this work as they have already been established in [18] by Elaydi and Sacker.

2.4 n-Periodic Dynamical Systems and the Cylinder Map

In [21], Franke and Selgrade introduced a time-independent, discrete dynamical system that captures the dynamics of a time-dependent, discrete dynamical system. Furthermore, they provide theorems that provide more structure to the understanding of periodic dynamical systems. The definitions and theorems in this section all appear in [21].

To produce an autonomous dynamical system from a time-dependent one, the authors let (X, d) be a metric space. They then define an *n-periodic dynamical system* as a finite sequence $\{f_0, f_1, \ldots, f_{n-1}\}$ where $f_i : X \to X$ for $i = 0, 1, \ldots, n-1$. This allows for the construction of an n-periodic nonautonomous difference equation given by

$$x_{t+1} = f(t \bmod n, x_t) = f_{t \bmod n}(x_t) \tag{2.6}$$

where $t \in \mathbb{Z}$. Then a forward orbit for $x_0 \in X$ under (2.6) is given by

$$\begin{aligned} \mathcal{O}^+(x_0) &= \{x_0, f_0(x_0), f_1(f_0(x_0)), \ldots, \\ &\quad f_{n-1} \circ \ldots \circ f_0(x_0), \\ &\quad f_0 \circ f_{n-1} \circ \ldots \circ f_0(x_0), \ldots\}. \end{aligned}$$

The corresponding autonomous difference equation is in on the *fibered cylinder*, $\mathcal{X}$, given by

$$\mathcal{X} = \{0, 1, \ldots, n-1\} \times X.$$

The metric on $\mathcal{X}$ is $d((i, x), (j, y)) = \delta_{ij} + d(x, y)$. Then for $i = 0, 1, \ldots, n-1$ and $(i, x) \in \mathcal{X}$, the autonomous *cylinder map*, $\mathcal{F} : \mathcal{X} \to \mathcal{X}$, is given by

$$\mathcal{F}(i, x) = (i + 1 \bmod n, f_i(x)).$$

Having established the cylinder map, the authors go on to give standard definitions in the study of difference equations as they apply to the cylinder map. Define the projection $\pi_X : \mathcal{X} \to X$ by $\pi_X(i, x) = x$. Since $\mathcal{X}$ is a finite number of copies of X, π_X is an open mapping.

Definition 2.4.1. A set $\Lambda \subset X$ is *invariant* under the time-periodic dynamical system if there is a set $\Gamma \subset \mathcal{X}$ with $\mathcal{F}(\Gamma) \subset \Gamma$ and $\pi_X(\Gamma) = \Lambda$.

The following lemma offers an equivalent definition for the invariance of a set that offers insight into into the behavior of periodic dynamical systems.

Lemma 2.4.1. *$\Lambda \subset X$ is invariant if and only if for each $x \in \Lambda$ there is an*

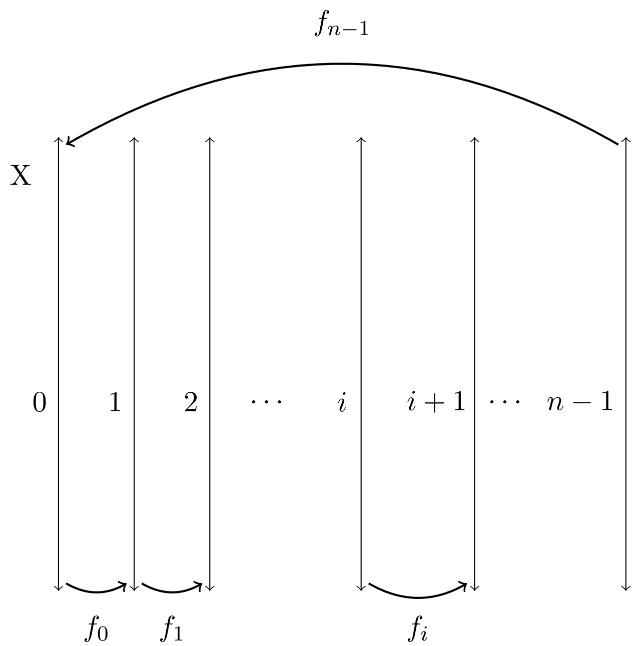


Figure 2.1: The fibered cylinder $\mathcal{X}$ and the cylinder map $\mathcal{F}$ for $\{f_0, f_1, \ldots, f_{n-1}\}$.

$i(x) \in \{0, 1, 2, \ldots, n-1\}$ *with*

$$(f_{(i(x)+k) \bmod n} \circ \ldots \circ f_{(i(x)+2) \bmod n} \circ f_{(i(x)+1) \bmod n} \circ f_{i(x) \bmod n})(x) \in \Lambda$$

for all integers $k \geq 0$.

Trapping regions will be used throughout this work. Franke and Selgrade offered the following definition for the trapping region of a periodic dynamical system.

Definition 2.4.2. A set $U \subset X$ is a *trapping region* for the time-periodic dynamical system if there is an open set $\mathcal{U} \subset \mathcal{X}$ with compact closure $\bar{\mathcal{U}}$ so that $\mathcal{F}(\bar{\mathcal{U}}) \subset \mathcal{U}$ and $\pi_X(\mathcal{U}) = U$.

Since the cylinder map, $\mathcal{F}$, is an autonomous map the set $\mathcal{U}$ is a trapping region for $\mathcal{F}$ in the traditional sense.

Definition 2.4.3. A set $\Lambda \subset X$ is an *attractor* for the time-periodic dynamical system if it has a trapping region U, with corresponding trapping region $\mathcal{U} \subset \mathcal{X}$, such that $\pi_X(\Gamma) = \Lambda$ where

$\Gamma = \bigcap_{n=0}^{\infty} \mathcal{F}^n(\bar{\mathcal{U}})$.

The following theorem provides a better understanding of the structure of attractors for periodic dynamical systems and appears in [21].

Theorem 2.4.1 (Structure Theorem)**.** *Let Λ be an attractor for the n-periodic dynamical system $\{f_0, f_1, \ldots, f_{n-1}\}$. Then $\Lambda = \bigcup_{i=0}^{n-1} \Lambda_i$, where Λ_i is an attractor for the map*

$$(f_{(i+n-1) \bmod n} \circ \ldots \circ f_{(i+1) \bmod n} \circ f_i \bmod n) : X \to X,$$

for $i = 0, 1, \ldots, n-1$.

The Structure Theorem asserts that an attractor for an n-periodic dynamical system must be a union of the attractors for the n-fold compositions of the maps that comprise the n-periodic dynamical system. As such, the theorem provides a method to find attractors for n-periodic dynamical systems. However, even if the individual maps in an n-periodic dynamical system are relatively simple to analyze individually, it may be very difficult to analyze the n-fold compositions. It is often the case that fixed points cannot be found algebraically for the n-fold compositions. Throughout this work, an effort is made to put conditions on the individual maps within a periodic dynamical system rather than on the system as a whole. This allows the conditions to be checked more readily.

The following theorem is closely related to the Structure Theorem and also appears in [21].

Theorem 2.4.2. *Let $\{f_0, f_1, \ldots, f_{n-1}\}$ be an n-periodic dynamical system on a complete, locally compact, metric space X. For $i = 0, 1, \ldots, n-1$, if each f_i is a contraction then each*

$$(f_{(i+n-1) \bmod n} \circ \ldots \circ f_{(i+1) \bmod n} \circ f_{i \bmod n})$$

is a contraction with a unique fixed point q_i and, for each of these compositions, there is an open set which is a trapping region. The collection of fixed points $\{q_0, q_1, \ldots, q_{n-1}\}$ is an attractor for the n-periodic system.

In this work, n-periodic dynamical systems based on the fibered cylinder and cylinder map are the preferred framework for the analysis of periodic dynamical systems because it easily allows for a new model at each time step.

2.5 Skew-Product Dynamical Systems and Geometric Cycles

Elaydi and Sacker provided an alternate framework for the analysis of periodic dynamical systems in [17]. Their construction is more consistent with the framework for the study of nonautonomous differential equations and even allows for the study of a general (not periodic) nonautonomous discrete-time dynamical system. The motivation for the work was to extend a previous theorem of Elaydi and Yakubu, but the work also allowed them to directly consider the Cushing and Henson conjectures.

Elaydi and Yakubu proved the following result in [19].

Theorem 2.5.1. *Let $f : X \to X$ be a continuous map on a connected metric space. If a periodic orbit c_k is globally asymptotically stable, then c_k must be a fixed point.*

The extension of this theorem to periodic dynamical systems provides significant insight into the possible orders of global attractors for periodic dynamical systems. Before providing the extension of the theorem, we discuss Sacker and Elaydi's skew-product dynamical system.

The concept of the skew-product dynamical system is originally due to Sell but was expanded on by Sell and Sacker. The following definition and subsequent discussion appears in [17] with slight alterations in notation.

Definition 2.5.1. Let X and Y be two topological spaces. A dynamical system

$$\pi = (\phi, \sigma) : X \times Y \times \mathbb{Z} \to X \times Y$$

is said to be a skew-product dynamical system if there exist continuous mappings $\phi : X \times Y \times \mathbb{Z} \to X$ and $\sigma : Y \times \mathbb{Z} \to Y$ such that

$$\pi(x, y, t) = (\phi(x, y, t), \sigma(y, t)),$$

where σ is a time-dependent dynamical system on Y.

If $\mathbb{Z}$ is replaced by $\mathbb{Z}^+$ then π is called a skew-product semi-dynamical system.

We now construct a skew-product semi-dynamical system for the nonautonomous difference equation

$$x_{t+1} = f(t, x_t). \tag{2.7}$$

Here, (2.7) is comparable to (2.6) but (2.7) may not be periodic. The authors first define σ. Let

$C = C(\mathbb{Z} \times X, X)$ be the space of continuous functions from $\mathbb{Z} \times X$ to X equipped with the topology of uniform convergence on compact subsets of $\mathbb{Z} \times X$. For $G \in C$, define $\sigma(G,t) = G_t$ as the shift map, $G_t(k,x) = G(k+t,x)$ and the *Hull* of G, $\mathcal{H}(G) \doteq Cl\,\{G_t : t \in \mathbb{Z}\}$ and set

$$Y = \mathcal{H}(f),$$

where f is as in (2.7). If we are dealing with an n-periodic dynamical system determined by an n-periodic difference equation

$$x_{t+1} = F(t \bmod n, x_t) \tag{2.8}$$

then $Y = \{F_0, F_1, \ldots, F_{n-1}\}$ where

$$F_i = \left\{f_{i \bmod n}, f_{(i+1) \bmod n}, \ldots, f_{(i+n-1) \bmod n}\right\}.$$

For $G \in Y$ the map ϕ is defined by

$$\phi(x_0, G, t) \doteq \Phi(t, G)x_0. \tag{2.9}$$

Hence, letting $G_0 = G$,

$$\pi(x, G_i, t) = (\Phi(t, G(i+t, \cdot))x, G_{i+t}).$$

The operator Φ is then given as follows:
For $G \in Y$ set $G_0 = G$. Then

$$\begin{aligned}
\Phi(0, G_0)x_0 &= id(x_0),\\
\Phi(t, G_0)x_0 &= \Phi(1, G_{t-1})\Phi(t-1, G_0)x_0\\
&= \Phi(1, G_{t-1})\Phi(1, G_{t-2})\ldots\Phi(1, G_0)x_0.
\end{aligned}$$

For example, if we are considering (2.8), then we may consider the n-periodic dynamical system

$$F_0 = \{f_0, f_1, \ldots, f_{n-1}\}.$$

Suppose that each f_i is a map on $\mathbb{R}$. Then

$$\begin{aligned}
\Phi(0,F)x_0 &= id(x_0), \\
\Phi(1,F)x_0 &= \Phi(1,F_0)x_0 \\
&= f_0(x_0), \\
\Phi(2,F)x_0 &= \Phi(1,F_1)\Phi(1,F_0)x_0 \\
&= (f_1 \circ f_0)(x_0), \\
&\vdots \\
\Phi(n,F)x_0 &= \Phi(1,F_{n-1})\Phi(1,F_{n-2})\ldots\Phi(1,F_1)\Phi(1,F_0)x_0 \\
&= (f_{n-1} \circ \ldots \circ f_1 \circ f_0)(x_0).
\end{aligned}$$

While skew-product semi-dynamical systems are useful in the study of periodic dynamical systems, we find the fibered cylinder and cylinder map to be a more intuitive setting for working with periodic dynamical systems. However, the following definition from [17] is very useful to avoid confusion when dealing with periodic dynamical systems.

Definition 2.5.2 (Geometric r-cycle)**.** Let $\{f_0, f_1, \ldots, f_{n-1}\}$ be an n-periodic dynamical system and let $r > 0$ be an integer. A *geometric r-cycle* is an ordered set of points

$$\mathcal{C} = \{c_0, c_1, \ldots, c_{r-1}\}, \quad c_i \in X$$

with the property that for $i = 0, 1, \ldots, r-1$

$$f_{(i+tr) \bmod n}(c_i) = c_{i+1 \bmod r} \quad \forall\ t \in \mathbb{Z}.$$

The notation in the above definition has been altered for consistency with an n-periodic dynamical system. The authors of [17] restrict $r \leq n$ but we allow r to take on any value in this work. Often, we will refer to geometric r-cycles simply as geometric cycles when the order is not important. Examples of geometric cycles and a further discussion follow in the next section.

With the definition of a geometric r-cycle we can state Elaydi and Sacker's extension of Theorem 2.5.1.

Theorem 2.5.2. *Let $\{f_0, f_1, \ldots, f_{n-1}\}$ be an n-periodic dynamical system where each f_i is a continuous map on a connected metric space X. Let $c_r = \{c_0, c_1, \ldots, c_r\}$ be a geometric r-cycle for the n-periodic dynamical system. If c_r is globally asymptotically stable then $r|n$.*

Elaydi and Sacker do not offer a formal definition of global asymptotic stability for a geometric cycle in [17], but the meaning can be understood by reading the examples given by the authors. A formal definition was given later in [2]. We offer an interpretation using orbits on the fibered cylinder in the following section. It should be mentioned that while Theorem 2.5.2 offers restrictions on possible orders of globally attracting geometric cycles, it offers no information concerning how to find a geometric cycle or determine if it is globally attracting.

Elaydi and Sacker explored the Cushing and Henson conjectures for (2.5) in both [17] and [18] using the skew-product dynamical system. However, their analysis essentially relies on considering arbitrary n-fold compositions. That is, for an n-periodic Beverton-Holt model they define

$$f_i(x) = \frac{rK_i x}{K_i + (r-1)x}, \tag{2.10}$$

for $i = 0, 1, \ldots, n-1$. After two iterations,

$$\Phi(2, f)x = x_2 = (f_1 \circ f_0)(x_0) = \frac{r^2 K_1 K_0 x_0}{K_1 K_0 + (r-1)M_1 x_0}$$

and after n iterations

$$\Phi(n, f)x = x_n = (f_{n-1} \circ f_{n-2} \circ \ldots \circ f_0)(x_0) = \frac{r^n K_{n-1} K_{n-2} \ldots K_0 x_0}{K_{n-1} K_{n-2} \ldots K_0 + (r-1)M_{n-1} x},$$

where M_k satisfies the second order linear difference equation

$$M_{k+1} = K_{k+1} M_k + r^{k+1} K_k K_{k-1} \ldots K_0, \; M_0 = 1.$$

They then define

$$H(x) = \Phi(n, f)x = (f_{n-1} \circ f_{n-2} \circ \ldots f_1 \circ f_0)(x_0)$$

and consider the autonomous difference equation $x_{n+1} = H(x_n)$ to prove the Cushing and Henson conjectures.

Notice, that this is synonymous with considering an n-periodic dynamical system

$$\mathcal{S} = \{f_0, f_1, \ldots, f_{n-1}\}$$

where f_i is given by (2.10). If $\mathcal{F}$ is the cylinder map for $\mathcal{S}$ then $\mathcal{F}^n(0, x)$ corresponds to $H(x)$. Figure 2.2 shows the fibered cylinder, a possible orbit, and autonomous map, H, for the given construction.

This technique relies heavily on finding a closed form expression for H as well as knowing that each f_i is monotone increasing and concave down. As such, it does not generalize to an arbitrary n-periodic dynamical system.

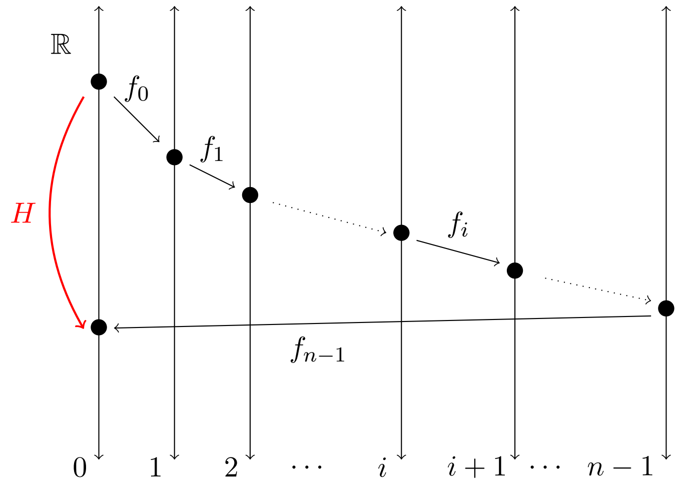


Figure 2.2: The fibered cylinder and autonomous map, H, for Elaydi and Sacker's work on the Beverton-Holt model.

2.6 Attracting Geometric Cycles

Throughout this work, we are generally interested in deciding when a geometric cycle is an attractor for an n-periodic dynamical system. In this section we give a formal definition of an attracting geometric cycle as well as examples and properties of geometric cycles. The reader may notice that the Structure Theorem can be directly applied to determine if a geometric cycle is an attractor for a periodic dynamical system. We reiterate, however, that the n-fold compositions that must be considered to apply the Structure Theorem directly may make the necessary analysis very difficult. Since relying on numerical results is often ill-advised in the study of difference equations, we prefer conditions that rely only on the individual maps in a periodic dynamical system.

Throughout Chapter 3, we are largely concerned with geometric 1-cycles. That is, if a geometric cycle of the form $\mathcal{C} = \{c\}$ for the n-periodic dynamical system $\{f_0, f_1, \ldots, f_{n-1}\}$. Notice that for such a cycle,

$$f_{(i+t) \bmod n}(c) = c$$

for $i = 0, 1, \ldots, n-1$ and all $t \in \mathbb{Z}^+$. Thus, c is fixed by each f_i. We therefore refer to such a geometric cycle as a *trivial geometric cycle*.

While geometric cycles play the role of equilibrium solutions for periodic nonautonomous dynamical systems, their characteristics may be notably different from the properties of periodic points of autonomous systems. Namely,

1. The elements of a geometric cycle may not be distinct.

2. Different geometric cycles may not be disjoint.

These behaviors and others are noted in [1]. Such characteristics are in stark contrast with the behavior of periodic points for autonomous difference equations.

The following example should aid in understanding geometric cycles and how they differ from periodic points.

Example 2.6.1. Consider the 3-periodic system

$$\mathcal{S} = \{f_0, f_1, f_2\}$$

where

$$f_0(x) = -x + 1,\ \ f_1(x) = -x + 2,\ \text{ and } f_2(x) = -x + 3.$$

Under this system, $g_0 = \{1, 0, 2\}$ is a geometric 3-cycle because

$$f_0(1) = 0,\ \ f_1(0) = 2,\ \ f_2(2) = 1.$$

Note that a cyclic permutation such as $\{0, 2, 1\}$, however, is not a geometric 3-cycle. Under $\mathcal{S}$, all $x_0 \neq 1$ lie on a geometric 6-cycle because

$$(f_2 \circ f_1 \circ f_0 \circ f_2 \circ f_1 \circ f_0)(x) = x.$$

An example of a geometric 6-cycle is $g_1 = \{0, 1, 1, 2, -1, 3\}$. Notice that 1 is repeated within g_1 and that $g_0 \bigcap g_1 = \{0, 1, 2\}$ but g_0 and g_1 are distinct geometric cycles. Figure 2.3 shows the orbits of 0 and 1 under $\mathcal{S}$.

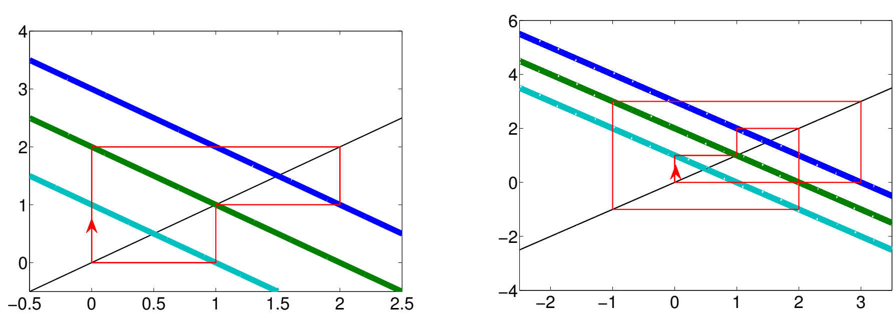


Figure 2.3: The geometric cycles of $x_0 = 1$ (Left) and $x_0 = 0$ (Right) under $\mathcal{S}$.

Figure 2.3 can be read much like a tradition cobweb or stair step diagram. In a traditional cobweb diagram, a vertical line is traced from the line $y = x$ to the graph of the functions being studied and then a horizontal line is drawn back to the line. This process is then repeated. For a periodic cobweb diagram, a vertical line is drawn from $y = x$ to f_0 and then a horizontal is drawn back to the line $y = x$. Then this process is repeated but f_0 is replaced by f_1 and so on.

To discuss the stability of periodic dynamical systems we consider the cylinder map as introduced in [21]. As it applies to this work the fibered cylinder, $\mathcal{X}$, is the Cartesian product of $\mathbb{R}$ with the distance metric and the discrete space $\{0, 1, \dots, n-1\}$,

$$\mathcal{X} = \{0, 1, \dots, n-1\} \times \mathbb{R},$$

using the product topology with the metric on $\mathcal{X}$ defined by

$$d((i, x), (j, y)) = \delta_{ij} + |x - y|.$$

Define the cylinder map, $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ corresponding to a periodic dynamical system $\{f_0, f_1, \dots, f_{n-1}\}$ as

$$\mathcal{F}(i, x) = (i + 1 \bmod n, f_i(x)).$$

A geometric r-cycle, $G = \{x_0, x_1, \dots, x_{r-1}\}$, for the periodic dynamical system, $\{f_0, f_1, \dots, f_{n-1}\}$, has a corresponding periodic orbit, $\mathcal{G}$, for the cylinder map, $\mathcal{F}$, of minimal period $l = lcm(n, r)$. Here, $\mathcal{G}$ is given by

$$\begin{array}{rcccc} \mathcal{G} = & \{(0, x_0), & (1, x_1), & \dots & (n-1, x_{n-1}), \\ & (0, x_n), & (1, x_{n+1}), & \dots & (n-1, x_{2(n-1)}), \\ & \vdots & \vdots & \vdots & \vdots \\ & (0, x_{(l/n-1)n}), & (1, x_{(l/n-1)n+1}), & \dots & (n-1, x_{l-1})\} \end{array}$$

where all subscripts are modulo r. In fact, the periodic orbit $\mathcal{G}$ is similar in structure to the algebraic expression for a geometric cycle used in [1].

Example 2.6.2. Here we consider an arbitrary dynamical system with a geometric cycle to aid in the understanding of an orbit under a periodic dynamical system versus the corresponding orbit on the fibered cylinder. Let $\mathcal{S} = \{f_0, f_1, f_2\}$ where $f_i : X \to X$ and assume there exists

$c_0, c_1 \in X$ such that

$$f_0(c_0) = c_1, \quad f_1(c_1) = c_0, \quad f_2(c_0) = c_1,$$
$$f_0(c_1) = c_0, \quad f_1(c_0) = c_1, \quad f_2(c_1) = c_0.$$

Then $\{c_0, c_1\}$ is a geometric 2-cycle for $\mathcal{S}$. However, if we consider the orbit of c_0 on the fibered cylinder then c_0 is a point of period $lcm(2,3) = 6$. Figure 2.4 gives a graphical representation of the cylinder map for the example. One may follow the orbit in Figure 2.4 by alternating between red and black arrows. ■

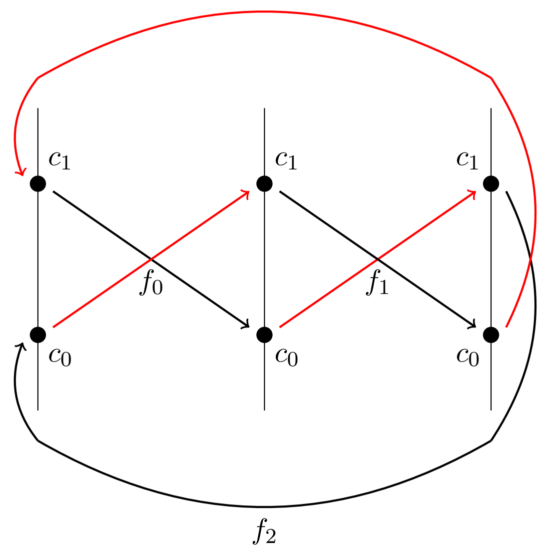


Figure 2.4: The orbit of a geometric 2-cycle for a period-3 dynamical system drawn on the fibered cylinder.

Since $\mathcal{G}$ is a periodic solution for the cylinder map the traditional definitions of stability and attraction apply. With this in mind we give the following definitions. The geometric cycle G is said to be *stable* if the corresponding orbit $\mathcal{G}$ of the cylinder map is stable under the definition given in Section 2.2. The geometric cycle G is *attracting* if $\mathcal{G}$ is attracting and G is *asymptotically stable* if it is stable and attracting.

2.7 Population Models and Enveloping

With the inspiration of the Cushing and Henson conjectures and the application of periodic dynamical systems to periodic systems occurring in nature, we wish to examine periodic dynamical systems of *population models*. Population models represent a specific class of continuous functions on the real line, of which the Beverton-Holt model is an example.

The behavior of population models has been widely studied by Cull in [8],[9], [10], and [11] and appeared in [16]. Here we use a less restrictive definition of population model than is common.

Definition 2.7.1 (Population Model)**.** A continuous function $f : [0, \infty) \to [0, \infty)$ is a *population model* if

(i) $f(0) = 0$ and f has a unique positive fixed point p,

(ii) $f(x) > x$ for $x \in (0, p)$ and $f(x) < x$ for $x \in (p, \infty)$,

(iii) $f(x) > 0$ when $x > 0$.

The standard definition of a population model replaces condition (iii) in Definition 2.7.1 with the following condition.

(iii) If $f'(x_m) = 0$ and $0 < x_m \leq p$ then $f'(x) > 0$ for $0 \leq x < x_m$ and $f'(x) < 0$ for $x > x_m$ and $f(x) > 0$ for $x > 0$.

However this condition is unnecessarily strict for our purposes so we revert to Definition 2.7.1. Figure 2.5 shows several functions satisfying Definition 2.7.1.

We refer to the space of population models by $\mathcal{P}$. When we refer to the fixed point for $f \in \mathcal{P}$ we are referring to the unique positive fixed point of f, since 0 is fixed for all $f \in \mathcal{P}$.

The following definition and theorem are due to Cull ([11]). Both are restated here for convenience with slight alterations for our purposes.

Definition 2.7.2 (Enveloping)**.** Let $f \in \mathcal{P}$ with fixed point p and let $x_- > p$. A continuous function $\phi : I \supseteq (0, x_-) \to \mathbb{R}$ *envelopes* f if and only if

(i) $\phi(x) > f(x)$ for $x \in (0, p)$ and $\phi(p) = p$

(ii) $\phi(x) < f(x)$ for $x > p$ whenever $\phi(x) > 0$ and $f(x) > 0$.

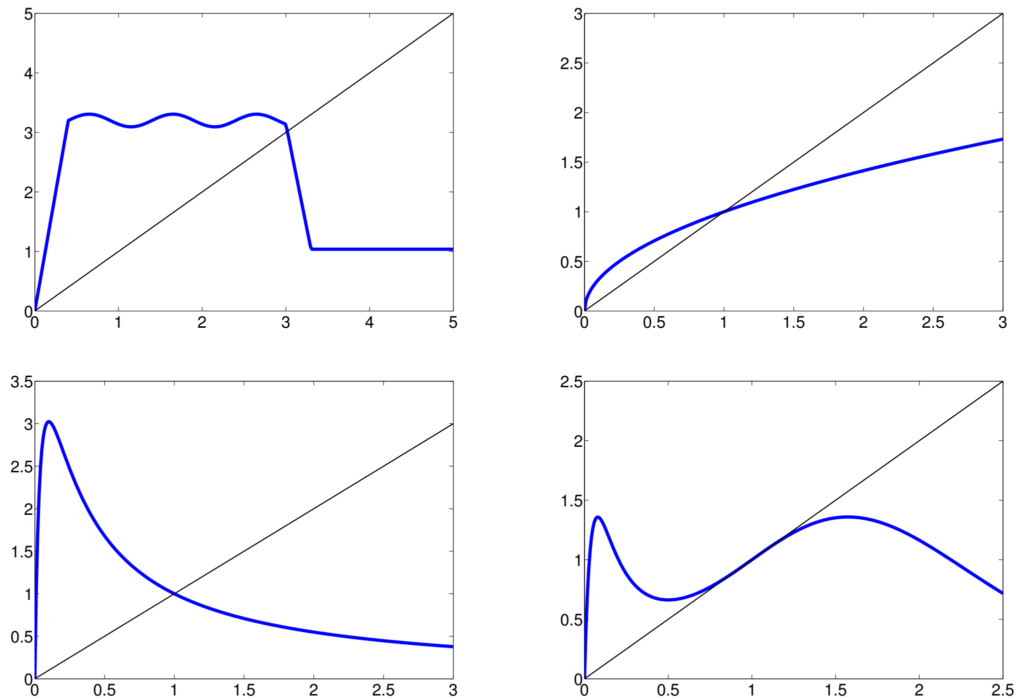


Figure 2.5: The graphs of several population models.

Figure 2.6 shows two enveloped population models. In [11], Cull provides several examples of population models and appropriate enveloping functions as well as a method to build enveloping functions.

The following theorem by Cull offers relatively simple conditions to assure that the fixed point of a population model is globally attracting. It will be the basis for the results in Chapter 3.

Theorem 2.7.1 (Cull's Theorem). *Let $\phi(x)$ be a monotone decreasing function which is positive on $(0, x_-)$ and $\phi(\phi(x)) = x$. Assume that f is a population model with $f(p) = p$ and that $\phi(x)$ envelopes $f(x)$. Then*

$$\lim_{k\to\infty} f^k(x) = p \quad \textit{for all } x \in (0, \infty).$$

As Cull frequently mentions, a wonderful aspect of this theorem is that enveloping is easy to understand and confirm if an appropriate enveloping function can be found. Therefore, a biologist or ecologist can apply enveloping with only a limited knowledge of difference equations. Other techniques like Liapunov theory are equally effective but it may be very difficult to

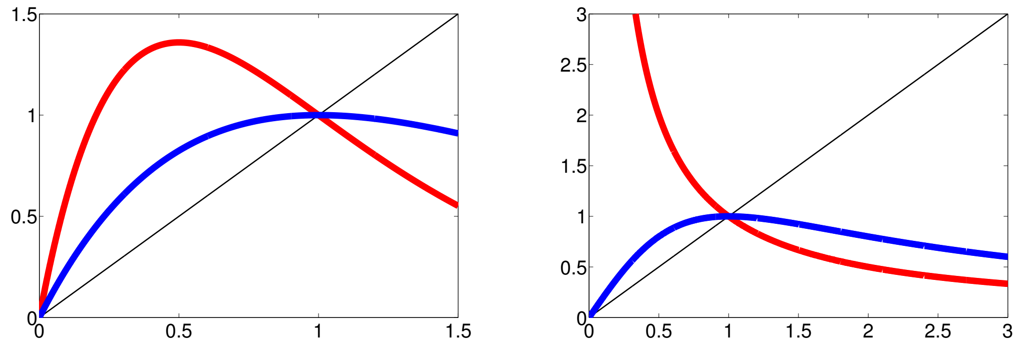


Figure 2.6: Two plots of population models (blue) with enveloping functions (red).

determine an appropriate Liapunov function. Chapters 3 and 4 of this work continue this theme of providing easily checked conditions that are sufficient to ensure the presence of a globally attracting geometric cycle.

It may be noted that Cull's Theorem was stated for maps with a fixed point at $p = 1$ but the proof is easily adapted for an arbitrary positive fixed point p. Cull provided a method to build enveloping functions based on linear fractional functions for several examples of population models. However, this method required that the map be rescaled so that the fixed point was at $p = 1$. For our purposes, where we may be considering several maps with different fixed points, such a rescaling is not possible. We will therefore not use Cull's construction for building enveloping functions. However, we will often utilize enveloping functions provided in [11] that have been altered to allow for a fixed point not at $p = 1$ without justification.

CHAPTER 3

Enveloping and Trivial Geometric Cycles

In this chapter we give conditions on an n-periodic dynamical system, $\{f_0, f_1, \ldots, f_{n-1}\}$, of population models that ensure the existence of a globally attracting trivial geometric cycle. These results allow the f_i in the periodic dynamical system to take any form but require that they all share a common fixed point. The motivation for this result is Cushing and Henson's first conjecture.

3.1 Inheritance of Enveloping

A wonderful aspect of Cull's Theorem is that enveloping is inherited under composition. That is, if two or more maps share an enveloping function ϕ then their composition is enveloped by ϕ as well. This result follows in Theorem 3.1.1 and can be used to understand the behavior of periodic dynamical systems of population models.

Theorem 3.1.1. *Let $f_0, f_1 \in \mathcal{P}$ share a common fixed point p. Suppose $\phi(x)$ is an enveloping function of f_0 and f_1 that is monotone decreasing, $\phi(x) > 0$ for $x \in (0, x_-)$, and $\phi(\phi(x)) = x$. Then $f_0(f_1(x))$ and $f_1(f_0(x))$ are population models and are enveloped by $\phi(x)$.*

Remark 3.1.1. Before beginning the proof we make a few comments concerning the nature of x_- in Definition 2.7.2. Note that either $x_- \in \mathbb{R}$ and $x_- > p$ or $x_- = \infty$. If x_- is finite then $\lim\limits_{x \to 0^+} \phi(x) = x_-$ because ϕ is self-inversing. For the purpose of our proof we want to extend the domain of ϕ in the case that x_- is finite using

$$\tilde{\phi}(x) = \begin{cases} -x + x_- & x \leq 0 \\ \phi(x) & 0 < x < x_- \\ -x + x_- & x \geq x_- \end{cases}.$$

Notice that if ϕ envelopes a function f then the extension $\tilde{\phi}$ still envelopes f and is monotone decreasing and self-inversing.

Proof. The functions $f_0(x)$ and $f_1(x)$ are interchangeable for the sake of the proof and so without loss of generality we will show $f_1(f_0(x))$ meets the conditions to be a population model and enveloped by ϕ. If x_- is finite then ϕ refers to the extension of ϕ. We explicitly state the assumptions on f_0 and f_1 for convenience:

(A1) $f_i(0) = 0$

(A2) $f_i(x) > 0$ for $x \in (0, \infty)$

(A3) $f_i(x) > x$ for $x \in (0, p)$

(A4) $f_i(x) < x$ for $x \in (p, \infty)$

(A5) $\phi(x) > f_i(x)$ for $x \in (0, p)$

(A6) $\phi(x) < f_i(x)$ for $x \in (p, x_-)$

for $i = 0, 1$. We must show that $f_1(f_0(x))$ meets conditions *(A1)*-*(A6)*.

(a) By condition *(A1)* we have $f_1(f_0(0)) = 0$.

(b) Since $f_i(x) > 0$ for $x \in (0, \infty)$ and for $i = 0, 1$ by *(A2)*, we must have $f_1(f_0(x)) > 0$ for $x \in (0, \infty)$.

(c) Let $x < p$ so that $f_0(x) > x$. If $f_0(x) < p$ then $f_1(f_0(x)) > f_0(x) > x$ by condition *(A3)*. If $f_0(x) > p$ we have $x < f_0(x) < \phi(x)$ and by the monotone decreasing nature of $\phi(x)$,

$$\phi(\phi(x)) = x < \phi(f_0(x)) < \phi(x).$$

By condition *(A6)* we have $\phi(f_0(x)) < f_1(f_0(x))$ and so

$$x < \phi(f_0(x)) < f_1(f_0(x)).$$

If $f_0(x) = p$ then $f_1(f_0(x)) = p$ and so $f_1(f_0(x)) = p > x$. Therefore $f_1(f_0(x)) > x$ for all $x \in (0, p)$.

(d) Let $x > p$ so that $f_0(x) < x$. If $f_0(x) > p$ then $f_1(f_0(x)) < f_0(x) < x$ by condition *(A4)*. If $f_0(x) < p$ then $\phi(x) < f_0(x) < x$ and so using the monotone decreasing nature of $\phi(x)$,

$$\phi(x) < \phi(f_0(x)) < \phi(\phi(x)) = x.$$

Since $f_0(x) < p$ we have $f_1(f_0(x)) < \phi(f_0(x)) < x$ by condition *(A5)*. Finally, if $f_0(x) = p$ then $f_1(f_0(x)) = p$ and so $x > f_1(f_0(x))$.

(e) Let $x < p$. Then $f_0(x) < \phi(x)$. If $f_0(x) < p$ then $x < f_0(x) < f_1(f_0(x))$ and since $\phi(x)$ is monotone decreasing

$$\phi(f_1(f_0(x))) < \phi(f_0(x)) < \phi(x).$$

Also, $f_1(f_0(x)) < \phi(f_0(x))$ by condition *(A5)* so $f_1(f_0(x)) < \phi(x)$.

If $f_0(x) > p$ we still have $\phi(x) > f_0(x)$ by *(A5)* while $f_1(f_0(x)) < f_0(x)$ by condition *(A4)* so $f_1(f_0(x)) < \phi(x)$. If $f_0(x) = p$ then $f_1(f_0(x)) = p$. Since $\phi(x)$ is monotone decreasing $p < \phi(x)$ for all $x \in (0, p)$ and so $f_1(f_0(x)) < \phi(x)$.

(f) Let $x > p$ so that $f_0(x) > \phi(x)$ and $f_0(x) < x$. By the monotone decreasing of $\phi(x)$,

$$\phi(x) < \phi(f_0(x)).$$

If $f_0(x) > p$ then $\phi(f_0(x)) < f_1(f_0(x))$ and so $\phi(x) < f_1(f_0(x))$.

If $f_0(x) < p$ then $f_1(f_0(x)) > f_0(x) > \phi(x)$. If $f_0(x) = p$ then $f_1(f_0(x)) = p$ and by the monotone decreasing of $\phi(x)$, $\phi(x) < p = f_1(f_0(x))$.

Note that cases (a)-(d) ensure that $f_1(f_0(x))$ is a population model and (e)-(f) ensure enveloping is maintained. □

Figure 3.1 shows a plot of $f_0(x) = xe^{0.5(1-x)}$, $f_1(x) = xe^{1.9(1-x)}$, and their enveloping function $\phi(x) = -x + 2$. The figure represents a typical plot of functions satisfying conditions *(A1)-(A6)* of Theorem 3.1.1 and many of the conditions satisfied by $f_1(f_0(x))$ can be realized by a careful examination of Figure 3.1.

Theorem 3.1.1 can be extended to n functions inductively and the result is stated as the following corollary.

Colrollary 3.1.1. *If f_i satisfies conditions (A1)-(A6) of Theorem 3.1.1 for $i = 0, \ldots n-1$ then*

$$F_i(x) = \left(f_{(i+n-1) \bmod n} \circ \ldots \circ f_{i \bmod n}\right)(x)$$

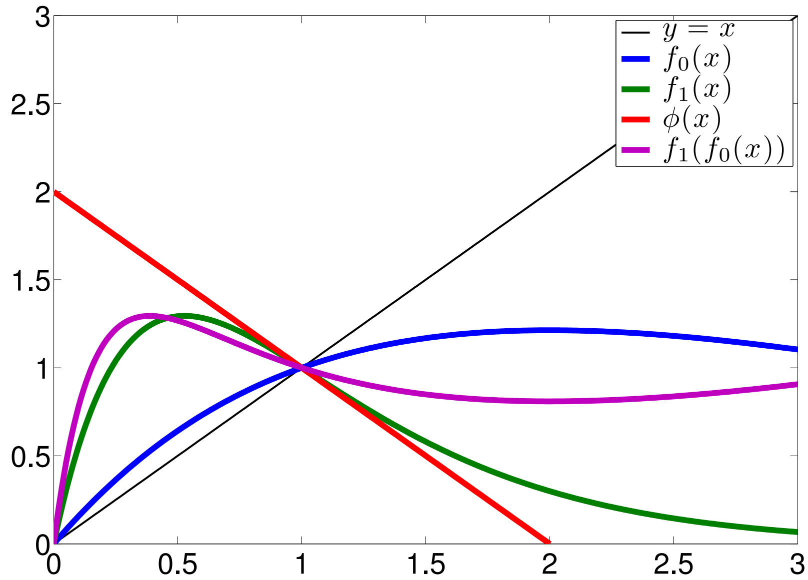


Figure 3.1: A plot of functions satisfying conditions *(A1)-(A6)* of Theorem 3.1.1 with their enveloping function and composition.

satisfies conditions (A1)-(A6). That is, $F_i(x)$ has a unique fixed point $x = p$, is a population model, and is enveloped by ϕ.

We use Theorem 3.1.1, Cull's Theorem, and Corollary 3.1.1 in the following section to establish results concerning the behavior of periodic dynamical systems.

3.2 Global Attractors for Periodic Dynamical Systems of Population Models

Here we consider periodic dynamical systems of the form $\{f_0, \dots f_{n-1}\}$ where each f_i is a population model enveloped by a monotone decreasing and self-inversing function, ϕ, and sharing a fixed point, p. We will establish that such systems have $\{p\}$ as a globally asymptotically stable attractor.

Theorem 3.2.1. *Suppose $\{f_0, f_1, \dots, f_{n-1}\}$ is a periodic dynamical system such that f_i satisfies conditions (A1)-(A6) of Theorem 3.1.1 so that $f_i(p) = p$ for $i = 0, 1, \dots, n-1$. Then the periodic* *dynamical system has a globally asymptotically stable trivial geometric cycle at $x = p$.*

We give two proofs of the Theorem 3.2.1, one using Cull's Theorem and another using a result from [33].

Proof. By Corollary 3.1.1 the periodic dynamical system will have a unique fixed point at $x = p$. That is, for $i \in \{0, 1, \ldots, n-1\}$,

$$F_i = \left(f_{(i+n-1) \bmod n} \circ \ldots \circ f_{i \bmod n}\right) : \mathbb{R}^+ \to \mathbb{R}^+$$

we have $F_i(p) = p$ and by Corollary 3.1.1, $F_i(x)$ satisfies conditions (A1)-(A6) of Theorem 3.1.1. We can equate $F_i(x)$ with the cylinder map $\mathcal{F}^n(i, x)$ which is restricted to the ith fiber of the cylinder space and has (i, p) as a fixed point. By Cull's Theorem,

$$\lim_{k\to\infty} F_i(x) = p \text{ for all } x \in (0, \infty).$$

Since p is globally attracting, p is also stable for $F_i(x)$ by the results of [32]. Then,

$$\lim_{k\to\infty} \left(\mathcal{F}^n\right)^k (i, x) = (i, p)$$

for all $x \in (0, \infty)$ and $i \in \{0, 1, \ldots, n-1\}$. Therefore (i, p) is a globally asymptotically stable periodic point of $\mathcal{F}$ and so $\{p\}$ is an attracting trivial geometric cycle of the periodic dynamical system. $\square$

The next proof relies on Theorem 2.1 from [33] which requires the following result.

Lemma 3.2.1. *If f is a function satisfying the properties of Theorem 3.1.1 then for $x_0 < p$, $x_0 < f^2(x_0)$ and for $x_0 > p$, $f^2(x_0) < x_0$.*

Proof. Suppose $f_0 = f_1$ in Theorem 3.1.1 and denote $f_0 = f_1 = f$. Then

$$f_0(f_1(x)) = f_1(f_0(x)) = f^2(x)$$

and the results of Theorem 3.1.1 guarantee that $x < f^2(x)$ for $0 < x < p$ and $f^2(x) < x$ for $x > p$. $\square$

We now give the second proof of Theorem 3.2.1.

Proof. For $i \in \{0, 1, \ldots, n-1\}$ define

$$F_i = \left(f_{(i+n-1) \bmod n} \circ \ldots \circ f_{i \bmod n}\right) : \mathbb{R}^+ \to \mathbb{R}^+.$$

By Corollary 3.1.1, $F_i(x)$ meets conditions (A1)-(A6) of Theorem 3.1.1. Thus, by Lemma 3.2.1, $F_i^2(x) > x$ for $x < p$ and $F_i^2(x) < x$ for $x > p$. Therefore, p is globally asymptotically stable for $F_i(x)$ by Theorem 2.1 from [33]. Then,

$$\lim_{k \to \infty} \left(\mathcal{F}^n\right)^k (i, x) = (i, p)$$

for all $x \in (0, \infty)$ and $i \in \{0, 1, \ldots, n-1\}$. Therefore (i, p) is a globally asymptotically stable periodic point of $\mathcal{F}$ and so $\{p\}$ is an attracting trivial geometric cycle of the periodic dynamical system. □

Example 3.2.1. Consider the 3-period dynamical system of Ricker type maps

$$\mathcal{S} = \left\{ f_0(x) = xe^{1-x},\ f_1(x) = xe^{1.5(1-x)},\ f_2(x) = xe^{2(1-x)} \right\}.$$

Each map has a fixed point $p = 1$ and it has been established in [11] that $\phi(x) = -x + 2$ is an enveloping function for each. The orbit of $x_0 = 0.5$ can be seen in Figure 3.2. As anticipated, the orbit rapidly approaches the fixed point $p = 1$. ■

Example 3.2.2. Consider now the 2-periodic dynamical system of Smith-Slatkin maps, $\{f_0, f_1\}$, where

$$f_0(x) = \frac{10x}{1 + 9x^{20/9}},$$

$$f_1(x) = \frac{11x}{1 + 10x^{22/10}}.$$

These functions have enveloping functions

$$\phi_0(x) = \frac{11/9 - (2/9)x}{2/9 + (7/9)x},$$

$$\phi_1(x) = \frac{6/5 - (1/5)x}{1/5 + (4/5)x}$$

respectively as prescribed in [11]. In this instance the maps share a fixed point at $p = 1$ and, according to Cull's Theorem, $p = 1$ is attracting for both maps. However, the periodic dynamical system does not have $\{1\}$ as a GAS trivial geometric cycle. If we consider the composition of

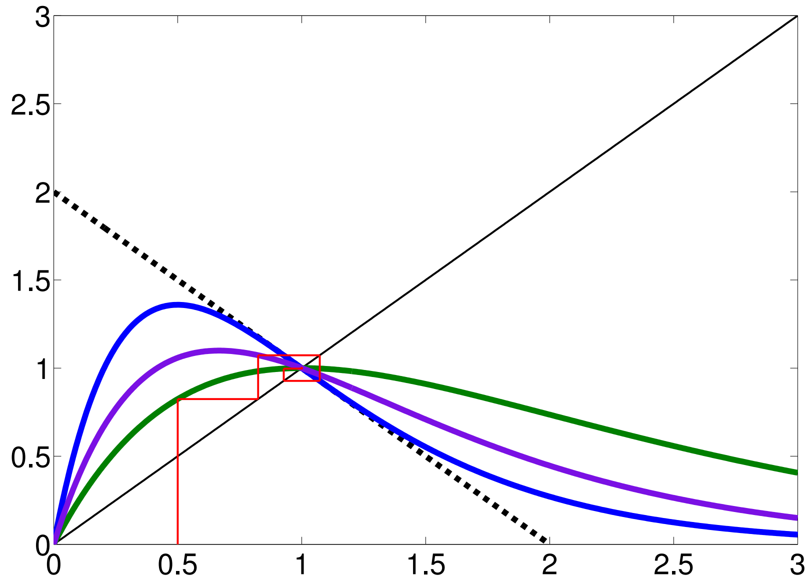


Figure 3.2: The orbit of a 3-periodic system $\mathcal{S}$ of Ricker maps f_0 (blue), f_1 (green), and f_2 (purple). Each is enveloped by $\phi(x) = -x + 2$. $\mathcal{S}$ has trivial geometric cycle $\{1\}$. Here $x_0 = 0.5$ and the first 50 iterations are shown.

the maps:

$$F(x) = f_1(f_0(x)) = \frac{110x}{\left(1 + 9x^{20/9}\right)\left(1 + 1000\left(10^{1/5}\right)\left(\frac{x}{1+9x^{20/9}}\right)^{11/5}\right)}.$$

It can be confirmed that $F(1) = 1$ and using the chain rule $F'(1) = 1$ so that $x^* = 1$ is a nonhyperbolic fixed point of the composition. Using a computer algebra system we can see that $F''(1) = 2/45$ indicating that $x^* = 1$ is a semistable fixed point of the composition (see Theorem 1.5 in [16]).

If we consider orbits under the periodic dynamical system for seed values less than and greater than the fixed point $x^* = 1$, we can see the orbit moving towards or away from the fixed point x^* respectively:

$$\mathcal{O}^+(0.5) = \{0.5, 1.707184, 0.561654, 1.605869, 0.601846\ldots\},$$

$$\mathcal{O}^+(1.05) = \{1.05, 0.951892, 1.050013, 0.951879, 1.050028\ldots\}.$$

In the orbits above, the odd entries are the orbit under F and the reader may note for a seed of $x_0 = 0.5$ the orbit under F increases monotonically but is bounded by the fixed point and therefore converges. The orbit of the seed $x_0 = 1.05$ also increases monotonically but without bound.

The even entries in the above orbits represent the orbits under $f_0(f_1(x))$. Notice that the even entries of the first orbit decrease monotonically and are bounded below by the fixed point. Conversely, the even entries in the second are decreasing monotonically but are not bounded by the fixed point.

Of particular interest in this example is that the functions f_0 and f_1 are so similar and the fixed point $p = 1$ is attracting for each. However, while f_0 and f_1 have enveloping functions, we claim they do not share an enveloping function and therefore Theorem 3.2.1 does not apply. The following lemma aids our claim.

Lemma 3.2.2. *If $f \in \mathcal{C}^2(\mathbb{R}^+)$ is a population model with fixed point p, $\phi \in \mathcal{C}^2(\mathbb{R}^+)$ is a monotone decreasing and self-inversing enveloping function for f, and $f'(p) = -1$ then $f'(p) = \phi'(p)$ and $f''(p) = \phi''(p)$.*

Proof. To begin we show that if ϕ is a monotone decreasing and self-inversing enveloping function of f, then $\phi'(p) = -1$. Using that $\phi'(x) < 0$ and $\phi(\phi(x)) = x$ and differentiating,

$$\begin{aligned} \frac{d}{dx}[x] &= \frac{d}{dx}[\phi(\phi(x))] \\ 1 &= \phi'(\phi(x))\phi'(x) \\ 1 &= \phi'(\phi(p))\phi'(p) = (\phi'(p))^2. \end{aligned}$$

Thus, $\phi'(p) = -1$ and so $f'(p) = \phi'(p)$.

Let $h(x) = \phi(x) - f(x)$. We wish to show that $h''(p) = 0$. For contradiction, suppose that $h''(p) = \bar{a}$ for $\bar{a} > 0$. Then there exists $\epsilon > 0$ and some $a > 0$ such that $h''(x) > a$ for $x \in (p, p + \epsilon)$. Then

$$h'(x) = \int_p^x h''(s)\ ds \geq \int_p^x a\ ds = a(x - p).$$

Thus $h'(x) > 0$ for $x \in (p, p + \epsilon)$. Consider now,

$$\frac{h(p + \epsilon) - h(p)}{(p + \epsilon) - p} = \frac{h(p + \epsilon)}{\epsilon} < 0$$

by the definition of enveloping. By the Mean Value Theorem, there exists $c \in (p, p + \epsilon)$ such that $h'(c) < 0$ giving a contradiction since we know $h'(x) > 0$.

Now suppose for contradiction that $h''(p) = -\bar{b}$ for $\bar{b} > 0$. Then there exists $\eta > 0$ and some $b > 0$ such that $h''(x) < -b$ for $y \in (p - \eta, p)$. Then

$$-h'(y) = \int_y^p h''(s)\ ds \leq \int_y^p -b\ ds = -b(p-y).$$

So $h'(y) \geq b(p-y)$ meaning $h'(y) > 0$ for $y \in (p-\eta, p)$. Again consider,

$$\frac{h(p) - h(p-\eta)}{p - (p-\eta)} = \frac{-h(p-\eta)}{\eta} < 0.$$

By the Mean Value Theorem, there exists $c \in (p-\eta, p)$ such that $h'(y) < 0$. This is a contradiction. Therefore, $h''(p) = 0$. $\square$

Recall, we claim that f_0 and f_1 do not share an enveloping function. Let ϕ_0 and ϕ_1 be any monotone decreasing, self-inversing functions of f_0 and f_1 respectively.

Proof. (Of Claim) Note that $f_0'(1) = -1$ and $f_1'(1) = -1$. Thus $f_0'(1) = \phi_0'(1)$ and $f_1'(1) = \phi_1'(1)$ meaning $f_0''(1) = \phi_0''(1)$ and $f_1''(1) = \phi_1''(1)$ by Lemma 3.2.2. However, $f_0''(1) \neq f_1''(1)$ and so $\phi_0''(1) \neq \phi_1''(1)$. Therefore $\phi_0 \neq \phi_1$. $\square$

$\blacksquare$

We end this section with an example that demonstrates the power of Theorem 3.2.1 in the application of periodic dynamical systems in which the maps have little in common.

Example 3.2.3. Consider the period-3 dynamical system of population models.

$$\mathcal{S} = \left\{ f_0(x) = xe^{1.5(1-x)},\ f_1(x) = \sqrt{x},\ f_2(x) = \frac{4x}{(1+x)^2} \right\}.$$

It can be confirmed that $p = 1$ is a fixed point and $\phi(x) = -x + 2$ is an enveloping function for each map in the periodic dynamical system. Therefore, Theorem 3.2.1 can be applied to ensure that $\{1\}$ is a globally attracting trivial geometric cycle for $\mathcal{S}$. Notice, that there is no need to consider the n-fold compositions for $\mathcal{S}$ to ensure that $\{1\}$ is globally attracting since the conditions for Theorem 3.2.1 apply to the maps that comprise the system.

Furthermore, the maps in $\mathcal{S}$ have little in common except for their fixed point and enveloping function. The maps f_0 and f_2 are both bounded while f_1 is monotone increasing and unbounded. All three maps have different derivatives at the fixed point. Yet none of these properties need to be considered to ensure $\{1\}$ is globally attracting. This is in contrast to Elaydi and Sacker's proof of the first Cushing and Henson conjecture. $\blacksquare$

CHAPTER 4

2-Periodic Systems of Population Models

In Chapter 3, we were concerned with finding conditions that assured the existence of a global attractor for a periodic dynamical system as motivated by the first Cushing and Henson conjecture. However, we assumed that the maps had a common fixed point. Cushing and Henson's conjecture assumed that the carrying capacity of the Beverton-Holt model varied over time. Since the Beverton-Holt model assumes the population's carrying capacity is the fixed point of the model, our simplifying assumption directly conflicts with Cushing and Henson's motivations. In this chapter, we consider periodic dynamical systems in which the maps do not share a common fixed point.

4.1 General Systems of Population Models

As the following example shows, a periodic dynamical system of maps that do not share a common fixed point may exhibit complex dynamics.

Example 4.1.1. Here we consider a 2-periodic dynamical system $\{f_0, f_1\}$ of Ricker type maps where f_i is given by

$$f_i(x) = xe^{2-s_i x},\ x \geq 0.$$

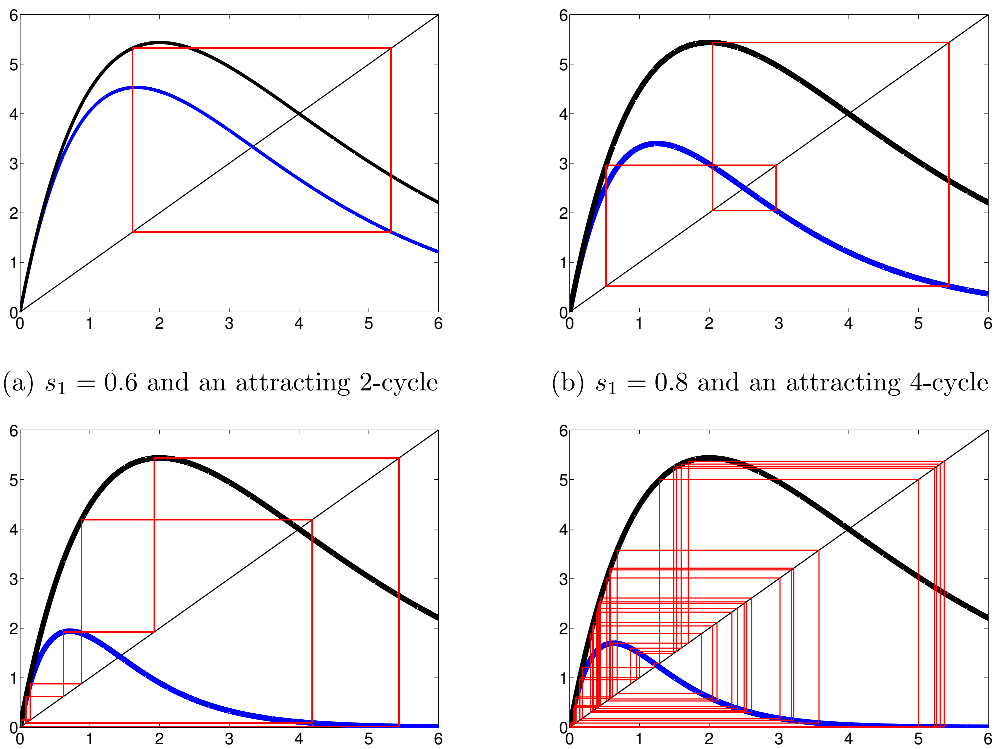
Maps of this form have a fixed point $p_i = 2/s_i$ and $f_i'(p_i) = -1$. We can envelope $f_i(x)$ with

$$\phi_i(x) = -x + \frac{4}{s_i}$$

since ϕ_i is tangent to f_i at p_i, f_i is concave down on $(0, p_i)$, and f_i is concave up on (p_i, ∞).

Figure 4.1 shows the given periodic dynamical system with several values for s_i where the maps fail to share an enveloping function. However, for s_0 close to s_1 the periodic dynamical system still has what appears to be a globally asymptotically stable geometric 2-cycle (Figure 4.1a). However, as we increase s_1 away from s_0 the system appears to go through period doubling route to chaos.

We note that in Figure 4.1b and Figure 4.1c the attracting geometric cycles cannot be globally attracting by Theorem 2.5.2, since the order of the geometric cycles do not divide the period of the system. In Chapter 5, it will be shown that as the value of s_1 is increased, the geometric 2-cycle in Figure 4.1a persists but becomes unstable. ■



(a) $s_1 = 0.6$ and an attracting 2-cycle

(b) $s_1 = 0.8$ and an attracting 4-cycle

(c) $s_1 = 1.4$ and an attracting 8-cycle

(d) $s_1 = 1.6$ and a potentially dense orbit.

Figure 4.1: A 2-periodic system of Ricker maps and their orbits. In all parts 5000 iterations are done and the last 50 are shown. We fix $s_0 = 0.5$ (black) and change s_1 (blue).

4.2 2-Periodic Dynamical Systems

The periodic dynamical system in Example 4.1.1 has a geometric cycle that appears to be globally attracting for many choices of s_0 and s_1. In fact, we will show that enveloping can be applied to ensure the existence of a globally asymptotically stable geometric 2-cycle for a periodic dynamical system like that in Example 4.1.1. The following lemmas establish the existence of a unique geometric 2-cycle for a periodic dynamical system and provide conditions to ensure that the cycle is locally attracting. Theorems 4.2.1 and 4.2.2 provide suitable conditions to ensure that the unique geometric 2-cycle is globally attracting.

It is very important here to note the subtle difference between local attraction for the fixed point of a map and local attraction for a geometric cycle of a periodic dynamical system. For a periodic dynamical system the order of the composition of the maps affects the order of the resulting cycle when such a cycle exists. As such, if $\{x_1^*, x_2^*\}$ is an attracting geometric cycle for $\{f_0, f_1\}$ and x_0 is near x_2^* then the resulting orbit may not approach $\{x_1^*, x_2^*\}$. Instead, we must have x_0 near x_1^* for the orbit to be attracted to the geometric cycle.

We will begin by providing conditions that ensure the composition of the maps from a 2-periodic dynamical system has a fixed point within a certain interval. We will apply this result to show that such a 2-periodic system has a geometric 2-cycle. The reader may want to refer to Figure 4.2 which labels a plot with the appropriate variables for the following lemmas and theorems.

Lemma 4.2.1. *Suppose $f_0, f_1 \in \mathcal{P}$ with distinct fixed points p_0 and p_1 respectively and there exists an open interval $\mathcal{I}$ with $p_0, p_1 \in \mathcal{I}$ such that $f_0(x) > f_1(x)$ for $x \in \mathcal{I}$. Suppose further that f_1 has a monotone decreasing, self-inversing enveloping function $\phi(x)$ and $f_1(x) < p_1$ for $x > p_1$. Then there exists $q \in (0, p_1)$ such that $f_0(q) = \phi(q)$ and for every such q, $f_1(f_0(x))$ has a fixed point in (q, p_1).*

Proof. We begin by proving the existence of $\bar{q}$ such that $\phi(\bar{q}) = f_0(\bar{q})$. Notice that since $f_0(x) > f_1(x)$ for $x \in \mathcal{I}$ we have $p_0 > p_1$ and so $f_0(p_1) > f_1(p_1) = p_1$. Define $h(x) = f_0(x) - \phi(x)$ which inherits continuity from f_0 and ϕ. Then

$$\lim_{x \to 0^+} h(x) = 0 - \lim_{x \to 0^+} \phi(x) < 0,$$

and

$$h(p_1) = f_0(p_1) - \phi(p_1) = f_0(p_1) - p_1 > 0.$$

Applying the Intermediate Value Theorem to h we see there exists $\bar{q}$ such that $f_0(\bar{q}) = \phi(\bar{q})$.

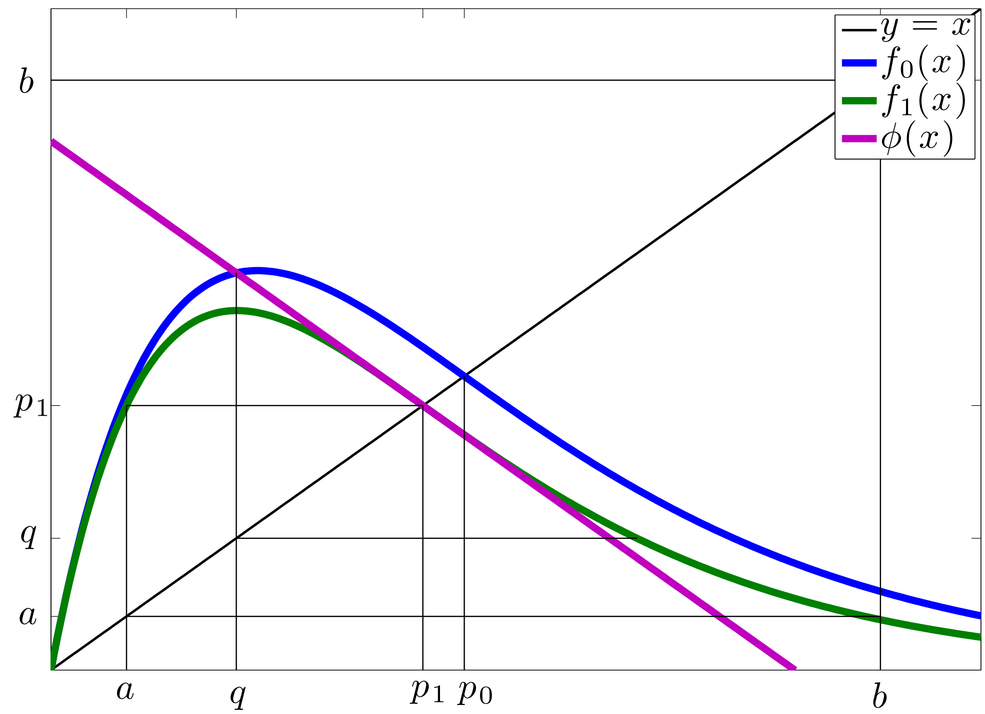


Figure 4.2: A plot of functions $f_0(x) = xe^{2-1.8x}$ and $f_1(x) = xe^{2-2x}$ satisfying the conditions of Lemmas 4.2.1 - 4.2.3 and Theorem 4.2.1. Here $p_1 = 1$, $p_0 = 10/9$, $a \approx 0.19$, $q \approx 0.50$, and $b \approx 2.23$.

The continuity of h gives the existence of a largest $\bar{q}$. Let q be the largest such $\bar{q}$ so that $f_0(x) > \phi(x) > f_1(x)$ on (q, p_1) and $f_0(x) > f_1(x)$ on $\mathcal{U} = (q, p_1) \cup \mathcal{I}$.

By assumption we have $f_1(x) < p_1$ for $x > p_1$ so $f_1(f_0(p_1)) < p_1$. Define $g(x) = x - f_1(f_0(x))$ and note that g inherits continuity from f_0 and f_1. Then $g(p_1) = p_1 - f_1(f_0(p_1)) > 0$.

We have that $q < p_1$ and since ϕ is monotone decreasing, $\phi(q) = f_0(q) > p_1$. By supposition $q = \phi(\phi(q)) = \phi(f_0(q))$. Then since $f_0(q) > p_1$, $\phi(f_0(q)) < f_1(f_0(q))$ implying that $f_1(f_0(q)) > q$. Then $g(q) = q - f_1(f_0(q)) < 0$.

Finally, by the Intermediate Value Theorem, there exists $t \in (q, p_1)$ such that $g(t) = 0$. That is, $f_1(f_0(x))$ has a fixed point in (q, p_1). $\square$

Having provided conditions to ensure that the composition of maps from a 2-periodic dynamical system will have a fixed point we move on to ensuring that the fixed point is attracting for the composition.

Remark 4.2.1. For the remainder of the section we use q to refer to the largest $\bar{q} \in (0, p_1)$ such that $f_0(\bar{q}) = \phi(\bar{q})$ and t is the largest fixed point of $f_1(f_0(x))$ in (q, p_1) as guaranteed by Lemma 4.2.1 unless otherwise stated.

Lemma 4.2.2. *Suppose $f_0 \in \mathcal{C}^0(\mathbb{R}^+) \bigcap \mathcal{C}^1(q, p_1)$ and $f_1 \in \mathcal{C}^0(\mathbb{R}^+) \bigcap \mathcal{C}^1(f_0(q, p_1))$ are as in Lemma 4.2.1. Also suppose that*

(i) $-1 < f_1'(x) < 1$ for $x \in f_0(q, p_1)$ and

(ii) $-1 < f_0'(x) < 1$ for $x \in (q, p_1)$.

Then $f_0(x) > p_1$ for $x \in (q, p_0)$,

$$\left| \frac{d}{dx} \left[f_1(f_0(x)) \right] \right| < 1 \text{ for } x \in (q, p_1),$$

and t is the only fixed point of $f_1(f_0(x))$ in (q, p_1).

Proof. To begin, we show that $f_0(x) > p_1$ for $x \in (q, p_0)$. Since q is the largest value in $(0, p_1)$ such that $\phi(q) = f_0(q)$ we must have $f_0(x) > \phi(x)$ for $x \in (q, p_1)$. For $x \in (q, p_1)$,

$$\phi(x) > \phi(p_1) = p_1$$

because ϕ is monotone decreasing. Thus

$$p_1 = \phi(p_1) < \phi(x) < f_0(x).$$

Then for $x \in (p_1, p_0)$,

$$p_1 < x < f_0(x)$$

because f_0 is a population model and p_0 is its unique fixed point. Therefore $f_0(x) > p_1$ for $x \in (q, p_0)$, which was the first desired result of the lemma.

The Chain Rule gives

$$\frac{d}{dx} \left[f_1(f_0(x)) \right] = f_1'(f_0(x)) f_0'(x).$$

Let $x \in (q, p_1)$ so that $-1 < f_0'(x) < 1$ by *(i)* and $-1 < f_1'(f_0(x)) < 1$ by *(ii)*. Thus we arrive at the second desired result of the lemma,

$$\left| \frac{d}{dx} \left[f_1(f_0(x)) \right] \right| = \left| f_1'(f_0(x)) f_0'(x) \right| < 1 \text{ for } x \in (q, p_1).$$

If t is not a unique fixed point in (q, p_1) then there exists $s \in (q, p_1)$ such that $f_1(f_0(s)) = s$

and $s < t$. Observe that $s, t \in (q, p_1)$ so $f_0'(s)$ and $f_0'(t)$ are defined. Also, since $f_0(s), f_0(t) > p_1$, $f_1'(f_0(s))$ and $f_1'(f_0(t))$ exist. Consider,

$$\frac{f_1(f_0(t)) - f_1(f_0(s))}{t-s} = \frac{t-s}{t-s} = 1$$

so by the Mean Value Theorem there exists $c \in (s,t)$ such that $\frac{d}{dx}\left[f_1(f_0(c))\right] = 1$. Since we know that

$$\left|\frac{d}{dx}\left[f_1(f_0(x))\right]\right| < 1$$

this is a contradiction and so t is unique. $\square$

The next example shows that the hypotheses of Lemma 4.2.2 are not necessary to obtain a geometric 2-cycle.

Example 4.2.1. Consider the 2-periodic dynamical system $\{f_0, f_1\}$ where

$$f_0(x) = xe^{2-1.5x} \text{ and } f_1(x) = xe^{2-2x}.$$

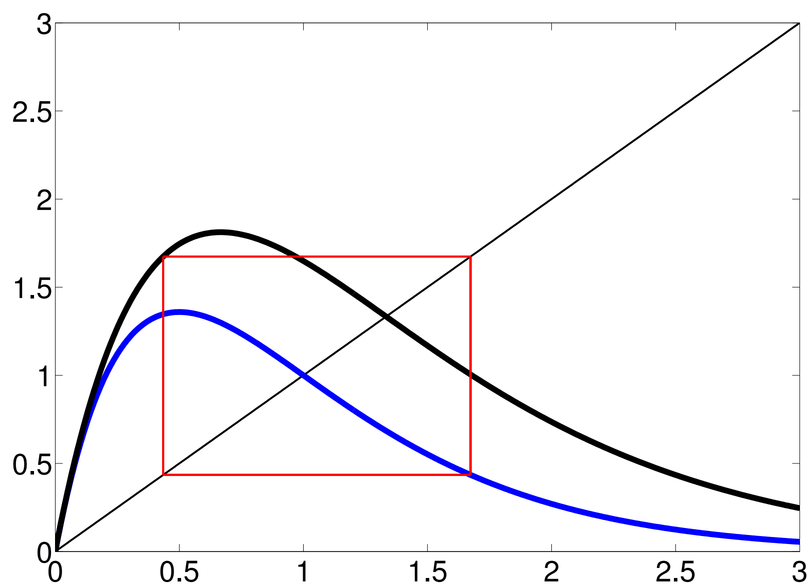


Figure 4.3: The 2-periodic system of Ricker maps from Example 4.2.1 defying Lemma 4.2.2 but displaying an attracting 2-cycle.

From Example 4.1.1 we can see that

$$f_0(4/3) = 4/3 = p_0,\ f_1(1) = 1 = p_1,\ \text{and}\ \phi_1(x) = -x + 2.$$

The unique solution to $\phi_1(x) = f_0(x)$ is $q \approx 0.3917$. Here the fixed point of $f_1(f_0(x))$ is $t \approx 0.4350$ and notice that $q < t < p_1$. We can also check that $\left|\frac{d}{dx}\left[f_1(f_0(t))\right]\right| \approx 0.8158$. However, $f_0'(t) = 1.3372$ which defies the conditions of Lemma 4.2.2. Figure 4.3 shows the periodic dynamical system after 300 iterations with $x_0 = 0.75$. The geometric 2-cycle that appears to be globally attracting is $\{t = 0.4350, 1.6738\}$. ■

Having established an attracting fixed point for $f_1(f_0(x))$ we proceed to using the result to show that the periodic dynamical system $\{f_0, f_1\}$ has a locally attracting geometric 2-cycle.

Lemma 4.2.3. *Consider the periodic dynamical system $\{f_0, f_1\}$ where f_0 and f_1 are as in Lemma 4.2.1 and Lemma 4.2.2. Then $\{f_0, f_1\}$ has a locally attracting geometric 2-cycle.*

Proof. Let x_1^* be the fixed point guaranteed by Lemma 4.2.1 so that

$$f_1(f_0(x_1^*)) = x_1^*.$$

Then by Lemma 4.2.2

$$\left|\frac{d}{dx}\left[f_1(f_0(x_1^*))\right]\right| < 1.$$

Denote $f_0(x_1^*) = f_0(f_1(f_0(x_1^*))) = x_2^*$ and note that $f_1(x_2^*) = x_1^*$. Then

$$f_0(f_1(x_2^*)) = f_0(x_1^*) = f_0(f_1(f_0(x_1^*))) = x_2^*$$

meaning x_2^* is a fixed point of $f_0(f_1(x))$.

Using the chain rule,

$$\frac{d}{dx}\left[f_1(f_0(x_1^*))\right] = f_1'(f_0(x_1^*))f_0'(x_1^*) = f_1'(x_2^*)f_0'(x_1^*)$$

and

$$\frac{d}{dx}\left[f_0(f_1(x_2^*))\right] = f_0'(f_1(x_2^*))f_1'(x_2^*) = f_0'(x_1^*)f_1'(x_2^*).$$

Observe that $x_2^* = f_0(x_1^*)$ and $x_1^* \in (q, p_1)$. Thus $x_2^* \in f_0(q, p_1)$ and $f_1'(x_2^*)$ exists. Also,

$$q < f_1(x_2^*) = f_1(f_0(x_1^*)) = x_1^* < p_1$$

and so $f_0'(f_1(x_2^*))$ exists. Thus $\left|\frac{d}{dx}\left[f_0(f_1(x_2^*))\right]\right| < 1$ by assumptions *(i)* and *(ii)* of Lemma 4.2.2. Since each 2-fold composition is locally attracted to a fixed point, the periodic dynamical system is locally attracted to a geometric 2-cycle. □

We need a final preliminary lemma to ensure a 2-periodic system of population models will have a globally asymptotically stable geometric 2-cycle.

Lemma 4.2.4. *Suppose f_0 and f_1 are population models satisfying the conditions of Lemma 4.2.3. Then there exists $a \in (0, q)$ such that $f_0(a) = p_1$.*

Proof. The enveloping function ϕ is monotone decreasing and $\phi(p_1) = p_1$. Since $q < p_1$, $\phi(q) > p_1$. Also, $f_0(0) = 0$ since f_0 is a population model. Define $h(x) = f_0(x) - p_1$ so that $h(0) < 0$ and $h(q) > 0$. By the Intermediate Value Theorem, there exists $a \in (0, q)$ such that $f_0(a) = p_1$. □

Remark 4.2.2. For Theorems 4.2.1 and 4.2.2 let a be the largest point in $(0, q)$ such that $f_0(a) = p_1$ which exists by continuity.

Theorem 4.2.1. *Suppose f_0 and f_1 are population models satisfying the conditions of Lemmas 4.2.1-4.2.4. Suppose there is only one $a \in (0, q)$ such that $f_0(a) = p_1$. Suppose there exists $\bar{b} \in (p_1, \infty)$ such that $f_1(\bar{b}) = a$ and let b be the least such $\bar{b}$. Suppose further that*

(i) $\max_{x \in \mathbb{R}^+} f_0(x) \le b$ *and* $\max_{x \in \mathbb{R}^+} f_1(x) \le b$,

(ii) $f_1(x) < p_1$ *on* (p_1, ∞),

(iii) $f_1(f_0(x)) > q$ *on* $[q, b]$,

(iv) $\phi(x) > f_0(x)$ *for* $x \in (0, q)$ *and* $\phi(x) < f_0(x)$ *for* $x \in (p_0, \infty)$.

Then the periodic dynamical system $\{f_0, f_1\}$ has a globally asymptotically stable geometric 2-cycle.

Assumption *(ii)* of Theorem 4.2.1 is actually already assumed in Lemma 4.2.1. However, the need for the assumption arises explicitly in the proof and so it is restated here for the reader's convenience. Also, assumption *(iii)* is stated here in a manner that is convenient for the proof. Assumption *(iii)* could be restated as $f_1(x) > q$ for $x \in f_0([q, b])$ and checked without considering the composition of the functions f_0 and f_1.

Proof. Let $F(x) = f_1(f_0(x))$. By Lemma 4.2.3, $\{f_0, f_1\}$ has a locally attracting geometric 2-cycle with one point in (q, p_1). To begin we will show that for $x_0 \in (0, \infty)$ there exists $i \in \mathbb{Z}^+$ such that $F^i(x_0) \in (q, p_1)$. Note that F must have a periodic point other than t for $\{f_0, f_1\}$ to have a geometric cycle other than $\{t, f_0(t)\}$.

(a) Suppose $x_0 \in (0, a]$. Then $f_0(x_0) > x_0$ because f_0 is a population model and $a < p_0$. Because a is unique, $f_0(x) < p_1$ for $x \in (0, a)$ and since $f_0(a) = p_1$ we have

$$x_0 < f_0(x_0) \leq p_1.$$

Then

$$x_0 < f_0(x_0) \leq f_1(f_0(x)) = F(x_0)$$

because f_1 is a population model and $f_0(x_0) \leq p_1$. So either $F^i(x_0) > a$ for some i or $\lim_{i\to\infty} F^i(x_0) = L$ and $L \leq a$. If $\lim_{i\to\infty} F^i(x_0) = L$ then $F(L) = L$ because F is continuous. But, we must have $F(L) > L$ because $L \leq a$. So the orbit of x_0 gets larger than a and in fact cannot converge to a value in the interval $(0, a]$. Also note that $F^i(x_0) \leq b$ by assumption *(i)*.

(b) Suppose $x_0 \in (a, q]$. Since $f_0(x) < \phi(x)$ on $(0, q)$ we have $f_0(x_0) < \phi(x_0)$ for $x_0 \in (a, q)$ and $f_0(x) > p_1$ on $(a, q]$ by the uniqueness of a. Then

$$p_1 < f_0(x_0) \leq \phi(x_0).$$

Since ϕ is monotone decreasing and self-inversing we have

$$x_0 = \phi(\phi(x_0)) \leq \phi(f_0(x_0)).$$

By the definition of enveloping and since $f_0(x_0) > p_1$ we also have

$$\phi(f_0(x_0)) < f_1(f_0(x_0)) < p_1.$$

Thus, $x_0 < F(x_0) \leq p_1$. So either there exists i such that $F^i(x_0) > q$ or $\lim_{i\to\infty} F^i(x_0) = L$ and $L \leq q$. The second case cannot happen by an argument identical to that in the previous interval. Thus there exists i such that $F^i(x_0) \in (q, p_1)$ for $x_0 \in (a, q]$. Since $x_0 < F(x_0)$ we have precluded a point of period 2 for F in $(0, q]$.

(c) Suppose $x_0 \in [p_1, p_0]$. By a result of Lemma 4.2.2 and since $f_0(x) \leq b$,

$$p_1 < f_0(x_0) \leq b$$

on $[p_1, p_0]$. Thus by *(ii)* and *(iii)*,

$$a < q < F(x_0) < p_1$$

and the orbit has either entered the desired interval, (q, p_1), or the interval $(a, q]$ which was the second case. In the second case we knew that the orbit stayed in $(a, q]$ until it entered $(q, p_1]$ so it is not possible that $\{f_0, f_1\}$ has a cycle with points in $(a, q] \cup [p_1, p_0]$.

(d) Suppose $x_0 \in (p_0, b]$. If $f_1(x) < p_0$ for all $x \in (0, \infty)$ then $F(x_0) = f_1(f_0(x_0)) < p_0$ and $q < F(x_0)$ by *(iii)*. In this instance, the orbit enters case (b) or case (c) after one iteration.

If $f_0(x_0) \geq p_1$ then $F(x_0) \leq p_1$ by *(ii)* and again $q < F(x_0)$ by *(iii)*. In this instance, the orbit enters the desired interval after one iteration.

If $f_0(x_0) < p_1$ then either $q < F(x_0) \leq p_0$ by *(ii)*, entering the desired interval or case (c), or $F(x_0) > p_0$. Note that in the latter case we have $x_0 > p_0$, $f_0(x_0) < p_1$, and $F(x_0) > p_0$. By *(iv)*,

$$\phi(x_0) < f_0(x_0)$$

and so

$$\phi(x_0) < f_0(x_0) < p_1 < x_0.$$

Applying ϕ again and using that ϕ is monotone decreasing and self-inversing,

$$\phi(x_0) < \phi(p_1) = p_1 < \phi(f_0(x_0)) < x_0.$$

Since $f_0(x_0) < p_1$ and ϕ is an enveloping function for f_1,

$$F(x_0) = f_1(f_0(x_0)) < \phi(f_0(x_0)) < x_0.$$

Then the orbit of x_0 under F is decreasing and cannot have a limit in $(p_0, b]$ by an argument similar to those in previous cases. Thus there exists $i \in \mathbb{Z}^+$ such that $F^i(x_0) < p_0$. Noting that $q < F(x_0)$ by *(iii)* we have that the orbit enters the desired interval or case (c).

(e) Finally, suppose $x_0 \in (b, \infty)$. Since $f_1(x) \leq b$ for all $x \in (0, \infty)$, $F(x_0) \leq b$ forcing the orbit into a previous case after one iteration.

Knowing now that all orbits enter (q, p_1) we now show that

$$\lim_{i \to \infty} F^i(x_0) = t$$

for $x_0 \in (q, p_1)$. From the results of Lemma 4.2.2 we have that $f_0(x_0) > p_1$ and so $F(x_0) < p_1$.

Also, by assumption *(iii)* of the theorem, $F(x_0) > q$. Thus the forward orbit of x_0 under F is contained within (q, p_1).

From Lemma 4.2.2,

$$\left|F'(x_0)\right| < 1.$$

Let t be the fixed point of F in (q, p_1) as in Lemma 4.2.1 and $x \in (q, p_1)$. We have

$$|F(t) - F(x)| < |t - x|$$

by the Mean Value Theorem. Thus any orbit in (q, p_1) converges to t. Therefore t is a globally attracting fixed point of $F(x)$ and by the arguments of Lemma 4.2.3, $\{f_0, f_1\}$ has a globally attracting geometric 2-cycle. □

Figure 4.2 shows the population models $f_0(x) = xe^{2-1.8x}$ and $f_1(x) = xe^{2-2x}$ and the enveloping function $\phi(x) = -x + 2$ that satisfy all the conditions of Lemma 4.2.1 - 4.2.3 as well as the conditions of Theorem 4.2.1. The results of the theorem can be realized by a careful analysis of the plot.

The conditions of Theorem 4.2.1 are sufficient to ensure the existence of a globally attracting geometric 2-cycle for a period-2 system of population models but the conditions are not necessary as the following example shows.

Example 4.2.2. Consider the period-2 dynamical system

$$\mathcal{S} = \left\{ f_0(x) = xe^{4-(2.1+2e^{2-2.1x})x}, f_1(x) = xe^{2-1.9x}. \right\}$$

The system, $\mathcal{S}$, has a globally attracting geometric 2-cycle as seen in Figure 4.4 and satisfies the conditions of Lemmas 4.2.1 through 4.2.3 with $\phi(x) = -x + \frac{40}{21}$. However, there exists more than one $x \in (0, q)$ such that $f_0(x) = p_1$ defying the conditions of Theorem 4.2.1. ■

The reader may note that Theorem 4.2.1 assumes $f_1(x) = a$ for some $x > p_1$. This condition may not hold if $f_1(x) > a$ for $x > p_1$. As such, we develop the following result to handle this situation.

Theorem 4.2.2. *Suppose f_0 and f_1 are population models satisfying the conditions of Lemmas 4.2.1-4.2.4. Suppose there is only one $a \in (0, q)$ such that $f_0(a) = p_1$. Suppose also*

(i) $a < f_1(x) < p_1$ *on* (p_1, ∞),

(ii) $f_1(x) > q$ *on* $f_0([q, p_1])$,

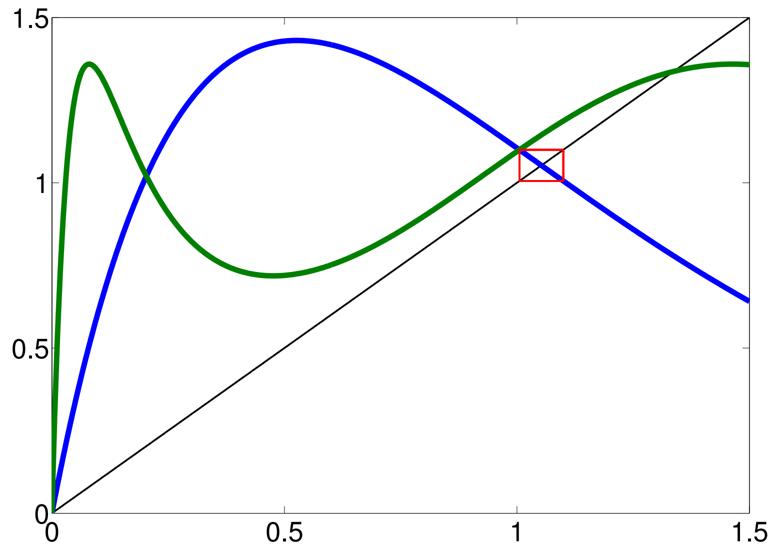


Figure 4.4: Plots of f_0 (green) and f_1 (blue) and the globally attracting 2-cycle for $\{f_0, f_1\}$ is Example 4.2.2.

(iii) $\phi(x) > f_0(x)$ *for* $x \in (0, q)$ *and* $\phi(x) < f_0(x)$ *for* $x \in (p_0, \infty)$.

Then the periodic dynamical system $\{f_0, f_1\}$ *has a globally attracting geometric 2-cycle.*

Proof. The proof for Theorem 4.2.2 is identical to that for Theorem 4.2.1 when $x_0 \in (0, q]$. We therefore omit these cases from the proof. Again, we let $F(x) = f_1(f_0(x))$.

(a) Suppose $x_0 \in [p_1, p_0]$. By the first result of Lemma 4.2.2, $p_1 < f_0(x)$ for $x \in [p_1, p_0)$. Also, $f_0(p_0) = p_0 > p_1$ so $f_0(x) > p_1$ for $x \in [p_1, p_0]$. By assumption *(i)*, $a < f_1(x) < p_1$ for $x > p_1$. Thus,

$$a < F(x_0) < p_1$$

and the orbit has entered the desired interval or $(a, q]$. From the proof of Theorem 4.2.1, that an iteration was not greater than p_1 so it is not possible that $\{f_0, f_1\}$ has a cycle with points in $(a, q] \cup [p_1, p_0]$.

(b) Suppose $x_0 \in (p_0, \infty)$. If $f_1(x) < p_0$ for all $x \in (0, \infty)$ then $F(x_0) < p_0$ for all $x \in (0, \infty)$. In this instance, the orbit enters the desired interval or $(0, q) \cup [p_1, p_0]$ after one iteration.

We may have $f_1(x_0) \geq p_0$. If $f_0(x_0) \geq p_1$ then $a < F(x_0) \leq p_1$ by *(i)* and the orbit has entered the desired interval or $(a, q]$ after one iteration.

If $f_1(x_0) \geq p_0$ and $f_0(x_0) < p_1$ then either $F(x_0) \leq p_0$ or $F(x_0) > p_0$. In the former case we have $a < F(x_0) \leq p_0$ and the orbit has entered the desired interval or $(a, q] \cup [p_1, p_0]$.

In the latter case, when $f_1(x_0) \geq p_0$, $f_0(x_0) < p_1$, and $F(x_0) > p_0$ we have

$$\phi(x_0) < f_0(x_0) < p_1 < x_0$$

by *(iii)*. Applying ϕ again and using the monotone decreasing nature of ϕ,

$$\phi(x_0) < \phi(p_1) = p_1 < \phi(f_0(x_0)) < \phi(\phi(x_0)) = x_0.$$

Since $f_0(x_0) < p_1 < p_0$ we have

$$\phi(f_0(x_0) > f_1(f_0(x_0)) = F(x_0)$$

by the definition of enveloping. Thus,

$$F(x_0) < \phi(f_0(x_0)) < x_0.$$

So either there exists $i \in \mathbb{Z}^i$ such that $F^i(x_0) < p_0$ or $\lim_{i \to \infty} F^i(x_0) = L$ for $L \in (p_0, \infty)$. In the latter case, $F(L) = L$ and $L > p_0$ because F is continuous. However, we have shown that $F(L) < L$ for $L \in (p_0, \infty)$. Also, note that $a < F^i(x_0)$ for $x_0 \in (p_1, \infty)$ by *(i)* so there cannot be a cycle with points in (p_0, ∞).

The remainder of the proof is identical to that for Theorem 4.2.1. $\square$

Since f_1 is assumed to be continuous and $f_1(x) < p_1$ for $x \in (p_1, \infty)$ in Lemma 4.2.3, either there exists $x > p_1$ such that $f_1(x) = a$ or $f_1(x) > a$ for all $x > p_1$. Therefore, Theorems 4.2.1 and 4.2.2 give conditions for a globally asymptotically stable geometric 2-cycle for either case for f_1 and therefore for many periodic dynamical systems $\{f_0, f_1\}$ meeting the conditions of Lemma 4.2.3.

Lemma 4.2.5. *Suppose $f_0, f_1 \in \mathcal{C}^1(\mathbb{R}^+) \cap \mathcal{P}$ satisfy the conditions of Lemma 4.2.3. Suppose f_0 is nondecreasing on $(0, q)$, $-1 < f_1'(x) < 0$ for $x \in f_0(q, p_1)$, and $-1 < f_0'(x) < 0$ for $x \in (q, p_1)$. Then $f_1(f_0(x)) > q$ for $x \in [q, p_1]$.*

Proof. If $-1 < f_0'(x) < 0$ for $x \in (q, p_1)$ then f_0 is decreasing on (q, p_1) and so $f_0(p_1) < f_0(q)$. Then since $-1 < f_1'(x) < 0$ for $x \in (f_0(p_1), f_0(q))$,

$$f_1(f_0(q)) < f_1(f_0(x)) \ \forall \ x \in (q, p_1).$$

Since $f_0(q) = \phi(q) > p_1$ we also have

$$\phi(f_0(q)) < f_1(f_0(q)),$$

giving

$$q = \phi(\phi(q)) = \phi(f_0(q)) < f_1(f_0(q)) < f_1(f_0(x)) \; \forall \; x \in (q, p_1). \qquad \square$$

Colrollary 4.2.1. *Suppose f_0 and f_1 are as in Lemma 4.2.3 and Lemma 4.2.5 and there exists unique $a \in (0, p_1)$ such that $f_0(a) = p_1$. If*

- *there exists a smallest $b > p_1$ such that $f_1(b) = a$ and $\max\limits_{x \in \mathbb{R}^+} f_0(x) \leq b$ or*
- *$f_1(x) > a$ for all $x > p_1$.*

Then $\{f_0, f_1\}$ has a globally asymptotically stable geometric 2-cycle.

Proof. Lemma 4.2.5 together with the conditions given in the corollary guarantee that $\{f_0, f_1\}$ satisfy all the conditions of Theorem 4.2.1 or Theorem 4.2.2. $\square$

The preceding results have all assumed that f_1, the "smaller" population model, has an enveloping function while f_0 may not. We change now to considering systems in which f_0 is enveloped while f_1 may not be. The results are similar in general but the respective proofs are different.

Here we redefine $q \in (p_0, \infty)$ as the smallest value such that $\phi(q) = f_1(q)$. When ϕ was the enveloping function of f_1 then there is always a point of intersection of ϕ and f_0. However, when ϕ is the enveloping function of f_0 there may not be a point of intersection between ϕ and f_1. We restrict ourselves here to considering cases when such a point of intersection does exist.

Remark 4.2.3. For the remainder of the section we use q to refer to the smallest $q \in (p_0, \infty)$ such that $f_1(q) = \phi(q)$.

Lemma 4.2.6. *Suppose $f_0, f_1 \in \mathcal{P}$ with fixed points p_0 and p_1 respectively and there exists open interval $\mathcal{I}$ with $p_0, p_1 \in \mathcal{I}$ such that $f_0(x) > f_1(x)$ for $x \in \mathcal{I}$. Suppose further that $f_0(x)$ has a monotone decreasing, self-inversing enveloping function $\phi(x)$ and $f_0(f_1(p_0)) > p_0$. Suppose there exist $q \in (p_0, \infty)$ be such that $\phi(q) = f_1(q)$. Then $f_0(f_1(x))$ has a fixed point in (p_0, q).*

Proof. Since $p_0, p_1 \in \mathcal{I}$ and $f_0(x) > f_1(x)$ on $\mathcal{I}$ we must have $p_0 > p_1$. By supposition, $f_1(p_0) < f_0(p_0) < p_0$ and $f_0(f_1(p_0)) > p_0$. Define $g(x) = x - f_0(f_1(x))$. Then $g(p_0) < 0$.

Consider now that since $\phi(q) = f_1(q)$ we have

$$\phi(\phi(q)) = \phi(f_1(q)).$$

Since ϕ is monotone decreasing we must have $\phi(q) < \phi(p_0) = p_0$ and so $\phi(q) = f_1(q) < p_0$. Thus, by the definition of enveloping,

$$f_0(f_1(q)) < \phi(f_1(q)) = \phi(\phi(q)) = q.$$

So $g(q) > 0$.

Finally, by the Intermediate Value Theorem, there exists $t \in (p_0, q)$ such that $g(t) = 0$ and so $f_0(f_1(x))$ has a fixed point in (p_0, q). $\square$

For the remainder of the section we use t to refer to the smallest fixed point of $f_0(f_1(x))$ in the interval (p_0, q).

Lemma 4.2.7. *Suppose $f_1 \in \mathcal{C}^0(\mathbb{R}^+) \bigcap \mathcal{C}^1(p_0, q)$ and $f_0 \in \mathcal{C}^0(\mathbb{R}^+) \bigcap \mathcal{C}^1(f_1(p_0, q))$ are as in Lemma 4.2.6 so that $f_0(f_1(x))$ has at least one fixed point $t \in (p_0, q)$. Also suppose that*

(i) $|f_1'(x)| < 1$ for $x \in (p_0, q)$ and

(ii) $|f_0'(x)| < 1$ for $x \in f_1(p_0, q)$.

Then

$$\left| \frac{d}{dx}[f_0(f_1(x))] \right| < 1 \text{ for } x \in (p_0, q)$$

and t is the unique fixed point of $f_0(f_1(x))$ in (p_0, q).

Proof. By the chain rule,

$$\frac{d}{dx}[f_0(f_1(x))] = f_0'(f_1(x))f_1'(x).$$

For $x \in (p_0, q)$, $|f_0'(f_1(x))| < 1$ and $|f_1'(x)| < 1$. Therefore,

$$\left| \frac{d}{dx}[f_0(f_1(x))] \right| < 1 \text{ for } x \in (p_0, q).$$

Now suppose for contradiction that there exists $s \in (p_0, q)$ such that $f_0(f_1(s)) = s$ and $s > t$. Then

$$\frac{f_1(f_0(s)) - f_1(f_0(t))}{s - t} = \frac{s - t}{s - t} = 1.$$

By the Mean Value Theorem there exists $c \in (t,s)$ such that $\frac{d}{dx}[f_0(f_1(c))] = 1$. This contradicts that $\left|\frac{d}{dx}[f_0(f_1(x))]\right| < 1$ and so t is a unique fixed point in (p_0, q). $\square$

Lemma 4.2.8. *Consider the periodic dynamical system $\{f_0, f_1\}$ where f_0 and f_1 are as in Lemmas 4.2.6 and 4.2.7. Then $\{f_0, f_1\}$ has a locally attracting geometric 2-cycle.*

Proof. The proof is identical to that of Lemma 4.2.3 with the roles of f_0 and f_1 reversed. $\square$

Theorem 4.2.3. *Suppose f_0 and f_1 are as in Lemmas 4.2.6-4.2.8. Suppose further*

(i) $p_0 < f_0(f_1(x))$ *on* $[p_1, q]$,

(ii) $f_0(f_1(x)) < q$ *on* $[0, q]$,

(iii) $\phi(x) > f_1(x)$ *for* $x \in (0, q)$ *and* $\phi(x) < f_1(x)$ *for* $x \in (q, \infty)$.

Then $\{f_0, f_1\}$ has a globally asymptotically stable geometric 2-cycle.

Proof. Define $F(x) = f_0(f_1(x))$. To begin we will show that for all $x \in \mathbb{R}^+$, there exists $i \in \mathbb{Z}^+$ such that $F^i(x) \in (p_0, q)$.

(a) Suppose $x_0 \in (0, p_1)$. If $f_1(x_0) < p_0$ then $x_0 < f_1(x_0) < F(x_0)$ since f_0 and f_1 are population models. If $f_1(x_0) = p_0$ then $F(x_0) = p_0 > x_0$.

Finally, if $f_1(x_0) > p_0$ then $F(x_0) < f_1(x_0)$ because f_0 is a population model. By the definition of enveloping and *(iii)*,

$$\phi(f_1(x_0)) < f_0(f_1(x_0)) = F(x_0) < f_1(x_0) < \phi(x_0). \tag{4.1}$$

Since ϕ is monotone decreasing and self-inversing we apply it to (4.1) to get,

$$x_0 = \phi(\phi(x_0)) < \phi(f_1(x_0)) < \phi(F(x_0)) < f_1(x_0). \tag{4.2}$$

So

$$x_0 < \phi(f_1(x_0)) < F(x_0) < f_1(x_0)$$

by (4.1) and (4.2). Noting that p_1 is not fixed under F, for all $x_0 \in (0, p_1]$ there exists $i \in \mathbb{Z}^+$ such that $F^i(x_0) > p_1$ and by *(ii)* $F^i(x_0) < q$.

(b) Suppose $x_0 \in [p_1, p_0]$. By assumption *(i)*, $p_0 < F(x_0) < q$.

(c) Suppose $x_0 \in (q, \infty)$. Since f_1 is a population model and $q > p_1$, $f_1(x_0) < x_0$. If $f_1(x_0) > p_0$ then $F(x_0) < f_1(x_0) < x_0$ because f_0 is a population model. If $f_1(x_0) = p_0$ we have $F(x_0) = p_0$ which was considered in case (b). If $f_1(x_0) < p_0$ we have $F(x_0) > f_1(x_0)$ because f_0 is a population model and

$$\phi(x_0) < f_1(x_0) < F(x_0) < \phi(f_1(x_0)).$$

Thus we may apply ϕ again to see

$$f_1(x_0) < \phi(F(x_0)) < \phi(f_1(x_0)) < x_0.$$

Recall, that since $f_1(x_0) < p_0$ we know $f_1(x_0) < F(x_0)$ giving,

$$f_1(x_0) < F(x_0) < \phi(f_1(x_0)) < x_0.$$

Thus, either the orbit enters the desired interval after one iteration or $F(x_0) < x_0$. Since q is not a fixed point of F, for all $x_0 \in (q, \infty)$ there exists $i \in \mathbb{Z}^+$ such that $F^i(x_0) < q$.

Having that there exists $i \in \mathbb{Z}^+$ such that $F^i(x_0) \in (p_0, q)$ for all $x_0 \in \mathbb{R}^+$ we now show that

$$\lim_{i \to \infty} F^i(x) = t$$

for $x \in (p_0, q)$. By assumption *(i)* we have that $p_0 < F(x_0) < q$ for $x_0 \in (p_0, q)$ so the forward orbit of x_0 under F is contained within (p_0, q).

By Lemma 4.2.7,

$$|F'(x)| < 1.$$

Let t be the fixed point of $F(x)$ in (p_0, q) as in Lemma 4.2.6 and $x \in (p_0, q)$. Then by the Mean Value Theorem,

$$\frac{|F(x) - F(t)|}{|x - t|} < 1$$

and so

$$|F(x) - F(t)| < |x - t|.$$

Thus any orbit with initial value in (p_0, q) converges to t. Therefore, t is a globally attracting fixed point of $F(x)$ and arguments similar to those of Lemma 4.2.8, $\{f_0, f_1\}$ has a globally asymptotically stable geometric 2-cycle. □

Remark 4.2.4. The reader may note that condition *(iii)* in Theorem 4.2.3 is very similar to enveloping. This is actually the inspiration for "weak enveloping" that appears in Chapter 5.

Like with Theorem 3.2.1, the main benefit of Theorems 4.2.1, 4.2.2, and 4.2.3 is that they do not require the functions in $\{f_0, f_1\}$ to be composed to determine the behavior of the system. All of the conditions require checking conditions on f_0 and f_1 separately.

4.3 General Period-n Systems

The results of the preceding sections are restrictive in that they either put severe conditions on the maps of a period-n dynamical system or they only apply to a period-2 system. Unfortunately, for a period-n system of population models in which the maps do not share a fixed point, it is unclear whether or not enveloping can be applied. Such systems often exhibit an attracting geometric cycle with the same period as that of the system and the individual maps may have enveloping functions but it is unclear what role the enveloping functions may play. The following example gives such a system.

Example 4.3.1. Let $\mathcal{S} = \{f_0, f_1, f_2, f_2, f_4\}$ be a 5-periodic system of maps referred to by Cull as Hassel type maps in [11]. Let

$$f_0(x) = \frac{2^{1.75}x}{(1+0.85x)^{1.75}}, \ f_1(x) = \frac{2x}{(1+x)}, \ f_2(x) = \frac{2^3x}{(1+1.1x)^3},$$

$$f_3(x) = \frac{2^{0.9}x}{(1+0.9x)^{0.9}}, \ f_4(x) = \frac{2^{2.5}x}{(1+1.2x)^{2.5}}.$$

Figure 4.5 shows an attracting geometric 5-cycle for $\mathcal{S}$. In [11], an enveloping function was established for models of this form. However, it is not clear if enveloping plays any role in this example. ■

It may be possible to establish the existence of a globally attracting geometric n-cycle for a period-n dynamical system of population models using only theorems for period-2 systems. The result of the next theorem is very useful in establishing the existence of such an attractor.

Theorem 4.3.1. *Suppose $\{f_0, \ldots, f_{n-1}\}$ is a periodic dynamical system of population models with a positive, globally asymptotically stable geometric n-cycle. Then*

$$(f_{(i+n-1) \bmod n} \circ \ldots \circ f_{(i+1) \bmod n} \circ f_{i \bmod n})(x)$$

is a population model for $i = 0, 1, \ldots, n-1$.

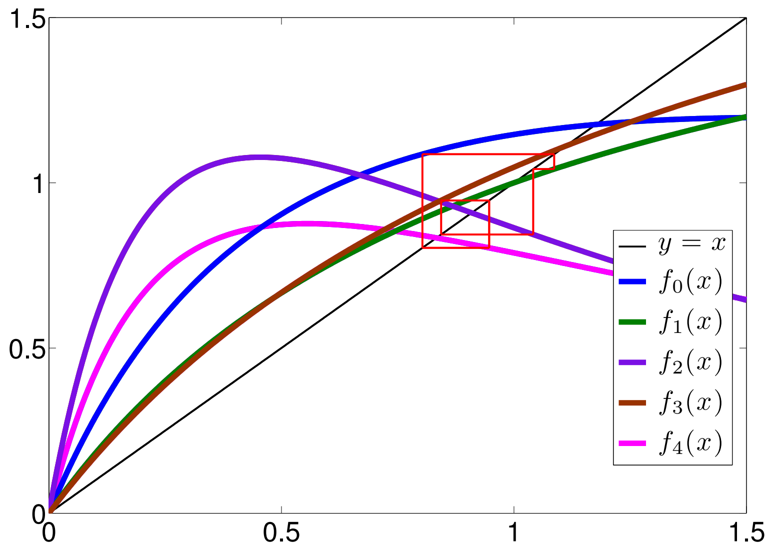


Figure 4.5: The attracting geometric 5-cycle for $\mathcal{S}$ in Example 4.3.1.

Proof. By the hypothesis of the theorem, f_j is a population model for $j = 0, 1, \ldots, n-1$, $f_j(0) = 0$ and each f_j is continuous. Define

$$F_i(x) = (f_{(i+n-1) \bmod n} \circ \ldots \circ f_{(i+1) \bmod n} \circ f_{i \bmod n})(x).$$

By continuity of each f_j there exists $\epsilon_i > 0$ such that $F_i(x) > x$ for $0 < x < \epsilon_i$. Since $\{f_0, \ldots, f_{n-1}\}$ has a positive, globally asymptotically stable geometric n-cycle, $F_i(x)$ has a globally asymptotically stable fixed point x_i^* which must be unique. As such we must have $F_i(x) > x$ for $0 < x < x_i^*$. Suppose for contradiction that there exists $y \in (x_i^*, \infty)$ such that $F_i(y) > y$. Then either there exists $\hat{x} \in (x_i^*, \infty)$ such that $F_i(\hat{x}) = \hat{x}$ which is a contradiction or $F_i(x) > x$ for all $x \in (x_i^*, \infty)$. However, if $F_i(x) > x$ for all $x \in (x_i^*, \infty)$ then the orbit of such an x cannot converge to x_i^* which contradicts the global attracting of x_i^*. Hence $F_i(x) < x$ for $x > x_i^*$. Therefore, $F_i(x)$ is a population model. $\square$

The benefit of Theorem 4.3.1 is that if two population models f_0 and f_1 meet the conditions of Theorems 4.2.1, 4.2.2, or 4.2.3 (or any theorem that establishes the existence of a globally attracting geometric 2-cycle for a 2-periodic system) then their compositions $f_0(f_1(x))$ and $f_1(f_0(x))$ are population models and there are bounds on the locations of the fixed points.

We now outline a technique with a general example that uses Theorem 4.3.1 to establish the existence of a globally attracting n-cycle for a period-n dynamical system. We restrict ourselves to a period-3 system but the method generalizes to any integer n.

Let $\mathcal{S} = \{f_0, f_1, f_2\}$. If it can be established that $\{f_0, f_1\}$ has a globally attracting 2-cycle using any of Theorems 4.2.1- 4.2.3 then $f_0(f_1(x))$ and $f_1(f_0(x))$ are both population models by Theorem 4.3.1. Define $F(x) = f_1(f_0(x))$ and consider now $\{F, f_2\}$. If a theorem can be applied to ensure that $\{F, f_2\}$ has a globally attracting geometric 2-cycle $\{t_0, t_2\}$ then we have

$$F(t_0) = t_2 \text{ and } f_2(t_2) = t_0.$$

Since $F(t_0) = f_1(f_0(t_0))$ we can define $t_1 = f_0(t_0)$. Thus,

$$f_0(t_0) = t_1,\ f_1(t_1) = t_2,\ \text{and} f_2(t_2) = t_0$$

and $\mathcal{S}$ must have a geometric 3-cycle. Furthermore, since $\{t_0, t_2\}$ was globally attracting for $\{F, f_2\}$, the geometric 3-cycle $\{t_0, t_1, t_2\}$ must be globally attracting for $\mathcal{S}$.

The method outlined here is relatively cumbersome. The conditions necessary for the theorems established in this chapter are restrictive and may be difficult to check. However, if improved theorems for determining the existence of a globally attracting 2-cycle can be found then the outlined method can be applied to a wide range of n-periodic dynamical systems to ensure the existence of globally attracting geometric cycle.

CHAPTER 5

n-Periodic Dynamical Systems

It is common practice in the study of autonomous difference equations (or of dynamical systems in general) to consider the behavior of a dynamical system that is in some way close to a system whose behavior is well understood. With this in mind, we begin now to analyze periodic dynamical systems that are perturbations of a system whose behavior is known. Of primary interest are periodic dynamical systems similar to a periodic dynamical system with a geometric cycle and periodic systems whose maps are close to an enveloped population model.

A typical definition of two maps $f, g : X \to X$ being $\mathcal{C}^0$ close on a set $Y \subset X$ would be f and g are $\mathcal{C}^0$-ϵ close on Y if $d(f(x), g(x)) < \epsilon$ for all $x \in Y$ (see [21]). With this definition, f and g are assumed to be identical on $X \setminus Y$ so that the dynamics under g are identical to the dynamics of f on $X \setminus Y$ and the dynamics are similar on Y.

In this work, however, we have properties of population models and concepts like enveloping to help describe global dynamics and therefore do not require such a restrictive definition. Furthermore, we desire a result that has application to the first Cushing and Henson conjecture which does not require the maps to be identical outside of some interval. Therefore, we desire a definition that ensures a perturbation, g, of a map, f, is close to f on an interval of interest, Y, and allow g to differ from f outside of Y. Since our work general concerns population models, we will restrict $X = \mathbb{R}$ and let $f, g \in \mathcal{C}^0(\mathbb{R})$. Given a compact interval $Y \subset \mathbb{R}$, define the equivalence relation $\sim$ on $\mathcal{C}^0(\mathbb{R})$ by

$$f \sim g \text{ if and only if } f(x) = g(x) \text{ for all } x \in Y.$$

Then we define a metric d on the space of equivalence classes, $\mathcal{C}^0(\mathbb{R})/\sim$, by

$$d([f],[g]) = \max_{Y} |f(x) - g(x)|.$$

We say f and g are $\mathcal{C}^0$-ϵ close on Y if $d([f],[g]) < \epsilon$ where it should be understood that the maximum is taken over Y. Furthermore, we refer to a $\mathcal{C}^0$-ϵ neighborhood of f on Y to mean all $g \in \mathcal{C}^0(\mathbb{R})$ such that $d([f],[g]) < \epsilon$. The benefit of this construction is that it is identical to the definition from [21] on the interval Y allowing us to use the local results, but this definition allows us to consider a much larger space of functions.

We will also have need to consider maps that have derivatives that are close on a compact set Y. Let $Y \subset Z \subset \mathbb{R}$ where Z is open and suppose $f, g \in \mathcal{C}^1(Z)$. We again define an equivalence relation, $\sim$, on $\mathcal{C}^1(Z)$ by

$$f \sim g \text{ if and only if } f(x) = g(x) \text{ and } f'(x) = g'(x) \text{ for all } x \in Y.$$

Then define metric d_1 on $\mathcal{C}^1(Z)/\sim$ by

$$d_1([f],[g]) = \max\left\{ d([f],[g]), \max_{Y} |f'(x) - g'(x)| \right\}.$$

We then say f and g are $\mathcal{C}^1$-ϵ close on Y if $d_1([f],[g]) < \epsilon$. Notice that if g is a $\mathcal{C}^1$-ϵ perturbation of f on a set Y then g must also be a $\mathcal{C}^0$-ϵ perturbation of f on Y.

We move now to a preliminary construction necessary for the rest of the chapter. In the preceding chapters, we have adhered to Definition 2.7.2 when working with enveloping. That is, we have allowed for the possibility that an enveloping function ϕ is not defined on all of $\mathbb{R}$ as was often the case in [11]. It is more convenient in the following sections for enveloping functions to be continuous on all of $\mathbb{R}$. As such, we give a method to extend any monotone decreasing and self-inversing enveloping function, ϕ, to all of $\mathbb{R}$ so that we may assume enveloping functions are continuous on $\mathbb{R}$.

In accordance with Definition 2.7.2 and our needs, assume that ϕ is a monotone decreasing and self-inversing enveloping function for $f \in \mathcal{P}$ with $f(p) = p$ so that $\phi : \mathcal{I} \supseteq (0, x_-) \to \mathbb{R}$. Recall that if x_- is finite than an appropriate monotone decreasing and self-inversing extension $\tilde{\phi}$ was provided in Remark 3.1.1. So suppose $x_- = \infty$ and note that this implies there exists $a \in [0, p)$ such that $\lim\limits_{x \to a^+} \phi(x) = \infty$. Let

$$M = \max_{[0,p]} f(x).$$

By the continuity of ϕ and the Intermediate Value Theorem, there exists $\gamma \in [0, p]$ such that

$\phi(\gamma) = M$. Furthermore, since ϕ is monotone decreasing, γ is unique. Define ψ by

$$\psi(x) = \begin{cases} -\frac{1}{\gamma}x + \phi(\gamma) + 1 & x < \gamma \\ \phi(x) & \gamma \leq x \leq \phi(\gamma) \\ -\gamma(x - \phi(\gamma) - 1) & x > \phi(\gamma) \end{cases}.$$

Then define the extension of ϕ, $\tilde{\phi}$, by

$$\tilde{\phi}(x) = \begin{cases} \min\{\phi(x), \psi(x)\} & x < \gamma \\ \phi(x) & \gamma \leq x \leq \phi(\gamma) \\ \min\{\phi(x), \psi(x)\} & x > \phi(\gamma) \end{cases}.$$

Then $\tilde{\phi}$ is monotone decreasing, self-inversing, and continuous on all of $\mathbb{R}$. By construction, $\tilde{\phi}(x) > f(x)$ for $x \in (0, \gamma)$ and $\tilde{\phi}(x) < f(x)$ for $x \in (\phi(\gamma), \infty)$. Hence, $\tilde{\phi}$ is an enveloping function for f. In most cases, ψ will serve as the desired extension of ϕ. However, examples can be constructed for which there exists $x \in (p, \infty)$ such that $\psi(x) = f(x)$.

Example 5.0.2. Consider the map referred to by Cull as a Hassel model in [11] given by

$$f(x) = \frac{(1+a)^b x}{(1+ax)^b}, \quad a, b > 0.$$

For the purpose of this example we fix $a = 5$ and $b = 2$. Cull shows in [11] that $\phi(x) = 1/x$ serves as an enveloping function for f over a wide range of values for a and b including $a = 5$ and $b = 2$. While $\phi(x) = 1/x$ is continuous on $(0, \infty)$ it is not defined on all of $\mathbb{R}$ so we desire to extend its domain. A quick check shows that f has an absolute maximum $M = 9/5$ at $x = 1/5$. Since $\phi(5/9) = 9/5$ we have $\gamma = 5/9$. Then the definition for $\tilde{\phi}$ is given by

$$\tilde{\phi}(x) = \begin{cases} -\frac{9}{5}x + \frac{14}{5} & x < \frac{5}{9} \\ \frac{1}{x} & \frac{5}{9} \leq x \leq \frac{9}{5} \\ -\frac{5}{9}x + \frac{14}{9} & \frac{9}{5} < x \end{cases}.$$

Plots of f, ϕ, and $\tilde{\phi}$ can be seen in Figures 5.1 and 5.2. ■

The reader may note that the method given for extending ϕ to $\tilde{\phi}$ is not unique. Furthermore,

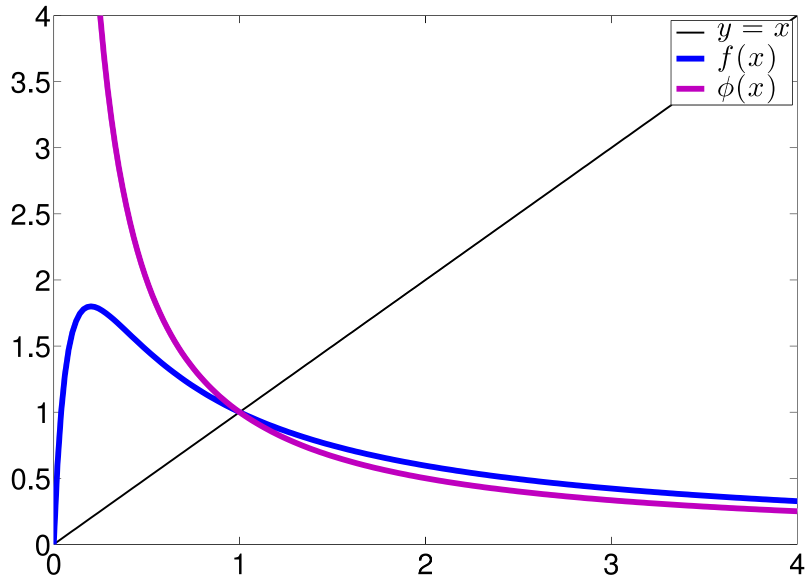


Figure 5.1: A plot of f and ϕ for Example 5.0.2.

the method is relatively conservative. That is, if

$$m = \max_{[0,\phi(0)]} \left| \tilde{\phi}(x) - f(x) \right|$$

there are extensions that would keep the value of m smaller or methods that may decrease the length of the interval $[0, \tilde{\phi}(0)]$. However, the given method of extension serves the needs of this work.

5.1 Perturbations and Enveloping

Cull's Theorem for determining the global stability of the fixed point of a population model, f, has conditions that are relatively easy to check making it a powerful theorem for applications. However, while it may be relatively easy to check if a given ϕ envelopes a given f, it is more difficult to find enveloping functions. In [11], multiple examples are provided along with methods for finding an appropriate enveloping function. Unfortunately, these examples are dependent on the explicit form of the population model. The following theorem provides conditions to ensure that a $\mathcal{C}^1$-ϵ perturbation of an enveloped population model will have an enveloping function. The theorem also shows that a simple translation of the original enveloping function will serve

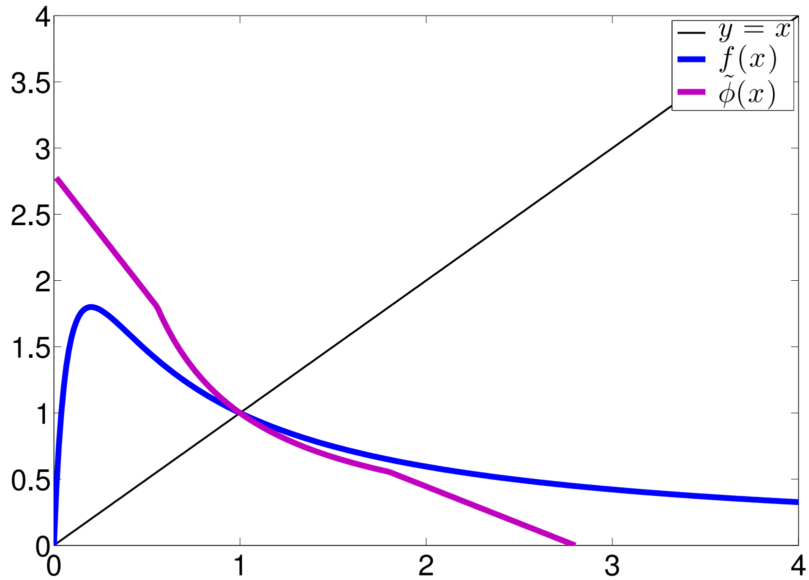


Figure 5.2: A plot of f and $\tilde{\phi}$ for Example 5.0.2.

as an enveloping function for the perturbed population model.

In the statement of the theorem and its proof, we may not assume that the perturbation of the population model is a completely arbitrary $\mathcal{C}^1$-ϵ perturbation. Instead, we are forced to perturb within the space of population models. The reason for this, as the following example demonstrates, is that an arbitrary $\mathcal{C}^1$-ϵ perturbation is not necessarily a population model.

Example 5.1.1. Consider

$$f(x) = \frac{x^4 - x^2 + 2x}{1 + x^3} = x + \frac{x(1-x)}{1+x^3}.$$

Notice that $f(0) = 0$ and $f(1) = 1$. If $x \in (0,1)$ then $\dfrac{x(1-x)}{1+x^3} > 0$ so $f(x) > x$. If $x \in (1, \infty)$ then $\dfrac{x(1-x)}{1+x^3} < 0$ so $f(x) < x$. Therefore, f is a population model. However, given any $\epsilon > 0$ there is an $\bar{x} > 1$ for which $|\bar{x} - f(\bar{x})| < \epsilon$. Therefore, on any compact interval $\mathcal{I}$ containing $\bar{x}$, a $\mathcal{C}^0$-ϵ neighborhood of f on $\mathcal{I}$ will contain some function that is not a population model. ■

Remark 5.1.1. The reader should note that we are proposing that if f is a population model and we wish to consider a C^0-ϵ perturbation, g, of f then we will require that g is also a population model. That is, the space of allowable perturbations of f is $\mathcal{P}$. This allows us to assume nothing further about f than f is a population model. One may ask if it might be better to place further restrictions on f so that for reasonable ϵ, *any* $\mathcal{C}^0$-ϵ perturbation of f *is* a population model. For instance, we could consider f to be in the space of bounded population models to avoid the issues of Example 5.1.1. However, this would immediately prevent our results from being applicable to the Cushing and Henson Conjectures (see Chapter 2) which require f to be unbounded. Furthermore, we would still need to assume that any $\mathcal{C}^0$-ϵ perturbation, g, of f fixes zero which is a notably problematic assumption. We would also be forced to assume that $g'(0) \geq 1$ to maintain the properties of a population model. Again, this is a closed property which is problematic for perturbation. We could instead assume that $g'(0) > 1$, but this assumption eliminates some population models from the space of allowable perturbations. In short, it appears that even if further restrictions are placed on f, assumptions must be made about the space of allowable perturbations and these new assumptions do not offer any more insight into the behavior of periodic systems of population models. We therefore choose to restrict our perturbations to the space of population models.

We begin with a preliminary lemma that will aid in the statement of the theorem that if $f \in \mathcal{P}$ has an enveloping function $\psi \in \mathcal{C}^0(\mathbb{R})$ and g is a $\mathcal{C}^1$-ϵ perturbation of f then g has an enveloping function that is a translation of ψ. That is, if $f(p) = p$ and $g(q) = q$ then $\phi(x) = \psi(x-(q-p))+(q-p)$. Within the proof, the case that $p > q$ and the case that $q > p$ splits the proof into two cases. To avoid confusion and redundancy, we state the theorem and give the proof with the assumption that $q > p$. A comment concerning the necessary adjustments follows the proof of the theorem.

Lemma 5.1.1. *Let $f \in \mathcal{P}$ with fixed point p and suppose $|f'(p)| < 1$ so that there exists $r \in (0, p)$ such that*

$$|f'(x)| \leq \lambda < 1 \text{ for all } x \in \mathcal{B}(p, r) = (a, b).$$

Let $0 < \epsilon < r(1-\lambda)/4$ and let $g \in \mathcal{P}$ be a $\mathcal{C}^1$-ϵ perturbation of f on $\bar{\mathcal{B}}(p, r)$ with fixed point q where $q > p$. Then $q - p < r/4$ and $a + 2(q - p) < p$.

Proof. By the assumptions on f, the Mean Value Theorem gives

$$f(x) \leq \lambda x + (1-\lambda)p \text{ for all } x \in (p, b).$$

Then

$$g(x) \leq \lambda x + (1-\lambda)p + \epsilon \tag{5.1}$$

because g is a $\mathcal{C}^0$-ϵ perturbation of f. Then substituting q into (5.1) and solving for $q - p$,

$$\begin{aligned} g(q) = q &\leq \lambda q + (1-\lambda)p + \epsilon \\ q - \lambda q - (1-\lambda)p &\leq \epsilon \\ q - p &\leq \frac{\epsilon}{1-\lambda} < \frac{r(1-\lambda)}{4(1-\lambda)} = \frac{r}{4}, \end{aligned} \tag{5.2}$$

where the last inequality is by assumption. Also,

$$a + 2(q-p) < a + 2\left(\frac{r}{4}\right) = a + \frac{r}{2} < p.$$

□

In the following theorem, we will keep the same assumptions on f as Lemma 5.1.1 and again take a $\mathcal{C}^1$-ϵ perturbation of f with $\epsilon < r(1-\lambda)/4$. In addition, f will be assumed to have an enveloping function ψ that is monotone decreasing. Since we now know that $a + 2(q-p) < p$, we point out that

$$0 < \min_{[0, a+2(q-p)]} (\psi(x) - f(x))$$

for the sake of the statement of the theorem.

Theorem 5.1.1. *Let $f \in \mathcal{P}$ with fixed point p and suppose $|f'(p)| < 1$ so that there exists $r \in (0, \min\{p, \psi(0) - p\})$ such that*

$$|f'(x)| \leq \lambda < 1 \text{ for all } x \in \mathcal{B}(p,r) = (a,b).$$

Suppose further that f is enveloped by a monotone decreasing, self-inversing $\psi \in \mathcal{C}^0(\mathbb{R})$ such that

$$\psi(x) = -x + 2p \text{ for all } x \in (a,b).$$

Let $g \in \mathcal{P}$ be a $\mathcal{C}^1$-ϵ perturbation of f on $[0, \psi(0)]$ with fixed point q where $q > p$. Let

$$\eta = \min\left\{ \min_{x \in [0, a+2(q-p)]} (\psi(x) - f(x)), \min_{x \in [b, \psi(0)]} (f(x) - \psi(x)) \right\}.$$

By uniform continuity of ψ on $[-r, \psi(-r)]$, there exists $0 < \delta < \eta/4$ such that

$$|\psi(x - \xi) - \psi(x)| \leq \eta/4 \text{ for all } |\xi| < \delta \text{ and } x \in [-(r+\eta), \psi(-(r+\eta))].$$

If

$$0 < \epsilon < \min\left\{\frac{1-\lambda}{2}, \frac{\eta}{4}, \delta(1-\lambda), \frac{r(1-\lambda)}{4}\right\}.$$

then g is enveloped by

$$\phi(x) = \psi(x - (q-p)) + (q-p).$$

Proof. To begin, consider that ψ on the interval (a, b) is given by $-x+2p$. Thus, on the interval $(a + (q-p), b + (q-p))$,

$$\begin{aligned}\phi(x) &= -(x - (q-p)) + 2p + (q-p) \\ &= -x + q - p + 2p + q - p \\ &= -x + 2q.\end{aligned}$$

By Lemma 5.1.1,

$$a + 2(q-p) < p < b < b + (q-p).$$

Hence, $\phi(x) = -x + 2q$ for $x \in (a + 2(q-p), b)$.

We now show that $-x+2q > g(x)$ for $x \in (a+2(q-p), q)$ and $-x+2q < g(x)$ for $x \in (q, b)$. Recall that g is a $\mathcal{C}^1$-ϵ perturbation of f so that

$$|g'(x)| \leq \lambda + \epsilon < \lambda + \frac{1-\lambda}{2} = \frac{1+\lambda}{2} < 1 \text{ for } x \in (a + 2(q-p), b).$$

Noting then that $g(q) = q$ and $\phi(q) = -q + 2q = q$, we have that

$$\begin{aligned}&-x + 2q > g(x) \text{ for } x \in (a + 2(q-p), q) \text{ and} \\ &-x + 2q < g(x) \text{ for } x \in (q, b).\end{aligned}$$

Let $x \in [0, a + 2(q-p)] \cup [b, \phi(0)]$. Then by (5.2) in Lemma 5.1.1,

$$x - \frac{\epsilon}{1-\lambda} < x - (q-p) < x,$$

and ψ is monotone decreasing so

$$\begin{aligned}|\phi(x)-\psi(x)| &= |\psi(x-(q-p))+(q-p)-\psi(x)| \\ &\leq \left|\psi\left(x-\frac{\epsilon}{1-\lambda}\right)-\psi(x)+\frac{\epsilon}{1-\lambda}\right| \\ &\leq \left|\psi\left(x-\frac{\epsilon}{1-\lambda}\right)-\psi(x)\right|+\left|\frac{\epsilon}{1-\lambda}\right|.\end{aligned}$$

Recall that by assumption $\epsilon < \delta(1-\lambda)$ so that $\dfrac{\epsilon}{1-\lambda} < \delta$. Then

$$\begin{aligned}\left|\psi\left(x-\frac{\epsilon}{1-\lambda}\right)-\psi(x)\right|+\left|\frac{\epsilon}{1-\lambda}\right| &< \frac{\eta}{4}+\delta \\ &< \frac{\eta}{4}+\frac{\eta}{4}=\frac{\eta}{2}.\end{aligned} \tag{5.3}$$

Notice that $d(f(x),g(x)) \leq \eta/4$ for $x \in [0,\psi(0)]$ since g is a $\mathcal{C}^1$-ϵ perturbation of f on $[0,\psi(0)]$. Also,

$$d(f(x),\psi(x)) \geq \eta \text{ for all } x \in [0,a+2(q-p)] \cup [b,\psi(0)]$$

by the definition of η. If we consider $x \in [0,a+2(q-p)]$, we desire to show that $\phi(x)-g(x)>0$. Here,

$$\eta \leq \psi(x)-f(x),$$

by definition. Then rewriting, we see

$$\eta \leq \psi(x)-\phi(x)+\phi(x)-g(x)+g(x)-f(x).$$

Then using the assumptions on f and g and (5.3) we have

$$\eta \leq \frac{\eta}{2}+\phi(x)-g(x)+\frac{\eta}{4}.$$

Rewriting again gives

$$\frac{\eta}{4} \leq \phi(x)-g(x).$$

Thus, $\phi(x)-g(x)>0$.

Similarly, suppose $x \in [b, \phi(0)]$. In this case

$$\begin{aligned}
\eta &\leq f(x) - \psi(x) \\
\eta &\leq f(x) - g(x) + g(x) - \phi(x) + \phi(x) - \psi(x) \\
&\leq \frac{\eta}{4} + g(x) - \phi(x) + \frac{\eta}{2} \\
\frac{\eta}{4} &\leq g(x) - \phi(x).
\end{aligned}$$

So $g(x) - \phi(x) > 0$.

Therefore, $\phi(x)$ is an enveloping function for $g(x)$. □

To adjust Lemma 5.1.1 and Theorem 5.1.1 to allow for $p > q$ one must change Lemma 5.1.1 to show that $b + 2(q - p) > p$ (when $p > q$) instead of $a + 2(q - p) < p$ (when $q > p$). Then throughout the proof of Theorem 5.1.1, the intervals $[0, a+2(q-p)]$, $[b, \psi(0)]$, and $(a+2(q-p), b)$ must be changed to $[0, a]$, $[b + 2(q - p), \psi(0)]$, and $(a, b + 2(q - p))$ respectively.

If f is a population model enveloped by monotone decreasing and self-inversing $\phi \in \mathcal{C}^0(\mathbb{R})$ and $\tilde{f}$ is a $\mathcal{C}^1$-ϵ perturbation of f on $[0, \phi(0)]$ that is also a population model, then Theorem 5.1.1 can be used to show that the fixed point of $\tilde{f}$ is globally attracting. Furthermore, Theorem 5.1.1 can be of great use in the application of Theorem 4.2.1 and Theorem 4.2.2 simultaneously. However, Theorem 5.1.1 is of little use in determining the behavior of an n-periodic system of population models if $n \geq 2$.

5.2 Perturbations of Periodic Dynamical Systems

We begin now to consider the asymptotic behavior of periodic dynamical systems that are perturbations of either single maps or of periodic dynamical systems with known behavior. We will first consider systems close to a "degenerate" periodic dynamical system, that is a periodic system consisting of only one map f. We can then form a true periodic dynamical system by perturbing f at certain time-steps. We begin with a preliminary lemma concerning vectors in $\mathbb{R}^n$.

Lemma 5.2.1. *Let $x = (x_0, x_1, \ldots x_{n-1}) \in \mathbb{R}^n$. Then there exists $\epsilon \in \mathbb{R}^n$ such that for all $y \in \mathcal{B}(0, \epsilon)$ the vector*

$$x + y = (x_0 + y_0, \ldots x_{n-1} + y_{n-1})$$

has the property that if $x_i \neq x_j$ then $x_i + y_i \neq x_j + y_j$.

Proof. By assumption, there exists i and j such that $x_i \neq x_j$, so we let η be the smallest nonzero

element of
$$\{|x_i - x_j| : 0 \leq i, j \leq n-1\}.$$
Take $\epsilon \in \mathbb{R}^n$ such that
$$\|\epsilon\|_\infty < \frac{\eta}{2}.$$
Let $y \in \mathcal{B}(0, \epsilon)$. Then
$$|x_i + y_i - (x_j + y_j)| > 0. \qquad \square$$

Using Lemma 5.2.1, we can say that given $x \in \mathbb{R}^n$ with at least two distinct entries there is an open dense set of perturbations of x that have at least two distinct entries. This will allow us to ensure that perturbations of geometric n-cycles, which are vectors in $\mathbb{R}^n$, still have period n.

Our next result allows us to perturb a continuous map f to a periodic system with a geometric n-cycle.

Theorem 5.2.1. *Suppose $f \in \mathcal{C}^0(\mathbb{R})$ has a unique fixed point p on $\mathcal{B}(p, r)$. Then given small $\epsilon > 0$ and any $n \in \mathbb{Z}^+$ there exists an n-periodic dynamical system,*
$$\mathcal{S} = \{f_0, f_1, \ldots, f_{n-1}\},$$
of $\mathcal{C}^0$-ϵ perturbations of f on a closed neighborhood of p that has a geometric n-cycle, $\mathcal{C} = \{c_0, c_1, \ldots, c_{n-1}\}$, such that $|c_i - p| \leq \epsilon$ for $i = 0, 1, \ldots, n-1$.

Proof. Let $r' \in (0, r)$, $U = \mathcal{B}(p, r)$, and $U' = \mathcal{B}(p, r') \subset U$. Let $\beta(x)$ be a $C^\infty(\mathbb{R})$ bump function such that
$$\beta(x) = \begin{cases} 0 & x \notin U \\ 1 & x \in U' \end{cases}.$$
Let $\epsilon > 0$ and define,
$$\kappa_\epsilon(x) = x + \epsilon\beta(x).$$
To ensure that κ_ϵ is invertible, we will ensure that κ_ϵ is monotone increasing and therefore one-to-one. Notice $\kappa_\epsilon'(x) = 1 + \epsilon\beta'(x)$ and since $\beta'(x)$ is bounded there is sufficiently small ϵ so that $\kappa_\epsilon'(x) > 0$. Hence, there exists ϵ so that κ_ϵ is increasing and therefore invertible.

Consider now the degenerate n-periodic dynamical system given by $\{f, f, \ldots, f\}$. This system has $\{p, p, \ldots, p\}$ as a trivial geometric cycle. We perturb this system using an invertible κ_ϵ to get an n-periodic dynamical system
$$\mathcal{S}_\epsilon = \left\{(\kappa_\epsilon \circ f), (f \circ \kappa_\epsilon^{-1}), f, \ldots, f\right\}.$$

This perturbed system has $\mathcal{C}_\epsilon = \{p, p+\epsilon, p, \ldots, p\}$ as a geometric-n cycle. By Lemma 5.2.1, there is an open dense subset of $\mathcal{C}_\epsilon$ in $\mathbb{R}^n$ of vectors that are geometric n-cycles for an appropriate $\mathcal{S}_\epsilon$. $\square$

In Theorem 5.2.1, the given function f is perturbed in a more traditional manner than the C^k-ϵ perturbations described at the start of this chapter. The perturbed function $g = (\kappa_\epsilon \circ f)(x)$ in the theorem is arbitrarily close to f on the interval U' but is identical to f outside of U. However, the theorem still holds in the C^0-ϵ sense that g only needs to be similar to f on the interval U for the local dynamics to behave as described by the theorem. This is an important distinction as the set of functions C^0-ϵ close to f is much larger than the set of functions ϵ close to f on U and identical to f on $\mathbb{R} \setminus U$.

Theorem 5.2.1 provides little insight into any given periodic dynamical system. Given any function, g, that is used in a periodic dynamical system it would be very difficult to produce an f and κ_ϵ so that $g = (\kappa_\epsilon \circ f)$. Furthermore, while we know restrictions can be made so that κ_ϵ is invertible, in general it would be very difficult to find an explicit expression for this inverse. The next example demonstrates some of the issues in trying to deal with κ_ϵ.

Example 5.2.1. Consider the Ricker type map $f(x) = xe^{(2-2x)}$ which has $f(1) = 1$. Cull's Theorem ensures that $x = 1$ is globally attracting. For our bump function, we let

$$\beta(x) = \begin{cases} e^{-\frac{1}{1-(x-1)^2}} & |x-1| < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Let $\kappa(x) = x + 0.3\beta(x)$ and note that $\kappa'(x) > 0.5$ so κ is invertible. Define the 3-periodic dynamical system $\{(\kappa \circ f), (f \circ \kappa^{-1}), f\}$. This periodic dynamical system has $\{1, 1 + 0.3e^{-1}, 1\}$ as a geometric cycle. However, it is very difficult to determine if this geometric 3-cycle is stable using either numerical or classic methods.

The following script was written in Maple 15 and run to analyze the long term behavior of the periodic system.

```
beta:=piecewise(abs(x-1)<1,exp(-1/(1-(x-1)^2)),0):
kappa:= x + 0.3 * beta:
f := x*exp(2-2*x):
x0 := 1.0:
for i from 1 to 2000 do
z := eval(f, x = x0):
x1 := eval(kappa, x = z):
```

```
z := fsolve[30](kappa = x1, x):
x2 := eval(f, x = z):
x3 := eval(f, x = x2):
x0 := x3:
if i > 1999 then
print(x1);
print(x2);
print(x3);
end if
end do:
```

When the numerical precision is set to the default 10 digits the script outputs the values $\{1.110364832, 0.9999990000, 1.000001000\}$. However, if the number of significant digits is increased to 16 then the script outputs $\{1.110363832351433, 1.000000000000000, 1.000000000000000\}$. If we were forced to use the 10 digits of numerical precision we may conclude that the geometric cycle is repelling since even starting on the cycle does not ensure the orbit stays on the cycle. The increased numerical precision leads us to believe the geometric cycle is stable, but the orbit of a seed other than $x_0 = 1.0$ does not seem to be approaching $\{1, 1 + 0.3e^{-1}, 1, 1\}$. Thus, we cannot assume the geometric cycle is attracting. However, as is demonstrated below, it is difficult to rigorously determine whether or not the geometric 3-cycle is stable.

In this example, there are three 3-fold compositions to consider. If we drop the subscript "ϵ" for convenience then the compositions are given by:

$$
\begin{aligned}
F_0(x) &= (f \circ (f \circ \kappa^{-1}) \circ (\kappa \circ f))(x) = f^3(x) \\
F_1(x) &= ((\kappa \circ f) \circ f \circ (f \circ \kappa^{-1}))(x) = (\kappa \circ f^3 \circ \kappa^{-1})(x) \\
F_2(x) &= ((f \circ \kappa^{-1}) \circ (\kappa \circ f) \circ f)(x) = f^3(x).
\end{aligned}
$$

Notice that $F_0(1) = 1$, $F_1(1 + 0.3e^{-1}) = 1 + 0.3e^{-1}$, and $F_2(1) = 1$. By the Structure Theorem, the geometric 3-cycle is attracting for the periodic dynamical system if the respective fixed points are attracting for the 3-fold compositions. However, $F_0'(1) = F_2'(1) = -1$ meaning that a more involved analysis is required. Using Theorem 3.1.1, we can conclude that both F_0 and F_2 are enveloped population models and therefore $p = 1$ is globally attracting for each.

Unfortunately, $F_1'(1 + 0.3e^{-1}) = -1$ as well, and F_1 may not be an enveloped population model. We cannot apply Theorem 5.1.1 to ensure that F_1 has an enveloping function because $F_0'(1) = -1$. Furthermore, it is not possible in this example to derive an explicit expression for κ^{-1} making it very difficult to approximate the necessary higher order derivatives of F_1 required to use the Schwarzian derivative (see Definition 1.3 and Theorem 1.6 in [16]). ■

Despite its limitations when working with a given periodic dynamical system, Theorem 5.2.1 does provide some insight into the nature of geometric cycles that result from the perturbation of a given periodic dynamical system. Let $\{f_0, f_1, \ldots, f_{n-1}\}$ be an n-periodic dynamical system with geometric n-cycle given by $\{c_0, c_1, \ldots, c_{n-1}\}$. If we perturb the periodic system in a manner similar to Theorem 5.2.1, we arrive at the perturbed system

$$\left\{f_0, f_1, \ldots, (\kappa_\epsilon \circ f_i), (f_{i+1} \circ \kappa_\epsilon^{-1}), \ldots, f_{n-1}\right\}.$$

The perturbed system has $\{c_0, c_1, \ldots, c_i, \kappa_\epsilon(c_{i+1}), c_{i+2}, \ldots, c_{n-1}\}$ as a geometric cycle but we cannot assume that the new geometric cycle has minimal period n. However, if the perturbation is small enough so that the resulting geometric cycle is within the neighborhood as specified by Lemma 5.2.1, then the resulting geometric cycle must have minimal period n. With this construction in mind, we know that given any n-periodic dynamical system, there is an open dense set of perturbations of the dynamical system that also have geometric n-cycles.

Do to the limitations of Theorem 5.2.1, we move now to more general theorems. It is well known in one-dimensional dynamics that if a one parameter family of maps has a fixed point for a given parameter value, then the persistence of the fixed point is guaranteed given some simple assumptions (i.e. $f'(x^*) \neq 1$) based on the Implicit Function Theorem (see [25]). We desire a version of this theorem that is applicable to periodic dynamical systems. That is, if an n-periodic dynamical system has a geometric n-cycle then, given certain assumptions, a perturbation of the parameters will result in a period-n dynamical system with a geometric n-cycle.

Theorem 5.2.2. *Let $n \in \mathbb{Z}^+$ and $i = 0, 1, \ldots, n-1$. Suppose*

$$f_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

is a set of maps that is jointly $\mathcal{C}^p$ such that the periodic dynamical system $\{f_0, f_1, \ldots, f_{n-1}\}$ has a geometric n-cycle for parameter values given by $y = (y_0, y_1, \ldots, y_{n-1})$. That is,

$$\begin{aligned} f_0(y_0; c_0) &= c_1 \\ f_1(y_1; c_1) &= c_2 \\ &\vdots \\ f_{n-1}(y_{n-1}; c_{n-1}) &= c_0. \end{aligned}$$

If
$$\prod_{i=0}^{n-1} f_i'(y_i; c_i) \neq 1 \tag{5.4}$$
then there exists an open neighborhood, U, of y such that $\{f_0, f_1, \ldots, f_{n-1}\}$ has a geometric n-cycle for all $z \in U$. Furthermore, there exists an open neighborhood, V, of $c = (c_0, c_1, \ldots, c_{n-1})$ and unique $\mathcal{C}^p$ function, $g : U \to V$, such that

$$\begin{aligned} f_0(z_0; g_0(z)) &= g_1(z) \\ f_1(z_1; g_1(z)) &= g_2(z) \\ &\vdots \\ f_{n-1}(z_{n-1}; g_{n-1}(z)) &= g_0(z). \end{aligned}$$

Proof. The proof is based on the Implicit Function Theorem as seen in [25]. Let $x \in \mathbb{R}^n$ where indexing begins at zero. Define $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$F_i(y_; x) = f_i(y_i; x_i) - x_{i+1 \bmod n}$$

for $i = 0, \ldots, n-1$ so that $F(y, c) = 0$. Define

$$\Delta = \left| \begin{bmatrix} \frac{\partial F_0}{\partial x_0} & \cdots & \frac{\partial F_0}{\partial x_{n-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_{n-1}}{\partial x_0} & \cdots & \frac{\partial F_{n-1}}{\partial x_{n-1}} \end{bmatrix} \right|.$$

Dropping the dependence on y_i and computing the appropriate derivatives,

$$\begin{aligned}
\Delta &= \left| \begin{bmatrix} f_0'(x_0) & -1 & 0 & \dots & 0 \\ 0 & f_1'(x_1) & -1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & f_{n-2}'(x_{n-2}) & -1 \\ -1 & 0 & \dots & 0 & f_{n-1}'(x_{n-1}) \end{bmatrix} \right| \\
&= \prod_{i=0}^{n-1} f_i'(x_i) + (-1)^n(-1)^{n-1} \\
&= \prod_{i=0}^{n-1} f_i'(x_i) - 1.
\end{aligned}$$

We may use the Implicit Function Theorem provided that $\Delta(y;c) \neq 0$, and so we assume

$$\prod_{i=0}^{n-1} f'(c_i) \neq 1.$$

The existence of $g \in C^p$ follows directly from the Implicit Function Theorem.

By the continuity of g, V can be taken arbitrarily close to c so that V is within the neighborhood as guaranteed by Lemma 5.2.1. Then all $g(z) \in V$ must have the property described in Lemma 5.2.1. That is, $g(z)$ is an n-cycle for a perturbation of $\{f_0, f_1, \dots, f_{n-1}\}$. ☐

Remark 5.2.1. In the proof of Theorem 5.2.2, we define $F_i(y;x) = f_i(y_i;x_i) - x_{i+1 \bmod n}$ but changing to $F_i(y;x) = x_{i+1 \bmod n} - f_i(y_i;x_i)$ still requires the assumption that

$$\prod_{i=0}^{n-1} f'(c_i) \neq 1.$$

Theorem 5.2.2 gives us insight into the persistence of a geometric n-cycle under the perturbation of the parameters of a periodic dynamical system. If Condition (5.4) is met for a given periodic dynamical system with an n-cycle and parameter values y, then parameter values within an open neighborhood of y will yield a geometric n-cycle. Furthermore, that n-cycle will also satisfy Condition (5.4). As such, if such an n-cycle is present, then it must persist until Condition (5.4) breaks.

Example 5.2.2. Consider the 2-periodic dynamical system

$$\left\{ f_0(x) = x^2, \quad f_1(x) = -\frac{1}{2}x - \frac{1}{2} \right\}.$$

We can check easily enough that $\{-1, 1\}$ is a geometric 2-cycle for the system. Since $f_0'(1) \cdot f_1'(-1) = 1$ this periodic dynamical system breaks Condition (5.4) of Theorem 5.2.2 and a slight change in the parameters may result in the dynamical system not having a geometric 2-cycle. If we consider instead the slightly altered system

$$\left\{g_0(x) = x^2, \quad g_1(x) = -\frac{1}{2}x + c\right\}$$

with $c < -\frac{1}{2}$ we can check the 2-fold compositions

$$\begin{aligned} g_1(g_0(x)) &= -\frac{1}{2}x^2 + c < x, \\ g_0(g_1(x)) &= \frac{1}{4}x^2 - cx + c^2 > x, \end{aligned}$$

to see that the perturbed system has no geometric 2-cycles. ■

Example 5.2.3. Here we reconsider Example 4.1.1. Specifically, we consider the 2-periodic dynamical system

$$\left\{f_0(x) = xe^{(2-0.5x)}, f_1(x) = xe^{(2-(0.5+\epsilon)x)}\right\}.$$

Previously, we saw the appearance of an attracting geometric 4-cycle as ϵ varied over the interval $[0.1, 0.3]$. However, Theorem 5.2.2 will allow us to show that the geometric 2-cycle still persists after the appearance of the geometric 4-cycle.

Using Theorem 5.2.1 we know for small enough ϵ the periodic dynamical system must have a geometric 2-cycle because f_0 has a fixed point. However, we cannot algebraically solve for the values of the geometric cycle in this example and are forced to rely on numeric results. To begin, notice that the geometric 2-cycle takes the form

$$\{c_0 = f_1(c_1), \ c_1 = f_0(c_0)\}.$$

Hence, Condition (5.4) in Theorem 5.2.2 that $f_0'(c_0)f_1'(c_1) \neq 1$ can be rewritten as

$$f_0'(c_0)f_1'(f_0(c_0)) \neq 1 \tag{5.5}$$

allowing us to ignore the value of c_1. We also note that c_0 is the real positive solution to

$$f_1(f_0(x)) = x \tag{5.6}$$

and depends on the value of ϵ. Figure 5.3 shows the numeric solutions of (5.6) and Figure 5.4 shows the values $f_0'(c_0)f_1'(f_0(c_0))$ to show that (5.5) holds. Notice in Figure 5.4 that the values

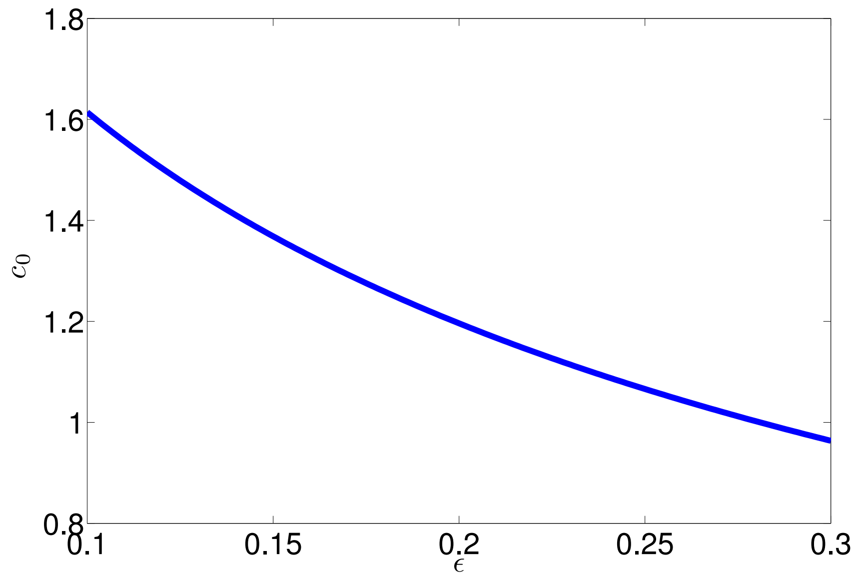


Figure 5.3: The solutions of $f_1(f_0(x)) = x$ for Example 5.2.3.

of the product are far away from the value 1. Thus, the geometric 2-cycle must persist in this example. ■

5.3 Perturbations of Population Models and Globally Attracting Geometric n-Cycles

We return now to our original goal of determining when periodic systems of population models have globally attracting geometric n-cycles. Here, we use a concept similar to Cull's enveloping functions called weak enveloping. Weak enveloping is applicable to a wider range of functions than population models so we begin by defining a new function space that contains population models as a subset.

Definition 5.3.1 (Pseudo-Population Model)**.** A continuous function $f : [0, \infty) \to [0, \infty)$ is a *pseudo-population model* if

(i) $f(0) = 0$

(ii) There exists $q, s \in (0, \infty)$ such that $f(x) > x$ for $x \in (0, q)$ and $f(x) < x$ for $x \in (s, \infty)$

(iii) $f(x) > 0$ for $x > 0$.

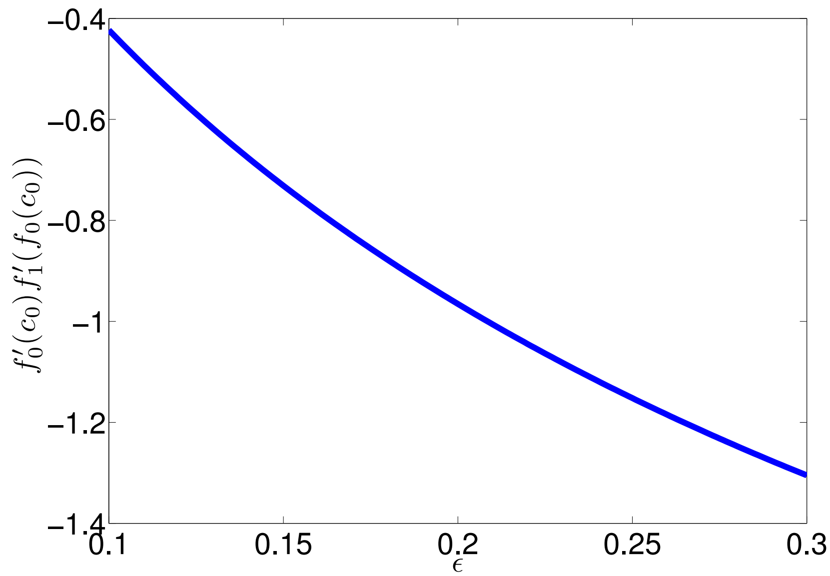


Figure 5.4: The values of $f_0'(c_0)f_1'(f_0(c_0))$ for Example 5.2.3.

When referring to Pseudo-Population Models, we refer to the interval $[q, s]$ in Definition 5.3.1 as the *crossing interval* for f. We will refer to the space of pseudo-population models as $\tilde{\mathcal{P}}$ and note that $\mathcal{P} \subset \tilde{\mathcal{P}}$. If $f \in \tilde{\mathcal{P}}$, note that f must have a fixed point in $[q, s]$ by the Intermediate Value Theorem. However, f may potentially have multiple fixed points in its crossing interval.

An important question of Chapter 3 was when is the composition of two population models a population model. A convenient property of pseudo-population models is that the composition of any two pseudo-population models is always a pseudo-population model. We state this result as the next theorem.

Theorem 5.3.1. *Suppose f_0 and f_1 are pseudo-population models. Then $f_1(f_0(x))$ is also a pseudo-population model.*

Proof. Obviously, $f_1(f_0(0)) = 0$ and $f_1(f_0(x)) > 0$ for $x > 0$. Suppose f_0 and f_1 have crossing intervals $[q_0, s_0]$ and $[q_1, s_1]$ respectively. Since both f_0 and f_1 are continuous, there exists $\epsilon > 0$ such that $f_1(f_0(x)) > x$ for all $x \in (0, \epsilon)$. We must now show that there exists x_M such that $f_1(f_0(x)) < x$ for all $x > x_M$. To begin, let $x > s_0$ so that $f_0(x) < x$. If $f_0(x) > s_1$ then $f_1(f_0(x)) < f_0(x) < x$. Let

$$M = \max_{[0,s_1]} f_1(x).$$

If $f_0(x) \leq s_1$ then $f_1(f_0(x)) \leq M$. We can choose $x_M > M$ so that if $x > \max\{x_M, s_0\}$ we have $f_1(f_0(x)) < x$. Therefore, $f_1(f_0(x))$ is a pseudo-population model. $\square$

An obvious distinction between a population model and a pseudo-population model is that while a population model is required to have a unique positive fixed point, a pseudo-population model may have multiple fixed points in its crossing interval. Obviously, if a pseudo-population has multiple fixed points then it does not have a globally attracting fixed point. Furthermore, enveloping is not applicable to a pseudo-population model with multiple fixed points. We desire then to create a more general concept of enveloping for pseudo-population models that will allow the map being enveloped to have multiple fixed points. It is not possible for this new weakened version of enveloping to imply the existence of a global attractor, but it can be used to show that the orbit of an arbitrary initial condition must enter an interval of interest. Ultimately, we will ensure the interval of interest is a trapping region to ensure the existence of a global attractor.

Definition 5.3.2 (Weak Enveloping)**.** Let $f \in \tilde{\mathcal{P}}$ with crossing interval $[q, s]$ and $\phi \in \mathcal{C}^0(\mathbb{R})$. Let $\alpha, \beta \in (0, \infty)$. Then ϕ *weakly envelopes* f on $U = [0, \infty) \setminus [\alpha, \beta]$ if

(i) $\alpha < q < s < \beta$,

(ii) $\phi(x) > f(x)$ for all $x \in (0, \alpha)$, and

(iii) $\phi(x) < f(x)$ for all $x \in (\beta, \infty)$.

Lemma 5.3.1. *Suppose $f \in \tilde{\mathcal{P}}$ is weakly enveloped by monotone decreasing and self-inversing ϕ on $U = [0, \infty) \setminus [\alpha, \beta]$. Then ϕ has a fixed point in (α, β).*

Proof. Let $\psi(x) = \phi(x) - x$ and note that ψ is continuous. Since $f \in \tilde{\mathcal{P}}$ we have that $f(\alpha) > \alpha$ and since ϕ weakly envelopes f we have that $\phi(\alpha) \geq f(\alpha) > \alpha$. Thus, $\psi(\alpha) > 0$. Similarly, $f(\beta) < \beta$ and $\phi(\beta) \leq f(\beta) < \beta$. Hence, $\psi(\beta) < 0$. By the Intermediate Value Theorem, there exists $x^* \in (\alpha, \beta)$ such that $\psi(x^*) = 0$. Therefore, $\phi(x^*) = x^*$ giving the fixed point of ϕ. $\square$

In the next theorem, we will refer to a nonautonomous dynamical system which is comprised of a finite number of pseudo-population models but is not necessarily periodic. To define this system, we begin with a set of n distinct pseudo-population models

$$\mathcal{S} = \{f_0, f_1, \ldots, f_{n-1}\}.$$

Let $g : \mathbb{Z}^+ \to \{0, 1, \ldots, n-1\}$ be a function. We denote our nonautonomous system by

$$\left\{f_{g(i)}\right\}_{i=0}^{\infty}. \tag{5.7}$$

An orbit for $x_0 \in \mathbb{R}^+$ under (5.7) is given by

$$\mathcal{O}^+(x_0) = \left\{x_0, x_1 = f_{g(0)}(x_0), x_2 = f_{g(1)}(x_1), \ldots, x_{i+1} = f_{g(i)}(x_i), \ldots, \right\}.$$

Notice that if (5.7) happens to be an n-periodic dynamical system then we may define g by

$$g(i) = i \bmod n.$$

While the "time" variable i completely determines which of the maps from $\mathcal{S}$ to use at a given "time step", we will often only be concerned with the time step of the dynamical system and not necessarily which map is being used. In other instances, we will need to refer to specific maps in $\mathcal{S}$ but will not be concerned with the time step at which it is being used. With this in mind, we introduce the notation $f_{i,j}$ to mean $f_{g(i)}$ where $g(i) = j$. Note that $f_{i,j}$ may not be defined for all choices of i and j. Using this notation, we may use $f_{i,\bullet}$ to emphasize that the time step is important but not the map being used. Similarly, we may use $f_{\bullet,j}$ to emphasize that the map being used is important but not the time step, and it should be assumed that $0 \leq j \leq n-1$. Using this notation,

$$f_{k,l} = f_{s,t} \text{ if and only if } l = t.$$

Furthermore, the above equation would imply that $g(k) = g(s) = l = t$.

Having defined a new type of nonautonomous dynamical system and associated notation, we state the next result.

Theorem 5.3.2 (Weak Enveloping Theorem)**.** *Suppose the nonautonomous dynamical system*

$$\left\{f_{g(i)}\right\}_{i=0}^{\infty}$$

corresponds to a set

$$\mathcal{S} = \{f_0, f_1, \ldots, f_{n-1}\}$$

of n distinct pseudo-population models with common crossing interval $[q, s]$ that are weakly enveloped on $U = [0, \infty) \setminus [\alpha, \beta]$ by a monotone decreasing and self-inversing ϕ. Let V be any open neighborhood of $[\alpha, \beta]$. Then for all $x_0 \in \mathbb{R}^+$, $\mathcal{O}^+(x_0) \cap V \neq \emptyset$, that is there exists k such *that*

$$(f_{k,\bullet} \circ f_{k-1,\bullet} \circ \ldots \circ f_{0,\bullet})(x_0) \in V.$$

Proof. Throughout the proof we will ignore the case that the orbit under the nonautonomous dynamical system enters $[\alpha, \beta]$ since this case gives the desired result. We will proceed by first showing that if $x_0 < \alpha$ then $x_i > x_0$ for all $i > 0$ and if $x_0 > \beta$ then $x_i < x_0$ for all i. Then we will show that the orbit enters any arbitrary open neighborhood of $[\alpha, \beta]$.

To begin, suppose if $x_i < \alpha$ then $f_{i,\bullet}(x_i) < \alpha$ for all i. Since each $f_{i,\bullet}$ is a pseudo-population model, $f_{i,\bullet}(x) > x$ for $x \in (0, \alpha)$ so

$$x_i < f_{i,\bullet}(x_i) < (f_{k+i,\bullet} \circ f_{k+i-1,\bullet} \circ \ldots \circ f_{i,\bullet})(x_i) < \alpha$$

for all i and $k \geq 1$.

Similarly, suppose if $x_i > \beta$ then $f_{i,\bullet}(x_i) > \beta$ for all i. Since each $f_{i,\bullet}$ is a pseudo-population model, $f_{i,\bullet}(x) < x$ for $x \in (\beta, \infty)$, and since $x_i > \beta$ we have

$$\beta < (f_{k+i,\bullet} \circ f_{k+i-1,\bullet} \circ \ldots \circ f_{i,\bullet})(x_i) < f_{i,\bullet}(x_i) < x_i$$

for all i and $k \geq 1$.

Suppose now that $x_i < \alpha$, $f_{i,\bullet}(x_i) > \beta$ for some i, and there exists $k \geq 1$ such that for all $0 \leq l < k$,

$$F_{i,l}(x_i) = (f_{l+i,\bullet} \circ f_{l+i-1,\bullet} \circ \ldots \circ f_{i,\bullet})(x_i) > \beta$$

and

$$F_{i,k}(x_i) = (f_{k+i,\bullet} \circ f_{k-1+i,\bullet} \circ \ldots \circ f_{i,\bullet})(x_i) < \alpha.$$

By Lemma 5.3.1, ϕ has a fixed point in (α, β) and ϕ is monotone decreasing so $\phi(x) < x$ for $x \in (\beta, \infty)$. Thus,

$$\phi(f_{i,\bullet}(x_i)) < f_{i,\bullet}(x_i). \tag{5.8}$$

Since ϕ is a weak enveloping function for $f_{i,\bullet}$ and $x_i < \alpha$ we have

$$x_i < f_{i,\bullet}(x_i) < \phi(x_i). \tag{5.9}$$

Applying the monotone decreasing and self-inversing nature of ϕ to (5.9) we have,

$$x_i < \phi(f_{i,\bullet}(x_i)) < \phi(x_i). \tag{5.10}$$

Combining (5.8), (5.9), and (5.10) we have,

$$x_i < \phi(f_{i,\bullet}(x_i)) < f_{i,\bullet}(x_i) < \phi(x_i). \tag{5.11}$$

By assumption, $f_{i,\bullet}(x_i) > \beta$. Hence,

$$\phi(f_{i,\bullet}(x_i)) < \phi(\beta) < \beta < f_{i,\bullet}(x_i). \tag{5.12}$$

Hence, we combine (5.11) and (5.12) to get

$$x_i < \phi(f_{i,\bullet}(x_i)) < \beta < f_{i,\bullet}(x_i) < \phi(x_i). \tag{5.13}$$

By assumption on $F_{k,i-1}(x_i)$ and the previous case we have

$$x_i < \phi(f_{i,\bullet}(x_i)) < \beta < F_{i,k-1}(x_i) < f_{i,\bullet}(x_i) < \phi(x_i). \tag{5.14}$$

Applying the monotone decreasing and self-inversing nature of ϕ to (5.14) we have

$$x_i < \phi(f_{i,\bullet}(x_i)) < \phi(F_{i,k-1}(x_i)) < \phi(\beta) < f_{i,\bullet}(x_i) < \phi(x_i). \tag{5.15}$$

Finally, note that $f_{k+i,\bullet}$ is a pseudo-population model that is weakly enveloped by ϕ and $F_{i,k-1}(x_i) > \beta$. Thus,

$$\phi(F_{i,k-1}(x_i)) < F_{i,k}(x_i) = (f_{k+i,\bullet} \circ F_{i,k-1})(x_i). \tag{5.16}$$

Using terms from (5.15) and combining them with (5.16) and the assumptions on $F_{i,k}$ we have,

$$x_i < \phi(F_{i,k-1}(x_i)) < F_{i,k}(x_i) < \alpha.$$

We now consider the case that $x_i > \beta$, $f_{i,\bullet}(x_i) < \alpha$ for some i and there exists $k \geq 1$ such that for all $0 \leq l < k$ we have

$$F_{i,l}(x_i) = (f_{i+l,\bullet} \circ f_{i+l-1,\bullet} \circ \ldots \circ f_{i,\bullet})(x_i) < \alpha$$

and

$$F_{i,k}(x_i) = (f_{k+i,\bullet} \circ \ldots \circ f_{i,\bullet})(x_i) > \beta.$$

Again, ϕ has a fixed point in (α, β) and is monotone decreasing so $\phi(x) > \alpha > x$ for $x \in (0, \alpha)$. Since $f_{i,\bullet}(x_i) < \alpha$

$$f_{i,\bullet}(x_i) < \phi(f_{i,\bullet}(x_i)). \tag{5.17}$$

Since ϕ is a weak enveloping function for $f_{i,\bullet}$ and $x_i > \beta$ we have

$$\phi(x_i) < f_{i,\bullet}(x_i) < x_i. \tag{5.18}$$

Applying the monotone decreasing and self-inversing nature of ϕ to (5.18) we have

$$\phi(x_i) < \phi(f_{i,\bullet}(x_i)) < x_i. \tag{5.19}$$

Thus, combining (5.17), (5.18), and (5.19), we have

$$\phi(x_i) < f_{i,\bullet}(x_i) < \phi(f_{i,\bullet}(x_i)) < x_i. \tag{5.20}$$

Using the assumption that $f_{i,\bullet}(x_i) < \alpha$ we have that

$$f_{i,\bullet}(x_i) < \alpha < \phi(f_{i,\bullet}(x_i)). \tag{5.21}$$

Then by (5.20) and (5.21) we have

$$\phi(x_i) < f_{i,\bullet}(x_i) < \alpha < \phi(f_{i,\bullet}(x_i)) < x_i. \tag{5.22}$$

By the previous cases, the assumptions on $F_{i,k-1}(x_i)$, and (5.22)

$$\phi(x_i) < f_{i,\bullet}(x_i) < F_{i,k-1}(x_i) < \alpha < \phi(f_{i,\bullet}(x_i)) < x_i. \tag{5.23}$$

Applying ϕ to (5.23),

$$\phi(x_i) < f_{i,\bullet}(x_i) < \phi(\alpha) < \phi(F_{i,k-1}(x_i)) < \phi(f_{i,\bullet}(x_i)) < x_i. \tag{5.24}$$

Finally, note that $f_{k+i,\bullet}$ is a pseudo-population model that is weakly enveloped by ϕ and $F_{i,k-1}(x_i) < \alpha$. Thus,

$$F_{i,k}(x_i) < \phi(F_{i,k-1}(x_i)). \tag{5.25}$$

Combining terms from (5.24), (5.25), and our assumptions on $F_{i,k}$ we have

$$\beta < F_{i,k}(x_i) < \phi(F_{i,k-1}(x_i)) < x_i.$$

Thus far, we have established that if $\mathcal{O}^+(x_0) \cap [\alpha,\beta] = \emptyset$ for $x_0 \in \mathbb{R}^+$ then $\mathcal{O}^+(x_0) \cap (0,\alpha)$ is a monotone increasing sequence and $\mathcal{O}^+(x_0) \cap (\beta,\infty)$ is a monotone decreasing sequence if we allow for empty or finite sequences. We desire to show that given $x_0 \in \mathbb{R}^+ \setminus [\alpha,\beta]$, $\mathcal{O}^+(x_0)$ enters any arbitrary open neighborhood of $[\alpha,\beta]$ and we will proceed by contradiction. Suppose

$x_0 \in \mathbb{R}^+ \setminus [\alpha, \beta]$ and the subsequence $\mathcal{O}^+(x_0) \cap (0, \alpha)$ has a supremum $a < \alpha$ or the subsequence $\mathcal{O}^+(x_0) \cap (\beta, \infty)$ has an infimum $b > \beta$ as guaranteed by monotonicity. That is, suppose for all $k \in \mathbb{Z}^+$

$$(f_{k,\bullet} \circ f_{k-1,\bullet} \circ \ldots \circ f_{0,\bullet})(x_0) \notin (a, b)$$

and $(a, b) \neq (\alpha, \beta)$. If $\left\{f_{g(i)}\right\}_{i=0}^{\infty}$ uses some maps from $\mathcal{S}$ only a finite number of times, then for $t^* \geq 0$ we consider

$$\mathcal{O}^+_{t^*}(x_0) = \mathcal{O}^+(x_0) \setminus \{x_0, x_1, \ldots, x_{t^*-1}\}$$

which is the orbit of x_{t^*} under the restricted dynamical system $\left\{f_{g(i)}\right\}_{i=t^*}^{\infty}$ in which all maps used appear an infinite number of times. Notice $\mathcal{O}^+_{t^*}(x_0)$ is a subsequence of $\mathcal{O}^+(x_0)$.

We proceed assuming that $\mathcal{O}^+_{t^*}(x_0) \cap (0, \alpha)$ is a sequence with supremum $a < \alpha$. Notice then that the monotonicity of $\mathcal{O}^+_{t^*}(x_0) \cap (0, \alpha)$ assures that a is the limit of $\mathcal{O}^+_{t^*}(x_0) \cap (0, \alpha)$. We consider two cases. The first is that there exists $N \in \mathbb{Z}^+$ such that for $x_i \in \mathcal{O}^+_{t^*}(x_0)$, $x_i < \alpha$ if $i > N$. The second case will be that no such N exists.

In the first case, we only consider terms in the orbit after the N-th iterate, and we assume that any $f_{\bullet,k}$ mentioned is used by the restricted system $\left\{f_{g(i)}\right\}_{i=M}^{\infty}$ where $M = \max\{N, t^*\}$. That is, $k \in U \subset \{0, 1, \ldots, n-1\}$. For all such $f_{\bullet,k}$ and a fixed $r \in (0, \alpha - a)$ let

$$\left\{x_i^k\right\} = \left\{x_i \in \mathcal{O}^+_M(x_0) \cap (0, \alpha) : x_{i+1} = f_{i,k}(x_i) = f_{\bullet,k}(x_i)\right\}.$$

Then for each k, $\left\{x_i^k\right\}$ is a subsequence of $\mathcal{O}^+_{t^*}(x_0) \cap (0, \alpha)$ so,

$$\lim_{i\to\infty} x_i^k = a.$$

Since each $f_{\bullet,k}$ is continuous,

$$\lim_{i\to\infty} f_{\bullet,k}(x_i^k) = f_{\bullet,k}\left(\lim_{i\to\infty} x_i^k\right) = f(a).$$

However, $f_{\bullet,k}\left(\left\{x_i^k\right\}\right)$ is also a subsequence of $\mathcal{O}^+_{t^*}(x_0) \cap (0, \alpha)$ so

$$\lim_{i\to\infty} f_{\bullet,k}\left(\left\{x_i^k\right\}\right) = a.$$

Thus, $f(a) = a$. Notice though that $a < \alpha < q$ so a is not fixed by any $f_{\bullet,k}$. This gives a contradiction so we move to our second case.

In the second case, we may assume that $\mathcal{O}^+_{t^*}(x_0) \cap (\beta, \infty)$ is a sequence which limits on b (in addition to the existing assumption that $\mathcal{O}^+_{t^*}(x_0) \cap (0, \alpha)$ is a sequence limiting on a). Then there are infinitely many

$$x_i \in \mathcal{O}^+_{t^*}(x_0) \cap (0, \alpha)$$

such that $x_{i+1} > b$. Therefore, there is an $f_{\bullet,k}$ used infinitely many times by $\left\{f_{g(i)}\right\}_{i=t^*}^{\infty}$ such that $x_i < a$ and $f_{i,k}(x_i) = b$. Again, let,

$$\left\{x_i^k\right\} = \left\{x_i \in \mathcal{O}_{t^*}^{+}(x_0) \cap (0, \alpha) : b < x_{i+1} = f_{i,k}(x_i) = f_{\bullet,k}(x_i)\right\}.$$

Then $x_i^k \to a$ as $i \to \infty$ and because $f_{\bullet,k}$ is continuous

$$\lim_{i\to\infty} f_{\bullet,k}\left(x_i^k\right) = f_{\bullet,k}\left(\lim_{i\to\infty} x_i^k\right) = f_{\bullet,k}(a).$$

Since $f_{\bullet,k}\left(\left\{x_i^k\right\}\right)$ is a subsequence of $\mathcal{O}_{t^*}^{+}(x_0) \cap (\beta, \infty)$, we know $f_{\bullet,k}\left(\left\{x_i^k\right\}\right) \to b$ as $i \to \infty$. Thus, $f_{\bullet,k}(a) = b$.

Because there are only finitely many maps in $\left\{f_{g(i)}\right\}_{i=t^*}^{\infty}$, there must be a pair, $f_{\bullet,k}$ and $f_{\bullet,l}$ that appears consecutively infinitely many times in the iteration of the system (where $f_{\bullet,k}(x_i) = x_{i+1} > b$ for $x_i \in \left\{x_i^k\right\}$). That is, if $f_{g(i)} = f_{i,k}$ then $f_{g(i+1)} = f_{i+1,l}$ for infinitely many i. Fix such a pair and let

$$F_k(x) = (f_{\bullet,l} \circ f_{\bullet,k})(x).$$

By the previous steps of the proof and our assumptions either

(I) $x_i < a < \alpha < \beta < b < x_{i+2} = F_k(x_i) < x_{i+1} = f_{i,k}(x_i)$ or

(II) $x_i < x_{i+2} = F_k(x_i) < a < \alpha < \beta < b < x_{i+1} = f_{i,k}(x_i)$.

We define

$$\left\{y_i^k\right\} = \left\{x_i \in \left\{x_i^k\right\} : x_{i+2} = F_k(x_i)\right\}.$$

This gives that $\left\{y_i^k\right\}$ is a subsequence of $\left\{x_i^k\right\}$. Since F_k is continuous, we know

$$\lim_{i\to\infty} F_k\left(y_i^k\right) = F_k\left(\lim_{i\to\infty} y_i^k\right) = F_k(a).$$

Case (I) and case (II) may each occur for infinitely many i. If case (I) occurs infinitely many times, then $F_k\left(\left\{y_i^k\right\}\right)$ is a subsequence of $\mathcal{O}_{t^*}^{+}(x_0) \cap (\beta, \infty)$ and so converges to b. If case (II) occurs infinitely many times, then $F_k\left(\left\{y_i^k\right\}\right)$ is a subsequence of $\mathcal{O}_{t^*}^{+}(x_0) \cap (0, \alpha)$ and converges to a. Thus,

(I) $f_{\bullet,k}(a) = b = F_k(a)$ or

(II) $a = F_k(a)$.

However, (I) contradicts that $f_{\bullet,l}$ is a pseudo-population model that does not fix b and (II) contradicts that we must have $a < F_k(a)$ as was shown in the first half of the proof.

A similar argument holds if we begin by assuming that $\mathcal{O}^+(x_0) \cap (\beta, \infty)$ is a sequence. Therefore, the subsequences $\mathcal{O}^+(x_0) \cap (0, \alpha)$ and $\mathcal{O}^+(x_0) \cap (\beta, \infty)$ cannot limit on values in $\mathbb{R}^+ \setminus [\alpha, \beta]$.

□

In autonomous examples, weak enveloping is most useful if either α or β is a point of intersection of ϕ and f. However, in nonautonomous examples we simply desire that the distance between α and β be as small as possible to give the strongest restrictions on the orbit of the dynamical system.

Example 5.3.1. Consider the Ricker model $f_r(x) = xe^{r(1-x)}$ for $r > 0$. The Ricker model is a known population model and therefore a pseudo-population model whose crossing interval can be chosen arbitrarily. The model is known to undergo period doubling route to chaos as r increases beyond 2. We set $r = 3$ and let $\phi(x) = -x + 2$. Here, $\phi(x)$ does not envelope $f_3(x)$ but it does weakly envelope on $(0, \infty) \setminus [0.14, 1.86]$. Therefore, any orbit under f_3 must either enter the interval $[0.14, 1.86]$ infinitely many times or come arbitrarily close to it infinitely many times.

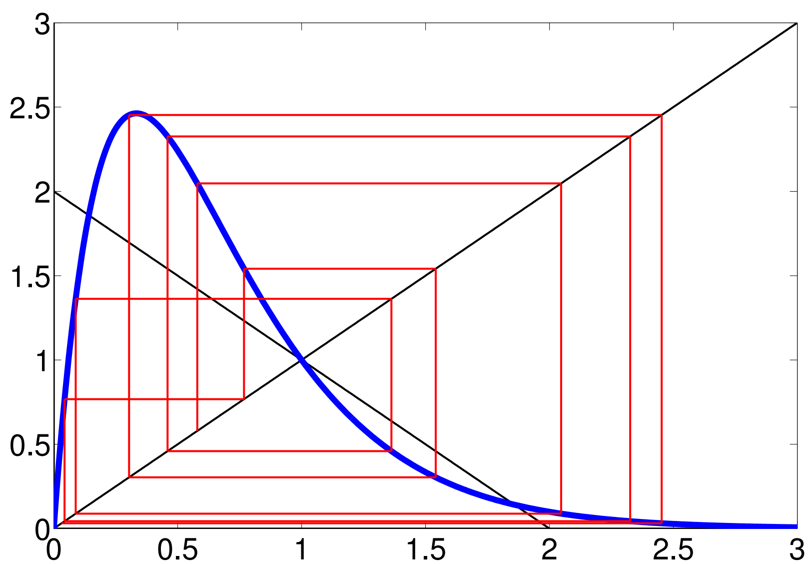


Figure 5.5: Several terms of the orbit of $x_0 = 0.5$ under $f_3(x)$ with weak enveloping function $\phi(x) = -x + 2$.

Figure 5.5 shows the 90th through 100th iterate of $x_0 = 0.5$ under $f_3(x)$. The orbit enters the interval $[0.14, 1.86]$ several times. ■

A desired result of Theorem 5.3.2 would be that there exists some k for which

$$(f_k \circ f_{k-1} \circ \ldots \circ f_0)(x) \in [\alpha, \beta].$$

Unfortunately, the following example shows that this is not always the case.

Example 5.3.2. Let $p > 0$ be given and choose a so that

$$0 < 2p - a < p < a.$$

Define the population model f with fixed point p by

$$f(x) = \begin{cases} \dfrac{\frac{p}{2} + \frac{3a}{4}}{p - \frac{a}{2}} x & 0 \le x < p - 1/2a \\ (-1/2)x + p + (1/2)a & p - (1/2)a \le x < 2p - a \\ -x + 2p & 2p - a \le x < a \\ (-1/2)x + 2p - (1/2)a & a \le x < p + (1/2)a \\ (3/2)p - (3/4)a & x \ge p + (1/2)a \end{cases}.$$

Then f is weakly enveloped on $U = (0, \infty) \setminus [2p - a, a]$ by $\phi(x) = -x + 2$. However, for $x_0 \in [p - 1/2a, 2p - a]$, $f(x_0) > a$ and $f^2(x_0) < 2p - a$. Thus, there is no k such that $f^k(x_0) \in [2p - a, a]$.

In fact, infinitely many examples of this form can be constructing by choosing the slopes of the respective line segments on $[p - (1/2)a, 2p - a)$ and $[a, p + (1/2))$ to be between -1 and 0. ■

Another restriction of Theorem 5.3.2 is that there can only be finitely many maps in the nonautonomous system. The next example gives a nonautonomous system of infinitely many population models for which weak enveloping does not ensure the orbit approaches an arbitrary open neighborhood of $[\alpha, \beta]$.

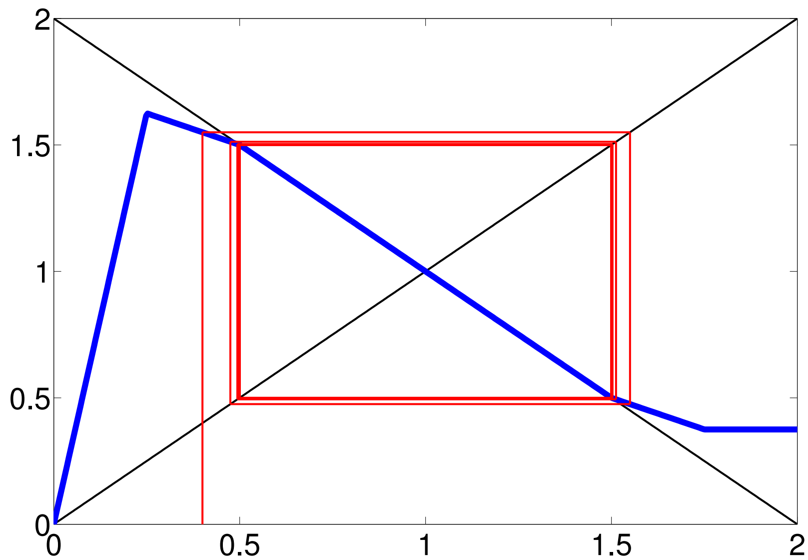


Figure 5.6: A weakly enveloped population model from Example 5.3.2 for which the orbit does not enter the interval $[\alpha, \beta]$.

Example 5.3.3. Consider the nonautonomous system of population models $\{f_i\}_{i=0}^{\infty}$ where

$$f_i(x) = \begin{cases} 2x & 0 \leq x \leq \dfrac{1}{2^i} \\ x + \dfrac{1}{2^i} & \dfrac{1}{2^i} < x \leq 5 - \dfrac{1}{2^i} \\ 5 & x > 5 - \dfrac{1}{2^i} \end{cases}$$

Each f_i is a population model with a fixed point at $x = 5$. Furthermore, each f_i can be properly enveloped by $\phi(x) = -x + 10$ which is monotone decreasing and self-inversing. We can therefore choose arbitrarily that ϕ weakly envelopes each f_i on $[0, \infty) \setminus [4.5, 5.5]$. If we consider the orbit of $x_0 = 1/2$ we see

$$\mathcal{O}^+\left(\frac{1}{2}\right) = \left\{\frac{1}{2}, 1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{4}, \ldots, \sum_{i=0}^{n} \frac{1}{2^i}, \ldots\right\}.$$

That is, the orbit of $x_0 = 1/2$ converges monotonically to 2. However, 2 is not contained in every open neighborhood of $[4.5, 5.5]$ which is contrary to the results of Theorem 5.3.2 for a nonautonomous system of a finite number of pseudo-population models. ■

Having established the Weak Enveloping Theorem, we move now to showing that a periodic dynamical system near an enveloped population model has a globally attracting geometric cycle. Specifically, we will start with an enveloped population model, f, with a hyperbolic fixed point and consider a periodic dynamical system comprised of functions that are in a $\mathcal{C}^1$-ϵ neighborhood of f. We will show that such a system has a globally attracting geometric n-cycle. We will begin with a preliminary lemma that establishes the existence of a trapping region that will contain the attracting geometric cycle.

Lemma 5.3.2. *Suppose $f \in \mathcal{P} \cap \mathcal{C}^1(\mathbb{R})$ with $f(p) = p$ and $|f'(p)| < 1$. Then there is an $r > 0$ for which $\mathcal{B}(p, r)$ is a trapping region for f. Furthermore, there is a $\mathcal{C}^0$-ϵ neighborhood of f on $\bar{\mathcal{B}}(p, r)$, $U \subset \mathcal{P}$, for which $\mathcal{B}(p, r)$ is a trapping region for every $f_i \in U$.*

Proof. Since $|f'(p)| < 1$ there exists $\lambda > 0$ so that $|f'(p)| \leq \lambda < 1$. Since $f \in \mathcal{C}^1(\mathbb{R})$, there exists $\delta > 0$ so that for all $x \in \mathcal{B}(p, \delta)$, $|f'(x)| \leq \lambda < 1$. Take $r \in (0, \delta)$ and let $V = \mathcal{B}(p, r)$. Because $|f'(x)| \leq \lambda$ for all $x \in V$, $f(\bar{V}) \subset V$. Next, take

$$\epsilon = 0.5 \min \left\{ d(f(x), (0, \infty) \setminus V | x \in \bar{V} \right\}$$

and let $U \subset \mathcal{P}$ be a $\mathcal{C}^0$-ϵ perturbation of f. By the results of Franke and Selgrade ([21]), every $f_i \in U$ has V as a trapping region. □

Remark 5.3.1. In Lemma 5.3.2, it is important to note that r was chosen arbitrarily in $(0, \delta)$. Therefore, it is possible to adjust the endpoints of the trapping region as needed within $\mathcal{B}(p, \delta)$.

Remark 5.3.2. In the proof of Lemma 5.3.2 we used the results of [21]. It should be noted that the definition of a C^0-ϵ perturbation was different in that work. However, the local dynamics are equivalent for both interpretations of a C^0-ϵ perturbation.

Lemma 5.3.2 offers just one option for ensuring the existence of an appropriate trapping region. The following lemma places further restrictions on the type of perturbation but gives a stronger result.

Lemma 5.3.3. *Suppose $f \in \mathcal{P} \cap \mathcal{C}^1(\mathbb{R})$ with $f(p) = p$ and $|f'(p)| < 1$. Then there exists $\delta > 0$ and $\epsilon > 0$ such that for all $r \in (0, \delta)$ any n-periodic dynamical system, $\{f_0, f_1, \ldots, f_{n-1}\}$, of maps such that each f_i is a $\mathcal{C}^1$-ϵ perturbation of f on $\bar{\mathcal{B}}(p, r)$ has an attracting geometric cycle, $\{c_0, c_1, \ldots, c_{n-1}\}$ where c_i is the fixed point of*

$$(f_{(i+n-1) \bmod n} \circ \ldots \circ f_{(i+1) \bmod n} \circ f_{i \bmod n})(x) = F_i(x).$$

Furthermore, there is an open neighborhood of each c_i that is a trapping region for $F_i(x)$.

Proof. Since $|f'(p)| < 1$ there exists $\lambda > 0$ so that $|f'(p)| \leq \lambda < 1$. Since $f \in \mathcal{C}^1(\mathbb{R})$, there exists $\delta > 0$ so that for all $x \in \mathcal{B}(p, \delta)$, $|f'(x)| \leq \lambda < 1$. From Chapter 2, we know for $r \in (0, \delta)$, f is a contraction on $\bar{\mathcal{B}}(p, r)$. Let

$$\epsilon < \frac{1 - \lambda}{2}.$$

Then any $\mathcal{C}^1$-ϵ perturbation of f on $\bar{B}(p, r)$ is also a contraction on $\bar{\mathcal{B}}(p, r)$. The rest follows from Theorem 2.4.2. $\square$

Lemma 5.3.3 guarantees the existence and local stability of a geometric cycle for a periodic dynamical system. However, we desire to prove the existence and global stability of a geometric cycle for a periodic dynamical system. The next lemma establishes weak enveloping for all $\mathcal{C}^0$-ϵ perturbations of an enveloped population model. We will then combine the local stability result with the weak enveloping result to establish the desired global result.

To avoid confusion between Definition 2.7.2 and Definition 5.3.2, we will refer to enveloping as it appears in Definition 2.7.2 are *proper enveloping* for the remainder of the section.

Lemma 5.3.4. *Suppose $f \in \mathcal{P}$ with $f(p) = p$ and that f is properly enveloped by monotone decreasing, self-inversing $\phi \in \mathcal{C}^0(\mathbb{R})$. If $a, b \in (0, \infty)$, $a < p < b$ then there exists $\mathcal{C}^0$-ϵ neighborhood of f on $[0, \phi(0)]$, U, for which every $f_i \in U$ is weakly enveloped on $(0, \infty) \setminus [a, b]$ by ϕ.*

Proof. Choose $a \in (0, p)$ and $b \in (p, \phi(0))$. Let

$$\epsilon < \min \left\{ \min_{[0,a]} (\phi(x) - f(x)), \min_{[b,\phi(0)]} (f(x) - \phi(x)) \right\}$$

and let $g \in \mathcal{P}$ be a C^0-ϵ perturbation of f. If $x \in (0, a)$ then

$$\begin{aligned} \phi(x) - g(x) &\geq \phi(x) - (f(x) + \epsilon) \\ &= \phi(x) - f(x) - \epsilon \\ &> 0 \end{aligned}$$

by the definition of ϵ. Similarly, if $x \in (b, \phi(0))$ then

$$\begin{aligned} g(x) - \phi(x) &\geq f(x) - \epsilon - \phi(x) \\ &= (f(x) - \phi(x)) - \epsilon \\ &> 0 \end{aligned}$$

by the definition of ϵ. Finally, if $x \in (\phi(0), \infty)$ then $\phi(x) < 0$ while $g(x) > 0$ because g is a population model. □

Having established the necessary preliminary results, we now give a theorem guaranteeing the existence of a globally attracting geometric cycle for a periodic system of population models. We choose to give the proof of the theorem in terms of Lemma 5.3.2 rather than Lemma 5.3.3.

Theorem 5.3.3. *Suppose $f \in P \cap \mathcal{C}^1(\mathbb{R})$ with $f(p) = p$ and $|f'(p)| < 1$. Suppose further that f is properly enveloped by monotone decreasing, self-inversing $\phi \in \mathcal{C}^0(\mathbb{R})$. Then there exists $r \in (0, p)$ and a $\mathcal{C}^1$-ϵ neighborhood on $[0, \phi(0)]$, $U \subset \mathcal{P}$, of f such that any n-periodic dynamical system $\{f_0, f_1, \ldots, f_{n-1}\}$ with $f_i \in U$ has a globally attracting geometric cycle in $\mathcal{B}(p, r)$ whose period divides n.*

Proof. By Lemma 5.3.2, we know that there is an $r > 0$ so that $\mathcal{B}(p, r)$ is a trapping region for all f_i in a $\mathcal{C}^1$-ϵ_1 neighborhood of f. Denote this neighborhood by U_1.

With the same $r > 0$, $|f'(x)| \leq \lambda < 1$ for all $x \in \mathcal{B}(p, r)$. Let

$$\epsilon_2 < \frac{1 - \lambda}{2}$$

and let U_2 be a $\mathcal{C}^1$-ϵ_2 neighborhood of f. Then for $x \in \mathcal{B}(p, r)$ and $f_i \in U_2$

$$\begin{aligned} |f_i'(x)| &\leq |f'(x)| + \left|\frac{1 - \lambda}{2}\right| \\ &\leq \lambda + \frac{1 - \lambda}{2} \\ &= \frac{\lambda + 1}{2} < 1. \end{aligned}$$

Let $\delta \in (0, r)$. By Lemma 5.3.4, there is a $\mathcal{C}^0$-ϵ_3 neighborhood of f, U_3 for which each $f_i \in U_3$ is weakly enveloped by ϕ on $(0, \infty) \setminus \bar{\mathcal{B}}(p, \delta)$. Let $U = U_1 \cap U_2 \cap U_3$ and consider the n-periodic dynamical system $\{f_0, f_1, \ldots, f_{n-1}\}$ where each $f_i \in U$.

Let

$$F_i(x) = (f_{i+n-1 \bmod n} \circ \ldots \circ f_{i+1 \bmod n} \circ f_{i \bmod n})(x).$$

Then $\mathcal{B}(p,r)$ is a trapping region for $F_i(x)$ and since $F_i(\bar{\mathcal{B}}(p,r)) \subset \mathcal{B}(p,r)$, $F_i(x)$ must have a fixed point p_i in $\mathcal{B}(p,r)$. Furthermore, for $x \in \mathcal{B}(p,r)$, $|F_i'(x)| < 1$ by the chain rule, so p_i is a unique fixed point of $F_i(x)$ in $\mathcal{B}(p,r)$ by the Mean Value Theorem. Then each $F_i(x)$ has an attracting fixed point in the interval $\mathcal{B}(p,r)$. The Structure Theorem asserts that if $\{f_0, \ldots f_{n-1}\}$ has an attractor in $\mathcal{B}(p,r)$ then the attractor is the union of the attractors of the F_i which are fixed points. Therefore, the attractor must be the geometric cycle $\{p_0, p_1, \ldots, p_{n-1}\}$ whose entries may not be distinct.

Since each f_i is weakly enveloped by ϕ on $(0,\infty)\setminus\bar{\mathcal{B}}(p,\delta)$ and the periodic dynamical system is made of a finite number of population models, the Weak Enveloping Theorem assures there exists $k \geq 1$ for each $x_0 \in (0,\infty)$ such that

$$(f_{i+k} \circ \ldots \circ f_i)(x_0) \in \mathcal{B}(p,r). \tag{5.26}$$

That is, $\{p_0, p_1, \ldots, p_{n-1}\}$ is a global attractor for the periodic dynamical system. Because the periodic dynamical system has a geometric cycle as a global attractor, the period of the geometric cycle must divide the period of the dynamical system. □

5.4 Perturbations of Nonhyperbolic Population Models

Theorem 5.3.3 and its preceding lemmas all require that $|f'(p)| < 1$. In this section we consider population models with derivative 1 or -1 at their fixed points as well as periodic systems made of perturbations of these functions. The results of the preceding section fail to hold for such periodic dynamical systems.

We begin with a result concerning a population model with $f'(p) = 1$.

Theorem 5.4.1. *Suppose $f \in \mathcal{P}$ such that $f'(p) = 1$. Then p is locally attracting.*

Proof. Since $f'(p) = 1$ and $f \in \mathcal{P}$ there exists $r > 0$ so that for all $x \in \mathcal{B}(p,r)$, $0 < f'(x) < 1$. If $x_0 \in (p-r, p)$ then $f(x_0) > x_0$ and $f(x_0) < p$. Thus, $\mathcal{O}^+(x_0)$ is monotone increasing and bounded by p and therefore must converge to p. Similarly, if $y_0 \in (p, p+r)$ then $f(y_0) < y_0$ and $f(y_0) > p$. Thus, $\mathcal{O}^+(x_0)$ is monotone decreasing and bounded below by p and therefore converges to p. □

Despite the fact that a population model with $f'(p) = 1$ is very predictable, a periodic dynamical system made of perturbations of such a population model may display particularly erratic behavior. The following example explores this phenomena.

Example 5.4.1. Consider $f(x) = \arctan(x)$. The function f is not a population model but it does have features like a population model. Namely, f has a unique fixed point at 0, $f(x) > x$ for $x < 0$, and $f(x) < x$ for $x > 0$. Hence, appropriate adaptations can be made to f to make it a population model. Since we are interested in the dynamics near the fixed point, we will analyze f as is. Note that $f'(0) = 1$.

Let

$$\beta(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\kappa_0(x) = x - \epsilon\beta(x)$$
$$\kappa_1(x) = x + \epsilon\beta(x).$$

Finally, let

$$f_0(x) = (\kappa_0 \circ f)(x),$$
$$f_1(x) = (\kappa_1 \circ f)(x),$$

and consider the 2-periodic dynamical system $\{f_0, f_1\}$. Unfortunately, the form of f_0 and f_1 make it difficult to prove that these functions have unique fixed points locally. Figure 5.7 shows typical plots of $f_0(x) - x$ and $f_1(x) - x$. The figure suggests that both f_0 and f_1 have unique fixed points and satisfy the conditions like those of a population model.

If we consider the maps $F_0(x) = f_1(f_0(x))$ and $F_1(x) = f_0(f_1(x))$ we find numerically that each map has three fixed points (see Figure 5.8). For $\epsilon > 0$ the map F_0 has two positive fixed points we name p_1 and p_2 and one negative fixed point n_1. The fixed point p_1 is the closest fixed point to 0. As symmetry would suggest, the fixed points of $F_1(x)$ are $-p_1$, $-p_2$, and $-n_1$ for $\epsilon > 0$. As the Structure Theorem suggests, the geometric cycles of $\{f_0, f_1\}$ consist of fixed points of F_0 and F_1. In this example, the typical geometric cycles are $\{p_1, -p_1\}$, $\{n_1, -p_2\}$, and $\{p_2, -n_1\}$. Numerical results indicate that both, $\{n_1, -p_2\}$ and $\{p_2, -n_1\}$ are locally attracting while $\{p_1, -p_1\}$ is repelling.

As the geometric 2-cycle $\{p_1, -p_1\}$ is the geometric cycle with terms closest to 0, this must be the geometric cycle as guaranteed by Theorem 5.2.1. However, the presence of the multiple attractors is not consistent with the results for hyperbolic population models. Furthermore, the results for hyperbolic population models would suggest that $\{p_1, -p_1\}$ would be an attractor

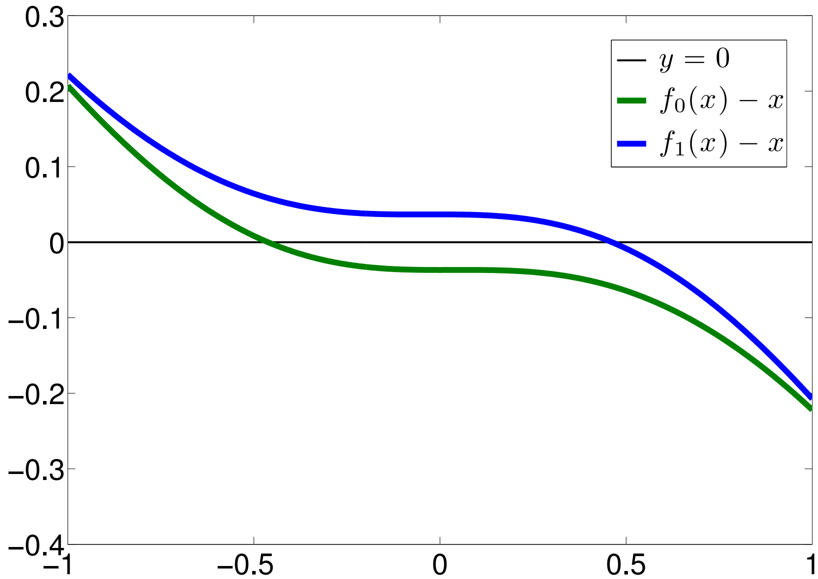


Figure 5.7: Typical plots of $f_0(x) - x$ and $f_1(x) - x$ for Example 5.4.1. Here $\epsilon = 0.1$.

because 0 was globally attracting for f.

Figures 5.9 through 5.11 give the bifurcation diagrams for $\{f_0, f_1\}$ with the initial conditions $x_0 = 0$, $x_0 = 0.1$, and $x_0 = -0.1$ respectively. For $\epsilon > 0$, the orbit of x_0 tends to the geometric cycle $\{n_1, -p_2\}$, the orbit of $x_0 = 0.1$ tends to $\{p_2, -n_1\}$, and the orbit of $x_0 = -.01$ tends to $\{n_1, -p_2\}$.

Figure 5.12 shows all three bifurcation diagrams on the same plot. If we note that $\{p_1, -p_1\}$ does not appear on the bifurcation diagram because it is repelling, then the diagram in Figure 5.12 appears to be a bi-directional pitchfork bifurcation were the attracting branches are geometric 2-cycles. The attractors when $\epsilon = 0$ should overlap exactly which is not the case in Figure 5.12. However, the spacing seems to decrease as the number of iterations is increased so this discrepancy is explained by transient behavior. ■

Example 5.4.1 clearly demonstrates that a version of Theorem 5.3.3 for a population model with $f'(p) = 1$ is not possible. The next example explores perturbations of a population model with $f'(p) = -1$ and demonstrates that perturbations of such models may not allow for globally attracting geometric cycles.

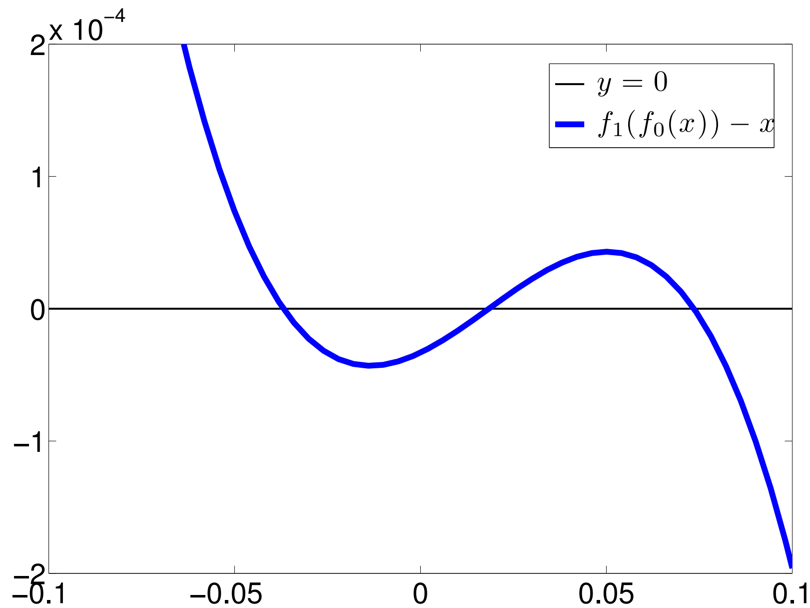


Figure 5.8: Typical plot of $f_1(f_0(x)) - x$ for Example 5.4.1. Here $\epsilon = 0.1$.

Example 5.4.2. Consider the linear-spline population model

$$f(x) = \begin{cases} 2x & 0 \le x \le 2 \\ -x+6 & 2 < x \le 5 \\ 1 & x > 5 \end{cases}$$

and for $0 < \epsilon < 0.5$ its $\mathcal{C}^0$-ϵ perturbation

$$g(x) = \begin{cases} 2x & 0 \le x \le 2+\epsilon \\ -x+6+3\epsilon & 2+\epsilon < x \le 5 \\ 1+3\epsilon & x > 5 \end{cases} .$$

Note that $f(3) = 3$ and $g(3 + \frac{3}{2}\epsilon) = 3 + \frac{3}{2}\epsilon$ and that both maps have derivative -1 at their respective fixed points. We consider the 2-periodic dynamical system $\mathcal{S} = \{f, g\}$ which demonstrates irregular dynamics which occur when the original population model has derivative -1 at its fixed point.

If we consider the orbit of $x_0 = 4$ under the periodic system we see

$$\mathcal{O}^+(x_0) = \{4, f(4) = 2, g(2) = 4, 2, 4, 2, 4, 2 \ldots\} .$$

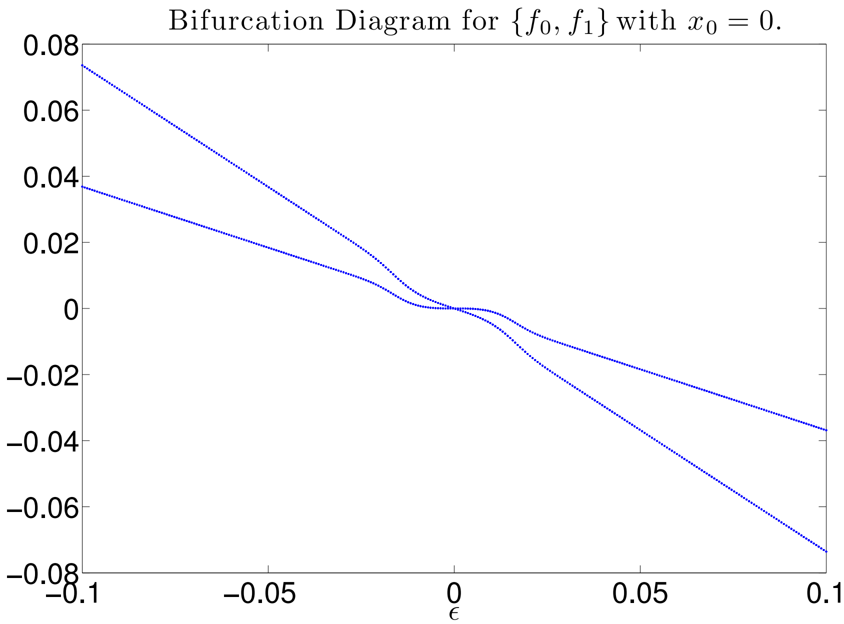


Figure 5.9: The bifurcation diagram for $\{f_0, f_1\}$ from Example 5.4.1 with $x_0 = 0$. ϵ ranges over 300 values in $[-0.1, 0.1]$. The system is iterated 250,000 times and the last 10 are shown for each ϵ.

Thus, $\mathcal{S}$ has a geometric 2-cycle. However, despite the value of ϵ, this 2-cycle cannot be made arbitrarily close to the fixed point of f. The same is true for the opposite ordering. That is, if $\mathcal{T} = \{g, f\}$ we can consider the orbit of $x_0 = 2$ under $\mathcal{T}$:

$$\mathcal{O}^+(2) = \{2, g(2) = 4, f(4) = 2, 4, 2, 4, \dots\}.$$

Furthermore, if we consider the orbit of $x_0 = 4 + 2\epsilon$ under $\mathcal{S}$ we have

$$\begin{aligned} f(4+2\epsilon) &= 2 - 2\epsilon \\ g(2-2\epsilon) &= 4 - 2\epsilon \\ f(4-2\epsilon) &= 2 + 2\epsilon \\ g(2+2\epsilon) &= 4 + \epsilon \\ f(4+\epsilon) &= 2 + \epsilon \\ g(2+\epsilon) &= 4 + 2\epsilon. \end{aligned}$$

That is, the map $F(x) = (g \circ f)(x)$ has a point of period 3. Similarly, the map $G(x) = (f \circ g)(x)$

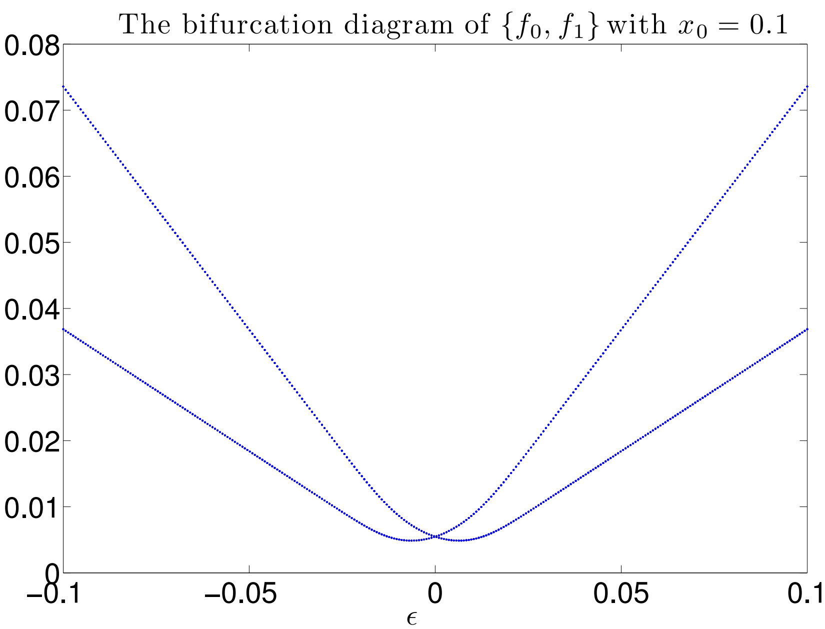


Figure 5.10: The bifurcation diagram for $\{f_0, f_1\}$ from Example 5.4.1 with $x_0 = 0.1$. ϵ ranges over 300 values in $[-0.1, 0.1]$. The system is iterated 250,000 times and the last 10 are shown for each ϵ.

has $2 + 4\epsilon$ as a point of period 3. Therefore, the periodic dynamical system, $\mathcal{S}$, has a geometric 6-cycle and therefore has a geometric cycle of every order greater than 6 in the extended Sharkovsky ordering (see [2]). Therefore, a theorem like Theorem 5.3.3 is not possible for a population model with $f'(p) = -1$. ■

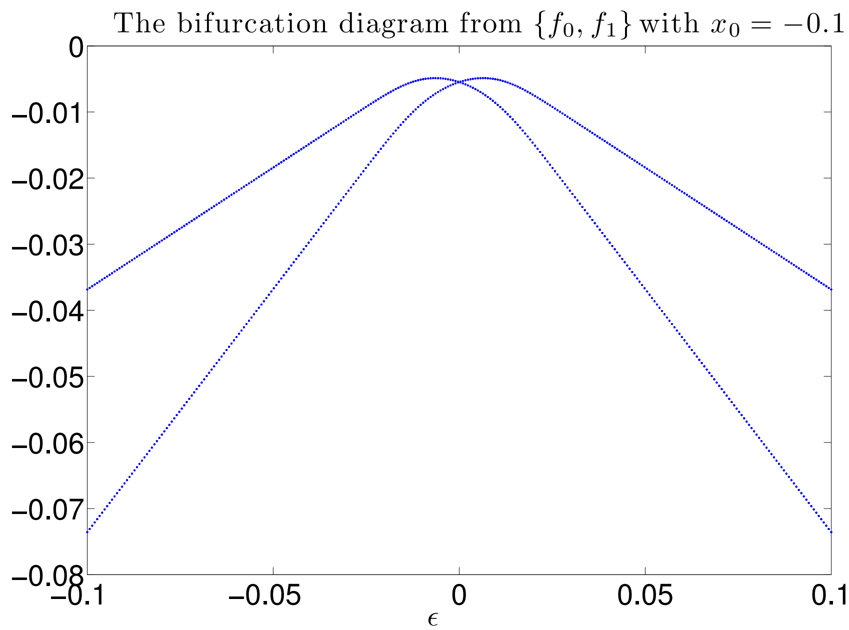


Figure 5.11: The bifurcation diagram for $\{f_0, f_1\}$ from Example 5.4.1 with $x_0 = -0.1$. ϵ ranges over 300 values in $[-0.1, 0.1]$. The system is iterated 250,000 times and the last 10 are shown for each ϵ.

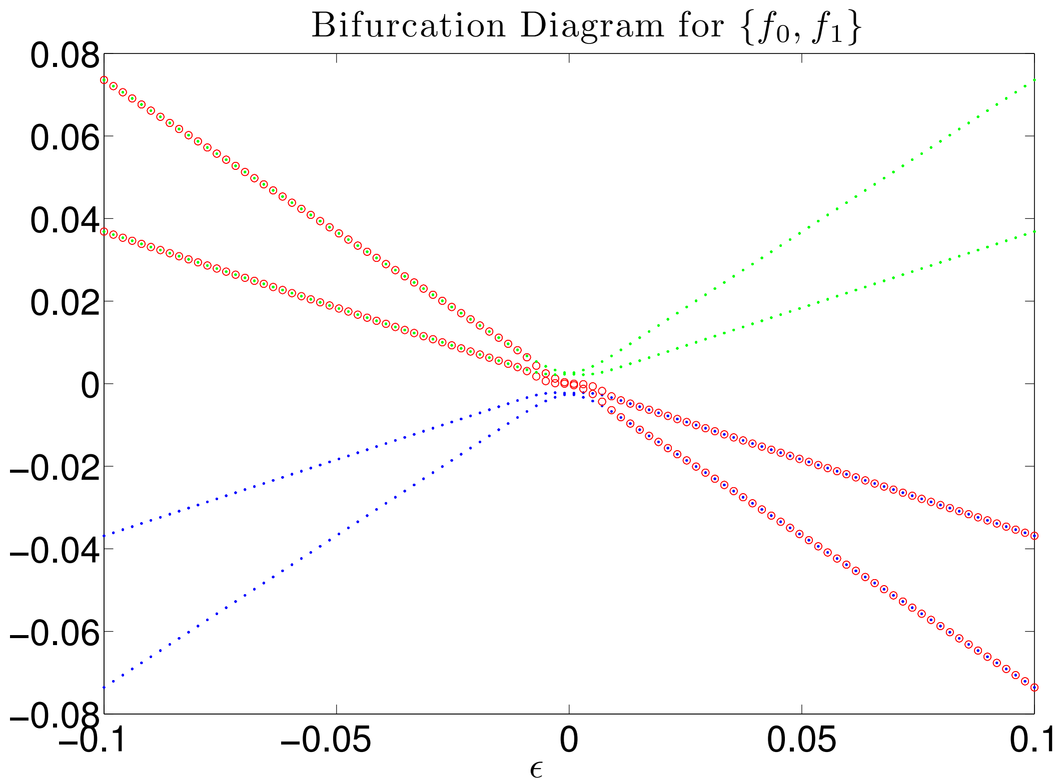


Figure 5.12: The bifurcation diagram for $\{f_0, f_1\}$ with initial conditions 0.1 (green dots), 0 (red circles), and -0.1 (blue dots). ϵ varies over 100 values values in $[-0.1, 0.1]$. For each ϵ value and initial condition the system is iterated 250,000 times with the last 10 iterates shown.

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