

# ABSTRACT

NORBROTHEN, EMMA MICHELE. On Classifying the Double Cosets  $H_k \backslash G_k / H_k$  of  $\mathrm{SL}(2, k)$ . (Under the direction of Aloysius Helminck.)

Symmetric spaces are defined as group  $G/H$ , where  $G$  is a reductive group over an algebraically closed field and  $H$  is the fixed point group of an involution  $\theta$ , and are important particularly in mathematics and physics. Recently the study of symmetric spaces has begun to expand to arbitrary fields and these generalizations are called symmetric  $k$ -varieties. Similarly, symmetric  $k$ -varieties are defined as the group  $G_k/H_k$ , where  $G_k$  and  $H_k$  are the  $k$ -points of  $G$  and  $H$ , and  $k$  is a field that is not necessarily algebraically closed.

A problem of importance in representation theory and algebraic group theory is to describe the action of  $H_k$  on the symmetric  $k$ -variety  $G_k/H_k$ , which can be seen as the double cosets  $H_k \backslash G_k / H_k$ . In the Riemannian symmetric space there is the Cartan decomposition  $G = HAH$  of the group  $G$ , where  $A$  is a maximal  $\theta$ -split torus of  $G$ . Additionally, in real Riemannian symmetric spaces, all  $A$  are  $H$ -conjugate. In symmetric  $k$ -varieties, the Cartan decomposition no longer holds and not all  $A$  are necessarily  $H_k$ -conjugate. In this thesis, we study the action of  $H_k$  on  $G_k/H_k$  for  $G = \mathrm{SL}(2, k)$  by studying the  $H_k$ -action on maximal  $\theta$ -split tori in  $G_k$ . In particular, we study the  $H_k$ -action on maximal  $\theta$ -split  $k$ -anisotropic tori, with an emphasis on the finite and  $p$ -adic fields.

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On Classifying the Double Cosets  $H_k \backslash G_k / H_k$  of  $\mathrm{SL}(2, k)$

by  
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## BIOGRAPHY

Emma Michele Norbrothen was born in Huntington, NY, to Trish Weller and Ken Norbrothen. Shortly after the birth of her brother, Ben, the family moved to Sarasota, FL. There, Emma learned many important life lessons, like how to find sand dollars and how to tie her shoes.

As Emma and Ben began elementary school, the family moved to MA. They started in Groton, and after a year, moved to Boxborough. Emma also learned many important life lessons in these towns, like the ideas that a Halloween costume should always be winter-ready and students quiet down when fed treats.

Several years later, Emma's parents divorced and separately moved to Acton, MA. This was a very good change for Emma, because it later meant that Matt and Karina Martel would join her family as her stepfather and sister. Moreover, Emma joined the Martel family, giving her the big-family experience she never had before and always wanted. Her dad got a cat, giving her the cat experience she never had before and always wanted.

After graduating high school, Emma migrated south to attend Gettysburg College in Gettysburg, PA. She majored in mathematics and minored in physics, and made excellent friends in both departments. During all four years at Gettysburg, Emma worked in the Office of Residence Life, making several more excellent friends, and appreciating another aspect of the college. Fall of her junior year, Emma studied abroad and attended Lancaster University in Lancaster, England. During her summers, Emma traveled to Coe College, in Cedar Rapids, IA, and Rose-Hulman Institute of Technology, in Terre-Haute, IN, to participate in Research Experience for Undergraduates programs. Throughout college, Emma learned many important lessons, like the value of traveling and the importance of asking questions.

Emma wanted to earn a PhD in mathematics in order to become a professor, and decided to attend North Carolina State University in Raleigh, NC. She chose to work with Loek Helminck, who supported her teaching ambitions, mentored her as a student mentor in another Research Experience for Undergraduates program, and, most importantly, fed her love for cosets with a research project. While at NCSU, Emma started dating Justin Wright, and before long the couple realized they were happier together than they had ever been before. In January of their final year, after Emma made Justin a particularly

special birthday lunch, Justin asked Emma for her hand in marriage.

Next, Emma and Justin will move to Meredith, NH, where they will begin their lives together and their careers as professors at Plymouth State University.

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# TABLE OF CONTENTS

<b>LIST OF TABLES . . . . .</b>	<b>viii</b>
<b>Chapter 1 Symmetric Variety Background . . . . .</b>	<b>1</b>
1.1 A Class of Reductive Symmetric Spaces . . . . .	2
1.2 Our Approach to Symmetric Varieties . . . . .	3
1.3 Riemannian Symmetric Spaces . . . . .	7
1.3.1 The Subgroup $H_k$ is Maximal Compact . . . . .	7
1.3.2 All Elements are Semisimple . . . . .	9
1.3.3 All Elements are $k$ -split . . . . .	11
1.3.4 Maximal $\theta$ -split Tori are Maximal $k$ -split . . . . .	12
1.3.5 All Maximal $(\theta, k)$ -split Tori are $H_k$ -conjugate . . . . .	13
1.3.6 The Weyl Group has Representatives in $H_k$ . . . . .	14
<b>Chapter 2 The <math>H_k</math>-orbits in <math>Q_k</math> Background . . . . .</b>	<b>18</b>
2.1 The $H_k$ -Action . . . . .	19
2.2 The $H_k$ -orbits in $Q_k$ over Algebraically Closed Fields . . . . .	20
2.3 The $H_k$ -orbits in $Q_k$ over General Fields . . . . .	21
2.4 The $k$ -split Tori . . . . .	23
2.5 Involutions over $\mathrm{SL}(2, k)$ . . . . .	25
2.6 The $(\theta, k)$ -split Tori in $\mathrm{SL}(2, k)$ over General Fields . . . . .	29
2.7 The $\theta$ -split $k$ -tori in $\mathrm{SL}(2, k)$ over General Fields . . . . .	31
<b>Chapter 3 Preliminary Results about <math>H_k</math>-conjugation Classes in <math>\mathfrak{sl}(2, k)</math> . . . . .</b>	<b>35</b>
3.1 Characterizing Tori . . . . .	35
3.2 The $H_k$ -conjugacy Classes of Type 2 Tori . . . . .	38
3.2.1 The $H_k$ -conjugacy Classes of Type 2 Tori when $k = \mathbb{R}$ . . . . .	39
3.2.2 The $H_k$ -conjugacy Classes of Type 2 Tori when $k = \mathbb{Q}$ . . . . .	39
3.2.3 The $H_k$ -conjugacy Classes of Type 2 Tori when $k = \mathbb{F}_p$ . . . . .	40
3.2.4 The $H_k$ -conjugacy Classes of Type 2 Tori when $k = \mathbb{Q}_p, p \neq 2$ . . . . .	41
<b>Chapter 4 Characterizing <math>H_k</math>-conjugacy Classes of Type 3 Tori . . . . .</b>	<b>43</b>
4.1 Characterizing $H_k$ -conjugacy Classes of Type 3 Tori . . . . .	43
4.2 The $H_k$ -conjugacy Classes of Type 3 Tori when $k = \mathbb{Q}$ . . . . .	52
<b>Chapter 5 Type 3 Tori in <math>\mathfrak{sl}(2, k)</math> when <math>k = \mathbb{F}_p</math> . . . . .</b>	<b>53</b>
5.1 Examples of Type 3 Tori . . . . .	53
5.2 The $H_k$ -conjugacy Classes of Type 3 Tori . . . . .	54
5.3 The $p \equiv 1, m \equiv N_p$ Case . . . . .	59
5.4 The $p \equiv 3, m \equiv 1$ Case . . . . .	62
5.5 The $p \equiv 1, m \equiv 1$ Case . . . . .	65



5.6	The $p \equiv 3, m \equiv N_p$ Case . . . . .	67
5.7	Example $H_k$ -conjugation Tables . . . . .	69
5.7.1	Example $H_k$ -The $H_k$ -conjugation Classes when $p \equiv 1$ and $m \equiv N_p$ . . . . .	69
5.7.2	Example $H_k$ -The $H_k$ -conjugation Classes when $p \equiv 3$ and $m \equiv 1$ . . . . .	72
5.7.3	Example $H_k$ -The $H_k$ -conjugation Classes when $p \equiv 1$ and $m \equiv 1$ . . . . .	75
5.7.4	Example $H_k$ -The $H_k$ -conjugation Classes when $p \equiv 3$ and $m \equiv N_p$ . . . . .	78
<b>Chapter 6</b>	<b>Type 3 Tori in <math>\mathfrak{sl}(2, k)</math> when <math>k = \mathbb{Q}_p, p \neq 2</math> . . . . .</b>	<b>81</b>
6.1	The $H_k$ -conjugation Classes when $p \neq 2$ . . . . .	81
<b>Chapter 7</b>	<b>The <math>H_k</math>-conjugacy Classes of <math>\theta</math>-split <math>k</math>-tori . . . . .</b>	<b>85</b>
7.1	The $H_k$ -conjugacy Classes of $\theta$ -split $k$ -tori when $k = \mathbb{Q}$ . . . . .	85
7.2	The $H_k$ -conjugacy Classes of $\theta$ -split $k$ -tori when $k = \mathbb{F}_p$ . . . . .	86
7.3	The $H_k$ -conjugacy Classes of $\theta$ -split $k$ -tori when $k = \mathbb{Q}_p, p \neq 2$ . . . . .	87
<b>REFERENCES</b>	. . . . .	<b>89</b>
<b>Appendix</b>	. . . . .	<b>92</b>
Appendix A	Example Computations . . . . .	93
A.1	Computing the $H_k$ -conjugacy Classes . . . . .	94
A.2	Computing the Conjugating Matrices in $H_k$ . . . . .	101

# LIST OF TABLES

Table 5.1	The $H_k$ -conjugation of $t_2$ to all tori in $\mathbb{F}_p$ when $p = 13$ and $m \equiv N_p$ . . . . .	62
Table 5.2	The $H_k$ -conjugation Classes when $p = 5$ and $m \equiv N_p$ . . . . .	70
Table 5.3	The $H_k$ -conjugation Classes when $p = 13$ and $m \equiv N_p$ . . . . .	70
Table 5.4	The $H_k$ -conjugation Classes when $p = 17$ and $m \equiv N_p$ . . . . .	71
Table 5.5	The $H_k$ -conjugation Classes when $p = 7$ and $m \equiv 1$ . . . . .	73
Table 5.6	The $H_k$ -conjugation Classes when $p = 11$ and $m \equiv 1$ . . . . .	73
Table 5.7	The $H_k$ -conjugation Classes when $p = 19$ and $m \equiv 1$ . . . . .	74
Table 5.8	The $H_k$ -conjugation Classes when $p = 5$ and $m \equiv 1$ . . . . .	75
Table 5.9	The $H_k$ -conjugation Classes when $p = 13$ and $m \equiv 1$ . . . . .	76
Table 5.10	The $H_k$ -conjugation Classes when $p = 17$ and $m \equiv 1$ . . . . .	77
Table 5.11	The $H_k$ -conjugation Classes when $p = 3$ and $m \equiv N_p$ . . . . .	79
Table 5.12	The $H_k$ -conjugation Classes when $p = 7$ and $m \equiv N_p$ . . . . .	79
Table 5.13	The $H_k$ -conjugation Classes when $p = 11$ and $m \equiv N_p$ . . . . .	80

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# Chapter 1

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## Symmetric Variety Background

Symmetric varieties, or the spaces of symmetries, are important in many sciences, particularly in mathematics and physics. They arose around 100 years ago in physics, and were traditionally studied over the real numbers using Lie theory and analysis by Cartan. Riemannian symmetric spaces were studied first, followed by all symmetric spaces over the reals, called affine symmetric spaces or real reductive symmetric spaces. Most of the representation theory of Riemannian symmetric spaces was developed by Harish-Chandra in [HC84]. The study of representations associated with general real reductive symmetric spaces involved the work of many mathematicians, including Brylinski, Delorme, Flensted-Jensen, Matsuki, Ōshima, Sekiguchi, Schlichtkrul, and van den Ban in [vdBS97, BD92, Del98, FJ80, OS80, OM84]. Vust, Richardson, Springer, Brion, and Helminck then began researching symmetric varieties over algebraically closed fields [Vus74, Ric82, Spr98, BH00]. Symmetric  $k$ -varieties emerged in the late 1980s as a natural extension, allowing symmetric spaces and their representations to be studied over fields other than the reals and complexes, like the finite and  $p$ -adic fields, see [HW93]. For results of symmetric  $k$ -varieties over the finite fields, see [Gro92] and [Lus90]. For results of symmetric  $k$ -varieties over  $p$ -adic fields, see [HHa, JLR93, RR96].

Symmetric  $k$ -varieties over a general field  $k$  occur in representation theory, geometry, singularity theory, the study of character sheaves, and the study of cohomology of arithmetic subgroups. Symmetric  $k$ -varieties over  $p$ -adic fields have particular importance in number theory, representation theory, geometry, and harmonic analysis. For examples,

see [HHb].

Let  $G$  be a reductive algebraic group defined over a field  $k$ , and take  $\theta \in \text{Aut}(G)$  to be an involution on  $G$ . The space  $Q = \{x\theta(x)^{-1} \mid x \in G\}$  is a subvariety of  $G$ , called the symmetric variety. Let  $H = G^\theta$  be the fixed point group of  $\theta$  in  $G$ . Then the homogeneous space  $G/H$  is isomorphic to the symmetric variety  $Q$ . The symmetric  $k$ -variety is the space  $Q_k = \{x\theta(x)^{-1} \mid x \in G_k\}$ , where  $G_k$  is the  $k$ -rational points of  $G$ . Similarly, taking  $H_k$  to be the  $k$ -rational points of  $H$ , we get that  $Q_k \simeq G_k/H_k$ . The definition of symmetric variety can be extended even further to general groups  $G$ , in which case, the spaces  $Q \simeq G/H$  are called the generalized symmetric space. The double cosets  $H_k \backslash G_k / H_k$  can be seen as  $H_k$  acting on the coset space  $G_k/H_k$ , or  $H_k$  acting on the symmetric variety  $Q_k$ . To study symmetric  $k$ -varieties, and then the  $H_k$ -action on symmetric  $k$ -varieties, we begin by reviewing symmetric varieties.

In this thesis, we study the double cosets  $H_k \backslash G_k / H_k$  over the rational, finite, and  $p$ -adic fields. To study these, we first review symmetric varieties. We begin by reviewing a classical approach to symmetric spaces and then a broader approach to symmetric varieties. To conclude our review of symmetric varieties, we discuss a prominent type of symmetric varieties, called the Riemannian symmetric space. Riemannian symmetric spaces are of particular importance due to their formulation and resulting characteristics.

## 1.1 A Class of Reductive Symmetric Spaces

Let  $k = \mathbb{R}$  and let  $V = k^n$  be an  $n$ -dimensional vector space over  $k$ . Let  $M_n(k)$  be the set of  $n \times n$  matrices with entries in  $k$ . Then

$$\text{GL}(n, k) = \{A \in M_n(k) \mid \det(A) \neq 0\}$$

is the set of invertible matrices in  $M_n(k)$ . Let  $\text{id} \in \text{GL}(n, k)$  be the identity matrix.

Let  $B$  be a nondegenerate bilinear form on  $V$ . Thus, for  $x, y \in V$ ,  $B(x, y) = x^T M y$  for some  $M \in \text{GL}(n, k)$ . Moreover, take  $B$  to be symmetric, thus  $M = M^T$ . Take some  $A \in M_n$ . Then  $B(Ax, y) = B(x, A'y)$  for some  $A'$ , and call  $A'$  the adjoint of  $A$ . We get that  $A' = M^{-1}A^T M$ , and note that when  $M = \text{id}$ , we get  $A' = A^T$ . The matrix  $A$  is orthogonal with respect to the bilinear form  $B$  if and only if  $AA' = A'A = \text{id}$ , meaning that  $B(Ax, Ay) = B(x, y)$  for all  $x, y \in V$ . The set of matrices  $A$  that are orthogonal with respect to  $B$  forms a group, called the orthogonal group,

$$\mathrm{O}(V, B) = \{A \in \mathrm{GL}(n, k) \mid A'A = \mathrm{id}\}.$$

Define a map  $\tau : \mathrm{GL}(n, k) \rightarrow \mathrm{GL}(n, k)$  by  $\tau(A) := AA'$ . Then the kernel of  $\tau$  is  $\ker(\tau) = \tau^{-1}(\mathrm{id}) = \mathrm{O}(V, B)$ . The image of  $\tau$  is  $\mathrm{Im}(\tau) = \{AA' \mid A \in \mathrm{GL}(n, k)\}$ , which we call  $Q(V, B) = \mathrm{Im}(\tau)$ . We get

$$\mathrm{GL}(n, k)/\mathrm{O}(V, B) \simeq Q(V, B),$$

and we call  $Q(V, B)$  a reductive symmetric space.

This approach to symmetric spaces was first constructed over the reals. These results can generalize to any field  $k$ .

## 1.2 Our Approach to Symmetric Varieties

The bilinear form construction to symmetric varieties is limited because it leads only to outer automorphisms in  $\mathrm{Aut}(G)$ . There are more ways to construct symmetric varieties. We choose to define symmetric varieties based on involutions over algebraic groups.

*Notation.* Let  $k$  be a field of characteristics not equal to 2, let  $G$  be a reductive algebraic group defined over  $k$ , and let  $G_k$  be the  $k$ -rational points of  $G$ . Let  $\mathrm{Aut}(G)$  be the set of all automorphisms of  $G$ .

**Definition 1.2.1.** An **involution** of  $G$  is an automorphism  $\theta \in \mathrm{Aut}(G)$  of  $G$  such that  $\theta^2 = \mathrm{id}$  yet  $\theta \neq \mathrm{id}$ . A  **$k$ -involution** of  $G$  is an involution of  $G$  that sends  $G_k$  to  $G_k$ .

Let  $\theta \in \mathrm{Aut}(G)$  be an involution of  $G$ , or a  $k$ -involution when  $k$  is not algebraically closed. Let  $H = G^\theta = \{g \in G \mid \theta(g) = g\}$  be the fixed point group of  $\theta$  in  $G$ , and let  $H_k$  be the  $k$ -rational points of  $H$ . By a result of Steinberg in [Ste68],  $H^0$  is reductive, which then implies that  $H$  itself is reductive.

**Definition 1.2.2.** For an involution  $\theta$ , the **symmetric variety** is defined to be the subvariety  $Q = \{x\theta(x)^{-1} \mid x \in G\}$ . This space is isomorphic to  $G/H$ , thus  $Q \simeq G/H$ , and both are referred to as the symmetric variety.

Both  $G/H$  and  $Q$  are considered the symmetric variety because of the following property.

**Proposition 1.2.1.** *There is an isomorphism between the generalized symmetric space  $G/H$  and  $Q$ .*

*Proof.* Let  $\tau : G \rightarrow G$  be defined by

$$\tau(g) := g * e = g\theta(g)^{-1}, \quad (1.1)$$

where  $g \in G$ . For  $x, y \in G$ ,

$$\tau(x) = \tau(y) \iff xy^{-1} = \theta(x)\theta(y)^{-1} \iff xy^{-1} \in H$$

Thus,  $\tau^{-1}(\tau(x)) = xH$  for any  $x \in G$ . Therefore,  $Q \simeq G/H$ .  $\square$

Richardson showed this is an isomorphism of varieties as well.

Assumed in the definition of symmetric variety is the idea that  $G$ ,  $H$ , and  $Q$  are taken over the algebraic closure of the base field  $k$ . Restricting these spaces to just their  $k$ -points changes the nature of these spaces considerably, warranting the definition below.

**Definition 1.2.3.** Let  $G_k$  and  $H_k$  be the  $k$ -rational points of  $G$  and  $H$ , respectively, and  $\theta \in \text{Aut}(G_k)$ . Let  $Q_k = \{x\theta(x)^{-1} \mid x \in G_k\}$ . Then  $Q_k \simeq G_k/H_k$  is called the **symmetric  $k$ -variety**, and is a generalization of symmetric spaces to general fields  $k$ .

When we take  $Q_k \simeq G_k/H_k$  to be the symmetric  $k$ -variety, this is an isomorphism of reductive varieties, making the proof of Proposition 1.2.1 more complicated. For the proof, see [Ric82].

**Definition 1.2.4.** Let  $G$  be a real reductive group. For an involution  $\theta$ , the **symmetric space** is defined to be subspace  $Q = \{x\theta(x)^{-1} \mid x \in G\}$ . This space is isomorphic to  $G/H$ , thus  $Q \simeq G/H$ , and both are referred to as the symmetric space.

There are many examples of symmetric spaces, symmetric varieties, and symmetric  $k$ -varieties. We first consider a well-known decomposition of  $\text{GL}(n, k)$ .

**Example 1.2.1.** Let  $k = \mathbb{R}$  and take  $G_k = \text{GL}(n, \mathbb{R})$ . Consider the involution

$$\theta(x) = (x^T)^{-1}$$

for all  $x \in G_k$ . Then  $H_k = \{x \in G_k \mid xx^T = \text{id}\} = \text{O}(n, \mathbb{R})$ . The symmetric  $k$ -variety is then  $Q_k = \{xx^T \mid x \in G_k\}$ . Thus, we get  $G_k/H_k \simeq Q_k$ . Written another way,  $G_k \simeq H_k Q_k$ ,

which is the decomposition of real invertible matrices into the product of orthogonal matrices and symmetric positive-definite matrices.  $\diamond$

Throughout this thesis,  $G$  is assumed to be a reductive algebraic group, including in the given definitions of symmetric variety, symmetric  $k$ -variety, and symmetric space. The notion of symmetric spaces can, in fact, extend to any group. For the following definition, temporarily drop the assumption that  $G$  is a reductive algebraic group.

**Definition 1.2.5.** Let  $G$  be a group. Let  $\theta \in \text{Aut}(G)$  an involution of  $G$ ,  $H = G^\theta$  the fixed point group of  $\theta$  in  $G$ , and  $Q = \{x\theta(x)^{-1} \mid x \in G\}$ , as before. Then  $Q \simeq G/H$ , and both are referred to as the **generalized symmetric space**.

We can define a generalized symmetric space for any group. In fact, any group can itself be seen as a generalized symmetric space.

**Example 1.2.2.** Let  $G$  be a group and consider the group  $G \times G$ . Take the involution  $\theta(x, y) = (y, x)$  for all  $(x, y) \in G \times G$ . Define an automorphism  $\tau : G \times G \rightarrow G \times G$  by  $\tau := (x, y) \cdot \theta(x, y)^{-1}$ . Then  $H = \ker(\tau) = \{(x, x) \mid x \in G\} \simeq G$  and

$$Q = \text{Im}(\tau) = \{(x, y) \cdot \theta(x, y)^{-1} \mid (x, y) \in G \times G\} \simeq \{(x, x^{-1}) \mid x \in G\}.$$

Notice that  $Q \simeq G$ , thus  $G$  itself can be viewed as a generalized symmetric space.  $\diamond$

We construct  $Q$  by acting  $\theta$  on  $G$ . The space defined below helps us classify how  $\theta$  acts on  $G$ .

**Definition 1.2.6.** Let  $G$  be a group and  $\theta \in \text{Aut}(G)$  an involution of  $G$ . The **extended symmetric space** of  $G$  is the space  $R = \{g \in G \mid \theta(g) = g^{-1}\}$ .

The action of  $\theta$  on an element  $x\theta(x)^{-1} \in Q$  is  $\theta(x\theta(x)^{-1}) = \theta(x)x^{-1} = [x\theta(x)^{-1}]^{-1}$ . Thus, the symmetric space  $Q$  is contained within  $R$ , hence the name. While we conventionally differentiate between symmetric spaces, symmetric  $k$ -varieties, and generalized symmetric spaces, we refer to the set  $R$  as the extended symmetric space in all situations. Also, if the group  $G$  has a topology and is connected, then  $Q$  is also connected because it is the image of  $\tau$  acting on  $G$ . Contained in  $Q$  is the identity, making  $Q$  the connected identity component of  $R$ , that is,  $Q = R^0$ . Thus,  $Q = R^0$  when  $k = \bar{k}$  is algebraically closed,  $k$  is the field of real numbers, or  $k$  is the field of  $p$ -adic numbers.

Throughout this thesis, we will study  $k$ -tori inside symmetric  $k$ -varieties, for they provide valuable information about the structure and behavior the symmetric  $k$ -varieties. As we will see throughout this chapter, there are many results about tori inside symmetric varieties, Riemannian symmetric spaces in particular, and many of these results do not translate to  $k$ -tori in symmetric  $k$ -varieties. Thus, we now focus on tori. For an arbitrary field  $k$ , we will refer to tori as  $k$ -tori to emphasize the field over which the tori are defined.

In a construction of  $Q$ , we act  $\theta$  on  $G$ . Thus, we are very interested in the action of  $\theta$  on tori in  $G$ , and this leads us to the following definitions.

**Definition 1.2.7.** Given an involution  $\theta \in \text{Aut}(G)$  where  $G$  is a group, a torus  $T$  is  $\theta$ -stable if  $\theta(T) = T$ .

An element inside the symmetric space  $Q \subseteq R$  has the property that  $\theta$  sends it to its inverse. Thus, we define the following property for tori.

**Definition 1.2.8.** Let  $T \subset G$  be a torus and  $\theta \in \text{Aut}(G)$  an involution. We say that  $T$  is  $\theta$ -split if  $\theta(t) = t^{-1}$  for every  $t \in T$ .

Note that  $\theta$ -split tori are  $\theta$ -stable by definition. Let  $T$  be a torus and let

$$T^+ = \{t \in T \mid \theta(t) = t\}^0; \quad (1.2)$$

$$T^- = \{t \in T \mid \theta(t) = t^{-1}\}^0. \quad (1.3)$$

Then  $T = T^+T^-$ , and  $T^+ \cap T^-$  is finite. Note that this is a group-theoretic version of an eigenspace decomposition.

A  $k$ -torus  $T$  can also be decomposed as  $T = T_a T_d$ , where  $T_a$  is  $k$ -anisotropic and  $T_d$  is  $k$ -split [Bor91].

**Definition 1.2.9.** Let  $G$  be a group defined over  $k$ ,  $T \subset G$  a torus, and  $\theta \in \text{Aut}(G)$  an involution. The torus  $T$  is  $(\theta, k)$ -split if it is both  $\theta$ -split and  $k$ -split.

As we review results about symmetric varieties, and Riemannian symmetric spaces in particular, we will see many results about  $(\theta, k)$ -split tori. In this thesis, we study  $\theta$ -split  $k$ -tori in general, including those that are  $k$ -anisotropic. Thus,  $(\theta, k)$ -split tori initially are of great importance to this thesis.



## 1.3 Riemannian Symmetric Spaces

Symmetric varieties first arose over the real numbers. Specifically, the first symmetric variety that mathematicians researched was the Riemannian symmetric space because it arises naturally from differential geometry. The Riemannian symmetric space has many nice properties, and often when studying symmetric  $k$ -varieties, we are in search of properties analogous to those in the Riemannian symmetric space, should such properties exist. Thus, we review Riemannian symmetric spaces, and most of these results can be found in [Hel01] and [HW93].

Throughout this section, assume  $k = \mathbb{R}$ . Let  $\mathfrak{g} = L(G)$ ,  $\mathfrak{h} = L(H)$ , and  $\mathfrak{q} = L(Q)$  be the Lie algebras of  $G$ ,  $H$ , and  $Q$ , respectively. Let  $G_k$  be a real Lie group.

**Definition 1.3.1.** Let  $\theta \in \text{Aut}(G)$  be an involution,  $H_k = G_k^\theta$  the fixed point group of  $\theta$  in  $G_k$ , and  $A_k$  a  $\theta$ -stable maximal  $k$ -split torus. Then  $\theta$  is a **Cartan involution** if the following three properties hold.

1. The subgroup  $H_k$  is maximal compact.
2. We have  $(k^*)^2 = (k^*)^4$ .
3. There exists a Cartan decomposition  $G_k = H_k A_k H_k$ .

**Definition 1.3.2.** Let  $G_k$  be a real Lie group. Take  $\theta \in \text{Aut}(G_k)$  to be a Cartan involution. Let  $H_k = G_k^\theta$  be the fixed point group of  $\theta$  in  $G_k$ , and take  $Q_k = \{x\theta(x)^{-1} \mid x \in G_k\}$ . Then  $Q_k \simeq G_k/H_k$  is the **Riemannian symmetric space**.

### 1.3.1 The Subgroup $H_k$ is Maximal Compact

We will review several important properties of Riemannian symmetric spaces, many of which stem from the property below.

**Property 1.** *In a Riemannian symmetric space  $Q_k$ , the fixed point group  $H_k$  is maximal compact.*

**Example 1.3.1.** Let  $G_k = \text{SL}(2, \mathbb{R})$  and take the involution  $\theta(x) = (x^T)^{-1}$ . Then the fixed point group is  $H_k = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1, a, b \in \mathbb{R} \right\}$ , which is maximal compact.  $\diamond$

We can conclude Property 1 from the following results in [Hel01].

**Proposition 1.3.1.** *Every semisimple Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$  has a real form which is compact.*

A real Lie algebra  $\mathfrak{g}$  is semisimple if and only if its killing form  $\kappa$  is nondegenerate. Take  $\mathfrak{k} \subset \mathfrak{g}$  to be a maximal compact subalgebra. Let  $\mathfrak{p} = \mathfrak{k}^\perp$ , and note that  $\kappa$  restricted to  $\mathfrak{p}$  is positive definite. Then  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  is the  $\pm 1$  eigenspace decomposition. Let  $\mathfrak{g}^\mathbb{C} \supset \mathfrak{g}$  be the complexification of  $\mathfrak{g}$  and  $\sigma$  the conjugation of  $\mathfrak{g}^\mathbb{C}$  that leaves  $\mathfrak{g}$  invariant.

**Lemma 1.3.1.** *Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be a real Lie algebra,  $\mathfrak{g}^\mathbb{C}$  its complexification, and  $\sigma$  the conjugation of  $\mathfrak{g}^\mathbb{C}$  that leaves  $\mathfrak{g}$  invariant. Then there exists a compact real form of  $\mathfrak{g}^\mathbb{C}$  that is  $\sigma$ -stable.*

To prove Lemma 1.3.1, define  $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ , and note that  $i\mathfrak{p}$  is now negative definite. Let  $\tau$  denote the conjugation of  $\mathfrak{g}^\mathbb{C}$  with respect to  $\mathfrak{u}$ . Then  $\sigma(\mathfrak{u}) = \mathfrak{u}$  if and only if  $\sigma\tau = \tau\sigma$ . Thus,  $\sigma\tau$  is of order 2, and moreover, it is a Cartan involution, leaving  $\mathfrak{g}^{\sigma\tau} = \mathfrak{k}$ .

Lemma 1.3.1 shows how to construct a Cartan involution. Combining Proposition 1.3.1 and Lemma 1.3.1 shows that each semisimple real Lie algebra has a Cartan decomposition. Moreover, one always has a maximal compact subalgebra, as shown below in Theorem 1.3.1, which proves Property 1 above.

**Theorem 1.3.1.** *Let  $\mathfrak{g}_0$  be a semisimple Lie algebra over  $\mathbb{R}$  which is the direct sum  $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ , where  $\mathfrak{k}_0$  is a subalgebra and  $\mathfrak{p}_0$  is a vector space. The following conditions are equivalent.*

1. *The decomposition  $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$  is a Cartan decomposition.*
2. *The mapping  $s_0 : T + X \rightarrow T - X$ , where  $T \in \mathfrak{k}_0$  and  $X \in \mathfrak{p}_0$ , is an automorphism of  $\mathfrak{g}_0$  and the symmetric bilinear form*

$$B_{s_0}(X, Y) = -B(X, s_0 Y)$$

*is strictly positive definite, that is,  $B < 0$  on  $\mathfrak{k}_0$  and  $B > 0$  on  $\mathfrak{p}_0$ .*

*If these conditions are satisfied,  $\mathfrak{k}_0$  is a maximal compactly imbedded subalgebra of  $\mathfrak{g}_0$ .*

Many of the other important properties of the Riemannian symmetric space follow from the result that  $H_k$  is maximal compact. As we will see in this section and throughout this thesis, not all of these properties generalize to symmetric  $k$ -varieties. In fact, the property the  $H_k$  is maximal compact does not necessarily generalize to symmetric  $k$ -varieties, as Examples 1.3.2 and 1.3.3 below demonstrate.

**Example 1.3.2.** Let  $G_k = \mathrm{SL}(2, \mathbb{C})$  and take the involution  $\theta(x) = (x^T)^{-1}$ . Then the fixed point group is  $H_k = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1, a, b \in \mathbb{C} \right\}$ , which is noncompact.  $\diamond$

Examples 1.3.1 and 1.3.2 differ only by the choice of the base field. In Example 1.3.1, we assumed  $k = \mathbb{R}$ , which is the base field in a Riemannian symmetric space. In Example 1.3.2, we went up to  $k = \bar{\mathbb{R}}$ , and this was enough to break the compactness quality of  $H_k$ .

*Notation.* We will use the notation  $\mathrm{Int}(A)$  to mean conjugation by  $A \in \mathrm{GL}(n, k)$ .

Note that  $\theta(g) = (g^T)^{-1}$  for all  $g \in G_k$ , the involution used to define a Riemannian symmetric space, is equivalent to  $\theta(g) = \mathrm{Int}(A)(g)$  with  $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  for all  $g \in G_k$ . Now we consider a Lie group with a different involution.

**Example 1.3.3.** Let  $G = \mathrm{SL}(2, \mathbb{R})$  and take the involution  $\theta = \mathrm{Int}(A)$  with  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Then the fixed point group is

$$H_k = \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} \mid x^2 - y^2 = 1 \right\} = \left\{ \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix} \mid \phi \in \mathbb{R} \right\},$$

which is noncompact. Thus, we can see that by simply switching to another involution, and thereby switching to a symmetric  $k$ -variety, we lose the property that  $H_k$  is maximal compact.  $\diamond$

### 1.3.2 All Elements are Semisimple

Below is a fundamental result of the compactness of the fixed point group  $H_k$ .

**Property 2.** *In a Riemannian symmetric space  $Q_k$ , all elements are semisimple.*

**Example 1.3.4.** Let  $G_k = \mathrm{SL}(2, \mathbb{R})$  and take  $\theta(g) = (g^T)^{-1}$  for all  $g \in G_k$ . Then the Riemannian symmetric space  $Q_k = \{xx^T \mid x \in G\}$  is the set of symmetric matrices with positive real eigenvalues, and thus consists of semisimple elements.  $\diamond$

Property 2 is a result of the following theorem, which was proven by Cartan. In [Hel01] and [HW93] it has been generalized to symmetric  $k$ -varieties such that the characteristic of  $k$  is not 0. Note that Property 1 satisfies the condition that  $H$  is anisotropic over  $\mathbb{R}$ .

**Theorem 1.3.2.** *Let  $G$  be a connected reductive algebraic group over  $k$  such that  $\mathrm{ch}(k) = 0$ ,  $Q = \{x\theta(x)^{-1} \mid x \in G\}$ , and  $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$  the decomposition of  $\mathfrak{g} = L(G)$  into eigenspaces of  $\theta$ . Suppose that  $H$  is anisotropic over  $k$ . Then the following conditions are true.*

1. *The symmetric  $k$ -variety  $Q_k$  consists of semisimple elements.*
2. *The symmetric  $k$ -variety  $\mathfrak{q}_k$  consists of semisimple elements.*

In fact, for a maximal compact subgroup  $H_k \subset G_k$ , the space  $G_k/H_k$  is going to consist of semisimple elements. This does not generalize to symmetric  $k$ -varieties over an arbitrary field  $k$ .

**Example 1.3.5.** As in Example 1.3.3, let  $k = \mathbb{R}$ ,  $G = \mathrm{SL}(2, \mathbb{R})$ , and  $\theta = \mathrm{Int}(A)$  with  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . In this case, the symmetric  $k$ -variety is

$$Q_k = \left\{ \begin{pmatrix} a^2 - b^2 & bd - ca \\ ca - bd & d^2 - c^2 \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\},$$

which contains non-symmetric matrices and non-semisimple elements. For instance, the values  $a = 1$ ,  $b = 2$ ,  $c = 0$ , and  $d = 1$  create the matrix  $\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$  with Jordan normal form  $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ , which is not semisimple.  $\diamond$

The elements in the Riemannian symmetric space  $Q_k$  are semisimple, and thus contained in a torus of  $G$ . Therefore, to study elements in  $Q_k \subset R$ , we can study  $\theta$ -split  $k$ -tori instead. In symmetric  $k$ -varieties, in addition to studying  $k$ -tori, we must also study the unipotent elements.

### 1.3.3 All Elements are $k$ -split

In a Riemannian symmetric space, semisimplicity implies the following condition.

**Property 3.** *All elements in a Riemannian symmetric space  $Q_k$  are  $k$ -split.*

By Property 2, all elements in the Riemannian symmetric space are semisimple, and now we see that they are also  $k$ -split.

**Example 1.3.6.** As in Example 1.3.4, let  $G_k = \mathrm{SL}(2, \mathbb{R})$  and take  $\theta(g) = (g^T)^{-1}$  for all  $g \in G_k$ , creating the Riemannian symmetric space  $Q_k = \{xx^T \mid x \in G\}$ . These elements are symmetric matrices with positive real eigenvalues and can be diagonalized over the base field. Hence they are  $k$ -split.  $\diamond$

By Property 1.3.1, the real Lie group  $G_k$  has an associated Cartan decomposition, allowing us to apply the following theorem.

**Theorem 1.3.3.** *Let  $H$  be compact, let  $A$  be a  $\theta$ -stable maximal  $k$ -split torus of  $G$ , and let  $G_k = H_k A_k H_k$ . Then the Riemannian symmetric space  $Q_k = \{x\theta(x)^{-1} \mid x \in G_k\}$  consists of  $k$ -split semisimple elements.*

This result is fundamental to this thesis, because in Riemannian symmetric spaces, semisimplicity implies that elements will split over  $\mathbb{R}$ . As we saw in Example 1.3.5, not all elements in symmetric  $k$ -varieties are semisimple. Further, in a symmetric  $k$ -variety, even those that are semisimple might not be  $k$ -split. For an example, we turn to Lie algebras.

**Example 1.3.7.** Let  $\mathfrak{g}_k = \mathfrak{sl}(2, \mathbb{R})$  and take  $\theta = \mathrm{Int}(A)$  with  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Then the symmetric  $k$ -variety is

$$\mathfrak{q}_k = \left\{ \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \mid a, b \in k \right\}.$$

The element  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  has minimum polynomial  $x^2 + 1$  and is hence semisimple, yet has eigenvalues  $\pm i$ , and thus is not split over  $\mathbb{R}$ .  $\diamond$

Elements in a Riemannian symmetric space are semisimple, which implies that they are also  $k$ -split. In a symmetric  $k$ -variety, there are both unipotent and semisimple elements, and the semisimple elements do not always split fully over the field. Thus, in a symmetric  $k$ -variety, there are many more components to consider.

### 1.3.4 Maximal $\theta$ -split Tori are Maximal $k$ -split

We know that elements in  $Q_k$  are semisimple,  $k$ -split, and  $\theta$ -split in a Riemannian symmetric space. This leads us to the following property.

**Property 4.** *In a Riemannian symmetric space  $Q_k$ , all maximal  $\theta$ -split tori are maximal  $k$ -split.*

**Example 1.3.8.** As in Example 1.3.1, let  $G_k = \mathrm{SL}(2, \mathbb{R})$  and  $\theta(g) = (g^T)^{-1}$  for all  $g \in G_k$  to create the Riemannian symmetric space  $Q_k = \{xx^T \mid x \in G\}$ . The torus

$$T = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in \mathbb{R}^* \right\} \subset Q_k$$

is contained in  $Q_k$  and is  $\theta$ -split. Also notice that  $T$  is maximal in  $G_k$ . In this form of the torus, we can see that all elements split over  $\mathbb{R}$ , too.  $\diamond$

Theorems 1.3.3 allows us to apply the following theorem, which results in Property 4 above.

**Theorem 1.3.4.** *Assume  $Q_k$  consists of  $k$ -split semisimple elements. Then the following are true.*

1. *All  $\theta$ -split tori of  $G$  are  $k$ -split.*
2. *All maximal  $\theta$ -split tori of  $G$  are maximal  $(\theta, k)$ -split.*
3. *Given any  $x \in Q_k$ , there is a maximal  $(\theta, k)$ -split torus of  $G$  containing  $x$ .*

All elements in the Riemannian symmetric space  $Q_k$  are semisimple and  $k$ -split, hence they are contained in  $k$ -split tori. We know that  $Q_k$  is  $\theta$ -split because  $Q_k \subset R$ , thus elements in  $Q_k$  are contained in  $(\theta, k)$ -split tori of  $G_k$ .

Once again, these properties do not hold in symmetric  $k$ -varieties. For an example, we turn to Lie algebra.

**Example 1.3.9.** As in Example 1.3.7, let  $\mathfrak{g}_k = \mathfrak{sl}(2, \mathbb{R})$  and take  $\theta = \mathrm{Int}(A)$  with  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Then the symmetric  $k$ -variety is

$$\mathfrak{q}_k = \left\{ \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \mid a, b \in k \right\}.$$

One of the toral subalgebras inside  $\mathfrak{q}_k$  is

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} \mid x \in k \right\}.$$

The Lie algebra equivalent of a  $\theta$ -split torus is a toral subalgebra such that  $\theta(t) = -t$  for all  $t \in \mathfrak{t}$ , and note that the toral subalgebra above is indeed  $\theta$ -split. The eigenvalues of the matrices inside  $\mathfrak{t}$  are  $\pm ix$ , hence the elements are semisimple and yet this maximal toral subalgebra does not split over  $\mathbb{R}$ .  $\diamond$

Theorem 1.3.4 shows us that because elements in  $Q_k$  are semisimple and  $k$ -split,  $\theta$ -split tori are  $k$ -split. Proposition 1.3.2 below shows reverses this, showing that  $k$ -split tori must be  $\theta$ -split as well.

**Proposition 1.3.2.** *Let  $H$  be compact and  $G_k = H_k A_k H_k$ , where  $A$  is a  $\theta$ -stable maximal  $k$ -split torus of  $G$ , as in the Riemannian symmetric space. Then  $A$  is maximal  $\theta$ -split.*

### 1.3.5 All Maximal $(\theta, k)$ -split Tori are $H_k$ -conjugate

In this thesis, we study maximal  $\theta$ -split  $k$ -tori. We review what is known about  $(\theta, k)$ -split tori, and then focus on  $\theta$ -split  $k$ -anisotropic tori. In particular, we analyze the  $H_k$ -conjugation of  $k$ -tori. The following result about  $H_k$  acting on  $\theta$ -split tori in Riemannian symmetric spaces is of fundamental importance to this thesis, because, once again, it does not hold over symmetric  $k$ -varieties.

**Property 5.** *In a Riemannian symmetric space  $Q_k$ , all maximal  $(\theta, k)$ -split tori are  $H_k$ -conjugate.*

**Example 1.3.10.** Let  $G_k = \mathrm{SL}(2, \mathbb{R})$  and  $\theta(g) = (g^T)^{-1}$  for all  $g \in G_k$ . In Example 1.3.3, we considered the torus

$$T = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in \mathbb{R}^* \right\}.$$

That is because all maximal  $(\theta, k)$ -split tori in  $G_k$  are conjugate to this form, allowing us to only consider the torus  $T$ .  $\diamond$

The following is a combination of several theorems in [HW93]. Property 1 shows us that the Riemannian symmetric space is created using a Cartan involution, allowing us to apply the theorem below.

**Proposition 1.3.3.** *Let  $H_k$  be  $k$ -anisotropic and  $G_k = H_k A_k H_k$ . Then any two maximal  $(\theta, k)$ -split tori are conjugate by an element of  $H_k^0$ .*

In symmetric  $k$ -varieties, maximal  $k$ -tori are not always  $H_k$ -conjugate.

**Example 1.3.11.** In [BH09], Beun and Helminck showed that there is an infinite number of  $H_k$ -conjugacy classes of  $(\theta, k)$ -split tori in  $\mathrm{SL}(2, \mathbb{Q})$ .  $\diamond$

We can extend Proposition 1.3.3 to create the theorem below. Let  $\theta \in \mathrm{Aut}(G)$  be an involution of  $G$ , though not necessarily a Cartan involution.

**Theorem 1.3.5.** *Let  $\theta$  be an involution of  $G$  defined over  $k$  and  $H^0$  the identity component of the  $\theta$  fixed point group,  $G^\theta$ . If  $G$  has a Cartan involution over  $k$ , then all maximal  $(\theta, k)$ -split tori are  $H_k$ -conjugate.*

Theorem 1.3.5 shows us that as long as the group  $G$  has a Cartan involution, all of its tori are conjugate by the fixed point group, even if the tori and the fixed point group are created using a non-Cartan involution. In symmetric  $k$ -varieties, this is far from true.

### 1.3.6 The Weyl Group has Representatives in $H_k$

**Property 6.** *Let  $Q_k$  be a Riemannian symmetric space and  $T$  a maximal  $\theta$ -split torus. The Weyl group  $W(T)$  has representatives in  $W_{H_k}(T)$ .*

**Example 1.3.12.** Let  $G_k = \mathrm{SL}(2, \mathbb{R})$  and take torus  $T = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in \mathbb{R}^* \right\}$ . Then the Weyl group of  $T$  is

$$W(T) = N(T)/T = \left\{ \mathrm{id}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

The fixed point group is  $H_k = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1 \right\}$ . Notice that the conditions  $a = 0$  or  $b = 0$  give us the following matrices when combined with the requirement  $a^2 + b^2 = 1$ .

$$\begin{aligned} a = 0 &\Rightarrow h_a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ b = 0 &\Rightarrow h_b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$



Notice that  $h_a, h_b \in W(T)$  and  $h_a, h_b \in N_H(T)$ , showing that both elements of  $W(T)$  are represented in  $W_{H_k}(T)$ .  $\diamond$

Borel and Tits give us the following theorems.

**Theorem 1.3.6.** *Let  $A$  be a maximal  $k$ -split torus. Then  $W(A)$  has representatives in  $W_{G_k}(A)$ .*

**Theorem 1.3.7.** *For an involution  $\theta$ , let  $A$  be a maximal  $\theta$ -split torus and  $H$  be its fixed point group in  $G$ . Then  $W(A)$  has representatives in  $W_H(A)$ .*

Property 6 is a result of the following theorem in [HW93], which is an extension of Theorems 1.3.6 and 1.3.7.

**Theorem 1.3.8.** *For a Cartan involution  $\theta$ , let  $A$  be a maximal  $(\theta, k)$ -split torus and  $H$  be its fixed point group in  $G$ . Then  $W(A)$  has representatives in  $W_{H_k}(A)$ .*

When working over a general field  $k$  and with a general involution  $\theta$ , elements in the Weyl group are not necessarily in the fixed point group. Below is an example of when it does not hold.

**Example 1.3.13.** Let  $G_k = \mathrm{SL}(2, \mathbb{F}_p)$  with  $p = 5$  and  $\theta = \mathrm{Int}(A)$  with  $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ . The fixed point group of  $\theta$  in  $G_k$  is

$$H_k = \left\{ \begin{pmatrix} x & y \\ 2y & x \end{pmatrix} \mid x^2 - 2y^2 = 1, x, y \in \mathbb{F}_p \right\}.$$

Take the torus  $T = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in \mathbb{F}_p^* \right\}$ , and the Weyl group of  $T$

$$W_{G_k}(T) = N_{G_k}(T)/T = \left\{ \mathrm{id}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Notice that the second Weyl group element has form  $\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$ , which does not fit the form of  $H_k$ . Thus, not all elements in  $W_{G_k}(T)$  are contained in  $H_k$ .  $\diamond$

Note that in Riemannian symmetric spaces and symmetric  $k$ -varieties when  $k$  is algebraically closed,  $W_G(T) = W_H(T)$ , for  $T$  a maximal torus. The Weyl group in  $H$  counts the number of times the  $H$ -orbit intersects the torus itself. [HW93]

**Proposition 1.3.4.** *Let  $H_k$  be maximal compact,  $T$  a maximal  $\theta$ -split torus, and  $t \in T$ . Then  $H_k \cdot t \cap T = W_{H_k}(T) \cdot t$ .*

*Proof.* Consider  $h_1 \cdot t = h_1 t h_1^{-1} \in H_k \cdot t \cap T$ , where  $h_1 \in H_k$ . All tori are  $H_k$ -conjugate, thus  $h_1 T h_1^{-1} = T_1$ , where  $T_1$  is another maximal torus in  $G_k$ . By construction,  $h_1 T h_1^{-1} \subset Z_{G_k}(h_1 t h_1^{-1})$  and  $T \subset Z_{G_k}(h_1 t h_1^{-1})$ . Thus, the tori  $T$  and  $T_1$  are  $H_k$ -conjugate in  $Z_{G_k}(h_1 t h_1^{-1})$  as well. Therefore, there exists an  $h_2 \in H_k \cap Z_{G_k}(h_1 t h_1^{-1})$  such that  $h_2 T_1 h_2^{-1} = T$ . Putting these together, we see that there exists an  $h \in N_{H_k}(T)$  such that  $h_1 \cdot t = h \cdot t$ , namely  $h = h_2 h_1$ . Thus,  $H_k \cdot t \cap T \subset W_{H_k}(T) \cdot t$ . The argument reverses, hence  $H_k \cdot t \cap T = W_{H_k}(T) \cdot t$ .  $\square$

**Example 1.3.14.** Let  $G_k = \mathrm{SL}(2, \mathbb{R})$  and take torus  $T = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in \mathbb{R}^* \right\}$ . Consider the fixed point group  $H_k$  and Weyl group  $W_{G_k}(T)$  from Example 1.3.12. We know that for  $t \in T$ ,  $H_k \cdot t \cap T = W_{H_k}(T) \cdot t$ . When we consider  $H_k \cdot t \cap T$ , we get the following relations:

$$\begin{aligned} abx &= abx^{-1} \\ (a^2x + b^2x^{-1})(a^2x^{-1} + b^2x) &= 1. \end{aligned}$$

The implication of these relations is that  $a = 0$ , or  $b = 0$ , or  $x = \pm 1$ . Combining these relations with the requirement that  $a^2 + b^2 = 1$ , we create  $h_a, h_b \in H_k$  as in Example 1.3.12, showing the  $H_k$ -orbits intersect the torus  $T$  twice.  $\diamond$

The proof of Proposition 1.3.4 uses the fact that in a Riemannian symmetric space, all tori are  $H_k$ -conjugate. Thus, this relation does not necessarily hold in a symmetric  $k$ -variety.

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## Chapter 2

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### The $H_k$ -orbits in $Q_k$ Background

The generalizations of symmetric spaces to an arbitrary field  $k$  are called symmetric  $k$ -varieties, which arise in many areas of mathematics and physics, and play a particularly important role in representation theory and number theory. In the past thirty years, the study of symmetric  $k$ -varieties has begun to expand, unveiling many open questions. Many of these open questions are about how the structure of symmetric spaces can be generalized to symmetric  $k$ -varieties, if at all. Some of the structure of symmetric  $k$ -varieties that is important to representation theory and number theory is how  $G_k$  and  $H_k$  act on  $Q_k \simeq G_k/H_k$ . In this thesis, we analyze how  $H_k$  acts on the symmetric  $k$ -variety  $G_k/H_k$ .

In [Ric82], Richardson analyzes the action of  $H$  on the symmetric space  $G/H$  when  $G$  is a group defined over an algebraically closed field. Helminck and Wang the transition to symmetric  $k$ -varieties by analyzing  $G_k/H_k$  over fields of characteristic  $p$ . Helminck and Schwarz study orbits of symmetric groups in symmetric  $k$ -varieties over algebraically closed fields and the real numbers [HS09, HS02, HS04, HS11]. The results of [Ric82] and [HS11] show that we can begin to study the movements of elements in  $G_k/H_k$  by analyzing  $\theta$ -split  $k$ -tori in  $G_k/H_k$ , where  $\theta$  is a  $k$ -involution on  $G_k$ . In [BH09], Beun and Helminck begin to analyze the  $\theta$ -split  $k$ -tori by considering  $\theta$ -split  $k$ -tori that are split over  $k$  as well. These  $k$ -tori are referred to as  $(\theta, k)$ -split tori. In this thesis, we analyze  $\theta$ -split  $k$ -anisotropic tori. Combined with the results of Helminck and Beun in [BH09], we get a characterization of all maximal  $\theta$ -split  $k$ -tori.

*Notation.* Let  $G$  be a reductive algebraic group over an algebraically closed field. Let  $\theta \in \text{Aut}(G)$  be an involution on  $G$ , thus  $\theta^2 = \text{id}$ , and let  $H = G^\theta$  be the fixed point group of  $\theta$  in  $G$ . The subvariety  $Q = \{x\theta(x)^{-1} \mid x \in G\}$  can be identified with the coset space  $G/H$ , and both are called the symmetric variety. Let  $\mathfrak{g} = L(G)$ ,  $\mathfrak{h} = L(H)$ , and  $\mathfrak{q} = L(Q)$  be the Lie algebras of  $G$ ,  $H$ , and  $Q$ , respectively. When working over a field that is not algebraically closed,  $G_k$ ,  $H_k$ ,  $\mathfrak{g}_k$ , and  $\mathfrak{h}_k$  denote the  $k$ -rational points of  $G$ ,  $H$ ,  $\mathfrak{g}$ , and  $\mathfrak{h}$ , respectively. Then  $Q_k = \{x\theta(x)^{-1} \mid x \in G_k\}$  and  $\mathfrak{q}_k = \{x - \theta(x) \mid x \in \mathfrak{g}_k\}$ .

## 2.1 The $H_k$ -Action

In this thesis, we examine  $H_k$  acting on the symmetric  $k$ -variety  $Q_k \simeq G_k/H_k$ . Note that  $G$  and  $H$  act on the symmetric variety  $G/H$  by left multiplication. There are two other actions by  $G$  and  $H$  on  $G/H \simeq Q$  that are important when working in symmetric spaces: conjugation and twisted conjugation.

*Notation.* Take  $g, x \in G$ . The action of conjugation of  $x$  by  $g$  is denoted by  $g \cdot x = gxg^{-1}$ .

**Definition 2.1.1.** Take  $g, x \in G$  and let  $\theta \in \text{Aut}(G)$  be an involution. The action of  $\theta$ -**twisted conjugation** of  $x$  by  $g$  is  $g * x = gx\theta(g)^{-1}$ .

Recall that  $Q = \{x\theta(x)^{-1} \mid x \in G\}$ , thus, we can view  $Q$  as  $G * e$ , the  $G$ -orbit of the identity. In particular, the  $\theta$ -twisted conjugation is an important mapping from  $Q$  to  $Q$ , because for  $g \in G$  and  $x \in Q$ , we find that  $g * x \in Q$  and  $g \cdot x \notin Q$ . Thus,  $\theta$ -twisted conjugation is a conjugation mapping that allows  $Q$  to go to  $Q$ . When conjugating by the fixed point group  $H$ , conjugation and  $\theta$ -twisted conjugation are the same action.

In general, for  $x \in G$ , we have the Jordan decomposition  $x = x_s x_u$ , where  $x_s$  is semisimple,  $x_u$  is unipotent, and  $x_s, x_u \in G$ . In [Ric82], Richardson shows that this decomposition holds even when restricted to the symmetric space.

**Theorem 2.1.1.** *Consider  $x = x_s x_u$ . Then  $x \in Q = \{x\theta(x)^{-1} \mid x \in G\}$  if and only if  $x_s, x_u \in Q$ .*

The proof of this theorem relies on the uniqueness of the Jordan decomposition. Richardson uses the map  $\tau : G \rightarrow G$ , defined by

$$\tau(g) := g * e = g\theta(g)^{-1},$$

as in 1.1, to show that  $x, x_s$ , and  $x_u$  are in  $Q$ . This theorem is significant because it shows that we can analyze elements in the symmetric space by first analyzing the semisimple elements and the unipotent elements. In [Ric82], Richardson gives some structure to these semisimple elements by linking them to  $\theta$ -split tori.

## 2.2 The $H_k$ -orbits in $Q_k$ over Algebraically Closed Fields

Recall that a  $\theta$ -split torus  $T$  is a torus such that  $\theta(t) = t^{-1}$  for all  $t \in T$ . In older literature, the concept that we define as a  $\theta$ -split torus is referred to as a  $\theta$ -anisotropic torus. Anisotropic is the algebra equivalent of compact. Thus, a torus that is  $\theta$ -split can also be compact, and a  $\theta$ -split toral subalgebra can also be anisotropic. For this reason, the terminology has shifted from  $\theta$ -anisotropic to  $\theta$ -split, and this shift begins in Springer and Helminck literature. In [Vus74], Vust shows that such tori exist.

**Lemma 2.2.1.** *In a nontrivial group  $G$ , there exists a nontrivial  $\theta$ -split torus.*

Let  $T$  be a  $\theta$ -split torus. Then  $\tau(T) \subseteq T$  because  $\theta(\tau(t)) = t^{-2} = \theta(t)t^{-1} = \tau(t)^{-1}$  for any  $t \in T$ . When  $G$  is over  $k$  an algebraically closed field, or  $k$  such that  $(k^*)^2 = (k^*)^4$ , these sets are equal. Note that  $\tau(T) \subset Q$  by definition, and when working over an algebraically closed field, this implies that  $\tau(T) = T \subseteq Q$ .

**Example 2.2.1.** Let  $k = \mathbb{F}_p$  for some prime  $p$  and let  $G = \mathrm{GL}(2, k)$ . Consider the torus

$$T = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x + y \neq 0, x, y \in \mathbb{F}_p \right\}$$

Then  $\tau(T) = \left\{ \begin{pmatrix} x^2 & 0 \\ 0 & y^2 \end{pmatrix} \mid x, y \in \mathbb{F}_p \right\}$ . Thus,  $\tau(T) \subsetneq T$ .  $\diamond$

In [Ric82], Richardson presents the theorem below. It provides important structure for studying semisimple elements inside of the symmetric space  $Q$  by linking semisimple elements to tori.

**Theorem 2.2.1.** *Let  $Q \simeq G/H$  be a symmetric space defined over an algebraically closed field. Let  $x \in Q$  be semisimple. Then  $x$  is contained in a  $\theta$ -split torus.*

Theorem 2.2.1 is fundamental to this thesis. Every semisimple element in  $Q$  is contained in a  $\theta$ -split torus, thus we can study semisimple elements by studying  $\theta$ -split tori, which will ultimately help us study general elements in  $Q$ .

When we study  $H$  acting on  $Q \simeq G/H$ , because of Theorem 2.1.1, we can study  $H$  acting on semisimple elements in  $Q$ . In order to study this action on semisimple elements, we can study the  $H$  action on  $\theta$ -split tori of  $G$  contained in  $Q$  by Theorem 2.2.1. In particular, a maximal  $\theta$ -split torus is a torus that is maximal when compared with other  $\theta$ -split tori. Throughout this thesis, let  $\mathcal{A}_\theta$  denote the set of maximal  $\theta$ -split tori in  $G$ , and let  $A \in \mathcal{A}_\theta$  be a maximal  $\theta$ -split torus. In the theorem below, Vust shows us how  $H$  acts on these maximal  $\theta$ -split tori.

**Theorem 2.2.2.** *Let  $A_1, A_2 \in \mathcal{A}_\theta$  be maximal  $\theta$ -split tori in  $G$ . Then there exists an  $h \in H^0$  such that  $h \cdot A_1 = A_2$ .*

This result is of fundamental importance in this thesis. Vust shows that all maximal  $\theta$ -split tori are  $H$ -conjugate when the field is algebraically closed. Over general fields, this result does not hold, that is, there is sometimes more than one  $H_k$ -conjugacy class of maximal  $\theta$ -split  $k$ -tori. In this thesis, we begin to determine when maximal  $\theta$ -split  $k$ -tori are  $H_k$ -conjugate, that is, how  $H$ -conjugacy classes break up into  $H_k$ -conjugacy classes.

## 2.3 The $H_k$ -orbits in $Q_k$ over General Fields

In the field of symmetric spaces, Riemannian symmetric spaces were first studied because they arise naturally over the reals. While Riemannian symmetric spaces are not technically defined over an algebraically closed field, many of their properties also hold in symmetric spaces when  $k$  is algebraically closed. Many of these properties do not necessarily hold in symmetric  $k$ -varieties over general fields.

One such property in the Riemannian symmetric space is that the fixed point group  $H_k$  is compact. In general fields  $k$ , the equivalent property is that  $H_k$  is  $k$ -anisotropic. When  $H_k$  is  $k$ -anisotropic, many of the additional properties of Riemannian symmetric spaces still hold.

Over general fields, including the reals,  $H_k$  is not always compact or  $k$ -anisotropic. Recall Examples 1.3.1 and 1.3.3, both of which illustrated groups over the reals. In Example 1.3.1 we used a Cartan involution, and the resulting fixed point group was

compact, whereas in Example 1.3.3 we used a non-Cartan involution, and the resulting fixed point group was noncompact. Therefore, throughout this thesis, assume  $H_k$  is not  $k$ -anisotropic, unless we explicitly state otherwise.

As we saw in Property 2, in a Riemannian symmetric space,  $Q_k$  consists only of semisimple elements. In [Hel10], Helminck shows that this property translates to symmetric  $k$ -varieties when  $H_k$  is  $k$ -anisotropic and the characteristic of the field  $k$  is zero.

**Theorem 2.3.1.** *Let  $k$  be a field such that  $\text{ch}(k) = 0$ . Let  $G$  be a connected reductive algebraic group over  $k$ . For an involution  $\theta \in \text{Aut}(G)$ , let  $Q = \{x\theta(x)^{-1} \mid x \in G\}$  be the symmetric space. For  $\mathfrak{g} = L(G)$ , we have the eigenspace decomposition  $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ , where  $\mathfrak{h} = L(H)$ . Assume  $H$  is  $k$ -anisotropic. Then the following conditions are true.*

1. *The symmetric  $k$ -variety  $Q_k$  consists of semisimple elements.*
2. *The symmetric  $k$ -variety  $\mathfrak{q}_k$  consists of semisimple elements.*

Below are some examples demonstrating that the condition that  $H_k$  is  $k$ -anisotropic is necessary. Recall that for  $g \in G$  and  $A \in \text{GL}(n, k)$ , the notation  $\text{Int}(A)(g)$  denotes  $A \cdot g$ .

**Example 2.3.1.** Let  $k = \mathbb{R}$  and  $G = \text{SL}(2, \mathbb{R})$ . Take the involution  $\theta(g) = (g^T)^{-1}$  for all  $g \in G$ . Then  $H_k = \text{SO}(2, \mathbb{R})$ , which is compact. The symmetric  $k$ -variety  $Q_k$  is the set of symmetric matrices with positive real eigenvalues, and thus consists of semisimple elements.  $\diamond$

**Example 2.3.2.** In the previous chapter, Example 1.3.3 demonstrated how the involution  $\theta(g) = \text{Int}(A)(g)$  with  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  over  $G = \text{SL}(2, \mathbb{R})$  led to a noncompact  $H_k$ . In Example 1.3.5, we saw that the resulting symmetric  $k$ -variety contained nonsemisimple elements like  $\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$ . Thus, we can begin to see that  $H_k$  being compact is necessary.  $\diamond$

In Example 2.3.1, the involution  $\theta(g) = (g^T)^{-1}$  for all  $g \in G$  can also be represented as  $\theta(g) = \text{Int}(A)(g)$  with  $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . A critical difference between the two examples is the lower left matrix entry 1 in Example 2.3.1 and  $-1$  in Example 2.3.2. We will expand on this important difference in the next section, in which we discuss involutions over  $\text{SL}(2, k)$ .

The following results carries over into symmetric  $k$ -varieties, as shown by Helminck [Hel].

**Theorem 2.3.2.** *Let  $k$  be any field of characteristic not 2. Let  $x \in Q$  be semisimple. Then  $x$  is contained in a  $\theta$ -split torus.*

Richardson originally presented this proof over algebraically closed fields, and in [Ric82], Springer presents a proof that holds over fields of characteristic  $p$ . The proof of this theorem over general fields is similar to the proof over algebraically closed fields.

## 2.4 The $k$ -split Tori

We study  $k$ -tori of  $G$  that are  $\theta$ -split. Historically,  $k$ -split tori were studied first, thus we now review some of results about  $k$ -split tori from Helminck and Wang in [HW93]. For these results, assume that  $k$  is an infinite field.

Recall from Property 1 in the previous chapter that in a Riemannian symmetric space, whenever we have a maximal compact subgroup of  $G$ , we automatically have a Cartan involution  $\theta$ . In symmetric  $k$ -varieties over general fields, we need to refine these concepts to properly define to the analogous involutions.

**Definition 2.4.1.** Let  $G$  be a reductive algebraic group defined over  $k$ ,  $\theta \in \text{Aut}(G)$  a  $k$ -involution, and  $A$  a maximal  $k$ -split torus of  $G$ . Then  $\theta$  is a **generalized Cartan involution** if the following three conditions are met.

1. The fixed point group  $H$  is  $k$ -anisotropic.
2. The field  $k$  satisfies  $(k^*)^2 = (k^*)^4$ .
3. The Cartan decomposition  $G_k = H_k A_k H_k$  holds.

The second condition states that the field also contains the square roots of its positive numbers. The third condition can be replaced using any of the conditions in Proposition 2.4.1, given below.

**Proposition 2.4.1.** *Assume that  $H$  is  $k$ -anisotropic and  $(k^*)^2 = (k^*)^4$ . Let  $A$  be a maximal  $k$ -split torus of  $G$ . Then the following conditions are equivalent.*

1. The group  $G_k = H_k^0 A_k H_k^0$ .



2. The group  $G_k = H_k A_k H_k$ .
3. The group  $G_k = U_k A_k H_k^0$  and  $Q_k$  consists of  $k$ -split semisimple elements.
4. The group  $G_k = U_k A_k H_k$  and  $Q_k$  consists of  $k$ -split semisimple elements.
5. The group  $G_k = Q_k H_k$  and  $Q_k$  consists of  $k$ -split semisimple elements.

To be a generalized Cartan involution, the associated fixed point group must be  $k$ -anisotropic, however, it is possible for  $H_k$  to be  $k$ -anisotropic without the involution being a Cartan involution. When  $H_k$  is  $k$ -anisotropic, we have the decomposition  $G_k = (H \cdot P)_k$ , where  $P$  is a minimal parabolic  $k$ -subgroup. When the involution is Cartan, we have the stronger decomposition  $G_k = H_k P_k$  and the Cartan decomposition  $G_k = H_k A_k H_k$ . Note that over the reals, the Cartan decomposition and the Iwasawa decomposition,  $G_k = H_k A_k U_k$ , where  $U$  is the unipotent radical of  $P$ , are equivalent.

Note that over the finite fields, no notion of Cartan involution exists. Some of the properties of Cartan involutions extend to involutions over finite fields.

**Proposition 2.4.2.** *Let  $G$  be a reductive Lie group over a field  $k$ . Let  $H$  be maximal  $k$ -anisotropic,  $\theta$  a Cartan involution and  $A$  a maximal  $k$ -split torus. Then the following are true.*

1. The torus  $A$  is  $(\theta, k)$ -split.
2. Maximal  $(\theta, k)$ -split tori are  $H_k^0$ -conjugate.
3. The Weyl group  $W_{G_k}(A) = W_{H_k}(A)$ .

The first two parts of Proposition 2.4.2 shows that when the field is infinite and the involution is a generalized Cartan involution,  $k$ -split tori are automatically  $(\theta, k)$ -split tori, and moreover, they are conjugate. The third part leads to the corollary below.

**Corollary 2.4.1.** *Let  $G$  be a reductive Lie group over a field  $k$ . Let  $H$  be maximal  $k$ -anisotropic,  $\theta$  a Cartan involution,  $A$  a maximal  $(\theta, k)$ -split torus, and  $a \in A$ . Then  $H \cdot a \cap W_{H_k} \cdot a$ .*

## 2.5 Involutions over $\mathrm{SL}(2, k)$

In this thesis, we study the  $H_k$ -action on the symmetric  $k$ -varieties  $Q_k \simeq G_k/H_k$  when  $G = \mathrm{SL}(2, k)$  and  $k$  is the rationals, finite fields, and  $p$ -adic fields. More specifically, we act  $H_k$  on  $\theta$ -split  $k$ -tori of  $G_k$  contained in  $Q_k$ . Thus, we first review some results about involutions over  $\mathrm{SL}(2, k)$ . We will follow the notations of Helminck, Wang, and Beun as shown in [HW02], [Beu08], and [BH09]. Throughout this section, let  $G = \mathrm{SL}(2, k)$  where  $k$  is a field of characteristic not equal to 2. Let  $\mathrm{Aut}(G)$  be the group of automorphisms of  $G$  and  $\mathrm{Int}(G)$  be the group of inner automorphisms of  $G$ .

**Definition 2.5.1.** Two involutions  $\theta, \phi \in \mathrm{Aut}(G)$  are  **$k$ -isomorphic** if and only if there is a  $\chi \in \mathrm{Int}(G)$  such that  $\chi^{-1}\theta\chi = \phi$ .

Involutions that are  $k$ -isomorphic are also called  $k$ -conjugate. In [HW02], Helminck and Wu classify all involutions over  $\mathrm{SL}(2, k)$  that are unique up to isomorphism.

**Lemma 2.5.1.** *All the  $k$ -isomorphism classes of involutions over  $G$  are of the form  $\mathrm{Int}(A)$  where  $A \in \mathrm{GL}(2, k)$  is of the form  $A = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ .*

The proof of this lemma relies on the fact that we can multiply the conjugating matrix  $A$  by a scalar in an extension field of  $k$  without affecting the involution, as shown in [HW02]. Helminck and Wu then classify which values of the matrix entry  $m \in k$  create isomorphic involutions. To better understand their classification, we first need the definition given below. Let  $k^*$  be the product group of all nonzero elements in the field  $k$ .

**Definition 2.5.2.** Let  $(k^*)^2$  be the set of all squares in  $k^*$ . The **square classes** in  $k^*$  are the representatives of the coset group  $k^*/(k^*)^2$ .

Square classes have been studied many mathematicians. In a sense, square classes measure the square-ness of a number. To understand square classes more, consider the following example.

**Example 2.5.1.** Let  $k = \mathbb{F}_p$  with  $p = 7$ . Then  $(k^*)^2 = \{1, 2, 4\}$  is the set of squares and the set of nonsquares is  $\{3, 5, 6\}$ . For simplicity, in  $\mathbb{F}_p$ , we typically represent the squares with 1 and the nonsquares with the smallest of the nonsquares, in this case, 3. This makes  $k^*/(k^*)^2 = \{\bar{1}, \bar{3}\}$ .  $\diamond$

Note that  $k^*/(k^*)^2$  is a group of cosets. In practice, we use coset representatives in calculations involving involutions of  $\text{SL}(2, k)$ . By abuse of notation, we often will drop the bar notation from a coset.

The following lemma about square classes can be found in [Ser73].

**Lemma 2.5.2.** *Let  $k$  be a finite field. Then  $|k^*/(k^*)^2| = 2$ .*

To see this, consider the mapping  $\phi : \mathbb{F}_p \rightarrow \mathbb{F}_p$  given by  $\phi(x) = x^2$ . Then  $\phi(\mathbb{F}_p^*) = (\mathbb{F}_p^*)^2$  is a normal subgroup and hence  $|\mathbb{F}_p^*/(\mathbb{F}_p^*)^2| = 2$ .

**Corollary 2.5.1.** *A square multiplied by a nonsquare is a nonsquare. A nonsquare multiplied by a nonsquare is a square.*

*Proof.* Let  $x_1^2$  be our square and  $N_p x_2^2$  and  $N_p x_3^2$  be our nonsquares. A square multiplied by a nonsquare gives us  $x_1^2 \cdot N_p x_2^2 = N_p (x_1 x_2)^2$  a nonsquare. A nonsquare multiplied by a nonsquare gives us  $N_p x_2^2 \cdot N_p x_3^2 = (N_p x_2 x_3)^2$  a square.  $\square$

The following lemma is useful for computations involving square classes. It can be found in [Ser73], and Wu gives a compact proof of it in [Wu02].

**Lemma 2.5.3.** *Let  $p$  be a prime and  $\mathbb{F}_p$  a finite field of order  $p$ . Then  $-1$  is a square if and only if  $p \equiv 1 \pmod{4}$ .*

*Proof.* Suppose  $x^2 = -1$  for some  $x \in \mathbb{F}_p$ . Then  $|x| = 4$ , and  $|\mathbb{F}_p^*| = p - 1$ , thus  $4|p - 1$ . Therefore,  $p$  must be equivalent to 1 mod 4. The argument reverses.  $\square$

The following corollary is the extension of Lemma 2.5.3 over  $\mathbb{Q}_p$  when  $p \neq 2$ .

**Corollary 2.5.2.** *Let  $p$  be an odd prime and consider  $\mathbb{Q}_p$ . Then  $-1$  is a square if and only if  $p \equiv 1 \pmod{4}$ . When  $p \equiv 3 \pmod{4}$ ,  $-1 \equiv N_p$ .*

*Proof.* Over  $\mathbb{Q}_p$ , the equivalent of  $-1$  is  $(p - 1), \overline{(p - 1)}$ . Thus,  $-1$  is a square if and only if  $p - 1 \equiv -1$  is a square in  $\mathbb{F}_p$ . This happens exactly when  $p \equiv 1 \pmod{4}$ . When  $p \equiv 3 \pmod{4}$ ,  $p - 1 \equiv -1 \equiv N_p$  in  $\mathbb{F}_p$ .  $\square$

**Lemma 2.5.4.** *Let  $p = 2$  and consider  $\mathbb{Q}_p$ . Then  $-1$  not a square.*

*Proof.* Over  $\mathbb{Q}_p$  when  $p = 2$ ,  $-1 = 1, \bar{1}$ . Let  $x = a_0 + a_1 p + a_2 p^2 + \cdots \in \mathbb{Q}_p$ . Then  $x^2 = a_0^2 + 2a_0 a_1 + (2a_0 a_2 + a_1^2)p^2 + \cdots = a_0^2 + a_1^2 p^2 + \cdots$ . Thus,  $x^2 \neq 1, \bar{1}$ .  $\square$

The following theorem shows how square classes impact conjugacy classes of involutions over  $\mathrm{SL}(2, k)$ .

**Theorem 2.5.1.** *Suppose  $\theta, \phi \in \mathrm{Aut}(G)$  are involutions such that  $\theta = \mathrm{Int}(A)$  with  $A = \begin{pmatrix} 0 & 1 \\ m_a & 0 \end{pmatrix}$  and  $\phi = \mathrm{Int}(B)$  with  $B = \begin{pmatrix} 0 & 1 \\ m_b & 0 \end{pmatrix}$ . Then  $\theta$  is conjugate to  $\phi$  if and only if  $m_b/m_a$  is a square in  $k^*$ .*

Theorem 2.5.1 is extremely important in this thesis. It shows us that isomorphism classes of involutions in  $\mathrm{SL}(2, k)$  depend exactly on the square classes in the field  $k$ .

**Example 2.5.2.** Recall Examples 1.3.1 and 1.3.3. In Example 1.3.1, the involution has  $m = -1$ , which is a nonsquare in  $\mathbb{R}$ , and the resulting fixed point group is compact. In Example 1.3.3, the involution has  $m = 1$ , which is a square in  $\mathbb{R}$ , and the resulting fixed point group is noncompact. The ratio of these  $m$ -values is  $-1$ , a nonsquare in  $\mathbb{R}$ . Thus, these involutions produce different results because they are not  $k$ -isomorphic.  $\diamond$

**Corollary 2.5.3.** *The number of isomorphism classes of involutions of  $\mathrm{SL}(2, k)$  is  $|k^*/(k^*)^2|$ .*

Further, we can exactly determine the form of the involutions. From Theorem 2.5.1, we know each involution has form  $\theta = \mathrm{Int}(A)$  with  $A = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ , and we can take  $m \in k^*$  to be a coset representative from  $k^*/(k^*)^2$ , giving us our  $|k^*/(k^*)^2|$  involutions. The following proposition shows us exactly how to determine these involutions, and is a compilation of results from [Mah81] and [HW02].

**Proposition 2.5.1.** *Let  $G = \mathrm{SL}(2, k)$  and  $\theta \in \mathrm{Aut}(G)$  be  $\theta = \mathrm{Int}(A)$  with  $A = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ . Let  $N_p \in k^*$  be the smallest nonsquare in the field  $k$ . Then we can take  $m$  to be a coset representative of the following square classes to create unique involutions over  $G$ .*

1. *Algebraically closed fields: In this case,  $k^*/(k^*)^2 \simeq \{1\}$ , creating 1 involution.*
2. *Real numbers,  $\mathbb{R}$ : In this case,  $k^*/(k^*)^2 \simeq \mathbb{Z}_2 \simeq \{1, -1\}$ , creating 2 involutions.*
3. *Rational numbers,  $\mathbb{Q}$ : In this case  $|k^*/(k^*)^2| = \infty$ , creating an infinite number of involutions.*
4. *Finite fields,  $\mathbb{F}_p$  for  $p \neq 2$ : In this case,  $k^*/(k^*)^2 \simeq \mathbb{Z}_2 \simeq \{1, N_p\}$ , creating 2 involutions.*

5. *p*-adic fields,  $\mathbb{Q}_p$  for  $p \neq 2$ : In this case,  $k^*/(k^*)^2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \{1, N_p, p, pN_p\}$ , creating 4 involutions.
6. *p*-adic fields,  $\mathbb{Q}_p$  for  $p = 2$ : In this case,  $k^*/(k^*)^2 = \{1, -1, 2, -2, 3, -3, 6, -6\}$ , creating 8 involutions.

An explanation of the square classes of  $\mathbb{Q}_p$  when  $p \neq 2$  is given below in Example 2.5.3. An explanation of the square classes of  $\mathbb{Q}_p$  when  $p = 2$  is given by Mahler in [Mah81].

**Example 2.5.3.** Let  $k = \mathbb{Q}_p$  with  $p \neq 2$ . Recall that  $\mathbb{Q}_p$  is the completion of  $\mathbb{Q}$  with respect to the  $p$ -norm, creating

$$\mathbb{Q}_p = \left\{ \sum_{i=-n}^{\infty} a_i p^i \mid a_i \in \{0, 1, \dots, p-1\}, a_{-n} \neq 0, n \in \mathbb{Z} \right\}.$$

To determine the square classes in  $\mathbb{Q}_p$ , we can consider  $x$  to be a  $p$ -adic integer without loss of generality. Thus, take  $x \in \mathbb{Q}_p$  to be  $x = a_0 + a_1 p + a_2 p^2 + \dots$  with  $a_0 \neq 0$  by definition. Then  $x^2 = a_0^2 + 2a_0 a_1 p + (2a_0 a_2 + a_1^2) p^2 + \dots$ . If  $x^2 = p$ , then  $a_0 = 0$ , a contradiction, thus  $p$  is not a square. Notice that the leading coefficient of  $x^2$  is a square. Thus  $N_p$  is also not a square in  $\mathbb{Q}_p$ . The leading coefficient also shows us that  $pN_p$  is not a square. Now, by way of contradiction, assume  $p$  and  $N_p$  are in the same coset, that is,  $p \equiv N_p$ . Then  $pN_p \equiv p^2$ , a square, which is a contradiction. Thus,  $p$  and  $N_p$  are in different cosets. By similar logic,  $pN_p$  is also in a different coset. Allowing 1 to represent the coset of squares, for  $p \neq 2$  we get

$$\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 \simeq \{1, N_p, p, pN_p\}. \quad (2.1)$$

◇

We know that any involution of  $G_k = \text{SL}(2, k)$  has the form  $\theta = \text{Int}(A)$  where  $A$  is of the form  $A = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$  and  $m$  is a representative of  $k^*/(k^*)^2$ . Then the fixed point group of  $\theta$  in  $G_k$  becomes

$$H_k = \left\{ \begin{pmatrix} x & y \\ my & x \end{pmatrix} \mid x^2 - my^2 \equiv 1, x, y \in k \right\}. \quad (2.2)$$

Recall from the previous chapter that many of the results about Riemannian symmetric spaces were because of the compactness of the fixed point group. In [Beu08], Beun determines when the fixed point group of  $\mathrm{SL}(2, k)$  is  $k$ -anisotropic for general  $k$ .

**Theorem 2.5.2.** *Let  $G = \mathrm{SL}(2, k)$  and  $\theta \in \mathrm{Aut}(G)$  be a  $k$ -involution of form  $\theta = \mathrm{Int}(A)$  where  $A = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$  and  $m$  is a representative of  $k^*/(k^*)^2$ . Then  $H_k$  is  $k$ -anisotropic if and only if  $m \not\equiv 1$ .*

Thus, an involution can not be a Cartan involution if  $m \equiv 1$ .

## 2.6 The $(\theta, k)$ -split Tori in $\mathrm{SL}(2, k)$ over General Fields

In [Beu08], Beun classifies the  $H_k$ -conjugacy classes of  $(\theta, k)$ -split maximal tori in  $\mathrm{SL}(2, k)$ . Her classifications depend on the theorem below from Helminck and Wang in [HW93], and note that the only restriction on  $k$  is that the characteristic is not 2.

**Theorem 2.6.1.** *Let  $A_1$  and  $A_2$  be maximal  $(\theta, k)$ -split  $k$ -tori, and let  $T_1 \subset A_1$  be a maximal  $k$ -split torus. Then there exists a  $g \in (H \cdot Z_G(T_1))_k$  such that  $gA_1g^{-1} = A_2$ .*

This theorem assumes the  $k$ -tori are  $k$ -split. Historically,  $k$ -split tori were studied before involutions were acted upon  $k$ -tori. In [BT65], Borel and Tits showed that  $k$ -split tori are  $G_k$ -conjugate if and only if minimal parabolic  $k$ -subgroups are  $G_k$ -conjugate. These parabolic subgroups play a major role in the study of algebraic groups, for instance, we have the Bruhat decomposition  $G_k = P_k W_{G_k}(A) P_k$ , where  $P_k$  is a minimal parabolic group and  $A$  is a maximal torus. There is no result analogous to that of Borel and Tits connecting  $k$ -anisotropic tori to parabolic subgroups, because usually there is an infinite number of  $G_k$ -conjugation classes. Therefore, Theorem 2.6.1, and hence the technique that Beun introduces, does not apply  $\theta$ -split  $k$ -tori that do not split over  $k$ , one of the main areas of focus of this thesis.

Let  $\mathcal{A}_\theta$  denote the set of maximal  $\theta$ -split  $k$ -tori in  $G_k$  and  $\mathcal{A}_{(\theta, k)}$  the set of maximal  $(\theta, k)$ -split tori in  $G_k$ . This notation has been adapted slightly from the notation Beun uses in [Beu08] to emphasize that maximal  $\theta$ -split  $k$ -tori that are not split over  $k$  will also be considered.

Beun uses Theorem 2.6.1 in the following way. Consider the torus

$$T = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in k^* \right\} \subset \mathrm{SL}(2, k). \quad (2.3)$$

and note that it is maximal in  $G_k$ . By construction,  $H_k$  acts on  $(H \cdot T)_k$  by left multiplication. By Theorem 2.6.1, there exists a  $g \in (H \cdot T)_k$  such that  $gTg^{-1} = T_i$ , where  $T_i$  is a  $(\theta, k)$ -split maximal torus. Thus, each element of  $(H \cdot T)_k$  corresponds to a  $(\theta, k)$ -split maximal torus by conjugating  $T$  in Equation 2.3. This implies that  $H_k$  acts on the set  $\mathcal{A}_{(\theta, k)}$  by acting on  $(H \cdot T)_k$ . Therefore, by classifying the left  $H_k$  cosets of  $(H \cdot T)_k$ , she classifies the  $H_k$ -conjugacy classes of  $\mathcal{A}_{(\theta, k)}$  simultaneously.

Below are the summaries of the  $H_k$ -conjugacy classes of maximal  $(\theta, k)$ -split  $k$ -tori in  $\mathrm{SL}(2, k)$  for algebraically closed, the real, the finite, and the  $p$ -adic fields.

**Lemma 2.6.1.** *Let  $k$  be algebraically closed. Then there is one  $H_k$ -conjugacy class of maximal  $(\theta, k)$ -split tori.*

**Lemma 2.6.2.** *Let  $k = \mathbb{R}$ . Then there is one  $H_k$ -conjugacy class of maximal  $(\theta, k)$ -split tori for each value of  $m$ .*

**Lemma 2.6.3.** *Let  $k = \mathbb{Q}$ . Then there is an infinite number of  $H_k$ -conjugacy classes of maximal  $(\theta, k)$ -split tori for each value of  $m$ .*

**Theorem 2.6.2.** *Let  $k = \mathbb{F}_p$  for  $p \neq 2$  and let  $N_p$  be the smallest nonsquare in  $k$ . Then we have the following number of  $H_k$ -conjugacy classes of maximal  $(\theta, k)$ -split tori.*

1. *When  $p \equiv 1$  and  $m \equiv N_p$ , there is 1 class of maximal  $(\theta, k)$ -split tori.*
2. *When  $p \equiv 1$  and  $m \equiv 1$ , there are 2 classes of maximal  $(\theta, k)$ -split tori.*
3. *When  $p \equiv 3$  and  $m \equiv 1$ , there is 1 class of maximal  $(\theta, k)$ -split tori.*
4. *When  $p \equiv 3$  and  $m \equiv N_p$ , there are 2 classes of maximal  $(\theta, k)$ -split tori.*

**Theorem 2.6.3.** *Let  $k = \mathbb{Q}_p$  for  $p \neq 2$  and let  $N_p$  be the smallest nonsquare in  $k$ . Then we have the following number of  $H_k$ -conjugacy classes of maximal  $(\theta, k)$ -split tori.*

1. *When  $m \equiv p, pN_p$ , regardless of  $p$ , there is 1 class of maximal  $(\theta, k)$ -split tori.*
2. *When  $p \equiv 1$  and  $m \equiv N_p$ , there is 1 class of maximal  $(\theta, k)$ -split tori.*

3. When  $p \equiv 1$  and  $m \equiv 1$ , there are 4 classes of maximal  $(\theta, k)$ -split tori.
4. When  $p \equiv 3$  and  $m \equiv 1$ , there are 2 classes of maximal  $(\theta, k)$ -split tori.
5. When  $p \equiv 3$  and  $m \equiv N_p$ , there are 2 classes of maximal  $(\theta, k)$ -split tori.

**Theorem 2.6.4.** *Let  $k = \mathbb{Q}_p$  for  $p = 2$  and let  $N_p$  be the smallest nonsquare in  $k$ . Then we have the following number of  $H_k$ -conjugacy classes of maximal  $(\theta, k)$ -split tori.*

1. When  $m \equiv \pm 2, \pm 3, \pm 6$ , regardless of  $p$ , there are 2 classes of maximal  $(\theta, k)$ -split tori.
2. When  $m \equiv \pm 1$ , regardless of  $p$ , there are 4 classes of maximal  $(\theta, k)$ -split tori.

## 2.7 The $\theta$ -split $k$ -tori in $\mathrm{SL}(2, k)$ over General Fields

In the previous chapter, we saw that in Riemannian symmetric spaces, every element is naturally semisimple element, and this implies that every element is also naturally  $\mathbb{R}$ -split. For general fields  $k$ , in symmetric  $k$ -varieties there are elements that do not split over the base field  $k$ . In [Beu08] and [BH09], Beun and Helminck classified the  $H_k$ -conjugacy classes of maximal  $(\theta, k)$ -split tori in  $\mathrm{SL}(2, k)$ . In this thesis, we study the  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori. We combine our results with those from Beun and Helminck to give a total count of  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori in  $\mathrm{SL}(2, k)$ . The following is a summary of our results.

First, note that all of this theory thus far is given at the Lie group level, and the corresponding results hold at the Lie algebra level. We choose to work over Lie algebras to make the computations less cumbersome, and the results can be lifted to Lie groups.

The maximal  $\theta$ -split  $k$ -anisotropic tori take two forms, Type 2 and Type 3 listed below.

$$\begin{aligned} \mathfrak{t}_2 &= \left\{ \begin{pmatrix} 0 & x \\ -mx & 0 \end{pmatrix} \mid x \in k \right\} = \left\langle \begin{pmatrix} 0 & 1 \\ -m & 0 \end{pmatrix} \right\rangle \\ \mathfrak{t}_3 &= \left\{ \begin{pmatrix} x & x\gamma \\ -mx\gamma & -x \end{pmatrix} \mid x \in k, \gamma \in k \text{ fixed} \right\} \\ &= \left\langle \begin{pmatrix} 1 & \gamma \\ -m\gamma & -1 \end{pmatrix} \mid \gamma \in k \text{ fixed} \right\rangle \end{aligned}$$



The Type 2 tori only have one  $H_k$ -conjugacy class, as shown in Lemma 3.2.1 below.

**Lemma 2.7.1.** *Fix an  $m \in k^*/(k^*)^2$ . Then there is exactly one  $H_k$ -conjugacy class of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 2.*

Type 3 tori are more complicated. There is a torus for every value of  $\gamma$  in  $\mathfrak{t}_3$  that makes the torus  $k$ -anisotropic. One of the defining characteristics of the  $H_k$ -conjugacy classes of these tori is whether or not a torus constructed with  $\gamma$  maps to a torus constructed with  $-\gamma$ . Theorem 4.1.1 addresses when this is possible.

**Theorem 2.7.1.** *Assume  $k = \mathbb{F}_p$  or  $k = \mathbb{Q}_p$ . Let  $\gamma \in \Gamma_{(k,m)}$  be a value that makes the maximal  $\theta$ -split  $k$ -tori of Type 3 not split over  $k$ . Then torus generated with  $\gamma$  is  $H_k$ -conjugate to the torus generated with  $-\gamma$  if and only if  $-m \equiv 1 - m\gamma^2$ .*

We then find two matrices in  $H$  that conjugate the generators of maximal  $\theta$ -split  $k$ -anisotropic Type 3 tori to other generators. The matrix that sends a generator with  $\gamma$  to a generator with  $-\gamma$  is

$$h_- = \pm \frac{1}{[-m(1 - m\gamma^2)]^{1/2}} \begin{pmatrix} m\gamma & 1 \\ m & m\gamma \end{pmatrix}. \quad (2.4)$$

The matrix that sends a generator with  $\alpha$  to the  $r^{th}$  multiple of a generator with  $\beta$  is

$$h_r = \frac{1}{[(1+r)^2 - m(\alpha - r\beta)^2]^{1/2}} \begin{pmatrix} 1+r & \alpha - r\beta \\ m(\alpha - r\beta) & 1+r \end{pmatrix}, \quad (2.5)$$

where  $\alpha \neq \pm\beta$ .

These matrices are particularly important in certain cases in  $\mathbb{F}_p$ , where we use the fact that  $\mathbb{F}_p^*/(\mathbb{F}_p^*)^2 \simeq \mathbb{Z}_2$  to determine the number of  $H_k$ -conjugacy classes in these certain cases, as shown below in Corollary 5.2.2.

**Corollary 2.7.1.** *Let  $k = \mathbb{F}_p$ , and let  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$  or  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ . Then there are exactly 2  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3.*

Sometimes we gave a lower bound, and sometimes an upper bound, on the number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split tori. The theorems below summarize our results over the rational, finite, and  $p$ -adic fields for  $p \neq 2$ .

**Theorem 2.7.2.** *Let  $k = \mathbb{Q}$  and consider  $\mathfrak{sl}(2, \mathbb{Q})$ . Then there is an infinite number  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori.*

Over the finite fields for  $p \neq 2$ , we give the following lower and upper bounds on number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split tori.

**Theorem 2.7.3.** *Let  $k = \mathbb{F}_p$  and consider  $\mathfrak{sl}(2, \mathbb{F}_p)$ . Then there are 4  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori when  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$  or when  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ .*

**Theorem 2.7.4.** *Let  $k = \mathbb{F}_p$  and consider  $\mathfrak{sl}(2, \mathbb{F}_p)$ . Then the number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split tori when  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$  or when  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$  is either 3 or 4.*

We also proved that the lower bound was correct for all primes under 50.

**Corollary 2.7.2.** *Let  $k = \mathbb{F}_p$  where  $p < 50$  is an odd prime, and consider  $\mathfrak{sl}(2, \mathbb{F}_p)$ . Then there are 3  $H_k$ -conjugacy classes of maximal  $\theta$ -split tori when  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$  or when  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ .*

Over the  $p$ -adics, we give the following minimum number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split tori.

**Proposition 2.7.1.** *Let  $k = \mathbb{Q}_p$  for  $p \neq 2$ . Then the following is a maximum number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori in  $\mathfrak{sl}(2, \mathbb{Q}_p)$ .*

1. *There are at most 16  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori when  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$ .*
2. *There are at most 17  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori when  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ .*
3. *There are at most 16  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori of when  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$ .*
4. *There are at most 14  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori of when  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ .*
5. *There are at most 16  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori of when  $m \equiv p, N_p$ , regardless of  $p$ .*

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## Chapter 3

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### Preliminary Results about $H_k$ -conjugation Classes in $\mathfrak{sl}(2, k)$

In this chapter, we lay the groundwork for discussing the  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori in  $\mathfrak{sl}(2, k)$ . The  $k$ -tori take on three different forms, which we call Type 1, Type 2, and Type 3. Type 1 tori are always split over the base field, hence we do not need to analyze them. Type 2 and Type 3 tori are sometimes split. Tori of Type 2 have a simpler form, thus the discussion of them extends to include their actual  $H_k$ -conjugation classes. Tori of Type 3 are more complicated, and the discussion overviews the general results. Specific cases and total results are discussed in the following chapters.

#### 3.1 Characterizing Tori

Throughout this thesis, let  $G = \mathrm{SL}(2, k)$ ,  $\mathfrak{g} = \mathfrak{sl}(2, k)$ , and  $\theta \in \mathrm{Aut}(G)$  an involution. Theorem 2.5.1 shows us that all  $k$ -involutions on  $\mathfrak{sl}(2, k)$  and  $\mathrm{SL}(2, k)$  have form  $\theta = \mathrm{Int} \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ , where  $m$  is a coset representative from  $k^*/(k^*)^2$ . By abuse of notation, when we use  $m$  in calculations, we will use the actual value of  $m$  in the field, instead of the coset representative. At the Lie group level, the fixed point group  $H_k = \{x \in \mathrm{SL}(2, k) \mid \theta(x) = x\}$  is shown below.

$$H_k = \left\{ \begin{pmatrix} x & y \\ my & x \end{pmatrix} \mid x^2 - my^2 = 1, x, y \in k \right\} \quad (3.1)$$

Note that for  $h_k \in H_k$ ,  $h_k^{-1} = \begin{pmatrix} x & -y \\ -my & x \end{pmatrix}$ . At the Lie algebra level, the fixed point group,  $\mathfrak{h}$ , and symmetric  $k$ -variety,  $\mathfrak{q}$ , are shown below.

$$\mathfrak{h}_k = \left\{ \begin{pmatrix} 0 & b \\ mb & 0 \end{pmatrix} \mid b \in k \right\} \quad (3.2)$$

$$\mathfrak{q}_k = \left\{ \begin{pmatrix} a & b \\ -mb & -a \end{pmatrix} \mid a, b \in k \right\} \quad (3.3)$$

In a sense,  $\mathfrak{q}$  represents elements that are  $\theta$ -split, meaning  $\theta(X) = -X$  for  $X \in \mathfrak{sl}(2, k)$ . Thus, we are studying  $\theta$ -split tori in  $\mathfrak{sl}(2, k)$ .

Maximal  $k$ -tori in  $\mathfrak{sl}(2, k)$  that are split over both  $\theta$  and  $k$  were classified by Stacy Beun in [BH09]. We study the maximal  $k$ -tori in  $\mathfrak{sl}(2, k)$  that are  $\theta$ -split  $k$ -anisotropic, meaning we are looking at toral elements whose eigenvalues are not all contained in the field  $k$ . Beun used the notation  $\mathcal{A}_\theta$  to denote the set of maximal  $(\theta, k)$ -split tori. We study all maximal  $\theta$ -split  $k$ -tori, thus we expand this notation.

*Notation.* Let  $\mathcal{A}_\theta$  denote the set of maximal  $\theta$ -split  $k$ -tori and let  $\mathcal{A}_{(\theta, k)}$  denote the set of all  $(\theta, k)$ -split tori. Similarly, let  $\mathfrak{a}_\theta$  denote the set of maximal  $\theta$ -split toral subalgebras over  $k$ , and let  $\mathfrak{a}_{(\theta, k)}$  denote the set of maximal  $(\theta, k)$ -split toral subalgebras over  $k$ .

Thus, we begin by studying tori in  $\mathfrak{a}_\theta - \mathfrak{a}_{(\theta, k)}$ . By abuse of notation, we may refer to toral subalgebras as tori when it is clear we are working over algebras over  $k$ .

To find the  $\theta$ -split tori, we take  $X, Y \in \mathfrak{q}$  and determine when  $[X, Y] = 0$ . This leaves us with three possible tori, which we refer to as Type 1, Type 2, and Type 3, respectively.

$$\mathfrak{t}_1 = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \mid x \in k \right\} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \quad (3.4)$$

$$\mathfrak{t}_2 = \left\{ \begin{pmatrix} 0 & x \\ -mx & 0 \end{pmatrix} \mid x \in k \right\} = \left\langle \begin{pmatrix} 0 & 1 \\ -m & 0 \end{pmatrix} \right\rangle \quad (3.5)$$

$$\begin{aligned} \mathfrak{t}_3 &= \left\{ \begin{pmatrix} x & x\gamma \\ -mx\gamma & -x \end{pmatrix} \mid x \in k, \gamma \in k \text{ fixed} \right\} \\ &= \left\langle \begin{pmatrix} 1 & \gamma \\ -m\gamma & -1 \end{pmatrix} \mid \gamma \in k \text{ fixed} \right\rangle \end{aligned} \quad (3.6)$$

Notice that each toral subalgebra is maximal in  $\mathfrak{sl}(2, k)$ . Tori of Type 1 and 2 only depend on  $x \in k$ , thus making them maximal in  $\mathfrak{sl}(2, k)$ . Tori of Type 3 depend on  $x \in k$  and  $\gamma$ , a fixed constant in  $k$ , also making these tori maximal. Thus, we are studying the  $T_a$  tori Borel described in [Bor91].

**Lemma 3.1.1.** *For distinct values of  $\gamma$ , Type 3 tori are distinct.*

*Proof.* Maximal  $k$ -tori in  $\mathfrak{sl}(2, k)$  are one-dimensional. Tori of Type 3 have at least one dimension because of the variable  $x$ . Moreover, Type 3 tori have exactly one dimension because  $\gamma$  is constant in each torus. To see this, take two tori  $t_\alpha, t_\beta \in \mathfrak{t}_3$  with  $\gamma$ -values  $\alpha$  and  $\beta$ , respectively. When

$$[t_\alpha, t_\beta] = 2 \begin{pmatrix} 0 & \beta - \alpha \\ \alpha - \beta & 0 \end{pmatrix} = 0$$

we must have  $\alpha = \beta$ .  $\square$

The following lemma shows that Type 2 and Type 3 are distinct and thus must be analyzed separately.

**Lemma 3.1.2.** *Type 2 and Type 3 are distinct.*

*Proof.* Let  $t_2 = \begin{pmatrix} 0 & 1 \\ -m & 0 \end{pmatrix}$  and  $t_3 = \begin{pmatrix} 1 & \gamma \\ -m\gamma & -1 \end{pmatrix}$ . Suppose  $t_2$  conjugates to  $rt_3$ , where  $r \in k^*$ , by some  $h = \begin{pmatrix} x & y \\ my & x \end{pmatrix} \in H_k$ . This forces  $x = 0$  and  $y = 0$ , which is a contradiction.  $\square$

The eigenvalues for tori of Type 1 are  $\pm x$ . Tori of this form always split over the field  $k$ , thus we will not consider this form because their conjugation classes have already been classified.

The eigenvalues for tori of Type 2 can be determined by  $\pm(-mx^2)^{1/2}$ . Thus, tori of Type 2 are  $k$ -split when  $-mx^2$  is a square in the field, which is not always the case. We consider tori of Type 2 such that  $-mx^2$  is not a square in  $k$ .

The eigenvalues for tori of Type 3 can be determined by  $\pm x(1 - m\gamma^2)^{1/2}$ . Thus, tori of Type 3 are  $k$ -split when  $1 - m\gamma^2$  is a square in the field, which depends on the combinations of  $m$  and  $\gamma \in k$ . We consider tori of Type 3 such that  $1 - m\gamma^2$  is not a square in  $k$ .

Note that  $1 - m\gamma^2$  depends on the choice of  $m \in k^*/(k^*)^2$ , which depends on the base field  $k$ . Thus the eigenvalues depend on the base field and choice of  $m$ . When the base field is  $\mathbb{F}_p$  or  $\mathbb{Q}_p$ ,  $m$  depends on the prime  $p$ . Thus, the eigenvalues in tori of Type 3 over  $\mathbb{F}_p$  and  $\mathbb{Q}_p$  depend on both  $p$  and  $m$ . We adopt the following notation. For a given field  $k$  and  $m \in k^*/(k^*)^2$ , let  $\Gamma_{(k,m)}$  denote the set of  $\gamma$ -values such that  $1 - m\gamma^2$  is not a square in  $k$ .

## 3.2 The $H_k$ -conjugacy Classes of Type 2 Tori

Tori of Type 2 tori have form

$$t_2 = \left\{ \begin{pmatrix} 0 & x \\ -mx & 0 \end{pmatrix} \mid x \in k \right\} = \left\langle \begin{pmatrix} 0 & 1 \\ -m & 0 \end{pmatrix} \right\rangle$$

Notice that the generator depends only on  $m$ .

**Lemma 3.2.1.** *Fix an  $m \in k^*/(k^*)^2$ . Then there is exactly one  $H_k$ -conjugacy class of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 2.*

*Proof.* There is only one generator for the form of Type 2 tori. Hence for each  $m$ , there is exactly one torus. Therefore, for each  $m$ , there will be exactly one conjugation class.

□

What remains to be shown is whether or not each torus is  $k$ -split. That is, we must determine when  $-mx^2$  is a square in each field, in particular, we need to determine when  $-m$  is a square in  $k$ .

### 3.2.1 The $H_k$ -conjugacy Classes of Type 2 Tori when $k = \mathbb{R}$

Maximal  $\theta$ -split  $k$ -anisotropic tori over the real numbers have been studied by Helminck and Schwarz. Some of their results from [HS11] are summarized here, though in an altered context to match the Type 2 and Type 3 notation adopted in this thesis. Over the reals, there are two square classes, creating two generators of maximal  $\theta$ -split  $k$ -tori, as shown below.

**Example 3.2.1.** When  $m \equiv -1$ , Type 2 tori are generated by  $\left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$ . The eigenvalues of this generator are  $\pm 1$ . When  $m \equiv 1$ , Type 2 tori are generated by  $\left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$ . The eigenvalues of this generator are  $\pm i$ .  $\diamond$

**Theorem 3.2.1.** *Let  $k = \mathbb{R}$  and consider  $\mathfrak{sl}(2, \mathbb{R})$ . Then we have the following number of  $H_k$ -conjugacy classes of  $\theta$ -split not  $k$ -split  $k$ -tori of Type 2.*

1. *There are no  $H_k$ -conjugacy classes of  $\theta$ -split not  $k$ -split  $k$ -tori of Type 2 when  $m \equiv -1$ .*
2. *There is 1  $H_k$ -conjugacy class of  $\theta$ -split not  $k$ -split  $k$ -tori of Type 2 when  $m \equiv 1$ .*

*Proof.* When  $m \equiv -1$ ,  $-m \equiv 1$ . This makes the eigenvalues of tori of Type 2 are  $\pm x$ , and all tori are  $k$ -split.

When  $m \equiv 1$ ,  $-m \equiv -1$ . Therefore, the eigenvalues are not in the base field, making the torus not  $k$ -split. Lemma 3.2.1 implies that, in this case, there is exactly one  $H_k$ -conjugacy class of maximal  $\theta$ -split not  $k$ -split  $k$ -tori of Type 2.  $\square$

As shown in Theorem 2.5.2, when  $m \equiv 1$ , the fixed point group is not compact. Thus, we already see an example of a symmetric  $k$ -variety in which the involution is not Cartan, and the Cartan decomposition no longer holds.

### 3.2.2 The $H_k$ -conjugacy Classes of Type 2 Tori when $k = \mathbb{Q}$

Over the rationals, there is an infinite number of square classes.

**Corollary 3.2.1.** *Let  $k = \mathbb{Q}$  and consider  $\mathfrak{sl}(2, \mathbb{Q})$ . Fix  $m \not\equiv -1 \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$ . Then there is exactly one  $H_k$ -conjugacy class of  $\theta$ -split  $k$ -anisotropic tori of Type 2.*

*Proof.* If  $m \not\equiv -1$ , then  $-m \not\equiv 1$ . Therefore, the eigenvalues of a Type 2 torus are not in the base field, making the torus not  $k$ -split. Lemma 3.2.1 implies that, in this case, there is exactly one  $H_k$ -conjugacy class of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 2.  $\square$

### 3.2.3 The $H_k$ -conjugacy Classes of Type 2 Tori when $k = \mathbb{F}_p$

Some examples nicely illustrate when  $-m$  is a square in  $\mathbb{F}_p$ .

**Example 3.2.2.** Let  $m \equiv 1$  and first consider  $p = 5$ . In this case,  $-m \equiv -1$  and because  $5 \equiv 1 \pmod{4}$ ,  $-1 \equiv 1$  in  $k^*/(k^*)^2$  by Lemma 2.5.3. Thus,  $-m$  is a square. This makes  $-mx^2$  a square, meaning that tori of Type 2 are diagonalizable when  $p = 5$  and  $m \equiv 1$ . Thus, there are no tori for us to consider.  $\diamond$

**Example 3.2.3.** Let  $m \equiv 1$ . When  $p = 5$ ,  $p \equiv 1 \pmod{4}$ , thus  $1 \equiv -1$  in  $k^*/(k^*)^2$ , and  $-m \equiv m$ . Tori have form  $\left\{ \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \mid x \in k \right\}$ , showing that these tori will split over all of  $k^*$ .  $\diamond$

**Example 3.2.4.** Now consider  $m \equiv 1$  and  $p = 7$ . Again,  $-m \equiv -1$ , however when  $p \equiv 3 \pmod{4}$ ,  $-1 \equiv N_p$ , and thus  $-m$  is a nonsquare. Therefore  $-mx^2$  will never be a square for  $x \in \mathbb{F}_p^*$ , and tori of this form do not diagonalize.

Calculation verifies that for all  $x \in \mathbb{F}_p^*$ ,  $-mx^2$  is a nonsquare. The set of squares in  $\mathbb{F}_p$  when  $p = 7$  is  $\mathbb{F}_p^2 = \{0, 1, 2, 4\}$  and  $-m\mathbb{F}_p^2 = \{0, 3, 5, 6\}$  when  $m \equiv 1$ . Comparing these, we see that no value of  $x \in \mathbb{F}_p^*$  makes  $-mx^2$  a square in the base field  $\mathbb{F}_p$ , that is, every value of  $x \in \mathbb{F}_p^*$  makes  $-mx^2$  a nonsquare. Therefore, when  $m \equiv 1$  and  $p = 7$ , we analyze tori of Type 2 for every value of  $x \in \mathbb{F}_p^*$ .  $\diamond$

In the past examples, we saw that the  $k$ -tori were  $k$ -split or not  $k$ -split for all values of  $x \in k^*$ . This is because there is exactly one torus over  $\mathbb{F}_p$  that encompasses all values  $x \in k^*$ . Thus, there is exactly one  $H_k$ -conjugation class of not  $k$ -split tori of Type 2.

**Theorem 3.2.2.** *Let  $k = \mathbb{F}_p$  and consider  $\mathfrak{sl}(2, \mathbb{F}_p)$ . Then we have the following number of  $H_k$ -conjugacy classes of  $\theta$ -split  $k$ -anisotropic tori of Type 2.*

1. *There is 1  $H_k$ -conjugacy class of  $\theta$ -split  $k$ -anisotropic tori of Type 2 when  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$ .*



2. There are no  $H_k$ -conjugacy classes of  $\theta$ -split  $k$ -anisotropic tori of Type 2 when  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$ .
3. There is 1  $H_k$ -conjugacy class of  $\theta$ -split  $k$ -anisotropic tori of Type 2 when  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ .
4. There no  $H_k$ -conjugacy classes of  $\theta$ -split  $k$ -anisotropic tori of Type 2 when  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ .

*Proof.* First consider the  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$  case. When  $p \equiv 1 \pmod{4}$ ,  $-1 \equiv 1$  in  $k^*/(k^*)^2$ , thus  $-m \equiv m$ . This makes the eigenvalues of tori of Type 2 are  $\pm x$ , and all tori are  $k$ -split.

Similarly, consider the  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$  case. When  $p \equiv 3 \pmod{4}$ ,  $-1 \equiv N_p$  in  $k^*/(k^*)^2$ , thus  $-m \equiv N_p^2$ . This makes the eigenvalues of tori of Type 2 are  $\pm N_p x$ , and all tori are  $k$ -split.

Now consider the  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$  case. When  $p \equiv 3 \pmod{4}$ ,  $-1 \equiv N_p$  in  $k^*/(k^*)^2$ , thus  $-m \equiv N_p$ . Therefore, the eigenvalues are not in the base field, making the torus not  $k$ -split. Lemma 3.2.1 implies that, in this case, there is exactly one  $H_k$ -conjugacy class of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 2.

Similarly, consider the  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$  case. When  $p \equiv 1 \pmod{4}$ ,  $-1 \equiv 1$  in  $k^*/(k^*)^2$ , thus  $-m \equiv N_p$ . Therefore, the eigenvalues are not in the base field, making the torus not  $k$ -split. Lemma 3.2.1 implies that, in this case, there is exactly one  $H_k$ -conjugacy class of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 2.  $\square$

### 3.2.4 The $H_k$ -conjugacy Classes of Type 2 Tori when $k = \mathbb{Q}_p$ , $p \neq 2$

Generalizing from the proof of Theorem 3.2.2, we have the following theorem about the number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 2.

**Theorem 3.2.3.** *Let  $k = \mathbb{Q}_p$  and consider  $\mathfrak{sl}(2, \mathbb{Q}_p)$ . Then we have the following number of  $H_k$ -conjugacy classes of  $\theta$ -split  $k$ -anisotropic tori of Type 2.*

1. There are 3  $H_k$ -conjugacy classes of  $\theta$ -split  $k$ -anisotropic tori of Type 2 when  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p, p, pN_p$ .

2. *There are no  $H_k$ -conjugacy classes of  $\theta$ -split  $k$ -anisotropic tori of Type 2 when  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$ .*
3. *There are 3  $H_k$ -conjugacy classes of  $\theta$ -split  $k$ -anisotropic tori of Type 2 when  $p \equiv 3 \pmod{4}$  and  $m \equiv 1, p, pN_p$ .*
4. *There no  $H_k$ -conjugacy classes of  $\theta$ -split  $k$ -anisotropic tori of Type 2 when  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ .*

*Proof.* First consider the  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$  case. When  $p \equiv 1 \pmod{4}$ ,  $-1 \equiv 1$  in  $k^*/(k^*)^2$ , thus  $-m \equiv m$ . This makes the eigenvalues of tori of Type 2 are  $\pm x$ , and all tori are  $k$ -split.

Similarly, consider the  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$  case. When  $p \equiv 3 \pmod{4}$ ,  $-1 \equiv N_p$  in  $k^*/(k^*)^2$ , thus  $-m \equiv N_p^2$ . This makes the eigenvalues of tori of Type 2 are  $\pm N_p x$ , and all tori are  $k$ -split.

Now consider the  $p \equiv 3 \pmod{4}$  and  $m \equiv 1, p, pN_p$  cases. When  $p \equiv 3 \pmod{4}$ ,  $-1 \equiv N_p$  in  $k^*/(k^*)^2$ , thus  $-m \equiv N_p, pN_p, p$ , respectively. Therefore, the eigenvalues are not in the base field, making the torus not  $k$ -split. Lemma 3.2.1 implies that, in this case, there is exactly one  $H_k$ -conjugacy class of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 2 for each value of  $m$ .

Similarly, consider the  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p, p, pN_p$  cases. When  $p \equiv 1 \pmod{4}$ ,  $-1 \equiv 1$  in  $k^*/(k^*)^2$ , thus  $-m \equiv N_p, p, pN_p$ . Therefore, the eigenvalues are not in the base field, making the torus not  $k$ -split. Lemma 3.2.1 implies that, in this case, there is exactly one  $H_k$ -conjugacy class of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 2 for each value of  $m$ .  $\square$

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## Chapter 4

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# Characterizing $H_k$ -conjugacy Classes of Type 3 Tori

In this chapter, we begin to analyze Type 3 tori and lay the framework that will allow us to study Type 3 tori over specific fields. Many of the properties and formulae we establish here will be useful in future chapters.

### 4.1 Characterizing $H_k$ -conjugacy Classes of Type 3 Tori

Type 3 tori have form

$$\begin{aligned} \mathbf{t}_3 &= \left\{ \begin{pmatrix} x & x\gamma \\ -mx\gamma & -x \end{pmatrix} \mid x \in k, \gamma \in k \text{ fixed} \right\} \\ &= \left\langle \begin{pmatrix} 1 & \gamma \\ -m\gamma & -1 \end{pmatrix} \mid \gamma \in k \text{ fixed} \right\rangle \end{aligned}$$

and their eigenvalues are  $\pm x(1 - m\gamma^2)^{1/2}$ . We consider  $k$ -anisotropic tori thus we need to determine when  $1 - m\gamma^2$  is not a square in the base field. This is dependent on choice of

the constant  $\gamma$  and the square class representative  $m$ , hence it is also heavily dependent on choice of field  $k$ . Throughout the rest of the thesis, we will adopt the following notation.

*Notation.* For a given choice of  $k$  and  $m$ , let  $\Gamma_{(k,m)}$  denote the set of values of  $\gamma$  that make a torus of Type 3  $k$ -anisotropic.

Some examples will help us gain intuition about  $1 - m\gamma^2$ .

**Example 4.1.1.** Consider  $k = \mathbb{F}_p$  for  $p = 7$ . Then the set of squares in  $\mathbb{F}_p$  when  $p = 7$  is  $\mathbb{F}_p^2 = \{0, 1, 2, 4\}$ . Let  $m \equiv 1$ . Thus,  $1 - m\gamma^2$  will take on the values  $1 - \mathbb{F}_p^2 = \{0, 1, 4, 6\}$ . We need to determine for which  $\gamma$  the value  $1 - m\gamma^2$  is a nonsquare, and hence,  $t_\gamma$  is  $k$ -anisotropic. We see that  $\mathbb{F}_p^2 \cap (1 - \mathbb{F}_p^2) = \{0, 1, 4\}$ , thus  $1 - m\gamma^2$  is a square for all the values of  $\gamma$  such that  $1 - m\gamma^2 = 0, 1$ , or  $4$ . Similarly,  $1 - m\gamma^2$  is not a square for the values of  $\gamma$  such that  $1 - m\gamma^2 = 6$ . This happens precisely when  $\gamma$  is 3 or 4. Thus, there are two values of  $\gamma$  that make a toral subalgebra of Type 3  $k$ -anisotropic when  $k = \mathbb{F}_p$ ,  $p = 7$ , and  $m \equiv 1$ .  $\diamond$

**Example 4.1.2.** Once again, consider  $k = \mathbb{F}_p$  for  $p = 7$ , only this time take  $m \equiv N_p$ . We see that the smallest nonsquare in  $\mathbb{F}_p$  is 3, thus take  $m = 3$ . Thus  $1 - m\gamma^2$  will have the values  $1 - 3\mathbb{F}_p^2 = \{1, 2, 3, 5\}$ . Comparing these values to the squares in  $\mathbb{F}_p$  when  $p = 7$ , we see that we need  $\gamma$ -values such that  $1 - m\gamma^2 = 3$  or  $5$ . This happens precisely when  $\gamma$  is 1, 2, 5, or 6. Thus, there are four values of  $\gamma$  that make a toral subalgebra of Type 3  $k$ -anisotropic when  $k = \mathbb{F}_p$ ,  $p = 7$ , and  $m \equiv N_p$ .  $\diamond$

In Example 4.1.1, observe that  $\pm 3 \in \Gamma_{(\mathbb{F}_p, N_p)}$ . Similarly, in Example 4.1.2, we have  $\pm 1, \pm 2 \in \Gamma_{(\mathbb{F}_p, N_p)}$ . This pattern generalizes to other fields, giving us the result below.

**Lemma 4.1.1.** *If  $\gamma \in \Gamma_{(\mathbb{F}_p, m)}$ , then  $-\gamma \in \Gamma_{(\mathbb{F}_p, m)}$ .*

*Proof.* If  $\gamma \in \Gamma_{(\mathbb{F}_p, m)}$ , then  $1 - m\gamma^2$  is a nonsquare. Then  $1 - m(-\gamma)^2 = 1 - m\gamma^2$  is also a nonsquare.  $\square$

Examples 4.1.1 and 4.1.2 demonstrate how the choice of  $m$  can affect the number of maximal  $\theta$ -split  $k$ -anisotropic tori. Also notice that different values of  $\gamma$  will generate different tori, thus we potentially have a different torus for every  $\gamma$  such that  $1 - m\gamma^2$  is a nonsquare.

**Lemma 4.1.2.** *There is a bijective correspondence between the values in  $\Gamma_{(k,m)}$  and the set of maximal  $\theta$ -split  $k$ -anisotropic tori in  $\mathfrak{sl}(2, k)$ .*

*Proof.* Suppose  $\mathfrak{t}_\alpha$  and  $\mathfrak{t}_\beta$  are toral subalgebras of Type 3, with generators  $t_\alpha$  and  $t_\beta$ , respectively, and suppose  $\mathfrak{t}_\alpha = \mathfrak{t}_\beta$ . Then  $t_\alpha = ct_\beta$  for some  $c \in k^*$ . Thus

$$\begin{pmatrix} 1 & \alpha \\ -m\alpha & -1 \end{pmatrix} = c \begin{pmatrix} 1 & \beta \\ -m\beta & -1 \end{pmatrix},$$

which shows that  $\alpha = \beta$ .

Let  $\mathfrak{t}_\alpha$  be a toral subalgebra of Type 3. Then  $1 - m\alpha^2$  is not a square in  $k$ . By definition,  $\alpha \in \Gamma_{(k,m)}$ .  $\square$

Every value of  $\gamma$  generates a unique torus of Type 3. We want to determine when these tori are  $H_k$ -conjugate. We analyze  $H_k$ -conjugacy by determining when a generator of one torus gets sent to a multiple of another generator by an element in  $H_k$ . Through the rest of this thesis, we will adopt the following notation.

*Notation.* Consider distinct values  $\alpha, \beta \in \Gamma_{(k,m)}$ . Consider the generator created by letting  $\gamma = \alpha$  and call it  $t_\alpha$ . Call the toral subalgebra  $t_\alpha$  generates  $\mathfrak{t}_\alpha$ . Similarly, let  $t_\beta$  be the generator formed when  $\gamma = \beta$ , and its toral subalgebra be  $\mathfrak{t}_\beta$ . Let  $h_k \in H_k$  be an element of the fixed point group. Let  $r \in k$  be a constant. We conjugate the generator  $t_\alpha$  to the  $r$ -multiple of  $t_\beta$  by  $h_k$ , that is,  $h_k t_\alpha h_k^{-1} = r t_\beta$ .

To determine when this conjugation is possible, we must determine when it is possible to find  $r \in k$  and  $h_k \in H_k$ . The following lemma shows us what  $r$ -values are possible.

**Lemma 4.1.3.** *Let  $\theta = \text{Int} \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ , and let  $H_k$  be its fixed point subgroup of  $G_k$ . Let  $\alpha, \beta \in \Gamma_{(k,m)}$  be distinct, and take  $\mathfrak{t}_\alpha, \mathfrak{t}_\beta \in \mathfrak{a}_3$  with generators  $t_\alpha$  and  $t_\beta$ , respectively. Take  $h_k \in H_k$ , and suppose  $h_k$  conjugates  $t_\alpha$  to the  $r$ -multiple of  $t_\beta$ . Then  $r = -1$  or  $r = \pm \left[ \frac{1 - m\alpha^2}{1 - m\beta^2} \right]^{1/2}$ .*

*Proof.* We want to determine when the tori of Type 3 with distinct  $\gamma$ -values are  $H_k$ -conjugate. Recall that the fixed point group is

$$H_k = \left\{ \begin{pmatrix} x & y \\ my & x \end{pmatrix} \mid x^2 - my^2 = 1, x, y \in k \right\},$$

and take  $h_k = \begin{pmatrix} x & y \\ my & x \end{pmatrix} \in H_k$ .

Suppose we take distinct  $\gamma$ -values  $\alpha$  and  $\beta$ . Hence consider distinct tori  $\mathfrak{t}_\alpha$  and  $\mathfrak{t}_\beta$  with generators  $t_\alpha$  and  $t_\beta$ , respectively. Then we want to find some  $h = \begin{pmatrix} x & y \\ my & x \end{pmatrix} \in H_k$  that conjugates the generator  $t_\alpha$  to a multiple of the generator  $t_\beta$ . Call this multiple  $r$ . Thus, we want

$$\begin{pmatrix} x & y \\ my & x \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ -m\alpha & -1 \end{pmatrix} \begin{pmatrix} x & y \\ my & x \end{pmatrix}^{-1} = r \begin{pmatrix} 1 & \beta \\ -m\beta & -1 \end{pmatrix}.$$

Rewritten, this gives us

$$\begin{aligned} \begin{pmatrix} x & y \\ my & x \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ -m\alpha & -1 \end{pmatrix} &= r \begin{pmatrix} 1 & \beta \\ -m\beta & -1 \end{pmatrix} \begin{pmatrix} x & y \\ my & x \end{pmatrix} \\ \begin{pmatrix} x - m\alpha y & \alpha x - y \\ my - m\alpha x & m\alpha y - x \end{pmatrix} &= r \begin{pmatrix} x + m\beta y & y + \beta x \\ -m\beta x - my & -m\beta y - x \end{pmatrix}. \end{aligned}$$

This creates the following two relations,

$$x - m\alpha y = r(x + m\beta y), \tag{4.1}$$

$$\alpha x - y = r(y + \beta x). \tag{4.2}$$

These relations become

$$x(1 - r) = my(\alpha + r\beta),$$

$$x(\alpha - r\beta) = y(1 + r).$$

Putting these together and solving for  $r$ , we find that

$$r = \pm \left[ \frac{1 - m\alpha^2}{1 - m\beta^2} \right]^{1/2}, \tag{4.3}$$

assuming  $r \neq \pm 1, \pm \alpha/\beta$ . Therefore, we need to consider the cases when  $r = \pm 1, \pm \alpha/\beta$  separately.

Consider  $r = 1$ . Combining Relations 4.1 and 4.2 yields  $y = \alpha x$ , making

$$h = x \begin{pmatrix} 1 & \alpha \\ m\alpha & 1 \end{pmatrix}. \quad (4.4)$$

Then  $\det h = x^2(1 - m\alpha^2)$ , which is a contradiction. By assumption  $1 - m\alpha^2 \neq 1$ , thus we can not choose an  $x$  such that  $\det h = 1$ . Therefore, the  $r = 1$  case is impossible.

Consider  $r = -1$ . Combining Relations 4.1 and 4.2 yields  $\alpha = \beta$  and  $x = m\alpha y$ . The condition that  $x = m\alpha y$  makes

$$h = y \begin{pmatrix} m\alpha & 1 \\ m & m\alpha \end{pmatrix}. \quad (4.5)$$

Then  $\det h = -my^2(1 - m\alpha^2)$ . By assumption  $1 - m\alpha^2 \neq 1$ , thus  $\det h \neq 1$ , thus  $\det h = 1$  when  $-m \equiv 1 - m\alpha^2$ . Therefore, the  $r = -1$  case is possible, depending on the choice of  $p$  and  $m$ . Individual cases will be discussed in future chapters. The condition that  $\alpha = -\beta$  means that  $r = -1$  will only apply to the specific cases when we are conjugating  $\mathbf{t}_\alpha$  to  $\mathbf{t}_{-\alpha}$ .

Consider  $r = -\alpha/\beta$ . This relation forces  $y = \alpha x$ . Then this case behaves exactly as the  $r = 1$  case, meaning that  $r = -\alpha/\beta$  is impossible.

Consider  $r = \alpha/\beta$ . This relation forces  $\alpha = -\beta$  and  $x = m\alpha y$ . The condition that  $\alpha = -\beta$  forces  $r = \alpha/\beta = -1$ , which we already know is a possible  $r$ -value.

Therefore,  $r = 1$  and  $r = \alpha/\beta$  are impossible  $r$ -values,  $r = \alpha/\beta$  is a redundant  $r$ -value that only applies when  $\alpha = -\beta$ , leaving  $r = -1$  and  $r = \pm \left[ \frac{1 - m\alpha^2}{1 - m\beta^2} \right]^{1/2}$  as the only possible  $r$ -values.  $\square$

As the theorem below demonstrates, when the  $\gamma$ -values are additive inverses of each other is a defining characteristic of the  $H_k$ -conjugacy classes.

**Theorem 4.1.1.** *Assume  $k = \mathbb{F}_p$  or  $k = \mathbb{Q}_p$ . Let  $\gamma \in \Gamma_{(k,m)}$  be a value that makes the maximal  $\theta$ -split  $k$ -tori of Type 3  $k$ -anisotropic. Then torus generated with  $\gamma$  is  $H_k$ -conjugate to the torus generated with  $-\gamma$  if and only if  $-m \equiv 1 - m\gamma^2$ .*

*Proof.* Assume  $h_k = \begin{pmatrix} x & y \\ my & x \end{pmatrix} \in H_k$  is a matrix that conjugates  $t_\gamma$  to  $rt_{-\gamma}$  for some  $r \in \mathbb{F}_p^*$ . This requires

$$x - m\gamma y = r(x - m\gamma y) \quad (4.6)$$

$$x\gamma - y = r(y - x\gamma) \quad (4.7)$$

Equation 4.7 implies that either  $r = -1$  or  $y = \gamma x$ . If  $y = \gamma x$ , then  $\det(h_k) = x^2(1 - m\gamma^2)$ . By assumption the  $t_\gamma$  is  $k$ -anisotropic, thus we know that  $1 - m\gamma^2 \not\equiv 1$ . This forces  $\det(h_k) \not\equiv 1$ , which contradicts the choice of  $h_k$ , thus  $y = \gamma x$  is impossible. Inserting  $r = -1$  into Equation 4.6, we get that  $x = m\gamma y$ . Now considering Equation 4.6, Lemma 4.1.3 shows us that  $r \neq 1$ , thus we again conclude  $x = m\gamma y$ .

Thus, in order for  $t_\gamma$  to conjugate  $rt_{-\gamma}$ , we must have  $r = -1$  and  $x = m\gamma y$ . This relation implies that  $\det(h_k) = -my^2(1 - m\gamma^2) \equiv 1$ . By the definition of square classes,  $y^2 \equiv 1$ . Thus, we need  $-m(1 - m\gamma^2) \equiv 1$ , or  $-m \equiv (1 - m\gamma^2)^{-1}$ . By Proposition 2.5.1, when  $k = \mathbb{F}_p$  or  $k = \mathbb{Q}_p$ , all cosets have order 2. Thus,  $-m(1 - m\gamma^2) \equiv 1$  when  $-m \equiv 1 - m\gamma^2$ . Therefore, the  $\mathfrak{t}_\gamma$  and  $\mathfrak{t}_{-\gamma}$  maximal  $\theta$ -split not  $k$ -split tori are not  $H_k$ -conjugate when  $-m \not\equiv 1 - m\gamma^2$  because then  $-m(1 - m\gamma^2)$  will not be a square.

Now we construct the exact  $h_k \in H_k$  that conjugates  $t_\gamma$  to  $t_{-\gamma}$  in the specific cases listed above. The relation  $x = m\gamma y$  forces  $\det(h_k) \equiv -m(1 - m\gamma^2)$ , and from the above argument, we know that that  $\det(h_k) \equiv 1$  exactly when  $-m \equiv 1 - m\gamma^2$ . Thus, we may assume  $-m(1 - m\gamma^2)$  is a square in  $k^*$ , and hence  $[-m(1 - m\gamma^2)]^{1/2}$  is in the base field  $k$ . Then the matrix

$$h_- = \pm \frac{1}{[-m(1 - m\gamma^2)]^{1/2}} \begin{pmatrix} m\gamma & 1 \\ m & m\gamma \end{pmatrix} \quad (4.8)$$

conjugates the generator  $t_\gamma = \begin{pmatrix} 1 & \gamma \\ -m\gamma & -1 \end{pmatrix}$  to the  $p - 1$  multiple of the generator  $t_{-\gamma} = \begin{pmatrix} 1 & -\gamma \\ m\gamma & -1 \end{pmatrix}$ . When calculating the  $h_- t_\gamma h_-^{-1}$ , the coefficients from  $h_-$  and  $h_-^{-1}$  become  $[-m(1 - m\gamma^2)]^{-1}$ , leaving us with



$$\begin{aligned}
h_- t_\gamma h_-^{-1} &= -m(1 - m\gamma^2)^{-1} \begin{pmatrix} m\gamma & 1 \\ m & m\gamma \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ -m\gamma & -1 \end{pmatrix} \begin{pmatrix} m\gamma & -1 \\ -m & m\gamma \end{pmatrix} \\
&= [-m(1 - m\gamma^2)]^{-1} \begin{pmatrix} m\gamma & 1 \\ m & m\gamma \end{pmatrix} \begin{pmatrix} 0 & m\gamma^2 - 1 \\ -m(m\gamma^2 - 1) & 0 \end{pmatrix} \\
&= [-m(1 - m\gamma^2)]^{-1} [m(1 - m\gamma^2)] \begin{pmatrix} -1 & \gamma \\ -m\gamma & 1 \end{pmatrix} \\
&= -t_{-\gamma}.
\end{aligned}$$

Thus,  $h_-$  conjugates  $t_\gamma$  to the  $p-1$  multiple of  $t_{-\gamma}$ . Therefore, the tori they generate are  $H_k$ -conjugate.  $\square$

The matrix  $h_-$  given in Equation 4.8 plays a critical role in the  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3. Notice that  $h_- \in H$ , regardless of the values of  $p$  and  $m$ . The values  $m\gamma$ , 1, and  $m$  are all in the base field  $k$ . The value  $[-m(1 - m\gamma^2)]^{-1/2}$  is in the algebraic closure  $\bar{k}$ . Thus,  $h_- \in H$ . Moreover,  $h_-$  will be in  $H_k$  when  $[-m(1 - m\gamma^2)]^{-1/2}$  is in the base field  $k$ , that is, when  $-m(1 - m\gamma^2)$  is a square in  $k$ . As Theorem 4.1.1 states, this happens exactly when  $-m \equiv 1 - m\gamma^2$ .

Now consider  $\alpha, \beta \in \Gamma_{(k,m)}$  such that  $\alpha \neq \pm\beta$ . When we want to analyze how  $\mathfrak{t}_\alpha$  conjugates to  $\mathfrak{t}_\beta$ , we will need the following lemma.

**Lemma 4.1.4.** *Let  $\alpha, \beta \in \Gamma_{(\mathbb{F}_p, m)}$  be values that makes the maximal  $\theta$ -split tori of Type 3  $k$ -anisotropic. Assume  $\alpha \neq \pm\beta$ . Then  $t_\alpha$  is  $H$ -conjugate to  $rt_\beta$  by*

$$h_r = \frac{1}{[(1+r)^2 - m(\alpha - r\beta)^2]^{1/2}} \begin{pmatrix} 1+r & \alpha - r\beta \\ m(\alpha - r\beta) & 1+r \end{pmatrix} \quad (4.9)$$

*Proof.* Theorem 2.2.2, there exists an element in  $H$  that conjugates  $\mathfrak{t}_\alpha$  to  $\mathfrak{t}_\beta$ . Suppose  $h_r \cdot t_\alpha = rt_\beta$  for some  $h_r = \begin{pmatrix} x & y \\ my & x \end{pmatrix} \in H$ . Rewriting this conjugation as  $h_r t_\alpha = rt_\beta h_r$  forces the relations

$$x - my\alpha = r(x + my\beta) \quad (4.10)$$

$$x\alpha - y = r(x\beta + y). \quad (4.11)$$

We can rewrite these equations as

$$x(1 - r) = ym(\alpha + r\beta) \quad (4.12)$$

$$x(\alpha - r\beta) = y(1 + r), \quad (4.13)$$

Note that  $r \neq -1$  and  $r \neq \alpha/\beta$  by assumption. Also note that  $r \neq 1$  and  $r \neq -\alpha/\beta$  by Lemma 4.1.3. Moreover, Equations 4.12 and 4.13 are equal exactly when  $1 - m\alpha^2 = r^2(1 - m\beta^2)$ . By assuming  $\alpha \neq \pm\beta$ , we know from Lemma 4.1.3 that  $r = \left[ \frac{1 - m\alpha^2}{1 - m\beta^2} \right]^{1/2}$ , thus  $1 - m\alpha^2 = r^2(1 - m\beta^2)$ . Therefore, Equations 4.12 and 4.13 are equivalent. We choose to solve Equation 4.13 for  $x$ ,

$$x = y \left( \frac{1 + r}{\alpha - r\beta} \right). \quad (4.14)$$

Plugging Relation 4.14 into  $h_1$ , we create the matrix

$$h_r = \frac{y}{\alpha - r\beta} \begin{pmatrix} 1 + r & \alpha - r\beta \\ m(\alpha - r\beta) & 1 + r \end{pmatrix} \quad (4.15)$$

To solve for  $y$ , we set  $\det(h_r) = \frac{y^2}{(\alpha - r\beta)^2}((1 + r)^2 - m(\alpha - r\beta)^2) = 1$ . This creates the matrix

$$h_r = \frac{1}{[(1 + r)^2 - m(\alpha - r\beta)^2]^{1/2}} \begin{pmatrix} 1 + r & \alpha - r\beta \\ m(\alpha - r\beta) & 1 + r \end{pmatrix} \quad (4.16)$$

Moreover,  $h_r$  conjugates  $t_\alpha$  to  $rt_\beta$ . Let  $c = \frac{1}{[(1 + r)^2 - m(\alpha - r\beta)^2]^{1/2}}$ .

$$\begin{aligned} h_r t_\alpha &= c \begin{pmatrix} 1 - m\alpha^2 + r(1 + m\alpha\beta) & r(\alpha + \beta) \\ -m(r(\alpha + \beta)) & -(1 - m\alpha^2 + r(1 + m\alpha\beta)) \end{pmatrix} \\ &= rc \begin{pmatrix} (r - rm\beta^2) + (1 + m\alpha\beta) & (\alpha + \beta) \\ -m((\alpha + \beta)) & -((r - rm\beta^2) + (1 + m\alpha\beta)) \end{pmatrix} \\ &= rt_\beta h_r \end{aligned}$$

Last, note that all entries of  $h_r$  are in the base field  $k$ , and only the coefficient  $c$  is potentially in the algebraic closure  $\bar{k}$ .  $\square$

By Lemma 4.1.4, the matrix  $h_r$  conjugates  $t_\alpha$  to  $rt_\beta$  when  $\alpha \neq \pm\beta$ . Recall that for a matrix  $h = \begin{pmatrix} x & y \\ my & x \end{pmatrix} \in H_k$ , the equation  $ht_\alpha = rt_\beta h$  forces the relations in Equations 4.10 and 4.10. We could have instead added these equations together to determine

$$x = y \left( \frac{1 + m\alpha + r(1 + m\beta)}{1 + \alpha - r(1 + \beta)} \right). \quad (4.17)$$

Equation 4.17 then creates the matrix  $h_1 \in H$

$$h_1 = c \begin{pmatrix} 1 + m\alpha + r(1 + m\beta) & 1 + \alpha - r(1 + \beta) \\ m(1 + \alpha - r(1 + \beta)) & 1 + m\alpha + r(1 + m\beta) \end{pmatrix}, \quad (4.18)$$

$$c = \frac{1}{[(1 + m\alpha + r(1 + m\beta))^2 - m(1 + \alpha - r(1 + \beta))^2]^{1/2}}$$

The matrix  $h_1$  in Equation 4.18 has the drawback that it is undefined when  $r = -(1 + m\alpha)/(1 + m\beta)$  and  $r = -(1 + \alpha)/(1 + \beta)$ , which is possible.

Moreover, Equation 4.18 is a multiple of Equation 4.16. Setting

$$\frac{1 + m\alpha + r(1 + m\beta)}{1 + r} = \frac{1 + \alpha - r(1 + \beta)}{\alpha - r\beta}, \quad (4.19)$$

we find that Equation 4.19 is true exactly when  $r^2(1 - m\beta^2) = 1 - m\alpha^2$ . Thus, matrices  $h_r$  and  $h_1$  in Equation 4.18 multiples of each other, only  $h_r$  is undefined when  $r = -1$  and  $h_1$  is undefined when  $r = -(1 + m\alpha)/(1 + m\beta)$  and  $r = -(1 + \alpha)/(1 + \beta)$ . We already know  $h_-$  works as a conjugating matrix for when  $r = -1$ , and hence  $\alpha = -\beta$ , thus we choose to work with  $h_r$  instead of  $h_1$ .

The matrix  $h_1$  is defined over  $r = -1$ , and hence  $\alpha = -\beta$ . Thus, when  $\alpha = \gamma$  and  $\beta = -\gamma$ ,  $h_1$  produces the matrix  $h_-$ .

We characterize when the Type 3 tori are  $H_k$ -conjugate for  $k = \mathbb{Q}$  in the next section and  $k = \mathbb{F}_p$  and  $k = \mathbb{Q}_p$  in the next two chapters of this thesis.

## 4.2 The $H_k$ -conjugacy Classes of Type 3 Tori when $k = \mathbb{Q}$

Recall that Type 3 tori have form

$$\begin{aligned} \mathbf{t}_3 &= \left\{ \begin{pmatrix} x & x\gamma \\ -mx\gamma & -x \end{pmatrix} \mid x \in k, \gamma \in k \text{ fixed} \right\} \\ &= \left\langle \begin{pmatrix} 1 & \gamma \\ -m\gamma & -1 \end{pmatrix} \mid \gamma \in k \text{ fixed} \right\rangle. \end{aligned}$$

Notice that the generator depends on the value of  $\gamma \in \mathbb{Q}$ . Therefore, for fixed  $m$ , we have an infinite number of generators, and an infinite number of tori. Moreover, the  $\mathbb{Q}^*/(\mathbb{Q}^*)^2$  is infinite, giving us the theorem below.

**Theorem 4.2.1.** *Let  $k = \mathbb{Q}$  and fix a value of  $m$ . Then there is an infinite number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3.*

*Proof.* Take two distinct values  $\alpha, \beta \in \Gamma_{(\mathbb{Q}, m)}$ , and note that  $\Gamma_{(\mathbb{Q}, m)}$  is infinite. Without loss of generality, assume  $\alpha \neq -\beta$ . Consider the generators  $t_\alpha$  and  $t_\beta$  they create. In order for  $t_\alpha$  to be  $H_k$ -conjugate to  $t_\beta$ , the value  $(1+r)^2 - m(\alpha - r\beta)^2$ , from the coefficient of  $h_r$  in Equation 4.16, would have to be a square, that is,  $(1+r)^2 - m(\alpha - r\beta)^2 \equiv 1$ . Over  $\mathbb{Q}$ , there is an infinite number of square classes, hence, over all possible  $\alpha$  and all possible  $\beta$ , there is an infinite number of equivalences of  $(1+r)^2 - m(\alpha - r\beta)^2$ . Thus, there is an infinite number of  $H_k$ -conjugacy classes.  $\square$

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# Chapter 5

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## Type 3 Tori in $\mathfrak{sl}(2, k)$ when $k = \mathbb{F}_p$

We study maximal  $\theta$ -split  $k$ -tori that do not split over the base field  $k$ . As we saw in Chapter 3, there are three different forms that maximal  $\theta$ -split  $k$ -tori can take, and two of those forms sometimes do not split over  $k$ . In this chapter, we classify tori of the third form and characterize their  $H_k$ -orbits over the finite fields for  $p \neq 2$ . As we saw in Chapter 3,  $k$ -tori with additive-inverse  $\gamma$ -values will play a major role in this characterization.

### 5.1 Examples of Type 3 Tori

As we saw earlier, tori of Type 3 are sometimes split over the field, thus we first construct some examples. Recall that Type 3 tori have form

$$\mathfrak{t}_3 = \left\{ x \begin{pmatrix} 1 & \gamma \\ -m\gamma & -1 \end{pmatrix} \mid x \in k, \gamma \in k \text{ fixed} \right\}$$

and their eigenvalues depend on  $\pm(1 - m\gamma^2)^{1/2}$ . We consider maximal  $\theta$ -split  $k$ -tori that are not split over the field, thus we need to determine when  $1 - m\gamma^2$  is not a square in the base field  $\mathbb{F}_p$ . That is, given a fixed value of  $m$ , we need to determine what values of  $\gamma \in \mathbb{F}_p$  make  $1 - m\gamma^2$  nonsquare.

**Example 5.1.1.** Let  $m \equiv N_p$ . When  $p = 5$ , the set of squares is  $\mathbb{F}_p^2 = \{0, 1, 4\}$ , thus we can take  $N_p = 2$ . This makes  $1 - 2\mathbb{F}_p^2 = 1 + 3\mathbb{F}_p^2 = \{1, 3, 4\}$ . Comparing these, we see that

$\mathbb{F}_p^2 \cap (1 - 2\mathbb{F}_p^2) = \{1, 4\}$ , and therefore  $1 - 2\gamma^2$  will be a nonsquare for values of  $\gamma$  such that  $1 - 2\gamma^2 = 3$ . This happens exactly when  $\gamma = 2$  or  $\gamma = 3$ , that is,  $\Gamma_{(\mathbb{F}_p, N_p)} = \{2, 3\}$ . Note that Type 3 tori have form

$$\mathbf{t}_3 = \left\{ x \begin{pmatrix} 1 & \gamma \\ -2\gamma & 4 \end{pmatrix} \mid x \in k, \gamma \in k \text{ fixed} \right\}$$

when  $p = 5$  and  $m \equiv N_p$ . Therefore,  $t_2 = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$  and  $t_3 = \begin{pmatrix} 1 & 3 \\ 4 & 4 \end{pmatrix}$  will each generate maximal  $\theta$ -split  $k$ -tori not split over  $\mathbb{F}_p$  when  $p = 5$  and  $m \equiv N_p$ .  $\diamond$

**Example 5.1.2.** Let  $m \equiv N_p$ . When  $p = 7$ , the set of squares is  $\mathbb{F}_p^2 = \{0, 1, 2, 4\}$ , hence  $N_p \equiv 3$ . Thus,  $1 - N_p\mathbb{F}_p^2 = 1 - 3\mathbb{F}_p^2 = \{1, 2, 3, 5\}$ . Comparing these, we see that  $\mathbb{F}_p^2 \cap (1 + \mathbb{F}_p^2) = \{1, 2\}$ , and therefore  $1 - 3\gamma^2$  will be a nonsquare for values of  $\gamma$  such that  $1 - 3\gamma^2$  yields 3 or 5. Note that  $1 - 3\gamma^2 = 3$  when  $\gamma$  equals 2 or 5 and  $1 - 3\gamma^2 = 5$  when  $\gamma$  equals 1 or 6. This shows us that  $\Gamma_{(\mathbb{F}_p, N_p)} = \{1, 2, 5, 6\}$ . Note that Type 3 tori have form

$$\mathbf{t}_3 = \left\{ x \begin{pmatrix} 1 & \gamma \\ -2\gamma & 6 \end{pmatrix} \mid x \in k, \gamma \in k \text{ fixed} \right\}$$

when  $p = 7$  and  $m \equiv N_p$ . Therefore  $t_1 = \begin{pmatrix} 1 & 1 \\ 4 & 6 \end{pmatrix}$ ,  $t_2 = \begin{pmatrix} 1 & 2 \\ 1 & 6 \end{pmatrix}$ ,  $t_5 = \begin{pmatrix} 1 & 5 \\ 6 & 6 \end{pmatrix}$ , and  $t_6 = \begin{pmatrix} 1 & 6 \\ 3 & 6 \end{pmatrix}$  will each generate tori not split over  $\mathbb{F}_p$  when  $p = 7$  and  $m \equiv N_p$ .  $\diamond$

## 5.2 The $H_k$ -conjugacy Classes of Type 3 Tori

In this chapter, we study maximal  $\theta$ -split  $k$ -anisotropic tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$ . The condition  $1 - m\gamma^2 \not\equiv 1$  is equivalent to the condition  $1 - m\gamma^2 \equiv N_p$  when  $k = \mathbb{F}_p$ . This provides us with the following result about the number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori.

**Proposition 5.2.1.** *Let  $k = \mathbb{F}_p$ . Then there is either exactly 1 or 2  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3.*

*Proof.* If all the tori are  $H_k$ -conjugate, then there is 1  $H_k$ -conjugacy class.

Now assume at least two tori are not  $H_k$ -conjugate. Let  $t_\alpha$  and  $t_\beta$  be the generators of  $\mathfrak{t}_\alpha$  and  $\mathfrak{t}_\beta$ , respectively. Assume  $t_\alpha$  is not  $H_k$ -conjugate to  $t_\beta$ . By Lemma 4.1.4,  $t_\alpha$  maps to  $t_\beta$  via

$$h_{r_1} = \frac{1}{[(1+r)^2 - m(\alpha - r\beta)^2]^{1/2}} \begin{pmatrix} 1+r & \alpha - r\beta \\ m(\alpha - r\beta) & 1+r \end{pmatrix},$$

where  $h_{r_1} \in H \setminus H_k$ . Thus, the value  $(1+r)^2 - m(\alpha - r\beta)^2$  is a nonsquare in  $k$ .

Now consider an arbitrary  $\gamma \in \Gamma_{(\mathbb{F}_p, m)}$  such that  $\gamma \neq \pm\alpha, \pm\beta$ . By Lemma 4.1.4,  $t_\gamma$  maps to  $t_\alpha$  via

$$h_{r_2} = \frac{1}{[(1+r)^2 - m(\gamma - r\alpha)^2]^{1/2}} \begin{pmatrix} 1+r & \gamma - r\alpha \\ m(\gamma - r\alpha) & 1+r \end{pmatrix},$$

where  $h_{r_2} \in H$ . If  $t_\gamma$  is  $H_k$ -conjugate to  $t_\alpha$ , then there are 2  $H_k$ -conjugacy classes. Now assume  $t_\gamma$  is not  $H_k$  conjugate to  $t_\alpha$ . Thus,  $h_{r_2} \in H \setminus H_k$ . This implies that the value  $(1+r)^2 - m(\gamma - r\alpha)^2$  is a nonsquare in  $k$ .

The matrix  $h_{r_1}h_{r_2}$  sends  $t_\gamma$  to  $t_\beta$ . Moreover, the value  $[(1+r)^2 - m(\alpha - r\beta)^2][(1+r)^2 - m(\gamma - r\alpha)^2] \equiv N_p^2$  is a square in  $k$ , thus  $h_{r_1}h_{r_2} \in H_k$ . Hence, if  $t_\gamma$  is not  $H_k$ -conjugate to  $t_\alpha$ , then  $t_\gamma$  is  $H_k$ -conjugate to  $t_\beta$ . Therefore, there are at most 2  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3 in  $\mathfrak{sl}(2, \mathbb{F}_p)$ .  $\square$

Lemma 4.1.1 shows that for every  $\gamma \in \Gamma_{(k, m)}$ ,  $-\gamma \in \Gamma_{(k, m)}$  as well. Whether or not these pairs of  $\gamma$ -values conjugate to each other is a defining characteristic of the  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori in  $\mathfrak{sl}(2, k)$  over  $k = \mathbb{F}_p$ .

**Theorem 5.2.1.** *Let  $\gamma \in \Gamma_{(\mathbb{F}_p, m)}$  be a value that makes the maximal  $\theta$ -split  $k$ -tori of Type 3  $k$ -anisotropic. Then torus generated with  $\gamma$  is  $H_k$ -conjugate to the torus generated with  $-\gamma$  if and only if*

1. *we have  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$ , or*
2. *we have  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ .*

*Proof.* By Theorem 4.1.1, the tori  $\mathfrak{t}_\gamma$  and  $\mathfrak{t}_{-\gamma}$  will be  $H_k$ -conjugate when  $-m \equiv 1 - m\gamma^2$ . When  $k = \mathbb{F}_p$  and the  $k$ -tori are not  $k$ -split, this means that  $1 - m\gamma^2 \equiv N_p$ . Thus, the  $k$ -tori will be  $H_k$ -conjugate when  $-m \equiv N_p$ . This is only possible when  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$  or when  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ . Therefore, the  $\mathfrak{t}_\gamma$  and  $\mathfrak{t}_{-\gamma}$  maximal  $\theta$ -split

$k$ -anisotropic tori are not  $H_k$ -conjugate when  $p \equiv 1$  and  $m \equiv 1$  or when  $p \equiv 3$  and  $m \equiv N_p$ .  $\square$

Theorem 5.2.1 is significant because it shows that sometimes at least two separate  $H_k$ -conjugacy classes exist, and this happens exactly when  $p \equiv 1$  and  $m \equiv 1$  or when  $p \equiv 3$  and  $m \equiv N_p$ . Recall from Section 1.3.5 that in Riemannian symmetric spaces all maximal  $\theta$ -split tori are  $k$ -split and  $H_k$ -conjugate, and again we see that this result does not translate to symmetric  $k$ -varieties.

**Lemma 5.2.1.** *Let  $k = \mathbb{F}_p$ , and let  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$  or  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ . Then  $\mathfrak{t}_\alpha$  is  $H_k$ -conjugate to  $\mathfrak{t}_\beta$  if and only if  $\mathfrak{t}_\alpha$  is  $H_k$ -conjugate to  $\mathfrak{t}_{-\beta}$ .*

*Proof.* Let  $t_\alpha$ ,  $t_\beta$ , and  $t_{-\beta}$  be the generators of  $\mathfrak{t}_\alpha$ ,  $\mathfrak{t}_\beta$ , and  $\mathfrak{t}_{-\beta}$ , respectively. Assume  $\mathfrak{t}_\alpha$  is  $H_k$ -conjugate to  $\mathfrak{t}_\beta$ , say  $h_1 \in H_k$  such that  $h_1 \cdot t_\alpha = t_\beta$ . By Theorem 5.2.1,  $h_2 \cdot t_\beta = t_{-\beta}$  for some  $h_2 \in H_k$ . Thus,  $h_2 h_1$  conjugates  $\mathfrak{t}_\alpha$  to  $\mathfrak{t}_{-\beta}$ , and  $h_2 h_1 \in H_k$ . The argument reverses.  $\square$

For all odd  $p$  under 50, we have computed the  $H_k$ -conjugacy classes of the maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3, and in every case we found that there is exactly one  $H_k$ -conjugacy class. For example computations, see Section A.1.

**Proposition 5.2.2.** *Let  $k = \mathbb{F}_p$  and  $p < 50$  an odd prime, and let  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$  or  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ . Then there is exactly 1  $H_k$ -conjugacy class of maximal  $\theta$ -split  $k$ -anisotropic tori.*

*Proof.* By Proposition 5.2.1, there is either 1  $H_k$ -conjugacy class or 2  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori. In Section A.1, we computed the  $H_k$ -conjugacy classes, and found exactly 1 for every odd prime  $p < 50$ .  $\square$

This result we generalize in the following conjecture.

**Conjecture 5.2.1.** *Let  $k = \mathbb{F}_p$ , and let  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$  or  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ . Then there is exactly 1  $H_k$ -conjugacy class of maximal  $\theta$ -split  $k$ -anisotropic tori.*

The following corollary shows how the generator  $t_\alpha$  is  $H_k$ -conjugate to the generator  $t_{-\beta}$  given that  $t_\alpha$  is  $H_k$ -conjugate to the generator  $t_\beta$ .

**Corollary 5.2.1.** *Let  $\alpha, \beta \in \Gamma_{(\mathbb{F}_p, m)}$  be values that makes the maximal  $\theta$ -split tori of Type 3  $k$ -anisotropic. Let  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$ , or let  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ . Let  $t_\alpha$  and  $t_\beta$  be generators of tori  $\mathfrak{t}_\alpha$  and  $\mathfrak{t}_\beta$ , respectively. If  $t_\alpha$  conjugates to  $rt_\beta$ , then  $t_\alpha$  conjugates to  $-rt_{-\beta}$ .*



*Proof.* By assumption, there exists  $h_1 \in H_k$  such that  $h_1 \cdot t_\alpha = rt_\beta$ . By Theorem 5.2.1, there exists  $h_2 \in H_k$  such that  $h_2 \cdot t_\beta = -t_{-\beta}$ . Thus,  $h_2 h_1 \in H_k$  maps  $t_\alpha$  to  $-rt_{-\beta}$ .  $\square$

When  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$  or  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ , the  $\mathfrak{t}_\alpha$  conjugates to both  $\mathfrak{t}_\beta$  and  $\mathfrak{t}_{-\beta}$ . By Theorem 5.2.1, we know that a similar result is not possible when  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$  or  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ . Theorem 5.2.2 below shows how this distinction divides the mappings of the maximal  $\theta$ -split  $k$ -anisotropic tori.

**Theorem 5.2.2.** *Let  $k = \mathbb{F}_p$ , and let  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$  or  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ . Suppose  $\alpha, \beta \in \Gamma_{(\mathbb{F}_p, m)}$  such that  $\alpha \neq \pm\beta$ . Then  $\mathfrak{t}_\alpha$  is  $H_k$ -conjugate to  $\mathfrak{t}_{-\beta}$  if and only if  $\mathfrak{t}_\alpha$  and  $\mathfrak{t}_\beta$  are not  $H_k$ -conjugate.*

*Proof.* By Theorem 2.2.2, we can take an element of  $H$  that conjugates  $t_\beta$  to  $-t_{-\beta}$ . By the discussion in the proof of Theorem 4.1.1, this element must be

$$h_- = \pm[-m(1 - m\beta^2)]^{-1/2} \begin{pmatrix} m\beta & 1 \\ m & m\beta \end{pmatrix},$$

from Equation 4.8, with  $\gamma = \beta$ . By Theorem 5.2.1, the value of  $-m(1 - m\beta^2)$  is a nonsquare in  $k$ . Thus,  $h_-$  is in  $H$  and not  $H_k$ .

Let  $t_\alpha, t_\beta$ , and  $t_{-\beta}$  be the generators of  $\mathfrak{t}_\alpha, \mathfrak{t}_\beta$ , and  $\mathfrak{t}_{-\beta}$ , respectively. By Lemma 4.1.4, we know

$$h_r = \frac{1}{[(1+r)^2 - m(\alpha - r\beta)^2]^{1/2}} \begin{pmatrix} 1+r & \alpha - r\beta \\ m(\alpha - r\beta) & 1+r \end{pmatrix} \quad (5.1)$$

from Equation 4.16, satisfies  $h_r \cdot t_\alpha = rt_\beta$ .

Combining these mappings, we see that  $h_r h_- \in H$  conjugates  $t_\alpha$  to  $-rt_\beta$ .

Now assume  $\mathfrak{t}_\alpha$  is not  $H_k$ -conjugate to  $\mathfrak{t}_\beta$ . This implies that  $h_r \notin H_k$ . As we know from Lemma 4.1.4, the matrix entries of  $h_r$  are in the base field, and only the coefficient  $[(1+r)^2 - m(\alpha - r\beta)^2]^{1/2}$  is potentially not in the base field. Therefore, assuming  $\mathfrak{t}_\alpha$  is not  $H_k$ -conjugate to  $\mathfrak{t}_\beta$  is equivalent to assuming  $(1+r)^2 - m(\alpha - r\beta)^2$  is a nonsquare in the base field. Now consider  $h_r h_-$ . We know  $-m(1 - m\beta^2) \equiv N_p$  by Theorem 5.2.1. By assumption,  $(1+r)^2 - m(\alpha - r\beta)^2 \equiv N_p$ . Thus,

$$-m(1 - m\beta^2)[(1+r)^2 - m(\alpha - r\beta)^2] \equiv N_p^2 \equiv 1.$$

Hence, we can take the square root of  $-m(1 - m\beta^2)[(1+r)^2 - m(\alpha - r\beta)^2]$  over the base

field  $k$ . Therefore, the matrix  $h_r h_- \in H_k$ , and  $\mathfrak{t}_\alpha$  is  $H_k$ -conjugate to  $\mathfrak{t}_{-\beta}$ . The argument reverses.  $\square$

Theorem 5.2.2 is significant to this thesis because of the following corollary, which shows how the maximal  $\theta$ -split  $k$ -anisotropic tori split into separate  $H_k$ -conjugacy classes.

**Corollary 5.2.2.** *Let  $k = \mathbb{F}_p$ , and let  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$  or  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ . Then there are exactly 2  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3.*

*Proof.* Let  $\beta \in \Gamma_{(\mathbb{F}_p, m)}$  and  $t_\beta$  be the generator of the toral subalgebra  $\mathfrak{t}_\beta$ , and similarly let  $t_{-\beta}$  be the generator of the toral subalgebra  $\mathfrak{t}_{-\beta}$ . By Theorem 5.2.1, when  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$  or  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ ,  $\mathfrak{t}_\beta$  and  $\mathfrak{t}_{-\beta}$  are not  $H_k$ -conjugate. This means that there are at least two  $H_k$ -conjugacy classes of tori. By Theorem 5.2.2, for every other  $\alpha \in \Gamma_{(\mathbb{F}_p, m)}$ ,  $\mathfrak{t}_\alpha$  is  $H_k$ -conjugate to either  $\mathfrak{t}_\beta$  and  $\mathfrak{t}_{-\beta}$ . Therefore, there are exactly 2  $H_k$ -conjugacy classes when  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$  or  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ .  $\square$

What remains is to determine the  $h_k \in H_k$  that conjugates  $t_\alpha$  to  $rt_\beta$ , for arbitrary  $\alpha, \beta \in \Gamma_{(\mathbb{F}_p, m)}$  when  $\mathfrak{t}_\alpha$  and  $\mathfrak{t}_\beta$  are  $H_k$ -conjugate. The following theorem and conjecture help provide formulae for such an element.

**Theorem 5.2.3.** *Suppose  $p$  is an odd prime under 50. Let  $\alpha, \beta \in \Gamma_{(\mathbb{F}_p, m)}$  such that  $\alpha \neq \pm\beta$ . Let  $t_\alpha$  and  $t_\beta$  be the generators of the maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3. Assume  $\mathfrak{t}_\alpha$  and  $\mathfrak{t}_\beta$  are  $H_k$ -conjugate. Then  $h_r$  is in  $H_k$ , where*

$$h_r = \frac{1}{[(1+r)^2 - m(\alpha - r\beta)^2]^{1/2}} \begin{pmatrix} 1+r & \alpha - r\beta \\ m(\alpha - r\beta) & 1+r \end{pmatrix} \quad (5.2)$$

*Proof.* In Lemma 4.1.4, we showed the  $h_r \in H$  conjugates  $t_\alpha$  to  $rt_\beta$  when  $\alpha \neq \pm\beta$ . The value of  $(1+r)^2 - m(\alpha - r\beta)^2$  was computed for every  $p < 50$ , and indeed  $(1+r)^2 - m(\alpha - r\beta)^2$  is a square in  $\mathbb{F}_p^*$ . Thus,  $[(1+r)^2 - m(\alpha - r\beta)^2]^{1/2}$  is in the base field  $\mathbb{F}_p$ . Hence,  $h_r \in H_k$ . For an example of these computations, see Section A.2.  $\square$

**Conjecture 5.2.2.** *Let  $\alpha, \beta \in \Gamma_{(\mathbb{F}_p, m)}$  such that  $\alpha \neq \pm\beta$ . Let  $t_\alpha$  and  $t_\beta$  be the generators of the maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3. Assume  $\mathfrak{t}_\alpha$  and  $\mathfrak{t}_\beta$  are  $H_k$ -conjugate. Then  $h_r \in H_k$ .*

### 5.3 The $p \equiv 1, m \equiv N_p$ Case

This case is characterized by the fact that  $-m \equiv N_p$ , as is the  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$  case, and for this reason, the two cases behave very similarly. Recall from Lemma 4.1.1 that if  $\gamma \in \Gamma_{(\mathbb{F}_p, m)}$ , then  $-\gamma \in \Gamma_{(\mathbb{F}_p, m)}$ . Moreover, by Theorem 5.2.1, when  $p \equiv 1$  and  $m \equiv N_p$ , we get that tori with  $\gamma$  and  $-\gamma$  are  $H_k$ -conjugate. We will see that this provides the critical link necessary to have just one  $H_k$ -conjugacy class of these maximal  $\theta$ -split  $k$ -anisotropic tori. Below is an example to demonstrate how Theorems 4.1.1 and 5.2.1 works.

**Example 5.3.1.** Let  $p = 5$  and  $m \equiv N_p$ , and specifically take  $m = 2$ . Then the set of  $\gamma$ -values that make the tori not  $k$ -split is  $\Gamma_{(\mathbb{F}_p, N_p)} = \{2, 3\}$ . Taking  $\gamma = 2$ , we get the torus

$$\left\langle \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Taking  $\gamma = 3$ , we get the torus

$$\left\langle \begin{pmatrix} 1 & 3 \\ 4 & 4 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1 & 3 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

We can see from the tori that transpose generators create tori with transpose elements. Theorem 4.1.1 provides the formula to create the  $h_k \in H_k$  that conjugates from an element in one torus to its transpose in the other torus. In this example, we can take  $\begin{pmatrix} 2 & 3 \\ 2 \cdot 3 & 2 \end{pmatrix} \in H_k$  and  $\begin{pmatrix} 3 & 2 \\ 2 \cdot 2 & 3 \end{pmatrix} \in H_k$  to conjugate the generator  $\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$  to the fourth multiple of the generator  $\begin{pmatrix} 1 & 3 \\ 4 & 4 \end{pmatrix}$ :

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} = 4 \begin{pmatrix} 1 & 3 \\ 4 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} = 4 \begin{pmatrix} 1 & 3 \\ 4 & 4 \end{pmatrix}.$$

Thus, when  $p = 5$  and  $m \equiv N_p$ , the only two maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3 over  $\mathbb{F}_p$  are indeed  $H_k$ -conjugate, creating one conjugacy class.  $\diamond$

This example is important because it shows that tori whose  $\gamma$ -values are additive inverses in  $\mathbb{F}_p$  are  $H_k$ -conjugate. Moreover, they conjugate to the  $p - 1$  multiple of each other. This is because  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$ , thus Theorem 5.2.1 applies. Formula 4.8 becomes

$$h_k = \pm [N_p(1 - N_p\gamma^2)]^{-1/2} \begin{pmatrix} N_p\gamma & 1 \\ N_p & N_p\gamma \end{pmatrix}. \quad (5.3)$$

Now we have a formula for determining  $h_k \in H_k$  that sends the torus with  $\gamma$  to the torus with  $-\gamma$ . Below is an example of how this works.

**Example 5.3.2.** Let  $p = 13$  and  $m \equiv N_p \equiv 2$ . In order for the tori to not be split over  $\mathbb{F}_p$ , we need  $1 - 2\gamma^2 \notin \mathbb{F}_p^2$ , and further calculation shows that  $\Gamma_{(\mathbb{F}_p, N_p)} = \{2, 4, 6, 7, 9, 11\}$  when  $p = 13$ .

Take  $\gamma = 4$ , for example. Using the formula above, we should be able to construct two elements of  $H_k$  to conjugate  $t_4 = \begin{pmatrix} 1 & 4 \\ 5 & 12 \end{pmatrix}$  to the twelfth multiple of  $t_9 = \begin{pmatrix} 1 & 9 \\ 8 & 12 \end{pmatrix}$ . Following the formula, we see that  $N_p(1 - N_p\gamma^2) = 2(2 \cdot 4^2 - 1) \equiv 10 \pmod{13}$ . Then  $[N_p(1 - N_p\gamma^2)]^{-1/2} = 10^{-1/2} = \pm 2 = 2, 11$ . Thus our formula yields two  $h_k \in H_k$ . Therefore, we can construct  $h_2$  and  $h_{11} \in H_k$  as follows:

$$h_2 = 2 \begin{pmatrix} 2 \cdot 4 & 1 \\ 2 & 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$$

$$h_{11} = 11 \begin{pmatrix} 2 \cdot 4 & 1 \\ 2 & 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} 10 & 11 \\ 9 & 10 \end{pmatrix}$$

We see that both  $\det(h_2) = 1$  and  $\det(h_{11}) = 1$ , confirming that  $h_2, h_{11} \in H_k$ . Moreover, each conjugates  $t_4$  to  $12t_9$ , as shown below.

$$\begin{aligned}
h_2 t_4 h_2^{-1} &= \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 5 & 12 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 5 & 12 \end{pmatrix} \begin{pmatrix} 3 & 11 \\ 9 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 12 & 4 \\ 5 & 1 \end{pmatrix} \\
&= 12t_9
\end{aligned}$$

$$\begin{aligned}
h_{11} t_4 h_{11}^{-1} &= \begin{pmatrix} 10 & 11 \\ 9 & 10 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 5 & 12 \end{pmatrix} \begin{pmatrix} 10 & 11 \\ 0 & 10 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 10 & 11 \\ 9 & 10 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 5 & 12 \end{pmatrix} \begin{pmatrix} 10 & 2 \\ 4 & 10 \end{pmatrix} \\
&= \begin{pmatrix} 12 & 4 \\ 5 & 1 \end{pmatrix} \\
&= 12t_9
\end{aligned}$$

We can turn to Table 5.3 to verify that Formula 5.3 would create the necessary  $h_k \in H_k$  to send  $t_\gamma$  to  $t_{-\gamma}$  for any given value of  $\gamma$  when  $p = 13$  and  $m \equiv N_p$ .  $\diamond$

These examples show that distinct maximal  $\theta$ -split  $k$ -anisotropic tori exist and at least some of these tori are  $H_k$ -conjugate. In Example 5.3.1,  $\Gamma_{(\mathbb{F}_p, N_p)} = \{2, 3\}$ . In  $\mathbb{F}_p$  when  $p = 5$ , 2 and 3 are additive inverses, and because  $m \equiv N_p$ , the tori they create are  $H_k$ -conjugate. Thus, Example 5.3.1 is an example in which all maximal  $\theta$ -split  $k$ -anisotropic tori are  $H_k$ -conjugate, and there is only one conjugacy class.

**Example 5.3.3.** Consider  $p = 13$  and  $m \equiv N_p \equiv 2$ . In this case, we find that  $\Gamma_{(\mathbb{F}_p, N_p)} = \{2, 4, 6, 7, 9, 11\}$ . We know by Theorem 5.2.1 that we can conjugate  $t_2$  to  $t_{11}$ ,  $t_4$  to  $t_9$ , and  $t_6$  to  $t_7$ . We wish to determine if all of these tori are  $H_k$ -conjugate. Note that when  $p = 13$  and  $m = 2$ , the fixed point group has form

Table 5.1: The  $H_k$ -conjugation of  $t_2$  to all tori in  $\mathbb{F}_p$  when  $p = 13$  and  $m \equiv N_p$

Mapping	Conjugating $h_k$
$t_2 \mapsto 11t_4$	$\begin{pmatrix} 4 & 12 \\ 11 & 4 \end{pmatrix}, \begin{pmatrix} 9 & 1 \\ 2 & 9 \end{pmatrix}$
$t_2 \mapsto 8t_6$	$\begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 10 & 11 \\ 9 & 10 \end{pmatrix}$
$t_2 \mapsto 5t_7$	$\begin{pmatrix} 5 & 5 \\ 10 & 5 \end{pmatrix}, \begin{pmatrix} 8 & 8 \\ 3 & 9 \end{pmatrix}$
$t_2 \mapsto 2t_9$	$\begin{pmatrix} 5 & 8 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 8 & 5 \\ 10 & 8 \end{pmatrix}$
$t_2 \mapsto 12t_{11}$	$\begin{pmatrix} 4 & 1 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 9 & 12 \\ 11 & 9 \end{pmatrix}$

$$H_k = \left\{ \begin{pmatrix} x & y \\ 2y & x \end{pmatrix} \mid x^2 - 2y^2 = 1, x, y \in \mathbb{F}_p \right\}$$

In fact, we can conjugate all of these generators to each other. As stated in Table 5.3, we can find an  $h_k$  that sends  $t_2$  to all other generators, as shown in Table 5.1 below.

Therefore, all maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3 over  $\mathbb{F}_p$  when  $p = 13$  and  $m \equiv N_p$  are  $H_k$ -conjugate, creating one conjugacy class.  $\diamond$

## 5.4 The $p \equiv 3, m \equiv 1$ Case

As we will see, this case behaves very similarly to the  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$  case. First, to gain some intuition, we consider a similar example.

**Example 5.4.1.** Let  $p = 11$  and  $m \equiv 1$ . Then  $\Gamma_{(\mathbb{F}_p, 1)} = \{2, 4, 7, 9\}$ . Taking  $\gamma = 2$ , we get the torus

$$\left\langle \begin{pmatrix} 1 & 2 \\ 9 & 10 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1 & 2 \\ 9 & 10 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 7 & 9 \end{pmatrix}, \begin{pmatrix} 3 & 6 \\ 5 & 8 \end{pmatrix}, \begin{pmatrix} 4 & 8 \\ 3 & 7 \end{pmatrix}, \begin{pmatrix} 5 & 10 \\ 1 & 6 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 6 & 1 \\ 10 & 5 \end{pmatrix}, \begin{pmatrix} 7 & 3 \\ 8 & 4 \end{pmatrix}, \begin{pmatrix} 8 & 5 \\ 6 & 3 \end{pmatrix}, \begin{pmatrix} 9 & 7 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 10 & 9 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Taking  $\gamma = 9$ , we get the torus

$$\left\langle \begin{pmatrix} 1 & 9 \\ 2 & 10 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1 & 9 \\ 2 & 10 \end{pmatrix}, \begin{pmatrix} 2 & 7 \\ 4 & 9 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 6 & 8 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 8 & 7 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 10 & 6 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 6 & 10 \\ 1 & 5 \end{pmatrix}, \begin{pmatrix} 7 & 8 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 8 & 6 \\ 5 & 3 \end{pmatrix}, \begin{pmatrix} 9 & 4 \\ 7 & 2 \end{pmatrix}, \begin{pmatrix} 10 & 2 \\ 9 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

We can see from the tori that transpose generators create tori with transpose elements.

What remains to be seen is how we conjugate from an element in one torus to its transpose

in the other torus. In this example, we can take  $\begin{pmatrix} 4 & 2 \\ 1 \cdot 2 & 4 \end{pmatrix} \in H_k$  and  $\begin{pmatrix} 7 & 9 \\ 1 \cdot 9 & 7 \end{pmatrix} \in H_k$

to conjugate the generator  $\begin{pmatrix} 1 & 2 \\ 9 & 10 \end{pmatrix}$  to the tenth multiple of the generator  $\begin{pmatrix} 1 & 9 \\ 2 & 10 \end{pmatrix}$ :

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 9 & 10 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 9 & 10 \end{pmatrix} \begin{pmatrix} 4 & 9 \\ 9 & 4 \end{pmatrix} \\ = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 0 & 6 \\ 5 & 0 \end{pmatrix} \\ = \begin{pmatrix} 10 & 2 \\ 9 & 1 \end{pmatrix} \\ = 10 \begin{pmatrix} 1 & 9 \\ 2 & 10 \end{pmatrix},$$

$$\begin{pmatrix} 7 & 9 \\ 9 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 9 & 10 \end{pmatrix} \begin{pmatrix} 7 & 9 \\ 9 & 7 \end{pmatrix}^{-1} = \begin{pmatrix} 7 & 9 \\ 9 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 9 & 10 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ 2 & 7 \end{pmatrix} \\ = \begin{pmatrix} 7 & 9 \\ 9 & 7 \end{pmatrix} \begin{pmatrix} 0 & 5 \\ 6 & 0 \end{pmatrix} \\ = \begin{pmatrix} 10 & 2 \\ 9 & 1 \end{pmatrix} \\ = 10 \begin{pmatrix} 1 & 9 \\ 2 & 10 \end{pmatrix}.$$

Thus, when  $p = 7$  and  $m \equiv 1$ , these two tori are conjugate.  $\diamond$

This another example of how tori whose  $\gamma$ -values are additive inverses in  $\mathbb{F}_p$  are  $H_k$ -conjugate. Moreover, they conjugate to the  $p - 1$  multiple of each other. This is because  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$ , thus Theorem 5.2.1 applies. Formula 4.8 becomes

$$h_k = \pm(\gamma^2 - 1)^{-1/2} \begin{pmatrix} \gamma & 1 \\ 1 & \gamma \end{pmatrix}, \quad (5.4)$$

Again we can see that distinct maximal  $\theta$ -split  $k$ -anisotropic tori exist and at least some of these tori are  $H_k$ -conjugate.

**Example 5.4.2.** Let  $p = 11$ ,  $m \equiv 1$ , and  $\Gamma_{(\mathbb{F}_{11}, 1)} = \{2, 4, 7, 9\}$  as before. We know the tori generated with 2 and 9 as their  $\gamma$ -values will be  $H_k$ -conjugate and, similarly, the tori generated with 4 and 7 as their  $\gamma$ -values will be  $H_k$ -conjugate. Thus, we have at most two  $H_k$ -conjugacy classes.

Let  $t_2 = \begin{pmatrix} 1 & 2 \\ 9 & 10 \end{pmatrix}$  and  $t_4 = \begin{pmatrix} 1 & 4 \\ 7 & 10 \end{pmatrix}$ . The elements  $\begin{pmatrix} 2 & 6 \\ 1 \cdot 6 & 2 \end{pmatrix}$  and  $\begin{pmatrix} 9 & 5 \\ 1 \cdot 5 & 9 \end{pmatrix}$  are in  $H_k$  and both conjugate  $t_2$  to  $3t_4$ . Therefore, in  $k = \mathbb{F}_p$  when  $p = 11$  and  $m \equiv 1$ , all maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3 are  $H_k$ -conjugate.  $\diamond$

## 5.5 The $p \equiv 1, m \equiv 1$ Case

When  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$ , we know that tori with additive inverses as their  $\gamma$ -values are  $H_k$ -conjugate, as stated in Theorem 5.2.1. The proof of this relied on  $-m \equiv N_p$ . When  $p \equiv 1$  and  $m \equiv 1$ , we get that  $-m = -1 \equiv 1$  in  $k^*/(k^*)^2$ , thus  $-m$  is now a square. Moreover, Theorem 4.1.1 shows that this does not happen when  $p \equiv 1$  and  $m \equiv 1$ . Below is an example to show that maximal  $\theta$ -split  $k$ -anisotropic tori with additive inverse  $\gamma$ -values are not  $H_k$ -conjugate.

**Example 5.5.1.** Take  $p = 5$  and  $m \equiv 1$ . Calculation shows that  $\mathbb{F}_p^2 = \{0, 1, 4\}$ . In order for the tori to be  $k$ -anisotropic, we need  $1 - \gamma^2 \notin \mathbb{F}_p^2$ . Thus, the set of  $\gamma$ -values such that the tori will not be split over  $\mathbb{F}_p$  is  $\Gamma_{(\mathbb{F}_p, 1)} = \{2, 3\}$  when  $p = 5$  and  $m \equiv 1$ . In this case that  $p = 5$  and  $m \equiv 1$ , Type 3 tori have form

$$\mathfrak{t}_3 = \left\{ x \begin{pmatrix} 1 & \gamma \\ -\gamma & 4 \end{pmatrix} \mid x \in k, \gamma \in k \text{ fixed} \right\}.$$



We have two values for  $\gamma$ , thus we have two tori.

$$\langle t_2 \rangle = \left\langle \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$\langle t_3 \rangle = \left\langle \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

What we want to determine is whether or not these two tori are  $H_k$ -conjugate.

As it turns out, there is no  $h_k \in H_k$  that conjugates  $t_1$  to a multiple of  $t_4$ . When  $p = 5$  and  $m \equiv 1$ ,

$$\begin{aligned} H_k &= \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} \mid x^2 - y^2 = 1, x, y \in k \right\} \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \right\} \end{aligned}$$

None of these  $h_k \in H_k$  conjugates  $t_2$  to any multiple of  $t_3$ . In fact, every  $h_k \in H_k$  conjugates  $t_2$  to itself. Therefore, these tori are not  $H_k$ -conjugate, and there are two distinct  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3 over  $\mathbb{F}_p$  when  $p = 5$  and  $m \equiv 1$ .  $\diamond$

**Example 5.5.2.** Take  $p = 13$  and  $m \equiv 1$ . Then  $\Gamma_{(\mathbb{F}_p, 1)} = \{3, 4, 5, 9, 10\}$ . As shown in Table 5.9, there are 2  $H_k$ -conjugacy classes,  $\{\mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_8\}$  and  $\{\mathbf{t}_5, \mathbf{t}_9, \mathbf{t}_{10}\}$ .  $\diamond$

Below is an example of how Theorem 5.2.2 works to predict the matrix that sends  $t_\alpha$  to  $rt_{-\beta}$  when  $t_\alpha$  is not  $H_k$ -conjugate to  $t_\beta$ .

**Example 5.5.3.** Let  $p = 13$  and  $m \equiv 1$ . Consider  $\alpha = 3$ ,  $\beta = 9$ , and  $-\beta = 4$ . We know from Example 5.5.2 above that  $\mathbf{t}_3$  is  $H_k$ -conjugate to  $\mathbf{t}_4$  and not  $\mathbf{t}_4$ .

Using Equation 4.3, we find that  $r = \pm 2$ . Using Equation 4.16, we find that  $\mathbf{t}_3$  conjugates to  $\mathbf{t}_9$  by the following matrices in  $H$ :

$$h_2 = \frac{1}{5^{1/2}} \begin{pmatrix} 3 & 11 \\ 11 & 3 \end{pmatrix},$$

$$h_{11} = \frac{1}{2^{1/2}} \begin{pmatrix} 12 & 8 \\ 8 & 12 \end{pmatrix}$$

Note that neither of these matrices are in  $H_k$  because 2 and 5 are not squares in  $\mathbb{F}_p$  when  $p = 13$ . Thus,  $\mathfrak{t}_3$  is not  $H_k$ -conjugate to  $\mathfrak{t}_9$ . Using Equation 4.8, we find that  $\mathfrak{t}_9$  conjugates to  $\mathfrak{t}_4$  by the following matrices in  $H$ :

$$h_- = \pm \frac{1}{2^{1/2}} \begin{pmatrix} 9 & 1 \\ 1 & 9 \end{pmatrix}.$$

Note that neither of these matrices are in  $H_k$ . Theorem 5.2.2 shows us that  $t_3$  will conjugates to  $t_4$ , under  $h_1 h_r$ . Composing these matrices, we get

$$h_- h_2 = \pm \frac{1}{10^{1/2}} \begin{pmatrix} 9 & 1 \\ 1 & 9 \end{pmatrix} \begin{pmatrix} 3 & 11 \\ 11 & 3 \end{pmatrix} = \pm \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \quad (5.5)$$

$$h_- h_{11} = \pm \frac{1}{4^{1/2}} \begin{pmatrix} 9 & 1 \\ 1 & 9 \end{pmatrix} \begin{pmatrix} 12 & 8 \\ 8 & 12 \end{pmatrix} = \pm \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix} \quad (5.6)$$

Notice that both of these matrices are in  $H_k$ , thus  $\mathfrak{t}_3$  is  $H_k$ -conjugate to  $\mathfrak{t}_4$ . We confirm with Table 5.9 that  $r = \pm 2$ ,  $h_- h_2$ , and  $h_- h_{11}$  are indeed the correct values for sending  $t_3$  to  $t_4$ .  $\diamond$

## 5.6 The $p \equiv 3, m \equiv N_p$ Case

As Theorems 5.2.1 and 5.2.2 suggest, the  $p \equiv 3$  and  $m \equiv N_p$  case behaves very similarly to the  $p \equiv 1$  and  $m \equiv 1$  case. Specifically, maximal  $\theta$ -split  $k$ -anisotropic tori with additive-inverse  $\gamma$ -values are not  $H_k$ -conjugate, forcing 2 distinct  $H_k$ -conjugacy classes.

**Example 5.6.1.** Take  $p = 7$  and  $m \equiv N_p \equiv 3$ . Calculation shows  $\Gamma_{(\mathbb{F}_p, N_p)} = \{1, 2, 5, 6\}$  when  $p = 7$  and  $m \equiv 3$ . In this case, Type 3 tori have form

$$\mathfrak{t}_3 = \left\{ x \begin{pmatrix} 1 & \gamma \\ -3\gamma & 6 \end{pmatrix} \mid x \in k, \gamma \in k \text{ fixed} \right\},$$

and because  $|\Gamma_{(\mathbb{F}_p, N_p)}| = 4$ , there are four such tori. What we want to determine is whether or not these tori are  $H_k$ -conjugate.

We find that there are 2  $H_k$ -conjugacy classes,  $\{\mathfrak{t}_1, \mathfrak{t}_5\}$  and  $\{\mathfrak{t}_2, \mathfrak{t}_6\}$ . The fixed point group is

$$\begin{aligned} H_k &= \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} \mid x^2 - 3y^2 = 1, x, y \in k \right\} \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 6 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 4 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 6 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 5 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix} \right\}. \end{aligned}$$

We can see from Table 5.12 that the second and fifth elements in the above list conjugate  $t_1$  to  $2t_5$ . The third and fourth elements in the above list conjugate  $t_1$  to  $5t_5$ . Thus, the tori  $\mathfrak{t}_1$  and  $\mathfrak{t}_5$  are  $H_k$ -conjugate.

Similarly, the third and fourth elements in the above list conjugate  $t_2$  to  $3t_6$  and the second and fifth elements in the above list conjugate  $t_2$  to  $4t_6$ . Thus, the tori  $\mathfrak{t}_2$  and  $\mathfrak{t}_6$  are  $H_k$ -conjugate.

None of these  $h_k \in H_k$  conjugates  $t_1$  to any multiple of  $t_2$ . Therefore, there are two distinct  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3 over  $\mathbb{F}_p$  when  $p = 7$  and  $m \equiv N_p$ .  $\diamond$

**Example 5.6.2.** Take  $p = 11$  and  $m \equiv N_p$ . Then  $\Gamma_{(\mathbb{F}_p, N_p)} = \{1, 4, 5, 6, 7, 10\}$ . As shown in Table 5.13, there are 2  $H_k$ -conjugacy classes,  $\{\mathfrak{t}_1, \mathfrak{t}_4, \mathfrak{t}_5\}$  and  $\{\mathfrak{t}_6, \mathfrak{t}_7, \mathfrak{t}_{10}\}$ .  $\diamond$

Below is an example of how Theorem 5.2.2 works to predict the matrix that sends  $t_\alpha$  to  $rt_{-\beta}$  when  $t_\alpha$  is not  $H_k$ -conjugate to  $t_\beta$ .

**Example 5.6.3.** Let  $p = 13$  and  $m \equiv 1$ . Consider  $\alpha = 5$ ,  $\beta = 10$ , and  $-\beta = 1$ . We know from Example 5.6.2 above that  $\mathfrak{t}_5$  is  $H_k$ -conjugate to  $\mathfrak{t}_1$  and not  $\mathfrak{t}_{10}$ .

Using Equation 4.3, we find that  $r = \pm 4$ . Using Equation 4.16, we find that  $\mathfrak{t}_5$  conjugates to  $\mathfrak{t}_{10}$  by the following matrices in  $H$ :

$$h_4 = \frac{1}{6^{1/2}} \begin{pmatrix} 5 & 9 \\ 7 & 5 \end{pmatrix},$$

$$h_7 = \frac{1}{7^{1/2}} \begin{pmatrix} 8 & 1 \\ 2 & 8 \end{pmatrix}$$

Note that neither of these matrices are in  $H_k$  because 6 and 7 are not squares in  $\mathbb{F}_p$  when  $p = 11$ . Thus,  $\mathbf{t}_5$  is not  $H_k$ -conjugate to  $\mathbf{t}_{10}$ . Using Equation 4.8, we find that  $\mathbf{t}_{10}$  conjugates to  $\mathbf{t}_1$  by the following matrices in  $H$ :

$$h_- = \pm \frac{1}{2^{1/2}} \begin{pmatrix} 9 & 1 \\ 2 & 9 \end{pmatrix}.$$

Note that neither of these matrices are in  $H_k$  because 2 is not a square in  $\mathbb{F}_p$  when  $p = 11$ . Theorem 5.2.2 shows us that  $t_5$  will conjugates to  $t_1$ , under  $h_1 h_r$ . Composing these matrices, we get

$$h_- h_4 = \pm \frac{1}{1^{1/2}} \begin{pmatrix} 9 & 1 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} 5 & 9 \\ 7 & 5 \end{pmatrix} = \pm \begin{pmatrix} 8 & 9 \\ 7 & 8 \end{pmatrix} \quad (5.7)$$

$$h_- h_7 = \pm \frac{1}{3^{1/2}} \begin{pmatrix} 9 & 1 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} 8 & 1 \\ 2 & 8 \end{pmatrix} = \pm \begin{pmatrix} 5 & 1 \\ 2 & 5 \end{pmatrix} \quad (5.8)$$

Notice that both of these matrices are in  $H_k$ , thus  $\mathbf{t}_5$  is  $H_k$ -conjugate to  $\mathbf{t}_1$ . We confirm with Table 5.13 that  $r = \pm 4$ ,  $h_- h_4$ , and  $h_- h_7$  are indeed the correct values for sending  $t_5$  to  $t_1$ .  $\diamond$

## 5.7 Example $H_k$ -conjugation Tables

The generator  $t_\alpha$  can be conjugated to the  $r^{th}$  multiple of the generator  $t_\beta$  by some  $\mathfrak{h}_k \in H_k$ . To create the necessary  $\mathfrak{h}_k$ , recall that every  $\mathfrak{h}_k = \begin{pmatrix} x & y \\ my & x \end{pmatrix}$ , and plug in corresponding  $m$  value and  $x$  and  $y$  values from the pairs  $(x, y)$  given in the table.

### 5.7.1 Example $H_k$ -The $H_k$ -conjugation Classes when $p \equiv 1$ and $m \equiv N_p$

When  $p = 5$ , we find that  $N_p = 2$ , thus  $m \equiv N_p = 2$ . In this case, we determine that  $\Gamma_{(\mathbb{F}_p, m)} = \{2, 3\}$ . Table 5.2 shows that there is one  $\theta$ -split, not  $k$ -split torus in  $\mathfrak{sl}(2, \mathbb{F}_p)$  for  $p = 5$ .

When  $p = 13$ , we find that  $N_p = 2$ , thus  $m \equiv N_p = 2$ . In this case, we determine that  $\Gamma_{(\mathbb{F}_p, m)} = \{2, 4, 6, 7, 9, 11\}$ . Table 5.3 shows that there is one  $\theta$ -split, not  $k$ -split torus in  $\mathfrak{sl}(2, \mathbb{F}_p)$  for  $p = 13$ .

When  $p = 17$ , we find that  $N_p = 3$ , thus  $m \equiv N_p = 3$ . In this case, we determine that  $\Gamma_{(\mathbb{F}_p, m)} = \{2, 5, 6, 7, 10, 11, 12, 15\}$ . Table 5.4 shows that there is one  $\theta$ -split, not  $k$ -split torus in  $\mathfrak{sl}(2, \mathbb{F}_p)$  for  $p = 17$ .

Table 5.2: The  $H_k$ -conjugation Classes when  $p = 5$  and  $m \equiv N_p$

$p = 5$	$\beta = 2$				$\beta = 3$			
$\alpha = 2$					$r$	4	$(x, y)$	$(2, 3), (3, 2)$
$\alpha = 3$	$r$	4	$(x, y)$	$(2, 2), (3, 3)$				

Table 5.3: The  $H_k$ -conjugation Classes when  $p = 13$  and  $m \equiv N_p$

$p = 13$	$\beta = 2$				$\beta = 4$			
$\alpha = 2$					$r$	11	$(x, y)$	$(4, 12), (9, 1)$
$\alpha = 4$	$r$	6	$(x, y)$	$(4, 1), (9, 12)$				
$\alpha = 6$	$r$	5	$(x, y)$	$(3, 11), (10, 2)$	$r$	9	$(x, y)$	$(3, 2), (10, 11)$
$\alpha = 7$	$r$	8	$(x, y)$	$(5, 8), (8, 5)$	$r$	10	$(x, y)$	$(4, 1), (9, 12)$
$\alpha = 9$	$r$	7	$(x, y)$	$(5, 5), (8, 8)$	$r$	12	$(x, y)$	$(3, 11), (10, 2)$
$\alpha = 11$	$r$	12	$(x, y)$	$(4, 12), (9, 1)$	$r$	2	$(x, y)$	$(5, 5), (8, 8)$
	$\beta = 6$				$\beta = 7$			
$\alpha = 2$	$r$	8	$(x, y)$	$(3, 2), (10, 11)$	$r$	5	$(x, y)$	$(5, 5), (8, 8)$
$\alpha = 4$	$r$	9	$(x, y)$	$(3, 11), (10, 2)$	$r$	4	$(x, y)$	$(4, 12), (9, 1)$
$\alpha = 6$					$r$	12	$(x, y)$	$(5, 8), (8, 5)$
$\alpha = 7$	$r$	12	$(x, y)$	$(5, 8), (8, 5)$				
$\alpha = 9$	$r$	4	$(x, y)$	$(4, 1), (9, 12)$	$r$	9	$(x, y)$	$(3, 2), (10, 11)$
$\alpha = 11$	$r$	5	$(x, y)$	$(5, 8), (8, 5)$	$r$	8	$(x, y)$	$(3, 11), (10, 2)$
	$\beta = 9$				$\beta = 11$			
$\alpha = 2$	$r$	2	$(x, y)$	$(5, 8), (8, 5)$	$r$	12	$(x, y)$	$(4, 1), (9, 12)$
$\alpha = 4$	$r$	12	$(x, y)$	$(3, 2), (10, 11)$	$r$	7	$(x, y)$	$(5, 8), (8, 5)$
$\alpha = 6$	$r$	10	$(x, y)$	$(4, 12), (9, 1)$	$r$	8	$(x, y)$	$(5, 5), (8, 8)$
$\alpha = 7$	$r$	3	$(x, y)$	$(3, 11), (10, 2)$	$r$	5	$(x, y)$	$(3, 2), (10, 11)$
$\alpha = 9$					$r$	6	$(x, y)$	$(4, 12), (9, 1)$
$\alpha = 11$	$r$	11	$(x, y)$	$(4, 1), (9, 12)$				

Table 5.4: The  $H_k$ -conjugation Classes when  $p = 17$  and  $m \equiv N_p$ 

$p = 17$	$\beta = 2$				$\beta = 5$			
$\alpha = 2$					$r$	13	$(x, y)$	$(8, 15), (9, 2)$
$\alpha = 5$	$r$	4	$(x, y)$	$(8, 2), (9, 15)$				
$\alpha = 6$	$r$	6	$(x, y)$	$(8, 15), (9, 2)$	$r$	10	$(x, y)$	$(8, 2), (9, 15)$
$\alpha = 7$	$r$	2	$(x, y)$	$(5, 5), (12, 12)$	$r$	9	$(x, y)$	$(7, 4), (10, 13)$
$\alpha = 10$	$r$	15	$(x, y)$	$(7, 4), (10, 13)$	$r$	8	$(x, y)$	$(2, 16), (15, 1)$
$\alpha = 11$	$r$	11	$(x, y)$	$(2, 1), (15, 16)$	$r$	7	$(x, y)$	$(7, 13), (10, 4)$
$\alpha = 12$	$r$	13	$(x, y)$	$(5, 12), (12, 5)$	$r$	16	$(x, y)$	$(2, 1), (15, 16)$
$\alpha = 15$	$r$	16	$(x, y)$	$(7, 13), (10, 4)$	$r$	4	$(x, y)$	$(5, 12), (12, 5)$
	$\beta = 6$				$\beta = 7$			
$\alpha = 2$	$r$	3	$(x, y)$	$(8, 2), (9, 15)$	$r$	9	$(x, y)$	$(5, 12), (12, 5)$
$\alpha = 5$	$r$	12	$(x, y)$	$(8, 5), (9, 2)$	$r$	2	$(x, y)$	$(7, 13), (10, 4)$
$\alpha = 6$					$r$	3	$(x, y)$	$(2, 1), (15, 16)$
$\alpha = 7$	$r$	6	$(x, y)$	$(2, 16), (15, 1)$				
$\alpha = 10$	$r$	11	$(x, y)$	$(5, 5), (12, 12)$	$r$	16	$(x, y)$	$(8, 15), (9, 2)$
$\alpha = 11$	$r$	16	$(x, y)$	$(5, 12), (12, 5)$	$r$	14	$(x, y)$	$(5, 5), (12, 12)$
$\alpha = 12$	$r$	5	$(x, y)$	$(7, 13), (10, 4)$	$r$	15	$(x, y)$	$(2, 16), (15, 11)$
$\alpha = 15$	$r$	14	$(x, y)$	$(2, 1), (15, 16)$	$r$	8	$(x, y)$	$(7, 4), (10, 13)$
	$\beta = 10$				$\beta = 11$			
$\alpha = 2$	$r$	8	$(x, y)$	$(7, 13), (10, 4)$	$r$	14	$(x, y)$	$(2, 16), (15, 11)$
$\alpha = 5$	$r$	15	$(x, y)$	$(2, 1), (15, 16)$	$r$	5	$(x, y)$	$(7, 4), (10, 13)$
$\alpha = 6$	$r$	14	$(x, y)$	$(5, 12), (12, 5)$	$r$	16	$(x, y)$	$(5, 5), (12, 12)$
$\alpha = 7$	$r$	16	$(x, y)$	$(8, 2), (9, 15)$	$r$	11	$(x, y)$	$(5, 12), (12, 5)$
$\alpha = 10$					$r$	6	$(x, y)$	$(2, 1), (15, 16)$
$\alpha = 11$	$r$	3	$(x, y)$	$(2, 16), (15, 11)$				
$\alpha = 12$	$r$	2	$(x, y)$	$(7, 4), (10, 13)$	$r$	12	$(x, y)$	$(8, 2), (9, 15)$
$\alpha = 15$	$r$	9	$(x, y)$	$(5, 5), (12, 12)$	$r$	3	$(x, y)$	$(8, 15), (9, 2)$
	$\beta = 12$				$\beta = 15$			
$\alpha = 2$	$r$	4	$(x, y)$	$(5, 5), (12, 12)$	$r$	16	$(x, y)$	$(7, 4), (10, 13)$
$\alpha = 5$	$r$	16	$(x, y)$	$(2, 16), (15, 1)$	$r$	13	$(x, y)$	$(5, 5), (12, 12)$
$\alpha = 6$	$r$	7	$(x, y)$	$(7, 4), (10, 13)$	$r$	11	$(x, y)$	$(2, 16), (15, 1)$
$\alpha = 7$	$r$	8	$(x, y)$	$(2, 1), (15, 16)$	$r$	15	$(x, y)$	$(7, 13), (10, 4)$
$\alpha = 10$	$r$	9	$(x, y)$	$(7, 13), (10, 4)$	$r$	2	$(x, y)$	$(5, 12), (12, 5)$
$\alpha = 11$	$r$	10	$(x, y)$	$(8, 15), (9, 2)$	$r$	6	$(x, y)$	$(8, 2), (9, 15)$
$\alpha = 12$					$r$	4	$(x, y)$	$(8, 15), (9, 2)$
$\alpha = 15$	$r$	13	$(x, y)$	$(8, 2), (9, 15)$				

### 5.7.2 Example $H_k$ -The $H_k$ -conjugation Classes when $p \equiv 3$ and $m \equiv 1$

When  $p = 3$  and  $m \equiv 1$ , all  $\theta$ -split tori are  $k$ -split, thus there are no cases for us to consider.

When  $p = 7$  and  $m \equiv 1$ , we find that  $\Gamma_{(\mathbb{F}_p, m)} = \{3, 4\}$ . Table 5.5 shows that there is one  $\theta$ -split,  $k$ -anisotropic tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$  for  $p = 7$ .

When  $p = 11$  and  $m \equiv 1$ , we find that  $\Gamma_{(\mathbb{F}_p, m)} = \{2, 4, 7, 9\}$ . Table 5.6 shows that there is one  $\theta$ -split,  $k$ -anisotropic tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$  for  $p = 11$ .

When  $p = 19$  and  $m \equiv 1$ , we find that  $\Gamma_{(\mathbb{F}_p, m)} = \{5, 6, 8, 9, 10, 11, 13, 14\}$ . Table 5.7 shows that there is one  $\theta$ -split,  $k$ -anisotropic tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$  for  $p = 19$ .



Table 5.5: The  $H_k$ -conjugation Classes when  $p = 7$  and  $m \equiv 1$

$p = 7$	$\beta = 3$				$\beta = 4$			
$\alpha = 3$					$r$	6	$(x, y)$	$(3, 1), (4, 6)$
$\alpha = 4$	$r$	6	$(x, y)$	$(3, 6), (4, 1)$				

Table 5.6: The  $H_k$ -conjugation Classes when  $p = 11$  and  $m \equiv 1$

$p = 11$	$\beta = 2$				$\beta = 4$			
$\alpha = 2$					$r$	3	$(x, y)$	$(2, 6), (9, 5)$
$\alpha = 4$	$r$	4	$(x, y)$	$(2, 5), (9, 6)$				
$\alpha = 7$	$r$	7	$(x, y)$	$(4, 2), (7, 9)$	$r$	10	$(x, y)$	$(2, 5), (9, 6)$
$\alpha = 9$	$r$	10	$(x, y)$	$(4, 9), (7, 2)$	$r$	8	$(x, y)$	$(4, 2), (7, 9)$
	$\beta = 7$				$\beta = 9$			
$\alpha = 2$	$r$	8	$(x, y)$	$(4, 9), (7, 2)$	$r$	10	$(x, y)$	$(4, 2), (7, 9)$
$\alpha = 4$	$r$	10	$(x, y)$	$(2, 6), (9, 5)$	$r$	7	$(x, y)$	$(4, 9), (7, 2)$
$\alpha = 7$					$r$	4	$(x, y)$	$(2, 6), (9, 5)$
$\alpha = 9$	$r$	3	$(x, y)$	$(2, 5), (9, 6)$				

Table 5.7: The  $H_k$ -conjugation Classes when  $p = 19$  and  $m \equiv 1$ 

$p = 19$	$\beta = 5$				$\beta = 6$			
$\alpha = 5$					$r$	7	$(x, y)$	$(6, 15), (13, 4)$
$\alpha = 6$	$r$	11	$(x, y)$	$(6, 4), (13, 15)$				
$\alpha = 8$	$r$	9	$(x, y)$	$(5, 10), (14, 9)$	$r$	6	$(x, y)$	$(9, 2), (10, 17)$
$\alpha = 9$	$r$	15	$(x, y)$	$(8, 5), (11, 14)$	$r$	10	$(x, y)$	$(9, 17), (10, 2)$
$\alpha = 10$	$r$	4	$(x, y)$	$(5, 9), (14, 10)$	$r$	9	$(x, y)$	$(6, 4), (13, 15)$
$\alpha = 11$	$r$	10	$(x, y)$	$(6, 15), (13, 4)$	$r$	13	$(x, y)$	$(5, 10), (14, 9)$
$\alpha = 13$	$r$	8	$(x, y)$	$(8, 14), (11, 5)$	$r$	18	$(x, y)$	$(8, 5), (11, 14)$
$\alpha = 14$	$r$	18	$(x, y)$	$(9, 2), (10, 17)$	$r$	12	$(x, y)$	$(8, 14), (11, 5)$
	$\beta = 8$				$\beta = 9$			
$\alpha = 5$	$r$	17	$(x, y)$	$(5, 9), (14, 10)$	$r$	14	$(x, y)$	$(8, 14), (11, 5)$
$\alpha = 6$	$r$	16	$(x, y)$	$(9, 17), (10, 2)$	$r$	2	$(x, y)$	$(9, 2), (10, 17)$
$\alpha = 8$					$r$	12	$(x, y)$	$(9, 17), (10, 2)$
$\alpha = 9$	$r$	8	$(x, y)$	$(9, 2), (10, 17)$				
$\alpha = 10$	$r$	11	$(x, y)$	$(8, 5), (11, 14)$	$r$	18	$(x, y)$	$(5, 10), (14, 9)$
$\alpha = 11$	$r$	18	$(x, y)$	$(6, 4), (13, 15)$	$r$	7	$(x, y)$	$(8, 5), (11, 14)$
$\alpha = 13$	$r$	3	$(x, y)$	$(5, 10), (14, 9)$	$r$	17	$(x, y)$	$(6, 4), (13, 15)$
$\alpha = 14$	$r$	2	$(x, y)$	$(6, 15), (13, 4)$	$r$	5	$(x, y)$	$(5, 9), (14, 10)$
	$\beta = 10$				$\beta = 11$			
$\alpha = 5$	$r$	5	$(x, y)$	$(5, 10), (14, 9)$	$r$	2	$(x, y)$	$(6, 4), (13, 15)$
$\alpha = 6$	$r$	17	$(x, y)$	$(6, 15), (13, 4)$	$r$	3	$(x, y)$	$(5, 9), (14, 10)$
$\alpha = 8$	$r$	7	$(x, y)$	$(8, 14), (11, 5)$	$r$	18	$(x, y)$	$(6, 15), (13, 4)$
$\alpha = 9$	$r$	18	$(x, y)$	$(5, 9), (14, 10)$	$r$	11	$(x, y)$	$(8, 14), (11, 5)$
$\alpha = 10$					$r$	8	$(x, y)$	$(9, 17), (10, 2)$
$\alpha = 11$	$r$	12	$(x, y)$	$(9, 2), (10, 17)$				
$\alpha = 13$	$r$	2	$(x, y)$	$(9, 17), (10, 2)$	$r$	16	$(x, y)$	$(9, 2), (10, 17)$
$\alpha = 14$	$r$	14	$(x, y)$	$(8, 5), (11, 14)$	$r$	17	$(x, y)$	$(5, 10), (14, 9)$
	$\beta = 13$				$\beta = 14$			
$\alpha = 5$	$r$	12	$(x, y)$	$(8, 5), (11, 14)$	$r$	18	$(x, y)$	$(9, 17), (10, 2)$
$\alpha = 6$	$r$	18	$(x, y)$	$(8, 14), (11, 5)$	$r$	8	$(x, y)$	$(8, 5), (11, 14)$
$\alpha = 8$	$r$	13	$(x, y)$	$(5, 9), (14, 10)$	$r$	10	$(x, y)$	$(6, 4), (13, 15)$
$\alpha = 9$	$r$	9	$(x, y)$	$(6, 15), (13, 4)$	$r$	4	$(x, y)$	$(5, 10), (14, 9)$
$\alpha = 10$	$r$	10	$(x, y)$	$(9, 2), (10, 17)$	$r$	15	$(x, y)$	$(8, 14), (11, 5)$
$\alpha = 11$	$r$	6	$(x, y)$	$(9, 17), (10, 2)$	$r$	9	$(x, y)$	$(5, 9), (14, 10)$
$\alpha = 13$					$r$	11	$(x, y)$	$(6, 15), (13, 4)$
$\alpha = 14$	$r$	14	$(x, y)$	$(6, 4), (13, 15)$				

### 5.7.3 Example $H_k$ -The $H_k$ -conjugation Classes when $p \equiv 1$ and $m \equiv 1$

When  $p = 5$  and  $m \equiv 1$ , we find that  $\Gamma_{(\mathbb{F}_p, m)} = \{2, 3\}$ . Table 5.8 shows that there are two  $\theta$ -split,  $k$ -anisotropic tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$  for  $p = 5$  when  $m \equiv 1$ . These tori are not conjugate, creating two different conjugacy classes, one with just the torus generated with  $\gamma = 2$  and one with just the torus generated with  $\gamma = 3$ .

When  $p = 13$  and  $m \equiv 1$ , we find that  $\Gamma_{(\mathbb{F}_p, m)} = \{3, 4, 5, 8, 9, 10\}$ . Table 5.9 shows that there are two  $\theta$ -split,  $k$ -anisotropic tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$  for  $p = 13$  when  $m \equiv 1$ . These tori are not conjugate, creating two different conjugacy classes. One conjugacy class consists of tori generated with  $\gamma$ -values 3, 4, and 8. The other conjugacy class consists of tori generated with  $\gamma$ -values 5, 9, and 10.

When  $p = 17$  and  $m \equiv 1$ , we find that  $\Gamma_{(\mathbb{F}_p, m)} = \{2, 5, 7, 8, 9, 10, 12, 15\}$ . Table 5.10 shows that there are two  $\theta$ -split,  $k$ -anisotropic tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$  for  $p = 17$  when  $m \equiv 1$ . These tori are not conjugate, creating two different conjugacy classes. One conjugacy class consists of tori generated with  $\gamma$ -values 2, 9, 10, and 12. The other conjugacy class consists of tori generated with  $\gamma$ -values 5, 7, 8, and 15.

Table 5.8: The  $H_k$ -conjugation Classes when  $p = 5$  and  $m \equiv 1$

$p = 5$	$\beta = 2$
$\alpha = 2$	

$p = 5$	$\beta = 3$
$\alpha = 3$	

Table 5.9: The  $H_k$ -conjugation Classes when  $p = 13$  and  $m \equiv 1$

$p = 13$	$\beta = 3$				$\beta = 4$				$\beta = 8$			
$\alpha = 3$					$r$	$\frac{2}{11}$	$(x, y)$	$\frac{(6, 3), (7, 10)}{(2, 4), (11, 9)}$	$r$	$\frac{3}{10}$	$(x, y)$	$\frac{(2, 9), (11, 4)}{(6, 10), (7, 3)}$
$\alpha = 4$	$r$	$\frac{6}{9}$	$(x, y)$	$\frac{(2, 9), (11, 4)}{(6, 10), (7, 3)}$					$r$	$\frac{5}{5}$	$(x, y)$	$\frac{(6, 3), (7, 10)}{(2, 4), (11, 9)}$
$\alpha = 8$	$r$	$\frac{4}{1}$	$(x, y)$	$\frac{(6, 3), (7, 10)}{(2, 4), (11, 9)}$	$r$	$\frac{5}{8}$	$(x, y)$	$\frac{(2, 9), (11, 4)}{(6, 10), (7, 3)}$				

$p = 13$	$\beta = 5$				$\beta = 9$				$\beta = 10$			
$\alpha = 5$					$r$	$\frac{5}{8}$	$(x, y)$	$\frac{(2, 4), (11, 9)}{(6, 3), (7, 10)}$	$r$	$\frac{4}{9}$	$(x, y)$	$\frac{(6, 10), (7, 3)}{(2, 9), (11, 4)}$
$\alpha = 9$	$r$	$\frac{5}{8}$	$(x, y)$	$\frac{(6, 10), (7, 3)}{(2, 9), (11, 4)}$					$r$	$\frac{6}{7}$	$(x, y)$	$\frac{(2, 4), (11, 9)}{(6, 3), (7, 10)}$
$\alpha = 10$	$r$	$\frac{3}{10}$	$(x, y)$	$\frac{(2, 4), (11, 9)}{(6, 3), (7, 10)}$	$r$	$\frac{2}{11}$	$(x, y)$	$\frac{(6, 10), (7, 3)}{(2, 9), (11, 4)}$				

Table 5.10: The  $H_k$ -conjugation Classes when  $p = 17$  and  $m \equiv 1$ 

$p = 17$	$\beta = 2$				$\beta = 9$			
$\alpha = 2$					$r$	$\frac{8}{9}$	$(x, y)$	$(3, 5), (14, 12)$ $(3, 12), (14, 5)$
$\alpha = 9$	$r$	$\frac{2}{15}$	$(x, y)$	$(3, 5), (14, 12)$ $(3, 12), (14, 5)$				
$\alpha = 10$	$r$	$\frac{4}{14}$	$(x, y)$	$(6, 16), (11, 1)$ $(4, 10), (13, 7)$	$r$	$\frac{2}{15}$	$(x, y)$	$(6, 1), (11, 16)$ $(4, 7), (13, 10)$
$\alpha = 12$	$r$	$\frac{5}{12}$	$(x, y)$	$(4, 7), (13, 10)$ $(6, 1), (11, 16)$	$r$	$\frac{6}{11}$	$(x, y)$	$(4, 10), (13, 7)$ $(6, 16), (11, 1)$
	$\beta = 10$				$\beta = 12$			
$\alpha = 2$	$r$	$\frac{4}{13}$	$(x, y)$	$(4, 7), (13, 10)$ $(6, 1), (11, 16)$	$r$	$\frac{7}{10}$	$(x, y)$	$(4, 10), (13, 7)$ $(6, 16), (11, 1)$
$\alpha = 9$	$r$	$\frac{8}{9}$	$(x, y)$	$(4, 10), (13, 7)$ $(6, 16), (11, 1)$	$r$	$\frac{3}{14}$	$(x, y)$	$(4, 7), (13, 10)$ $(6, 1), (11, 16)$
$\alpha = 10$					$r$	$\frac{6}{11}$	$(x, y)$	$(3, 5), (14, 12)$ $(3, 12), (14, 5)$
$\alpha = 12$	$r$	$\frac{3}{14}$	$(x, y)$	$(3, 12), (14, 5)$ $(3, 5), (14, 12)$				

$p = 17$	$\beta = 5$				$\beta = 7$			
$\alpha = 5$					$r$	$\frac{3}{14}$	$(x, y)$	$(3, 5), (14, 12)$ $(3, 12), (14, 5)$
$\alpha = 7$	$r$	$\frac{6}{11}$	$(x, y)$	$(3, 12), (14, 5)$ $(3, 5), (14, 12)$				
$\alpha = 8$	$r$	$\frac{3}{14}$	$(x, y)$	$(4, 10), (13, 7)$ $(6, 16), (11, 1)$	$r$	$\frac{8}{9}$	$(x, y)$	$(4, 7), (13, 10)$ $(6, 1), (11, 16)$
$\alpha = 15$	$r$	$\frac{7}{10}$	$(x, y)$	$(4, 7), (13, 10)$ $(6, 1), (11, 16)$	$r$	$\frac{4}{13}$	$(x, y)$	$(4, 10), (13, 7)$ $(6, 16), (11, 1)$
	$\beta = 8$				$\beta = 15$			
$\alpha = 5$	$r$	$\frac{6}{11}$	$(x, y)$	$(4, 7), (13, 10)$ $(6, 1), (11, 16)$	$r$	$\frac{5}{12}$	$(x, y)$	$(4, 10), (13, 7)$ $(6, 16), (11, 1)$
$\alpha = 7$	$r$	$\frac{2}{15}$	$(x, y)$	$(6, 16), (11, 1)$ $(6, 16), (11, 1)$	$r$	$\frac{4}{13}$	$(x, y)$	$(6, 1), (11, 16)$ $(4, 7), (13, 10)$
$\alpha = 8$					$r$	$\frac{2}{15}$	$(x, y)$	$(3, 12), (14, 5)$ $(3, 5), (14, 12)$
$\alpha = 15$	$r$	$\frac{8}{9}$	$(x, y)$	$(3, 12), (14, 5)$ $(3, 5), (14, 12)$				

#### 5.7.4 Example $H_k$ -The $H_k$ -conjugation Classes when $p \equiv 3$ and $m \equiv N_p$

When  $p = 3$ , we find that  $N_p = 2$ , thus  $m \equiv N_p = 2$ . In this case, we determine that  $\Gamma_{(\mathbb{F}_p, m)} = \{1, 2\}$ . Table 5.11 shows that there are two  $\theta$ -split,  $k$ -anisotropic tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$  for  $p = 3$  when  $m \equiv N_p$ . These tori are not conjugate, creating two different conjugacy classes, one with just the torus generated with  $\gamma = 1$  and one with just the torus generated with  $\gamma = 2$ .

When  $p = 7$ , we find that  $N_p = 3$ , thus  $m \equiv N_p = 3$ . In this case, we determine that  $\Gamma_{(\mathbb{F}_p, m)} = \{1, 2, 5, 6\}$ . Table 5.12 shows that there are two  $\theta$ -split,  $k$ -anisotropic tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$  for  $p = 7$  when  $m \equiv N_p$ . These tori are not conjugate, creating two different conjugacy classes. One conjugacy class consists of tori generated with  $\gamma$ -values 1 and 5. The other conjugacy class consists of tori generated with  $\gamma$ -values 2 and 6.

When  $p = 11$ , we find that  $N_p = 2$ , thus  $m \equiv N_p = 2$ . In this case, we determine that  $\Gamma_{(\mathbb{F}_p, m)} = \{1, 4, 5, 6, 7, 10\}$ . Table 5.13 shows that there are two  $\theta$ -split,  $k$ -anisotropic tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$  for  $p = 11$  when  $m \equiv N_p$ . These tori are not conjugate, creating two different conjugacy classes. One conjugacy class consists of tori generated with  $\gamma$ -values 1, 4, and 5. The other conjugacy class consists of tori generated with  $\gamma$ -values 6, 7, and 10.

Table 5.11: The  $H_k$ -conjugation Classes when  $p = 3$  and  $m \equiv N_p$

$p = 3$	$\beta = 1$
$\alpha = 1$	

$p = 3$	$\beta = 2$
$\alpha = 2$	

Table 5.12: The  $H_k$ -conjugation Classes when  $p = 7$  and  $m \equiv N_p$

$p = 7$	$\beta = 1$				$\beta = 5$			
$\alpha = 1$					$r$	$\frac{2}{5}$	$(x, y)$	$(2, 1), (5, 6)$
								$(2, 6), (5, 1)$
$\alpha = 5$	$r$	$\frac{3}{4}$	$(x, y)$	$(2, 1), (5, 6)$				
		$\frac{4}{5}$		$(2, 6), (5, 1)$				

$p = 7$	$\beta = 2$				$\beta = 6$			
$\alpha = 2$					$r$	$\frac{3}{4}$	$(x, y)$	$(2, 6), (5, 1)$
						$\frac{4}{5}$		$(2, 1), (5, 6)$
$\alpha = 6$	$r$	$\frac{2}{5}$	$(x, y)$	$(2, 6), (5, 1)$				
		$\frac{5}{6}$		$(2, 1), (5, 6)$				

Table 5.13: The  $H_k$ -conjugation Classes when  $p = 11$  and  $m \equiv N_p$

$p = 11$	$\beta = 1$				$\beta = 4$				$\beta = 5$			
$\alpha = 1$					$r$	$\frac{4}{7}$	$(x, y)$	$\frac{(3, 2), (8, 9)}{(5, 1), (6, 10)}$	$r$	$\frac{3}{8}$	$(x, y)$	$\frac{(5, 10), (6, 1)}{(3, 9), (8, 2)}$
$\alpha = 4$	$r$	$\frac{3}{8}$	$(x, y)$	$\frac{(3, 9), (8, 2)}{(5, 10), (6, 1)}$					$r$	$\frac{2}{9}$	$(x, y)$	$\frac{(5, 1), (6, 10)}{(3, 2), (8, 9)}$
$\alpha = 5$	$r$	$\frac{4}{7}$	$(x, y)$	$\frac{(5, 1), (6, 10)}{(3, 2), (8, 9)}$	$r$	$\frac{5}{6}$	$(x, y)$	$\frac{(3, 9), (8, 2)}{(5, 10), (6, 1)}$				

  

$p = 11$	$\beta = 6$				$\beta = 7$				$\beta = 10$			
$\alpha = 6$					$r$	$\frac{5}{6}$	$(x, y)$	$\frac{(3, 2), (8, 9)}{(5, 1), (6, 10)}$	$r$	$\frac{4}{7}$	$(x, y)$	$\frac{(5, 10), (6, 1)}{(3, 9), (8, 2)}$
$\alpha = 7$	$r$	$\frac{2}{9}$	$(x, y)$	$\frac{(5, 10), (6, 1)}{(3, 9), (8, 2)}$					$r$	$\frac{3}{8}$	$(x, y)$	$\frac{(3, 2), (8, 9)}{(5, 1), (6, 10)}$
$\alpha = 10$	$r$	$\frac{3}{8}$	$(x, y)$	$\frac{(5, 1), (6, 10)}{(3, 2), (8, 9)}$	$r$	$\frac{4}{7}$	$(x, y)$	$\frac{(3, 9), (8, 2)}{(5, 10), (6, 1)}$				



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## Chapter 6

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### Type 3 Tori in $\mathfrak{sl}(2, k)$ when $k = \mathbb{Q}_p$ , $p \neq 2$

In this chapter, we classify tori of the third form and characterize their  $H_k$ -orbits over the  $p$ -adic fields for  $p \neq 2$ . As we saw in Chapter 3,  $k$ -tori with additive-inverse  $\gamma$ -values will play a major role in this characterization. Some of the methods introduced in Chapter 5 can generalize.

#### 6.1 The $H_k$ -conjugation Classes when $p \neq 2$

Suppose we take distinct  $\gamma$ -values  $\alpha$  and  $\beta$ . Hence consider distinct tori  $\mathfrak{t}_\alpha$  and  $\mathfrak{t}_\beta$  with generators  $t_\alpha$  and  $t_\beta$ , respectively. Then we want to find some  $h_k \in H_k$  that conjugates the generator  $t_\alpha$  to the  $r$  multiple of the generator  $t_\beta$ , where  $m$  is a representative of the a square class in  $k^*/(k^*)^2$ . Recall that when  $k = \mathbb{Q}_p$  and  $p \neq 2$ , the set of square classes is  $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 = \{1, N_p, p, pN_p\}$ . By assumption, the tori and their generators are not split over  $\mathbb{Q}_p$ . Thus, we consider  $\gamma \in \Gamma_{(\mathbb{Q}_p, m)}$  such that  $1 - m\gamma^2 \equiv N_p, p, pN_p$ .

We know from the discussion in Lemma 4.1.3 that  $r = -1$  or  $r = \pm \left[ \frac{1 - m\alpha^2}{1 - m\beta^2} \right]^{1/2}$  when  $t_\alpha$  and  $t_\beta$  are  $H_k$ -conjugate.

**Theorem 6.1.1.** *Let  $\gamma \in \Gamma_{(\mathbb{Q}_p, m)}$  be a value that makes the maximal  $\theta$ -split  $k$ -tori of Type 3  $k$ -anisotropic. Then torus generated with  $\gamma$  is  $H_k$ -conjugate to the torus generated with  $-\gamma$  if and only if*

1. when  $m \equiv 1$ ,

(a) we have  $p \equiv 3 \pmod{4}$ , and  $1 - m\gamma^2 \equiv N_p$ , or

2. when  $m \equiv N_p$ ,

(a) we have  $p \equiv 1 \pmod{4}$ , and  $m \equiv N_p$ , or

3. when  $m \equiv p$ ,

(a) we have  $p \equiv 1 \pmod{4}$ , and  $1 - m\gamma^2 \equiv p$ , or

(b) we have  $p \equiv 3 \pmod{4}$ , and  $1 - m\gamma^2 \equiv pN_p$ , or

4. when  $m \equiv pN_p$ ,

(a) we have  $p \equiv 1 \pmod{4}$ , and  $1 - m\gamma^2 \equiv pN_p$ , or

(b) we have  $p \equiv 3 \pmod{4}$ , and  $1 - m\gamma^2 \equiv p$ .

*Proof.* By Theorem 4.1.1, the tori  $\mathbf{t}_\gamma$  and  $\mathbf{t}_{-\gamma}$  are  $H_k$ -conjugate when  $-m \equiv 1 - m\gamma^2$ . Recall from Corollary 2.5.2 that  $-1 \equiv 1$  when  $p \equiv 1 \pmod{4}$  and  $-1 \equiv N_p$  when  $p \equiv 3 \pmod{4}$ .

If  $m \equiv 1$ , then  $-m \equiv -1$ . When  $p \equiv 1 \pmod{4}$ ,  $-1 \equiv 1$ . Thus,  $t_\gamma$  and  $t_{-\gamma}$  will be  $H_k$ -conjugate when  $1 - m\gamma^2 \equiv 1$ , making  $t_\gamma$  and  $t_{-\gamma}$   $k$ -split and not of consideration in this theorem. When  $p \equiv 3 \pmod{4}$ ,  $-1 \equiv N_p$ . Thus,  $t_\gamma$  and  $t_{-\gamma}$  will be  $H_k$ -conjugate only when  $1 - m\gamma^2 \equiv N_p$ .

If  $m \equiv N_p$ , then  $-m \equiv -N_p$ . When  $p \equiv 1 \pmod{4}$ ,  $-N_p \equiv N_p$ . Thus,  $t_\gamma$  and  $t_{-\gamma}$  will be  $H_k$ -conjugate only when  $1 - m\gamma^2 \equiv N_p$ . When  $p \equiv 3 \pmod{4}$ ,  $-N_p \equiv 1$ . Thus,  $t_\gamma$  and  $t_{-\gamma}$  will be  $H_k$ -conjugate when  $1 - m\gamma^2 \equiv 1$ , making  $t_\gamma$  and  $t_{-\gamma}$   $k$ -split and not of consideration in this theorem.

If  $m \equiv p$ , then  $-m \equiv -p$ . When  $p \equiv 1 \pmod{4}$ ,  $-p \equiv p$ . Thus,  $t_\gamma$  and  $t_{-\gamma}$  will be  $H_k$ -conjugate only when  $1 - m\gamma^2 \equiv p$ . When  $p \equiv 3 \pmod{4}$ ,  $-p \equiv pN_p$ . Thus,  $t_\gamma$  and  $t_{-\gamma}$  will be  $H_k$ -conjugate only when  $1 - m\gamma^2 \equiv pN_p$ .

If  $m \equiv pN_p$ , then  $-m \equiv -pN_p$ . When  $p \equiv 1 \pmod{4}$ ,  $-pN_p \equiv pN_p$ . Thus,  $t_\gamma$  and  $t_{-\gamma}$  will be  $H_k$ -conjugate only when  $1 - m\gamma^2 \equiv pN_p$ . When  $p \equiv 3 \pmod{4}$ ,  $-pN_p \equiv p$ . Thus,  $t_\gamma$  and  $t_{-\gamma}$  will be  $H_k$ -conjugate only when  $1 - m\gamma^2 \equiv p$ .  $\square$

Like Theorem 5.2.1 over  $\mathbb{F}_p$ , Theorem 6.1.1 is significant because it shows that at least two  $H_k$ -conjugacy classes occur in many situations.

**Lemma 6.1.1.** *Let  $k = \mathbb{Q}_p$ , and the combination of  $m$ ,  $p$ , and  $1 - m\gamma^2$  be one of those outlined in Theorem 6.1.1. Then  $\mathfrak{t}_\alpha$  is  $H_k$ -conjugate to  $\mathfrak{t}_\beta$  if and only if  $\mathfrak{t}_\alpha$  is  $H_k$ -conjugate to  $\mathfrak{t}_{-\beta}$ .*

*Proof.* Let  $t_\alpha$ ,  $t_\beta$ , and  $t_{-\beta}$  be the generators of  $\mathfrak{t}_\alpha$ ,  $\mathfrak{t}_\beta$ , and  $\mathfrak{t}_{-\beta}$ , respectively. Assume  $\mathfrak{t}_\alpha$  is  $H_k$ -conjugate to  $\mathfrak{t}_\beta$ , say  $h_1 \in H_k$  such that  $h_1 \cdot t_\alpha = t_\beta$ . By Theorem 5.2.1,  $h_2 \cdot t_\beta = t_{-\beta}$  for some  $h_2 \in H_k$ . Thus,  $h_2 h_1$  conjugates  $\mathfrak{t}_\alpha$  to  $\mathfrak{t}_{-\beta}$ , and  $h_2 h_1 \in H_k$ . The argument reverses.  $\square$

For any  $\gamma \in \Gamma_{(\mathbb{Q}_p, m)}$ , the value  $1 - m\gamma^2$  can be equivalent to  $N_p$ ,  $p$ , or  $pN_p$ . Thus, maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3 can have eigenvalues that depend on the square roots of  $N_p$ ,  $p$ , or  $pN_p$ . Moreover, these maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3 can not be  $H_k$ -conjugate unless their  $1 - m\gamma^2$  values are equivalent. For this reason, we adopt the following notation.

*Notation.* Let  $\Gamma_{N_p}$ ,  $\Gamma_p$ , and  $\Gamma_{pN_p}$  be the subsets of  $\Gamma_{(\mathbb{Q}_p, m)}$  containing exactly the  $\gamma \in \Gamma_{(\mathbb{Q}_p, m)}$  such that  $1 - m\gamma^2 \equiv N_p$ ,  $p$ , and  $pN_p$ , respectively. We use the notation  $\Gamma_m$  to denote either  $\Gamma_{N_p}$ ,  $\Gamma_p$ , or  $\Gamma_{pN_p}$ , arbitrarily.

This notation is useful in the application of Lemma 6.1.1 below.

**Corollary 6.1.1.** *Let  $k = \mathbb{Q}_p$  for  $p \neq 2$ . Consider  $\Gamma_m \subset \Gamma_{(\mathbb{Q}_p, m)}$ , the set of all  $\gamma \in \Gamma_{(\mathbb{Q}_p, m)}$  of equivalent  $1 - m\gamma^2$ . Then there are at most 4  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic Type 3 tori with  $\gamma \in \Gamma_m$ .*

*Proof.* Let  $\alpha, \beta, \gamma, \delta, \varepsilon \in \Gamma_m$ . Assume  $t_\alpha, t_\beta, t_\gamma$ , and  $t_\delta$  are not  $H_k$ -conjugate, thus there are at least 4  $H_k$ -conjugation classes.

We know from Lemma 4.1.4 that  $t_\alpha$  is  $H$ -conjugate to  $t_\beta$  by a matrix  $h_1$  of the form

$$\frac{1}{[(1+r)^2 - m(\alpha - r\beta)^2]^{1/2}} \begin{pmatrix} 1+r & \alpha - r\beta \\ m(\alpha - r\beta) & 1+r \end{pmatrix}. \quad (6.1)$$

Assume that  $t_\alpha$  and  $t_\beta$  are not  $H_k$ -conjugate, that is, assume the value

$$(1+r)^2 - m(\alpha - r\beta)^2 \quad (6.2)$$

is not equivalent to 1 and hence not a square. Suppose further that  $(1+r)^2 - m(\alpha - r\beta)^2 \equiv N_p$ . Thus,  $t_\alpha$  and  $t_\beta$  are in separate  $H_k$ -conjugation classes.

Similarly, consider the matrices of the form in Formula 6.1 that send  $t_\alpha$  to  $t_\gamma$  and  $t_\delta$  and label them matrices  $h_2$  and  $h_3$ , respectively. Further assume that  $(1+r)^2 - m(\alpha - r\gamma)^2 \equiv p$  and  $(1+r)^2 - m(\alpha - r\delta)^2 \equiv N_p$ . Thus,  $t_\alpha$  is not  $H_k$ -conjugate to  $t_\gamma$  or  $t_\delta$ .

Now consider the matrix  $h_4 \in H$ , like Formula 6.1, that sends  $t_\varepsilon$  to  $t_\alpha$ . The value  $(1+r)^2 - m(\varepsilon - r\alpha)^2$  will be equivalent to either  $1, N_p, p$ , or  $pN_p$ . Thus,  $t_\varepsilon$  will be  $H_k$ -conjugate to  $t_\alpha, t_\beta, t_\gamma$ , or  $t_\delta$ , respectively. Therefore, there are at most 4  $H_k$ -conjugation classes of tori with  $\gamma$ -values in  $\Gamma_m$ .  $\square$

Below, we give a count of the maximum number of  $H_k$ -conjugacy classes of all  $\theta$ -split  $k$ -anisotropic tori.

**Proposition 6.1.1.** *Let  $k = \mathbb{Q}_p$  for  $p \neq 2$ . Then there are at most 12  $H_k$ -conjugation classes of tori of Type 3.*

*Proof.* Tori of Type 3 are  $k$ -anisotropic when their  $\gamma$ -values are in  $\Gamma_m$ . There are 3 values of  $m$  that make these tori  $k$ -anisotropic. By Corollary 6.1.1, there are at most 4  $H_k$ -conjugation classes for each value of  $m$ . Therefore, there are at most 12  $H_k$ -conjugation classes of Type 3 tori in  $\mathbb{Q}_p$  when  $p \neq 2$ .  $\square$

Note that Proposition 6.1.1 implies that there is a finite number of  $H_k$ -conjugation classes of tori of Type 3 over  $\mathbb{Q}_p$  when  $p \neq 2$ .

---

## Chapter 7

---

### The $H_k$ -conjugacy Classes of $\theta$ -split $k$ -tori

In this chapter, give a total count, or in some cases, minimum number, of  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori in  $\mathfrak{sl}(2, k)$  for  $k = \mathbb{Q}, \mathbb{F}_p, \mathbb{Q}_p$ . In [Beu08], Beun gives a count of the maximal  $(\theta, k)$ -split tori. In Chapter 3, we give a count of the maximal  $\theta$ -split  $k$ -anisotropic Type 2 tori as well as Type 3 when  $k = \mathbb{Q}$ . In Chapters 5 and 6, we give a count of the maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3. The following theorems summarize all of these results.

#### 7.1 The $H_k$ -conjugacy Classes of $\theta$ -split $k$ -tori when $k = \mathbb{Q}$

**Theorem 7.1.1.** *Let  $k = \mathbb{Q}$  and consider  $\mathfrak{sl}(2, \mathbb{Q})$ . Then there is an infinite number  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori.*

*Proof.* In Theorem 2.6.3, Beun shows that there is an infinite number of  $H_k$ -conjugacy classes of maximal  $(\theta, k)$ -split tori. In Theorem 3.2.1, we show that there is only one class of maximal  $\theta$ -split  $k$ -anisotropic Type 2. In Theorem 4.2.1, we show that there is an infinite number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic Type 3. Thus, in total, there is an infinite number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split tori in  $\mathfrak{sl}(2, \mathbb{Q})$ .  $\square$

## 7.2 The $H_k$ -conjugacy Classes of $\theta$ -split $k$ -tori when $k = \mathbb{F}_p$

Corollary 5.2.2 gives us an exact count of the  $H_k$ -conjugacy classes of  $\theta$ -split  $k$ -anisotropic tori when  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$  or  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ . Combined with Beun's results in [Beu08], we can now give an exact count of the total number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori over  $\mathfrak{sl}(2, \mathbb{F}_p)$  when  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$  or when  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ .

**Theorem 7.2.1.** *Let  $k = \mathbb{F}_p$  and consider  $\mathfrak{sl}(2, \mathbb{F}_p)$ . Then there are 4  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori when  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$  or when  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ .*

*Proof.* Theorem 3.2.2 shows that there are no  $H_k$ -conjugacy classes of  $\theta$ -split  $k$ -anisotropic tori of Type 2 in  $\mathfrak{sl}(2, \mathbb{F}_p)$ . Corollary 5.2.2 shows that there are 2  $H_k$ -conjugacy classes of  $\theta$ -split  $k$ -anisotropic tori of Type 3 in  $\mathfrak{sl}(2, \mathbb{F}_p)$ . By Lemma 3.1.2, these are distinct tori. Theorem 2.6.2 shows that there are 2  $H_k$ -conjugacy classes of  $(\theta, k)$ -split  $k$ -tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$ . Tori with eigenvalues in the base field can not be  $H_k$ -conjugate to tori with eigenvalues not in the base field. Therefore, there are 4 classes of  $\theta$ -split  $k$ -tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$ .  $\square$

There are two possibilities for the number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori when  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$  or  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ . Proposition 5.2.1 shows that there is either 1 or 2  $H_k$ -conjugacy classes. Combined with Beun's results in [Beu08], we have the following theorem that provides a lower bound and upper bound on the number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$ .

**Theorem 7.2.2.** *Let  $k = \mathbb{F}_p$  and consider  $\mathfrak{sl}(2, \mathbb{F}_p)$ . Then the number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split tori when  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$  or when  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$  is either 3 or 4.*

*Proof.* By Proposition 5.2.1, there is either 1  $H_k$ -conjugacy class or 2  $H_k$ -conjugacy classes of  $\theta$ -split  $k$ -anisotropic tori of Type 3. Proposition 5.2.2 shows that there is 1  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$  of Type 3. By Lemma 3.1.2, these are distinct tori. Theorem 2.6.2 shows that there is 1  $H_k$ -conjugacy class of maximal  $(\theta, k)$ -split  $k$ -tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$ . Tori with eigenvalues in the base field can

not be  $H_k$ -conjugate to tori with eigenvalues not in the base field. Therefore, there are 3 or 4  $H_k$ -conjugacy classes of  $\theta$ -split  $k$ -tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$ .  $\square$

By Proposition 5.2.2, there is exactly 1 class when  $p < 50$  is an odd prime. Conjecture 5.2.1 generalizes this result. Combined with Beun's results in [Beu08], we have the following conjecture and theorem about the exact number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori over  $\mathfrak{sl}(2, \mathbb{F}_p)$  for when  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$  or when  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ .

**Conjecture 7.2.1.** *Let  $k = \mathbb{F}_p$  and consider  $\mathfrak{sl}(2, \mathbb{F}_p)$ . Then there are 3  $H_k$ -conjugacy classes of maximal  $\theta$ -split tori when  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$  or when  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ .*

**Corollary 7.2.1.** *Let  $k = \mathbb{F}_p$  where  $p < 50$  is an odd prime, and consider  $\mathfrak{sl}(2, \mathbb{F}_p)$ . Then there are 3  $H_k$ -conjugacy classes of maximal  $\theta$ -split tori when  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$  or when  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ .*

*Proof.* Proposition 5.2.2 shows that there is 1  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$  of Type 3. Theorem 7.2.2 then implies that there is exactly 3  $H_k$ -conjugacy classes of  $\theta$ -split  $k$ -tori in  $\mathfrak{sl}(2, \mathbb{F}_p)$ .  $\square$

### 7.3 The $H_k$ -conjugacy Classes of $\theta$ -split $k$ -tori when $k = \mathbb{Q}_p$ , $p \neq 2$

**Proposition 7.3.1.** *Let  $k = \mathbb{Q}_p$  for  $p \neq 2$ . Then the following is a maximum number of  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori in  $\mathfrak{sl}(2, \mathbb{Q}_p)$ .*

1. *There are at most 16  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori when  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$ .*
2. *There are at most 17  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori when  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ .*
3. *There are at most 16  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori of when  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$ .*
4. *There are at most 14  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori of when  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ .*

5. *There are at most 16  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori of when  $m \equiv p, N_p$ , regardless of  $p$ .*

*Proof.* First, consider the case when  $p \equiv 1 \pmod{4}$  and  $m \equiv 1$ . Theorem 2.6.3 shows that there are 4  $H_k$ -conjugacy classes of maximal  $(\theta, k)$ -split tori. Theorem 3.2.3 shows that there are no  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 2. Proposition 6.1.1 shows that there are at most 12  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3. Therefore, there are at most 16 classes of maximal  $\theta$ -split  $k$ -tori in  $\mathfrak{sl}(2, \mathbb{Q}_p)$ .

Consider the case when  $p \equiv 3 \pmod{4}$  and  $m \equiv 1$ . Theorem 2.6.3 shows that there are 2  $H_k$ -conjugacy classes of maximal  $(\theta, k)$ -split tori. Theorem 3.2.3 shows that there are 3  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 2. Proposition 6.1.1 shows that there are at most 12  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3. Therefore, there are at most 17  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori in  $\mathfrak{sl}(2, \mathbb{Q}_p)$ .

Consider the case when  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$ . Theorem 2.6.3 shows that there is 1  $H_k$ -conjugacy class of maximal  $(\theta, k)$ -split tori. Theorem 3.2.3 shows that there are 3  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 2. Proposition 6.1.1 shows that there are at least 5  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3. Therefore, there are at most 16  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori in  $\mathfrak{sl}(2, \mathbb{Q}_p)$ .

Consider the case when  $p \equiv 3 \pmod{4}$  and  $m \equiv N_p$ . Theorem 2.6.3 shows that there are 2  $H_k$ -conjugacy class of maximal  $(\theta, k)$ -split tori. Theorem 3.2.3 shows that there are no  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 2. Proposition 6.1.1 shows that there are at most 12  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -anisotropic tori of Type 3. Therefore, there are at most 14  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori in  $\mathfrak{sl}(2, \mathbb{Q}_p)$ .

The cases when  $m \equiv p, N_p$ , regardless of  $p$ , behave the same as the  $p \equiv 1 \pmod{4}$  and  $m \equiv N_p$ , for a total of at most 16  $H_k$ -conjugacy classes of maximal  $\theta$ -split  $k$ -tori in  $\mathfrak{sl}(2, \mathbb{Q}_p)$ .  $\square$



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## APPENDIX

---

# Appendix A

---

## Example Computations

Many of the computations over  $\mathbb{F}_p$  were done in Maple<sup>TM</sup>. We include two examples of code, the first of which shows the computation of  $H_k$ -conjugacy classes, the second shows the computation of the conjugating matrices  $h_k \in H_k$ .

In each example, note that the only user input required is the values of  $p$  and  $m$ . A list of squares in  $\mathbb{F}_p$  is output line (3), and from this we can determine  $N_p$  as well. One of the main results of both examples is the computations of all possible  $t_\alpha$  to  $rt_\beta$  conjugations.

Throughout this thesis, we use the notation  $t_\alpha$  to denote the Type 3 total subalgebra generator with  $\gamma$ -value  $\alpha$  that gets mapped to the  $r^{th}$  multiple of the Type 3 total subalgebra generator with  $\gamma$ -value  $\beta$ . For simplicity, in this code,  $\alpha$  is represented by  $a$  and  $\beta$  is represented by  $b$ . Note that in output line (4), *EigensList* is not a list of actual eigenvalues, but a list of values whose square roots would be the eigenvalues of the matrix  $t_\alpha$ , should they exist. In line (5), we determine which  $\gamma$ -values lead to eigenvalues in the base field and which  $\gamma$ -values lead to nonsplit generators.

In line (10), we define the expressions necessary for the  $t_\alpha \rightarrow rt_\beta$  computations. The functions *reln1* and *reln2* are used for computing the matrix entries, and the function *reln3* defines the determinant of the matrix  $h_k$ , which is simply called  $h$  in this code, as in line (8). Then, in line (11), we create several loops to check the resulting relations from the equation  $h_k t_\alpha h_k^{-1} = rt_\beta$ . The loops store data when a conjugation over the base field  $\mathbb{F}_p$  exists, and this data is stored in the list *solutions*. Each datum returned has the

form  $[a, b, r, x, y]$ , where  $a$  is the  $\alpha$ -value,  $b$  is the  $\beta$ -value,  $r$  is the multiple of  $t_\beta$  to which  $t_\alpha$  is sent, and  $x$  and  $y$  define the entries of the matrix  $h_k$ .

## A.1 Computing the $H_k$ -conjugacy Classes

The following shows example code for the  $H_k$ -conjugacy class computations in  $\mathfrak{sl}(2, k)$  for  $k = \mathbb{F}_p$  when  $p = 23$  and  $m \equiv Np \equiv 5$ .

After computing all  $H_k$ -conjugations in **(11)**, we select the first  $\gamma$ -value listed in *Gammas* and find every other  $\gamma$ -value whose torus conjugates to  $\mathfrak{t}_\gamma$ . These values are stored in *Class1* in line **(12)**. Any  $\gamma$ -values whose tori do not conjugate to  $\mathfrak{t}_\gamma$  are stored in *NotClass1*. The process then repeats in line **(13)**, where the first  $\gamma$ -value in *NotClass1* is selected and we find every other  $\gamma$ -value whose torus conjugates to  $\mathfrak{t}_\gamma$ . These values are stored in *Class2*.

In the event that there is only one  $H_k$ -conjugacy class, the set *NotClass1* is empty, making the loop in line **(13)** inoperable. When this loop is executed, the output will be “**Error, invalid subscript selector**” because *NotClass1Set* is empty.

Last, in line **(14)** we check that every  $\gamma$ -value was sorted into either *Class1* or *Class2*. We tested all primes under 50, and *Neither1or2* was always returned empty, meaning that there are at most two  $H_k$ -conjugacy classes.

```
[ > with(LinearAlgebra[Modular]) :
    with(ListTools) :
```

```
[ > p := 23;
    m := 5;

                                p := 23
                                m := 5

(1)
```

```
[ > # Define the field

    Fp := [ ] :
    for i from 0 to p - 1 do
        Fp := [op(Fp), i]
    od:
    Fp;

                                [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]

(2)
```

```
[ > #Find the squares in the field

    SqsListShort := [ ] :
    SqsSet := { } :
    for i from 0 to p - 1 do
        SqsSet := [op(SqsSet), i^2 mod p] :
    od:
    SqsListAll := convert(SqsSet, list) :
    > SqsListShort := MakeUnique(SqsListAll);
                                SqsListShort := [0, 1, 4, 9, 16, 2, 13, 3, 18, 12, 8, 6]

(3)
```

```
[ > #Find all the  $1-ma^2$  values, those that will split & those that won't

    EigensList := [ ] :
    templist := [ ] :
    for i from 0 to p - 1 do
        templist := [op(templist), 1 - m · i^2 mod p] :
    od:
    EigensList := MakeUnique(templist) :
    EigensList;

                                [1, 19, 4, 2, 13, 14, 5, 9, 3, 10, 7, 17]

(4)
```

> #Sort the  $1-ma^2$  values into splits & nonsplits

```
SplitsSet := { } :
SplitsList := [ ] :
NonSplitsSet := { } :
NonSplitsList := [ ] :
```

```
EigensSet := convert(EigensList, set) :
SqsSetShort := convert(SqsListShort, set) :
SplitsSet := intersect(EigensSet, SqsSetShort);
NonSplitsSets := EigensSet minus SplitsSet;
SplitsList := convert(SplitsSet, list);
NonSplitsList := convert(NonSplitsSets, list);
```

```
SplitsSet := {1, 2, 3, 4, 9, 13}
NonSplitsSets := {5, 7, 10, 14, 17, 19}
SplitsList := [1, 2, 3, 4, 9, 13]
NonSplitsList := [5, 7, 10, 14, 17, 19]
```

(5)

> #Find the elements that lead to nonsplits

```
Gammas := [ ] :
for i from 0 to p - 1 do
  if member(1 - m · i2 mod p, NonSplitsList) = 'true' then Gammas := [op(Gammas), i] fi;
od;
Gammas;
```

```
[1, 5, 6, 9, 10, 11, 12, 13, 14, 17, 18, 22]
```

(6)

> # Define the matrices and resulting corresponding conjugation relations

```
unassign('a');
t := Matrix([ [1, a], [-1 · m · a, -1] ]);
```

$$t := \begin{bmatrix} 1 & a \\ -5a & -1 \end{bmatrix}$$

(7)

```
> unassign('x','y');
h := Matrix([ [x, y], [m · y, x] ]);
hinv := Matrix([ [x, -y], [-1 · m · y, x] ]);
```

$$h := \begin{bmatrix} x & y \\ 5y & x \end{bmatrix}$$

(8)



$$hinv := \begin{bmatrix} x & -y \\ -5y & x \end{bmatrix} \quad (8)$$

```
> simplify(h.(t.hinv));
```

$$\begin{bmatrix} x^2 - 10yax + 5y^2 & -2xy + ax^2 + 5ay^2 \\ 10xy - 25ay^2 - 5ax^2 & -5y^2 + 10yax - x^2 \end{bmatrix} \quad (9)$$

```
> reln1 := (a, x, y) → x^2 - 2·m·a·x·y + m·y^2;
   reln2 := (a, x, y) → a·x^2 - 2·x·y + m·a·y^2;
   reln3 := (x, y) → x^2 - m·y^2;
```

$$\begin{aligned} reln1 &:= (a, x, y) \rightarrow x^2 - 2m a x y + m y^2 \\ reln2 &:= (a, x, y) \rightarrow a x^2 - 2 x y + m a y^2 \\ reln3 &:= (x, y) \rightarrow x^2 - m y^2 \end{aligned} \quad (10)$$

```
> #Find the conjugation classes

unassign('x','y','a','r');
solutions := { } :
for i from 1 to nops(Gammas) do
for b from 1 to nops(Gammas) do
for r from 1 to p do
for x from 0 to p - 1 do
for y from 0 to p - 1 do

a := Gammas[i] :
if a ≠ Gammas[b] then
if reln1(a, x, y) mod p = r and reln2(a, x, y) mod p = (Gammas[b]·r mod p) and reln3(x,
y) mod p = 1 then solutions := [op(solutions), ([a, Gammas[b], r, x, y])] fi;
fi;

od;
od;
od;
od;
od;
od;
solutions;
#Each datum returned has form [a, b, r, x, y].

[[1, 11, 4, 4, 7], [1, 11, 4, 19, 16], [1, 11, 19, 10, 11], [1, 11, 19, 13, 12], [1, 13, 11, 10, 12],
[1, 13, 11, 13, 11], [1, 13, 12, 4, 16], [1, 13, 12, 19, 7], [1, 14, 8, 9, 4], [1, 14, 8, 14, 19],
[1, 14, 15, 9, 19], [1, 14, 15, 14, 4], [1, 17, 6, 11, 22], [1, 17, 6, 12, 1], [1, 17, 17, 8, 10],
[1, 17, 17, 15, 13], [1, 18, 7, 8, 13], [1, 18, 7, 15, 10], [1, 18, 16, 11, 1], [1, 18, 16, 12,

```

(11)

22], [5, 6, 9, 11, 22], [5, 6, 9, 12, 1], [5, 6, 14, 8, 10], [5, 6, 14, 15, 13], [5, 9, 11, 4, 7],  
 [5, 9, 11, 19, 16], [5, 9, 12, 10, 11], [5, 9, 12, 13, 12], [5, 10, 5, 4, 16], [5, 10, 5, 19, 7],  
 [5, 10, 18, 10, 12], [5, 10, 18, 13, 11], [5, 12, 6, 9, 19], [5, 12, 6, 14, 4], [5, 12, 17, 9, 4],  
 [5, 12, 17, 14, 19], [5, 22, 10, 8, 13], [5, 22, 10, 15, 10], [5, 22, 13, 11, 1], [5, 22, 13, 12,  
 22], [6, 5, 5, 8, 13], [6, 5, 5, 15, 10], [6, 5, 18, 11, 1], [6, 5, 18, 12, 22], [6, 9, 9, 4, 16],  
 [6, 9, 9, 19, 7], [6, 9, 14, 10, 12], [6, 9, 14, 13, 11], [6, 10, 2, 9, 4], [6, 10, 2, 14, 19], [6,  
 10, 21, 9, 19], [6, 10, 21, 14, 4], [6, 12, 7, 4, 7], [6, 12, 7, 19, 16], [6, 12, 16, 10, 11], [6,  
 12, 16, 13, 12], [6, 22, 4, 11, 22], [6, 22, 4, 12, 1], [6, 22, 19, 8, 10], [6, 22, 19, 15, 13],  
 [9, 5, 2, 10, 12], [9, 5, 2, 13, 11], [9, 5, 21, 4, 16], [9, 5, 21, 19, 7], [9, 6, 5, 10, 11], [9,  
 6, 5, 13, 12], [9, 6, 18, 4, 7], [9, 6, 18, 19, 16], [9, 10, 10, 11, 1], [9, 10, 10, 12, 22], [9,  
 10, 13, 8, 13], [9, 10, 13, 15, 10], [9, 12, 11, 8, 10], [9, 12, 11, 15, 13], [9, 12, 12, 11,  
 22], [9, 12, 12, 12, 1], [9, 22, 3, 9, 4], [9, 22, 3, 14, 19], [9, 22, 20, 9, 19], [9, 22, 20, 14,  
 4], [10, 5, 9, 10, 11], [10, 5, 9, 13, 12], [10, 5, 14, 4, 7], [10, 5, 14, 19, 16], [10, 6, 11, 9,  
 4], [10, 6, 11, 14, 19], [10, 6, 12, 9, 19], [10, 6, 12, 14, 4], [10, 9, 7, 11, 22], [10, 9, 7,  
 12, 1], [10, 9, 16, 8, 10], [10, 9, 16, 15, 13], [10, 12, 8, 8, 13], [10, 12, 8, 15, 10], [10,  
 12, 15, 11, 1], [10, 12, 15, 12, 22], [10, 22, 2, 4, 16], [10, 22, 2, 19, 7], [10, 22, 21, 10,  
 12], [10, 22, 21, 13, 11], [11, 1, 6, 4, 16], [11, 1, 6, 19, 7], [11, 1, 17, 10, 12], [11, 1, 17,  
 13, 11], [11, 13, 3, 8, 13], [11, 13, 3, 15, 10], [11, 13, 20, 11, 1], [11, 13, 20, 12, 22],  
 [11, 14, 2, 11, 22], [11, 14, 2, 12, 1], [11, 14, 21, 8, 10], [11, 14, 21, 15, 13], [11, 17, 10,  
 4, 7], [11, 17, 10, 19, 16], [11, 17, 13, 10, 11], [11, 17, 13, 13, 12], [11, 18, 4, 9, 19],  
 [11, 18, 4, 14, 4], [11, 18, 19, 9, 4], [11, 18, 19, 14, 19], [12, 5, 4, 9, 4], [12, 5, 4, 14,  
 19], [12, 5, 19, 9, 19], [12, 5, 19, 14, 4], [12, 6, 10, 4, 16], [12, 6, 10, 19, 7], [12, 6, 13,  
 10, 12], [12, 6, 13, 13, 11], [12, 9, 2, 11, 1], [12, 9, 2, 12, 22], [12, 9, 21, 8, 13], [12, 9,  
 21, 15, 10], [12, 10, 3, 8, 10], [12, 10, 3, 15, 13], [12, 10, 20, 11, 22], [12, 10, 20, 12, 1],  
 [12, 22, 6, 4, 7], [12, 22, 6, 19, 16], [12, 22, 17, 10, 11], [12, 22, 17, 13, 12], [13, 1, 2, 4,  
 7], [13, 1, 2, 19, 16], [13, 1, 21, 10, 11], [13, 1, 21, 13, 12], [13, 11, 8, 8, 10], [13, 11, 8,  
 15, 13], [13, 11, 15, 11, 22], [13, 11, 15, 12, 1], [13, 14, 7, 11, 1], [13, 14, 7, 12, 22],  
 [13, 14, 16, 8, 13], [13, 14, 16, 15, 10], [13, 17, 11, 9, 19], [13, 17, 11, 14, 4], [13, 17,  
 12, 9, 4], [13, 17, 12, 14, 19], [13, 18, 9, 10, 12], [13, 18, 9, 13, 11], [13, 18, 14, 4, 16],  
 [13, 18, 14, 19, 7], [14, 1, 3, 9, 19], [14, 1, 3, 14, 4], [14, 1, 20, 9, 4], [14, 1, 20, 14, 19],  
 [14, 11, 11, 8, 13], [14, 11, 11, 15, 10], [14, 11, 12, 11, 1], [14, 11, 12, 12, 22], [14, 13,  
 10, 11, 22], [14, 13, 10, 12, 1], [14, 13, 13, 8, 10], [14, 13, 13, 15, 13], [14, 17, 5, 10,  
 12], [14, 17, 5, 13, 11], [14, 17, 18, 4, 16], [14, 17, 18, 19, 7], [14, 18, 2, 10, 11], [14,  
 18, 2, 13, 12], [14, 18, 21, 4, 7], [14, 18, 21, 19, 16], [17, 1, 4, 11, 1], [17, 1, 4, 12, 22],  
 [17, 1, 19, 8, 13], [17, 1, 19, 15, 10], [17, 11, 7, 4, 16], [17, 11, 7, 19, 7], [17, 11, 16, 10,  
 12], [17, 11, 16, 13, 11], [17, 13, 2, 9, 19], [17, 13, 2, 14, 4], [17, 13, 21, 9, 4], [17, 13,  
 21, 14, 19], [17, 14, 9, 4, 7], [17, 14, 9, 19, 16], [17, 14, 14, 10, 11], [17, 14, 14, 13, 12],  
 [17, 18, 5, 8, 10], [17, 18, 5, 15, 13], [17, 18, 18, 11, 22], [17, 18, 18, 12, 1], [18, 1, 10,

```

8, 10], [18, 1, 10, 15, 13], [18, 1, 13, 11, 22], [18, 1, 13, 12, 1], [18, 11, 6, 9, 4], [18, 11,
6, 14, 19], [18, 11, 17, 9, 19], [18, 11, 17, 14, 4], [18, 13, 5, 4, 7], [18, 13, 5, 19, 16],
[18, 13, 18, 10, 11], [18, 13, 18, 13, 12], [18, 14, 11, 4, 16], [18, 14, 11, 19, 7], [18, 14,
12, 10, 12], [18, 14, 12, 13, 11], [18, 17, 9, 11, 1], [18, 17, 9, 12, 22], [18, 17, 14, 8, 13],
[18, 17, 14, 15, 10], [22, 5, 7, 8, 10], [22, 5, 7, 15, 13], [22, 5, 16, 11, 22], [22, 5, 16, 12,
1], [22, 6, 6, 11, 1], [22, 6, 6, 12, 22], [22, 6, 17, 8, 13], [22, 6, 17, 15, 10], [22, 9, 8, 9,
19], [22, 9, 8, 14, 4], [22, 9, 15, 9, 4], [22, 9, 15, 14, 19], [22, 10, 11, 10, 11], [22, 10,
11, 13, 12], [22, 10, 12, 4, 7], [22, 10, 12, 19, 16], [22, 12, 4, 4, 16], [22, 12, 4, 19, 7],
[22, 12, 19, 10, 12], [22, 12, 19, 13, 11]]

```

> #Find all tori that conjugate to the first torus.

```

Class1 := [Gammas[1]] :
NotClass1Set := {} :
GammasSet := {op(Gammas)} :

HkConjugate := 'false':
for j from 1 to nops(solutions) do

  if Gammas[1] = solutions[j][2]
    then HkConjugate := 'true':
    else HkConjugate := 'false':
  fi:

  if HkConjugate = 'true'
    then Class1 := [op(Class1), solutions[j][1]] :
    else NotClass1Set := {op(NotClass1Set), solutions[j][1]} :
  fi:

od;
Class1 := MakeUnique(Class1);
NotClass1Set := GammasSet minus {op(Class1)};
Class1 := [1, 11, 13, 14, 17, 18]
NotClass1Set := {5, 6, 9, 10, 12, 22}

```

(12)

> #Find all tori that conjugate to the first torus in NotClass1Set.

```

NotClass1 := convert(NotClass1Set, list) :
Class2 := [NotClass1[1]] :

HkConjugate := 'false':
for j from 1 to nops(solutions) do

```

```

if NotClass1[1] = solutions[j][2]
  then HkConjugate := 'true';
  else HkConjugate := 'false';
fi;

if HkConjugate = 'true'
  then Class2 := [op(Class2), solutions[j][1]] :
fi;

od;
Class2 := MakeUnique(Class2);

                                     Class2 := [5, 6, 9, 10, 12, 22]

```

**(13)**

```

> #Check that everything fell into the first class or the second.

Neither1or2 := GammasSet minus convert(Class1, set) minus convert(Class2, set);
                                     Neither1or2 := { }

```

**(14)**

## A.2 Computing the Conjugating Matrices in $H_k$

The following shows example code for the  $h_k \in H_k$  computations in  $\mathfrak{sl}(2, k)$  for  $k = \mathbb{F}_p$  when  $p = 13$  and  $m \equiv Np \equiv 2$ .

The matrix  $h_r$  conjugates the generator  $t_\alpha$  to the generator  $rt_\beta$  when  $\alpha \neq -\beta$ . As discussed Section 4.1,  $h_1$  in Equation 4.18 also conjugates  $t_\alpha$  to  $rt_\beta$ , and is an almost equivalent, though less applicable, formula. The following computations use both formulas. We first compute  $h_1$ , and when the computations return  $h_1 = 0$ , we compute  $h_r$ , which is named  $h_2$  in the code. We check these formulas against the data found in line (11), which is stored in the list *solutions*. Lines (15) and (16) create the expressions necessary for the  $x$  and  $y$  entries of both  $h_1$  and  $h_2$ .

In line (17), we filter out the cases when  $\alpha = -\beta$ , because we know from Theorem 4.1.1 the form of the conjugating matrix  $h_k$ . Then, in lines (18) - (21), we create a series of nested loops using the  $a$ ,  $b$ , and  $r$  data from *solutions* to test if the entries of  $h_1$  and  $h_2$  created are 0 and if the determinants of  $h_1$  and  $h_2$  are indeed squares. When the  $h_1$  entries are zero, the data is sent through the  $h_2$  tests. The created  $x$  and  $y$  entries for  $h_1$  and  $h_2$  are then checked against the known  $x$  and  $y$  data from *solutions*. If they match, the data is stored under *h1works* or *h2works*; if they do not match, the data is stored under *h1doesntwork* or *h2doesntwork*. As is shown in the example, the lists *h1doesntwork* or *h2doesntwork* remain empty.

Last, in line (22), the results are checked by showing that every datum from *solutions* did indeed get stored in list for a zero case, and nonsquare determinant case, an  $h_i$  that works, or an  $h_i$  that does not work.

```
[ > with(LinearAlgebra[Modular]) :
    with(ListTools) :
```

```
[ > p := 13;
    m := 2;

                                p := 13
                                m := 2

(1)
```

```
[ > # Define the field

    Fp := [ ] :
    for i from 0 to p - 1 do
        Fp := [op(Fp), i]
    od:
    Fp;

                                [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]

(2)
```

```
[ > #Find the squares in the field

    SqsListShort := [ ] :
    SqsSet := { } :
    for i from 0 to p - 1 do
        SqsSet := [op(SqsSet), i^2 mod p] :
    od:
    SqsListAll := convert(SqsSet, list) :
    > SqsListShort := MakeUnique(SqsListAll);
                                SqsListShort := [0, 1, 4, 9, 3, 12, 10]

(3)
```

```
[ > #Find all the  $1-ma^2$  values, those that will split & those that won't

    EigensList := [ ] :
    templist := [ ] :
    for i from 0 to p - 1 do
        templist := [op(templist), 1 - m · i^2 mod p] :
    od:
    EigensList := MakeUnique(templist) :
    EigensList;

                                [1, 12, 6, 9, 8, 3, 7]

(4)
```

> #Sort the  $1-ma^2$  values into splits & nonsplits

```
SplitsSet := { } :
SplitsList := [ ] :
NonSplitsSet := { } :
NonSplitsList := [ ] :
```

```
EigensSet := convert(EigensList, set) :
SqsSetShort := convert(SqsListShort, set) :
SplitsSet := intersect (EigensSet, SqsSetShort);
NonSplitsSets := EigensSet minus SplitsSet;
SplitsList := convert(SplitsSet, list);
NonSplitsList := convert(NonSplitsSets, list);
SplitsSet := {1, 3, 9, 12}
NonSplitsSets := {6, 7, 8}
SplitsList := [1, 3, 9, 12]
NonSplitsList := [6, 7, 8]
```

(5)

> #Find the elements that lead to nonsplits

```
Gammas := [ ] :
for i from 0 to p - 1 do
  if member(1 - m·i2 mod p, NonSplitsList) = 'true' then Gammas := [op(Gammas), i] fi;
od;
Gammas;
```

[2, 4, 6, 7, 9, 11]

(6)

> # Define the matrices and resulting corresponding conjugation relations

```
unassign('a');
t := Matrix([[1, a], [-1·m·a, -1]]);
```

$$t := \begin{bmatrix} 1 & a \\ -2a & -1 \end{bmatrix}$$

(7)

```
> unassign('x','y');
h := Matrix([[x, y], [m·y, x]]);
hinv := Matrix([[x, -y], [-1·m·y, x]]);
```

$$h := \begin{bmatrix} x & y \\ 2y & x \end{bmatrix}$$

(8)

$$hinv := \begin{bmatrix} x & -y \\ -2y & x \end{bmatrix} \quad (8)$$

```
> simplify(h.(t.hinv));
```

$$\begin{bmatrix} x^2 - 4yax + 2y^2 & -2xy + ax^2 + 2ay^2 \\ 4xy - 4ay^2 - 2ax^2 & -2y^2 + 4yax - x^2 \end{bmatrix} \quad (9)$$

```
> reln1 := (a, x, y) → x^2 - 2·m·a·x·y + m·y^2;
   reln2 := (a, x, y) → a·x^2 - 2·x·y + m·a·y^2;
   reln3 := (x, y) → x^2 - m·y^2;
```

$$\begin{aligned} reln1 &:= (a, x, y) \rightarrow x^2 - 2m a x y + m y^2 \\ reln2 &:= (a, x, y) \rightarrow a x^2 - 2 x y + m a y^2 \\ reln3 &:= (x, y) \rightarrow x^2 - m y^2 \end{aligned} \quad (10)$$

```
> #Find the conjugation classes

unassign('x','y','a','r');
solutions := { } :
for i from 1 to nops(Gammas) do
for b from 1 to nops(Gammas) do
for r from 1 to p do
for x from 0 to p - 1 do
for y from 0 to p - 1 do

a := Gammas[i] :
if a ≠ Gammas[b] then
if reln1(a, x, y) mod p = r and reln2(a, x, y) mod p = (Gammas[b]·r mod p) and reln3(x,
y) mod p = 1 then solutions := [op(solutions), ([a, Gammas[b], r, x, y])] fi;
fi;

od;
od;
od;
od;
od;
solutions;
#Each datum returned has form [a, b, r, x, y].

[[2, 4, 11, 4, 12], [2, 4, 11, 9, 1], [2, 6, 8, 3, 2], [2, 6, 8, 10, 11], [2, 7, 5, 5, 5], [2, 7, 5, 8, 8],
[2, 9, 2, 5, 8], [2, 9, 2, 8, 5], [2, 11, 12, 4, 1], [2, 11, 12, 9, 12], [4, 2, 6, 4, 1], [4, 2, 6, 9,
12], [4, 6, 9, 3, 11], [4, 6, 9, 10, 2], [4, 7, 4, 4, 12], [4, 7, 4, 9, 1], [4, 9, 12, 3, 2], [4, 9,
12, 10, 11], [4, 11, 7, 5, 8], [4, 11, 7, 8, 5], [6, 2, 5, 3, 11], [6, 2, 5, 10, 2], [6, 4, 3, 3, 2],
```



```
[6, 4, 3, 10, 11], [6, 7, 12, 5, 8], [6, 7, 12, 8, 5], [6, 9, 10, 4, 12], [6, 9, 10, 9, 1], [6, 11,
8, 5, 5], [6, 11, 8, 8, 8], [7, 2, 8, 5, 8], [7, 2, 8, 8, 5], [7, 4, 10, 4, 1], [7, 4, 10, 9, 12], [7,
6, 12, 5, 5], [7, 6, 12, 8, 8], [7, 9, 3, 3, 11], [7, 9, 3, 10, 2], [7, 11, 5, 3, 2], [7, 11, 5, 10,
11], [9, 2, 7, 5, 5], [9, 2, 7, 8, 8], [9, 4, 12, 3, 11], [9, 4, 12, 10, 2], [9, 6, 4, 4, 1], [9, 6, 4,
9, 12], [9, 7, 9, 3, 2], [9, 7, 9, 10, 11], [9, 11, 6, 4, 12], [9, 11, 6, 9, 1], [11, 2, 12, 4, 12],
[11, 2, 12, 9, 1], [11, 4, 2, 5, 5], [11, 4, 2, 8, 8], [11, 6, 5, 5, 8], [11, 6, 5, 8, 5], [11, 7, 8,
3, 11], [11, 7, 8, 10, 2], [11, 9, 11, 4, 1], [11, 9, 11, 9, 12]]
```

```
> # To check h1 and h2, we need to define modular inverses, square roots, and inverse square roots
inverse := [ ]:
```

```
for i from 1 to p - 1 do
for j from 1 to p - 1 do
```

```
if i·j mod p = 1 then inverse := [op(inverse), j] fi;
```

```
od;
od;
inverse;
```

```
[1, 7, 9, 10, 8, 11, 2, 5, 3, 4, 6, 12] (12)
```

```
> SqRoots := [ ]:
```

```
for i from 2 to nops(SqsListShort) do
```

```
TempSqs := [SqsListShort[i]]:
```

```
for j from 1 to p - 1 do
```

```
if j·j mod p = SqsListShort[i] then TempSqs := [op(TempSqs), j] fi;
```

```
od:
```

```
SqRoots := [op(SqRoots), TempSqs]:
```

```
TempSqs := [ ]:
```

```
od:
```

```
SqRoots;
```

```
#Each datum returned has form [square, low root, high root].
```

```
[1, 1, 12], [4, 2, 11], [9, 3, 10], [3, 4, 9], [12, 5, 8], [10, 6, 7]] (13)
```

```
> InvSqRoots := [ ]:
```

```
for i from 1 to nops(SqRoots) do
```

```

InvSqRoots := [op(InvSqRoots), [SqRoots[i][1], inverse[SqRoots[i][2]],
inverse[SqRoots[i][3]]]]

```

**od:**

```

InvSqRoots;

```

```

[[1, 1, 12], [4, 7, 6], [9, 9, 4], [3, 10, 3], [12, 8, 5], [10, 11, 2]]

```

(14)

> #Create the h1 and h2 x and y entries and determinants

```

unassign('a','b','x','y','r') :

```

```

h1x := (a, b, r) → 1 + m·a + r·(1 + m·b) ;

```

```

h1y := (a, b, r) → 1 + a - r·(1 + b);

```

```

h1det := (a, b, r) → (1 + m·a + r·(1 + m·b))2 - m·(1 + a - r·(1 + b))2;

```

```

h1x := (a, b, r) → 1 + m a + r (1 + m b)

```

```

h1y := (a, b, r) → 1 + a - r (1 + b)

```

```

h1det := (a, b, r) → (1 + m a + r (1 + m b))2 - m (1 + a - r (1 + b))2

```

(15)

> h2x := (r) → 1 + r;

```

h2y := (a, b, r) → a - r·b;

```

```

h2det := (a, b, r) → (1 + r)2 - m·(a - r·b)2;

```

```

h2x := r → 1 + r

```

```

h2y := (a, b, r) → a - b r

```

```

h2det := (a, b, r) → (1 + r)2 - m (a - b r)2

```

(16)

> #Before checking h1 and h2, weed out the cases when additive inverse gammas are conjugate.

Send the non additive inverse cases to SolutionsToCheck1Part1.

```

AddInverses := [ ] :

```

```

SolutionsToCheck1Part1 := [ ] :

```

```

for i from 1 to nops(solutions) do

```

```

a := solutions[i][1];

```

```

b := solutions[i][2];

```

```

if (a + b) mod p = 0 then AddInverses := [op(AddInverses), solutions[i]]

```

```

else SolutionsToCheck1Part1 := [op(SolutionsToCheck1Part1), solutions[i]]

```

```

fi;

```

**od:**

```

AddInverses;

```

*SolutionsToCheck1Part1;*

[[2, 11, 12, 4, 1], [2, 11, 12, 9, 12], [4, 9, 12, 3, 2], [4, 9, 12, 10, 11], [6, 7, 12, 5, 8], [6, 7, 12, 8, 5], [7, 6, 12, 5, 5], [7, 6, 12, 8, 8], [9, 4, 12, 3, 11], [9, 4, 12, 10, 2], [11, 2, 12, 4, 12], [11, 2, 12, 9, 1]]  
 [[2, 4, 11, 4, 12], [2, 4, 11, 9, 1], [2, 6, 8, 3, 2], [2, 6, 8, 10, 11], [2, 7, 5, 5, 5], [2, 7, 5, 8, 8], [2, 9, 2, 5, 8], [2, 9, 2, 8, 5], [4, 2, 6, 4, 1], [4, 2, 6, 9, 12], [4, 6, 9, 3, 11], [4, 6, 9, 10, 2], [4, 7, 4, 4, 12], [4, 7, 4, 9, 1], [4, 11, 7, 5, 8], [4, 11, 7, 8, 5], [6, 2, 5, 3, 11], [6, 2, 5, 10, 2], [6, 4, 3, 3, 2], [6, 4, 3, 10, 11], [6, 9, 10, 4, 12], [6, 9, 10, 9, 1], [6, 11, 8, 5, 5], [6, 11, 8, 8, 8], [7, 2, 8, 5, 8], [7, 2, 8, 8, 5], [7, 4, 10, 4, 1], [7, 4, 10, 9, 12], [7, 9, 3, 3, 11], [7, 9, 3, 10, 2], [7, 11, 5, 3, 2], [7, 11, 5, 10, 11], [9, 2, 7, 5, 5], [9, 2, 7, 8, 8], [9, 6, 4, 4, 1], [9, 6, 4, 9, 12], [9, 7, 9, 3, 2], [9, 7, 9, 10, 11], [9, 11, 6, 4, 12], [9, 11, 6, 9, 1], [11, 4, 2, 5, 5], [11, 4, 2, 8, 8], [11, 6, 5, 5, 8], [11, 6, 5, 8, 5], [11, 7, 8, 3, 11], [11, 7, 8, 10, 2], [11, 9, 11, 4, 1], [11, 9, 11, 9, 12]] (17)

> #Check the calculations for h1 that do not create the zero matrix and the h1 determinant is a square, send these solutions to SolutionsToCheck1Part2. Send the zero cases to SolutionsToCheck2Part1.

*h1doesntwork := [ ]:*

*SolutionsToCheck1Part2 := [ ]:*

*SolutionsToCheck2Part1 := [ ]:*

**for** *i* **from** 1 **to** nops(*SolutionsToCheck1Part1*) **do**

*a* := *SolutionsToCheck1Part1*[*i*][1];

*b* := *SolutionsToCheck1Part1*[*i*][2];

*r* := *SolutionsToCheck1Part1*[*i*][3];

**if** *h1x*(*a*, *b*, *r*) **mod** *p* = 0 **and** *h1y*(*a*, *b*, *r*) **mod** *p* = 0

**then** *SolutionsToCheck2Part1* := [op(*SolutionsToCheck2Part1*),

*SolutionsToCheck1Part1*[*i*]]

**elif** member(*h1det*(*a*, *b*, *r*) **mod** *p*, *SqsListShort*) = 'false'

**then** *h1doesntwork* := [op(*h1doesntwork*), *SolutionsToCheck1Part1*[*i*]]

**else** *SolutionsToCheck1Part2* := [op(*SolutionsToCheck1Part2*), *SolutionsToCheck1Part1*[*i*]]

**fi:**

**od:**

*h1doesntwork;*

*SolutionsToCheck1Part2;*

*SolutionsToCheck2Part1;*

```

[ ]
[[2, 6, 8, 3, 2], [2, 6, 8, 10, 11], [2, 7, 5, 5, 5], [2, 7, 5, 8, 8], [2, 9, 2, 5, 8], [2, 9, 2, 8, 5], [4,
6, 9, 3, 11], [4, 6, 9, 10, 2], [4, 7, 4, 4, 12], [4, 7, 4, 9, 1], [4, 11, 7, 5, 8], [4, 11, 7, 8, 5],
[6, 2, 5, 3, 11], [6, 2, 5, 10, 2], [6, 4, 3, 3, 2], [6, 4, 3, 10, 11], [6, 9, 10, 4, 12], [6, 9, 10,
9, 1], [6, 11, 8, 5, 5], [6, 11, 8, 8, 8], [7, 2, 8, 5, 8], [7, 2, 8, 8, 5], [7, 4, 10, 4, 1], [7, 4,
10, 9, 12], [7, 9, 3, 3, 11], [7, 9, 3, 10, 2], [9, 2, 7, 5, 5], [9, 2, 7, 8, 8], [9, 6, 4, 4, 1], [9,
6, 4, 9, 12], [9, 7, 9, 3, 2], [9, 7, 9, 10, 11], [9, 11, 6, 4, 12], [9, 11, 6, 9, 1], [11, 4, 2, 5,
5], [11, 4, 2, 8, 8], [11, 6, 5, 5, 8], [11, 6, 5, 8, 5], [11, 9, 11, 4, 1], [11, 9, 11, 9, 12]]
[[2, 4, 11, 4, 12], [2, 4, 11, 9, 1], [4, 2, 6, 4, 1], [4, 2, 6, 9, 12], [7, 11, 5, 3, 2], [7, 11, 5, 10,
11], [11, 7, 8, 3, 11], [11, 7, 8, 10, 2]]

```

(18)

> #Check that the calculations for h1 actually work.

```
h1works := [ ]:
```

```
for i from 1 to nops (SolutionsToCheck1Part2) do
```

```
  a := SolutionsToCheck1Part2[i][1];
```

```
  b := SolutionsToCheck1Part2[i][2];
```

```
  r := SolutionsToCheck1Part2[i][3];
```

```
  x := SolutionsToCheck1Part2[i][4];
```

```
  y := SolutionsToCheck1Part2[i][5];
```

```
  SolutionWorks := 'false';
```

```
for j from 1 to nops (InvSqRoots) do
```

```
  if InvSqRoots[j][1] = h1det(a, b, r) mod p
```

```
    and InvSqRoots[j][2] · h1x(a, b, r) mod p = x and InvSqRoots[j][2] · h1y(a, b, r) mod p = y
    then SolutionWorks := 'true'
```

```
  elif InvSqRoots[j][1] = h1det(a, b, r) mod p
```

```
    and InvSqRoots[j][3] · h1x(a, b, r) mod p = x and InvSqRoots[j][3] · h1y(a, b, r) mod p = y
    then SolutionWorks := 'true'
```

```
  fi;
```

```
od:
```

```
if SolutionWorks = 'true'
```

```
  then h1works := [op(h1works), SolutionsToCheck1Part2[i]]
```

```
  else h1doesntwork := [op(h1doesntwork), SolutionsToCheck1Part2[i]]
```

```
fi;
```

```
od:
```

```
h1works;
```

```

h1doesntwork;

[[2, 6, 8, 3, 2], [2, 6, 8, 10, 11], [2, 7, 5, 5, 5], [2, 7, 5, 8, 8], [2, 9, 2, 5, 8], [2, 9, 2, 8, 5], [4,
  6, 9, 3, 11], [4, 6, 9, 10, 2], [4, 7, 4, 4, 12], [4, 7, 4, 9, 1], [4, 11, 7, 5, 8], [4, 11, 7, 8, 5],
  [6, 2, 5, 3, 11], [6, 2, 5, 10, 2], [6, 4, 3, 3, 2], [6, 4, 3, 10, 11], [6, 9, 10, 4, 12], [6, 9, 10,
  9, 1], [6, 11, 8, 5, 5], [6, 11, 8, 8, 8], [7, 2, 8, 5, 8], [7, 2, 8, 8, 5], [7, 4, 10, 4, 1], [7, 4,
  10, 9, 12], [7, 9, 3, 3, 11], [7, 9, 3, 10, 2], [9, 2, 7, 5, 5], [9, 2, 7, 8, 8], [9, 6, 4, 4, 1], [9,
  6, 4, 9, 12], [9, 7, 9, 3, 2], [9, 7, 9, 10, 11], [9, 11, 6, 4, 12], [9, 11, 6, 9, 1], [11, 4, 2, 5,
  5], [11, 4, 2, 8, 8], [11, 6, 5, 5, 8], [11, 6, 5, 8, 5], [11, 9, 11, 4, 1], [11, 9, 11, 9, 12]]
  [ ]

```

(19)

```

> #Check that the h2 calculations do not create zero, the determinant is a square, and send to
  SolutionsToCheck2Part2.

h2zeros := [ ] :
h2doesntwork := [ ] :
SolutionsToCheck2Part2 := [ ] :

for i from 1 to nops(SolutionsToCheck2Part1) do

  a := SolutionsToCheck2Part1[i][1];
  b := SolutionsToCheck2Part1[i][2];
  r := SolutionsToCheck2Part1[i][3];

  if h2x(a, b, r) mod p = 0 and h2y(a, b, r) mod p = 0
    then h2zeros := [op(h2zeros), SolutionsToCheck2Part1[i]]

  elif member(h2det(a, b, r) mod p, SqsListShort) = 'false'
    then h2doesntwork := [op(h2doesntwork), SolutionsToCheck2Part1[i]]

  else SolutionsToCheck2Part2 := [op(SolutionsToCheck2Part2), SolutionsToCheck2Part1[i]]

fi;
od;
h2zeros;
h2doesntwork;
SolutionsToCheck2Part2;

  [ ]
  [ ]

[[2, 4, 11, 4, 12], [2, 4, 11, 9, 1], [4, 2, 6, 4, 1], [4, 2, 6, 9, 12], [7, 11, 5, 3, 2], [7, 11, 5, 10,
  11], [11, 7, 8, 3, 11], [11, 7, 8, 10, 2]]

```

(20)

```

> #Check that the calculations for h2 actually work.

h2works := [ ] :

for i from 1 to nops (SolutionsToCheck2Part2) do

  a := SolutionsToCheck2Part2[i][1];
  b := SolutionsToCheck2Part2[i][2];
  r := SolutionsToCheck2Part2[i][3];
  x := SolutionsToCheck2Part2[i][4];
  y := SolutionsToCheck2Part2[i][5];
  SolutionWorks := 'false';

  for j from 1 to nops (InvSqRoots) do

    if InvSqRoots[j][1] = h2det(a, b, r) mod p
      and InvSqRoots[j][2] · h2x(r) mod p = x and InvSqRoots[j][2] · h2y(a, b, r) mod p = y
      then SolutionWorks := 'true'

    elif InvSqRoots[j][1] = h2det(a, b, r) mod p
      and InvSqRoots[j][3] · h2x(r) mod p = x and InvSqRoots[j][3] · h2y(a, b, r) mod p = y
      then SolutionWorks := 'true'

    fi;
  od;

  if SolutionWorks = 'true'
    then h2works := [op(h2works), SolutionsToCheck2Part2[i]]
    else h2doesntwork := [op(h2doesntwork), SolutionsToCheck2Part2[i]]
  fi;

od;
h2works;
h2doesntwork;
[[2, 4, 11, 4, 12], [2, 4, 11, 9, 1], [4, 2, 6, 4, 1], [4, 2, 6, 9, 12], [7, 11, 5, 3, 2], [7, 11, 5, 10,
11], [11, 7, 8, 3, 11], [11, 7, 8, 10, 2]]

[ ]
(21)

```

```

> #Double check that every solution made it through the pipeline somehow.

solutionsSet := convert(solutions, set) :
AddInversesSet := convert(AddInverses, set) :
h1worksSet := convert(h1works, set) :
h1doesntworkSet := convert(h1doesntwork, set) :
h2zerosSet := convert(h2zeros, set) :
h2worksSet := convert(h2works, set) :

```

$$\begin{aligned}
 & h2doesntworkSet := \text{convert}(h2doesntwork, set) : \\
 & solutionsSet \textbf{ minus } AddInversesSet \textbf{ minus } h1worksSet \textbf{ minus } h1doesntworkSet \\
 & \quad \textbf{ minus } h2zerosSet \textbf{ minus } h2worksSet \textbf{ minus } h2doesntworkSet, \\
 & \quad \quad \quad \{ \}
 \end{aligned}
 \tag{22}$$

[>