

## ABSTRACT

CORLEY, ANDREW KENT. A Design Study of Co-splitting as Situated in the Equipartitioning Learning Trajectory. (Under the direction of Dr. Jere Confrey.)

The equipartitioning learning trajectory (Confrey, Maloney, Nguyen, Mojica & Myers, 2009) has been hypothesized and the proficiency levels have been validated through much prior work. This study solidifies understanding of the upper level of co-splitting, which has been redefined through further clinical interview work (Corley, Confrey & Nguen, 2012) in response to performances at that level on the field test (Pescosolido, 2010). In addition, it provides new perspective on the relationships between lower levels within the trajectory, and co-splitting and the upper levels. It describes a teaching experiment on co-splitting with sixteen rising fourth-, fifth-, and sixth-graders, and examines their development of that construct. The primary research questions focused on relating students' progress of development and strategies for co-splitting to the lower levels of the learning trajectory, their knowledge of multiplication and division, and the tasks and parameters used in the curriculum. The results were analyzed and synthesized with respect to the findings for each of the research questions. The study shows that all of the lower levels of the equipartitioning learning trajectory are utilized by students interacting with co-splitting to various degrees, and it is critical that a strong, conceptual understanding of multiplication and division be developed prior to, and in parallel with, co-splitting. The research establishes a precedent for making connections between levels, and between trajectories, in a learning trajectories based instruction (Sztajn, Confrey, Wilson & Edgington, 2012) model, and it also sets the stage for similar study of the other upper levels in the equipartitioning learning trajectory, such as equipartitioning multiple wholes and the distributivity property of equipartitioning.

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A Design Study of Co-splitting as Situated in the Equipartitioning Learning Trajectory

by  
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## **DEDICATION**

In loving memory of my maternal grandparents,  
Robert Bachellor Albrecht and Eleanor Francis Albrecht.

## BIOGRAPHY

Andrew “Drew” Kent Corley was born on December 14, 1980 in Richmond, Virginia. He graduated in 1998 from Varina High School. He then graduated from the University of South Carolina – Columbia in May 2002 with a Bachelor of Science degree in mathematics. He continued his education at the University of South Carolina – Columbia immediately following this and graduated with a Master of Teaching – Secondary Education with a focus in mathematics in 2003, and concurrently attained his South Carolina teaching certificate for mathematics in grades 7-12. Upon graduation, Drew took a position as a high school mathematics teacher at Spring Valley High School in Columbia, SC beginning in the fall of 2003. He taught at Spring Valley for six years, and attained National Board Certification in Mathematics – Early Adolescence from the National Board for Professional Teaching Standards in 2008. During his tenure at Spring Valley, Drew sponsored several extracurricular clubs, and was an active Student Council advisor as well as a Varsity and Junior Varsity women’s softball coach.

Drew returned to school in August 2009 to pursue a Doctor of Philosophy degree in Mathematics Education at North Carolina State University and joined the GISMO (Generating Increased Science and Mathematics Opportunities) research team at the Friday Institute for Educational Innovation as a graduate research assistant. During the first year of his doctoral studies, he worked on the North Carolina Integrated Mathematics project, and then on the Diagnostic E-learning Trajectories Approach (DELTA) and the Learning Progress Profiles Synchronized for Mobile Devices (LPPSync) projects, integrating learning

trajectories, technology, and diagnostic assessment systems, for the subsequent two years. In September 2012, Drew accepted a position as Mathematics Curriculum Developer at Wireless Generation (now Amplify Learning) in Durham, NC.

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## CHAPTER 1: INTRODUCTION

This dissertation study is intended to inform a growing body of work on the equipartitioning learning trajectory identified by the DELTA (Diagnostic E-Learning Trajectories Approach) research team<sup>1</sup> at North Carolina State University. The larger work situates itself in the realm of learning trajectory research and has focused on the content domain of rational number reasoning. The present study adds to this body of work by further elaborating one of the upper levels of the trajectory so as to better understand the functionality of the upper end of the trajectory. As a result, it provides insight into how the subsequent levels of a learning trajectory are visibly and viably related to the earlier ones. In the remainder of this chapter, I demonstrate the timeliness of this study in relation to other research in the field.

### Three Trends in Mathematics Education Research

This dissertation brings together three crucial trends in mathematics education to offer a pragmatic way forward to improve instruction in a key topic. By using the concept of *learning trajectories* as a way to describe theoretically and empirically how to describe students' progress, the dissertation reports on work understanding students' learning of equipartitioning to provide a foundation for *ratio and proportion reasoning*. A mechanism

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<sup>1</sup> The DELTA research team is led by Dr. Jere Confrey and Dr. Alan Maloney at North Carolina State University. Former and current members of the DELTA research team include Dr. P. Holt Wilson, Dr. Gemma Mojica, Dr. Kenny Nguyen, Dr. Cyndi Edgington, Dr. KoSze Lee, Dr. Nicole Panorkou, Marielle Myers, Ryan Pescosolido, Ayanna Franklin, Zuhal Yilmaz, Nadia Monroe, Will McGowan, Tamar Avineri, and Jennifer Nickell. The pronouns "we" and "our," as well as references to the research group acknowledge work that has been conducted in collaboration with other members of the team and/or was built on Dr. Confrey's seminal works on splitting and equipartitioning.

for practically applying these ideas is that of an interactive *diagnostic assessment* system, which locates student progress on the trajectory in equipartitioning at the upper levels.

### *Learning Trajectories*

Learning progressions<sup>2</sup> or learning trajectories (LTs) were introduced into the mathematics education literature almost 20 years ago, and have recently gained greater attention in the field. In particular, with the current, rolling implementation of the Common Core State Standards for Mathematics (CCSSM, Council of Chief State School Officers (CCSSO), 2010) that stress the importance of learning progressions and trajectories (Confrey & Maloney, in press). The DELTA research team has identified 18 learning trajectories in CCSSM for grades K-8 (Confrey, Nguyen & Maloney, 2011; see [www.turnonccmath.net](http://www.turnonccmath.net)), and has taken on a subsequent large-scale effort around those standards and the LTs that consisted of identifying each standard within an appropriate LT, ordering the standards within each LT, and *unpacking* (Confrey et al., 2012) the sets of standards that constitute each LT. The unpacking of the standards for each LT contains five overarching themes: 1) conceptual principles linked to research, 2) diverse representations, strategies, and misconceptions, 3) meaningful distinctions and multiple models, 4) coherent structure within and between LTs, and 5) bridging standards as suggested additions to identify and address gaps. These standards and the unpackings, along with an increasing research base will result in the learning trajectory terminology becoming more prevalent for all stakeholders both in and out of schools and classrooms alike. Therefore, the notion of learning trajectories could

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<sup>2</sup> The term learning progressions is more common in science education literature and a brief overview of its history and the distinctions that exist between learning progressions and trajectories follow in Chapter 2. However, I will refer to either idea, in all other instances in this paper, simply as learning trajectories.

progress beyond limited conversations among university-level education researchers or resting as publications in the little-read practitioner journals to a bridge that many have sought for a long time – one that successfully links research and practice.

Despite the growing momentum that LT research has been gaining, along with the emergence of consistencies in and agreement on how that research should be conducted, what it should look like, and the expected outcomes, there are still some difficulties and areas of concern. For instance, Gotwals and Alonso (2012) caution that this type of work consists of four overlapping strands – developing LTs, developing assessments, modeling and interpreting student performance, and using LTs – which should not be pursued independently, but are at the same time less complex and easier to manage and understand when that is the case. This study focused primarily on student performance and using LTs, but scratches the surface on some of the interrelationships between all of those areas. It offers an early and rudimentary work that can be studied itself to better understand how such research can and should be conducted, and it provides one example of using LTs to develop assessments and curriculum. In particular, how to construct and interpret valid assessments around LTs, and what pedagogy based on LTs looks like and how to provide useful professional development are both newer and less-researched areas that will need a great deal more attention moving forward. A few studies (e.g., Wilson, 2009; Steedle & Shavelson, 2009; Confrey, Rupp, Maloney & Nguyen, in review) have provided different perspectives on potential means for assessment in the area, but all of these need further investigation.

### *Diagnostic Assessment*

The DELTA research group has been moving towards the use of an Interactive Diagnostic Assessment System (IDAS; Confrey, Hasse, Maloney, Nguyen & Varela, 2011) for LT-based assessments trajectories. We aim to establish a precedent for an IDAS to deliver and score diagnostic information on novel tasks of varying difficulties, in real time, that will provide information to students and teachers about individual student's and whole class' proficiencies on the levels of a particular LT, and situates a student along the trajectory based on those proficiencies. This information can then be used to inform instructional interventions and next steps in the learning process, as well as monitoring and reporting on progress through a related set of “big ideas” over time. We have developed a prototype IDAS – Learning Progress Profiles Synchronized for Network Devices (LPPSync) – that allows students to interact with the equipartitioning LT. The system has been designed around an LT-based approach to diagnostic assessment that employs technology, and it collects data in the form of diagnostic measures within item generation environments (IGEs) that can be used to determine to what degree students understand a concept.

Our view of an IDAS consists of two notions: 1) the assessments should be created from validated empirical evidence of how a student grows cognitively while engaging with an idea; and 2) the system records data in real-time and rapidly generates reports that can be used to inform a potential next piece of instruction or intervention. We believe that such a system has the potential to reduce the demand on teachers not only in scoring assessments, but also in interpreting the results and making informed decisions as to how to act on those

results. In addition, it has the potential to allow students to make more ownership of their learning and understanding to the point that they become partners with teachers in determining and enacting next appropriate steps.

Within the LPPSync IDAS, the levels of an LT (16 for equipartitioning) are grouped into the IGEs, which we call e-Packets (8 for equipartitioning). These e-Packets allow for both the dynamic generation of items and subsequent gathering of data, as well as means for students to engage in additional relevant experiences to enable further progress through the LT. Item generation is based on creating a sample space that systematically supports the production of items of various difficulties based on prior empirical study of a particular construct (and not just by simply introducing larger numbers or novel number types – integer, ratio, etc.). This creates an often expanded, yet finite set of combinations of parameters that are then each assigned to particular levels of difficulty. These can be generated randomly so that each assessment is unique for each student, yet theoretically equivalent for each administration for scoring and comparison purposes; thus strengthening the validity of the constructs and the inferences that can be drawn from the resulting data.

### *Ratio and Proportion*

In middle grades, Ratio and Proportion, and Percent is another set of the big ideas for which the DELTA research group has identified an LT in the unpacking of CCSSM. Considerable research has suggested that mastery of ratio and proportion is fundamental for success in high school and more advanced mathematics courses later (Hiebert & Behr, 1989; Harel & Confrey, 1994). Likewise, it has also been well documented that ratio and

proportion are difficult concepts for children to understand (e.g., see Behr, Harel, Post & Lesh, 1992). Yet, some prior research has shown that young students have early intuitions that could be harnessed as a foundation for strengthening their development of ratio reasoning. For instance, Confrey conjectured (1988) and has demonstrated (1994, 1995) a fundamental relationship among splitting, multiplication and division, and ratio. In dealing with part-whole relationships, relative to compensation and covariation, Irwin (1996) showed that students could prove successful in solving tasks and provide reasonable justification by the second grade. For instance, some of the students in that study were able to make a claim of covariance in that if one part was increased or decreased while the other part remained the same, then the whole would be increased or decreased by the same amount. Also, some students could make a compensatory argument that an increase in one part would require a relative decrease in another part if the whole were to remain the same. However, both of those notions are at their roots additive, and do not lend themselves to inferences about the development of an understanding of multiplicative relationships.

Similarly, it has been shown that students around this same age are capable of developing a qualitative understanding that does not involve the formation an actual ratio and may not incorporate numeric values at all, called protoratio reasoning (Resnick & Singer, 1993). Protoquantitative ratio reasoning (Resnick & Greeno, 1990) – solving problems without quantification of values – and the related protoratio reasoning – solving ratio-like problems as coordinating two additive compositions (Singer, Kohn & Resnick, 1997) – as they are described, also both “account only for the development of additive properties of

measure numbers” (Singer & Resnick, 1992, p. 231). The first of these two constructs does not properly translate to ratio reasoning tasks and their multiplicative structure, as it is noted that the first quantifications are often additive in the sense of identifying individual differences for two quantities said to be in a ratio, and therefore may cultivate a misconception for students in later experiences. The second construct also refers to an additive notion, but one that we would argue is quite different, and also quite a sophisticated form of ratio reasoning when applied properly; therefore, it may not represent a concept of protoratio but rather a concept of advanced ratio reasoning. The additive sense here is one of adding ratios, as “so much of this for so much of that,” and therefore a coordinated adding of amounts to each quantity that preserve the ratio relationship. It is my conjecture that co-splitting as a strategy and construct of equipartitioning offers the form of *multiplicative protoratio* that is lacking in these other descriptions and from students’ early learning experiences that build the foundations of ratio reasoning.

Although the formal introduction to ratio and proportion occurs in grade six in CCSSM, the DELTA research group suggests in our research-based unpacking of the standards that the foundations for ratio and proportional reasoning begin much earlier in students’ learning of equipartitioning, division, multiplication, and fractions (e.g., Clark, Berenson & Cavey, 2003; Nabors, 2003; Confrey & Carrejo, 2005). Within this research, I believe that the construct of co-splitting represents a missing multiplicative form of protoratio reasoning that can occur before, or even completely devoid of, formal instruction

on division and multiplication through leveraging the concepts of equipartitioning and students' experience with fair-sharing tasks.

### Statement of Purpose

The purpose of this dissertation study was to investigate what it means to progress through the levels of the equipartitioning LT in a classroom setting provided an explicit curriculum, to develop an understanding of the necessity of preceding levels or how they act as tools for interactions with later levels, and to better determine an equipartitioning concept of ratio, or multiplicative protoratio, as developed by students who are proficient with upper levels of the equipartitioning LT. The focal points of the trajectory at these upper levels had been hypothesized by Confrey, Maloney, Nguyen, Mojica, and Myers (2009) as equipartitioning multiple wholes, related forms of composition and distribution, co-splitting, and finally reaching the generalization that  $a$  objects shared among  $b$  sharers will always result in a fair share of  $a/b$ <sup>3</sup>. A teaching experiment with three specific aims was carried out to investigate more deeply and learn more about one of these ideas – co-splitting. The first aim was to describe the progressive nature of understanding students develop through sequential interactions with the levels of the equipartitioning LT, and how later ideas are connected to earlier ideas. The second aim was to identify ways in which students applied and incorporated knowledge of multiplication and division in working with equipartitioning problems, particularly those tasks relevant to co-splitting. The third, and final, aim was to determine whether there was a difference in student strategies when the parameters (number

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<sup>3</sup> Details about each of these upper-level constructs will be elaborated in Chapter 3: Methodology.

of objects shared and number of sharers) in co-splitting tasks varied between problems where lesser values of each quantity were involved in relation to the other quantity.

### Research Questions

Specifically, the three research questions for this dissertation study were:

1. How do the lower levels of the equipartitioning learning trajectory have an impact on student interactions with and success on the upper levels – particularly co-splitting?
2. How does students' knowledge of multiplication and division interplay with their learning and understanding of equipartitioning at the upper levels of the learning trajectory?
3. How does students' strategy use and performances on co-splitting tasks differ when the number of objects is greater than the number of sharers versus when the number of objects is less than the number of sharers?

The first research question sought to indicate which lower levels, if any, of equipartitioning are necessary and sufficient for progressing through the LT. A secondary result of investigating this question would be either further validation of the ordering of the proficiency levels within the equipartitioning LT, or indications that the current order needed to be studied further in similar contexts to inform potential alterations. The second research question intended to offer a initial description of the relationship between two sets of big ideas – multiplication and division, and equipartitioning – that are hypothesized to be strongly interrelated, and because these concepts are introduced concurrently at various levels in CCSSM and grade-level based curricula. The third research question aimed to

provide a comparative study of student strategy use on upper-level equipartitioning tasks based on their parameterizations, seeking to identify any discrepancies in difficulty or levels sophistication of strategy use within co-splitting.

### Plan for Thesis

Following this introduction, Chapter Two presents a synthesis of literature justifying the need for this study, developing in detail the three trends in mathematics education. The context of the research and the methods used to collect and analyze data will be discussed in Chapter Three. An analysis of the data and findings from the study pertaining to the research questions will be presented in Chapter Four. Finally, Chapter Five discusses the overall findings from the study, including implications and limitations of the research, and draws conclusions based on those findings and makes recommendations for future research.

## CHAPTER 2: REVIEW OF RELATED LITERATURE

In this chapter, I describe the two theoretical frameworks that informed and motivated the study. This begins with a brief history of learning progressions and learning trajectories, which includes the development and use of LTs in mathematics education research and practice particularly. Then I provide a comparison of two conceptions of splitting as an operation, and emphasize the development of the splitting construct relevant to equipartitioning and the identification of co-splitting. This is followed by summaries of relevant existing literature on other topics related to the research questions and conjectures of the study – division and multiplication, the roles of context and quantity in problem solving, ratio and proportional reasoning, and equipartitioning. In the last section, a clear delineation of the DELTA LT for equipartitioning and the progression of work that led to this study are provided.

### Theoretical Frameworks

There are two different, yet interrelated, frameworks underlying the design of the teaching experiment in this study of the levels of the equipartitioning LT. The first is the theoretical framework of learning trajectories as both a means to investigate cognition and student learning as well as inform the design of tasks, curricula, and professional development on content and pedagogy. The second is the conceptual framework of splitting, based on Confrey's body of work that began with the original splitting conjecture (1988; see also Confrey, 1994, 1995), and led to the development of the equipartitioning learning trajectory and the identification of the co-splitting construct.

### *Learning Trajectory Framework*

Learning trajectories, or learning progressions, come from a constructivist foundation in the development of understandings in mathematics and science. The phrase “hypothetical learning trajectory” was first defined by Simon (1995) as “the teacher’s prediction as to the path by which learning might proceed” (p. 135), from lesser to more sophisticated ideas and understandings over time. His choice of the term hypothetical indicates that an actual learning trajectory is not necessarily knowable *a priori*. Furthermore, Simon claimed that a learning trajectory is made up of three components, “the learning goal, the learning activity, and the thinking and learning in which students might engage” (p. 133). Within his Mathematics Teaching Cycle, which Simon proposed based on learning trajectories, he acknowledged the need for and the importance of attention to each of the following: research on students’ mathematical thinking, relevant and innovative curricula, and professional development for teachers. His descriptions of both the learning trajectory and the cycle consider that the trajectory itself could vary based on the individual student’s enacted path, as well as interpretations of individual teachers and researchers, and the subsequent experiences provided to the students.

Clements and Sarama (2004) adapted Simon’s description of a learning trajectory, stating that a learning trajectory describes “children’s thinking and learning in a specific mathematical domain … [and] those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking” (p. 83). They describe two aspects of the learning trajectory – the developmental progression aspect and the

instructional task aspect. The developmental progression consists of children's thinking and how they learn within a mathematical domain. This is related to a conjectured route through benchmark levels of thinking as fostered by a set of instructional tasks hypothesized to promote the development of a concept.

Battista (2004), focusing on the notion of learning trajectories through the lens of cognition-based assessment, emphasized a “levels-model for a topic [that] describes not only cognitive plateaus, but also what students can and cannot do, students’ conceptualizations and reasoning, cognitive obstacles that obstruct learning progress, and mental processes needed both for functioning at a level and for progressing to higher levels” (p. 187). Therefore, a learning trajectory is a path by which students reach these plateaus – indicated by research as landmarks students typically pass through – in learning a topic. Confrey (2006) later echoed one of Simon’s early sentiments about learning trajectories, that there may be more than one trajectory, differing for individual students, within a single concept domain. If this view is taken to encompass a singular trajectory of that concept writ large, then the trajectory in that sense embodies what Confrey described as a larger conceptual corridor in which students operate (see Figure 1). In other words, students may encounter different obstacles or boundary objects throughout their learning of a topic; and their paths between landmarks, or plateaus, would need to include all of the landmarks, but those paths need not be identical or sequential. It is important to point out that each of these descriptions of learning trajectories, although focusing on the learning of concepts by students, are not meant to be understood devoid of tasks and curriculum, nor of teachers and instruction.

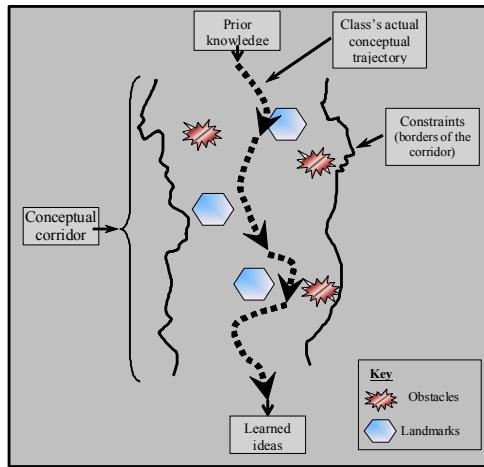


Figure 1. Conceptual corridor and one possible conceptual trajectory (Confrey, 2006).

The DELTA research group has attempted to unify several of the ideas with respect to learning trajectories, and I adopt their definition of a learning trajectory for this study:

... a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction, and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time. (Confrey et al., 2009, p. 2)

For the DELTA research group, a learning trajectory is comprised of a variety of proficiency levels, each associated with a set of possible outcomes that are based on prior, related literature and the results of careful, empirical research with students of the appropriate ages. In addition, the tasks associated with each proficiency level can be varied based on a set of task classes, which are also defined through empirical research and syntheses of existing literature in the targeted mathematical domain. Together, these possible outcomes and tasks

classes result in problems of varied difficulty and sample spaces, and help to establish both the depth and breadth of student understanding necessary at each level of the trajectory.

A growing number of learning trajectories and progressions have been developed for a wide array of topics in mathematics and science; for example, in addition to the eighteen trajectories defined by the DELTA research group in CCSSM (Hexagon Map of the Common Core State Standards in Mathematics, 2011), there exist trajectories for measurement and flexible arithmetic (Gravemeijer, Bowers & Stephan, 2003), composition of geometric figures (Clements, Wilson & Sarama, 2004), scientific modeling (Schwarz et al., 2009), carbon-cycling in socio-ecological systems (Mohan, Chen & Anderson, 2009), and force and motion (Alonzo & Steedle, 2009), to name a few. There appears to be a general consensus among this literature that the undertaking of such work must be grounded in theory, and that empirical research is necessary to solidify that theory (e.g., Gravemeijer, 1999; Clements & Sarama, 2004; Shavelson & Kurpius, 2012). A large contingent of researchers also agree that utilizing the teaching experiment methodology in which the learning trajectory framework was first hypothesized by Simon is the most promising and effective way to study and refine these conceptual learning trajectories for children (e.g., Confrey, Maloney, Nguyen & Corley, 2012; Steffe, 2004; Krajcik, Sutherland, Drago & Merritt, 2012).

### *Splitting Framework*

Two conceptions of splitting as an operation have been widely used and cited in recent literature, and although they share some large similarities, there are also subtle

differences. Confrey (1988) first presented splitting as a “multiplicative interpretation of partitive division” (p. 255) in a similar way that repeated addition is a multiplicative interpretation of quotitive division. In that sense, Confrey claimed that the concept of splitting is not universally an action of division, as the term is often used, but rather an action of reproduction and its action leading to it as an operation was contrasted to counting. Splitting is defined as “an action of creating simultaneously multiple versions of an original, an action which is often represented by a tree diagram” (Confrey, 1994, p. 292). Confrey and colleagues have developed a sound research base around these ideas, and within that made several other assertions about the nature of the splitting construct: 1) related actions start with the unit, one (Confrey, 1988), 2) those actions are one-to-many, such as in fair sharing (Confrey & Smith, 1994), and 3) this primitive concept, in addition to being related to multiplication and division, is also connected to rate of change, exponential growth, and similarity (Confrey, 1994; Confrey & Smith, 1994, 1995). For this study, I adopt Confrey’s definition of splitting, as it was this research that has led to the development of the equipartitioning learning trajectory and informed my work on co-splitting.

Confrey’s interpretation of splitting predicated, and differs from, that used by Steffe (2002), Hackenberg (2007), and others, wherein a great deal of focus is placed on the unit (and coordination of units of units). They differ in three ways: 1) the consideration of splitting as an action or operation, 2) the justification and interpretation of the result(s) of splitting, and 3) the other mathematical operations believed to be built upon splitting. Each considers splitting as an operation that is composed of two actions. For Confrey, splitting is

associated with reassembly as “times as many” just as division is associated with multiplication. She argued the constant relationship of share to whole linked these subsequent operations with ratio, such that division, multiplication, and ratio co-define each other (Confrey & Scarano, 1995). Steffe and colleagues (e.g., Olive & Steffe, 2001) refer to partitioning and iteration (Steffe, 2002). First, there is a clear distinction between partitioning and equipartitioning, in that the prior does not inherently assume equal-sized groups or parts, whereas the latter does so explicitly. While partitioning actions can be multiplicative – when they are actions of equipartitioning – iteration is necessarily additive. Therefore, Steffe’s assumed simultaneity of these actions in his splitting scheme is difficult to imagine.

Second, Hackenberg (2007) states that a task requiring students to determine a length given another length that is “ $n$  times” the unknown length, implies iteration, yet is solved by splitting. I do not contest that such a task should invoke splitting, nor that the task and splitting are both multiplicative, but to identify the task as one calling for iteration narrows the interpretation of multiplication to only that of repeated addition. Within equipartitioning, the result of a splitting action cannot be one single part or share, as implied by this and other tasks used by Steffe and colleagues (for example, sharing a candy bar among eight people by only marking one share; Steffe, 2002) but rather a ratio of a single unit to the whole as  $n$  times as large or many.

A single share can be identified, but the criteria for equipartitioning make it explicit that *all* shares are created by the splitting action. The abstraction that all shares do not need to be physically identified or represented in order to constitute the size of one share can

certainly occur mentally; however, in requiring students to perform a physical action representing the splitting operation, this reduces the task to one of estimation that can only be justified as correct or incorrect through additive iteration to reconstruct the whole. Under this interpretation of splitting, the means for a student to justify whether the given whole is in fact  $n$  times the part they determined is through iteration of that one part. This is then not actually demonstrating that the whole was split, but rather a new unit was created (here from the whole, but I would offer that perhaps that is one sufficient way of defining the unit but not a necessary one). This discussion is further enhanced by the elaboration of equipartitioning in the section that follows.

Consider a comparison of the splitting operations as defined by Confrey and Steffe in the context of sharing a rectangular cake among a group of six people, which is similar to the tasks mentioned thus far. Under Confrey's definition of splitting as an operation, a student would split the cake into six equal parts (possibly using a composition of parallel or perpendicular actions) and the result would be one piece of cake for each person. The student could justify that all of the shares are equal (the same size) because each piece resulted from the same six-split and therefore represents one-sixth of the whole cake. In other words, the split is complete and the relative sizes are known, and to channel Steffe (2002), "without further action or operation" (p. 289). Performing this physically, some pieces may be exactly one-sixth, while others may not. The student would know the size of all six parts from the split (even if misapplied) and be able to judge whether each is the same size and therefore one-sixth of the whole. If any are not (which does not necessarily imply all are not) one-

sixth, then the student could employ mechanisms of compensation to adjust how the split was physically carried out, assuming they are operating at a level of understanding in which it is known the result *should* have been six equal parts produced simultaneously – a recursive, not iterative, notion.

Under Steffe's definition of splitting as an operation, the student would identify or create one piece of cake, and to follow the conventions described and established with his related tasks, it would come from the whole, given cake. (One could also argue that it could come from an entirely different cake, as could the iterated parts that follow, but that tangential to the point of this exercise.) The student would then iterate the one piece of cake five times to justify the other shares as being the same size, which one would presume the student is also actually creating those other pieces at each iteration. This would result in the first several shares being the same size, but it would not be known until the last iteration (assuming the student's piece was close to one-sixth) whether the final result was actually six shares of exactly that size that are equivalent to the whole cake, or whether the last piece would be smaller because there was not enough cake left to create a sixth full share, or whether there was cake left over after six shares were created. Here, too, the student had the intention of creating a piece that was one-sixth of the whole, and also would have compensation mechanisms for determining how to adjust the pieces, if necessary. However, the student would need to go back and either reduce or enlarge the size of their first piece and then iterate this newly-determined piece five times again, still unaware what would result from the last iteration with respect to the size of the original whole.

Through this example, three distinct differences in the two notions of the splitting operation become apparent: 1) Confrey's notion is truly multiplicative, simultaneous, and recursive – one share does not exist unless all shares exist, and each is exactly the same one- $n$ th of the whole, 2) Steffe's notion does not account for a means by which the initial one- $n$ th parts could actually be realized with certainty or do justice to the claim of simultaneity of partitioning and iteration, and 3) the reversibility (Piaget, 1970) of splitting as an operation is understood by Confrey's notion to be multiplicative, whereas it is understood by Steffe's notion to be additive. Therefore, with respect to the third difference between the two notions, Confrey's splitting builds multiplication, division, fraction, and ratio as co-defined, while Steffe's splitting builds multiplication as repeated addition, from counting, addition, and subtraction. It should be noted, however, that the aims of the two research bases discussed here with respect to splitting are quite different and could account for some of the fundamental differences in their definitions of the splitting operation and their interpretations of splitting actions. Steffe and colleagues have largely focused on the coordination of levels of units in students' development of fractions, and particularly non-part-whole interpretations of non-unit fractions leading to improper fractions. On the other hand, Confrey and colleagues have largely focused on students' development of ratio and proportional reasoning, including scaling, similarity, and exponentiation, which all involve the coordination of two or more quantities or dimensions, and that may or may not consist of levels of units that are commensurable. It is for these reasons that Confrey's definition of

splitting and her explanation of the splitting construct inform equipartitioning and co-splitting in this study.

### Models for Division and Multiplication

In this section, I will not present a full review of the extensive body of literature on division and multiplication, but rather a more focused synthesis of a portion of that literature that is directly related defining models<sup>4</sup> for division and multiplication. Fischbein, Deri, Nello, and Marino (1985) identified two primitive models for division – partitive and quotitive – and only one primitive model for multiplication – repeated addition – although they allude to the existence of other models, such as arrays. Confrey (1988, 1994) has offered splitting as the basis for another model for multiplication that is an interpretation of partitive division in the sense that repeated addition is an interpretation of quotitive division. She identified young children's use of "1/nth of" as a form of expression of that action. Schwartz (1988) noted two ways in which quantities can be composed through mathematical operations – referent preserving (two like quantities produce a third like quantity) and referent transforming (two like, or unlike, quantities produce a third quantity unlike neither of the first two). Furthermore, Schwartz claimed that referent transforming operations, such as division and multiplication, draw attention to another distinction between types of quantities – extensive and intensive. In simplest terms, extensive quantities refer to a countable amount (or size) and intensive quantities refer to a ratio of relative amount (or size). For example, consider the task cited in the section above on splitting, in which six

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<sup>4</sup> This section still only reflects a portion of the literature on models for division and multiplication, but the references align well with the larger consensus of that other literature (e.g., English & Halford, 1995) and were chosen to offer a coherent structure as situated in the chapter.

people were fairly sharing one whole cake, the amounts of both people and cake, six and one respectively, are extensive quantities. The fair share of one person resulting from splitting, stated as one-sixth of the cake per person, is an intensive quantity.

In order to not overgeneralize or oversimplify the notion, consider another example of fairly sharing two and one-half cakes among six people. First, notice the amount of cake, still an extensive quantity is not a whole number, but yet can still be counted. The fact that we can count one-half a cake introduces further elaboration on the term referent (i.e., the referent whole, or referent unit). Clearly, the resulting fair share – five-twelfths of a cake – would be the same if the task were stated as fairly sharing five half-cakes among six people. However, a fair share of five-twelfths of a cake per person is based on the referent being a whole cake; whereas if when the task was presented as five half-cakes it was also re-unitized to assume a half-cake as the referent, then the fair share could be stated as five-sixths of a half-cake per person. Nonetheless, in both instances the relative size of the fair share – an intensive quantity – is one-sixth of another referent whole (total cakes) per person.

In the unpacking of the CCSSM standards (Confrey et al., 2012), these distinctions have been built upon and identified three models for division/ multiplication– referent transforming, referent preserving, and referent composing. Unlike many prior theorists, she argued that division and multiplication as inverse operations must share the same underlying models. The notion of referent transforming here is very much in line with that offered by Schwartz because there are always two unlike quantities and the result is a third quantity unlike either of the first two, but the distinction is that one of the three is also always an

intensive quantity. This type of division and multiplication is best realized in fair sharing, equal groups, and rate problems. The notion of referent preserving is markedly different from that offered by Schwartz because although there are always two like quantities, the third is not like those other two; rather, the third quantity is always a scalar – a quantity without any associated unit. This type of division and multiplication is best realized in scaling problem, but can also be realized in equal groups and unit conversion problems. In both of these latter cases, the scalar results from juxtaposing the relative size of two like quantities (feet and inches, eggs and dozens of egg) and an understanding that there is a constant intensive quantity between the units of those quantities and the unit of the third quantity (e.g., see Kaput, 1985). For example, recognizing that thirty-six inches is three times as many inches as twelve inches, and understanding that the relationship between inches and feet is that there are always twelve inches per one foot; therefore, to determine the number of feet in thirty-six inches, one multiplies the implied given of one *foot* times the scalar of three to yield a result of three *feet*.

The notion of referent composing division and multiplication addresses some of the other potential models alluded to by Fischbein et al. (1985), and also partially aligns to the other referent transforming types of problems, not incorporated into the model by the same name here, in which there are two like quantities producing a third unlike quantity. In this type of division and multiplication, the first two (like or unlike) quantities produce a third unlike quantity that is specifically not representative of a rate (i.e., the *per* language does not apply). This type of division and multiplication is best realized in area and Cartesian product

problems, or any problems that can be represented by an array structure. For example, two lengths producing area, or the number of male-female couples possible from a group of men and a group of women. The area example is most easily understood when the units of length are the same (inches x inches = square inches), but it should be understood that the “likeness” of the quantities is their both representing length, not their both having the same unit (e.g., inches x feet = area, and inches x centimeters = area; both would produce uncommon measures of area but could easily be converted to a common measure of area). A contextually similar example of a referent composing relationship in measurement with unlike quantities would be a measure of energy (i.e., feet x pounds = foot-pounds).

Ratio and proportion, and thus splitting, equipartitioning and co-splitting, problems rely most heavily on students’ application and understanding of the referent transforming model of division and multiplication, as it is described in the unpacking of CCSSM (Confrey et al., 2012). As such, this requires students to coordinate quantities (and their units) in appropriate ways and then apply the appropriate form of multiplicative reasoning – perhaps an act of multiplication or division, or perhaps something more intuitive, such as equipartitioning or splitting.

In the next section, I present a brief summary of literature that identifies several problematic factors in students’ abilities to solve contextual problems.

#### Context and Quantity in Multiplicative Problem Solving

In this section, I will briefly summarize two separate yet related bodies of literature that have implications for any study of the mathematical reasoning of students while

engaging with problem-solving tasks. First, there are the issues centered around the values or quantities used in the problem, which can vary by size, number type, and unit. Second, there are the issues centered around the context of the problem, which can either be familiar, unfamiliar or misleading and ambiguous. All of these greatly impact students' success in solving a problem, and therefore could have implications for any inferences made from episodes of problem solving in which either or (often) both are involved.

A great deal has been noted with regard to the interplay between number type and operation – based on intuitive models – and how those relationships may impact student problem-solving ability (e.g., see Goldin & McClintock, 1984; Caldwell & Goldin, 1987). The most common example being related to the early conceptions of multiplication makes bigger and division makes smaller, which is upheld in the whole numbers and then breaks down when moving to rational numbers (e.g., Bell, Swan & Taylor, 1981; Hart, 1981). Nesher (1988) investigated students' creation and description of multiplicative problems by considering a dimensional analysis of intensive, extensive, and scalar quantities across three multiplicative problem types: mapping rule, compare, and Cartesian. The results showed no clear consensus of students privileging one type over another, thus indicating that not all informal and intuitive understandings of multiplicative problems are the same, or even necessarily ordered. Additionally, Mulligan (1992) observed that students' strategies were representative of both the problem type and the size of the values of the quantities involved, suggesting that more than one intuitive model could exist for a student and the model applied may differ based on changes to either the problem type or the size of the numbers.

It has also been noted that even among adults the appropriate operation – multiplication or division – to solve a contextual problem is often difficult to ascertain, a problem that becomes even thornier when the problems involve fractions (e.g., Armstrong & Bezuk, 1995). In problems involving division with decimal quantities, Bell, Fischbein, and Greer (1984) observed similar reversals of operations (the students would apply multiplication), even when qualitatively the context of the problem was understood. Therefore, it does appear that there is often an interaction between a students' intuitive models for operations, such as division and multiplication, and their approaches to solving such problems that is affected by both the values of the quantities involved and the contextual elements of how the problems are presented. Likewise, similar effects based on the values involved would be expected with respect to students' understanding of intensive and extensive quantities (Kaput & West, 1994) and rate type (Heller, Post & Behr, 1985) when moving to proportional reasoning problems. For example, Behr, Harel, Post, and Lesh (1987) hypothesized three structural variables – order, unit of measure, and partitionability – that account for over five hundred different possible variations in missing value proportion problems. A similar additive misconception exists in ratio problems and has received due diligence with respect to research (e.g., Hart, 1984), yet it is much more complex and more difficult to confront.

The contextual elements of a problem situation can be problematic for students in multiple ways, beyond that which can be inferred from the discussion above. The semantics (Kouba, 1989) of the problem's structure can greatly influence the way in which a student

solves a problem, and thus has an impact on the operation chosen (which may or may not be correct or productive) and the model for that operation. Aside from verbal interpretations and linguistic issues, there is also the issue of familiarity and motivation associated with contextual word problems. The two sides of the argument are certainly in conflict, and both widely supported, yet it does not appear there is a universal solution. On the one side, the seminal works of Lave (1988), Nunes, Carraher, and Schliemann (1993), and others that have followed, have shown that “real-world” mathematics make the concepts accessible to students and often afford greater success than traditional “school” mathematics. While on the other side, the process of translating those situations into mathematical problems that can be solved, before even attempting to solve them may present inherent difficulties for numerous reasons: 1) it requires an act of modeling – both formulating and interpreting (Niss, Blum & Galbraith, 2007), 2) interpreting the context in light of unknown, or additionally known, information about it may interfere (e.g., Inoue, 2005; Boaler, 2002; Kazemi, 2002), and 3) lack of interest in, or accessibility of, the context may introduce additional challenge for students of differing genders, class, and ethnicities (e.g., Boaler, 1994, Cooper & Harries, 2009; Lubienski, 2000).

Within the domain of equipartitioning and co-splitting, I conjecture that the introduction of carefully scaffolded problems in which the quantities are clearly represented and the number types are altered at appropriate intervals based on the research that has informed the learning trajectory, difficulties with respect to quantity will be minimized. This is not in an effort to remove the possibility of any related misconceptions, but rather to allow

student to confront those misconceptions on their own, in a controlled environment, before they become too deeply rooted; many strategies and ideas subsequently labeled as misconceptions are merely alternative conceptions early. These can be viable up to a point, and then when a problematic is introduced, the conception is revised to accommodate or assimilate the new situation. I also conjecture that there is no appropriate way in which to introduce or discuss equipartitioning and co-splitting problems that is devoid of context. When the premise of the problem is fair sharing, trying to strip the problem of context to make it purely mathematical would make it much more difficult and possibly even confusing to students of the target age – early and late elementary grades. For example, consider fairly sharing six among four, without naturally adhering a noun to each value.

### Ratio and Proportional Reasoning

In this section, I summarize and discuss the existing literature on the development of ratio and proportional reasoning and their constituent parts, tracing a path from the necessity of early multiplicative reasoning, to the role and makeup of ratio in the larger multiplicative conceptual field. A critical issue that our research group constantly attempts to address and combat is a singular treatment and development of understanding for multiplication as repeated addition. Therefore, the timing and means by which students are able to also acquire an understanding of multiplication that is logically and inherently multiplicative in nature are crucial components for circumventing later possible misconceptions and deficits. The review concludes with some studies that have shown promising evidence that such a treatment of

multiplicative reasoning evinces a path to ratio that is accessible to students at the elementary level if the proper approaches and tasks are exploited.

First, consider the ratio comparison tasks studied by Noelting (1980a, b), in which students were asked whether drink A or drink B would have more orange flavor, or if they would taste the same, in which both drinks consisted of a number of cups of orange juice and a number of cups of water. For example, the task of comparing a drink mixture (A) consisting of two cups of orange juice and three cups of water with a drink mixture (B) consisting of four cups of orange juice and six cups of water. The characteristics of the parameters (the values of the two quantities – cups of orange juice and cups of water) in these tasks were assigned to stages, and then identified by a name representative of the strategies (corresponding to their explanations by the students and researcher) necessary to solve the tasks successfully (see Table 1). The median ages of accession for each of these strategies were calculated and the ordering appeared to hold consistent with the stages, but more importantly the frequency of correct responses declined across all children as the difficulty of the strategy required by the parameters increased (e.g., fewer children responded correctly to items at level IIIA than level IIB).

The first level that represents multiplicative reasoning (IIA) deals with being able to identify equivalence classes of a 1:1 ratio, or in other words children working with one-to-one correspondence. This strategy would break down for children at this level when the item presented did not actually contain equal amounts of both quantities, and they would therefore abandon multiplicative comparison for additive comparisons of the remaining amounts after

the one-to-one correspondence was exhausted. In order for children to identify the equivalence of any ratio other than a 1:1 using multiplicative or ratio reasoning, two successful strategies at the following level (IIB) were identified: one *between* strategy and one *within* strategy. The between strategy, which Noelting called covariation, referred to children who simultaneously compared the multiplicative change in the quantity of juice and the multiplicative change in the quantity of water to determine if these changes were equivalent, thus resulting in equivalent mixtures. The within strategy, which Noelting called division, referred to children who simultaneously compared the multiplicative relationship between the number of cups of juice and number of cups of water in one mixture with the corresponding multiplicative relationship between those same two quantities in the second mixture. This was typically done using division, and thereby children could determine whether the two resulting fractions sharing an invariant unit of 1 (i.e.,  $\frac{6}{4} : 1$  versus  $\frac{3}{2} : 1$ ), were equivalent.

*Table 1.* Stages in development of the concept of ratio (adapted from Noelting, 1980a).

Stage	Name	Age of Accession	Typical Item	Characteristics
<b>0</b>	Symbolic	(2;0)	(1,0) vs. (0,1)	Identification of elements
<b>IA</b>	Lower Intuitive	(3;6)	(4,1) vs. (1,4)	Comparison of first terms only
<b>IB</b>	Middle Intuitive	(6;4)	(1,2) vs. (1,5)	Like first terms, comparison of second terms
<b>IC</b>	Higher Intuitive	(7;0)	(3,4) vs. (2,1)	Inverse relation between terms in the ordered pairs
<b>IIA</b>	Lower Concrete Operational	(8;1)	(1,1) vs. (2,2)	Equivalence class of (1,1) ratio
<b>IIB</b>	Higher Concrete Operational	(10;5)	(2,3) vs. (4,6)	Equivalence class of any ratio
<b>IIIA</b>	Lower Formal Operational	(12;2)	(1,3) vs. (2,5)	Ratios with two corresponding terms multiple of one another
<b>IIIB</b>	Higher Formal Operational	(15;10)	(3,5) vs. (5,8)	Any ratio

These strategies also ramified to the next level (IIIA) where children could successfully apply those strategies to ratios that were not equivalent (both quantities of juice and water are not the same multiple of one another) but included *one pair* of corresponding quantities (either juice or water) that were still multiples of one another. According to Noelting, reaching this level merely required the extra step of then comparing the unlike quantities once the multiples were equated (e.g., claiming that  $2 : 5$  is greater in the first quantity or lesser in the second quantity than  $1 : 3$ , because doubling  $1 : 3$  yields  $2 : 6$  and 5 is less than 6). However, Noelting asserts that to reach the highest level and successfully compare any ratio (specifically including those where neither pair of quantities is a multiple of the other) children must have an understanding of fractions, to the extent of finding common denominators, or percent (and one could infer, decimals).

Seminal works by Vergnaud (1983, 1988) on what he called multiplicative structures and the an encompassing multiplicative conceptual field (MCF) have been instrumental in focusing the way researchers in the field of mathematics education discuss and study topics related to the concepts of multiplication and division, including ratio and proportion.

Vergnaud (1988) first described the conceptual field of multiplicative structures as consisting “of all situations that can be analyzed as simple or multiple proportion problems and for which one usually needs to multiple or divide” (p. 141). In a later explication of the MCF, Vergnaud (1994) went on to state that:

The situations and problems that offer a sound experiential reference for MCF are not purely mathematical, especially at the elementary and early secondary levels. The child’s early experience of buying goods and sharing sweets, and his or her first

understanding of speed, concentration, density, similarity, or probability are essential. (pp. 42-43)

The examples given in the accompanying research by Vergnaud (1983, 1988, 1994) elicited several problem types – simple proportions such as isomorphism of measures, or product of measures, concatenation of two simple proportions, and multiple proportions – including describing two types of division. However, the analysis of the mathematics behind how one solves those problems focuses on the underlying functional nature of multiplicative structures as can be represented symbolically by a proportional function, or in other words, an equation demonstrating a direct variation relationship between two quantities (i.e.,  $f(x) = ax$ ). This is clearly an appropriate means for describing the mathematics, but it is not the way young children would necessarily represent a solution or even first develop an understanding of the core concepts of multiplication and division. In fact, because additive structures and counting are given preference in school curricula, students' opportunities to develop multiplicative structures as a parallel conceptual field that can develop simultaneously is greatly reduced (e.g., see Confrey, 1994).

As noted earlier, Fischbein, Deri, Nello, and Marino (1985) claimed that the “primitive model associated with multiplication is *repeated addition* … although other models, such as the rectangular pattern, have been proposed for teaching multiplication” (p. 6). The authors go on to hypothesize that two primitive models exist for division: partitive – sharing or dividing a collection of objects or a single object into equal parts or groups, and quotitive – determining the number of times a smaller quantity is contained in a larger quantity. The second of these is also referred to as measurement division, whereby the

dividend must be larger than the divisor, which would support a model of *repeated subtraction*, and is therefore also established through actions of counting. It is not argued that division and multiplication are inverse operations, but then this series of claims indicated that there was not a primitive model of multiplication that was the inverse of partitive division.

There also exist situations where the repeated addition model of multiplication breaks down (e.g., Cartesian products), and situations that require differentiation between partitive and quotitive division models (e.g., quantity per group versus number of groups). Beyond the problem situation itself dictating an appropriate operational model, the context and the numbers involved as parameters have also been found to affect students' success in solving problems that are relatively similar otherwise. For instance, problems involving rate are typically more difficult than those solely involving partitioning (Bell, Fischbein & Greer, 1984), and when division is the expected operation, the model depends on the relative sizes of the divisor, dividend, and quotient (Fischbein et al., 1985). Regardless of the context, students show an inherent affinity for the number 2 as a factor in multiplicative situations (doubling or halving), and they find increased success when working with problems where the parameters are conducive to using 2 as an operator (e.g., Spinillo & Bryant, 1991).

Streetland (1984, 1985) has argued for an earlier introduction of ratio reasoning, claiming that the learning of ratio could incorporate schemas of anticipatory action and close modeling of the student's perception of reality, which often relate to scaling, density, multiplication, or probability. Added to that list has been the context of fair sharing. Confrey (1994) has argued that the cognitive underpinnings for acquiring the concept of ratio lie

within an understanding of splitting and equipartitioning that can occur at the early elementary grades. Confrey (1988) first proposed the operation of splitting as an intuitive and primitive model of multiplication that relates to partitive division. Therein, splitting refers to the operation of forming equal-sized groups, or equipartitioning<sup>5</sup>, which can be seen in activities involving sharing, folding, or geometric similarity. Therefore, as an action or operation, there are no constraints on the quantities in splitting and equipartitioning as there are in the contexts of quotitive division. In this work, she argued for the independence of splitting structures and counting structures, suggesting that from splitting, students could learn multiplication, division, and ratio as a triplet, and that fractions could be introduced as a subset of ratio reasoning.

In a three-year teaching experiment with third- to fifth-graders, she demonstrated the feasibility of such an approach (Confrey & Scarano, 1995). These students developed ratio constructs as they worked with multiplication, division, slope, and similarity. They were introduced to ratio simultaneously with division and multiplication in third grade, paying careful attention to the quantities involved as operators (beginning with 2's, 4's, ... and progressing in a natural order of accessibility) and the relationships between partitive and quotitive division utilizing array structures. In fourth grade, the students were introduced to least common multiple, greatest common factor, proportion, operations within rational numbers (developing fractions as a type of ratio), and geometric similarity in two-dimensional space. At this same time, the students invented the idea of describing the

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<sup>5</sup> We distinguish between the terms “partitioning,” as is used by some other authors to indicate the creation of any type of parts in part-part-whole relations (e.g., Steffe, 2002; Hackenberg, 2007), and “equipartitioning,” which emphasizes the creation of *equal-sized* groups.

smallest whole number ratio as “the little recipe,” which Confrey and Scarano (1995) labeled as the “ratio unit” (p. 6) – that which we refer to now as the base ratio. Confrey (1995) went on to contrast this with the idea of the “unit ratio” – quantity per 1 – which also plays a significant, but different role, in student reasoning. Further results from this same longitudinal study showed that when students are challenged to find a missing value for a large number (the height of a tree given its shadow and the height and shadow of a ruler), students build up, and apply repeated doubling and halving to solve the problem. This led her to show that problems involving slope and/or similarity of triangles can lead children to discovery of the *base ratio* and the *unit ratio*, which are precursors to understanding fractions (Confrey & Carrejo, 2005).

Finally, in fifth grade, extensions on all of the previously covered ideas were made to decimals, percent, and algebraic symbols. It was found that these students were able to work with a variety of representations (Venn diagrams, tables of values, dot drawings, ratio boxes, graphs, etc.) to reason about and successfully solve several types of problems, including scaling and mixture problems. A more detailed account and analysis of different students working with these latter two types of problems are presented later in this review.

When applying this work to the development of a set of learning trajectories for rational number reasoning, Confrey and Maloney (2010) recognized that the fundamental construct of equipartitioning could be used to synthesize a number of prior studies on disparate cases, sharing a collection, sharing a whole, and sharing multiple wholes. A large body of work on equipartitioning has been predominantly conducted within the context of

fair sharing (Squire & Bryant, 2002; Confrey et al., 2009), which relies on the cases of sharing collections (Pepper & Hunting, 1998; Confrey et al., 2009) and partitioning a whole (Pothier & Sawada, 1983). Confrey and colleagues have also conducted extensive research around these ideas, showing their importance to the development of the multiplicative conceptual field via their relation not only to multiplication through ratio and proportion (Confrey & Scarano, 1995), but also to geometric similarity (Confrey, 1994) and functions, specifically exponential and logarithmic functions (Confrey & Smith, 1995; Smith & Confrey, 1994). Therefore, it is believed that the vital notions of base ratios and unit ratios can be approached through these types of settings as well.

In a different setting, Lamon (1993) conducted clinical interviews with sixth-grade students, prior to any formal instruction of ratio, on four types tasks: well-chunked measures, part-part-whole, associated sets, and stretchers and shrinkers. Well-chunked measures problems involved comparing an *intensive measure* resulting from two *extensive measures*; for example, comparisons of speed as composed of miles and hours. The terms intensive and extensive measures are used in the sense of referent transforming compositions as noted by Schwartz (1988), discussed in a later section. Part-part-whole problems involved the counts of two or more quantities comprising a larger set. For example, the amounts of boys and girls would represent the parts in a class of students as the whole. Associated sets problems involved a rate of two measures given within the problem context, resulting in an intensive measure that is not in itself well known. For example, the quantity “pizzas per person” would be defined in an associated sets problem, whereas unit price (e.g., dollars per pizza) would

represent an unknown quantity in a well-chunked measures problem. Lastly, stretchers and shrinkers (Dienes, 1967; Braunfeld, 1968) problems involved the ratio-preserving coordination of two quantities referring to the same object. For example, in coordinating the lengths and widths of a set of similar rectangles, the ratio of length to width (or width to length) is preserved.

Lamon (1993) found that the students in this study used ratio and proportion constructs – particularly using ratios as units (composed of composite units) – more predominantly in associated-sets problems, where the inherent ratios consisted of two discrete quantities. For example, the problem of determining whether the girls or the boys will get more pizza if 7 girls fairly share 3 pizzas and 3 boys fairly share 1 pizza, where the number of people represents one discrete quantity and the number of pizzas represents the other discrete quantity. In contrast, students used what she called preproportional reasoning strategies – relying on modeling with pictures, charts, and manipulatives, and only showing signs of some (but not full) relative thinking – most often when presented with well-chunked measures and part-part-whole problems. Likewise, only four of the twenty-four students in the study used what Lamon called “constructive” strategies – those that involved some form of relative thinking or numerical or functional relationships – on stretcher and shrinker problems, and the majority of the rest of the students relied on visual or additive strategies for these types of problems. It appears then that the best access point for students to develop proportional reasoning and a quantitative understanding of ratio within their multiplicative conceptual field is through associated sets problems. Here students can explore both easily

recognizable multiplicative changes and relations, such as those present in the recipe problems described by Confrey and Scarano (1995), as well as those that are not as easily recognized as multiplicative and typically lend to protoratio reasoning.

In all of the problems used by Lamon, with the exception of one stretcher and shrinker problem in the context of similar rectangles, the students were required to make a comparison of ratios. Underlying the ability of a student to make an assertion that one ratio quantity is greater than or lesser than another, there is an inherent notion of equivalence of ratios, as was seen earlier in Noelting (1980a, b); and this was not addressed by Lamon. Also, Lamon identifies the use of ratio as a unit as a qualitative proportional reasoning strategy; only when students apply algebraic symbols to represent proportions and identify and understand the functional relationships does she confer quantitative proportional reasoning (the most sophisticated strategy identified). Although this allows for a progression through the strategies in increasing sophistication, it discredits the strategy of using ratios as units for comparison through iteration or recursion. The claim being that such a strategy is qualitative simply for lack of symbolic forms and an understanding of function.

In an attempt to describe the early development of ratio and proportion as part of the larger multiplicative conceptual field, Lo and Watanabe (1997) conducted an extensive teaching experiment with a fifth-grade student. The work stemmed from an interesting strategy used by the student when asked how many candies could be bought with 15 quarters if 12 quarters would buy 28 candies in a previous study (Lo & Watanabe, 1993). His strategy involved attempting to sort the quarters and candies into the same number of groups

(eventually finding four groups of 3 quarters and 7 candies), and then using one of those groups to aid in constructing the targeted values by combining an additional one of those such groups with the original totals: adding 3 to 12, which made 15, and consequently adding 7 to 28, which made 35. The authors referred to such a strategy as “building-up,” or a ratio-unit (base ratio) strategy, and noted that there were connections to multiplication, division, and fractions; and such a strategy is clearly quantitative, in contrast to Lamon’s description presented above.

However, the student lacked formal skills in those areas and further study was pursued to determine how the strategy would develop once knowledge of those other concepts was acquired, and what difficulties he would encounter in attempts to generalize the strategy in a variety of contexts. It was found that application and articulation of the strategy in new ratio contexts with varying parameters enhanced the student’s understanding of division and multiplication. Consequently, it was also noted that during times when the student struggled to successfully apply his strategy, there was distinct evidence that lacking knowledge of those same operations could be cited as the culprit, rather than inapplicability or a misuse of the strategy itself. These results confirm Confrey’s earlier findings and validate her approach to introduce the topics of division, multiplication, and ratio simultaneously. Therefore, tasks that would evoke similar such strategies may provide insight to early ratio reasoning. Furthermore, it appears these tasks are fitting for students to utilize their knowledge of equipartitioning and employ the co-splitting construct and the generalization that  $a$  objects shared among  $b$  sharers will always produce a share of  $a/b$ .

## Equipartitioning

In this section I will trace the work and literature that reflect the development of the equipartitioning learning trajectory as it follows from earlier work in ratio and proportion, and has led to the current study. To build a foundation for ratio within the multiplicative conceptual field (Vergnaud, 1988, 1994) that includes, and extends beyond, the topics of multiplication, division, ratio, and fractions, the DELTA research group has identified an LT for equipartitioning (Figure 2). Equipartitioning is defined as:

... cognitive behaviors that have the goal of producing equal-sized groups (from collections) or equal-sized parts (from continuous wholes), or equal-sized combinations of wholes and parts, such as is typically encountered by children initially in constructing “fair shares” for each of a set of individuals. (Confrey et al., 2009)

The equipartitioning LT developed when Confrey identified four cases of sharing that had existed independently in the literature previously summarized. The four cases were:

Case A – discrete collections, Case B – single continuous wholes, Case C – multiple continuous wholes for fewer groups than wholes, and Case D – multiple continuous wholes for more groups than wholes. In studying students learning the four cases, the DELTA research group came to recognize a set of core proficiencies associated with the different cases, and this led to the articulation of the LT. The third and fourth cases (C and D) have since been combined in the group’s work, accounting for the differences through the outcome spaces of the applicable levels of the equipartitioning LT.

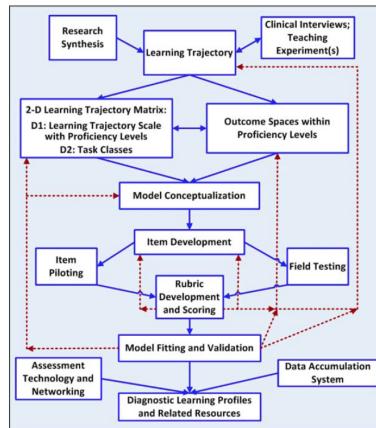
		Equipartitioning Learning Trajectory Matrix (grades K-5)												
		Task Classes →												
		Proficiency Levels												
A	B	C	D	E	F	G	H	I	J	K	L	M		
Collections														
2-split (Rect/Circle)														
2 <sup>n</sup> split (Rect)														
2 <sup>n</sup> split (Circle)														
Even split (Rect)				X										
Odd split (Rect)					X									
Even split (Circle)						X								
Odd split (Circle)							X							
Arbitrary integer split								X						
$p = n + 1; p = n - 1$									X					
$p$ is odd, and $n = 2^k$										X				
$p >> n, p$ close to $n$											X			
all $p, \text{all } n$ (integers)												X		
16 Generalize: $a$ among $b = a/b$														
15 Distributive property, multiple wholes														
14 Co-splitting	X													
13 Compositions of splits, mult. wholes														
12 Equipartition multiple wholes														
11 Assert Continuity principle								X						
10 Property of Equality of Equipartitioning	X	X	X	X	X	X								
9 Redistribution of shares (quantitative)	X													
8 Factor-based changes (quantitative)	X		X											
7 Compositions of splits; factor-pairs	X	X	X	X	X	X	X	X	X	X				
6 Qualitative compensation	X		X	X										
5 Re-assemble: $n$ times as much	X	X	X	X	X	X	X	X	X					
4 Name a share w.r.t. the referent unit	X	X		X	X	X	X	X	X					
3 Justify the results of equipartitioning	X	X	X	X	X	X	X	X	X					
2 Equipartition single wholes		X	X	X	X	X	X	X	X					
1 Equipartition Collections	X													

Figure 2. Learning trajectory for equipartitioning - proficiency levels and task classes.

The DELTA methodology for the construction, refinement, and validation of a learning trajectory in a certain cognitive domain consists is a multi-tiered and multi-faceted process, potentially involving several iterations and cycles within various stages. The full complexity of the methodology is shown in Figure 3 below; however, the larger stages of the process are briefly summarized as follows:

1. Synthesis of relevant research literature in the domain.
2. Clinical interviews and/or teaching experiments to fill in gaps.
3. Initial delineation of hypothesized proficiency levels and outcome spaces, ordered from lowest to highest levels of sophistication.

4. Cycles of item development, model fitting, and validation by qualitative and quantitative methods – informed by conducting further clinical interviews and teaching experiments, or field testing.
5. Creating diagnostic assessments for the validated trajectory, and learning progress profiles based on data from those assessments.



*Figure 3. DELTA methodology for trajectory construction (Maloney & Confrey, 2010).*

The learning trajectory for equipartitioning begins in early primary grades before the introduction of the other topics previously mentioned within the multiplicative conceptual field, and as indicated by prior literature (e.g., Confrey & Scarano, 1995), can inform them as well. We have also pulled out the few standards that support this trajectory in CCSSM, and those standards span from second grade to fifth grade. This aligns well with our beliefs and prior research on the topic, although we place some of the lower levels in the trajectory as early as kindergarten.

### *Proficiency Levels and Task Classes*

In the third stage of the DELTA methodology for LT development, proficiency levels and outcome spaces are hypothesized – and both of which are later validated through various forms of empirical study. The identification of task classes is corollary to identification of the outcome spaces and the creation of relevant tasks and items for assessment of a proficiency level. I will briefly describe each of the levels, and their inherent task classes, and then a more thorough delineation of the level of focus in this study – co-splitting – follows.

*Level one – Equipartitioning collections.* To demonstrate proficiency with equipartitioning collections, students must be able to split and distribute a set of discrete objects among a number of sharers so that the whole (all objects) is exhausted and each sharer receives a fair share (equal amount of objects). Students must then also be able to name the fair share as a count of objects per sharer. At this level, the fair is always a whole number of objects, which implies that the number of objects is greater than the number of sharers and a multiple of the number of sharers, and proficiency can be demonstrated by dealing the objects to the sharers in a form of one-to-one correspondence.

*Level two – Equipartitioning (single) wholes.* To demonstrate proficiency with equipartitioning wholes, students must be able to split and distribute a continuous object (whole) among a number of sharers so that the whole is exhausted and each sharer receives a fair share (equal-size parts of the object). At this level, the whole is either a rectangle or a circle, and proficiency can be demonstrated by physically splitting, by cutting or marking, the whole.

*Level three – Justification.* To demonstrate proficiency with justification, students must be able to explain how they know the result of an equipartitioning action on both collections and wholes results in a fair share for each sharer. This can be done for collections by citing a one-to-one correspondence of objects to sharers as a result of dealing, counting, or creating an array, and for wholes through folding, measuring, or citing symmetry or congruence.

*Level four – Naming.* To demonstrate proficiency with naming, students must be able to name the fair share after an equipartitioning action on collections or single and multiple wholes with respect to the share's relative size to the whole collections or a single whole – as a unit fraction.

*Level five – Reassembly.* To demonstrate proficiency with reassembly, students must be able to state the multiplicative relationship between a fair share and the whole (either a collection of discrete or continuous objects – multiple wholes, or a single continuous whole) as the whole being  $n$  times as many or  $n$  times as much as the fair share, when there are  $n$  sharers.

*Level six – Qualitative compensation.* To demonstrate proficiency with qualitative compensation, students must be able to state whether changes in the number of sharers will result in a fair share of a size that is greater or lesser than the size of the fair before those changes, while the number of objects remains the same.

*Level seven – Composition of splits (single wholes).* To demonstrate proficiency with composition of splits on single wholes, students must be able to create and identify the result

of consecutive splits on the same whole, whereby each split after the first acts recursively on the results of the preceding split.

*Level eight – Quantitative compensation.* To demonstrate proficiency with quantitative compensation, students must be able to state that multiplicative changes (by  $m$  or  $1/m$ ) in both the number of sharers and the fair share will result in either a fair share of a size that is  $m$  or  $1/m$  times as large as the fair share before the change or a number of sharers that is  $m$  or  $1/m$  times as large as the number of sharers before the change, while either the number of objects or number of sharers remains the same.

*Level nine – Reallocation.* To demonstrate proficiency with reallocation, students must be able to determine the new fair share for a collection previously shared among one number of sharers after one or more of those sharers is removed, and the size of the collection remains the same. Although this could be done by recombining the collection and solving the problem as a level one problem, in order to demonstrate reallocation, the students must only operate on the shares left behind by the departing sharers, redistributing that portion of the collection among the remaining sharers.

*Level ten – Property of equality of equipartitioning (peeq).* To demonstrate proficiency with PEEQ, students must be able to recognize and assert that the result of any  $n$ -split on a whole, regardless of geometric properties, will result in  $n$  equal-sized shares, each representing one- $n$ th of the whole.

*Level eleven – Continuity principle of equipartitioning.* To demonstrate proficiency with the Continuity Principle of equipartitioning, students must be able to assert that a whole

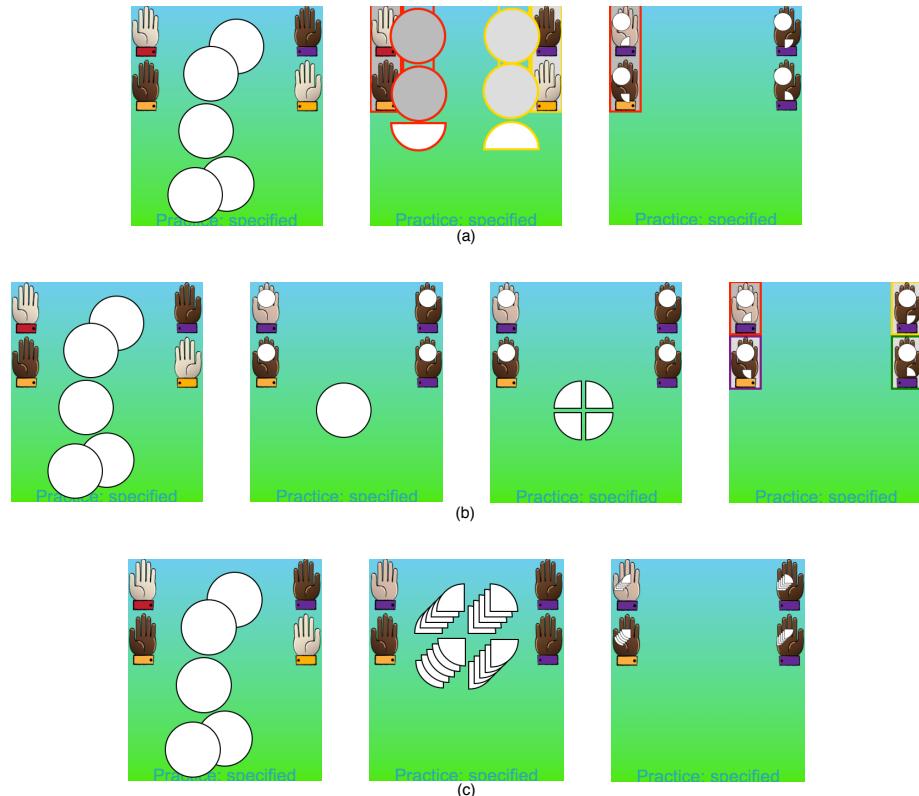
can be equipartitioned for any natural number of sharers, although proficiency does not require them to be able to physically demonstrate how to carry out the act of sharing.

*Level twelve – Equipartitioning multiple wholes.* To demonstrate proficiency with multiple wholes, students must be able to determine the fair share of more than one continuous whole among a number of sharers (that is not a factor of the number of wholes) through a combination of dealing and coordinated splitting actions. Three critical strategies have been identified to demonstrate this: 1) *co-splitting* (Figure 4a; per level 14); 2) *deal and split* (Figure 4b; per level 15) –students deal the objects in rounds of one-to-one correspondence until no further complete rounds can be dealt, and then split the remaining objects to share among the number of sharers (only applicable when the number of objects is greater than the number of sharers); and 3) *split-all* (Figure 4c; per level 16) – students perform a split equal to the number sharers on all objects, and deal one part from each object to each sharer.

*Level thirteen – Composition of splits (multiple wholes).* To demonstrate proficiency with composition of splits on multiple wholes, students must be able to create and identify the result of consecutive splits on multiple wholes, whereby each split after the first acts recursively on the results of the preceding split, but all initial splits need not act on all of the wholes, nor result in parts of a whole (such as with collections).

*Level fourteen – Co-splitting.* To demonstrate proficiency with co-splitting, students must be able to assert that if both the number of objects and the number of sharers are split

by the same factor, then the resulting share will be the same, and that the result will contain a number of identical groups equal to that factor.



*Figure 4. Equipartitioning multiple wholes strategies: (a) co-splitting, (b) deal and split, and (c) split-all.*

*Level fifteen – Distributive property of equipartitioning (multiple wholes).* To demonstrate proficiency with the Distributive Property of equipartitioning, students must be able to create fair shares for multiple wholes by fracturing the set of wholes into natural-number subsets that can each be equipartitioned for all of the sharers. They must also

recognize that the combination of those fracturing and equipartitioning actions results in a fair share that is the same size as results from sharing the wholes without fracturing the set.

*Level sixteen – Generalization.* To demonstrate proficiency with generalization, students must be able to assert that the fair share when sharing  $a$  objects among  $b$  sharers is  $a/b$  objects per sharer, without performing any physical act of splitting, equipartitioning, or sharing. They should also relate this unit ratio of objects per sharer with the base ratio of  $c$  objects and  $d$  sharers, resulting from co-splitting, where  $c/d$  is the simplest form of  $a/b$ .

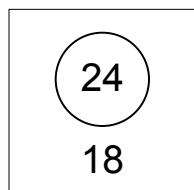
### *Co-splitting*

A primitive form of “covariation” was initially identified as an upper-level construct in equipartitioning, and the concepts of covariation and equipartitioning have both been posited to be linked to ratio and proportional reasoning, with supporting evidence to this claim from prior research (Confrey, 2008). The way that covariation was being defined in equipartitioning however was far more narrow than is used in other research, which includes the coordination of changes that are additive, exponential, or other (e.g., Rizutti, 1991; Thompson, 1994; Confrey & Smith, 1995; Carlson, Jacobs, Coe, Larsen & Hsu, 2002). Therefore, the term co-splitting (Corley, Confrey & Nguyen, 2012) was conceived to make explicit the nature of relationships in the tasks and problem-types utilized in our work, and to draw a more direct connection to splitting (Confrey, 1988, 1994, 1995), which we believe to be its basis. Co-splitting, thus, is defined as:

... a specific form a covariation, and constitutes cognitive behaviors that establish a ratio relationship between two quantities, such that any multiplicative change in one quantity is coordinated with the same multiplicative change (and in the same

direction: increasing or decreasing) in the other quantity. (modified from Corley et al., 2012)

Streefland (1991) explored young students' investigations of tasks he referred to as "distribution situations," where one particular type of problem involved arrangements of (equivalent) distribution situations. Students were asked to determine how a given number of people and number of pizzas could be fairly distributed across multiple tables so that all receive the same size share. His representation for the problem showed a circle containing the number of pizzas drawn above the number of people sharing (see Figure 5, which will henceforth be called a "Streefland diagram"), and was used in allowing students to explore equivalent ratios with the use of a tree diagram. These types of tasks are similar to those that we refer to as co-splitting in our work, and therefore, we adopted this representation for presenting similar co-splitting situations to students in paper-and-pencil tasks and in the co-splitting item-generation environment (IGE) in our IDAS – LPPSync.



*Figure 5. Streefland diagram for distributing or sharing 24 pizzas among 18 people.*

It is also important to note that within the scope of equipartitioning, the referent whole (which could be a single continuous whole or a collection of discrete or continuous wholes) must persist in context. In other words, the whole must be exhausted in order to successfully solve an equipartitioning problem, and it also cannot be altered to contain more

objects (or sharers) than was given. In co-splitting, both the numbers of objects and sharers are given, but the number of groups is not necessarily equivalent to the number of sharers, which implies a subtle change in the meaning of the whole, to include the number of sharers more explicitly as well. For example, consider the co-splitting task of determining an equivalent way to share 24 pizzas among 18 people such that every person gets a fair share of all of the pizza and all of the pizzas and people are not in one, singular group. One could see the merit in a student being able to state that each person in a group of 12 pizzas and 9 people would receive the same fair share, or that each person in a group of 36 pizzas and 27 people would as well. However, the task is stated so as to elicit an action of equipartitioning, of which the previous statements are not, in order to determine a satisfactory response, such as *three groups of 12 pizzas and 9 people.* (Note that the second originally offered possible response is not valid because there were only 24 pizzas and 18 people in the referent whole as defined by the task.) Therefore, the ability to determine an equivalent ratio from a given ratio through a co-splitting action, devoid of the recursive nature within equipartitioning, is considered a level of ratio reasoning that is beyond the scope of equipartitioning, which merely establishes the foundation for acquiring such an understanding.

An equipartitioning field test was conducted with 5000 students using paper-and-pencil items, and after disaggregating by grade level, it was found that for students in grades 3-8, the learning trajectory level associated with an item was a good predictor of item difficulty (Pescosolido, 2010). The overall scores were quite low, posited to be due in part to the lack of a curricular treatment for equipartitioning in typical classroom curricula, with a

mean score of 0.34 in a range from 0 to 1. However, it was also documented that co-splitting tasks (still referred to as covariation at the time) were markedly some of the most difficult, even for older students. The percent of overall correct responses to items at the co-splitting level of the trajectory, which were given to students in grades 2-6 based on the booklet design used, was only 17.8%.

In an attempt to further explain the relative difficulty of these and other items associated with atypical scores, Yilmaz (2011) conducted and analyzed interviews to find common themes in student responses as they related to characteristics of the items. She compared the performances of older (grades 6-7) and younger (grades 2-4) students on co-splitting tasks (again, still under the covariation name and understanding of the construct), and she also explored the differences in their strategies. The five strategies seen and their distributions seemed to indicate that younger students gravitated towards unsuccessful and additive strategies (including identifying and using individual differences between quantities), whereas the older students much more frequently relied on splitting and covariation strategies. It was not surprising that older students also performed slightly better than younger students on such tasks and used the more complex strategies. However, the age/grade gap between these two groups could have been pivotal, in that the upper elementary grades (4-5) is exactly where fluency in multiplication and division are believed to develop, thereby making it a logical assumption that most middle-grades students would be operating at or above that level of fluency. In Yilmaz's descriptions of the results, there appeared to be a gap in the terminology and explanations of the different strategies that

mimicked the gap in grade levels between younger and older students. This suggests that intermediate strategies may exist that did not surface with her tasks and/or the participants in the two age ranges in the study.

There were two lingering concerns around these studies of the equipartitioning learning trajectory, and co-splitting items in particular: 1) the way in which the items presented the problem did not explicitly or accurately support the current understanding of co-splitting within equipartitioning, and 2) there was no comparison group of students in either the field test or the clinical interviews that had received exposure to a curricular treatment of any of the levels of equipartitioning. The first of these concerns was that items included required students to determine equivalent ratios that were larger and smaller than a given ratio, therefore not preserving the quantities as “wholes” and as is indicative of equipartitioning tasks. This was partially addressed by the creation of the IGE for co-splitting in LPPSync, which now encapsulates the current definition of co-splitting within the equipartitioning LT.

Both concerns were also partially addressed by recent clinical interviews (Corley et al., 2012) with third-graders who had been introduced to equipartitioning of collections and wholes in a two-week pilot teaching experiment that also utilized LPPSync in an actual classroom setting. In these interviews, paper-and-pencil items that carried the same problem stem as is presented in the LPPSync IGE, and utilizing Streefland diagrams, were used. The analysis of the data resulted in a more detailed set of strategies (see Table 2) that were also ranked in order of theoretical sophistication. The delineation of these strategies focuses on

the ways in which students determine a co-split to be used for a given set of parameters, as is indicated by their choice of a number of groups in a fair sharing problem or how those groups are formed through splitting actions. The second concern is also intended to be further addressed and informed by the current study, in that the students in this teaching experiment had received a full treatment of the lower levels of the equipartitioning learning trajectory that precede the level of focus – co-splitting.

A set of curriculum materials (Confrey, 2011) based on the equipartitioning learning trajectory was created and consisted of seven Packets in order to align with the original seven e-Packets created for LPPSync. During the year preceding the current study, our team conducted a design study in a classroom-like setting with a group of second-, third-, and fourth-graders from a local community center. This two-week experiment covered the first four Packets of the curriculum and also utilized the four corresponding e-Packets on LPPSync. Retrospective analysis of the teaching experiment and use of an IDAS in this type of setting informed the team about students' interactions with several key ideas at the lower levels of the trajectory (Confrey et al., 2012). The results from these analyses were used to inform revisions to the curriculum and activities for the lower levels of the trajectory that were made prior to this study.

*Table 2.* Student strategies on co-splitting tasks.

Strategy	Description	Student Exemplar
Guess	Students determine the number of groups randomly – do not specify mathematical reasoning (often an initial approach)	
Incremental Adjustment	Students determine the number of groups by adding/subtracting tables from previously already attempted numbers	
Factor-based Adjustment	Students determine the number of groups by multiplying/dividing tables from previously already attempted numbers	
Split and Check	Students determine the number of groups by splitting (partitive) one quantity, and then check the same split on the other quantity	
Partitive Co-split	Students determine the number of groups by splitting (partitive) both quantities	
Inverse Co-split	Students determine the number of groups by splitting (doubling or n times) both quantities	
Combining Like Ratio Units	Students determine the number of groups by combining like unit ratios, ratio units, or compound ratio units, to form equivalent fair shares	
Hypothesized: Combining Unlike Ratio Units	Students determine the number of groups by combining unlike unit ratios, ratio units, or compound ratio units, to form equivalent fair shares	<i>Continuing from the above scenario – Students could create two tables: one with 3 pizzas and 2 people and the other with 9 pizzas and 6 people (by combining one table with 3 pizzas and 2 people and the table with 6 pizzas and 4 people).</i>

Although the curriculum presents a variety of other contexts to students (e.g., paper-folding) as equipartitioning problems, the first and most prevalent of these is the context of fair sharing. The notion is appealing because it is accessible even to young children who have

had experiences with sharing in their everyday lives. Realistically however, the context limits the values of the quantities that can be used such that at least one quantity (the one representing the number of sharers) must likely be a whole number. More broadly, the nature of equipartitioning contains the goal of creating equal-sized groups, and groups are a countable quantity that can only take on whole-number values. This has not been seen to hinder students from being able to solve the problems, or from determining and working with both the base ratio (smallest whole-number pair of values representing an equivalent ratio) and the unit ratio for sharers (equivalent ratio per one sharer) in context. The limitation is only realized in that where there are typically two unit ratios associated with a splitting problem, while there is only one realistic unit ratio in a fair sharing problem, which *is* the fair share (i.e., objects per one sharer). One possible inscription we have developed for working with these types of problems is the fair-share box (see Figure 6a), where one cell always contains a 1 and the multiplicative relationship between rows can be represented as the product of two whole numbers. This precedes the multiplication and division box (where one cell still must contain a 1, but the multiplicative relationship between rows can now be represented as the product of a whole number and a rational number; see Figure 6b), the ratio box (where all of the cells can take on any values and the multiplicative relations can be represented as the product of two rational numbers; see figure 6c), and the ratio table (in which all of the criteria of a ratio box apply, but it can include more than just two pairs of related values – more than two rows; see Figure 6d).

In working with these types of fair-sharing problems, students often make statements involving the fair share (e.g., each person gets 3), but they typically do not equate the relationship in the fair share as representing another equivalent fair-sharing problem itself (i.e., 3 objects fairly shared among 1 sharer *is the same as* 12 objects fairly shared among 4 sharers, as compared to 12 objects fairly shared among 4 sharers results in 3 objects per sharer). The focus is on describing a fair-sharing problem in a different way – referencing the fair share of each sharer as opposed to stating the situation (and ratio relationship) as given, including the total numbers of objects and sharers – and therefore the fact that the two situations are equivalent is not explicit. There is also another limitation to fair-sharing situations that make them different from ratio tasks, in that the parameters are closed. In other words, the given numbers of objects and sharers are what is available to work with and not only must they all be used, but no additional objects or sharers can be introduced (e.g., the equivalence of fair shares between 6 objects shared among 4 sharers and 12 objects shared among 8 sharers is not an applicable result when given only the 6 objects and 4 sharers to work with).

In splitting tasks in general, neither of these limitations on the quantities exists and therefore a greater variety of problems can be presented with a wider array of possible solutions. In particular, students could be asked to find individual pairs of values that are equivalent to a presented pair of values. These singular sets of numbers as solutions no longer require that the original totals still be included in some way, and solutions could then extend beyond those original totals to include larger numbers. For instance, when asked to

determine and represent an equivalent situation to 8 candy bars costing 6 dollars, a student could respond with 4 candy bars costing 3 dollars (and need not consider that two sets of those would be necessary to fully replicate the given situation, as would be required in a fair sharing problem). Likewise, the student could also respond with 16 candy bars costing 12 dollars.

Number of Objects	Number of Sharers
12	4
3	1

(a)

Number of Objects	Number of Sharers
6	4
$\frac{3}{2}$	1

(b)

Number of Objects	Number of Sharers
$\frac{1}{3}$	$\frac{1}{18}$
$\frac{3}{2}$	$\frac{1}{4}$

(c)

Number of Objects	Number of Sharers
12	8
6	4
3	2
$\frac{3}{2}$	1
1	$\frac{2}{3}$
2	$\frac{4}{3}$

(d)

Figure 6. (a) Fair-share Box for 12 objects and 4 sharers. (b) Multiplication and Division Box for 6 divided by 4. (c) Ratio Box for equivalent ratios of  $\frac{1}{3} : \frac{1}{18}$  and  $\frac{3}{2} : \frac{1}{4}$ . (d) Ratio table for subset of ratios equivalent to 12 : 8.

Problems such as these attend more to the notion of determining equivalent forms of ratios; and by using ratio boxes or ratio tables, students should be asked to interpret the multiplicative relationships between each row (the co-splitting relationship) as well as those between each column (the correspondence relationship). Such activities are not included in

the equipartitioning curriculum, but it is believed that if students become proficient at the upper levels of equipartitioning as defined by the learning trajectory, then a move to solving these types of problems would not be a difficult one. In addition, the types of inscriptions mentioned would be a good means to aid students in their understanding.

It is my conjecture that there is a difference between these moves beyond equipartitioning into splitting and early ratio reasoning, and the move to a full understanding of ratio. Identifying equivalent ratios is the basis on which ratio comparison can then be developed and the ideas of density and unit rate can be explored. It is with the addition of comparison, particularly with these last two concepts, that a student becomes proficient with formal ratio reasoning. Graphing ratio relationships on the coordinate plane is a useful inscription for exploring ratio comparisons, and would also aid students in relating these ideas to slope and similarity later, but is not a useful method for operating with equipartitioning and co-splitting problems as the sense of a number of equivalent groups is not evident.

### Summary

In this chapter, I have provided the theoretical frameworks that informed the design of this study, and reviewed and synthesized the areas of literature relevant to the study – division and multiplication, contextual problem solving, ratio and proportional reasoning, and equipartitioning. It is from these bodies of literature that areas in need of further investigation were identified in order to progress the work forward, and within which I

situate the study of co-splitting. In the next chapter, I will describe the methods that were chosen and used to carry out the study.

## CHAPTER 3: METHODOLOGY

### Overview

This research was conducted as a case study. Creswell (2007) describes case study research as a methodology for studying an issue through exploration within a bounded system, such as a setting or context, involving multiple sources of data to report descriptions and themes. In this study, the procedures for case study research laid out by Stake (1995) and Yin (2009) were considered, by first identifying this research as an instrumental case study, followed by extensive data collection for holistic analysis, and then interpreting the meaning of the case. The use of a qualitative case study aligns with the specific implications for such research in the field of education – the value of context-dependent knowledge, an understanding that formal generalization is often not possible, nor desired, and the importance of generating and testing hypotheses within a bounded system – exemplified by Merriam (1988, 1998). As this research focused on the study of a single case, and that case was a group (or class) of students participating in a teaching experiment, there were other methodological implications specific to this type of case to be considered.

According to the National Research Council (2002) report *Scientific Research in Education*, the research question should drive the design of a study. The report also suggests that when exploring and examining a theoretical mechanism to explain how a specific topic in mathematics is learned by students through instruction, a design study or design experiment (Brown, 1992; Collins, 1992; Cobb, Confrey, diSessa, Lehrer & Schauble, 2003),

or a teaching experiment (Lesh & Kelly, 2000; Schoenfeld, 2002) is appropriate. It goes on to characterize these studies as a means to generate hypotheses.

Cobb et al. (2003) described five “crosscutting features” of design experiments: 1) they are highly interventionist, 2) they employ an iterative design, 3) the purpose is to develop theories about the process of learning and that which has been designed to support that learning, 4) they are both prospective and reflective, wherein the theories are scrutinized and if refuted, alternative paths for learning are embraced, and 5) theories developed during the experiment are “humble,” because they deal with processes for domain-specific learning and are subject to the activity of design. The “humble theories,” or conjectures, herein were directly linked to the research questions and they reflect any assumptions made about the starting points for learning. These conjectures were specific to the content domain of equipartitioning, but also to the design of the study itself, and therefore they were subject to scrutiny and modification both while the experiment was taking place and after its completion. In describing best practices for conducting a design experiment, Cobb et al. (2003) stated:

As part of the process of preparing for a classroom design experiment, the research[er] also specifies [his or her] assumptions about the intellectual and social starting points for the envisioned forms of learning. To achieve the instructional agenda, the [researcher] identifies current student capabilities, current practices, and other resources on which it might be able to build. In relatively well-researched domains, the [researcher] can draw on the literature to develop conjectures about students’ initial interpretations and understandings. However, in less researched areas, the [researcher] typically needs to conduct pilot work to document these understandings and, thus, the consequences of students’ prior instructional histories. (p. 11)

The first of these points – specifying assumptions – was addressed in preparation for the experiment through a reliance on past work of the research team in studying the equipartitioning LT and validation of the order of the proficiency levels resulting from that work. It was therefore assumed that students who had sufficient experiences with all of the lower levels of the LT would be intellectually situated at the appropriate starting point to engage with topics at the upper levels that constituted the intended forms of learning in the experiment. The second point – identifying student capabilities, and instructional practices and resources – was addressed in three ways:

- 1) The students participated in a workshop that introduced them to all of the prerequisite lower levels.
- 2) It was assumed that the topics covered by the equipartitioning curriculum in the workshop and teaching experiment are not widely taught in classrooms, and thus the students had no prior, outside experiences with the content.
- 3) The LPPSync IDAS was a resource for classroom instruction and assessment, and its use was an intended part of the design.

Finally, to address the third point – students' initial interpretations and understandings of the content – the clinical interviews leading to and findings from the paper-and-pencil field test on equipartitioning (Pescosolido, 2010; Confrey, Rupp, Maloney & Nguyen, in review), and co-splitting in particular (Corley et al., 2012), were used to inform the study.

In this study, some ideas from the form of design research introduced by Confrey and Lachance (2000) called transformative and conjecture-driven teaching experiments were also

adopted. Although, the authors expressed seeing “the teaching experiment as a planned intervention that takes place over a significant period of time in a classroom where a continuing course of instruction is taught” (p. 239), which portrays a much greater scope of longitudinal work than the present study, it was believed that several ideas could be gleaned from this methodology that were applicable and useful in thinking about the framing of the study. First, it was stated that the conjecture(s) underlying the study should have a dual focus, in two separate but related dimensions: mathematical content and pedagogy. Furthermore, the conjecture(s) in such a teaching experiment must also be situated in a theory, which should inform and help to structure the activities and methodologies in the experiment. Lastly, the experiment should involve a dialectical relationship between each conjecture and the major components of instruction: the curriculum, method of instruction, role of the teacher, and methods of assessment.

The mathematical content dimension of this study consisted of posited foundations for ratio reasoning as situated in equipartitioning – specifically the co-splitting construct. The pedagogical dimension consisted of a learning trajectories approach to instruction (Sztajn, Confrey, Wilson & Edgington, 2012; Wilson, 2010). The cognitive theory of learning trajectories framed the overall study, and the curriculum (Confrey, 2011) was one our research group had created based on the equipartitioning LT. This curriculum consists of eight units (which we call Packets) that cover all sixteen levels of the trajectory, and is sequenced based on the eight corresponding e-Packets in our LPPSync IDAS (see Figure 7), which were developed first. Each of the eight Packets contains an overview of the

Mathematical Emphasis and Vocabulary, and one or more Lessons. All of the lessons begin with a Launch activity to be facilitated by the teacher, followed by another activity for students to Explore on their own or in small groups, a set of questions for class discussion to Summarize, and often an additional Extension or Supplemental Activity, or an Additional Practice project. Each lesson was designed for one or two instructional days depending on the depth of coverage.

<i>Proficiency Levels</i>	<i>Packet #s</i>							
	P1	P2	P3	P4	P5	P6	P7	P8
<b>16 - Generalization</b>								<b>P8</b>
<b>15 - Distribution</b>								<b>P8</b>
<b>14 - Co-Splitting</b>							<b>P7</b>	<b>P8</b>
<b>13 - Composition of Splits (MW)</b>							<b>P7</b>	<b>P8</b>
<b>12 - Multiple Wholes (MW)</b>								<b>P8</b>
<b>11 - Continuity Principle</b>						<b>P6</b>		
<b>10 - Property of Equality</b>						<b>P6</b>		
<b>9 - Reallocation</b>				<b>P4</b>				
<b>8 - Quantitative Compensation</b>				<b>P4</b>	<b>P5</b>			<b>P8</b>
<b>7 - Composition of Splits (SW)</b>					<b>P5</b>			
<b>6 - Qualitative Compensation</b>				<b>P4</b>	<b>P5</b>			<b>P8</b>
<b>5 - Reassembly</b>	<b>P1</b>	<b>P2</b>	<b>P3</b>	<b>P4</b>	<b>P5</b>			<b>P8</b>
<b>4 - Naming</b>	<b>P1</b>	<b>P2</b>	<b>P3</b>	<b>P4</b>	<b>P5</b>			<b>P8</b>
<b>3 - Justification</b>	<b>P1</b>	<b>P2</b>		<b>P4</b>	<b>P5</b>	<b>P6</b>		<b>P8</b>
<b>2 - Single Wholes (SW)</b>		<b>P2</b>						
<b>1 - Collections</b>	<b>P1</b>							

*Figure 7. Mapping equipartitioning levels to LPPSync e-Packets.*

Prior to the teaching experiment, the students participated in a three-day workshop (hereafter, the “workshop”), lasting ninety minutes each day and in which I acted as teacher-researcher. The workshop covered Packets 1-5 of the curriculum (levels 1-11 of the

equipartitioning LT), in order to gain an adequate understanding of the lower levels as prerequisites. Therefore, these Packets were covered at a much more rapid pace than was intended by the design of the curriculum, but the topics were also intended for younger students and thus the approach was deemed appropriate.

The teaching experiment for this study originally aimed to investigate levels 12-16 of the trajectory; but due to the contingencies of the methodology, which adjust the pace based on the emerging learning, the enacted teaching experiment ultimately focused on just one level – co-splitting (level 14) – and utilized the corresponding e-Packet on LPPSync. Levels 12 and 13 address equipartitioning multiple wholes and composition of splits on multiple wholes, respectively, and are covered as part of the eighth, and final, curriculum Packet. Therefore, these levels were not planned to be treated before co-splitting during the teaching experiment, and were subsequently dropped when the decision was made to focus only on Packet 7 and co-splitting for the entire two weeks.

I acted as teacher-researcher during both the workshop and the teaching experiment, facilitating the learning of the students by guiding them through the curriculum. The LPPSync IDAS was utilized to afford the students with opportunities to engage in activities that would further their understanding of the topics through practice on relevant problems, as well as provide the teacher-researcher with real-time information about the students' development and success. Instructional activities were varied and consisted of whole group discussion, small group activities, and individual practice. Students took paper-and-pencil pre- and post-tests on all eight packets in the curriculum, given at the end of the preceding

workshop and at the end of the two-week teaching experiment, as well as two paper-and-pencil assessments of multiplication and division knowledge, and a diagnostic assessment using e-Packet 7 (Co-splitting) in LPPSync (see Appendix A for full versions of all paper-and-pencil assessments).

#### Sample Selection

The sample of students was solicited as a convenience sample from a local organization hosting a weekday camp for children in the area throughout the summer of 2012. Our research team worked in conjunction with the director of the organization to advertise the opportunity and recruit potential participants, both targeting and capping the enrollment at twenty-four participants for several reasons: 1) to represent the size of a typical upper-elementary grades class, 2) to be able to gather an adequate amount of data, and 3) to account for attrition while maintaining a manageable number of students. Parent or guardian consent for both participation and video recording of the sessions was obtained per the Institutional Review Board's (IRB) policies, with relevant approval of the study from the IRB. The sample intended to include approximately equal numbers of students from each of the three targeted grade levels, but the actual breakdown of participants was 12 rising fourth-graders, 6 rising fifth-graders, and 6 rising sixth-graders; however, there was by chance, an overall equal number of females and males. Considerations of neither the students' prior educational background and experiences, nor their ethnicity or family's socio-economic status were used for the purpose of variation or any other, in selecting the sample.

Consent and intent to participate was obtained from twenty-four students initially, but four did not show up at all for the workshop, and were therefore dropped from the remainder of the study. In addition, two students chose to drop out during the workshop, and three others displayed sporadic attendance over the three days of the workshop (missing one or two of the three days). These latter three would not have met the assumptions for the two-week teaching experiment sufficiently, but they chose not to continue on their own and were therefore dropped from the study. This left the sample size at 15 students for the teaching experiment, but unexpectedly, five additional students were not able to attend the second week of the teaching experiment after having attended the first week. This was due to various reasons related to the experiment being conducted over summer vacation (such as family vacations, enrollment in other camps, or returning home from out of town for the start of the school year). Therefore, ten students completed both the workshop and the full two-week teaching experiment – four rising fourth-graders (two female and two male), four rising fifth-graders (one female and three male), and two rising sixth-graders (one female and one male).

Students at these specific grade levels were selected for several reasons. First, the equipartitioning LT as a whole is posited to span instructionally from kindergarten to fifth-grade. Second, the uppermost level of the equipartitioning LT is aligned with a fifth grade standard, 5.NF.3 in CCSSM (CCSSO, 2010). Lastly, without asking too much from participants during their summer vacation, it was believed that in order to ensure the students would have the mental capacities to take in all of the lower levels of the LT during the three-day workshop, students any younger would not have been as well-suited.

Although co-splitting is at the upper end of the trajectory, it has been shown that third-grade students with some prior experience with equipartitioning are capable of operating at that level (Corley et al., 2012). Therefore, the decision needed to be made as to whether co-splitting would be pulled up to higher-grades students, or equipartitioning multiple wholes would be pushed down to lower-grades students. It was decided to pull co-splitting up and work with older students for the reasons mentioned above; even though, these students would have most likely had a formal introduction to division and multiplication (and the sixth graders should be expected to possess some form of mastery of those operations). This was not expected to affect the overall learning of these students in a negative way, but it was recognized that it may interfere with their acquisition of some conceptual ideas of equipartitioning, and it could also make it difficult to differentiate where in the trajectory students were operating if they relied heavily on division and multiplication facts. I believed that students could develop an understanding of all the ideas in the trajectory prior to division and multiplication (or at least in parallel, for the upper levels). I believe that the decision to work with the older students still fits well within the purpose of this study, even though it implies that no inferences can be made about how lower level (or lower grade) students, without formal experiences with division and multiplication, would perform with upper levels of the LT.

The site and sample for this teaching experiment exhibited some challenges. First, the teaching experiment was conducted in the context of a summer program, and as a result the attendance was voluntary despite the team having obtained parental and student

commitments to attendance in advance. As a result, the study had significant amounts of missing data on assessments and a reduced sample size over time. These problems were anticipated to a degree, but the placement of the study in a setting where students were likely to be of lower socio-economic status was selected to reach students who would benefit the most from additional instruction on the topic. Secondly, the teaching experiment was conducted outside of a regular classroom environment, and therefore the students interpreted the stakes as low. Thus, the effort they chose to put forth suffered on many occasions and the seriousness with which they took their learning was minimal at the beginning. A large effort was devoted to building improved classroom norms and an orientation towards discourse. Although it took a while, and the students needed to be encouraged to participate and share ideas, and reminded of rules and procedures regularly, the group as a whole made major strides over the course of the thirteen days in the program.

### Research Questions

The purpose of this research was multifaceted, and at two levels of grain size. At the more detailed level, the study set out to investigate students' movement through the equipartitioning LT in a teaching experiment setting, to inform understanding of student strategies used at the upper levels of the equipartitioning LT as related to the lower levels, and to articulate differences seen in those strategies based on the parameters of curricular and diagnostic items. At the broader level, the study aimed to provide insight for both LT research in general, and specifically, the development of LTs and related models for learning trajectories-based instruction. Furthermore, the study was intended to identify the challenges

in implementing such instruction from a design research perspective, and translate those issues for use in the actual classroom.

The three research questions formulated to address these purposes were as follows:

1. How do the lower levels of the equipartitioning learning trajectory have an impact on student interactions with and success on the upper levels – particularly co-splitting?
2. How does students' knowledge of multiplication and division interplay with their learning and understanding of equipartitioning at the upper levels of the learning trajectory?
3. How does students' strategy use and performances on co-splitting tasks differ when the number of objects is greater than the number of sharers versus when the number of objects is less than the number of sharers?

#### Initial Conjectures

Multiple conjectures were formulated around each of the research questions, and each of these was revised when appropriate during and after the teaching experiment. Some emergent conjectures were also formulated during the teaching experiment based on observations of the students during the teaching experiment. In this section, I describe the initial conjectures and the rationale behind each for the three research questions.

#### *Research Question 1 Conjectures*

It was believed that students who were proficient with the lower levels (1-11) of the equipartitioning LT would be adequately prepared to become subsequently proficient with the upper levels (12-16), and specifically co-splitting (level 14). Particular sets of knowledge

and skills acquired at the lower levels were also believed to be useful and pertinent for achieving success and sophistication in working with the upper levels. Table 3 shows the predicted relationships between each of the lower levels and the major upper level idea of co-splitting that was the focus of the curriculum in this teaching experiment. Knowledge of the existence and nature of these connections is important for the study of LTs as further validation (not statistical validation) of the ordering and appropriateness of each level, as well as for informing the development of curriculum materials around an LT.

*Table 3. Conjectured relationships between equipartitioning lower levels and co-splitting.*

Proficiency Level (#)	Relevance to Co-Splitting ( $n$ objects and $p$ people)
Equipartitioning Collections (1)	In justification of equivalent fair sharing situations: When $n > p$ , students may deal in rounds and one-to-one correspondence to determine the fair share.
Equipartitioning Single Wholes (2)	In justification of equivalent fair sharing situations: When $n < p$ , or once the remaining objects becomes less than $p$ , students may share each whole for the number of sharers to determine the fair share.
Justification (3)	In justification of equivalent fair sharing situations: When more than one situation with identical quantities for objects and sharers are in question, students may relate justifying fair shares by saying “they all got the same” with “every group of the same size got the same.”
Naming as a Relational (4)	In justification of equivalent fair sharing situations: Students may determine the fair share for each and use their knowledge of the names of fractional parts in so naming it to assert equivalence.
Reassembly (5)	In justification of equivalent fair sharing situations: When the given situation is being compared to several situations with lesser numbers of objects and sharers, students may claim that they can all be reassembled to create the given situation. In creating equivalent situations: Students may partially reassemble groups of smaller situations to determine new ones with intermediate values.
Qualitative Compensation (6)	In creating equivalent situations: Students may first determine whether the quantities should be greater or lesser in moving from one situation to another.
Composition of Splits (7)	In justification of equivalent fair sharing situations: Students may relate equivalent fair-sharing situations that have been arrived at through a composition of splits to the original situation as being the same as a single split.
Quantitative Compensation (8)	In creating equivalent situations: Students may determine the number of objects and the number of sharers that will create an equivalent fair-sharing situation based on co-splitting and then go on to determine values based on multiplicative changes related to the splits used.
Reallocation (9)	In creating equivalent situations: Students may combine already known equivalent fair-sharing situations by reallocating the quantities from one or more fair-sharing situations to the others, equipartitioning each of the quantities and then dealing them evenly.
Property of Equality of Equipartitioning (10)	None predicted.
Continuity Principle (11)	In justification of equivalent fair sharing situations: Students may rely on the notion that a single whole can be fairly shared for any number of sharers and therefore not need to determine the fair share itself.

*Research Question 2 Conjectures*

Consistent with the belief that concepts in the equipartitioning LT develop prior to and in parallel with the concepts of multiplication and division, it was believed that students of all ability levels with respect to multiplication and division would be able to successfully solve co-splitting problems, based on their understandings of, and proficiency with, lower level ideas in the trajectory alone. However, at the time of interaction with the upper levels, such as co-splitting (standard 5.NF.A.3 in CCSSM), students should have been introduced to multiplication and division formally (third grade in CCSSM), and therefore it was believed they would employ some of that knowledge in solving co-splitting tasks. It was also believed that students would rely on multiplication and division facts more frequently as an entry point into the task as the values of the parameters involved in co-splitting problems increased, and particularly when the problems were presented without a context.

I conjectured that multiplication and division facts would be used during the teaching experiment predominantly as justification of co-splitting solutions, and not necessarily as a means of determining those solutions, based on what had similarly been seen in prior clinical interviews (Corley et al., 2012). I also believed that students displaying weaker conceptual understandings of multiplication and division would still use facts as frequently as students with stronger conceptual understandings, but in a noticeably more rote and procedural way to justify solutions for co-splitting problems; and when facts were unavailable to students, they would struggle to attain solutions at all. The understanding of the interplay between content domains and LTs, especially based on their treatments in CCSSM, is critical in structuring

student experiences and harnessing appropriate prior knowledge, and recognizing when parallel development is needed.

### *Research Question 3 Conjectures*

We believed based on the equipartitioning field test and the pilot clinical interviews that tasks where the number of objects was greater than the number of sharers were the only type for which co-splitting was applicable, because co-splitting (then covariation) was originally identified and witnessed as a strategy for equipartitioning multiple wholes problems, and it was only seen on these such tasks. Therefore, our previous clinical interviews on co-splitting (Corley et al., 2012) also only used tasks with that configuration of parameters. With further study of co-splitting, I conjectured that students could apply similar strategies to problems in which the number of objects was fewer than the number of sharers. However, students using a deal and split strategy (see Figure 4b, pg. 42) in the previous clinical interviews often based their chosen split on the number of sharers and then proceeded to deal the objects to “test” whether the same split was viable for the objects. Thus, in a task with fewer objects than sharers – for example, 4 objects among 6 sharers – this would not work unless the students considered splitting the objects first, which contextually could cause some dissonance. Therefore, it was believed students would adjust the deal and split strategy to focus on the number of objects first, for one of two reasons: 1) the students believe that if 4 can be split, 6 can be split (an application of the Continuity Principle of equipartitioning) – regardless of whether the 4 represents the sharers, or in this example, the objects, or 2) the students interpret the task as similar to equipartitioning collections and envision establishing

the 4 first and then dealing the 6 to those groups – again, regardless of whether the 4 represents the sharers or the objects.

I also observed from pilot clinical interviews that students with some experience on the lower levels of the equipartitioning LT tended to want to actually determine a single person's fair share, which corresponds to the unit ratio. They did this not only for the given situations (e.g., 12 objects shared among 8 sharers results in one and one-half objects per sharer) in co-splitting tasks, but also for all equivalent situations they created (e.g., two groups of 6 objects shared among 4 sharers both also result in one and one-half objects per sharer). However, it was often used to justify how they knew the fair shares would be identical across a variety of arrangements in those particular tasks. Only infrequently did students recognize that from the unit ratio, they could create other equivalent arrangements by combining unit ratios; that is, they could then compose any number of sharers, from 1 to the given total systematically (each also associated with the determined number of objects), to form new equivalent arrangements.

For co-splitting tasks with different contexts, and questions that include asking for the base ratio explicitly (not by name, but as the desired type of answer – the smallest pair of whole number values for each quantity, I predicted that students would be able to determine those values on demand and not just by chance. I also predicted that this new treatment would help them recognize the significance of the base ratio in finding equivalent situations, and perhaps draw more attention to the similar significance of the unit ratio. The implications of how students interact with these different parameter variations, problem types, and

contexts inform other upper levels of the equipartitioning LT, particularly multiple wholes, and whether those concepts and co-splitting actually develop in parallel.

### Methods

A design experiment was conducted with rising fourth-, fifth-, and sixth-grade students in the form of a two-week teaching experiment (hereafter, the “teaching experiment”) that covered the upper level of co-splitting from the equipartitioning LT. Prior to the teaching experiment, the students participated in a three-day workshop that covered the prerequisite lower levels of the equipartitioning LT, utilizing curriculum (Confrey, 2011) Packets 1-5. After a one-week break, the teaching experiment began. During the teaching experiment, individual clinical interviews were conducted with selected students (discussed later) to elicit further information of individual student progress and understanding.

The workshop culminated with the students taking a paper-and-pencil pre-test, covering all eight of the equipartitioning curriculum packets, and therefore all 16 levels of the equipartitioning LT. An identical paper-and-pencil post-test was administered for comparison at the end of the two-week teaching experiment. The teaching experiment continued for ninety minutes each day, over ten days. I fulfilled the role of the teacher for the teaching experiment as well, but two other members of the research team acted as classroom assistants. These assistants dealt with management and logistic issues, such as distributing materials and handling technological issues, and interacted with students in their small groups, to ensure all received support equal to that which would be afforded by a more lengthy treatment of the lower levels of the trajectory.

As stated previously, an underlying conjecture of this research was that students should have adequate knowledge of and experience with all of the topics and skills identified at the lower levels. In case students' mastery of the content from the workshop was not sufficient, an option to segment the students into two groups during the teaching experiment was considered. It proved unnecessary, in part due to a natural attrition of participating students over the teaching experiment.

Curriculum Packets 7 and 8 were expanded and modified to provide for a more thorough treatment of the topics based on information gathered from both the field test and recent clinical interviews. A preliminary schedule was devised for the both workshop and the teaching experiment, but it was modified as needed based on the pace of learning of the group. Initially, I planned to use the two curriculum Packets in their entirety, introducing the students to the Co-Splitting Packet during week one, and the Equipartitioning Multiple Wholes Packet during week two.

The first major instructional modification affecting the teaching experiment came after the last day of the workshop, when only the curriculum materials designed for levels 1-10 of the equipartitioning LT had been covered. The research team and I decided that the first day of the teaching experiment should be used to cover the remaining lower level – the Property of Equality of equipartitioning (PEEQ; level 11), as it was important to the assumptions of the study for the students to get a full treatment of all the lower-level topics. To utilize all of the time we had with the students and provide a smooth, yet definite, transition between workshop topics and teaching experiment topics, the remainder of that

first day culminated with the students working in groups to create videos on flip-cams about key aspects of the lower levels.

Another major instructional decision came after the sixth day of the teaching experiment. At that point, the students were still fully engaged with the concept of co-splitting and had not yet displayed desired levels of understanding by and large. Therefore, I decided that the equipartitioning multiple wholes pieces of the curriculum (related to levels 12, 13, 15, and 16 of the equipartitioning LT) and corresponding e-Packet 8 on LPPSync would not be used during the remainder of the teaching experiment. Therefore, equipartitioning multiple wholes was eliminated from the overall study although it was explored tangentially through clinical interviews with some students over the course of the second week, and with all of the students one day before class began.

The co-splitting curriculum Packet representing the focus of this study and the corresponding LPPSync e-Packet are described in the section that follows.

#### *Curriculum Packet 7: Co-Splitting*

There are also two main distinctions between co-splitting tasks and those students encountered in earlier packets for fairly sharing collections and single wholes: 1) there is more than one whole, but it is not considered a collection because each of the wholes is continuous (able to be split), and 2) these tasks do not require determination of the fair share directly, but rather the creation of two or more fair-sharing situations that are equivalent, through the invariance of the fair share. Proficiency with co-splitting implies that students are able to determine fair-sharing situations that are equivalent to a given situation by splitting

simultaneously both of the quantities of objects and sharers. While these tasks do involve fair-sharing multiple wholes among two or more sharers, as is the fundamental task for the upper-level construct of equipartitioning multiple wholes, co-splitting does not require that students find one person's fair share (or the unit ratio of "per person").

All of the tasks in this Packet, and the corresponding level of the equipartitioning LT, are designed such that the larger-valued quantity (objects or sharers) is not evenly divisible (in the whole numbers) by the smaller-valued quantity. The original curriculum Packet only contained one lesson, and the Launch activity required students to determine multiple ways to allocate a number of fish and a number of food pellets into bowls such that each fish gets the same fair share as it would if all of the fish and food were placed in one bowl. The students were also asked to justify how they determined the number of bowls and justify that each fish would be able to get the same fair share. The Explore activity continued the exploration of this type of problem, having students determine more than one other way to arrange a variety of different numbers of fish and food pellets.

*Modifications to co-splitting curriculum.* The original curriculum was updated for the teaching experiment. All language was revised to use the term co-splitting, properly and consistent, in place of the earlier term "covariation" which was too broad. Additional tasks were added to provide examples where the number of objects is fewer than the number of sharers, but still adhering to the rule that the greater quantity was not divisible by the lesser quantity (in the whole numbers). These types of problems were added because they had not yet been studied within the clinical interviews based on the present definition of co-splitting.

Additional tasks were added asking students to find the smallest-valued combination of objects and sharers that are both whole numbers, in order to target explicitly their knowledge of the base ratio. This was done because although students have been observed to find such values in working with co-splitting tasks, the nature of the tasks had not placed any importance on the base ratio. Therefore, students did not recognize the base ratio as being any different from other sets of values making up part of an equivalent arrangement. It was believed that students might alter their strategies when it was made more explicit for them to determine this pair of values, or that they may even discover how to use the base ratio to determine other equivalent arrangements. Lessons were created around all of these newly added tasks and contexts (see Appendix B for full versions of all student worksheets), which are discussed further below.

*Co-splitting tasks.* The context of the first set of tasks presented to the students was that of pellets being shared among fish in bowls, and the idea of overcrowding was emphasized to motivate the need for more than one bowl (Figure 8). The first parameters were 10 pellets to be shared among 4 fish, and these parameters were chosen because co-splitting tasks where there are more objects than sharers have also been shown to accommodate a wide array of possible strategies, making scaffolding easier. For instance, if necessary, the students are able to deal the objects to the sharers in order to determine equivalent fair sharing situations or to justify the equivalence of the fair shares. The parameters were also chosen because 2-splits on co-splits proved to be most intuitive, just as 2-splits have been the easiest in single splits; and that would be the only viable co-split resulting in whole number

quantities (fish and pellets). In initial equipartitioning tasks for collections or wholes, the use of 4 as a parameter can be confounding in helping students to realize the multiplicative structure of the concept, since both 2 plus 2 equals 4 and 2 times 2 equals 4. However, this is easier to confront in co-splitting tasks because the confounding additive structure is unlikely to apply across both quantities. Therefore, these parameters were purposefully chosen for the first task so as to elicit the expression of additive misconceptions if they existed. I planned to address these early in order to help students recognize co-splitting tasks as demanding the coordination of both quantities.

You have a group of fish in one bowl and it is too crowded. You need to split the fish into more than one bowl and still feed them a fair share of the food pellets.

**Directions:** Draw a possible combination in the space provided and then answer the questions that follow.

1. There are 10 pellets and 4 fish to be fed.
  - a. How many bowls did you use?
  - b. How many pellets are in each bowl?
  - c. How many fish are in each bowl?

*Figure 8.* Sample co-splitting task: Sharing pellets among fish in bowls.

This first task was more heavily scaffolded than all later tasks in that the number of groups to be used (i.e., the split) was pre-determined for the students in order to get students accustomed to the idea of co-splitting. Here, the emphasis was not on finding a number of bowls but merely on maintaining an equivalence of fair shares between two different situations.

The next parameters were 24 pellets to be shared among 18 fish, a task which I constructed late in the day to align with a closing activity I planned involving the students snack. The other parameters used with this context included 12 pellets being shared among 8 fish, 18 pellets being shared among 9 fish, and 16 pellets being shared among 12 fish. These were all chosen because they each shared more than one common factor, and in all of the problems those common factors were the same twice over (2 and 2 or 4, and 3 and 3 or 9). A third set of parameters with 18 pellets and 9 fish was chosen because it did not involve a no 2-split, and it yielded an arrangement with just 1 fish in each bowl, linked to sharing collections and anticipating the unit ratio.

The context used with the second set of tasks involved sharing pizzas among people at tables (Figure 9). This context was taken directly from the LPPSync, and again the need for more than one table was set using the idea that a restaurant may not have tables large enough to seat a relatively large number of people, or hold a relatively large number of whole pizzas. The first parameters were 28 pizzas and 12 people, which still contained the possibility of either 2-co-splits or 4-co-splits, but potentially resulted in odd values for both pizzas and people at each table. This transition in media also introduced splitting trees as both a means to solve the problems as well as organize multiple solutions to the same problem, as this represented the first time students were asked to determine more than one arrangement within the original task statement. Although the students already recognized that these types of problems could have more than one possible solution, they had mostly been

working only from the original parameters and essentially determining each arrangement as a separate problem and not forming connections among solutions of their own or of others.

The first problem with pizzas and people was presented on LPPSync to get students familiar with the IGE, but students found it relatively difficult and whether due to the change in context, the novel technology, or a combination of both. The second parameters were 24 pizzas and 18 people, which provided an opportunity to see if any differences in strategies or solutions arose when students worked with the same parameters, more experience but the task is placed a different context. This problem was presented on a worksheet alone, but then students were given the next problems as worksheets and also allowed to use LPPSync to complete their work and determine solutions. The third set of parameters, for which students could use both the worksheet and LPPSync, were 35 pizzas and 15 people. These values were chosen because there was only one possibility for determining a solution within the whole numbers – a 5-co-split. The next set of parameters on were 30 pizzas being shared among 24 people, which still offered the possibility of 2-co-splits, 3-co-splits, or 6-co-splits as the students had seen before but with different values.

The third context presented to the students was that of cookies being shared among people at tables (see Figure 10), only a subtle alteration to the previous context with pizzas and people at tables. The first parameters were 27 cookies and 18 children, which only allowed for the possibility of 3-co-splits or 9-co-splits. The second parameters were 24 cookies and 30 people, which now represented the first case where the number of objects was less than the number of sharers. The reason I expected this problem to be more difficult was

because students found problems involving multiple wholes more difficult to share when there were fewer wholes than people. Prior work suggested it was due to the absence of a strategy of dealing initial rounds.

A sports team arrives at a restaurant to order pizzas. The owner cannot provide a table large enough to fit all of the players and pizza they ordered. You must help the owner figure out how to seat some team members and serve some pizzas at more than one table, so that all of the players have a seat and all of the pizzas are served (without cutting them first – *whole pizzas only*), and every player at all of the tables gets the same share of pizza.

**Directions:** Show how the players could sit at multiple tables and the waiters could deliver all of the pizzas to those tables so everyone still gets a fair share of all of the pizza.

1. There are 28 pizzas needed for 12 players to get their fair share.
  - a. How many tables will you use? Explain your thinking.
  - b. Show *one way* how all of the players could be seated and pizzas could be served.
  - c. How many players are at each table? Explain your thinking.
  - d. How many pizzas are at each table? Explain your thinking.

*Figure 9.* Sample co-splitting task: Sharing pizzas among people as tables.

The last parameters used in this context were 36 cookies and 30 people – another set that allowed for the use of 2-co-splits, 3-co-splits, and 6-co-splits. These values were also chosen to: 1) allow students to apply repeated reasoning from previous examples, 2) keep the values within the range of accessible facts, without limiting the number of possible solutions, and 3) not overemphasize 2-co-splits as the only possible approach.

A group of children want to fairly share a package of large cookies. There is not a table in the room big enough for all of them. The teacher wants to give out all of the cookies right away so that no one will have to get up from their seats. Find other ways that the children could sit at more than one table and the teacher can pass out all of the cookies.

*Remember:* Each table must fairly share only the cookies given to them, and the teacher wants all of the children to be able to get the same fair share.

1. There are 18 children to fairly share 27 cookies.

*Figure 10.* Sample co-splitting task: Sharing cookies among people at tables.

No inherent context was presented for the fourth set of tasks presented to the students, and they were allowed to either work in an abstract way with just the numbers and any chosen representation or to create their own context, which could be one of those already presented. The focus of this set of tasks was on the use of the splitting trees and the recursive nature of compositions of splits; that is, that performing a second split on the result of a first split acts on *every* part resulting from the first split, as in perpendicular splitting actions on a single whole (e.g., creating 6 parts for a 2-split composed with a 3-split). In addition to the initial values for objects and sharers, the students were now also given that the desired solution involved a certain split (co-split) and/or resulted in groups that contained a certain number of objects or sharers, for each of the four tasks as follows: a) 24 objects, 18 sharers, 3-split followed by 2-split (Figure 11a); b) 30 objects, 36 sharers, 12 sharers in each group after first co-split, 5 objects in each group after second co-split (Figure 11b); c) 18 objects, 30 sharers, 6 objects in each group after first co-split, 5 sharers in each group after second co-

split (Figure 11c); and d) 15 objects, 35 sharers, 7 sharers in each group after first co-split, 9 objects in one group not resulting from a co-split (Figure 11d).

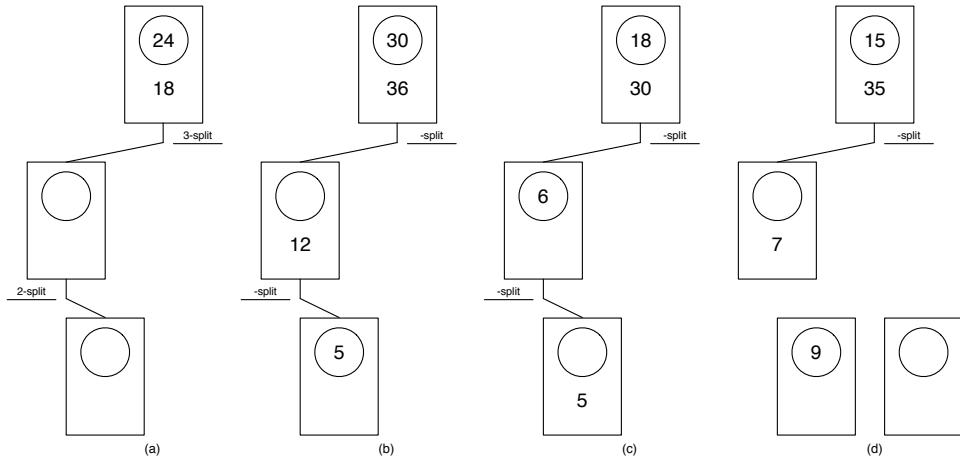


Figure 11. Sample co-splitting task: Tree diagrams.

The context of the fifth, and final, set of tasks presented to the students was that of orange drink mixtures being created from cups of orange juice and cups of water in pitchers (Figure 12), and similar to the pizza and cookie problems, the idea that pitchers large enough to contain the entire mixture may not exist was used to motivate using more than one pitcher. These types of tasks are similar to those used in previous studies of ratio and proportion (e.g., Noelting, 1980a,b; Kaput & West, 1994), and they provided an opportunity to observe the students' culminating, equipartitioning concept of ratio that was being developed over the course of the teaching experiment through all of their work with co-splitting tasks. There was no implied object or sharer in these tasks, but the diagrams portrayed the cups of orange juice as the objects and the cups of water as the sharers based on similar representations for pizzas and people in LPPSync. The students were also explicitly asked for the first time in these

tasks to determine the smallest whole number of cups of orange juice and whole number of cups of water that could go in one pitcher and still taste the same – targeting the base ratio.

The first parameters were 20 cups of orange juice and 8 cups of water, and with each of these problems a physical representation of both sets of cups was included along with the Streefland-like diagram resembling a pitcher. The next parameters were 16 cups of orange juice and 20 cups of water, then 36 cups of orange juice and 12 cups of water, and finally, 10 cups of orange juice and 25 cups of water. The third parameters with 36 and 12 cups involved a base ratio that was also the unit ratio, and the second and fourth parameters involved lesser values on top of the Streefland diagram, to give the impression of fewer objects than sharers.

An orange drink is made from a mixture of orange juice and water. Each recipe combines a certain number of cups of the orange juice with a certain number of cups of the water.

**Directions:** Show how the orange drink could be made in more than one pitcher, so that the orange drink in every pitcher tastes the same.

1. The orange drink recipe calls for 20 cups of orange juice and 8 cups of water. What are the smallest numbers of cups of orange juice and water that can go in a pitcher so that the recipe tastes the same?

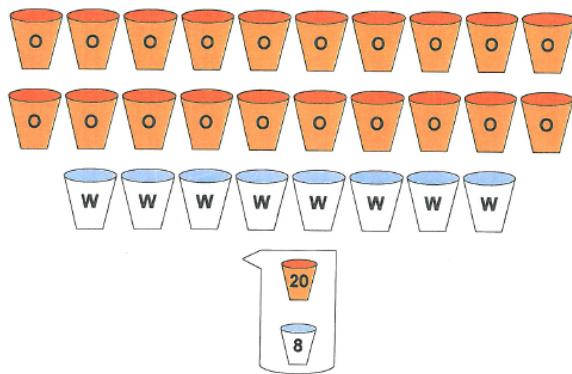


Figure 12. Sample co-splitting task: Orange drink mixtures.

#### LPPSync e-Packet 7: Co-splitting

The IGE for e-Packet 7 in LPPSync involves tasks in which students are given a number of objects to be fairly shared by two or more sharers, represented by icons on the screen as a number of pizzas and a number of people all seated at one table in the form of a Streetland diagram (see Figure 13a). Students are then asked to create one or more other arrangements using more than one table, whereby all of the people can be seated and all of the pizzas distributed such that all of the people still get the same fair share. This can be done in one of three ways: 1) splitting a table of pizzas and people into a number of identical smaller tables that each contain lesser amounts of both pizzas and people (Figure 13b), 2)

combining one or more tables of pizzas and people into one table that contains the sum of all of the pizzas and people from the tables being combined (Figure 13c), or 3) creating one or more tables independently with potentially varying numbers of pizzas and people at each table (Figure 13d).

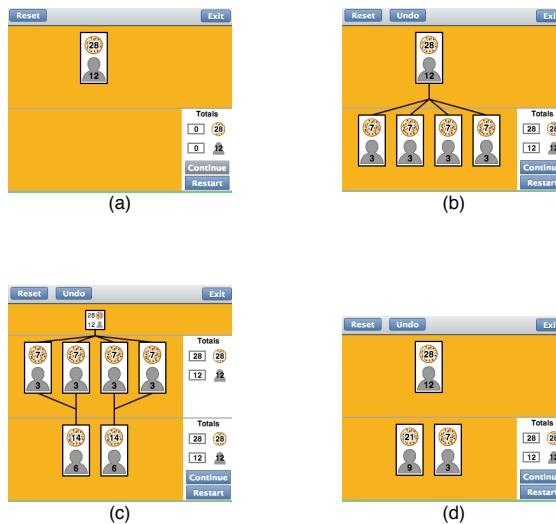


Figure 13. Screenshots from LPPSync e-Packet 7 (Co-splitting): (a) modified Streefland diagram, (b) the splitting tool, (c) the combining tables tool, and (d) the table stamp tool.

In any of these instances, the values of both pizzas and people at the tables that students can enter are limited to whole numbers. In the diagnostic assessment, students are given six problems, sampled from three levels of difficulty – two easy, two medium, and two hard. They are given feedback after completion on their abilities to construct equivalent fair-sharing situations using the same given numbers of pizzas and people. The parameters defining each level of difficulty are shown in Table 4.

*Table 4.* Sample space for e-Packet 7 (co-splitting).

$n \neq a \cdot p$ for any $a$	Easy	Medium	Hard
Number of Pizzas ( $n$ ) (minimum–maximum)	12–36	12–36	12–36
Number of People ( $p$ ) (minimum–maximum)	8–24	8–24	8–24
Number of Common Factors	1+	2+	2+
Equivalent Arrangements Required	1	2	3

### *Clinical Interviews*

Throughout the course of the teaching experiment, individual clinical interviews (Opper, 1977) were conducted with seven students in total. Each day interviews were conducted, the same interview protocol was used with two students (if available) that had demonstrated distinct differences in their strategy use or ability levels during instruction on previous days were selected for each interview. The purpose of the interviews was to explore the research questions more in depth and gain a better understanding of individual students' thinking by probing them further on unique ideas or strategies they offered during class, or providing them a space where theirs was the only voice, giving more opportunities to share their thinking. The interviews were semi-structured, using premade protocols that consisted of the tasks and prompts of interest (Appendix C). The protocols contained questions that would allow for a greater understanding with regards to the research questions, but as they were created during the teaching experiment rather than in advance, questions were added and altered each day based on events during instruction and the particular students being

interviewed. In order to gain an in depth understanding of emergent ideas, the flexibility to pursue other lines of thought during the interview was also part of the format. Each interview was limited to 45 minutes in duration, so as to focus the sessions and keep the students' engaged; and no student was asked to complete more than one interview session.

### Data Sources

The primary data sources for this study consisted of video recordings of class sessions and clinical interviews, collected student work, and notes compiled by observers from the research team and myself. I kept personal notes, as the teacher-researcher and from the point of view of the classroom teacher, on a daily basis while the experiment was ongoing to document the story lines of emerging themes relevant to the research questions and any others of interest that arose. These included instructional decisions made based on an evolving understanding of the learning processes of the students as it unfolded with respect to co-splitting, suggested changes to the curriculum Packets and LPPSync e-Packet, and suggested changes for future instruction. These notes also reflected alternate pathways to learning taken by the students, previously seen or not, and revisions made to the conjectures.

Immediately following each day of the teaching experiment, a debriefing session was held with all members of the research team present to observe that day's lesson to discuss implications for the next day's instructional plan and larger implications for the trajectory, curriculum Packets, and corresponding LPPSync e-Packets. Together, an additional set of notes on similar themes, plus any others the group believed merited discussion, was compiled. The teaching experiment sessions and clinical interviews were also video-recorded

for retrospective analysis. I watched the videos and transcribed the audio from the camera that followed me throughout, to provide a view of the class through the teacher's eyes and ears; and the clinical interviews were transcribed as well. I then overlaid both sets of notes with the transcripts, and considered any happenings relevant to the research questions, as well as those not captured in the notes. Finally, I synthesized all of the findings and generated a summary that includes a discussion of assessments, including pre- and post-test results, which were analyzed as discussed below. This discussion includes data from the paper-and-pencil tests, along with data from a diagnostic assessment session on co-splitting using LPPSync, and two other brief assessments given during the teaching experiment to acquire an understanding of students' multiplication and division knowledge. The same number and distribution of items across LT levels was used on the paper-and-pencil pre-test and post-test. All of the assessments were scored by me, as the teacher and as would be typical in a classroom setting. Most items were scored dichotomously as 0 or 1 for incorrect or correct, and all were scored according to a strict rubric. The results were not used to show growth, but rather as static statements of student knowledge at different time points during the study. Therefore, no additional or outside scorers were used, and for these reasons no sense of inter-scorer reliability was needed.

### Data Analysis

The analysis of this study aimed to answer the research questions, as well as inform the design of the corresponding curriculum, and provides suggestions for improvement. This was done through qualitative analysis and description of the "learning ecology" (Cobb et al.,

2003) that was created in the classroom setting where the curriculum and instruction were based on a learning trajectories approach to learning. As noted earlier, Cobb et al. (2003) stated that design experiments are both prospective and reflective. The authors go on to state that “it is therefore important that the [researcher] generates a comprehensive record of the ongoing design process … to support the retrospective analysis of the experiment” (p. 12). The data analysis in this study primarily reflects the two-week teaching experiment, and only references occurrences within, or understandings from, the workshop directly when data from those days were considered useful to make distinctions in students’ performances or strategy-use on co-splitting tasks. All other occurrences during the workshop were considered prior knowledge of the students to meet the assumptions entering the study, and therefore not part of the study itself.

After the conclusion of the teaching experiment, I took several passes through the data and transcripts to identify and code informative moments with respect to the research questions. Most codes were not created a priori, but developed as I inspected the data; however, each day and task (including parameters) were coded first to act as bracketing points and information to be overlaid with the findings within and across, and I also intended to code instances that related to each of the lower levels of the LT prior to studying the data, and in response to the first research question. On the first full pass through the data, I recognized the distinction between the uses of the lower levels as either occurring as part of the students’ work and strategies or as part of the justifications and explanations, so those cases were coded differently for each level. Additionally, I recognized several strategies and

instances related to equipartitioning multiple wholes and other upper levels of the LT as well, so those were coded accordingly. On the second full pass through the data, I coded the values of the splits used by students, which then led to other codes related to the second research question about multiplication and division knowledge such as recognized misconceptions and errors. On a third, and final, full pass through the data, I coded the amounts of correct and incorrect solutions offered for co-splitting tasks by pairs of parameters, and also coded implied and inferred levels of understanding of both co-splitting and multiplication and division by individual student. Each of the codes informed the analysis and synthesis of the findings, and I went back to the data as I formulated my conclusions to look for any other relevant pieces of information I had missed or forgotten that would either support or bring into question those conclusions.

Although this study employed qualitative methods, some quantitative data was included for the assessments, and therefore some statistical analyses were conducted. This was not a direct study of the curriculum itself, and thus there was no intention of making any comparative claims about the curriculum or the use of the learning trajectory in such a setting. Rather the data were included as a means for adding to the landscape of the overall setting and providing information about the student-participants. For all of the assessments, individual data are reported and one-variable statistics were calculated for the class as a whole. Additionally, for both of the pairs of assessments – multiplication and division, and the pre- and post-test – classic statistical analyses were conducted as proxies for student learning. These were done using appropriate parametric and non-parametric statistical tests –

paired *t*-test and Wilcoxon Signed-Rank Test – because the data were paired, the sample size was small, and the normality of the distributions were questionable. These simply aimed to inform whether the scores were significantly different between the tests.

### Summary

This chapter included an overview of the methodology with a description of the teaching experiment. It presented three research questions and the related initial conjectures. Finally, the specific methods were articulated, including a description of the sample, the data collected, and the means of analysis. In the next chapter, I will present the results of the teaching experiment as they pertained to these research questions and conjectures, and in the final chapter, I will state my conclusions and discuss the limitations and implications of the study, including future directions for this line of research.

## CHAPTER 4: FINDINGS

### Introduction to Chapter

In the first section of this chapter, after some initial definitions, each research question is reviewed and followed with a brief presentation of the findings. In the second section, relevant quantitative results are presented. In the third section, the teaching experiment is described day-by-day, highlighting the critical moments that illustrate how students develop and refine an understanding of the co-splitting construct. Those critical moments represent either the first instance of a strategy or understanding by one or more students, instances that led to further understandings built on those previously identified, or instances that indicate a prolonged misconception or alternative conception. This is followed by a description of the clinical interviews that often stemmed from such critical moments and the overall results for each protocol used. In the fourth and final section, the research questions are restated and the related findings are summarized.

Throughout the chapter, several terms – arrangement, base ratio, fair share, fair-sharing situation, group, object, quantity, ratio unit, sharer, sub-group, unit ratio, value, and whole – are used in technically explicit ways to help describe the activities of the students, and these technical definitions are presented in Appendix D for reference. In the majority of instances in this chapter, the use of these terms and related language comes from the *perspective* of the researcher, in the sense of “voice and perspective” described by Confrey (1998). The use of these terms in perspective are meant to neither replace, nor detract from, the *voices* of the students, but rather to provide a uniform means for discussing the theoretical

and epistemological interpretations of the students' voices, and to thereby facilitate discussion. Confrey (1998) stated that presentation of both student voice and interpreter (researcher) perspective should be included in analysis and interpretation of work with students, such as clinical interviews and teaching experiments. In addition, she cited the importance of acknowledging how the process of interpreting student voice alters the perspective of the researcher. Language in perspective for research on a learning trajectory allows one to relate students' thinking to the descriptions of the proficiency levels in the trajectory and the corresponding outcome spaces. In the analysis of this teaching experiment, students' voices as presented through both their utterances and actions, which are related to researcher perspective to show how the students perceive the ideas and how their thinking develops, while also relating those to more sophisticated thinking and understanding.

Of greatest importance here is the use of the term co-splitting, which was unknown to students entering the teaching experiment. Within this chapter, the term often appears from the perspective of the researcher, but on occasion through the students' voices. In fact, the creation and refinement of a definition for co-splitting by the students became a key aspect of the teaching experiment. Nonetheless, when I refer to *co-splitting* in this chapter, it is as:

. . . the operations or actions of forming equal-sized groups, or equipartitioning, that establish a ratio relationship between two quantities, such that any multiplicative change in one quantity is coordinated with the same multiplicative change in the other quantity. Moreover, these multiplicative changes always occur in the same direction: increasing or decreasing. (Corley et al., 2012)

Co-splitting tasks always involve two quantities between which a multiplicative relationship exists that represents the fair share in the framework of equipartitioning. That

relationship is defined by the values given for each quantity as the parameters of a problem. The general premise of these tasks is to identify ways in which the values for both quantities can be split or partitioned such that the relationship is preserved. Therefore, three requirements for solutions to co-splitting tasks within equipartitioning are: 1) the values for each quantity that make up parts of a solution are always less than the original value of each quantity, 2) a solution consists of more than one pair of values, across which the sums of the values for each quantity are equivalent to the original corresponding quantity (i.e., the “whole” must be exhausted), and 3) the multiplicative relationship between every pair of values representing the two quantities must be preserved. Note that this does not require all pairs of values to be identical as counts, but it does require them to be equivalent as ratios.

Within equipartitioning, co-splitting specifically refers to actions that simultaneously generate pairs of values that represent the same multiplicative relationship as that given by the parameters, and account for the total amounts of each quantity given. Although these actions generally create identical pairs of values, when combined with other co-splitting actions and the notion of distributivity of multiplication over addition, the results may take on the form of non-identical, yet ratio-equivalent pairs of values. The distinction between these types of tasks and ratio tasks is the notion of a fixed original whole (accounting for both quantities) that cannot be expanded, yet must be exhausted. Therefore, a solution claiming equivalency to 12 objects being shared among 8 sharers is not 3 objects and 2 sharers, but rather *four groups of 3* objects and 2 sharers.

## Findings Related to Research Questions

The research questions addressed by the teaching experiment are reviewed below, followed by the initial conjectures, and a brief description of the relevant findings for each. A more detailed discussion of the findings with interspersed connections to the research questions is presented in the next section of this chapter. Then, a concise summary of the findings for each research question is presented in the final section.

### *Research Question 1*

*How do the lower levels of the equipartitioning learning trajectory have an impact on student interactions with and success on the upper levels – particularly co-splitting?* The primary initial conjecture was that students proficient with the lower levels of the equipartitioning LT would be prepared to become proficient with the upper levels of co-splitting and multiple wholes, and that particular knowledge and skills from the lower levels would pertain to students' relative success on, and levels of sophistication in working with, the upper levels. The findings largely supported this conjecture through evidence that identified the relevance of each lower level of the trajectory. Although several students took almost the full two weeks to show abilities aligned with proficiency in co-splitting, every student that completed the teaching experiment was able to do so at some point. Little can be said about the relevance of the lower levels to equipartitioning multiple wholes, as that part of the curriculum was dropped in lieu of a deeper, more substantial, coverage of co-splitting. However, some students offered solutions similar to those for equipartitioning multiple

wholes during the teaching experiment (although it was not intended by the tasks), and some evidence was seen to warrant the conjecture is still viable but in need of further study.

### *Research Question 2*

*How does students' knowledge of multiplication and division interplay with their learning and understanding of equipartitioning at the upper levels of the learning trajectory?*

It had been seen in previous clinical interviews (Corley et al., 2012) that students would often cite multiplication and division facts in their justifications for both determining a number of groups used in a co-splitting problem, and stating the equivalence of fair shares between groups or situations. However, most of the students in the clinical interview study had not been formally introduced to multiplication or division, or were concurrently receiving instruction on those topics. In contrast, the students in this teaching experiment all had been introduced to multiplication and division, and the older students were expected to be quite fluent with those concepts. This aligns better with what is expected based on CCSSM, in which multiplication and division are introduced in third grade and co-splitting lies in fifth grade (Standard 5.NF.3). Therefore, the interrelationships among these concepts are critical, but the implications are not yet fully known. It was conjectured that for those students who were well versed and fluent with multiplication and division, a heavy reliance on facts and procedural knowledge would be used to work on co-splitting problems. However, it was also believed that their conceptual understandings of multiplication and division, and procedural fluency, would benefit from work on co-splitting.

### *Research Question 3*

*How does students' strategy use and performances on co-splitting tasks differ when the number of objects is greater than the number of sharers versus when the number of objects is less than the number of sharers?* In previous co-splitting clinical interviews (Corley et al., 2012), only tasks in which the number of objects was greater than the number of sharers were used. This was a purposeful decision based on items in the equipartitioning field test and the even earlier clinical interviews that informed those items; it was believed that these were the only types of problems for which co-splitting would be used by students. Now that co-splitting has been better defined within equipartitioning, it was conjectured that students might also use similar strategies on problems in which the number of objects is less than the number of sharers. However, students in the previous clinical interviews who employed a deal and split multiple wholes strategy on co-splitting tasks, often based their chosen split on the number of sharers, and then they would deal the objects to "test" whether the same split was viable. In tasks where the number of objects is less than the number of people, this strategy would not work because there would not be enough objects to complete even one full round of dealing. It was conjectured that some students would alter the deal and split strategy to focus on the number of objects first, and that this might arise for one of two reasons: 1) the students thought that finding a viable split for the smaller quantity would afford a better chance for the split to also work on the larger quantity (no matter which quantity – objects or sharers – the smaller value represented), or 2) from experiences with fair-sharing collections and establishing a one-to-one correspondence between objects and

sharers, students would always envision dealing as distributing the larger quantity among the smaller quantity.

### Quantitative Data Analysis

There were five assessments used to gauge different aspects of students' knowledge and understanding at various time points throughout the study. Two of the assessments targeted knowledge related to multiplication and division, while the other three assessments targeted equipartitioning knowledge, and one of those focused solely on co-splitting utilizing LPPSync. These assessments were implemented as indicators of student understanding at different times to provide a quantitative backdrop of the ranges of student ability-levels for the rich, qualitative analysis of the teaching experiment that follows; when appropriate, they also served as proxy measures of learning. The results of these assessments are presented in this section, along with a brief summary of each, and comparative analyses of the two pairs of related assessments.

#### *Multiplication, Division, and Fractions Assessment*

On the second day of the workshop, the students took a multiplication, division, and fractions assessment, consisting of eleven items, some containing multiple parts. The DELTA research group constructed the assessment, and the items ranged from recalling rote facts to solving word problems. Point values were assigned to each item and I scored all of the tests. The purpose of this assessment was to inform claims made about students' use of multiplication and division as prior knowledge, and about their fluency with fractions as is

used in naming fair shares, in working with equipartitioning items. The scores of those students present on the second day of the workshop are presented in Table 5.

The raw scores were based on the points awarded for each item, and the percents are simply the raw scores divided by the 18 total points possible. The mean for the raw scores on the assessment was 8.27 (45.94%) with a standard deviation of 4.00 (22.21%). Since the later multiplication and division assessment, administered near the end of the teaching experiment, did not include fraction items, a truncated score (and corresponding percentage), labeled M/D only, was computed using only the nine multiplication and division items, omitting the fraction items in order to attain a comparative score for that later assessment. The mean for the truncated scores was 6.00 (66.67%) with a standard deviation of 2.31 (25.66%). In general, students' performances were much higher on the multiplication and division items, with the most commonly missed item being the only multi-step problem, that required students to multiply one factor by another factor, and then divide by a third factor.

*Table 5. Multiplication, division, and fractions assessment results.*

Name	Raw Score (18)	Percent	M/D only (9)	M/D only Percent
HI	13.5	75.00	9	100.00
YW	5	27.78	5	55.56
LB	6.5	36.11	5	55.56
RD	4	22.22	4	44.44
MH	14.5	80.56	8	88.89
XE	1	5.56	1	11.11
TA	6.5	36.11	4	44.44
AU	6	33.33	5	55.56
OZ	11	61.11	6	66.67
BE	8.5	47.22	7	77.78
XY	11.5	63.89	8	88.89
RR	12	66.67	9	100.00
BK	7.5	41.67	7	77.78
All Students	<b>8.27</b>	<b>45.94</b>	<b>6</b>	<b>66.67</b>

### *Equipartitioning Pre-Test*

Students took a pretest on the equipartitioning LT at the end of the workshop in order to establish a baseline for studying the introduction of the upper levels. The test consisted of fifteen items, the last four of which covered the upper levels (12-16) of the equipartitioning LT. The test was constructed by the DELTA research group, using items from the equipartitioning field test, paper forms of LPPSync IGE items, and items similar to those in the curriculum. The students were given approximately one hour to complete the test; most items contained multiple parts and required written justification. Point values were assigned to each item and I scored all of the tests; partial credit was awarded based on criteria laid out in a rubric. The total number of possible points for the test was 47, and any items left blank, whether skipped or not finished due to time, were marked as incorrect, resulting in a score of 0 for that item. The purpose of this test was three-fold: 1) to determine which, if any, students did not meet the assumption of prerequisite knowledge for the teaching experiment, 2) to inform grouping of students during the teaching experiment, and 3) to act as a baseline and comparative score against which the identical post-test could be interpreted. The scores of those students present on the third day of the workshop are presented in Table 6.

The raw scores were based on the points awarded for each item, and the percents are simply the raw scores divided by the 47 total points possible. The mean for the raw scores was 14.57 (30.99%) with a standard deviation of 4.85 (10.33%). Since the workshop did not cover all of the items that had been included on the pre-test, a truncated score (and corresponding percentage), labeled Workshop Levels, was computed using only items 1-10,

which addressed the levels of the LT actually covered. The mean for the truncated scores was 14 (38.89%) with a standard deviation of 4.73 (13.14%). In general, students' performances progressively diminished at the upper levels of the LT. Students showed the greatest success on items covering levels 1-4, and the least success on items covering levels 7, 8, 12, and 14 (which includes co-splitting and multiple wholes).

*Table 6.* Equipartitioning pre-test results.

Name	Raw Score (47)	Percent	Workshop Levels (36)	Workshop Levels Percent
XI	20.5	43.62	19.5	54.17
HI	21	44.68	19.5	54.17
YW	17	36.17	16.5	45.83
LB	20	42.55	20	55.56
RD	10.5	22.34	10.5	29.17
MH	13	27.66	13	36.11
XE	3.5	7.45	3.5	9.72
TA	14.5	30.85	13.5	37.50
AU	8.5	18.09	6.5	18.06
MV	13	27.66	13	36.11
OZ	17.5	37.23	15.5	43.06
BE	17	36.17	17	47.22
RR	12.5	26.60	12.5	34.72
BK	12	25.53	12	33.33
JW*	18	38.30	17.5	48.61
<b>All Students</b>	<b>14.57</b>	<b>30.99</b>	<b>14</b>	<b>38.89</b>

\*Student did not continue beyond the workshop to the teaching experiment

However, it should also be noted that across the fifteen students who were present to take the test, there were 209 (out of a possible 555) opportunities to respond left blank, coded NR for no response, and scored as 0 points awarded – a non-response rate of approximately 38%. Most of these instances (178) were seen on items 8-15. This could indicate that many of these were due to a lack of enough time to respond to later questions on the test, or it could

indicate the students' general attitudes towards test taking and a lack of perseverance on tests, especially on what they perceived as a low-stakes assessment during the summer.

One weakness in the design of the test was that since the students did not all appear to have enough time to complete the test and the items were ordered by LT level, so a failure to respond at the higher levels cannot be confidently attributed to a student's lack of ability to solve the items. Neither can one assume that students actually would have been able to successfully answer those questions if attempted. As a result, a slight modification was made to the post-test and is discussed below.

#### *Multiplication and Division Assessment*

On the eighth day of the teaching experiment, students took a second multiplication and division-related assessment. In this assessment, all of the items were presented in the form of facts; the assessment also contained some items requiring the students to skip count on to a list of given numbers, but no fraction items were included. These changes were made for two reasons: 1) the research question that these assessments could inform is about the interrelationships between multiplication and division, and equipartitioning – not fractions, and 2) the first multiplication and division-related assessment introduced the confounding factors of reading and problem-solving abilities through the use of word problems.

Each of the twenty items was assigned a value of one point, for a total of 20 possible points, and I scored all of the items as correct or incorrect only. One purpose of this assessment was to determine whether students were able to skip count more successfully than they could cite rote multiplication facts. A second purpose was to gauge whether students'

inabilities to solve some co-splitting problems may be related to a weak knowledge base of multiplication and division facts when they rely on those as a means to solve such problems. The scores of those students present on the eighth day, along with their corresponding scores on the multiplication and division items on the prior assessment, are presented in Table 7.

The raw scores were based on the points awarded for each item, and the percents are simply the raw scores divided by the 20 total points possible. The columns labeled M/D only refer to the multiplication and division items from the earlier assessment. The mean for the raw scores on this assessment was 14.88 (74.38%) with a standard deviation of 5.84 (29.21%). In general, students' performances were high, and higher than on the earlier, related assessment. Looking at the responses of the six highest-scoring students, there were only three items on which more than one incorrect response was given (excluding non-responses). Two students incorrectly identified the second of the next two numbers (28 and 35) for skip counting based on the sequence "7, 14, 21, . . .", two students provided an incorrect quotient for  $40 \div 8$ , and three students provided an incorrect product for  $4 \cdot 9$ . The same two students accounted for 6 of these 8 incorrect responses, and had raw scores of 15 and 16 (out of 20), respectively. There was a noticeable difference in the students' success between skip-counting and multiplication and division items, with more skip-counting items answered correctly, including non-responses (76/90 versus 56/90).

*Table 7.* Multiplication and division assessment results.

Name	Raw Score (20)	Percent	M/D only (10)	M/D only Percent
MH	20	100.00	10	100.00
XE	4	20.00	0	0.00
TA	18	90.00	8	80.00
AU	16	80.00	7	70.00
OZ	15	75.00	4	40.00
BE	18	90.00	8	80.00
RR	8	40.00	4	40.00
BK	20	100.00	10	100.00
<b>All Students</b>	<b>14.88</b>	<b>74.38</b>	<b>5.88</b>	<b>65.28</b>

Two students scored significantly lower than the other students on this assessment.

One of those low-performing students did not respond to 11 out of the 20 items, only providing one other incorrect response on the 9 items to which that student did respond.

Upon further investigation, the items on which this student did not respond were not merely the last 11 items on the assessment, which may have implied not having enough time.

Therefore, it could be inferred that the student chose to only respond to items for which she believed she could provide a correct response. The other low-performing student was only able to successfully identify the next two numbers in order for the first two skip-counting problems, and incorrectly responded to all other items. Upon further investigation, the student's responses to the last 11 items of the assessment were 1, 2, 3, . . . , 11. This could have been due to the student not having an adequate amount of time to determine reasoned responses for these last items, and a sense that giving any response, even a seemingly uneducated guess, provided a better chance of getting a correct answer than leaving an item blank. It also could have been due to the student's attitude towards the test, or test-taking in

general, and a lack of concern for effort. Further discussion of comparisons between the results on the two multiplication and division assessments are elaborated in a later section.

#### *Equipartitioning Post-Test*

On the ninth day of the teaching experiment, a paper-and-pencil post-test that was theoretically similar to the pre-test was administered. The items were identical in form, with changes in basic parameters for some items to minimize recall from the pre-test (see Table 8). These changes were made by selecting parameters from equivalent ranges of the samples spaces designed for each level of the LT as is used by LPPSync. Each item contained the same number of parts as its corresponding item in the pre-test, and identical point values were assigned to each item. In addition, the order of the items was altered, so that the co-splitting and equipartitioning multiple wholes items appeared first this time, in hopes that students working sequentially would provide responses to these items even if they did not have adequate time to complete the test. The scores of those students present on the ninth day of the teaching experiment, along with corresponding scores on the pre-test, are presented in Table 9, with co-splitting and multiple wholes items starred (\*).

*Table 8.* Comparison of pre- and post-test items and parameters.

Pre-Test (Item) Parameters	Post-Test (Item) Parameters
(1) 15 shared among 3	(5) 20 shared among 5
(2) Whole shared for 6	(6) Whole shared for 9
(3) Different-sized wholes shared for 8	(7) Different-sized wholes shared for 8
(4) 24 shared among 4	(8) 30 shared among 3
(5) 7 objects is $1/5^{\text{th}}$	(9) 8 objects is $1/6^{\text{th}}$
(6) $1/8^{\text{th}}$ equals $1/8^{\text{th}}$	(10) $1/4^{\text{th}}$ equals $1/4^{\text{th}}$
(7) 30 shared among 6 to among 3	(11) 30 shared among 6 to among 2
(8) One-third as many tables, same people	(12) One-half as many tables, same people
(9) 5-split by 4-split	(13) 5-split by 3-split
(10) 4 parts, 24 blocks; half as many	(14) 4 parts, 24 blocks; half as many
(11) 6-split by 3-split	(15) 3-split by 8-split
(12)* 36 shared among 24	(1)* 36 shared among 24
(13)* 20 shared among 12	(2)* 20 shared among 12
(14)* 5 shared among 8	(3)* 5 shared among 8
(15)* 35 shared among 26	(4)* 35 shared among 26

The raw scores were based on the points awarded for each item, and the percents are simply the raw scores divided by the 47 total points possible. The mean for the raw scores was 12.69 (26.99%) with a standard deviation of 7.01 (14.93%). In order to offer an adequate, albeit proxy, measure of learning during the teaching experiment on the upper levels of the equipartitioning LT, a truncated score (and corresponding percentage), labeled Upper Levels, was computed using only items 1-4, which now addressed the five uppermost levels of the LT (12-16). The mean for the truncated scores on the post-test was 4.00 (36.36%) with a standard deviation of 3.05 (27.70%). In general, half of the students performed better than on the pre-test, and half performed worse. The overall performances on co-splitting, and multiple wholes, were as high as those on collections and wholes, with most of the drop-off coming from the middle, more conceptual levels. However, the middle-level items were now also the last items on the assessment.

*Table 9.* Equipartitioning post-test results.

Name	Raw Score (47)	Percent	Upper Levels (11)	Upper Levels Percent	Pre-Test (47)	Pre-Test Percent
XI	14.5	30.85	9	81.82	20.5	43.62
XE	4.5	9.57	1.5	13.64	3.5	7.45
TA	10.5	22.34	1	9.09	14.5	30.85
AU	18	38.30	5.5	50.00	8.5	18.09
MV	19.5	41.49	1.5	13.64	13	27.66
OZ	21	44.68	7	63.64	17.5	37.23
BE	12	25.33	5	45.45	17	36.17
BK	1.5	3.19	1.5	13.64	12.5	26.60
<b>All Students</b>	<b>12.69</b>	<b>26.99</b>	<b>4.00</b>	<b>36.36</b>	<b>13.38</b>	<b>28.46</b>

A similar non-response rate to that of the pre-test was seen on the post-test, in that 161 (out of a possible 555) opportunities to respond were left blank, coded NR for no response, and scored as 0 points awarded – a non-response rate of approximately 29%. The students' performances across all upper level items, including co-splitting, exceeded their overall performances on both the pre-test and the post-test. Half of the students scored lower on the post-test than on the pre-test, and the overall average on the post-test was lower than the average on the pre-test. Three hypothesized, possible explanations for this latter outcome are: 1) the students' knowledge decreased, 2) the students were resistant to persevering through the entire assessment, due to having the more difficult items first on the post-test, which led to students give up and never attempt the easier items, or 3) the students were resistant to participating in school-like activities in a summer program. Based on observations in the class showing that these students were capable of successfully solving problems of these types, these results could plausibly represent a resistance to taking the assessment itself, rather than an actual loss of understanding. The same two students that

performed much more poorly on the second multiplication and division assessment than the other students, also performed much more poorly on the post-test, which provides some evidence that the inferred resistance may be warranted, at least for those two students. Nonetheless, four other students did also perform better on the post-test than on the pre-test. Further discussion of comparisons between the results on the pre- and the post-test are elaborated in a later section.

#### *Co-Splitting Diagnostic Assessment*

On the tenth day of the teaching experiment, the students were asked to complete an entire, six-question diagnostic assessment on co-splitting using LPPSync. The problems were all presented in the familiar context of a number of pizzas being shared among a number of people. The diagnostic consisted of two problems from each of three difficulty levels, requiring one, two, and three solutions (arrangements), respectively. The parameters for the problems differed for each student, but all problems at each difficulty level were randomly assigned by the system from theoretically equivalent sample spaces. These sample spaces include problems in which the relative amounts of pizzas and people can be greater or lesser. The scores were calculated by the system, with each possible solution at all difficulty levels assigned equal value, and thus the percent score is based on the number of correct solutions given, out of 12 required solutions. Half-credit is assigned for providing the given parameters at one table as a solution. The scores of those students present on the tenth day of the teaching experiment, and who completed the entire diagnostic assessment, are presented in Table 10. Unfortunately, although the system was designed to render a score for unfinished

assessments after a 24-hour period, marking unanswered items as incorrect, there was a malfunction in the technology carrying this action out, and the data was lost.

The raw scores were based on the number of correct solutions submitted, including the submission of the given parameters as a solution. The percents are simply the raw scores divided by the 12 total points possible. The mean for the raw scores was 6.5 (54.17%) with a standard deviation of 3.87 (32.27%). Two students did not complete the diagnostic assessment entirely, and therefore their scores were not captured and are not in the table.

*Table 10. Co-splitting diagnostic assessment results.*

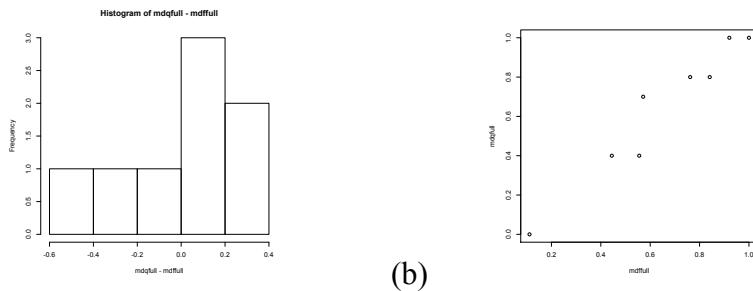
Name	Raw Score (12)	Percent
MH	12	100.00
XE	1.5	12.50
TA	9	75.00
AU	6	50.00
MV	7.5	62.50
OZ	3	25.00
<b>All Students</b>	<b>6.5</b>	<b>54.17</b>

### *Comparisons of Assessments*

The two multiplication and division-related assessments contained different numbers of items therefore, the percentages were used for comparisons between those assessments, rather than the raw scores. I isolated the scores for only the items involving the multiplication and division of whole numbers for both assessments and the scores were virtually the same. However, it should also be noted that these sets of items were not identical, nor were the tests constructed to be truly equivalent measures using similar forms and only varied parameters. Although the paired data appeared relatively normal (see Figures 14a and 14b), I conducted

both a parametric and a, more robust, non-parametric statistical test of the differences between the percentage scores on the multiplication and division items on both assessments.

The results of both the two-sample  $t$ -test ( $t=0.1417$ ,  $df=7$ ,  $p=0.8913$ ) and the Wilcoxon signed rank test ( $V=16$ ,  $p=0.8438$ ) showed that there were no statistically significant differences in the students' scores on those items between the two assessments.



*Figure 14.* Multiplication and division assessments: (a) histogram distribution, and (b) q-q plot.

The scores on the second, full multiplication and division assessment were much higher overall across all students taking both assessments (74.38% compared to 45.94%). These results could be attributed to a combination of two things: 1) the first assessment involved word problems, whereas the second was all facts, which could have made the second assessment easier or more accessible; and 2) the skip-counting tasks on the second assessment were easier and more accessible than the fractions tasks on the first assessment. Eight students were present to take both assessments, all of which completed the entire teaching experiment, and therefore received full instruction on equipartitioning and co-splitting; five students scored higher (with an average increase of 17.11%) on the second

assessment across the multiplication and division items only, and consequently, three students scored lower (with an average decrease of 32.59%). However, removing the score of the one student who did not provide any correct responses to multiplication and division items on the second assessment, the average score on those items across all other students was 72.86%, as compared to 66.67% across all students for the similar items on the first assessment. This could indicate that some form of increased skill with multiplication and division resulted from experiences with co-splitting for five of the other seven students present to take both of these assessments.

The raw scores for the pre- and post-tests were used for comparisons between those assessments, because the assessments were theoretically identical and scored out the same total number of possible points (47); there were also similar non-response rates, with the commonality being by item location within a given assessment, and not by level of the LT. The scores on the pre-test were slightly higher overall across all students taking each assessment (13.38 compared to 12.69). As with the multiplication and division assessments, although the paired data appeared relatively normal (see Figures 15a and 15b), I conducted both a parametric, and a more robust, non-parametric statistical test of the differences between the pre-test and the post-test. This could be attributed to the fact that the earliest items, those most responded to on the pre-test covered the lowest levels of the trajectory, and therefore the most basic proficiencies of equipartitioning, whereas the earliest items on the post-test included co-splitting as well as the other upper levels not covered at all. The results of both the two-sample *t*-test ( $t=0.2760$ ,  $df=7$ ,  $p=0.7905$ ) and the Wilcoxon signed rank test

( $V=19.5$ ,  $p=0.8885$ ) showed that these slight differences in the students' scores between the pre-test and the post-test were not statistically significant. Eight students were present to take both assessments, all of which completed the entire teaching experiment, and therefore received full instruction on equipartitioning and co-splitting; four students scored higher (with an average increase of 6.5) on the second assessment across the multiplication and division items only, and consequently, three students scored lower (with an average decrease of 5.125). Nonetheless, for both the pre-test and the post-test, the percentage scores across all students present to take each assessment were higher for the workshop levels than the pre-test as a whole (38.89% versus 30.99%) and higher for the upper levels than the post-test as a whole (36.36% versus 26.99%); and even higher yet for co-splitting (48.75%) than the upper levels on the post-test. These results could indicate a recency effect, but also could represent some forms of learning based on the instruction and student experiences from the treatments.

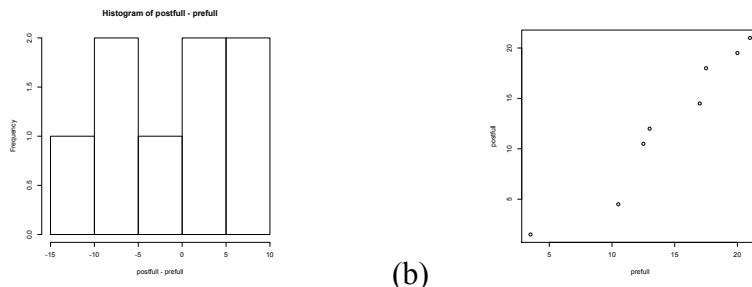


Figure 15. Equipartitioning pre- and post-tests: (a) histogram distribution, and (b) q-q plot.

I would place more weight and confidence into the results of the multiplication and division-related assessments than the others as pertaining to the study because they were both fully completed and aptly represent the students' knowledge and understanding of that which

they measured. However, the results of all of the assessments used during the teaching experiment, and especially the pre- and post-test scores, should be considered under the caveat that the experiment was conducted during the summer and outside of a real classroom where certain norms and accountability for performance and effort are inherent. Not only were the assessments low stakes for the students, with no grades or repercussions attached for either success or failure, but also they were quite lengthy and mentally taxing for such an environment as that in which the experiment was conducted. The amount of data that could be collected for all assessments was hindered by inconsistent attendance and attrition of students as there was no requirement for them to participate each day, despite having recruited all of them with parental consent and verbal agreements to attend for the duration and see the experiment to completion. Unfortunately, these students had other obligations and opportunities – some perhaps more appealing, such as field trips, athletic events, and conflicting summer camps – to choose from over the course of the teaching experiment, which not only pulled them away during the allotted instructional times, but also made it relatively impossible to convince them to dedicate much, if any, outside time to make up missed assessments.

#### Description of Teaching Experiment

In this section, I elaborate on day-to-day occurrences that provide evidence of findings relative to the research questions. Each day and clinical interview session offers their own set of relevant information, and the descriptions herein proceed chronologically by day, followed by a section on information gathered from the clinical interviews. The

connections between events on these days, showing changes or progress in the thinking and development of ideas across the class and within individual students are discussed in the chapter that follows. Two of the ten days from the two-week teaching experiment were omitted from the descriptions: *Day 1* – covered PEEQ, the remaining lower level (11) from the workshop, and *Day 10* –administered co-splitting diagnostic assessment on LPPSync and distributed awards.

I use the first person in this section to refer to myself, as the teacher-researcher, along with my thinking relative to instructional decisions and corresponding actions during times of instruction. The information is presented from a first-person point-of-view, as if one had followed the teacher around the classroom. Therefore, it only contains discussions from the groups whom I engaged in conversation, and those responses offered and shared in whole-class settings. Thus, no generalizations can be made as to the understanding of all students based on a single given response; however, I use the phrases “all students” or “all of the students” when there appeared to be a general consensus, gathered from at least a majority of the students nodding their heads or agreeing in choral responses. Some exchanges with students were omitted if they were deemed as redundant or non-informative. In the excerpted direct quotes, when multiple students were involved, I refer to the first to speak as Student 1, and then name the students in a similar manner as they enter the conversation (Student 2, Student 3, etc.). This process was repeated for each piece of dialogue; therefore, the name “Student 1” does not necessarily always represent the same student responding. Student IDs were not used in the narrative descriptions or the quoted excerpts because it was not the

intent of this study to trace the thinking of any one particular student or try to show growth for any individual, but rather to describe the teaching experiment as an episodic view into a classroom with a collective group of students interacting with a concept over time.

### *Day 2*

The students' first experience with co-splitting was presented on the second day, in the context of 10 fish in a single fish bowl with 4 pellets of food to share and eat. I explained (superficially) that this was the exact amount of food required for 10 fish, and it was assumed they would share the pellets fairly. I then claimed that it might be too crowded for that many fish in just one bowl, so the students were to determine another arrangement for the fish and pellets using two bowls, such that every fish would still be able to get their fair share of food.

The students quickly determined 5 fish and 2 pellets in each of the two bowls. I then asked them to justify the equivalence of fair shares between the two bowls:

Teacher: How do we know these fish, these two fish (on left) are going to get the same fair share of food as these two fish (on right)?

Student: Because they're both equal.

Teacher: What's equal?

Student: The pellets.

This last response shows possible evidence of a common early misconception, that the equivalence of ratios can be explained by attending to only one of the two quantities.

However, it was unclear whether this was how the student was thinking, or whether the equality of the numbers of fish in the two bowls was being assumed as a given. I chose to leave this open, as students would have further opportunities to address it later.

Co-splitting tasks explicitly require students to preserve the fair share in determining the groups in their arrangements. Adherence to this requirement can be justified in one of two ways: 1) demonstrating that all groups represent the same fair share, and then demonstrating that one of those groups represents the same fair share as the given situation, which by a transitive relationship implies that all groups would, or 2) systematically demonstrating that each group represents the same fair share as the given situation, which also by a transitive relationship implies that each group represents the same fair share. We typically ask students to justify their results in the first way, and the example above demonstrates the first part of that process – equivalence of fair shares among created groups. An example of justifying the equivalence of fair shares between a created group and the given situation follows:

Teacher: And do you think the fish down here in this bowl, now that we separated them, are they still going to get the same fair share that they were supposed to get originally?

...

Student: Because there's five pellets for the two fish, and ten pellets for four fish.

Teacher: So is that the same or not?

Student: No.

Teacher: Not the same why? ... Just because the numbers are different?

Student: Wait! Yes it is. Because, uh, five ... Five plus five ... Equals ten. And two plus two equals four.

This shows the possibility of another misconception developing for the equivalence of ratios, that summation alone will preserve the ratio. In this misconception, students believe that an equivalence of ratios is maintained between two quantities simply if the sums of the quantities are preserved. For example, if the two bowls instead contained 7 fish and 3 fish, and 2 pellets in each bowl, then students may claim that the fair shares would still be the

same because  $7 + 3 = 10$  and  $2 + 2 = 4$ . In this case, the sums are preserved, but the shares would not be the same. Similarly, I left this general issue open for students to address later.

The next problem involved 24 pellets being shared among 18 fish. This time the students were not given a number of bowls to use and were asked to determine that on their own, as well as determining how to arrange the fish and pellets so that the fish would still get the same fair share. Students worked with partners, and the use of four different numbers of bowls (2, 3, 4, and 6) were attempted across all of the students. Those students who came up with a successful answer quickly were challenged to determine another possible arrangement, and a few came up with more than one correct response.

The majority of students chose to use two bowls first, and all were able to determine a correct solution. It is not assumed that these students were using co-splitting yet, but they justified and referred to their use of 2-co-splits in several ways: as additive, or as multiplicative – both as halves and doubles. Students were only seen using three bowls as an incremental adjustment strategy from either successfully working with two bowls or unsuccessfully working with four bowls, cycling through consecutive numbers in either forward or reverse counting order. The students who initially attempted to use four bowls correctly concluded that it would not work (in the whole numbers), and all went on to eventually determine at least one correct response. Some students attempted to use six bowls from the start, justifying their choice by citing 6 as a common factor in multiplication facts ( $4 \cdot 6 = 24$ , and  $3 \cdot 6 = 18$ ). Clearly, reasoning with co-splitting tasks is dependent on facility with multiplication (and division) facts. In their explanations of how their values were

determined, the students referenced such facts often. However, despite most of the students claiming that the situations they had determined were equivalent to the original here, they were unable to give explicit justifications of this equivalence, multiplicatively or other.

### *Day 3*

On the third day, the students continued working with similar problems, beginning with 12 pellets being shared among 8 fish. Two members of one group stated that they were going to attempt four bowls and twelve bowls, respectively. They both attempted to deal one quantity at a time using manipulatives to determine an answer, but both had difficulty completing this process until I pointed them back to the problems from the previous day. One of the students then immediately reverted back to attempting to find a solution using two bowls instead, and was quickly able to do so. I chose to not push them any further to determine whether this made them believe their previous choices would work, and thus did not get an explanation as to why four and twelve bowls were chosen originally.

Another student in a different group said that she was using a form of multiplication (called “lattice”) to determine answers. This indicated that even at this early stage, she believed multiplication was related to co-splitting and could lead to a solution. Her choice to use multiplication could have been due to references to multiplication facts in previous examples, or it could have represented another form of sense making she internalized for these relatively new problems. However, she was attempting to multiply 12 by 4, which she explained was because there were 4 fish in each bowl, and 12 pellets. She did not say how many bowls she was using, but there were two groups of 4 manipulatives separated out on

her desk, so two bowls were inferred. She believed that the result of the multiplication, which she correctly determined to be 48, would represent the total number of pellets.

When I directed her back to the manipulatives, she then said she needed to divide in order to determine the pellets, and chose to divide 12 by 4, because there were 4 fish in each bowl. She successfully determined this result as 3 pellets, but based on the schema under which she was operating, assigned 3 pellets to each of the two bowls. This division should have represented either sharing the 12 pellets among four bowls, or sharing all 12 pellets among the 4 fish (perhaps in just one bowl), thus resulting in 3 pellets for each bowl or fish, neither of which matched her representation with the manipulatives. After placing 3 pellets in each bowl and realizing there were still 6 left over, she modified her claim and stated that 6 pellets should go in each bowl (with 4 fish already in each), because 6 plus 6 equals 12. Throughout this discussion she referred to multiplication, division, and ultimately addition, showing that her understanding of co-splitting and ability to justify related actions were still in an early and confused state. She also gave no indication of recognizing how the 6 pellets could be related back to the 3 pellets stated initially, for each pair of fish in each bowl, thus also resulting in 6 pellets total in each bowl.

Two other students indicated that the amounts of both the fish and the pellets should be the same – perhaps recalling the notion of fair sharing from equipartitioning collections, in which all shares were represented as identical groups of objects per every one sharer – thus parsing the task into first sharing the 8 fish among two bowls, and then separately sharing the 12 pellets among two bowls. They relied on dealing to play this out, first creating two equal

groups of 4 fish each. However, they were then unable to determine the number of pellets in a similar manner, perhaps confusing “who” the sharers were in this context (bowls, not individual fish). This perhaps also demonstrates that the students were not yet associating the desired split with the number of groups (bowls here) when those groups contained more than one sharer – a major difference between co-splitting tasks and equipartitioning collections tasks. The students continued on to give 1 pellet to every fish, resulting in 4 pellets leftover. They indicated understanding this was not a solution because they needed to use all of the pellets, recalling the criteria for equipartitioning collections. Both students then realized that they could use another 2-split on the 4 remaining pellets, and dealt those accordingly using a reallocation strategy, to make 6 total in each bowl.

Each group was asked to present their work for this problem, and several different types of reasoning came out from the explanations as illustrated in the excerpt below:

Teacher: How did you decide to do two bowls? . . . Here [on your work], it says because half of eight is four, so that's how you figured out your fish, right? How did you figure out your pellets? Is there a similar reason?

Student 1: Well, half of twelve is six.

. . .  
Teacher: So you did multiplication?

Student 2: Yeah.

Teacher: And how did that help you figure out your answer?

Student 2: Because they, there are four fish in two bowls, and six pellets for each fish, because six plus six equals twelve.

. . .  
Student 3: Twelve pellets, you can divide those equally into two bowls and four fish . . . And since you need twelve pellets, eight fish, so I divided those equally among two bowls. Four fish in each bowl, and each bowl will get six pellets.

Teacher: And what does it mean to divide equally?

Student 3: Divide equally means to, like, since eight fish are divided equally, I got four, so there's two bowls.

The first strategy described – halving – is a reference to splitting, which is the most intuitive for students. Then, the third strategy represents a way of mathematizing splitting, by referring to division, which is a similar schema for students, but articulated in a more sophisticated manner; and it would be expected for students to make this transition with a factor of 2 first, as was the case here. The second strategy was initially identified by the student as multiplication, which is quite different from the other two strategies in that it is a reversal, similar to reassembly in equipartitioning collections or wholes. However, when trying to articulate this verbally, the student was not yet able to explain the conclusion through multiplication alone, and resorted back to describing this reversal process as addition.

Some students had made it to a second problem, 18 pellets being shared among 9 fish, and the only solution offered used 3-splits. The student that had referred to this process as division in the discussion of the first problem (the third strategy above) referred to the process as division again. I asked whether this was similar to, or different from, that group's strategy for the first problem, and she recognized it as similar to their 2-splits strategy, claiming they were both division.

No groups provided more than one possible solution for either of the problems, so I checked to see whether they had previously experienced math problems in which more than one correct solution was possible. Most students said they had, or acknowledged it is a possibility, citing examples from the previous day and from simplifying fractions. I then asked all of the students to think about the 12 pellets and 8 fish problem again, in order to stimulate another possible solution besides two bowls. Three bowls and four bowls were

suggested, and the class decided to start with three. This is another prime example of the incremental adjustment strategy, whereby the number of bowls (corresponding to the splits) attempted proceeds in counting order.

Before allowing the students to work, I questioned whether all students believed three bowls was a good choice, and got mixed responses. One student said she knew three bowls would not work because 3 could not divide 8 evenly, and there would be a remainder. This response implied a belief that there should be the same number of fish in each bowl, so I asked why that was important. Another student said that having the same number of fish in each bowl would ensure that each fish gets the same number of pellets, which essentially restates what it means to fairly share. Since three bowls did not appear to work, the class moved to using four bowls, and one student claimed there would be 3 pellets in each bowl, and also 2 fish in each of the four bowls. She said that she came up with those values by counting, while another student offered that the values resulted from multiplication, again demonstrating that the students were not yet able to justify using the reversal of fair sharing as multiplication, despite showing a strong belief that this was the operation being used to determine the shares in many instances.

When then asked how a fish from each of two of the four bowls could get the same fair share, a student suggested it was because there were the same numbers of both fish and pellets in each bowl. Another student questioned whether the shares would be the same because the fish may be different sizes, which I acknowledged but stated that in these problems, all fish are assumed to be the same size, eat the same amount of food, and be

capable of fairly sharing. This second student then said the fish would get the same fair share because each fish would get one and one-half pellets. This demonstrated a leap in reasoning beyond the other students at that point in two ways: 1) she explicitly cited the fair share for each fish, which in the language of “perspective” is a call out to the unit ratio, and 2) this was the first instance that non-whole number values for either of the quantities was suggested.

After this discussion, I asked whether a fish in one of the four bowls would get the same fair share as a fish in the original bowl. This same student stated that if each of the eight fish got one pellet, then there would be four pellets left, which could each be split in half (creating eight halves), thus also resulting in one and one-half pellets for each fish in the original bowl – the same fair share as determined for each fish in the four bowls. Due to this drastic leap in reasoning, all of the other students had grown quiet; perhaps this was not only due to a lack of understanding, but it may also have indicated a larger stigma relative to school, whereby these students assume the “smart kid” (also assumed to be someone else) will carry the cognitive load, therefore choosing to ignore or bow out of the conversation.

In an attempt to not lose the whole class, I acknowledged these rational-valued answers as correct but then asked whether there was another possible arrangement. After suggestions of five and six bowls, a consensus was reached to try six bowls. One group of students determined this would work for the pellets, but when trying to deal the fish one at a time to each bowl, there were 2 left over. I asked whether the leftover fish could just be placed into any two of the six bowls – one fish in each – and the students agreed the shares would not be the same in that case. No student suggested splitting the fish further (into

fractional parts), perhaps because they were only willing to consider splitting the pellets to fractional values in context; in other words, a partial fish was not part of a viable solution.

Another student then suggested eight bowls – one bowl for each fish; however, none of the students could describe how to physically distribute the pellets for this arrangement either. Combined with what was described above, this confirms that these other students were not ready to create solutions involving fractional values or unit ratios yet, nor understand them beyond 2-splits. The student who had determined the actual fair shares earlier expanded her reasoning here and claimed that since the same total numbers of fish and pellets existed in the eight bowls as existed in the one original bowl, then each fish would still *have to* get the same one and one-half pellets – alluding to the preservation of the fair share (or unit ratio) across arrangements. I then asked the students to justify whether this worked and how they could make sense of it. One student stated that since two halves make a whole, the fish and each one's corresponding one and one-half pellets could be seen in groups of two fish and three pellets, which would be the same as before in the four bowls. This explanation indicated the first early use of a reverse co-splitting strategy.

I then formally introduced the term “co-splitting” and asked the students what they believed it meant. Several students contributed, mostly offering their understandings of splitting, how to name splits, and referring to both quantities at different points, which built an early notion of the concept that I was satisfied with. However, I decided not to define co-splitting formally here. The students were then introduced to a new context of pizzas being shared among people through the Co-splitting LPPSync e-Packet (see Figure 16). The same

premise of determining other arrangements applied, in which tables became the new grouping mechanism (in place of the bowls).

All of the students were given the problem of 28 pizzas being shared among 12 people on which to work. The first answer shared was a solution using 4-splits, explaining the choice as based on the number of pizzas first, and justified by both division and multiplication. However, the division fact was stated as 28 divided by 7. This is typical of early justifications provided by students, in which they do not give due diligence to the terms in a fact with regards to units in the context of the problem. They seem to rather carelessly state any facts that involve the triad of values (objects, sharers, and groups) to support their result. Similarly, the student then stated that 12 divided by 4 determined 3 people at each table, now correctly identifying the 4 as representing tables. When I asked all of the students to justify the equivalence of fair shares between these four tables, it was stated that the fair shares were the same because there were the exact same values for both quantities at all of the tables, but there was no evidence that any students were able to justify the fair shares by the relationship between the quantities within a given table, or this was taken by them as a given, as the fair share.

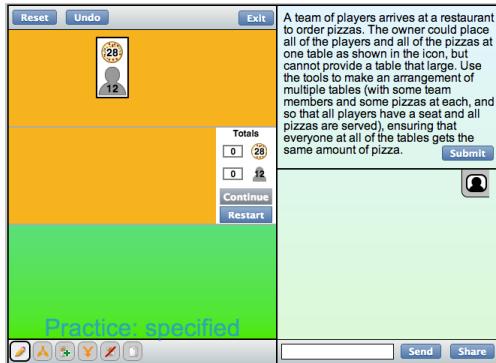


Figure 16. Co-splitting task in LPPSync IGE.

#### Day 4

On the fourth day, I decided to continue with the context of pizzas and people, since the LPPSync problems had appeared to be difficult for the students the day before, but now providing the students with manipulatives of cutout people and pizzas with which to work. The first problem was 8 pizzas being shared among 6 people, and I asked how 2-splits could be used on those quantities, thus resulting in two tables, to get the students to focus more on the splitting actions in co-splitting problems. One student stated that 3 people could be placed at each table, and then each of those tables could be given 4 pizzas.

The students then worked on the problem of 28 pizzas being shared among 12 people, and I asked them to determine as many possible arrangements as they could. One group of three students attempted to split the quantities between two tables, identifying 6 people for each table. Then for the pizzas, a student dealt one pizza to each person, using 12 of the pizzas. Dealing as a strategy is typical at this stage of understanding of fair sharing to verify whether the chosen split is viable for the number of objects (or sharers). However, in this case, the student dealt the pizzas to the people rather than to each of the two tables, as if

trying to determine the fair share, or unit ratio. It appeared this strategy was employed as a means to reduce the number of objects to coordinate with the tables. The students could not complete this act of fair sharing, as they were not yet fully capable of determining the fair share in such a situation. The following conversation ensued among the group and myself:

Teacher: Well what happens if we only do one each?

Student 1: Everybody won't get the same. . . . But they would get only one pizza.

....  
Teacher: So that's not fair?

Student 2: Yeah.

Student 1: It is fair!

Teacher: But what's wrong?

Student 2: We still got pizzas. . . . Give them another one.

....  
Student 1: Give them like three pizzas! Three pizzas.

Teacher: So if we gave each one three pizzas, how many would be on this table?

Student 1: Three, six, nine, . . . , eighteen.

Teacher: So there's eighteen here [on the left table], and there's eighteen here [on the right table]. How many would that be?

Student 2: There's four leftover.

Student 1: Eighteen plus eighteen is . . .

Student 3: I know what half of twenty-eight is.

Teacher: What's eighteen plus eighteen?

....  
Student 3: Thirty-six.

Student 1: Yeah! But that would be too much.

....  
Student 3: But half of twenty-eight is fourteen.

Teacher: So thirty-six, if we do three each, that's too much. Two each, we got four left, so that's not enough.

Student 3: Half!

Teacher: So what do we . . . Okay, half a pizza? (Student 3 nods) So could, do we have enough to give each of them half of another pizza.

Student 3: Yes.

....  
Student 2: (places 1 leftover pizza on the left table) They both get, the two tables get two.

Teacher: So you put two more here [at the left table]?

Student 2: Yeah.

Teacher: And then what?

Student 2: (places 1 more leftover pizza on the left table and the last two leftover pizzas on the right table) Two more there.

Teacher: Now is that fair?

Student 2: Yes.

Teacher: So how many pizzas are on each table now?

Student 2: Fourteen.

Student 1: Fourteen plus fourteen (writes on white board) is . . .

Student 2: Twenty-eight!

The first student expected the dealing to come out evenly, perhaps because her thought to deal was based on the divisibility of 28 pizzas by the 2 tables, which was confounded by dealing to the people rather than the tables. The second student reiterated that it was not fair, because there were still pizzas left, which showed an application of one of the criteria for equipartitioning, that the “collection” or “whole” (all of the pizzas) must be exhausted. She suggested giving each person 2 pizzas, but after dealing a full second round, saw that there were in fact still 4 pizzas left, confirming the notion that dealing the pizzas in a one-to-one correspondence with the people would not come out even. When the third student stated that half of 28 is 14, none of the students offered any indication of what that meant in context, perhaps again because they were trying to coordinate the pizzas with the people and not the tables as groups. Picking up with their reasoning from 2 pizzas for each person and 4 pizzas leftover, the first student decided to place 2 of the remaining pizzas on each of the two tables, and the students determined there were 14 pizzas at both tables. When asked whether that was fair, the first student said it was, because there were the same amounts at both tables. The group then abandoned determining how many pizzas each person would get and simply

took this as their answer, again indicating they were not ready to determine the fair share itself, but very capable of using a form of co-splitting to determine a viable arrangement.

In another group of two students working on the same problem, one initially stated that she guessed five tables, while the other said that 4 times 3 is 12, so there would be 4 people at each of three tables. Both students then attempted to deal using manipulatives to work out their answers. The first student found that five tables would not allow for the people to come out evenly and therefore began working along with the second student, who had already dealt out the people as represented by her multiplication fact. The first student, perhaps realizing why they were stuck on the pizzas – since 28 is not divisible by 3 – suggested taking one table away to make it just two tables. After doing so, they used a reallocation strategy to adjust the numbers of people at each table, going from three groups with 4 people in each, to two groups with 6 people in each. They were then able to deal the pizzas pair-wise, one to each table simultaneously, until all of the pizzas were used and there were 14 pizzas at each table to go with the 6 people at each. The class was brought together to discuss this strategy and the following conversation then took place:

Teacher: How do we know that everybody at these two tables [with fourteen pizzas and six people each] is still going to get a fair share?

Student 1: Because half of twenty-eight is fourteen, and half is twelve is six.

Teacher: He said “*half* of twenty-eight is fourteen, *half* of twelve is six” (writes both statements on the board), and why would we be thinking *half* in this case?

Student 2: Well, if you see the twenty-eight pizzas and the twelve people, you would think because in the problem it says it’s not enough for everybody to be seated, so you would need about like, like half, like two tables to make it so each would be half.  
 Teacher: Okay, so when we decided we were going to use two tables, we did what kind of split?

...

Student 3: A 2-split.

Teacher: A 2-split. And when we did a 2-split, what were we actually splitting? What got split here?

Student 4: The twelve pizzas, the twelve people and the twenty-eight pizzas.

Teacher: Both of them?! . . . Yes or no?

Student 4: Yes.

Teacher: So we split both the pizzas and the people. And did we do the same split on both of them? Are they both 2-splits? . . . I did half of each, right? So is that a 2-split on both pizzas and people?

Students: Yes.

Teacher: How would I know somebody sitting here [at a table with fourteen pizzas and six people] is going to get the same fair share of pizza as if they were sitting here, at all one big table [the original table with twenty-eight pizzas and twelve pizzas]? . . .

Student 5: Well, if it was like two tables mixed together to make one big table, it would be those two halves mixed in with the twenty-eight pizzas and twelve people.

Teacher: So, I think you're saying I could . . . think of them as being pushed together like that (teacher adjusts drawing on the board; see Figure 17). . . . Anybody else have another way to think about how we know they're going to get the same amount of pizza?

Student 6: Because they're going to get the same amount because, if you add the two together, like the twenty-eight, and each one would get fourteen.

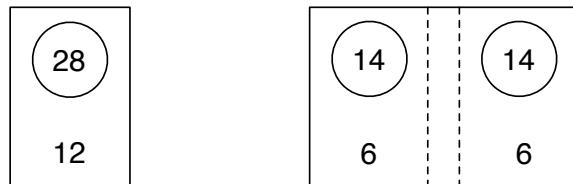


Figure 17. Combining ratios (tables) using streetland diagrams.

When I asked why it was believed that each person at the two tables would still get a fair share, the first student responded that *half* of 28 is 14 and *half* of 12 is 6. Several students were then able to describe taking *half* of a quantity as a 2-split. I then asked what was actually being split and the fourth student replied that *both* the pizzas and the people were being split. This represented the first, informal, definition of co-splitting offered by the students. In response to whether a person at one of the two tables would get the same fair

share as the people sitting at the original table (12 people and 28 pizzas), the fifth student said they would because the original table was the same as just pushing the two tables together. The sixth student also said they would, but because it could be thought of as just adding the numbers together ( $14 + 14 = 28$  and  $6 + 6 = 12$ ). This discrepancy shows that some students were moving towards an understanding of co-splitting that justified fair shares while others still had a misconception about summation preserving the ratio.

I then asked whether anyone had determined another arrangement, and two groups stated they had done it with four tables, each containing 3 people and 7 pizzas. They identified this as a 4-split and recognized the split was the same as the number of tables. One student said all of the people at the four smaller tables would get the same fair share because the numbers of pizzas and people were the same at both. To probe further, I asked them whether *both* the numbers of people and pizzas *needed* to be the same for the share to be the same. The student said they did, but could not elaborate, and no one else responded, so I decided to pose a question presenting a possible counterexample: would everyone get the same fair share if there were two tables, one containing 3 people and 5 pizzas, and one containing 3 people and 7 pizzas? Several students agreed that they would not, and one said it was because the pizzas were uneven, and the people with 7 pizzas would get more.

Returning to the original problem, I then asked whether a person at one of the four tables would get the same fair share as a person at the one original table, with all 28 pizzas and 12 people. One student responded that they would, because 7 times 4 is 28 and 3 times 4 is 12. I then asked how that told her the fair shares would be the same, and she said it was

because they were equal. In order to clarify what she meant by “equal,” I suggested that 6 times 2 also equals 12, and asked why it was important to think about 3 times 4 rather than 6 times 2. She stated that it was because there were *four* tables with 3 people at each, and made similar statements about 7 times 4, concluding that the 4 in both multiplication facts represented the tables, just as it did in the diagram, and just as it represented the split.

I then presented the students with a new problem to work on in their groups – 24 pizzas among 18 people – and they were encouraged to determine more than one solution. All of the groups were able to come up with one solution rather quickly, most using two or three tables. Several students indicated focusing on one quantity first, and then checking the other quantity for a determined number of tables. One student dealt 1 pizza to all 6 people at each of three tables, and then split the 6 remaining pizzas among the three tables by adding 2 more pizzas to each (not assigning the pizzas to any one person), which was a similar strategy to one the student had previously used. A few groups also determined a solution involving six tables, and two groups determined all three ways – two, three, and six tables.

All of the students then worked on the problem of 35 pizzas being shared among 15 people to determine as many solutions as possible using LPPSync. One student attempting five tables using the splitter tool had copied the 35 pizzas and 15 people to the tables, which resulted in 175 pizzas and 75 people total. The following conversation ensued:

Teacher: That’s a lot of pizzas. How many are we supposed to use total?  
Student: Thirty-five.

Teacher: Thirty-five. So can we figure out how to make it be only thirty-five pizzas? How many tables did you do? . . . So how did you come up with five tables?  
Student: Because five times seven is thirty-five.

Teacher: Okay, five times seven is thirty-five, so what should that seven be doing?

Student: Seven, tables?

Teacher: We could try seven tables, but if we have five tables already and five times seven makes thirty-five . . . So how many pizzas at each table? We want thirty-five.

Student: Seven.

Teacher: Right now you have seventy-five people. We only need fifteen.

Student: So, less tables?

Teacher: We could try less tables, but do you think there's a way to make it work with five tables, to get fifteen people total? . . . What if we only had one person at each table, how many people would that be?

Student: Seven.

Teacher: How many tables are there?

Student: Five.

Teacher: So if I had one person at each table, I have how many people?

Student: Six. Eight.

Teacher: (using manipulatives) So one [person] goes on each, [which] makes five. Is that enough? (student shakes his head no) So how many [people] could we try?

Student: Ten.

Teacher: Ok, if I have ten at each table, how many does that make? . . .

Student: Fifty.

Teacher: We only want fifteen.

Student: I don't know. Five? (student attempts five on LPPSync)

Teacher: So does it need to be more or less than five?

Student: Less! . . . So, two [at each makes] ten. One [at each makes] five.

Teacher: But we need fifteen.

Student: Three [at each makes] fifteen.

Teacher: Why does that make fifteen?

Student: Because . . . I don't know.

Teacher: What did you tell me about the thirty-five and the five tables? . . . You put seven [pizzas] here because why?

Student: Seven times five is thirty-five.

Drew: Okay, so why would we, why do you think we would put three [people] there?

Student: Because . . . five times three equals fifteen.

When asked why he had chosen to work with five tables in the first place, the student stated a seemingly relevant fact. However, when I asked what 7 represented in the problem context, since nothing on the screen in LPPSync indicated a 7, the student thought maybe he should switch to seven tables. Eventually he said there should be 7 pizzas at each table, but in

moving to the people he began guessing again rather than going for another fact, starting with 10 people at each table, and then 5 at each. This seemed to indicate he had forgotten that the 5 from the tables should still be part of the multiplication fact, but also that he had some sense of multiplication with 5s being related to numbers such as 5 and 10. Eventually, he stumbled onto what was actually being multiplied, going from 5 people at each table (making 25 total), to 2 people at each table (10 total), then 1 person at each table (5 total), and finally, 3 people at each table, making the desired 15 total. Although this student was attuned to the multiplication facts being related to the problem in some way, he was only able to cite the proper facts after going through a sequence of guessing and checking.

Another student described arriving at this same solution of 7 pizzas and 3 people at each of five tables using the multiple wholes strategy of deal and split (Figure 18). First, she dealt 2 pizzas to each of the 3 people, resulting in 1 remaining pizza. Then, she split the remaining pizza into thirds, so that one-third could be dealt to each of the 3 people. However, despite the fact that her work showed two and one-third pizzas for each person, she wrote this as the fraction two-thirds, so I took the opportunity to work with her on how to represent mixed numbers, until she was able to represent her answer properly.

When asked whether the people at each of the five tables would get the same share as the people at the one original table, another student said they would because they had done a 5-split. Unfortunately, I did not follow up to determine whether this was representative of PEEQ reasoning – all 5-splits yield equivalent results – or just a response used because the term “splits” had been used previously. A few students attempted to use seven tables, but

determined that a 7-split only worked for the pizzas and not for the people, which I used to re-emphasize that for co-splitting, the same split had to be used on *both* quantities. No one attempted to use three tables, which perhaps indicates that students were considering the factors of both quantities, recognizing that thirty-five is not divisible by three, even though fifteen is. No students were seen using a 2-split either, which shows it is not always the first split they attempt, at least not when odd values are given values for both quantities.

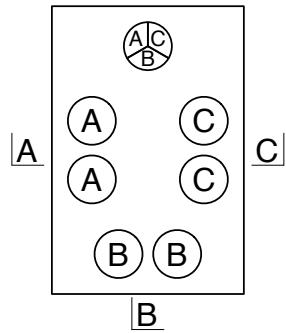
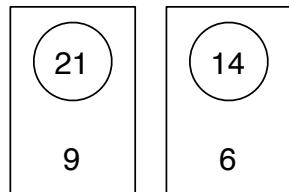


Figure 18. Deal and split strategy for determining fair share: 7 objects among 3 sharers.

During whole class discussion, I claimed to have determined a solution using two tables, and questioned whether the students believed me. Most students said no, 2-splits wouldn't work because the values were odd – specifically, that nothing (in the whole numbers) is half of 35 or half of 15. I then represented my solution on the board (Figure 19) and asked whether I had used all of both quantities. The students agreed I had, so I then asked whether a person at each table would get the same fair share, to which most students responded they would. One student said it was because the table with more people also had more pizzas – a qualitative compensation argument. When asked whether this was still the

same fair share as the original table, only some students said it was, and one still justified it as being the same simply because the values for both quantities summed to the original totals.



*Figure 19.* Teacher-presented solution to co-splitting task: 35 objects among 15 sharers.

Another student then asked if it could be done with three tables; although she called it a 3-split, she meant three tables with different values, but it is unknown whether she fully understood that splits had to result in equal amounts. I assisted the class in trying to determine whether this was possible by recording the suggested values on the board, beginning with 5 people at each table and then 10, 10, and 15 pizzas at those tables. Due to time constraints, the remainder of the conversation had to be saved for the next day.

#### *Day 5*

Extending the conversation from the fourth day, I presented one response for further discussion to begin the fifth day, asking the students how they knew a person at a table with 7 pizzas and 3 people (the result of a 5-co-split) would get the same fair share as a person at a table with 35 pizzas and 15 people. One student said the shares were the same because the tables were equal. Similarly to when another student used the term “equal” previously, I asked him to elaborate on what he meant. The student stated that 7 times the 5 tables is 35 pizzas, and 3 times the 5 tables is 15 people. I questioned further as to why that would mean

that the fair shares were the same, and he said it meant you could put the five tables together to make the one table. It was not clear how this student or the rest of the class was making sense of this combining of tables action – simply as sums, or as groups of ratio units – or whether they believed that the tables being combined *needed* to be identical (the result of a co-split).

In an attempt to clarify, I presented another example, first showing one table with 4 pizzas and 2 people, and then as a possible equivalent arrangement of two tables – one with 1 pizza and 1 person, and one with 3 pizzas and 1 person. I asked whether the people at the one table would get the same fair share as the people at the two tables, and the students gave mixed responses. One student who said the shares were not the same indicated it was because there were different numbers of people (1 and 3) at the two tables. I suggested pushing the two tables together, as the other student had just done. The students then said that the shares were the same, and a different student claimed it was because in each representation every person would get 2 pizzas. I asked whether this was solely because the two tables could be pushed together to make the same arrangement as the one table. Again, there were mixed responses, but one student now indicated that it did not necessarily mean that the shares were the same this time because the two tables being combined were not themselves identical, as they were in the previous problem. To close this discussion, I asked how the one table with 4 pizzas and 2 people could be arranged differently using more than one table that *would* still maintain the fair share, and all students quickly and correctly offered a 2-co-split as their

response. The students appeared to have reached a (temporary) conclusion that pushing tables together *was* sufficient justification, but *only* when the tables were identical.

I then presented the students with the problem of 12 pizzas being shared among 9 people and asked them to determine another possible arrangement. While the students were working, I constructed three possible responses on the board, labeled A, B, and C, as shown in Figure 20 below. In their own work, a few students unsuccessfully attempted to use a 2-co-split, but several other students were able to come up with constructed response A, using a 3-co-split. I drew their attention to this and asked them to justify the equivalence to the original table; they did so using multiplication facts ( $3 \cdot 3 = 9$  and  $4 \cdot 3 = 12$ ). We then moved on to discussing the other two constructed responses (B and C):

Student 1: Because the second table has 3 and the last table has the same number of people but 2. And it's just not equal, because ... and the second to last one has 2 and it's only one person.

...

Student 2: This is five pizzas, this three pizzas, this two pizzas, and this two more pizzas. Okay, so add all of this together and you'll get twelve.

Teacher: So if we add them all together it's the same number of pizzas?

Student 2: Yes.

Teacher: What about the people?

Student 2: Then it's two, and then one, and then two. So now you've got to add four, two, one, and two together, and you get nine people.

Teacher: Ok, so if I can add them both up to the same numbers it means it's a fair, same fair share?

Student 2: Yep!

Teacher: Does anybody disagree?

Student 1: Because, yeah you can add them up and then it gets the same numbers, but that doesn't mean each person will get the same amount at the end.

...

Student 2: Add all of these [values for the four tables ( $12 + 9$ )] together and you get this entire picture [of the total values for the original table (21)].

Students: No, you don't.

Student 2: Yes, you do!

...  
 Student 1: Since there's four people at that one table, how are they going to split that in half? Because ... And the second table, there's two people and three pizzas. I understand they could split that one in half, but the other is not equal.

Teacher: Ok, so maybe we should look at it a different way. . . . Does *every* person at that table (12:9; circles all 9 people) get the same fair share as *every* person over here (the four 4:3 tables; circles all 9 people)?

Students: No. Yes. I don't know.

Teacher: Do the same people still say yes that said yes?

Students: Yes.

Teacher: Same people still say no that said no?

Students: I'm between. Yes.

...  
 Student 2: I don't agree. . . . Because we're not talking about how they're going to split it, we're talking about if they're equal!

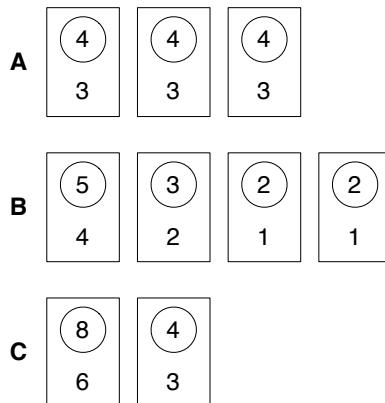


Figure 20. Teacher-presented solutions for co-splitting task: 12 objects among 9 sharers.

In addressing whether constructed response B represented the same fair share, most students said that it did not and the first student claimed that this was because each table had different numbers of pizzas and people at them. The second student claimed that it was the same fair share because the values for each quantity could be summed to their respective original totals. What ensued represented the first time during the teaching experiment that a

conversation filled with constructive argumentation took place between students. The first student responded back that just because the quantities could be added did not mean that the shares would be the same in the end, and went on to describe the fair shares (unit ratios) at each of the 4 tables, as one and something that was not half, one and one-half, and two. The second student still did not agree, and (incorrectly) claimed that the question at hand was not *how* they were going to share it, but rather whether the shares were the same.

A third student indicated that she believed the only time the fair shares would be the same was when the tables all had the same values for both quantities (i.e., from a co-split). A fourth student related this point to the argument from the second student about knowing what the fair shares were, and physically showed how to find the fair shares using constructed response A using a deal and split strategy. I then asked whether this was the same fair share as the original table and the students said it was, but did not offer any valid justifications. I chose to revert back to constructed response B at that point and asked what the fair share for each table in that response *should* be, if it were to be the same as constructed response A and the original table. The students were able to identify the fair share as also needing to be one and one-third pizzas for each person at each table there as well, which was not true as constructed response B was presented. The second student still held firmly to her summing strategy and remained unconvinced. I decided to move on and discuss it further with her individually, but too much time had elapsed to address constructed response C.

The students were then asked to work on the problem of 30 pizzas being shared among 24 people on LPPSync. One student stated that she was trying to first think of

multiples that would equal the 30 pizzas, and then trying to think of multiples that result in the same amounts of people. This was the first articulation by a student of co-splitting as an action of coordinating the splits between both quantities. However, in her work, it was still evident that she was focusing on the quantities one at a time, the 30 first by trying 5 tables and then 10 tables, and then checking whether it worked for the people (24). Despite her verbal claim, this is not truly a co-splitting action, in which she would be thinking of the 30 and 24 simultaneously. She went on to try 8 tables next, and then 3 tables, which indicates perhaps she flexibly switched to working with the number of people first.

Most students ended up with three tables and 8 people at each to start. A couple other students found an arrangement involving four tables, each with 6 people and  $7 \frac{1}{2}$  pizzas at them. When asked about the half pizzas, one student claimed that each person would get one and one-half pizzas. I had her draw this to see that there would not be enough pizzas; nonetheless, she was still able to state confidently that every 6 people would get  $7 \frac{1}{2}$  pizzas. Another student in a different group had used 3 tables followed by using 6 tables. I asked him a series of questions to explore whether he had used a composition of splits or thought about the multiplicative relationships between these arrangements and was relating them to quantitative compensation. The conversation went as follows:

Teacher: Can you see a relationship between your first answer (3-co-split) and your second answer (6-co-split)?

Student: Yes, a little bit. . . . That they both add up to equal this [one original table].

Teacher: How many tables did you use here [in your first answer]?

Student: Three.

Teacher: How many tables did you use here [in your second answer]?

Student: Six.

Teacher: How many pizzas [in your first answer]?

Student: Ten.  
Teacher: How many pizzas [in your second answer]?  
Student: Five.  
Teacher: And what's . . . ?  
Student: Oh, wait! I used half.  
Teacher: What do you mean half?  
Student: Because, half of ten is five. And half of eight is four. . . . I just did more tables.  
Teacher: But how many tables did you use?  
Student: Six.  
Teacher: Which is what compared to three?  
Student: Three more.  
Teacher: Is there another way we could say it?  
Student: Three times as many.  
Teacher: Three times as many? How many times? What does it mean to be three times as much?  
Student: Only two.  
Teacher: Two times as much?  
Student: Because two times three is six.  
Teacher: Okay. So we did two times as many tables. And what about the pizzas and people?  
Student: Half.  
Teacher: Half! Pretty cool?! (Student nods yes)

Although the student was not necessarily employing the reasoning I thought, he was clearly capable of understanding not only the connections between his arrangements, but also could incorporate ideas from the lower levels of equipartitioning into that understanding. Most students did not necessarily relate one answer to another without being prompted first, but when asked, they were typically able to provide a coherent and sufficient multiplicative response. The class came up with four different arrangements for the task (2-, 3-, 4-, and 6-co-splits), which were all presented and agreed upon as correct, but time ran out before there could be discussion of justification questions for equivalence of arrangements or fair shares.

*Day 6*

On the sixth day, I changed the context slightly and presented the students with the problem of 27 cookies being shared among 18 people, still implying tables as the grouping mechanism. Several responses were given: two, three, and nine tables. One student explained that she had determined 9 people would be at each of two tables, and then stated that 9 times 2 equals 18 and 9 times 3 equals 27. This pair of multiplication facts indicated that she was not associating the numbers in her facts with the quantities accurately, and did not recognize the disconnect of a 9 appearing in both of her facts, rather than a 2 representing two tables.

I asked the class whether it was possible to split 27 cookies between two tables, and several students began attempting additive facts with whole numbers, but all decided that none of those would be fair, or even. One student suggested that the total number of cookies would need to be even for that to work, which indicated that a whole number of cookies at each table was considered a requirement of the task, although the use of only whole-number values had never been explicitly stated as a requirement. It was not clear whether the inability to perform a 2-split on 27 was because most students were not ready to determine answers in which halves and other partial amounts of objects were used, or whether they simply believed cookies could not be split into fractional parts as easily as pizzas contextually.

The students then keyed in on the suggestion of three tables, and one student stated each table would have 9 cookies and 6 people. When asked how she determined three tables, she gave the facts of 27 divided by 3 and 18 divided by 3. To help clarify the misconception with the facts presented by the other student earlier, I asked what the 3s represented in each

fact, and it was correctly identified that they represented the tables. I then asked whether the people at the three tables would get the same fair share as the people at the original table.

Most students agreed they would, but again no justifications were given. The students then worked in groups, or alone, to determine another way to fairly share 27 cookies among 18 people besides the one given table or the three-table answer already determined.

One student said she had the “perfect idea” of nine tables, but it would only work for the 18, which could mean her multiplication and division facts for 9s were weak, as 27 is divisible by 9. When asked why it was perfect for the people but not for the cookies, she responded that the cookies would need to be split in half. Perhaps she said half because she was still thinking about 27 being an odd number, but in conjunction with the two tables from earlier and not the nine tables she was suggesting. She was eventually able to determine how to use the correct facts to make nine tables with 3 cookies and 2 people at each. Another example of an underdeveloped sense of using facts to solve these problems was displayed by a different student, who had written  $27 \div 4$ , but when asked what the result of that division would tell him, he said he would need to determine the answer first. Here the student was unable to relate the factors to the context until a result had been determined to then help identify what each number represented. Weaknesses with multiplication and division facts relevant to the parameters in this problem were much more evident than had been seen on previous problems with at least one even value. Other students unsuccessfully attempted several other numbers of tables, including seven and five, perhaps because 27 is odd.

I then displayed both the 3-co-split and the 9-co-split answers on the board and asked the students to consider just one table from the 3-co-split (9 cookies and 6 people) as a new problem. Several students determined an equivalent arrangement using another 3-co-split, resulting in three tables with 3 cookies and 2 people, and we had the following discussion:

Teacher: Does that one [with three 3:2 tables] work? What split is that?

Student 1: Three-split.

Teacher: And how did you know that was going to work?

Student 2: Just use your multiples.

Teacher: What multiples?

Student 2: Three and . . .

Student 1: Well, I did addition, because three plus three is three, and plus three equals nine; and then if you count by twos: two, four, six.

Drew: So it adds up to this [9:6 table]. What if I did two tables (draws a 7:4 table and a 2:2 table) that were like that? Does that work?

Student 1: Yeah, because it adds up to these two numbers (nine and six).

Teacher: So is everybody at this table [with seven and four] going to get the same share as everybody at that table [with two and two]?

Students 1 and 2: No.

Teacher: Why not?

Student 2: Because that one has seven and that one has two. So that one has more than that, and that one has less than that.

Teacher: So all that matters is the cookies?

Student 1: And the people.

...  
Teacher: So what does the fact that it adds up tell us? Because these (7:4 and 2:2) add up, but you said this one (3:2, 3:2, 3:2) works and this one (7:4, 2:2) doesn't.

Student 1: Because [in] these [with three and two], each one has a fair share, because . . . there's the same amount of people at each table.

Teacher: Okay. So the only way we can use this (addition) is if they're all the same? (Student 1 nods yes) If they're not the same, that [addition] doesn't mean anything. Is that what you're saying? (Student 1 nods yes)

Although the first student claimed it would work by citing multiplication facts, the second said she had used addition. This brought into question whether the summation misconception still existed for the justification of equivalence. In other words: was this student using

addition here because the numbers were relatively small, or was she actually beginning to develop an early understanding of ratio as “so much of this for so much of that”? When I presented the counter-example, the first student said the fair shares would not be the same, and the people with 7 cookies would get more. She then went on to claim that addition only worked when all of the tables being added were identical. This may simply indicate that adding does work after this stipulation is met, or perhaps it shows a belief that as one quantity increases by a set amount, the other quantity also increases by a set amount.

When going over all possible ways to share 9 cookies among 6 people, one student suggested another way of using two tables that contained different values for the corresponding quantities at each: one table with 6 cookies and 4 people, and one table with 3 cookies and 2 people. The discussion that follows, of whether this was a correct arrangement, represents the first time students showed clear signs of transgressing summative reasoning:

Student 1: Another way is two tables. Four people. . . (teacher writes as a 2-split, with four at both tables) . . . But can you make that a two? And on top of that two you put three. On top of that four you put six.

Teacher: Does everybody think these people [at the two tables] are going to get the same fair share as these people [at the one original table]?

...  
Student 2: Well, four plus two is equal six. But six plus three equals . . . nine. But I think, but the thing is, I don't think it's right because that table right there with the six and the four, they won't [get] same amount of pizza. Well, they will get the same amount, but like . . .

Student 1: They will.

Student 2: But it's equal. They, like, get like a different kind of number of pizza than the others.

Student 1: They both get the same number of pizzas.

Teacher: So, one way people have been checking these is by just adding the number of pizzas and the number of people. Does everybody see how six plus three equals nine; four plus two equals six? . . . So that seems okay, but look at this one (draws 7:5

and 2:1 on the board). Seven plus two equals nine; five plus one equals six. And are these people [all] going to get the same fair share?

Students: No.

Teacher: Because this person [at the table with two and one] gets how many cookies?

Student 2: One.

Student 3: Two.

Teacher: Two cookies. This [other table with seven and five] is five people, right? So if they *were* going to get the same, how many cookies *should* there be?

Student 3: Two.

Teacher: Two for each person, so how many cookies should there be?

Student 2: Ten.

Teacher: [So we don't] have enough cookies for them to get the same fair share. [Student 1], you said you could prove that these people were going to get the same.

Student 1: First, each one's going to get one and a half. . . . If you add it all up it equals up to 9 cookies.

Teacher: Add what up?

Student 1: One and one-half.

Teacher: How many of these [one and one-halves] am I going to write?

Student 1: Six.

Teacher: And how did you know that was the same down here [at the two tables with 6:4 and 3:2]?

Student 1: Well, I know that there's, there's the same amount; if you add four and two, it's the same amount of people.

...

Teacher: Well how about everybody think about it. Just picture in your head, there's a table with two people and three cookies. Could they share those?

Student 4: They can't, because . . . one person will have to get two and one person would get one.

Teacher: Do they have to? This one could go here [and this one could go here]. What could we do with that last one?

Students: Split. Half.

Student 3: Split it.

Teacher: Split it in half. So how much does each person get?

Students: One and one-half.

Teacher: Same thing that [Student 1] said up here.

One student suggested that the people would all get the same amount, referring to the fair share (unit ratio), and yet another suggested that the tables were equal, but a different kind of equal, which represented an early statement of equivalence of ratios. I utilized this chance to

help nullify the summation misconception more permanently by presenting one table with 7 cookies and 5 people and another table with 2 cookies and 1 person, and claiming these were equivalent to the original table. The students agreed they would not be the same, based on the fair shares. Then, going back to the original problem, the student that had referred to the unit ratio previously said that each person should get one and one-half cookies, so I asked how she determined one and one-half. She responded that it added up to the original total, referring to all of the one and one-halves, but could not explain mathematically where one and one-half was coming from in the first place.

The groups then all set out to determine whether one and one-half cookies was in fact the share for the people at each of the tables with 6 cookies and 4 people and the table with 3 cookies and 2 people, which almost all of them were able to do for both using deal and split strategies. One student still had trouble naming the shares, and at one point had re-unitized to half cookies, claiming that the fair share was 3 for each person. I asked the class how using a 3-co-split could be thought of as the same as one table with 6 cookies and 4 people and another table with 3 cookies and 2. One student explained how he had already worked it out that the table with 6 cookies and 4 people was the same as two tables with 3 cookies and 2 people at each, through using a 2-co-split.

I then asked what could be done with one of the other 9 and 6 tables (since there were three total), leading the students a bit by reminding them that the one 9 and 6 table they just did work using a 3-co-split. The students echoed back to do the same thing, which could be

interpreted as PEEQ reasoning in co-splitting, in that the same split on the same “wholes” will yield the same result, and then I guided them as follows:

Teacher: I did a a 9:6 table with a 3-split, and I got 3:2, 3:2, 3:2. Now, I want to know what I could do with a 9:6 table.

Student 1: Split it in half.

Student 2: Do the same thing.

Teacher: In half?

Student 3: Nine.

Student 4: You could do the same thing.

Teacher: What's the same thing mean?

Student 2: Same thing we just did.

Teacher: Well what do we call that?

Student 2: 3-split.

Teacher: 3-split. Agree? . . . So if I do a 3-split here [on the second 9:6 table], what could I do with this [third] 9:6 table?

Student 4: Same.

Teacher: Which is a . . . ?

Students: 3-split.

Teacher: How many [3:2] tables [did we] end up with total?

Students: Nine.

Teacher: Do they all have the same numbers?

Students: Yes.

Teacher: So if [we] have nine tables that all have the same numbers, what is that the same as?

Student 5: Twenty-seven pizzas and eighteen people.

Teacher: Okay. It makes my total of twenty-seven cookies and eighteen people. But how else could I think of getting to this result? Here (27:18 to 9:6 to 3:2) we had to do two steps, right? I had to do a 3-split, then 3-splits. How could I go straight from 27:18 to nine tables with all the same thing?

Student 4: You could divide the numbers from the tables. Like nine divided by the number of tables, which is three. You could've got three. And then if you do six divided by three, it equals two. And you could've done that.

Teacher: Okay. From here (9:6s), right? We can divide by three – that's doing our 3-split. But what if I wanted to skip this step? Nine tables obviously works. Could I go straight to nine tables from here (27:18)?

Student 5: So twenty-seven would equal the nine and eighteen would equal the six, so if you skip that one . . . all those threes would equal the twenty-seven. And all those twos would equal the eighteen.

Teacher: Good. How about everybody look at this side of the board (the 9-split that was drawn earlier); the very first answer you gave me. How many tables?

Students: Nine.

Teacher: Were they all the same?

Students: Yes.

Teacher: Are they all exactly the same as this (the nine 3:2s from the consecutive 3-splits)?

Students: Yes.

Teacher: And what did we say we did [the first time]?

Students: 9-split.

Getting to the point of this exercise, I asked the students how they could justify those two processes (a 3-co-split followed by another 3-co-split and a 9-co-split) yielding the same result, based on the composition of co-splits. There were mixed responses about adding and multiplying, but it seemed that the students who had cited addition were still somehow focusing on the sums of all the individual values for the quantities, not the splits. Finally, a student suggested that 3 times 3 is 9, and I connected this back with composition of splits on a single whole.

I then presented students with the first problem that specifically addressed the third research question, in which the number of objects was fewer than the number of sharers – 24 cookies shared among 30 people. I wrote the parameters for several previous problems on the board to remind the students of them, and asked what was different about this new problem. After several unproductive (albeit true) statements, I directed their focus to the values within each of the problems, noting which numbers were larger. The students then had the collective realization that there were more people than objects in the current problem. Finally, and perhaps too directly, I asked whether that meant they would need to change the way they thought about sharing. At first they indicated it might, but when the question was posed a different way – whether 24 cookies could actually be shared among 30 people – the students

said that sharing was possible (an application of the Continuity Principle of equipartitioning), and they believed it was still really the same type of sharing.

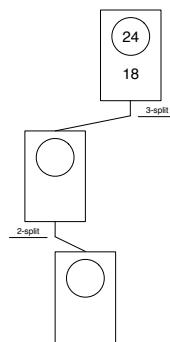
Several students quickly came up with *both* correct 2- *and* 3-co-splits, showing their increased adeptness in solving co-splitting problems. This also indicated that the variation in parameters did not affect the students' ability to determine correct solutions and multiple arrangements. A few students performed 6-co-splits, and some used a composition of co-splits, following a 3-co-split with a 2-co-split. One student did not realize she had already determined this same answer from a 6-co-split until after completely working through the composition, indicating she did not expect the results to be the same. Going over similar answers with the whole class, I pushed the students towards an abstraction of composition of co-splits and asked what the result of a 7-co-split followed by a 4-co-split would be. The students were not able to come up with the answer right away, but with some guidance they eventually cited the multiplication fact  $7 \cdot 4 = 28$ , which I related to a 28-co-split in context.

#### *Day 7*

To begin the seventh day, I checked to see how the students would articulate the meaning of co-splitting again. Several responses were given: divided equally, splitting into groups, splitting in half, fairly sharing, 2-split, 3-split, etc. Based on the variety of responses but a seeming lack of acknowledging both quantities, I decided to add an example of a co-splitting problem (12 objects being shared among 8 sharers) to help the students talk about the concept. One student then stated that the objects had to be split for the people. I re-

emphasized that the original question was about co-splitting, so “the ‘objects’ need to be split and . . .,” to which a student added, the “people” needed to be split as well.

I then presented the students with another variation on co-splitting problems, in which the situations were only presented as Streetland diagrams, and there was no context necessarily given, but the objects and sharers were assumed by familiar positions in the diagrams – objects on top in the circle and sharers on the bottom. Additionally, indications of some splits and values, along with some blank diagrams, were given, as shown in Figure 21 below. Although the splits were not given explicitly as co-splits, students seemed to make that interpretation anyway based on the presence of the two quantities. The purpose of these tasks was to determine whether, and how, the students could work with the recursive nature of co-splitting, and if they were able to determine solutions without filling in all of the tables at any given level of co-splitting. If they could do so, it would indicate an understanding that all of the resulting tables should contain equal values for both quantities.



*Figure 21.* Tree diagram presentation of co-splitting task.

The first set of parameters was 24 objects shared among 18 sharers, with the initial split indicated to be a 3-co-split and the second split a 2-co-split. Most students drew in the other tables for the splits, and they cited various reasons, including not being able to do it in their heads. One student said, “Look, and *they gave it away* because they said 3-split, so I, if I didn’t draw that it wouldn’t look right!” Some students only drew in tables on the third level from the given table at the second level and not from the others they had added. I checked to make sure one student understood the resulting arrangement still needed to represent the original values by asking how two tables with 4 objects could represent all 24 objects:

Teacher: How many pizzas do we have on this [third] level right now?

Student: Four.

Teacher: And [with the other table you have drawn]?

Student: Oh, all together?

Teacher: Yeah.

Student: Eight.

Teacher: So how could we show that it’s twenty-four?

Student: You go all the way back. This one [two 4:3 tables] is the same as this one [8:6 table]. So if you have all of these [other 8:6 tables], it would be two, four, six. There would be six of these [4:3] boxes, will equal up to [those three 8:6 boxes], will equal up to that [one 24:18 box].

The student responded by saying that there would be two such tables where she had drawn them, as well as two more for each of the other two tables above with 8 objects and 6 sharers each, thus making it the same “all the way back up.” Another student was still struggling with either the representation or splitting in general, but it was not clear which, so I asked her what would go in the blank for “\_ • 3 = 24,” instead of how to split 24 objects into three groups, to which she responded “eight.” Then, similarly when asked about the 18 sharers, she said “six,” because 6 times 3 is 18, before I finished writing out the fill-in-the-blank fact.

Going over this problem with the whole class, I did not draw all of the tables but only filled in the values required by the tasks on the students' worksheets for each level, as students suggested them. Then, after the students indicated that there should be six tables at the third level, I asked how that was possible since a 2-co-split determined the values. One student stated it was because the six tables resulted from all three tables at the previous level, indicating a composition of splits, and that the 2-co-split was for each of the three tables.

The next task was implied as 36 objects among 30 sharers; however, this time the students were also required to determine the initial co-split based on the fact that a table at the second level was shown to have 12 objects. One student, self-admittedly for most problems throughout the teaching experiment, relied on using manipulatives or pictorial representations in order to deal objects and sharers. In determining the values for third-level tables, she represented 12 objects and 10 sharers and indicated that a line could just be drawn down the middle – a 2-co-split. She had already represented two tables coming from each of the three tables at the second level, which led her to say that there would then be six such tables. I asked whether this made sense for her work to result in six tables when she had just done a 3-co-split followed by a 2-co-split, and she said it did because 3 times 2 makes 6.

One student generated a combining ratios strategy to build-up from the resulting tables in a previous level, and not in an inverse co-splitting manner. The first answer he gave like this was for the problem of 35 objects and 15 sharers, in which he determined a solution using two tables – one with 21 objects and 9 sharers, and one with 14 objects and 6 sharers. I praised the student for his unique solution and on the spot, made up a special, bonus problem

for him (40 objects shared among 25 people, with the second level defined by a 5-co-split, and the third level containing two tables, not resulting from a co-split). Within minutes, he replicated his strategy and correctly solved the problem. He latched on to this newfound strategy and applied it correctly and often for several other problems. There were no instances in which he misapplied the strategy, but it was not clear whether it might have actually hindered him from determining more straightforward solutions, such as those resulting from multiple levels of co-splitting. Clearly, his strategy would have been more difficult to determine without being given an initial co-split, and it is unclear whether he was able to determine multiple co-splits on any problems prior to determining the strategy.

At the end of the seventh day, I had the students go over their answers and asked them about the equivalence of fair shares between levels, to which the only responses given involved stating multiplication and division facts. It was unclear whether this was due to the recent emphasis on such facts to describe compositions of co-splits, or because the students were beginning to recognize multiplicative justifications as the *only* fully explanatory way to respond; or, whether it was just a coincidence. I then allowed the student who had determined the combining ratios strategy to share his work with the other students, and asked how an equivalence of fair shares could be claimed for these types of solutions, in which the tables had different values for each quantity. Most students seemed to be getting tired at this point in the day, but they did agree that grouping sets of ratio units that were already identical, and thus represented the same fair share, would still maintain the fair share.

*Day 8*

The eighth day, I presented the students with the challenge of determining as many unique solutions as possible for the co-splitting problem of 48 objects being shared among 36 sharers. Similarly to the problems from the previous day, no specific context was given, and the students worked both individually and with their groups. As new answers were determined, I gave the students a sheet of poster paper to copy their answer and work onto, and then post it for display on the board.

The class collectively determined *nine* different solutions, or variations in work leading to solutions. Only one student attempted a 2-co-split as the first strategy; the first three submitted solutions were a 6-co-split, a 3-co-split, and another 6-co-split represented as a composition of a 3- and a 2-co-split. The one student trying the 2-co-split was able to determine 24 objects for each table, but then struggled to determine the number of people for each table. This showed the student's skills were still weak, thus providing further insight as to why he may have been the only one to employ the most basic strategy. Several strategies and justifications that have been mentioned previously were seen throughout; so only those that were new or are noted as having some other importance are discussed below.

Several students utilized Streefland diagrams to determine solutions resulting from a composition of splits. After a 4-co-split was posted and acknowledged as a solution by one student, a different student immediately attempted to try an 8-co-split, perhaps either because of the evenness of the parameters, or applying a multiplicative adjustment strategy. However, he assigned 4 sharers to each of his tables and when asked what that would make in total, he

correctly identified half as 16, but then claimed that it would make 36 when including the other half. Although he erred on his multiplication fact of 16 times 2, he motioned to show that he was essentially envisioning the 8-co-split as two 4-co-splits. This coincides with a multiplicative adjustment strategy, attempting an 8-co-split immediately after seeing that a 4-co-split worked, but shows even more flexibility in that he had really done four 2-co-splits. After I pointed out the mismatch of his total number of sharers (32) and the actual total number of sharers (36), he quickly revised his values to show four and one-half sharers at each table. In later discussion, the students all recognized that the result of this 8-co-split was different from all of the other solutions displayed because it involved non-whole number values. It became evident they did not recognize this rational value was for the sharers, and thus would have been referring to four and one-half people, which would not make sense contextually; however, it was still a viable co-splitting action and was acknowledged as such.

Also using a composition of co-splits, another student claimed she was trying to use a 4-co-split to determine another solution, in spite of the fact that she was the one who had already determined a 4-co-split. It turned out she was referring to a 4-co-split on a 3-co-split, so it *was* actually a different solution. I asked how many tables she expected to end up with, and at first she displayed a misconception for composition of splits, claiming that there would be seven tables (because four plus three is seven). She eventually resolved this, but it took physically drawing out every table to determine there would be twelve tables, and then filling in every value and checking the sums before she could state for certain it was a viable solution. Another student justified the same result of twelve tables as a composition of splits

by citing the multiplication fact of four times three, but then also added that this was because she was making four new tables *for each* of the three existing tables. A 12-co-split without a composition of splits was also offered as a solution by a third student.

A few students also created tables that were not the results of splits, and therefore containing different values at some of the tables, including the student who had been working with the combining ratio units strategy. One solution he posted was built off of a 3-co-split where the tables each had 16 objects and 12 sharers, suggesting two tables – one of which was copied from the 3-co-split (16 objects and 12 sharers) and the other of which was a combination of the other two tables (32 objects and 24 sharers). When asked whether this would still represent the same fair share during the whole class discussion, a different student used the notion of combining the physical tables to both justify the result as having the same fair share as well as explain where the 32 and 24 came from in the first place.

During the discussion of all of the students' solutions, I asked which contained the smallest whole-number values for both objects and sharers, which the students correctly identified as the 12-co-split. They also went on to acknowledge that this was the largest co-split used. Following this, I asked whether the students believed the tables resulting from that 12-co-split could be split any further and still have whole-number values. The students said it could not, and one student stated that the 4 could be split but not the 3. Another student added that this was true because it would result in one and one-half (people), describing a 2-split. It appeared though, that this was not because of the fractional value for a quantity that may be representing people, but rather because the question was pertaining to whole-number

values in general. No one suggested a 4-co-split, which would have had a similar result, or a 3-co-split, which also would have resulted in a fractional value, but for the objects.

For the remaining portion of class, the students worked in groups to create videos in which they responded to several prompts about co-splitting, and each group had a different example problem to work with. Several students still identified co-splitting in general with particular splits – creating halves, thirds, etc. – and especially 2-splits. Some students were later heard using more appropriate language, such as splitting into groups or equal groups.

One student working with the example of 45 objects being shared among 15 sharers immediately went to a solution involving three tables, based on the multiplication fact of 3 times 5 equals 15, but he could not determine the corresponding fact for 45. He finally did figure out that 3 times 15 equals 45, and when asked whether everyone at all of the tables would get the same fair share, he first said that they just needed to be halved, but then corrected himself to say it was a 3-split, clarifying that it was just like doing a 3-split on each of the pizzas individually. He then carried this out to demonstrate that the split-all strategy for equipartitioning multiple wholes would show the shares (the unit ratios) were the same.

### *Day 9*

On the ninth day, I presented the last co-splitting context of drink mixtures made from two distinct ingredients – orange juice and water. The first problem involved 20 cups of orange juice to be mixed with 8 cups of water, and the students were first asked whether they believed the mixture would taste more “orangey” or more “watery.” Most students said it would either taste more watery or taste equally of both ingredients. To help establish a

baseline I asked the students what drinks of just cups of orange juice, and drinks of just cups of water, would taste like, and they said just orange juice and just water. Then, I restated the question for mixing 20 cups of orange juice and 8 cups of water, and the students now said the mixture would taste more “orangey,” because there were more cups of orange juice.

The tasks here were posed for the students to determine a way the mixture could be created using more than one pitcher, *and* using the smallest whole numbers of cups of each ingredient (targeting the base ratio) in each of those pitchers so that all of the drink mixtures would still taste the same. It was left to be discovered through their work whether the students would introduce or interpret the meaning of the same fair share in this sense, and whether any of them would associate it with being the fair share of one ingredient to the other – in other words, containing the same unit ratio of one ingredient to the other.

A few students quickly suggested 10 as a response, referring to the orange juice, and one added that it could be done with 4 cups of water and 10 cups of orange juice. The students had not explicitly said so, but the number of pitchers was implied to be two. I showed how this could be represented by drawing a line down the middle of the physical representation of all the cups of both orange juice and water, which were pre-organized into an array on the students’ sheets (see Figure 12, p. 77). I then asked whether the mixtures in each of those two pitchers would taste the same, or if one would be more orangey or more watery. One student said it would taste the same because it was half and half. I asked “half and half of what,” and the student said it was half of the orange juice and half of the water.

I then asked whether the mixture in one of the smaller pitchers would taste the same as the mixture in the one larger pitcher. A few students said it would not, and one added it was because there were “too many” cups of orange juice and only 8 cups of water in the larger pitcher, so it would taste more orangey. Although he cited the number of cups of water, his judgment was based solely on the overall numbers of cups of orange juice rather than actually coordinating the two quantities. I asked the class whether the water mattered, but received another solution from another student in response instead – 5 cups of orange juice and 2 cups of water. Again, it was not stated outright that he was using four pitchers. It was also not clear whether this answer stemmed from the original problem or was further work from the first solution given, as a composition of splits.

I asked how many pitchers of 5 cups of orange juice and 2 cups of water would be needed to still use all of the original amounts and the student who offered the solution said four pitchers. I used the representation again to show this as a composition of co-splits by halving each of the original halves for both quantities. Then, I asked whether one of these smaller pitchers would taste the same as the original. The students gave mixed responses, but no one who said yes gave a reason. Instead of asking for reasons from those who said no, I presented the problem in the context pizzas and people, in which 20 pizzas being shared among 8 people had been 2-co-split twice to get four tables of 5 pizzas and 2 people each. I asked those who said the orange drink mixtures would not taste the same whether the people at a table with 5 pizzas would get the same fair share as the people at the table with 20 pizzas, and everyone responded boisterously, yes, they would get the same fair share.

Some students still disagreed that the pitchers would taste the same in the orange juice problem however, and the same student from the first discussion cited the fact that there were simply more cups of orange juice again. It could have been that he was thinking of water as not having any taste at all, and therefore the amount of water did not matter, or that the taste of the orange juice would not be neutralized by the taste of the water in a one-to-one manner because orange juice has a stronger fruity or sour taste than water has some opposite taste – both possibilities have been noted in previous studies with drink mixtures involving lemons or oranges and sugar water or water. Another student however, who also said it would taste the same, now recognized it was the same as the diagram drawn for pizzas, and claimed it would taste the same because 5 times 4 is 20 and 2 times 4 is 8. When asked to restate his idea, he shifted wording to say that each smaller pitcher was the same (equal values for each quantity), so they would taste the same, and they would taste the same as the larger pitcher but just have different numbers of cups.

A few other students, including the one who had been referring only to orange juice, were still not convinced. That particular student stated that for the pitcher with 20 cups of orange juice you would definitely taste more orange juice, because the smaller pitchers (with 10 cups of orange juice) had 4 less cups of water. I drew a physical representation of a new pitcher that showed the 20 cups of orange juice but only 4 cups of water, and all of the students agreed it would not only taste different, but specifically more orangey, than 20 cups of orange juice and 8 cups of water. I then responded to the student's comparison for the cups of water between the two pitchers, and asked whether there was another way to talk about the

differences in the cups of water between the two pitchers, besides four less cups. He quickly switched to multiplicative reasoning and said it was also half as many cups.

I asked what could be done to the 20 cups of orange juice in the pitcher with 4 cups of water that would make it taste the same as the pitcher with 20 cups of orange juice and 8 cups of water, since there was half the water. The student suggested taking half of the orange juice, to indicate a pitcher with 10 cups of orange juice and 4 cups of water. Although this was already on the board elsewhere, the students considered these as a new set of tasks. I then asked again whether this “newly created” pitcher with 10 cups of orange juice and 4 cups of water would taste the same as the pitcher with 20 and 8 cups respectively, and the students who had been disagreeing gave in a little, to say that “it kind of would.”

When I asked what the difference would be, the orange juice only student now said they were the same, in that both were half. He then added why the taste would still be a little different, because if it were just 20 cups of each, then that would taste the same. This indicated that some students were struggling more with the “sameness” of the contextual elements of the problem rather than the mathematics of equivalence in co-splitting. This contextual difficulty was reinforced by the fact that the same student could derive the solution strictly from the numbers and had some sense that it should maintain a form of equality – that it was the same, only with different values. However, he could not articulate a belief that it would taste the same because he associated that with “one-to-one-ness” of the quantities – equal parts of orange juice and water. Realizing this focus on the relationship within one pitcher, and not the relationship between two pitchers, I described differently how

the problem could look in context. I suggested actually mixing the 20 cups of orange juice and 8 cups of water together in the one big pitcher first, stirring it all around to be mixed evenly first, and then pouring half of what was in that pitcher into each of two smaller pitchers. When asked how much orange juice would be in one of the smaller pitchers if it were done this way, a few students said there would be 10 cups. Similarly for water, a few students said there would be 4 cups. I reiterated that it was all being mixed together first and then poured, and the students agreed this would not change the taste. They then also said the smaller pitchers would taste the same as the larger one. Finally, I asked about the taste of the pitchers with 5 cups of orange juice and 2 cups of water, if those were created by mixing first as well, and the students said the taste would still be the same there also.

To really see whether they were now getting it, I asked how a pitcher with 5 cups of orange juice and 2 cups of water would be the same as a pitcher with 10 cups of orange juice and 4 cups of water. One student said it was because half went into the smaller pitcher. To make sure the students were still in fact thinking about co-splitting and the original quantities (and not just one continuous, homogeneous solution), I asked whether the mixture from the pitcher with 5 cups of orange juice and 2 cups of water could be poured into a smaller pitcher and split again. One student said it would only work for the water, indicating a 2-split and a focus on the individual quantities, because the problem asked to only use whole cups.

Satisfied, I presented another problem, involving 16 cups of orange juice and 20 cups of water. One student immediately made the claim that it would taste “more watery.” It was not clear whether the student was thinking of his comparison statement within the new

problem (more water than orange juice) or whether he was in fact making the ratio comparison between problems, or whether he would have even distinguished between those two types of comparisons. I then asked how he would solve the new problem, and he said half of 16 is 8, half of 8 is 4, and half of 4 is 2. When utilizing this sequence of co-splits to determine other solutions, he tried an additional 2-co-split (from pitchers of 4 cups of orange juice and 5 cups of water) but only applied the 2-split to the orange juice, making pitchers of 2 cups of orange juice and 5 cups of water. I asked why the 4 had changed but the 5 had stayed the same, and the student said it was because the 5 could not go any smaller. This indicated he wanted to only use whole numbers as the task asked, but he was still not fully understanding the requirement of co-splitting to apply the same split to *both* quantities – here only applying it where it was convenient to maintain both quantities as whole-number values.

Another student was also asked whether he thought the new mixture in this second problem would taste more orangey or more watery, and he said more watery because there was more water than orange juice (making the within comparison). He was then asked whether his solution, a 2-co-split, would taste the same as the original mixture in this problem, and he said it would. I asked which mixture would be more orangey, the one from the first problem or the one from the second problem (the between comparison). He claimed that the one from the first problem would be more orangey, because there were more cups of orange juice, but it was unclear whether he meant more cups of orange juice as a count (20 versus 16), or whether he was actually thinking about the ratios of orange juice to water in the two mixtures. I pointed out that the second problem also had more cups of water than the

first problem (8 versus 20), but he still claimed that the second mixture would be more watery, now adding that the second was more watery (within) and the first was more orangey (within), which was a nice qualitative comparison between mixtures. Unfortunately, there was not time to come back to him once there was a third problem to refer to, and ask the same question about two mixtures that were either both more orangey or both more watery.

Working with the third problem – 36 cups of orange juice and 12 cups of water – a student claimed he was stuck because 36 was “not a good number” (perhaps too big, or too many factors). I asked about the 12, and he said it *was* a good number, but he couldn’t do anything with it because the 36 was “in the way.” On one hand this could be interpreted as a step backwards in terms of his ability to determine an answer, but on the other hand it shows growth in working with the concept of co-splitting, because he knew that he had to deal with *both* quantities, one way or another. He eventually determined four pitchers of 9 cups of orange juice and 3 cups of water through consecutive 2-co-splits. Only a few students were able to go further and come up with pitchers containing 3 cups of orange juice and 1 cup of water. This represents the unique case where the base ratio is also a unit ratio, but no students drew any other connections other than they were the smallest whole number pair of values. No problems were used in other contexts where the base ratio was a unit ratio, so it is uncertain whether there would be a different interpretation when the “sharers” are better defined and it represents a fair share.

*Clinical Interviews*

Three different clinical interview protocols were used during the teaching experiment, with some modifications being made while talking with a student under the semi-structured format, and if the protocol was used again, those changes were incorporated to be available for later students. Interviews were conducted on four days, with two students each day (except for one day when only one new student was available), and the same initial protocol for each pair of students interviewed on the same day. The first two interview protocols required students to further engage with co-splitting tasks using a fair sharing premise and providing them with manipulatives, in order to gather more information on individual understandings and strategies that may have been lost or missed in the whole-class setting, and at different times during the study. Four students completed these interviews – two on day three, and two on day four. The third interview protocol required students to engage with equipartitioning multiple wholes tasks using a fair sharing premise as well, but was created once the decision was made to drop that portion of the curriculum and because some multiple wholes strategies were already being seen on co-splitting tasks. Three different students completed these interviews – one on day seven, and two on day eight. The selections of students for each interview session were first and foremost out of convenience, as they did not take place during the allotted time for the teaching experiment, but occurred later in the afternoon of those days, so not all students were still on site and available. Then, out of those available on a given day, I selected the two students who had not been interviewed

previously that I believed would provide the most varied sets of data and information based on subjective (and objective) observations during the teaching experiment lessons.

The premade tasks used for the co-splitting interviews were 12 objects among 8 sharers, 16 objects among 6 sharers, and 18 objects among 12 sharers, with the possibility of reversing any of the parameters so that there were fewer objects than sharers to address the third research question. During the interviews for some students, I also added in the tasks of 6 objects among 4 sharers and 16 objects among 12 sharers, to see if any multiplicative reasoning was applied in the sense of quantitative compensation related to other tasks, with the same possibilities of reversal. Each of the students was able to determine at least one solution for each task, and 2-co-splits were the most commonly used. All of the students used the manipulatives to represent most of their solutions, but not all necessarily did so as dealing or using one-to-one correspondence as had been seen in previous work with other students who had not been given any co-splitting instruction.

On the task of 18 among 12, one student used an incremental adjustment strategy to successfully determine 2-, 3-, 4-, and 6-co-splits, ruling out a 5-co-split along the way. He also thought of trying a 12-co-split, but was not able to determine a solution using only whole-number values and therefore claimed it was not possible. Similarly, on the same task, another student found the successful co-splits and ruled out the unsuccessful ones all the way to the 12-co-split, and then claimed that was all that needed to be attempted despite there being more than 12 objects, because everything would be less than one. For the tasks that were added in which one or both of the quantities were a multiple of a previously presented

task, the students did offer some qualitative and quantitative compensation arguments when they were successfully able to determine solutions, stating that the shares or quantities would be half or twice those in the previous problems. In justifying their responses, some students were able to state that each group formed by the co-splits would represent the same fair share as one another because of the equality of the values, but one student was still questioning whether it was the same as the fair share for the original given amounts throughout, despite that being a required premise of actually determining a solution to the tasks.

The premade tasks for the multiple wholes interviews utilized the sample space from LPPSync that was developed from previous fieldwork, but also included some of the same pairs of parameters as the co-splitting interviews. Tasks that involve exactly one more object than sharer have proven to be the easiest in the past, and these are most likely to elicit a deal and split strategy; therefore, most students were given either 4 objects among 3 sharers, 5 objects among 4 sharers, or 7 objects among 6 sharers to start. For those students who were successful with 4 among 3, the next tasks they were given were 7 objects among 3 sharers and then 8 objects among 3 sharers. Tasks in which the number of objects is less than the number of sharers have been seen to be most likely to elicit the split-all strategy, so 5 objects among 6 sharers and 2 objects among 5 sharers were included. Finally, one task explicitly to relate to co-splitting was included – 6 objects among 4 sharers. As with the co-splitting tasks and protocol, the option to follow a task up with another in which the value for one quantity was a multiple of that used in the previous task.

All of the students were eventually able to solve every problem presented to them, and many previously seen strategies were also witnessed here: deal and split, split-all, and co-splitting. Somewhat surprisingly, no students immediately related these tasks to co-splitting and tried such a strategy first, although not all parameters were ideal for co-splitting either. Nonetheless, the students definitely understood a clear difference in the goals of the multiple wholes tasks – to determine the fair share – and the co-splitting tasks from class – to determine equivalent arrangements. One student however, did demonstrate a form of co-splitting for 6 among 8 after dealing 8 halves (from 4 whole objects), one to each sharer first, and then described sharing the last 2 objects among the 8 sharers as sharing 1 object among two sets of 4 sharers. None of the students were able to add fractions that resulted in either a proper or improper fraction to name the fair shares, but all were able to name the shares as sums of fractional (and sometimes, whole) parts (e.g., each person gets a half and a fourth). Further discussion of the connections among multiple wholes, co-splitting, and the lower levels of the equipartitioning LT are presented in the final chapter.

### Summary of Findings

#### *Research Question One*

*How do the lower levels of the equipartitioning learning trajectory have an impact on student interactions with and success on the upper level of co-splitting?* In interacting with the upper-level equipartitioning construct of co-splitting, the students in this teaching experiment relied on experiences with equipartitioning collections and employed dealing strategies early on. They particularly attended to the ideas of systematicity and one-to-one

correspondence when dealing quantities with provided manipulatives or as demonstrated by representations they constructed. This was sometimes done using composite units when the amounts of the quantities involved were large, and the size of those units was often determined in reference to multiplication or division facts. However, when relying on dealing to determine whether it would be possible for the second quantity to be evenly distributed after determining a number of groups (bowls, tables, etc.), the students typically dealt in a one-to-one correspondence with the first quantity, although some students learned to adjust this strategy and deal in a one-to-one correspondence with the groups.

Students then progressed to employ ideas of both qualitative and quantitative compensation, along with composition of splits, as they became more adept at solving co-splitting problems, but prior to abandoning the use of manipulatives altogether. The students called upon their knowledge of equipartitioning single wholes in naming and determining fair shares, and to justify equivalence in co-splitting situations. Some students called upon PEEQ reasoning when reaching more sophisticated forms of understanding of co-splitting, by realizing that same splits on same “wholes” (now a combination of objects and sharers) yield identical results, thereby deeming it unnecessary to write out the full results of every co-split in an arrangement in order to justify equivalence. The students implied the possibility of fair sharing in all co-splitting tasks as an application of the Continuity Principle of equipartitioning. This belief was clear when multiple wholes strategies of split-all and deal and split were applied as forms of co-splitting to identify the unit ratio ( $a p$ -split for  $n$  objects shared among  $p$  sharers). A few students used the concept of reallocation to aid them in

building up values through a combination of adjustment and partitive or inverse co-splitting strategies when confronted with the task of determining more than one equivalent fair-sharing situation for a given co-splitting problem.

The students in this study were not accustomed to explaining and justifying their solutions, particularly not by reconstructing the mathematics used to arrive at those solutions. This was most visible when students were asked to justify that a given solution to a co-splitting problem preserved the fair share. For equipartitioning collections and wholes, a sophisticated justification required the student to cite a strategy indicative of equipartitioning actions to produce a result beyond simply referring to that result as equivalent in name or form. Similarly, for co-splitting, a sophisticated justification requires the student to assert that if both the number of objects and the number of sharers are adjusted by the same factor or split, then the share remains the same – again focusing on the strategy and co-splitting action. When students did move beyond trivial statements about the fair shares being the same, it was apparent that the common misconception existed that the equivalence of fair shares can be given by an summative relationship, noting only that the total amounts (sums) for both quantities remained the same, no matter the distribution of the partial amounts (addends).

### *Research Question Two*

*How does students' knowledge of multiplication and division interplay with their learning and understanding of equipartitioning at the upper levels of the learning trajectory?*

Students with a greater understanding and who were more fluent in multiplication and division relied on those ideas more heavily when attempting to solve co-splitting tasks, and

especially referred to facts more often in their explanations of reasoning and justifications. Students who had weaker skill sets with regards to multiplication and division tended to rely on additive notions (summations) more often and those lasted deeper into the teaching experiment. Those students also tended to guess at a number of groups to attempt more often, and after having done so, relied heavily on dealing in order to determine the results of those splits. However, the relative success on co-splitting tasks did not differ much between these two types of students over the course of the teaching experiment.

The students in the teaching experiment frequently cited multiplication and division facts directly in their work on co-splitting problems, both in justifying the number of groups used and in justifying the equivalence of fair shares between groups and situations. There were several instances in which students articulated using a fact, but incorrectly stated the result. These instances show that either the students were not adequately fluent with their facts, or the students were hasty in their application of those facts and made careless errors. Consequentially, it reinforced the ideas that students often struggle with contextual and word problems, and in using estimation skills to check answers, in that they were not attuned to the context of the co-splitting problems to help them realize their mistakes outright. This was especially true when students displayed the common misconception of using the improper operation (multiplication or division) when calculating their values, at times resulting in greater amounts of some quantities than constituted the given values for each.

Three types of students emerged with respect to their relative abilities with multiplication and division combined with their work on co-splitting tasks. The first type of

student is very fluent with multiplication and division facts, relying almost solely on those to work out co-splitting problems, and was even capable of determining answers that involved fractional values, including the unit ratio on a few occasions. However, these students were less likely to develop a deep conceptual understanding of co-splitting as a construct, perhaps because their fluency with multiplication and division provided them with an already familiar means of determining solutions. These students operated in a very procedural sense and learned to solve co-splitting problems in that manner, without showing a concern for truly understanding conceptually and becoming fluent with co-splitting, but rather simple being pleased to get an answer and move on to the next problem, for which the process would repeat itself and no greater sense of confidence would be developed. This is typical of struggling mathematics students in many classrooms, especially as conceptual understanding becomes increasingly important based on the content.

The second type of student was only moderately fluent with multiplication and division facts, and generally only realized a relevant fact after having solved co-splitting problems by means of equipartitioning and dealing. Nonetheless, these students were seen to develop the most complete conceptual understanding of the co-splitting construct, possibly for a couple reasons. First, they may have needed to understand the goal of co-splitting problems in greater depth and through equipartitioning in order to gain an entry point to the problems because they did not have the same sense of familiarity as the students more fluent with multiplication and division. Second, their recognition of the multiplicative relationships through facts after having determined an answer rather than before may have acted as

justification as to why their answers worked and allowed them to make more sense of the construct of co-splitting, instead of simply being a means to an end in finding those answers.

At least for these students, there was evidence of the value of co-development of multiplication and division and co-splitting, as their fluency with facts appeared to be strengthened throughout the teaching experiment in working on co-splitting tasks.

The third type of student was not fluent at all with multiplication and division, and was seen to still have difficulty finding solutions to co-splitting tasks even at the end of the teaching experiment. Although they were able to determine some solutions to co-splitting problems, these students often struggled to determine more than one solution and frequently needed guidance in understanding the goal of the problems and how to begin working on them. The inability of these students to cite multiplication and division facts combined with their struggling to determine solutions to co-splitting problems and understand the construct seems to represent the opposite extreme of the first type of student and indicates a broader deficit operating multiplicatively and reasoning in such ways in general. It should be noted that these students also displayed the weakest understandings of equipartitioning throughout the workshop and the teaching experiment, and were largely the weakest performers on all assessments throughout. This could simply reflect a lack of educational acumen or lower test-taking abilities, but also could be due in part to little or no foundation being in place for working in a multiplicative space.

Despite multiplication and division facts being used regularly in discussion and even written on the board for all students to see at times, there was not one instance observed

where even the most fluent students indicated a strategy of simply writing out and solving a fact as their first step in solving co-splitting problems. It can not be said with certainty that none of the students thought of these facts ahead of time and just did not write or state them, and there were definite indications that was happening sometimes, but it appears that the use of facts for these students was thought of as a means for justifying a solution once it had been determined by some other method or combination of methods.

### *Research Question Three*

*How does students' strategy use and performances on co-splitting tasks differ when the number of objects is greater than the number of sharers versus when the number of objects is less than the number of sharers?* Overall, for tasks where the number of objects was less than the number of sharers, most students in this teaching experiment used the same broad strategies as for those tasks where the number of objects was greater than the number of sharers. Those students who relied more heavily on dealing strategies ran into minor difficulties, needing to adjust their strategy in terms of the order in which they worked with objects or sharers. This could have led to some uneasiness if students were confronted with the contextual notion of what it would mean to deal sharers to objects; however, most students did not seem to pay much attention to which values went with which quantities as they were working on the problems. Rather, they would first work with whichever value had led them to choose the number of groups they had, and then only in explaining their results in words would they cite the units of the quantities that implied objects or sharers. When the number of objects was less, the students did display a sense that the shares would involve

parts of objects, but were less likely to attempt to determine the actual fair share, thereby paying more attention to groups and how objects were shared within identical groups.

### Summary of Chapter

This chapter elaborated the findings of the study organized around the three research questions and related conjectures. The limited quantitative results were and discussed followed by a rich, qualitative description of the critical moments on each day of the teaching experiment. Lastly, summaries of the findings pertaining to each research question were provided. In next, and final, chapter I discuss the importance and implications of these findings with respect to the larger frameworks of the study, relate conclusions based on these findings, and discuss the limitations of the work.

## CHAPTER 5: DISCUSSION AND CONCLUSIONS

In this chapter, I address the research questions that motivated the study, providing responses based on the findings of the teaching experiment and by revisiting the related conjectures. I describe modifications made to the conjectures, and identify any newly formed conjectures during the teaching experiment. Then, I discuss the implications of this study for the field, particularly with respect to students' development of the construct of co-splitting. I connect this with further considerations of the significance for classroom instruction based on an LT for equipartitioning, as it relates to multiplication, division, and ratio. Lastly, I present the limitations of the study and outline a scope of future work in these areas. The next section strives to answer the three research questions.

### Discussion

#### *Research Question 1*

*How do the lower levels of the equipartitioning learning trajectory have an impact on student interactions with and success on the upper level construct of co-splitting?* The initial conjectures hypothesized and inferred relationships among the lower, preceding levels of the equipartitioning LT and co-splitting. The findings of this teaching experiment showed similar results to previous clinical interviews to substantiate many of these relationships. However, it is important to acknowledge that within classroom discourse and through interpretation of student work, it is difficult to elicit such connections directly. Therefore, many of these connections and relationships are represented in perspective – interpretations by the researcher – of the students' voices and actions.

Each of the lower levels of the equipartitioning LT appears to be related to co-splitting in one of two ways for students: 1) directly, as part of a strategy for determining solutions to co-splitting tasks, or 2) indirectly, as part of their reasoning and sense-making in justifying those responses. The levels that this study emerged as informing a student's process for developing more sophisticated co-splitting strategies were: equipartitioning collections (1), equipartitioning single wholes (2), composition of splits on single wholes (7), quantitative compensation (8), reallocation (9), and the Continuity Principle of equipartitioning (11). The levels that this study showed to manifest in a student's justification for the equivalence of fair-sharing situations, or the preservation of the fair share in such situations, and the development of more sophisticated reasoning are: justification (3), naming as a relational (4), reassembly (5), qualitative compensation (6), and the Property of Equality of equipartitioning (PEEQ; 10).

Of these proficiencies, some appeared to be more predominant than others, thereby suggesting that certain levels are necessary for students to interact successfully with the co-splitting construct at varying degrees of sophistication, while the others are labeled as useful, but not necessary. There are also three distinct types of levels at the lower end of the equipartitioning LT from an observation standpoint. The first type are the most easily observable, and I refer to those as *action* levels, as they only manifest through physical actions that can be seen as a student works through solving a problem. Therefore, if asked to describe their process of determining a solution, students would likely be able to describe it as using of one of those lower level actions, perhaps even by name if that terminology was

well-established in classroom discourse. The second type are more of reflective abstractions (Piaget, 1985) that appear in what I refer to as *justification* levels, and are not likely to be recognized by name by students, but the levels are still evident in explanations of the student's thinking after determining a solution (but also before and during), as they are often applications of the actions. This relies on students' abilities to recount and articulate their actions and explain their reasoning before acting, or remember their reasoning after acting, which are both difficult, especially in student populations for whom explanation has not been the norm. The third type are what I refer to as the *property* levels, which manifest in a similar manner to the justification levels, but not as an application of actions, yet rather as a means for establishing a basis for the applicability of those actions and their results. Therefore, these require even greater inferences by an observer to recognize their relevance and utility.

The students in the teaching experiment were observed to most frequently display and rely on behaviors relevant to the first two levels of the LT – the actions of equipartitioning collections and single wholes. These connections were seen both as being referenced in students' explanations of their arrangements as equivalent, which matched the initial conjecture, but also more than was expected in the physical acts of using manipulatives and creating diagrams to determine equivalent arrangements in co-splitting tasks. It should be noted that these two levels, along with reallocation and composition of splits, actually require mastering physical actions (splitting and dealing) in order to display proficiency. Because co-splitting and multiple wholes are also partially demonstrated as a physical manipulation, the connections become more easily observable.

Although I also witnessed the other two physical actions being used during the teaching experiment, their mastery did not appear to be as necessary for success with co-splitting tasks as the first two. Composition of splits was seen being used in co-splitting tasks not for justifying the equivalence of arrangements as conjectured, but rather in the explanation of the number of groups created when working through multiple levels of co-splitting to determine equivalent arrangements. This was partially due to the questions and tasks that explicitly called out this relationship. Reallocation was only observed on a few occasions, in making adjustments with manipulatives to determine another equivalent arrangement from either a successful or unsuccessful solution, or as explanations of those types of adjustments verbally, based on diagrams. There was a clear distinction as to when and how these latter two actions were used, compared to equipartitioning collections and wholes. Composition of splits and reallocation were only seen as the students' strategies became more sophisticated and as they began determining multiple solutions. The use of equipartitioning collections and wholes were evident immediately in co-splitting tasks, while students were operating at the most basic level of understanding; thus the claim about the necessity of these two levels.

Additionally, the students demonstrated clear connections to the justification levels of justification and naming in their verbal responses to questions about arrangements and solutions to co-splitting tasks, as was conjectured based on previous observations. These two lower, justification levels were also present from the earliest experiences of the students with co-splitting, while other conceptual levels were only seen later in the teaching experiment as

the students displayed more sophisticated strategies and expressed deeper understandings of co-splitting. For instance, qualitative and quantitative compensation were only illustrated once the students began determining multiple equivalent arrangements for the same task.

Unlike at the lower levels where there is typically a transition period between students discussing changes in fair shares qualitatively before quantitatively, in co-splitting the two appeared to emerge simultaneously. Students were more likely to offer quantitative judgments on their own first, and only referred to the qualitative when prompted in that manner. This is not surprising due to the quantitative nature of co-splitting tasks and the fact that all of these students were assumed to have reached proficiency at both lower levels in the trajectory (with quantitative compensation essentially subsuming qualitative compensation).

The justification level of reassembly was seen throughout, both in creating equivalent fair sharing situations and in justifying the equivalence between arrangements, as conjectured. However, these observations also led to some additional insight about reassembly. First, students were able to verbalize things indicative of the concept of reassembly early on in justifying equivalence, which represented the first occasions in which students recognized groups within arrangements not just as the tables or bowls, but also as ratio units that could be multiplied, or later added together, to form new combinations while preserving the fair share. It was then through this recognition that students were able to reverse this grouping notion and build on reassembly as a means to create equivalent arrangements in solving co-splitting tasks by going from more groups to fewer groups (inverse co-splitting), and eventually for one student, to flexibly move from any number of

original groups to any other number of equivalent groups (combining ratio units). Second, students appeared to use these more sophisticated applications of reassembly alongside their use of quantitative compensation almost exclusively, as if a form of reassembly acts as an inverse to quantitative compensation, just as the lower level of reassembly acts as an inverse to splitting through equipartitioning. The importance of reassembly and the multiplicative language that it instills in students early on as they move to the upper levels of equipartitioning may have been underestimated and certainly appears to be necessary for students demonstrate the most sophisticated actions of co-splitting. It should be noted, however, that reassembly could also be classified as an action level when students demonstrate physically combining fair shares to reconstruct the “whole,” but within this study such actions were not seen except for as strategies of reverse co-splitting, which are therefore not considered part of the lower level of reassembly.

The Continuity Principle of equipartitioning states that a continuous whole can be fairly shared for any number of sharers. Extrapolating from that belief, one could then say that any number of continuous wholes can be shared among any number of sharers by applying the Principle repeatedly, which leads to the split-all strategy for multiple wholes. Therefore, one inference could be that every time the students set out to solve a co-splitting task, and unless they explicitly state otherwise, they believe that the given number of objects can be fairly shared among the given number of sharers, thus applying the Continuity Principle of equipartitioning. However, to provide a more concrete example of its relevance to co-splitting, take the first instance that the students were presented with a co-splitting task

where the number of objects was less than the number of sharers. In order to believe that fair sharing in such a situation was possible, students were required to adjust their thinking from every sharer being assigned at least one object to understanding that the share would now be less than one object. Thus, they had to believe that at least some objects would be split, or all objects could be split, in different or the same ways, and then be fairly shared for the sharers.

The Property of Equality of equipartitioning (PEEQ) implies that the same split performed on the same “whole” will result in the same size share (for example, a vertical 2-split on a rectangle and a diagonal 2-split on a congruent rectangle). This was the only lower level for which no initial conjectures were hypothesized about a relationship to co-splitting. I believe that the connections should be reconsidered as PEEQ is revised and perhaps redefined yet again moving forward in this research. I was at first surprised at the way in which PEEQ was applied to co-splitting, and then more surprised that I had not recognized the connection immediately. Before justifying the equivalence of the fair shares between a determined arrangement and the given situation, students often justify the equivalence of the fair shares among the groups within that determined arrangement. Arrangements determined by a co-splitting strategy result in groups with identical numbers of sharers and identical numbers of objects. It was in those instances that some students applied a form of PEEQ reasoning by stating that the shares would be the same in each group because all of the groups contained those equivalent amounts of each respective quantity; in other words, they were equivalent because they resulted from the same co-split. This was even more evident when students were working with the orange drink mixture problems when they were not

actually required to show all of the groups and could state that even groups not represented would get the same fair share as the groups represented, based on that reasoning.

A potential confounding factor in these results is that the source of some relationships between certain lower levels and co-splitting could be misconstrued relative to ages and prior learning experiences of the students. It is impossible to account for all of the other mathematical ideas brought from their prior experiences that may or may not have surfaced during the shortened time frame of the workshop on equipartitioning. An ideal curricular treatment of the LT would unfold over the course of several years and begin with students of a younger age, prior to or in parallel with other related concepts. This idea, for one set of topics in particular – multiplication and division – was in part addressed and explored by the second research question.

*Revisiting the conjectures.* With respect to the conjectures related to the first research question, it did appear that students who were proficient at the lower levels were in fact adequately prepared for moving to the upper level of co-splitting. Since students varied in their levels of proficiency, they demonstrated varied levels of success throughout the teaching experiment. Overall, they were all able to determine solutions to co-splitting tasks on multiple occasions, and most provided evidence of also attaining related conceptual understanding. Lower level actions (equipartitioning collections and wholes) proved to be the most critical to this success, and those were the levels that emerged as influential to student strategies even at the earliest stages of determining solutions to co-splitting tasks. However, these levels alone did not appear sufficient for reaching a more sophisticated understanding

later when more advanced strategies and flexibility in operating with co-splitting were witnessed. Then, other action levels (reallocation and composition) along with the ideas of reassembly, qualitative compensation, and quantitative compensation were seen to be extremely important. Although students at this age are not adept at being able to explain their strategies and thought processes in general, the lower levels of justification and naming proved to be critical in building a foundation for students to discuss their work and establish a greater conceptual understanding of the co-splitting construct.

In addition, there were relationships seen between co-splitting and every lower level of the trajectory. This includes, PEEQ, where no relationship was hypothesized as part of the initial conjectures. The revised relationships between each of the lower levels and co-splitting, based on previous observations and what was seen during this teaching experiment, are presented in Table 11.

*Table 11.* Comparison of original and revised conjectured relationships between lower levels and co-splitting.

Proficiency Level (#)	Original Conjecture Relevance to Co-Splitting ( <i>n</i> objects and <i>p</i> people)	Revised Conjecture Relevance to Co-Splitting ( <i>n</i> objects and <i>p</i> people)
Equipopartitioning Collections (1)	In justification of equivalent fair sharing situations: When $n > p$ , students may deal in rounds and one-to-one correspondence to determine the fair share.	In creating and justifying equivalent fair sharing situations, students deal in rounds to show that a quantity is evenly distributed among a chosen number of groups.
Equipopartitioning Single Wholes (2)	In justification of equivalent fair sharing situations: When $n < p$ , or once the remaining objects becomes less than <i>p</i> , students may share each whole for the number of sharers to determine the fair share.	In justifying equivalent fair sharing situations, students demonstrate how whole objects can be shared among multiple groups or multiple sharers in identifying the amount of objects in a group or the fair share of one sharer in a group.
Justification (3)	In justification of equivalent fair sharing situations: When more than one situation with identical quantities for objects and sharers are in question, students may relate justifying fair shares by saying “they all got the same” with “every group of the same size got the same.”	In justifying equivalent fair sharing situations, students use experiences justifying fair shares to reference strategies and results using learned language and similar new language.
Naming as a Relational (4)	In justification of equivalent fair sharing situations: Students may determine the fair share for each and use their knowledge of the names of fractional parts in so naming it to assert equivalence.	In justifying equivalent fair sharing situations where wholes are shared among multiple groups or sharers, students reference the names of the parts based on experiences from equipartitioning wholes.
Reassembly (5)	In justification of equivalent fair sharing situations: When the given situation is being compared to several situations with lesser numbers of objects and sharers, students may claim that they can all be reassembled to create the given situation. In creating equivalent situations: Students may partially reassemble groups of smaller situations to determine new ones with intermediate values.	In justifying equivalent fair sharing situations, students recognize subgroups in the results of co-splitting and make claims that the shares are the same because the groups that were split are the same but just contain more of both quantities – the same multiple of each.
Qualitative Compensation (6)	In creating equivalent situations: Students may first determine whether the quantities should be greater or lesser in moving from one situation to another.	In justifying equivalent fair sharing situations, and within quantitative compensation argument, students explain that the shares are the same because there are either less or more of each quantity but the situations are the same.
Composition of Splits (7)	In justification of equivalent fair sharing situations: Students may relate equivalent fair-sharing situations that have been arrived at through a composition of splits to the original situation as being the same as a single split.	In creating equivalent fair sharing situations, students recognize the recursive nature of co-splitting across equivalent groups and identify the number of identical subgroups that will be formed by co-splitting without needing to represent all of those subgroups physically.
Quantitative Compensation (8)	In creating equivalent situations: Students may determine the number of objects and the number of sharers that will create an equivalent fair-sharing situation based on co-splitting and then go on to determine values based on multiplicative changes related to the splits used.	In creating equivalent fair sharing situations, students determine the number of objects and the number of sharers that result from co-splitting based on multiplicative relationship between the numbers of groups and subgroups.
Reallocation (9)	In creating equivalent situations: Students may combine already known equivalent fair-sharing situations by reallocating the quantities from one or more fair-sharing situations to the others, equipartitioning each of the quantities and then dealing them evenly.	In creating equivalent fair sharing situations, students combine already known equivalent fair-sharing situations by reallocating the quantities from one or more groups to the others, equipartitioning each of the quantities and then dealing them evenly; or students perform similar actions to resolve solutions that were unsuccessful.
Property of Equality of Equipopartitioning (10)	None predicted.	In justifying equivalent fair sharing situations where identical groups of both quantities have been created by co-splitting, students apply PEEQ reasoning to state that the fair shares will be the same within each group because they all resulted from the same co-split.
Continuity Principle (11)	In justification of equivalent fair sharing situations: Students may rely on the notion that a single whole can be fairly shared for any number of sharers and therefore not need to determine the fair share itself.	In creating and justifying equivalent fair sharing situations, students rely on the notion that a single whole can be fairly shared for any number of sharers, expanding this to any number of wholes, and realizing that they do not need to determine the fair share itself.

### *Research Question 2*

*Does students' knowledge of multiplication and division interplay with their understanding and learning of the upper levels of the equipartitioning learning trajectory?*

During the workshop on the lower levels of equipartitioning, the students did cite some multiplication and division facts, but the use of facts was for the most part limited to factors within 10. Students also struggled with the notions of reassembly and quantitative compensation; although most students were able to adopt the “times as many” language, the references to multiplication and division in these situations were predominantly limited to doubling and halving. In working with composition of splits problems, the vast majority of students exhibited behavior consistent with one of two groups: 1) those that were *fully capable* of determining the resulting number of parts after a composition of splits, regardless of the values of the splits, and 2) those that were *not able* to determine the result of a composition of splits in any instance where the product of the splits was different than the sum of the splits.

The relative success of students on lower level concepts, with most demonstrating abilities to complete tasks at every level during the workshop, indicates that multiplication and division are not necessarily prerequisite for those levels. This is expected according to Confrey’s splitting conjecture, as she argued that some of these lower level concepts can develop, and should be developed, prior to a formal introduction to multiplication and division. However, the relative success of these same students was much more varied for co-splitting, which indicates that some degree of fluency with multiplication and division may in

fact be necessary for that upper level of equipartitioning; and this aligns appropriately with the placement of the concepts in CCSSM. It also suggests that relationships across various LTs are complex and require some overlap, such that the division and multiplication LT may ideally begin prior to the completion of equipartitioning.

One significant misconception that hindered students who were either moderately or extremely fluent with multiplication and division in applying that knowledge to co-splitting tasks was the confusion of factors within the facts they were citing in context. This does not indicate an issue with their multiplication and division skills, but rather issues with their multiplicative understanding of co-splitting tasks, how the quantities are related to the underlying third variable of groups, and how those groups are determined. The number of groups should be determined as being the same as a chosen split, which would then be represented by a common factor between the numbers of objects and sharers in facts. However, students often indicated the use of multiplication or division facts to determine a number of groups and equivalent fair-sharing situations based on the value of one quantity (perhaps sometimes also considering the value of the other quantity), but then they would then become confused, despite determining an appropriate fact. This resulted because in establishing the number of groups they chose the groups based on the inappropriate factor – not the one representing the split – that led to an unsuccessful solution.

For example, in the co-splitting tasks of sharing 35 objects among 15 sharers, one student in this teaching experiment cited the multiplication fact of 7 times 5 equaling 35, which in a successful solution would have implied also considering that 15 is divisible by 5,

but he attempted to create 7 groups rather than 5. This type of misconception has an implication for further study of the concept of co-splitting, which was recognized during the analysis of this teaching experiment. It raises the question: what is evidence of understanding co-splitting as a construct versus evidence of performing co-splitting as an action? The student in the example above was eventually able to determine a correct solution of an arrangement using five tables, but only through a sequence of guessing and checking using similar multiplication facts. I would say that the student was certainly capable of performing a co-splitting action once the facts allowed him to determine the correct number of groups (the co-split). However, based on the way in which he arrived at that number of groups, I would not say that his overall solution strategy was one of co-splitting. This implies that the use of coordinated multiplication and division appears to be one less sophisticated, yet productive, strategy for solving co-splitting problems.

*Revisiting the conjectures.* With respect to the conjectures related to the second research question, there appeared to be relationships between students' multiplication and division knowledge and their performances on co-splitting tasks, evident in their rates of success, their strategies, and their conceptual understanding. Students of different ability levels for multiplication and division were able to successfully solve co-splitting problems to varying degrees and from various entry points. Most surprising was that students who displayed the greatest conceptual understanding were not those with the highest levels of multiplication and division knowledge. Similarly, students with a weaker conceptual understanding of multiplication and division relied more on the use of manipulatives to solve

co-splitting problems. As the values of the parameters involved in co-splitting problems increased, there were some differences seen in students' success, and those with stronger multiplication and division abilities were successful more frequently. Lastly, variations in context (or lack thereof) did not have an impact on the students' use of multiplication and division facts, but students began to cite facts more frequently after the use of facts and similar knowledge was modeled by myself, as the teacher, and even more so when it was modeled by peers.

### *Research Question 3*

*How do students' strategy use and performances on co-splitting tasks differ when the number of objects is greater than the number of sharers versus when the number of objects is lesser than the number of sharers?* The relative difficulty of co-splitting tasks and the types of strategies that would be elicited when the number of objects was less than the number of sharers had not been investigated prior to this study. The conjectures around this research question were based on empirical evidence from related equipartitioning multiple wholes tasks, which revealed more student difficult and failure when the number of objects was less than the number of sharers. Based on this teaching experiment, this did not appear to hold true for co-splitting, as students were capable of determining solutions and equivalent arrangements for co-splitting tasks to the same degrees of success regardless of the relative sizes of the quantities within. However, the tasks in which the number of objects was less than the number of sharers were purposefully delayed during the teaching experiment until after students had ample experience with tasks in which the number of objects was greater

than the number of sharers. Therefore, it cannot be decided whether these variations in the relationship between the two quantities would have been equally accessible from the onset, or whether the order of presentation was in fact a relevant variable.

*Revisiting the conjectures.* With respect to the conjectures related to the third research question, the students did not drastically change their strategies when they were presented with tasks in which the number of objects was less than the number of sharers from those strategies that they had been using when the number of objects was greater than the number of sharers. Throughout the teaching experiment, overall the students did not appear to develop a strong tie to operating with one of the quantities (objects or sharers) before the other, but rather appeared to determine a number of groups, split, or co-split to use based on a recognized relationship between that number of groups and one quantity, moreover, for the first of the quantities that a relationship was recognized. In other words, if a student recognizes the multiplication fact of six times *four* equals twenty-four, then they may choose to attempt *four groups* in a problem involving twenty-four sharers, and then again in a different problem involving twenty-four objects, regardless of the amount of the other quantity in the respective problems. Therefore, the impact of fewer objects than sharers was minimal in terms of the students' identification of splits and strategies. Even when dealing was utilized, the students did not necessarily associate each value with each quantity up front, and thus there was no consideration of the size of the share or the order in which to operate on the two quantities.

## Conclusions

This section expands upon the answers to the research questions and focuses on three primary areas to draw conclusions with respect to each based on the results: curricular implications, the relationship between co-splitting and equipartitioning multiple wholes, and co-splitting as multiplicative protoratio. The first of these speaks to the methodology of the study as a teaching experiment designed around a construct within an LT. The second addresses the other major upper-level concept of the equipartitioning LT not covered in the study that we believe is related to and develops in parallel with co-splitting. The third projects beyond the equipartitioning LT to another topic – ratio – that is built directly on equipartitioning, and describes the gap in multiplicative reasoning often seen previously that co-splitting addresses. Following this section, I make explicit the limitations of the study that could have had an impact on the results and my conclusions, and then I explicate the directions I believe this line of research should take building off of my study.

### *Curricular Implications*

The students in this teaching experiment were identified to have met the assumption of being proficient at the lower levels of the equipartitioning LT prior to being introduced to co-splitting tasks. Although what constituted proficiency was not formally quantified, this judgment was based on all students having experienced the lower levels and shown the ability to solve tasks at each level during the workshop and the first day of the teaching experiment. This does not imply equal proficiency among them, as evidenced in the results of the pre-test. Ideally, claims about students relative to proficiency at a level of an LT would be

based on multiple pieces of information, which was not possible with a single assessment such as the pre-test. The pre- and post-test format used in this teaching experiment for making these statements about the students' proficiency with levels of the equipartitioning LT, and particularly the lower prerequisite levels, was not ideal. In light of the brief treatment of the lower levels in the workshop, even for slightly older students, the tests were still too lengthy, and not sufficient to gather rich, informative data about the students' performances and understanding at any given level. Rather a diagnostic assessment that targets the skills and concepts within a level would be best suited. The LPPSync system is designed for that purpose, but time constraints did not allow for the students to complete all of the appropriate diagnostic assessments to cover every lower level of the trajectory. Therefore, this information had to be gathered more loosely for the purposes of this study from the empirical evidence witnessed during the workshop, which I still believe was adequate, albeit not ideal. It also points to the need for a means to give initial diagnostics across proficiency levels effectively, a future challenge.

The results of this study show that students at the upper elementary grades are capable of understanding the concept of co-splitting and operating at the upper levels of the equipartitioning LT when they have the necessary prerequisite skills from the lower levels. However, despite most students being capable of solving individual problems in a collaborative classroom setting, there was clearly a broad spectrum of understanding at those lower levels that may explain a similar spectrum of understanding of co-splitting. Although the study initially intended to cover all of the upper levels of the LT, it became clear during

the teaching experiment that was not feasible in the two weeks of time allotted with this group of students, and in that setting. It is not clear whether this would be true for all groups of students at these grade levels, but it seems fair to anticipate that fifth-grade students of average ability and with a prior curricular treatment of the lower levels of equipartitioning would be capable of successfully solving co-splitting problems and attaining an understanding of the concept as many of these students were. However, the concept of equipartitioning multiple wholes would need to be afforded an equal coverage, unless the two were introduced simultaneously, and then perhaps three weeks would be adequate.

Research on the use of learning trajectories in an instructional model (Sztajn et al., 2012; Wilson, 2010) portrays successful implementation as largely based on assumptions of the teacher possessing an in-depth understanding of the nuances of learning trajectories themselves, as well as the specific content domain of a given LT. In this teaching experiment, I, as teacher-researcher, met this assumption through extensive and detailed experiences with various stages of the development and refinement of both the equipartitioning LT itself, particularly the level of co-splitting, and the creation and revision processes for the curriculum. Although these intimate experiences perhaps set me apart from a typical teacher, who would at best have their experiences through professional development opportunities, I do not believe it biases the results. Rather, good teaching practices, well-designed professional development, and experience teaching under an LT based instruction model (LTBI; Sztajn et al., 2012) can impart a necessary, and equally viable, level of understanding to enact successful learning opportunities for students.

One critical aspect to teaching and learning in accordance with LTs is a proper sequencing of concepts and tasks that allow for a gradual development of understanding within a content domain such that the pace is not so quick as to leave students behind or frustrated, while also empowering those students who are ready to move forward. Throughout the description of the results of this teaching experiment, it can be seen that several students were at different stages of understanding each day, and with each new task type. It also may have appeared as if some students took steps backward in their learning at times, but across all of the students, progress was being made in a forward direction constantly. Some students grasped certain aspects of co-splitting quicker than others, but through repetitive tasks and regular discourse, each was afforded the opportunity to make progress at their own pace, and learn from peers who may have already gained further understanding. This is not to say that all of the students ended in the same place, or with the same understandings, either, but rather that if I, as the teacher-researcher, had presented the material at a more rapid pace, the risk of more students feeling dejected or bored could have been realized. In the following paragraphs, I trace the general understandings of the students across the class throughout the teaching experiment, and note it as an example of how different levels of understanding among individual students can exist harmoniously.

Early in the teaching experiment, many students shared the misconception that only one quantity mattered in preserving the fair share for co-splitting problems. This then led to several students taking on another misconception once they began to attend to both quantities – that the fair share was preserved when the respective “wholes” (or totals) of each quantity

were preserved by sums, regardless of the multiplicative relationships between quantities. This latter misconception was carried throughout the entire teaching experiment by a few students, despite revisiting the idea on many occasions and a large other portion of the class having explained the preservation of the fair share using ratio-like language. However, all of the students were able to come up with at least one solution even by the end of the first day, and continued to be able to determine varying amounts of solutions to new tasks on their own throughout. The students who continued to wrestle with the summation misconception of preserving fair shares were able to determine solutions because they were not the ones generating solutions in which the misconception would interfere; these students were only able to determine solutions resulting from partitive co-splitting actions and therefore resulting in identical groups (same number of objects and same number of sharers), for which exhausting the totals of both quantities does in fact preserve the fair share.

For most students, their first solutions were determined by using 2-co-splits, both because the problems presented allowed for it, and also likely because it is known to be the most intuitive and easiest means of determining solutions to splitting problems. The use of 2-co-splits exclusively as the only successful strategy was seen for a few students well into the first week of the teaching experiment, while at the same time, some students were already able to come up with multiple solutions by the second day. The possibility for both of these cases was because every task involved even values for both quantities for most of the first two days, but also many tasks were presented in which the quantities shared a common factor of 6 or 12, so that 3-co-splits and 4-co-splits, possibly from consecutive 2-co-splits, could

also lead to successfully determined arrangements. The first problem that did not have a possible solution within the whole numbers through 2-co-splits was not presented until the third day.

Manipulatives were made available to the students throughout the teaching experiment, and some relied on using those for the entire two weeks. Although one student made the leap to determining the unit ratio and cited an actual fair share involving fractional values much earlier than anticipated, it was the availability of, and level of comfort with, the manipulatives that later allowed other students, who may not have been able to get there numerically, to eventually gain an understanding of how to determine the fair shares (and fractional values in other correct solutions). Similarly, the use of the manipulatives helped some students make sense of equivalent arrangements that were not the results of partitive co-splits and did not have groups with identical values for each of the two quantities across them. These students developed a strategy of dealing the second quantity (typically objects) to the groups rather than trying to associate it with the first quantity (typically sharers). Only one student was able to regularly determine such arrangements numerically, but even some students who had abandoned the use of manipulatives for determining co-splits came back to using them in these situations, and were able to make progress and participate in conversations around those types of solutions.

By the end of the first week, there was evidence that all of the students were thinking about methods for solving co-splitting problems that were multiplicative, and in explaining their results and justifying the equivalence of fair shares in their arrangements, students

would refer to division, halving, or multiplication, yet others still used additive structures. It was not until well into the second week that a more formal, and shared language was developed. This required me, as the teacher, to help the students communicate and articulate their thinking, both through modeling the language and introducing a heavy reliance on referring to the actions of splitting while talking about the co-splitting tasks. Even still, students carried their own individual mechanisms for solving and explaining problems as division and multiplication and the likes, but when prompted to translate that thinking into splitting language, they became comfortable and adept at doing so.

*Contextual problems.* Equipartitioning problems are most readily and easily portrayed in a fair-sharing context. Therefore, despite evidenced claims in other areas that context may interfere with students' abilities to solve problems, it is nearly impossible to introduce the notions of equipartitioning and fair sharing, that would not in itself be cumbersome, without a context. This is also then true for co-splitting problems, especially in introductory experiences for students; however, the fair-sharing context has limitations. First, for the two quantities involved, there is the need to be able to understand an implied sharer and an implied object. Second, and because of this then, the sharer is often a person or some other living being, for which contextually and realistically fractional amounts would not exist or do not make sense. The co-splitting construct itself is not limited to these situations in the same ways though: it can be applied just the same when there is no inherent understanding of objects versus sharers, and it can result in solutions in which either quantity takes on fractional values.

Although all of the students in this study displayed a level of understanding of fair sharing and equipartitioning that was believed to be suitable for the introduction of co-splitting, the notions as presented in co-splitting tasks seemed to take on new meanings for many of the students. On several occasions there was an apparent tension for the students between creating an equivalent fair-sharing situation and justifying that equivalence by referencing or determining the actual fair share. When no apparent, natural sharer existed between the two quantities, such as in the orange drink mixture context, students did not demonstrate any motivation to move towards determining the fair share (or unit ratio).

The use of the orange drink mix context made another concern apparent as well. I had to spend a great deal of time discussing the context with the students before they were able to convince themselves that mixtures involving different total numbers of cups of ingredients could ever taste the same. It is not clear whether this was due to the fact that there was no apparent sharer in these situations, and that a determination of cups of orange juice per one cup of water (or cups of water per one cup of orange juice) was an arbitrary one, or rather because the notion of a mixture tasting the same based on the ratio of its ingredients is foreign to students at this age.

This implies that a context less likely to be interpreted as a fair-sharing situation is more closely aligned with later ratio reasoning tasks, and is more difficult for students to understand conceptually. Therefore, within the development of equipartitioning, fair sharing tasks *are* the appropriate introductory situations for co-splitting and any curriculum designed to build these understandings should organize the task types in such a way. Furthermore,

students most readily understand fair-sharing tasks (including for co-splitting) in context when the sharers are living beings, such as people or animals. This appears to be true regardless of the relative sizes of the two quantities (more or less of one or the other). I suggest that students first engage with co-splitting tasks where the sharers are easily referenced and identifiable in context. Additional studies will need to be conducted in order to determine whether the objects should initially be limited to discrete objects, not considered appropriate to be split in context (and how this can be presented as a contextual stipulation early on that will be relaxed later), or whether the objects should be continuous wholes that are considered appropriate to be split in context from the onset. Likewise, additional studies will be needed to determine whether the order matters in which students experience tasks where the number of objects is greater than the number of sharers versus tasks where the number of objects is less than the number of sharers. These studies would be best informed by also gathering additional information about the relationship between co-splitting and equipartitioning multiple wholes, and how each of those contribute to movement further up the LT and lead to generalization.

#### *Relationship to Equipartitioning Multiple Wholes*

Within the equipartitioning LT, the proficiency level for equipartitioning multiple wholes precedes the proficiency level for co-splitting; however, our research team believes that the two concepts develop in parallel. The curriculum used in this study and two previous teaching experiments was created based on the design of the LPPSync IDAS for equipartitioning, in which multiple levels were combined and assessed within each e-Packet

so as to minimize assessment administration load. Those e-Packets were sequenced in an alignment to the levels of the trajectory, but due to overlap and the nature of some levels fitting better in similar IGEs than others, decisions had to be made that break the linear structure of the proficiency levels in the LT. Therefore, the co-splitting e-Packet and corresponding curriculum precedes the multiple wholes e-Packet and corresponding curriculum.

Although this teaching experiment originally intended to cover all of the upper levels of the trajectory, including multiple wholes, the decision was made during the first week to forego pushing the students beyond co-splitting, as they were not fully proficient with that concept within that time frame. This allowed for a greater depth of understanding with respect to how these students interacted with co-splitting problems and developed their reasoning around that construct. At the same time, it did not provide for any evidence to support or contradict our claims about the parallel development of the upper level concepts within the trajectory. Nor did it provide any further understanding of the connections of the lower levels of the trajectory to any other upper levels besides co-splitting.

Nonetheless, there were some instances during the teaching experiment that provide anecdotal evidence about the relationship between co-splitting and equipartitioning multiple wholes and the abilities of students to operate at those levels. Equipartitioning multiple wholes tasks involve the fair sharing of more than one continuous whole (the objects) among two or more sharers, and students are presented with the task of determining the fair share (for one sharer). For example, the task of determining the fair share in the scenario of 6

pizzas being shared among 4 people. This scenario itself could constitute a co-splitting task, but the difference between co-splitting and equipartitioning multiple wholes tasks lies in the desired responses and the processes by which those responses are determined. In co-splitting, the desired response is *an equivalent arrangement* (of which there are many), which involves partitioning two quantities in such a way that the fair share is preserved across all partitions, while accounting for the original amounts of each quantity. However, there need not be any designation, implied or otherwise, of which quantity represents objects and which quantity represents sharers. In equipartitioning multiple wholes, the desired response is *the fair share* (one of the two possible unit ratios), which involves determining the amount of objects per one sharer when the original amount of objects have been exhausted and equally distributed across all sharers; therefore, which quantity represents objects and which quantity represents sharers must be known or declared.

Co-splitting is one of several strategies that could be employed to determine the solution to an equipartitioning multiple wholes problem. In the example stated above, a student could use, for instance, a 4-co-split (or consecutive 2-co-splits) to determine that the fair share is one and one-half pizzas per person. It is believed that this strategy would be seen more often when the values of both quantities share a common whole-number factor, such as in this example. Another known, productive strategy is “deal and split” (only possible when the number of objects is greater than the number of sharers), in which the student deals the objects in rounds to each sharer until a further round cannot be completed evenly, and then splits the remaining objects in such a way that they can be dealt in a similar manner and all

parts of the original objects are exhausted – a form of re-unitizing and considered a precursor to the distributive property of multiplication over addition. A third known, productive strategy is “split-all,” in which the student shares each continuous whole (object) with all of the sharers as if each were an individual equipartitioning single wholes problems, thereby performing the same split (equivalent to the number of sharers) on every object.

One connection between the two concepts that was seen throughout the teaching experiment was when students were asked to justify the equivalence of the fair shares in their co-splitting arrangements. A few students throughout the teaching experiment, and more so towards the end, determined the fair share for a sharer in each different group of their arrangements in order to provide this justification. For example, in an arrangement involving four bowls resulting from a 4-co-split in the problem of 12 pellets being shared among 8 fish, one student identified the fair share in each of the four bowls as being one and one-half pellets for each of the two fish, using the deal and split strategy for multiple wholes. Similarly, she determined that each of the 8 fish in the original bowl would also get a fair share of one and one-half pellets, thus reasoning that the arrangement determined with four bowls would allow all of the fish to get the same fair share.

This type of strategy was predominantly seen used successfully when the fractional parts of the fair shares were halves, but it was also seen on two other occasions when the fractional parts of the fair shares were thirds. In the first of these instances, the groups involved 4 pizzas and 3 people at each of three tables, and the student used the deal and split strategy to determine the fair share as one and one-third pizzas per person. Prior observations

have shown that problems involving a number of objects that is exactly one more than the number of sharers prove to be easier for students to solve, and the deal and split strategy is the most commonly used in those cases, as was true in both examples described here thus far. The second instance in which a student determined a fair share involving thirds was for the problem involving 15 pizzas being shared among 45 people, and he had determined an arrangement involving three tables with 5 pizzas and 15 people at each. In claiming that the fair shares were the same at all of the tables, he suggested performing a 3-split on each pizza at every table, so that each person would get one-third of a pizza. The split-all strategy would imply performing a 15-split on each of the pizzas and then dealing those such that each person would get five-fifteenths of a pizza; however, this student's strategy is similar yet perhaps even more sophisticated, in that his chosen split to perform on all of the pizzas in each group represented the most efficient re-unitizing (so that the fractional parts could be identified in simplest form). This would only work when amounts of each of the two quantities involved are not relatively prime, such as was the case here.

Despite the fact that very few students demonstrated the capability of solving equipartitioning multiple wholes problems during the teaching experiment through their justifications, it should not be considered as evidence that co-splitting is easier or necessarily emerges before multiple wholes, as the two different types desired responses were not both explicitly sought after. Even though the multiple wholes part of the curriculum was dropped from the teaching experiment, I chose to conduct clinical interviews that did target the equipartitioning multiple wholes level of the LT with three students, who had each shown

varying ability levels with respect to co-splitting. All three students were able to solve several equipartitioning multiple wholes tasks and name the resulting fair share as either a mixed number (including whole numbers and proper fractions) or a sum of proper fractions and/or whole numbers. The students in these clinical interviews were provided with manipulatives of paper cutouts to represent objects and sharers, and were given a pencil and scissors to mark and cut the manipulatives further if desired.

These students predominantly used deal and split strategies, and other similar strategies related to the Distributive Property of equipartitioning. Surprisingly, being that the students were in the midst of a teaching experiment on co-splitting, not once did any of these students employ a co-splitting strategy, of which they had all demonstrated they were capable on co-splitting tasks. The closest resemblance to co-splitting within the clinical interviews was actually when one student was working with the task of sharing 5 cakes among 6 people, where the number of objects was less than the number of sharers. She began by re-unitizing and changing the task to sharing 10 half-cakes among 6 people. After dealing one half to each person, and realizing there were not enough halves to go around again, she exchanged the remaining 4 half-cakes for 2 new whole cakes to essentially start over with the task of sharing 2 cakes among 6 people. Having previously shared 1 cake among 3 people using a deal and split strategy for the larger task of sharing 4 cakes among 3 people, she very quickly and excitedly stated that each cake could be shared with each group of 3 people that had been laid out in two rows – showing how this would actually look by exchanging the 2 whole cakes for 6 third-cakes that had already been cut, and then dealing them to the 6 people. This

represented co-splitting (and a form of composition of co-splits) in that she first performed a 2-split on the 6 people by visualizing them as two groups of 3 people, and likewise performed a 2-split on the 2 cakes by assigning 1 whole cake to each of those groups. Furthermore, she then performed a 3-split on the 3 people in each group to identify the individuals, and likewise performed a 3-split on the 1 cake assigned to each group of 3 people, by creating third-cakes.

There were some indications within the teaching experiment that equipartitioning multiple wholes may have been more accessible to these students than co-splitting in certain instances because: 1) equipartitioning multiple wholes strategies were invented and applied by the students with no prompting or instruction, and 2) those strategies were often the ones students reverted back to when newly learned co-splitting strategies broke down. The accounts of the clinical interviews and the instances within the teaching experiment related to multiple wholes all go to show that students in the teaching experiment were at some level simultaneously capable of solving both co-splitting and equipartitioning multiple wholes problems. It also goes to show that the students quite clearly interpret a difference in the goals of the tasks for co-splitting and equipartitioning multiple wholes problems. It is believed that the connection between the two constructs could be made through a well thought-out curricular treatment of both concepts, with appropriately sequenced tasks. It is also intended in the LT that as students reach the uppermost level of generalization, they recognize *both* co-splitting and equipartitioning multiple wholes (through application of the

Distributive Property of equipartitioning) as viable strategies for determining and justifying that  $a$  objects shared among  $b$  sharers, and both will result in the same  $a/b$  objects per sharer.

There was also preliminary anecdotal evidence of connections between some of the lower levels of the equipartitioning LT and multiple wholes from the clinical interviews in this study, just as these connections were evident for co-splitting in the teaching experiment. The clearest connections were to equipartitioning collections, witnessed in the dealing actions with manipulatives, and equipartitioning single wholes, witnessed in the physical splitting of objects for a given, or selected, number of sharers. One student, through the sequence of tasks for 7 pancakes shared among 3 people, then 7 pancakes shared among 6 people, and then 7 pancakes shared among 12 people, demonstrated a form of quantitative compensation by continuously halving the objects representing the shares from each previous task. Although it was not stated explicitly, and was also perhaps aided by the sharers being laid out in an array that obviated the fact that there were *twice* as many sharers each time, the student quickly recognized that the shares would be *half* of those from the previous task each time. Further connections to other lower levels of the equipartitioning LT may have been present, but were not as noticeable, and will require a more in depth study before claims can be made. This is also true for any further substantiation of claims as to connections between co-splitting and equipartitioning multiple wholes, and others that may exist.

#### *Co-splitting as Multiplicative Proportion*

It was quite difficult to determine whether a student was actually arriving at a solution to a co-splitting task by coordinating the two quantities from the start or whether they were

operating on each quantity separately, as splitting problems, once a number of groups was determined based on only one of the quantities. This was partially due to the students' inabilities to articulate their processes both during and after solving a problem, but it was also influenced heavily by the use of manipulatives and dealing, which can only be feasibly carried out one quantity at a time. Nonetheless, the students in the teaching experiment all showed an understanding that both quantities mattered, and that when splitting was a strategy (including co-splitting), the same split needed to be performed on both quantities. This establishes a richer understanding of early ratio reasoning than additive notions of protoratio and protoquantitative ratio, in which students only attend to one quantity at a time and/or to differences in amounts between quantities, for comparison of ratio tasks. For example, citing the "more-ness" of one quantity with respect to the other or the whole, and going on to identify how much more in an additive (or subtractive) manner.

In order to fully understand where the students in this teaching experiment stand on the spectrum of developing ratio reasoning, comparison of ratios *would* need to be considered, but comparison of ratios does not lie within the equipartitioning LT and is one of the lower levels of the Ratio and Proportion, and Percent LT identified in CCSSM by the DELTA research group, which is a middle grades trajectory. Much of the foundation for the later ratio LT lies within the upper levels of equipartitioning, and co-splitting in particular. There is some overlap between the LTs with respect to determining and identifying equivalent ratios that begins in co-splitting tasks, and then is formalized and abstracted in ratio when the "wholes" no longer need be exhausted to make claims of equivalence. There

was some evidence in the teaching experiment though, particularly with the orange drink mixture problems, that students were also already beginning to consider ratio comparisons as well, and referring to different mixtures as more or less orangey, and tasting the same. Although the context of these tasks was problematic for some students and required a great deal of setup, they were beneficial in demonstrating the proximity of the upper levels of equipartitioning to more formal ratio reasoning. It seems that the notion of correspondence between the quantities in co-splitting problems naturally arises to allow these conversations, and that correspondence is identified early on within the equipartitioning LT as the fair share. It then strengthens as students move further along the trajectory towards co-splitting, where equivalence across both quantities (including without equality of values) becomes of greater importance. Whether mixture problems such as these belong in an equipartitioning curriculum or should be delayed until the sixth grade when CCSSM first addresses ratio is debatable, but from a research perspective, they proved useful in this teaching experiment.

Additive misconceptions have been seen in students' first experiences with both ratio comparison tasks and co-splitting tasks. Unlike ratio comparison tasks however, which do not set students up well to elicit early multiplicative forms of reasoning, co-splitting tasks establish a means for students to determine early versions of equivalent ratios that is entirely multiplicative. This begins with relating two pairs of values for two quantities as preserving or indicating the same fair share through identical multiplicative changes in both quantities (co-splitting), which can then be related to the same preservation of the fair share by combining ratios – pairs of values already established to represent the same fair share for the

quantities. Most of these relationships are initially determined through partitive co-splitting actions that produce lesser values for each quantity, and the relationships to greater values are realized through inverse co-splitting actions and the justification of equivalence using multiplicative reasoning. Beyond equipartitioning, and into ratio, students then are able to apply co-splitting to situations in which the “whole” (numbers of objects and sharers) is not fixed, and therefore determining equivalent ratios involving greater values than those given as well as smaller values without needing to actually account for the number of groups. From that point, students could be introduced to comparison of ratios, with much stronger and better established multiplicative reasoning skills that would not revert back to a focus on additive differences, that are only upheld when the ratio relationship can be represented as 1:1, but rather be enhanced by thinking of different fair shares as representing a multiplicative difference in relative size that in turn becomes unit rate.

### *Summary of Conclusions*

Overall, this teaching experiment has permitted the clarification of the meaning of the proficiency of co-splitting within the equipartitioning LT in two critical ways. First, it has shown that the LT itself supports the positioning of co-splitting at the upper levels with dependencies on the lower levels to various degrees. Second, it has shown many ways in which students of varying ability levels learning the concept can be successful, adding in new concepts and skills gradually as it is mastered. The study has also provided insight into the viability of an LT approach to curriculum and instruction, in the way in which it has been shown that the ideas involved develop gradually and can support diverse methods and paces

by students. Furthermore, the study has shed some light on the importance of coordinating across trajectories, to see when a trajectory overlaps with a concurrent one, such as equipartitioning and division and multiplication, and when a new trajectory emerges from a prior one, such as ratio reasoning emerging from equipartitioning. Finally, the study established the need for further research to understand the development of equipartitioning multiple wholes more completely, and to determine what relationships between multiple wholes and co-splitting emerge as leading towards the highest levels of generalization within the equipartitioning LT.

### Limitations

There were three main limitations to this teaching experiment as a case study on co-splitting. First, and foremost, were the restrictions of the setting of the teaching experiment, which was not ideal because it was conducted with a group of students across grade levels over the summer. This had an impact on the findings in that it made it difficult to make broad claims about a typical, more structured classroom environment, and it resulted in a great deal of missing data and cohesion of the classroom structure. Second, the level of exposure to the lower levels that the students received was very limited and rushed, as those concepts should span four prior grades, and previous teaching experiments did not even cover all of the lower levels in a two-week period, let alone three days. Third, and related to these first two, the assumed levels of proficiency among the students was not very reliable and was based on their exposure through the workshop rather than the inadequate measure of the pre-test that contained large amounts of missing data and limited data at each level. Nonetheless, there

were also obvious advantages in establishing the importance and viability of the types of reasoning to students who were largely unaccustomed to the forms of interactions required by the teaching experiment and the value to providing these to under-served populations of students.

#### Recommendations for Future Research

The progression of work by the DELTA research group that led to this dissertation study has included many iterative phases of the following: a) the development of an LT for equipartitioning, identifying levels and outcome spaces, b) validation of the hypothesized levels and their ordering through a field-test, and c) investigation of anomalies in those results addressed by further clinical interviews and refinement of level meanings and outcome spaces. Part of the last phases of this latter stage of development has been the refinement of definition of co-splitting, and revisions to the trajectory based on clinical interviews I conducted. The larger project has now reached the next stages of development by embarking on two subsequent efforts – creating diagnostic assessments and conducting teaching experiments. This study set out to add to the information gathered from two previous teaching experiments with the lower levels of the LT, that will inform future teaching experiments, curriculum and assessment development, and the process of LT development for use in instruction writ large. The next steps in this line of research would be to design and refine more robust curriculum materials around the levels already studied in depth. Then, to conduct further, empirical validation of the ordering of the levels of the LT through additional studies like this one of other upper level constructs, such as

equipartitioning multiple wholes and the Distributive Property of equipartitioning, and the equipartitioning LT as a whole, and longitudinally across grade levels. And lastly, to conduct studies that inform the interrelationships between the LTs identified in CCSSM.

Two notions stand out as critical for future research on the equipartitioning LT in a classroom or curricular setting related to this study and co-splitting. First, equipartitioning collections and single wholes could be considered co-splitting problems, where meeting the criterion of creating the appropriate number of shares (often implicit in collections problems) determines the co-split to be equal the number of sharers. For example, in sharing a collection of  $n$  objects among  $p$  people, students could recognize the sharing action of dealing those objects to the people as a  $p$  co-split. This is because the goal of the problems is to determine the fair share, and thus  $p$  groups of size 1 are the intended (and again, implicit) outcome. Therefore, both the objects and the people are being  $p$ -split, such that  $p$  groups with 1 sharer are created, and each of those groups receives the same fair share of  $n/p$  objects. It would be interesting to see how an informal introduction to this notion when equipartitioning collections is introduced, and wholes the same, would be received and understood by the students. Furthermore, to see whether it provided a nice reference point when co-splitting is introduced and how the students relate the ideas under such a treatment.

Secondly, the students in this study were introduced to the fair-share box during the workshop as a means of coordinating their responses for equipartitioning collections and wholes problems. Even at these lower levels, students are also able to use the fair share box in responding to reassembly questions to state the multiplicative relationships between fair

sharing situations. However, it was not referenced during the teaching experiment on co-splitting, because there is no inherent way to represent the number of groups in the fair share box. As was seen during the teaching experiment, upper-elementary students already employ multiplication and division facts in working with co-splitting tasks, and can relate the common factor between a pair of facts with the split and the number of groups. Therefore, if afforded the introduction of the division and multiplication box as an extension of the fair share box, they would have a means to represent this as well. In the multiplication and division box, the multiplicative relations between and within fair sharing situations can be identified with an operation and a value along an arrow outside the box indicating the directionality of the operation. It does not seem like a drastic leap to then coordinate those values with the splits and the number of groups, as students already showed they were capable in other representations. This would, however, require students to understand the recursive nature of co-splitting, as was necessary in the way in which the splitting diagram problems were presented in this study. It would be interesting to see how students would use this notation, and whether it would impede them from determining and understanding solutions involving a combining ratio units strategy.

Within any future studies of co-splitting, careful attention should be paid to the contexts used and explorations of other contexts that are not as readily interpretable as fair sharing need to be conducted. In addition, the parameters should be varied to better understand the relationship between the relative sizes of the two quantities as portrayed in those contexts as well as in representations used. This also establishes the need for

investigation into the use of particular representations and tools, and perhaps the invention and introduction of new ones, for operating on co-splitting tasks. Lastly, with respect to the values and parameters in co-splitting problems, it should be studied when the appropriate time is in a sequence of tasks and instructions to introduce and require solutions that represent the base ratio, and how it can be leveraged, as well as solutions involving fractional values, and particularly unit ratios, and likewise how they can be leveraged.

### Closing Remarks

This research found that students who received some prior curricular exposure to the lower levels of the equipartitioning LT were capable of successfully interacting with the co-splitting construct and related tasks at the upper levels of the LT. The performances of these students, as witnessed during the teaching experiment and that have been described herein, far exceeded those performances of other students in previous work on equipartitioning, and on the field test. The relationships between multiplication and division and equipartitioning, and co-splitting in particular, were quite apparent in this study. However, given the complicated nature of those relationships and how they are realized and then utilized, understanding them (and others like them – connecting other LTs) is only in a state of infancy. The work of developing and refining learning trajectories is an extremely iterative process, and as that process expands to the creation of curriculum, assessments, and pedagogy that are all linked and centered around learning trajectories, more and more iterations will be required, even for a well-established LT such as equipartitioning.

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**APPENDICES**

## Appendix A

### Assessments

#### *Multiplication, Division, and Fractions*

1. 36 votes were cast for the class president. All 4 candidates got the same number of votes. How many votes did each candidate get?
2. You have a ribbon that is 32 inches long. An eight-inch piece of ribbon is needed to make a pretty bow. How many bows can you make if you use all of your ribbon?
3. The floor of a small room is 8 feet by 5 feet. The area of a square tile is 1 square foot. How many tiles are needed to cover the floor? Show your work.
4. A bus has 12 rows with 4 students in each row. The bus makes 3 stops and the same number of students gets off at each stop. How many students get off at each stop?
5. Write down the answer for each multiplication or division problem below.

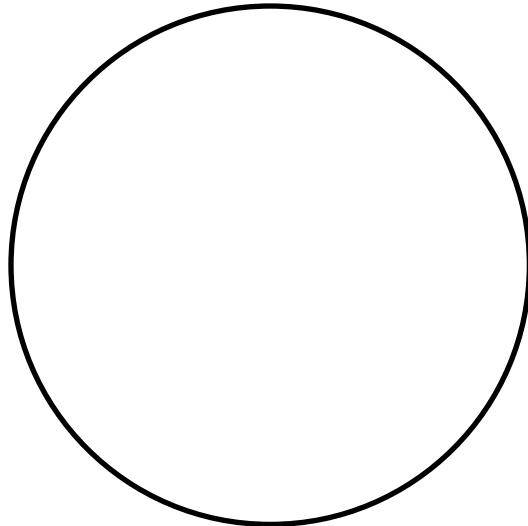
$$\begin{array}{r} 3 \\ \times 4 \\ \hline \end{array}$$

$$\begin{array}{r} 7 \\ \times 5 \\ \hline \end{array}$$

$$\begin{array}{r} 6)18 \\ \hline \end{array}$$

$$\begin{array}{r} 9 \\ \times 0 \\ \hline \end{array}$$

6. Shade  $\frac{3}{5}$  of the circle.



For problems 7-10, circle the statement that is correct.

7.

$$\frac{1}{4} > \frac{3}{4}$$

$$\frac{1}{4} = \frac{3}{4}$$

$$\frac{1}{4} < \frac{3}{4}$$

8.

$$\frac{6}{8} > \frac{3}{4}$$

$$\frac{6}{8} < \frac{3}{4}$$

$$\frac{6}{8} = \frac{3}{4}$$

9.

$$\frac{3}{8} < \frac{3}{5}$$

$$\frac{3}{8} > \frac{3}{5}$$

$$\frac{3}{8} = \frac{3}{5}$$

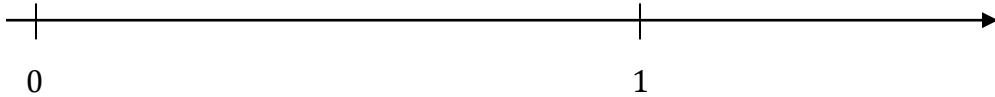
10.

$$\frac{2}{3} < \frac{7}{12}$$

$$\frac{2}{3} = \frac{7}{12}$$

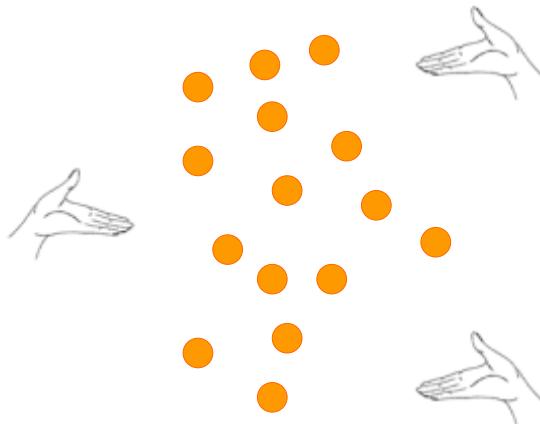
$$\frac{2}{3} > \frac{7}{12}$$

11. Draw dots to show where  $\frac{3}{4}$ ,  $\frac{1}{2}$ ,  $\frac{5}{4}$ , and  $\frac{3}{5}$  are located on the number line below. Label each of your dots with the number it represents.

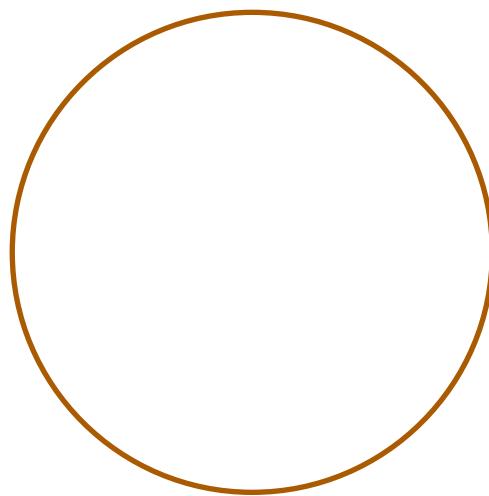


*Equipartitioning Pre-Test*

1. Three (3) friends, Fred, Rebecca, and Tim have a small package of butterscotch candies that they want to share. Show how the three friends could share the candies fairly.

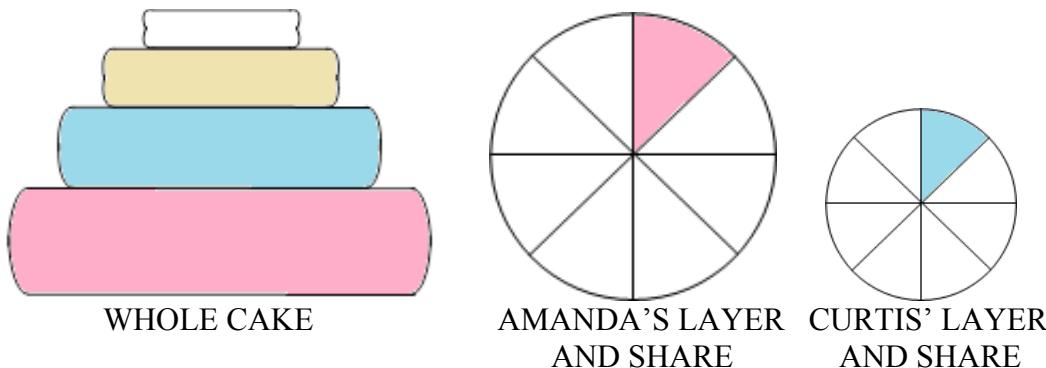


- a. How do you know that each person got a fair share?  
b. How many candies did each person get?
2. Six (6) students were selected to attend a pizza party for performing well on their math test. The drawing below represents a pizza. Show how the six students could share a pizza fairly.

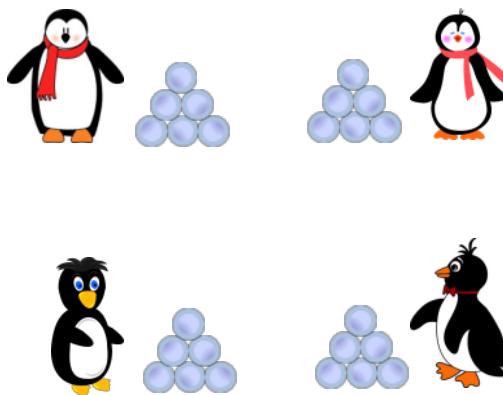


- a. How do you know that each student got a fair share?  
b. Compare the size of one student's share to the size of the whole pizza. How much of the whole pizza is one student's share?  
c. Compare the size of the whole pizza to the size of one student's share.

- d. The whole pizza is \_\_\_\_\_ times as large as one student's share.
3. Amanda and Curtis' wedding cake was made up of four different size circular layers. Each of them ate a fair share from two different layers of the wedding cake, as shown in the picture below.



- a. Compare the size of Amanda's share to the *size of her layer* (in pink) of the whole cake. How much of her layer of cake is Amanda's share?
- b. Compare the size of Curtis' share to the *size of his layer* (in blue) of the whole cake. How much of his layer of cake is Curtis' share?
- c. Did Amanda and Curtis get the same fair share of their layers of cake? Why or why not?
- d. Did Amanda and Curtis get the same fair share of the whole cake (all layers)? Why or why not?
4. Four (4) penguins have fairly shared twenty-four (24) snowballs to use in the big snowball fight at the annual Winter Festival, as shown in the picture below.



- a. How many snowballs are in each penguin's share?

- b. Compare the size of one penguin's share to the size of the whole collection of snowballs. How much of the whole collection is one penguin's share?
- c. Compare the size of the whole collection of snowballs to the size of one penguin's share. How many times as large is the whole collection?

5. Bill and some of his cousins all fairly shared a bag of marshmallows to make S'Mores around the bonfire at their family reunion.

If one fair share is  $1/5^{\text{th}}$  of the whole bag and Bill has 7 marshmallows, how many marshmallows were in the whole bag? *Show your work.*

6. Indicate whether the following sentence is ALWAYS true, SOMETIMES true, or NEVER true by drawing a *circle* around the word that matches your answer below.

$1/8^{\text{th}}$  of one collection is **equal** to  $1/8^{\text{th}}$  of a different collection.

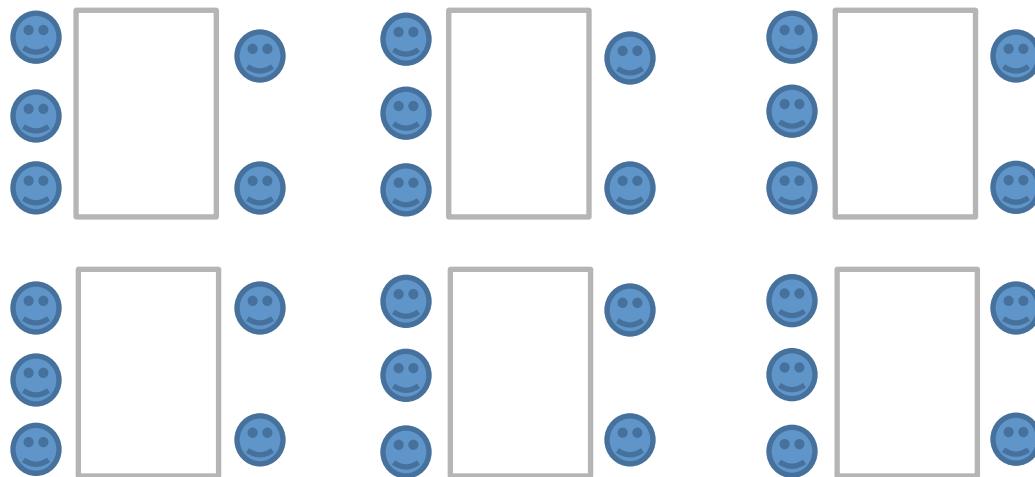
ALWAYS

SOMETIMES

NEVER

7. A group of 30 people have been seated at 6 tables in a restaurant so that every table is seating a fair share of all the people. Three (3) tables need to be removed.

(You may use the diagram below to help answer the following questions, if necessary)



- How many people will be at each table (after removing the 3 tables)? How do you know?
- Indicate your response coloring in the bubble beside the statement you agree with and fill in the blank to complete the statement.
- If there are now one-third as many tables as before and still the same number of people, then compared to the number of people at each table before, after removing tables there will be:

\_\_\_\_ times as many people at each table.

- \_\_\_\_\_ more people at each table.  
 \_\_\_\_\_ less people at each table.  
 the same number of people at each table.
8. Indicate whether the sentence is ALWAYS true, SOMETIMES true, or NEVER true when completed by the statements that follow. Indicate your responses by drawing a *circle* around the word that matches each answer.

Some tables are removed from a restaurant so that there are one-third as many tables as before, but there is still the same number of people.

Compared to the number of people at each table before, *after removing the tables*, there could be \_\_\_\_\_.

- d. ... 4 more people at each table.

ALWAYS      SOMETIMES      NEVER

- e. ... three times as many people at each table.

ALWAYS      SOMETIMES      NEVER

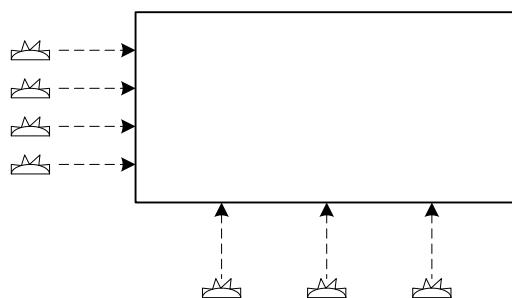
- f. ... less people at each table.

ALWAYS      SOMETIMES      NEVER

- g. ... the same number of people at each table.

ALWAYS      SOMETIMES      NEVER

9. A factory makes wooden building blocks by sending large rectangular pieces of wood through two sets of parallel saw blades, one after the other, as shown below.

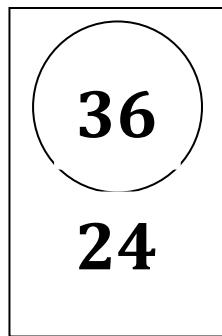


- a. How many total blocks would be created after the large block of wood shown above runs through both sets of saw blades as indicated by the picture? Explain how you got your answer.
10. Consider block making machines like those from the previous problem and two new and different situations below. Give your responses in the space provided.
- Another machine splits large wooden rectangles into 4 equal parts by a first set of parallel saw blades. Determine the number of parts the second set of saw blades will need to split the wood into so that you end up with 24 total blocks. Explain how you got your answer.
  - Suppose there are two machines making blocks. The first set of saw blades splits the wood into the same number of parts on both machines. The second set of saw blades on the second machine splits the wood into *half as many parts* as the second set of saw blades on the first machine.
- Compare the size of the final blocks created by the first machine to the final blocks created by the second machine. Explain how you got your answer.
11. Suppose you had a rectangular piece of paper. If you were to fold it into 6 equal parts horizontally and then into 3 equal parts vertically, how many total parts would be created? Explain how you got your answer.
- Describe as many other ways to fold the paper as you can think of that would produce the same number of parts.

12. Twenty-four players arrive at a restaurant to order pizzas. If the owner had a very large table, he could place all of the players and all of the pizzas at one table as shown in the picture below. The number in the circle (36) shows the total number of pizzas and the number written below shows the number of players (24) seated at the table.

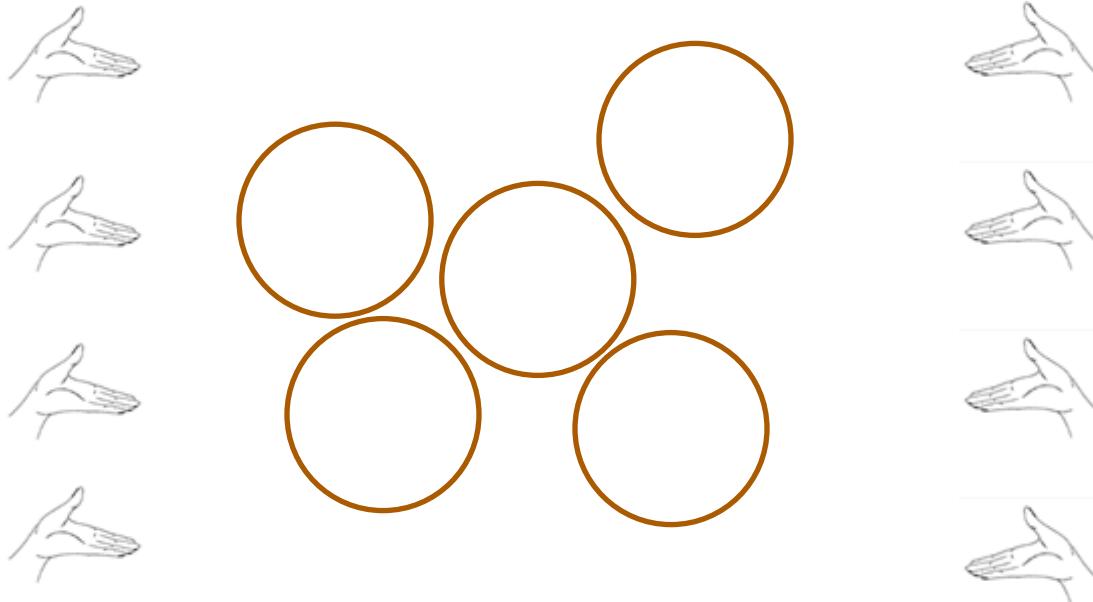
The owner has lots of tables in the back but he does not have one that can seat the 24 players at one table.

In the spaces provided, *show three different arrangements* of tables and pizzas such that all of the 24 players are seated at tables and they are all going to get their fair share of the 36 pizzas.



13. Sebastian has 20 fish in one large tank and feeds them 12 pellets of food every day, which all of the fish share fairly. He wants to separate the fish into smaller tanks.
- What is the smallest number of fish that can go into each tank if they must all still be able to get the same fair share of food pellets and the food pellets cannot be broken into smaller pieces? *Show your work.*
  - How many food pellets make up a fair share for each of Sebastian's fish? *Show your work.*

14. Eight (8) children want to fairly share five (5) pies. Show how you would help the children fairly share the pies using the figures below.



- a. How *many* pies does each child get?
  - b. How *much of all* of the pies does each child get?
13. Twenty-six (26) friends want to fairly share thirty-five (35) cupcakes.
- a. How *many* cupcakes will each person get?
  - b. How *much of all* of the cupcakes will each person get?

*Multiplication and Division Quick Check*

Fill in the next two numbers.

1. 2, 4, 6, \_\_\_\_\_, \_\_\_\_\_

2. 5, 10, 15, 20, \_\_\_\_\_, \_\_\_\_\_

3. 4, 8, 12, 16, \_\_\_\_\_, \_\_\_\_\_

4. 3, 6, 9, 12, \_\_\_\_\_, \_\_\_\_\_

5. 7, 14, 21, \_\_\_\_\_, \_\_\_\_\_

Multiply.

6.  $5 \times 6 =$  \_\_\_\_\_

7.  $7 \times 4 =$  \_\_\_\_\_

8.  $8 \times 3 =$  \_\_\_\_\_

9.  $4 \times 9 =$  \_\_\_\_\_

10.  $2 \times 12 =$  \_\_\_\_\_

Divide.

11.  $32 \div 4 =$  \_\_\_\_\_

12.  $18 \div 2 =$  \_\_\_\_\_

13.  $35 \div 5 =$  \_\_\_\_\_

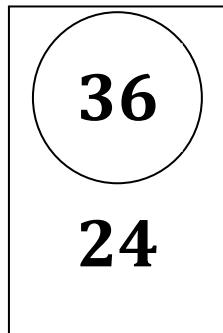
14.  $24 \div 3 =$  \_\_\_\_\_

15.  $40 \div 8 =$  \_\_\_\_\_

*Equipartitioning Post-Test*

1. Twenty-four players arrive at a restaurant to order pizzas. If the owner had a very large table, he could place all of the players and all of the pizzas at one table as shown in the picture below. The number in the circle (36) shows the total number of pizzas and the number written below shows the number of players (24) seated at the table.

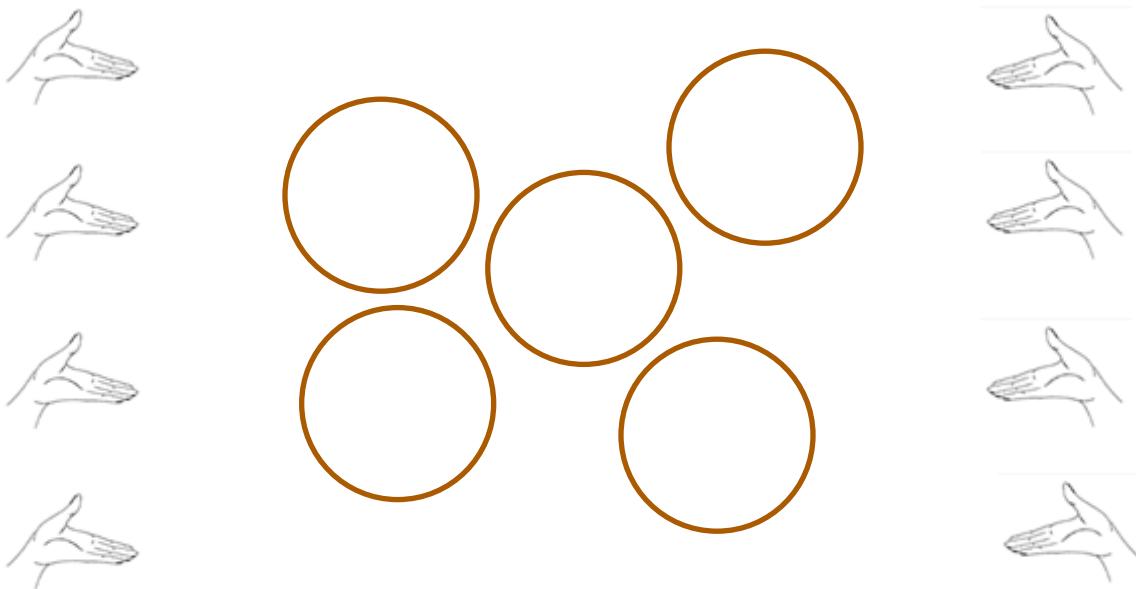
The owner has lots of tables in the back but he does not have one that can seat the 24 players at one table.



In the three spaces provided on the next page, *show three different arrangements* of tables and pizzas such that all of the 24 players are seated at tables and they are all going to get their fair share of the 36 pizzas.

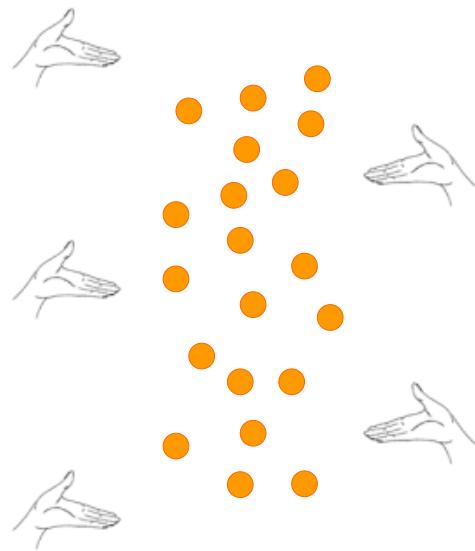
2. Sebastian has 20 fish in one large tank and feeds them 12 pellets of food every day, which all of the fish share fairly. He wants to separate the fish into smaller tanks to give them more room to swim around.
  - a. What is the smallest number of fish that can go into each tank if they must all still be able to get the same fair share of food pellets and the food pellets cannot be broken into smaller pieces? *Show your work.*
  - b. How many food pellets make up a fair share for each of Sebastian's fish? *Show your work.*

3. Eight (8) children want to fairly share five (5) pies. *Show* how you would help the children fairly share the pies (represented by the circles) using the diagram below.



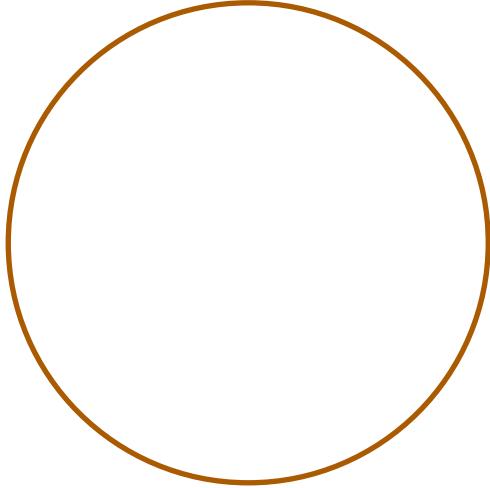
- a. How *many* pies does each child get? Explain how you got your answer.
  - b. How *much of all* of the pies does each child get?
4. Twenty-six (26) friends want to fairly share thirty-five (35) cupcakes.
- a. How *many* cupcakes will each person get? Explain how you got your answer.
  - b. How *much of all* of the cupcakes will each person get?

5. Five (5) friends, Stephen, Gwen, and Marcus have a small package of butterscotch candies that they want to share. Use the picture below to *show* how the three friends could share the candies fairly.



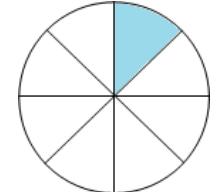
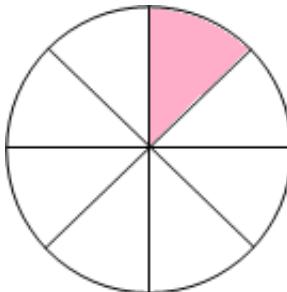
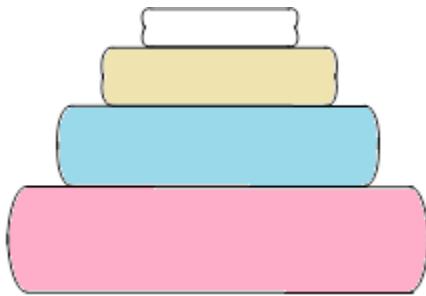
- a. How do you know that each person got a fair share?
- b. How many candies did each person get?

6. Nine (9) students were selected to attend a pizza party for performing well on their math test. The drawing below represents a pizza. *Show* how the nine students could share a pizza fairly.



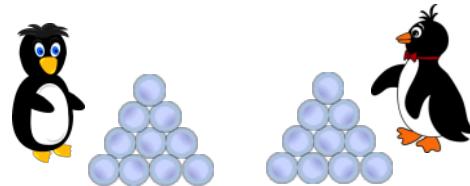
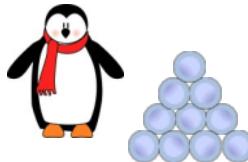
- a. How do you know that each student got a fair share?
- b. *How much* of the whole pizza is one person's share?
- c. The whole pizza is *how many times larger* than one person's share?

7. Amanda and Curtis' wedding cake was made up of four different size circular layers. Amanda ate the fair share marked from the pink layer and Curtis ate the fair share marked from the blue layer as shown in the picture below.



- a. Describe the size of Amanda's share *of her layer* (in pink).
- b. Describe the size of Curtis's share *of his layer* (in blue).
- c. Did Amanda and Curtis get the same fair share *of each of their layers* of cake? Why or why not?
- d. Did Amanda and Curtis get the same fair share *of the whole cake (all layers)*? Why or why not?

8. Three (3) penguins have fairly shared twenty-four (24) snowballs to use in the big snowball fight at the annual Winter Festival, as shown in the picture below.



- a. How many snowballs are in each penguin's share?
  - b. *How much* of the whole collection of snowballs is one penguin's share?
  - c. The whole collection of snowballs is *how many times larger* than one penguin's share?
9. Bill and some of his cousins all fairly shared a bag of marshmallows to make S'Mores around the bonfire at their family reunion.

If one fair share is  $1/6^{\text{th}}$  of the whole bag and Bill has 8 marshmallows, how many marshmallows were in the whole bag? *Show your work.*

10. Indicate whether the following sentence is ALWAYS true, SOMETIMES true, or NEVER true by drawing a *circle* around the word that matches your answer below.

**$1/4^{\text{th}}$  of one collection is equal to  $1/4^{\text{th}}$  of a different collection.**

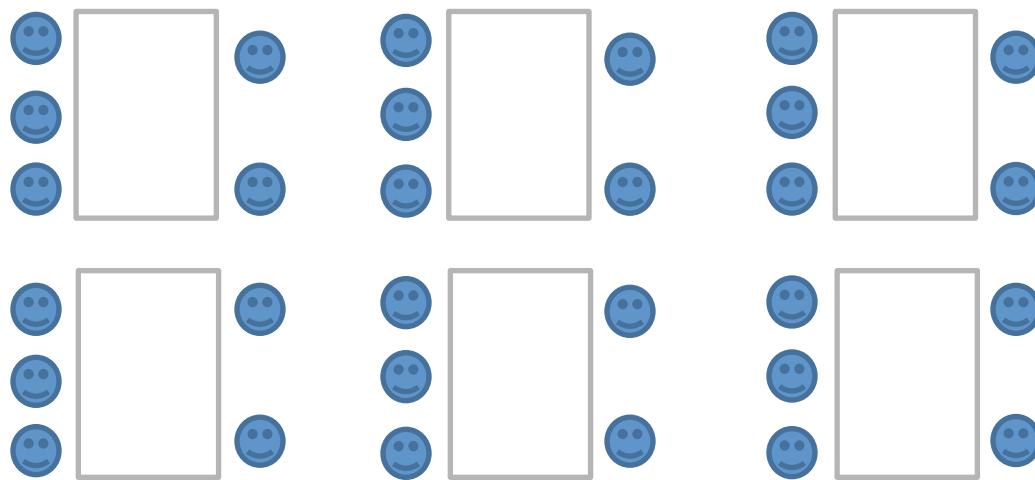
ALWAYS

SOMETIMES

NEVER

Give an example to explain your choice.

11. Thirty (30) people have been seated fairly at six (6) tables in a restaurant, as shown in the diagram below.



Four (4) tables need to be removed to create more floor space for entertainment. The people still need to be seated fairly among the remaining tables to make service easier.

How many people will be at each table after removing the 3 tables?

Explain how you got your answer and then justify it as many ways as you can. (You may use the diagram, if necessary.)

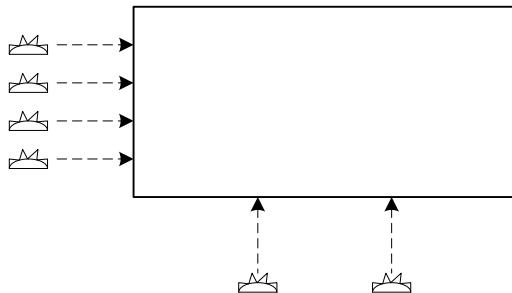
12. Indicate whether the sentence below is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true when completed by the statements that follow. Indicate your responses by drawing a *circle* around the word that matches each answer.

Some tables are removed from a restaurant so that there is one-half as many tables as before, but there is still the same number of people.

Compared to the number of people at each table before, *after removing the tables* there could be \_\_\_\_\_:

- ... 4 more people at each table.  
ALWAYS      SOMETIMES      NEVER
- ... two times as many people at each table.  
ALWAYS      SOMETIMES      NEVER
- ... less people at each table.  
ALWAYS      SOMETIMES      NEVER
- ... the same number of people at each table.  
ALWAYS      SOMETIMES      NEVER

13. A factory makes wooden building blocks by sending large rectangular pieces of wood through a machine that has two sets of parallel saw blades, which cut the wood one after the other, as shown below.



How many total blocks would be created after the large block of wood runs through both sets of saw blades as indicated by the picture above? Explain how you got your answer.

14. Consider block making machines like those from the previous problem and two new and different situations below. Give your responses in the space provided.

- a. Another machine splits large wooden rectangles into 4 equal parts by a first set of parallel saw blades. Determine the number of parts the second set of saw blades will need to split the wood into so that you end up with 24 total blocks. Explain how you got your answer.
- b. Suppose there are two machines making blocks. The first set of saw blades splits the wood into the same number of parts on both machines. The second set of saw blades on the second machine splits the wood into *half as many parts* as the second set of saw blades on the first machine.

Compare the size of the final blocks created by the first machine to the final blocks created by the second machine. Explain how you got your answer.

15. Suppose you had a rectangular piece of paper. If you were to fold it into 3 equal parts horizontally and then into 8 equal parts vertically, how many total parts would be created? Explain how you got your answer.

Describe as many other ways to fold the paper as you can think of that would produce the same number of total parts.

## Appendix B

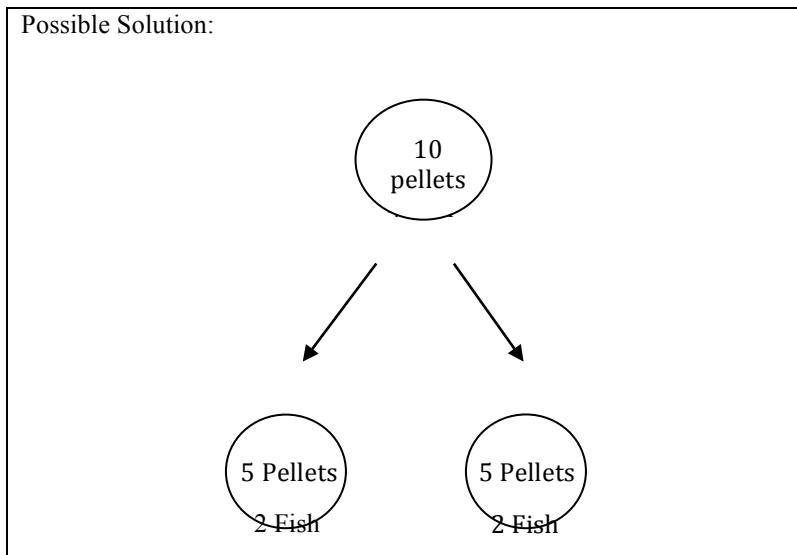
### Student Worksheets

#### *Fish and Bowls*

You have a group of fish in one bowl and it is too crowded. You need to split the fish into more than one bowl and still feed them a fair share of the food pellets.

**Directions:** Draw a possible combination in the space provided and then answer the questions that follow.

1. There are 10 pellets and 4 fish to be fed.



- a. How many bowls did you use?
  - b. How many pellets are in each bowl?
  - c. How many fish are in each bowl?
2. There are 12 pellets and 8 fish to be fed.
    - a. How many bowls did you use?
    - b. How many pellets are in each bowl?
    - c. How many fish are in each bowl?

- d. Explain how you know every fish got a fair share of the pellets.
3. There are 18 pellets and 9 fish to be fed.
- How many bowls did you use?
  - How many pellets are in each bowl?
  - How many fish are in each bowl?
  - Explain how you know every fish got a fair share of the pellets.
4. There are 16 pellets and 12 fish to be fed.

# of Pellets	# of Fish

- Explain how you know every fish got a fair share of the pellets.
5. There are 27 pellets and 18 fish to be fed.

# of Pellets	# of Fish

- Explain how you know every fish got a fair share of the pellets.



6. There are 24 pellets and 30 fish to be fed.

# of Pellets	# of Fish

- a. How many bowls did you use?
  - b. How many pellets are in each bowl?
  - c. How many fish are in each bowl?
  - d. Explain how you know every fish got a fair share of the pellets.
7. There are 30 pellets and 36 fish to be fed.

# of Pellets	# of Fish

- a. How many bowls did you use?
- b. How many pellets are in each bowl?
- c. How many fish are in each bowl?
- d. Explain how you know every fish got a fair share of the pellets.

8. There are 15 pellets and 35 fish to be fed.

# of Pellets	# of Fish

- a. How many bowls did you use?
  - b. How many pellets are in each bowl?
  - c. How many fish are in each bowl?
  - d. Explain how you know every fish got a fair share of the pellets.
9. There are 18 pellets and 30 fish to be fed.

# of Pellets	# of Fish

- a. Explain how you know every fish got a fair share of the pellets.
- b. Do you think there are more ways to place the fish and pellets in more than one bowl? How many do you think there is?
- c. Explain to your partner how you could find another way, and write down what you would say or give an example below. Tell how you knew how many bowls to use and how you knew each fish would be able to get a fair share of the pellets.

### *Pizzas and People*

A sports team arrives at a restaurant to order pizzas. The owner cannot provide a table large enough to fit all of the players and pizza they ordered. You must help the owner figure out how to seat some team members and serve some pizzas at more than one table, so that all of the players have a seat and all of the pizzas are served (without cutting them first – *whole pizzas only*), and every player at all of the tables gets the same share of pizza.

**Directions:** Show how the players could sit at multiple tables and the waiters could deliver all of the pizzas to those tables so everyone still gets a fair share of all of the pizza?

1. There are 28 pizzas needed for 12 players to get their fair share.
  - a. How many tables will you use? Explain your thinking.
  - b. Show *one way* how all of the players could be seated and pizzas could be served.
  - c. How many players are at each table? Explain your thinking.
  - d. How many pizzas are at each table? Explain your thinking.
2. There are 24 pizzas needed for 18 players to get their fair share.
  - a. How many tables will you use? Explain your thinking.
  - b. Show *one way* how all of the players could be seated and pizzas could be served.
  - c. How many players are at each table? Explain your thinking.
  - d. How many pizzas are at each table? Explain your thinking.
3. There are 16 pizzas needed for 12 players to get their fair share.
  - a. Show *one way* how all of the players could be seated and pizzas could be served.
  - b. Explain how you know every player still gets their same fair share of all the pizza at the tables you created.
4. There are 27 pizzas needed for 18 players to get their fair share.  
What is the smallest pairing of players and whole pizzas that can be at a table where each player still receives their same fair share?  
*Show your work.*
5. There are 24 pizzas needed for 30 players to get their fair share.

- a. Show *as many ways* as you can how all of the players could be seated and pizzas could be served.
  - b. Explain how you know every player got a fair share of the pizza for each of the different sets of tables you created.
6. There are 30 pizzas needed for 36 players to get their fair share.
    - a. Show *as many ways* as you can how all of the players could be seated and pizzas could be served.
    - b. Explain how you know every player got a fair share of the pizza for each of the different sets of tables you created.
  7. There are 15 pizzas needed for 35 players to get their fair share.
    - a. Show *as many ways* as you can how all of the players could be seated and pizzas could be served.
    - b. How much pizza does each person get?
  8. There are 18 pizzas needed for 30 players to get their fair share.
    - a. Show *one way* how all of the players could be seated and pizzas could be served.
    - b. Explain to a friend how you know how many tables to use and how you knew each player gets a fair share of the pizza.
    - c. How many ways do you think there are to seat this team? Could you come up with them all? Show *as many ways* as you can how all of the players could be seated and pizzas could be served on the next page.

*Cookies and People*

A group of children want to fairly share a package of large cookies. There is not a table in the room big enough for all of them. The teacher wants to give out all of the cookies right away so that no one will have to get up from their seats.

**Directions:** Find other ways that the children could sit at more than one table and the teacher can pass out all of the cookies.

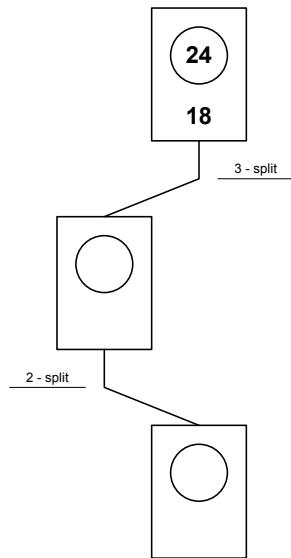
*Remember:* Each table must fairly share only the cookies given to them, and the teacher wants all of the children to be able to get the same fair share.

1. There are 18 children to fairly share 27 cookies.
2. There are 30 children to fairly share 24 cookies.
3. There are 36 children to fairly share 30 cookies.
4. There are 35 children to fairly share 15 cookies.
5. There are 30 children to fairly share 18 cookies.

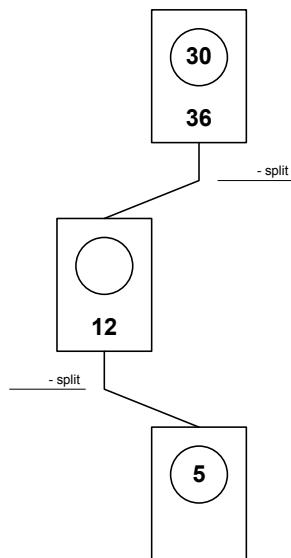
### *Co-splitting Trees*

**Directions:** Fill in the missing values in each box and label the splits, so that each level is equivalent to all of those above. *Remember:* Co-splitting means that the same split is done to both numbers, and they will always create a group of all equal boxes.

1.

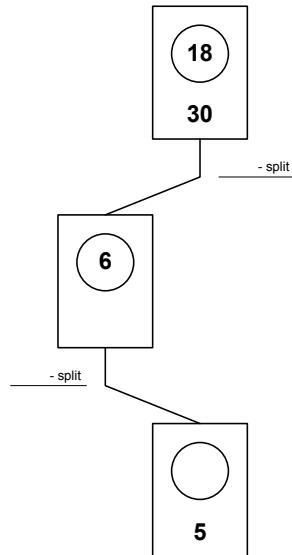


2.



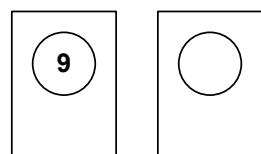
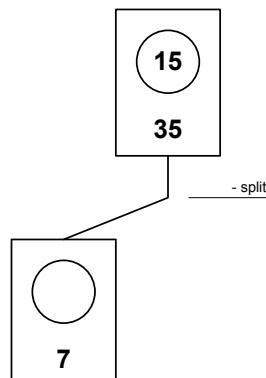
How many boxes will there be like the bottom one in the tree?  
Explain how you know.

3.



How many boxes will there be like the bottom one in the tree?  
Explain how you know.

4. *Hint:* Co-splits do not always have to be used and there can be equivalent (same fair share) boxes with different numbers in them!

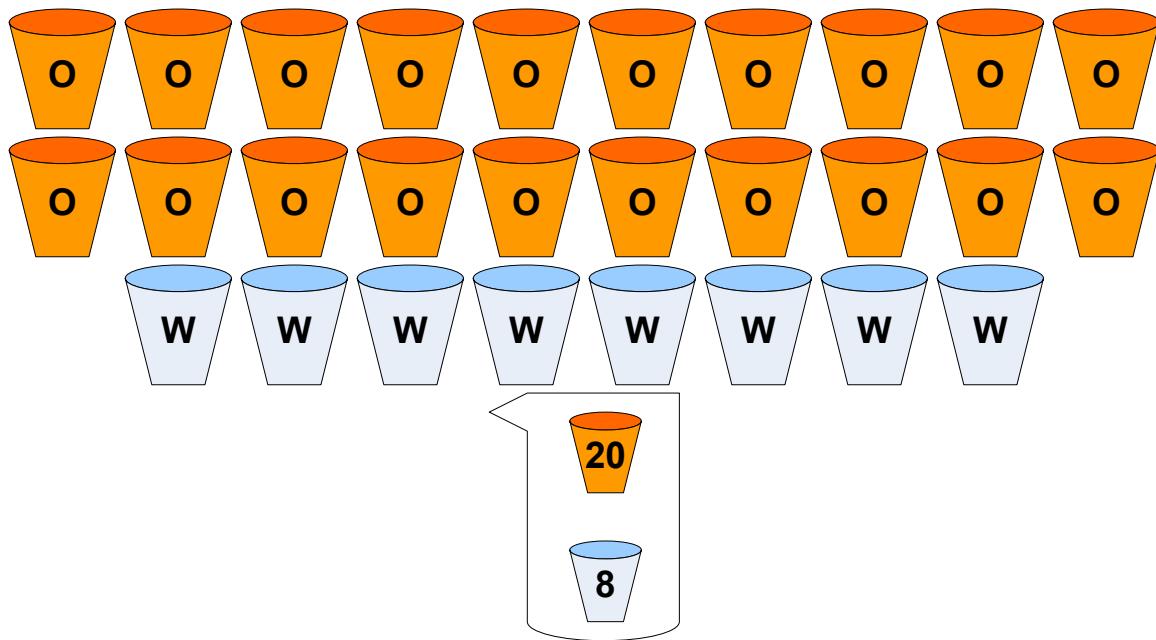


### *Orange Drink Mixtures*

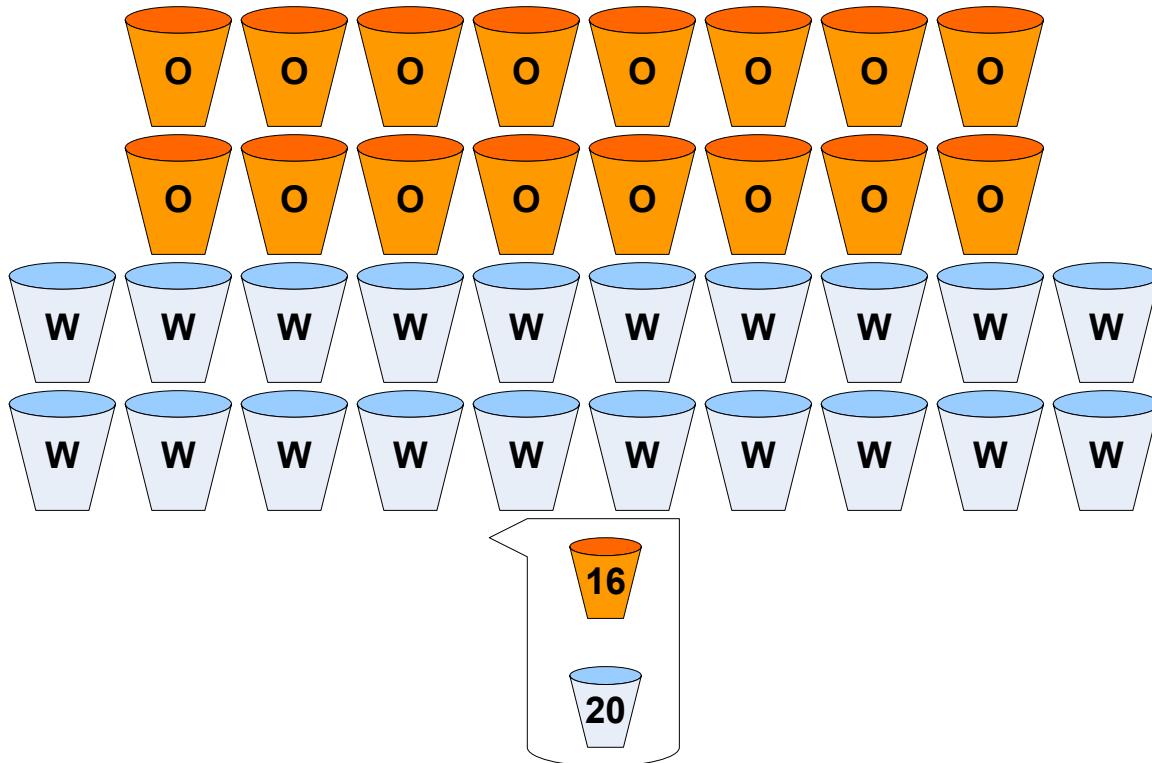
An orange drink is made from a mixture of orange juice and water. Each recipe combines a certain number of cups of the orange juice with a certain number of cups of the water.

**Directions:** Show how the orange drink could be made in more than one pitcher, so that the orange drink in every pitcher still tastes the same.

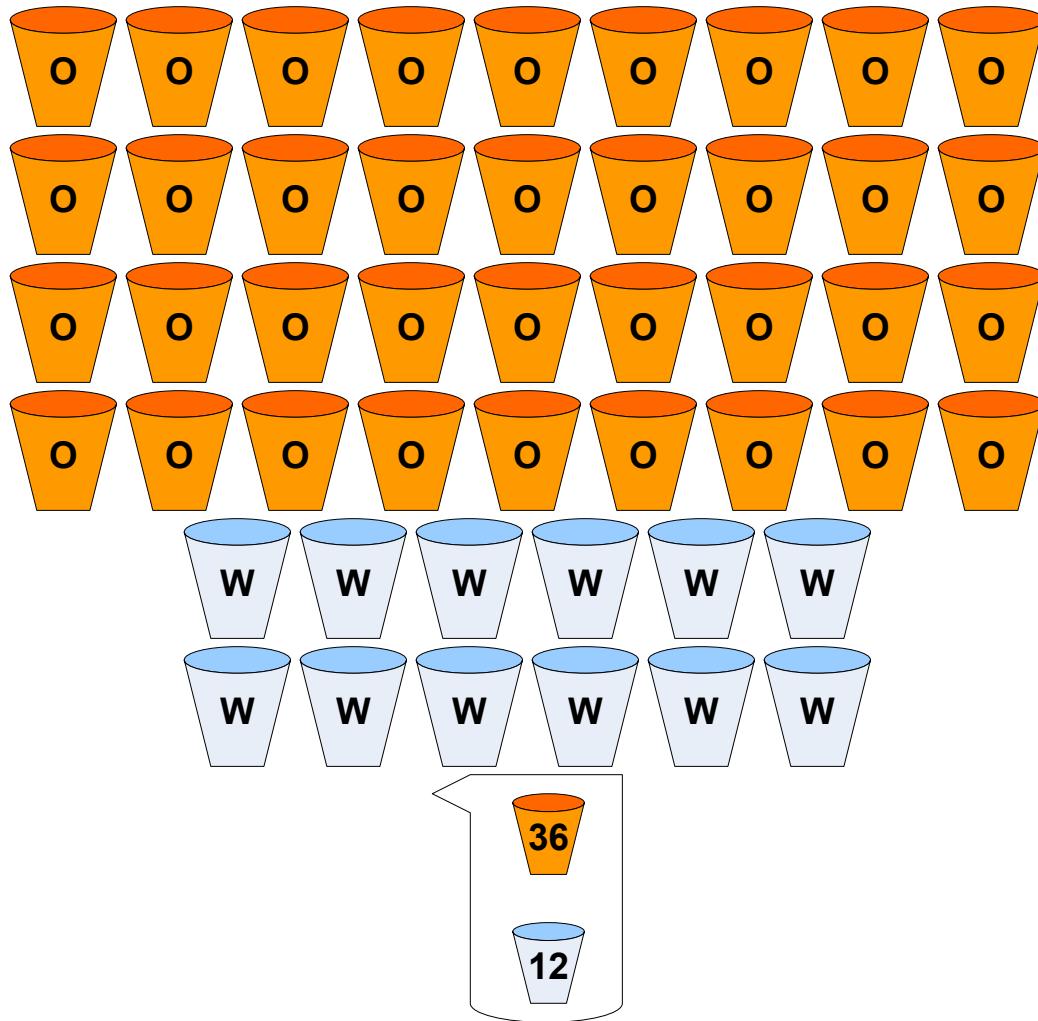
1. The orange drink recipe calls for 20 cups of orange juice and 8 cups of water. What are the smallest numbers of cups of orange juice and water that can go in a pitcher so that the recipe tastes the same?



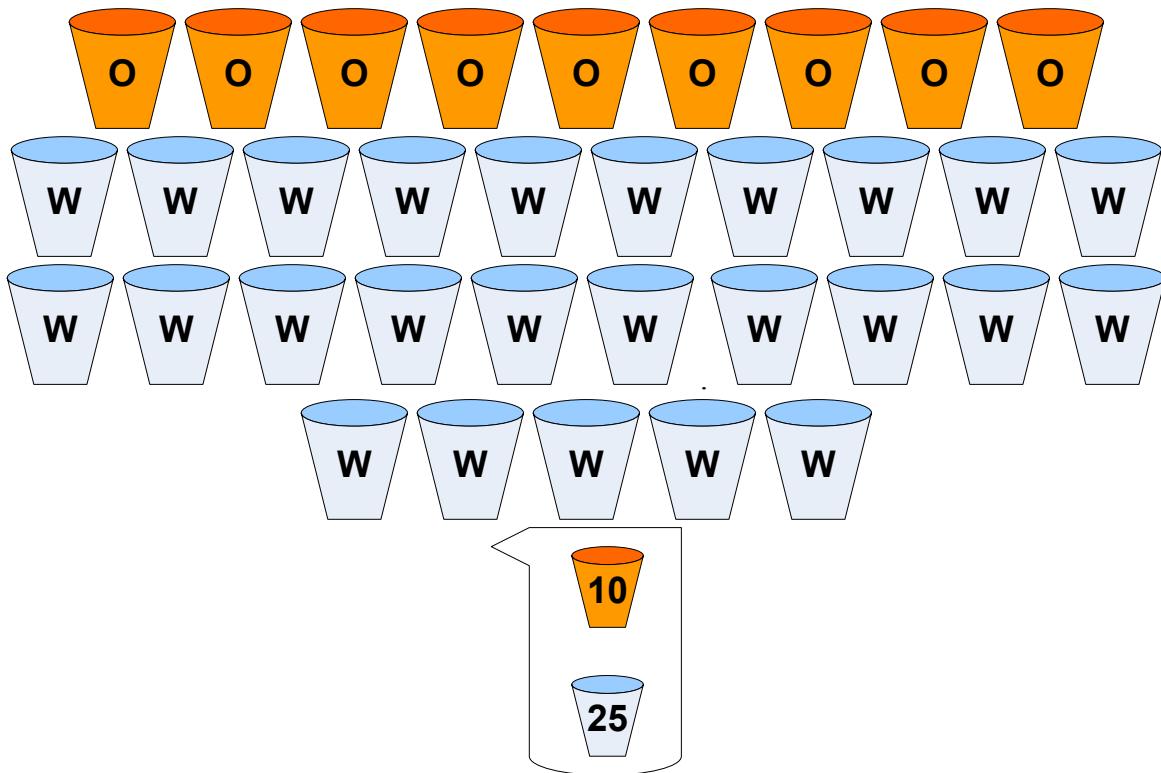
2. The orange drink recipe calls for 16 cups of orange juice and 20 cups of water. What are the smallest numbers of cups of orange juice and water that can go in a pitcher so that the recipe tastes the same?



3. The orange drink recipe calls for 36 cups of orange juice and 12 cups of water. What are the smallest numbers of cups of orange juice and water that can go in a pitcher so that the recipe tastes the same?



4. The orange drink recipe calls for 10 cups of orange juice and 25 cups of water. What are the smallest numbers of cups of orange juice and water that can go in a pitcher so that the recipe tastes the same?



## Appendix C

### Clinical Interview Protocols

#### *Clinical Interview 1 (co-splitting)*

Thank you for coming back to help us out more this afternoon. The reason we are having some of you come back individually is so we can hear more about what *just you* are thinking and give you more chances to talk to us about it, which you may not get to do when the whole class is in the room. We are going to work on some problems similar to those we have been doing in the morning, and we will only go for about 30-45 minutes. If you get tired or do not want to continue for any reason before that time is up, just let me know and we will stop. We are not concerned with right and wrong answers here, but we really just want to hear how you are thinking about the problems and the ways you go about solving them. I will ask you to tell me what you are doing and thinking as you are working, and then also ask you some more questions after you are done working on each problem.

Do you have any questions for me?

**Materials:**

Manipulatives – Paper cutouts (circles and rectangles), colored counters, and colored blocks. Scissors. Scratch paper and pencils.

First, ask the student to tell you their favorite animal, and then collaborate with him or her to determine a logical food source that can be represented by the manipulatives and would make sense for partial amounts to exist.

Introduce the manipulatives, and identify what each will represent – objects, sharers, and grouping mechanisms. Also, indicate that paper and pencil are available, as well as scissors, if he or she needs to draw, write, or cut anything.

**Tasks:**

Determine a way that all of the animals and food can be split into more than one group so that the animals still get the same fair share of food.

Determine another way (*repeat until all solutions determined or student cannot determine any further solutions*).

**Sequence of Parameters:**

More objects than sharers

*Task 1 – 16 objects shared among 6 sharers (if too difficult: 12 among 8)*

*Task 2 – 18 objects shared among 12 sharers*

*\*for each or any task, you may build on previous parameters by adjusting to multiples or factors of one or both quantities*

Probes:

How did you determine the number of groups to use?

How did you determine the amounts of animals and food in each group?

Do the animals in each group get the same fair share of the food?

Do the animals in one group get the same fair share of the food as if they were all in one group?

*Clinical Interview 2 (co-splitting)*

Thank you for coming back to help us out more this afternoon. The reason we are having some of you come back individually is so we can hear more about what *just you* are thinking and give you more chances to talk to us about it, which you may not get to do when the whole class is in the room. We are going to work on some problems similar to those we have been doing in the morning, and we will only go for about 30-45 minutes. If you get tired or do not want to continue for any reason before that time is up, just let me know and we will stop. We are not concerned with right and wrong answers here, but we really just want to hear how you are thinking about the problems and the ways you go about solving them. I will ask you to tell me what you are doing and thinking as you are working, and then also ask you some more questions after you are done working on each problem.

Do you have any questions for me?

Materials:

Manipulatives – Paper cutouts (circles and rectangles), colored counters, and colored blocks. Scissors. Scratch paper and pencils.

First, ask the student to tell you their favorite animal, and then collaborate with him or her to determine a logical food source that can be represented by the manipulatives and would make sense for partial amounts to exist.

Introduce the manipulatives, and identify what each will represent – objects, sharers, and grouping mechanisms. Also, indicate that paper and pencil are available, as well as scissors, if he or she needs to draw, write, or cut anything.

Tasks:

Determine a way that all of the animals and food can be split into more than one group so that the animals still get the same fair share of food.

Determine another way.

**Sequence of Parameters:**

More sharers than objects

*Task 3 – 12 objects shared among 16 sharers*

*Task 4 – 6 objects shared among 16 sharers*

*Task 5 – 12 objects shared among 18 sharers*

*\*for each or any task, you may build on previous parameters by adjusting to multiples or factors of one or both quantities*

**Probes:**

How did you determine the number of groups to use?

How did you determine the amounts of animals and food in each group?

Do the animals in each group get the same fair share of the food?

Do the animals in one group get the same fair share of the food as if they were all in one group?

***Clinical Interview 3 (multiple wholes)***

Thank you for coming back to help us out more this afternoon. The reason we are having some of you come back individually is so we can hear more about what *just you* are thinking and give you more chances to talk to us about it, which you may not get to do when the whole class is in the room. We are going to work on some problems that are kind of like what we have been doing in the morning, but they are slightly different; and we will only go for about 30-45 minutes. If you get tired or do not want to continue for any reason before that time is up, just let me know and we will stop. We are not concerned with right and wrong answers here, but we really just want to hear how you are thinking about the problems and the ways you go about solving them. I will ask you to tell me what you are doing and thinking as you are working, and then also ask you some more questions after you are done working on each problem.

Do you have any questions for me?

**Materials:**

Manipulatives – Paper cutouts (circles and rectangles). Scissors. Scratch paper and pencils.

First, ask the student to tell you their favorite type of pie, cake, or pancake that can be represented by the cutout-circle manipulatives and would establish that for that type of food, it would make sense for partial amounts to exist. Introduce rectangle-cutout manipulatives as people.

Also, indicate that paper and pencil are available, as well as scissors, if he or she needs to draw, write, or cut anything.

**Tasks:**

Determine a way that each person can be given a fair share of all the objects.

**Sequence of Parameters:**

More sharers than objects

*Task 1 – 4 objects shared among 3 sharers (5 among 4 if too difficult)*

*Task 2 – 7 objects shared among 3 sharers*

*Task 3 – 8 objects shared among 3 sharers*

*Task 4 – 6 objects among 4 sharers*

*Task 5 – 3 objects among 4 sharers*

*Task 6 – 5 objects among 6 sharers*

*Task 7 – 2 objects among 5 sharers*

*\*for each or any task, you may build on previous parameters by adjusting to multiples or factors of one or both quantities*

**Probes:**

How did you determine how many objects each person would get?

Did everyone get a fair share? How do you know?

## Appendix D

### Acronyms, Terms, and Definitions

*Arrangement* – a set of groups of the two quantities in a fair-sharing situation offered as a solution to a co-splitting task.

*Base Ratio* – the ratio unit in which the values for each quantity constitute the smallest, relatively prime, whole-number pair.

*CCSSM* – Common Core State Standards for Mathematics

*DELT A* – Diagnostic E-learning Trajectories Approach (research group)

*Fair share* – this term is used to refer to the unit ratio of objects per one sharer in an equipartitioning problem. The determination of a fair share is the goal of many types of equipartitioning problems (fair-sharing problems in which equipartitioning is the desired strategy), such as equipartitioning collections, single wholes, or multiple wholes.

*(Fair-sharing) Situation* – a parameterized scenario in which two quantities are implied to be in a ratio with one another.

*Group* – in co-splitting, and equipartitioning multiple wholes, problems, this term refers to the inherent third variable, which is equivalent to the split or co-split. The number of groups used in a solution is not fixed or pre-specified, whereas the number of objects and sharers are, as given in the presentation of the problem. There is often more than one possible number of groups that can yield a successful solution; however, the number of groups is limited to whole-number values, as it corresponds to the split or co-split. In the context of

problems used in the teaching experiment, bowls, tables, pitchers, etc. represent the groups, and each group must contain a number of both objects and sharers.

*IDAS* – Interactive Diagnostic Assessment System

*IGE* – Item Generation Environment

*LPPSync* – Learning Progress Profiles Synchronized for Mobile Devices

*LT* – Learning Trajectory (also stands in for Learning Progression)

*Object* – the contextual quantity, such as pellets, pizzas, cookies, etc., being shared among a number of sharers in an equipartitioning problem. It constitutes one of two quantities given as parameters in the context of equipartitioning problems.

*Quantity* – equipartitioning problems always involve two quantities, typically objects and sharers. The terms quantity and quantities are used to refer to one or both of these in the general sense, where no specific value is necessarily assigned to either.

*Ratio Unit* – any combination of values for both quantities that is used either multiplicatively or additively in order to maintain, or preserve, the invariance of the ratio. Often used in re-unitizing the parameters.

*Sharer* – the contextual quantity, such as fish, people, etc., that is sharing a number of objects in an equipartitioning problem. It constitutes one of the two quantities given as parameters in the context of equipartitioning problems.

*Sub-group* – this term is used to distinguish between the use of groups and the identification of sets of both objects and sharers that comprise such groups, and maintain an equivalence of

fair shares among the objects and sharers within each sub-group. Students often identify sub-groups as a means to justify the equivalence of fair shares when using inverse co-splitting.

*Unit Ratio* – a ratio unit in which the value for one of the quantities is 1.

*Value* –the amount (or count) of either of the quantities involved in an equipartitioning problem. The values for objects and sharers are not limited to whole numbers theoretically; however, in context, it is typically understood that the sharers would take on only whole-number values.

*Whole* –specifically a continuous whole, or object; one that can be conceivably split into an infinite number of equipartitions, based on the Continuity Principle of equipartitioning. (*Multiple wholes*, in an equipartitioning problem, refer to a set of wholes that are all continuous.)