ABSTRACT

JIANG, MING. Travelling Wave Solutions, Periodic and Chaotic Solutions of a PDE Approximation of Coupled Chua's Circuits. (Under the direction of Xiao-Biao Lin.)

We study a singularly perturbed system of partial differential equations that models a one-dimensional array of coupled Chua's circuits. The PDE system is a natural generalization of the FitzHugh-Nagumo's equation. In part I of the paper, we show that similar to the FitzHugh-Nagumo's equation, the system can have periodic solutions formed alternatively by fast and slow flows. First, asymptotic method is used on the singular limit of the fast/slow systems to construct a formal periodic solution. Then, dynamical systems method is used to obtain an exact solution near the formal periodic soluion. Also, we show that the system can have a pair of heteroclinic orbits that form a closed loop connecting two equilibrium points. The dominant eigenvalues of the equilibrium points are complex numbers. Using the idea of Silnikov, we show that in a neighborhood of the heteroclinic loop, all the solutions are one-to-one correspond to to two sequence of symbols. Thus there are infinitely many homoclinic, heteroclinic, periodic and chaotic orbits nearby.

Travelling Wave Solutions, Periodic and Chaotic Solutions of a PDE Approximation of Coupled Chua's Circuits

by
Ming Jiang

A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh, North Carolina

2013

APPROVED BY:

Michael Shearer	Zhilin Li
Stephen Schecter	Xiao-Biao Lin Chair of Advisory Committee

DEDICATION

To my parents. It's been almost four years since I went back to China and got together with them, thank you for the understanding and encouragement. I hope that they are happy and healthy!

BIOGRAPHY

The author was born in Hubei, a small town in center part of China. He was inspired by his high school math teacher with lots of amazing techniques while solving problems. As a result he was determined to go to the department of Mathematics in Nankai University in China. In college, he got exposed to the mathematics atmosphere under the influence of Dr Shiing-Shen Chern, and learned to be rigorous when working in mathematics. He went to graduate school at North Carolina State University under the direction of Dr Xiao-Biao Lin and worked in singular perturbation and dynamical systems. The author thanks Dr Lin for all his help and instruction!

ACKNOWLEDGEMENTS

It is my honor to thank the many people who have helped me through this work.

First of all, I would like to thank my thesis advisor Dr Xiao-Biao Lin for his help. Throughout the whole time, he provided me with very careful, detailed academic support, invaluable guidance, as well as encouragement, which is very important to the completion of this thesis work.

Besides my advisor, I would like to sincerely thank the rest of my thesis committee: Dr. Shearer, Dr. Franke, Dr. Schecter and Dr. Ghosh for their encouragement and suggestion during the prelim exam.

Among all the memories of studying and life at NCSU. Dr. Martin, Dr. Chertock, Dr. Shearer, and Dr. Li are among the best teachers that any student can have. Their instruction for doing mathematics and their teaching style will benefit me way beyond this thesis work.

Also I would like to thank my officemate Amanda Landi for proof-reading the thesis and making the the sentences more smoothly, her suggestions are very helpful.

Thanks to Ms Denise Seabrooks for helping me with all of my paper-work.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

Introduction

1.1 Introduction of the background and problem set-up

Complex dynamical networks are everywhere, such as the Internet, power networks, neural networks, literature search networks, etc. The research on all the above complex networks has a lot to do with our daily life, and would possibly result in the development of other science direction. It becomes more and more important with the development of the information technology and biological science.

Chua's circuit is a simple electronic circuit that can have sophisticated behaviors like traveling wave solutions, periodic and chaotic solutions. This circuit consists of a nonlinear resistor N_R, linear inductor L, resistor R, and capacitors C_1, C_2. See Figure 1.1. The system of equations can be written as:

$$
\begin{aligned}
C_1 \frac{dV_{C_1}}{dt} &= (V_{C_2} - V_{C_1})/R - G(V_{C_1}) \\
C_2 \frac{dV_{C_2}}{dt} &= (V_{C_1} - V_{C_2})/R + i_L \\
L \frac{di_L}{dt} &= -V_{C_2}
\end{aligned}
$$

where G is usually, but not limited to, a piecewise-linear function. Sometimes G is even a nonlinear cubic function that shows the nonlinearity of the diode.

Systems of coupled cells with reactions and mass, energy or electric charge transfer often serve as standard models for investigating the phenomena occurring in the transformation and transport processes in living cells, tissues, neuron networks, and ecosystems, as well as in all forms of

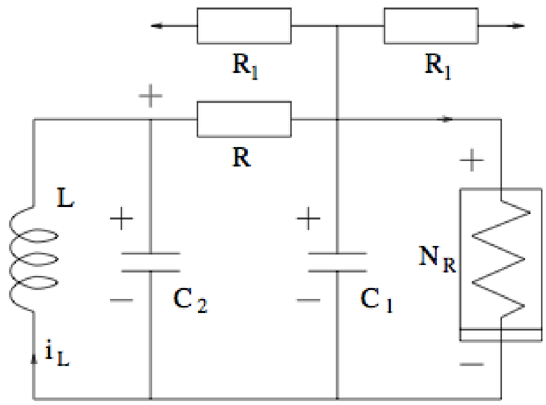


Figure 1.1: Chua's circuit

chemical, biochemical and biological reactors. In the continuous limit, it is possible to get a reaction-diffusion type model which exhibits all of the classical properties of an autowave process.

Also it is recognized that brains are nonlinear networks composed of chaotic systems, such as the Cellular Neural Networks (CNNs), which are time-continuous nonlinear dynamical systems. Therefore, it's important to investigate the dynamical behaviors of the simplest class of neural networks which exhibit chaos.

The coupled system is an example of CNNs, as described by Chua in his book [7]. According to Chua, all the CNNs have much in common as each cell can be a model from a biological, neurological, chemical or electronic system. Compared to other systems, electrical circuit networks are simpler to build, therefore, provide a practical, low cost method to simulate the other networks. We study traveling wave solutions to this CNN system since the existence of such solutions is one of the most prominent features of the network. We notice that our system is one of the simplest generalizations of FitzHugh-Nagumo equation, which is a second order bistable PDE coupled with linear first order ODE. The slow system we consider has two complex eigenvalues while in FitzHugh-Nagumos system the one-dimensional slow system has only one real eigenvalue.

In this paper, we consider an array of Chua's circuits connected by resistors R_1. We use k as the index for the kth circuit so that we have a system of equations as:

$$
\begin{aligned}
C_1 \frac{dV_{C_1}^k}{dt} &= (V_{C_2}^k - V_{C_1}^k)/R - G(V_{C_1}^k) - (V_{C_1}^{k-1} - 2V_{C_1}^k + V_{C_1}^{k+1})/R_1 \\
C_2 \frac{dV_{C_2}^k}{dt} &= (V_{C_1}^k - V_{C_2}^k)/R + i_L^k \\
L \frac{di_L^k}{dt} &= -V_{C_2}^k
\end{aligned}
$$

where G is the conductance of Chua's diode. By change of variables: $\alpha = 1/(C_1 R)$, combine $u + g(u)/\alpha$ as $h(u)$, the above system can be transformed into the following dimensionless form, which we rewrite for each circuit cell k $(k = 1, 2 \cdots l)$ as:

$$
\begin{aligned}
\dot{u}_k &= \alpha(y_k - h(u_k)) + \bar{D}(u_{k-1} - 2u_k + u_{k+1}) \\
\dot{y}_k &= u_k - y_k + z_k \\
\dot{z}_k &= -\beta y_k
\end{aligned} \tag{1.1}
$$

where $u_0(t) = u_1(t), u_l(t) = u_{l+1}(t)$. h is defined as follows in some of the previous work:

$$
h(u) = \begin{cases} m_1(u - u_-) & u \leq u_1 \\ -m_0 u & u_1 \leq u \leq u_2 \\ m_2(u - u_+) & u \geq u_2 \end{cases}
$$

and our cubic function in this paper:

$$
h(u) = mu(u - c)(u + c), m > 0, c > 0
$$

See [31] and figure 1.2 when $m = 1/30, c = 3$ for a later numerical example. I use this symmetric

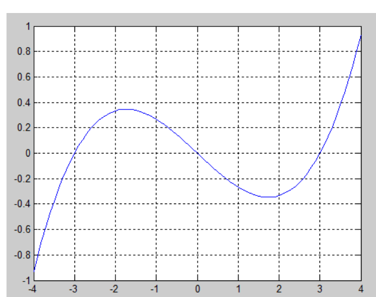


Figure 1.2: Graph of $y = h(u)$

cubic function as a special case, because I can use the Tschirnhaus transformation so that a generic cubic function $h(u) = au^3 + bu^2 + cu + d$ with three real roots can be rewritten as $h(t) = a(t^3 + pt + q)$ in order to get rid of the second order term. Here I substitute u by $t - b/3a$, and:

$$\begin{aligned} p =& \frac{3ac - b^2}{3a^2} \\ q =& \frac{2b^3 - 9abc + 27a^2 d}{27a^3}. \end{aligned}$$

For the limiting systems, we can shift the y variable so that we can get rid of the constant term and obtain a symmetric nonlinear function.

In this paper, we mainly consider the original system with the same scaling as P.P.C. See [31] Let $\epsilon = 1/\alpha, \delta x = \sqrt{\epsilon}, u_k(\bar{t}) = u(\bar{t}, k\Delta x)$. We approximate (1.1) by the following PDE:

$$\begin{aligned} \epsilon u_{\bar{t}} &= (y - h(u)) + \epsilon^2 D u_{xx} \\ y_{\bar{t}} &= u - y + z \\ z_{\bar{t}} &= -\beta y \end{aligned} \tag{1.2}$$

(1.2) can be rewritten as a four dimensional system of ODEs as follows. Let the traveling wave solution be $w = w(x - s\bar{t}) = w(t)$, where $w = (u, v, y, z)$ and $\dot{w} = dw/dt$, we have the so-called slow system, in which t is the slow time scale:

$$\begin{aligned} \epsilon \dot{u} &= v/D && (1.3) \\ \epsilon \dot{v} &= h(u) - sv/D - y \\ -s\dot{y} &= u - y + z \\ -s\dot{z} &= -\beta y \end{aligned}$$

Let $\tau = t/\epsilon$ and $w' = dw/d\tau$, then we have the so-called fast system, in which τ is the fast variable:

$$\begin{aligned} u' &= v/D && (1.4) \\ v' &= h(u) - sv/D - y \\ sy' &= \epsilon(-u + y - z) \\ sz' &= \epsilon \beta y \end{aligned}$$

with equilibria $P_\pm = (u, v, y, z) = (u_\pm, 0, 0, -u_\pm), P_0 = (0, 0, 0, 0)$.

In order to analyze different types of formal solutions, we obtain the reduced limiting problem of the dynamics of (1.3), (1.4) when $\epsilon = 0$ on the slow and fast time scales, respectively:

$$\begin{aligned} 0 &= v \\ 0 &= h(u) - y \\ -s\dot{y} &= u - y + z \\ -s\dot{z} &= -\beta y \end{aligned} \tag{1.5}$$

$$\begin{aligned} u' &= v/D \\ v' &= h(u) - sv/D - y \\ y' &= 0 \\ z' &= 0 \end{aligned} \tag{1.6}$$

Acosta considered the case when $\beta < 0$ in which saddle points on the slow manifolds are studied, cf [1], while in reality, β should be positive as we study here. We can show that for a suitable wave speed s and β value, there exists a unique traveling wave solution connecting the equilibria $P_{\pm}$. The idea of such internal layer solution is not new in singular perturbation methods [25], [22], [21], [18], [9]. They are based on singular perturbation and heteroclinic bifurcations. There are also geometric singular perturbation methods, such as: [10], [11], [17]. The geometric method involves the construction of some invariant manifolds and their foliations.

There are works about the periodic orbits and aperiodic orbits for a single uncoupled circuit. For example, in [14] the Brouwer's fixed-point theorem is applied to prove the existence of periodic solution for a Chua's circuit with smooth nonlinearity.

1.2 Outline of the thesis

In this paper, we use analytical method in singular perturbation and obtain different types of solutions for the systems (1.3), (1.4).

In Chapter Two, we introduce exponential dichotomies on singular perturbation problems, see [4], [30].

In Chapter Three, we construct different possible formal solutions when $\epsilon = 0$. One of the situations is the generalization of the solution of FitzHugh-Nagumo's system in [25].

In Chapter Four, we construct the approximated solution by the asymptotic matched expansion of the system, since it is known that formal series expansions in singular perturbation problems often provide accurate approximations of exact solutions. We can actually obtain expansion up to any order by solving the recursive differential/algebraic equations, but the 0th order is enough for the proof of the existence of the periodic solution.

In Chapter Five, we first prove there exists a periodic solution to the linear variational system of the correction function. We find a generalized solution that allows a gap at $\tau = 0$ in singular layers along a fixed direction. The size of the gaps are expressed by the Melnikov functions. We use the Melnikov integral to eliminate the gap by shifting the y values in the gap function g^i. The shifting of the y values results in the updated domains for the solutions on the outer layers. Then we obtain solutions $(\hat{U}_i, \hat{Y}_i)$ on the updated domain that satisfy the jump conditions exactly with no gap at $\tau = 0$. Now that we prove the existence of correction solution to the linear variational system, the exact solution of the original nonlinear system follows by contraction mapping.

In Chapter Six, we obtain a chaotic(nonperiodic) solution for the original system based on a pair of heteroclinic solutions. We find a generalized solution that allows a gap at $\tau = 0$ along a fixed direction. We use the Melnikov integral to eliminate the gap by shifting the y values in the gap function g^k. Then we obtain solutions $\hat{U}_k$ on the updated domain that satisfy the jump conditions JU_k with no gap at $\tau = 0$. However, the JY_k on the updated domain is not satisfied exactly. After we define the Y_k^{i+1}, the difference of jump errors $E(JY_k^{i+1})$ is reduced by a multiple of a small number in the i-th iteration, due to the contraction caused by the stable spiral near the equilibrium points. Therefore, the exact solution can be obtained after iterations. We show that there are infinitely many chaotic solutions, each solution uniquely corresponds to a sequence of symbols corresponding to the rotation numbers around the equilibrium points.

Chapter 2

Preliminary Results

We introduce some properties for linear systems of differential equations.

2.1 Exponential dichotomies

Consider the linear homogeneous differential equation:

$$\dot{x} = A(t)x \tag{2.1}$$

Here $A : I \to R^{n\times n}$ is continuous, where $I \in R$ is a finite or infinite interval. Let $\Phi(t,s)$ be the principal matrix solution of (2.1).

Definition 2.1.1. We say that (2.1) has an exponential dichotomy on I if there exist positive constants K, α and projections $P_s(t) + P_u(t) = I_n$ such that for $t, s \in I$ we have:
(i)$\Phi(t,s)P_s(s) = P_s(t)\Phi(t,s)$
(ii)$|\Phi(t,s)P_s(s)| \leq Ke^{-\alpha}(t-s), s \leq t$
(iii)$|\Phi(t,s)P_u(s)| \leq Ke^{-\alpha}(s-t), t \leq s$

Next we consider the adjoint system:

$$\dot{x} + A^{\star}(t)x = 0 \tag{2.2}$$

which has an exponential dichotomy on J if (2.1) has exponential dichotomy on J with the same constants K and α. Let $\Phi^*(s,t) := (\Phi(t,s)^{-1})^*$ be the principal matrix solution of the adjoint equation. Also the projections are $P_s^{\star}(t)$ and $P_u^{\star}(t)$ to the stable and unstable subspaces.

The projections $P_s(t)$ and $P_u(t)$ are unique only if $J = R$. Solutions on the unstable (stable) subspaces of $\Phi^*(s,t)$ decay exponentially if solved forward (backward) in time. We are interested in the exponential dichotomy where K is not too large and $\alpha(b-a)$ is not too small so that $Ke^{-\alpha(b-a)} << 1$. See [8, 29] for details.

Lemma 2.1.1. *If $A(t) \to A^{\pm}$ as $t \to \pm\infty$ and $Re\sigma(A^{\pm}) \neq 0$, then* (2.1) *has an exponential dichotomy on $R^{\pm}$ with projection matrix $P_s^{\pm}(t)$ satisfying $P_s^{\pm}(t) \to P^{\pm}$ as $t \to \pm\infty$, where $P^{\pm}$ are projection matrices relative to the exponential dichotomies of $\dot{x} = A^{\pm}x$.*

Proof. See [29] □

Lemma 2.1.2. *Assume that $|A(t)| \leq M$ for all $t \in J$ and $A(t)$ has d eigenvalues with real part $Re\lambda \leq -\alpha \leq 0$, and $(n-d)$ eigenvalues with real part $Re\lambda \geq \alpha \geq 0$. Then the system $x' = A(\epsilon\tau)x$ has an exponential dichotomy for $t = \epsilon\tau \in J$ with constant $K(\epsilon)$ and exponent $\alpha - \delta(\epsilon)$, if ϵ is sufficiently small. Moreover, as $\epsilon \to 0$, $K(\epsilon)$ remain bounded and $\alpha - \delta(\epsilon) \to 0$; the projections $P_s(\tau), P_u(\tau)$ approach the spectral projections to the stable, unstable eigenspace of $A(\epsilon\tau)$.*

Proof. See [8] □

Let $C_b^k(R, R^n)$ denotes the Banach space of bounded, continuous, vector-valued functions with derivatives up to k-th order exist, bounded and continuous on R. For $x \in C(R, R^n)$, $\gamma \leq 0, j \in Z^+$, we define:

$$
\begin{aligned}
||x||_{(\gamma,j)} &= \{\sup_{t\in R} |x(t)|e^{\gamma|t|}(1+|t|^{-j})\} \\
B(\gamma,j) &= \{x : ||x||_{(\gamma,j)} \leq \infty\} \\
B^1(\gamma,j) &= \{x \in C^1(R,R^n), x \in B(\gamma,j) : ||\dot{x}||_{(\gamma,j)} \leq \infty\}
\end{aligned}
$$

Then we have $B(\gamma,j), B^1(\gamma,j)$ as Banach spaces with norms $||x||_{(\gamma,j)}, ||\dot{x}||_{(\gamma,j)} + ||x||_{(\gamma,j)}$. Note that when $\gamma = j = 0$, then $B(\gamma,j) = C_b^0(R,R^n), B^1(\gamma,j) = C_b^1(R,R^n)$.

Recall that a linear map L: $E \to F$ is **Fredholm** if and only if: (a) dim N(L)$< \infty$, (b) R(L) is closed and codim R(L)$< \infty$, where N(L) and R(L) denote the nullspace and range of L. The index of L is defined to be index L:= dim N(L)-codim R(L).

Lemma 2.1.3. *If* (2.1) *has exponential dichotomy on both half lines, with $\alpha > 0$ as the exponent. $0 < \gamma < \alpha$. Consider the linear operator L: $B^1(\gamma,j) \to B(\gamma,j)$ defined by: $(Lx)(t) = \dot{x}(t) - A(t)x(t)$, then L is Fredholm with*

$$
indexL = \dim RP_u^-(0) - \dim RP_u^+(0)
$$

Moreover, $g \in N(L)$ if and only if $g(0) \in RP_s^+(0) \cap RP_u^-(0)$. And $f \in R(L)$ if and only if $\int_{-\infty}^{\infty} \phi^\star(t) f(t) dt = 0$ for all bounded solutions $\phi(t)$ of the adjoint system (2.2).

Proof. For a special case where $B(\gamma, j) = C_b^0(R, R^n), B^1(\gamma, j) = C_b^1(R, R^n)$, the proof can be found in [29] Lemma 4.2. A general case has the similar proof. □

Lemma 2.1.4. *Consider*

$$\dot{x} - A^L(t)x = f^L(t) t \leq 0 \tag{2.3}$$

$$\dot{x} - A^R(t)x = f^R(t) t \geq 0 \tag{2.4}$$

Assume that (2.3) *has exponential dichotomy on R_-, R_+ respectively with the same constants α, K. Let $P_s^{L,R}(t), P_u^{L,R}(t)$ be the projections that define the dichotomies. $rank P_u^L(0) = rank P_u^R(0)$ and $RP_u^L(0) \oplus RP_s^R(0) = R^n$. Then for any $\eta \in R^n$ and $f^{L,R} \in B(\gamma, j)$, there exists a unique solution $x \in B(\gamma, j)$ to* (2.3) *such that*

$$x^R(0) - x^L(0) = \eta$$

$$|x|_{\gamma,j} \leq C(|\eta| + |f^L|_{\gamma,j} + |f^R|_{\gamma,j})$$

Proof. See [29]. □

2.2 Fredholm Property for linearized system

Assume that $T(t,s)$ has exponential dichotomies on $(-\infty, 0]$ and $[0, \infty)$ and
$\dim \mathcal{R}P_u(0-) = \dim \mathcal{R}P_u(0+) = k^+$.
$\mathcal{R}P_u(0-) \cap \mathcal{R}P_s(0+) = \text{span}\{\phi(0)\}^\perp$
where $\dot{q}(t)$ is the only bounded solution (up to constant multiple) to $\dot{x} = A(t)x$. Then the adjoint equation also has a unique bounded solution $\psi(t)$ where
$\psi(0) \in \mathcal{R}P_u^*(0+) \cap \mathcal{R}P_s^*(0-) = (\mathcal{R}P_u(0-) + \mathcal{R}P_s(0+))^\perp$.

Lemma 2.2.1. *For a given $f \in C[a,b]$ and $(\phi_s, \phi_u) \in (\mathcal{R}P_s(a), \mathcal{R}P_u(b))$ consider the nonhomogeneous boundary value problem:*

$$\dot{x} - A(t)x = f(t), \quad a \leq t \leq b, \ a < 0 < b \tag{2.5}$$

$P_s(a)x(a) = \phi_s, \quad P_u(b)x(b) = \phi_u$

The system has a unique C^1 solution $x(t)$ with $x(0) \perp \phi(0)$ if and only if

$$\int_a^b \langle \psi(t), f(t) \rangle dt + \langle \psi(a), \phi_s \rangle - \langle \psi(b), \phi_u \rangle = 0. \tag{2.6}$$

If (2.6) does not hold, then let the left hand side be G. There exists a unique piecewise C^1 solution $x \in C^1[a,0] \cap C^1[0,b]$ for (2.1.3) with $x(0\pm) \perp \phi(0)$ such that
$x(0-) - x(0+) = G\ \psi(0)$.
$|G| \leq C(\|f\| + e^{-\alpha|a|}|\phi_s| + e^{-\alpha|b|}|\phi_u|)$.

Proof. See [24]. □

2.3 Liapunov-Schmidt Reduction

Liapunov-Schmidt Reduction can be applied to study solutions to nonlinear equations when the implicit function theorem does not work. It allows to reduce infinite-dimensional equations in Banach spaces to finite-dimensional equations.

Let $f(x, \lambda) = 0$ be the given nonlinear equation, X, Λ and Y are Banach space. Λ is the parameter space. $f(x, \lambda)$ is the C^p-map from a neighborhood of some point $(x_0, \lambda_0) \in X \times \Lambda$ to Y and the equation is satisfied at this point $f(x_0, \lambda_0) = 0$.

For the case when the linear operator $f_x(x, \lambda)$ is invertible, the implicit function theorem assures that there exists a solution $x(\lambda)$ satisfying the equation $f(x(\lambda), \lambda) = 0$ at least locally close to λ_0.

On the other hand, when the linear operator$f_x(x, \lambda)$ is non-invertible, the Lyapunov-Schmidt reduction can be applied in the following way.

One assumes that the operator $f_x(x, \lambda)$ is a Fredholm operator. $\ker f_x(x_0, \lambda_0) = X_1$ and X_1 has finite dimension. The range of this operator Range$f_x(x_0, \lambda_0) = Y_1$ has finite co-dimension and is a closed subspace in Y.

Let us split Y into the direct sum $Y = Y_1 \oplus Y_2$, where $\dim Y_2 < \infty$. Let Q be the Projection onto Y_1. Let us consider also the direct sum $X = X_1 \oplus X_2$.

Applying the operators Q and $I - Q$ to the original equation, one obtains the equivalent system:

$$Qf(x,\lambda)=0$$
$$(I-Q)f(x,\lambda)=0$$

Let $x_1 \in X_1$ and $x_2 \in X_2$, then the first equation: $Qf(x_1+x_2,\lambda)=0$ can be solved with respect to x_2 by applying the implicit function theorem to the operator $Qf(x_1+x_2,\lambda) : X_2\times(X_1\times\Lambda) \to Y_1$ (now the conditions of the implicit function theorem are fulfilled).

Thus, there exists a unique solution $x_2(x_1,\lambda)$ satisfying $Qf(x_1+x_2(x_1,\lambda),\lambda)=0$. Now substituting $x_2(x_1,\lambda)$ into the second equation, one obtains the final finite-dimensional equation $(I-Q)f(x_1+x_2(x_1,\lambda),\lambda)=0$. Indeed, the last equation is now finite-dimensional, since the range of $(I-Q)$ is finite-dimensional. This equation is now to be solved with respect to x_1, which is finite-dimensional, and parameters λ.

Chapter 3

Formal traveling wave solutions

In this section, we study the limiting systems (1.5), (1.6) to obtain formal solutions when $\epsilon = 0$.

3.1 Regular slow-fast-slow type of solution

(A): First, we consider the slow flows on the y-z plane by analyzing (1.5). We only consider formal solutions in between $S_\pm$ because of the symmetry of $h(u)$. We can see that the first two equations of (1.5) $v = 0, u = h^{-1}(y)$ give us the three slow manifolds:

$$S_- = \{w : v = 0, y < y_m, u = h_-^{-1}(y)\}$$
$$S_0 = \{w : v = 0, -y_m < y < y_m, u = h_0^{-1}(y)\}$$
$$S_+ = \{w : v = 0, y > -y_m, u = h_+^{-1}(y)\}$$

See Figure 1.2, which consists of the equilibrium points of the fast system (1.4). On these manifolds, we have the equations for the y-z variables:

$$\begin{aligned} -s\dot{y} &= h^{-1}(y) - y + z \\ -s\dot{z} &= -\beta y \end{aligned}$$

Let $h^{-1}(y) - y = ky + c, k(y) = 1/h'(u) - 1 = 1/h'(h^{-1}(y)) - 1$, where c is a constant, so that we have:

$$\begin{aligned} \dot{y} &= -(ky + c + z)/s \\ \dot{z} &= \beta y/s \end{aligned}$$

with equilibria $P_{Y\pm} = (0, \mp c)$. The characteristic polynomial for y is:

$$s^2 r^2 + ksr + \beta = 0.$$

Notice that the product of two roots of the characteristic polynomial is $\beta/s^2 > 0$, $h'(u) > 0$ on $S_\pm$. Therefore, $k > 0$ if $0 < h'(u) < 1$ and $-1 < k < 0$ if $h'(u) > 1$. Thus we have:
Case 1: $P_\pm$ are stable spirals if $4\beta > k^2, sk > 0$.
Case 2: $P_\pm$ are unstable spirals if $4\beta > k^2, sk < 0$.
Case 3: $P_\pm$ are stable nodes if $4\beta < k^2, sk > 0$.
Case 4: $P_\pm$ are unstable nodes if $4\beta < k^2, sk < 0$.

(B). We construct a fast heteroclinic solution in between $P_{U\pm} = (\pm c, 0)$ when $\epsilon = 0$.
Consider the fast flow on the u-v plane in (1.6), where we have constants y, z and a second order differential equation of u:

$$Du'' + su' - h(u) + y = 0$$

with the characteristic equation $Dr^2 + sr - h'(u) = 0$, whose determinant $\Delta = s^2 + 4Dh'(u)$ is positive on $S_\pm$. So we have two eigenvalues with opposite signs, i.e. we have fast heteroclinic solution from saddle to saddle.

Note that $h'(u)$ only has influence on the behaviors of equilibria on slow flows(spirals/nodes) because the coefficient k depends on y. But $h'(u)$ has no influence on the behaviors of equilibria for the fast heteroclinic solution since we always have $\Delta > 0$ on $S_\pm$.

According to what we have above, we obtain the Table (3.1) that describes the behavior of the flows when β is large enough.

Based on the analysis of the limiting slow and fast system, we look for a formal solution when $\epsilon = 0$ in the form of "Slow $\xrightarrow{\text{Fast}}$ Slow" type. Here $\longrightarrow$ means the heteroclinic solution on fast flow. Considering there are cases where a slow flow can't go from a stable spiral to an unstable spiral, we have the following possibilities:

Table 3.1: Summary of behavior of equilibria in β large case

		S_-	S_0	S_+
Fast Flow	$s > 0$	saddle	stable spiral	saddle
	$s < 0$	saddle	unstable spiral	saddle
Slow Flow	$s > 0$	stable spiral	unstable spiral	stable spiral
	$s < 0$	unstable spiral	stable spiral	unstable spiral

β **large case**($4\beta > k^2$)

1) $s > 0$

case A: P_-(stable spiral) $\xrightarrow{\text{saddle to saddle}}$ P_+ (stable spiral).

There exists a unique traveling wave solution if $4\beta > k^2, k > 0$. See Figure 3.1.

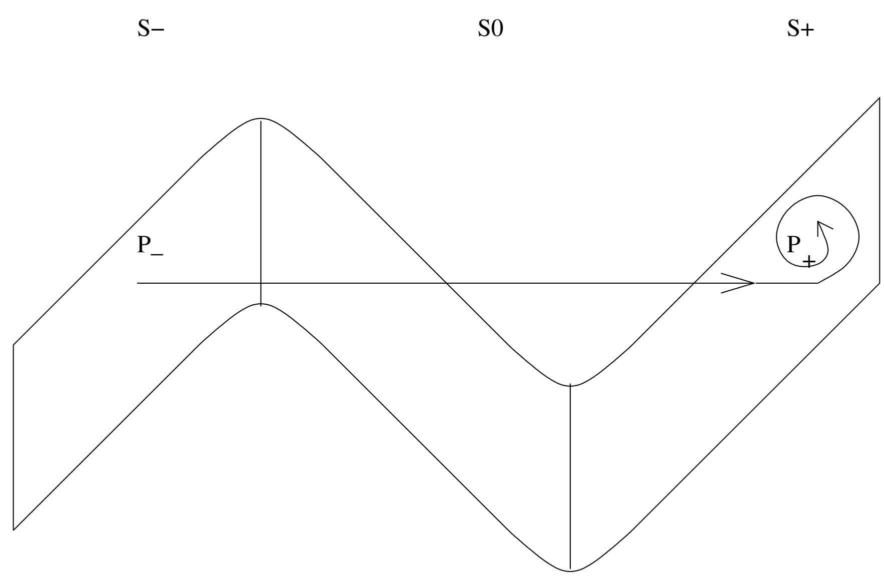


Figure 3.1: Traveling wave solution with positive wave speed and stable spirals on $S_\pm$

case B: P_+ (stable spiral) $\xrightarrow{\text{saddle to saddle}}$ P_- (stable spiral)

There exists a unique traveling wave solution if $4\beta > k^2, k > 0$. See Figure 3.2.

2) $s < 0$

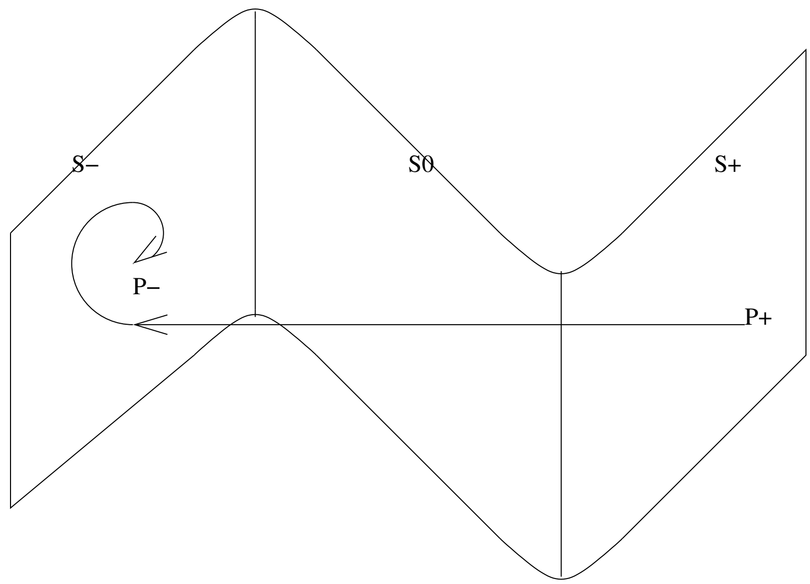


Figure 3.2: Traveling wave solution that starts from and hits stable spirals on $S_\pm$ with positive wave speed

Table 3.2: Summary of behavior of equilibria in β small case

		S_-	S_0	S_+
Fast Flow	$s > 0$	saddle	stable spiral	saddle
	$s < 0$	saddle	unstable spiral	saddle
Slow Flow	$s > 0$	stable node	unstable node	stable node
	$s < 0$	unstable node	stable node	unstable node

case A: P_- (unstable spiral) $\overset{\text{saddle to saddle}}{\longrightarrow}$ P_+ (unstable spiral)
There exists a unique traveling wave solution if $4\beta > k^2, k > 0$. See Figure 3.3.

case B: P_+ (unstable spiral) $\overset{\text{saddle to saddle}}{\longrightarrow}$ P_- (unstable spiral)
There exists a unique traveling wave solution if $4\beta > k^2, k > 0$. See Figure 3.4.

β **small case**$(4\beta > k^2)$

Based on the above Table (3.2), we have the following posiibilities:
1) $s > 0$
case A: P_-(stable node) $\overset{\text{saddle to saddle}}{\longrightarrow}$ P_+ (stable node).

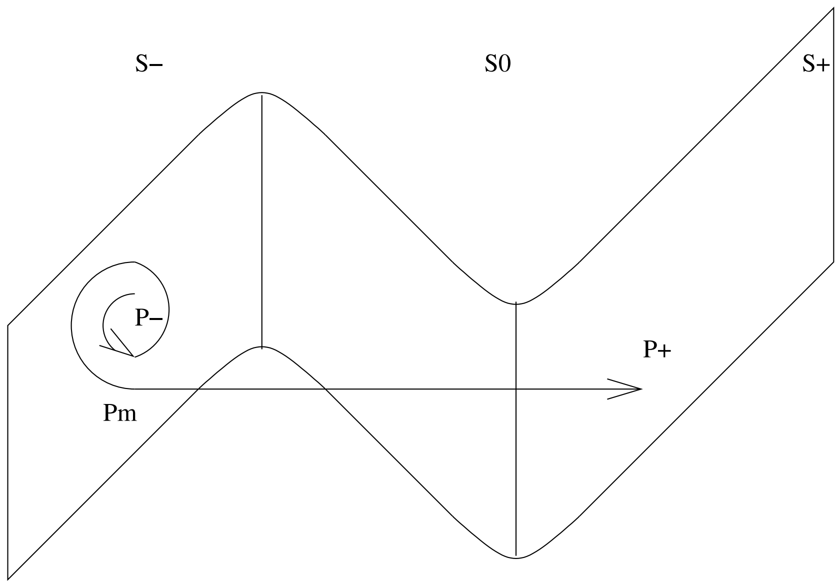


Figure 3.3: Traveling wave solution that starts from and hits unstable spirals on $S_{\pm}$ with negative wave speed

There exists a unique traveling wave solution if $4\beta < k^2, k > 0$. See Figure 3.5.

case B: P_+ (stable node) $\overset{\text{saddle to saddle}}{\longrightarrow}$ P_- (stable node)
There exists a unique traveling wave solution if $4\beta < k^2, k > 0$. See Figure 3.6.

2) $s < 0$

case A: P_- (unstable node) $\overset{\text{saddle to saddle}}{\longrightarrow}$ P_+ (unstable node)
There exists a unique traveling wave solution if $4\beta < k^2, k > 0$. See Figure 3.7.

case B: P_+ (unstable node) $\overset{\text{saddle to saddle}}{\longrightarrow}$ P_- (unstable node)
There exists a unique traveling wave solution if $4\beta < k^2, k > 0$. See Figure 3.8.

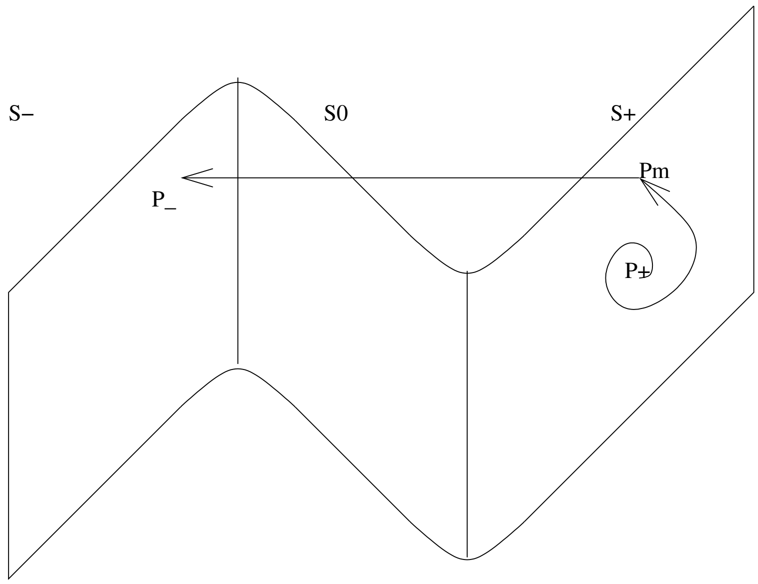


Figure 3.4: Traveling wave solution with negative wave speed and unstable spirals on $S_{\pm}$

3.2 Periodic solution in a degenerated PPC model as the generalization of the FitzHugh-Nagumo equation

Consider the slow flow (1.3) with the change of variable $z = \beta = 0, U = -Du, V = -v, Y = y$, so that we have

$$
\begin{aligned}
U' &= V \\
V' &= -sV/D - h(-\frac{U}{D}) + Y \\
Y' &= -\epsilon(U - Y)/s
\end{aligned}
$$

Let $\theta = -s/D$, then we have

$$
\begin{aligned}
U' &= V \\
V' &= \theta V - H(U) + Y \\
\theta Y' &= \epsilon_1(U - Y)
\end{aligned}
$$

which is in the form of the FitzHugh-Nagumo equation with wave speed θ, $\gamma = 1, \epsilon_1 = \epsilon/D$. Now we can treat the FitzHugh-Nagumo equation as the degenerated case of the PPC model,

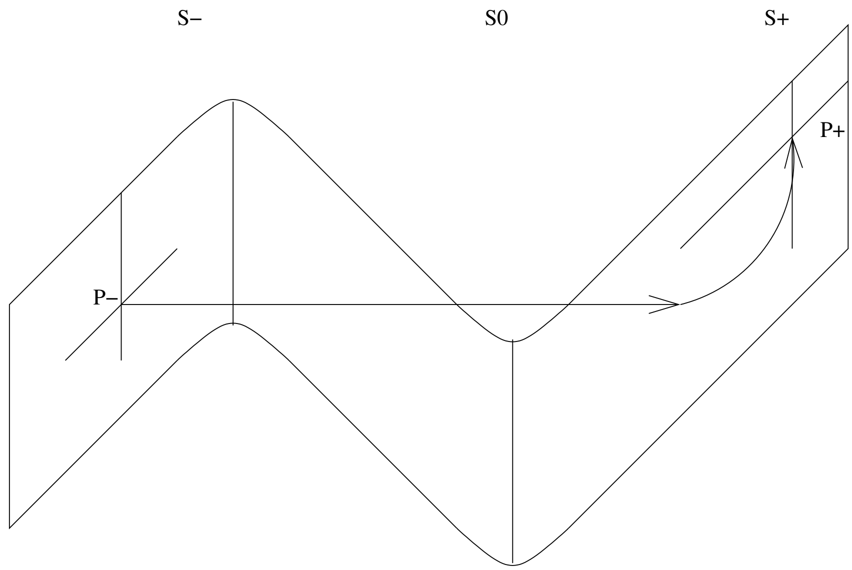


Figure 3.5: Traveling wave solution with positive wave speed and stable nodes on $S_\pm$

so that we have a periodic solution when $z = \beta = 0$. See [25].

Notice that if y decreases/increases on both manifolds $S_\pm$ in between the two foldlines $y = y_m, y = y_M$, there will be no solution since we need opposite orientation in order to form a loop. Consider the case when $s > 0, \beta = 0$, $z = z_0$ is a constant(not necessarily zero).
(A): On S_-, we need $\dot{y} < 0$, that is $u - h_-(u) + z_0 > 0$, while on S_+, we need $\dot{y} > 0$, that is $z_0 < h_+(u) - u = g_+(u)$. On S_+, we only consider $c/\sqrt{3} < u < 2c/\sqrt{3}$ in between the two foldlines, also $g'_+(u) < 0$ when $c/\sqrt{3} < u < \sqrt{c^2 + 1/m}/\sqrt{3}$. Therefore, we have:

$$z_0 < h_+(u) - u = g_+(u) < g(2c/\sqrt{3}) = 2\sqrt{3}c^3m/9 - 2c/\sqrt{3}$$

Similarly, we have $z_0 > -g(2c/\sqrt{3})$ on S_-. There is no intersection for z_0 because $g(2c/\sqrt{3}) < 0$.

(B): On S_-, we need $\dot{y} > 0$, that is $u - h_-(u) + z_0 < 0$, while on S_+ we need $\dot{y} < 0$, that is $z_0 > h_+(u) - u = g_+(u)$. On S_+, we only consider $c/\sqrt{3} < u < 2c/\sqrt{3}$ in between the two foldlines, also $g'_+(u) < 0$ when $c/\sqrt{3} < u < \sqrt{c^2 + 1/m}/\sqrt{3}$. Therefore, we have:

$$z_0 > h_+(u) - u = g_+(u) > g(c/\sqrt{3}) = -2\sqrt{3}c^3m/9 - c/\sqrt{3}$$

Similarly, we have $z_0 < -g(c/\sqrt{3})$ on S_-, the intersection gives us the region for z_0 in which

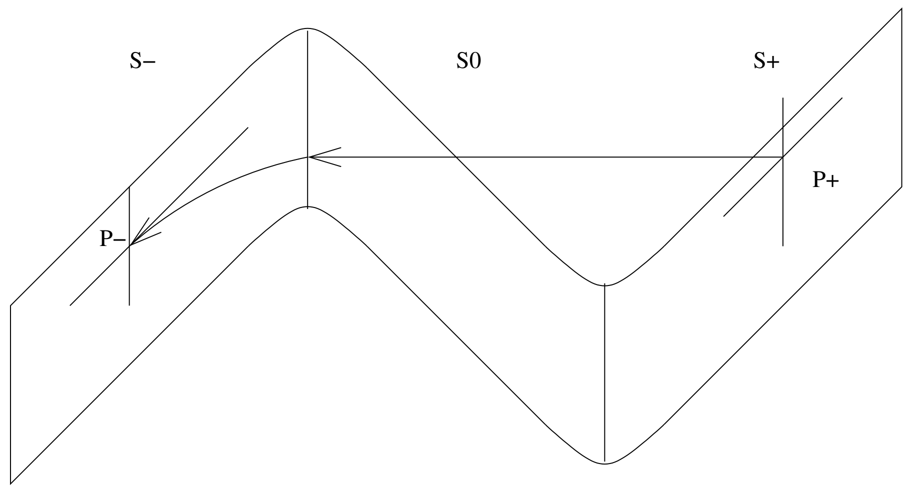


Figure 3.6: Traveling wave solution with positive wave speed and stable nodes on $S_\pm$

there are periodic solutions:

$$g(c/\sqrt{3}) < z_0 < -g(c/\sqrt{3})$$

See Figure 3.9.

3.3 A pair of formal heteroclinic solutions

According to the regular cases in section 3.1, we have by symmetry that if $s < 0$ there is a pair of formal heteroclinic solutions, which go back and forth from S_- to S_+ with both equilibria, those equilibria are unstable spirals on slow manifolds. See Figure 3.10. For $s > 0$, both equilibria are stable spirals. We have a pair of formal heteroclinic solutions, we omit the graph here.

3.4 Formal Periodic Solution

β large case

The two equilibria on the slow manifolds are stable/unstable spirals. The slow flows take only part of the spiral on $S_\pm$ and are connected by the fast heteroclinic flows in between. So we have a periodic solution that consists of two pieces of fast flows and two pieces of slow flows. See Figure 3.11.

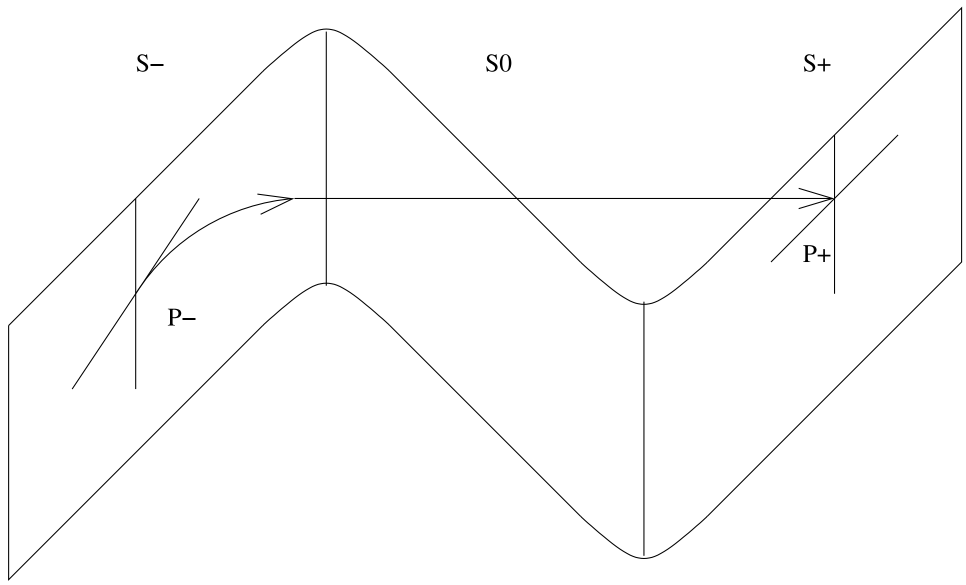


Figure 3.7: Traveling wave solution with negative wave speed and unstable nodes on $S_\pm$

β **small case**

We consider the case where two equilibria are stable nodes on $S_\pm$ for $sk > 0$. From Figure 3.12, we can see that it is possible that two solutions from different stable nodes intersect and form a closed loop (not unique) in their overlapped region within the largest loop R, as shown in Figure 3.13. If $sk < 0$, we have a similar case when we have unstable nodes on $S_\pm$. The largest loop is the one that contains both equilibria points, we can figure out the detailed solution curves with $P_\pm$ as initial points.

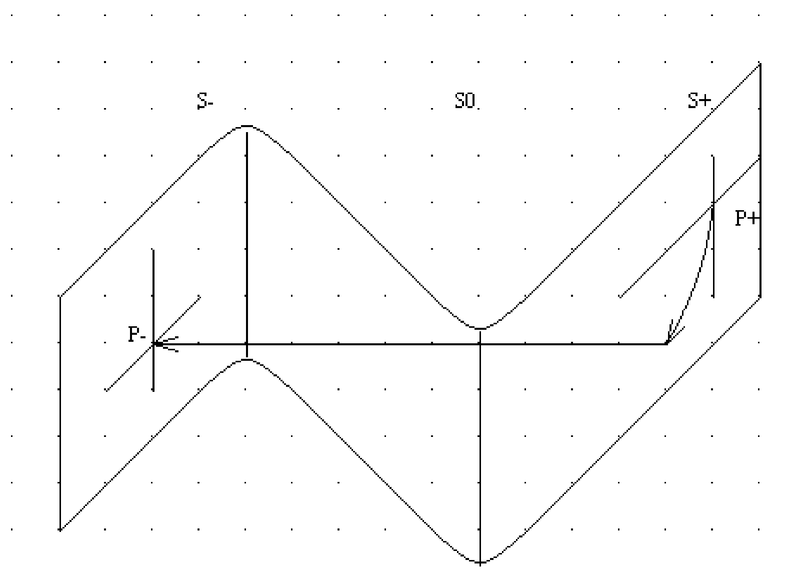


Figure 3.8: Traveling wave solution with negative wave speed and unstable nodes on $S_\pm$

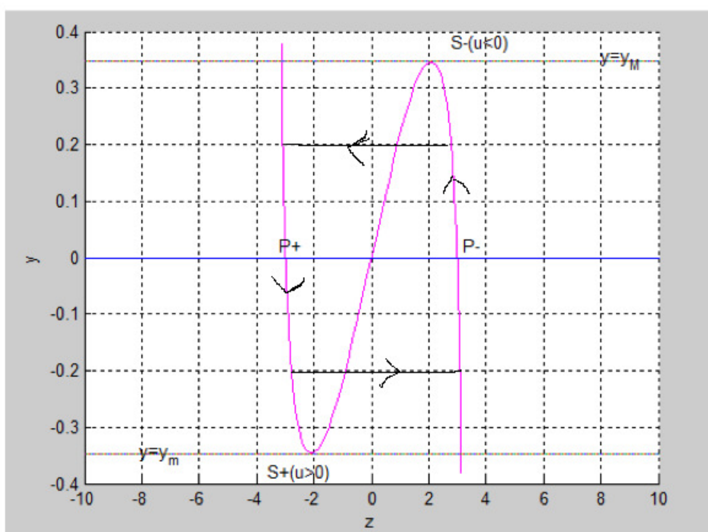


Figure 3.9: Region bounded by $y = y_m$ and $y = y_M$ in which there exists periodic solution for $\beta = 0$

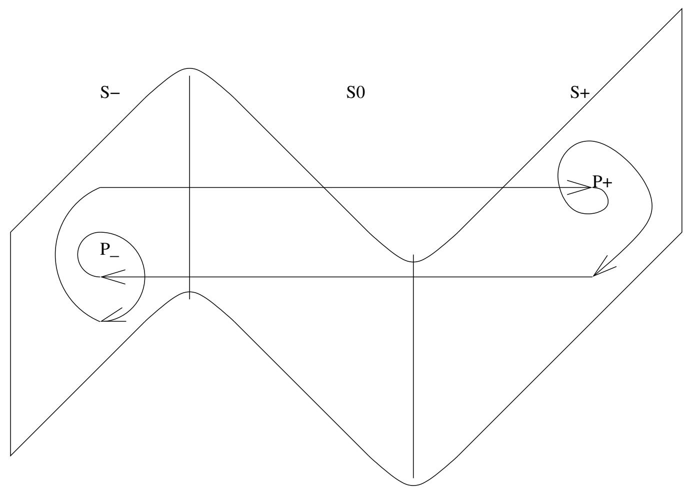


Figure 3.10: A formal pair of heteroclinic solutions

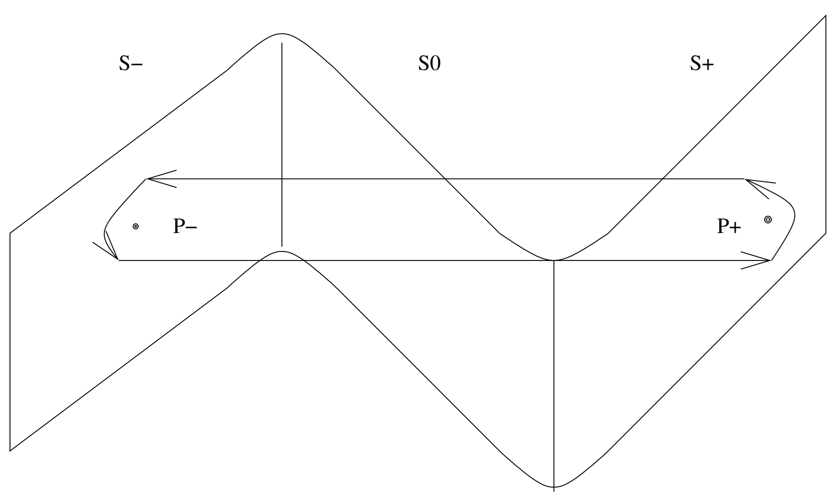


Figure 3.11: Formal periodic solution when $\epsilon = 0$

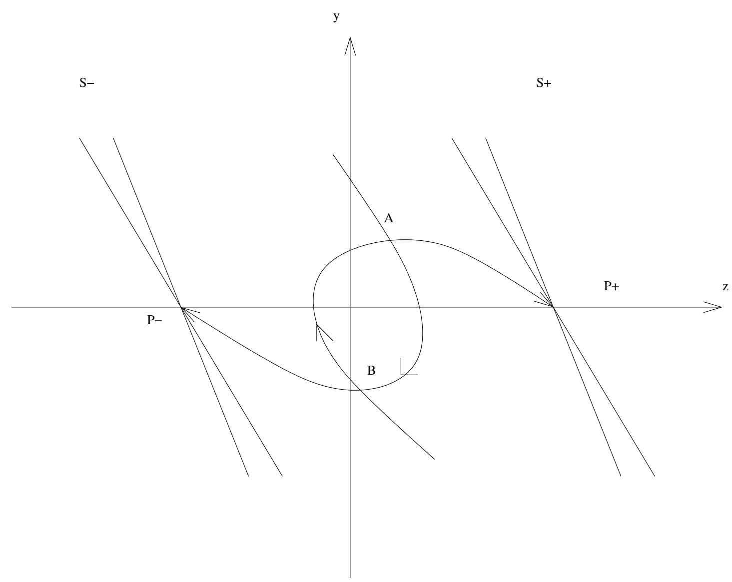


Figure 3.12: Periodic solution projected on y-z plane as two pieces of part of the stable nodes

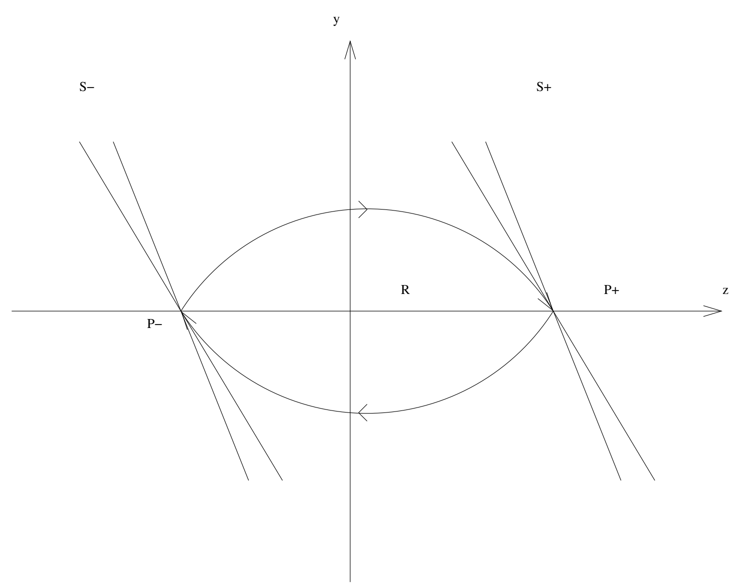


Figure 3.13: Largest region for periodic solution projected on y-z plane as part of the stable nodes

Chapter 4

Construction of the approximated solution for periodic solution

In this chapter we construct the approximated solution based on the solution to the 0th order asymptotic expansions when $\epsilon = 0$.

4.1 Formal systems of the asymptotic expansions in the outer and inner layers

Consider the asymptotic expansions of $w = (u, v, y, z)$ in the inner and outer layers with different time scales:

$$w^{in}(\tau, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j w_j^{in}(\tau)$$

$$w^{out}(t, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j w_j^{out}(t)$$

Let $I_i = \{t | t \in [\alpha_i, \beta_i]\}$, I_{2l+1} are singular layers, I_{2l} are regular layers, $l \in Z$. Plug the above expansion into the fast and slow systems and compare the coefficient of powers of ϵ, we can have a system of algebraic or differential equations as follows:

In the outer layers where $w(t) = (u, v, y, z)$, we have the $\epsilon^j, j \geq 0$ the order expansions as:

$$
\epsilon^0 \begin{cases} 0 & = & v_0^{out} \\ 0 & = & sv_0^{out}/D - y_0^{out} + h(u_0^{out}) \\ -s\dot{y}_0^{out} & = & u_0^{out} - y_0^{out} + z_0^{out} \\ -s\dot{z}_0^{out} & = & -\beta y_0^{out} \end{cases} \tag{4.1}
$$

$$
\epsilon^1 \begin{cases} \dot{u}_0^{out} & = & v_1^{out}/D \\ \dot{v}_0^{out} & = & sv_1^{out}/D + h'(u_0^{out})u_1^{out} - y_1^{out} \\ -s\dot{y}_1^{out} & = & u_1^{out} - y_1^{out} + z_1^{out} \\ -s\dot{z}_1^{out} & = & -\beta y_1^{out} \end{cases} \tag{4.2}
$$

$$
\epsilon^j \begin{cases} \dot{u}_{j-1}^{out} & = & v_j^{out}/D \\ \dot{v}_{j-1}^{out} & = & -sv_j^{out}/D + h'(u_0^{out})u_j^{out} - y_j^{out} + h_j \\ -s\dot{y}_j^{out} & = & u_j^{out} - y_j^{out} + z_j^{out} \\ -s\dot{z}_j^{out} & = & -\beta y_j^{out} \end{cases} \tag{4.3}
$$

In the inner layer where $w(\tau) = (u, v, y, z)$, we have the $\epsilon^j, j \geq 0$ the order expansions as:

$$
\epsilon^0 \begin{cases} u_0' & = & v_0/D \\ v_0' & = & -sv_0/D - y_0 + h(u_0) \\ y_0' & = & 0 \\ z_0' & = & 0 \end{cases} \tag{4.4}
$$

$$
\epsilon^1 \begin{cases} \begin{pmatrix} u_1' \\ v_1' \end{pmatrix} & = & \begin{pmatrix} 0 & 1/D \\ h(u_0) & -s/D \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} 0 \\ -y_1 \end{pmatrix} \\ -sy_1' & = & u_0 - y_0 + z_0 \\ -sz_1' & = & -\beta y_0 \end{cases} \tag{4.5}
$$

$$\epsilon^j \begin{cases} \begin{pmatrix} u_j' \\ v_j' \end{pmatrix} & = \begin{pmatrix} 0 & 1/D \\ h'(u_0) & h_j \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix} + \begin{pmatrix} 0 \\ h_j - y_j \end{pmatrix} \\ -sy_j' & = u_{j-1} - y_{j-1} + z_{j-1} \\ -sz_j' & = -\beta y_{j-1} \end{cases} \tag{4.6}$$

where $h_j, j > 1$ is a polynomial of lower order terms.

4.2 Asymptotic matching conditions

The asymptotic expansion needs a condition to make sure that the inner expansion and outer expansion are conformable with each other, which reduces to the matching condition as follows:

For smooth outer expansion, $w^{out}(t, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j w_j^{out}(t)$, we have the inner expansion of the outer solution after substituting $t = \epsilon\tau$:

$$\tilde{w}^{out}(t, \epsilon) = w^{out}(\epsilon\tau, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \tilde{w}_j^{out}(\tau)$$

where the $\tilde{w}_j^{out}(\tau)$ is computable from the Taylor expansion of $w^{out}(t)$.

Similarly, we do the expansion in ϵ for smooth function $w^{in}(\tau, \epsilon)$ in the inner layer:

$$w^{in}(\tau, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j w_j^{in}(\tau).$$

Next, we have our matching conditions:

$$\lim_{\tau \to \pm\infty} w_j^{in}(\tau) - \tilde{w}_j^{out}(\tau) = 0. \tag{4.7}$$

Here 'out' stands for 'R' if $\tau \to \infty$, if the outer solution is to the right of the inner solution, i.e. $R = in + 1$; 'out' stands for 'L' if $\tau \to -\infty$, if the outer solution is to the left of the inner solution, i.e. $L = in - 1$.

4.2.1 Formal expansion of ϵ^0th order

When $\epsilon = 0$, we solve for the periodic solution $w = (U, Y) = ((u, v), (y, z))$ for (1.5), (1.6).
In singular/inner layer, $w^0(\tau) = (U^0(\tau), Y^0(\epsilon\tau))$, where $\tau \in [-\infty, \infty], i = 2l + 1$
In regular/outer layer, $w^0(t) = (U^0(t/\epsilon), Y^0(t))$, where $t \in [\alpha_i, \beta_i] = [a_l, b_l], i = 2l$.

(1) Below we provide a Lemma that guarantees the existence of fast heteroclinic solutions $U^i, i = 1, 3$. Plug the constant y,z values into the first two equations of (1.6), we have a system for the fast heteroclinic flow:

$$\begin{aligned} u_0' &= v_0/D \\ v_0' &= -sv_0/D - \bar{y}_0 + h(u_0) \end{aligned} \tag{4.8}$$

which can be rewritten as the second order ODE:

$$Du_0'' = v_0' = -su_0' - \bar{y}_0 + h(u_0) \tag{4.9}$$

with characteristic equation:

$$Dr^2 + sr - h'(u) = 0$$

Multiply the above equation by u_0' and integrate from $-\infty$ to ∞, we have:

$$\bar{y}_0(u_0(\infty) - u_0(-\infty)) = -[\int_{-\infty}^{\infty} (u_0')^2 d\tau]s + \int_{u_0(-\infty)}^{u_0(\infty)} h(u_0)du_0 \tag{4.10}$$

For the heteroclinic solution A connecting $(-c, 0)$ to $(c, 0)$, (4.10) gives us:

$$2c\bar{y}_0^A = -[\int_{-\infty}^{\infty} (u_0')^2 d\tau]s + \int_{-c}^{c} h(u_0)du_0$$

Similarly, for the heteroclinic solution B connecting $(c, 0)$ to $(-c, 0)$, we have:

$$-2c\bar{y}_0^B = -[\int_{-\infty}^{\infty} (u_0')^2 d\tau]s + \int_{c}^{-c} h(u_0)du_0$$

In order to maintain the same s for the two heteroclinic solutions A and B, we need:

$$\bar{y}_0^A + \bar{y}_0^B = [\int_{-c}^{c} h(u_0)du_0]/c \tag{4.11}$$

In particular, when $h(u)$ has symmetry in the form of $\int_{-c}^{c} h(u_0)du_0 = 0$. Then we need $\bar{y}_0^A + \bar{y}_0^B = 0$ in order to maintain the same s for the two heteroclinic solutions.

On the other hand, we recall that we only consider the region in between the two foldlines, i.e. $-y_m \leq \bar{y}_0 \leq y_m$. When $h(u)$ has symmetry, then we have

Lemma 4.2.1. *As $y_0 \to y_m$ or $y_0 \to -y_m$, the wave speed $s^*(y_0)$ monotonically approaches $-s_m$ or s_m. Moreover, s_m is the minimum wave speed for the existence of a connection from the turning point of S to the saddle point on S^+, with the same parameter $y_0 = -y_m$.*

Proof. The result was proved in [3]. □

Lemma 4.2.2. *For each $y_0 \in (-y_m, y_m)$, there is a $s_0 = s^*(y_0)$ such that system (4.8) has a unique heteroclinic solution A $(\hat{u}, \hat{v})$ connecting $(u_0^-(y), 0)$ to $(u_0^+(y), 0)$ for $y = y_0^-(s)$, and $\int_{u_0^-(y_0^-(0))}^{u_0^+(y_0^-(0))}[h(u) - y_0^-(0)] = 0$. See figure 4.1. Also system (4.8) has a unique heteroclinic solution B connecting $(u_0^+(y), 0)$ to $(u_0^-(y), 0)$ for $y = y_0^+(s)$.*

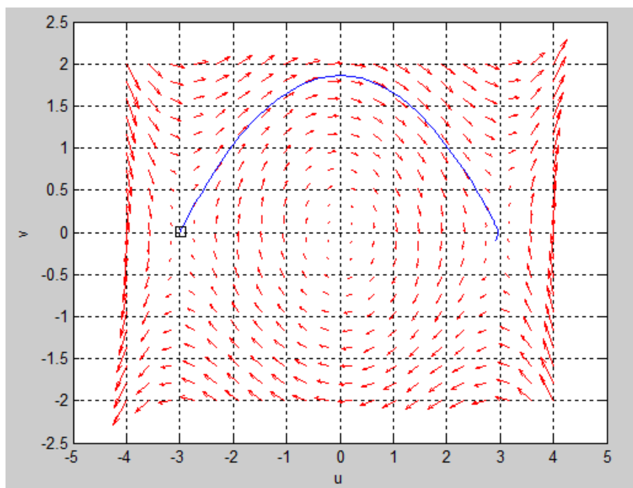


Figure 4.1: Heteroclinic solution A connecting P_- to P_+

Proof. When $s = 0$, integrate the Hamiltonian system (4.9) so that $\int_{u_0(-\infty)}^{u_0(\infty)}[h(u_0) - \bar{y}_0] = 0$ and we can solve for a unique heteroclinic solution $(\hat{u}, \hat{v})$ connecting $(u_0(-\infty), 0)$ and $(u_0(\infty), 0)$. Similarly, there is a unique heteroclinic solution B connecting $(u_0(\infty), 0)$ and $(u_0(-\infty), 0)$. We can verify that $(\hat{u}', \hat{v}')$ is a solution of the linear variational system of the u-v equation of (1.6):

$$\begin{aligned} \Phi_1' &= \Phi_2/D \\ \Phi_2' &= h'(\hat{u})\Phi_1 - s\Phi_2/D \end{aligned} \tag{4.12}$$

We can show that (s, y) has to satisfy a bifurcation function $g(s, y) = 0$, whose solution near $(\hat{s}, \hat{y})$ corresponds to a unique heteroclinic solution near $(\hat{u}, \hat{v})$. Moreover, we can compute that:

$$\frac{\partial g(s,y)}{\partial s} = \int_{-\infty}^{\infty} [\hat{v}(\tau)]\Psi_2(\tau)d\tau \tag{4.13}$$

where (Ψ_1, Ψ_2) is the unique bounded solution to the adjoint system of (4.12). Also we can find out:

$$\frac{\partial g(s,y)}{\partial s} = \int_{-\infty}^{\infty} [\hat{v}(\tau)]^2 e^{-s\tau} d\tau > 0 \tag{4.14}$$

Therefore, $g(s, y) = 0$ has a solution $y = y_0(s)$ locally, with $\frac{\partial y_0}{\partial s} < 0$ on heteroclinic solution A, and $\frac{\partial y_0}{\partial s} < 0$ on heteroclinic solution B when $s > 0$, we can complete the proof of the Lemma by the continuous dependence of homotopy continuation, starting from the case when $s = 0$. $\square$

Remark: we can manipulate the generic nonlinear cubic function $y = h(u)$ with three distinct real roots so that it is in the form of $y = au^3 + bu$ by shifting u to left/right. Also we can shift the constant y_0 so that the cubic function becomes symmetric(odd). For simplicity, we set $h(u) = mu(u + c)(u - c)$ which is symmetric with respect to the origin.

Remark: the symmetry of $h(u)$ in (4.10) gives us the fact that in order for the wave speed s for the two heteroclinic solutions A and B to be the same, we need $y_0^A = -y_0^B$.

(2) We solve for the two slow flows on $S_\pm$, which become a closed loop if projected on the y-z plane. In the outer layers, the first two equations of (1.5) give us two algebraic equations $v = 0, y = h(u)$. We solve for them and plug into the other two equations and obtain:

$$\begin{aligned} -s\dot{y}_0^i &= h^{-1}(y_0^i) - y_0^i + z_0^i \\ -s\dot{z}_0^i &= -\beta y_0^i \end{aligned}$$

which can be rewritten as:

$$\dot{Y}_0^i = A_0 Y_0^i + f_0^i \tag{4.15}$$

where $i = 2, 4$.

For local solution in a small region near the z axis where $|y| < \delta, y = h(u)$ is approximately linear as $y = h'(u_\pm)(u - u_\pm)$. Then we can solve for u in terms of y linearly, so that (4.15) becomes:

$$\begin{aligned} \dot{y} &= -(ky + u_\pm + z)/s \\ \dot{z} &= \beta y/s \end{aligned}$$

where $k = 1/h'(u_\pm) - 1$ is a constant. Therefore, according to the eigenvalue analysis as well as the symmetry, we have a closed loop formed by two pieces of the stable(unstable) spirals near z-axis. For the global solution in a larger region, in order to obtain a closed loop of an increasing and a decreasing curve on $S_\pm$ respectively, we want to consider only the region in between the two foldlines $y = y_m$ and $y = y_M$ in the y-z plane, so that there exist fast heteroclinic solutions to connect the slow flows in between $S_\pm$.

We plot the vector field of (4.15) in Figure 4.2 to analyze the slow flow on $S_\pm$. In this figure,

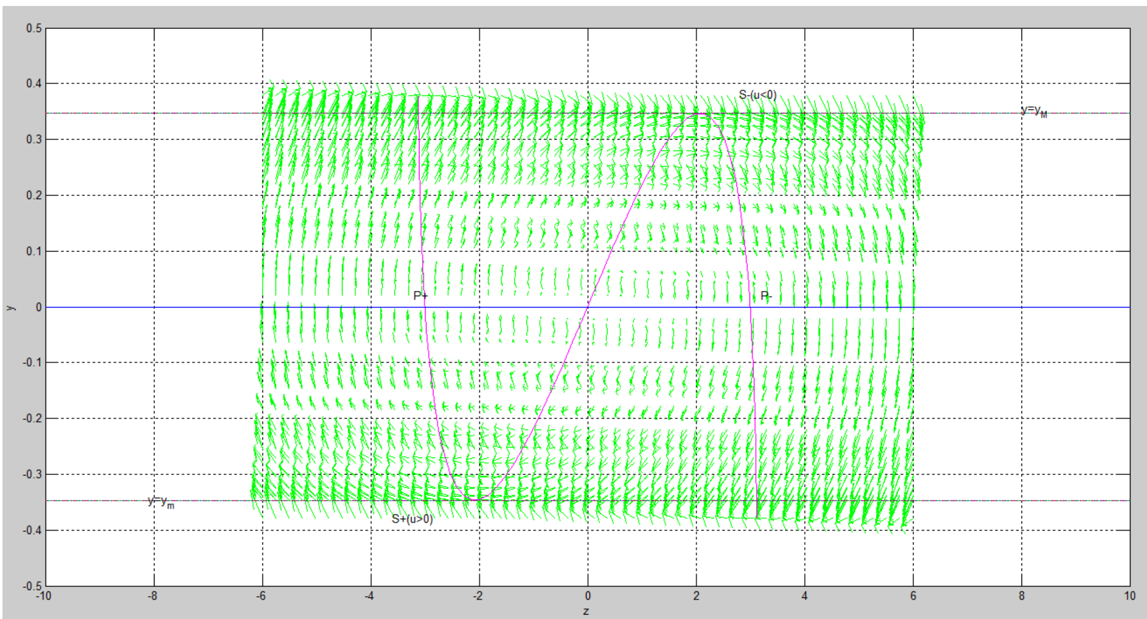


Figure 4.2: Vector field of the slow system on $S_\pm$

$\dot{y} = 0$ on the pink cubic null curve $z = y - u = h(u) - u$, which contains all the turning points of each spiraling orbit on $S_\pm$. Also notice that $\dot{z} = 0$ on the blue z-axis. The two null curves divide the entire plane into several regions in which the vector field is displayed by the green arrows.

Next we want to locate the one piece of slow flows on the z-y plane given the other piece. See Figure 4.3. We define the ascending curves $Y_a(t) = (z_a(t), y_a(t)), -T_1^a < t < T_2^a, \quad T_1^a, T_2^a > 0$ Recall the condition $\bar{y}_0^A + \bar{y}_0^B = 0$ in order to maintain the same s for the two heteroclinic solutions. We define the descending curves $Y_d(t)$ that are symmetric to $Y_a(t)$ with respect to the origin.

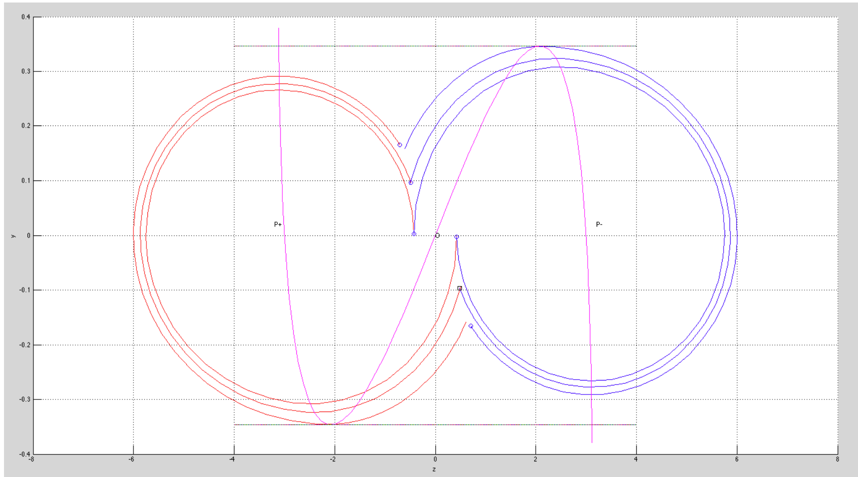


Figure 4.3: For the nonlinearity $h(u)$ and $\beta = 100$, several closed loops that consist of Y_a (red) on S^+ and Y_d(blue) on S^- are depicted. The largest loop touches the line $y = \pm y_m$. The smallest loop has junction points on $y = 0$.

Theorem 4.2.3. *For different y_0 values, we define the descending curve $Y_d(t)$ that is symmetric to $Y_a(t)$ with respect to the origin, they give us a closed loop on the z-y plane, where the largest loop touches the line $y = \pm y_m$. The smallest loop has junction points on $y = 0$.*

Now that we have described how to obtain the existence of periodic solution when $\epsilon = 0$, we want to obtain the equations the periodic solution has to satisfy when $\epsilon = 0$. Assume that (1.5) has exponential dichotomy on I^2, I^4. Define the operator P_i that maps the starting point to the ending point in each interval according to the Y solution of (1.5), i.e.

$$Y_0^i(\beta_i) = P_i(Y_0^i(\alpha_i)), i = 1, 2, 3, 4$$

The last two equations of (1.6) implies that $Y_0^i(\tau), i = 1, 3$ must be constants:

$$\begin{aligned} Y_0^i(\tau) &= Y_0^i(a_{i+1}) = Y_0^i(a_i), i = 1, 3 \\ Y_0(a_{i+1}) &= P_i(Y_0(a_i)) = \Phi_0^i(a_{i+1}, a_i)Y_0(a_i) + \int_{a_i}^{a_{i+1}} \Phi_0^i(a_{i+1}, s)f^i(s)ds, i = 2, 4 \end{aligned} \tag{4.16}$$

Also define the Poincare mapping $P = P_4P_3P_2P_1$, so that

$$\begin{aligned} P(Y_0(a_1)) = {} & \Phi_0^4(a_1, a_4)\Phi_0^2(a_3, a_2)Y_0(a_1) + \Phi_0^4(a_1, a_4) \int_{a_2}^{a_3} \Phi_0^2(a_3, s)f^2(s)ds \\ & + \int_{a_4}^{a_1} \Phi_0^4(a_1, s)f^4(s)ds \end{aligned}$$

According to the assumption (H1): 1 is not an eigenvalue of the map $\Phi^4(a_1, a_4)\Phi^2(a_3, a_2)$, so we can solve for the initial value $Y_0(a_1)$ uniquely as a fixed point for P, i.e. $P(Y_0(a_1)) = Y_0(a_1)$. Now we have the Y^i solution, we plug them back into the two algebraic equations to solve for $U^i, i = 2, 4$ solutions. Now we have the solution for (4.1), (4.4) as the 0th order expansion. See Figure 3.11.

Remark: when $h(u)$ is not symmetric, we need (4.11) for the restriction for the closed periodic loop on the z-y plane, which will be investigated in the future.

Remark: the 0th order expansion is actually enough for the existence of an exact solution, but we can go further to higher order expansions for a more accurate approximation solution, which is not discussed here.

4.3 Approximations, jump and residual errors

We truncate the formal series:

$$w^{out}(t,\epsilon) = \sum_0^\infty \epsilon^j w_j^{out}(t,\epsilon), \quad w^{in}(\tau,\epsilon) = \sum_0^\infty \epsilon^j w_j^{in}(\tau)$$

to form an approximation solution when $\epsilon > 0$.

We divide the entire domain as the following to investigate the periodic solution w_j when ϵ gets sufficiently small. Let $I_i = \{t | t \in [\alpha_i, \beta_i]\}$, I_{2l+1} are singular layers, I_{2l} are regular layers. For any $m \geq 0, 0 < \lambda < 1$(in this paper, we have $m = 0$). We define the approximation when $\epsilon > 0$ of order ϵ^m to be:

$$w_{ap} = \begin{cases} \sum_{j=0}^m \epsilon^j w_j^{out}(t) & t \in [\alpha_i, \beta_i] = [a_l + \epsilon^\lambda, b_l - \epsilon^\lambda], i = 2l \\ \sum_{j=0}^m \epsilon^j w_j^{in}(\tau) & \tau \in [\alpha_i, \beta_i] = [-\epsilon^{\lambda-1}, \epsilon^{\lambda-1}], i = 2l+1 \end{cases}$$

4.3.1 Jump errors

Estimates for the jump error $J_{wi} = w_{ap}(\alpha_{i+1}, \epsilon) - w_{ap}(\beta_i, \epsilon) = O(\epsilon^\lambda)$ can be obtained by comparing outer and inner approximations with the inner expansion of outer layers.

4.3.2 Residual errors

In the outer layers, the residual errors $RE^i(t), i = 2, 4$ are defined by:

$$RE^i(t) = \begin{pmatrix} \epsilon\dot{u}_{ap}(t) - v_{ap}(t)/D \\ \epsilon\dot{v}_{ap}(t) - sv_{ap}(t)/D + y_{ap}(t) - h(u_{ap}(t)) \\ -s\dot{y}_{ap}(t) - u_{ap}(t) + y_{ap}(t) - z_{ap}(t) \\ -s\dot{z}(t) + \beta y_{ap}(t) \end{pmatrix}$$

In the inner layers, the residual errors $RE^i(t), i = 1, 3$ are defined by:

$$RE^i(\tau) = \begin{pmatrix} u'_{ap}(\tau) - v_{ap}(\tau)/D \\ v'_{ap}(\tau) - sv_{ap}(t)/D + y_{ap}(t) - h(u_{ap}(t)) \\ sy'_{ap}(\tau) - \epsilon[-u_{ap}(\tau) + y_{ap}(\tau) - z_{ap}(\tau)] \\ sz'_{ap}(\tau) + \epsilon\beta y_{ap}(\tau) \end{pmatrix}$$

Lemma 4.3.1. *The residual errors, uniformly with respect to t or τ, satisfy the following*

$$RE^{2,4}(t) = \begin{pmatrix} O(\epsilon) \\ O(\epsilon) \\ 0 \\ 0 \end{pmatrix}, RE^{1,3}(\tau) = \begin{pmatrix} O(\epsilon^\lambda) \\ O(\epsilon^\lambda) \\ O(\epsilon) \\ O(\epsilon) \end{pmatrix}$$

Details of the calculation is omitted here.

Chapter 5

Existence of Periodic Solution

In this chapter we prove the existence of periodic solution near the approximated solution that consists of two pieces of slow flows as part of the spirals connected by two pieces of fast flows.

5.1 Periodic solution for linearized variational system

Now we construct the linear variational system for the correction function $w = (U, Y)$, where we take different time scales in different layers, as defined in the previous chapter.

1. When $\epsilon = 0$, we have the pair of formal heteroclinic solutions. See Figure 3.11
In singular layer, $w^0(\tau) = (U^0(\tau), Y^0(\epsilon\tau))$, where $\tau \in [-\infty, \infty], i = 2l + 1$
In regular layer, $w^0(t) = (U^0(t/\epsilon), Y^0(t))$, where $t \in [\alpha_i, \beta_i] = [a_l, b_l], i = 2l$.

2. When $\epsilon > 0$
(i) For w^{ap}, $N = \epsilon^{\lambda-1}$.
In singular layer, $w^{ap}(\tau) = (U^{ap}(\tau), Y^{ap}(\epsilon\tau))$, where $\tau \in [\alpha_i, \beta_i] = [-\epsilon^{\lambda-1}, \epsilon^{\lambda-1}]$,
$i = 2l + 1, 0 < \lambda < 1$
In regular layer, $w^{ap}(t) = (U^{ap}(t/\epsilon), Y^{ap}(t))$, where $t \in [\alpha_i, \beta_i] = [a_l + \epsilon^{\lambda}, b_l - \epsilon^{\lambda}]$,
$i = 2l, 0 < \lambda < 1$
(ii) For $w = w^{ex} - w^{ap}$:
$w(\tau) = (U(\tau), Y(\epsilon\tau))$, $\tau \in [\bar{\alpha}_i, \bar{\beta}_i] = [\alpha_i, \beta_i] = [-\epsilon^{\lambda-1}, \epsilon^{\lambda-1}], i = 2l + 1$,
$w(t) = (U(t/\epsilon), Y(t))$, $t \in [\bar{\alpha}_i, \bar{\beta}_i] = [a_l + \epsilon^{\lambda} + \delta a_l, b_l - \epsilon^{\lambda} + \delta b_l], i = 2l$, where $\delta a_l, \delta b_l$ are real polynomials in ϵ.

5.1.1 Jump conditions for the linearized variational system

Define jump conditions at the junction of two adjacent intervals for the approximation solution to be:

$$\begin{aligned} J_{U1} &= U_2^{ap}(\alpha_2/\epsilon) - U_1^{ap}(\beta_1), J_{U4} = U_1^{ap}(\alpha_1) - U_4^{ap}(\beta_4/\epsilon) \\ J_{U3} &= U_4^{ap}(\alpha_4/\epsilon) - U_3^{ap}(\beta_3), J_{U2} = U_3^{ap}(\alpha_3) - U_2^{ap}(\beta_2/\epsilon) \end{aligned} \tag{5.1}$$

$$\begin{aligned} J_{Y1} &= Y_2^{ap}(\alpha_2) - Y_1^{ap}(\epsilon\beta_1), J_{Y4} = Y_1^{ap}(\epsilon\alpha_1) - Y_4^{ap}(\beta_4) \\ J_{Y3} &= Y_4^{ap}(\alpha_4) - Y_3^{ap}(\epsilon\beta_3), J_{Y2} = Y_3^{ap}(\epsilon\alpha_3) - Y_2^{ap}(\beta_2) \end{aligned} \tag{5.2}$$

See Figure 5.1.

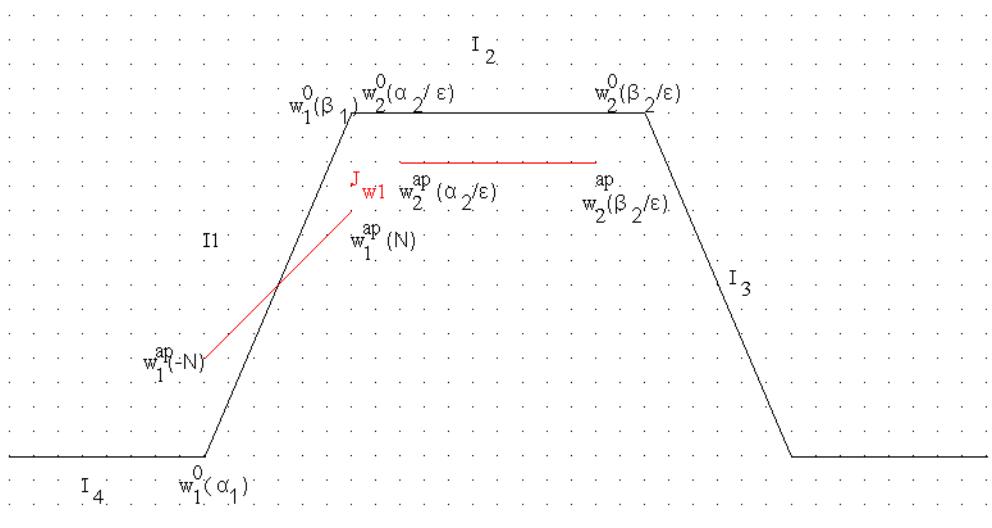


Figure 5.1: Jump error for approximation solutions

Also define the jump conditions for the approximation solution in between two slow flows as:

$$JY_1 = Y_2^{ap}(\alpha_2) - Y_4^{ap}(\beta_4), JY_2 = Y_4^{ap}(\alpha_4) - Y_2^{ap}(\beta_2) \tag{5.3}$$

where

$$
\begin{aligned}
JY_1 &= J_{Y1} + J_{Y4} + Y_1^{ap}(\epsilon\alpha_1) - Y_1^{ap}(\epsilon\beta_1) \\
JY_2 &= J_{Y2} + J_{Y3} + Y_3^{ap}(\epsilon\beta_3) - Y_3^{ap}(\epsilon\alpha_3)
\end{aligned}
\tag{5.4}
$$

Now we look for the jump conditions at the junction of two adjacent intervals for the correction solution U, Y respectively:

On $\bar{I}_3$, we look for the exact solution $U_3^{ex} = U_3^{ap} + U_3(\tau)$ such that:

$$U_3^{ex}(\bar{\beta}_3) = U_4^{ex}(\bar{\alpha}_4/\epsilon), U_3^{ex}(\bar{\alpha}_3) = U_2^{ex}(\bar{\beta}_2/\epsilon)$$

Define:

$$
\begin{aligned}
\Delta U_4(\alpha_4/\epsilon) = U_4^{ex}(\bar{\alpha}_4/\epsilon) - U_4^{ap}(\alpha_4/\epsilon), \Delta U_2(\beta_2/\epsilon) = U_2^{ex}(\bar{\beta}_4/\epsilon) - U_2^{ap}(\beta_2/\epsilon) \\
\Delta U_3(\beta_3) = U_3^{ex}(\bar{\beta}_3) - U_3^{ap}(\beta_3), \Delta U_3(\alpha_3) = U_3^{ex}(\bar{\alpha}_3) - U_3^{ap}(\alpha_3)
\end{aligned}
$$

Thus we have the jump conditions for correction function U as:

$$\Delta U_4(\alpha_4/\epsilon) - \Delta U_3(\beta_3) = -J_{U3}, \Delta U_3(\alpha_3) - \Delta U_2(\beta_2/\epsilon) = -J_{U2}$$

where we have:

$$
\begin{aligned}
\Delta U_4(\alpha_4/\epsilon) &= [U_4^{ex}(\bar{\alpha}_4/\epsilon) - U_4^{ap}(\bar{\alpha}_4/\epsilon)] + [U_4^{ap}(\bar{\alpha}_4/\epsilon) - U_4^{ap}(\alpha_4/\epsilon)] \\
&= U_4(\bar{\alpha}_4/\epsilon) + \delta U_4(\alpha_4/\epsilon)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\Delta U_2(\beta_2/\epsilon) &= [U_2^{ex}(\bar{\beta}_2/\epsilon) - U_2^{ap}(\bar{\beta}_2/\epsilon)] + [U_2^{ap}(\bar{\beta}_2/\epsilon) - U_2^{ap}(\beta_2/\epsilon)] \\
&= U_2(\bar{\beta}_2/\epsilon) + \delta U_2(\beta_2/\epsilon)
\end{aligned}
$$

$$\Delta U_3(\beta_3) = U_3^{ex}(\beta_3) - U_3^{ap}(\beta_3) = U_3(\beta_3)$$

$$\Delta U_3(\alpha_3) = U_3^{ex}(\alpha_3) - U_3^{ap}(\alpha_3) = U_3(\alpha_3)$$

Therefore we have the jump conditions:

$$\begin{aligned} U_4(\bar{\alpha}_4/\epsilon) - U_3(\beta_3) &= -J_{U3} - \delta U_4(\alpha_4/\epsilon) = -\bar{J}_{U3} \\ U_3(\alpha_3) - U_2(\bar{\beta}_2/\epsilon) &= -J_{U2} + \delta U_2(\beta_2/\epsilon) = -\bar{J}_{U2} \end{aligned} \tag{5.5}$$

Also on $\bar{I}_1$, we look for the exact solution $U_1^{ex} = U_1^{ap} + U_1(\tau)$ such that:

$$U_1^{ex}(\bar{\beta}_1) = U_2^{ex}(\bar{\alpha}_2/\epsilon), U_1^{ex}(\bar{\alpha}_1) = U_4^{ex}(\bar{\beta}_4/\epsilon)$$

Define

$$\Delta U_2(\alpha_2/\epsilon) = U_2^{ex}(\bar{\alpha}_2/\epsilon) - U_2^{ap}(\alpha_2/\epsilon), \Delta U_4(\beta_4/\epsilon) = U_4^{ex}(\bar{\beta}_4/\epsilon) - U_4^{ap}(\beta_4/\epsilon)$$
$$\Delta U_1(\beta_1) = U_1^{ex}(\bar{\beta}_1) - U_1^{ap}(\beta_1), \Delta U_1(\alpha_1) = U_1^{ex}(\bar{\alpha}_1) - U_1^{ap}(\alpha_1)$$

Thus we have the jump conditions for correction function U as:

$$\Delta U_2(\alpha_2/\epsilon) - \Delta U_1(\beta_1) = -J_{U1}, \Delta U_1(\alpha_1) - \Delta U_4(\beta_4/\epsilon) = -J_{U4}$$

where we have:

$$\begin{aligned} \Delta U_2(\alpha_2/\epsilon) &= [U_2^{ex}(\bar{\alpha}_2/\epsilon) - U_2^{ap}(\bar{\alpha}_2/\epsilon)] + [U_2^{ap}(\bar{\alpha}_2/\epsilon) - U_2^{ap}(\alpha_2/\epsilon)] \\ &= U_2(\bar{\alpha}_2/\epsilon) + \delta U_2(\alpha_2/\epsilon) \end{aligned}$$

Similarly,

$$\begin{aligned} \Delta U_4(\beta_4/\epsilon) &= [U_4^{ex}(\bar{\beta}_4/\epsilon) - U_4^{ap}(\bar{\beta}_4/\epsilon)] + [U_4^{ap}(\bar{\beta}_4/\epsilon) - U_4^{ap}(\beta_4/\epsilon)] \\ &= U_4(\bar{\beta}_4/\epsilon) + \delta U_4(\beta_4/\epsilon) \end{aligned}$$

$$\Delta U_1(\beta_1) = U_1^{ex}(\beta_1) - U_1^{ap}(\beta_1) = U_1(\beta_1)$$

$$\Delta U_1(\alpha_1) = U_1^{ex}(\alpha_1) - U_1^{ap}(\alpha_1) = U_1(\alpha_1)$$

Therefore we have the jump conditions:

$$\begin{aligned} U_2(\bar{\alpha}_2/\epsilon) - U_1(\beta_1) &= -J_{U1} - \delta U_2(\alpha_2/\epsilon) = -\bar{J}_{U1} \\ U_1(\alpha_1) - U_4(\bar{\beta}_4/\epsilon) &= -J_{U4} + \delta U_4(\beta_4/\epsilon) = -\bar{J}_{U4} \end{aligned} \tag{5.6}$$

On $\bar{I}_3$, we look for the exact solution $Y_3^{ex} = Y_3^{ap} + Y_3(t)$ such that:

$$Y_3^{ex}(\epsilon\bar{\beta}_3) = Y_4^{ex}(\bar{\alpha}_4), Y_3^{ex}(\epsilon\bar{\alpha}_3) = Y_2^{ex}(\bar{\beta}_2)$$

Define:

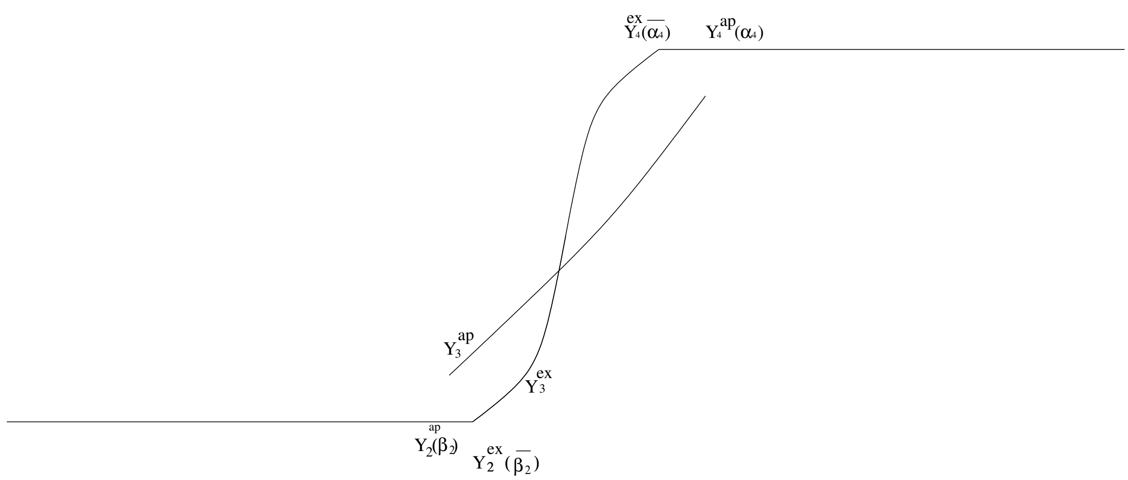


Figure 5.2: Jump condition for Y

$$\begin{aligned} \Delta Y_4(\alpha_4) = Y_4^{ex}(\bar{\alpha}_4) - Y_4^{ap}(\alpha_4), \Delta Y_2(\beta_2) = Y_2^{ex}(\bar{\beta}_4) - U_2^{ap}(\beta_2) \\ \Delta Y_3(\epsilon\beta_3) = Y_3^{ex}(\epsilon\bar{\beta}_3) - Y_3^{ap}(\epsilon\beta_3), \Delta Y_3(\epsilon\alpha_3) = Y_3^{ex}(\epsilon\bar{\alpha}_3) - Y_3^{ap}(\epsilon\alpha_3) \end{aligned}$$

Thus we have the jump conditions for correction function Y as:

$$\Delta Y_4(\alpha_4) - \Delta Y_3(\epsilon\beta_3) = -J_{Y3}, \Delta Y_3(\epsilon\alpha_3) - \Delta Y_2(\beta_2) = -J_{Y2} \tag{5.7}$$

where we have:

$$\begin{aligned}\Delta Y_4(\alpha_4) &= [Y_4^{ex}(\bar{\alpha}_4) - Y_4^{ap}(\bar{\alpha}_4)] + [Y_4^{ap}(\bar{\alpha}_4) - Y_4^{ap}(\alpha_4)] \\ &= Y_4(\bar{\alpha}_4) + \delta Y_4(\alpha_4)\end{aligned}$$

Similarly,

$$\begin{aligned}\Delta Y_2(\beta_2) &= [Y_2^{ex}(\bar{\beta}_2) - Y_2^{ap}(\bar{\beta}_2)] + [Y_2^{ap}(\bar{\beta}_2) - Y_2^{ap}(\beta_2)] \\ &= Y_2(\bar{\beta}_2) + \delta Y_2(\beta_2)\end{aligned}$$

$$\Delta Y_3(\epsilon\beta_3) = Y_3^{ex}(\epsilon\beta_3) - Y_3^{ap}(\epsilon\beta_3) = Y_3(\epsilon\beta_3)$$

$$\Delta Y_3(\epsilon\alpha_3) = Y_3^{ex}(\epsilon\alpha_3) - Y_3^{ap}(\epsilon\alpha_3) = Y_3(\epsilon\alpha_3)$$

Therefore we have:

$$\begin{aligned}Y_4(\bar{\alpha}_4) - Y_3(\epsilon\beta_3) = -J_{Y3} - \delta Y_4(\alpha_4) = -\bar{J}_{Y3} \\ Y_3(\epsilon\alpha_3) - Y_2(\bar{\beta}_2) = -J_{Y2} + \delta Y_2(\beta_2) = -\bar{J}_{Y2}\end{aligned} \tag{5.8}$$

Also on $\bar{I}_1$, we look for the exact solution $Y_1^{ex} = Y_1^{ap} + Y_1(t)$, such that:

$$Y_1^{ex}(\epsilon\bar{\beta}_1) = Y_2^{ex}(\bar{\alpha}_2), Y_1^{ex}(\epsilon\bar{\alpha}_1) = Y_4^{ex}(\bar{\beta}_4)$$

Define

$$\begin{aligned}\Delta Y_2(\alpha_2) = Y_2^{ex}(\bar{\alpha}_2) - Y_2^{ap}(\alpha_2), \Delta Y_4(\beta_4) = Y_4^{ex}(\bar{\beta}_4) - Y_4^{ap}(\beta_4) \\ \Delta Y_1(\epsilon\beta_1) = Y_1^{ex}(\epsilon\bar{\beta}_1) - Y_1^{ap}(\epsilon\beta_1), \Delta Y_1(\epsilon\alpha_1) = Y_1^{ex}(\epsilon\bar{\alpha}_1) - Y_1^{ap}(\epsilon\alpha_1)\end{aligned}$$

Thus we have the jump condition for correction function Y as:

$$\Delta Y_2(\alpha_2) - \Delta Y_1(\epsilon\beta_1) = -J_{Y1}, \Delta Y_1(\epsilon\alpha_1) - \Delta Y_4(\beta_4) = -J_{Y4} \tag{5.9}$$

where we have:

$$\begin{aligned}\Delta Y_2(\alpha_2) &= [Y_2^{ex}(\bar{\alpha}_2) - Y_2^{ap}(\bar{\alpha}_2)] + [Y_2^{ap}(\bar{\alpha}_2) - Y_2^{ap}(\alpha_2)] \\ &= Y_2(\bar{\alpha}_2) + \delta Y_2(\alpha_2)\end{aligned} \tag{5.10}$$

Similarly,

$$\begin{aligned}\Delta Y_4(\beta_4) &= [Y_4^{ex}(\bar{\beta}_4) - Y_4^{ap}(\bar{\beta}_4)] + [Y_4^{ap}(\bar{\beta}_4) - Y_4^{ap}(\beta_4)] \\ &= Y_4(\bar{\beta}_4) + \delta Y_4(\beta_4)\end{aligned}$$

$$\Delta Y_1(\epsilon\beta_1) = Y_1^{ex}(\epsilon\beta_1) - Y_1^{ap}(\epsilon\beta_1) = Y_1(\epsilon\beta_1)$$

$$\Delta Y_1(\epsilon\alpha_1) = Y_1^{ex}(\epsilon\alpha_1) - Y_1^{ap}(\epsilon\alpha_1) = Y_1(\epsilon\alpha_1)$$

Therefore we have:

$$\begin{aligned}Y_2(\bar{\alpha}_2) - Y_1(\epsilon\beta_1) = -J_{Y1} - \delta Y_2(\alpha_2) = -\bar{J}_{Y1} \\ Y_1(\epsilon\alpha_1) - Y_4(\bar{\beta}_4) = -J_{Y4} + \delta Y_4(\beta_4) = -\bar{J}_{Y4}\end{aligned} \tag{5.11}$$

$$\begin{aligned}\delta(Y_2(\beta_2/\epsilon)) = Y_2^{ap}(\bar{\beta}_2/\epsilon) - Y_2^{ap}(\beta_2/\epsilon) = Y_2^{ap'}(\beta_2/\epsilon)\delta\beta_2/\epsilon + O(|\delta\beta_2/\epsilon|^2) \\ \delta(Y_4(\beta_4/\epsilon)) = Y_4^{ap}(\bar{\beta}_4/\epsilon) - Y_4^{ap}(\beta_4/\epsilon) = Y_4^{ap'}(\beta_4/\epsilon)\delta\beta_4/\epsilon + O(|\delta\beta_4/\epsilon|^2) \\ \delta(Y_2(\alpha_2/\epsilon)) = Y_2^{ap}(\bar{\alpha_2}/\epsilon) - Y_2^{ap}(\alpha_2/\epsilon) = Y_2^{ap'}(\alpha_2/\epsilon)\delta\alpha_2/\epsilon + O(|\delta\alpha_2/\epsilon|^2) \\ \delta(Y_4(\alpha_4/\epsilon)) = Y_4^{ap}(\bar{\alpha_4}/\epsilon) - Y_4^{ap}(\alpha_4/\epsilon) = Y_4^{ap'}(\alpha_4/\epsilon)\delta\alpha_4/\epsilon + O(|\delta\alpha_4/\epsilon|^2)\end{aligned} \tag{5.12}$$

5.1.2 Linear variational system for the correction function

We want to construct the linear variational system for the correction function $w = (U, Y)$ such that $w^{ex} = w^{ap} + w$, according to the definition of residual error:

$$\begin{aligned}p(\tau) = U^{ap'} - F(U^{ap}, Y^{ap}, s) \\ q(t) = \dot{Y}^{ap} - G(U^{ap}, Y^{ap}, s)\end{aligned}$$

In order to best describe the behavior of the fast/slow orbits, we use different time scales for U and Y. We obtain the system of equations that the correction functions satisfy:

$$\begin{aligned} U'(\tau) &= F(U^{ex}, Y^{ex}, s) - [F(U^{ap}, Y^{ap}, s) + p(\tau)] \\ \dot{Y}(t) &= G(U^{ex}, Y^{ex}, s) - [G(U^{ap}, Y^{ap}, s) + q(t)] \end{aligned} \tag{5.13}$$

which can be rewritten as:
(i) For $i = 2l$,

$$\begin{aligned} U_i'(\tau) &= F_U U_i + F_Y Y_i - p_i(\tau) + (F^{ex} - F^{ap} - F_U U_i - F_Y Y_i) \\ &= F_U U_i + F_Y Y_i + \bar{P}_i = F_U V_i + \bar{P}_i \\ \dot{Y}_i(t) &= G_U U_i + G_Y Y_i - q_i(t) + (G^{ex} - G^{ap} - G_U U_i - G_Y Y_i) \\ &= G_U U_i + G_Y Y_i + \bar{Q}_i \end{aligned} \tag{5.14}$$

After a change of variables, $V_i(\tau) = U_i(\tau) + F_U^{-1}(\epsilon\tau) F_Y(\epsilon\tau) Y_i(\epsilon\tau), i = 1, 2, 3, 4$, where $F_U(\epsilon\tau) = F_U(U_{ap}(\epsilon\tau), Y_{ap}(\epsilon\tau))$, we further reduce the (5.14) to be in terms of (V_i, Y_i):

$$\begin{aligned} V_i'(\tau) &= F_U(\epsilon\tau) V_i(\tau) + \epsilon \frac{d}{dt}[F_U^{-1} F_Y Y] + \bar{P}_i = F_U(\epsilon\tau) V_i(\tau) + \bar{\bar{P}}_i \\ \dot{Y}_i(t) &= (G_Y - G_U F_U^{-1} F_Y) Y_i + G_U V_i + \bar{Q}_i \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} \bar{Q}_i &= -q_i(t) + (G^{ex} - G^{ap} - G_U U - G_Y Y) \\ &= G(U^{ex}, Y^{ex}, s) - G(U^{ap}, Y^{ap}, s) - D_w G(U^{ap}, Y^{ap}, s) w(t) \\ &+ [D_w G(U^{ap}, Y^{ap}, s, \epsilon) - D_w G(U^{ap}, Y^{ap}, s, \epsilon = 0)] w(t) - q_i(t) \\ &= O(|w_i|^2 + \epsilon |w_i| + |q_i(t)|) \end{aligned}$$

$$
\begin{aligned}
\bar{P}_i &= -p^i(\tau) + (F^{ex} - F^{ap} - F_U U - F_Y Y) \\
&= [U' + (F_U^{-1}(\epsilon\tau)F_Y(\epsilon\tau)Y(\tau))'] - F_U(U(\tau) + F_U^{-1}(\epsilon\tau)F_Y(\epsilon\tau)Y(\tau)) \\
&= U' + (F_U^{-1}(\epsilon\dot{\tau)}F_Y(\epsilon\tau))Y + (F_U^{-1}(\epsilon\tau)F_Y(\epsilon\tau))\dot{Y} - (F_U U + F_Y Y) \\
&= [F(U^{ex}, Y^{ex}, s) - F(U^{ap}, Y^{ap}, s) - p(\tau)] + (F_U^{-1}(\epsilon\tau)F_Y(\epsilon\tau))'Y \\
&+ (F_U^{-1}(\epsilon\tau)F_Y(\epsilon\tau))\epsilon[G(U^{ex}, Y^{ex}, s) - G(U^{ap}, Y^{ap}, s) - q(\tau)] \\
&- D_w F(U^{ap}, Y^{ap}, s) + [D_w F(U^{ex}, Y^{ex}, s, \epsilon) - D_w F(U^{ap}, Y^{ap}, s, \epsilon = 0)]w \\
&= O(|w_i|^2 + \epsilon|w_i| + |\epsilon q^i(t)| + |p^i|)
\end{aligned}
\tag{5.16}
$$

$$
\begin{aligned}
\bar{\bar{P}}_i &= \bar{P}_i + \epsilon\frac{d}{dt}[F_U^{-1}F_Y Y] \\
&= \bar{P}_i + \epsilon(F_U^{-1}(\epsilon\dot{\tau)}F_Y(\epsilon\tau))Y + \epsilon(F_U^{-1}(\epsilon\tau)F_Y(\epsilon\tau))\dot{Y} \\
&= \bar{P}_i + \epsilon|w| + \epsilon O(G_U U + G_Y Y + \bar{Q}_i) \\
&= \bar{P}_i + \epsilon|w| + \epsilon O(\bar{Q}_i) \\
&= \bar{P}_i + \epsilon\bar{Q}_i = C\bar{P}_i
\end{aligned}
$$

(ii) For $i = 2l + 1$

$$
\begin{aligned}
U_i'(\tau) &= F_U U_i + F_Y Y_i + \bar{P}^i \\
\dot{Y}_i(t) &= \bar{Q}^i
\end{aligned}
\tag{5.17}
$$

where

$$
\bar{Q}_i = G(U^{ex}, Y^{ex}, s) - G(U^{ap}, Y^{ap}, s) - q(t) = O(|w| + |q|) \tag{5.18}
$$

$$
\begin{aligned}
\bar{P}^i &= F(U^{ex}, Y^{ex}, s) - F(U^{ap}, Y^{ap}, s) - p(\tau) - D_w F(U^{ap}, Y^{ap}, s)w \\
&+ [D_w F(U^{ex}, Y^{ex}, s, \epsilon) - D_w F(U^{ap}, Y^{ap}, s, \epsilon = 0)]w \\
&= O(|w|^2 + |p| + \epsilon^{\lambda}|w|)
\end{aligned}
\tag{5.19}
$$

With the above construction, we introduce the Melnikov integral to solve for a generalized solution.

5.1.3 Introduction of the Melnikov integral for the generalized solution

Based on the approximated solution w^i_{ap} when $s = s_0$, we want to find a correction solution to the linearized variational system. Notice the homogeneous part of (5.20) has exponential dichotomy on $[-N, 0], [0, N]$ respectively, but not on $[-N, N]$ due to the non-transversal intersection on $RP^i_u(0-)$ and $RP^i_s(0+)$ at $\tau = 0$ for $i = 1, 3$. In fact, $RP^i_u(0-) + RP^i_s(0+) = R^1$. We need to take care of the non-transversal intersection issue for the $U = (u, v)$ equation by introducing the Melnikov integral.

Define operator $\mathcal{F} : U \to H(\tau)$ with $\tau \in I^{1,3} = [-N, N], N = \epsilon^{\lambda-1}$ and boundary value $U(\pm N)$ given.

$$H(\tau) = \mathcal{F}(U) = U'(\tau) - A(\tau)U(\tau), A(\tau) = D_U F(U^i_{ap}, Y^i_{ap}) \tag{5.20}$$

We have the following lemma for the existence of a generalized solution that allows a gap at $\tau = 0$:

Lemma 5.1.1. *Assume that $\mathcal{F}$ is of codimension one, $\dot{U}^i_{ap}(\tau), i = 1, 3$ is the unique nonzero bounded solution to the equation $U^{i'}(\tau) - D_U F^i(U^i_{ap}, Y^i_{ap}, s = s_0)U^i(\tau) = 0$, and ψ^i is the unique nonzero bounded solution to the adjoint equation of the previous homogeneous equation, then $\mathcal{F}$ is Fredholm with index being 0. The range of $\mathcal{F}$ is of codimension one and the kernel of $\mathcal{F}$ is one dimensional for each $H^i(\tau)$. There exists a unique generalized solution U^i for system (5.20) such that $U^i(0) \perp Ker\mathcal{F}$, i.e. $U^i(0) \perp \dot{U}^i_{ap}(0)$ and U^i has a gap at $\tau = 0$ along the given direction d^i:*

$$U^i(0+) - U^i(0-) = g^i d^i$$

Also we have estimate for g^i, U^i as:

$$|g^i| \le C(|P^i_s(-N)U^i(-N)|e^{-\alpha N} + |P^i_u(N)U^i(N)|e^{-\alpha N} + |H^i|) \tag{5.21}$$

$$|U^i| \le C(|\phi^i_1| + |\phi^i_2| + |H^i|) \tag{5.22}$$

Proof. Here we introduce the Melnikov function.

Let $S^i(t, s)$ be the principal matrix solution of the homogeneous part of equation $U^{i'} = AU^i + H^i$, which has exponential dichotomies on $(-N, 0]$, and $[0, N)$, but no exponential dichotomy on $(-N, N)$ because:

$$RP^i_u(0-) + RP^i_s(0+) = R^{n-1}$$

Let $\psi^i(0) \perp RP^i_u(0-) + RP^i_s(0+), M^i = \{x| < \psi^i(0), x >= 0\}, \dim M^i = n - 1$.

We define $d^i = \psi^i(0)/(||\psi^i(0)||^2) \in X, \phi_1^i = P_s^i(-N)U^i(-N), \phi_2^i = P_u^i(N)U^i(N)$ such that $< d^i, \psi^i(0) >= 1, span\{d^i\} \bigoplus M^i = R^n$ and we define the following solution:

$$\begin{aligned} U^i(\tau) &= S^i(\tau,-N)P_s^i(-N)U^i(-N) + \int_{-N}^{\tau} S^i(\tau,s)P_s^i(s)H^i(s)ds \\ &\quad + S^i(\tau,0-)P_u^i(0-)U^i(0-) + \int_0^{\tau} S^i(\tau,s)P_u^i(s)H^i(s)ds, \tau < 0 \end{aligned} \tag{5.23}$$

$$\begin{aligned} U^i(\tau) &= S^i(\tau,N)P_u^i(N)U^i(N) + \int_{N}^{\tau} S^i(\tau,s)P_u^i(s)H^i(s)ds \\ &\quad + S^i(\tau,0+)P_s^i(0+)U^i(0+) + \int_0^{\tau} S^i(\tau,s)P_s^i(s)H^i(s)ds, \tau > 0 \end{aligned} \tag{5.24}$$

We project them onto the stable and unstable spaces, and there are exponential dichotomies on $[-N,0],[0,N]$ respectively:

$$P_s^i(0-)U^i(0-) = \int_{-N}^{0-} S^i(0-,s)P_s^i(s)H^i(s)ds + S^i(0-,-N)P_s^i(-N)U^i(-N)$$

$$P_u^i(0+)U^i(0+) = \int_{N}^{0+} S^i(0+,s)P_u^i(s)H^i(s)ds + S^i(0+,N)P_u^i(N)U^i(N)$$

define $\phi_3^i = P_u^i(0-)U^i(0-), \phi_4^i = P_s^i(0+)U^i(0+)$, $\psi^i(s) = T^\star(s,0)\psi^i(0)$, where $T^\star(s,t)$ is the adjoint of $T(t,s)$. After subtracting ϕ_3^i and ϕ_4^i we have:

$$\begin{aligned} \phi_4^i - \phi_3^i &= [I - P_u^i(0+)]U^i(0+) - [I - P_s^i(0-)]U^i(0-) \\ &= g^i(Y,N)d^i + P_s^i(0-)U^i(0-) - P_u^i(0+)U^i(0+) \\ &= g^i(y^1(0),y^3(0),N)d^i + [\int_{-N}^{0-} S^i(0-,s)P_s^i(s)h^i(s)ds + S^i(0-,-N)P_s^i(-N)U^i(-N)] \\ &- [\int_{N}^{0+} S^i(0+,s)P_u^i(s)h^i(s)ds + S^i(0+,N)P_u^i(N)U^i(N)] \end{aligned}$$

which leads to the estimate:

$$|g^i| + |\phi_3^i| + |\phi_4^i| \leq C(e^{-\alpha N}|\phi_1^i| + e^{-\alpha N}|\phi_2^i| + |H^i|e^{-\eta N}) \tag{5.25}$$

Based on (5.23), (5.24), (5.25), we obtain the estimate (5.22) for U^i. Since $\phi_4^i - \phi_3^i \in RP_u^i(0-)+$

$RP_s^i(0+)$, thus $< \psi^i(0), \phi_4^i - \phi_3^i >= 0$. As a result,

$$
\begin{aligned}
g^i =& < \psi^i, \int_{-N}^{0-} S^i(0-,s)P_s^i(s)H^i(s)ds > - < \psi^i, \int_{N}^{0+} S^i(0+,s)P_u^i(s)H^i(s)ds > \\
&+ < \psi^i, S^i(0-,-N)P_s^i(-N)U^i(-N) > - < \psi^i, S^i(0+,N)P_u^i(N)U^i(N) > \\
=& \int_{-N}^{N} < \psi^i(s), H^i(s) > ds + < \psi^i(-N), P_s^i(-N)U^i(-N) > - < \psi^i(N), P_u^i(N)U^i(N) >
\end{aligned}
\tag{5.26}
$$

Estimate (5.21) follows from (5.26) and $\psi^i(\tau) \leq e^{-\alpha|\tau|}$. □

In order to analyze the dependence of the gap g^i with respect to the y values, we consider the derivative of (5.20): $U_{y_j}^{i'} = (DF)U_{y_j}^i + H_{y_j}^i$ for $i, j = 1, 3$, with the principal matrix solution $T^i(t,s)$ to its homogeneous part, given the prescribed boundary conditions at $\tau = \pm N$:

$$
\begin{aligned}
U_{y_j}^i(0-) &= \int_{-N}^{0} T^i(0,s)P_s^i(s)H_{y_j}^i(s)ds + P_u^i(0-)U_{y_j}^i(0-) + T^i(0,-N)P_s^i(0-)U_{y_j}^i(-N) \\
U_{y_j}^i(0+) &= \int_{N}^{0} T^i(0,s)P_u^i(s)H_{y_j}^i(s)ds + P_s^i(0+)U_{y_j}^i(0+) + T^i(0,N)P_u^i(0+)U_{y_j}^i(N) \\
U_{y_j}^i(0-) - U_{y_j}^i(0+) &= \int_{-N}^{0} T(0,s)P_s^i(s)H_{y_j}^i(s)ds + P_u^i(0-)U_{y_j}^i(0-) + T^i(0,-N)P_s^i(0-)U_{y_j}^i(-N) \\
&\quad - [\int_{N}^{0} T(0,s)P_u^i(s)H_{y_j}^i(s)ds + P_s^i(0+)U_{y_j}^i(0+) + T^i(0,N)P_u^i(0+)U_{y_j}^i(N)]
\end{aligned}
$$

where

$$
\begin{gathered}
P_u^i(0-)U_{y_j}^i(0-) \in RP_u(0-), \\
P_s^i(0+)U_{y_j}^i(0+) \in RP_s(0+), \\
RP_u(0-) + RP_s(0+) = R^{n-1}
\end{gathered}
$$

Let $\psi(0) \perp RP_u(0-) + RP_s(0+)$, we have $< \psi(0), P_u^i(0-)U_{y_j}^i(0-) - P_s^i(0+)U_{y_j}^i(0+) >= 0$.

Moreover,

$$
\begin{aligned}
&< \psi(0), \int_N^0 T(0,s) P_u^i(s) H_{y_j}^i(s) ds > \\
&= \int_N^0 < \psi(0), T(0,s) P_u^i(s) H_{y_j}^i(s) > ds \\
&= \int_N^0 < T^*(s,0)\psi(0), P_u^i(s) H_{y_j}^i(s) > ds \\
&= \int_N^0 < \psi(s), P_u^i(s) H_{y_j}^i(s) > ds
\end{aligned}
$$

Similarly $< \psi(0), \int_{-N}^0 T(0,s) P_s^i(s) H_{y_j}^i(s) ds >= \int_{-N}^0 < \psi(s), P_s^i(s) H_{y_j}^i(s) > ds$
Now define $\psi(s) = T^\star(s,0)\psi(0)$ such that $\psi' = -(DF)^T \psi$, according to Lemma 5.1.1 we have:

$$
\begin{aligned}
\frac{\partial g_i}{\partial y_j} &= < \psi(0), U_{y_j}^i(0-) - U_{y_j}^i(0+) > \\
&= \int_{-N}^N < \psi(s), H_{y_j}^i(s) > ds + < \psi(N), P_u^i(0+) U_{y_j}^i(N) > + < \psi(-N), P_s^i(0-) U_{y_j}^i(-N) >
\end{aligned}
\tag{5.27}
$$

In our case,

$$
A = F_U(U,Y) = \begin{pmatrix} 0 & 1/D \\ h'(U) & -s/D \end{pmatrix}, H_i = \begin{pmatrix} 0 \\ h(U) - h'(U)U^i - y^i \end{pmatrix}
$$

$\frac{\partial H^i}{\partial y_j} = \delta_{ij}(0,-1)^T$, where δ_{ij} is the Kronecker delta.

Let $X(\tau)$ be the fundamental matrix solution to $U' = AU$. If (u^i, v^i) is the heteroclinic fast solution of (5.20), then the first column of X is $(u^{i'}, v^{i'})^T$. Let the other linear independent column be $(p,q)^T$. We claim that $(X^{-1})^T$ is the fundamental matrix solution to the adjoint equation: $Y' + A^T Y = 0$.
In fact, $XX^{-1} = I$ implies $(X^{-1})^T X^T = I$. Differentiate both sides with respect to τ, and we have:

$$
(X^{-1})^{T'} X^T + (X^{-1})^T X^{T'} = 0
$$

i.e.

$$
(X^{-1})^{T'} X^T + (X^{-1})^T X^T A^T = 0
$$

$$
(X^{-1})^{T'} X^T + A^T = 0
$$

$$(X^{-1})^{T'} + A^T (X^T)^{-1} = 0$$

Since $(X^T)^{-1} = (X^{-1})^T$, we have $(X^{-1})^T$ is the fundamental matrix solution to the adjoint equation. Simple computation shows that:

$$(X^T)^{-1}(\tau) = \frac{1}{\Delta} \begin{pmatrix} q & -v^i \\ -p & u^i \end{pmatrix} (\tau)$$

where $\Delta = \det(X)(\tau) = ce^{-s\tau/D}$. Without loss of generality, by normalizing $(p, q)^T$, we can have $\det(X(0)) = c = 1$. Thus by taking the second column of $(X^T)^{-1}(\tau)$ so that we have $\psi = e^{s\tau/D}(-v^{i'}, u^{i'})^T$, which is the solution to the adjoint equation: $\psi' + A^T\psi = 0$. Therefore,

$$\int_{-N}^{N} < \psi, H^i_{y_j} > d\tau = \delta_{ij} \int_{-N}^{N} -\psi_2(\tau) d\tau = -\delta_{ij} \int_{u^i(-N)}^{u^i(N)} e^{s\tau/D} du^i \tag{5.28}$$

Notice that the integral is non-zero for $i \neq j$, and $\lim_{N\to\infty} \psi(\pm N) \to 0$, according to (5.27), (5.28), we have an almost diagonal matrix:

$$G_y = \{\frac{\partial g^i}{\partial y_j}\}_{i,j=1,3} \approx \begin{pmatrix} G_{11} & 0 \\ 0 & G_{33} \end{pmatrix} \tag{5.29}$$

$$\frac{\partial g_i}{\partial y_j} \approx \begin{cases} G_{ii} \neq 0 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

G_y is almost a diagonal matrix. Therefore g^i mainly depends on y^i and does depend on $y_j(0)$ weakly for $i \neq j$.

5.1.4 Eliminate the gaps of the generalized solution by manipulating the Melnikov function

When $\epsilon = 0$, we have a closed periodic solution. When $\epsilon > 0$, we have a generalized solution U^i with gaps at $\tau = 0$ according to Lemma 5.1.1. Notice that in the original u, v equations in (1.3), $\dot{U}$ depends on the y value on the right hand side. Only by changing the y value, can we affect the gap g. Moreover, according to (1.4), $Y' = \epsilon G(U, Y)$, therefore we obtain $Y^i(t) = Y^i(0) + \int_0^t \epsilon G(U^i, Y^i) dt = Y^i(0) + O(\epsilon)|G|_{L_1}, t \in I^i, i = 1, 3$. We use $(y_1(0), y_3(0))$ as the parameter y for g^i. We define the equal gap surfaces $\Sigma_i \in R^2$ as in the following Figure 5.3:

$$\Sigma_i(k_i, \epsilon = \epsilon_0) = \{(y_i, z_i) | g^i(y_1(0), y_3(0), \epsilon_0) = k_i\}$$

On which the y value is fixed, and the gap is the constant k_i given ϵ.

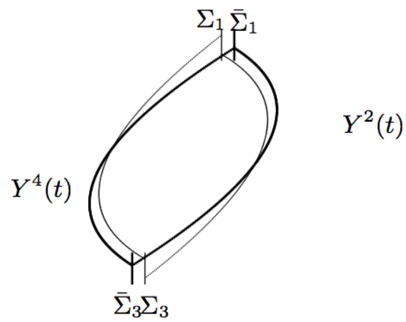


Figure 5.3: Equal gap surfaces Σ_i and the shifted equal gap surfaces $\bar{\Sigma}_i$

Recall (5.29), we know that g^i mainly depends on $y_i(0)$, but does depend on $y_j(0)$ weakly for $j \neq i$. We shift Σ_i along the direction $\vec{n} = (1, 0)$ by amount δy_i (can be negative) in order to make the fast solution U^i transversal at $\tau = 0$ and eliminate the gaps. Thus the new shifted equal gap surfaces are defined as:

$$\bar{\Sigma}_i(\epsilon = \epsilon_0) = \{(y_i, z_i)|g^i(\bar{y}_1(0), \bar{y}_3(0), \epsilon_0) = k_i + \Delta k_i = 0\}$$

We prove the following lemma 5.1.3 by using the main theorem in [20]:

Theorem 5.1.2. *Suppose that F: $\bar{D} \to \bar{D}$ is a compact continuous map which is continuously Frechet differentiable on D. Suppose that (a) for each $x \in D$, 1 is not an eigenvalue of F'(x), and (b) for each $x \in \partial D, x \neq F(x)$. Then F has a unique fixed point.*

Definition 5.1.1. The **winding number** of a contour Γ about a point z_0, denoted as $I_\Gamma(z_0)$ gives the number of times the Γ curve passes (counterclockwise) around a point z_0. Counterclockwise winding is assigned a positive winding number, while clockwise winding is assigned a negative winding number. The winding number is also called the index or degree, which can be used in the following lemma.

Lemma 5.1.3. *Let $\Omega_1(\delta y_i) = \{(y_1, y_3) : |y_i - y_i(0)| \leq \delta y_i, i = 1, 3\}$ be a rectangle in R^2. $H : \Omega_1 \to R^2$*

$$H(y_1, y_3) = (g^1(y_1, y_3), g^3(y_1, y_3))$$

is continuous with $g^i > 0 (< 0)$ if $y_i = y_i(0) + \delta y_i (y_i = y_i(0) - \delta y_i)$. Then there exists a unique $\hat{y} \in \Omega_1$ with $H(\hat{y}) = 0$.

Proof. Let $H_1(y_1, y_3) = (y_1 - \delta y_1 g^1/|g^1|, y_3 - \delta y_3 g^3/|g^3|)$ be the compact continuous map on Ω_1. The existence of a fixed point for H_1 has been obtained by Brouwer fixed-point theorem, winding numbers or index theory, see [24]. For the additional uniqueness, we try to verify for theorem 5.1.2: (a) 1 is not an eigenvalue of $H_1' = \frac{\partial H_1}{\partial y} = \begin{pmatrix} 1 - \frac{\delta y_1 \partial g_1}{|g^1| \partial y_1} & -\frac{\delta y_1 \partial g_1}{|g^1| \partial y_3} \\ -\frac{\delta y_3 \partial g_3}{|g^3| \partial y_1} & 1 - \frac{\delta y_3 \partial g_3}{|g^3| \partial y_3} \end{pmatrix}$ because $\det(G_y) \neq 0$ according to (5.29). (b) for each $x \in \partial D, x \neq H_1(x)$ because $g^i > 0 (< 0)$ if $y_i = y_i(0) + \delta y_i (y_i = y_i(0) - \delta y_i)$. i.e. $g^i \neq 0$ on ∂D, therefore there exists a unique fixed point $H_1(y_1, y_3) = (y_1, y_3)$, which indicates that there is (y_1, y_3) such that $H(y_1, y_3) = (0, 0)$.

In order to estimate $|y_1 - \hat{y}_1|$, according to the Mean-Value Theorem of several variables:

$$\begin{aligned} g^1(y_1, y_3) &= g^1(y_1, y_3) - g^1(\hat{y}_1, \hat{y}_3) \\ &=< \frac{\partial g_1}{\partial y_1}, \frac{\partial g_1}{\partial y_3} >< y_1 - \hat{y}_1, y_3 - \hat{y}_3 > \\ &\approx< G_{11}, 0 >< y_1 - \hat{y}_1, y_3 - \hat{y}_3 > \\ &= G_{11}(y_1 - \hat{y}_1) \end{aligned}$$

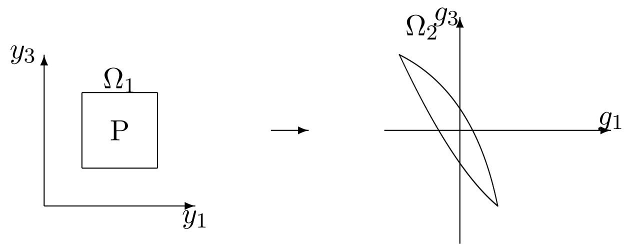
Therefore, $|y_1 - \hat{y}_1| = O(|g^1|)$. Similarly, $|y_3 - \hat{y}_3| = O(|g^3|)$.

Another proof of the existence is to use the winding numbers. We define the function H: $\Omega_1 \in R^2 \to \Omega_2 = O_\delta(0) \in R^2$ as:

$$H(y_1, y_3) = (g^1(y_1, y_3), g^3(y_1, y_3)),$$

where

$$\Omega_1(\delta y_i) = \{(y_1, y_3) : |y_i - y_i(0)| \leq \delta y_i, i = 1, 3\}$$



Ω_2 is the image of H, which maps the rectangle area Ω_1 to a neighborhood of the origin Ω_2. Notice that the winding number of the boundary $\partial\Omega_1$ around $(y_1(0), y_3(0))$, the center of Ω_1 is 1. Now assume for contradiction that $0 \notin \Omega_2$, then the winding number of $H(\partial\Omega_1)$ around the origin is zero. Because homotopy does not change the winding number, we have the contradiction $1 \neq 0$. Therefore the origin $O \in \Omega_2$.

Next, we decrease δy_i, the size of Ω_1. By homotopy, the boundary of Ω_2 shrinks correspondingly to a point. During the shrinking process there must exist a size $\hat{\delta y_i}$, when $\partial\Omega_2$ contains the origin, because $O \in \Omega_2$. Therefore, there exists a $\hat{y} \in \partial\Omega_1(\hat{\delta y_i}) \subset \Omega_1(\delta y_i)$ such that $H(\hat{y}) = (0, 0)$. $\square$

5.1.5 Periodic correction solution for the linearized variational system

We try to find the correction solutions by the following Lemmas:

Lemma 5.1.4. *Let L be the linear operation from Banach space E_1 to E_2, and T from E_2 to E_1 is an approximate right inverse operation of L if $|I - LT| < 1$. Then the equation $Lx = y$ has a solution $x = T\sum_{j=0}^{\infty}(I - LT)^j y$.*

Lemma 5.1.5. *In regular layers $[\alpha_i, \beta_i], i = 2, 4$, there exists a unique solution to the following system of equations:*

$$\dot{V}_i - F_U V_i = h_i(\tau) \tag{5.30}$$

with $S^i(t, s)$ to be the principal matrix of the system above and the jump conditions:

$$\begin{aligned} J_{V1} &= V_2(\alpha_2/\epsilon) - V_1(\beta_1), J_{V4} = V_1(\alpha_1) - V_4(\beta_4/\epsilon) \\ J_{V3} &= V_4(\alpha_4/\epsilon) - V_3(\beta_3), J_{V2} = V_3(\alpha_3) - V_2(\beta_2/\epsilon) \end{aligned} \tag{5.31}$$

and estimate:

$$|V_2| + |V_4| \leq C_1(|h_2| + |h_4| + \Sigma_{i=1}^4 |J_{Vi}|) \tag{5.32}$$

Proof. Because of the hyperbolicity of the coefficient matrix F_U, we know that the slow varying system has exponential dichotomy on $I^i, i = 2, 4$ with corresponding projections Q_s^i, Q_u^i. Also, we have the following decomposition:

$$RP_u^1(\beta_1) \oplus RP_s^2(\alpha_2/\epsilon) = R^2, RP_u^2(\beta_2/\epsilon) \oplus RP_s^3(\alpha_3) = R^2$$
$$RP_u^3(\beta_3) \oplus RP_s^4(\alpha_4/\epsilon) = R^2, RP_u^4(\beta_4/\epsilon) \oplus RP_s^1(\alpha_1) = R^2$$

Based on the decomposition above, we can split the jump conditions as following:

$$J_{V1} = Q_s^2 J_{V1} - (-Q_u^1 J_{V1}), J_{V4} = Q_s^1 J_{V4} - (-Q_u^4 J_{V4})$$
$$J_{V3} = Q_s^4 J_{V3} - (-Q_u^3 J_{V3}), J_{V2} = Q_s^3 J_{V2} - (-Q_u^2 J_{V2})$$

We give the stable component of each jump as the initial value for the solution after the jump, and the negated unstable component of the jump as the backward initial value for the solution before the jump, as in figure 5.4. That is to say, the solution between two jumps takes the negated unstable component of the latter jump as the backward initial value, and the stable component of the previous jump as the forward initial value. Therefore we define:

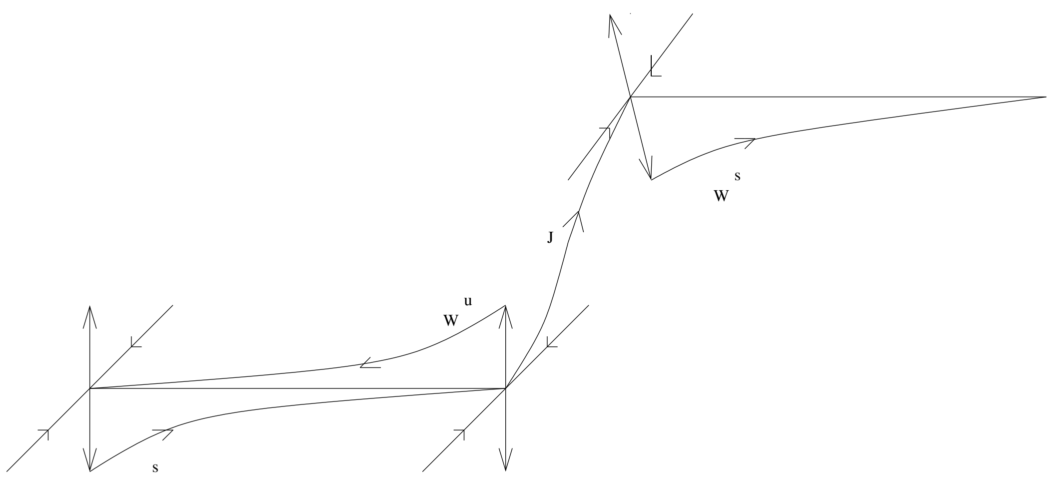


Figure 5.4: Decomposition of the jump errors for defining the solutions V_i

$$
\begin{aligned}
V_2^1(\tau) = S^2(\tau, \alpha_2/\epsilon)Q_s^2 J_{V1} &+ \int_{\alpha_2/\epsilon}^{\tau} S^2(\tau, s)Q_s^2(s)h^2(s)ds \\
-S^2(\tau, \beta_2/\epsilon)Q_u^2 J_{V2} &+ \int_{\beta_2/\epsilon}^{\tau} S^2(\tau, s)Q_u^2(s)h^2(s)ds \\
&\qquad \tau \in [\alpha_2/\epsilon, \beta_2/\epsilon]
\end{aligned}
$$

$$
\begin{aligned}
V_4^1(\tau) = S^4(\tau, \alpha_4/\epsilon)Q_s^4 J_{V3} &+ \int_{\alpha_4/\epsilon}^{\tau} S^4(\tau, s)Q_s^4(s)h^4(s)ds \\
-S^4(\tau, \beta_4/\epsilon)Q_u^4 J_{V4} &+ \int_{\beta_4/\epsilon}^{\tau} S^4(\tau, s)Q_u^4(s)h^4(s)ds \\
&\qquad \tau \in [\alpha_4/\epsilon, \beta_4/\epsilon]
\end{aligned}
$$

$$
\begin{aligned}
V_1^1(\tau) = S^1(\tau, \alpha_1)Q_s^1 J_{V4} &+ \int_{\alpha_1}^{\tau} S^1(\tau, s)Q_s^1(s)h^1(s)ds \\
-S^1(\tau, \beta_1)Q_u^1 J_{V1} &+ \int_{\beta_1}^{\tau} S^1(\tau, s)Q_u^1(s)h^1(s)ds \\
&\qquad \tau \in [\alpha_1, \beta_1]
\end{aligned}
$$

$$
\begin{aligned}
V_3^1(\tau) = S^3(\tau, \alpha_3)Q_s^3 J_{V2} &+ \int_{\alpha_3}^{\tau} S^3(\tau, s)Q_s^3(s)h^3(s)ds \\
-S^3(\tau, \beta_3)Q_u^3 J_{V3} &+ \int_{\beta_3}^{\tau} S^3(\tau, s)Q_u^3(s)h^3(s)ds \\
&\qquad \tau \in [\alpha_3, \beta_3]
\end{aligned}
$$

The above solutions satisfy (5.30), but satisfy the jump conditions almost accurately, for example:

$$\begin{aligned}
J_{V1}^1 &= V_2^1(\alpha_2/\epsilon) - V_1^1(\beta_1) \\
&= [Q_s^2 J_{V1} - S^2(\alpha_2/\epsilon, \beta_2/\epsilon) Q_u^2 J_{V2} + \int_{\beta_2/\epsilon}^{\alpha_2/\epsilon} S^2(\tau, s) Q_u^2(s) h^2(s) ds] \\
&- [S^1(\beta_1, \alpha_1) Q_s^1 J_{V4} + \int_{\alpha_1}^{\beta_1} S^1(\tau, s) Q_s^1(s) h^1(s) ds - Q_u^1 J_V(\Sigma_1+)] \\
&= J_{V1} + E^1(J_{V1}) \\
E^1(J_{V1}) &= -S^2(\alpha_2/\epsilon, \beta_2/\epsilon) Q_u^2 J_{V2} + \int_{\beta_2/\epsilon}^{\alpha_2/\epsilon} S^2(\alpha_2/\epsilon, s) Q_u^2(s) h^2(s) ds \\
&- S^1(\beta_1, \alpha_1) Q_s^1 J_{V4} + \int_{\alpha_1}^{\beta_1} S^1(\beta_1, s) Q_s^1(s) h^1(s) ds
\end{aligned}$$

with estimate:

$$\begin{aligned}
|E^1(J_{V1})| \leq C_2(e^{\alpha(\alpha_2 - \beta_2)/\epsilon} J_{V2} + \int_{\beta_2/\epsilon}^{\alpha_2/\epsilon} e^{\alpha(\alpha_2/\epsilon - s)} |h_2| ds) \\
+ C_1(e^{-\alpha(\beta_1 - \alpha_1)} J_{V4} + \int_{\alpha_1}^{\beta_1} e^{-\alpha(\beta_1 - s)} |h_1| ds)
\end{aligned}$$

Notice that $e^{-\alpha(\beta_2 - \alpha_2)/\epsilon}, e^{-\alpha(\beta_1 - \alpha_1)}$ as well as the two integral terms added together are all of $O(\epsilon)$. Thus $|E^1(JV_1)| = O(\epsilon \sum_1^4 |JV_i|)$ is small compared to the given jump conditions by multiplying ϵ in each iteration process. Therefore, we can define the solution $V_i^k(\tau), \tau \in [\alpha_i, \beta_i]$ recursively, with $-E^k(JV_i)$ as the jump condition in the next iteration. According to Lemma 5.1.4, the jump condition will be satisfied by $V_i(\tau) = \Sigma_{k=1}^{\infty} V_i^k(\tau)$ after the iteration process.

Next we give an estimate of the solution, for example:

$$\begin{aligned}
|V_2(\tau)| &\leq e^{-\alpha(\tau - \alpha_2/\epsilon)} J_{V1} + |h_2| \int_{\tau}^{\alpha_2/\epsilon} e^{-\alpha(\tau - s)} ds + e^{\alpha(\tau - \beta_2/\epsilon)} J_{V2} + |h_2| \int_{\beta_2/\epsilon}^{\tau} e^{\alpha(\tau - s)} ds \quad (5.33) \\
&\leq C_1(|h_2| + |J_{V1}| + |J_{V2}|)
\end{aligned}$$

$$\begin{aligned}
|V_1^1(\tau)| &\leq e^{-\alpha(\tau - \alpha_1)} J_{V4} + |h_1| \int_{\tau}^{\alpha_1} e^{-\alpha(\tau - s)} ds \\
&+ e^{\alpha(\tau - \beta_1)} J_{V1} + |h_1| \int_{\beta_1}^{\tau} e^{\alpha(\tau - s)} ds \\
&\leq e^{-\alpha(\tau - \alpha_1)} J_{V4} + e^{\alpha(\tau - \beta_1)} J_{V1} + (e^{\alpha(\tau - \beta_1)} - e^{-\alpha(\tau - \alpha_1)}) |h_1| / \alpha \\
&\leq C_2(|h_1| + |J_{V4}| + |J_{V1}|)
\end{aligned}$$

These above estimates result in the estimates for the solution V^i. $\square$

Lemma 5.1.6. *In regular layers $[\alpha_i, \beta_i], i = 2, 4$, assume (H1): the linear composite map $\Phi^4(\beta_4, \alpha_4)\Phi^2(\beta_2, \alpha_2)$ is non-degenerate, that is to say: 1 is not an eigenvalue of the map $\Phi^4(\beta_4, \alpha_4)\Phi^2(\beta_2, \alpha_2)$, then there exists a unique solution to the following system of equations with jump conditions:*

$$\begin{aligned} \dot{Y}_i &= (G_Y - G_U F_U^{-1} F_Y) Y_i + h_i(t) \\ Y_2(\alpha_2) - Y_4(\beta_4) &= JY_1 \\ Y_4(\alpha_4) - Y_2(\beta_2) &= JY_2 \end{aligned} \tag{5.34}$$

with estimate:

$$|Y_2| + |Y_4| \leq C(|JY_1| + |JY_2| + |h_2| + |h_4|) \tag{5.35}$$

Proof. The solution to (5.34) is:

$$Y_i(t) = \Phi^i(t, \alpha_i) Y_i(\alpha_i) + \int_{\alpha_i}^{t} \Phi^i(t, s) h_i(s) ds$$

Thus,

$$Y_i(\beta_i) = \Phi^i(\beta_i, \alpha_i) Y_i(\alpha_i) + \int_{\alpha_i}^{\beta_i} \Phi^i(\beta_i, s) h_i(s) ds$$

Plug this into the above jump conditions, we have:

$$\begin{aligned} Y_2(\alpha_2) - \Phi^4(\beta_4, \alpha_4) Y_4(\alpha_4) &= JY_1 + \int_{\alpha_4}^{\beta_4} \Phi^4(\beta_4, s) h_4(s) ds \\ Y_4(\alpha_4) - \Phi^2(\beta_2, \alpha_2) Y_2(\alpha_2) &= JY_2 + \int_{\alpha_2}^{\beta_2} \Phi^2(\beta_2, s) h_2(s) ds \end{aligned} \tag{5.36}$$

which can be rewritten as:

$$AX = B$$

where

$$A = \begin{pmatrix} I_2 & -\Phi^4 \\ -\Phi^2 & I_2 \end{pmatrix}, X = \begin{pmatrix} Y_2(\alpha_2) \\ Y_4(\alpha_4) \end{pmatrix},$$
$$B = \begin{pmatrix} JY_1 + \int_{\alpha_4}^{\beta_4} \Phi^4(\beta_4, s) h_4(s) ds \\ JY_2 + \int_{\alpha_2}^{\beta_2} \Phi^2(\beta_2, s) h_2(s) ds \end{pmatrix}$$

According to assumption (H1), we have $\det(A) = \det(I_2)\det(I_2 - (-\Phi^2)(-\Phi^4)) = \det(I_2 - \Phi^2\Phi^4) \neq 0$, therefore we can solve for a unique X as the initial value for the solution $Y_i(t), i = 2, 4$. Next we give an estimate of the solution by $X = A^{-1}B$:

$$|Y_i(\alpha_i)| \leq C_1(|JY_1| + |JY_2| + |h_2| + |h_4|), i = 2, 4$$

Then by substituting the initial value into the solution by method of variation of parameter, we obtain:

$$|Y_2| + |Y_4| \leq C(|JY_1| + |JY_2| + |h_2| + |h_4|)$$

□

We want to verify by the following Lemma the assumption (H1), which is true in general, especially when the solutions on the slow manifolds are close to the equilibria $P_\pm$ in our case.

Lemma 5.1.7. *Suppose $A \in R^{n\times n}$ is an invertible real matrix. If $|B - A| < 1/|A^{-1}|$, then B is invertible.*

Proof. Observe that $B = A + (B - A) = A[I + A^{-1}(B - A)]$. If $x \neq 0$, we have

$$|A^{-1}(B - A)x| \leq |A^{-1}||B - A||x| < |x|$$

this indicates that $[I + A^{-1}(B - A)]x \neq 0$, therefore $Bx \neq 0$ since A is invertible. For square matrices, invertibility is equivalent to the condition that $\{x : Bx = 0\} = \{0\}$, so B is invertible.

□

Notice that if we have two pieces of stable spirals close to the equilibrium points on $S_\pm$, then $|\Phi^4(\beta_4, \alpha_4)\Phi^2(\beta_2, \alpha_2)| = |(\Phi^4(\beta_4, \alpha_4)\Phi^2(\beta_2, \alpha_2) - I) - (-I)| << 1 = 1/|I^{-1}|$, which means that 1 is not an eigenvalue of $\Phi^4(\beta_4, \alpha_4)\Phi^2(\beta_2, \alpha_2)$ according to Lemma 5.1.7. If the two pieces of stable spirals are not close to the equilibrium points on $S_\pm$, we can show that in most cases that 1 is not an eigenvalue.

Before proving the following theorem, I give a general outline of the proof: (1) I first construct a generalized solution (U_i, Y_i) that allows a gap at $\tau = 0$ but satisfies the jump conditions exactly by Lemma 5.1.5, 5.1.6, and 5.1.1. During this step, we eliminate the residual error caused by dropping the $\epsilon\frac{d}{dt}[F_U^{-1}F_Y\bar{Y}]$ term with iteration method. (2) We use the Melnikov integral to eliminate the gap by shifting the y values in the gap function $g^i, i = 1, 3$ by Lemma 5.1.3. The change of y value results in the updated domains for the solutions on the outer layers. Then I obtain a solution $(\hat{U}_i, \hat{Y}_i)$ on the updated domain as the exact solutions that satisfy the jump conditions exactly with no gap at $\tau = 0$. (3) I give estimates for the exact solutions.

Theorem 5.1.8. *Assume (H1) hold, then there exists unique periodic solutions* (U_i, Y_i) *in* $[\bar{\alpha}_i, \bar{\beta}_i], i = 1, 2, 3, 4,$ *to the linear variational system (5.14), (5.17) with jump conditions* $\bar{J}_{wi} = (\bar{J}_{Ui}, \bar{J}_{Yi}), i = 1, 2, 3, 4$ *according to* (5.5), (5.6), (5.8), (5.11)*:*

$$\begin{aligned} Y_4(\bar{\alpha}_4) - Y_3(\epsilon\beta_3) &= -\bar{J}_{Y3} \\ Y_3(\epsilon\alpha_3) - Y_2(\bar{\beta}_2) &= -\bar{J}_{Y2} \end{aligned} \tag{5.37}$$

$$\begin{aligned} Y_2(\bar{\alpha}_2) - Y_1(\epsilon\beta_1) &= -\bar{J}_{Y1} \\ Y_1(\epsilon\alpha_1) - Y_4(\bar{\beta}_4) &= -\bar{J}_{Y4} \end{aligned} \tag{5.38}$$

$$\begin{aligned} U_4(\bar{\alpha}_4/\epsilon) - U_3(\beta_3) &= -\bar{J}_{U3} \\ U_3(\alpha_3) - U_2(\bar{\beta}_2/\epsilon) &= -\bar{J}_{U2} \end{aligned} \tag{5.39}$$

$$\begin{aligned} U_2(\bar{\alpha}_2/\epsilon) - U_1(\beta_1) &= -\bar{J}_{U1} \\ U_1(\alpha_1) - U_4(\bar{\beta}_4/\epsilon) &= -\bar{J}_{U4} \end{aligned} \tag{5.40}$$

and phase condition $\dot{w}_i^{ap}(0) \perp w_i(0)$*, with estimate:*

$$\begin{aligned} &\Sigma_{i=1}^4 |U_i| + \Sigma_{i=1}^4 |Y_i| \\ &\leq C(|\bar{P}_2| + |\bar{P}_4| + |\bar{Q}_2| + |\bar{Q}_4| + |\bar{P}_1| + |\bar{P}_3| + |\bar{Q}_1|_{L_1} + |\bar{Q}_3|_{L_1} + \Sigma_{i=1}^4 |\bar{J}_{wi}|)) \end{aligned} \tag{5.41}$$

Proof. The linear variational system of equations is autonomous. So if $w(t)$ is a solution then $w(t+k)$ is also a solution, where k is a constant. Without loss of generality, after a proper time shift we assume that at time t=0 the solution is in a cross section T_i that is transverse to the flow as in the phase condition $w_i(0) \in T_i$, where $T_i := \{x | \langle \dot{w}_i^{ap}(0), x \rangle = 0\}$.

(1) We solve for the generalized solutions $U_i(\tau), Y_i(t), i = 1, 2, 3, 4$ that allow gaps for $U_i(0), i = 1, 3$ and satisfy the phase condition.

First the system (5.14) is coupled on $I^i, i = 2, 4$, we need to use the change of variables to have it decoupled as in (5.15).

Notice that the homogeneous part of the equation $V' = A^i(\tau)V + \bar{P}^i, i = 2, 4$, which is the V equation in (5.15), has an exponential dichotomy on $I^i, i = 2, 4$. Thus according to Lemma

5.1.5, there exist solutions $\bar{V}_i$ that satisfy $|\bar{V}_i| \leq C(JV_i + |\bar{P}_i|)$. Here

$$\begin{aligned}
\bar{V}_4(\alpha_4/\epsilon) - \bar{V}_3(\beta_3) &= -J_{V3} \\
\bar{V}_3(\alpha_3) - \bar{V}_2(\beta_2/\epsilon) &= -J_{V2} \\
\bar{V}_2(\alpha_2/\epsilon) - \bar{V}_1(\beta_1) &= -J_{V1} \\
\bar{V}_1(\alpha_1) - \bar{V}_4(\beta_4/\epsilon) &= -J_{V4}
\end{aligned} \tag{5.42}$$

The homogeneous part of the Y equation in (5.15): $\dot{Y}_i = MY_i, i = 2, 4$ has an exponential dichotomy on I^i, where

$$M(t) = G_Y - G_U F_U^{-1} F_Y = -\frac{1}{s} \begin{pmatrix} -1 + 1/h'(u_i^0(t)) & 1 \\ -\beta & 0 \end{pmatrix}$$

Thus, if $\sigma(M_\pm) \neq 0$, where $M_+ = M(\beta_2/\epsilon), M_- = M(\alpha_2/\epsilon)$, then according to Lemma 2.1.1, Lemma 5.1.6, the Y equation in (5.15) has an exponential dichotomy on I^i. Therefore the solution to the Y equation in (5.15) after plug in $\bar{V}_i$ satisfies the jump conditions:

$$\begin{aligned}
JY_1 &= \bar{Y}_2(\alpha_2) - \bar{Y}_4(\beta_4) \\
&= J_{Y1} + J_{Y4} + \bar{Y}_1(\epsilon\beta_1) - \bar{Y}_1(\epsilon\alpha_1) \\
&= J_{Y1} + J_{Y4} + \int_{\epsilon\alpha_1}^{\epsilon\beta_1} \bar{Q}_1(t)dt \\
&= J_{Y1} + J_{Y4} + |\bar{Q}_1|_{L_1} \\
JY_2 &= \bar{Y}_4(\alpha_4) - \bar{Y}_2(\beta_2) \\
&= J_{Y2} + J_{Y3} + |\bar{Q}_3|_{L_1}
\end{aligned} \tag{5.43}$$

with estimate:

$$|\bar{Y}_i| \leq C(|\bar{V}_i(\tau)| + JY_i + |\bar{Q}_i|), i = 2, 4$$

Now we define $\bar{U}_i(\tau) = \bar{V}_i(\tau) - F_U^{-1}(\epsilon\tau)F_Y(\epsilon\tau)\bar{Y}_i(\epsilon\tau)$, which satisfies

$$\bar{U}_i' = F_U\bar{U}_i + F_Y\bar{Y}_i + \bar{P}_i - \frac{d}{d\tau}[F_U^{-1}F_Y Y]$$

Notice that the first equation of (5.14) is not satisfied by $\bar{U}^i$ because of the error term $\frac{d}{d\tau}[F_U^{-1}F_Y Y]$, we have the estimate of the $\frac{d}{d\tau}[F_U^{-1}F_Y Y]$ as:

$$|\frac{d}{d\tau}[F_U^{-1}F_Y\bar{Y}]| = \epsilon|\frac{d}{dt}[F_U^{-1}F_Y\bar{Y}]| = K\epsilon(|\bar{V}_i| + |\bar{Q}^i| + |JY_i|) \leq K\epsilon(|\bar{P}^i| + |\bar{Q}^i| + |JY_i| + |JV_i|)$$

Now we have $(\bar{U}_i, \bar{Y}_i), i = 2,4$ is a good approximation with residual errors $O(\epsilon(|\bar{P}^i|+|\bar{Q}^i|+|J_i|))$ in the U equation, therefore we can obtain the generalized solution $U_i(\tau), Y_i(t), i = 2,4$ to (5.14), (5.17) by iteration process.

On the other hand, according to (5.17), we have:

$$\begin{aligned} Y_i(t) = S^i(t,\alpha_i)Y_i(\alpha_i) + \int_{\alpha_i}^{t} S^i(t,s)\bar{Q}_i(s)ds \\ + S^i(t,\beta_i)Y_i(\beta_i) + \int_{\beta_i}^{t} S^i(t,s)\bar{Q}_i(s)ds, i = 1,3 \end{aligned}$$

where $Y_i(\alpha_i), Y_i(\beta_i), i = 1,3$ are given by passing the boundary values of $Y^{2,4}$ through the jump conditions:

$$\begin{aligned} Y_4(\alpha_4) - Y_3(\epsilon\beta_3) &= -J_{Y3} \\ Y_3(\epsilon\alpha_3) - Y_2(\beta_2) &= -J_{Y2} \\ Y_2(\alpha_2) - Y_1(\epsilon\beta_1) &= -J_{Y1} \\ Y_1(\epsilon\alpha_1) - Y_4(\beta_4) &= -J_{Y4} \end{aligned} \tag{5.44}$$

We have the estimate:

$$|Y_i(t)| \leq C|\bar{Q}_i|_{L_1}$$

Next we plug in the $Y_i, i = 1,3$ into the U_i equations to solve:

$$U_i'(\tau) = F_U U_i + h_i(\tau) \tag{5.45}$$

with boundary conditions $U_i(\alpha_i), U_i(\beta_i), i = 1,3$ given by:

$$\begin{aligned} U_4(\alpha_4/\epsilon) - U_3(\beta_3) &= -J_{U3} \\ U_3(\alpha_3) - U_2(\beta_2/\epsilon) &= -J_{U2} \\ U_2(\alpha_2/\epsilon) - U_1(\beta_1) &= -J_{U1} \\ U_1(\alpha_1) - U_4(\beta_4/\epsilon) &= -J_{U4} \end{aligned} \tag{5.46}$$

and the bounded forcing term $h_i(\tau) = F_Y(Y_i(0) + \int_0^{\epsilon\tau} \bar{Q}_i(s)ds) + \bar{P}^i$. Next we give estimates for U_i based on (5.22):

$$|U_i(\tau)| \leq C(|U_i(\alpha_i)| + |U_i(\beta_i)| + |h_i(\tau)|) \leq C(|J_w| + |\bar{P}_i| + |\bar{Q}_i|_{L1[-\epsilon^\lambda,\epsilon^\lambda]})$$

Now we have the generalized solutions $U_i, Y_i, i = 1,2,3,4$ that allow a gap at $\tau = 0$ for $U_i, i =$

$1, 3$.

(2) Here we want to obtain the exact solutions $\hat{Y}_i, \hat{U}_i, i = 1, 2, 3, 4$. In order to solve for the exact $\hat{Y}_i, \hat{U}_i, i = 2, 4$, we apply Lemma 5.1.3 by modifying the $y_i(0), i = 1, 3$ in the g^i function. After this shift of the equal gap surface, we obtain a $\hat{y}_i$ such that $g(\hat{y}_i(0)) = 0$. We replace the y value in $Y_i(0)$ by $\hat{y}_i(0)$ such that $\hat{Y}_i(0) = (\hat{y}_i(0), z_i(0)), i = 1, 3$. Now $\hat{Y}_i(t), i = 2, 4$ need extra time Δt_i to get to the shifted equal gap surface, Δt_i can be used to update the domain of the solutions for $\hat{w}_i(t)$, $t \in [\bar{\alpha}_i, \bar{\beta}_i] = [\alpha_i + \Delta\alpha_i, \beta_i + \Delta\beta_i], i = 2, 4$, on which we repeat the procedure in step (1) to obtain $\hat{V}_i(\tau)$ on the updated domain with the following jump conditions according to (5.5), (5.6), (5.8), (5.11):

$$\begin{aligned} \hat{V}_4(\bar{\alpha}_4/\epsilon) - V_3(\beta_3) &= -\bar{J}_{V3} \\ V_3(\alpha_3) - \hat{V}_2(\bar{\beta}_2/\epsilon) &= -\bar{J}_{V2} \end{aligned}$$

$$\begin{aligned} \hat{V}_2(\bar{\alpha}_2/\epsilon) - V_1(\beta_1) &= -\bar{J}_{V1} \\ V_1(\alpha_1) - \hat{V}_4(\bar{\beta}_4/\epsilon) &= -\bar{J}_{V4} \end{aligned}$$

By the change of variable, we have $\hat{U}_i(\tau)$ which eliminates the gap and is transversal at $\tau = 0$. Now we have the updated jump conditions satisfied:

$$\begin{aligned} \hat{U}_4(\bar{\alpha}_4/\epsilon) - U_3(\beta_3) &= -\bar{J}_{U3} \\ U_3(\alpha_3) - \hat{U}_2(\bar{\beta}_2/\epsilon) &= -\bar{J}_{U2} \end{aligned}$$

$$\begin{aligned} \hat{U}_2(\bar{\alpha}_2/\epsilon) - U_1(\beta_1) &= -\bar{J}_{U1} \\ U_1(\alpha_1) - \hat{U}_4(\bar{\beta}_4/\epsilon) &= -\bar{J}_{U4} \end{aligned}$$

After plugging in the $\hat{V}_i(\tau), i = 2, 4$ into the Y equation of (5.15), based on (H1) 1 is not an eigenvalue of $\Phi^2(\bar{\beta}_2, \bar{\alpha}_2)\Phi^4(\bar{\beta}_4, \bar{\alpha}_4)$, which is true according to the continuous dependence of semigroup to t. We obtain $\hat{Y}_i(t), i = 2, 4$ by Lemma 5.1.6 with the following jump conditions of Y(similar to (5.43)) satisfied:

$$\begin{aligned} \hat{Y}_2(\bar{\alpha}_2) - \hat{Y}_4(\bar{\beta}_4) &= \hat{J}Y_1 \\ \hat{Y}_4(\bar{\alpha}_4) - \hat{Y}_2(\bar{\beta}_2) &= \hat{J}Y_2 \end{aligned} \tag{5.47}$$

Next we obtain $\hat{Y}_i, \hat{U}_i, i = 1, 3$ as in step (1). Now we have the exact solution $\hat{Y}_i(t), \hat{U}_i(\tau), i = 1, 2, 3, 4$.

(3) Now we give an estimate of $\hat{Y}_i(t), \hat{U}_i(\tau), i = 1,2,3,4$. According to (5.33), we consider (5.15) and have:

$$|\hat{V}_2| \leq C(|\bar{\bar{P}}_2| + |J_{V1}| + |J_{V2}|) \leq C(\bar{P}_2 + |J_{V1}| + |J_{V2}|)$$
$$|\hat{V}_4| \leq C(|\bar{\bar{P}}_4| + |J_{V3}| + |J_{V4}|) \leq C(\bar{P}_4 + |J_{V3}| + |J_{V4}|)$$

According to the change of variables $\hat{U}_i(\tau) = \hat{V}_i(\tau) - F_U^{-1}(\epsilon\tau)F_Y(\epsilon\tau)\hat{Y}_i(\epsilon\tau)$, we obtain estimate for $\hat{U}_i$ $i = 2,4$: $|\hat{U}_i| \leq C(\bar{Q}_i + \Sigma_{i=1}^2 |JY_i| + |\hat{V}_i|)$, therefore

$$|\hat{U}_2| \leq C(\bar{Q}_2 + \Sigma_{i=1}^2 |JY_i| + \bar{P}_2 + |J_{V1}| + |J_{V2}|)$$
$$|\hat{U}_4| \leq C(\bar{Q}_4 + \Sigma_{i=1}^2 |JY_i| + \bar{P}_4 + |J_{V3}| + |J_{V4}|)$$

We apply Lemma (5.1.6) with jump condition (5.47) and forcing term $\hat{h}_i = G_U\hat{V}_i(\tau) + \bar{Q}_i$, we have the system for solving the initial value $AX = B$ with:

$$B = \begin{pmatrix} JY_1 + \int_{\alpha_4/\epsilon}^{\beta_4} \Phi^4(t,s)\hat{h}_4(s)ds \\ -JY_2 + \int_{\alpha_2/\epsilon}^{\beta_2} \Phi^2(t,s)\hat{h}_2(s)ds \end{pmatrix}$$

Now we obtain exact solutions for $\hat{Y}_i(t), i = 2,4$, with estimate according to (5.35)

$$\begin{aligned} |\hat{Y}_i(t)| &\leq C_1(|JY_1| + |JY_2| + |\hat{h}_2| + |\hat{h}_4|) \\ &\leq C_1(|JY_1| + |JY_2| + \bar{P}_2 + \bar{Q}_2 + \bar{P}_4 + \bar{Q}_4 + \sum_1^4 |J_{Vi}|) \end{aligned} \tag{5.48}$$

Now we obtain the estimate of $\hat{Y}_i(t), \hat{U}_i(\tau), i = 1,2,3,4$

□

5.2 Periodic solution for the original system

Now we prove that there is a periodic solution for the original nonlinear system, based on the previous Theorem 5.1.8 about the existence of the periodic solution to the linear variational system. We prove that a contraction mapping has a fixed point as the correction solutions and the extra time $(\{w_i\}_{i=1}^4, \{\zeta_i\}_{i=1}^4)$.

Theorem 5.2.1. *Suppose* (U_i^{ap}, Y_i^{ap}) *is given as the approximation solution that satisfies (1.5), (1.6), with period* $\omega^{ap} = \Sigma_{l=1}^2[\epsilon\beta_{2l-1} - \epsilon\alpha_{2l-1}] + \Sigma_{l=1}^2[\beta_{2l} - \alpha_{2l}]$. *Also assumptions (H1) is* *satisfied. Then there exists a unique exact periodic solution w^{ex} of (1.3), (1.4) with period $\omega^{ex} = \Sigma_{i=1}^{4}(\bar{\beta}_i - \bar{\alpha}_i)$ such that*

$$|U^{ex} - U^{ap}| + |Y^{ex} - Y^{ap}| = O(\epsilon^{\lambda}). \tag{5.49}$$

Proof. Define the residual error:

$$\begin{aligned} p^i(\tau) &= U^{ap\prime} - F(U^{ap}, Y^{ap}, s) \\ q^i(t) &= \dot{Y}^{ap} - G(U^{ap}, Y^{ap}, s) \end{aligned}$$

with estimates:

$$\begin{aligned} p^i(\tau) &= O(\epsilon^{\lambda}) \quad i = 1,3 & \qquad p^i(\tau) &= O(\epsilon) \quad i = 2,4 \\ q^i(t) &= O(\epsilon^{0}) \quad i = 1,3 & \qquad q^i(t) &= O(\epsilon) \quad i = 2,4 \end{aligned} \tag{5.50}$$

Moreover $|q^i(t)|_{L_1(-\epsilon^{\lambda},\epsilon^{\lambda})} = O(\epsilon^{\lambda}), i = 1,3$.

We define the jump conditions according to (5.38), (5.37), (5.39), (5.40):

$$w_{i+1}(A_{i+1}) - w_i(B_i) + w_{i+1}^{ap(1)}(A_{i+1})\Delta A_{i+1} - w_i^{ap(1)}(B_i)\Delta B_i = \mathcal{J}_i, i = 1,2,3,4 \tag{5.51}$$

where $w^{(1)} = (dU/d\tau, dY/dt)^T$, and

$$A_i = \begin{cases} \alpha_i & i = 1,3 \\ \alpha_i/\epsilon & i = 2,4 \end{cases} \quad B_i = \begin{cases} \beta_i & i = 1,3 \\ \beta_i/\epsilon & i = 2,4 \end{cases}$$

$$\begin{aligned} \mathcal{J}_i &= \mathcal{J}_i(J_{wi}, w, \Delta A_{i+1}, \Delta B_i) \\ &= w_{i+1}(A_{i+1}) - w_i(B_i) - [w_{i+1}(A_{i+1} + \Delta A_{i+1}) - w_i(B_i + \Delta B_i)] \\ &- w_{i+1}^{ap}(A_{i+1} + \Delta A_{i+1}) + w_i^{ap}(B_i + \Delta B_i) + w_{i+1}^{ap}(A_{i+1}) - w_i^{ap}(B_i) \\ &- J_{wi} + w_{i+1}^{ap(1)}(A_{i+1})\Delta A_{i+1} - w_i^{ap(1)}(B_i)\Delta B_i \\ &= O(|w_{i+1}^{(1)}||\Delta A_{i+1}| + |w_i^{(1)}||\Delta B_i| + |w_{i+1}^{ap(2)}||\Delta A_{i+1}|^2 + |w_i^{ap(2)}||\Delta B_i|^2 + |J_{wi}|) \\ &= O((|w_{i+1}| + |\bar{P}_{i+1}| + |\bar{Q}_{i+1}|)|\Delta A_{i+1}| + (|w_i| + |\bar{P}_i| + |\bar{Q}_i|)|\Delta B_i| + |\Delta A_{i+1}|^2 + |\Delta B_i|^2 + |J_{wi}|) \end{aligned}$$

where $J_{wi} = w^{ap}_{i+1}(A_{i+1}) - w^{ap}_i(B_i)$, with estimate:

$$J_{wi} = O(\epsilon^\lambda) \tag{5.52}$$

(5.51) can be rewritten as:

$$w_{i+1}(A_{i+1}) - w_i(B_i) + \zeta_i\gamma_i = \mathcal{J}_i, i = 1, 2, 3, 4 \tag{5.53}$$

$\gamma_i = (F(A_{i+1}), G(A_{i+1}))^T, \zeta_i = \delta(A_{i+1})$ for $i = 1, 3$
$\gamma_i = (F(B_i), G(B_i))^T, \zeta_i = -\delta(B_i)$ for $i = 2, 4$
Recall that $\delta(A_i) = \delta(B_i) = 0, i = 1, 3$. Next we look for solutions $(\{w_i\}^4_{i=1}, \{\zeta_i\}^4_{i=1})$ of system: (5.15), (5.17), (5.53), in the Banach space:

$$(\{w_i\}^4_{i=1}, \{\zeta_i\}^4_{i=1}) \in \Pi^4_{i=1}C^1[\alpha_i - \delta, \beta_i + \delta] \times \Pi^4_{i=1}R$$

with the norm defined as

$$|| \cdot || = \sup_{1\leq i\leq 4} |w_i| + \sup_{1\leq i\leq 4} |\zeta_i|$$

Define an open subset $\mathcal{O}(\delta)$ as:

$$\mathcal{O}(\delta) = \{(\{w_i\}^4_{i=1}, \{\zeta_i\}^4_{i=1}) : ||\{w_i\}, \{\zeta_i\}|| < \delta\}$$

According to Theorem (5.1.8), we have:

$$\begin{aligned}(\{w_i\}^4_{i=1}, \{\zeta_i\}^4_{i=1}) &= \mathcal{A}^{-1}(\{\bar{P}_i, \bar{Q}_i\}^4_{i=1}, \{\mathcal{J}_i\}^4_{i=1},) \\ &= \mathcal{L}(\{w_i\}^4_{i=1}, \{\zeta_i\}^4_{i=1}, \{p_i\}^4_{i=1}, \{q_i\}^4_{i=1}, \{J_{wi}\}^4_{i=1})\end{aligned} \tag{5.54}$$

where $\mathcal{L}$ is a mapping in the Banach space: $\Pi^4_{i=1}C^1[\bar{\alpha}_i, \bar{\beta}_i] \times \Pi^4_{i=1}R$ to itself.
Now we look for a fixed point for the mapping $\mathcal{L}$. According to (5.16) to (5.19) and (5.41), if $(\{w_i\}^4_{i=1}, \{\zeta_i\}^4_{i=1}) \in \mathcal{O}(\delta)$, then we have

$$\begin{aligned}&|\bar{P}_2| + |\bar{P}_4| + |\bar{Q}_2| + |\bar{Q}_4| + |\bar{P}_1| + |\bar{P}_3| + (|\bar{Q}_1|_{L_1} + |\bar{Q}_3|_{L_1}) \\ &\leq K(\delta^2 + \epsilon\delta + \epsilon^\lambda)\end{aligned} \tag{5.55}$$

Now we can verify that given a very small δ, we can choose a sufficiently small ϵ so that the right hand side of (5.54) lies in $\mathcal{O}(\delta)$. Also we can directly show that $\mathcal{L}$ is an contraction mapping in $\mathcal{O}(\delta)$. Therefore, there exists a unique fixed point for (5.54). Also the estimate (5.49) follows according to (5.41), (5.52), (5.50), (5.55) and Theorem (5.1.8). □

Chapter 6

Chaotic Solutions

In the previous chapter we obtain a periodic solution with two slow flows on $S_\pm$ being two segments of monotone curves on the z-y plane. In this chapter we construct a solution near a pair of heteroclinic solutions, the solution spirals around the two equilibrium points with prescribed number of rotations on $S_\pm$ on the z-y plane.

6.1 Formal solutions when $\epsilon = 0$.

First of all, we construct a formal solution composed of two fast orbits and two slow orbits pieced together alternatively. Those two slow orbits and two fast orbits form a formal solution. See Figure 6.1.

Based on Lemma 4.2.2, we have a pair of fast solutions on $(-\infty, \infty)$ when $\epsilon = 0$ with the same wave speed $s > 0$, and they are on the surfaces of opposite small y_0 values. On the other hand, in order to determine the slow orbits on $S_\pm$, we notice that $0 < h'(u) < 1$, so that we have a pair of stable(unstable) spirals as slow orbits with the same $s > 0 (s < 0)$, if β is sufficiently large, $4\beta > k^2$. The two slow orbits start from points P_i or P_j on $y = \pm y_0$ and ends on each other. See Figure 6.2.

According to the projection of the solution in Figure 6.2 onto the y-z plane, we see that the stable spirals start from $P_j(P_i)$ and approach the other equilibrium (goes beyond $P_j(P_i)$) as $t \to \infty$. Recall that we only consider the two slow flows on $S_\pm$ within the two foldlines $y = -y_m$ and $y = y_m$. We observe from Figure 6.3 that there are solutions starting at a point within the two foldlines(say P_-), but do not approach $P_\pm$(say P_+) as $t \to \pm\infty$. Therefore we consider the domain of influence of $P_\pm$(or domain of attraction of $P_\pm$), that contains all the initial points

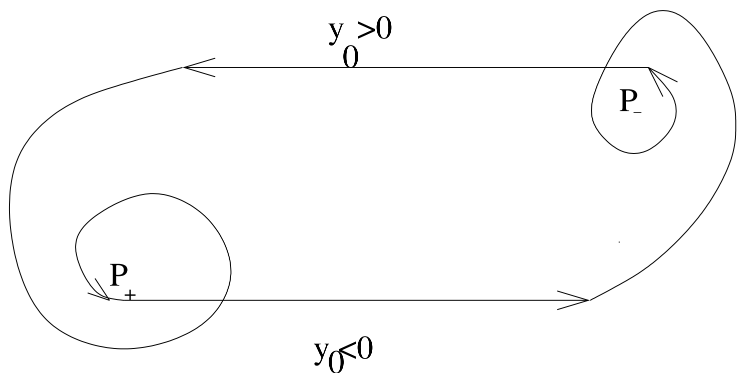


Figure 6.1: Formal solutions when $\epsilon = 0$

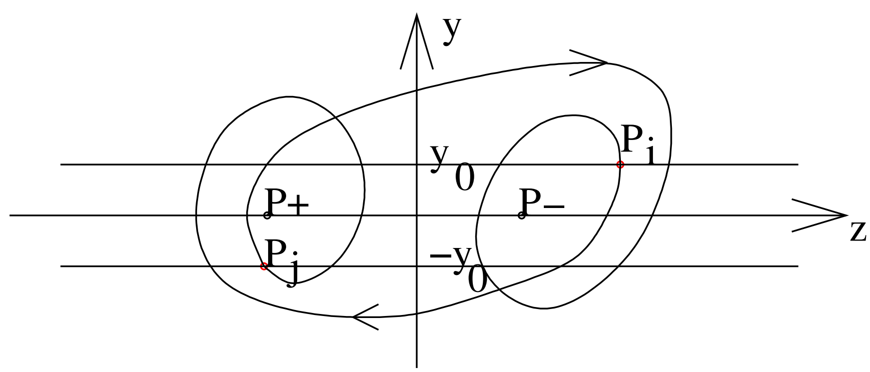


Figure 6.2: Projection of formal solutions onto z-y plane when $\epsilon = 0$

(y_0, z_0) passing which the solutions exist for all $t \leq 0$(or for all $t \geq 0$), and will approach $P_{\pm}$ as $t \to -\infty$(or as $t \to \infty$). See Figure 6.3, the domain of attraction of stable spirals on $S_{\pm}$ is bounded by the two foldlines.

Theorem 6.1.1. *For different values of β, equilibrium $P_{\pm}$ are stable spirals if $4\beta > k^2, sk > 0$(unstable spirals if $4\beta > k^2, sk < 0$). Moreover,*
for $s > 0, t > 0$, the contour of domain of attraction of stable spirals on $S_{\pm}$ is the solution with initial point at $A_{\pm} = (\pm(2\sqrt{3}c^3m/9 + c/\sqrt{3}), \pm(2\sqrt{3}c^3m/9))$ in the z-y plane;
for $s < 0, t < 0$, the contour of domain of influence of unstable spirals on $S_{\pm}$ is the solution with initial point at $A_{\pm} = (\pm(2\sqrt{3}c^3m/9 + c/\sqrt{3}), \pm(2\sqrt{3}c^3m/9))$ in the z-y plane.

Proof. We observe from Figure 6.3 that the vector field is horizontal on the initial points $A_{\pm}$, where the solution is tangent to the foldlines. See Figure 6.4 for 3D view. Take a point to the

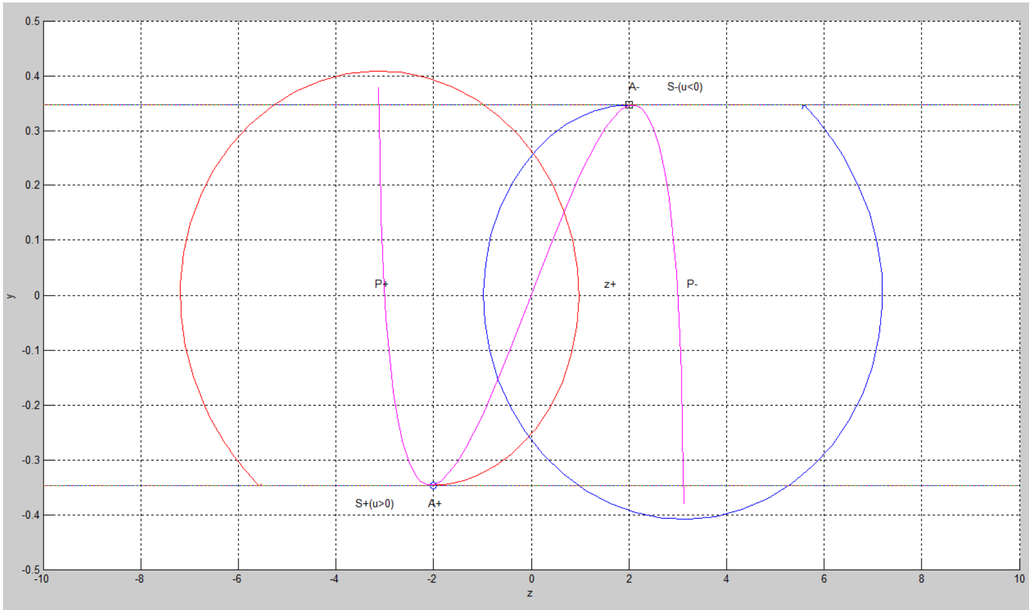


Figure 6.3: Domain of attraction of stable spirals on $S_\pm$

left of B_+ on the foldline $y = y_m$, the vector field points left and up. But the solution starts there will go back down and hit the foldline $y = y_m$, it can't go across the foldline, therefore will not be able to go towards P_+. To the right of A_+ on the foldline $y = y_m$, the vector field points left and down. In fact, when $u = -c/\sqrt{3}, y = -2\sqrt{3}c^3m/9, z > -2\sqrt{3}c^3m/9 + c/\sqrt{3}$, we have $-s\dot{y} = u - y + z > 0$, therefore $\dot{y} < 0$ for $s > 0$, so the solutions starting there will go below the foldlines when $t > 0$ and can't approach P_+ as $t \to \infty$. These indicate that the solutions start on $A_\pm$ are the contours of domain we want.

$\square$

Notice that it is possible that P_- is not contained in the domain of attraction of P_+. Therefore, we need the following theorem to guarantee the existence of the formal solution.

Theorem 6.1.2. *The existence of the heteroclinic solution $q(\tau)$ from an equilibrium point P_- connected by fast flow to the stable spirals towards P_+ is guaranteed, if and only if the equilibrium point P_- is contained in the domain of attraction on S_+ in between the foldlines when projected in y-z plane. There is a critical value $\beta = \beta_0$ when the contours of domain of attraction(stable spirals) exactly hit $P_\pm$, See figure 6.6. Also when $\beta \geq \beta_0$ one equilibrium point is contained in the domain of attraction of the other equilibrium point. See Figure 6.5.*

Proof. Consider $\beta > k^2/4, sk > 0$ where we have stable spirals on $S_\pm$, also consider $\frac{dy}{dz} = \frac{u-y+z}{-\beta y}$

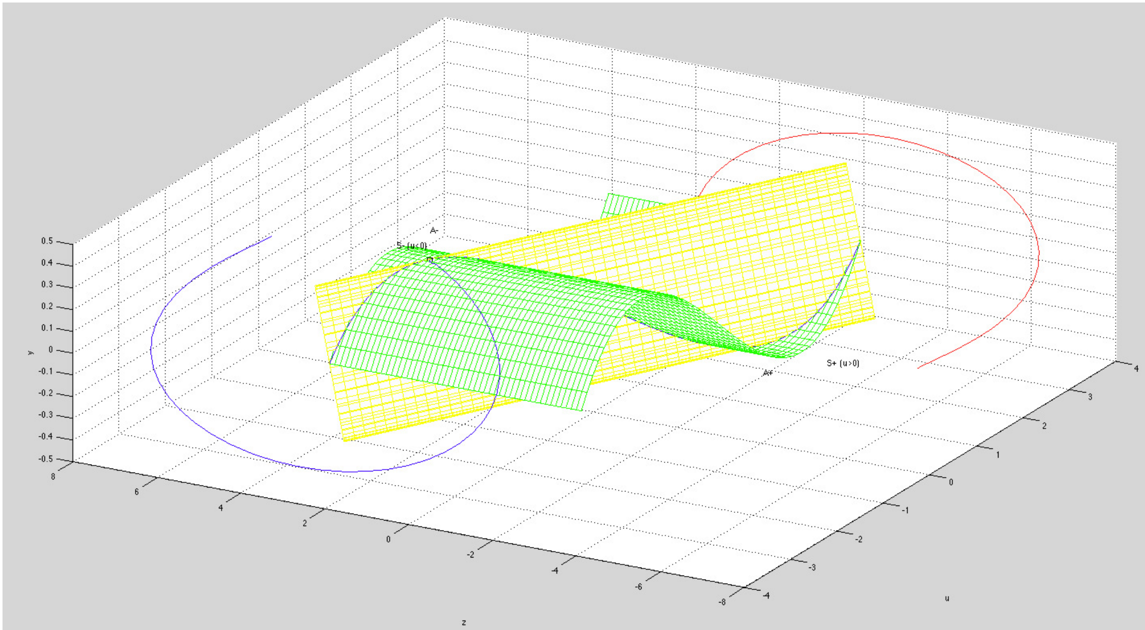


Figure 6.4: Domain of influence of unstable spirals on $S_{\pm}$ in 3D

in the region $D = \{(z, y)|u - y + z > 0, y < 0\}$ such that $\frac{dy}{dz}$ decreases when β increases on the point (z,y). Therefore we compare the differential equations $\frac{dy_i}{dz} = f_i(\beta_i, z), i = 1, 2$ with different vector fields, by comparison theorem in [2] we have: if $\beta_2 > \beta_1$, then $f_2(\beta_2, z) < f_1(\beta_1, z)$ and $y_2 < y_1$. So if the domain of attraction doesn't contain P_-, we can increase β value continuously so that the contour of the domain of attraction around P_+ hits the z-axis exactly at P_- when $\beta = \bar{\beta}$. Now we take $\beta_0 = max\{\bar{\beta}, k^2/4\}$, then when $\beta \geq \beta_0$ one equilibrium point is contained in the domain of attraction of the other equilibrium point. □

Now that we have the formal solutions as in Figure 6.1, we can obtain an ordered sequence of solutions $\bar{q}_k(\tau), k \in Z$ in between P_j, P_i by repeating the formal solutions.

6.2 Existence of exact chaotic solution when $\epsilon > 0$

6.2.1 Approximation solutions and jump conditions

We first define the counting surface to be $Z_+ = \{(z, y)|y = y_0\}, Z_- = \{(z, y)|y = -y_0\}$, in order to keep track of the intersections of the solutions with $Z_{\pm}$ around $P_{\pm}$. Based on the formal solution $\bar{q}_k(\tau)$ obtained from the previous section when $\epsilon = 0$, we define $Q_i(Q_j), i_2 \geq$

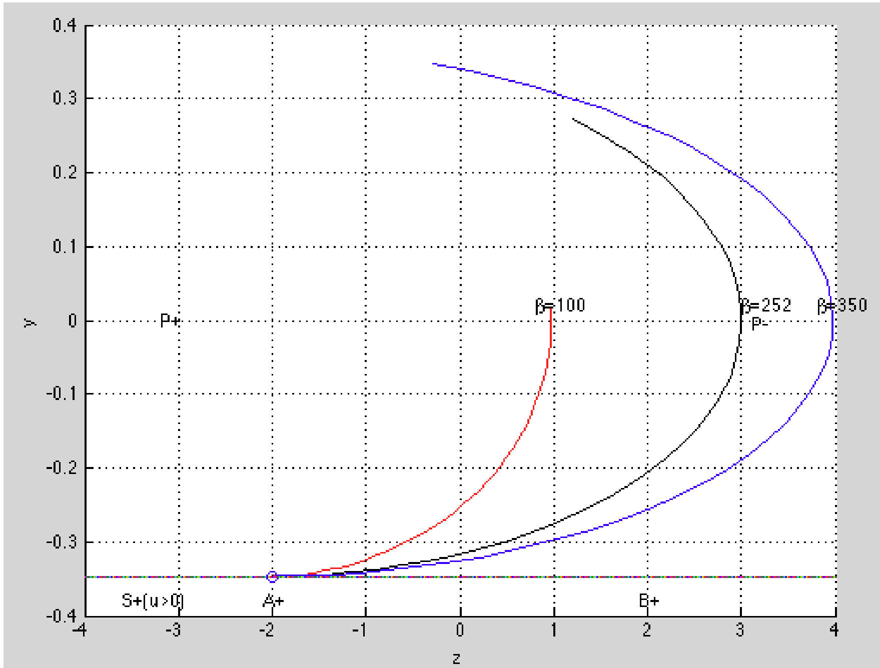


Figure 6.5: The domain of attraction becomes larger as β increases

$i \geq i_1, j_2 \geq j \geq j_1$ to be the points, where $\bar{q}_k(\tau)$ intersects with $Z_\pm$ for the i-th (j-th) time, and are close enough to $P_-(P_+)$ for $i \geq i_1, j \geq j_1$. See Figure 6.7.

Given symbol (j, i), we define a family of approximated solutions $q^k_{ji}(t)$ when $\epsilon > 0$ by truncating the formal solutions:

$$q^{ji}_k = \begin{cases} q^{slow}(t) & t \in [\alpha_l, \beta_l] = [\epsilon^{1/2}, \gamma_k], l = 2 \\ q^{fast}(\tau) & \tau \in [\alpha_l, \beta_l] = [-\gamma_k/\epsilon, \epsilon^{-1/2}], l = 1 \end{cases}$$

$q^{ji}_k(t), t \in [-\gamma_k, \gamma_k]$ starts at $q_k(-\gamma_k) = P_j \in O(Q_j)$ a neighborhood of Q_j on S_+ and ends at $P_i = q_k(\gamma_k) \in O(Q_i)$, a neighborhood of Q_i on S_-. Also notice that $q^{ji}_k(t)$ intersects with Z_- i times on S_- as the perturbation of the formal solution $\bar{q}_k(\tau)$.

When we construct the exact chaotic solution, the points $W_i(W_j)$ where the chaotic solution intersects with the $Z_-(Z_+)$ for the i-th(j-th) time on the z-y plane must be close to the equilibria $P_-(P_+)$. Therefore, we should only consider the $W_i(W_j)$ points within the neighborhoods $O(P_\pm) = \{Y | dist(Y, P_\pm) < \delta\}$ to be relevant for counting, see Figure 6.8. Next we define the extended neighborhood: $\mathcal{O}_- = \{O(Q_1), O(Q_2), O(Q_3), \ldots, O(Q_{\bar{i}-1}), O(P_-)\}$ such that $W_i \in O(Q_i) \subset O(P_-)$, for $i \geq \bar{i}$ on S_-. Similarly we define $\mathcal{O}_+$ for W_j on S_+.

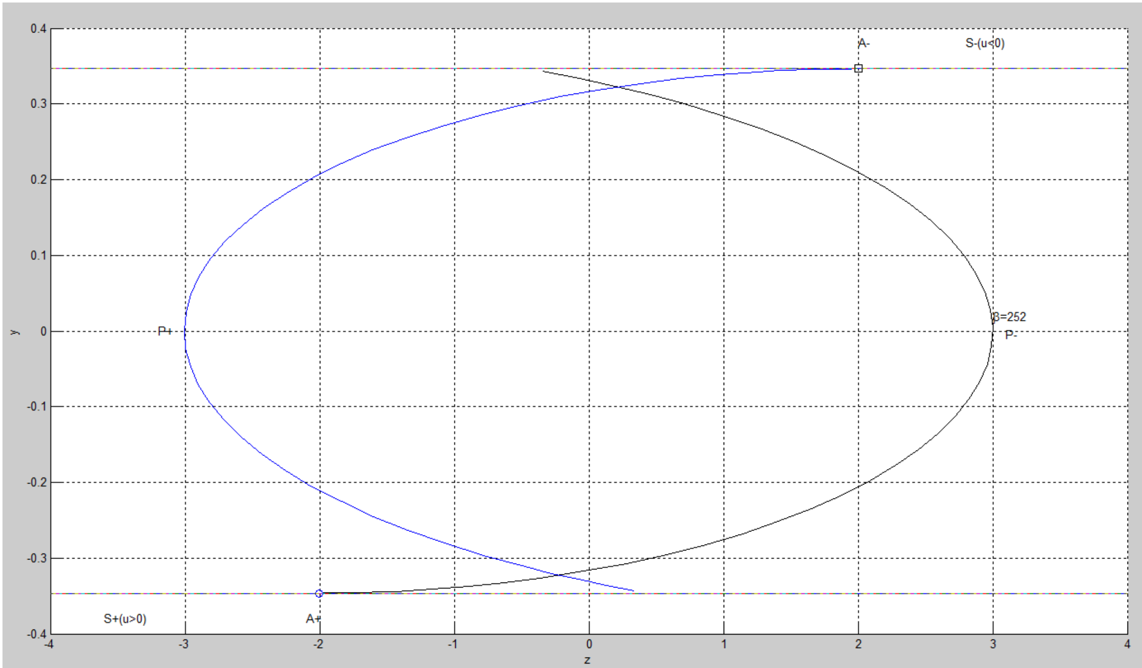


Figure 6.6: Stable spirals around one equilibrium that exactly hit the other equilibrium at $P_{\pm}$

Given $i(j)$, the number of intersections of the solution with $Z_{-}(Z_{+})$ on $S_{-}(S_{+})$, we want to construct an exact chaotic solution w^{ex} when $\epsilon > 0$ based on the *ordered* approximated solutions $q_k^{ap}(t) = q_k^{ji}, t \in [-\gamma_k, \gamma_k], k \in Z$, such that $w_k^{ex} = q_k^{ap} + w_k$, where $w = (U, Y)$ is the correction solution. Notice that the approximation q_k^{ji} intersects with Z_{-} exactly i times, the exact solution near q_k^{ji} must intersect with Z_{-} exactly i times as well.

Before we obtain the correction solution, we figure out the jump conditions at the junction of two adjacent solutions for the correction solutions. First we define the jump conditions at the junction of two adjacent solutions for the approximated solutions:

$$U_k^{ap}(-\gamma_k/\epsilon) - U_{k-1}^{ap}(\gamma_{k-1}/\epsilon) = JU_k, \tag{6.1}$$

$$Y_k^{ap}(-\gamma_k) - Y_{k-1}^{ap}(\gamma_{k-1}) = JY_k \tag{6.2}$$

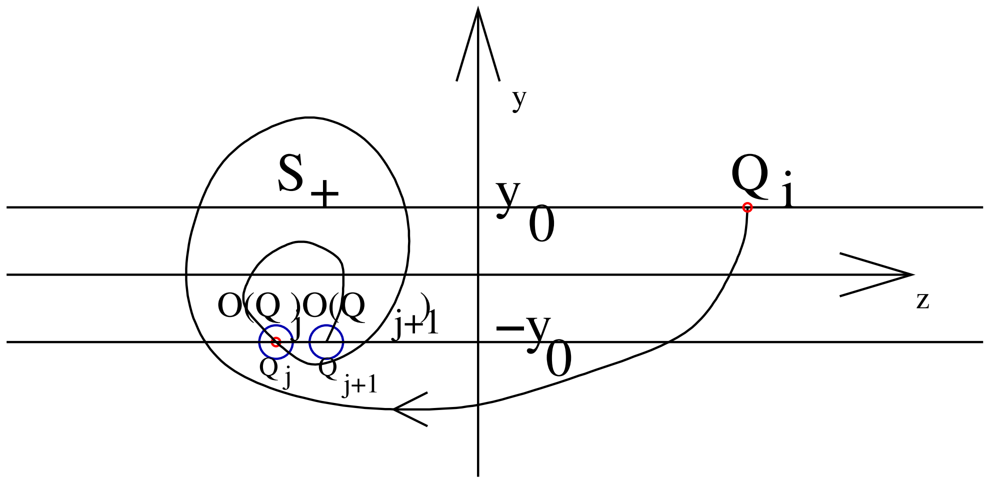


Figure 6.7: The formal solution $\bar{q}_k$ intersects with the Z_+ at points Q_j on S_+ for the j-th time

In order to obtain an exact solution where $Y_k^{ex}(-\gamma_k) = Y_{k-1}^{ex}(\bar{\gamma}_{k-1})$, we notice that:

$$\begin{aligned}
&Y_k^{ex}(-\gamma_k) - Y_k^{ap}(-\gamma_k) = Y_k(-\gamma_k)\\
&Y_{k-1}^{ex}(\bar{\gamma}_{k-1}) - Y_{k-1}^{ap}(\gamma_{k-1})\\
&= [Y_{k-1}^{ex}(\bar{\gamma}_{k-1}) - Y_{k-1}^{ap}(\bar{\gamma}_{k-1})] + [Y_{k-1}^{ap}(\bar{\gamma}_{k-1}) - Y_{k-1}^{ap}(\gamma_{k-1})]\\
&= Y_{k-1}(\bar{\gamma}_{k-1}) + \delta Y_{k-1}(\gamma_{k-1})
\end{aligned}$$

Subtracting the above two equations gives us:

$$Y_k(-\gamma_k) - Y_{k-1}(\bar{\gamma}_{k-1}) = -JY_k + \delta Y_{k-1}(\gamma_{k-1}) = -\hat{J}Y_k \tag{6.3}$$

Similarly we have:

$$U_k(-\gamma_k/\epsilon) - U_{k-1}(\bar{\gamma}_{k-1}/\epsilon) = -JU_k + \delta U_{k-1}(\gamma_{k-1}/\epsilon) = -\hat{J}U_k \tag{6.4}$$

Lemma 6.2.1. *Let $\Omega_1 = \{y = (\ldots y_{-1}, y_0, y_1 \ldots) : |y_i - y_i(0)| \le \delta y_i, i \in Z\}$ be a rectangle in R^∞. $H : \Omega_1 \to R^\infty$*

$$H(y) = (\ldots, g^{-1}(y), g^0(y), g^1(y), \ldots)$$

is continuous with $g^i > 0 (< 0)$ if $y_i = y_i(0) + \delta y_i (y_i = y_i(0) - \delta y_i)$. Then there exists a $\hat{y} \in \Omega_1$ with $H(\hat{y}) = 0$.

Proof. Let $H_1(y) = (\ldots, y_{-1} - \delta y_{-1} g^{-1}(y)/|g^{-1}|, y_0 - \delta y_0 g^0(y)/|g^0|, y_1 - \delta y_1 g^1(y)/|g^1|, \ldots)$ be

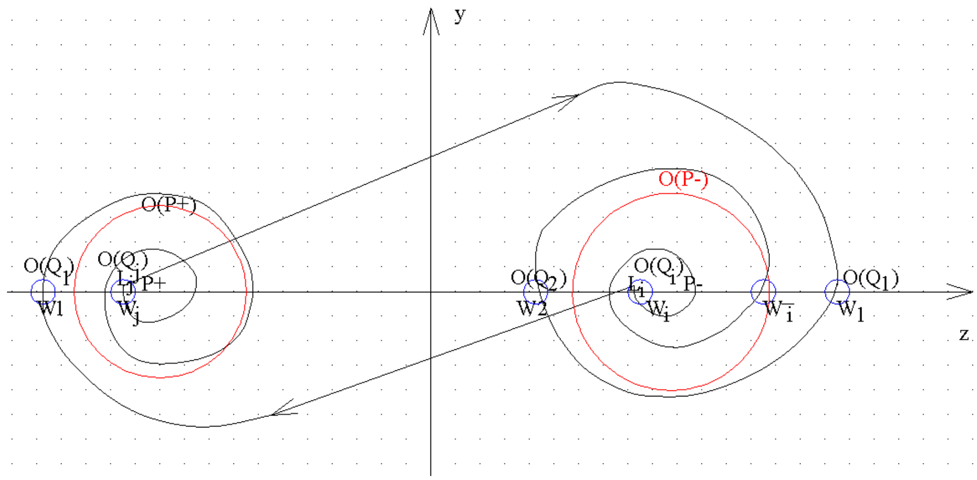


Figure 6.8: The chaotic solution intersects with the Z_- at points W_i on S_-.

a mapping from Ω_1 to Ω_1. There exists a fixed point $\hat{y} \in \Omega_1$ for H_1 by Schauder fixed point theorem. Therefore, there exists a $\hat{y} \in \Omega_1$ with $H(\hat{y}) = 0$. $\square$

6.2.2 Proof of existence of the solution to the linear variational system

We give an outline of the proof. (1) We first obtain a generalized solution $w_k, k \in Z$, that allows a gap at $\tau = 0$ but satisfies the jump conditions by Lemma 5.1.5 and 5.1.1. During this step, we eliminate the residual error caused by dropping the $\epsilon \frac{d}{dt}[F_U^{-1} F_Y \bar{Y}]$ term with iteration method. (2) We use the Melnikov integral to eliminate the gap by shifting the y values in the gap function g^k by Lemma 6.2.1, the change of y value results in the updated domains for the solutions. Then we obtain solutions $\hat{U}_k$ on the updated domain as the exact solutions that satisfy the jump conditions JU_k exactly with no gap at $\tau = 0$. However, the JY_k on the updated domain is not satisfied exactly. After we define the Y_k^{i+1}, the difference of jump errors $E(JY_k^{i+1})$ is reduced by a multiple of a small number in the i-th iteration, due to the contraction caused by the stable spiral near the equilibrium points. Therefore, the exact solution can be obtained after iterations.

Theorem 6.2.2. *There exists unique solution $w_k = (U_k(\tau), Y_k(t)), t \in [-\gamma_k, \bar{\gamma}_k]$ to the linear variational system (5.14) with jump conditions $J_k = (JU_k, JY_k)$ according to* (6.3), (6.4)*:*

$$U_k(-\gamma_k/\epsilon) - U_{k-1}(\bar{\gamma}_{k-1}/\epsilon) = -\hat{JU}_k, \tag{6.5}$$

$$Y_k(-\gamma_k) - Y_{k-1}(\bar{\gamma}_{k-1}) = -\hat{JY}_k \tag{6.6}$$

and phase condition $\dot{q}_k^{ap}(0) \perp w_k(0)$. We also have the estimate:

$$|U_k| + |Y_k| \leq C(|\bar{P}_k| + |\bar{Q}_k| + |\hat{J}_k|)$$

Proof. The linear variational system of equations is autonomous. So if $w(t)$ is a solution then $w(t+k)$ is also a solution, where k is a constant. Without loss of generality, after a proper time shift we assume that at time t=0 the solution is in a cross section T_i that is transverse to the flow as in the phase condition $w_i(0) \in T_i$, where $T_i := \{x | \langle \dot{q}_i^{ap}(0), x\rangle = 0\}$.

(1) We want to solve for a generalized solution $(U_k(\tau), Y_k(t)), t \in [-\gamma_k, \gamma_k]$.

We first solve for $\bar{V}_k(\tau), \tau \in [-\gamma_k/\epsilon, \gamma_k/\epsilon]$ in (5.15) according to Lemma 5.1.5, with the $\epsilon \frac{d}{dt}[F_U^{-1} F_Y Y]$ term dropped and $H_k = \bar{P}_k, \bar{V}_k(-\gamma_k/\epsilon) - \bar{V}_{k-1}(\gamma_k/\epsilon) = -JV_k$, also we obtain estimates $|\bar{V}_k(\tau)| \leq C(JV_k + |\bar{P}_k|)$.

Next we solve for $\bar{Y}_k(t)$ with the initial condition $\bar{Y}_k(-\gamma_k) = \bar{Y}_{k-1}(\gamma_{k-1}) - JY_k$ after we plug in $\bar{V}_k(\tau)$ into the Y equation of (5.15). We also have the estimates $|\bar{Y}_k(t)| \leq C(|\bar{V}_k(\tau)| + JY_k + |\bar{Q}_k|)$.

Now we define $\bar{U}^k(\tau) = \bar{V}_k(\tau) - F_U^{-1}(\epsilon\tau) F_Y(\epsilon\tau) \bar{Y}^k(\epsilon\tau)$ which satisfies:

$$\bar{U}_k' = F_U \bar{U}_k + F_Y \bar{Y}_k + \bar{P}_k - \frac{d}{d\tau}[F_U^{-1} F_Y \bar{Y}] \tag{6.7}$$

Notice that the first equation of (5.14) is not satisfied by $\bar{U}^k$ because of the error term $\frac{d}{d\tau}[F_U^{-1} F_Y \bar{Y}]$ in (6.7), we have the estimate of the $\frac{d}{d\tau}[F_U^{-1} F_Y \bar{Y}]$ as:

$$|\frac{d}{d\tau}[F_U^{-1} F_Y \bar{Y}]| = \epsilon|\frac{d}{dt}[F_U^{-1} F_Y \bar{Y}]| = K\epsilon(|\bar{V}^k| + |\bar{Q}^k| + |JY^k|) \leq K\epsilon(|\bar{P}^k| + |\bar{Q}^k| + |J_k|)$$

Now we have $(\bar{U}^k, \bar{Y}^k)$ is a good approximation with residual errors $O(\epsilon(|\bar{P}^k| + |\bar{Q}^k| + |J_k|))$ in the U equation. Therefore, we can obtain the generalized solution $U_k(\tau), Y_k(t)$ to (5.14) with jump conditions (6.5) by iteration process. We have the estimates for $U_k(\tau)$ as:

$$|U_k| \leq C(|V_k| + |Y_k|) \leq C(|\bar{P}_k| + |\bar{Q}_k| + |J_k|)).$$

(2) However the above solution $U_k(\tau), Y_k(t)$ is still a generalized solution that allows a gap at $\tau = 0$ for U as:

$$U^i(0+) - U^i(0-) = g^i d^i$$

according to Lemma 5.1.1. Here the gap is defined as:

$$g^k(y(0)) \\ = \int_{-\gamma_k}^{\gamma_k} < \psi^k(s), H^k(s) > ds + < \psi^k(-\gamma_k), P_s^k(-\gamma_k)U^k(-\gamma_k) > - < \psi^k(\gamma_k), P_u^k(\gamma_k)U^k(\gamma_k) > \tag{6.8}$$

where $y(0) = (\ldots y^{-1}(0), y^0(0), y^1(0)\ldots)$. Considering q_k is the perturbation of the heteroclinic solution $q(t)$, we have $G_y = \{\frac{\partial g^k}{\partial y_j}\}$ is almost a diagonal matrix similar to (5.29). Therefore g^k mainly depends on $y^k(0)$. According to the higher dimensional Intermediate Value Theorem Lemma 6.2.1, we obtain $\hat{y}(0)$ with $\hat{y}^k(0) \in [y^k(0)-\delta y^0, y^k(0)+\delta y^0]$ such that $g^k(\hat{y}(0)) = 0, k \in Z$. Now we have eliminated the gaps for U at at $\tau = 0$ by shifting the equal gap surface and the y values.

(3) After the change of y values, $\hat{Y}_k(t)$ needs extra time Δt_k to get to the shifted equal gap surface, which can be used to update the domain of the solutions $\hat{w}_k(t)$, $t \in [-\gamma_k, \bar{\gamma}_k], \bar{\gamma}_k = \gamma_k + \Delta\gamma_k$. We repeat the procedure on $[-\gamma_k, \bar{\gamma}_k]$ in step (1) to obtain $\hat{V}_k(\tau)$ on the updated domain. By the change of variables, we have $\hat{U}_k(\tau)$, which eliminates the gap at $\tau = 0$ and satisfies the jump condition:

$$-J\hat{V}_k = \hat{V}_k(-\gamma_k/\epsilon) - \hat{V}_{k-1}(\bar{\gamma}_{k-1}/\epsilon)$$

Next we want to obtain the $\hat{Y}_k$ solutions that satisfy the linear variational system and the jump condition by iteration method. First, we compare the jump errors of two adjacent Y^1 solutions Y_k^1, Y_{k-1}^1: $-JY_{k-1}^1 = Y_k^1(-\gamma_k) - Y_{k-1}^1(\gamma_{k-1})$ with the updated jump errors on the updating the domains:

$$-JY_{k-1}^2 = Y_k^1(-\gamma_k) - Y_{k-1}^2(\bar{\gamma}_{k-1})$$

We obtain the difference of the above jump errors caused by the extra time $\Delta\gamma_{k-1}$ to be

$$E(JY_{k-1}^2) = JY_{k-1}^2 - JY_{k-1}^1 = [Y_{k-1}^2(\bar{\gamma}_{k-1}) - Y_{k-1}^1(\gamma_{k-1})]$$

In order to reduce the difference of the jump errors $E(JY_{k-1}^2)$, we define the initial value $Y_k^2(-\gamma_k)$ for $Y_k^2(t)$ based on $Y_k^2(-\gamma_k) - Y_k^1(-\gamma_k) := E(JY_k^1) = Y_k^1(\bar{\gamma}_k) - Y_k^0(\gamma_k)$, where $Y_k^0(t)$ is the Y solution obtained in step (1).

$$Y_k^1(t) = Y_k^0(-\gamma_k)S_k(t, -\gamma_k) + \int_{-\gamma_k}^{t} S_k(t, -\gamma_k)H^k(s)ds, -\gamma_k \leq t \leq \bar{\gamma}_k$$

For the general iteration process, see Figure 6.9. We compare the jump errors in the i-th iteration of two adjacent Y^i solutions Y_k^i, Y_{k-1}^i: $-JY_{k-1}^i = Y_k^i(-\gamma_k) - Y_{k-1}^i(\gamma_{k-1}^i)(= BD)$ with the updated jump errors on the updating the domains:

$$-JY_{k-1}^{i+1} = Y_k^i(-\gamma_k) - Y_{k-1}^{i+1}(\bar{\gamma}_{k-1}^i)(= BC)$$

We define the difference of the above jump errors to be:

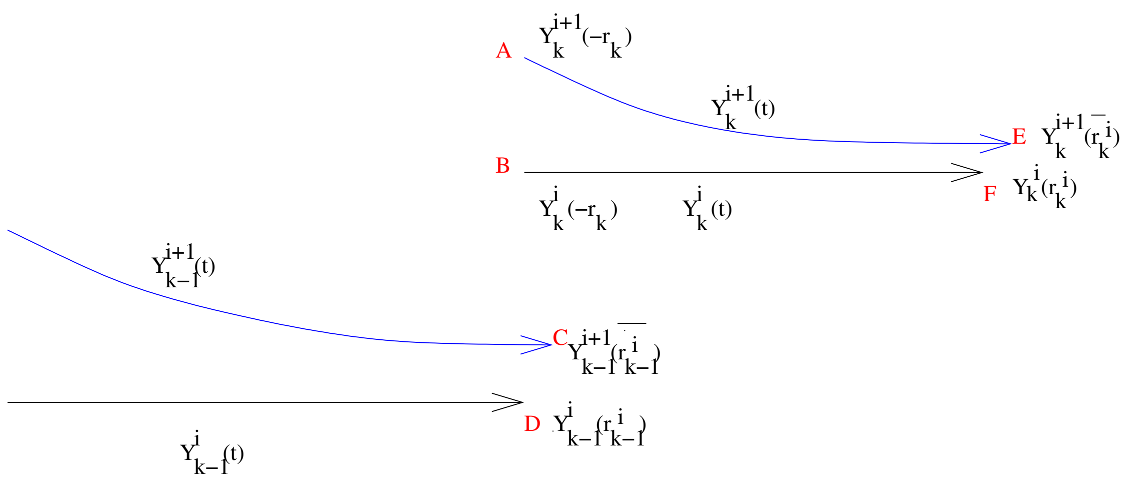


Figure 6.9: The jump errors $JY_{k-1}^i(= BD)$ and updated jump errors $JY_{k-1}^{i+1}(= BC)$ result in the difference of the jump errors $E(JY_{k-1}^{i+1})(= CD)$, which is to be reduced in the next iteration.

$$E(JY_{k-1}^{i+1}) = JY_{k-1}^{i+1} - JY_{k-1}^i = [Y_{k-1}^{i+1}(\bar{\gamma}_{k-1}^i) - Y_{k-1}^i(\gamma_{k-1}^i)](= CD)$$

In order to reduce the difference of the jump errors $E(JY_k^{i+1})(= EF)$, we define the initial value $Y_k^{i+1}(-\gamma_k)$ for $Y_k^{i+1}(t)$ based on $AB = Y_k^{i+1}(-\gamma_k) - Y_k^i(-\gamma_k) := E(JY_k^i) = Y_k^i(\bar{\gamma}_k^i) - Y_k^{i-1}(\gamma_k^i)$, from the $E(JY_k^i)$ in the i-th iteration (previous ith EF).

$$Y_k^{i+1}(-\gamma_k) = Y_k^i(-\gamma_k) + E(JY_k^i)$$

Notice that the Y_k^i solutions are changed to

$$\bar{Y}_k^{i+1}(t) = Y_k^{i+1}(-\gamma_k)S_k(t, -\gamma_k) + \int_{-\gamma_k}^{t} S_k(t, -\gamma_k)H^k(s)ds, -\gamma_k \leq t \leq \gamma_k^i$$

Accordingly, the nonlinear term H in (5.20) that involves Y_k^i is changed. Therefore, we need to repeat step (2) to eliminate the gaps for the generalized solutions at $\tau = 0$ and update the domain as $-\gamma_k \leq t \leq \bar{\gamma}_k^i$, where $\bar{\gamma}_k^i = \gamma_k^i + \Delta\gamma_k^i$. Now we define for $i \geq 1$:

$$Y_k^{i+1}(t) = Y_k^{i+1}(-\gamma_k)S_k(t, -\gamma_k) + \int_{-\gamma_k}^{t} S_k(t, -\gamma_k)H^k(s)ds, -\gamma_k \leq t \leq \bar{\gamma}_k^i$$

when compared with:

$$Y_k^i(t) = Y_k^i(-\gamma_k)S_k(t, -\gamma_k) + \int_{-\gamma_k}^{t} S_k(t, -\gamma_k)H^k(s)ds, -\gamma_k \leq t \leq \gamma_k^i$$

We take the sup norm of the difference of the jump errors and define:

$$\Delta_{i+1} = \sup_{k\in Z} |E(JY_k^{i+1})| = \sup_{k\in Z} |Y_k^{i+1}(\bar{\gamma}_k^i) - Y_k^i(\gamma_k^i)|$$

We give an estimate for the difference of the jump error $E(JY_k^{i+1})(= |EF|$ in Figure (6.9)):

$$\begin{aligned} |E(JY_k^{i+1})| &= [Y_k^{i+1}(\bar{\gamma}_k^i) - Y_k^i(\gamma_k^i)] \\ &= Y_k^i(-\gamma_k)[S_k(\bar{\gamma}_k^i, -\gamma_k) - S_k(\gamma_k^i, -\gamma_k)] + E(JY_k^i)S_k(\bar{\gamma}_k, -\gamma_k) \\ &+ \int_{-\gamma_k}^{\bar{\gamma}_k^i} S_k(\bar{\gamma}_k^i, -\gamma_k)H^k(s)ds - \int_{-\gamma_k}^{\gamma_k^i} S_k(\gamma_k^i, -\gamma_k)H^k(s)ds \\ &\leq Ce^{-2\alpha\gamma_k}|E(JY_k^i)| \end{aligned} \tag{6.9}$$

Here we use the fact that there are stable spirals near equilibria on $S_\pm$ with eigenvalues $-\alpha \pm i\beta, \alpha > 0$ for the linear variational system, so that the principal matrix solution $S_k(t, -\gamma_k) \leq Ce^{-2\alpha(t+\gamma_k)}$ for large enough t.

The existence of an exact solution $\hat{w}_k, k \in Z$, follows by iteration method. In fact, after the i-th iteration, $E(JY_k^i)$ gets reduced by a multiple of an exponentially small number. Therefore, $\Delta_{i+1} = \sup_{k\in Z} |E(JY_k^{i+1})| \leq C_i\Delta_i$ by (6.9), where $C_i << 1$. Moreover, Δ_i gets reduced by a multiple of an exponentially small number. Therefore, $\sum_i |Y_k^{i+1} - Y_k^i| \leq \sum_i \Delta_i < \infty$, and we have $\lim_{i\to\infty} Y_k^i = \hat{Y}_k$. Now we obtain the correction solutions $(\hat{U}_k, \hat{Y}_k)$ to the linear variational system (5.14) with jump conditions (6.5) satisfied. Estimates of the correction solutions follow similarly to those in step (1).

$\square$

Theorem 6.2.3. *In a small neighborhood of q_k, there exists a unique exact chaotic solution*

w^{ex}*, which satisfies* (1.3)*,* (1.4) *with estimates:*

$$|w_k^{ex} - q_k| \le K\epsilon^\lambda \quad 0 < \lambda < 1$$

We can prove this theorem similarly to theorem 5.2.1 by contraction mapping.

6.3 Symbolic dynamics

We want to make correspondence of the solution w^{ex} to a sequence of symbols.
Theorem 6.3.1. *The chaotic solution* w^{ex} *that intersects with* $Z_\pm$ *exactly finite i(j) times on* $S_-(S_+)$ *corresponds to a sequence of symbols* $\{(i,j)\}_{i\ge\bar{i},j\ge\bar{j}}$.

Proof. Given any chaotic solution near the heteroclinic solutions with $W_i \in \mathcal{O}_-(W_j \in \mathcal{O}_+)$, we set $y = 0$ and solve for the t values on which W_i points fall into $O(P_-)$. We can keep track of the symbols (i,j) by putting the W_i points in order according to the orientation of the spirals. For clockwise orientation, W_1 is defined to be the intersection point with the largest z value within $O(P_-)$; W_2 is defined to be the intersection point with the smallest z value within $O(P_-)$; W_3 is defined to be the intersection point with the second largest z value within $O(P_-)$; W_4 is defined to be the intersection point with the second smallest z value within $O(P_-)$, etc. We count up to $W_i(W_j)$.

On the other hand, given sequence of symbols $\{(i,j)\}_{i\ge\bar{i},j\ge\bar{j}}$, we want to obtain a chaotic solution with $W_i \in \mathcal{O}_-(W_j \in \mathcal{O}_+)$ which intersects with the $Z_\pm$ for the prescribed number of times. We first construct the approximated solutions $q_{ji}(q_{ij})$ on $S_-(S_+)$ when $\epsilon = 0$, which intersect with the $Z_\pm$ only i(j) times, then according to Theorem 6.2.3, we obtain the chaotic solution near the approximated solutions corresponding to the symbols.

□

REFERENCES

[1] A. Acosta (1998). Traveling wave solutions in coupled Chua's circuits, *preprint*, 1-39.

[2] M. Budincevic (2010). A comparison theorem of differential equations, *Novi Sad J. Math*, 40(1), 55-56.

[3] J. Billingham and D. J. Needham, A note on the properties of a family of travelling-wave solutions arising in cubic autocatalysis, *Dynamics and Stability of Systems*, 6 (1991), 33-49.

[4] F. Battelli (1994), Heteroclinic orbits in singular systems: a unifying approach, *J. Dyn. Diff. Eqns*, 6, 147-173.

[5] Brouwer fixed-point theorem. *Wikipedia*

[6] S. N. Chow, X. Lin, K. Lu (1991). Smooth Invariant Foliations in Infinite Dimensional Spaces. *J. Differential Equations*, 94, 266-291.

[7] L. O. Chua (1998). CNN: A paradigm for complexity. *World Scientific Series on Nonlinear Science*, Series A - Vol. 31.

[8] W. A. Coppel (1978). Dichotomies in stability theory. *Lectures notes in Math. Springer-Verlag, Berlin*, 629.

[9] A. Doelman and V. Rottschafer (1997). Singularly perturbed and nonlocal modulation equations for systems with interacting instability mechanisms, *J. Nonlinear Sci*, 7, 371-409.

[10] N. Fenichel (1971). Persistence and smoothness of invariant manifolds for flows. *Indiana Univ. Math. Journal*, 21, 193-226.

[11] N. Fenicheln (1979). Geometric singular perturbation theory for differential equations, *J. Differential Equations* 31, 53-98.

[12] N. Fenichel (1977). Asymptotic stability with rate conditions II, *Indiana Univ. Math. J*, 26, 81-93.

[13] P. C. Fife (1974). Transition layers in singular perturbed problems. *J. Differential Equations*, 15, 77-105.

[14] Fu Zhang (2005). Existence of a periodic solution in a Chua's circuit with smooth nonlinearity. *Differential and Integral Equations*, 18(1), 83-120.

[15] J. K. Hale (1980). Ordinary Differential Equations. *Pure and applied mathematics, a series of texts and monographs, XXI.*

[16] J. K. Hale and X. B. Lin (1998). Multiple internal layer solutions generated by spatially oscillatory perturbations. *Journal of Differential Equations*, 154(2), 364-418.

[17] C. Jones (1995). Geometric singular perturbation theory. *Lectures notes in Math. Springer-Verlag, Berlin*, 106(9), 44-118.

[18] C. Jones and J. Rubin (1998). Existence of standing pulse solutions of an inhomogeneous reaction-diffusion system, *J. Dyn. Diff. Eqns*, 10, 1-35.

[19] C. Jones and N. Kopell (1994). Tracking invariant manifolds with differential forms in singularly perturbed systems. J. Differential Equations, 108, 64-88.

[20] R. B. Kellogg (1976). Uniqueness in the Schauder Fixed Point Theorem. *Proceedings of the AMS*, 60(1), 207-210.

[21] G. Kovacic and S. Wiggins (1992). Orbits homoclinic to resonances, with an application to chaos in a model of the forced and damped sine-Gorden equations, *Physica* D 57, 185-225.

[22] M. Krupa, B. Sandstede and P. Szmolyan (1997). Fast and slow waves in the FitzHugh-nagumo equation, *J. Diff. Eqns*. 133, 49-97.

[23] X. B. Lin (1989). Shadowing lemma and singularly perturbed boundary value problems. *SIAM J. Appl. Math*, 49, 26-54. 38

[24] X. B. Lin (1990). Using Melnikov's methods to solve Silnikov's problems. *Proc. Royal Soc. Edinburgh*, 116A, 295-325.

[25] X. B. Lin (1990). Heteroclinic bifurcation and singularly perturbed boundary value problems, *J. Diff. Eqns*. 84, 319-382.

[26] X-B. Lin (2001). Construction and asymptotic stability of structurally stable internal layer solutions, *Transactions of the American Mathematical Society*, 353(8), 2983-3043.

[27] X-B. Lin, Lin's method. *http://www.scholarpedia.org/article/Lin's method*

[28] J. Mallet-Paret (1999).The Fredholm alternative for functional differential equations of mixed type, *J. Dyn. Diff. Eq*. 11, 1-47.

[29] K. J. Palmer (1984). Exponential dichotomies and transversal homoclinics points. *J. Differential Equations*, 55, 225-256.

[30] K. J. Palmer (1986). Transversal heteroclinic points and Cherry's example of a nonintegrable hamiltonian system. *J. Diff. Eqs*. 65, 321-360.

[31] V. Perez-Munuzuri, V. Perez-Villar and L. O. Chua (1993). Traveling wave front and its failure in a one-dimensional array of Chua's circuit. *Chua's circuit. A paradigm for Chaos, World Scientific Singapore*, 336-350.

[32] Lawrence Perko (1996). Differential Equations and Dynamical Systems, *Springer Verlag*

[33] P. Szmolyan (1991). Transversal heteroclinic and homoclinic orbits in singular perturbation problems. *J. Differential Equations*, 92, 252-281.

[34] P. Szmolyan. M. Wechselberger (2001). Canards in R^3, *Journal of Differential Equations*, 177(2), 419-453.