BUCH, NILS. Inventory Models Without Explicit Fixed Ordering Costs. (Under the direction of Russell King and Donald Warsing.)

Our research investigates new methods for solving single and multiple period, finite-horizon, inventory problems when the number of orders allowed during the horizon is limited. Existing literature has solved the question of how many goods to order when there is only one order available to make at the beginning of the horizon, as well as how many goods to order when there is an explicit and known fixed ordering cost incurred upon the placing of an order. Our models extend the problem to the case when the fixed ordering costs are either unknown or very difficult to estimate.

The classical "newsvendor problem" in the management sciences solves for the optimal ordering quantity when there is a single period with a known single demand distribution. Our first research question is how to extend this classical model when there exist two opportunities for ordering within the horizon. We develop analytical formulations to solve this extension optimally.

An extension of the single-period case with one replenishment is a general multiple-period model with multiple (restricted) replenishment opportunities, without an explicitly stated fixed ordering cost. To solve this problem, we employ a Markov Decision Process-based solution methodology, where the costs of over-ordering and under-ordering are weighed against each other to determine the optimal ordering quantity for any period, when the decision maker has a given number of orders remaining. We also investigate an approximation to solve this problem near-optimally. We extend this model to account for price markdown opportunities.

Our research also solves a related problem when the demand distribution parameters are unknown. We evaluate the extent to which additional orders can alleviate some of the risks involved in ordering under these assumptions. To solve this problem, we employ a joint
simulation and Markov Decision Process solution methodology that utilizes both optimal and near-optimal solutions for the known demand case in a simulation environment consisting of unknown seasonal demand. We show the profit difference in the known and unknown demand parameter cases when multiple orders are available.
Inventory Models Without Explicit Fixed Ordering Costs

by
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introduction

1.1 Description of the general problem

The problem considered in this research is a periodic-review, finite-horizon inventory control problem with multiple replenishment opportunities, uncertain demand, and lost sales. We call the problem and associated model in its most realistic form the “general problem”. Throughout this work, we consider simplifications or variants of the general problem to build the necessary intuition and knowledge base to build a model and solution technique for a problem closely representing the general problem. One application of the general problem is experienced by retailers of apparel products. Throughout this work we will couch the problem in this context, however, developments in this research have applications to a large variety of other domains. The following is a brief introduction and description of the major elements of the general problem that we discuss in more detail throughout this work.
1.1.1 Season length

Traditionally, the problem of determining ordering quantities in a seasonal apparel retail setting is solved using a single-period approach. This approach is adequate in a newsvendor-like environment where one order is placed at the beginning of the season, and there are no opportunities to reorder. These assumptions, however, don't hold in our general problem. As an example, the benefits and applicability of quick response in apparel retailing, which requires a modeling approach with multiple periods throughout the horizon, have been well documented (see e.g. Hunter et al. [13] for an overview). For this and several other reasons which we explain in detail throughout the paper, we take a multiple period approach to the problem. As such, the selling season, which we shall indistinguishably also call the horizon, is comprised of periods $t = 1, \cdots, T$, with period 1 representing the first period of the horizon, and period $T$ representing the end of the horizon.

1.1.2 Demand and forecast reestimation

As is typical in retail settings, demand is forecast as an aggregate seasonal quantity. We define $ESD$ as the expected seasonal demand. This is the mean, or expected, value of demand over the entire horizon. $ESD$ is also the sum of the means of demand in every period. Since we focus on consumers in a retail environment, demand is assumed to occur throughout the horizon and not at a single point in time. Therefore, we need to define demand in each period. To do so, we follow the convention outlined in Eppen and Iyer [9], where a “percent done curve” gives the proportion of $ESD$ expected in each period. We call this the seasonality curve to reflect that it also outlines a seasonality shape over the horizon. We define $PR_t$ as the expected proportion of $ESD$ in period $t$. In this work, we make the assumption that $PR_t$ is known (see e.g. Hunter et al. [14] for justification). A plot of $PR_t$ over the horizon can show various expected seasonality shapes. Some products may exhibit a traditional lifecycle trend with ramp-up, maturity, and ramp-down phases. Others may exhibit a tendency to be very popular initially followed by declining sales as the season progresses, such as technology
products. Others, like swimwear, see most of their sales in the latter parts of the season [20].

In the general problem, uncertainty manifests itself in two ways. First, $ESD$ may be either deterministic (defined by a single value) or stochastic (defined by a distribution function). As we describe in this work, apparel products, especially fashion-oriented ones, can be highly “hit-or-miss”. We define $\overline{ESD}$ as the mean of the distribution of $ESD$ if it is stochastic. $\overline{ESD}$ is often management’s point forecast of demand, while the variance and shape of $ESD$ is determined by management’s past forecasting performance. A similar approach was outlined in Cachon and Terwiesch [3]. If management is historically accurate in their forecasts and the product is a staple good with little variability (such as plain white socks), then the variance of $ESD$ is minimal. In some models, we make the assumption that $ESD$ is deterministic, i.e. it has zero variance. We call these models “known demand parameter” models. On the other hand, fashion-oriented products coupled with a management team that has historically not been able to forecast well lead to high variance in $ESD$, which we call “unknown demand parameter” models.

Secondly, the distribution of demand in each period also has inherent variability. This variability still exists under the assumption of known demand parameters, or zero variance around $ESD$. We define $D_t$ as the demand in period $t$, and $F_{D_t}$ as its distribution. $F_{D_t}$ is centered at $\overline{ESD} \cdot PR_t$ (note: in the case of unknown demand parameters, the estimate of $F_{D_t}$ changes with updated information as we discuss next). Its variance can be estimated from past sales data.

Along with the notion of stochastic $ESD$ and multiple replenishment opportunities comes the possibility that the retailer can reestimate the forecast of demand, i.e. management has the ability to reestimate $\overline{ESD}$ as the season progresses. Newer demand information, such as point-of-sale (POS) data gathered throughout the horizon can be used to make (presumably) better ordering decisions as the season progresses. We show that a relatively simple reestimating scheme using exponential smoothing works sufficiently well in our experimentation, consistent with similar literature (see Hunter et al [14]).
1.1.3 Basic costs

At a broad level, inventory models based solely on minimizing costs need several types of costs as inputs. We should note at this point that the word “cost” is used in broad manner, to include monetary costs and difficult-to-quantify costs such as management burden. The first type is the cost related to placing an order. It is comprised of a fixed cost which is assessed anytime an order is made and a variable cost which is a function of the number of units purchased. The second type of cost reflects the downside of carrying and maintaining inventory and is typically referred to as holding cost. Holding costs are what keep a decision maker from ordering too many goods. They can represent opportunity costs to the firm, driven by investing in inventory rather than pursuing other investment opportunities, as well as the cost of having inventory left over after demand has ceased. Holding costs can be defined as the actual cost of maintaining inventory. Excess inventory at the end of the horizon often has a value (negative cost) associated with it, which is often referred to as the salvage value. Lastly there is a cost associated with not having enough inventory, a lost sales penalty. This factor is what drives a cost-minimizing firm to invest in at least some inventory. This cost is assessed when demand exceeds on-hand supply. It can have fixed and variable portions as well, but the fixed portion is typically ignored. We can now define our basic cost terms in their simplest form. Define:

- $c$: Unit raw materials cost,
- $v$: Unit salvage value per item in inventory at the end of the horizon,
- $h$: Unit holding cost per unit in inventory at the end of every period,
- $p$: Unit lost sales (goodwill) penalty.

Fixed ordering costs

If we look at the ordering costs in detail, we see there is a fixed component which is typically representative of the burdens involved in ordering. Most inventory models challenge decision
makers to quantify, in dollar terms, how much it “costs” to place an order, regardless of order size. Some aspects of answering that challenge are relatively straightforward, e.g. the price of a truck shipment, regardless of the payload, is easily quantifiable. Other aspects of fixed ordering costs, however, are not as easily quantifiable. Take, for example, any salaried employee working on orders for a firm. If this employee is currently preparing two orders per day, and we ask how much it would “cost” the employee to prepare three orders per day, it may be difficult to answer that question. The employee’s salaried pay would not increase as long as the extra work does not require over-time pay. Perhaps Silver et al. [25] says it best when they state that, “Clearly, one could spend months trying to nail down the [fixed ordering cost].”

Another aspect of fixed ordering costs not captured by traditional inventory models is the idea that they may change depending on the number of orders made. Consider, as an example, a firm which has arbitrarily decided that two orders are sufficient, but adding a third would significantly burden their employees. The cost of the third order should have a higher fixed cost than the first two. Conversely, it is simple to think of examples where a firm’s per-order fixed ordering costs decrease as the number of orders made in a season increases. Take for example a salaried employee who is responsible for placing orders, and is deciding between two or three total orders over the season. If we divide the employee’s salary by the number of orders made, the per-order costs decrease. It would therefore seem that there are variable aspects to the fixed ordering cost. In any case, it is more intuitive to let a firm arbitrarily weigh the benefits and “costs” of each additional order.

Costing situations such as the examples presented above often appear in apparel retailing. Speaking with industry leaders in this field, we asked what the fixed costs of ordering are for them, and they were unable to answer the question with a specific number. Instead, their answer was more qualitative. They were more likely to specify the number of orders they are comfortable with over a horizon than they were to specify a fixed dollar amount. With this in mind, our ultimate model does not require the determination of a fixed ordering cost. This
alternative approach is both easier for decision makers to understand and easier for them to quantify. Our approach involves calculating the marginal benefit of additional replenishment opportunities, and allowing decision makers to arbitrarily decide for themselves if this benefit outweighs their perceived burden of placing those additional orders. Using this method avoids the notoriously difficult task of quantifying in monetary terms a series of arbitrary factors which may not necessarily be directly quantifiable.

1.1.4 Lead time

In the general problem, there is a lead time between the time an order is placed and the time at which product from that order is available to sell. Retail apparel presents an interesting challenge in regards to lead time assumptions. Certainly, there is a sense of long lead times from overseas products that lend themselves to a single newsvendor-type solution approach. On the other hand, the development of quick response (QR) suppliers has enabled much shorter lead times that enable intra-season ordering. One can also make the assumption that a retailer is ordering from a domestic distribution center or central warehouse such that lead times are negligibly short. In any case, the general problem needs to be general enough to be able to include an aspect of lead time.

1.1.5 Revenue and markdowns

A typical pure cost model discussed in Section 1.1.3 fails to capture a very important aspect of reality in retailing. That is, revenue may change over time through the use of discounts or clearance sales. Define \( r_t \) as the per unit revenue for which a product is sold in period \( t \). As mentioned, \( r_t \) may fluctuate based on the current markdown level. If we let \( r_0 \) be some initial base selling price, then we can redefine \( r_t = r_0 \cdot \lambda_t \), where \( 0 \leq \lambda_t \leq 1 \) is the markdown level in period \( t \). In the ultimate model, \( \lambda_t \) is a pricing decision.

Another aspect of modeling demand in the general problem is that of price elasticity. In the general problem, markdown opportunities clearly have an effect on demand for a product. In
retail apparel there is usually an option to move the item to a clearance rack, which we shall call a clearance move or a clearance sale. A clearance sale is different from a markdown in its goal. The goal of a markdown is to stimulate demand in the hopes of reducing inventory at a faster pace. The goal of a clearance is to rid the firm of all inventory as fast as possible. With this in mind, our general problem has multiple markdown levels with varying degrees of demand stimulation, as well as a clearance option. The clearance option reduces the unit revenue significantly and guarantees (by assumption) all remaining inventory is sold immediately. While we do not consider the mechanics of price elasticity in detail, it is important to note that this effect should be carefully incorporated into any applied model based on marketing and sales data. It is worth noting that the effect that pricing has on demand needs to be incorporated into our demand reestimation scheme to reflect the fact that $ESD$ changes with pricing decisions.

1.1.6 Lost sales

The context of our general problem dictates that a lost sales model be developed instead of one with backorders. In a typical retail setting, it is unlikely, if not infeasible, for a customer to order a certain fashion item that is no longer on the store shelves. Instead, these customers typically pick a substitute product, or they will simply not buy the product and perhaps go to another store to find what they are looking for. Modeling lost sales environments is certainly more challenging than modeling backorder models, but is necessary in this context to have the general problem represent reality as closely as possible.

1.1.7 Multiple products

The general problem is not centered around a firm selling only one product. As is a more realistic scenario, the firm sells multiple, related products, or stock keeping units (SKU’s). While we do not directly consider multiple products in this work, it is left as an important part of the continued expansion of this line of research. We note that the demand levels for various
products are often related. Demand may be either substitutable or complementary depending on the exact context. Substitutable goods have a negative effect on one another, such that increased demand for one SKU causes a decrease in demand for another. Conversely, demand could also have a complementary relationship, where demand for goods rise and fall together. Therefore, pricing decisions do not only affect one product alone.

1.2 Description of models considered

In working towards the goal of the general problem, we model some different simplifying variants of it. In some cases, we are able to present analytical solutions which aid in building intuition about the problem. In other cases, we present the results of numerical experimentation and analysis to describe certain phenomena and interesting behavior. Here, we give a brief description of all the variants presented in this work. This section serves as a reference location of all the variants of the problem we consider. This section is particularly useful in cross-referencing our variants of the general problem to similar existing models in the literature which we discuss in the literature review.

- Chapter 2- Single period problem with one replenishment opportunity
- Chapter 3- Multiple period, multiple replenishment opportunities with known demand parameters
- Chapter 4- Multiple period, multiple replenishment opportunities with unknown demand parameters
- Chapter 5- Multiple period, multiple replenishment opportunities with known demand parameters, including markdown

Each model considered builds upon a previous model and adds a layer of complexity. Each case presented provides its own set of interesting results that need to be understood and analyzed in order to move towards an understanding of the general problem and its results.
1.3 General model formulation

In this section, we present the notation and profit model for the general problem. First, we present the model parameters found in the general problem: Costs:

- \( t \): period, \( t = 1, 2, \cdots, T \), where \( T \) is the end of horizon
- \( J \): Total number of markdown (pricing) levels
- \( K \): Total number of orders that can be placed over the horizon
- \( I \): Inventory level
- \( ESD \): Expected seasonal demand
- \( \overline{ESD} \): Mean of \( ESD \), when \( ESD \) is stochastic
- \( \mu_{a/f} \): Average of the historical ratio of actual demand to forecast demand of the firm
- \( \sigma_{a/f} \): Standard deviation of the distribution of historical actual demand to forecast demand ratios of the firm
- \( PR_t \): Seasonality proportion, i.e. the percent of \( ESD \) expected in period \( t \). Assumed known
- \( \delta \): The percent off the original selling price (\( r \)) when markdown is triggered
- \( e_{\delta} \): The elasticity of demand (mean multiplier) under markdown level \( \delta \). Assumed known.
- \( F_{D_t} \): Distribution function of demand in period \( t \), under markdown level \( m \). \( E[F_{D_t}] = \overline{ESD} \cdot PR_t \cdot e_{\delta} \).

We can now present the general model formulation, under the objective of profit maximization. Define \( y \) as the inventory in the beginning of the season including starting inventory and
order amount which is assumed to arrive before demand occurs. First define an expected single period profit equation, as a function of a given inventory position $y$, markdown decisions $\delta$, and some specified demand realization $x$, specifically,

$$L_t(y, \delta, x) = r(1 - \delta)\min(x, y) - h \cdot (y - x)^+ + p(y - x)^-.$$  

(1.1)

Then, we can define the recursion equation:

$$V_t(I, k, \delta) = \max_{(y \geq I, m')} \int_{x=0}^{\infty} -c(y - I) + L_t(y, m') + V_{t+1}(y - x, k', m'),$$

(1.2)

where $\delta = 0$ if $m' = 0$ (no markdown made), and $\delta$ = the specified markdown percentage if $m' = 1$, signifying a markdown is made. We clarify the model in the presence of markdowns in detail in Chapter 5.

We note that if $y$ is greater than or equal to $I$, then an order is made and $k' = k - 1$. Otherwise, $k' = k$. This highlights an important assumption of the model that $k$ is decremented if an order is made, regardless of order quantity. This assumption supports the notion that the additional burden of ordering additional products in any period is marginal compared to the burden of replenishing just one product over no replenishment.

The recursive equations can be solved given an initial state, and some assumption of terminating conditions, such as the salvage value of the remaining inventory.

In Chapter 2 we consider simplifying models, where it is assumed that there is only one distribution of demand which defines the entire horizon. Furthermore, we assume that there is only one initial order and at most one replenishment order, i.e. $K = 2, T = 1$. We find analytic solutions for the optimal initial and replenishment order quantities for varying assumptions regarding the demand distribution.

In Chapter 3, we consider more general cases when $T > 2$ and there exist a finite number of orders $K$ available to make during the horizon. We assume that $ESD$ is deterministic. As analytic solutions become intractable in these cases, we develop and analyze the results of an
exhaustive experimental design to evaluate the value of additional ordering opportunities under numerous cost and demand conditions.

In Chapter 4, we relax the assumption of a deterministic $ESD$. We consider two cases: (1) when $ESD$ is specified, but unknown to the decision maker, and (2) when $ESD$ follows some distribution. Therefore, not only is there stochasticity in the periodic demand, there is also an inherent unknown mean demand over the entire horizon (subsequently, in each period). Products can either have high expected sales with some probability if the product turns out to be very popular, or low expected sales if it is not well received by the public. Following the convention introduced in Cachon and Terwiesch [3], we assume that we can use management's forecasting history to form a distribution around the mean seasonal demand. Using this distribution, we simulate the performance of varying demand reestimation procedures, and evaluate the value of additional orders in this uncertain environment. In each simulated season, the expected seasonal demand is sampled from the distribution of $ESD$.

In Chapter 5, we incorporate the option of marking down the selling price of the product. This option mirrors real retail decision making where markdowns are an integral part of the inventory (and marketing) decision processes. We show the value of markdowns and compare this value to the value of replenishment opportunities.

Finally, in Chapter 6, we summarize and conclude the work.
2.1 Introduction

In a traditional newsvendor-type single-period inventory model, there is only one opportunity to order a good. In an apparel retailing setting, which we focus on here, this assumption is traditionally applicable. Along with the advent of quick response (QR) retailing in the 1980s, opportunities for additional replenishment orders during the selling season became a possibility. For an introduction to QR retailing and the benefits thereof, see e.g. Hunter et al [13], and Iyer and Bergen [16]. In a general sense, QR transforms a retailer's ordering process from a simple one-order-only approach to something more complicated involving multiple periods and/or ordering opportunities. The definition of a "period" and how ordering epochs and demand distribution assumptions relate to this definition are critical in differentiating our model from existing work in this domain. A period in our model consists of a block of time defined by one demand distribution. In the model we consider here, there is one order made at
or before the beginning of the season, and there exists the opportunity for one replenishment order within the single period. In this model, all variation and/or uncertainty in the demand is captured by the demand distribution's variance.

This paper considers a single-period inventory problem, where it is assumed that there is a known distribution of demand for the entire season. We call this a single-period problem, even though we assume there may be a replenishment opportunity (for a total of two orders) during the horizon. In this model, a decision maker decides upon an initial order quantity and a replenishment order quantity before the beginning of the season. These order quantities are based on a given distribution of demand over the entire horizon. The initial order is delivered at the beginning of the season. If and when this initial order quantity is depleted during the course of the selling season, a distribution of remaining demand is formed. At that time, the replenishment order quantity is delivered instantaneously.

While we recognize that retailers do not always behave optimally, we assume that the optimal newsvendor ordering quantity is employed when only one order is available. We then compare our single-replenishment alternative against this assumed optimal behavior to develop a lower bound on profit improvement.

Our model includes standard newsvendor overage and underage costs for the retailer consisting of raw materials costs, revenue, and salvage value. The salvage value is derived from the retailer marking down the product from its original revenue to a level which assures it will be sold. Additionally, we assume any unmet demand is lost, and the retailer is charged a lost sales penalty per unit short. The retailer's objective is to maximize profit. The supplier's benefit, which we will report as profit, consists of the sum of the raw materials costs from the retailer.

In Section 2.3, we assume there is no profit sharing arrangement between the retailer and the supplier. This section serves as a basis to demonstrate the value of the replenishment opportunity to both parties. In Section 2.4, we assume there is a compensation scheme set up whereby the retailer shares the benefit coming from the replenishment opportunity with
the supplier. This sharing is sometimes necessary to entice the supplier into a replenishment arrangement since he would otherwise make more profit under a traditional newsvendor arrangement.

A critical assumption in this research is that the timing of the (potential) depletion of the initial order does not affect the replenishment order quantity decision. The demand distribution is not “updated”, i.e. it doesn't change based on what happens with the initial order, as we assume the demand distribution has been known since the beginning of the season. In other words, there are no implied assumptions regarding the distribution of demand within the season. By making the claim that there are no assumptions regarding the distribution of demand within a season, we are implicitly assuming a season with all demand occurring right at the beginning is equally as likely as a season with all demand occurring in the last \( \varepsilon \) time unit, a season with demand constant throughout, or any other conceivable “distribution” of demand throughout the season. Our assumption removes one layer of complexity that is involved in demand re-estimation, which is making assumptions of how demand is distributed across time during the season.

2.2 Literature Review

The model presented in this section contains aspects of models from the existing literature. The literature in this area makes varying assumptions regarding the definition of a “period”, and whether those periods are driven by a demand distribution or an ordering epoch. The papers we discuss here all have in common the opportunity for at least one replenishment order within the horizon, and have no more defined time intervals with their own demand distribution than they have replenishment/ordering opportunities. In other words, these are not what could easily be classified as multi-period inventory models with some form of fixed ordering costs where ordering is typically not done every period.

Four closely related existing works are those of Carlson [4, 5], and Lau and Lau [18, 19]. Carlson [4] models a situation in which one has the choice of one or two replenishment op-
opportunities in addition to the initial order. In his experimentation, Carlson utilizes a triangular
distribution to model seasonal demand, i.e. one demand distribution over the entire horizon.
He sets up a two parameter linear equation and solves his model using dynamic programming,
thus linking an initial order quantity with a replenishment order quantity. Carlson’s model
assumes the replenishment order (and delivery) occurs immediately when (and if) the initial
order is depleted. Carlson’s model is somewhat unclear in that his result assumes the replen-
ishment quantity is fixed and independent of the timing of the depletion of the initial order
quantity; however, he comments that in implementation, there is a “suitable time adjustment”
of the replenishment quantity. In effect, there is a disconnect between the analytic results and
the implementation. Carlson [4] provides no detail as to how his suitable time adjustment
is to be made. With some thought, it is apparent that any time adjustment will implicitly
assume some form of demand re-estimation. This is contrary to his assumption of a single
known demand distribution over the season. To illustrate the disconnect, imagine the results
of Carlson’s ordering policy give an ordering pair \((X_1, X_2)\), where \(X_1\) is an initial order quantity
and \(X_2 - X_1\) is the replenishment quantity. The formula in getting this policy assumes that if
and when \(X_1\) are depleted, \(X_2 - X_1\) will be ordered. According to the formula, if \(X_1\) is depleted
immediately after the start of the season or it is depleted with one second left in the horizon,
the same amount \(X_2 - X_1\) will be ordered. In implementation, however, Carlson’s [4] “time-
adjustment” linearly scales \(X_2 - X_1\) depending on how much time has passed in the season.
For example, if \(X_1\) is immediately depleted at the start of the season, the full amount \(X_2 - X_1\)
would be ordered. If, on the other hand, \(X_1\) is depleted at the end of the season, an amount
close to 0 would be ordered. Anywhere in between, the reorder quantity can be interpolated.
The confusion over Carlson’s [4] model is apparent when Lau and Lau [18] write that Carlson
“formulated the replenishment problem as one in which both \(X_1^*\) and \(X_2^*\) are to be determined
(and hence fixed) at the start of the season \([\cdots]\)”. Lau and Lau [18] also write about Carlson’s
[4] model that “a second order (always of \([X_2]\)) will be placed, regardless of how early or late in
the season the \(X_1\) units are depleted”. It is unclear whether Carlson’s numerical results are
a function of his analytical assumptions or his implementation's assumptions. Carlson's [4] implementation makes the assumption that demand is constant across the horizon; ergo, it would be a simple exercise in linear interpolation to find the actual seasonal demand. If this assumption were true, demand would be fully known after the depletion of $X_1$ (under the assumption of continuous review which is not specified in Carlson's papers).

Our single period model differs from Carlson's [4] in two ways. First, we do in fact restrict ourselves to a $(Q_1, Q_2)$ pair, where the (potential) timing of the depletion of $Q_1$ does not affect the ordering quantity $Q_2$, where $Q_1$ and $Q_2$ are the initial and replenishment order quantities, respectively. Our formula and implementation agree on this assumption, making the formula and its analytic results truly optimal under the assumptions. Our single-period model is a true single-period model with no form of demand re-estimation, either implicitly or explicitly. We make no assumptions as to the rate, whether constant or not, of demand over the season. In our model, the only thing known about demand is some given distribution over the entire season. For any given realization of demand from this distribution, we assume every possible “distribution” of that amount of demand is equally likely. This assumption is valid if the given seasonal distribution is, in fact, all that is known or given. If we are not given any information of how demand will occur throughout the horizon, then we simply cannot assume any such realization for any demand updating scheme. Secondly, our model differs from Carlson [4, 5] in that we provide analytic solutions to the optimal ordering quantities under varying demand distribution assumptions, whereas he uses a dynamic programming approach to provide problem-specific integer solutions.

Lau and Lau [18] set up a model which is carried over to Lau and Lau [19]. In their model, the “period” (season) is broken up into two “subperiods”, with the dividing point (between subperiod 1 and subperiod 2) fixed at the beginning of the season. The two subperiods are not necessarily equal in length. Demand in each subperiod is independent of the other subperiod. In their model, only the first ordering decision needs to be made before the beginning of the season. The intra-season replenishment order quantity decision doesn't need to be made
until the beginning of the second subperiod. They propose a profit equation that calculates the expected profit of the first subperiod plus the expected profit of the second subperiod. The latter half of their profit equation assumes there is an expected optimal replenishment quantity *given* an initial order quantity. While the replenishment order quantity isn't binding, one needs to know how one *would* order for the replenishment given an initial order quantity and a first subperiod demand realization. This result is then used to choose an optimal initial order quantity. Lau and Lau [18, 19] assume unmet demand in either the first or the second subperiod is lost. They show that the optimal second subperiod ordering decision is an order up to (a newsvendor-type critical fractile) policy. Depending on the remaining inventory at the beginning of the second subperiod, one would either order up to a given amount or order nothing if on hand inventory is above that amount. They show profit equations for each of the scenarios that can play out in the season (positive or zero inventory after the first subperiod). They state that the complete seasonal profit equation is a cubic function of the initial order quantity, and is therefore impractical to solve analytically. They go on to solve for it numerically and demonstrate numerical results for a series of problem instances.

Lau and Lau [19] differs from Lau and Lau [18] in three ways: by including a fixed cost of re-ordering, by varying the timing of the reorder opportunity $t_2$, and by considering different ordering costs in the two subperiods. In their first analysis, Lau and Lau ([18]), $t_2$ (the beginning of subperiod 2) is fixed at the beginning of the season. Due to the nature of the Normal distribution which they assume, closed form solutions are intractable, hence they provide numerical results. Their numerical results show that the expected profit of ordering twice is greater than that of ordering only once.

In their second analysis, Lau and Lau [19] investigate the effect of varying and optimizing $t_2$, though it is still required to be fixed at the beginning of the horizon. In order to investigate the effects of varying $t_2$, one needs to specify how the total seasonal demand mean and standard deviation vary as $t_2$ is shifted around in the season. Let $\mu$ and $\sigma$ be the seasonal demand Normal distribution parameters, and $\mu_1$ and $\sigma_1$ the Normal distribution parameters
of demand in the first subperiod. They then define a variable \( r = \frac{\mu_1}{\mu} = \frac{\sigma_1}{\sigma} \). They observe that as the standard deviation of the seasonal demand increases, \( t^*_2 \) noticeably decreases (comes earlier in the season). In their explanation thereof, they implicitly back away from the independence assumption, in that some assumptions need to be made as to how demand variance changes as \( t_2 \) changes. When the season is broken up into two subperiods, and one's information about demand distributions does not match these subperiods, then some assumptions regarding the rate/distribution of demand (especially variance) over time need to be made, which negates the independence assumption. Lau and Lau provide three submodels with different assumptions regarding the distribution of variance over the horizon.

In their third analysis, Lau and Lau [19] consider the case of a fixed ordering cost imposed if the replenishment opportunity is used. They modify the second period's order-up-to level to incorporate a threshold level, where one orders up to \( S \) if the added (expected) benefit of doing so compensates for the fixed ordering costs. In their conclusion, Lau and Lau [19] mention another variant of the problem, which allows the second order to be made (and arrive) at \textit{or anytime after} \( t_2 \), but provide no other detail.

Our model partly differs from Lau and Lau [18, 19] in the same way that it differs from Carlson [4, 5]. First, we provide analytic results for given demand distribution types. Secondly, we base our ordering decisions under the assumption that the \textit{only} knowledge available is a seasonal demand distribution. This assumption is supported by our communications with retail apparel firms that are more willing to forecast at an aggregate, or seasonal, level, rather than a day to day or week to week forecast of demand. Our model also differs from Lau and Lau [18, 19] in that our results more closely resemble a newsvendor environment in that there is only one period (or subperiod), whereas Lau and Lau [18] claim their model is a “single-period” model, but assume there are two distributions of demand within the single period. We should note that both our model and Lau and Lau [18, 19] are not truly newsvendor problems as we assume demand can occur at any time within the season, and there is only one such distribution.
A key difference between our single period model and the models by both Carlson [4, 5], and Lau and Lau [18, 19], as well as most other traditional newsvendor problems, is that we extend our model to account for an instantaneous holding charge for goods in inventory over time, which is ignored in most style goods and newsvendor problems. Newsvendor models assume demand occurs at one instantaneous point in time, which makes holding costs irrelevant, and therefore they don’t influence the optimal order decision. In our context, however, the initial order quantity determines (in an expected sense) the timing of the replenishment decision. A higher initial order quantity leads to a later replenishment decision and vice versa. Not surprisingly, we find that the introduction of holding costs does affect the optimal inventory policy. Therefore, we conclude it is unrealistic to ignore holding costs in a QR-type environment. This assumption comes with some tradeoffs (namely we are forced to make some assumptions regarding the rate of demand, which may not be realistic).

Wolfe [27] develops a method to determine the second period order quantity $Q_2$, given an (external to the model) initial order quantity, $Q_1$. The objective is to achieve a management mandated ending inventory level instead of maximizing profits or minimizing costs.

Hillier and Lieberman [11] outline a procedure for finding the optimal ordering policy in a two period inventory problem with demand characterized by a known distribution in both periods. Similar to Lau and Lau [18, 19], but unlike our model, Hillier and Lieberman [11] assume equal length subperiods. In their model, $C_1 = \min_{y_1 \geq x_1} \left\{ c(y_1 - x_1) + L(y_1) + E[C_2(x_2)] \right\}$, and $C_2 = \min_{y_2 \geq x_2} \left\{ c(y_2 - x_2) + L(y_2) \right\}$, where $y_i$ denotes the optimal order up to level for period $i$, and $x_i$ is the starting inventory at the beginning of period $i$. $x_2 = y_1 - D_1$, for realized first period demand $D_1$. $L(y_1)$ is the one period expected holding and backorder charges. Hillier and Lieberman allow for backordering any unmet period 1 demand, but not period 2 unmet demand, which is lost. They assess a negative salvage value (deemed “holding cost”) on unsold goods at the end of both periods, with no salvage opportunity at the end of period 2. The value of $y_2$ is determined by the classical newsvendor approach. They demonstrate that $C_1(x_1)$ has a unique minimum after substituting the newsvendor cost function, $C_2$ as a
function of $y_1$ and $D_1$ into $C_1$. They present a closed form solution when demand is assumed to follow the Uniform or the Exponential distribution.

Cheaitou et al [6] propose a two-period model characterized by independent demand distributions in both periods. Their model allows excess product to be salvaged at the end of both periods, as well as backlogging of unsatisfied demand.

### 2.3 No Compensation Model

In this section, we develop our models without consideration of any profit sharing schemes between the supplier and the retailer. The results of this section serve primarily to demonstrate the value of the replenishment opportunity to the retailer and the value (whether positive or negative) to the supplier. We consider a seasonal good with an assumed known total seasonal demand distribution. An initial order of $Q_1$ units is made before the beginning of the season, and a replenishment order quantity $Q_2$ is specified. If and when $Q_1$ is depleted, a replenishment order of $Q_2$ units is (must be) delivered with negligible lead time. So while the amount of the replenishment order, $Q_2$, is specified at the beginning of the season, we note that the replenishment order is not delivered if $Q_1$ is never depleted. Additionally, in the models we discuss in this section, the supplier is not compensated in any way for items not ordered. We reiterate our assumptions regarding the rate of demand over the course of the period/season. In our model, we assume that demand has no particular rate. That is, given a particular realization of seasonal demand, we make no assumption of when or how it occurs during the season.

This single-period model with replenishment assumes:

- the distribution, and its parameters, of total *seasonal* demand are known;

- there are two ordering opportunities: one initial order before the beginning of the season, and one replenishment order which is delivered when (and only if) the initial order is depleted during the horizon.
The problem of interest is how much should be ordered for the initial order and how much should be replenished if needed.

An analogous description of the problem would lead to identical results. Consider a variation of our problem where $Q_1$ is placed well in advance of the selling season and shipped via a slow and cheap mode. Shortly before the beginning of the season, the retailer is told from marketing that updated information indicates that demand will be either lower or higher than $Q_1$. The retailer then has a choice to order once again, with these products arriving immediately (or before cumulative demand reaches $Q_1$).

In this section we:

- develop analytic results when possible, and numerical when intractable, for various discrete and continuous demand distributions; and
- quantify the benefit of a replenishment opportunity for the retailer.

In developing the model, we take into account three possible inventory scenarios which can play out during the season under our assumptions. In Figure 2.1, the initial order of $Q_1$ is never depleted, and the firm salvages remaining goods at the end of the season. In the second scenario (Figure 2.2), $Q_1$ is depleted, causing a reorder of $Q_2$, which is not depleted. This scenario leads to the salvage of inventory at the end of the horizon with no lost sales. In the final scenario (Figure 2.3), both $Q_1$ and $Q_2$ are depleted. This is the only scenario that results in lost sales.

An interesting by product of our assumptions, and one that aids in the building of the model, is that the specification of $Q_2$ is not restrictive or hindering to the retailer. In other words, whether $Q_2$ needs to be specified at the beginning of the season, or if it is specified during the season immediately before it is delivered does not change the optimal replenishment order quantity.

Let us first consider the replenishment order quantity. Under our assumption of a known distribution of total seasonal demand, there is no reason or benefit of revising the demand
Figure 2.1: No replenishment

Figure 2.2: Replenishment with salvage
distribution, i.e. forecast, once $Q_1$ is depleted. This dictates an application of the law of total probability/Bayes’ rule. This method forms the only truly accurate distribution of remaining demand from the time of the replenishment to the end of the horizon without changing the assumptions of an initial known seasonal demand distribution. Let $g(\xi')$ be the probability of observing $\xi'$ units of demand between the replenishment time and the end of the horizon. Also, define $f(\xi)$ and $F(\xi)$ as the probability density and cumulative probability distribution, respectively, of observing $\xi$ units of demand over the entire season. Then $g(\xi') = f(\xi' + Q_1 | \xi \ge Q_1) = \frac{f(Q_1 + \xi')}{1 - F(Q_1)}$. The optimal replenishment ordering decision is merely an application of the simple newsvendor problem, whose solution is widely known as $\int_0^{Q_2} g(\xi')d\xi' = \frac{r + p - c}{r + p - s}$, where $r =$ unit revenue, $p =$ unit shortage penalty, $c =$ unit (materials) cost, and $s =$ unit salvage value.

To aid in building the models that follow, we make the distinction here that $c_1$ is the unit materials price of the initial order, and $c_2$ its replenishment order counterpart. As such, the second order decision is more specifically $\int_0^{Q_2} g(\xi')d\xi' = \frac{r + p - c_2}{r + p - s}$. In order to solve for an optimal $Q_1$, we need to know what decision would be made if the firm were to run out of $Q_1$. As stated, this decision is a true newsvendor problem under our assumptions. Thus, we know

Figure 2.3: Replenishment and lost sales
the decision that would be made if \( Q_1 \) were depleted sometime during the season. Using the law of total probability, it can be shown (Appendix Section A.1) that the optimal replenishment order quantity is

\[
Q_2^* = F^{-1} \left\{ F(Q_1) + \frac{r + p - c_2}{r + p - s} \left[ 1 - F(Q_1) \right] \right\} - Q_1
\]

(2.1)

Intuitively, one would order the critical single period newsvendor fractile of remaining demand. We see the replenishment order quantity is merely a function of \( Q_1 \) and constants, so we use \( Q_2 \) and \( Q_2|Q_1 \) interchangeably. A similar result would occur if we solved a traditional newsvendor problem with a conditional demand distribution instead of the unconditional one. A total seasonal profit equation as a function of the single variable \( Q_1 \) can now be developed. Let \( Z(Q_1) \) be the expected seasonal profit with an initial order of \( Q_1 \) units. As mentioned previously, if and when the initial order is depleted, a replenishment of \((Q_2|Q_1)\) units will be made. It is only in this case that the decision maker will pay material costs (variable ordering costs) for \( Q_2 \), i.e.

\[
E[Z(Q_1)] = -c_1 Q_1 + \int_0^{Q_1} \left[ r \xi + s (Q_1 - \xi) \right] f(\xi) d\xi + \int_{Q_1}^{Q_1+Q_2} \left[ -c_2 Q_2 + r \xi + s (Q_1 + Q_2 - \xi) \right] f(\xi) d\xi + \int_{Q_1+Q_2}^{\infty} \left[ -c_2 Q_2 + r (Q_1 + Q_2) - p (\xi-(Q_1+Q_2)) \right] f(\xi) d\xi
\]

(2.2)

Due to the fact that all integrands in Eq. 2.2 are linear and therefore form a concave combination, the equation is also concave and we can find the optimal initial order quantity \( Q_1 \) which
maximizes Eq. 2.2 at the point where:

\[-c - c * Q_2 \frac{d}{dQ_1} + Q_2 \frac{d}{dQ_1} \cdot F(Q_1)(c - s) + s F(Q_1 + Q_2) \left(1 + Q_2 \frac{d}{dQ_1}\right) + Q_2 f(Q_1)(c - s) + (r + p) \left(1 + Q_2 \frac{d}{dQ_1}\right) (1 - F(Q_1 + Q_2)) = 0\]  \hspace{1cm} (2.3)

Eq. 2.3 assumes an unbounded, continuous demand distribution. Suitable changes are easily made should any of these assumptions not hold. A quick glance at Eq. 2.2 shows that the replenishment model's expected profit can be no worse than the expected newsvendor profit, since setting \( Q_2 = 0 \) results in the latter.

### 2.3.1 Results for uniformly distributed demand

Under the assumption that seasonal demand is distributed according to a Uniform distribution with parameters \((a, b)\), combining Eq. 2.1 and Eq. 2.2 yields an analytical solution for \( Q_1^* \). Let \( z = \frac{r + p - c_2}{r + p - s} \), the standard newsvendor critical fractile representing a type 1 service level.

\[Q_1^* = \frac{-c_1 (b - a) - sa + b (r + p) - b z (r + p - c_2)}{r + p - s - z (r + p - c_2)}\]  \hspace{1cm} (2.4)

Under the aforementioned assumptions, if and when these \( Q_1 \) units are depleted, a replenishment order of \((Q_2|Q_1)\) units is made. For the Uniform distribution with parameters \((a, b)\), \((Q_2|Q_1)^*\) can also be derived analytically.

\[(Q_2|Q_1)^* = z (b - Q_1)\]  \hspace{1cm} (2.5)

The replenishment order is simply the newsvendor critical fractile of remaining maximum possible demand after \( Q_1 \) units have been sold.

It can be shown that when the lower limit of demand \( a = 0 \), Eq. 2.4 and Eq. 2.5 specify an
optimal solution $Q_1 = Q_2 = b \frac{z}{1+z}$. In other words, the optimal policy specifies that the two ordering decisions are identical, with probability of replenishment $1 - F(Q_1) = \int_{Q_1}^{b} \frac{1}{b-a} \, dx = \frac{1}{b} \left( b - b \frac{z}{1+z} \right) = 1 - \frac{z}{(1+z)}$. However, this does not hold when $a > 0$.

We also consider a numerical analysis of varying problem instances with parameter bounds listed in Table 2.1.

Table 2.1: Experimental set

<table>
<thead>
<tr>
<th></th>
<th>low</th>
<th>middle</th>
<th>high</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$, revenue</td>
<td>1.75</td>
<td>2.5</td>
<td>3.25</td>
</tr>
<tr>
<td>$s$, salvage</td>
<td>0</td>
<td>0.4</td>
<td>0.8</td>
</tr>
<tr>
<td>$p$, lost sales penalty</td>
<td>0</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>$c_1 = c_2$, materials cost</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For each problem instance, we compare the expected seasonal profit of a newsvendor single-order solution to that of our proposed replenishment opportunity model. In this
Figure 2.5: Comparing expected total units ordered over, 27 instances, Uniform distribution analysis, $F(\xi) \sim U(10, 100)$. Across these instances, we find the average percentage increase in seasonal profit by allowing the option of a replenishment to be 15.4%, with a median improvement of 13.2%. A histogram of the profit improvement is shown in Figure 2.4.

An interesting result is the comparison of total units ordered. For the newsvendor problem with uniform demand between $(a, b)$, total units ordered is simply $a + \frac{r + p - c_1}{r + p - s} (b - a)$. When a replenishment order opportunity is allowed, the expected number of total units ordered is $Q_1 + Q_2 (1 - F(Q_1))$. Numerically, we can find the percent change in total expected units ordered. These results are summarized in Figure 2.5. The results show that in low service level cases, the replenishment opportunity results in more goods being ordered compared to the single order newsvendor case. Problem instances with increasing service level show a trend that the replenishment opportunity “allows” fewer units to be ordered on average compared to the newsvendor solution, while still increasing profits and maintaining service levels.

Comparing the newsvendor order quantity with $Q_1$ for the replenishment problem, we find that the average difference over the problem instances tested was 36%. In other words, with the possibility of a replenishment opportunity during the season, $Q_1$ decreases by an
average of 36%. If \( Q_1 \) is depleted in the replenishment case and a subsequent order of \( Q_2 \) is made, the average increase in the total units ordered during the season is 13% over the simple newsvendor order quantity.

We conduct an analysis of variance (ANOVA) test to determine which of the factors are most important in determining the value of the replenishment opportunity. In addition, we conduct a similar ANOVA test against the percentage increase in the number of goods sold with the replenishment opportunity, as well as the percentage decrease in lost sales with the replenishment opportunity. These results are highlighted in Table 2.2. It is worth noting that revenue and salvage value were considered significant at the 0.05 significance level, whereas the lost sales penalty, \( p \), was not. This results stems from the fact that the lost sales penalty is small compared to the lost revenue of missing a sale. In other words, missing the revenue is enough of a deterrent for the model to avoid missing sales.
Figure 2.6: ANOVA results for % profit increase with replenishment, salvage value term

Visually, we see in Figure 2.6-Figure 2.14, the same percentage changes as shown collectively in Table 2.2. The figures are separated by cost levels.
Figure 2.7: ANOVA results for % profit increase with replenishment, lost sales penalty term
Figure 2.8: ANOVA results for % profit increase with replenishment, revenue term
Figure 2.9: ANOVA results for % sales increase with replenishment, revenue term
Figure 2.10: ANOVA results for % sales increase with replenishment, lost sales penalty term
Figure 2.11: ANOVA results for % sales increase with replenishment, salvage value term
Figure 2.12: ANOVA results for % lost sales decrease with replenishment, revenue term
Figure 2.13: ANOVA results for % lost sales decrease with replenishment, lost sales penalty term
Figure 2.14: ANOVA results for % lost sales decrease with replenishment, salvage value term
2.3.2 Results for Exponentially distributed demand

In this section, we assume that the seasonal demand distribution is exponential with a known rate parameter $\lambda$. With this information, we can build a profit equation similar to the Uniform distribution in section 2.3.1. The memoryless property of the Exponential distribution manifests itself in this model in an interesting, potentially unrealistic, manner. To illustrate this behavior, imagine $Q_1$ have been ordered initially and depleted. Thus, our best estimate of demand for the remainder of the season, $g(\xi')$, is $f(\xi' + Q_1 | \xi \geq Q_1)$, where $f(\xi)$ is the seasonal demand pdf, and $g(\xi')$ is the pdf of remaining demand. In the case of the Exponential distribution, $g(\xi') = f(\xi' + Q_1 | \xi \geq Q_1) = f(\xi'')$ by assumptions of the memoryless property of the Exponential distribution. Thus, our estimate of remaining demand after running out of $Q_1$ is identical to our original estimate of total seasonal demand, regardless of how much demand we’ve already seen. Analytical results can be obtained for $(Q_2^* | Q_1^*)$ and $Q_1^*$. Eq. 2.6 and Eq. 2.7 assume $c_1 = c_2 = c$.

\[
(Q_2 | Q_1)^* = \frac{1}{-\lambda} \ln \left[ \frac{c-s}{r+p-s} \right] \quad (2.6)
\]

\[
Q_1^* = \frac{1}{-\lambda} \ln \left[ \frac{1}{1 - \ln \left[ \frac{c-s}{r+p-s} \right]} \right] \quad (2.7)
\]

We immediately spot similarities to the elements of the classical newsvendor fractile, $\frac{c_o}{c_o + c_u}$, where $c_o = c - s$, and $c_u = r + p - c$. Additionally, the memoryless property of the Exponential distribution is evident by noticing that $Q_2$ is independent of $Q_1$.

2.3.3 Results for Normally distributed demand

While the Normal distribution does not lend itself to closed form analytic solutions, it does provide valuable insight into the behavior of the model for a well-behaved, unimodal, unbounded distribution with central tendency. In Figure 2.15 and Figure 2.16 we plot the value of $Q_1^*$ and $Q_2^*$ as the standard deviation, $\sigma$, of the distribution increases. When $\sigma = 0$, the decision
maker should clearly order the deterministic demand amount, \( \mu \). Figure 2.15 and Figure 2.16 show the optimal ordering behavior with low (\$0) and high (approaching \( c \)) salvage value, respectively. In either case, \( Q^*_2 \) is more sensitive to changes in \( \sigma \) than \( Q^*_1 \). However, with a lower overage cost (i.e. \( s = 0 \)), the initial order is more likely to decrease rather than increase with increasing \( \sigma \).

### 2.3.4 Results for Negative Binomial demand distribution

The Negative Binomial distribution, unlike the normal distribution, is a discrete distribution which is bounded below by zero. This more accurately models demand for our purposes, especially within the context of fashion retailing. It also allows us to analyze the effect of changes in the variance of demand. The three distributions we consider for illustrative purposes have a mean of 55. We use the same 27 cost instances as in Section 2.3.1, shown in Table 2.1. The 3 different distributions and their corresponding Negative binomial shape parameter are shown in Figure 2.17, which also specifies the negative binomial probability of success parameter.

An interesting analysis here is to compare the effect on the order quantity when a replenish-
Figure 2.16: Optimal order quantities with high $s$, $D \sim N(30, \sigma)$, $c = 1$, $r = 2$, $p = .3$

Figure 2.17: Three Negative binomial distributions considered, mean=55
The replenishment opportunity is available. Figure 2.18 shows the expected (total) order quantity, weighted by the probability that a replenishment is triggered, \( Q_1 + Q_2 (1 - F (Q_1)) \) for the 27 instances outlined above. As seen, when service level is below 75%, the expected total order quantity increases when a replenishment is allowed compared to the newsvendor order quantity. Alternatively, when service level is above 75%, the expected total order quantity decreases when a replenishment is allowed. The replenishment reduces overall underage costs in low service level cases, while lowering overage costs in higher service level cases. This effect is magnified for higher variance cases. As seen in Figure 2.18, there are numerous cases where the retailer will order fewer total units in expectation when given the benefit of a replenishment opportunity. In these cases the supplier would make less profit, and presumably would never enter into a replenishment contract to begin with. In such cases, the retailer will need to compensate the supplier to entice it to enter into a replenishment contract with the retailer. It is this situation that we consider in the next section.

Figure 2.19 shows the value of the replenishment opportunity for the 27 instances for the retailer, the supplier, and as a total profit between the retailer plus the supplier. As seen, there
are some instances where the total supply chain profit is negative, where the retailer alone would benefit from the replenishment opportunity.

## 2.4 Compensation Model

There are two primary motivations for the analysis in this section. The first, as explained in the previous section, is that the supplier might not be interested in a replenishment contract without being compensated in some way by the retailer. Secondly, it stands to reason that items purchased for the replenishment order are, or should be, in some way more expensive than items purchased in the initial order. This can be due to many factors, including but not limited to logistics differences or contract specifications.

The first retailer-supplier compensation arrangement we consider imposes a premium on goods ordered during the replenishment order by having the retailer pay an up-front cost for the option of placing a replenishment order sometime during the season. This cost is non-refundable, and can be viewed as a way to compensate the supplier to reserve pro-
duction capacity. The second compensation arrangement, representing the classical fixed ordering cost, is to impose a fixed cost of $A$ in addition to the unit costs, paid by the retailer to the supplier, if and when a replenishment order is triggered. This method is applicable to the case of a large supplier who has ample production capacity to meet a relatively smaller retailer’s replenishment demand, but nevertheless requires a (fixed) setup fee or compensation for expedited shipping. The third compensation arrangement we consider is a rather straightforward arrangement of simply setting $c_2 \geq c_1$, which we call the higher unit cost arrangement. This arrangement is perhaps the simplest, and as we will show, is also the most flexible. The fourth compensation arrangement we consider is the idea that a “down payment” of $k \cdot c_2 \cdot Q_2$, $0 \leq k \leq 1$, must be made at the beginning of the season. If $Q_1$ is depleted during the season, the decision maker must pay the remaining amount $c_2 \cdot (1 - k) \cdot Q_2$. The down payment is lost if $Q_2$ are never ordered. This arrangement could be used in the case of a supplier that reserves production capacity on a “per unit” basis. We present general formulations for each of the approaches considered, and numerical results that shed insight into the strengths and weaknesses of each in the eyes of the supplier and retailer. For each arrangement, we also consider “negotiation ranges”, which determine the minimum and maximum level of the parameter, such that both supplier and retailer maintain at least their respective newsvendor profits. These ranges are important, because as we point out, offering a replenishment opportunity without compensation often leads to lower supplier profits. In some cases, we see lower supply chain profits (summation of supplier and retailer profits) when a replenishment opportunity is offered. In these cases, there is no possible incentive (across our compensation arrangements) that entices both the retailer and supplier to enter into a replenishment contract, and presumably, a newsvendor-type contract is maintained. These scenarios may be considered non-Pareto optimal, in that there is no scheme in which both retailer and supplier benefit. Throughout this analysis, it is assumed that the retailer and supplier are separate entities, such that no centralized planning can be achieved. In all cases, the retailer sets his optimal order quantity to maximize his own profits. We assume
the supplier fully knows, or is able to predict, the retailers optimal order quantities under any scenario.

As shown in Figure 2.5, the retailer’s expected order quantity can either increase or decrease when going from a traditional newsvendor approach to a replenishment contract. For the Uniform distribution instances in Figure 2.5, low newsvendor critical fractile instances actually result in a higher order quantity, while problem instances with medium to high service levels result in fewer goods being ordered in expectation in the replenishment contract than a newsvendor setting. If we consider the supplier’s profit as simply the margin multiplied by the expected units bought by the retailer, it is easy to see why the supplier needs no incentive to enter into a replenishment contract for low service level cases. On the other hand, when the supplier stands to lose significant revenue in high service level cases, he is reluctant to enter into a replenishment contract without adequate compensation. Under the very reasonable assumption which we make throughout this analysis, namely that the retailer can choose between a replenishment contract or a traditional newsvendor contract before the season begins, his expected profit is bounded below by the newsvendor expected profit.

### 2.4.1 Option compensation arrangement

Under this arrangement, \( A \) is paid by the supplier to the retailer before the beginning of the season. This payment, similar to an American call option, gives the retailer the right, but not the obligation to request a replenishment order of any amount sometime during the season. The supplier is required to deliver all goods requested. An upper bound on \( A \), or the most that the retailer is willing to offer to the supplier to entice him to enter into a replenishment contract is simply the expected gain the retailer expects from the replenishment opportunity, or the difference between Eq. 2.2 evaluated at \( Q_1^* \) and the newsvendor expected profit. A lower bound for this option requires a similar calculation for the supplier with the caveat that the lower bound may actually be negative, such as in the low service level instances shown in Figure 2.5, where expected order quantities increase under a replenishment contract. In these
cases, it may be possible that the supplier benefits more from the replenishment opportunity than the retailer.

**Uniform distribution**

Figure 2.20 shows the benefit of the replenishment opportunity (compared to the newsvendor solution) for both the supplier and retailer, which form the upper and lower bound of the option price for the 27 uniformly distributed sample instances ordered by their respective newsvendor critical fractiles. In cases where the overall supply chain benefit is negative, there is no option price that is high enough to increase the supplier’s expected profits, while being low enough to not make the replenishment prohibitively expensive for the retailer. A closer look at these cases reveals an interesting insight. These cases are high critical fractile instances, with a relatively large underage cost compared to overage costs. These cases, by virtue of a high newsvendor critical fractile, have large newsvendor order quantities. This seems to be a great situation for the supplier, who is “guaranteed” high profits in the newsvendor case. If the retailer were to be given a replenishment opportunity, the expected total order quantity drastically reduces, reducing the supplier’s profits considerably. While still beneficial, the replenishment opportunity for the retailer is not that valuable because the high service level dictates a relatively high $Q_1$ (compared to lower service levels) and lower probability of a replenishment being triggered.

**Negative binomial distribution**

Figure 2.21 shows the retailer’s maximum option payment that the retailer would be willing to pay to enter into the option contract, as well as the minimum that the supplier would accept to enter into the contract. As the figure shows, there are a number of high service level cases where the retailer and supplier will not be able to find a suitable option payment they can both agree to.
Figure 2.20: Retailer’s and supplier’s profit change (in %), representing their maximum and minimum option price requirements, 27 instances with Uniform demand distribution.

Figure 2.21: Retailer’s maximum and supplier’s minimum option price to enter into replenishment contract, 27 instances with Negative binomial distribution and medium variance.
2.4.2 Fixed cost compensation arrangement

This compensation arrangement is similar to the options method, except the fixed payment from the retailer to the supplier is incurred if and only if \( Q_1 \) is depleted and subsequently a replenishment order for \( Q_2 \) units is placed. This model can also be seen as a more generalized and traditional version of the model presented in Section 2.3 to include a burden (fixed cost in this case) of the retailer utilizing the replenishment opportunity. In this section, we assess this fixed replenishment cost in the traditional manner, where it is assumed there is an explicit and known monetary expense associated with placing an order. If \( Q_1 \) is indeed depleted intra-season, then we are assessed a fixed ordering cost \( A \). With this fixed replenishment order cost we would expect the model to shy away from reordering, and therefore, order more initially. Indeed, this intuition holds. Let \( Q'_1 \) be the optimal initial order quantity assuming a fixed order cost would be assessed if a replenishment order were triggered. Under this assumption, the formula for the reorder amount, \( Q_2 \) remains unaffected by the fixed reorder cost, although \( Q_2 \) will decrease as \( Q_1 \) increases due to the nature of Eq. 2.1.

**Uniform distribution**

Continue to let \( Q_1^* \), from Eq. 2.4, be the retailer’s optimal initial order quantity under a replenishment contract without a fixed replenishment order cost, and \( Q_2^* \) from Eq. 2.5 be the retailer’s optimal replenishment order quantity without a fixed replenishment cost. Again, let

\[
z = \frac{r + p - c_2}{r + p - s}.
\]

\[
Q'_1 = Q_1^* + \frac{A}{r + p - s - z} \frac{r + p - c_2}{r + p - s} \tag{2.8}
\]

\[
\left( Q'_2 | Q'_1 \right) = Q_2^* - z \frac{A}{r + p - s - z} \frac{r + p - c_2}{r + p - s} \tag{2.9}
\]

The result indicates that with a fixed cost to place a replenishment order, one would order proportionately more goods initially compared to a replenishment model with no fixed costs.
to order. The probability of a reorder is therefore lowered. However, if a reorder is triggered, the reduction in \( Q_2' | Q_1' \) would only make up for a fraction of the increased initial order.

Under our assumption that the supplier receives the full amount, \( A \), if and only if a replenishment order is triggered, his expected profit becomes \( c_1 Q_1' + (A + c_2 \cdot Q_2') \frac{b-Q_1'}{b-a} \). Figure 2.22 shows the negotiating range specified by the supplier's lower bound and retailer's upper bound of the fixed replenishment cost for the 27 uniformly distributed instances with \( c_1 = c_2 = c = 1 \). Again we see that some high service level instances lead to a non-Pareto solution.

**Negative binomial distribution**

Consider Figure 2.23 which shows the supply chain profit (retailer plus supplier) of the same instances as Table 2.1 with mean demand 55 and medium variance (negative binomial success parameter .1). In this case, the figure shows the range of negotiation when the retailer and supplier could agree to a fixed cost of replenishment. Of note is that the negotiating range generally decreases as the newsvendor critical fractile, or service level, increases. We note that
there are some cases, particularly in higher variance cases, when the two supply chain players will not be able to agree to a fixed replenishment cost.

2.4.3 Downpayment compensation arrangement

This compensation arrangement involves setting $c_2 \geq c_1$, as well as requiring a non-refundable portion of $c_2 \cdot Q_2$ to be paid at the beginning of the season. Suppliers may prefer the downpayment as it can be considered a payment for them to reserve the capacity necessary to produce/deliver $Q_2$ units at a later time. Let $(1 - k)$ be the proportion of $c_2 \cdot Q_2$ that is required as a downpayment at the beginning of the season. If and when $Q_1$ is depleted during the season, the retailer pays $c_2 \cdot Q_2 \cdot k$ to receive a replenishment order of $Q_2$ units.

Uniform distribution

Let $z = \frac{r + p - k c_2}{r + p - s}$, which is the newsvendor critical fractile facing the retailer at the time of the replenishment. This simple newsvendor problem at the replenishment time after $Q_1$ units

Figure 2.23: Negotiation range for fixed replenishment cost from retailer to supplier, 27 instances with Negative binomial distribution and medium variance
Figure 2.24: Negotiation range for down payment method, 27 instances with Negative binomial distribution and medium variance

have been depleted yields the standard solution that \( Q_2^* = z(b - Q_1) \). We find that:

\[
Q_1^* = \frac{(-c_1 + c_2(1-k)z)(b-a) + b(r+p) - zb(r+p-kc_2)}{r+p-s-z(r+p-kc_2)}
\]  

\( (2.10) \)

**Negative binomial distribution**

In Figure 2.24, we show the maximum down payment proportion that a retailer would be willing to pay and the minimum down payment proportion that a supplier would accept to enter into a replenishment contract. As the figure shows, there are several instances which would lead to an impossible negotiation for the replenishment opportunity using this method. These cases tend to be higher service level cases.

**2.4.4 Higher unit cost compensation arrangement**

A simple way to place a premium on replenished goods is to let \( c_2 \geq c_1 \), such that the retailer is discouraged from ordering as much in the replenishment as he would if \( c_1 = c_2 \). Indeed, this
approach has been analyzed before in the contract literature. Contrary to our model where the retailer and supplier act in their own interest, Donohue [7] evaluates supply contracts with two supply modes (cheap initial and expensive replenishment) where the goal is to achieve channel optimal solutions. They find that a contract of the form $c_1, c_2, x$, where $x$ is a supplier buyback price given to the retailer for leftover goods at the end of the season, can be made to achieve channel optimal behavior between the retailer and supplier. Similar to our model, Donohue [7] retains a single period/ single demand distribution nature. Contrary to our model, however, Donohue [7] makes no assumptions regarding the form of demand updating. He specifically writes in a footnote:

> How market information is gathered and translated into a second-stage forecast is not the focus of our study. Therefore, we use a general model in our analysis, which makes few assumptions on the form of this new market information or its relationship to final demand.

This higher unit cost method has one distinct advantage over the other three methods we consider here. This method, by virtue of being a variable function of the order quantity, $Q_2$, is the only method of the four that guarantees convergence to the newsvendor method when the premium becomes too high for the retailer. The proof is rather straightforward. Consider the replenishment order quantity decision $Q_2^* = F^{-1}\left[F(Q_1) + \frac{r+p-c_2}{r+p-s} [1-F(Q_1)]\right] - Q_1$. As $c_2 \to r+p$, we find that $Q_2 \to 0$. The $Q_1$ decision remaining is a simple newsvendor profit equation, guaranteeing a newsvendor order quantity and expected profit. The benefits of this property are that the retailer and supplier are guaranteed to agree on at least one value of $c_2$, where they both achieve at least their respective newsvendor profit, occurring at $c_2 = r+p$.

**Uniform distribution**

Using the same basic assumptions of section 2.3.1, and letting $c_2 = k c_1$, and $z = \frac{r+p-c_2}{r+p-s}$, it is easy to show that
\[ Q_1^* = \frac{-c_1(b-a) - sa + b(r+p) - b(c_2)z(r+p-c_2)}{r+p-s-z(r+p-c_2)} \] (2.11)

Greater insight into \( Q_1^* \) and \( Q_2^* | Q_1^* \) can be gleaned if we let \( s = 0 \). In that case:

\[ Q_1^* = b - (b-a) \frac{1}{k} \left[ \frac{r+p}{2(r+p)-c_1} \right] \] (2.12)

and

\[ Q_2^* | Q_1^* = (b-a) \frac{1}{k} \left[ \frac{r+p-c_2}{2(r+p)-c_2} \right] \] (2.13)

Comparing Eq. 2.12 to its earlier \( c_1 = c_2 \) counterpart Eq. 2.4, it can be seen that they are proportional to one another by a factor of \( \frac{1}{k} \), the purchase price ratio, when \( s = 0 \). A decision maker would like to decrease the probability of a reorder and avoid the higher \( c_2 \). If a reorder is required, the decision maker would order less at the higher \( c_2 \) than in the regular replenishment case.

**Negative binomial distribution**

Similar to the uniform case, we evaluate here the effect of increasing \( c_2 \) on the retailer’s order quantities and the profit of both the supplier and retailer. The negative binomial distribution allows us to additionally model the effect of altering the variance of the demand distribution. Figure 2.25 shows the negotiating range, i.e. the supplier determined minimum and retailer determined maximum level of \( c_2 \).

Figure 2.25 also shows how the discrete nature of the negative binomial distribution affects results such as the negotiation range for \( c_2 \). The total expected order quantity is not necessarily monotonically decreasing as \( c_2 \) increases. Since supplier profits go up as more units are purchased from the supplier, it is clear how supplier profits themselves can fluctuate as the order quantity fluctuates.
Figure 2.25: Negotiation range for higher replenishment unit cost method, 27 instances with Negative binomial distribution and medium variance

### 2.4.5 Comparing methods

We have shown that a retailer can do no worse than his newsvendor profit given a replenishment opportunity. The supplier, however, can either gain or lose profit when entering a replenishment contract. As such, the supplier might not enter into the replenishment contract without some form of compensation. As we have shown, there are several methods available to retailers and suppliers entering into a replenishment contract which apply a premium on goods ordered in the replenishment opportunity, and this might entice the supplier to enter in the contract. We have outlined a procedure which can allow retailers and suppliers to know the range of a certain method’s parameters such that both retailer and supplier gain from the replenishment opportunity. The ranges where the retailer and supplier agree are highly situationally dependent, and as such we don’t attempt to locate the exact set of parameters that achieves some definition of optimality.

An interesting analysis, which our procedures are able to illustrate, is the comparison of two particular methods. As an example, Figure 2.26 and Figure 2.26 show two comparisons of the
Figure 2.26: Retailer profit equivalency between fixed cost and higher unit cost compensation arrangement comparison, Uniform distribution

parameters of two different methods, that achieve the same retailer profit. This type of analysis is useful for a retailer deciding between two potential supplier agreements. Each line in the figure represents one problem instance. The line is formed by matching the parameters on each axis that achieve the same profit in their respective environments. Where each line ends shows the maximum of each parameter where the retailer still benefits from the replenishment opportunity. If either the x-axis or y-axis parameter extends beyond this maximum, the retailer would achieve higher profits from the newsvendor solution than the replenishment solution. Furthermore, a decision maker can quickly and easily see which parameters from one method achieve higher (or lower) profits than a given parameter from a different method, helping the retailer in potential negotiation situations where one method needs to be compared to another method.
Figure 2.27: Retailer profit equivalency between fixed cost and down payment compensation arrangements, Uniform distribution

### 2.5 Other Embellishments

In this section, we examine embellishments of the single-replenishment problem facing a retailer. We examine the Uniform distribution when salvage value is negligible, as is often the case when deep discounts are needed to move product which are not popular with the customers. We also examine other limiting behavior of the Uniform distribution to gain insight into the effect of the parameters on the optimal initial and replenishment ordering quantities.
2.5.1 Uniform distribution

Zero salvage value simplification

Under the assumption that the goods in question have no salvage value, $Q_1$ and $(Q_2|Q_1)$ simplify to

$$Q_1^* = b - (b - a) \frac{r + p}{2(r + p) - c}$$

$(Q_2^*|Q_1^*) = (b - a) \frac{r + p - c}{2(r + p) - c}$

With no salvage value, underage costs are represented by $r - c + p$ and overage costs by $c$ only. To gain a better understanding of the relationship between $Q_1$ and $(Q_2|Q_1)$, we can look at their behavior at the extreme points of the variables.

Penalty cost limit behavior

We can also examine the behavior of $Q_1$ and $Q_2$ when the lost sales penalty, $p$ approaches 0.

$$\lim_{p \to 0} Q_1 = b - (b - a) \frac{r}{2r - c}$$

$$\lim_{p \to 0} Q_2 = (b - a) \frac{r - c}{2r - c}$$

Not unexpectedly, Eq. 2.16 and Eq. 2.17 show that when $s = p = 0$, $r$ must be greater than $c$ for the newsvendor results to hold (and for any business to be sustainable).

When $p$ becomes prohibitively expensive ($p \to \infty$), as shown in Eq. 2.18 and Eq. 2.19, one would expect to never risk a stock out. In the case of the Uniform distribution with a replenishment option, the initial order would be for exactly the expected value of demand, leaving a 50% chance of a replenishment being triggered. If the replenishment is indeed triggered, the replenishment order quantity would be exactly the amount of maximum possible remain-
ing demand, leaving no chance for a stock out. Most interesting is the optimal initial order quantity $Q_1$. For the initial order, underage costs are zero since there is another opportunity to order. However, overage costs are $c$. This mismatch does not cause the model to under order (in expectation) for the initial order. Instead, the optimal initial order quantity is such that the probability of replenishment is exactly equal to the probability of not replenishing and facing overage costs.

$$\lim_{p \to \infty} Q_1 = \frac{a + b}{2}$$

$$\lim_{p \to \infty} Q_2 = \frac{b - a}{2}$$

**Unit order cost limit behavior**

Assuming that $c_1 = c_2 = c$, Eq. 2.18 and Eq. 2.19 hold when $c \to 0$. This implies that when $c = 0$ there is no reason to risk a stockout as overage costs are zero. Again, we see the same breakup of $Q_1$ and $Q_2$, where the expected value of demand is ordered in the initial order, and if a replenishment is triggered, the maximum possible remaining demand is ordered. Also not surprisingly, when $c \to \infty$, there is no reason to stay in business. $c \to r$ provides better insight into the behavior of the results.

$$\lim_{c \to r} Q_1 = b - (b - a) \frac{r + p}{2p + r}$$

$$\lim_{c \to r} Q_2 = (b - a) \frac{p}{2p + r}$$

The results indicate that, as the profit margin shrinks, $Q_1$ remains between $a$ and average demand, $\frac{1}{2}(b + a)$. Similarly, $\lim_{c \to r} Q_2 \leq \frac{1}{2}(b - a)$. In other words, as the profit margin shrinks one would never initially order nor reorder more than half of maximum demand. Additionally, as $c \to r$ in the limit and $p \ll r$, then $Q_1 \to a$ and $(Q_2|Q_1) \to 0$. Conversely, if $p \gg r$, then $Q_1 \to \frac{a + b}{2}$ and $(Q_2|Q_1) \to \frac{1}{2}(b - a)$. 

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2.5.2 Holding cost model

Another generalizing aspect to the model is the inclusion of a holding cost assessed on the amount of inventory. While the newsvendor problem ignores this cost because the single period time frame is negligible, in our context the time frame of an entire season makes holding costs no longer negligible. Additionally, a holding cost can simply be viewed as a negative salvage value in a true, single-period, short-selling-season newsvendor problem. In our model, however, the season is longer, and it stands to reason that a cost for maintaining inventory over the season should be imposed as a function of the length of time that item has gone unsold. The assumptions of the model in this section include all of those discussed in Section 2.3, as well as a holding cost on each item in inventory per unit time.

With the introduction of holding costs, we need to make provisions regarding the timing of the depletion of the initial order, and the general shape (i.e. rate) of demand over time. Here we reiterate our divergence from classical newsvendor assumptions. Contrary to the assumptions made in Section 2.3, we need to explicitly incorporate into the model the fact that a higher initial order will, in expectation at least, cause depletion to occur at a later point in the season. Additionally, we need to make assumptions regarding the level of inventory over time to be able to accurately assess holding costs. To accommodate this, we make an assumption of a constant demand rate throughout the season. Though unknown until the very end of the season, let $\xi$ be the total seasonal demand observed, and assume that the demand rate was constant throughout the season. The decision maker does not assume that demand is occurring at a constant rate. A similar assumption was made in [4, 5]. As we mention earlier, if the decision maker knew this to be the case, then the true seasonal demand could be known with certainty at the point $Q_1$ is depleted. In an extreme example, we could order only 1 unit initially, and depending when it runs out, determine the total seasonal demand with certainty. Due to this unrealistic nature of the problem, we assume the decision maker does not know the demand rate is constant. Define

- $T =$ the season length, without loss of generality assume unit length
• as in Figure 2.1-Figure 2.3:
  
  - \( t_1 \) = the time of the initial order depletion (if \( Q_1 \) is depleted, \( t_1 = T \frac{Q_1}{\xi} \))
  
  - \( t_2 \) = the time of the reorder depletion (if \( Q_2 \) is also depleted, \( t_2 = T \frac{Q_2 + Q_1}{\xi} \))

• \( h \) = holding cost per item if it were held for the entire season

• \( Z^1(Q_1, Q_2) = \) profit equation for entire season

• \( Z^2(Q_2|Q_1, t_1) = \) profit equation covering the time from \( t_1 \) until the end of the season.

Eq. 2.22 shows the seasonal profit equation which incorporates traditional newsvendor elements of overage and underage costs, as well as holding costs in the three seasonal inventory scenarios.

\[
Z^1(Q_2, Q_1) = -c Q_1 + \int_0^{Q_1} \{ r \xi + s (Q_1 - \xi) \} f(\xi) d\xi \\
+ \int_{Q_1}^{Q_1+Q_2} \{ r \xi + s (Q_1 + Q_2 - \xi) f(\xi) \} d\xi + \int_{Q_1+Q_2}^{\infty} \{ r (Q_1 + Q_2) - p (x - Q_1 - Q_2) \} f(\xi) d\xi \\
- h \int_0^{Q_1} \left( Q_1 - \frac{1}{2} \xi \right) f(\xi) d\xi - h \int_{Q_1}^{Q_1+Q_2} \left\{ \frac{1}{2} Q_1 \frac{t_1}{T} + \left( Q_2 - \frac{1}{2} (\xi - Q_1) \right) \frac{T - t_1}{T} \right\} f(\xi) d\xi \\
- h \int_{Q_1+Q_2}^{\infty} \left( h \frac{1}{2} Q_1 \frac{t_1}{T} + \frac{1}{2} Q_2 \frac{t_2 - t_1}{T} \right) f(\xi) d\xi \quad (2.22)
\]

There are two slightly different methods to solve for an optimal \( Q_1 \). One approach, referred to as Method 1, is to make the replenishment \( (Q_2) \) decision first, similar to the approach in Section 2.3. Imagine the decision maker has just run out of \( Q_1 \) units at time \( t_1 \). She knows the distribution of remaining demand \( g(\xi) \) which is dependent on \( Q_1 \) only. (Since we are not updating demand in an advanced way, the timing of the depletion, \( t_1 \), does not affect \( g(\xi) \)).
Her profit equation for the replenishment decision is

\[
Z^2(Q_2|Q_1, t_1) = -cQ_2 + \int_0^{Q_2} \left\{ r\xi' + s(Q_2 - \xi') \right\} f(\xi') d\xi' + \int_{Q_2}^{b-Q_1} \left\{ rQ_2 - p(\xi' - Q_2) \right\} f(\xi') d\xi' - \int_0^{Q_2} h\left(Q_2 - \frac{1}{2}\xi'\right) \frac{T - t_1}{T} f(\xi') d\xi' - \int_{Q_2}^{b-Q_1} h\frac{1}{2}Q_2 \frac{t_2 - t_1}{T} f(\xi') d\xi'
\]

(2.23)

Using Eq. 2.23, we could solve for the optimal decision given \(Q_1\) was depleted at time \(t_1\). However, before the beginning of the season, we don't know exactly when \(Q_1\) will be depleted. To find \(E[Q_2|Q_1]\), we can iterate over all possible values of \(t_1\) for a given \(Q_1\). We know that the depletion time, \(t_1\), will occur between \(\frac{Q_1}{b} T\) and \(T\), where \(b\) is the maximum seasonal demand. When \(b \to \infty\), then the range reduces to between 0 and \(T\). For the purposes of this section, we assume that a finite upper limit of demand, \(b\), exists. Let \(y(t_1|Q_1)\) be the probability of depleting \(Q_1\) at time \(t_1\). If the upper limit of demand, \(b = \infty\), then \(0 \leq t_1 \leq T\). Hence our best estimate of the decision maker’s replenishment profit equation given we ran out of \(Q_1\), and will replenish \(Q_2\) is

\[
E\left\{Z^2[Q_2|Q_1]\right\} = \int_{\frac{Q_1}{b} T}^{T} \int_0^{Q_2} \left\{ -cQ_2 + \int_0^{Q_2} \left\{ r\xi' + s(Q_2 - \xi') \right\} f(\xi') d\xi' + \int_{Q_2}^{b-Q_1} \left\{ rQ_2 - p(\xi' - Q_2) \right\} f(\xi') d\xi' - \int_0^{Q_2} h\left(Q_2 - \frac{1}{2}\xi'\right) \frac{T - t_1}{T} f(\xi') d\xi' - \int_{Q_2}^{b-Q_1} h\frac{1}{2}Q_2 \frac{t_2 - t_1}{T} f(\xi') d\xi' \right\} y(t_1|Q_1) d\xi' d\xi' d t_1
\]

(2.24)

(2.25)

In Eq. 2.25, the following notation is used:

- \(g(\xi') = f(\xi = Q_1 + \xi'|\xi \geq Q_1)\), the probability distribution of remaining demand given we ran out of \(Q_1\)
\[ y(t_1|Q_1) = f(\xi = \frac{Q_1}{T} | T - Q_1 \leq \xi \leq b) \]

\[ t_2 = Q_2 \frac{T-t_1}{\xi} + t_1 \]

Maximizing Eq. 2.25 in \( Q_2 \), and substituting it into Eq. 2.22 can provide an optimal \( Q_1 \).

In another approach, referred to as Method 2, one could set \( \frac{\partial Z(Q_2,Q_1)}{\partial Q_2} = 0 \), set \( \frac{\partial Z(Q_2,Q_1)}{\partial Q_1} = 0 \) and iterate between the two formulations until they converge on an optimal \( Q_1 \) and \( Q_2 \) pair.

In that case, \( Q_2 \) is an expected amount that takes the expected timing of depleting \( Q_1 \) into account. In the implementation, however, the decision maker would order \( Q_1 \) initially and if and when these \( Q_1 \) are depleted, she would reevaluate at that time to determine an optimal \( Q_2 \). This is because \( t_1 \) does affect the replenishment quantity decision under the holding cost assumptions of our model, as shown in Eq. 2.23. Following this approach, it can be shown that \( Q_1^*|Q_2 \) occurs at the point where:

\[
(2.26) \quad r + p - c - (r + p + h - s)F(Q_1 + Q_2) + Q_2 f(Q_1)(c - s) \\
+ hQ_2 \int_{Q_1}^{Q_1+Q_2} \frac{1}{\xi} f(\xi) d\xi - hQ_1 \int_{Q_1+Q_2}^{\infty} \frac{1}{\xi} f(\xi) d\xi = 0
\]

Similarly, \( Q_2^*|Q_1 \) occurs at the point where

\[
(2.27) \quad r + p - c - (r + p + h - s)F(Q_1 + Q_2) + (c + h - s)F(Q_1) \\
+ hQ_1 \int_{Q_1}^{Q_1+Q_2} \frac{1}{\xi} f(\xi) d\xi - hQ_2 \int_{Q_1+Q_2}^{\infty} \frac{1}{\xi} f(\xi) d\xi = 0
\]

**Uniformly distributed demand with holding costs, lower limit zero simplification**

For uniformly distributed seasonal demand with lower bound \( a = 0 \), let

\[ y(t_1) = \frac{1}{T-Q_1} \]

\[ g(\xi) = \frac{1}{b-Q_1} \]
\[ f(\xi) = \frac{1}{b} \]

**Method 1**

To solve for \( E[Q_2^*|Q_1] \), we use Eq. 2.25. Maximizing this equation yields:

\[
\frac{1}{2} \left[ hQ_2 \left[ \ln(b - Q_1) - \ln(Q_2) \right] + h \frac{1}{2} (b - Q_1) \right] \left( 1 - \frac{Q_1}{b} \right) = \\
- \left( \frac{Q_1}{b} \right) \left[ -c b + cQ_1 + sQ_2 + (r + p)(b - Q_1 - Q_2) - hQ_2 - hQ_2 \ln(b - Q_1) - \ln(Q_2) \right]. \tag{2.28}
\]

Without loss of generality, let

- \( a = 0 \), and
- \( T = 1 \)

To solve for \( Q_1^* \), one would need to enumerate all possible \( Q_1 \). For each possible value of \( Q_1 \), find \( E[Q_2^*|Q_1] \) numerically. Then, substitute that \((Q_1, Q_2)\) pair into Eq. 2.22. We choose the \( Q_1 \) that achieves the maximum value of Eq. 2.22. Unfortunately, this method requires enumerating all possible discrete values of \( Q_1 \).

**Method 2**

This iterative method involves optimizing Eq. 2.22 in \( Q_1 \) and \( Q_2 \) independently. For a uniform seasonal demand distribution with parameters \((0, b)\) and season length \( T = 1 \), \( \frac{\partial Z_1^*(Q_1, Q_2)}{\partial Q_2} \) yields Eq. 2.29.

\[
\begin{align*}
&b \left( r + p - c \right) - (r + p + h - s) Q_2 - (r + p - c) Q_1 \\
&+ hQ_1 \left[ \ln(Q_1 + Q_2) - \ln(Q_1) \right] - hQ_2 \left[ \ln(b) - \ln(Q_1 + Q_2) \right] = 0 \tag{2.29}
\end{align*}
\]
Likewise, optimizing Eq. 2.22 in $Q_1$ for a Uniform distribution with parameters $(0, b)$ yields Eq. 2.30, which can be solved numerically to find $Q_1^*|Q_2$.

$$
\begin{align*}
    b \left( r + p - c \right) - \left( r + p + h - c \right) Q_2 - \left( r + p + h - s \right) Q_1 \\
    h Q_2 \left[ \ln(Q_1 + Q_2) - \ln(Q_1) \right] - h Q_1 \left[ \ln(b) - \ln(Q_1 + Q_2) \right] = 0
\end{align*}
$$

(2.30)

Numerical solution techniques are easily attainable to solve for an optimal $Q_1$ and $Q_2$. With a starting value of $Q_2$, this method will converge to an optimal $(Q_1, Q_2)$ pair. Figure 2.28 shows how quickly Method 2 can converge to an optimal solution by iterating back and forth between Eq. 2.30 and Eq. 2.29 for a numerical example.

Figure 2.29 shows the optimal behavior of $Q_1$ and $Q_2$ as a function of one another for a simple numerical example.

Let $h = ic$, where $i$ is a carrying rate charged per dollar in inventory per season. The behavior of the optimal solution as the holding charge varies provides managerial insight into the problem. Figure 2.30 shows how the optimal solution changes as the holding charge increases from $i = 0$ (the assumptions of Section 2.3) to $i = 0.2$. As Figure 2.30 shows, $Q_1$ is more sensitive to a change in holding cost.
Figure 2.29: Numerical example of $Q_2^*$ and $Q_1^*$ as functions of each other

Figure 2.30: Sensitivity of optimal solution to holding cost, $h$
2.6 Conclusion

We have shown analytical and numerical results for numerous versions, under varying assumptions, of an extension of the newsvendor problem where one additional replenishment opportunity is allowed during the season. Our assumption that there is only one known distribution of demand throughout the season is unique within the literature. Our models incorporate the stochastic nature of single period seasonal products. An area of future research is to expand the level of uncertainty to include uncertainty regarding the parameters of the distribution of total seasonal demand. Additionally, future research can evaluate the benefit of more than one replenishment opportunity.
3.1 Introduction

We examine a multiple period, finite-horizon inventory model with limited replenishment opportunities when the distribution of demand and its parameters are known. We assume the parameters of the demand distribution in every period are known, and there exists an opportunity to place an order (under certain restrictions) at the beginning of each period. Traditional inventory models quantify the implicit and explicit costs of placing an order through a fixed ordering cost. This cost is often assumed to be zero, leading to base stock type inventory policies, or assumed to be some positive value, leading to \((s, S)\)-type inventory policies. We present here another approach which we argue is a more intuitive method of representing the burden involved with placing replenishment orders. This alternative approach, which distinguishes our work from the existing literature, is one in which no direct monetary cost of placing orders is considered. Instead, the model takes into account the
maximum number of times management is willing to replenish their inventory (including the initial order and all replenishments) over the horizon. The model is then able to quantify the value of each additional replenishment opportunity, allowing the decision maker to weigh the benefits of each additional replenishment opportunity against their willingness to place orders. In Section 3.2 and Section 3.3, we present the model to solve this problem and a detailed analysis of the cost function. In Section 3.4 we present numerical results that shed insight into the value of reordering opportunities. Finally, in Section 3.5, we present a simple, yet powerful, heuristic for the optimal ordering quantities.

3.2 Definitions and Notation

Define:

- $c$, variable cost per unit;
- $p$, lost sales penalty per unit;
- $h$, inventory holding cost per unit;
- $v$, salvage value per unit (end of season only);
- $t = 1 \ldots T$, number of periods remaining in the horizon, where $T$ is the total number of periods;
- $F_t(\cdot)$, known cumulative distribution of demand in period $t$, with density $f_t(\cdot)$;
- $K$, maximum number of orders allowed in the season, including initial order and subsequent replenishment;
- $k = 0, 1 \ldots K$, number of orders unused (remaining);
- $I$, inventory in the beginning of each period, before order is placed and received; and
• \( y \), inventory in the beginning of each period, after order is receiver, but before demand occurs. We assume instantaneous delivery of the order quantity \( y - I \).

The sequence of events in each period is depicted in Figure 3.1. At the beginning of each period, inventory is counted, and a decision to order or not is made. If an order is placed, it is assumed to arrive instantaneously. Demand then occurs, after which costs and revenue are calculated. If the season is concluded, the salvage value is calculated based on the remaining inventory.

### 3.3 Profit Model

Before we present our profit maximizing problem, we present here an analogous cost minimization problem. The purpose of this section is to draw similarities between our model and the classical multiple period inventory models in the literature with explicit fixed ordering costs. We assume there is only one product being considered.

We model the profit maximization problem as a dynamic program with two state variables,
$I$ and $k$, where $I$ is the beginning of period inventory before any orders are made or received, and $k$ is the number of orders remaining in the season. Define $L_n(x)$, shown in Eq. 3.1, as the single period expected profit assessed at the end of each period after demand is realized.

$$L_t(x) = \int_{-\infty}^{\infty} \left[ r \min(x, \xi) - h(x - \xi)^+ + p(x - \xi)^- \right] f_t(\xi) d\xi$$  \hspace{1cm} (3.1)

First, let $C^k_t(I)$ be the expected profit-to-go from period $t$ until the end of the horizon, when there are $k$ orders remaining, and current inventory, before any orders are made and received, is $I$.

Define $Y^k_t(x)$ and $Z^k_t(x)$ which define the expected profit-to-go if an order is placed or not placed, respectively, in period $t$ when $k$ orders are remaining and current inventory at the beginning of the period is $x$.

$$Y^k_t(x) = L_n(x) + \int_{-\infty}^{\infty} C^{k-1}_{n-1}(x - \xi)^+ f_t(\xi) d\xi, \forall k \geq 1 \hspace{1cm} (3.2)$$

$$Z^k_t(x) = L_n(x) + \int_{-\infty}^{\infty} C^k_{n-1}(x - \xi)^+ f_t(\xi) d\xi, \forall k \geq 1 \hspace{1cm} (3.3)$$

When $k = 0$, the option of ordering does not exist. Hence, $Y^0_t = Z^0_t$. A decision maker with inventory $I$ (before any orders are made and arrive), in period $t$ determines whether there exists any $y > I$, such that $Y^k_t(y) - c(y - x) > Z^k_t(I)$. If so, they should order up to the $y > I$ which maximizes $Y^k_t(y)$.

We can represent this decision mathematically as follows.

$$C^k_t(x) = \max \left\{ Z^k_t(x), \max_{y > x} \left[ Y^k_t(y) - c(y - x) \right] \right\} \hspace{1cm} (3.4)$$

To summarize the ordering decision: if $k > 0$, the decision is whether there exists a point $y > I$, such that $Y^k_t(y) > Z^k_t(I)$. If there is, order up to $y$. If not, do not order. If $k = 0$, do not order.
We can represent the decision facing the decision maker visually. An arbitrary example is shown in Figure 3.2 and Figure 3.3, showing two different cases when an order would be made and not make, respectively. The figures show one line and a separate single point. The solid line represents $Y_t^k(I)$, $\forall y > I$, while the single point represents $Z_t^k(I)$. In the classical inventory problem with fixed costs (see e.g. Scarf [24]), the vertical distance between $Z_t^k(I)$ and the first point on the solid line at $Y_t^k(I + \epsilon)$ would be equal to the first ordering cost. This is not the case in our model. In fact, the vertical distance between the two points at $I$ represents the actual “fixed cost” to the decision maker of placing an order which is the cost of “losing” a replenishment opportunity.

We now demonstrate the decision facing a decision maker in every period in which $k > 0$. We analyze two arbitrary cases for illustrative purposes only, when $k > 0$ and $I = 5$. In Figure 3.2, an inventory manager could raise the expected profit to go by ordering up to 15. In doing so, he exhausts an order opportunity, but therefore has the opportunity to maximizes profits by ordering up to 15. Conversely, consider the case when $I = 25$ depicted in Figure 3.3. No point on the orderin line for inventory $\geq 25$ yields a higher profit than the single point representing $Z_t^k(25)$. As such, the best decision is to not order in this case.

Through extensive numerical experiments, we have observed only optimal inventory policies of the form $(s_t, S_t)$, though a formal proof is left for future research. Similar to traditional finite-horizon, periodic-review inventory problems with explicit fixed costs (see e.g. Scarf [24]), our $C_t^k(\cdot)$ function can have local maxima and minima. It remains to be formally proven for our model that these local maxima and minima do not affect the optimality of the $(s_t, S_t)$ inventory policy structure, although this has been the case throughout our experiments.
Figure 3.2: Ordering is best

Figure 3.3: Not Ordering is best
3.4 Numerical Experiments

3.4.1 Demand seasonality

In the apparel retailing context we consider here, there is an innate seasonality in most, if not all, products. We assume that the seasonality shape can be defined by the proportion of total seasonal demand expected in each period. This assumption follows the same spirit as Eppen and Iyer [9]. It is often assumed that a typical product goes through a ramp-up phase, maturity phase, and ramp-down phase, represented in this work as “centered demand”. Other products, however (e.g. swimsuits) are introduced in late winter to early spring when demand for swimsuits is relatively weak. As the spring and summer months approach, demand for swimsuits picks up and reaches a peak towards the end of the selling season. We term these type of products “back-heavy” goods. There are other products that start their lifecycle with peak demand. These products, typically technology items, tend to see declining sales as the selling horizon progresses, eventually being phased out completely, which we term “front-heavy” goods. These three different seasonality shapes are summarized in Figure 5.6-Figure 5.8. We find that the value of replenishment opportunities changes significantly as a function of the product’s seasonality shape.
Figure 3.5: Centered demand

Figure 3.6: Back-heavy demand
Table 3.1: Experimental parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Low</th>
<th>Medium</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>$2.25$</td>
<td>$2.75$</td>
<td>$3.25$</td>
</tr>
<tr>
<td>$h$</td>
<td>$0.00$</td>
<td>$0.02$</td>
<td>$0.04$</td>
</tr>
<tr>
<td>$p$</td>
<td>$0.00$</td>
<td>$0.15$</td>
<td>$0.30$</td>
</tr>
<tr>
<td>seasonality</td>
<td>front-heavy</td>
<td>back-heavy</td>
<td>center</td>
</tr>
<tr>
<td>$p_{nb}$</td>
<td>.1</td>
<td>.2</td>
<td>.3</td>
</tr>
</tbody>
</table>

3.4.2 Experimental design

An extensive set of numerical experiment was conducted. A key question of interest is how various model parameters affect the value of replenishment opportunities, including the cost parameters, the demand function, and the seasonality shape. Also of interest is the insight that can be gained from studying the optimal policy structure and characteristics under varying model parameters. Table 3.1 shows the levels of each of the parameters studied. These numbers represent scaled values of actual data provided by a retail partner for a particular garment.
Demand was assumed to follow a negative binomial distribution in every period. Though the model, as presented in Section 3.3, assumed continuous demand, there is a trivial conversion to account for this discrete demand distribution for the purposes of numerical illustration. The negative binomial distribution takes two parameters, \( r_{nb} \) and \( p_{nb} \), where \( r_{nb} \) is defined as the number of customers who walk in the store, but don’t buy (failures) until the experiment is stopped and \( p_{nb} \) is defined as the probability that a customer who walks in buys the garment (success probability). The random variable of the distribution is defined as the number of customers who walk in and buy the product before the experiment ends. This distribution is convenient to use in our setting as it is bounded below by zero and it is discrete. From our assumption, we know the mean demand in every period, \( t \), given by assumed total seasonal demand, \( ESD \), and seasonality proportion \( PR_t \). We fix \( p_{nb} \) to be the same in every period, and adjust \( r_{nb} \) to give the desired mean in every period. A higher \( p_{nb} \) results in a lower variance of the periodic distribution for any given \( r_{nb} \), so were able to test the effects of varying demand variability.

An analysis of variance (ANOVA) was carried out on the significance of the revenue, holding cost, lost sales penalty, seasonality and \( p_{nb} \) terms of the model. The response variable in this case is the value of the first replenishment opportunity, measured as the percent improvement over the single-order newsvendor profit. Results are shown in Figure 3.7.
The ANOVA results in Figure 3.7 indicate all tested parameters were significant in determining the benefit of additional ordering opportunities. Additionally, there were several significant interaction effects, including some interaction effects.

Table A.3, in the appendix, shows the details of profit in each of the 81 parameter instances considered in this analysis when none, one, or two replenishment opportunities are available, as well as the associated improvement over the no-replenishment solution when one or two replenishments are available.

Figure 3.8-Figure 3.12 show the main effects of holding cost, lost sales penalty, revenue, seasonality shape, and demand variability on the value of one replenishment opportunity. These figures confirm the ANOVA results. Visually, it is clear that these factors effect the value of the replenishment opportunity, measured as a percentage gain in profit over the no-replenishment solution.
Figure 3.9: Main effect of lost sales penalty on value of a single replenishment opportunity
Figure 3.10: Main effect of holding cost on value of a single replenishment opportunity
Figure 3.11: Main effect of demand seasonality on value of a single replenishment opportunity
Figure 3.12: Main effects of demand variability on value of a single replenishment opportunity
Figure 3.13: Average fill rate across all instances

Figure 3.13 shows the improvement in customer demand fill rate with one or two replenishments over the no-replenishment solution. The first replenishment results in approximately a one basis point improvement on average, while a second replenishment increases the fill rate approximately another half basis point for all seasonality shapes. As expected, the marginal improvement is diminishing as more replenishments are considered. We note the exceptional service level in these examples is due to our assumption that the demand distribution is known.
3.5 Heuristic Solution

In this section we present an heuristic solution approach for determining both the reorder level \((s_t)\) and the order up to level \((S_t)\) in each period \(t\) for the inventory model presented above. To demonstrate the effectiveness of the heuristic, we compare the profit of using the heuristic to the profit under optimal control for 250 instances where each parameter is randomly generated from between the minimum and maximum limits shown in Table 3.2.

Table 3.2: Range of parameters from which 250 random instances were selected for numerical results

<table>
<thead>
<tr>
<th></th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>(r)</td>
<td>1.5</td>
<td>3</td>
</tr>
<tr>
<td>(p)</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(v)</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(nb_p)</td>
<td>0.04</td>
<td>0.2</td>
</tr>
</tbody>
</table>
For both $s_t$ and $S_t$, the heuristic originates from the newsvendor critical fractile of $\frac{r+p-c}{r+p-v}$ with suitable adjustments. The classical newsvendor problem assumes no holding costs, and only a single ordering opportunity, which clearly differ from our model. The newsvendor model can be rewritten as $\frac{c_u}{c_u+c_o}$, where $c_a$ is the cost associated with over-ordering one unit beyond demand, and $c_u$ the cost of under-ordering one unit below demand. In the case of the classical newsvendor, inventory is then brought up to a level $S$, such that $\int_0^S f(x) \, dx = \frac{r+p-c}{r+p-v}$, where $f(x)$ is the probability density function of demand over the horizon. In our multi-period model, we adapt the newsvendor solution to say that we should order if demand falls below some single-period fractile, and order up to an amount equal to another fractile, with provisions for how much demand is remaining in the horizon and how many replenishments are remaining. For $s_t$, we concern ourselves only with the demand probability density function for the current period $f_t(\cdot)$. Then, we set the critical fractile for the heuristic such that $\int_0^{s_t} f_t(x) \, dx = \frac{r+p-c}{r+p+h}$. We make two alterations from the classic newsvendor in this case. First, we remove the salvage value $v$ from the fractile as there is no salvage value until the final period and as such, it should not be included in the calculations until that time. Secondly, we add $h$ into the overage costs, under the assumption that any extra inventory will be carried for at least one period, and should be penalized for at least one period. We experimented with more sophisticated methods involving estimating the number of periods that an overbought item will be held for, but found they were not better than this simple assumption of holding overage for one period. When $s_t$ is fixed by the heuristic, and a Markov Decision Process is used to find the optimal $S_t$ corresponding to that sub-optimal policy, our numerical results show an average performance loss compared to optimal of less than 0.30%, or less than one third of one percent for all three seasonality shapes and when there are none, one, or two replenishment opportunities in the season. We note that $s_t$ does not affect the results when no replenishment is available. The results for this policy, which we denote as the $\left(s_t, S_t^*|\hat{s}_t\right)$ policy, are summarized in Table 3.3, where $s_t$ is approximated, and the best corresponding $S_t$ is chosen by an MDP.
Table 3.3: Results for the $(\hat{s}_r, S^*_r|\hat{s}_l)$ policy for 250 random instances

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Average</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Front-heavy, 2 replenishments</td>
<td>0.25%</td>
<td>0.00%</td>
<td>1.50%</td>
</tr>
<tr>
<td>Front-heavy, 1 replenishment</td>
<td>0.09%</td>
<td>0.00%</td>
<td>0.82%</td>
</tr>
<tr>
<td>Front-heavy, no-replenishment</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>Centered, 2 replenishments</td>
<td>0.25%</td>
<td>0.00%</td>
<td>1.50%</td>
</tr>
<tr>
<td>Centered, 1 replenishment</td>
<td>0.09%</td>
<td>0.00%</td>
<td>0.82%</td>
</tr>
<tr>
<td>Centered, no-replenishment</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>Back-heavy, 2 replenishments</td>
<td>0.25%</td>
<td>0.00%</td>
<td>2.03%</td>
</tr>
<tr>
<td>Back-heavy, 1 replenishment</td>
<td>0.06%</td>
<td>0.00%</td>
<td>0.68%</td>
</tr>
<tr>
<td>Back-heavy, no-replenishment</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>
As Table 3.3 shows, the deviation from optimal for the \( \hat{s}_t, S_t^*|\hat{s}_t \) heuristic policy is fairly small. We see averages below one third of one percent, and maximums no higher than 2.03% profit loss. Generally speaking, it is not critical to the overall profitability of the product in question to find the optimal value of \( s_t \). Define \( \hat{S}_t \) as an estimate of \( S_t \). The calculation of \( S_t \) is similar with that of \( s_t \), but requires the computation of a new demand distribution. Define \( h_{t,k}(x) \) as the probability density function of demand for the next \( \lceil \frac{T-t}{k} \rceil \) periods, where \( T \) is the total number of periods in the horizon, \( t \) is the current period, and \( k \) is the number of orders remaining. The fractile of \( \frac{r+p-c}{r+p+h(T-t)} \) is used, and \( \hat{S}_t \) is set such that \( \int_0^{\hat{S}_t} h_{t,k}(x) \, dx = \frac{r+p-c}{r+p+h(T-t)} \).

Again, we make provisions for the last period to include holding cost, such that the fractile in the last period is \( \frac{r+p-c}{r+p-s} \).

Define the \( \{\hat{S}_t, \hat{s}_t\} \) policy when the respective heuristics are used for both \( s_t \) and \( S_t \). As shown in Table 3.4, when the heuristic is used for \( S_t \) as well, it becomes more costly than using the \( s_t \) heuristic only, with averages up to 1.92% profit loss, and maximums up to 7%. It is important, however, to remember how much profit gain there is by allowing replenishment opportunities, such that a heuristic policy with replenishment is still more profitable than the no-replenishment optimal solution. Additionally, it can be implemented and solved in any simple spreadsheet packages since it does not require solving an MDP.

### Table 3.4: Results of \( \{\hat{S}_t, \hat{s}_t\} \) policy, profit loss from optimal for 250 random instances

<table>
<thead>
<tr>
<th></th>
<th>average</th>
<th>min</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>front-heavy, 2 replenishments</td>
<td>0.54%</td>
<td>0.04%</td>
<td>3.32%</td>
</tr>
<tr>
<td>front-heavy, 1 replenishment</td>
<td>0.50%</td>
<td>0.01%</td>
<td>2.93%</td>
</tr>
<tr>
<td>front-heavy, no-replenishment</td>
<td>0.83%</td>
<td>0.00%</td>
<td>6.75%</td>
</tr>
<tr>
<td>centered, 2 replenishments</td>
<td>0.86%</td>
<td>0.14%</td>
<td>4.11%</td>
</tr>
<tr>
<td>centered, 1 replenishment</td>
<td>1.31%</td>
<td>0.15%</td>
<td>5.76%</td>
</tr>
<tr>
<td>centered, no-replenishment</td>
<td>0.85%</td>
<td>0.00%</td>
<td>7.00%</td>
</tr>
<tr>
<td>back-heavy, 2 replenishments</td>
<td>2.16%</td>
<td>0.26%</td>
<td>8.15%</td>
</tr>
<tr>
<td>back-heavy, 1 replenishment</td>
<td>2.03%</td>
<td>0.36%</td>
<td>8.46%</td>
</tr>
<tr>
<td>back-heavy, no-replenishment</td>
<td>0.85%</td>
<td>0.00%</td>
<td>7.13%</td>
</tr>
</tbody>
</table>
In Table 3.4, we show the profit gain of the heuristic policy with replenishment over the no-replenishment optimal policy. These results show that the average profit loss resulting from the heuristic policy is minimal, with very high upside benefits. In other words, in most cases, just one replenishment increases profit when the heuristic is used over the no-replenishment optimal solution.

The correlation between salvage value and the profit gain of a heuristic policy with single replenishment over the no-replenishment optimal policy are shown in Figure 3.14. As the figure shows, the instances with no profit gain all share a very high salvage value. The insight to management is that the heuristic might not be ideal when salvage value is almost as high as the unit cost of the good. In these cases, management would be wise to forego the heuristic and instead spend the resources for finding optimal policies.
Figure 3.15: Profit gain vs. salvage value for the single replenishment heuristic policy over no-replenishment optimal policy for back-heavy demand

In Figure 3.15, we repeat that analysis for goods with back-heavy demand and a single replenishment orders. The results are similar to that of Figure 3.14, in that instances with high salvage cost lead to reduced profit when using the heuristic policy. We also notice a general upward shift in the value of the replenishment using the heuristic reiterating that products with back-heavy demand make better use of the replenishment opportunity by using it to avoid holding costs.
3.6 Summary

We have presented a Markov Decision Process formulation that determines the optimal ordering quantities for a finite horizon inventory policy with a known demand distribution and known parameters. The formulation takes into account the fact that fixed ordering costs are inherently difficult to quantify. We therefore replace the explicit fixed ordering cost with a restriction on the number of times an order may be placed. Our formulation allows us to directly quantify the value of additional ordering opportunities. A manager can then compare each additional ordering opportunity to the marginal benefit thereof to determine whether or not replenishment opportunities, and how many, are worthwhile. Once this determination has been made, the model can then be used as an optimal ordering guideline.

We then present a heuristic approach to solving this model which does not require solving a Markov Decision Process. Our results show a negligible average percentage loss of using the heuristic in most cases, and we characterize the cases in which the heuristic does not perform as well.
A Multiple Period Inventory Model with Restricted Replenishment and Unknown Demand Distribution Parameters

4.1 Introduction

The model presented here takes a different approach to solving the finite-horizon, periodic-review inventory model typically seen in the literature and engineering textbooks for a single product. We replace the traditional fixed ordering cost with a limit on the number of times an order may be placed across the horizon. The model replaces a monetary amount with this ordering limit, thereby retaining the concept that orders are not free and should not be “wasted” unless an order is warranted. Our model is also applicable to the fixed costs involved in a production environment. While the cost of a setup is perhaps difficult to quantify in monetary terms, certainly a manager has some sense of the maximum number of changeovers, setups or replenishments his staff and resources can handle in any given time period. Our approach, we believe, is more intuitive from a managerial perspective as it does not require decision makers to state the explicit fixed ordering cost they face for every order. Additionally, our model
can assess the explicit benefit of replenishment opportunities, which a decision maker can then weigh against his qualitative assessment of the burden involved in reordering to decide how many orders he should plan for across the horizon. We prove that having additional replenishments never lead to higher costs, but show through numerical experimentation that the benefit of replenishment opportunities is marginally decreasing in the number of orders.

In this work we explore three aspects of this problem. The first two aspects can broadly be differentiated by a difference in assumptions regarding the expected seasonal demand ($ESD$), which is the sum of the mean demand parameters across every period. First, we present a basic model where the parameters of the demand distribution are known, such that the mean and variance of $ESD$ as specified numerically. We then add an element of realism in that the parameters of $ESD$ are themselves stochastic. Finally, we add an element often found in retail management involving “inventory coverage”, where the first order might be suboptimal, but is management driven to allow a product to achieve a certain store-visibility target (often arbitrarily) set by management. In Section 4.3, we present the basic formulation of a finite-horizon inventory model with stochastic demand, but where the parameters of the distribution of demand in every period are assumed known. In addition to the known parameter case, we extend the model to account for the situation where the demand parameters are not known with certainty and must be forecast and reestimated during the horizon. We consider the unknown demand case in two parts, in Section 4.4 and Section 4.5. In Section 4.4, we assume that the distribution of demand in each period is fixed, but its mean is unknown to the decision maker. We then test the effect of the initial forecast error on profit and the value of replenishment opportunities. In reality, the demand distribution parameters in any given period cannot be assumed fixed, instead they are inherently stochastic. However, this provides insight into the value of replenishments when a decision maker under- or over estimates the expected seasonal demand. In the second part (Section 4.5) we assume that there is an underlying distribution which defines the mean parameter of the demand distribution in every period, i.e. the mean demand parameter in every period is itself defined by a stochastic
distribution. The uncertainty in the mean creates a distribution of the potential forecast errors for any given forecast of demand. We present a method of estimating this distribution based on the past forecasting ability of the decision maker, and show how the value of multiple replenishment opportunities changes when the variance of the distribution of the \textit{expected seasonal demand} increases. We also show how different seasonality patterns affect the value of replenishment opportunities for both cases. Finally, in Section 4.5.1, we consider the notion of “inventory coverage”. Inventory coverage was explained by fashion industry representatives we contacted as a simple heuristic used for the initial order quantity when one or more replenishments are available in the season. Typically under this heuristic, management sets a coverage amount, which is a percentage of expected demand to be covered with the initial order, e.g. if expected seasonal demand is 1000, and the coverage percentage of that firm for that type of product is 70%, then the initial order would be for 700 units. The reason for the coverage is specific to retail because the firms claim that a product would sell better over the course of a season if that product’s store presence was larger. In other words, consumers will feel more attracted to a product if that product is prominently displayed in the store. For that reason the initial order is restricted from being too small to give the product the best chance of consumer adoption and success. If the product is popular, then replenishments can be used to satisfy demand. If the product is a failure, 70% is arbitrarily chosen as “not too much investment” and the clearance rack can be employed to sell off excess inventory.

In our analysis, we compare the 70% coverage policy to optimal policies to find the value of using more advanced ordering techniques over traditional heuristics when one or more replenishments are used.

4.2 Literature Review

One way of viewing our work is to consider it as a multi-period extension of the newsvendor problem with replenishment, see e.g. Bardford and Sugrue [2], and Lau and Lau [18, 19]. These papers present numerical and analytical results when, unlike our model, \textit{only one}
additional order may be made during the season. They split the horizon into two periods with
an explicitly known distribution of demand in each period. Similar to our updating scheme
presented in Section 4.3, these papers include methods of utilizing past demand to better
predict future demand.

In a recent study, Milner and Kouvelis [21] examine the effects of product/demand char-
acteristics, e.g. classifying products as standard, fashion, or innovative products— on the
benefit of timing flexibility and quantity flexibility. Their models, unlike ours, consider only
the single replenishment case. Similar to their research, we recognize the inherent hit-or-miss
nature of fashion products, as well as the inherent seasonality patterns they undergo.

Iglehart [15] models the dynamic inventory problem with unknown exponentially dis-
tributed demand. Azoury [1] considers the case of an unknown demand distribution parama-
ter with Bayesian updating of the parameter. In this paper, she demonstrates that an \( N \) period
inventory problem can be reduced to one with a one dimensional state space. As in Eppen
and Iyer [9], she represents demand through the use of a (one) sufficient statistic. Eppen and
Iyer [9] differs from Dvoretzky et al [8], Scarf [23, 24], Iglehart [15], and Murray and Silver [17]
in that they consider lost sales and make no specific assumptions about the type of demand
distribution. They consider a style goods problem in which the decision made is how much
to "dump down" to an external outlet store at the end of every period. The outlets cannot
be used to satisfy the retailer's demand after products have been dumped down. They track
total demand over the course of a selling season according to “percent-done” curves, which
are assumed to accurately measure the expected percent of total seasonal demand to have
been observed from the beginning of the season up to any time (regardless of the magnitude
of demand). Using observed sales data and following a percent-done curve, the model can
estimate total demand remaining until the end of the season, and dump down inventory in
excess of total expected demand left. To represent the magnitude of demand, they assume that
the demand follows one of several known demand processes. A probability distribution over
these demand processes represents a decision maker's belief what the magnitude of demand
is. These probabilities, along with a percent-done curve and the knowledge of past sales data are used to form an estimate of remaining demand. They present a backwards dynamic programming formulation, as well as a newsvendor approximation scheme which is shown to produce satisfactory results. In this newsvendor approach, the marginal single period purchase cost is represented by a weighted function, \( c' = c + h_1 f_1 + h_2 f_2 + \ldots + h_T f_T \), where \( h_i \) represents the holding cost in period \( i \), \( T \) the end of the horizon, and \( f_i \) the expected percent of inventory sold in period \( i \) (from the percent done curve). A similar weighted function is used to represent the single period salvage cost and holding cost.

More recently, Wang et al. [26] consider a problem where the decision maker receives continually improving forecast information during the planning horizon before the beginning of the sales season. In their model, forecasts evolve according to a Martingale Model of Forecast Evolution, and ordering costs get progressively more expensive as the season approaches. The decision maker may make any number of orders before the beginning of the season. An extension of their model includes the allowance of a positive fixed ordering cost. Revenue in their model is fixed at a certain value.

Several papers from Hunter et al [14, 13, 12], as well as Nuttle et al [22], that have studied the benefit of replenishment on a seasonal product, without the explicit determination of fixed ordering costs, in a simulation environment. Our model differs from these papers in that we consider an optimal formulation of the problem, while their simulation approach tested non-optimal, yet very reasonable, replenishment policies.

Our work differs from the existing literature in that we make no explicit determination of the fixed ordering costs involved with placing an order, while still restricting the number of orders that may be made and maintaining optimal or near optimal solutions. While our model does incorporate a simple demand learning technique, we note that the choice of updating scheme has little effect on the value of replenishment opportunities which we are measuring. To the best of our knowledge, no other paper has analyzed a finite-horizon inventory model with lost sales and no explicit fixed ordering cost that limits the number of
orders that may be made during the horizon in the face of uncertain demand while utilizing a
demand reestimation scheme.

4.3 Model Formulation and Demand Reestimation

We present here a formulation for solving a multiple-period, finite-horizon inventory model
in which the underlying demand distribution and its parameters are not known with cer-
tainty. We present two methods of representing this uncertainty, and we present a way to use
management’s initial forecast of demand in combination with a demand updating scheme to
determine ordering quantities throughout the horizon. Uncertainty in the parameters of the
underlying demand distribution are a reality in many industries. Consider, for example, the
fashion retailing industry. When a new product, or new style, etc., is introduced, managers
certainly hope that it will be successful, but history shows that some products simply never
take off, and are unsuccessful. Alternatively, some products are such successes that managers
find themselves overwhelmed with demand. We call this the “hit of miss” nature of products.

In addition to our assumption that the parameters of the demand distribution are them-
selves unknown, we assume that products follow one of three specified lifecycles. First, we
define a typical product goes through a ramp-up phase, maturity phase, and ramp-down
phase, represented in the work as having “centered” demand. Other products, e.g. swimsuits,
are introduced in late winter to early spring when demand for swimsuits is relatively weak. As
the spring and summer months approach, demand for swimsuits picks up and reaches a peak
towards the end of the selling season. We term these type of products “back-heavy” goods.
Last, there are other products that start their lifecycle with peak demand. These products,
including technology items, tend to see declining sales as the selling horizon progresses,
eventually being phased out completely, which we classify “front-heavy” goods. These three
different seasonality shapes, which we term front-heavy, back-heavy, and centered demand
are summarized in Figure 4.1, Figure 4.2, Figure 4.3 respectively.
Figure 4.1: Front-heavy seasonality
Figure 4.2: Centered seasonality
Figure 4.3: Back-heavy seasonality
In the same spirit as [9], we recognize that while the seasonality shape (i.e. lifecycle - front-heavy, centered, or back-heavy) of a product may be known or assumed known with certainty, the magnitude of sales (i.e. the mean parameter of the demand distribution in every period) is not necessarily known with certainty, and this is where the inherent uncertainty of the demand distribution parameters comes from. It is difficult for even the best forecasting software to have the foresight to know whether a product will be popular or not with the general public, even though the relative shape of demand over time, i.e. the seasonality, looks similar in either case. We consider two ways - a simple and a more realistic method - of approaching this aspect of uncertainty.

We define $ESD$ as the expected seasonal demand, which is the sum of the means of the demand distribution in every period. Define $PR_t$ as the (assumed) known proportion of $ESD$ which occurs, in expectation, in period $t$. Taken together, the proportions ($PR_t$) and the estimate of expected seasonal demand ($ESD$) provide the mean of the distribution of demand in every period (or the estimate of the mean in every period – which we discuss later in detail). We assume that demand in every period is distributed according to a negative binomial distribution. This distribution is appropriate in our situation because of a number of its properties. First, it is discrete which accurately describes retail demand. Secondly, it is bounded below by zero, which is appropriate for demand. Thirdly, the distribution tends towards central tendency with the high demand that we consider here. The negative binomial distribution takes two input parameters. The first input parameter is the probability of failure which we use to represent the probability that a customer does not buy a good $p_{nb} \in (0, 1)$. The second parameter, $r_{nb} > 0$, is a predetermined number of failures (or unsatisfied customers) which must occur for the experiment to be stopped. The two parameters of the negative binomial distribution fully determine the mean and variance of the distribution. Alternatively, given a mean and one of the parameters, the distribution is also fully defined.

The sequence of events in each period is depicted in Figure 4.4. At the beginning of each period, inventory is counted, and a decision to order or not is made based on the current
Figure 4.4: Timeline of events within a period

estimate of $ESD$. Demand then occurs, after which costs and revenue is calculated. Finally, the estimate of $ESD$ is updated based on the actual demand experienced in the period.
Inherent in this analysis is the idea that a company is able to update their forecast of a certain product’s ESD as more demand is observed. This updating allows inventory managers to learn the true underlying magnitude of seasonal demand as the season progresses, and be able to use that information in subsequent replenishments if available. Define $\hat{ESD}_t$ as the estimate of $ESD$ after $t-1$ observations of demand. We set $\hat{ESD}_1 = \overline{ESD}$, where $\overline{ESD}$ is the initial estimate of $ESD$ from management.

While other research has solved similar problems through the use of Bayesian updating (see e.g. [9]), our research focuses on a much simpler form of updating that reflects actual industry practice. We use simple exponential smoothing of past observations of demand to update $\hat{ESD}_t$, where $\alpha$ represents its smoothing coefficient parameter. Our updating scheme is shown in Eq. 4.1.

$$\hat{ESD}_t = \alpha \frac{\sum_{i=1}^{t} D_i}{\sum_{i=1}^{t} PR_i} + (1-\alpha) \hat{ESD}_{t-1}, \quad t = 2, \ldots, T$$ (4.1)

Note that $\overline{ESD} = \hat{ESD}_1$ is exogenous to the model and must be decided upon by a decision maker. $\hat{ESD}_2$ is the first updated and endogenous estimate of $ESD$. In every period $t$, the ordering (or not ordering) decision is based on the $\hat{ESD}_t$ being the true underlying demand, as if it were known with certainty. Importantly, our model does not base its ordering decision on a prior and posterior probability distribution over the possible values of $ESD$, which is the case with Bayesian updating.

In our first analysis, presented in Section 4.4, we look at the effect on profit and the value of replenishment opportunities when the forecast error is known, i.e. when $\frac{\hat{ESD}_1}{ESD}$ is a specific value, and we compare the effect of under- or over-estimating $ESD$. In a Monte Carlo simulation environment, we simulate many seasons, where $ESD$ and $\overline{ESD}$ are fixed in all cases. In our second analysis, shown in Section 4.5, we assume that $ESD$ follows a distribution, i.e. $ESD \sim N(\overline{ESD}, \sigma_{ESD})$, similar to the procedure presented in Cachon and Terwiesch [3]. We explain how to estimate that distribution and how to use it in a Monte Carlo simulation environment to find the value of additional replenishment opportunities.
under various seasonality settings, and the effect of forecast error variance. For the analysis in Section 4.5, we simulate thousands of seasons, where each simulation samples a different $ESD$ from the given distribution, more closely reflecting reality where a decision maker provides his best estimate of demand, but realizes he may have under- or over-estimated with a certain probability.

In both sections Section 4.4, and Section 4.5, the ordering policies at any given time are based on $\hat{ESD}_t$, as if that estimate were true and known. This heuristic is suboptimal, but we note that this is how ordering decisions are made in many industries and businesses, and it provides a lower bound for the value of replenishment opportunities. In the concluding remarks, we show how much more profit could be generated if the true $ESD$ were in fact known and an updating scheme were unnecessary.

For any given $ESD$, the ($ESD$-specific) optimal ordering policy is derived from solving a Markov Decision Process (MDP). We model the problem as a dynamic program with two state variables, $I$ and $k$, where $I$ is the current inventory level, and $k$ is the number of orders remaining to use. Define $L_t(y)$ as the single-period expected cost assessed at the end of each period, where $y$ is the order up to level, i.e. inventory plus order quantity, and $h$ and $p$ are the per unit holding and lost sales penalty costs respectively, and $r$ is the unit revenue per good sold. For convenience, let $f_t$ define the probability density function of demand in period $t$, and define $c$ as the unit cost to purchase a good. Thus, we have

$$L_t(y) = \int_{-\infty}^{\infty} \left[ r \min(\xi, y) - h(y - \xi)^+ + p(y - \xi)^- \right] f_t(\xi) d\xi$$  \hfill (4.2)

Let the state specific expected profit-to-go from period $t$ until the end of the horizon.
\[ V^k_t(I) = \max_{y \geq I} \left[ -c(y - I) + L_t(y) + \begin{cases} \int V^k_{t+1}(y - \xi)^+ f_t(\xi) d\xi & \text{if } y = I \\ \int V^{k-1}_{t+1}(y - \xi)^+ f_t(\xi) d\xi & \text{if } y > I \end{cases} \right] \] 

\( \forall (t, k > 0), \)

and \( V^k_{t=0}(I) = L_t(I) + \int V^0_{t+1}(I - \xi)^+ f_t(\xi) d\xi \) 

In Eq. 4.2 and Eq. 4.3, the demand functions \( f_t(\xi) \), are specific to a given sample of \( ESD \). The objective for a specific \( ESD \) (or, in the case of estimated demand, \( \hat{ESD} \)) therefore, is to minimize \( V^k_{t=0}(0) \), given the terminating salvage value condition \( V^k_T(I) = sI \) \( \forall (k, I). \)

### 4.4 Results for a Specified Forecast Error

For the analysis in this section, we examine the impact of various forecast errors on the profitability of a product, and the value of additional replenishment opportunities for that product. We examine the case when \( \frac{\hat{ESD}_{t}}{ESD} = 0.5, 0.75, 1, 1.25, \) and \( 1.5 \), representing a range of underestimating by half to overestimating by 1.5 times actual expected seasonal demand. In addition to the three different seasonality shapes and the range of forecast errors, we evaluate the value of replenishment opportunities for a range of cost parameters. These parameters are outlined in Table 4.1

Table 4.1: Range of parameters from which 250 random instances were selected for numerical results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>1.5</td>
<td>3</td>
</tr>
<tr>
<td>( p )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( s )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( n b_p )</td>
<td>0.04</td>
<td>0.2</td>
</tr>
</tbody>
</table>
In Figure 4.5, we compare the profit of differing ordering policies for a front-heavy product. One ordering policy we consider is the MDP-optimal policy, where a decision maker orders the optimal amount, according to the MDP formulation (Eq. 4.3), of his then current estimate of $ESD$. Another ordering policy we examine is a 0.7 coverage ordering policy, where a decision maker places his first order equal to $0.7 \hat{ESD}_1$, where $\hat{ESD}_1$ ranges from 0.5$ESD$ to 1.5$ESD$. For the 0.7 coverage policy, all subsequent replenishments are placed according to the MDP-optimal solution. The difference between this ordering policy and the always MDP-optimal policy, is the comfort that decision makers receive from a known initial coverage amount, which was expressed to us as being an important element of their marketing plan. Part of our analysis shows how much profit is being lost by these type of plans (compared to optimal). The final ordering policy we consider is for no replenishment, and can be viewed as the “current” system in many industries, where there is only one initial order with no replenishment opportunities, and that single order is based on the newsvendor solution of $\hat{ESD}_1$. 

Figure 4.5: Front-heavy demand, average profit under different ordering schemes and replenishment opportunities
Figure 4.6: Front-heavy demand, close up view, average profit under different ordering schemes and replenishment opportunities

In Figure 4.5, we see three straight lines and three curved lines. The straight lines correspond to optimal ordering policies with no forecast error. In other words, for a given number of replenishment opportunities, these lines represent the maximum profit attainable in a perfect world. As shown, the coverage policies approach these lines when two things happen: first, when the forecast is good, and second, when there are numerous replenishments. We show a close-up view of Figure 4.5 in Figure 4.6.

A similar analysis of center seasonality products is shown in Figure 4.7 and Figure 4.8, as well as for back-heavy products in Figure 4.9 and Figure 4.10.

4.5 Results for a Stochastic Forecast Error

In this section, we assume that the forecast error is not known. Instead, there is a given (and known) distribution of the forecast error— which in turn provides a distribution of the actual ESD which might occur in the season for a product given a point forecast from management. To develop this distribution of the forecast error, we follow the approach outlined in Cachon
Figure 4.7: Center seasonality demand, average profit under different ordering schemes and replenishment opportunities

Figure 4.8: Center seasonality demand, close up view, average profit under different ordering schemes and replenishment opportunities
Figure 4.9: Back-heavy demand, average profit under different ordering schemes and replenishment opportunities

Figure 4.10: Back-heavy demand, close up view, average profit under different ordering schemes and replenishment opportunities
and Terwiesch [3] to develop a representation of the forecast error found in many businesses, which helps us determine the distribution of $E_{SD}$. Cachon and Terwiesch present a case study from the athletic apparel company O’Neal. They use historical forecast error information of products with similar characteristics to the one to be forecasted in order to construct a distribution of expected demand based on a single point forecast estimate for a new product. They argue that a company’s current ability to predict demand is directly related to their past ability to predict demand. Hence, a distribution of past $\frac{A_{F}}{F}$ ratios from similar products provides an adequate measure of the expected demand of a new product given a company’s point forecast for this new product. Furthermore, Cachon and Terwiesch fit a Normal distribution to the distribution of historical $\frac{A_{F}}{F}$ ratios. They found that the historical expected $\frac{A_{F}}{F}$ ratio at O’Neal is 0.9975. The reader should be aware that this impressive number is an average ratio, that mathematically is the result of a high and low number cancelling out in expectation. In fact, the same data indicate an absolute average forecast error of 31%, which is not as impressive. The data concludes that the $\frac{A_{F}}{F}$ ratios are unimodal and centered, but still highly variable on both sides of the mean.

In this work, we adapt the conclusions of the work in Cachon and Terwiesch. We assume a retailer has the ability to, on average, predict the demand of a new product very well, i.e. with mean forecast error of zero, or $E\left[\overline{ESD}_{1} - ESD\right] = 0$. However, differences between their estimates and the true $ESD$ of a specific product may vary, sometimes significantly. This variance comes from two sources. The first we term a “hit-or-miss factor”. Some products are inherently more likely than others to be a major success (hit) or not grab the public’s attention (miss), but there is no way of knowing which way the pendulum will swing before the beginning of the season. Secondly, the amount of marketing resources to learn about demand will affect the average forecast error. Regardless of where the variance comes from, we assume a company is able to analyze their past performance of similar products to come up with their own distribution of historical forecast errors. We examine three different distributions representing past forecast error: low, medium, and high variance, where the latter can be
viewed as a product that is most likely to be hit or miss. These distributions are shown in Figure 4.11.

The solution methodology of this model is as follows. For a given set of parameters: revenue \( r \), lost sales penalty \( p \), salvage value \( s \), negative binomial parameter \( n b \), a point forecast from management \( ESD \), and a distribution of past forecasting ability with mean 0 and standard deviation \( \sigma \), we construct a distribution of \( ESD \) and simulate 2000 seasons in a Monte Carlo fashion. Each simulation samples \( ESD \) from a Normal distribution with mean \( \overline{ESD} \) and variance \( \sigma_{ESD} \). Using that sampled \( ESD \), a distribution of demand in every period is formed, i.e. demand in period \( t \) is distributed according to a negative binomial distribution with mean \( ESD \left( PR_t \right) \), and given parameter \( n b \). A specific value of demand for each period is sampled from each of those distributions. The decision maker makes an initial order, and the season progresses. Throughout the season, demand is re-estimated at every time that a new observation of demand is observed. This reestimation technique is described in detail in Section 4.3. This analysis also allows us to compare the value of additional replenishment opportunities to the value of information.
4.5.1 Coverage policies

From our contact with retail industry managers, we learned that “coverage policies” of around 0.7 were almost always employed on a new product if multiple orders were to be used, no matter the expected forecast variance of the products. While our analysis in Section 4.5 compares optimal ordering policies with a 0.7 coverage policy, a question remains what the optimal, or best, coverage number should be (assuming such a non-optimal policy is to be implemented for marketing or other business purposes). In this section, we examine various aspects of the coverage policy. Specifically, we examine the profit improvement of the standard 0.7 coverage over a one-order newsvendor solution, as well as the profit of other coverage policies in a multi order environment.

In Figure 4.12, Figure 4.13, and Figure 4.14, we show the expected seasonal profit of various replenishment schemes, including optimal ordering, or employing coverage policies, and compare those to the profit under traditional no-replenishment ordering schemes. We see that the a replenishment opportunity, even under a sub-optimal coverage policy, greatly increases profits over any no-replenishment scheme.

In Figure 4.15, we show the profit gain over the one-order newsvendor solution when a 0.7 coverage policy is employed. For this analysis, we set $\hat{ESD}_1 = ESD$, but allow the actual $ESD$ to be sampled from the 3 different distributions of low, medium, and high variance. The figure shows the relative value of additional orders for different number of replenishments (2 and 3 total orders) and different seasonality shapes (front-heavy, centered, and back-heavy). The highest value of reorders, under this coverage scheme, comes from the two front-heavy seasonalities cases with two and three total orders. The next two cases are for 3 orders center and 3 orders back, followed by 2 orders center and 2 orders back. We see that a front-heavy product has the most to gain from additional ordering opportunities under a 0.7 coverage scheme, for the same reasons as in the optimal ordering policy case. We also see the relative increase in this value when the variance of $ESD$, or forecast error, is higher.

An interesting question we examine next is what the best coverage policy actually is. In
**Figure 4.12:** Front-heavy demand, stochastic forecast error, average profit under different ordering schemes and replenishment opportunities

**Figure 4.13:** Center seasonality demand, stochastic forecast error, average profit under different ordering schemes and replenishment opportunities
Figure 4.14: Back-heavy demand, stochastic forecast error, average profit under different ordering schemes and replenishment opportunities

Figure 4.15: 0.7 coverage policies, % profit gain over newsvendor approximation, $E\hat{S}D_1 = \overline{ESD}$
In other words, if we look at the first order quantity under optimal control, what percentage of $ESD$ is this number. This analysis is shown in Figure 4.16. We see there is no clear factor that determines the optimal coverage percentage. While one-replenishment cases generally prefer higher coverages than two-replenishment cases, the front-heavy two-replenishment case has a higher percentage than other two-replenishment cases. For a given number of orders, however, the sequence of front-heavy, centered, followed by back-heavy seasonality in decreasing coverage holds true.

In Figure 4.17, Figure 4.18, and Figure 4.19, we show the effect on profit that various coverage policies have. We test coverage policies of 0.3, 0.5, 0.7, and 0.9. Of note in these figures is the relative robustness of 0.7, giving credence to it being used in industry. Only in the back heavy case with three orders is there a clear negative effect from using 0.7 as opposed to a lower coverage percentage.

In our final analysis, we consider the effect of the coverage policy for a specific $ESD$ variance. In Figure 4.20, we look at the profitability of various coverage policies when two-replenishments are available and the $ESD$ variance is low. In Figure 4.21, we repeat a similar
Figure 4.17: Profit analysis of various coverage policies, front-heavy demand

Figure 4.18: Profit analysis of various coverage policies, centered demand
Figure 4.19: Profit analysis of various coverage policies, back-heavy demand

analysis when $ESD$ variance is high. An interesting result here is that high-variance cases, regardless of seasonality, prefer a lower coverage percentage than their low-variance counterparts. In high variance cases, replenishment opportunities allow a decision maker to under-commit initially in case demand is extremely low, but use the replenishments to make up the difference in a high $ESD$ season.

### 4.6 Summary and Conclusion

To summarize the findings, we list below the average value of two managerial aspects to the problem— the value of perfect information, and the value of additional replenishment opportunities. To calculate the average, we compare the alternative cases to the “base case” which we assume is the no-replenishment, newsvendor solution of the problem using management’s point forecast of demand and the known periodic stochasticity of demand to form what would be management’s estimate of the distribution of demand over the horizon. We compare the alternatives and the base cases in an environment of distributed $ESD$ with low, medium, and
Figure 4.20: Two-replenishments, low ESD variance, coverage effect on profit

Figure 4.21: Two-replenishments, high ESD variance, coverage effect on profit
high variance— which we assume is reflective of reality. The summary is shown in Table 4.2, Table 4.3, and Table 4.4. The results indicate that additional replenishment opportunities yield profit improvements in line with, if not exceeding, the value of perfect information. Management should view these results and compare the burdens/costs involved with placing additional orders with gathering additional marketing information. There will undoubtedly be some organizations where one is easier to come by than the other, but these results give a clear indication that additional replenishment opportunities can make up for the lack of information in most cases, regardless of forecast error variance or seasonality type.
Table 4.2: Front-heavy seasonality, value of replenishments and information summary

<table>
<thead>
<tr>
<th></th>
<th>Low variance</th>
<th>Medium variance</th>
<th>High variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value of perfect information</td>
<td>1.6%</td>
<td>15.9%</td>
<td>43.9%</td>
</tr>
<tr>
<td>Value of 1 additional replenishment</td>
<td>7.3%</td>
<td>19.3%</td>
<td>40.4%</td>
</tr>
<tr>
<td>Value of 1 additional replenishment and perfect information</td>
<td>7.8%</td>
<td>23.0%</td>
<td>52.6%</td>
</tr>
<tr>
<td>Value of 2 additional replenishments</td>
<td>9.1%</td>
<td>23.3%</td>
<td>49.0%</td>
</tr>
<tr>
<td>Value of 2 additional replenishments and perfect information</td>
<td>9.4%</td>
<td>24.7%</td>
<td>54.8%</td>
</tr>
<tr>
<td></td>
<td>Low variance</td>
<td>Medium variance</td>
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<tr>
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<td>--------------</td>
</tr>
<tr>
<td>Value of perfect information</td>
<td>0.9%</td>
<td>10.5%</td>
<td>29.3%</td>
</tr>
<tr>
<td>Value of 1 additional replenishment</td>
<td>5.6%</td>
<td>13.6%</td>
<td>28.0%</td>
</tr>
<tr>
<td>Value of 1 additional replenishment and perfect information</td>
<td>6.8%</td>
<td>16.9%</td>
<td>36.6%</td>
</tr>
<tr>
<td>Value of 2 additional replenishments</td>
<td>8.1%</td>
<td>17.6%</td>
<td>36.6%</td>
</tr>
<tr>
<td>Value of 2 additional replenishments and perfect information</td>
<td>10.1%</td>
<td>20.3%</td>
<td>40.5%</td>
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</table>
Table 4.4: Back-heavy seasonality, value of replenishments and information summary

<table>
<thead>
<tr>
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<th>Low variance</th>
<th>Medium variance</th>
<th>High variance</th>
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<tbody>
<tr>
<td>Value of perfect information</td>
<td>1.0%</td>
<td>10.9%</td>
<td>29.6%</td>
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<tr>
<td>Value of 1 additional replenishment</td>
<td>3.8%</td>
<td>13.0%</td>
<td>29.2%</td>
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<tr>
<td>Value of 1 additional replenishment and perfect information</td>
<td>7.6%</td>
<td>18.2%</td>
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<td>Value of 2 additional replenishments</td>
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<td>36.0%</td>
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<tr>
<td>Value of 2 additional replenishments and perfect information</td>
<td>10.0%</td>
<td>20.8%</td>
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</table>
In this work we present a new method for determining optimal order quantities in a finite-horizon, periodic-review inventory setting where an explicit determination of fixed ordering costs is not available and need not be estimated. Instead of asking a manager to determine his fixed ordering costs, we are able to present him with the cost benefits of ordering multiple times, and allow him to arbitrarily compare his burden of ordering with the potential benefits generated by additional orders. We then extend this model to account for uncertainty in the initial estimate of demand. First, we assume that the initial forecast error is given, and in a second extension, we assume that the forecast error itself follows a certain distribution. In this second extension, we present an inventory ordering model for the case when the expected seasonal demand ($ESD$) is not known with certainty, but a distribution of potential $ESD$ values can be formed by using the distribution of past actual to forecast ($\frac{A}{F}$) $ESD$ ratios. We present results and discuss insights for three cases of the distribution of the forecast error, i.e. the distribution of $ESD$. The three distributions of $ESD$ we consider are all unimodal and symmetric about $\overline{ESD}$ with varying levels of variance, where $\overline{ESD}$ is the mean of the distribution and can be considered managements point forecast. The distribution about $\overline{ESD}$ is then derived from their past forecasting ability which is captured in the distribution of $\frac{A}{F}$ ratios.
5.1 Introduction

Markdowns are an unfortunate reality, albeit an integral part of inventory management for products with short selling seasons. The option of implementing a markdown provides a retailer the means to generate end-of-season sales and move excess inventory that resulted from lower than expected demand. Markdowns allow a retailer to dynamically align supply and demand. This section evaluates the expected increase in seasonal profits for a product when there exists an option for marking down the selling price during the season. We assume that demand in each period for a single given product follows a known distribution with known parameters. Consistent with the behavior of retailers, we restrict markdowns to the latter portion of the season, which we will define later. Once marked down, the price may not increase for the remainder of the horizon. Additionally, once a product has been marked down from its initial selling price, no more replenishment orders for that product are made. We
assume that the lost sales penalty after the markdown is zero. This assumption stems from a
conventional understanding that products on the “clearance” rack are not expected to last long,
and certainly not expected to be there when a customer is looking for something in particular.
As such, there is no loss of goodwill if demand exceeds inventory after the markdown has
occurred. We assume a retailer places an initial order long before the start of the season at a
given cost $c$ per unit ordered. As noted in Gutgeld and Beyer (1995) [10], some retailers limit
the initial order to less than 70% of the total projected volume. As the season progresses, the
retailer may choose to replenish his inventory in any period $t$, which assumes a negligible
lead time, either from a central distribution center, warehouse, or through expedited shipping
means. Any goods remaining in inventory at the end of the season are “jobbed off” (i.e. sold
at a steep discount) to a secondary market, such as an outlet center, for a salvage value of $s$
per unit after the end of the horizon. Based on our conversations with subject matter experts
in the fashion retailing industry, we restrict the time that the markdown may occur to the
last third of the selling season, e.g. the markdown may occur only in periods 7,8,9, or 10 in a
ten-period season.

In Section 5.2, we first present the problem of interest, which is a finite-horizon inventory
model with restricted replenishment opportunities and the option of markdown. After an
introduction and formulation of this model, we turn our attention to an analytical subproblem
of the markdown decision found in the actual realistic problem in Section 5.3. The results
and insights from this analytical analysis will then be used to further the discussion of the
multi-period model where we then show numerical results of the value of replenishment
opportunities and the value of the markdown opportunity. We will also demonstrate how the
analytical results in Section 5.3 can be used to justify when management should not markdown
on the basis of inventory reduction alone (there may be other reasons to markdown, e.g.
marketing or using one product as a loss leader, which we do not consider).
5.2 A Multi-Period, Finite-Horizon Inventory Model with Markdown

In this section, we outline a mathematical model to solve the finite-horizon inventory problem with an option for markdown. This model allows us to find optimal ordering and markdown policies. We use these optimal policies to solve for the value of markdowns and compare that value to the value of replenishment opportunities within the horizon.

5.2.1 Formulation

The timeline of events is shown in Figure 5.1. Before the beginning of the season an initial order is placed, and this order arrives at the beginning of period $t = 1$. At the beginning of each subsequent period, $t = 2, \ldots, T$, an ordering and/or markdown decision is made and, if necessary, an order is placed. Demand then occurs. If a markdown has not been executed yet, goodwill loss is calculated if demand exceeds beginning of period inventory. Holding costs are subsequently calculated on the number of units of inventory carrying over into the next period.

Figure 5.1: Timeline of events
Instead of quantifying a fixed component of ordering costs, we limit the number of replenishments which may be placed during the selling horizon. This allows us to examine the value of additional ordering opportunities and let the decision maker weigh the value of additional orders to a fixed cost (if it is known or can be estimated) or to some arbitrary burden associated with placing an order. We define $K$ as the total number of orders allowed (including the initial order) during the horizon.

Let $m$ denote the markdown level in the current period, which is either 0 if a markdown has not been made yet, or 1 if it has. We denote $p_m$ as the lost sales penalty given by the firm to represent the goodwill loss of an unsatisfied customer under markdown level $m$ in the current period as:

$$p_m = \begin{cases} p & \text{if } m = 0 \\ 0 & \text{if } m = 1 \end{cases}.$$  

Similarly, we define the revenue under markdown level $m$ as follows:

$$r_m = \begin{cases} r & \text{if } m = 0 \\ r(1 - \delta) & \text{if } m = 1 \end{cases},$$

where $\delta$ is the “percent off” markdown.

We note that if $m = 1 \Rightarrow m_{t+1} = 1$, i.e. once a markdown is made, it cannot be undone.

Define $f_{t,m}(\xi)$ as the demand density function in period $t$ under markdown level $m$. We denote the single-period expected profit in period $t$, with an inventory level after any orders arrive, of $x$, a current markdown level $m$ as $L_{t,m}(x)$, shown in Eq. 5.1.

$$L_{t,m}(x) = \int_{-\infty}^{\infty} \left[ r_m \min(x, \xi) - h(x - \xi)^+ + p_m (x - \xi)^- \right] f_{t,m}(\xi) d\xi$$  \hspace{1cm} (5.1)

Let $V_{t,m}^k(I)$ be the expected maximum profit in the beginning of period $t$ with $k$ orders remaining when the system is in markdown level $m$. In each period, the decision maker
chooses an order quantity $Q$ if $k > 0$. Let $k' = k - 1$ if $Q > 0$ and $k' = k$ if $Q = 0$, where $k'$ is the number of orders remaining in the next period. The decision maker also chooses to markdown or not if $m = 0$. Let $m'$ represent the markdown decision, such that $m' = m$ if no markdown decision was made, or $m' = 1$ if either a decision to markdown was made in the current period or had been made in a previous period. We denote the known distribution function of demand in period $t$, under markdown level $m$ as $f_{t,m}$. We discuss later the relationship between $f_{t,0}$ and $f_{t,1}$, which depends on demand elasticity and the magnitude of $\delta$. The recursive profit equations of the model are shown in Eq. 5.2-Eq. 5.4.

\[
V_{t,m}^k(I) = \max_{Q \geq 0, m' \geq m} \left[ L_{t,m'}(I + Q) + \int_x \left[ -cQ + V_{t+1,m'}^k(I + Q - \xi)^+ \right] f_{t,m'}(\xi) d\xi \right],
\]
\[\forall (t = 1, \cdots, T, \ k > 0) \quad (5.2)\]

\[
V_{t,m}^k(I) = \max_{m' \geq m} \left[ L_{t,m'}(I) + \int_x \left[ + V_{t+1,m'}^k(I - \xi)^+ \right] f_{t,m'}(\xi) d\xi \right],
\]
\[\forall (t = 1, \cdots, T, \ k = 0) \quad (5.3)\]

\[
V_{T+1,m}(I) = sI \quad \forall (k, m) \quad (5.4)
\]

Eq. 5.4 represents the salvage value of leftover inventory, where $s$ is the per unit salvage value.

### 5.2.2 Modeling demand

Define $ESD$ as the expected seasonal demand, which is the sum of each period’s mean demand, and define $PR_t$ such that $\mu_{t,m=0} = ESD \cdot PR_t$, i.e. the mean demand in period $t$ is
a proportion $PR_t$ of $ESD$ if no markdown has been made. A plot of $PR_t$ across the horizon represents a seasonality curve, or product lifecycle, of a particular product. Let $f_{t,0}$ be the negative binomial demand density function, with mean $\mu_{t,0}$ and success probability $nb_p$. The periodic mean demand $\mu_t = PR_t \cdot ESD$ and success parameter $nb_p$ completely defines the distribution of demand in each period.

The negative binomial distribution has several properties which makes it conducive to modeling demand in a retail setting. First, it is bounded below by zero, secondly it is discrete, and lastly it is approximately normally shaped for the higher values of $\mu_{t,m}$ we are considering. In our problem context, we assume returns are insignificant compared to sales, and our products are sold in units of one.

When the selling price is marked down to a lower price, the mean demand naturally increases. This inverse relationship is captured in the elasticity of demand to changes in price. This relationship can be calculated from past products with similar characteristics. In our work we consider a simple model of demand elasticity, such that $f_{t,1} \sim Negbin(\mu_{t,0}e, y)$, where $e$ is the elasticity of demand corresponding to markdown level “percent off” $\delta$, such that $r_1 = r_0(1-\delta)$. Since we consider only one markdown level, there is no need to define the relationship between $e$ and $\delta$, and we assume such a relationship can be provided by the decision maker. In our analysis, we consider different relationships of these two parameters.

Before showing numerical results of the model presented in this section, we first take a step back and examine a simplified variant of the markdown problem outside of our problem context. As we will show, this simplification can be directly applied to our model and allow us to quickly determine in which cases the markdown decision needs not to be considered.

### 5.3 Analytical Results

In this section, we look at the general markdown problem facing a decision maker and show some simplified analytical expressions which not only demonstrate managerial insight into the markdown problem, but also provide bounds that can guide a decision maker in the
markdown process within our previously described model. To show these insights, we use a combination of analytical formulations and numerical examples.

Consider a ten-period season with below average sales such that there is a high level of inventory remaining in the beginning of period eight. Management suspects that inventory, \( I \), is so high, that it is doubtful it will be sold in the next three periods. In this case, there is very little value in any potential reordering opportunity as it is highly unlikely that demand will exceed inventory in the next three periods. Management is left with the decision of whether to markdown or not. The decision maker must evaluate two possible choices under this setting. Choice 1 is to markdown, increasing demand, but decreasing unit revenue and sacrificing some salvage value. Choice 2 is to not markdown, accept the original demand and unit revenue, and get some salvage value (where \( s < r (1 - \delta) \)) at the end of the season from all the remaining excess inventory. The expected profit, \( E [P_1(I)] \) and \( E [P_2(I)] \) of these two choices is shown in Eq. 5.5 and Eq. 5.6 respectively. In this section, we drop the \( t \) subscript for ease of exposition. The following results should be seen independently from the periodic demand problem shown above. After this presentation of analytical results, we will then tie the results back to our periodic demand, restricted replenishment problem.

\[
E [P_1(I)] = \int_0^I r x + s (I - \xi) f_0(\xi) d\xi + \int_1^\infty r I - p (\xi - I) f_0(\xi) d\xi \tag{5.5}
\]

\[
E [P_2(I)] = \int_0^I r (1 - \delta) \xi + s (I - \xi) f_1(\xi) d\xi + \int_1^\infty r (1 - \delta) I f_1(\xi) d\xi \tag{5.6}
\]

The markdown decision can be reduced to a simple inequality: If:

\[
\Delta(I) = E [P_1(I)] - E [P_2(I)] < 0, \tag{5.7}
\]
5.3.1 Deterministic demand

In Figure 5.2, we plot $\Delta(I)$ for a random problem instance to illustrate the concept. In Figure 5.2 and the analysis to follow, we make the simplifying assumption that demand is deterministic (i.e. $\sigma = 0$). When the plot is below zero on the y-axis, $\Delta < 0$, and it is favorable to markdown. On the x-axis, we plot inventory level.

As Figure 5.2 shows, there are two regions where a markdown is preferable to not marking down. These two regions occur at the two tails of graph, where inventory is very low and where inventory is very high. A detailed analytical look at these two extremes, i.e. where $I = 0$, and $I \to \infty$, reveals an exact formulation of the value of the markdown option at these two locations. In other words, we can solve analytically the increase in profit that a markdown would provide over not marking down for these inventory levels. This analysis for $I = 0$ is presented in Eq. 5.8, and for $I \to \infty$ in Eq. 5.9. For the case where $I \to \infty$, it is worth noting

Figure 5.2: $\Delta$ as a function of inventory for one example with $r = 2.49$, $p = 0.39$, $s = 0.41$, $\delta = .5$, $\mu = 100$, $e = 3$, $D \sim N(100, 0)$

then markdown. Otherwise, do not markdown.
this value provides a bound on the value of the markdown.

\[ \Delta (I = 0) = \int_{x=0}^{I} r x + s (I - x) f(x) \, dx + \int_{I}^{\infty} r I - p (x - I) f(x) \, dx - \int_{x=0}^{I} (1 - \delta) r x + s (I - x) g(x) \, dx + \int_{I}^{\infty} (1 - \delta) r I g(x) \, dx \]

\[ = r I - p (\mu - I) - (1 - \delta) r I \]
\[ = -p \mu \] (5.8)

The result of Eq. 5.8 states that penalty costs could be saved if a markdown were executed at 0 inventory because, as we noted earlier, in our model \( p_1 = 0 \). We note that Eq. 5.8 is only theoretically true if \( k = 0 \) (there were no more orders available). If orders are available, and inventory is zero, the markdown decision will clearly not be entertained by management and an order would be placed.

\[ \lim_{I \to \infty} \Delta (I) \Rightarrow \]

\[ = \int_{x=0}^{I} r x + s (I - x) f(x) \, dx + \int_{I}^{\infty} r I - p (x - I) f(x) \, dx - \int_{x=0}^{I} (1 - \delta) r x + s (I - x) g(x) \, dx + \int_{I}^{\infty} (1 - \delta) r I g(x) \, dx \]

\[ = r I + s (I - \mu) - [(1 - \delta) r e \mu + s (I - e \mu)] \]
\[ = \mu (r - s)(1 - e) + re \mu \delta \] (5.9)
We note that not all problem instances result in a figure like the one shown in Figure 5.2, where a portion of the graph falls below the x-axis. In fact, we can analytically derive the exact set of circumstances which must occur for \( \lim_{I \to \infty} \Delta(I) < 0 \), where a markdown would at least be considered for some inventory level. If the opposite holds, \( \Delta(I) > 0 \quad \forall I \), then a markdown should never be considered. Looking at Eq. 5.9, we see that the notable parameters are \( r, s, e, \) and \( \delta \). Some of these parameters, in certain settings, might be exogenous, while in other settings, they could be set by the decision maker. With that in mind, we present Figure 5.3 and Figure 5.4. These graphs show the relationship of the parameters to the value of a markdown. Specifically, Figure 5.3 shows the threshold value of the \( \frac{s}{r} \) ratio for any given markdown level \( \delta \) to make a markdown potentially viable with \( e \) fixed. Figure 5.4 shows what the elasticity needs to be for any given \( \frac{s}{r} \) ratio to make a markdown potentially profitable with \( e(\delta) = 2 \) fixed. These results are useful for management to determine what levels of pricing \( (r) \) and/or markdown \( \delta \) are required for the markdown to potentially be profitable. The lines represent a frontier, below which are all points where the combination of parameters lead to no-markdown outcomes.
We can also draw analytical conclusions regarding the upward and downward slopes of $\Delta$ using simple geometry and the results of Eq. 5.8 and Eq. 5.9. For the region between $I = 0$ and $I = \mu$, the slope can be calculated to be $r \left(1 - \delta\right) + p$, whereas the slope of the section between $I = \mu$ and $I = e\mu$ can be calculated as $s - r \left(1 - \delta\right)$. We make a simple assumption that the tails of $f$ and $g$ do not overlap with any meaningful probability. This is certainly the case with known demand, and is not practically prohibitive for higher values of $e \left(\delta\right)$ that we consider.

Furthermore, with the simple substitution of $I = \mu$ for the deterministic demand case, we can solve for the maximum point of the curve in Figure 5.2, calculated in Eq. 5.10. This is the point where the markdown is least useful. Clearly, this point is where inventory equals the exact value of (known) demand. When inventory moves away from this maximum point (in either direction), a quick glance at the slopes calculated above show why a markdown starts to become more appealing.

Figure 5.4: Minimum markdown level required to make markdown worthwhile, $s/r = 0.12$
\[
\Delta(I = \mu) = \int_{x=0}^{I} r x + s(1-x) f(x) \, dx + \int_{I}^{\infty} r I - p(x-I) f(x) \, dx \\
- \left\{ \int_{x=0}^{I} (1-\delta) r x + s(1-x) g(x) \, dx + \int_{I}^{\infty} (1-\delta) r I g(x) \, dx \right\} \\
= r\mu + r\mu - [0 + (1-\delta) r\mu] \\
= r\mu - r\mu (1-\delta) \\
= r\mu \delta. \tag{5.10}
\]

The deterministic demand case also allows us to examine the inventory level \( I \) where \( \Delta(I) = 0 \). When inventory is greater than this threshold, the markdown is preferable to not marking down. Below the inventory threshold, it is more profitable to not mark down. There are two such thresholds, but we are interested in the larger of the two as this is the point where a markdown is made for the purpose of reducing inventory, not avoiding lost sales penalties. The inventory threshold value is calculated from the maximum point, \( r\mu \delta \) and the downward slope of \( s - r(1-\delta) \), and is at the point where:

\[
\begin{align*}
 r\mu \delta + [s - r(1-\delta)](I-\mu) &= 0 \\
 r\mu \delta + s I - s\mu - (r - r\delta)(I-\mu) &= 0 \\
 s I - s\mu - r I + r\mu + r\delta I &= 0 \\
 I &= \frac{\mu(s-r)}{s-r+r\delta}. \tag{5.11}
\end{align*}
\]

### 5.3.2 Stochastic demand

We now look at the case when demand is not deterministic, and instead follows a given probability distribution with known parameters. This clearly reflects the reality of demand more closely. In this section, we derive analytic results and insights into the markdown
decision when demand is inherently stochastic, and apply these results in Section 5.2 where stochastic demand is assumed in every period of a finite horizon.

Of particular interest is that $\Delta(0)$ and $\lim_{I \to \infty} \Delta(I)$, and to an extent the slopes discussed previously do not change with the introduction of demand uncertainty (in the form of positive standard deviation). This result is shown visually in Figure 5.5. For simplicity, let us denote $\Delta_{\text{deterministic}}$ when referring to the deterministic case and $\Delta_{\text{stochastic}}$ when referring to the probabilistic case.

An analytically powerful result is that $\Delta_{\text{deterministic}}(I)$ provides bounds to the probabilistic case. We can show this with a few simple assumptions. First, note that for non-overlapping demand distributions $f_0$ and $f_1$, $\Delta_{\text{deterministic}}(I) = \Delta_{\text{stochastic}}(I)$ along the upward and downward slopes. Secondly, note that $\Delta(0)$ and $\lim_{I \to \infty} \Delta(I)$ are also the same in the known and stochastic demand cases. It remains to be shown that when $I = \mu$, $\Delta_{\text{deterministic}}(I) > \Delta(I)$; and when $I = e \mu$, $\Delta_{\text{deterministic}}(I) < \Delta_{\text{stochastic}}(I)$. These propositions are shown in proposition 5.3.1 and proposition 5.3.2, respectively. These results, along with the fact that $\Delta_{\text{stochastic}}(I)$
is easily shown to be convex, will show that the known case (which is easily computable) provides bounds on the value of a markdown for all cases.

Similar to the discussion following Eq. 5.9, we note that if \( \Delta(I) > 0 \), then no amount of inventory will make the markdown profitable. Now, combined with the fact that \( \Delta(I) \) is a limit for the stochastic case presented here, it reasons that \( \Delta_{\text{deterministic}} > 0 \) is an accurate way of quickly checking if markdowns should ever be considered for a particular problem instance.

In proposition 5.3.1, we turn our attention to the top region of Figure 5.5, where inventory is near the expected value of demand. We show that the known case is an upper bound to the unknown case, or that \( \Delta_{\text{deterministic}}(I) \geq \Delta_{\text{stochastic}}(I) \).

**Proposition 5.3.1.** Here we show that \( \Delta_{\text{stochastic}} \leq \Delta_{\text{deterministic}} \) when evaluated at \( I = \mu \).

\[
\Delta_{\text{stochastic}}(I = \mu) \leq \Delta_{\text{deterministic}}(I = \mu)
\]

\[
\Rightarrow \int_{x=0}^{\mu} r x + s (\mu - x) f(x) dx + \int_{\mu}^{\infty} r \mu - p (x - \mu) f(x) dx - \\
\left\{ \int_{x=0}^{\mu} (1-\delta) r x + s (\mu - x) g(x) dx + \int_{\mu}^{\infty} (1-\delta) r \mu g(x) dx \right\} \leq r \mu \delta
\]

\[
\Rightarrow \int_{x=0}^{\mu} r x + s (\mu - x) f(x) dx + \int_{\mu}^{\infty} r \mu - p (x - \mu) f(x) dx - (1-\delta) r \mu \leq r \mu - r \mu (1-\delta)
\]

\[
\Rightarrow \int_{x=0}^{\mu} r x + s (\mu - x) f(x) dx + \int_{\mu}^{\infty} r \mu - p (x - \mu) f(x) dx \leq r \mu
\]

Proposition 5.3.1 boils down to a trivial statement that profit with known demand is higher than the expected profit under some unknown, stochastic demand with the same expected value. This is clearly true, as uncertain outcomes on both the high and low side of expected demand have cost consequences.

In proposition 5.3.2, we show a similar analysis for the case when \( I = e\mu \). If we can show that \( \Delta_{\text{stochastic}}(I = e\mu) \geq \Delta_{\text{deterministic}}(I = e\mu) \), then coupled with the earlier results that \( \Delta_{\text{deterministic}}(I = 0) = \Delta_{\text{stochastic}}(I = 0) \), \( \lim_{I \to \infty} \Delta_{\text{deterministic}}(I) = \lim_{I \to \infty} \Delta_{\text{stochastic}}(I) \), the assumption that the slopes are overlapping, and the assumption of convex profit curves, we can
say that the easily computed deterministic case provides a bound for the value of a markdown. In other words, a markdown will never be of more value (negative point on the graph), than in the deterministic case, for any \( I \).

**Proposition 5.3.2.** This proposition mirrors Eq. 5.3.1, with \( \Delta_{\text{stochastic}} \leq \Delta_{\text{deterministic}} \) evaluated at \( I = e \mu \).

\[
\Delta_{\text{stochastic}}(I = e \mu) \geq \Delta_{\text{deterministic}}(I = e \mu)
\]

\[
\Rightarrow \int_{x=0}^{I} r x + s (I - x) f(x) \, dx + \int_{I}^{\infty} r I - p (x - I) f(x) \, dx - \\
\left\{ \int_{x=0}^{I} (1 - \delta) r x + s (I - x) g(x) \, dx + \int_{I}^{\infty} (1 - \delta) r I g(x) \, dx \right\} \geq r (1 - \delta) e \mu
\]

\[
\Rightarrow r \mu + s (e \mu - \mu) - \left\{ \int_{x=0}^{e \mu} (1 - \delta) r x + s (e \mu - x) g(x) \, dx + \int_{I}^{\infty} (1 - \delta) r e \mu g(x) \, dx \right\} \geq \\
r \mu + s (e \mu - \mu) - r (1 - \delta) e \mu
\]

\[
\Rightarrow \int_{0}^{e \mu} \left[ r (1 - \delta) x + s (e \mu - x) \right] g(x) \, dx + \int_{e \mu}^{\infty} r (1 - \delta) e \mu g(x) \, dx \leq r (1 - \delta) e \mu
\]

Proposition Eq. 5.3.2 states that if a markdown has been made and lost sales penalty is no longer imposed, the profit with known demand is higher than the expected profit of some unknown, stochastic demand distribution, which is trivially true.

It is worth noting that the threshold value derived in Eq. 5.11 for the known demand case is a very close approximation to the threshold value in stochastic cases. While we do not quantify this relationship here, when the two distributions of pre- and post-markdown demand have negligible overlap, the sloping, linear area between \( I = \mu \), and \( I = e \mu \) are sufficiently close to each other and generally, this is the area where the threshold is likely to occur.
5.4 Value of a Markdown Opportunity

In this section, we return to our original problem of a finite-horizon inventory problem with restricted replenishment and the opportunity for markdown. We employ here the lessons learned from the analytical results found earlier, mainly under what set of conditions does a markdown not need to be considered, and a simpler decision process can be solved. To aid in this, let us define \( \Delta'(I) = -\lim_{I \to \infty} \frac{\Delta_{\text{deterministic}}(I)}{\mu} = r(1 - e + e \delta) - s(1 - e) \). Since \( \Delta' \) is the negative of, and is proportional to, \( \Delta_{\text{deterministic}}(I) \), a markdown is non-optimal when \( \Delta'(I) < 0 \).

For these results, we assume negligible lead time for reorders placed during the selling season. This assumption closely represents a small retailer ordering from a much larger distributor, or a retail shop ordering from a central corporate distribution center.

Of interest are three questions: 1) What is the value of additional replenishment opportunities? 2) What is the value of the markdown option? 3) What are the key cost parameters which affect these results?

To answer these questions, we find the optimal policy and corresponding optimal profit of our model using a Markov Decision Process. Given a set of cost parameters (an instance), and a potential markdown level, we find the optimal profit when \( K = 1, 2, \) and \( 3 \), where \( K \) is the total number of orders (which includes the preseason order and any subsequent replenishments) that a decision maker may utilize during the season.

We consider three different seasonality shapes in this analysis. Seasonality shapes are constructed from the seasonality proportions \( (PR, \forall t) \). The three shapes we consider are front-heavy, centered, and back-heavy. A visual representation of these seasonality types are shown in Figure 5.6, Figure 5.7, and Figure 5.8.

Recall that once a decision to markdown has been made, no more orders may be made, and the selling price will not change from that point forward. We are interested in finding the effects of the selling price \( r \), goodwill penalty \( p \), end of horizon salvage value \( s \), and per period holding cost \( h \) on the value of replenishments when a markdown opportunity exists. The raw materials cost \( c \) and all subsequent replenishment materials costs are arbitrarily scaled to $1.
Figure 5.6: Front-heavy seasonality shape

Figure 5.7: Center seasonality shape

Figure 5.8: Back-heavy seasonality shape
We generate 250 random problem instances with parameters randomly (uniformly) selected from the ranges listed in Table 5.1, and use these same 250 instances for front-heavy, centered, and back-heavy demand.

Table 5.1: Experimental design parameters range, $\delta = 0.5$, $e(\delta) = 2$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>revenue, $r$</td>
<td>1.5</td>
<td>3</td>
</tr>
<tr>
<td>lost sales penalty, $p$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>salvage value, $s$</td>
<td>0</td>
<td>$\min(1, r (1 - \delta))$</td>
</tr>
<tr>
<td>holding cost, $h$</td>
<td>0</td>
<td>0.03</td>
</tr>
<tr>
<td>$nb_p$</td>
<td>0.04</td>
<td>0.2</td>
</tr>
</tbody>
</table>
Figure 5.9: Front-heavy seasonality, correlation between $\Delta'$ and the value of the markdown, no-replenishment

In Figure 5.9, Figure 5.10, and Figure 5.11 we plot the value of the markdown (as a percentage gain from the same instance with no markdown opportunity) when $K = 1$. We plot this value as a function of $\Delta'$. When $\Delta'$ is negative, the value of a markdown is negative (not profitable) as $I \to \infty$, and therefore won't be used for any inventory level. Figure 5.9, Figure 5.10, and Figure 5.11 show one dot for each of the 250 random instances.

There are several results to be gleaned from Figure 5.9, Figure 5.10, and Figure 5.11. First, the value of a markdown is zero when $\Delta'$ is negative. In fact, we see for some cases, especially back-heavy goods, that the value of the markdown is very often zero. The reason for this can be explained partly by Figure 5.12. In our experimentations, we found that for problems with no-replenishment, the order quantity for products with front-heavy demand is higher than the order quantity with back-heavy demand. The magnitude of these differences is shown in Figure 5.12. Of course, a decision maker wants to position the firm to be able to take advantage of above average demand, while not leaving himself with excess inventory if demand is below average. We propose that the front-heavy demand allows a decision maker
Figure 5.10: Centered seasonality, correlation between $\Delta'$ and the value of the markdown, no-replenishment

Figure 5.11: Back-heavy seasonality, correlation between $\Delta'$ and the value of the markdown, 1 order
to make this gamble more so than back-heavy goods. Consider the front-heavy seasonality and a decision maker who orders more than expected in the hopes of capturing high sales. If the gamble pays off, and there are high sales, he is satisfied. If demand is below average (50% chance of happening), the firm simply carries the excess goods for the remainder of the season and salvages them or marks them down at the end. The risk of this strategy is excess holding costs 50% of the time. On the other hand, in a back-heavy situation, the holding costs will be incurred no matter what, since the bulk of the sales towards the end. If demand is low, this information won't be known until the very end, when any markdown decision is unlikely to help since it is too late. For this reason, the front-heavy goods order more, and are able to better use the markdown opportunity to their advantage.

Figure 5.9, Figure 5.10, and Figure 5.11 also show that there are many cases when a markdown should not be used under optimal control. This result is at odds with many industries, especially fashion retailing, where it is assumed that the product will be “marked down” at some point, and certainly a look at the local shopping center would confirm that most products have some sort of discount attached to them. We reconcile these differences by noting that markdowns are used as tools for purposes other than simply reducing leftover stock at the end of the season, or maximizing profit in the face of excess inventory. Rather, it would seem that there are other reasons to use markdowns, such as driving traffic into the store, or because it is expected by the customers who want a “good deal”.

We complete the analysis with Figure 5.13- Figure 5.18 which show the value of the markdown when one or two replenishments are available during the horizon. The figures show that the value of the markdown significantly declines when multiple orders are available. In fact, multiple orders negate the value of the markdown to negligible levels for most problem instances when the product follows a back-heavy seasonality. These results are magnified in Figure 5.16, Figure 5.17, and Figure 5.18, which show the value of the markdown and the correlation to $\Delta'$ when three total orders are available to the decision maker.

We have shown that in a stochastic environment with multiple replenishment opportuni-
Figure 5.12: Order quantity difference between products with front-heavy and back-heavy demand (front-back), correlated to $\Delta'$, 1 order, 250 random instances

Figure 5.13: Front-heavy seasonality, correlation between $\Delta'$ and the value of the markdown, 2 orders
Figure 5.14: Centered seasonality, correlation between $\Delta'$ and the value of the markdown, 2 orders

Figure 5.15: Back-heavy seasonality, correlation between $\Delta'$ and the value of the markdown, 2 orders
Figure 5.16: Front-heavy seasonality, correlation between $\Delta'$ and the value of the markdown, 3 orders

Figure 5.17: Centered seasonality, correlation between $\Delta'$ and the value of the markdown, 3 orders
ties, where the distribution of demand and its parameters are known, the value of replenishment orders is greater than the value of a 50% markdown option. We now consider multiple demand elasticity parameters to show that the relative value of the markdown increases as elasticity increases. Figure 5.19 shows the value of the markdown opportunity across 250 random instances chosen from Table 5.1. We note that some problem instances resulted in a zero value of the markdown. These cases were included in the averages presented in Figure 5.19.

What stands out in Figure 5.19 is the relative magnitude difference in the value of the markdown with no replenishments versus one or more replenishments. We see that markdowns are relatively more valuable to a decision maker with no replenishments. As the number of replenishments increases, a markdown becomes less valuable because additional replenishments can be used more effectively to hedge against uncertain demand, minimizing high end-of-season inventory levels and the need to markdown.

In Figure 5.20, we show the benefit of additional replenishment opportunities in an en-
environment with markdown opportunities. We see a decision maker currently only utilizing one order with markdowns can increase his profits by adding just one replenishment during the season. Figure 5.20 shows the value of additional replenishment opportunities without markdown and with the option of markdown for elasticity values ranging from two to five. This figure also shows the value of replenishment opportunities alone, without markdown, and how much additional profit can be made with replenishment opportunities and markdowns together.

5.5 Conclusion

In this chapter we have shown both analytical results and numerical results that explore the value of a markdown and compare it to the value of replenishment opportunities for a decision maker with a seasonal product. We have shown how the seasonality of the product affects these values, as well as the parameters themselves. We show that the value of a markdown can be negligible for certain problem instances, and we define these instances analytically. We also
provide analytical results which define the maximum and minimum value of a markdown with known demand, and we show how these results provide bounds to a problem with stochastic demand.

Our work makes the assumption that demand is either fully known, or it follows a known stochastic distribution. Even the latter case can be argued to be unrealistic, as seasonal products are often “hit or miss”. We leave for future work the possibility of unknown parameters of the demand distribution itself. For example, we can define a distribution of $ESD$, which would implicitly define a distribution of the mean demand in every period. We surmise the value of the markdown would increase under these conditions.

Our analysis shows that the probability, or likelihood, of a markdown following an optimal policy according to the given model is relatively small. This result is in direct conflict with what is observed in practice where markdowns are a way of life. We note that markdowns also serve a marketing purpose which is not reflected in our model. Our model takes into account the value of markdowns only as a tool to reduce the amount of excess or leftover
inventory at the end of the horizon, when that inventory could produce more profit if sold under increased demand instead of being salvaged. In practice, we see markdown “schedules”, representing management that feel markdowns are a necessity no matter the relative success of a product. Management in many stores, especially fashion retail stores as an example, feel that consumers expect markdowns, and as such plan for markdowns in the latter parts of the season as a general rule, not a decision.
In this work, we propose multiple models that have in common that no explicit fixed costs need to be specified. We first consider a single period model, similar to the classical newsvendor problem, with the addition of one replenishment opportunity during the season. The classical "newsvendor problem" in the management sciences solves for the optimal ordering quantity when there is a single period with a known single demand distribution. Our first research question is how to extend this classical model when there exist two opportunities for ordering within the horizon. We develop analytical formulations to solve this extension optimally.

In Chapter 3, we propose an extension of the single-period case with one-replenishment to a more general multiple-period model with multiple (restricted) replenishment opportunities. This model does not require the specification of a fixed ordering cost, yet takes into account the inherent burden involved with placing orders. To solve this problem, we employ a Markov Decision Process based solution methodology, where the costs of over-ordering and under-ordering are weighed against each other to determine the optimal ordering quantity for any period, when the decision maker has a given number of orders remaining. We also propose an
approximation to solve this problem and show how well the approximation performs across different instance types. In Chapter 4, we extend this model to allow a decision maker to mark down the selling price of the product which in turn increases demand.

In Chapter 5, we solve a related problem when the demand distribution parameters are unknown. We propose two methods of specifying the uncertainty of the demand distribution parameters. We show to what extent additional orders can alleviate some of the risks involved in ordering under these assumptions. To solve this problem, we employ a joint simulation and Markov Decision Process solution methodology that utilizes both optimal and near-optimal solutions for the known demand case in a simulation environment consisting of unknown true demand. We show the profit difference known and unknown demand parameter cases when multiple orders are available.
REFERENCES


APPENDIX
A.1 Chapter 2 appendix

General replenishment order quantity. If a replenishment is called for, \( Q_1 \) has been depleted. Define \( f(x) \) and \( F(x) \) as the probability density and cumulative density functions, respectively, of total seasonal demand. Since \( Q_1 \) has already been observed, the distribution of total seasonal demand can be amended as

\[
 f(x| x \geq Q_1) = \frac{f(x)}{1 - F(Q_1)} \quad \forall \quad x \geq Q_1. 
\]

Let \( y = Q_1 + Q_2 \), which is the total order up to quantity if \( Q_2 \) units are ordered for the replenishment. The decision maker’s replenishment profit equation, as a function of \( Q_2 \) becomes:

\[
-c_2 Q_2 + \int_{Q_1}^{y} \left[ r(x - Q_1) + s(y - x) \right] \frac{f(x)}{1 - F(Q_1)} \, dx + \int_{y}^{\infty} \left[ rQ_2 - p(x - y) \right] \frac{f(x)}{1 - F(Q_1)} \, dx \quad (A.1)
\]
Taking the derivative with respect to $Q^2$, setting equal to zero, and solving for $Q^2$ yields:

$$-c_2 + \int_{Q_1}^{y} s \frac{f(x)}{1 - F(Q_1)} + r(Q_2) \frac{f(x)}{1 - F(Q_1)} + \int_{y}^{\infty} (r + p) \frac{f(x)}{1 - F(Q_1)} - \left( rQ_2 \frac{f(x)}{1 - F(Q_1)} \right) = 0$$

$$-c_2 + \frac{s}{1 - F(Q_1)} \left[ F(y) - F(Q_1) \right] + \frac{r + p}{1 - F(Q_1)} (1 - F(y)) = 0$$

$$(r + p - s) F(y) = -c (1 - F(Q_1)) + r + p - s F(Q_1)$$

$$(r + p - s) F(y) = (r + p - c_2) + F(Q_1)(c_2 - s)$$

$$F(y) = \frac{r + p - c_2}{r + p - s} + F(Q_1)$$

$$F(y) = \frac{r + p - c_2}{r + p - s} + F(Q_1) \left[ 1 - \frac{r + p - c_2}{r + p - s} \right]$$

$$F(y) = \frac{r + p - c_2}{r + p - s} [1 - F(Q_1)] + F(Q_1)$$

$$y^* = F^{-1} \left\{ \frac{r + p - c_2}{r + p - s} [1 - F(Q_1)] + F(Q_1) \right\}$$

$$Q_2^* = F^{-1} \left\{ \frac{r + p - c_2}{r + p - s} [1 - F(Q_1)] + F(Q_1) \right\} - Q_1$$

$Q_2$ uniformly distributed demand. From above, we have $Q_2^* = F^{-1} \left\{ \frac{r + p - c_2}{r + p - s} [1 - F(Q_1)] + F(Q_1) \right\} - Q_1$. With a uniformly distributed demand distribution between $a$ and $b$, with constant probability density $\frac{1}{b-a}$, we can write:

$$Q_2 = F^{-1} \left[ \frac{b - Q_1}{b - a} + \frac{Q_1 - a}{b - a} \right] - Q_1$$

$$Q_2 = F^{-1} \left[ \frac{zb - zQ_1 + Q_1 - a}{b - a} \right] - Q_1$$
We can solve for the inverse of \( \frac{z b - zQ_1 + Q_1 - a}{b - a} \), referred to as \( x \), as follows:

\[
\begin{align*}
  x - a &= \frac{z b - zQ_1 + Q_1 - a}{b - a} \\
  x - a &= z b - zQ_1 + Q_1 - a \\
  x &= z b - zQ_1 + Q_1
\end{align*}
\]

Substituting \( x \), we can continue:

\[
Q_2 = z b - zQ_1 + Q_1 - Q_1
\]

\[
Q_2 = z (b - Q_1)
\]

\( \square \)

*Uniformly distributed* \( Q_1 \). From above, we have \( Q_2 = z (b - Q_1) \). Substituting into the profit equation, we have:

\[
E[Z(Q_1)] = -c_1 Q_1 + \int_0^{Q_1} [r \xi + s (Q_1 - \xi)] f(\xi) d\xi
\]

\[
+ \int_{Q_1 + z(b - Q_1)}^{Q_1 + z(b - Q_1)} [-c_2 (z (b - Q_1)) + r \xi + s (Q_1 + z (b - Q_1) - \xi)] f(\xi) d\xi
\]

\[
+ \int_{Q_1 + z(b - Q_1)}^{\infty} [-c_2 (z (b - Q_1)) + r (Q_1 + z (b - Q_1)) - p (\xi - (Q_1 + z (b - Q_1)))] f(\xi) d\xi
\]

Taking the derivative with respect to $Q_1$ yields:

$$-c_1 + \int_a^{Q_1} (s) f(x) \, dx + (rQ_1) f(Q_1)$$
$$+ \int_{Q_1 + z(b-zQ_1)}^{b} (c_2z + s - sz) f(x) \, dx$$
$$- [-c_2z + c_2zQ_1 + rQ_1 + s(zb - zQ_1)] f(Q_1)$$
$$+ \int_{Q_1 + z(b-zQ_1)}^{b} [c_2z + r - rz + p -pz] f(x) \, dx = 0$$

Which reduces further to:

$$-c_1 + s \frac{1}{b-a} (Q_1-a) + rQ_1 \frac{1}{b-a} + \frac{1}{b-a} (c_2z + s - sz)(zb - zQ_1)$$
$$+ \frac{1}{b-a} [c_2z(b - c_2zQ_1 - rQ_1 - s(zb - zQ_1))]$$
$$\frac{1}{b-a} (c_2z + r - rz + p -pz)(b - Q_1 - zb + zQ_1)$$

After multiplying every term by $\frac{1}{b-a}$, and distributing some, we find:

$$-c_1(b-a) + sQ_1 - sa + rQ_1 - rQ_1$$
$$+ zb \left(c_2z + s - sz + c_2 - s - c_2z - r + rz - p -pz \right)$$
$$-zQ_1 \left(c_2z + s - sz + c_2 - s - c_2z - r + rz - p -pz \right)$$
$$+ b \left(c_2z - r - rz + p -pz \right)$$
$$-Q_1 \left(c_2z - r - rz + p -pz \right) = 0$$

Grouping like terms, we have:

$$-c_1(b-a) - sa + br + bp - Q_1r - Q_1p$$
$$+(zb - zQ_1)(c_2z + s - sz + c_2 - s - c_2z - r + rz - p +pz + c_2 - r -p) = 0$$
After some cancellation, we are left with:

\[-c_1(b - a) - Q_1(r + p - s) - sa + b(r + p)\]
\[+ (zb - zQ_1)[z(r + p - s) - 2(r + p - c_2)] = 0\]

Recognizing that \(z(r + p - s) = \frac{r + p - c_2}{r + p - s}(r + p - s) = r + p - c_2\), we simplify:

\[-c_1(b - a) - Q_1(r + p - s) - sa + b(r + p)\]
\[+ (-zb + zQ_1)(r + p - c_2) = 0\]

Rearranging terms, we have the result:

\[Q_1^* = \frac{-c_1(b - a) - sa + b(r + p - z(r + p - c_2))}{r + p - s - z(r + p - c_2)}\]

\(a = 0\) uniformly distributed demand \(Q_1\) and \(Q_2\). Let \(c_1 = c_2 = c\), and \(z = \frac{r + p - c}{r + p - s}\). We have
shown that \( Q_1^* = \frac{-c_1(b-a)-s_a+b(r+p)-bz(r+p-c_2)}{r+p-s-z(r+p-c_2)} \). With \( a = 0 \), this reduces as follows:

\[
Q_1^* = \frac{-c b + b(r + p) - b z (r + p - c)}{r + p - s - z(r + p - c)}
\]

\[
Q_1^* = \frac{b(r + p - c) - b z (r + p - c)}{(r + p - s)^2 - (r + p - c)^2}
\]

\[
Q_1^* = \frac{b(r + p - c)(1 - z)}{r + p - s - z(r + p - c)}
\]

\[
Q_1^* = \frac{b(r + p - c)(1 - z)}{r + p - s - z(r + p - c)}
\]

\[
Q_1^* = \frac{b(r + p - c)(1 - z)}{r + p - s - z(r + p - c)}
\]

\[
Q_1^* = \frac{b(r + p - c)(1 - z)}{r + p - s - z(r + p - c)}
\]

Recognizing that \( 1 + z = \frac{(r+p-s)+(r+p-c)}{r+p-s} = \frac{2r+2p-s-c}{r+p-s} \),

\[
Q_1^* = \frac{b(r + p - c)(1 - z)}{(c - s)(1 + z)}
\]

Recognizing that \( \frac{1-z}{1+z} = \frac{r+p-s-(r+p-c)}{r+p-s} = \frac{c-s}{2r+2p-s-c} \),

\[
Q_1^* = \frac{b(r+p-c)}{r+p-s} = b \frac{z}{1+z}
\]

Recall that \( (Q_2|Q_1)^* = z(b - Q_1) \). If we substitute \( Q_1 \) from above: \( (Q_2|Q_1)^* = z \left( b - b \frac{z}{1+z} \right) = z b \frac{1}{1+z} \).
fixed cost to replenish, uniformly distributed demand. Without a fixed cost to order, recall that:

\[
E[Z(Q_1)] = -c_1 Q_1 + \int_0^{Q_1} [r \xi + s (Q_1 - \xi)] f(\xi) d\xi \\
+ \int_{Q_1}^{Q_1 + Q_2} [-c_2 Q_2 + r \xi + s (Q_1 + Q_2 - \xi)] f(\xi) d\xi \\
+ \int_{Q_1 + Q_2}^{\infty} [-c_2 Q_2 + r (Q_1 + Q_2) - p (\xi - (Q_1 + Q_2))] f(\xi) d\xi
\]

and recall that:

\[
Q_1^* = \frac{-c_1 (b - a) - s a + b (r + p) - b z (r + p - c_2)}{r + p - s - z (r + p - c_2)}
\]

With a fixed cost to replenish, \$A, which is assessed only if and when a replenishment order is placed, the profit equation is slightly altered as follows:

\[
E[Z(Q_1')] = -c_1 Q_1' + \int_0^{Q_1'} [r \xi + s (Q_1' - \xi)] f(\xi) d\xi \\
+ \int_{Q_1'}^{Q_1' + Q_2} [-A - c_2 Q_2 + r \xi + s (Q_1' + Q_2 - \xi)] f(\xi) d\xi \\
+ \int_{Q_1' + Q_2}^{\infty} [-A - c_2 Q_2 + r (Q_1' + Q_2) - p (\xi - (Q_1' + Q_2))] f(\xi) d\xi
\]

Taking the derivative is much like the proof of \(Q_1^\ast\) without a fixed cost to order with the addition of one term:

\[
-c_1 + \int_a^{Q_1'} (s) f(x) dx + r Q_1' f(Q_1') \\
+ \int_{Q_1'}^{Q_1' + z b - z Q_1'} (c_2 z + s - s z) f(x) dx \\
- [-A - c_2 z b + c_2 z Q_1' + r Q_1' + s (z b - z Q_1')] f(Q_1') \\
+ \int_{Q_1' + z b - z Q_1'}^b [c_2 z + r - r z + p - p z] f(x) dx = 0
\]
This single term is carried throughout the analysis, which otherwise mirrors the proof without the fixed cost to replenish. Eventually, we end up with

\[ A - c_1(b - a) - Q_1'(r + p - s) - sa + b(r + p) \]
\[ + (z b - z Q_1') [z(r + p - s) - 2(r + p - c_2)] = 0 \]

and finally:

\[ Q_1'^* = \frac{A - c_1(b - a) - sa + b(r + p - z(r + p - c_2))}{r + p - s - z(r + p - c_2)} \]

Which can be rewritten as:

\[ Q_1'^* = Q_1^* + \frac{A}{r + p - s - z(r + p - c_2)} \]

Where \( Q_1^* \) is the optimal order quantity without a fixed replenishment cost. Solving for the optimal replenishment order quantity, we have:

\[ Q_2'^* = z \left( b - Q_1'^* \right) \]
\[ Q_2'^* = z \left( b - \left( Q_1^* + \frac{A}{r + p - s - z(r + p - c_2)} \right) \right) \]
\[ Q_2'^* = z \left( b - Q_1^* - \frac{A}{r + p - s - z(r + p - c_2)} \right) \]
\[ Q_2'^* = z \left( b - Q_1^* - z \frac{A}{r + p - s - z(r + p - c_2)} \right) \]
\[ Q_2'^* = Q_1^* - z \frac{A}{r + p - s - z(r + p - c_2)} \]

\[ \square \]

*Uniformly distributed demand with downpayment contract.* When a downpayment is made for \( Q_2 \) units before the beginning of the season, the replenishment order decision mirror that
of the regular replenishment order decision, except only $kc_2$ is left to pay as the per unit raw materials cost instead of the regular $c_2$. As such, the replenishment solution mirrors that of the regular replenishment solution of $Q^*_2 = z(b - Q_1^*)$, with the slight change that $z = \frac{r + p - kc_2}{r + p - c_2}$ now.

The new profit equation for the season looks very similar to the regular profit equation, except for the new definition of $z$, adding $(1 - k)c_2Q_2$, and changing $c_2$ to $kc_2$ within the integrals. The new profit equation then becomes:

$$E \left[ Z(Q_1) \right] = -c_1Q_1 - (1 - k)c_2Q_2 \int_0^{Q_1} [r\xi + s(Q_1 - \xi)] f(\xi) d\xi$$

$$+ \int_{Q_1}^{Q_1 + Q_2} [-kc_2Q_2 + r\xi + s(Q_1 + Q_2 - \xi)] f(\xi) d\xi$$

$$+ \int_{Q_1 + Q_2}^{\infty} [-kc_2Q_2 + r(Q_1 + Q_2) - p(Q_1 + Q_2 - \xi)] f(\xi) d\xi$$

Converting $Q_2$ to $z(b - Q)$, and taking the derivative:

$$-c_1 + (1 - k)c_2z + \int_a^{Q_1} (s f(x)) d x + (rQ_1) f(Q_1)$$

$$+ \int_{Q_1}^{Q_1 + z(b - Q)} (kc_2z + s - sz) f(x) d x$$

$$- [-kc_2z b + kc_2zQ_1 + rQ_1 + s(zb - zQ_1)] f(Q_1)$$

$$+ \int_{Q_1 + z(b - Q)}^{b} [kc_2z + r - rz + p - pz] f(x) d x = 0$$

Which reduces further to:

$$-c_1 + (1 - k)c_2z + s \frac{1}{b - a}(Q_1 - a) + rQ_1 \frac{1}{b - a} + \frac{1}{b - a} (kc_2z + s - sz)(zb - zQ_1)$$

$$+ \frac{1}{b - a} [kc_2z b - kc_2zQ_1 - rQ_1 - s(zb - zQ_1)]$$

$$\frac{1}{b - a} (kc_2z + r - rz + p - pz)(b - Q_1 - zb + zQ_1)$$
After multiplying every term by $\frac{1}{b-a}$, and distributing some, we find:

$$
\begin{align*}
&\quad [-c_1 + c_2(1-k)z](b-a) + sQ_1 - sQ_1 + rQ_1 - rQ_1 \\
&+ z\left(kc_2z + s - sz + c_2 - s - kc_2z - r + rz - p - pz\right) \\
&- zQ_1\left(kc_2z + s - sz + kc_2 - s - kc_2z - r + rz - p - pz\right) \\
&+ b\left(kc_2z - r - rz + p - pz\right) \\
&- Q_1\left(kc_2z - r - rz + p - pz\right) = 0
\end{align*}
$$

Grouping like terms, we have:

$$
\begin{align*}
&\quad [-c_1 + c_2(1-k)z](b-a) - sa + br + bp - Q_1r - Q_1p \\
&+ (z b - zQ_1)\left(kc_2z + s - sz + kc_2 - s - kc_2z - r + rz - p + pz + kc_2 - r - p\right) = 0
\end{align*}
$$

After some cancellation, we are left with:

$$
\begin{align*}
&\quad [-c_1 + c_2(1-k)z](b-a) - Q_1\left(r + p - s\right) - sa + b\left(r + p\right) \\
&+ (z b - zQ_1)\left[z\left(r + p - s\right) - 2\left(r + p - kc_2\right)\right] = 0
\end{align*}
$$

Recognizing that $z\left(r + p - s\right) = \frac{r + p - kc_2}{r + p - s}\left(r + p - s\right) = r + p - kc_2$, we simplify:

$$
\begin{align*}
&\quad [-c_1 + c_2(1-k)z](b-a) - Q_1\left(r + p - s\right) - sa + b\left(r + p\right) \\
&+ (-z b + zQ_1)\left(r + p - kc_2\right) = 0
\end{align*}
$$

Rearranging terms, we have the result:

$$
Q_1^* = \frac{[-c_1 + c_2(1-k)z](b-a) - sa + b\left(r + p - z\left(r + p - kc_2\right)\right)}{r + p - s - z\left(r + p - kc_2\right)}
$$

\[\square\]
Table A.1: Comparison of newsvendor solution results to single period replenishment solution

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A.2 Chapter 3 appendix
Table A.2: Profit and value of replenishment opportunity results for 81 parameter instances

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Table A.3: Profit and value of replenishment opportunity results for 81 parameter instances

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