ABSTRACT

FLEISHER, DANIEL NICHOLAS. Wakimoto Modules, FMS Bosonization and Integrable Hierarchies. (Under the direction of Bojko Bakalov.)

Partial differential equations (PDEs) are of central importance to many areas of mathematics, physics and other disciplines. Solving them is an extremely difficult and often wholly intractable task. In the past fifty years, the systematic algebraic study of a certain special class of equations called *integrable PDEs* has assumed a prominent position in mathematics and physics. More recently, the representation theory of infinite dimensional Lie algebras has been discovered to be intimately connected to so called *soliton solutions* of certain hierarchies of integrable PDEs.

In this thesis, we investigate at the algebraic level some of this representation theory in the form of *vertex algebras* and their structure theory. We prove certain uniqueness results and give a unified treatment of a number of previous results. We then use one of these results to produce a new hierarchy of integrable, non-autonomous PDEs. The appearance of non-autonomous PDEs appears to be a novel result. This is due to the fact that we use the *boson-boson correspondence*, where previously the *boson-fermion correspondence* was always employed.

Wakimoto Modules, FMS Bosonization and Integrable Hierarchies

by
Daniel Nicholas Fleisher

A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh, North Carolina

2013

APPROVED BY:

_____   _____
Naihuan Jing   Kailash Misra

_____   _____
Ernest Stitzinger   Bojko Bakalov
                                                                  Chair of Advisory Committee

DEDICATION

For Bubby and for Poppy, may their memories be a blessing.

BIOGRAPHY

Daniel Nicholas Fleisher was born in Cary, North Carolina on August 2nd 1985. He enjoyed math and science as a young child and was very fortunate to have parents who actively facilitated these interests. In the first decade or so of his life, he realized that physical reality could be broken down into progressively more fundamental components: the crust of the earth was made of rocks, rocks were made of chemicals, chemicals were made of matter and matter was made of mathematics. This is how he settled on mathematics as a discipline and it has thusfar been very rewarding.

Towards the end of the second decade of his life, he began to realize that the totality of reality was greater than the sum of its physical components: the universe appeared to be filled with cracks through which wretched, cyclopean horrors leered. His spare time and much of his leftover mental energy was (and is) spent probing these Lovecraftian depths for an understanding of the aspects of reality which are not readily tamed by reason.

ACKNOWLEDGEMENTS

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Chapter 1

Introduction

The algebraic study of integrable, non-linear partial differential equations is now nearly fifty years old. Beginning with the paper of Gardner-Greene-Kruskal-Miura [1], little was known of the systematic study of such PDEs - as Dickey puts it, the Korteweg-de Vries (KdV) equation, now ubiquitous in areas as distant as quantum field theory, "seemed to be merely an unassuming equation of mathematical physics describing waves in shallow water [2]."

Perhaps at this point, it was not the "algebraic" study of anything as we today would understand it, but one year later, the landmark paper of Lax [3] appeared, which gave the subject its first manifestly Lie algebraic appearance in the form of *Lax pairs*. These objects allow us to turn many sometimes dreadful looking PDEs into something as simple as "time-evolution = Lie bracket".

Fast-forwarding ten or twenty years, vertex operators and, later, vertex (operator) algebras gave us a new and powerful language with which to describe certain infinite dimensional Lie algebras, their representations and generalizations [4] [5] [6].

More specifically, consider an integral lattice L and construct the lattice vertex algebra

V_L associated to L. This vertex algebra is a tensor product of the Fock space generated by the negative part of the infinite dimensional Heisienberg algebra arising from L and the twisted group algebra associated to L. In the case when L is one dimensional, one finds (infinite dimensional) representations of the Heisenberg and Virasoro Lie algebras. When L is the root lattice of a simply laced Lie algebra of finite type, the lattice vertex algebra gives a representation of the associated affine Kac-Moody algebra at level one [7].

More generally, we consider a (possibly twisted) representation M of the vertex algebra V_L (possibly V_L itself). Given an operator $\Omega \in \mathrm{End}(V_L \otimes V_L)$ which commutes with the diagonal action of V_L on $M \otimes M$, we consider the equation

$$\Omega(\tau \otimes \tau) = 0, \quad \tau \in M.$$

A natural source of such situations is the representation theory of affine Kac-Moody algebras. Here Ω is the Casimir operator and the highest weight representations are well understood. After a number of rather non-trivial changes of variables, one is able to identify an infinite nested hierarchy of non-linear partial differential equations *in addition to* a number of particularly nice solutions to these equations called *solitons*. This process has been used in a number of cases already to produce for example the Kadomtsev-Petviashvili (KP) and non-linear Schrödinger (NLS) hierarchies. These are instances of Kac-Wakimoto hierarchies using representations of the affine algebras $\widehat{\mathfrak{gl}_\infty}$, and $\widehat{\mathfrak{sl}_2}$ respectively.

Kac-Wakimoto hierarchies associated to highest weight representations of affine algebras have been studied in e.g [8], [9]. What is less understood - or, perhaps, not understood at all - are such hierarchies associated to other, non-highest weight representations. In this thesis, we give a detailed description of certain *Wakimoto representations* (which are not

highest weight) in terms of Friedan-Martinec-Shenker bosonization [10]. This gives us a unified lattice vertex algebraic treatment of a number of previous results ([11], [12], [13]) which is both useful and aesthetically appealing. Furthermore we prove a certain uniqueness of these realizations, something which is not clear from the free field realizations.

We then use this bosonization to construct integrable hierarchies associated to representations of $\widehat{\mathfrak{sl}_2}$ at *arbitrary level* - that is, the level becomes a *parameter* in the equations. This appears to be a new phenomenon in the area, as hitherto all representations used in this process were of a fixed level. Throughout this thesis, we will always work over the field of complex numbers.

Chapter 2

Finite and Affine Lie Algebras

2.1   Review of classical Lie theory

We begin with some background, preliminary notions and definitions. One can easily give the definition of a Lie algebra axiomatically and proceed entirely from there, but we find it instructive to briefly mention their origin.

What are today called *Lie groups* arose in the late nineteenth century through Sophus Lie's study of differential equations with continuous symmetries. In modern terminology, a Lie group is a smooth (or topological) manifold with a compatible group structure - or, alternatively, a group structure in the category of smooth manifolds with smooth maps. Lie groups are manifestly nonlinear objects and their study - in particular, that of their representation theory - highly nontrivial.

It turns out that a great deal of important information about a Lie group is encoded in its "infinitesimal symmetries"; that is, in elements very close to the identity. Elementary differential geometry tells us that these "infinitesimal symmetries" are really the *tangent space* at the identity. There are a number of ways of constructing this rigorously; for

example, as the set of left-invariant vector fields on the Lie group, but as background it suffices to merely say that the tangent space of a Lie group at its identity is the associated *Lie algebra.*

There is a natural product on the Lie algebra, called the *bracket*, which in the concrete case of (left-invariant) vector fields on the Lie group, is just the Lie bracket of vector fields. This product is neither commutative nor associative, and its properties are a direct result of the definition of the Lie bracket. At this point, however, we will move on to a purely axiomatic discussion.

Definition 2.1.1. *A Lie algebra is a vector space $\mathfrak{g}$ over $\mathbb{C}$ with a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denoted $[\cdot, \cdot]$, such that for $\alpha \in \mathbb{C}; x, y, z \in \mathfrak{g}$,*

1. *$[\alpha x + y, z] = \alpha[x, z] + [y, z]$,*

2. *$[x, y] + [y, x] = 0$, and*

3. *$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$.*

The first axiom simply means that the bracket is bilinear and the second, that it is skew-symmetric. The third axiom is called the Jacobi identity and is in some sense the most important.

Like with other algebraic objects, one defines substructures of and structure preserving maps in a natural way.

Definition 2.1.2. *A Lie subalgebra (or just subalgebra) of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ which is closed under addition, scalar multiplication and the bracket. A Lie algebra homomorphism is a linear map between two Lie algebras $\phi : \mathfrak{g} \to \mathfrak{k}$ preserving the bracket.*

Again, as in other areas of algebra, one is often primarily interested in classifying the the "primary" or "irreducible" Lie algebras.

Definition 2.1.3. *An ideal of a Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{i} \subset \mathfrak{g}$ such that $[\mathfrak{i}, \mathfrak{g}] \subset \mathfrak{i}$. A simple Lie algebra is one whose only ideals are itself and the trivial ideal, $\{0\}$. Furthermore, a semisimple Lie algebra is one which can be written as the direct sum of simple subalgebras.*

The classification of simple finite dimensional Lie algebras over $\mathbb{C}$ is one of the great mathematical successes of the past 150 years. There is a plethora of wonderful texts describing the classification (see e.g. [14]), so the discussion here will be quite brief so that it will introduce only what we need to proceed to the next part of the thesis.

Let $\mathfrak{g}$ be a Lie algebra, $x \in \mathfrak{g}$ and ad_x the linear operator on $\mathfrak{g}$ given by $\mathrm{ad}_x(y) = [x, y]$. This is called the adjoint map; one can formulate the Jacobi identity as the requirement that the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ given by $\mathrm{ad}(x) = \mathrm{ad}_x$ be a representation of $\mathfrak{g}$. Recall that a *representation* of a Lie algebra is, like with other structures, a (left) $\mathfrak{g}$-module - so, the adjoint map gives a representation of $\mathfrak{g}$ on $\mathfrak{g}$ itself, called the *adjoint representation*.

Let V be a vector space over $\mathbb{C}$ and x a linear operator on V. Recall that x is called *nilpotent* if $x^N = 0$ for some N; recall also that x is called *semisimple* if its minimal polynomial has distinct roots. Since our base field is algebraically closed, this last condition is equivalent to x being diagonalizable. An element $x \in \mathfrak{g}$ is called *ad-semisimple* if the map ad_x is semisimple (diagonalizable) as a map on $\mathfrak{g}$.

Definition 2.1.4. *For $\mathfrak{g}$ a Lie algebra not consisting entirely of ad-nilpotent elements, a toral subalgebra of $\mathfrak{g}$ is a subalgebra spanned by semisemimple elements.*

It is known (e.g. [H]) that such a subalgebra is abelian. We now fix a *maximal* toral subalgebra $\mathfrak{h}$ - that is, a toral subalgebra not properly contained in any other. It should be noted that since, for the time being, we are only interested in (semi)simple Lie algebras over $\mathbb{C}$, when maximal toral subalgebras and *Cartan subalgebras* coincide, we can use

the terms interchangeably. (A Cartan subalgebra is a nilpotent subalgebra equal to its normalizer.)

Since $\mathfrak{h}$ is abelian, $\mathrm{ad}_\mathfrak{h}$ is a set of commuting, diagonalizable linear operators on $\mathfrak{g}$, meaning they are simultaneously diagonalizable. To wit, we have the direct sum decomposition,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where, for $\alpha \in \mathfrak{h}^* \backslash \{0\}$, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x, \text{for all } h \in \mathfrak{h}\}$ and Φ is the set of all $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq \{0\}$. The elements of the set Φ are called *roots* of $\mathfrak{g}$.

An important tool is the *Killing form*, defined by $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}_x \mathrm{ad}_y)$ for $x, y \in \mathfrak{g}$. It is clear from this definition that the Killing form is bilinear and that the properties of the trace function and the Jacobi identity give that it is symmetric and invariant - that is, $\kappa(x, y) = \kappa(y, x)$ and $\kappa([x, y], z) = \kappa(x, [y, z])$. Furthermore, since $\mathfrak{g}$ is (semi)simple, the Killing form is nondegenerate.

We know (from e.g. [14]) that the restriction of the Killing form to $\mathfrak{h}$ remains nondegenerate. This then lets us construct the canonical isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$: $\phi \in \mathfrak{h}^*$ is mapped to t_ϕ, where $\phi(h) = \kappa(t_\phi, h)$ for all $h \in \mathfrak{h}$. This isomorphism allows us to define the Killing form on $\mathfrak{h}^*$ in the obvious way, $(\alpha, \beta) := \kappa(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$.

Moving to the abstract for the moment, we give a few important definitions. A *Euclidean space* is a vector space V with a positive definite, symmetric bilinear form $(\cdot, \cdot)$. For $\alpha, \beta \in V$, the *reflection* of β about α is given by $\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$.

Definition 2.1.5. *A root system of V is a finite spanning set $\Phi \subset V$ such that:*

1. *If $\alpha \in \Phi$ then the only multiples of α in Φ are $\pm \alpha$, in particular, $0 \notin \Phi$.*

2. *For all $\alpha \in \Phi$, we have $\sigma_\alpha(\Phi) = \Phi$.*

3. *if* $\alpha, \beta \in \Phi$ *then* $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Definition 2.1.6. *A* base *of a root system* Φ *is a subset* $\Delta \subset \Phi$ *such that every* $\alpha \in \Phi$ *can be written as a linear combination of elements of* Δ *where the coefficients are either entirely positive or negative. The elements of a base are called* simple roots. *The elements of* Φ *expressed as a sum of simple roots with positive (resp. negative) coefficients are called the* positive *(resp.* negative*) roots, denoted by* Φ_+ *(resp.* Φ_-*).*

Definition 2.1.7. *The subgroup of* $GL(V)$ *generated by the reflections* σ_α *(*$\alpha \in \Phi$*) is called the* Weyl group *of* Φ.

As it turns out, the classification of simple, finite dimensional Lie algebras (over $\mathbb{C}$) can be reduced to the classification of root systems. We will not cover this in detail, instead giving the necessary data - and only that of the *simply laced* Lie algebras.

For $N = 1, 2, \ldots$, the Lie algebra A_N is most commonly realized as the set of $N+1$ by $N+1$ traceless matrices, $\mathfrak{sl}_{N+1}$. Defining the elements $\epsilon_i \in \mathfrak{h}^*$ by $\varepsilon_i(\text{diag}(a_1, \ldots, a_{N+1})) = a_i$, the roots of A_N are $\alpha_{i,j} = \varepsilon_i - \varepsilon_j$ $(1 \leq i \neq j \leq N+1)$; the simple roots are $\alpha_i = \alpha_{i,i+1}$ $(1 \leq i \leq N+1)$. The Weyl group is then simply S_{N+1}, the symmetric group on acting by permutations of $\{\epsilon_1, \ldots, \epsilon_{N+1}\}$.

For $N = 4, 5, \ldots$, the Lie algebra D_N is commonly realized as the set of $2N$ by $2N$ skew-symmetric matrices, $\mathfrak{so}_{2N}$. The roots of D_N are given by $\pm\alpha_i \pm \alpha_j$ $(1 \leq i < j \leq N)$, where the signs are independent of each other.

Since it is the algebra we will be most concerned with in this thesis, we review explicitly the algebra $A_1 = \mathfrak{sl}_2$. As a vector space, $\mathfrak{sl}_2 = \text{span}_\mathbb{C}\{e, f, h\}$. It can be realized explicitly by 2×2 matrices,

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the bracket is given by the commutator $[x, y] = xy - yx$. One can easily check the commutation relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

The root system of $\mathfrak{g} = \mathfrak{sl}_2$ is rank-1: $\Phi = \{\alpha, -\alpha\}$ and we have the *triangular decomposition*,

$$\mathfrak{g} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{h} \oplus \mathfrak{g}_\alpha.$$

Definition 2.1.8. *Given a Lie algebra $\mathfrak{g}$, the* universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ *is the quotient of the tensor algebra of $\mathfrak{g}$,*

$$T(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \dots,$$

by the two-sided ideal generated by

$$x \otimes y - y \otimes x - [x, y], \quad (x, y \in \mathfrak{g}).$$

It is clear that there is an equivalence between $\mathcal{U}(\mathfrak{g})$-modules and $\mathfrak{g}$-modules: products of elements in $\mathcal{U}(g)$ simply act successively - precisely, the categories of $\mathcal{U}(\mathfrak{g})$-modules and $\mathfrak{g}$-modules are isomorphic.

2.2 Affine Kac-Moody algebras

There are two equivalent ways of defining *(untwisted) affine Kac-Moody algebras*. One can give the notion of a *generalized Cartan matrix* and break them down into (three) cases, of which one is the affine. Alternatively, one can give an affine algebra as the central extension of the loop algebra of a simple Lie algebra of finite type. We will use the latter definition, as it is explicit and more suggestive of later constructions.

Definition 2.2.1. *Let $\mathfrak{g}$ be a Lie algebra. The algebra $\widetilde{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ is called the* loop *algebra. A typical element of $\widetilde{\mathfrak{g}}$ is of the form $a \otimes t^n$ which we write as a_n, and the bracket is given by $[a_n, b_m] = [a, b]_{n+m}$.*

It is well known that the loop algebra $\widetilde{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ has a one-dimensional central extension

$$\bar{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K,$$

and bracket of this new Lie algebra is given by

$$[x_n + aK, y_m + bK] = [x, y]_{m+n} + m\delta_{m,-n}(x|y)K,$$

where $(\cdot|\cdot)$ is a symmetric, invariant bilinear form (e.g. the Killing form). Adjoining the degree operator d which $[d, x_n] = nx_n$, we can now make the final definition.

Definition 2.2.2. *The Lie algebra*

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

is called the affine Kac-Moody algebra *associated to $\mathfrak{g}$ [15].*

Returning to the relevant example of $\mathfrak{sl}_2$, we describe explicitly its associated affine algebra.

Recall the decomposition of the finite algebra $\mathfrak{sl}_2$ above,

$$\mathfrak{g} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{h} \oplus \mathfrak{g}_\alpha = \mathrm{span}\{f\} \oplus \mathrm{span}\{h\} \oplus \mathrm{span}\{e\}.$$

There is an analogous decomposition of the affine algebra $\mathfrak{g}$,

$$\widehat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}_+},$$

where

$$\hat{\mathfrak{n}}_- = t^{-1}\mathbb{C}[t^{-1}] \otimes \mathrm{span}\{e, h\} + \mathbb{C}[t^{-1}] \otimes \mathrm{span}\{f\}$$

$$\hat{\mathfrak{h}} = h + \mathbb{C}K + \mathbb{C}d$$

$$\hat{\mathfrak{n}}_+ = t\mathbb{C}[t] \otimes \mathrm{span}\{f, h\} + \mathbb{C}[t] \otimes \mathrm{span}\{e\}$$

The representation theory of affine algebras is exceedingly rich. One of the most important representations of affine algebras are *Verma modules*.

Definition 2.2.3. *Let $\lambda \in \widehat{\mathfrak{h}}^*$ be a weight (linear functional) and F_λ be the 1-dimensional module where $\hat{\mathfrak{h}}$ acts as multiplication by λ and $\hat{\mathfrak{n}}_+$ acts trivially. We define the* Verma *module M_λ to be the induced module*

$$M_\lambda = \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{\mathfrak{n}}_+ \oplus \hat{\mathfrak{h}})} F_\lambda.$$

By the Poincaré -Birkhoff-Witt theorem (see e.g. [14]), we know then that this module is isomorphic as a vector space to $\mathcal{U}(\hat{\mathfrak{n}}_-)$.

Verma modules are an example of *induced modules*. Essentially, an induced module is one for which there is already an action of some given submodule, and then the rest of the module is made to act by left multiplication (concatenation). There is a more general construction, that of a *generalized Verma module* which are essential for later constructions. We give the definition in general.

Definition 2.2.4. *For an affine Kac-Moody algebra* $\widehat{\mathfrak{g}}$, *the* generalized Verma module $V_k(\widehat{\mathfrak{g}})$ *is the induced module*

$$\mathrm{Ind}_{\mathfrak{g}[t]+\mathbb{C}K}^{\widehat{\mathfrak{g}}} 1,$$

where the action of $\mathfrak{g}[t]$ *is trivial and* K *acts as multiplication by* $k \in \mathbb{C}$ *on the 1-dimensional vector space spanned by* 1. *The rest of* $\widehat{\mathfrak{g}}$ *acts by left multiplication.*

Chapter 3

Vertex algebras

The notion of a *vertex operator* dates back to the dual resonance model of string theory in the late 1960s. They made their appearance on the mathematical stage in a series of papers on the representation theory of infinite dimensional Lie algebras, notably in [5], [4], [7] to name a few. At some point thereafter if became apparent that these constructions were related to the monstrous moonshine conjectures (c.f. [16]). The axiomatic definition of a *vertex algebra* was given by Borcherds in his famous 1986 paper [6], which he then used several years later to prove the monstrous moonshine conjectures.

Since this breakthrough, vertex (operator) algebras have been tied and indispensable to many areas of mathematics and physics. Notably in the latter field, in [17] it became clear that vertex algebras were very closely related to 2-dimensional conformal quantum field theory. The clearest application of vertex algebras is to that area out of which they were born: the representation theory of infinite dimensional Lie algebras, and it is precisely this application which we shall exploit to generate hierarchies of integrable PDEs.

3.1 Introduction and examples

Consider the affine algebra $\widehat{\mathfrak{g}}$. Given $a \in \mathfrak{g}$ we call the formal power series

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

the *current* associated to $a \in \mathfrak{g}$ (z is a formal variable).

Now, given $a, b \in \mathfrak{g}$, we can write *all* of the commutation relations between the a_m and b_n succinctly; it is not difficult to show that

$$[a(z), b(w)] = \delta(z - w)[a, b](w) + \partial_w \delta(z - w)(a|b)K$$

where

$$\delta(z - w) = \sum_{n \in \mathbb{Z}} z^n w^{-n-1}$$

is the *formal delta function*. An important fact about the delta function is that

$$(z - w)^{n+1} \partial_w^n \delta(z - w) = 0.$$

This fact immediately implies that for $a, b \in \mathfrak{g}$,

$$(z - w)^2 [a(z), b(w)] = 0.$$

Whenever two currents satisfy this property for sufficiently large powers of $z - w$, we say that they are *mutually local*. This definition allows us to give the definition of a vertex algebra.

Definition 3.1.1. *A* vertex (super)algebra *consists of the following data: a vector (su-*

per)space V *called the* space of states, *a linear map* $Y(\cdot, z) : V \to End(V)[\![z, z^{-1}]\!]$ *called the* state-field correspondence, *a(n even) vector* $|0\rangle$ *called the* vacuum vector *and an (even) endomorphism* $T : V \to V$ *called the* infinitesimal translation operator. *These data are subject to the following requirements:*

$$Y(a, z)b \in V(\!(z)\!) = V[\![z]\!][z^{-1}],$$

$$[T, Y(a, z)] = \partial_z Y(a, z),$$

$$Y(|0\rangle, z) = Id_V, \quad Y(a, z)|0\rangle|_{z=0} = a,$$

and for $a, b \in V$, $a(z)$ *and* $b(z)$ *are mutually local.*

In general, elements of $U[\![z, z^{-1}]\!]$ are called *formal distributions* with coefficients in U. Also, the requirement that

$$Y(a, z)b \in V(\!(z)\!),$$

means that $a_{(n)}b = 0$ for $n \gg 0$. Such formal distributions are called *fields*.

Before proceeding to some other very important examples, we introduce some (physically suggestive) notation. If we consider z and w as complex numbers, we can express

$$\frac{1}{z - w}$$

as two different geometric series in the domains $|z| > |w|$ and $|w| < |z|$,

$$\frac{1}{z}\left(1 + \frac{w}{z} + \frac{w^2}{z^2} + \dots\right)$$

and

$$-\frac{1}{w}\left(1 + \frac{z}{w} + \frac{z^2}{w^2} + \dots\right)$$

respectively. Observe that the difference of these two expressions is none other than the the formal delta function $\delta(z - w)$, so we can write

$$\delta(z - w) = \iota_{z,w}\left(\frac{1}{z - w}\right) - \iota_{w,z}\left(\frac{1}{z - w}\right),$$

where $\iota_{z,w}f(z, w)$ denotes the power series expansion of f over the domain $|z| > |w|$. It is then also clear that taking divided derivatives, we have the general formula,

$$\partial_w^{(j)}\delta(z - w) = \iota_{z,w}\left(\frac{1}{(z - w)^{j+1}}\right) - \iota_{w,z}\left(\frac{1}{(z - w)^{j+1}}\right),$$

where $\partial_w^{(j)} = \frac{\partial_w^j}{j!}$.

Another important notion is that of *normal ordering*. The idea allows many sums which would otherwise not be well defined to be utilized in a mathematically consistent manner.

Definition 3.1.2. *Given two fields $a(z)$ and $b(z)$, we define their* normally ordered product *to be*

$$:a(z)b(z): := a(z)_+b(z) + b(z)a(z)_-,$$

where

$$a(z)_+ = \sum_{j<0} a_{(j)}z^{-j-1}$$

and

$$a(z)_- = \sum_{j\geq 0} a_{(j)}z^{-j-1}$$

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The vertex algebraic analogue of the bracket in a Lie algebra can again be simply thought of as a "bracket" of fields,

$$[a(z), b(w)] = \sum_{m,n} [a_{(m)}, b_{(n)}] z^{-m-1} w^{-n-1}.$$

This is not the the easiest notation for computation, nor is the one most commonly used by physicists. The more common notation requires a bit of mind-bending and abuse of notation, so for the sake of brevity, we introduce it by fiat.

Definition 3.1.3. *The* operator product expansion *(OPE) of two fields $a(z)$, $b(w)$ is,*

$$a(z)b(w) = \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}} + :a(z)b(w):,$$

where $c^j(w)$ are some other fields. Furthermore, we write

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}},$$

for the singular part of the OPE.

Then, for local fields $a(z), b(z)$, we have ([18]),

$$[a(z), b(w)] = \sum_{j=0}^{N-1} c^j(w) \partial_w^{(j)} \delta(z-w).$$

A vertex algebra even simpler than the affinization of a simple Lie algebra $\mathfrak{g}$ is the affinization of a 1-dimensional (even) vector space, $\mathfrak{s}$, with inner product $(\cdot|\cdot)$, thought of as an abelian Lie algebra. This is called the *oscillator algebra* or *Heisenberg algebra* and is of central importance.

Let α be an element spanning $\mathfrak{s}$ and $\widehat{\mathfrak{s}} = \mathfrak{s}[t, t^{-1}] \oplus \mathbb{C}K$ the affinization of $\mathfrak{s}$. It is clear from above that we have

$$[\alpha(z), \alpha(w)] = (\alpha|\alpha)\partial_w\delta(z - w)K,$$

since the bracket of α with itself is zero. Alternatively, in our new notation, we have

$$\alpha(z)\alpha(w) \sim \frac{(\alpha|\alpha)K}{(z - w)^2}.$$

A field with this OPE is called a *free boson.*

An important vertex algebra which can be built from the Heisenberg algebra is the *Virasoro algebra.* Let $\alpha(z)$ be a Heisenberg field (equivalently, free boson) with $(\alpha|\alpha) = 1$ and define $L(z)$ to be the field $\frac{1}{2} :\alpha(z)^2:$. It is an easy exercise to show that the OPE of $L(z)$ with itself is

$$L(z)L(w) \sim \frac{\partial L(w)}{z - w} + \frac{2L(w)}{(z - w)^2} + \frac{\frac{1}{2}}{(z - w)^4}.$$

It is also fairly clear that this is equivalent to the commutation relations of the Virasoro (Lie) algebra.

The Virasoro algebra is often regarded as the *stress-energy tensor* of a given conformal field theory, and as such, one expects any reasonable theory to contain it. Hence, we have the following definition.

Definition 3.1.4. *We call a vector ν in a vertex algebra V a* conformal vector *if its corresponding field $Y(\nu, z)$ is a Virasoro field. Such a vertex algebra V with a conformal vector is called a* conformal vertex algebra.

3.2 Lattice vertex algebras

One of the most important examples of vertex algebras is the *lattice vertex algebra*, which dates back to Borcherds's original paper [6].

Recall that a *lattice* is a free abelian group L with inner product $(\cdot|\cdot)$. We say L is *integral* (resp. *even*) if for all $x, y \in L$ we have $(x|y) \in \mathbb{Z}$ (resp. $(x|x) \in 2\mathbb{Z}$).

Let $L, (\cdot|\cdot)$ be an even integral lattice. Let $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ be its complexification and $\widehat{\mathfrak{h}}$ the affinization of $\mathfrak{h}$ as an abelian Lie algebra. We call $S(\mathfrak{h}[t^{-1}]t^{-1})$ the *bosonic Fock space* associated to L. Next we define $\mathbb{C}_\varepsilon[Q] = \text{span}_{\mathbb{C}}\{e^\alpha | \alpha \in Q\}$ - the ε-twisted group algebra, with $e^\alpha e^\beta = \varepsilon(\alpha, \beta)e^{\alpha+\beta}$, where $\varepsilon \colon Q \times Q \to \{\pm 1\}$ is a bimultiplicative map with $\varepsilon(\alpha, \alpha) = (-1)^{(\alpha|\alpha)/2}$. Now we can give the formal definition.

Definition 3.2.1. *Given an integral lattice L, the* lattice vertex algebra, V_L, *associated to L is given as follows. The space of states is $V_L = S(\mathfrak{h}[t^{-1}]t^{-1}) \otimes \mathbb{C}_\varepsilon[Q]$ and the state-field correspondence is given by*

$$Y((h_1 t^{-n_1-1})(h_2 t^{-n_2-1})\ldots \otimes e^\alpha, z) =: \partial^{(n_1)} h_1(z) \partial^{(n_2)} h_2(z) \cdots e^\alpha(z) :,$$

for $h_1, h_2, \ldots \in \mathfrak{h}$, $n_1, n_2, \ldots \geq 0$, $\alpha \in L$ and where $e^\alpha(z)$ is the vertex operator *given by*

$$e^\alpha(z) = e^\alpha z^{\alpha_0} \exp\left(\sum_{j<0} \frac{z^{-j}}{-j}\alpha_j\right) \exp\left(\sum_{j>0} \frac{z^{-j}}{-j}\alpha_j\right)$$

The OPEs in a lattice vertex algebra are given by ([18])

$$h(z)h'(w) \sim \frac{(h|h')}{(z-w)^2},$$

$$h(z)e^\alpha(w) \sim \frac{(h|\alpha)e^\alpha(w)}{z-w},$$

$$e^\beta(z)h(w) \sim \frac{-(h|\beta)e^\alpha(w)}{z-w},$$

and

$$e^\alpha(z)e^\beta(w) \sim \sum_{n\in\mathbb{Z}_+} \sum_{\substack{k_1+2k_2+\dots=n \\ k_i,n\in\mathbb{Z}_+}}$$

$$\varepsilon(\alpha,\beta)\frac{(z-w)^{n+(\alpha|\beta)}}{(1!)^{k_1}k_1!(2!)^{k_2}k_2!\dots} :\alpha(w)^{k_1}(\partial\alpha(w))^{k_2}\dots e^{\alpha+\beta}(w):$$

More generally, for a (possibly non-even) integral lattice L, one can define a vertex superalgebra V_L with parities

$$p(e^\alpha) = |\alpha|^2 \mod 2\mathbb{Z}.$$

3.3   Boson-fermion and boson-boson correspondences

One of the most elegant and powerful results in the representation theory of infinite dimensional Lie algebras is the *boson-fermion correspondence*. Within the language of Lie algebras, the result somewhat difficult to understand; when we switch to the language of vertex algebras, however, the result is rather beautiful: it is simply the isomorphism between two vertex algebras.

Let V_L be the lattice vertex algebra associated to the simplest possible lattice $L = \mathbb{Z}\alpha$ with $(\alpha|\alpha) = 1$. One easily checks the following OPEs,

$$e^{\pm\alpha}(z)e^{\pm\alpha}(z) \sim 0, \quad e^{\pm\alpha}(z)e^{\mp\alpha}(z) \sim \frac{1}{z-w},$$

which means that these are charged free fermions. It is also straightforward to check that

$$:e^{\alpha}(z)e^{-\alpha}(z):\,= \alpha(z).$$

What these two statements together provide is an expression of the charged free fermions in terms of free bosons and vice-versa: this is the essence of the boson-fermion correspondence [18].

The boson-boson correspondence originates from an important paper of Friedan-Martinec-Shenker [10], and we will often refer to it simply as *FMS-bosonization*. Unlike the boson-fermion correspondence, it is not an isomorphism of vertex algebras. Instead, it is a homomorphism from the vertex algebra generated by a pair of charged free bosons into a lattice vertex algebra: it is a *bosonization* (embedding into a lattice vertex algebra) of the charged free bosons.

The charged free bosons are a pair of even fields $a^{+}(z), a^{-}(z)$ with OPEs

$$a^{+}(z)a^{-}(w) \sim \frac{1}{z-w}, \quad a^{\pm}(z)a^{\pm}(w) \sim 0.$$

Consider the lattice $L = \mathbb{Z}\alpha + \mathbb{Z}\beta$ with $(\alpha|\beta) = 0$ and $(\alpha|\alpha) = -(\beta|\beta) = 1$ [19] [20]. The following lemma is clear from direct computation.

Lemma 3.3.1. *The map given by*

$$a^{+}(z) \mapsto e^{\alpha+\beta}(z), \quad a^{-}(z) \mapsto - :\alpha(z)e^{-\alpha-\beta}(z):$$

gives a realization of a pair of charged free bosons in the lattice vertex algebra V_L

Proof. Note that,

$$a^+(z)a^+(w) \sim 0,$$

is trivial, since $\alpha + \beta$ is isotropic, so we only need to verify the non-trivial OPEs,

$$a^+(z)a^-(w) \sim \frac{1}{z-w}, \quad a^-(z)a^-(w) \sim 0.$$

We have

$$a^+(z)a^-(w) = e^{\alpha+\beta}(z)(-:\alpha(w)e^{-\alpha-\beta}(w):)$$
$$\sim \frac{(\alpha+\beta|\alpha)}{z-w} + \text{h.o.t.} = \frac{1}{z-w},$$

where "h.o.t." stands for "higher order terms in $z-w$," and

$$a^-(z)a^-(w) = (:\alpha(z)e^{-\alpha-\beta}(z):)(:\alpha(w)e^{-\alpha-\beta}(w):)$$
$$\sim \left(\frac{(\alpha| -\alpha-\beta)\alpha(w) - (-\alpha-\beta|\alpha)\alpha(z)}{z-w} \right.$$
$$\left. + \frac{(\alpha|\alpha) - (\alpha|-\alpha-\beta)(-\alpha-\beta|\alpha)}{(z-w)^2} \right) e^{-2(\alpha+\beta)} + \text{h.o.t.}$$
$$= 0$$

as desired. □

Chapter 4

Bosonic lattice realizations of $\widehat{\mathfrak{sl}}_N$ and $\widehat{\mathfrak{so}}_{2N}$

4.1 Motivation

We review the Wakimoto realization of $\widehat{\mathfrak{sl}}_2$ at level k given in [11] in the notation of [21].
Assume that we have a pair of charged free bosons a^+, a^- and a free boson b with non-zero
OPEs given by

$$a^+(z)a^-(w) \sim \frac{1}{z-w}, \quad b(z)b(w) \sim \frac{2k}{(z-w)^2}.$$

Then we have a representation of $\widehat{\mathfrak{sl}}_2$ at level k given by

$$e(z) = a^-(z),$$

$$h(z) = -2 : a^+(z)a^-(z) : + b(z),$$

$$f(z) = -2 : a^+(z)^2 a^-(z) : + k\partial_z a^+(z) + : a^+(z)b(z) :,$$

where the normal ordering of several terms is from the right. To construct (interesting) integrable PDEs from these representations, one can apply FMS-bosonization to give these fields a representation in terms of vertex operators. Doing so produces formulas of the general form

$$e(z) =: A(z)e^{\delta}(z):,$$

$$h(z) = B(z),$$

$$f(z) =: C(z)e^{-\delta}:,$$

where $|\delta|^2 = 0$ and $A(z), B(z), C(z)$ are Heisenberg fields. This is interesting when we consider the restrictions the notion of *conformal weight* has on the possible forms of the fields $e(z)$ and $f(z)$.

Definition 4.1.1. *Let H be a diagonalizable derivation of a vertex algebra V called a Hamiltonian. We say a field (or formal distribution) $a(z)$ has* conformal weight Δ *if*

$$(\mathrm{ad}_H - \Delta - z\partial_z)a(z) = 0.$$

We know from Proposition 5.5 in [18] that in a lattice vertex algebra, a Heisenberg field $h(z)$ have conformal weight 1, and a vertex operator $e^{\alpha}(z)$ has conformal weight $\frac{1}{2}(\alpha|\alpha)$. Recall also Proposition 2.6, which tells us that given two fields $a(z), b(z)$ of conformal weights Δ, Δ' respectively, that $: a(z)b(w) :$ has conformal weight $\Delta + \Delta'$. Furthermore, in the OPE of $a(z)b(w)$, the coefficient of the $\frac{1}{(z-w)^{j+1}}$ term has conformal weight $\Delta + \Delta' - j - 1$.

Now, a vertex algebra giving a representation of $\widehat{\mathfrak{sl}_2}$ at level k has as one of its defining

OPEs,

$$e(z)f(w) \sim \frac{h(w)}{z-w} + \frac{k}{(z-w)^2}.$$

Looking at the conformal weights of each side of this, we deduce,

$$\Delta(e) + \Delta(f) = 2.$$

Let Q be a root system of a simply laced (type ADE) finite dimensional Lie algebra. The Frenkel-Kac construction [7] tells us that the lattice vertex algebra V_Q gives a representation of the associated affine algebra at level $k = 1$. The Chevalley generators can be written easily as

$$e_i(z) = e^{\alpha_i}(z), \quad f_i(z) = e^{-\alpha_i}(z), \quad h_i(z) = \alpha_i(z),$$

where i ranges over the rank (number of simple roots) of the algebra.

For the relevant case of $\widehat{\mathfrak{sl}_2}$ this has the form

$$e(z) = e^{\alpha}(z), \quad h(z) = \alpha(z), \quad f(z) = e^{-\alpha}(z).$$

We see that all of the Chevalley generators of $\widehat{\mathfrak{sl}_2}$ in both the Wakimoto and Frenkel-Kac realizations have conformal weight 1, but what is different about them is, in the fields $e(z)$ and $f(z)$ how this comes about. In the Frenkel-Kac construction,

$$\Delta(e(z)) = \Delta(e^{\alpha}(z)) = \frac{1}{2}(\alpha|\alpha) = 1,$$

and in the Wakimoto realization,

$$\Delta(e(z)) = \Delta(:A(z)e^\delta(z):) = \Delta(:A(z)) + \Delta(e^\delta(z):) = 1 + 0.$$

So we see that in this sense, the Wakimoto, "square-length zero" case is a natural ana-logue of the Frenkel-Kac, "square-length one" construction. This leads to a very natural question: what other "square-length zero" constructions are there and can we give some sort of classification?

4.2   Bosonic $\widehat{\mathfrak{sl}_2}$

We now consider the "square-length 0" case for $\widehat{\mathfrak{sl}_2}$. We consider a lattice L of rank 3 containing an element δ with $(\delta|\delta) = 0$ such that

$$\mathbb{C} \otimes_{\mathbb{Z}} L = \mathfrak{h} = \operatorname{span}\{\delta, \chi, \gamma\}.$$

We wish to classify embeddings of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}_2}$ into the lattice vertex algebra V_L of the form

$$e(z) \mapsto :\chi(z)e^\delta(z):,$$

$$f(z) \mapsto :\gamma(z)e^{-\delta}(z):,$$

and $h \in \mathfrak{h}$.

Theorem 4.2.1. *The above embedding fixes the lattice L, whose inner products are given by,*

	δ	χ	γ
δ	0	1	-1
χ	1	1	k+1
γ	-1	k+1	1

That is, there is a unique lattice L with inner products given above for every choice of $k \in \mathbb{C}$, whose associated lattice vertex algebra gives a representation of $\widehat{sl_2}$ at level k. Furthermore, h is also determined by the OPEs and we have

$$h = k\delta + \chi - \delta.$$

Proof. We prove the theorem by direct computation. We compute the OPEs

$$e(z)f(w) \sim \frac{h(w)}{z-w} + \frac{k}{(z-w)^2}, \qquad h(z)h(w) \sim \frac{2k}{(z-w)^2},$$

$$h(z)e(w) \sim \frac{2e(w)}{z-w}, \qquad h(z)f(w) \sim \frac{-2f(w)}{z-w},$$

$$e(z)e(w) \sim 0, \qquad f(z)f(w) \sim 0.$$

Then,

$$e(z)f(w) =: \chi(z)e^\delta(z) :: \gamma(w)e^{-\delta}(w):$$

$$\sim \left(\frac{-(\chi|\delta)\gamma(w) - (\gamma|\delta)\chi(z)}{z-w} + \frac{(\chi|\gamma) + (\delta|\chi)(\delta|\gamma)}{(z-w)^2}\right)$$

$$\cdot (1 + (z-w)\delta(w) + \text{h.o.t.})$$

$$\sim \frac{((\chi|\gamma) + (\delta|\chi)(\delta|\gamma))\delta(w) - (\chi|\delta)\gamma(w) - (\gamma|\delta)\chi(w)}{z-w}$$

$$+ \frac{(\chi|\gamma) + (\delta|\chi)(\delta|\gamma)}{(z-w)^2}$$

implies that

$$k = (\chi|\gamma) + (\delta|\chi)(\delta|\gamma)$$

and

$$h = k\delta(z) - (\chi|\delta)\gamma - (\gamma|\delta)\chi.$$

Similar computations for $e(z)e(w)$ and $f(z)f(w)$ give

$$(\chi|\chi) = (\delta|\chi)^2, \quad (\gamma|\gamma) = (\delta|\gamma)^2.$$

Now,

$$h(z)e(w) = h(z) :\chi(w)e^\delta(w):$$
$$\sim \frac{(h|\delta) :\chi(w)e^\delta(w):}{z - w} + \frac{(h|\chi)}{(z - w)^2},$$

which tells us that

$$(h|\delta) = 2, \quad (h|\chi) = 0.$$

A nearly identical computation for $h(z)f(w)$ gives

$$(h|\gamma) = 0.$$

Then,

$$(h|h) = (h|k\delta - (\chi|\delta)\gamma - (\gamma|\delta)\chi) = k(h|\delta) = 2k,$$

as required.

Expanding $(h|\delta) = 2$ gives

$$2 = (h|\delta)$$

$$= (k\delta - (\delta|\chi)\gamma - (\delta|\gamma)\chi|\delta)$$

$$= -2(\delta|\chi)(\delta|\gamma),$$

which implies

$$(\delta|\chi)(\delta|\gamma) = -1.$$

Similarly, from $(h|\chi) = 0$, we obtain

$$0 = (h|\chi)$$

$$= (k\delta - (\delta|\chi)\gamma - (\delta|\gamma)\chi|\chi)$$

$$= k(\delta|\chi) - (\delta|\chi)(\gamma|\chi) - (\delta|\gamma)(\chi|\chi)$$

$$= (\delta|\chi)(k + 1 - (\chi|\gamma)),$$

which gives

$$(\chi|\gamma) = k + 1.$$

Gathering all the lattice equations so far obtained,

$$(\delta|\chi)(\delta|\gamma) = -1, \quad (\chi|\gamma) = k + 1,$$

$$(\chi|\chi) = (\delta|\chi)^2, \quad (\gamma|\gamma) = (\delta|\gamma)^2,$$

we notice that we are free to rescale $\chi \mapsto c\chi$ and $\gamma \mapsto \frac{1}{c}\gamma$, which allows us to fix $(\delta|\chi) = 1$. This then immediately fixes all the other inner products and we obtain the

desired result. $\qquad\square$

Note that the Gramm matrix associated to this lattice has determinant $-2k - 4$, so that when the level is critical (i.e. $k = -2$) the Gramm matrix is singular and the rank of the lattice is then two. In this case, we can set $\gamma = -\chi$.

Furthermore, one should note that since this embedding into a lattice of rank 3 is unique and so it is equivalent to FMS-bosonization of the modules described in [11], though the form we give is somewhat more aesthetically appealing and easier to compute with.

4.3 Bosonic $\widehat{\mathfrak{sl}}_N$

This square-length zero ansatz extends to affine vertex algebras associated to some higher rank affine algebras, though as we will see the level is no longer free. Let L now be the lattice

$$L = \bigoplus_{i=1}^{N-1} \mathbb{Z}\delta_i \oplus \bigoplus_{i=1}^{N-1} \mathbb{Z}\chi_i \oplus \bigoplus_{i=1}^{N-1} \mathbb{Z}\gamma_i,$$

with $(\delta_i | \delta_j) = 0$ for all i, j, and consider the embedding of the affine Kac-Moody algebra $\widehat{sl}_N$ $(N \geq 3)$ into the lattice vertex algebra V_L of the form

$$e_i(z) \mapsto \, :\chi_i(z)e^{\delta_i}(z):,$$

$$f_i(z) \mapsto \, :\gamma_i(z)e^{-\delta_i}(z):,$$

and $h_i \in \mathfrak{h}$, where $\{e_i, f_i, h_i\}$ are the Chevalley generators of $\mathfrak{sl}_N$ We begin with the next simplest case after $\widehat{\mathfrak{sl}}_2$.

Theorem 4.3.1. *The above lattice in the case of $N = 3$ gives a representation of the*

algebra $\widehat{sl_3}$ at level $k = -1$. In particular, this embedding is unique, the lattice is of rank 5 and has inner products given by

	δ_1	χ_1	γ_1	δ_2	γ_2
δ_1	0	1	-1	0	0
χ_1	1	1	0	0	0
γ_1	-1	0	1	1	0
δ_2	0	0	1	0	-1
γ_2	0	0	0	-1	1

and $\chi_2 = \gamma_1$.

Proof. First we should note the rather conspicuous fact that the ansatz calls for a lattice of rank 6 and the theorem states that the lattice is rank 5. As we shall see shortly, one of the lattice elements in the ansatz becomes superfluous. We know from the previous theorem that each triple $\{e_i(z), h_i(z), f_i(z)\}$ gives a representation of $\widehat{sl_2}$ at level k. This takes care of most of the values of the inner product - we need only determine the cross-terms, viz. $(\delta_i | \chi_j), (\delta_i | \gamma_j), (\chi_i | \gamma_j), (\chi_i | \chi_j), (\gamma_i | \gamma_j)$ with $i \neq j$. We consider first the two OPEs

$$e_1(z) f_2(w) \sim 0,$$

$$e_2(z) f_1(w) \sim 0.$$

These computations are identical to those in the previous theorem and from them we obtain four equations,

$$(\delta_1 | \gamma_2) \chi_1 + (\delta_2 | \chi_1) \gamma_2 = 0,$$

$$(\delta_2 | \chi_1)(\delta_1 | \gamma_2) + (\chi_1 | \gamma_2) = 0,$$

$$(\delta_2 | \gamma_1) \chi_2 + (\delta_1 | \chi_2) \gamma_1 = 0,$$

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$$(\delta_1|\chi_2)(\delta_2|\gamma_1) + (\chi_2|\gamma_1) = 0.$$

The first and third equations taken together allow us to consider a number of possible cases. Let us consider first the case where all the elements χ_i, γ_j are linearly independent, so that $(\delta_i|\gamma_j) = (\delta_i|\chi_j) = 0$ for all $i \neq j$. Consider now the OPEs,

$$h_1(z)e_2(w) \sim \frac{-e_2(w)}{z - w}, \quad h_2(z)e_1(w) \sim \frac{-e_1(w)}{z - w},$$

whence we get the lattice equations,

$$(h_1|\delta_2) = -1 = (h_2|\delta_1).$$

Substituting in the forms of h_i we know from the case of $\widehat{\mathfrak{sl}_2}$, namely

$$h_i = k\delta_i + \chi_i - \gamma_i$$

we find

$$(\delta_2|\gamma_1) - (\delta_2|\chi_1) = 1,$$

$$(\delta_1|\gamma_2) - (\delta_1|\chi_2) = 1,$$

which clearly cannot hold if all the terms are zero. Hence we know that there is at least one dependency in the previous equations. Assume then that there are two dependencies, that is,

$$\chi_1 = c_1\gamma_2, \quad \chi_2 = c_2\gamma_1.$$

Substituting these into the previous equations we obtain

$$c_1 + \frac{1}{c_2} = c_2 + \frac{1}{c_1} = 1,$$

which imply that at least one c_i is zero. We have shown that both of them cannot be, so we deduce that precisely one is. By symmetry we are allowed to choose $c_2 = 0$, so $\chi_2 = \gamma_1$ - hence the reduction in the rank of the lattice - which then allows us to determine all of the other inner products from the equations above. In particular, note that the OPE,

$$e_1(z)e_2(w) \sim \frac{e_{\alpha_1+\alpha_2}(w)}{z-w},$$

not only gives us the precise form of $e_{\alpha_1+\alpha_2}(w)$, but also,

$$(\chi_1|\chi_2) + (\delta_1|\chi_2)(\delta_2|\chi_1) = 0,$$

which gives,

$$(\chi_1|\gamma_1) = 0,$$

and so $k = -1$ as was stated. $\qquad\square$

This result generalizes to the case of $\widehat{\mathfrak{sl}}_N$ in a rather straightforward manner.

Theorem 4.3.2. *There is a unique representation of $\widehat{\mathfrak{sl}}_N$ at level $k = -1$ in the lattice*

vertex algebra associated to the rank-$(2N - 1)$ lattice given by

$$(\delta_i | \delta_j) = 0, \quad (\chi_i | \chi_j) = \delta_{ij}, \quad (\delta_i | \chi_j) = \begin{cases} 1, & j = i \\ -1, & j = i + 1 \\ 0, & otherwise. \end{cases}$$

Proof. We first consider the case of $\widehat{\mathfrak{sl}_4}$. The unreduced lattice from the original ansatz is spanned by $\{\delta_i, \chi_i, \gamma_i\}_{i=1,2,3}$. It can be thought of as a copy of $\widehat{\mathfrak{sl}_3}$ attached to a copy of $\widehat{\mathfrak{sl}_2}$, whose underlying lattices are whose lattice is $\{\delta_i, \chi_i, \gamma_i\}_{i=1,2}$ and whose lattice is $\{\delta_i, \chi_i, \gamma_i\}_{i=3}$, respectively.

We know from the previous theorem that since there is an $\widehat{\mathfrak{sl}_3}$ subalgebra, the level must again be -1 and that $\chi_2 = \gamma_1$. The question now is, if we divide $\widehat{\mathfrak{sl}_4}$ up differently, and consider the copy of $\widehat{\mathfrak{sl}_3}$ whose underlying lattice is $\{\delta_i, \chi_i, \gamma_i\}_{i=2,3}$, are we free to choose either $\chi_3 = \gamma_2$ or $\chi_2 = \gamma_3$?

Assume then that $\chi_2 = \gamma_3$. Combining this with $\chi_2 = \gamma_1$, we have $\gamma_1 = \gamma_3$. This poses a problem, as we see

$$e_1(z) f_3(w) \sim 0$$

implies

$$(\delta_1 | \gamma_3) \chi_1 + (\delta_2 | \chi_1) \gamma_3 = 0,$$

$$(\delta_2 | \chi_1)(\delta_1 | \gamma_3) + (\chi_1 | \gamma_3) = 0.$$

Substituting γ_1 in for γ_3 in the first equation, we have

$$(\delta_1 | \gamma_1) \chi_1 + (\delta_2 | \chi_1) \gamma_1 = 0,$$

which implies $\chi_1 = 0$, which is impossible. We then understand that once we choose $\chi_2 = \gamma_1$, this fixes all of the later equalities between the χ and γ. The general case of $\widehat{\mathfrak{sl}_N}$ follows immediately from this algorithm, where nodes are added one at a time in the above process.

$\square$

4.4   Bosonic $\widehat{\mathfrak{so}_{2N}}$

Now we turn our attention to the case of $\widehat{\mathfrak{so}_{2N}}$. In this case, the original ansatz is insufficient and we must allow for the possibility that the fields $e_i(z), f_i(z)$ be sums of the form

$$e_j(z) = \sum_{i=1}^{N} :\chi_{j,i}(z) e^{\delta_{j,i}}(z):, \quad f_j(z) = \sum_{i=1}^{N} :\gamma_{j,i}(z) e^{-\delta_{j,i}}(z): .$$

The reason for this is that unlike in the case of $\widehat{\mathfrak{sl}_N}$ where there are in a sense two "interacting" copies of $\widehat{\mathfrak{sl}_3}$ interacting at a time, here we have three copies:

We do not prove that this is the only such realization of $\widehat{\mathfrak{so}_{2N}}$, nor do we proceed directly from a more general ansatz. Instead, we simply recall the result found in [12], [13] and perform FMS bosonization to see that the result is of the form above.

First, we modify the notation of FMS bosonization slightly. Let $a^+(z), a^-(z)$ be a pair of charged free bosons. We replace $\alpha + \beta$ with δ and α with x, so that the bosonization is

$$a^+(z) \mapsto e^{\delta}(z),$$

$$a^-(z) \mapsto - :x(z) e^{\delta}(z): .$$

Now we recall the relevant result of [13], considering the representation the 1-toroidal

(=affine) algebra $\widehat{\mathfrak{so}}_{2N}$. We translate their results into the above language. Let $a_{i,j}^{+}, a_{k,l}^{-}$ be pairs of charged free bosons, that is:

$$a_{i,j}^{+}(z)a_{k,l}^{-}(w) \sim \frac{\delta_{i,k}\delta_{j,l}}{z-w},$$

and the other OPEs are zero.

The Chevalley generators are given by

$$e_i(z) =\, :a_{i,1}^{+}(z)a_{i+1,1}^{-}(z): \, - \, :a_{i+1,2}^{+}(z)a_{i,2}^{-}(z):,$$

$$f_i(z) =\, :a_{i+1,1}^{+}(z)a_{i,1}^{-}(z): \, - \, :a_{i,2}^{+}(z)a_{i+1,2}^{-}(z):,$$

$$h_i(z) =\, :a_{i,1}^{+}(z)a_{i,1}^{-}(z): \, - \, :a_{i+1,1}^{+}(z)a_{i+1,1}^{-}(z): \, - \, :a_{i,2}^{+}(z)a_{i,2}^{-}(z): \, + \, :a_{i+1,2}^{+}(z)a_{i+1,2}^{-}(z):,$$

for $1 \leq i \leq N-1$. The remaining generators are given by,

$$e_N(z) =\, :a_{N-1,1}^{+}(z)a_{N,2}^{-}(z): \, - \, :a_{N,1}^{+}(z)a_{N-1,2}^{-}(z):,$$

$$f_N(z) =\, :a_{N,2}^{+}(z)a_{N-1,1}^{-}(z): \, - \, :a_{N-1,2}^{+}(z)a_{N,1}^{-}(z):,$$

$$h_N(z) =\, :a_{N-1,1}^{+}(z)a_{N-1,1}^{-}(z): \, + \, :a_{N,1}^{+}(z)a_{N,1}^{-}(z): \, - \, :a_{N-1,2}^{+}(z)a_{N-1,2}^{-}(z): \, - \, :a_{N,2}^{+}(z)a_{N,2}^{-}(z):,$$

Applying FMS bosonization to this we obtain

$$e_i(z) =\, :x_{i,2}e^{\delta_{i+1,2}-\delta_{i,2}}: \, - \, :x_{i+1,1}e^{\delta_{i,1}-\delta_{i+1,1}}:,$$

$$f_i(z) =\, :x_{i+1,2}e^{\delta_{i,2}-\delta_{i+1,2}}: \, - \, :x_{i,1}e^{\delta_{i+1,1}-\delta_{i,1}}:,$$

$$h_i(z) = x_{1,1} - \delta_{1,1} - (x_{2,1}-\delta_{2,1}) - (x_{1,2}-\delta_{1,2}) + (x_{2,2}-\delta_{2,2})$$

for $1 \leq i \leq N - 1$ and,

$$e_N(z) =: x_{N-1,2} e^{\delta_{N,1} - \delta_{N-1,2}} : - : x_{N,2} e^{\delta_{N-1,1} - \delta_{N,2}} :,$$

$$f_N(z) =: x_{N,1} e^{\delta_{N-1,2} - \delta_{N,1}} : - : x_{N-1,1} e^{\delta_{N,2} - \delta_{N-1,1}} :,$$

$$h_N(z) = x_{1,1} - \delta_{1,1} + (x_{2,1} - \delta_{2,1}) - (x_{1,2} - \delta_{1,2}) - (x_{2,2} - \delta_{2,2})$$

We can see from these formulas that the level is $k = -2$ and the refined ansatz, allowing

for sums, includes this result.

Chapter 5

Integrable Hierarchies

5.1 Introduction

Our object of study for the remainder of the thesis is the lattice vertex algebra given previously which gives a representation of $\widehat{\mathfrak{sl}}_2$ at arbitrary level $k \in \mathbb{C}$. Recall that this is the lattice vertex algebra associated to the rank-3 lattice L spanned by an isotropic element $\delta \in Ł$ and two others, χ, γ, such that

$$L \otimes_{\mathbb{Z}} \mathbb{C} = \operatorname{span}\{\delta, \chi, \gamma\}$$

with

$$(\chi|\chi) = (\gamma|\gamma) = (\delta|\chi) = -(\delta|\gamma) = 1,$$

$$(\chi|\gamma) = c = k + 1, \quad (\delta|\delta) = 0,$$

where $c \in \mathbb{C}$. Note that at the critical level ($k = -2$), the rank of the lattice reduces and we can set $\gamma = -\chi$.

5.2 Computing the hierarchy

We know that the modes of

$$e(z) =: \chi(z)e^{\delta}(z) :,$$

$$f(z) =: \gamma(z)e^{-\delta}(z) :,$$

$$h(z) = k\delta(z) + \chi(z) - \gamma(z),$$

give a representation of the affine algebra $\widehat{\mathfrak{sl}}_2$ at level k.

As has already been mentioned, it is central - no pun intended - that the operator

$$\Omega = \sum_{n \in \mathbb{Z}} (e_n \otimes f_{-n} + f_n \otimes e_{-n} + \frac{1}{2}h_n \otimes h_{-n}) + K \otimes d + d \otimes K$$

commute with the diagonal action of $\widehat{\mathfrak{sl}}_2$. That is, we need the following lemma.

Lemma 5.2.1. *We have*

$$[\Delta(x_m), \Omega] = 0,$$

for all $x \in \widehat{\mathfrak{sl}}_2, m \in \mathbb{Z}$, where

$$\Delta(x_m) = x_m \otimes 1 + 1 \otimes x_m$$

Proof. It is clear that $\{e_n, f_n, h_n, K, d\}$ and $\{f_{-n}, e_{-n}, \frac{1}{2}h_{-n}, d, K\}$ are dual bases of $\widehat{\mathfrak{sl}}_2$.

So we can write

$$\Omega = \sum_{n \in \mathbb{Z}} a_n^i \otimes b_{-n}^i + K \otimes d + d \otimes K,$$

where $\{a^i\}, \{b^i\}$ are dual bases of $\mathfrak{sl}_2$. We compute directly.

$$[x_m \otimes 1 + 1 \otimes x_m, \Omega] = \sum_{\substack{n \in \mathbb{Z} \\ i \in I}} [x, a^i]_{m+n} \otimes b^i_{-n} + a^i_n \otimes [x, b^i]_{m-n}$$

$$+ \sum_{\substack{n \in \mathbb{Z} \\ i \in I}} \delta_{m,-n} mk(x|a^i) \otimes b^i_{-n} + \delta_{m,n} mk(x|b^i) a^i_n \otimes 1$$

$$- mx_m \otimes k - mk \otimes x_m.$$

The calculation proceeds in two parts, one for the first line and another for the second and third. Making the substitution $n = m + n$ in the second summand in the first line, we have

$$\sum_{\substack{n \in \mathbb{Z} \\ i \in I}} [x, a^i]_{m+n} \otimes b^i_{-n} + a^i_{m+n} \otimes [x, b^i]_{-n}.$$

Now, we fix n. We will prove that

$$\sum_{i \in I} [x, a^i] \otimes b^i + a^i \otimes [x, b^i] = 0.$$

Making use of the isomorphism $\mathfrak{g} \otimes \mathfrak{g} \to \text{End}(\mathfrak{g})$ given by $x \otimes y \mapsto (x|\cdot)y$ we write this as

$$\sum_{i \in I} ([x, a^i]|y) b^i + (a^i|y)[x, b^i],$$

where $y \in \mathfrak{g}$ is arbitrary. Now we use the invariance of the bilinear form on first term and take the constants $(a^i|y)$ into the bracket in the second term,

$$\sum_{i \in I} -(a^i|[x, y]) b^i + [x, (a^i|y) b^i].$$

Finally, we use a basic fact from linear algebra,

$$\sum (a^i|y)b^i = y,$$

and we have the whole sum reducing to

$$-[x,y] + [x,y] = 0,$$

as desired. Now we turn our attention to the other part of the original sum. This is quite evidently zero, however, using the aforementioned fact from linear algebra.

$\square$

Let $\tau \in V_L$ be an arbitrary element; that is,

$$\tau = \sum_{m \in \mathbb{Z}} \tau_m q^m,$$

where $\tau_m \in \mathbb{C}[\![x_i, y_i, t_i]\!]$ and $q = e^\delta$.

The Fock space here is given by

$$B = \mathbb{C}[\![x_i, y_i, t_i]\!][q, q^{-1}]$$

where the fields $\chi(z), \gamma(z)$ and $\delta(z)$ act as, for $n > 0$

$$\chi_{(n)} = \frac{\partial}{\partial x_n} + c\frac{\partial}{\partial y_n} + \frac{\partial}{\partial t_n}, \quad \chi_{(-n)} = nx_n, \quad \chi_{(0)} = q\frac{\partial}{\partial q},$$

$$\gamma_{(n)} = c\frac{\partial}{\partial x_n} + \frac{\partial}{\partial y_n} - \frac{\partial}{\partial t_n}, \quad \gamma_{(-n)} = ny_n, \quad \gamma_{(0)} = -q\frac{\partial}{\partial q},$$

$$\delta_{(n)} = \frac{\partial}{\partial x_n} - \frac{\partial}{\partial y_n}, \quad \delta_{(-n)} = n t_n, \quad \delta_{(0)} = 0.$$

The fields $e(z), f(z), h(z)$ given above we can write explicitly via,

$$\chi(z) = \sum_{i\in\mathbb{Z}} \chi_{(i)} z^{-i-1}$$

$$= \chi(z)_+ + \chi(z)_-$$

$$= \left(\sum_{i>0} i x_i z^{i-1}\right) + \left(q\frac{\partial}{\partial q}z^{-1} + \sum_{i>0}\left(\frac{\partial}{\partial x_i} + c\frac{\partial}{\partial y_i} + \frac{\partial}{\partial t_i}\right)z^{-i-1}\right),$$

$$\gamma(z) = \sum_{i\in\mathbb{Z}} \gamma_{(i)} z^{-i-1}$$

$$= \gamma(z)_+ + \gamma(z)_-$$

$$= \left(\sum_{i>0} i y_i z^{i-1}\right) + \left(-q\frac{\partial}{\partial q}z^{-1} + \sum_{i>0}\left(c\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i} - \frac{\partial}{\partial t_i}\right)z^{-i-1}\right),$$

$$e^{\pm\delta}(z) = z^{\pm\delta_0} e^{\pm\delta} \exp\left(\pm \sum_{i<0} \delta_{(i)} \frac{z^{-i}}{-i}\right) \exp\left(\pm \sum_{i>0} \delta_{(i)} \frac{z^{-i}}{-i}\right)$$

$$= q^{\pm1} \exp\left(\pm \sum_{i>0} t_i z^i\right) \exp\left(\pm \sum_{i>0} (\tilde{\partial}_{x_i} - \tilde{\partial}_{y_i}) z^{-i}\right)$$

$$= q^{\pm1} \sum_{i,j\in\mathbb{Z}} S_i(t) S_j(\tilde{\partial}_x - \tilde{\partial}_y) z^{i-j},$$

where in the last set of equations we have made a number of adjustments: we have

made the substitution $q = e^{\delta}$, we use the notation $\tilde{\partial}_{x_i} = \frac{1}{i}\frac{\partial}{\partial x_i}$ and have made use of Schur polynomials to expand the exponentials. On that last note, recall the following definition.

Definition 5.2.2. *The elementary Schur polynomials, $S_n(x_1, x_2, \ldots)$ are given in terms of the generating series,*

$$\exp\left(\sum_{n \geq 1} x_n z^n\right) = \sum_{n \in \mathbb{Z}} S_n(x) z^n,$$

where $S_n(x) = 0$ for $n < 0$.

Explicitly, the elementary Schur polynomials can be computed via the formula

$$S_n(x_1, x_2, \ldots) = \sum_{n = i_1 + 2i_2 + \ldots} \frac{x^{i_1}}{i_1!}\frac{x^{i_2}}{i_2!} \cdots,$$

so for example

$$S_0(x) = 1, \quad S_1(x) = x_1, \quad S_2(x) = \frac{x_1^2}{2} + x_2, \quad S_3(x) = \frac{x_1^3}{6} + x_1 x_2 + x_3.$$

Before getting into specifics, we review the procedure we are about to perform. We will, to a large extent, follow the procedure of the Japanese school [22], which produces *Hirota bilinear equations*.

Definition 5.2.3. *Given a differential operator $D(\partial_{x_1}, \partial_{x_2}, \ldots)$ and two functions $f(x_1, x_2, \ldots), g(x_1, x_2, \ldots)$, we define the Hirota bilinear equation $Df \cdot g = 0$ to be,*

$$Df \cdot g = D(\partial_{u_1}, \partial_{u_2}, \ldots)(f(x_1 + u_1, x_2 + u_2, \ldots)g(x_1 - u_1, x_2 - u_2, \ldots))|_{u_1 = u_2 = \ldots = 0} = 0.$$

It is convenient to observe that

$$\Omega = \mathrm{Res}_z \, z\Omega(z) + K \otimes d + d \otimes K,$$

where

$$\Omega(z) = e(z) \otimes f(z) + f(z) \otimes e(z) + \frac{1}{2}h(z) \otimes h(z).$$

We will use a slightly altered version of this, taking the residue without first multiplying $\Omega(z)$ by z and so replacing $K \otimes d + d \otimes K$ with $-K \otimes T - T \otimes K$.

Consider now the $e(z) \otimes f(z)$ term in $\Omega(z)$. This is the most complicated term and $f(z) \otimes e(z)$ is nearly identical, so it is useful to start here. From the definition we have

$$e(z) \otimes f(z) = \colon\!\chi(z)e^{\delta}\!\colon \otimes \colon\!\gamma(z)e^{-\delta}\!\colon,$$

but since the operators on either side of the tensor product act on respective sides of $\tau \otimes \tau$, we can rewrite this as,

$$\colon\!\chi'(z)e^{\delta'}(z)\!\colon\!\colon\!\gamma''(z)e^{-\delta''}(z)\!\colon$$

where, for example,

$$\chi'(z) = \sum_{n\in\mathbb{Z}} \chi'_{(n)} z^{-n-1},$$

acts of the first factor in

$$\tau'\tau'' = \sum_{m,n\in\mathbb{Z}} \tau_m(x')\tau_n(x'')(q')^m(q'')^n.$$

Since all the modes $\delta'_{(n)}$ commute with $\delta''_{(m)}$, we can combine the exponentials above,

$$e^{\delta'}(z)e^{-\delta''}(z) = e^{\delta'-\delta''}(z),$$

which explicitly on the Fock space is,

$$q'(q'')^{-1} \sum_{i,j\in\mathbb{Z}} S_i(t'-t'')S_j(\tilde{\partial}_{x'} - \tilde{\partial}_{y'} - \tilde{\partial}_{x''} + \tilde{\partial}_{y''})z^{i-j}.$$

We can now write out the term $e(z) \otimes f(z)$ contributes to $\Omega(z)$ in full,

$$\sum_{\substack{m,n\in\mathbb{Z}\\i,j>0}} (ijx'_iy''_j)S_m(t'-t'')S_n(\tilde{\partial}_{x'} - \tilde{\partial}_{y'} - \tilde{\partial}_{x''} + \tilde{\partial}_{y''})z^{i+j+m-n-2}$$

$$- \sum_{\substack{m,n\in\mathbb{Z}\\i>0}} (ix'_i)S_m(t'-t'')S_n(\tilde{\partial}_{x'} - \tilde{\partial}_{y'} - \tilde{\partial}_{x''} + \tilde{\partial}_{y''})(q''\partial_{q''})z^{i+m-n-2}$$

$$+ \sum_{\substack{m,n\in\mathbb{Z}\\i,j>0}} (ix'_i)S_m(t'-t'')S_n(\tilde{\partial}_{x'} - \tilde{\partial}_{y'} - \tilde{\partial}_{x''} + \tilde{\partial}_{y''})(c\partial_{x''_j} + \partial_{y''_j} - \partial_{t''_j})z^{i-j+m-n-2}$$

$$+ \sum_{\substack{m,n\in\mathbb{Z}\\j>0}} (jy''_j)S_m(t'-t'')S_n(\tilde{\partial}_{x'} - \tilde{\partial}_{y'} - \tilde{\partial}_{x''} + \tilde{\partial}_{y''})(q'\partial_{q'})z^{j+m-n-2}$$

$$+ \sum_{\substack{m,n\in\mathbb{Z}\\i,j>0}} (jy''_j)S_m(t'-t'')S_n(\tilde{\partial}_{x'} - \tilde{\partial}_{y'} - \tilde{\partial}_{x''} + \tilde{\partial}_{y''})(\partial_{x'_i} + c\partial_{y'_i} + \partial_{t'_i})z^{-i+j+m-n-2}$$

$$+ \sum_{m,n\in\mathbb{Z}} S_m(t'-t'')S_n(\tilde{\partial}_{x'} - \tilde{\partial}_{y'} - \tilde{\partial}_{x''} + \tilde{\partial}_{y''})(q'\partial_{q'})(q''\partial_{q''})z^{m-n-2}$$

$$+ \sum_{\substack{m,n\in\mathbb{Z} \\ i,j>0}} S_m(t'-t'')S_n(\tilde{\partial}_{x'}-\tilde{\partial}_{y'}-\tilde{\partial}_{x''}+\tilde{\partial}_{y''})$$

$$(\partial_{x'_i}+c\partial_{y'_i}+\partial_{t'_i})(c\partial_{x''_j}+\partial_{y''_j}-\partial_{t''_j})z^{-i-j+m-n-2}.$$

We now make an important change of variables,

$$x'_i = x_i + \bar{x}_i, \ \ x''_i = x_i - \bar{x}_i,$$

$$\partial_{x'_i} = \frac{1}{2}\left(\partial_{x_i}+\partial_{\bar{x}_i}\right), \ \ \partial_{x''_i} = \frac{1}{2}\left(\partial_{x_i}-\partial_{\bar{x}_i}\right)$$

and similarly for y',y'' and t',t''. This turns the above sum into

$$\sum_{\substack{m,n\in\mathbb{Z} \\ i,j>0}} ij(x_i+\bar{x}_i)(y_j-\bar{y}_j)S_m(2\bar{t})S_n(\tilde{\partial}_{\bar{x}}-\tilde{\partial}_{\bar{y}})z^{i+j+m-n-2}$$

$$- \sum_{\substack{m,n\in\mathbb{Z} \\ i>0}} (x_i+\bar{x}_i)S_m(2\bar{t})S_n(\tilde{\partial}_{\bar{x}}-\tilde{\partial}_{\bar{y}})(q''\partial_{q''})z^{i+m-n-2}$$

$$+\frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z} \\ i,j>0}} (x_i+\bar{x}_i)S_m(2\bar{t})S_n(\tilde{\partial}_{\bar{x}}-\tilde{\partial}_{\bar{y}})\frac{1}{2}(c(\partial_{x_j}-\partial_{\bar{x}_j})+(\partial_{y_j}-\partial_{\bar{y}_j})-(\partial_{t_j}-\partial_{\bar{t}_j}))z^{i-j+m-n-2}$$

$$+ \sum_{\substack{m,n\in\mathbb{Z} \\ j>0}} (y_j-\bar{y}_j)S_m(2\bar{t})S_n(\tilde{\partial}_{\bar{x}}-\tilde{\partial}_{\bar{y}})(q'\partial_{q'})z^{j+m-n-2}$$

$$+\frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z} \\ i,j>0}} (y_j-\bar{y}_j)S_m(2\bar{t})S_n(\tilde{\partial}_{\bar{x}}-\tilde{\partial}_{\bar{y}})\frac{1}{2}((\partial_{x_i}+\partial_{\bar{x}_i})+c(\partial_{y_i}+\partial_{\bar{y}_i})+(\partial_{t_i}+\partial_{\bar{t}_i}))z^{-i+j+m-n-2}$$

$$+ \sum_{m,n\in\mathbb{Z}} S_m(2\bar{t})S_n(\tilde{\partial}_{\bar{x}}-\tilde{\partial}_{\bar{y}})(q'\partial_{q'})(q''\partial_{q''})z^{m-n-2}$$

$$+\frac{1}{4}\sum_{\substack{m,n\in\mathbb{Z}\\i,j>0}}S_m(2\bar{t})S_n(\tilde{\partial}_{\bar{x}}-\tilde{\partial}_{\bar{y}})((\partial_{x_i}+\partial_{\bar{x}_i})+c(\partial_{y_i}+\partial_{\bar{y}_i})+(\partial_{t_i}+\partial_{\bar{t}_i}))$$

$$(c(\partial_{x_j}-\partial_{\bar{x}_j})+(\partial_{y_j}-\partial_{\bar{y}_j})-(\partial_{t_j}-\partial_{\bar{t}_j}))z^{-i-j+m-n-2}.$$

There is one more change of variables to make before we can begin considering PDEs from this, and that is due to Taylor's formula. Recall from [18] the formula,

$$P(\tilde{\partial}_y)\tau(x+y)\tau(x-y)=P(\tilde{\partial}_u)\tau(x+y+u)\tau(x-y-u)|_{u=0}$$
$$=P(\tilde{\partial}_u)\exp(\sum_{j\geq1}y_j\partial_{u_j})\tau(x+u)\tau(x-u)|_{u=0}.$$

We perform the exact same process on the above sum, with $\partial_{\bar{x}_i}=\partial_{u_i},\partial_{\bar{y}_i}=\partial_{v_i},\partial_{\bar{t}_i}=\partial_{w_i}$. So, making this substitution and then taking residue we finally have

$$\sum_{\substack{m\in\mathbb{Z}\\i,j>0}}ij(x_i+\bar{x}_i)(y_j-\bar{y}_j)S_m(2\bar{t})S_{i+j+m-1}(\tilde{\partial}_u-\tilde{\partial}_v)$$

$$-\sum_{\substack{m\in\mathbb{Z}\\i>0}}(x_i+\bar{x}_i)S_m(2\bar{t})S_{i+m-1}(\tilde{\partial}_u-\tilde{\partial}_v)(q''\partial_{q''})$$

$$+\frac{1}{2}\sum_{\substack{m\in\mathbb{Z}\\i,j>0}}(x_i+\bar{x}_i)S_m(2\bar{t})S_{i-j+m-1}(\tilde{\partial}_u-\tilde{\partial}_v)(c(\partial_{x_j}-\partial_{u_j})+(\partial_{y_j}-\partial_{v_j})-(\partial_{t_j}-\partial_{w_j}))$$

$$+\sum_{\substack{m\in\mathbb{Z}\\j>0}}(y_j-\bar{y}_j)S_m(2\bar{t})S_{j+m-1}(\tilde{\partial}_u-\tilde{\partial}_v)(q'\partial_{q'})$$

$$+\frac{1}{2}\sum_{\substack{m\in\mathbb{Z}\\i,j>0}}(y_j-\bar{y}_j)S_m(2\bar{t})S_{m-i+j-1}(\tilde{\partial}_u-\tilde{\partial}_v)((\partial_{x_i}+\partial_{u_i})+c(\partial_{y_i}+\partial_{v_i})+(\partial_{t_i}+\partial_{w_i}))$$

$$+\sum_{m\in\mathbb{Z}}S_m(2\bar{t})S_{m-1}(\tilde{\partial}_u-\tilde{\partial}_v)(q'\partial_{q'})(q''\partial_{q''})$$

$$+ \frac{1}{4} \sum_{\substack{m \in \mathbb{Z} \\ i,j>0}} S_m(2\bar{t}) S_{m-i-j-1}(\tilde{\partial}_u - \tilde{\partial}_v)((\partial_{x_i} + \partial_{u_i}) + c(\partial_{y_i} + \partial_{v_i}) + (\partial_{t_i} + \partial_{w_i}))$$

$$(c(\partial_{x_j} - \partial_{u_j}) + (\partial_{y_j} - \partial_{v_j}) - (\partial_{t_j} - \partial_{w_j})).$$

It is clear that terms in the above sum such as,

$$\frac{1}{2} \sum_{\substack{m \in \mathbb{Z} \\ i,j>0}} (x_i + \bar{x}_i) S_m(2\bar{t}) S_{i-j+m-1}(\tilde{\partial}_u - \tilde{\partial}_v)(c(\partial_{x_j} - \partial_{u_j}) + (\partial_{y_j} - \partial_{v_j}) - (\partial_{t_j} - \partial_{w_j})),$$

require us to explicitly reduce the number of parameters on which $\tau(x_i, y_i, t_i)$ depends. Such cutoffs introduce a certain amount of arbitrariness but are necessary to explore particular PDEs.

Before looking at specific reductions, we examine the other terms of $\operatorname{Res}\Omega(z)$. It is clear how to obtain $f(z) \otimes e(z)$ from what we have already calculated (simply switch the single-primed and double-primed terms), so let us look at $\frac{1}{2}h(z) \otimes h(z)$. Recall that $h_{(j)} = k\delta_{(j)} + \chi_{(j)} - \gamma_{(j)}$, so the following is clear,

$$\text{Res}\, h'(z)h''(z) = \text{Res} \sum_{i,j\in\mathbb{Z}} h'_{(i)} h''_{(j)} z^{-i-j-2}$$

$$= \sum_{j\in\mathbb{Z}} h'_{(-j-1)} h''_{(j)}$$

$$= \frac{1}{2} \sum_{j\neq 0,-1} (h_{(-j-1)} + \bar{h}_{(-j-1)})(h_{(j)} - \bar{h}_{(j)})$$

$$+ \frac{1}{2}(h_{(-1)} + \bar{h}_{(-1)})(2q''\partial_{q''}) + \frac{1}{2}(h_{(-1)} - \bar{h}_{(-1)})(2q'\partial_{q'})$$

$$= \frac{1}{2} \sum_{j\in\mathbb{Z}} (h_{(-j-1)} + \bar{h}_{(-j-1)})(h_{(j)} - \bar{h}_{(j)})$$

$$+ 2(m+n)(t_1 + x_1 - y_1) + 2(n-m)(\bar{t}_1 + \bar{x}_1 - \bar{y}_1).$$

All that is left is $-K \otimes T - T \otimes K$, but it is easy to see that since $T = L_{-1}$, this is

$$k \sum_{i\geq 0} \chi_{(-i-1)}\chi_{(i)} + \gamma_{(-i-1)}\gamma_{(i)} + \delta_{(-i-1)}\delta_{(i)}.$$

5.3   Reduction $\tau = \tau(x_1, y_1, t_1)$

We make the reduction in the parameters of τ by

$$\tau = \tau(x_1, 0, 0, \ldots, y_1, 0, 0, \ldots, t_1, 0, 0) = \tau(x, y, t).$$

With the reduction in variables in mind, it is clear that the only modes of $h(z)$ which act non-trivially are $h_{(-1)}, h_{(0)}, h_{(1)}$. Taking residue means that the only contributing terms are,

$$\frac{1}{2}(h''_{(-1)} h'_{(0)} + h'_{(-1)} h''_{(0)}),$$

which reduces to

$$(m+n)(t+x-y)+(n-m)(\bar{t}+\bar{x}-\bar{y}).$$

The last terms we consider are those from $-K \otimes T - T \otimes K$, which simplify to

$$k\big((m+n)(x-y)+(m-n)(\bar{x}-\bar{y})\big).$$

We are now in principle ready to tackle the task of pulling off coefficients from the dummy variables $(q')^m(q'')^n \bar{x}_i \bar{y}_j \bar{t}_k$. The first thing about these equations which appears to differ from early work is that the coefficient on $(q')^m(q'')^n$ is non-zero, and it gives us a first-degree PDE in three variables:

$$\big[xy(\partial_u - \partial_v)+(m-1)y-(n+1)x\big]\tau_{m-1}\cdot\tau_{n+1}$$

$$+\big[-xy(\partial_u - \partial_v)+(n-1)y-(m+1)x\big]\tau_{m+1}\cdot\tau_{n-1}$$

$$+\big[(m+n)(kt+x-y)+k(m+n)(x-y)\big]\tau_m\cdot\tau_n = 0.$$

Letting $m=n=0$ and simplifying this a bit, we have

$$\big[2xy(\partial_u - \partial_v)-2(x+y)\big]\tau_{-1}\cdot\tau_1 = 0.$$

The next coefficient we examine is that on $(q')^m(q'')^n \bar{t}_1$:

$$\Big[xy(\partial_u - \partial_v)^2 - 2(n+1)x(\partial_u - \partial_v)+x\big(c(\partial_x - \partial_u)+(\partial_y - \partial_v)-(\partial_t - \partial_w)\big)$$

$$+2(m-1)y(\partial_u - \partial_v)+y\big((\partial_x + \partial_u)+c(\partial_y + \partial_v)+(\partial_t + \partial_w)\big)$$

$$+2(m-1)(n+1)+\big(xy(\partial_u-\partial_v)+(m-1)y-(n+1)x\big)\partial_w\Big]\tau_{m-1}\cdot\tau_{n+1}$$

$$+\Big[-xy(\partial_u-\partial_v)^2-2(m+1)x(\partial_u-\partial_v)+x\big(c(\partial_x+\partial_u)+(\partial_y+\partial_v)-(\partial_t+\partial_w)\big)$$

$$+2(n-1)y(\partial_u-\partial_v)+y\big((\partial_x-\partial_u)+c(\partial_y-\partial_v)+(\partial_t-\partial_w)\big)$$

$$-2(n-1)(m+1)+\big(-xy(\partial_u-\partial_v)+(n-1)y-(m+1)x\big)\partial_w\Big]\tau_{m+1}\cdot\tau_{n-1}$$

$$+\Big[(n-m)+\big((m+n)(kt+x-y)+k(m+n)(x-y)\partial_w\Big]\tau_m\cdot\tau_n=0.$$

As before, we set $m=n=0$, but now this coefficient turns out to be trivial! Similarly, the coefficients on $(q')^m(q'')^n\bar{x}_1$ and $(q')^m(q'')^n\bar{y}_1$ are, respectively,

$$\Big[y(\partial_u-\partial_v)-(n+1)+\big(xy(\partial_u-\partial_v)+(m-1)y-(n+1)x\big)\partial_u\Big]\tau_{m-1}\cdot\tau_{n+1}$$

$$+\Big[y(\partial_u-\partial_v)+(m+1)+\big(-xy(\partial_u-\partial_v)+(n-1)y-(m+1)x\big)\partial_u\Big]\tau_{m+1}\cdot\tau_{n-1}$$

$$+\Big[(n-m)+\big((m+n)(kt+x-y)+km+k(m+n)(x-y)\big)\partial_u\Big]\tau_m\cdot\tau_n=0,$$

and

$$\Big[-x(\partial_u-\partial_v)-(m-1)+\big(xy(\partial_u-\partial_v)+(m-1)y-(n+1)x\big)\partial_v\Big]\tau_{m-1}\cdot\tau_{n+1}$$

$$+\Big[-x(\partial_u-\partial_v)+(n-1)+\big(-xy(\partial_u-\partial_v)+(n-1)y-(m+1)x\big)\partial_v\Big]\tau_{m+1}\cdot\tau_{n-1}$$

$$+\Big[(n-m)-kn+\big((m+n)(kt+x-y)+k(m+n)(x-y)\big)\partial_v\Big]\tau_m\cdot\tau_n=0,$$

which are both also trivial when we set $m=n=0$. We now examine the coefficients on

$(q')^m(q'')^n\bar{t}_1^2$, and $(q')^m(q'')^n\bar{x}_1\bar{t}_1$. On the first we have:

$$\left[\frac{1}{3}xy(\partial_u - \partial_v)^3 - (n+1)x(\partial_u - \partial_v)^2 + x(\partial_u - \partial_v)(c(\partial_x - \partial_u) + (\partial_y - \partial_v) - (\partial_t - \partial_w))\right.$$

$$+(m-1)y(\partial_u - \partial_v)^2 + y(\partial_u - \partial_v)((\partial_x + \partial_u) + c(\partial_y + \partial_v) + (\partial_t + \partial_w))$$

$$+2(m-1)(n+1)(\partial_u - \partial_v) + \Big(xy(\partial_u - \partial_v)^2 - 2(n+1)x(\partial_u - \partial_v)$$

$$+x(c(\partial_x - \partial_u) + (\partial_y - \partial_v) - (\partial_t - \partial_w)) + 2(m-1)y(\partial_u - \partial_v)$$

$$+y((\partial_x + \partial_u) + c(\partial_y + \partial_v) + (\partial_t + \partial_w)) + 2(m-1)(n+1)\Big)\partial_w$$

$$\left.+\frac{1}{2}\Big(xy(\partial_u - \partial_v) + (m-1)y - (n+1)x\Big)\partial_w^2\right]\tau_{m-1}\cdot\tau_{n+1}$$

$$+\left[-\frac{1}{3}xy(\partial_u - \partial_v)^3 - (m+1)x(\partial_u - \partial_v)^2 - x(\partial_u - \partial_v)(c(\partial_x + \partial_u) + (\partial_y + \partial_v) - (\partial_t + \partial_w))\right.$$

$$+(n-1)y(\partial_u - \partial_v)^2 - y(\partial_u - \partial_v)((\partial_x - \partial_u) + c(\partial_y - \partial_v) + (\partial_t - \partial_w))$$

$$-2(n-1)(m+1)(\partial_u - \partial_v) + \Big(-xy(\partial_u - \partial_v)^2 - 2(m+1)x(\partial_u - \partial_v)$$

$$+x(c(\partial_x - \partial_u) + (\partial_y - \partial_v) - (\partial_t - \partial_w)) + 2(n-1)y(\partial_u - \partial_v)$$

$$+y((\partial_x + \partial_u) + c(\partial_y + \partial_v) + (\partial_t + \partial_w)) - 2(n-1)(m+1)\Big)\partial_w$$

$$\left.+\frac{1}{2}\Big(-xy(\partial_u - \partial_v) + (n-1)y - (m+1)x\Big)\partial_w^2\right]\tau_{m+1}\cdot\tau_{n-1}$$

$$+\left[((n-m))\partial_w + \frac{1}{2}((m+n)(kt + x - y) + k(m+n)(x - y))\partial_w^2\right]\tau_m\cdot\tau_n = 0.$$

Again, we set $m = n = 0$ and simplify,

$$\Big[\frac{2}{3}xy(\partial_u - \partial_v)^3 + 2xy(\partial_u - \partial_v)^2\partial_w + xy(\partial_u - \partial_v)\partial_w^2 - 2(x + y)(\partial_u - \partial_v)^2$$

$$-4(x - y)(\partial_u - \partial_v)\partial_w - (x + y)\partial_w^2 - 4(\partial_u - \partial_v + \partial_w)\Big]\tau_{-1} \cdot \tau_1 = 0.$$

Next we examine the coefficient on $(q')^m(q'')^n\bar{x}_1\bar{t}_1$,

$$\Big[2y(\partial_u - \partial_v)^2 - 2(n + 1)(\partial_u - \partial_v) + \big(c(\partial_x - \partial_u) + (\partial_y - \partial_v) - (\partial_t - \partial_w)\big)$$

$$+2y(m - 1)(\partial_u - \partial_v) + y\big((\partial_x + \partial_u) + c(\partial_y + \partial_v) + (\partial_t + \partial_w)\big)$$

$$+\Big(xy(\partial_u - \partial_v)^2 - 2(n + 1)x(\partial_u - \partial_v) + x\big(c(\partial_x - \partial_u) + (\partial_y - \partial_v) - (\partial_t - \partial_w)\big)$$

$$+2(m - 1)y(\partial_u - \partial_v) + y\big((\partial_x + \partial_u) + c(\partial_y + \partial_v) + (\partial_t + \partial_w)\big) + 2(m - 1)(n + 1)\Big)\partial_u$$

$$+\Big(y(\partial_u - \partial_v) - (n + 1)\Big)\partial_w + \Big(xy(\partial_u - \partial_v) + (m - 1)y - (n + 1)x\Big)\partial_u\partial_w\Big]\tau_{m-1} \cdot \tau_{n+1}$$

$$+\Big[2y(\partial_u - \partial_v)^2 + 2(m + 1)(\partial_u - \partial_v) + \big(c(\partial_x + \partial_u) + (\partial_y + \partial_v) - (\partial_t + \partial_w)\big)$$

$$+2y(n - 1)(\partial_u - \partial_v) - y\big((\partial_x - \partial_u) + c(\partial_y - \partial_v) + (\partial_t - \partial_w)\big)$$

$$\Big(-xy(\partial_u - \partial_v)^2 - 2(m + 1)x(\partial_u - \partial_v) + x\big(c(\partial_x + \partial_u) + (\partial_y + \partial_v) - (\partial_t + \partial_w)\big)$$

$$+2(n - 1)y(\partial_u - \partial_v) + y\big((\partial_x - \partial_u) + c(\partial_y - \partial_v) + (\partial_t - \partial_w)\big) - 2(n - 1)(m + 1)\Big)\partial_u$$

$$+\Big(y(\partial_u - \partial_v) + (m + 1)\Big)\partial_w + \Big(-xy(\partial_u - \partial_v) + (n - 1)y - (m + 1)x\Big)\partial_u\partial_w\Big]\tau_{m+1} \cdot \tau_{n-1}$$

$$+\Big[\big((n - m)\big)\partial_u + \big((n - m)\big)\partial_w + \big((m + n)(kt + x - y) + k(m + n)(x - y)\big)\partial_u\partial_w\Big]\tau_m \cdot \tau_n = 0.$$

Once again taking $m = n = 0$ and simplifying, we have,

$$\Big[2xy(\partial_u - \partial_v)^2\partial_u + 2xy(\partial_u - \partial_v)\partial_u\partial_w + 4y(\partial_u - \partial_v)^2 - 4(x+y)(\partial_u - \partial_v)\partial_u$$

$$+2y(\partial_u - \partial_v)\partial_w - 2(x+y)\partial_u\partial_w - 4(\partial_u - \partial_v)$$

$$+2\big(c(\partial_x - \partial_u) + (\partial_y - \partial_v) - (\partial_t - \partial_w)\big) - 4\partial_u - 2\partial_w\Big]\tau_{-1} \cdot \tau_1 = 0.$$

Now we have three Hirota-type equations which we simplify. Beginning with the first, we write

$$xy(\partial_u - \partial_v)\tau_{-1} \cdot \tau_1 = (x+y)\tau_{-1}\tau_1.$$

Keeping in mind that in the right-hand side of this equation, the functions τ_{-1} and τ_1 are really of the form

$$\tau_{-1}(x+u, y+v, t+w)|_{u=v=w=0}, \quad \tau_1(x-u, y-v, t-w)|_{u=v=w=0},$$

we can write expressions for $xy(\partial_u - \partial_v)^2$ and $xy(\partial_u - \partial_v)^3$ which appear in other equations (and are otherwise difficult to deal with):

$$xy(\partial_u - \partial_v)^2\tau_{-1} \cdot \tau_1 = (x+y)(f_x + 2fh_x - f_y - 2fh_y),$$

$$xy(\partial_u - \partial_v)^3\tau_{-1} \cdot \tau_1 = (x+y)$$

$$(f_{xx} + 4f_xh_x + 2fh_x^2 - 2f_{xy} - 4f_xh_y - 4f_yh_x - 2fh_xh_y + f_{yy} + 4f_yh_y + 2fh_y^2),$$

where we have introduced a new set of variables,

$$f = \frac{\tau_{-1}}{\tau_1}, \quad h = \log \tau_1.$$

The equation associated to $\bar{t}_1^2$ then becomes

$$\frac{2}{3}(x+y)(f_{xx} + 4f_x h_x + 2f h_x^2 - 2f_{xy} - 4f_x h_y - 4f_y h_x - 2f h_x h_y + f_{yy} + 4f_y h_y + 2f h_y^2)$$

$$+2(x+y)(f_{xt} + 2f_x h_t + 2f_t h_x + 2f h_x h_t) + (x+y)(f_{tt} + 4f_t h_t + 2f h_t^2)$$

$$-2xy(f_x + 2f h_x - f_y - 2f h_y) - 4\frac{xy(x+y)}{x-y}(f_t + 2f h_t) - (f_{tt} + f_t^2)$$

$$+(x+y)(f_{xy} + 2f h_{xy} - 4f_x - 4f_y + 4f_t) = 0,$$

and that associated to $\bar{x}_1 \bar{t}_1$ is

$$2(x+y)(f_{xx} + 4f_x h_x + 2f h_x^2) + 2(x+y)(f_{xt} + 2f_x h_t + 2f_t h_x + 2f h_x h_t)$$

$$+4x(x+y)(f_x + 2f h_x - f_y - 2f h_y) - 4xy(f_x + f h_x) + 2x(x+y)(f_t + f h_t)$$

$$-2xy(f_{xt} + 2f_{xt}) - 4(x+y)f + 2c(f_x + f h_x) + 2(f_y + f h_y) + 2(f_t + f h_t) - 4f_x - 2f_t = 0$$

At the critical level - that is, when $\gamma = -\chi$, $y = -x$ and $k = -2$ - these equations simplify dramatically.

$$\left[-4x^2 \partial_u \right] \tau_{-1} \cdot \tau_1 = 0$$

$$\left[-\frac{4}{3}x^2 \partial_u^3 - 4x^2 \partial_u^2 \partial_w - 2x^2 \partial_u \partial_w^2 - 8\partial_u \partial_w - 8\partial_u - 4\partial_w \right] \tau_{-1} \cdot \tau_1 = 0,$$

and

$$\Big[-4x^2\partial_u^3 - 4x^2\partial_u^2\partial_w - 4x\partial_u^2 - 4x\partial_u\partial_w - 12\partial_u - 2\partial_w$$

$$+2\big(k(\partial_x - \partial_u) - (\partial_t - \partial_w)\big)\Big]\tau_{-1} \cdot \tau_1 = 0.$$

We then obtain the following system of non-autonomous PDEs involving the functions $f(x,t), g(x,t), h(x,t)$

$$4x^2 f_x = 0,$$

$$x^2\Big(\frac{4}{3}f_{xxx} + f_{xx}h_x + 8f_xh_{xx} + 4fh_x^2 + 4f_{xxt} + 16f_xh_{xt} + 8f_th_{xx}$$

$$+2f_{xtt} + 8f_th_{xt} + 4f_xh_{tt}\Big) + 16fh_{xt} + 8f_{xt} + 8f_x + 4f_t = 0,$$

$$x^2\big(4f_{xxx} + 3f_{xx}h_x + 24f_xh_{xx} + 12fh_x^2 + 4f_{xxt} + 16f_xh_{xt} + 8f_th_{xx}\big)$$

$$x\big(8fh_{xx} + 4f_{xx} + 8fh_{xt} + f_{xt}\big) - 12f_x - 2f_t + 4kfh_x - 2fh_t = 0.$$

The first equation again allows us to simplify the system drastically, and we obtain

$$f_t = -x^2(fh_x^2 + 2f_th_{xx} + 2f_th_{xt}) - 4fh_{xt},$$

$$h_t = \frac{1}{10}\Big(x^2(6h_x^2 - 4(\log f)_t h_{xx}) + x(4h_{xx} + 4h_{xt}) - f_t\Big).$$

Chapter 6

Conclusion and future work

In this thesis, we have raised more questions than were answered. We now review some of these questions, as well as other planned lines of work and open problems.

6.1 Vertex algebras

As mentioned above, the "square-length 0" case is not immediately available via FMS-bosonization in the case of E_N. It is possible that a more general ansatz exists - one involving vertex operators whose boson has square-length 0, but not reducible to a result of FMS-bosonization - but this remains unclear.

It may be required to invoke the use of more general sums of fields, perhaps like those found in e.g. [19] or [23]. This paper includes a very interesting result which appears to be related to the *symplectic fermions* [24]. Somewhat tangentially, it is not clear that this representation (of $\widehat{sl_2}$ at level $k = -\frac{4}{3}$) is unique - this is an interesting open problem which we hope to explore, as well as the associated integrable system.

Also recall the classification ansatz for $\widehat{\mathfrak{sl}_N}$: we required that $(\delta_i|\delta_j) = 0, \forall i, j$, but this

may be too strong an assumption. In the case of $\widehat{\mathfrak{sl}_3}$ for example, it is not immediately apparent that $(\delta_1 | \delta_2) = -1$ is forbidden by the equations.

6.2   Bosonic $\mathfrak{gl}_\infty$ and Gelfand-Dickey hierarchies

It is widely known that the charged free fermions give a representation of $\mathfrak{gl}_\infty$ at level $k = 1$, and that applying the boson-fermion correspondence produces the KP hierarchy. It is also known that the restriction of this representation of $\mathfrak{gl}_\infty$ to the subalgebra $\widehat{sl_N}$ produces the Nth Gelfand-Dickey hierarchy.

It is somewhat less widely known that the charged free bosons give a representation of $\mathfrak{gl}_\infty$ at level $k = -1$ [25]. Recently, the boson-boson correspondence was applied to this representation and the corresponding integrable hierarchy was investigated [26].

What remains unclear and will require further research is how the hierarchy investigated in this thesis is related to that found in [26], and if there is a reduction of "bosonic" $\mathfrak{gl}_\infty$ to the $\widehat{\mathfrak{sl}_N}$ algebras described in this thesis which produce "bosonic Gelfand-Dickey".

6.3   Twisted hierarchy

An important fact about the representation of $\widehat{\mathfrak{sl}_2}$ at level k given in Chapter 4 is that there is an isometry of the underlying lattice and hence of the lattice vertex algebra, which gives a *twisted* representation of $\widehat{\mathfrak{sl}_2}$ again at level k. We can use this twisted representation to produce another, different integrable hierarchy.

The first equation in this hierarchy is of order six and is again non-autonomous. This twisted hierarchy is in a way analogous to the KdV hierarchy, in that they both arise from representations of $\widehat{\mathfrak{sl}_2}$ which are twisted by a Coxeter element.

6.4   Integrable hierarchies with parameters

A perplexing fact that we saw in Chapter 4 is that we were able to produce representation of $\widehat{\mathfrak{sl}_N}$ at arbitrary level only for $N = 2$, and for general N only level $k = -1$ was found. It is known that charged free bosons can be utilized to produce representations of $\widehat{\mathfrak{sl}_N}$ at the critical level $k = -N$ [21], but it remains unclear under what more general assumptions such results can be expanded. To wit, we wish to be able to produce representations of $\widehat{\mathfrak{sl}_N}$ (and simply laced algebras in general) at arbitrary level, bosonize them and produce other integrable hierarchies where the level enters into the equations as a free parameter.

6.5   Vertex coalgebras

As was mentioned in section, this whole business involving vertex algebras, their representations and hierarchies of integrable PDEs appears to be potentially more general than the setting of affine algebras and their Casmimir operators. In principle, all that is required is a vertex algebra, a Fock space representation, and an operator which commutes with the diagonal action of the vertex algebra on the tensor product of the module with itself. The crux of this, then is the existence of a "diagonal action" - that is, if a *coproduct*.

There are notions of *vertex coalgebras* and *vertex bialgebras* [27] [28], but it is not clear if this is what is necessary. It will be very interesting to see what sort of apparatus allows for the most general treatment of this entire programme.

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