ABSTRACT

AL-SAGGAF, MAJID. Three Essays in CCAPM: Durable and Nondurable Consumption with Long-Run Risk. (Under the direction of Denis Pelletier.)

We study consumption-based capital asset pricing models with the assumption of nonseparability in the utility function between durable and nondurable consumption goods. As a result, our models are going to have asset pricing implications not only from the dynamic of nondurable consumption, but also from the dynamic of durable consumption. We have looked into four different specifications for the dynamics of nondurable consumption, durable consumption, and dividends. We use both calibrated and estimated parameters to test the validity of our model with the four different settings. We show that our models can generate moments for riskless assets return, equity premium, and the logarithm of price-dividend ratio which corresponds to the same moments in the empirical data. We achieve the best results with the following specification: (i) long-run risk in the dynamic of durable consumption. (ii) long-run risk as well as mean reversion in the dynamic of nondurable consumption.
Three Essays in CCAPM: Durable and Nondurable Consumption with Long-Run Risk

by
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APPROVED BY:

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_________________________  ________________________
Karlyn Mitchell             Nora Traum
DEDICATION

To my parents, wife, and children.
**BIOGRAPHY**

Majid was born in Riyadh, Saudi Arabia. In 2001, he received a B.A. from the Economics department, at the Faculty of Economics and Administration, King Abdulaziz University. Thereafter, Majid planned to study his master degree in Applied Economics at Illinois State University. After his first semester in Illinois State University, he was awarded a governmental scholarship from his country Saudi Arabia. Majid successfully finished his master degree on May 2006. Next, Majid was accepted in the Ph.D. program at North Carolina State University. Majid worked on his Ph.D. thesis under the supervision of Prof. Denis Pelletier. He was awarded his Ph.D. on May, 2014. Majid has research interests on consumption-based capital asset pricing models and macro-finance related analysis.
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My completion of this project could not have been accomplished without the support and encouragement of my parents, my brother, my sisters, and my children. Special thanks to my loving mother who I owe all my success in this life. The countless times of overseas phone calls I received from my mother will not be forgotten.

Finally, to my caring, loving, and supportive wife, Taghreed: my deepest gratitude. Your encouragement when the times got rough are much appreciated and highly noted. During my research time, your willing to provide management of our household activities brought great comfort and relief.
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Chapter 1

Mean Reversion in Nondurable Consumption and Long-Run Risk in Durable Consumption

1.1 Introduction

Many studies done on consumption-based capital asset pricing models (CCAPM) can be categorized by two distinguishing features. The first feature is whether the utility function is separable between durable and nondurable consumption. The second feature is whether equity price movements are consistent with forward-looking or backward-looking behavior models.

For the first feature, the past literature on CCAPM has almost completely ignored the role of consumption of durable goods when trying to explain key movements in asset prices. For instance, in the habit formation (HF) based paper by Campbell and Cochrane (1999), the growth of nondurable consumption and dividends are modeled as iid processes, but the persistent external habit on past consumption of only nondurable goods is essentially what causes movements in asset prices. Moreover, the Long-Run Risk (LRR) model by Bansal and Yaron
(2004) proposes two channels through which it can justify a wide-range of asset pricing phenomena. Namely, long-run risk in the expected component of nondurable consumption growth and long-run risk in consumption uncertainty. That is, in the LRR model, predictability of asset prices, dividends, and risk prima, rely solely on nondurable consumption and its volatility. Therefore, both papers, which are considered to a large extent as leading papers in CCAPM literature, have assumed separability between consumption of durable and nondurable goods.

Lately, very few papers have introduced the non-separability assumption between durable and nondurable consumption. For example, Yogo (2006) and Yang (2011) besides very few others have assumed non-separability between durable and nondurable consumption. Yogo (2006) uses a model with the Epstein and Zin (1989) utility function and assumes that a representative agent gets utility from consuming both durable and nondurable goods. His model can successfully explain not only cross-sectional variation but also time variation in expected stock returns. Along this line, Yang (2011) finds that the growth of durable consumption exhibits a highly persistent component (long-run risk part). By relaxing the separability assumption between the two goods and using Epstein-Zin preferences, his model is able to explain several asset market phenomena such as the mean of the risk-free rate and the equity premium as well as their volatilities.

For the second feature, the literature on CCAPM has either assumed backward-looking or forward-looking behavior models. For instance, HF models depend on the so-called habit process formed from historical consumption of nondurable goods to explain future equity price movements. On the other hand, LRR models rely on the expected component of consumption growth along with its volatility to explain asset prices movements, which in turn might help in
forecasting future growth and uncertainty as mentioned in Bansal et al (2009). Therefore, HF
is a backward-looking model whereas the LRR is a forward-looking model.

Regardless of which features a model has, the recent CCAPM literature has deviated
from assuming a random walk process for the dynamics of the growth rate of consumption and
has emphasized instead the existence of persistent time-varying consumption risk. For example,
papers on LRR models introduce time-varying risk through the use of expected persistent
components in the dynamics of consumption goods. Similarly, in HF models, time-varying
risk is introduced by either a ratio $\frac{C_t}{X_t}$ or as the difference $C_t - X_t$ between consumption $C_t$
and the level of habit $X_t$. Pakos (2004) finds a similarity between the behavior of the stock
of durable goods and habit formation; to be specific, habit formation that is measured by
the distance of current consumption from a reference level and introduced in several works
such as: Constantinides (1990), Abel (1990) and Campbell and Cochrane (1999). Durability
in the stock of durable goods and persistence in habit formation make both processes slowly
moving variables and therefore exhibit long-run risk. One reason for this deviation was to
account for the empirical time-varying risk premium. Moreover, the random walk assumption
for consumption dynamics can be problematic whether we use the traditional power utility
preferences or Epstein and Zin utility preferences. In the case of power utility, the assumption
of random walk consumption would result in an equity premium puzzle. That is, the model
will not be capable of generating the observed excess return from investing in risky assets over
the risk-free return one can get from investing in government bonds. To solve this problem
Kandel and Stambaugh (1991) suggest using a high value for risk aversion. However, setting an
unrealistically high value for risk aversion would result in the well-known risk-free puzzle. That
is, the very high risk aversion implies that people care too much about smoothing consumption over time. This will lead them to borrow more money to reduce the discrepancy between future and present consumption. However, if that is really what is happening, then we should observe an increasing trend for the real interest rate overtime, but actually we don’t. In the case of Epstein-Zin utility, Kocherlakota (1990) finds that models that uses Epstein-Zin utility, for example LRR models, and assumes an iid process for consumption can no longer have distinctive roles for risk aversion on the one hand and the elasticity of intertemporal substitution on the other. That is, Epstein-Zin utility coincides with the power utility and risk aversion become the reciprocal of the elasticity of intertemporal substitution.

The model developed in this chapter uses a consumption-based asset pricing model in an Epstein-Zin framework where the role of elasticity of intertemporal substitution is separated from that of risk aversion. We also assume non-separability between durable and nondurable consumption. Following Yang (2011), we model durable consumption growth to have a highly persistent expected component (which represents long-run risk) and time-varying volatility which varies negatively with the expected component. Unlike Yang (2011) who models nondurable consumption growth as an iid process, our model for nondurable consumption growth follows a mean reverting process. Although, the representative agent in our model is concerned with the past history of nondurable consumption (via mean reversion) as well as the future expected component of durable goods (via long-run risk), the consumption-based asset pricing model proposed in this paper is both forward and backward-looking.

In the recent CCAPM literature, many papers focus on separating the roles of the elasticity of intertemporal substitution and risk aversion (use Epstein-Zin framework) and the
calibration technique instead of the estimation technique to account for stock market implications. However, the papers of Bansal and Yaron (2004) and Yang (2011) stand out. Table 1.1 reports part of the asset pricing implications for both papers. Both papers use a calibrated value of 1.5 for the elasticity of intertemporal substitution (EIS). The results of both papers are based on 10,000 simulated samples and with monthly observations that match the length of the actual data they use. The paper by Bansal and Yaron (2004) reports only the mean of the simulated observations, while Yang (2004) reports the mean, the 5\textsuperscript{th} percentile, and 95\textsuperscript{th} percentile of the simulated observations. The model of Bansal and Yaron (2004), which relies only on nondurable consumption (consumption separability assumption), has results that are a close match to the empirical data for some moments, but not all. For example, with an EIS value of 1.5 and a risk aversion value of 7.5, their model is able to generate a close match to the empirical mean of the price-dividend ratio, but a lower value for its volatility than that of the empirical data. However, with these calibrated values, their model generates a higher value for the mean of the risk-free rate, and a lower value for the mean of the excess return than those of the empirical counterparts. When the risk aversion is set to 10, their model has still higher values for the mean of the risk-free rate, but lower value for the mean of the excess return than the empirical counterparts. Also, with these calibrated values, their model implied mean of the price-dividend ratio becomes lower than the empirical mean and is not a close match anymore.

In contrast, the model in Yang (2011) relies apparently on durable and nondurable consumption (non-separability assumption) in maximizing utility for the agent’s problem. However, his model’s analytical solution depends solely on one state variable, the expected component of the growth rate of durable consumption, even though nondurable consumption is a choice vari-
able embedded in his model’s utility function. Unlike Bansal and Yaron’s model, Yang’s model suggests that asset prices correlate with only one state variable. Therefore, Yang’s model, most probably, wouldn’t be a good choice when looking to match cross-sectional return on assets. It is an oversimplifying view of how asset prices should be related to the dynamics of consumption goods and dividends. In addition, with an EIS value of 1.5 and a risk aversion value of 10, Yang’s model is able to match the mean and volatility of the excess return. However, his model fails to match the mean and volatility of the risk-free rate and price-dividend ratio.

Table 1.1: Asset pricing implications of the models in Bansal and Yaron (2004) and Yang (2010)

| Data (1929-1998) | Model (EIS=1.5)  |
| --- | --- | --- | --- |
| Moments | Mean | SE | Mean | Mean |
| $E[r_f]$ | 0.0086 (0.004) | 0.0144 | 0.0093 |
| $\sigma[r_f]$ | 0.0097 (0.002) | 0.0044 | 0.0057 |
| $E(r_m - r_f)$ | 0.0633 (0.002) | 0.0401 | 0.0684 |
| $\sigma[r_m]$ | 0.1942 (0.003) | 0.1781 | 0.1865 |
| $E[p - d]$ | 3.28 (0.009) | 3.22 | 2.99 |
| $\sigma[p - d]$ | 0.29 (0.001) | 0.18 | 0.21 |

| Panel 2: Yang (2011) |
| Data (1952-2007) | Model (EIS=1.5)  |
| --- | --- | --- | --- | --- | --- |
| Moments | Mean | SE | Mean | 5th | 95th |
| $E[r_f]$ | 0.0168 (0.005) | 0.0172 | 0.0170 | 0.0175 |
| $\sigma[r_f]$ | 0.0223 (0.002) | 0.0002 | 0.0001 | 0.0006 |
| $E[p - d]$ | 3.47 (1.480) | 3.077 | 2.8903 | 3.226 |
| $\sigma[p - d]$ | 0.354 (0.034) | 0.195 | 0.133 | 0.280 |

In terms of the model’s consumption predictability by the log of the price-dividend ratio, Table 1.2 shows that the model of Bansal and Yaron (2004) generates $R^2$s which increase
Table 1.2: Predictability of consumption growth by the logarithm of the price-dividends ratio

<table>
<thead>
<tr>
<th>Horizon year</th>
<th>Bansal &amp; Yaron (2004)</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data</td>
<td></td>
</tr>
<tr>
<td></td>
<td>slope</td>
<td>std err</td>
</tr>
<tr>
<td>1</td>
<td>0.04</td>
<td>(0.03)</td>
</tr>
<tr>
<td>3</td>
<td>0.03</td>
<td>(0.05)</td>
</tr>
<tr>
<td>5</td>
<td>0.02</td>
<td>(0.04)</td>
</tr>
</tbody>
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when going from a one-year to a three-year horizon but decrease from a three-year to a five-year horizon. On the other hand, the $R^2$s for the empirical data are decreasing as we move to a higher horizon. Since Yang’s model assumes a random walk for the growth of nondurable consumption, predictability of total consumption by the log price-dividend is not applicable.

Table 1.3: Predictability of excess return by the logarithm of the price-dividends ratio

<table>
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<tbody>
<tr>
<td></td>
<td>Data</td>
<td>Model</td>
</tr>
<tr>
<td></td>
<td>slope</td>
<td>SE</td>
</tr>
<tr>
<td>1</td>
<td>-0.08</td>
<td>(0.07)</td>
</tr>
<tr>
<td>3</td>
<td>-0.37</td>
<td>(0.16)</td>
</tr>
<tr>
<td>5</td>
<td>-0.66</td>
<td>(0.21)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Horizon year</th>
<th>Panel 2: Yang (2011)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data</td>
</tr>
<tr>
<td></td>
<td>slope</td>
</tr>
<tr>
<td>1</td>
<td>-0.096</td>
</tr>
<tr>
<td>3</td>
<td>-0.188</td>
</tr>
<tr>
<td>5</td>
<td>-0.264</td>
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</table>

For the predictability of the excess return on risky assets by the log price-dividend ratio, Table 1.3 reports the results for Bansal and Yaron (2004) in the first panel and the results for Yang (2011) in the second panel. The model of Bansal and Yaron (2004) generates $R^2$s that are to some extent lower than their empirical counterparts, whereas Yang (2011) is successful in replicating the empirical slopes and $R^2$s for one-year, three-year, and five-year horizons.
Therefore, we have two goals which we hope to achieve in this chapter. The first goal is to have a CCAPM that can generate better matching for the moments of the return on assets or at least similar to what Bansal and Yaron (2004) and Yang (2011) models could generate. However, the model that we are going to build should internalize news not only form nondurable consumption good, but also news from durable consumption good. That is, our analytical solution should depend on at least one state variable from each type of consumption good. The second goal is to get better predictability of total consumption and excess return on risky assets by the log of the price-dividend ratio or at least similar to these two papers.

The main focus of this chapter is trying to match features of key data such as: consumption and dividend dynamics, mean and volatility of the excess market return, risk-free rate, and the log of the price-dividend ratio. To do so, we first use the Epstein-Zin utility function and the budget constraint to derive the relevant intertemporal marginal rate of substitution of our model. Second, we specify the dynamics for durable consumption growth rate, nondurable consumption growth rate, and dividends growth rate. Third, we use the utility function to derive an expression for total consumption growth rate. Fourth, using calibrated parameters, we simulate monthly observations for the growth rates of durable consumption, nondurable consumption, and dividends. After that, we use time aggregation to generate observations at the annual frequency for these growth rates and eventually compare their moments to their corresponding empirical observations. Fifth, we use the standard approximations utilized in Campbell and Shiller (1988) to solve for the return on wealth as well as for the market portfolio return. The solutions to these returns are assumed to rely on the two state variables specified in our model, namely, the expected component of durable consumption and the mean reversion
of nondurable consumption. The state variables will be extracted from the simulated growth rate of consumption goods in the previous fourth step. Lastly, we compute through simulations the first and second moments of the model-implied risk-free rate and excess market return and compare them to the moments of their empirical counterparts.

1.2 The model

1.2.1 Utility function and total consumption

The Generalized Elasticity of Substitution (GES) utility function is non-separable between durable and the nondurable consumption in each period, which can be written as:

$$\Omega_t = \left[ (1 - \alpha)C_t^{\frac{1}{\rho}} + \alpha D_t^{\frac{1}{\rho}} \right]^{\frac{1}{1 - \frac{1}{\rho}}}$$  

where $C_t$ denotes nondurable consumption and $D_t$ denotes the service flow of durable consumption. The consumer gains utility each period from consumption of nondurable goods and from service flows of durable goods. The parameter $\alpha$ determines the relative weight of the utility from durable consumption versus the nondurable one, while $\rho$ captures the elasticity of substitution between the two goods. Plugging (1.1) in the Epstien-Zin (1989) framework in which the utility can be written recursively as:

$$U_t = \left[ (1 - B)\Omega_t^{1 - \frac{1}{\Psi}} + B \left( E_t U_{t+1}^{1 - \gamma} \right)^{\frac{1 - \frac{1}{\Psi}}{1 - \gamma}} \right]^{\frac{1}{1 - \frac{1}{\Psi}}}$$  

s.t

$$W_{t+1} = R_{g,t+1}(W_t - G_t)$$  

where the time preference parameter is $B$, the EIS is $\Psi$, and the risk aversion parameter is $\gamma$.

The budget constraint in equation (1.3) links total consumption expenditures $G_t$ and wealth
$W_{t+1}$ through the gross return on wealth $R_{g,t+1}$ by the fact that $W_{t+1}$ represents the total future stream of consumption claims as defined in Yang (2011). Epstein-Zin (1989) proxy the return on wealth by the return on the stock market. However, in this paper we make a direct use of the return on wealth and also distinguish between the gross return on wealth $R_{g,t+1}$ and the gross return on the stock market $R_{m,t+1}$ as done in Bansal and Yaron (2004) and many other previous papers. Details on this distinction is provided in the analytical solution section. Further, the return on wealth can be thought of as the return on an asset that generates aggregate consumption as its dividends.

For the moment, we consider the GES utility function embedded in the Epstein-Zin preferences, but later on our model will be based on an approximation of the GES utility by a Cobb-Douglas utility and a justification for this approximation will be given by the end of this subsection. Based on that, we derive an expression for total expenditures $G_t$ for the GES utility and for the approximated Cobb-Douglas utility function $\Omega_t = C_t^{1-\alpha}D_t^\alpha$. We also take the price of nondurable goods as numéraire. On the other hand, the price of the one-period service flow from durable goods is not the purchase price, but rather is the user cost denoted by $F$ (see Yogo (2006), Yang (2011)). Therefore, $FD_t$ represents the cost of the service flow of durable consumption:

$$F = \frac{\partial \Omega_t}{\partial D_t}$$

which is equal to $\frac{C_t}{D_t^{\frac{1}{\alpha}}}$ in the GES case and to $\frac{C_t}{D_t}$ in the case of the Cobb-Douglas approximation. Accordingly, expenditures on the service flow for durable goods $FD_t$ in the GES case and the Cobb-Douglas case will be equal to $\frac{C_t}{D_t^{\frac{1}{\alpha}}} - \frac{1}{\alpha} D_t^{\frac{1}{\alpha}} C_t$ and $\frac{C_t}{D_t}$ respectively. Unlike the GES case, expenditures on the service flow of durable goods, in the Cobb-Douglas
case, is proportional to nondurable consumption. Thus, we can write the total consumption expenditures $G_t$ in terms of nondurable goods as follows$^1$:

$$G_t = C_t + \frac{\alpha}{1 - \alpha} C_t \tag{1.5}$$

$$= \frac{1}{1 - \alpha} C_t \tag{1.6}$$

It also follows that the growth rate of total consumption is equal to growth rate of nondurable consumption:

$$\Delta g_{t+1} = \Delta c_{t+1}. \tag{1.7}$$

Throughout the paper, all lower case letters represent the log of the corresponding variable.

Equation (1.7) shows that the growth of total consumption is equal to the growth of nondurable consumption, which we will make use of when we derive the analytical solution later. Overall, the model in this paper and the derivation for the intertemporal marginal rate of substitution (IMRS) are consistent with the work in Yang (2011) except for the fact that we assume mean reversion for the nondurable growth process, whereas Yang (2011) assumes a random walk process for it. Moreover, the share of durable goods in the GES case will be equal to:

$$\frac{\partial \Omega_t}{\partial D_t} \frac{D_t}{G_t} = \frac{\alpha D_t^{1-\frac{1}{\rho}}}{(1-\alpha)C_t^{1-\frac{1}{\rho}} + \alpha D_t^{1-\frac{1}{\rho}}} \tag{1.8}$$

Equation (1.8) indicates that the share of durable consumption in this case will not be constant and, therefore, induces a composition risk that will not allow us to have a closed form solution to our model. To be keep it simple, we focus on the implications of the model under the special case where $\rho = 1$ as in Yang (2011). As a result, the share of durable consumption will be

$^1$expressions for consumption expenditures $G_t$ and its growth $\Delta g_{t+1}$ for the GES case is provided in Appendix A.
constant and equal to $\alpha$. Going back to the Cobb-Douglas utility function, the constant share of durable goods $\alpha$ can be derived as follows:

$$\frac{\partial \Omega_t / \partial D_t}{\partial \Omega_t / \partial C_t} = \frac{\alpha C_t^{1-\alpha} D_t^{\alpha-1}}{C_t^{1-\alpha} D_t^{\alpha}} = \alpha.$$  \hspace{1cm} (1.9)

### 1.2.2 IMRS

Following Yogo (2006), Yang (2011) and others, the intertemporal marginal rate of substitution for this economy, using the Epstein-Zin framework, can be written as:

$$M_{t+1} = \left[ B \frac{\alpha \nu_{t+1}}{\nu_t} R_{g,t+1}^{1-\frac{1}{\beta}} \right]^{1-\frac{1}{\nu}} \left[ \nu_t \right]^{\theta \left( \frac{1}{\nu} - \frac{1}{\beta} \right)} R_{g,t+1}^{\theta - 1},$$  \hspace{1cm} (1.10)

$$= \left[ B \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\nu}} \left( \frac{D_{t+1}/D_t}{C_{t+1}/C_t} \right)^{\alpha \theta \left( \frac{1}{\nu} - \frac{1}{\beta} \right)} \right]^{1-\frac{1}{\beta}} R_{g,t+1}^{\theta - 1},$$  \hspace{1cm} (1.11)

where,

$$\theta = \frac{1-\gamma}{1-\frac{1}{\nu}} \text{ and } \nu_t = \left[ 1 - \alpha + \alpha \left( \frac{D_t}{C_t} \right)^{1-\frac{1}{\beta}} \right]^{\frac{1}{1-\frac{1}{\beta}}}.$$

A detailed derivation of the pricing kernel, which shows how to go from equation (1.10) to equation (1.11), is provided in Appendix A.

When $\rho = 1$ then the GES can be approximated by a Cobb-Douglas with $\nu_t = \left( \frac{D_t}{C_t} \right)^{\alpha}$. For further details on this approximation look at Appendix B. The intertemporal marginal rate of substitution used in our model takes the following form:

$$M_{t+1} = B^{\theta} \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\nu}} \left( \frac{D_{t+1}/D_t}{C_{t+1}/C_t} \right)^{\alpha \theta \left( \frac{1}{\nu} - \frac{1}{\beta} \right)} R_{g,t+1}^{\theta - 1}. \hspace{1cm} (1.12)$$

According to Yang (2011), the choice of $\rho = 1$ is reasonable and not very far from what previous papers have estimated, assuming non-separability between durable and nondurable consumption. For example, Ogaki and Reinhart (1998) find $\rho$ to be about 1.2 whereas, Yogo
(2006) gets an estimate of about 0.8. Following equation (1.12) the logarithm of the pricing kernel can be written as:

\[ m_{t+1} = \theta \log(B) - \frac{\theta}{\psi} \Delta c_{t+1} + \theta(1 - \frac{1}{\psi}) \alpha (\Delta d_{t+1} - \Delta c_{t+1}) + (\theta - 1) r_{g,t+1}. \]  

(1.13)

1.2.3 Dynamics of growth processes

We develop a new process for the dynamics of nondurable consumption. We assume that the growth of nondurable consumption \( \Delta c_{t+1} \) consists of two parts, namely, a mean reverting process \( (\Delta s_{t+1}) \) plus a random walk process with a drift \( (\Delta \bar{c}_{t+1}) \), and it is presented in the following expression:

\[ \Delta c_{t+1} = \Delta \bar{c}_{t+1} + \Delta s_{t+1} \]  

(1.14)

with

\[ \Delta \bar{c}_{t+1} = \mu_{c} + \omega_{c,t} \sigma_{c} \epsilon_{c,t+1}, \]  

(1.15)

\[ \Delta s_{t+1} = (\phi_{s} - 1) (s_{t} - \bar{s}) + \omega_{s,t} \sigma_{s} \epsilon_{s,t+1}, \]  

(1.16)

\[ \omega_{s,t} = \sqrt{1 - \zeta_{c}(s_{t} - \bar{s})}. \]  

(1.17)

For the first part, \( \Delta \bar{c}_{t+1} \), the parameter \( \mu_{c} \) represents the mean of nondurable consumption growth and \( \omega_{s,t} \sigma_{c} \) represents its standard deviation which varies positively with the growth rate of past nondurable consumption growth \( s_{t} \). All the error terms \( \epsilon_{c,t+1}, \epsilon_{s,t+1} \) are assumed to be i.i.d. \( N(0,1) \). From equation (1.17), we can write an expression for \( s_{t+1} \):

\[ s_{t+1} = (1 - \phi_{s}) \bar{s} + \phi_{s} s_{t} + \omega_{s,t} \sigma_{s} \epsilon_{s,t+1}. \]  

(1.18)

It is assumed that the persistence parameter \( \phi_{s} \) has a range of \( 0 < \phi_{s} < 1 \), so the higher is \( \phi_{s} \) the more persistent would the process be. Furthermore, the more \( s_{t} \) is above its long-run mean \( \bar{s} \),
the better would be the economic condition.

For the growth of durable consumption, $\Delta d_{t+1}$, we follow the model that has been developed in Yang (2011). His model matches closely the features of the empirical durable consumption. The model can be written as the following:

$$\Delta d_{t+1} = \mu_d + x_t + \omega_{d,t} \sigma_d \epsilon_{d,t+1}, \quad (1.19)$$

with

$$x_{t+1} = \phi_x x_t + \omega_{x,t} \sigma_x \epsilon_{x,t+1}, \quad (1.20)$$

$$\omega_{d,t} = \sqrt{1 - \zeta x_t}. \quad (1.21)$$

The growth of durable consumption has a long-run risk which is represented by the predictable component in durable consumption $x_t$. The parameters $\mu_d$ is the mean of durable consumption growth. The range for the persistence parameter is $0 < \phi_x < 1$, and $x_{t+1}$ follows an AR(1) that is being generated with time-varying volatility $\sigma_x \omega_{x,t}$. Both time-varying volatilities $\sigma_d \omega_{d,t}$ (short-run risk) and $\sigma_x \omega_{x,t}$ (long-run risk), go to zero as $x_t \to 1$. As it appears from equation (1.21), $\omega_{d,t}$ is inversely related to $x_t$. Also, the error terms $\epsilon_{d,t+1}$ and $\epsilon_{x,t+1}$ are assumed to be iid $N(0, 1)$. As a result, high $x_t$ today is associated with low volatility $\omega_{d,t}$ and the future durable consumption growth is expected to increase, but low $x_t$ today is associated with high volatility $\omega_{d,t}$ and an expectation for future durable consumption growth to shrink.

To see how persistent the yearly growth rates of both nondurable and durable consumption are, you can look at Figure 1.2.3. It is clear from the graph that the real annual growth rate of durables is more persistent than the real annual growth rate of nondurables. We also find that the first three-order autocorrelations for the real annual growth rate of durable
1.2.4 Dividends Growth

Bansal and Yaron (2004) demonstrate that many asset market movements can be explained mainly by two critical assumptions. One of these assumptions is that the dividend growth process should be exposed to the persistent expected component of nondurable consumption growth. The second assumption is that the volatility of dividends relies largely on the volatility of nondurable consumption growth. Therefore, the growth of dividends, in their paper, relies on the exposure to these two state variables, namely, the expected component of nondurable consumption growth and its volatility. On the other hand, Yang (2011) used only the expected component of the growth of durable consumption. Consequently, dividend growth process depends on the exposure to the persistent expected component of durable consumption growth. Unlike models with power utility preferences and random walk process for dividend consumption in our sample are 0.69, 0.42, and 0.11, whereas the first third-order autocorrelations for the real annual growth rate of nondurable consumption are 0.45, 0.107, and 0.004.

Figure 1.1: Real annual growth rates of durable and nondurable consumption
growth, with the Epstein-Zin preferences and the assumed exposure of dividends growth to the state variables, these two papers are able to match the observed volatility of dividend growth, the high equity return volatility and the equity risk premium.

In our model, we have two state variables, namely $x_t$ which represents the persistent expected component in the durable consumption growth process, and $(s_t - \bar{s})$ which represents mean reversion in nondurable consumption growth. Following Yang (2011), we assume that the dividends growth has a shock with volatility $\sigma_y$. Accordingly, dividends growth $\Delta y_{t+1}$ process can be written as:

$$\Delta y_{t+1} = \mu_y + \lambda_x x_t + \lambda_\sigma (s_t - \bar{s}) + \sigma_y \epsilon_{y,t+1}, \epsilon_{y,t+1} \sim N(0,1).$$  \hspace{1cm} (1.22)

The mean of dividend growth is $\mu_y$, the parameter $\lambda_x$ measures the exposure of dividend growth to the expected component of durable consumption growth, and the parameter $\lambda_\sigma$ measures the exposure of dividend growth to mean reversion element in nondurable consumption growth. Overall, the five shocks $\epsilon_{c,t+1}, \epsilon_{x,t+1}, \epsilon_{d,t+1}, \epsilon_{s,t+1}, \epsilon_{y,t+1}$ are assumed to be mutually independent. Bansal and Yaron (2004) and Yang (2011) have showed that not allowing for correlations between these shocks is not very restrictive because assuming such correlations will not lead to major qualitative and quantitative changes in the model’s asset pricing implications. Therefore, the model’s performance depends entirely on the expected component of durable consumption growth $x_t$ and the mean reverting persistence term of nondurable consumption growth $(s_t - \bar{s})$. Figure 1.2.5 shows the real annual growth rates of dividends.
1.2.5 Analytical solution

As mentioned before, the return on wealth is equivalent to the return on an asset that generates aggregate consumption $G_t$ as its dividends. Let $P_{g,t}$ denote the price of that asset. As a consequence, the price-total consumption ratio $\frac{P_{g,t}}{G_t}$, in this setting, corresponds to the price-dividend ratio $\frac{P_{m,t}}{D_t}$ that we get for the stock market. Since, the state variables of my model are $x_t$ and $(s_t - \bar{s})$, we assume that both the log of price-total consumption ratio $z_t$ and the log of price-dividend ratio $z_{m,t}$ are linearly related to these state variables. Following Bansal and Yaron (2004), we make the following conjectures:

$$z_t \approx A_{g,0} + A_{g,1} x_t + A_{g,2} (s_t - \bar{s}). \quad (1.23)$$

$$z_{m,t} \approx A_{y,0} + A_{y,1} x_t + A_{y,2} (s_t - \bar{s}). \quad (1.24)$$

Based on these conjectures, expressions for $A_{g,0}$, $A_{g,1}$, $A_{g,2}$, $A_{y,0}$, $A_{y,1}$, and $A_{y,2}$ in terms of the model parameters are derived in appendix D. With the calibrated parameters provided in Table 1.4, $A_{g,1} = \frac{(1 - \frac{1}{\sigma})\alpha}{1 - A_{\sigma}^{10}} > 0$ and this indicates that the expected component of durable consumption
has a positive relation with the log of the price-consumption ratio $z_t$, and hence the intertemporal substitution effect dominates the income effect\(^2\). That is, a higher expected component of durable consumption gives the agent an incentive to buy more assets and consequently lead to a rise in wealth-consumption ratio $\frac{P_{G,t+1} + G_t}{C_t}$. Whereas $A_{g,2} = \frac{(\sigma - 1)(1 - \frac{1}{s})}{1 - A_1 \sigma} < 0$ indicates that the higher the past nondurable consumption growth over its long-run mean the lower the log of price-consumption ratio would be. Therefore, the wealth effect dominates the intertemporal substitution effect. Overall, with the calibrated parameters we choose for our model, $A_{g,2}$ is negative but has a small value of -.05 and $A_{y,2}$ has a value of 1.25. Furthermore, we find that $A_{y,1} > A_{g,1}$ and $|A_{y,2}| > |A_{g,2}|$, so both the expected component of durable consumption growth $x_t$ and the mean reversion term $(s_t - \bar{s})$ of nondurable consumption growth have positive relationships with the price-dividend ratio. Most importantly, the price-dividend ratio is exposed to higher long-run risk from the past history of nondurable goods as well as from the expected future component of durable consumption than the price-consumption ratio does.

Campbell (1996), Bansal and Yaron (2004) and many others make a distinction between the unobserved return to consumption claim $R_{g,t+1}$ and the return to dividend claim $R_{m,t+1}$, which is also referred to as the market return on the portfolio. In their paper, using log-linear approximations for both returns, following Campbell and Shiller (1988), the return on consumption and the return on dividend claims are equal to:

\begin{align*}
    r_{g,t+1} &= A_0 + A_1 z_{t+1} + \Delta g_{t+1} - z_t, \\
    r_{m,t+1} &= A_{m0} + A_{m1} z_{m,t+1} + \Delta y_{t+1} - z_{m,t}. 
\end{align*}

\(^2\)Bansal and Yaron (2004) provide a similar argument between the expected component of non-durable consumption growth and price-consumption ratio and they conclude a domination of income effect over substitution effect.
Here, $r_{g,t+1} = \log(R_{g,t+1})$ and $z_t = \log(P_{g,t}/G_t)$ is the log of price-total consumption ratio, whereas $P_{g,t}$ is the price of the total future stream of total consumption. Bansal and Shaliastovich (2006) argued that $P_{g,t} + G_t$ represents an agent’s wealth to consumption ratio, so movements in the price-consumption ratio ($z_t = \log(P_{g,t}/G_t)$) lead to movements in the wealth-consumption ratio. To clarify this argument, we should remember that an agent invests in assets that deliver future aggregate consumption as its dividends. Therefore, the gross return $R_{g,t+1}$ on such an asset by definition is equal to:

$$R_{g,t+1} = \frac{P_{g,t+1} + G_{t+1}}{P_{g,t}}.$$  \hspace{1cm} (1.27)

Let $W_t = P_{g,t} + G_t$ and substituting for wealth in the budget constraint introduced earlier as $W_{t+1} = R_{g,t+1}(W_t - G_t)$, with some rearrangement we can reach back to equation (1.27).

Similarly, $z_{m,t} = \log(P_{m,t}/Y_t)$ represents the log of the price-dividends ratio and the gross return on the stock market is $R_{m,t+1} = \frac{P_{m,t+1} + D_{t+1}}{P_{m,t}}$. Further, the log-linear approximation and the solution to the parameters $A_0, A_1, A_{m,0}$, and $A_{m,1}$ are provided in appendix C. Having exogenous processes for $\Delta g_{t+1}, \Delta y_{t+1}, x_{t+1}$, and $s_{t+1}$, the price-consumption ratio $z_t$ and $z_{m,t}$ lead us to have a complete solution to both the return on consumption claim $r_{g,t+1}$ as well as the return on the market portfolio $r_{m,t+1}$. The analytical expression to the return on consumption claim can be expanded to:

$$r_{g,t+1} = A_0 + A_1(A_{g,0} + A_{g,1}x_{t+1} + A_{g,2}(s_{t+1} - \bar{s})) + \frac{\Delta g_{t+1} - \Delta g_{t}}{\Delta t} - (A_{g,0} + A_{g,1}x_t + A_{g,2}(s_t - \bar{s})).$$  \hspace{1cm} (1.28)

In a similar way, the return on the market portfolio can be expanded to:

$$r_{m,t+1} = A_{m,0} + A_{m,1}(A_{y,0} + A_{y,1}x_{t+1} + A_{y,2}(s_{t+1} - \bar{s})) + \Delta y_{t+1} - (A_{y,0} + A_{y,1}x_t + A_{y,2}(s_t - \bar{s})).$$  \hspace{1cm} (1.29)
1.2.6 Innovation of IMRS

As of now, we have expressions for IMRS and the dynamics of durable consumption, nondurable consumption, the return on wealth \( r_{\phi, t+1} \), and the growth of total consumption expenditures \( \Delta g_{t+1} \). That is, by substituting equations (1.14), (1.16), (1.17), (1.18), (1.19), (1.20), (1.21), (1.7), and (1.28) into the intertemporal equation (1.13), the innovation of the intertemporal of marginal rate of substitution can be written as:

\[
m_{t+1} - E_t[m_{t+1}] = \lambda_{md} \sigma_d \varepsilon_d, t+1 - \lambda_{mc} \sigma_c \varepsilon_c, t+1 - \lambda_{ms} \sigma_s \varepsilon_s, t+1 - \lambda_{mx} \sigma_x \varepsilon_x, t+1,
\]

(1.30)

\[
\lambda_{md} = \theta (1 - \frac{1}{\psi}) \alpha \omega_{d, t},
\]

\[
\lambda_{mc} = \left( \theta \left( \frac{1}{\psi} + (1 - \frac{1}{\psi}) \alpha \right) + (1 - \theta) \right) \omega_{c, t},
\]

\[
\lambda_{ms} = \left( \theta \left( \frac{1}{\psi} + (1 - \frac{1}{\psi}) \alpha \right) + (1 - \theta)(A_1 A_{\phi, 2} + 1) \right) \omega_{s, t},
\]

\[
\lambda_{mx} = (1 - \theta) A_1 A_{\phi, 1} \omega_{x, t}.
\]

The first term represents the short-run risk generated by the shock of durable consumption growth with a price of \( \lambda_{md} \). The second term shows the exposure of the pricing kernel to the short-run risk of \( \Delta \tilde{c}_{t+1} \), the first part in the nondurable growth dynamic, with a price of \( \lambda_{mc} \). The third term represent the short-run risk of \( \Delta s_{t+1} \), the second part in the nondurable growth dynamic, with a price of \( \lambda_{ms} \). The fourth term represents the short-run risk generated by the shock of the expected component of durable consumption \( x_t \). Notice that the higher the persistent parameter \( \phi_x \), the higher will be \( A_{\phi, 1} = \frac{\alpha (1 - \frac{1}{\psi})}{(1 - A_1 \phi_x)} \) and eventually result in higher prices of risk \( \lambda_{ms} \) and \( \lambda_{mx} \).
1.2.7 Risk-free rate

As it appears from Appendix D, the analytical solutions for the model’s coefficients $A_{g,0}, A_{g,1}, A_{g,2}, A_{y,0}, A_{y,1}, A_{y,2}$ depend only on the linear terms of $x_t$ and $(s_t - \bar{s})$ and their volatilities. In comparison to Bansal and Yaron (2004), their analytical solutions rely on, besides the expected component of nondurable consumption, the time varying volatility of nondurable consumption growth. Therefore, the inclusion of such volatility is essential to their analytical solutions, to the ability of their model to generate time varying volatilities for the price-dividend ratio and price-consumption ratio, and eventually to match the volatility of the empirical data on the risk-free rate and equity return rate. The risk-free rate has the following form:

$$r_{f,t} = -E_t[m_{t+1}] - 0.5\text{var}_t[m_{t+1}]$$

In this paper, we use two state variables $x_t$ and $(s_t - \bar{s})$ and only $x_t$ has the conditional volatility $\omega_{x,t}$ which alone can generate enough time-varying volatility to match that of the actual data. However, $\omega_{x,t}^2 = (1 - \zeta x_t)$, is by itself linearly related to $x_t$ and not a separate state variable. In other words, the conditional volatility $\omega_{x,t}$ can generate time-varying implications in our model only through the expected component of durable consumption $x_t$. Therefore, we follow Yang (2011) and ignore the very small variance part of the pricing kernel in deriving the risk free rate. Ignoring this part will not lead to a dramatic change in the quantitative implications of the model. Therefore, the expression for the risk free rate will be:

$$r_{f,t} = -E_t[m_{t+1}]$$
By ignoring the second part of this expression the risk free rate can be written as:

\[
\begin{align*}
    r_{f,t} &= -\theta \log(B) + \frac{\theta}{\psi} (\mu_c + (\phi_z - 1) (s_t - \bar{s})) - \theta (1 - \frac{1}{\psi}) \alpha (\mu_c + x_t - \mu_c - (\phi_z - 1) (s_t - \bar{s})) \\
    &\quad - (\theta - 1) \left[ A_0 + A_1 [A_{g,0} + A_{g,1} \phi_x x_t + A_{g,2} \phi_z (s_t - \bar{s})] + \\
    &\quad \mu_c + (\phi_z - 1) (s_t - \bar{s}) - [A_{g,0} + A_{g,1} x_t + A_{g,2} (s_t - \bar{s})] \right] \\
    &\quad E_t[r_{g,t+1}]
\end{align*}
\]

(1.31)

with a variance equal to:

\[
\begin{align*}
    \text{var}_t(r_{f,t}) &= \Gamma_x \text{var}(x_t) + \Gamma_x \text{var}((s_t - \bar{s})) \\
    \Gamma_x &= \left( \theta \left( \frac{1}{\psi} - 1 \right) \alpha + (1 - \theta) (A_1 A_{g,1} \phi_x - A_{g,1}) \right)^2 \\
    \Gamma_x &= \left( (\phi_x - 1) \left( \frac{\theta}{\psi} + \theta (1 - \frac{1}{\psi}) \alpha \right) + (1 - \theta) (A_1 A_{g,2} \phi_x + (\phi_x - 1) - A_{g,2}) \right)^2
\end{align*}
\]

To derive the relative equity premium, we assume that the risky asset return \( r_{m,t+1} \) and the pricing kernel \( m_{t+1} \) are conditionally log-normal. Therefore, the equity premium for all of our models is going to take the following form:

\[
E(r_{m,t+1} - r_{f,t}) = -\text{cov}_t(r_{m,t+1} - E_t[r_{m,t+1}], m_{t+1} - E_t[m_{t+1}]) - 0.5 \text{var}_t(r_{m,t+1})
\]

1.3 Data

The Center for Research in Security Prices database is our source for stock market data. For the stock market return we use the return on a value-weighted portfolio of NYSE, AMEX, and Nasdaq. The risk-free rate is constructed from the three-month Treasury-bill rate. Annual growth of dividends and the log price-dividend ratio are constructed using Shiller’s data on real dividends and prices of the S&P composite index. Both the risky return, the risk free rate, as well as dividend growth have been adjusted for inflation and used in real terms. For
consumption data, we use Bureau of Economic Analysis (BEA) data on the annual sum of real personal expenditures on nondurables and services as a measure for nondurable consumption. Unlike nondurable goods, durable goods may provide service flows for more than one period, and therefore, personal expenditures should not be considered as a good proxy for their service flow. We follow Yang (2011) in assuming that the service flow of durable goods is proportional to the household annual realized stock of durable goods. We take the year-end real net stock of durable goods held by households to represent the service flow of durable consumption. Net here refers to the depreciation rate of the stock of durable goods, which BEA takes it into account when collecting data on the household stock of durable goods. As it is conventional in the literature, all data on consumptions are considered in per capita terms. Regarding sample length, Bansal and Yaron (2004) use a long sample period (1930-2008) and argues that it better captures all the relevant macroeconomic outcomes. Notice that they only rely on nondurable data and, therefore, assume separability between the two consumption goods. On the other hand, Yang (2011), Yogo (2006), and Ogaki and Reinhart (1998) choose their sample period to start after 1951 even though earlier data are available. This choice is based on the fact that the immediate period after the war had been associated with unusual rapid restocking of durable goods and thus resulted in unusually high durable consumption growth. In order for our model to produce valid results it should not depend on these unusual periods; the sample period that we choose is 1952-2010. Moreover, we choose to use annual frequency data for two reasons. The first is to account for a longer span, since higher-frequency data are available only for a shorter time span. Shiller and Perron (1985) show that a longer span of low-frequency data is more appropriate for measuring low-frequency movements (like persistent components) compared to
a shorter span of high-frequency data (Bansal et al., 2012). Second, we measure the service flow of durable consumption by the real net stock of durable goods following Mankiw (1985), which is only available at the annual frequency.

### 1.4 Simulation

Following previous studies in the literature, we assume that the decision interval of the agent is monthly. Then, we aggregate the simulated monthly data and construct annual growth rates and annual asset returns, which in turn makes the model-implied data comparable to the observed annual data. Statistical inference is based on 10,000 simulated samples with 59 x 12 monthly observations that match the length of the actual data. We report the mean, and the 5th and 95th percentiles of the simulated Monte-Carlo distribution.

<table>
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<th>Table 1.4: Calibrated parameters of the first model</th>
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<td>Preferences</td>
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<td>Dividend growth</td>
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The calibrated parameters of the model is reported in Table 1.4. For the dynamic of durable consumption we set $\zeta_d = 220$. This allows us to achieve a skewness of -0.4048 for the model implied growth rate of durable consumption, which is close to the skewness of the empirical data of -0.4076. Also, we set $\zeta_c = .53$ to enable the model of nondurable dynamic to
generate reasonable autocorrelation moments. Notice that the expected component of durable consumption and the growth rate of past nondurable consumption \( s_t \) are assumed to be highly persistent processes, therefore \( \phi_x \) and \( \phi_s \) are set to 0.99 and 0.98 respectively. Moreover, we set the value for the elasticity of intertemporal substitution EIS to 1.2 and the value for risk aversion to 1.09. Although, the preference parameters are calibrated, the calibrated magnitudes of the risk aversion and EIS are empirically controversial. On the one hand, Hansen and Singleton (1982), Attanasio and Weber (1989), and others estimate the EIS to be larger than 1. On the other hand, Hall (1988) and Campbell (1999) estimate its value to be well below 1.

Bansal and Yaron (2004) argued that the value of EIS should be larger than one and the estimated EIS in Hall’s paper, as an example, has a severe downward bias in their estimates of EIS. Similarly, for the risk aversion parameter \( \gamma \), an estimation of relative risk aversion for a value less than one has been supported in the work of Lucas (1972), Schluter and Mount (1976), and Hansen and Singleton (1983). Mehra and Prescott (1985) argue that a reasonable maximum value for risk aversion is 10. My choices for the values of EIS and risk aversion ensure a better match with model implied moments. I set the time preference parameter around 0.997 which is close to the time preference value of 0.998 as calibrated by Bansal and Yaron (2004) and Yang (2011). This value of time preference is needed to match the means of the empirical risk-free rate and log price-dividend ratio. For the durable consumption share parameter \( \alpha \), we go with the calibration of Yang (2011) and set it at 0.5. Also, the log of long-run mean of past year consumption \( \bar{s} \) is set equal to the mean of the ratio of \( \log(\frac{C_t}{C_{t-1}}) \). Table 1.5, reports model simulated moments for durable growth, nondurable growth, and dividends growth as well as the their data counterpart.
Table 1.5: Growth rates of the first model

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SE</td>
</tr>
<tr>
<td>Durable consumption growth</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_t[\Delta d]$</td>
<td>0.0377</td>
<td>(0.004)</td>
</tr>
<tr>
<td>$\sigma[\Delta d]$</td>
<td>0.0171</td>
<td>(0.001)</td>
</tr>
<tr>
<td>$AC1(\Delta d)$</td>
<td>0.6910</td>
<td>(0.135)</td>
</tr>
<tr>
<td>$AC2(\Delta d)$</td>
<td>0.4330</td>
<td>(0.140)</td>
</tr>
<tr>
<td>$AC3(\Delta d)$</td>
<td>0.1150</td>
<td>(0.161)</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.407</td>
<td>(0.188)</td>
</tr>
<tr>
<td>Nondurable consumption growth</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_t[\Delta c]$</td>
<td>0.0196</td>
<td>(0.002)</td>
</tr>
<tr>
<td>$\sigma[\Delta c]$</td>
<td>0.0099</td>
<td>(0.001)</td>
</tr>
<tr>
<td>$AC1(\Delta c)$</td>
<td>0.4531</td>
<td>(0.215)</td>
</tr>
<tr>
<td>$AC2(\Delta c)$</td>
<td>0.1097</td>
<td>(0.154)</td>
</tr>
<tr>
<td>$AC3(\Delta c)$</td>
<td>0.0047</td>
<td>(0.121)</td>
</tr>
<tr>
<td>Dividends growth</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_t[\Delta y]$</td>
<td>0.0109</td>
<td>(0.009)</td>
</tr>
<tr>
<td>$\sigma[\Delta y]$</td>
<td>0.0390</td>
<td>(0.005)</td>
</tr>
<tr>
<td>$AC1(\Delta y)$</td>
<td>0.5390</td>
<td>(0.169)</td>
</tr>
</tbody>
</table>

The results in Table 1.5 show that all the moments of durable consumption growth are within the simulated 5%-95% percentiles, even the third-order autocorrelation. Similarly, the mean and standard deviation of the simulated model of nondurable growth replicates closely the corresponding empirical data’s moments and autocorrelations up to the third-order are within the simulated 5<sup>th</sup>-95<sup>th</sup> percentile tails.

The model for dividend growth also can generates mean and volatility that are very close and the empirical moments are within the range of simulated 5th-95th percentiles. To be specific, the mean of the simulated 10,000 samples is a close match and its volatility is a

---

3In Table 1.5, the first two columns provides GMM estimations for the sample data 1952-2010. Standard errors are Newey and West (1987) corrected using 3 lags. The statistics for the model are based on 1,0000 simulations each with 708 monthly observations which time-aggregated to an annual frequency. The third column shows the mean over the simulations. The forth and fifth columns represents the 5% and 95% estimated percentiles of the simulated Monte-Carlo distribution.
little bit higher than what is found in the empirical data, but they are still within the simulated percentiles. Also, the first-order autocorrelation of dividend growth is not captured by the simulated model. Overall, the assumed growth rate models seem suitable to replicate sample data characteristics.

Table 1.6 reports the model implied moments for the risk-free rate, equity premium return, and price-dividend ratio. Most of these model implied moments are a close match to the empirical data’s moments. The model implied mean of the risk-free rate is 1.58% which is not far from the actual mean of 1.61%. Furthermore, the empirical mean for the risk-free rate is within still the simulated 5\textsuperscript{th}-95\textsuperscript{th} percentiles\footnote{In Table 1.6, the first two columns provide GMM estimations for the sample data 1952-2010. Standard errors are Newey and West (1987) corrected using 3 lags. The statistics for the model are based on 1,0000 simulations each with 708 monthly observations which time-aggregated to an annual frequency. All model implied moments are based on the analytical solutions provided in the paper. The third column shows the mean over the simulations. The fourth and fifth columns represent the 5\% and 95\% estimated percentiles of the simulated Monte-Carlo distribution.}. The equity premium in the empirical data is around 6.7%, much higher than the mean value from the simulation and even falls outside the 5\textsuperscript{th}-95\textsuperscript{th} percentile interval. However, the volatility of the equity return is very close to the one from simulation and within the range of the simulated 5\textsuperscript{th}-95\textsuperscript{th} percentiles. For the price-dividend ratio, in addition to the mean and standard deviation, we provide the first, second, and third-order autocorrelations. The empirical first-order autocorrelation is within the simulated 5\textsuperscript{th}-95\textsuperscript{th} percentile interval. The second and third-order autocorrelation are close to the 95\textsuperscript{th} percentile of simulated samples.
Table 1.6: Asset pricing implications of the first model

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th></th>
<th>Model</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SE</td>
<td>Mean</td>
<td>5th</td>
</tr>
<tr>
<td>$E[r_f]$</td>
<td>0.0161</td>
<td>(0.005)</td>
<td>0.0158</td>
<td>0.0145</td>
</tr>
<tr>
<td>$\sigma[r_f]$</td>
<td>0.0165</td>
<td>(0.017)</td>
<td>0.0167</td>
<td>0.0156</td>
</tr>
<tr>
<td>$E(r_m - r_f)$</td>
<td>0.0670</td>
<td>(0.020)</td>
<td>0.0362</td>
<td>0.0200</td>
</tr>
<tr>
<td>$\sigma[r]$</td>
<td>0.1448</td>
<td>(0.013)</td>
<td>0.1464</td>
<td>0.1272</td>
</tr>
<tr>
<td>$E[p - d]$</td>
<td>3.4960</td>
<td>(0.102)</td>
<td>3.4979</td>
<td>3.2451</td>
</tr>
<tr>
<td>$\sigma[p - d]$</td>
<td>0.3080</td>
<td>(0.062)</td>
<td>0.2653</td>
<td>0.1892</td>
</tr>
<tr>
<td>$AC1[p - d]$</td>
<td>0.9100</td>
<td>(0.306)</td>
<td>0.8323</td>
<td>0.7187</td>
</tr>
<tr>
<td>$AC2[p - d]$</td>
<td>0.8400</td>
<td>(0.290)</td>
<td>0.6503</td>
<td>0.4351</td>
</tr>
<tr>
<td>$AC3[p - d]$</td>
<td>0.7830</td>
<td>(0.269)</td>
<td>0.5023</td>
<td>0.2244</td>
</tr>
</tbody>
</table>

1.5 Total consumption predictability

Once again our main assumption in this chapter states that durable consumption growth exhibit a long-run risk $x_t$ and the growth of nondurable consumption follows a mean reverting process $(s_t - \tilde{s})$. Since the price-dividend ratio depends on these two state variables, it should be a valid predictor of total consumption growth. However, we know from equation (1.7) that the growth rate of total consumption $\Delta g_{t+1}$ is equivalent to the growth rate of nondurable consumption $\Delta c_{t+1}$. Therefore, we regress the sum of the growth rate of nondurable consumption $\sum_{j=1}^{5} \Delta c_{t+j}$ on the logarithm of the price-dividend ratio $\log(P_{m,t}/Y_t)$ plus a constant. We perform this regression for $j = 1, 3, 5$.

Table 1.7 reports the evidence on nondurable consumption growth predictability from the empirical data and that implied by the model. Data estimates and their $R^2$s show an increasing slope coefficient with the horizon. The model-implied estimated coefficients have also this increasing trend as we move toward longer horizons, but their magnitudes are lower than those of the empirical data counterparts. The data estimated $R^2$ increases across longer
horizons, however, the model implied $R^2$ also increases over longer horizons, but they are higher compared to those of the empirical data. Overall, these results provide strong evidence that nondurable consumption growth can be predicted and should not treated as a random walk process.

Table 1.7: Predictability of nondurable consumption growth by the log of price-dividend ratio

<table>
<thead>
<tr>
<th>Horizon year</th>
<th>Data slope</th>
<th>SE</th>
<th>$R^2$</th>
<th>Model slope</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.0037</td>
<td>0.004</td>
<td>0.013</td>
<td>-0.0014</td>
<td>0.126</td>
</tr>
<tr>
<td>3</td>
<td>-0.0184</td>
<td>0.009</td>
<td>0.069</td>
<td>-0.0034</td>
<td>0.194</td>
</tr>
<tr>
<td>5</td>
<td>-0.0246</td>
<td>0.012</td>
<td>0.076</td>
<td>-0.0048</td>
<td>0.219</td>
</tr>
</tbody>
</table>

1.6 Return on risky assets predictability

Another natural implication that our model can provide is the predictability of the future return on risky assets by the log of the price-dividend ratio. That is, the models’ implied return on risky assets as well as the log of the price-dividend ratio also depend on the same state variables. Therefore, we regress the sum of return on risky assets $\sum_{j=1}^{5} r_{m,t+j}$ on the log of the price-dividend ratio $\log(P_{m,t}/Y_t)$ plus a constant. Also, we are going to provide estimates for the predictability of one-year, three-year, and five-year horizons.

Table 1.8 shows that the slope, for the one-year horizon, is lower than the estimated data slope, but still within one standard error. The $R^2$ for the one-year horizon is close to the $R^2$ of empirical data. For longer horizons, the model implied slopes as well as their $R^2$ are almost a perfect match to their estimated data counterparts. The model succeeds in matching
the empirical data’s predictability of the return on risky assets by the log price-dividend ratio over the different horizons.

<table>
<thead>
<tr>
<th>Horizon year</th>
<th>Data slope</th>
<th>SE</th>
<th>$R^2$</th>
<th>Model slope</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.116</td>
<td>0.059</td>
<td>0.06</td>
<td>-0.0604</td>
<td>0.025</td>
</tr>
<tr>
<td>3</td>
<td>-0.254</td>
<td>0.085</td>
<td>0.15</td>
<td>-0.2425</td>
<td>0.096</td>
</tr>
<tr>
<td>5</td>
<td>-0.347</td>
<td>0.108</td>
<td>0.17</td>
<td>-0.3752</td>
<td>0.151</td>
</tr>
</tbody>
</table>

1.7 Conclusion

In this chapter we investigate how well the implications of a consumption-based asset pricing model, assuming non separability between durable and nondurable consumption in the utility function, match the data. We use the Epstien-Zin (1989) framework which allows us to have distinctive roles for the elasticity of intertemporal substitution and risk aversion. Our specification of the nondurable consumption growth process contains a mean reverting process, while durable consumption growth exhibits long-run risk. In other words, we try to introduce long-run risk into the dynamics of durable consumption by the inclusion of an expected component (forward-looking factor). Whereas, the dynamics of nondurable consumption growth relies on the past behavior of the growth rate of nondurable consumption relative to its long run mean (backward-looking factor). Our proposed model is able to generate moments that match closely in most cases their empirical moments counterparts for key variables such as: consumption and dividend dynamics, mean and volatility of the risk-free rate, and the log of
the price-dividend ratio. One obvious drawback of the model is its inability to replicate the empirical data’s mean equity premium. Finally, we show that the log of the price-dividend ratio can predict total consumption growth. The model implication for this predictability shows that the estimated slope is to a large extent reasonable, but the goodness-of-fit at various horizons are higher than predicted by empirical data.
Chapter 2

Three Additional Models on the Dynamics of Durable and Nondurable Consumptions

2.1 Introduction

In this chapter, we are going to introduce some changes to the CCAPM model in the first chapter in order to improve its asset pricing implications especially on the return of the stock market. These changes will be mainly on the dynamics of consumption goods as well as the dynamic of dividends. We know from the first chapter that the solution to the coefficients of our model \( A_{g,0}, A_{g,1}, A_{g,2}, A_{y,0}, A_{y,1}, A_{y,2} \) depend on the state variables. Since the state variables of our new specifications will be different than those of the first model, we will have new analytical solutions to our models. That is, changing the dynamics of consumption goods lead us to different derivation for the analytical solutions and eventually different solutions to the parameters of our models. On the other hand, some of the components we derived in the first model will remain the same in our new models. Namely, the GES utility function, total consumption, and IMRS derived in Sections 1.2.1 and 1.2.2 respectively will be the same. With different dynamics for both consumption goods, it allows us to determine if such changes can
resolve the drawbacks associated with our model in the first chapter. Some might argue that a value of risk aversion around 1, which we calibrated for our model in the first chapter, is implausibly low and that the empirically relevant range of risk aversion is from 3 to 10. More importantly, the mean of the empirical equity premium can’t be matched with the specifications of our first model.

Overall, we have two motivations to do the second chapter. The first motivation is to determine whether we can resolve all or at least some of these drawbacks associated with our first model. To do this, we are going to assume different specifications than what we have assumed earlier in the first chapter. The second motivation is to have more than one model, so when we use the estimation technique in the third chapter we can compare our models and choose the best among them. This will allow us to say how close our calibrated values are to their estimated counterparts. To be specific, we will investigate three more models which will be variants of the model in the first chapter. In the second model, the dynamics for the growth rates of nondurable and durable consumption will each exhibit a persistent expected component. In the third model, we will model the growth rate of nondurable to be a random walk process while the dynamic for the growth rate of durable has a persistent expected component. Finally, the fourth model will be an adjusted version of the first model. To be specific, we are going to add an expected component (LRR) to the dynamic of nondurable consumption. In addition, the growth rate of dividends will be leveraged claims on the expected component of both the growth rates of durable and nondurable consumption and also on the mean reversion part of the growth rate of nondurable consumption.
2.2 The second model

We assume that both the growth rates of durable and nondurable consumption exhibit persistent expected components. The notation for the durable consumption dynamic will take the following form:

\[ \Delta d_{t+1} = \mu_d + x_{d,t} + \omega_{d,t} \varepsilon_{d,t+1}, \]  
\[ (2.1a) \]

with the assumption of an AR (1) for the expected component \( x_{d,t} \):

\[ x_{d,t+1} = \phi_d x_{d,t} + \omega_{d,t} \varepsilon_{d,t+1}, \]  
\[ (2.2) \]

\[ \omega_{d,t} = \sqrt{1 - \psi_d x_{d,t}}. \]  
\[ (2.3) \]

Similarly, the nondurable consumption dynamic will take the following form:

\[ \Delta c_{t+1} = \mu_c + x_{c,t} + \omega_{c,t} \varepsilon_{c,t+1}, \]  
\[ (2.4a) \]

with a similar AR (1) process for the expected component of the growth rate of nondurable consumption \( x_{c,t} \):

\[ x_{c,t+1} = \phi_c x_{c,t} + \omega_{c,t} \varepsilon_{c,t+1}, \]  
\[ (2.5) \]

\[ \omega_{c,t} = \sqrt{1 - \psi_c x_{c,t}}. \]  
\[ (2.6) \]

where \( \varepsilon_{d,t+1}, \varepsilon_{x_{d,t+1}}, \varepsilon_{c,t+1}; \text{and } \varepsilon_{c_{c,t+1}} \sim N(0,1) \).

In response to a rise in the expected component \( x_{d,t} (x_{c,t}) \) for durable good consumption (nondurable good consumption), the growth rate of durable consumption (nondurable consumption) increases and as a result consumers buy more assets. In this case the intertemporal substitution effect dominates the income effect. We still have two state variables in the
second model as in the first model. However, here we assume that both durable and nondurable consumption have long-run risk in their processes. The mean parameters for the growth rates of durable and nondurable consumption denoted by $\mu_d$ and $\mu_c$ respectively. The parameters that control the persistence of the expected components of durable consumption $x_{d,t}$ and nondurable consumption $x_{c,t}$ are $\phi_d$ and $\phi_c$ respectively. Both of these persistence parameters have a range of (0,1). To have time-varying volatility for the growth rates of both durable and nondurable consumptions, we include the terms $\omega_{d,t}$ and $\omega_{c,t}$. These terms will allow us to have the implication of time-varying risk premiums.

### 2.2.1 Dividend growth rate

The dividend growth process is assumed to rely on the expected components of the growth rates of durable and nondurable consumption. We also assume that the dividend growth has a shock with volatility $\sigma_y$. Most importantly, we assume that the growth rate of dividends is a levered claim on the volatility of the growth rate of nondurable consumption. That is, the uncertainty in the growth rate of nondurable consumption should have a strong impact on the uncertainty in the stock market. Accordingly, the dividend growth rate $\Delta y_{t+1}$ process takes the following form:

$$\Delta y_{t+1} = \mu_y + \lambda_d x_{d,t} + \lambda_c x_{c,t} + \phi_d \omega_{d,t} \sigma_d x_{d,t+1} + \phi_c \omega_{c,t} \sigma_c x_{c,t+1} + \sigma_y \epsilon_{y,t+1}. \quad (2.7)$$

The mean of the growth rate of dividends is denoted by $\mu_y$. The leverage effect parameter for the expected component of the growth rate of durable consumption $x_{d,t}$ is $\lambda_d$. The leverage effect parameter for the expected component of the growth rate nondurable consumption $x_{c,t}$ is $\lambda_c$. Also, the leverage effect parameter for the conditional volatility of the growth
rate of durable consumption \( \omega_{d,t} \sigma_d \) is \( \phi_{dd} \). The leverage effect parameter for the conditional volatility of the growth rate of nondurable \( \omega_{c,t} \sigma_e \) is \( \phi_{ce} \). In total, all these leverage ratio effects, as defined in Bansal and Yaron (2004), will ensure extra source of risks to the growth rate of dividends \( \Delta y_{t+1} \) relative to the growth rate of total consumption \( \Delta g_{t+1} \). Therefor, the price-dividends ratio and the return on risky assets will be more volatile than the price-consumption ratio and the return on consumption. As a consequence, the equity premium can be justified because the return on risky assets is going to be associated with extra risk relative to the risk-free rate of return. Note that we still assume that the four shocks of our second model \( \epsilon_{c_{t+1}}, \epsilon_{x_{c_{t+1}}}, \epsilon_{z_{d_{t+1}}}, \epsilon_{x_{d_{t+1}}} \) and \( \epsilon_{a_{t+1}} \) are i.i.d.

2.2.2 Analytical solution

We are going to use the linear approximation of Campbell and Shiller (1988) to make our conjectures for the solutions to the log of the price-total consumption ratio \( z_t \) and the log of the price-dividend ratio \( z_{m,t} \). For the second model our state variables are \( x_{d,t} \) and \( x_{c,t} \). These linear approximations can be written as the following:

\[
\begin{align*}
  z_t & \approx A_{g,0} + A_{g,1} x_{d,t} + A_{g,2} x_{c,t}. \\
  z_{m,t} & \approx A_{y,0} + A_{y,1} x_{d,t} + A_{y,2} x_{c,t}.
\end{align*}
\]

Based on these conjectures, expressions for \( A_{g,0}, A_{g,1}, A_{g,2}, A_{y,0}, A_{y,1}, \) and \( A_{y,2} \) in terms of the model parameters are derived in Appendix E. As a result, the model’s expression for the return on wealth will take the following form:

\[
r_{g,t+1} = A_0 + \left( A_{g,0} + A_{g,1} x_{d,t+1} + A_{g,2} x_{c,t+1} \right) + \underbrace{\Delta y_{t+1}}_{z_t} - \underbrace{\left( A_{g,0} + A_{g,1} x_{d,t} + A_{g,2} x_{c,t} \right)}_{z_t}.
\] (2.8)
Along this line, the return on the market portfolio $r_{m,t+1}$ will take the following form:

$$
r_{m,t+1} = A_{m,0} + A_{m,1}(A_{y,0} + A_{y,1}x_{d,t+1} + A_{y,2}x_{c,t+1}) + \Delta y_{t+1} - (A_{y,0} + A_{y,1}x_{d,t} + A_{y,2}x_{c,t}).
$$

(2.9)

### 2.2.3 Innovation of IMRS

The IMRS for this model is the same as in the first chapter. However, our dynamics for the growth rates of nondurable consumption have changed. Therefore, the innovation of IMRS will take a slightly different form. Substituting equations (2.1a), (2.2), (2.3), (2.4a), (2.5), (2.6), (2.8), and (1.7), into the intertemporal equation (1.13), the volatility of the intertemporal marginal rate of substitution can be written as:

$$
m_{t+1} - E_t[m_{t+1}] = \lambda_{md,2} \sigma_{d,c_{t+1}} - \lambda_{mc,2} \sigma_{c,c_{t+1}}$$

$$
-\lambda_{mxc,2} \sigma_{xc,c_{t+1}} - \lambda_{mxd,2} \sigma_{xd,c_{t+1}}.
$$

(2.10)

where

$$
\lambda_{md,2} = \theta(1 - \frac{1}{\psi}) \alpha \omega_{d,t},
$$

$$
\lambda_{mc,2} = \left(\theta \left(\frac{1}{\psi} + (1 - \frac{1}{\psi}) \alpha \right) + (1 - \theta)\right) \omega_{c,t},
$$

$$
\lambda_{mxc,2} = (1 - \theta)A_1 A_{g,2} \omega_{c,t},
$$

$$
\lambda_{mxd,2} = (1 - \theta)A_1 A_{g,1} \omega_{d,t}.
$$

### 2.2.4 Risk-free rate

For the second model, we use two state variables $x_{d,t}$ and $x_{c,t}$. The conditional volatilities $\omega_{d,t}$ and $\omega_{c,t}$ of the growth rates by themselves are linearly related to the state variables
$x_{d,t}$ and $x_{c,t}$ respectively. These conditional volatilities $\omega_{d,t}$ and $\omega_{c,t}$ can generate time-varying implications to the second model only through the expected components of durable $x_{d,t}$ and nondurable consumption $x_{c,t}$ respectively. Therefore, we ignore the very small variance part of the pricing kernel in our derivation for the risk free rate. The expression for the risk free rate will be:

$$r_{f,t} \approx -E_t[m_{t+1}].$$

Which can be explicitly written as:

$$r_{f,t} = -\theta \log(B) + \frac{\theta}{\psi} (\mu_c + x_{c,t}) - \theta(1 - \frac{1}{\psi})\alpha (\mu_d + x_{d,t} - \mu_c - x_{c,t})$$

$$- (\theta - 1) \left[ A_0 + A_1 [A_{g,0} + A_{g,1}\phi_d x_{d,t} + A_{g,2}\phi_c x_{c,t}] + \mu_c + x_{c,t} - [A_{g,0} + A_{g,1}x_{d,t} + A_{g,2}x_{c,t}] \right] E_t[r_{g,t+1}].$$

and its volatility is:

$$\text{var}(r_{f,t}) = \Gamma_{d,2}\text{var}(x_{d,t}) + \Gamma_{c,2}\text{var}(x_{c,t}),$$

where

$$\Gamma_{d,2} = \left( \theta \left( \frac{1}{\psi} - 1 \right) \alpha + (1 - \theta) \left( A_1 A_{g,1}\phi_d - A_{g,1} \right) \right)^2;$$

$$\Gamma_{c,2} = \left( \frac{\theta}{\psi} + \theta(1 - \frac{1}{\psi})\alpha + (1 - \theta) \left( A_1 A_{g,2}\phi_c - A_{g,2} \right) \right)^2.$$  

2.3 The third model

We start with our specification for the growth rate of durable consumption. We keep our model dynamics for the growth rate of durables the same as in the first and second model.
Thus, the dynamic equation for durable consumption should take the following form:

$$\Delta d_{t+1} = \mu_d + x_{d,t} + \omega_d t \sigma_d \epsilon_{d,t+1}, \quad (2.11)$$

with

$$x_{d,t+1} = \phi_d x_{d,t} + \omega_d t \sigma_d \epsilon_{d,t+1}; \quad (2.12)$$

$$\omega_d = \sqrt{1 - \zeta_d x_{d,t}}; \quad (2.13)$$

where $\epsilon_{d,t+1}, \epsilon_{d,t+1} \sim N(0, 1)$.

Definitions for these parameters have been given above in the previous section. On the other hand, we assume that the growth rate of nondurable consumption follows a random walk with a drift. Moreover, we assume that the volatility of the growth rate of nondurable consumption follows a mean reverting process. We are inspired by the work of Bansal and Yaron (2004) only in modeling the time-varying volatility of the growth rate of nondurable consumption. However, we do not include an expected component as a state variable as they do in their model. As such, the dynamic of nondurable consumption can be written as the following:

$$\Delta c_{t+1} = \mu_c + \sigma_{c,t} \epsilon_{c,t+1}, \quad (2.14)$$

with

$$\sigma_{c,t+1}^2 = \sigma_c^2 + \phi_c (\sigma_{c,t}^2 - \sigma_c^2) + \sigma_{c,c} \epsilon_{c,c,t+1}; \quad (2.15)$$

where $\epsilon_{c,t+1} \sim N(0, 1)$ and $\epsilon_{c,c,t+1} \sim N(0, 1)$.

The mean of the growth rate of nondurable is $\mu_c$. Unlike our previous models, the volatility of the growth rate of nondurable $\sigma_{c,t}$ is time-varying and not constant anymore. Also, the variance of the dynamics of nondurable consumption is a reverting process around its mean.
The persistence of this time-varying variance is controlled by the parameter $\phi_{\sigma c}$, which has a range of $0 < \phi_{\sigma c} < 1$. The higher this parameter is the more persistent the dynamic of the variance would be. This time-varying variance also has a volatility denoted by the term $\sigma_{\sigma c}$.

Our third model still depends on two state variables. As before, the expected component of durable consumption $x_{d,t}$ is considered as the first state variable. While, the time-varying volatility of the growth rate of nondurable consumption $\sigma_{c,t}$ is the second state variable.

2.3.1 Dividend growth rate

The dividend growth process of our third model is going to rely on the expected component of the growth rate of durable consumption as well as its conditional volatility, the time-varying volatility of the growth rate of nondurable consumption, and on an additional shock with volatility $\sigma_y$. To be specific, the growth rate of dividend is a levered claim on the expected component of the growth rate of durable consumption $x_{d,t}$, the conditional volatility of the growth rate of durable consumption $\sigma_{d,t}$, and the time-varying volatility of the growth rates of nondurable consumption $\sigma_{c,t}$. The dynamic of dividend growth rate $\Delta y_{t+1}$ can be written as:

$$\Delta y_{t+1} = \mu_y + \lambda_d x_{d,t} + \rho_c \sigma_{c,t} \epsilon_{c,t+1} + \rho_d \omega_{d,t} \sigma_{d,t} \epsilon_{d,t+1} + \sigma_y \epsilon_{y,t+1},$$

(2.16)

where $\epsilon_{y,t+1} \sim N(0, 1)$.

The mean of the growth rate of dividend is $\mu_y$. The leverage effect parameter for the expected component of the growth rate of durable consumption $x_{d,t}$ is $\lambda_d$. The leverage effect parameter for the time-varying volatility of the growth rate nondurable consumption $\sigma_{c,t}$ is $\rho_c$. The leverage effect parameter for the conditional volatility of the growth rate of durable
consumption \( \omega_{d,t} \sigma_d \) is \( \rho_d \)

### 2.3.2 Analytical solution

As for the third model, the state variables are \( x_{d,t} \) and \( \sigma_{c,t}^2 \). Therefore, we make the assumption that both the log of price-total consumption ratio \( z_t \) and the log of price dividend ratio \( z_{m,t} \), following the linear approximation of Campbell and Shiller (1988), are linearly related to these state variables. These conjectures can be written as:

\[
    z_t \approx A_{g,0} + A_{g,1} x_{d,t} + A_{g,2} \sigma_{c,t}^2.
\]

\[
    z_{m,t} \approx A_{y,0} + A_{y,1} x_{d,t} + A_{y,2} \sigma_{c,t}^2.
\]

Based on these conjectures, expressions for the coefficients \( A_{g,0}, A_{g,1}, A_{g,2}, A_{y,0}, A_{y,1}, \) and \( A_{y,2} \) in terms of the model parameters are derived in Appendix F. The relative return on wealth of the third model, will take the following form:

\[
    r_{g,t+1} = A_0 + A_1 (A_{g,0} + A_{g,1} x_{d,t+1} + A_{g,2} \sigma_{c,t+1}^2) + \frac{\Delta g_{t+1}}{\Delta x_{t+1}} (A_{g,0} + A_{g,1} x_{d,t} + A_{g,2} \sigma_{c,t}^2). \tag{2.17}
\]

Similarly, the return on the stock market will be linearly related to the log of price dividend ratio \( z_{m,t} \) and dividend growth, and this can be given in the following equation:

\[
    r_{m,t+1} = A_{m,0} + A_{m,1} (A_{y,0} + A_{y,1} x_{d,t+1} + A_{y,2} \sigma_{c,t+1}^2) + \frac{\Delta y_{t+1}}{\Delta x_{t+1}} (A_{y,0} + A_{y,1} x_{d,t} + A_{y,2} \sigma_{c,t}^2). \tag{2.18}
\]

### 2.3.3 Innovation of IMRS

The innovation to the IMRS of our third model will also be different than those of the first and second models. As a result, substituting equations (2.11), (2.12), (2.13), (2.14), (2.15),
(2.17), and (1.7), into the intertemporal equation (1.13), the innovation of the intertemporal of marginal rate of substitution can be written as:

\[ m_{t+1} - E_t[m_{t+1}] = \lambda_{md,3} \sigma_d \epsilon_{d,t+1} - \lambda_{mc,3} \sigma_c \epsilon_{c,t+1} - \lambda_{mxc,3} \sigma_m \sigma_c \epsilon_{c,t+1} - \lambda_{mxd,3} \sigma_m \sigma_d \epsilon_{d,t+1}, \]

with

\[
\begin{align*}
\lambda_{md,3} &= \theta(1 - \frac{1}{\psi}) \omega_{d,t}, \\
\lambda_{mc,3} &= \left( \theta \left( \frac{1}{\psi} + (1 - \frac{1}{\psi}) \alpha \right) + (1 - \theta) \right), \\
\lambda_{mxc,3} &= (1 - \theta) A_1 A_{g,2}, \\
\lambda_{mxd,3} &= (1 - \theta) A_1 A_{g,1} \omega_{d,t}.
\end{align*}
\]

We want to point out that all our three models have time-varying innovations to their IMRS, and therefore, their source of risks are varying over time.

### 2.3.4 Risk-free rate

For the third model, we are going to account for the variance part of the pricing kernel in deriving the risk free rate. Unlike our previous models, the time-varying variance of nondurable consumption growth \( \sigma_{c,t}^2 \) is a state variable by itself. Therefore, we can’t ignore the variance part as we did before in the first and second models. As a result, the expression for the risk free rate will be written as:

\[ r_{f,t} \approx -E_t[m_{t+1}] - 0.5 \text{var}_t[m_{t+1}]. \]
Which can be explicitly expanded as:

\[
    r_{f,t} = -\theta \log(B) + \frac{\theta}{\psi} \mu_c - \theta (1 - \frac{1}{\psi}) \alpha (\mu_d + x_{d,t} - \mu_c) + \\
    (1 - \theta) \left[ A_0 + A_1 \left[ A_{g,0} + A_{g,1} \phi_d x_{d,t} + A_{g,2} \phi_\sigma \sigma_{c,t}^2 \right] + \mu_c - \left[ A_{g,0} + A_{g,1} x_{d,t} + A_{g,2} \sigma_{c,t}^2 \right] \right] - \text{var}(m_{t+1}),
\]

and

\[
    \text{var}(m_{t+1}) = [m_{t+1} - E_t[m_{t+1}]]^2.
\]

Also, the volatility of the risk-free rate takes the following form:

\[
    \text{var}(r_{f,t}) = \Gamma_{d,3} \text{var}(x_{d,t}) + \Gamma_{c,3} \text{var}(\sigma_{c,t}^2),
\]

where

\[
    \Gamma_{d,3} = \left( \theta \left( \frac{1}{\psi} - 1 \right) \alpha + (1 - \theta) (A_1 A_{g,1} \phi_d - A_{g,1}) \right)^2,
\]

\[
    \Gamma_{c,3} = \left( (1 - \theta) (A_1 A_{g,2} \phi_c - A_{g,2}) - .5((\theta - 1) - \frac{\theta}{\psi} - \theta(1 - \frac{1}{\psi}) \alpha) \right)^2.
\]

### 2.4 Simulation

Our data are the same as we used with our first model in Chapter 1. CRSP is the main source for stock market data. Data on real dividends and prices of the S&P composite index are taken from Shiller’s dataset. Both risky return, risk free rate, as well as dividend growth have been adjusted for inflation and used in real terms. For data on consumption, we use BEA data on both nondurables and services and on the durables stock. All our data are in real terms, in annual frequency, and taken for the period 1952-2010. For further details please refer to Section 1.3 in the first chapter. As in the first chapter, we simulate at the monthly
frequency and then time aggregate at the annual frequency before computing some moments of the growth rates and asset returns. That is, we simulate 10,000 samples for 59 x 12 monthly observations that match the length of the empirical data. The calibrated parameters of our second and third models are reported in Table 2.1.

<table>
<thead>
<tr>
<th>Table 2.1: Calibrated parameters of the second and third models</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The 2nd model</strong></td>
</tr>
<tr>
<td>Durable growth</td>
</tr>
<tr>
<td>$\mu_d, \phi_d, \sigma_d, \sigma_{sd}, \zeta_d$</td>
</tr>
<tr>
<td>Preferences</td>
</tr>
<tr>
<td>$\gamma, \psi, B, \alpha$</td>
</tr>
<tr>
<td>Nondurable growth</td>
</tr>
<tr>
<td>$\mu_c, \phi_c, \sigma_c, \sigma_{sc}, \zeta_c$</td>
</tr>
<tr>
<td>Dividend growth</td>
</tr>
<tr>
<td>$\mu_y, \lambda_d, \lambda_c, \phi_{dc}, \phi_{cc}, \sigma_y$</td>
</tr>
<tr>
<td><strong>The 3rd model</strong></td>
</tr>
<tr>
<td>Durable growth</td>
</tr>
<tr>
<td>$\mu_d, \phi_d, \sigma_d, \sigma_{sd}, \zeta_d$</td>
</tr>
<tr>
<td>Preferences</td>
</tr>
<tr>
<td>$\gamma, \psi, B, \alpha$</td>
</tr>
<tr>
<td>Nondurable growth</td>
</tr>
<tr>
<td>$\mu_c, \sigma_c, \phi_{ac}, \sigma_{ac}$</td>
</tr>
<tr>
<td>Dividend growth</td>
</tr>
<tr>
<td>$\mu_y, \lambda_d, \rho_c, \rho_d, \sigma_y$</td>
</tr>
</tbody>
</table>

Our calibrated value for EIS of 1.5 is consistent with the EIS value that has been used in previous papers such as Bansal and Yaron (2004), Yang (2011) and many others. The second model’s risk aversion value of 10.8 is a reasonable value, and it is very close to the choice of Bansal and Yaron (2004) of 10. The calibrated risk aversion of 17 for the third model is a little higher than the suggested maximum value of 10 by Mehra and Prescott (1985). This high risk aversion is needed to achieve better matching for moments of the equity premium.
Table 2.2 reports the moments of durable consumption growth, nondurable consumption growth, and dividend growth. Most of these moments are within the simulated 5%-95% percentiles and the means of the simulated 10,000 samples are close to the empirical counterparts. However, the first order autocorrelation of dividend growth is the only exception because it does not fall within the range of the simulated 5th-95th percentile interval. Overall, the assumed growth rates for both the 2nd and 3rd models do a good job in replicating the empirical data’s moments.

Table 2.2: Growth rates of the second and third models

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th>2nd Model</th>
<th>3rd Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SE</td>
<td>Mean</td>
</tr>
<tr>
<td>Durable consumption growth</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[\Delta d]$</td>
<td>0.0377 (0.004)</td>
<td>0.0378</td>
<td>0.0202</td>
</tr>
<tr>
<td>$\sigma[\Delta d]$</td>
<td>0.0171 (0.001)</td>
<td>0.0181</td>
<td>0.0123</td>
</tr>
<tr>
<td>$AC1(\Delta d)$</td>
<td>0.6910 (0.135)</td>
<td>0.6424</td>
<td>0.3985</td>
</tr>
<tr>
<td>$AC2(\Delta d)$</td>
<td>0.4330 (0.140)</td>
<td>0.5151</td>
<td>0.2126</td>
</tr>
<tr>
<td>$AC3(\Delta d)$</td>
<td>0.1150 (0.161)</td>
<td>0.4074</td>
<td>0.0803</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.407 (0.188)</td>
<td>-0.3859</td>
<td>-1.0383</td>
</tr>
<tr>
<td>Nondurable consumption growth</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[\Delta c]$</td>
<td>0.0196 (0.002)</td>
<td>0.0196</td>
<td>0.0160</td>
</tr>
<tr>
<td>$\sigma[\Delta c]$</td>
<td>0.0099 (0.001)</td>
<td>0.0114</td>
<td>0.0095</td>
</tr>
<tr>
<td>$AC1(\Delta c)$</td>
<td>0.4531 (0.215)</td>
<td>0.2584</td>
<td>0.0401</td>
</tr>
<tr>
<td>$AC2(\Delta c)$</td>
<td>0.1097 (0.154)</td>
<td>0.0857</td>
<td>-0.1449</td>
</tr>
<tr>
<td>$AC3(\Delta c)$</td>
<td>0.0047 (0.121)</td>
<td>0.0138</td>
<td>-0.2154</td>
</tr>
<tr>
<td>Dividends growth</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[\Delta y]$</td>
<td>0.0109 (0.009)</td>
<td>0.0109</td>
<td>-0.0086</td>
</tr>
<tr>
<td>$\sigma[\Delta y]$</td>
<td>0.0392 (0.005)</td>
<td>0.0393</td>
<td>0.0304</td>
</tr>
<tr>
<td>$AC1(\Delta y)$</td>
<td>0.5390 (0.169)</td>
<td>0.1725</td>
<td>-0.0779</td>
</tr>
</tbody>
</table>

Turning to Table 2.3, we see that most of the moments can be successfully generated for both the 2nd and 3rd models with the exception of some moments. Both models have a low
value for the volatility of risk-free rates. However, this small volatility value is close to what previous papers report. Yang (2011) argues that this result is what we should expect to have because the volatility of the ex-ante real risk-free rate is expected to be lower than the realized ex-post real rate volatility. That is, the realized rate would contain inflation and therefore, should be more volatile. Also, both the 2\textsuperscript{nd} and 3\textsuperscript{rd} models match closely the moments of the equity premium. On the other hand, the second model generates a higher value for the mean of the log dividend-price ratio than that of the empirical data. Also, the third-order autocorrelation of the log dividend-price ratio is not within the range of the simulated 5\textsuperscript{th}-95\textsuperscript{th} percentile interval for both models. All in all, both models are to a high extent reasonable in replicating many moments of the empirical data.

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th>2\textsuperscript{nd} Model</th>
<th>3\textsuperscript{rd} Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SE</td>
<td>Mean 5\textsuperscript{th}</td>
</tr>
<tr>
<td>$E[rf]$</td>
<td>0.0161</td>
<td>(0.005)</td>
<td>0.0160</td>
</tr>
<tr>
<td>$\sigma[rf]$</td>
<td>0.0165</td>
<td>(0.017)</td>
<td>0.0074</td>
</tr>
<tr>
<td>$E[r - rf]$</td>
<td>0.0670</td>
<td>(0.020)</td>
<td>0.0670</td>
</tr>
<tr>
<td>$\sigma[r]$</td>
<td>0.1448</td>
<td>(0.013)</td>
<td>0.1582</td>
</tr>
<tr>
<td>$E[p - d]$</td>
<td>3.4960</td>
<td>(0.102)</td>
<td>4.3464</td>
</tr>
<tr>
<td>$\sigma[p - d]$</td>
<td>0.3080</td>
<td>(0.062)</td>
<td>0.2153</td>
</tr>
<tr>
<td>$AC1[p - d]$</td>
<td>0.9100</td>
<td>(0.306)</td>
<td>0.8468</td>
</tr>
<tr>
<td>$AC2[p - d]$</td>
<td>0.8400</td>
<td>(0.290)</td>
<td>0.6583</td>
</tr>
<tr>
<td>$AC3[p - d]$</td>
<td>0.7830</td>
<td>(0.269)</td>
<td>0.5054</td>
</tr>
</tbody>
</table>

### 2.5 Adjustments to the first model

We try to modify our first model in order to improve its implications. To be specific, we assume that the dynamic process of the growth rate of nondurable consumption exhibits not
only mean reversion process, but also a long-run risk element. Therefore, this adjusted model is going contain three state variables. Two state variables in the dynamic of the growth rate of nondurable consumption and one state variable in the dynamic of the growth rate of durable consumption. We also are going to assume a different specification for the dynamic of dividend growth. The dynamic of the growth rate of nondurable consumption will take the following form:

\[ \Delta c_{t+1} = \Delta \tilde{c}_{t+1} + \Delta s_{t+1}, \]

with

\[
\begin{align*}
\Delta \tilde{c}_{t+1} &= \mu_c + x_{c,t} + \omega_z \sigma_c \varepsilon_{c,t+1}, \\
x_{c,t+1} &= \phi_c x_c t + \omega_c \sigma_c \varepsilon_c c_t + 1; \\
\omega_c &= \sqrt{1 - \zeta c \sigma c t}, \\
\Delta s_{t+1} &= (\phi_s - 1)(s_t - \bar{s}) + \omega_s \sigma_s \varepsilon_{s,t+1}, \\
\omega_s &= \sqrt{\zeta_s (1 - (s_t - \bar{s}))},
\end{align*}
\]

where \( \varepsilon_{c,t+1}, \varepsilon_{cc,t+1}, \) and \( \varepsilon_{s,t+1} \sim N(0, 1). \)

This adjusted model keeps the same definitions for most of the parameters as in the first model. The only additional part that we add is the long-run risk element \( x_{c,t}. \) The range for the persistence parameter \( \phi_c \) is given as \( 0 < \phi_c < 1. \) The variable \( x_{c,t+1} \) follows an AR(1) process that is being generated by a time-varying volatility of \( \sigma_c \omega_c t. \) For further information on the definitions of the other parameters, please refer to Section 1.2.3.
The dynamic of the growth rate of durable consumption is the same as before and can be rewritten as:

$$\Delta d_{t+1} = \mu_d + x_d,t + \omega_d,t \sigma_d \epsilon_{d,t+1},$$

with

$$x_{d,t+1} = \phi_d x_{d,t} + \omega_d,t \sigma_d \epsilon_{d,t+1},$$

$$\omega_d,t = \sqrt{1 - \zeta_d x_d,t},$$

where $\epsilon_{d,t+1}$ and $\epsilon_{x_d,t+1} \sim N(0, 1)$.

The dividend growth will also be modified for this adjusted model and can be written as:

$$\Delta y_{t+1} = \mu_y + \lambda_d x_{d,t} + \lambda_c x_{c,t} + \lambda_s(s_t - \overline{s}) + \rho_d \omega_{s,t} \sigma_d \epsilon_{c,t+1} + \rho_d \omega_{d,t} \sigma_d \epsilon_{d,t+1}.$$
The solution of the coefficients \(A_{g,0}, A_{g,1}, A_{g,2}, A_{y,0}, A_{y,1},\) and \(A_{y,2}\) will be the same as in the first chapter. As for the coefficients \(A_{g,3}\) and \(A_{y,3}\), their solutions are provided in Appendix G.

### 2.6 Simulation of the adjusted first model

We still use the same data and same methodology to simulate the growth rates of this model as done previously with our previous models. Table 2.4 reports the calibrated parameters that we are going to use for the adjusted model. For risk aversion, we choose a value of 10.9 for the parameter \(\gamma\), which is close the maximum value of 10 suggested by Mehra and Prescott (1985). All these parameters are chosen in order to enable our model to match closely estimated moments of the empirical data. Table 2.5 reports the moments of the growth rates of durable consumption, nondurable consumption, and dividends. We see results comparable to the results obtained with the second and third models. Unlike the first model, the adjusted first model can now replicate both the volatility as well as the first three autocorrelations of the growth rate of nondurable consumption. However, this adjusted model still could not match the first order autocorrelation of the growth rate of dividends.

<table>
<thead>
<tr>
<th>Durable growth</th>
<th>(\mu_d)</th>
<th>(\phi_d)</th>
<th>(\sigma_d)</th>
<th>(\sigma_{zd})</th>
<th>(\zeta_{zd})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\gamma)</td>
<td>0.0031</td>
<td>0.99</td>
<td>0.0022</td>
<td>2.5800e-04</td>
<td>220</td>
</tr>
<tr>
<td>Preferences</td>
<td>(\psi)</td>
<td>(B)</td>
<td>(\alpha)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nondurable growth</td>
<td>(\mu_c)</td>
<td>(\sigma_c)</td>
<td>(\zeta_c)</td>
<td>(\phi_c)</td>
<td>(\sigma_s)</td>
</tr>
<tr>
<td>(\phi_c)</td>
<td>0.0016</td>
<td>0.0289</td>
<td>0.01</td>
<td>0.98</td>
<td>8.8914e-04</td>
</tr>
<tr>
<td>Nondurable growth</td>
<td>(\sigma_{sc})</td>
<td>(\zeta_{sc})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\phi_c)</td>
<td>0.95</td>
<td>2.5800e-04</td>
<td>200</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dividend growth</td>
<td>(\mu_y)</td>
<td>(\lambda_d)</td>
<td>(\lambda_c)</td>
<td>(\lambda_s)</td>
<td>(\rho_c)</td>
</tr>
<tr>
<td>(\rho_{y})</td>
<td>9.08e-04</td>
<td>1.0549</td>
<td>0.9961</td>
<td>0.9356</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 2.4: Calibrated parameters of the adjusted first model
Moreover, we are able to resolve the biggest problem that we have encountered with our first model. Now, the adjusted model can replicate the mean excess return and its volatility as reported in Table 2.6. Unlike our second and third model, this model could also achieve a closer match for the volatility of the risk-free rate. The first three autocorrelations are still lower than the empirical counterpart.

Table 2.5: Growth rates of the adjusted first model

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th>Model</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SE</td>
<td>Mean</td>
<td>5th</td>
<td>95th</td>
</tr>
<tr>
<td>Durable consumption growth</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[\Delta d]$</td>
<td>0.0377 (0.004)</td>
<td>0.0378</td>
<td>0.0196</td>
<td>0.0531</td>
<td></td>
</tr>
<tr>
<td>$\sigma[\Delta d]$</td>
<td>0.0171 (0.001)</td>
<td>0.0189</td>
<td>0.0124</td>
<td>0.0280</td>
<td></td>
</tr>
<tr>
<td>$AC1(\Delta d)$</td>
<td>0.6910 (0.135)</td>
<td>0.6925</td>
<td>0.4705</td>
<td>0.8600</td>
<td></td>
</tr>
<tr>
<td>$AC2(\Delta d)$</td>
<td>0.4330 (0.140)</td>
<td>0.5552</td>
<td>0.2619</td>
<td>0.7905</td>
<td></td>
</tr>
<tr>
<td>$AC3(\Delta d)$</td>
<td>0.1150 (0.161)</td>
<td>0.4388</td>
<td>0.1054</td>
<td>0.7273</td>
<td></td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.407 (0.188)</td>
<td>0.4031</td>
<td>-1.0676</td>
<td>0.1776</td>
<td></td>
</tr>
<tr>
<td>Nondurable consumption growth</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[\Delta c]$</td>
<td>0.0196 (0.002)</td>
<td>0.0196</td>
<td>0.0152</td>
<td>0.0238</td>
<td></td>
</tr>
<tr>
<td>$\sigma[\Delta c]$</td>
<td>0.0099 (0.001)</td>
<td>0.0131</td>
<td>0.0109</td>
<td>0.0154</td>
<td></td>
</tr>
<tr>
<td>$AC1(\Delta c)$</td>
<td>0.4531 (0.215)</td>
<td>0.2567</td>
<td>0.0241</td>
<td>0.4721</td>
<td></td>
</tr>
<tr>
<td>$AC2(\Delta c)$</td>
<td>0.1097 (0.154)</td>
<td>0.1187</td>
<td>-0.1104</td>
<td>0.3479</td>
<td></td>
</tr>
<tr>
<td>$AC3(\Delta c)$</td>
<td>0.0047 (0.121)</td>
<td>0.0472</td>
<td>-0.1823</td>
<td>0.2823</td>
<td></td>
</tr>
<tr>
<td>Dividends growth</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[\Delta y]$</td>
<td>0.0109 (0.009)</td>
<td>0.0109</td>
<td>-0.0088</td>
<td>0.0279</td>
<td></td>
</tr>
<tr>
<td>$\sigma[\Delta y]$</td>
<td>0.0392 (0.005)</td>
<td>0.0451</td>
<td>0.0381</td>
<td>0.0530</td>
<td></td>
</tr>
<tr>
<td>$AC1(\Delta y)$</td>
<td>0.5390 (0.169)</td>
<td>0.7706</td>
<td>0.6963</td>
<td>0.8381</td>
<td></td>
</tr>
</tbody>
</table>
Table 2.6: Asset pricing implications of the adjusted first model

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data Mean</th>
<th>SE</th>
<th>Model Mean</th>
<th>5th</th>
<th>95th</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[\rf]$</td>
<td>0.0160</td>
<td>0.005</td>
<td>0.0160</td>
<td>0.0122</td>
<td>0.0196</td>
</tr>
<tr>
<td>$\sigma[\rf]$</td>
<td>0.0165</td>
<td>0.017</td>
<td>0.0121</td>
<td>0.0108</td>
<td>0.0136</td>
</tr>
<tr>
<td>$E[r - rf]$</td>
<td>0.0670</td>
<td>0.020</td>
<td>0.0670</td>
<td>0.0597</td>
<td>0.0756</td>
</tr>
<tr>
<td>$\sigma[r]$</td>
<td>0.1448</td>
<td>0.013</td>
<td>0.1636</td>
<td>0.1434</td>
<td>0.1862</td>
</tr>
<tr>
<td>$E[p - d]$</td>
<td>3.4960</td>
<td>0.102</td>
<td>3.4832</td>
<td>3.2294</td>
<td>3.6977</td>
</tr>
<tr>
<td>$\sigma[p - d]$</td>
<td>0.3080</td>
<td>0.062</td>
<td>0.2699</td>
<td>0.1885</td>
<td>0.3911</td>
</tr>
<tr>
<td>$AC1[p - d]$</td>
<td>0.9100</td>
<td>0.306</td>
<td>0.8360</td>
<td>0.7205</td>
<td>0.9235</td>
</tr>
<tr>
<td>$AC2[p - d]$</td>
<td>0.8400</td>
<td>0.290</td>
<td>0.6541</td>
<td>0.4304</td>
<td>0.8289</td>
</tr>
<tr>
<td>$AC3[p - d]$</td>
<td>0.7830</td>
<td>0.269</td>
<td>0.5056</td>
<td>0.2200</td>
<td>0.7448</td>
</tr>
</tbody>
</table>

2.7 Total consumption predictability

Since the price-dividend ratio depends on the state variables for all of our models, it should be a valid predictor of total consumption growth. Further, we make use of the expression $\Delta g_{t+1} = \Delta c_{t+1}$ that we derived in the first chapter. We regress the sum of the growth rate of nondurable consumption $\sum_{j=1}^{5} \Delta c_{t+j}$ on the logarithm of the price-dividend ratio $\log(P_{m,t}/Y_t)$ plus a constant. Estimates for this regression involve one-year, three-year, and five-year horizons, so $j = 1, 3, 5$. Note that our third model is not included since we assume that the dynamic of nondurable growth $\Delta c_{t+1}$ is a random walk.

Table 2.7: Predictability of consumption growth by the logarithm of the price-dividend ratio

<table>
<thead>
<tr>
<th>Horizon year</th>
<th>Data slope</th>
<th>SE</th>
<th>$R^2$</th>
<th>2nd Model slope</th>
<th>$R^2$</th>
<th>Adj. 1st Model slope</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.003</td>
<td>0.004</td>
<td>0.01</td>
<td>0.001</td>
<td>0.02</td>
<td>-0.001</td>
<td>0.03</td>
</tr>
<tr>
<td>3</td>
<td>-0.01</td>
<td>0.009</td>
<td>0.07</td>
<td>0.001</td>
<td>0.04</td>
<td>-0.003</td>
<td>0.06</td>
</tr>
<tr>
<td>5</td>
<td>-0.02</td>
<td>0.012</td>
<td>0.08</td>
<td>0.001</td>
<td>0.05</td>
<td>-0.005</td>
<td>0.09</td>
</tr>
</tbody>
</table>
Table 2.7 reports evidence on total consumption growth predictability for both the empirical data and for the second and adjusted models. The slope of empirical data has a decreasing trend as we go over longer horizons. On the other hand, $R^2$s of the empirical data increase with the horizon. Both the second and the adjusted model match closely the empirical data’s $R^2$s. Unlike the second model, the adjusted model has the correct signs for the slope estimates and their magnitudes are a little bit lower than those of the empirical data.

### 2.8 Excess return predictability

Another natural implication of our models is that they can predict the equity excess return using the log of price-dividend ratio. That is, the models’ implied equity premium as well as the log of the price-dividend ratio depend on the state variables and that is true for all of the three models that we introduced in the second chapter. As in chapter 1, we regress the sum of the returns on risky assets $\sum_{j=1}^{5} r_{m,t+j}$ on the logarithm of the price-dividend ratio $\log(P_{m,t}/Y_t)$ plus a constant. Here, we are going to provide estimates for the predictability of one-year, three-year, and five-year horizons.

<table>
<thead>
<tr>
<th>Horizon</th>
<th>Data</th>
<th>$2^{nd}$ Model</th>
<th>$3^{rd}$ Model</th>
<th>Adj. $1^{st}$ Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>year</td>
<td>slope SE</td>
<td>slope $R^2$</td>
<td>slope $R^2$</td>
<td>slope $R^2$</td>
</tr>
<tr>
<td>1</td>
<td>-0.11 (0.09)</td>
<td>0.06</td>
<td>-0.09 (0.19)</td>
<td>0.03</td>
</tr>
<tr>
<td>3</td>
<td>-0.25 (0.09)</td>
<td>0.14</td>
<td>-0.35 (0.35)</td>
<td>0.11</td>
</tr>
<tr>
<td>5</td>
<td>-0.34 (0.11)</td>
<td>0.16</td>
<td>-0.53 (0.43)</td>
<td>0.19</td>
</tr>
</tbody>
</table>
Table 2.8 shows that the slopes for the one-year horizon, of the second model and third models are very close to the slope of the empirical data. Both of these slopes are within one standard error. The slope for the adjusted model is still within one standard error. On the other hands, the slope of the adjusted model better matches the slope of data estimates for the longer horizons than the other two models. The $R^2$ for the one-year horizon is lower for all three models than the $R^2$ of the empirical data. For the three-year horizon, the second model and the third model have a closer match for $R^2$ than the adjusted model. However, as we account for longer horizons, the $R^2$ of the adjusted model becomes the closest to the $R^2$ of the empirical data.

2.9 Conclusion

In this chapter, we use the same CCAPM model as in the first model in Chapter 1. Unlike the first model, we have introduced three different specifications for the growth rates of durable consumption, nondurable consumption, and dividends. For these growth rates, all new models are able to provide almost a perfect match to moments of the empirical data, but with one exception; the first-order autocorrelation of the growth rate of dividends. For the risk aversion parameter, the second and the adjusted first models have a calibrated values around 11 in comparison to 17 for the third model. In addition, the EIS calibrated values for both the second and third models are 1.5 compared to an EIS of 1.1 for the adjusted first model. Also, the asset pricing implications of these models are close to the empirical counterparts. On the other hand, all these models are not able to replicate the volatility of the risk-free rate, but the models’ generated sample means are not far from the empirical data and the adjusted
model’s sample mean is the closest. Moreover, the empirical mean of the log of price-dividends is matched perfectly by the adjusted first model, whereas the third could barely includes it within the simulated 5th-95th interval. The second model is not able to match the mean of the log of price-dividends. For the predictability of total consumption growth, the adjusted model provides better results for the slopes across the different horizons than the second model, whereas, the third model cannot predict this series. The adjusted model also provides decent predictions for the future return on risky assets. Overall, the adjusted model outperforms the other two models for the calibrated parameters values employed.
Chapter 3

Estimations of CCAPM Models

To further confirm the validity of our models we used in Chapters one and two, we are going to estimate their parameters instead of relying on calibration. There are several techniques that can be used to estimate our models. We decided to employ the indirect inference estimation technique, which was initially introduced in Gourieroux, Monfort, and Renault (1993). When the likelihood function is intractable, but simulating data from the model is a possibility, as in our case, the indirect inference technique would be a good choice.

Under the indirect inference technique, we need to consider an auxiliary model and provide a method through which we estimate the parameters of this auxiliary model. The goal is then to find values for the parameters of our (structural) model such that data simulated from the model give auxiliary parameter estimates as close as possible to the auxiliary parameter estimates obtained with the true data.

3.1 Indirect inference

We use moments as the auxiliary model for the indirect inference estimation. As such, the indirect inference criterion function will try to choose the vector of (structural) parameters
\( \eta \) in order to bring the moments from simulated data closer to moments from empirical data. In our case, \( \eta \) contains the risk aversion parameter \( \gamma \), the EIS parameter \( \psi \), the time preference parameter \( B \), and the share of durable consumption parameter \( \alpha \). Moreover, we use the same set of moments that we used to assess the models in Chapter 1 and 2: namely, the mean of the risk-free rate and its volatility, the mean of the return on risky assets and its volatility, and the mean of the log of the price-dividend ratio and its volatility, and also its first three autocorrelations. Most importantly, we use indirect inference to search over the optimal combination of our four parameters, contained in \( \eta \), in order to solve the following minimization problem:

\[
\eta = \arg\min_{\eta} \pi_T(\eta, \beta),
\]

where

\[
\pi_T(\eta, \beta) = \left[ \beta_\ast - \frac{1}{H} \sum_{h=1}^{H} \beta^h_T(\eta) \right] \Lambda_T \left[ \beta_\ast - \frac{1}{H} \sum_{h=1}^{H} \beta^h_T(\eta) \right]
\]

and \( \beta_\ast \) denotes the vector of observed data moments. The vector \( \beta^h_T \) denotes the moments computed with the \( h^{th} \) simulated sample of the length \( T \) (to match the length of the actual data). The variable \( \Lambda_T \) denotes a positive definite symmetric weighting matrix.

Following the notation in Gourieroux, Monfort, and Renault (1993), we define the binding function as:

\[
b(\eta) = \arg\min_{\beta} \Pi_\infty(\eta, \beta),
\]

where

\[
\Pi_\infty = \plim \Pi_T(\eta, \beta).
\]

The general form of the asymptotic variance-covariance matrix \( \Gamma \) for this optimization problem
is given by:

\[
\Gamma = \left(1 + \frac{1}{H}\right) \left[ \frac{\partial b'}{\partial \eta} (\eta_0) \Lambda^{-1} \frac{\partial b}{\partial \eta} (\eta_0) \right]^{-1}
\]

\[
\frac{\partial b'}{\partial \eta} (\eta_0) \Lambda J_0^{-1} (I_0 - K_0) J_0^{-1} \Lambda \frac{\partial b}{\partial \eta} (\eta_0)
\]

\[
\left[ \frac{\partial b'}{\partial \eta} (\eta_0) \Lambda^{-1} \frac{\partial b}{\partial \eta} (\eta_0) \right]^{-1},
\]

where

\[
\Lambda = J_0 (I_0 - K_0)^{-1} J_0
\]

(3.1)

For the particular form of an indirect inference problem, \( J_0 \) is an identity matrix. The partial derivatives \( \frac{\partial b}{\partial \eta} (\eta_0) \) can be consistently estimated by numerical derivatives evaluated at \( \hat{\eta} \). Also, the matrix \( (I_0 - K_0) \) represents the long-run variance of the moments used for the estimation. We estimate it using the Newey and West (1987) variance-covariance matrix estimator with the Bartlett kernel and bandwidth equal to \( T^{1/2} \) as suggested in Hayashi (2000).

### 3.2 Kalman Filter

Before we can simulate annual data for the growth rates and then construct annual data on asset returns and price-dividend ratios, we need to estimate the parameters appearing in the processes for nondurable consumption, durable consumption, and dividends. We do this with the Kalman filter.

After obtaining these parameter estimates through the kalman filter, we can easily simulate data from these models as we did earlier in our calibration parts. Then, we can calculate moments (auxiliary model parameters) from these simulated data that correspond to their counterparts in the empirical data. Finally, we evaluate the Indirect Inference objective
function. Details of the state-space formulation used for the Kalman filter for each model are given below.

We start by writing down the state-space representations of all our four models. In general, we can define the state-space representation for the system of equations as the following:

$$\xi_{t+1} = F\xi_t + V_{t+1}, \quad (3.2)$$
$$Y_t = H\xi_t + W_{t+1}, \quad (3.3)$$

where $V_{t+1} \sim N(0, Q)$ and $W_{t+1} \sim N(0, R)$.

Equation (3.2) is referred to as the state equation and equation (3.3) represents the observation equation. Accordingly, the state-space specifications of the first model are listed in the following box.

$$\begin{align*}
\xi_t &= \begin{bmatrix} x_{d,t}^t \\ \mu_y - \bar{\mu}_y \end{bmatrix} \\
F &= \begin{bmatrix} \phi_x & 0 \\ 0 & \phi_z \end{bmatrix} \\
V_t &= \begin{bmatrix} \omega_{d,t}^2 \sigma_d^2 \xi_{d,t+1} \\ \sigma_d^2 \xi_{d,t+1} \end{bmatrix} \\
Q &= \begin{bmatrix} \omega_{d,t}^2 \sigma_d^2 & 0 \\ 0 & \sigma_z^2 \end{bmatrix} \\
Y_t &= \begin{bmatrix} \Delta d_{t+1} - \mu_d \\ \Delta c_{t+1} - \mu_c \\ \Delta y_{t+1} - \mu_y \end{bmatrix} \\
H &= \begin{bmatrix} 1 & 0 \\ 0 & (\phi_z - 1) \\ \lambda_x & \lambda_y \end{bmatrix} \\
W_t &= \begin{bmatrix} \omega_{d,t}^2 \sigma_d^2 \xi_{d,t+1} \\ \omega_{s,t}^2 \sigma_s^2 \xi_{s,t+1} + \sigma_z^2 \xi_{s,t+1} \\ \sigma_y^2 \xi_{y,t+1} \end{bmatrix} \\
R &= \begin{bmatrix} \omega_{d,t}^2 \sigma_d^2 & 0 & 0 \\ 0 & \omega_{s,t}^2 \sigma_s^2 + \sigma_z^2 & 0 \\ 0 & 0 & \sigma_y^2 \end{bmatrix}
\end{align*}$$

Similarly, the state-space specifications of the second model is displayed in the following page.
Before writing the space-state representation of the third model, we need to use the Taylor expansion for the observed equation of the growth of nondurable consumption. That is, the Taylor expansion will enable us to estimate parameters of the growth of nondurable consumption by the Kalman filter. The original model of the growth of nondurable consumption is:

\[ \Delta c_{t+1} = \mu_{c} + \sigma_{c,t} \epsilon_{c,t+1}, \]  

(3.4)

with

\[ \sigma_{c,t+1}^{2} = \sigma_{c}^{2} + \phi_{lc}(\sigma_{c,t}^{2} - \sigma_{c}^{2}) + \sigma_{oc} \epsilon_{c,c,t+1}. \]  

(3.5)

The second equation 3.5, which represents the evolution of the state variable \( \sigma_{c,t}^{2} \), will remain the same, whereas the first equation 3.4 will be altered. By subtracting \( \mu_{c} \) from both sides, taking logs of both sides, and then squaring both sides we get the following equation:

\[ \ln((\Delta c_{t+1} - \mu_{c})^{2} = \ln \sigma_{c}^{2} + \ln \epsilon_{c,c,t+1}. \]  

(3.6)
Now, we can use a Taylor expansion as follows:

\[
X_{t+1} = f(\tilde{X}) + f'(\tilde{X})(X_t - \tilde{X}),
\]

\[
\ln \sigma_{\tilde{e},t}^2 \approx \ln \sigma_{\tilde{e}}^2 + \frac{1}{\sigma_{\tilde{e}}^2} (\sigma_{\tilde{e},t}^2 - \sigma_{\tilde{e}}^2).
\]  

(3.7)

Substituting equation (3.7) into equation (3.6) gives the following:

\[
\ln(\Delta c_{t+1} - \mu_e)^2 = \ln \sigma_{\tilde{e}}^2 + \frac{1}{\sigma_{\tilde{e}}^2} (\sigma_{\tilde{e},t}^2 - \sigma_{\tilde{e}}^2) + \ln \epsilon_{\tilde{e},t+1}^2,
\]

where

\[
\ln \epsilon_{\tilde{e},t+1}^2 \sim N(-1.27, 4.93).
\]

Therefore, the state-space specification of the third model can be written as the following:

\[
\begin{align*}
\xi_t &= \begin{bmatrix} x_{d,t} \\ \sigma_{\tilde{e},t}^2 - \sigma_{\tilde{e}}^2 \end{bmatrix} \\
F &= \begin{bmatrix} \phi_{d} & 0 \\ 0 & \phi_{\sigma e} \end{bmatrix} \\
V_t &= \begin{bmatrix} \omega_{d,t} \sigma_{\tilde{e},t} \epsilon_{d,t} + 1 \\ \sigma_{\sigma e} \sigma_{\tilde{e},t} \epsilon_{\tilde{e},t+1} + 1 \end{bmatrix} \\
Q &= \begin{bmatrix} \omega_{d,t}^2 \sigma_{\tilde{e}}^2 & 0 \\ 0 & \sigma_{\sigma e}^2 \end{bmatrix} \\
Y_t &= \begin{bmatrix} \Delta d_{t+1} - \mu_d \\ \ln(\Delta c_{t+1} - \mu_e)^2 - \ln \sigma_{\tilde{e}}^2 \\ \Delta y_{t+1} - \mu_y \end{bmatrix} \\
H &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sigma_{\tilde{e}}^2 & 0 \\ \lambda_d & 0 \end{bmatrix} \\
W_t &= \begin{bmatrix} \omega_{d,t}^2 \sigma_{\tilde{e}}^2 \epsilon_{d,t+1} + 1 \\ \ln \epsilon_{\tilde{e},t+1}^2 \\ \rho_d \omega_{d,t}^2 \sigma_{\tilde{e}}^2 \epsilon_{d,t+1} + \rho_d \sigma_{\tilde{e},t} \epsilon_{\tilde{e},t+1} + \sigma_{y} \epsilon_{y,t+1}, \epsilon_{y,t+1} \end{bmatrix} \\
R &= \begin{bmatrix} \rho_{d}^2 \sigma_{\tilde{e},t}^2 & 0 & 0 \\ 0 & 4.93 & 0 \\ \rho_{d}^2 \sigma_{\tilde{e},t}^2 & \rho_{d}^2 \sigma_{\tilde{e},t}^2 & \sigma_y^2 \end{bmatrix}
\end{align*}
\]
The state-space specification for the adjusted model takes the following form:

\[
\begin{align*}
\xi_t &= \begin{bmatrix} x_{d,t} \\ x_{c,t} \\ (s_t - \bar{s}) \end{bmatrix} \\
F &= \begin{bmatrix} \phi_d & 0 & 0 \\ 0 & \phi_c & 0 \\ 0 & 0 & \phi_s \end{bmatrix} \\
V_t &= \begin{bmatrix} \omega_{d,t} \sigma_d \epsilon_{d,t} + \omega_{s,t} \sigma_s \epsilon_{s,t} + 1 \\ \omega_{c,t} \sigma_c \epsilon_{c,t} + 1 \\ \omega_{s,t} \sigma_s \epsilon_{s,t} + 1 \end{bmatrix} \\
Q &= \begin{bmatrix} \omega^2_{d,t} \sigma^2_d & 0 & 0 \\ 0 & \omega^2_{c,t} \sigma^2_c & 0 \\ 0 & 0 & \omega^2_{s,t} \sigma^2_s \end{bmatrix} \\
Y_t &= \begin{bmatrix} \Delta d_{t+1} - \mu_d \\ \Delta c_{t+1} - \mu_c \\ \Delta y_{t+1} - \mu_y \end{bmatrix} \\
H &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & (\phi_s - 1) \\ \lambda_d & \lambda_c & \lambda_s \end{bmatrix} \\
W_t &= \begin{bmatrix} \omega_{d,t} \sigma_d \epsilon_{d,t} + 1 \\ \omega_{c,t} \sigma_c \epsilon_{c,t} + 1 + \omega_{s,t} \sigma_s \epsilon_{s,t} + 1 \\ \rho_d \omega_{s,t} \sigma_s \epsilon_{s,t} + 1 + \rho_d \omega_{d,t} \sigma_d \epsilon_{d,t} + 1 \end{bmatrix} \\
R &= \begin{bmatrix} \omega^2_{d,t} \sigma^2_d & 0 & 0 \\ 0 & \omega^2_{c,t} \sigma^2_c + \omega^2_{s,t} \sigma^2_s & 0 \\ \rho_d^2 \omega^2_{s,t} \sigma^2_d & \rho_d \omega^2_{s,t} \sigma^2_s & \rho_s \omega^2_{s,t} \sigma^2_c \end{bmatrix}
\end{align*}
\]
Table 3.1: Estimated vs. calibrated parameters of the first model

<table>
<thead>
<tr>
<th>Durable growth</th>
<th>( \mu_d )</th>
<th>( \phi_d )</th>
<th>( \sigma_d )</th>
<th>( \sigma_x )</th>
<th>( \zeta_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>calibrated</td>
<td>0.0031</td>
<td>0.99</td>
<td>0.0022</td>
<td>2.58e-04</td>
<td>220</td>
</tr>
<tr>
<td>estimated</td>
<td>0.0031</td>
<td>0.99</td>
<td>0.0022</td>
<td>2.58e-04</td>
<td>219.99</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Preferences</th>
<th>( \gamma )</th>
<th>( \psi )</th>
<th>( B )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>calibrated</td>
<td>1.09</td>
<td>1.2</td>
<td>0.9966</td>
<td>0.5</td>
</tr>
<tr>
<td>estimated</td>
<td>1.0925</td>
<td>1.791</td>
<td>0.9960</td>
<td>0.2315</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Nondurable growth</th>
<th>( \mu_e )</th>
<th>( \sigma_e )</th>
<th>( \zeta_e )</th>
<th>( \phi_x )</th>
<th>( \sigma_x )</th>
<th>( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>calibrated</td>
<td>0.0016</td>
<td>0.0025</td>
<td>0.53</td>
<td>0.98</td>
<td>8.66e-06</td>
<td>0.02</td>
</tr>
<tr>
<td>estimated</td>
<td>0.0016</td>
<td>3.41e-04</td>
<td>0.0100</td>
<td>0.98</td>
<td>0.0029</td>
<td>0.02</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dividend growth</th>
<th>( \mu_y )</th>
<th>( \lambda_{\pi} )</th>
<th>( \lambda_{x} )</th>
<th>( \sigma_y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>calibrated</td>
<td>0.002</td>
<td>1.05</td>
<td>0.944</td>
<td>1.03e-04</td>
</tr>
<tr>
<td>estimated</td>
<td>9.14e-04</td>
<td>1.8</td>
<td>0.5</td>
<td>0.0028</td>
</tr>
</tbody>
</table>

3.3 Estimation results of the first model

Our data are the same data that we used in the first and second chapters. Thus, the results of our models with calibrated parameters are comparable to the results of our models with estimated parameters. For further details please refer to Section 1.3 in Chapter 1. As in Chapter 1, we simulate at a monthly frequency and then time aggregate to an annual frequency to calculate our moments of the growth rates and annual asset returns.

We simulate 100 samples with a total of 59 \times 12 monthly observations that match the length of the empirical data. Table 3.1 reports both calibrated and estimated parameters. They are very close in values with a few exceptions. The parameters for the growth rate of dividends are not close match. Also, the estimated EIS \( \psi \) is bigger than the calibrated EIS and the estimated value of the durable goods share, \( \alpha \), is lower than its calibrated value of 0.5.

Table 3.2 displays the moments of the growth rates from the empirical data and those implied
by the first model. For the growth rate of durable consumption, the model implied moments follow closely those of the empirical data.

Table 3.2: Growth rate of the first model using estimated parameters

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SE</td>
</tr>
<tr>
<td>Durable consumption growth</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_t[\Delta d]$</td>
<td>0.0377 (0.004)</td>
<td>0.0385 0.0230 0.0508</td>
</tr>
<tr>
<td>$\sigma[\Delta d]$</td>
<td>0.0171 (0.001)</td>
<td>0.0182 0.0129 0.0256</td>
</tr>
<tr>
<td>$AC1(\Delta d)$</td>
<td>0.6910 (0.135)</td>
<td>0.6771 0.4895 0.8534</td>
</tr>
<tr>
<td>$AC2(\Delta d)$</td>
<td>0.4330 (0.140)</td>
<td>0.5330 0.2660 0.7788</td>
</tr>
<tr>
<td>$AC3(\Delta d)$</td>
<td>0.1150 (0.161)</td>
<td>0.4198 0.1071 0.7161</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.407 (0.188)</td>
<td>-0.3566 -0.9967 0.2165</td>
</tr>
<tr>
<td>Nondurable consumption growth</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_t[\Delta c]$</td>
<td>0.0196 (0.002)</td>
<td>0.0199 0.0198 0.0200</td>
</tr>
<tr>
<td>$\sigma[\Delta c]$</td>
<td>0.0099 (0.001)</td>
<td>0.0012 0.0011 0.0013</td>
</tr>
<tr>
<td>$AC1(\Delta c)$</td>
<td>0.4531 (0.215)</td>
<td>0.1806 -0.0230 0.3505</td>
</tr>
<tr>
<td>$AC2(\Delta c)$</td>
<td>0.1097 (0.154)</td>
<td>0.1674 0.0159 0.3180</td>
</tr>
<tr>
<td>$AC3(\Delta c)$</td>
<td>0.0047 (0.121)</td>
<td>0.1264 -0.0651 0.3150</td>
</tr>
<tr>
<td>Dividends growth</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_t[\Delta y]$</td>
<td>0.0109 (0.009)</td>
<td>0.0045 -0.0219 0.0099</td>
</tr>
<tr>
<td>$\sigma[\Delta y]$</td>
<td>0.0390 (0.005)</td>
<td>0.0470 0.0386 0.0570</td>
</tr>
<tr>
<td>$AC1(\Delta y)$</td>
<td>0.5390 (0.169)</td>
<td>0.7496 0.6566 0.8259</td>
</tr>
</tbody>
</table>

On the other hand, the models of the growth rate of both nondurable consumption and dividends mostly fail to generate moment estimates close to their empirical counterparts. Table 3.3 shows clearly that the first model fails to match the return on the risk-free asset. Also, the estimated excess return is the greatest mismatch for the first model whether we calibrated or estimated the parameters. On the bright side, the rest of the moments implied by the first model are very close to what we observe with the data.
Table 3.3: Asset pricing implications of the first model using estimated parameters

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th></th>
<th></th>
<th>Model</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SE</td>
<td>Mean</td>
<td>5th</td>
<td>95th</td>
<td>5th</td>
<td>95th</td>
</tr>
<tr>
<td>$E_t[\text{rf}]$</td>
<td>0.0161</td>
<td>(0.005)</td>
<td>0.0284</td>
<td>0.0265</td>
<td>0.0305</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma[\text{rf}]$</td>
<td>0.0165</td>
<td>(0.017)</td>
<td>0.0174</td>
<td>0.0155</td>
<td>0.0194</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_t[r - rf]$</td>
<td>0.0670</td>
<td>(0.020)</td>
<td>-0.0097</td>
<td>-0.0126</td>
<td>-0.0075</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma[r]$</td>
<td>0.1448</td>
<td>(0.013)</td>
<td>0.1315</td>
<td>0.1260</td>
<td>0.1621</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_t[p - d]$</td>
<td>3.496</td>
<td>(0.102)</td>
<td>3.4968</td>
<td>3.2799</td>
<td>3.6742</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma[p - d]$</td>
<td>0.308</td>
<td>(0.062)</td>
<td>0.2605</td>
<td>0.1917</td>
<td>0.3486</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$AC1[p - d]$</td>
<td>0.910</td>
<td>(0.306)</td>
<td>0.8786</td>
<td>0.8462</td>
<td>0.9050</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$AC2[p - d]$</td>
<td>0.840</td>
<td>(0.290)</td>
<td>0.7784</td>
<td>0.7155</td>
<td>0.8215</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$AC3[p - d]$</td>
<td>0.783</td>
<td>(0.269)</td>
<td>0.6935</td>
<td>0.6159</td>
<td>0.7568</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.4: Estimated vs. calibrated parameters of the second model

<table>
<thead>
<tr>
<th></th>
<th>Durable growth</th>
<th>Nondurable growth</th>
<th>Dividend growth</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mu_d$</td>
<td>$\phi_d$</td>
<td>$\sigma_d$</td>
<td>$\sigma_{x_d}$</td>
</tr>
<tr>
<td>calibrated</td>
<td>0.0031</td>
<td>0.990</td>
<td>0.0025</td>
<td>2.40e-04</td>
</tr>
<tr>
<td>estimated</td>
<td>0.0031</td>
<td>0.989</td>
<td>0.0025</td>
<td>2.39e-04</td>
</tr>
<tr>
<td>Preferences</td>
<td>$\gamma$</td>
<td>$\psi$</td>
<td>$B$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>calibrated</td>
<td>10.8</td>
<td>1.5</td>
<td>0.9995</td>
<td>0.5</td>
</tr>
<tr>
<td>estimated</td>
<td>10.804</td>
<td>1.015</td>
<td>0.9982</td>
<td>0.2958</td>
</tr>
<tr>
<td></td>
<td>$\mu_e$</td>
<td>$\phi_e$</td>
<td>$\sigma_e$</td>
<td>$\sigma_{x_e}$</td>
</tr>
<tr>
<td>calibrated</td>
<td>0.0016</td>
<td>0.93</td>
<td>0.0024</td>
<td>2.95e-04</td>
</tr>
<tr>
<td>estimated</td>
<td>0.0016</td>
<td>0.93</td>
<td>0.0024</td>
<td>3.51e-04</td>
</tr>
<tr>
<td></td>
<td>$\lambda_d$</td>
<td>$\lambda_c$</td>
<td>$\phi_{dd}$</td>
<td>$\phi_{cc}$</td>
</tr>
<tr>
<td></td>
<td>1.7</td>
<td>1.7</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>$\mu_y$</td>
<td>$\lambda_c$</td>
<td>$\phi_{dd}$</td>
<td>$\phi_{cc}$</td>
</tr>
<tr>
<td>calibrated</td>
<td>9.100e-04</td>
<td>1.7</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>estimated</td>
<td>9.145e-04</td>
<td>1.7011</td>
<td>0.5978</td>
<td>0.5998</td>
</tr>
</tbody>
</table>

3.4 Estimation results of the second model

Table 3.4 shows that the calibrated and estimated parameters are a very close match for most parameters, but with two exceptions. For example, the estimated share of durable goods $\alpha$ is lower than the calibrated $\alpha$. Also, the EIS is estimated way lower than the calibrated EIS.
Table 3.5: Growth rates of the second model using estimated parameters

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean SE</td>
<td>Mean 5th</td>
</tr>
<tr>
<td><strong>Moments</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Durable consumption growth</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[\Delta d]$</td>
<td>0.0377 (0.004)</td>
<td>0.0374</td>
</tr>
<tr>
<td>$\sigma[\Delta d]$</td>
<td>0.0171 (0.001)</td>
<td>0.0178</td>
</tr>
<tr>
<td>$AC1(\Delta d)$</td>
<td>0.6910 (0.135)</td>
<td>0.6171</td>
</tr>
<tr>
<td>$AC2(\Delta d)$</td>
<td>0.4330 (0.140)</td>
<td>0.4806</td>
</tr>
<tr>
<td>$AC3(\Delta d)$</td>
<td>0.1150 (0.161)</td>
<td>0.3797</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.407 (0.188)</td>
<td>-0.378</td>
</tr>
<tr>
<td><strong>Nondurable consumption growth</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[\Delta c]$</td>
<td>0.0196 (0.002)</td>
<td>0.0194</td>
</tr>
<tr>
<td>$\sigma[\Delta c]$</td>
<td>0.0099 (0.001)</td>
<td>0.0128</td>
</tr>
<tr>
<td>$AC1(\Delta c)$</td>
<td>0.4531 (0.215)</td>
<td>0.3125</td>
</tr>
<tr>
<td>$AC2(\Delta c)$</td>
<td>0.1097 (0.154)</td>
<td>0.1000</td>
</tr>
<tr>
<td>$AC3(\Delta c)$</td>
<td>0.0047 (0.121)</td>
<td>0.0269</td>
</tr>
<tr>
<td><strong>Dividends growth</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[\Delta y]$</td>
<td>0.0109 (0.009)</td>
<td>0.0101</td>
</tr>
<tr>
<td>$\sigma[\Delta y]$</td>
<td>0.0392 (0.005)</td>
<td>0.0399</td>
</tr>
<tr>
<td>$AC1(\Delta y)$</td>
<td>0.5390 (0.169)</td>
<td>0.1683</td>
</tr>
</tbody>
</table>

Table 3.5 reports the moments of the growth rates from the empirical data and the same moments implied by the second model. The second model can, in most cases, produce moments that match closely their empirical counterparts for all the growth rates. The only exceptions are the volatility of the growth rate of nondurables and the first-order autocorrelation of the growth rate of dividends. However, the simulated intervals for both of these moments are not far from their empirical moments.

Table 3.6 reports a higher mean for the return on the risk-free asset with a lower volatility of about one percent than their empirical counterparts. On the other hand, the model shows some improvement in the reported mean of excess returns compared to the first model. Most importantly, the empirical excess return is within the simulated percentile interval. All other moments are a close match to the empirical counterparts.
Table 3.6: Asset pricing implications of the second model using estimated parameters

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SE</td>
</tr>
<tr>
<td>$E[r_f]$</td>
<td>0.0161 (0.005)</td>
<td>0.0405</td>
</tr>
<tr>
<td>$\sigma[r_f]$</td>
<td>0.0165 (0.017)</td>
<td>0.0097</td>
</tr>
<tr>
<td>$E[r - r_f]$</td>
<td>0.0670 (0.020)</td>
<td>0.0634</td>
</tr>
<tr>
<td>$\sigma[r]$</td>
<td>0.1448 (0.013)</td>
<td>0.1498</td>
</tr>
<tr>
<td>$E[p - d]$</td>
<td>3.496 (0.102)</td>
<td>3.4988</td>
</tr>
<tr>
<td>$\sigma[p - d]$</td>
<td>0.308 (0.062)</td>
<td>0.1922</td>
</tr>
<tr>
<td>$AC1[p - d]$</td>
<td>0.910 (0.306)</td>
<td>0.8412</td>
</tr>
<tr>
<td>$AC2[p - d]$</td>
<td>0.840 (0.290)</td>
<td>0.6451</td>
</tr>
<tr>
<td>$AC3[p - d]$</td>
<td>0.783 (0.269)</td>
<td>0.4884</td>
</tr>
</tbody>
</table>

Table 3.7: Estimated vs. calibrated parameters of the third model

<table>
<thead>
<tr>
<th>Durable growth</th>
<th>$\mu_d$</th>
<th>$\phi_d$</th>
<th>$\sigma_d$</th>
<th>$\sigma_{se}$</th>
<th>$\zeta_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>calibrated</td>
<td>0.0031</td>
<td>0.99</td>
<td>0.0022</td>
<td>2.58e-04</td>
<td>220</td>
</tr>
<tr>
<td>estimated</td>
<td>0.0031</td>
<td>0.99</td>
<td>0.0022</td>
<td>2.58e-04</td>
<td>220</td>
</tr>
<tr>
<td>Preferences</td>
<td>$\gamma$</td>
<td>$\psi$</td>
<td>$\beta$</td>
<td>$\alpha$</td>
<td></td>
</tr>
<tr>
<td>calibrated</td>
<td>17</td>
<td>1.5</td>
<td>0.9912</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>estimated</td>
<td>16.61</td>
<td>0.735</td>
<td>0.9992</td>
<td>0.795</td>
<td></td>
</tr>
<tr>
<td>Nondurable growth</td>
<td>$\mu_c$</td>
<td>$\sigma_c$</td>
<td>$\phi_{ac}$</td>
<td>$\sigma_{ac}$</td>
<td></td>
</tr>
<tr>
<td>calibrated</td>
<td>0.0016</td>
<td>0.0032</td>
<td>0.99</td>
<td>9e-09</td>
<td></td>
</tr>
<tr>
<td>estimated</td>
<td>0.0016</td>
<td>0.0029</td>
<td>0.99</td>
<td>1.29e-04</td>
<td></td>
</tr>
<tr>
<td>Dividend growth</td>
<td>$\mu_y$</td>
<td>$\lambda_d$</td>
<td>$\rho_c$</td>
<td>$\rho_d$</td>
<td>$\sigma_y$</td>
</tr>
<tr>
<td>calibrated</td>
<td>9.0833e-04</td>
<td>1.0452</td>
<td>1.2</td>
<td>0.78</td>
<td>0.0092</td>
</tr>
<tr>
<td>estimated</td>
<td>9.1453e-04</td>
<td>1.701</td>
<td>1.2</td>
<td>0.7797</td>
<td>0.0092</td>
</tr>
</tbody>
</table>

3.5 Estimated results of the third model

We can see the calibrated and estimated parameters in Table 3.7. Most of the estimated parameters of the third model are close to their corresponding calibrated parameters. However, the estimated EIS is less than one and far from the calibrated EIS of 1.5. In addition,
the share of durable consumption $\alpha$ is about 0.8, which is very far from the calibrated $\alpha$ of 0.5.

The estimated leverage effect parameter $\lambda_d$ on the expected component of durable consumption $x_{d,t}$ is larger than the calibrated $\lambda_d$.

Table 3.8 reports the estimated moments of the growth rates for the third model. Only the estimated moment of the growth rate of durable consumption is a close match to the empirical counterparts. The estimated moments for the growth rates of both nondurable consumption and dividends are poorly matched and fall outside the 5$^{th}$-$95^{th}$ percentile intervals.

Table 3.8: Growth rates of the third model using estimated parameters

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data Mean</th>
<th>Data SE</th>
<th>Model Mean</th>
<th>5$^{th}$</th>
<th>95$^{th}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[\Delta d]$</td>
<td>0.0377</td>
<td>(0.004)</td>
<td>0.0378</td>
<td>0.0197</td>
<td>0.0543</td>
</tr>
<tr>
<td>$\sigma[\Delta d]$</td>
<td>0.0171</td>
<td>(0.001)</td>
<td>0.0190</td>
<td>0.0117</td>
<td>0.0314</td>
</tr>
<tr>
<td>$AC1(\Delta d)$</td>
<td>0.6910</td>
<td>(0.135)</td>
<td>0.7051</td>
<td>0.4724</td>
<td>0.8760</td>
</tr>
<tr>
<td>$AC2(\Delta d)$</td>
<td>0.4330</td>
<td>(0.140)</td>
<td>0.5705</td>
<td>0.2639</td>
<td>0.8099</td>
</tr>
<tr>
<td>$AC3(\Delta d)$</td>
<td>0.1150</td>
<td>(0.161)</td>
<td>0.4546</td>
<td>0.1226</td>
<td>0.7624</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.407</td>
<td>(0.188)</td>
<td>-0.4098</td>
<td>-1.0254</td>
<td>0.1937</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data Mean</th>
<th>Data SE</th>
<th>Model Mean</th>
<th>5$^{th}$</th>
<th>95$^{th}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[\Delta c]$</td>
<td>0.0196</td>
<td>(0.002)</td>
<td>0.0202</td>
<td>-0.0053</td>
<td>0.0415</td>
</tr>
<tr>
<td>$\sigma[\Delta c]$</td>
<td>0.0099</td>
<td>(0.001)</td>
<td>0.0908</td>
<td>0.0694</td>
<td>0.1157</td>
</tr>
<tr>
<td>$AC1(\Delta c)$</td>
<td>0.4531</td>
<td>(0.215)</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$AC2(\Delta c)$</td>
<td>0.1097</td>
<td>(0.154)</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$AC3(\Delta c)$</td>
<td>0.0047</td>
<td>(0.121)</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data Mean</th>
<th>Data SE</th>
<th>Model Mean</th>
<th>5$^{th}$</th>
<th>95$^{th}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[\Delta y]$</td>
<td>0.0109</td>
<td>(0.009)</td>
<td>0.0092</td>
<td>-0.0237</td>
<td>0.0446</td>
</tr>
<tr>
<td>$\sigma[\Delta y]$</td>
<td>0.0392</td>
<td>(0.005)</td>
<td>0.1149</td>
<td>0.0871</td>
<td>0.1465</td>
</tr>
<tr>
<td>$AC1(\Delta y)$</td>
<td>0.539</td>
<td>(0.169)</td>
<td>-0.0117</td>
<td>-0.2597</td>
<td>0.2222</td>
</tr>
</tbody>
</table>

Only the estimated moment of the growth rate of durable consumption is a close match to the empirical counterparts. The estimated moments for the growth rates of both nondurable consumption and dividends are poorly matched and fall outside the 5$^{th}$-$95^{th}$ percentile intervals.

The results reported in Table 3.9 shows a very high model implied mean for the return on riskless assets, but its volatility matches what we see in the empirical data. On the other
hand, the third model also generates a very high mean of excess return compared to the data’s mean excess return. Even more, the mean excess return of the empirical data is outside the simulated percentiles.

Table 3.9: Asset pricing implications of the third model using estimated parameters

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th></th>
<th></th>
<th>Model</th>
<th>5th</th>
<th>95th</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SE</td>
<td>Mean</td>
<td>5th</td>
<td>95th</td>
<td></td>
</tr>
<tr>
<td>$E[r_f]$</td>
<td>0.0161 (0.005)</td>
<td>0.0647</td>
<td>0.0348</td>
<td>0.0852</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma[r_f]$</td>
<td>0.0165 (0.017)</td>
<td>0.0172</td>
<td>0.0109</td>
<td>0.0247</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[r - r_f]$</td>
<td>0.0670 (0.020)</td>
<td>0.1361</td>
<td>0.1049</td>
<td>0.1775</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma[r]$</td>
<td>0.1448 (0.013)</td>
<td>0.1707</td>
<td>0.1523</td>
<td>0.1927</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[p - d]$</td>
<td>3.496 (0.102)</td>
<td>3.4969</td>
<td>3.3209</td>
<td>3.6506</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma[p - d]$</td>
<td>0.308 (0.062)</td>
<td>0.1769</td>
<td>0.1036</td>
<td>0.2965</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$AC1[p - d]$</td>
<td>0.910 (0.306)</td>
<td>0.8468</td>
<td>0.7194</td>
<td>0.9480</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$AC2[p - d]$</td>
<td>0.840 (0.290)</td>
<td>0.6630</td>
<td>0.4011</td>
<td>0.8655</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$AC3[p - d]$</td>
<td>0.783 (0.269)</td>
<td>0.5134</td>
<td>0.1445</td>
<td>0.8069</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 3.10: Estimated vs. calibrated parameters of the adjusted first model

<table>
<thead>
<tr>
<th></th>
<th>( \mu_d )</th>
<th>( \phi_d )</th>
<th>( \sigma_d )</th>
<th>( \sigma_{se} )</th>
<th>( \zeta_{se} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Durable growth</td>
<td>calibrated</td>
<td>0.0031</td>
<td>0.99</td>
<td>0.0022</td>
<td>2.58e-04</td>
</tr>
<tr>
<td></td>
<td>estimated</td>
<td>0.0031</td>
<td>0.99</td>
<td>0.0022</td>
<td>2.97e-04</td>
</tr>
<tr>
<td>Preferences</td>
<td>calibrated</td>
<td>10.9</td>
<td>1.1</td>
<td>0.99663</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>estimated</td>
<td>10.9004</td>
<td>1.0994</td>
<td>0.9967</td>
<td>0.348</td>
</tr>
<tr>
<td>Nondurable growth</td>
<td>calibrated</td>
<td>0.0016</td>
<td>0.0289</td>
<td>0.01</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>estimated</td>
<td>0.0016</td>
<td>0.0289</td>
<td>0.01</td>
<td>0.98</td>
</tr>
<tr>
<td>Nondurable growth con't</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>calibrated</td>
<td>0.95</td>
<td>2.58e-04</td>
<td>200</td>
<td></td>
</tr>
<tr>
<td></td>
<td>estimated</td>
<td>0.949</td>
<td>2.57e-04</td>
<td>200</td>
<td></td>
</tr>
<tr>
<td>Dividend growth</td>
<td>calibrated</td>
<td>9.08e-04</td>
<td>1.055</td>
<td>0.996</td>
<td>0.935</td>
</tr>
<tr>
<td></td>
<td>estimated</td>
<td>9.14e-04</td>
<td>1.900</td>
<td>0.955</td>
<td>0.450</td>
</tr>
</tbody>
</table>

3.6 Estimated results of the adjusted first model

Table 3.10 shows that the calibrated and estimated parameters are an almost perfect match with only three exceptions: The share of durable goods \( \alpha \), the estimated leverage effect parameter \( \lambda_d \) on the expected component of durable consumption \( x_{d,t} \), is larger than the calibrated \( \lambda_d \), and the estimated leverage effect parameter \( \lambda_s \) on the mean reversion part \((s_t - \bar{s})\) of nondurable consumption is lower than the calibrated \( \lambda_s \).

Table 3.11 reports the simulated moments of the growth rates for the adjusted first model. The model’s moments follow closely those of the empirical data. However, the empirical data first-order autocorrelation of the growth rate of nondurable consumption is not within the simulated percentiles, but it is still very close to the 95th percentile. On the other hand,
Table 3.11: Growth rates of the adjusted first model using estimated parameters

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th>Model</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean   SE</td>
<td>Mean 5th   95th</td>
<td>Mean 5th   95th</td>
<td></td>
</tr>
<tr>
<td>Durable consumption growth</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[Δd]$</td>
<td>0.0377 (0.004)</td>
<td>0.0367 0.0153</td>
<td>0.0575</td>
<td></td>
</tr>
<tr>
<td>$σ[Δd]$</td>
<td>0.0171 (0.001)</td>
<td>0.0219 0.0136</td>
<td>0.0331</td>
<td></td>
</tr>
<tr>
<td>$AC1(Δd)$</td>
<td>0.6910 (0.135)</td>
<td>0.7389 0.5104</td>
<td>0.8821</td>
<td></td>
</tr>
<tr>
<td>$AC2(Δd)$</td>
<td>0.4330 (0.140)</td>
<td>0.6057 0.3679</td>
<td>0.8302</td>
<td></td>
</tr>
<tr>
<td>$AC3(Δd)$</td>
<td>0.1150 (0.161)</td>
<td>0.4824 0.2224</td>
<td>0.7744</td>
<td></td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.407 (0.188)</td>
<td>-0.534 -1.273</td>
<td>0.1517</td>
<td></td>
</tr>
<tr>
<td>Nondurable consumption growth</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[Δc]$</td>
<td>0.0196 (0.002)</td>
<td>0.0198 0.0153</td>
<td>0.0237</td>
<td></td>
</tr>
<tr>
<td>$σ[Δc]$</td>
<td>0.0099 (0.001)</td>
<td>0.0131 0.0112</td>
<td>0.0153</td>
<td></td>
</tr>
<tr>
<td>$AC1(Δc)$</td>
<td>0.4531 (0.215)</td>
<td>0.2570 0.0546</td>
<td>0.4502</td>
<td></td>
</tr>
<tr>
<td>$AC2(Δc)$</td>
<td>0.1097 (0.154)</td>
<td>0.1033 -0.0808</td>
<td>0.3066</td>
<td></td>
</tr>
<tr>
<td>$AC3(Δc)$</td>
<td>0.0047 (0.121)</td>
<td>0.0265 -0.1879</td>
<td>0.2427</td>
<td></td>
</tr>
<tr>
<td>Dividends growth</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[Δy]$</td>
<td>0.0109 (0.009)</td>
<td>-0.0061 -0.0262</td>
<td>0.0140</td>
<td></td>
</tr>
<tr>
<td>$σ[Δy]$</td>
<td>0.0392 (0.005)</td>
<td>0.0468 0.0390</td>
<td>0.0574</td>
<td></td>
</tr>
<tr>
<td>$AC1(Δy)$</td>
<td>0.5390 (0.169)</td>
<td>0.7839 0.7213</td>
<td>0.8542</td>
<td></td>
</tr>
</tbody>
</table>

The empirical data first-order autocorrelation of dividends is not only not captured by the simulated percentiles, but also far from the simulated percentiles.

The results reported in Table 3.12 shows that the adjusted model can produce moments that follow closely their corresponding moments of the empirical data. The standard deviation of the return on riskless assets is lower than what we get from the empirical data, but is still very close. Moreover, the simulated standard deviation for excess return is a little bit higher and not far from what get with the empirical data.
Table 3.12: Asset pricing implications of the adjusted first model using estimated parameters

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data Mean (SE)</th>
<th></th>
<th>Model Mean 5&lt;sup&gt;th&lt;/sup&gt;</th>
<th>Model 95&lt;sup&gt;th&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[r_f]$</td>
<td>0.0160 (0.005)</td>
<td>0.0155</td>
<td>0.0116</td>
<td>0.0185</td>
</tr>
<tr>
<td>$\sigma[r_f]$</td>
<td>0.0165 (0.017)</td>
<td>0.0118</td>
<td>0.0108</td>
<td>0.0132</td>
</tr>
<tr>
<td>$E[r - rf]$</td>
<td>0.0670 (0.020)</td>
<td>0.0668</td>
<td>0.0614</td>
<td>0.0725</td>
</tr>
<tr>
<td>$\sigma[r]$</td>
<td>0.1448 (0.013)</td>
<td>0.1843</td>
<td>0.1512</td>
<td>0.2145</td>
</tr>
<tr>
<td>$E[p - d]$</td>
<td>3.496 (0.102)</td>
<td>3.4961</td>
<td>3.1955</td>
<td>3.7785</td>
</tr>
<tr>
<td>$\sigma[p - d]$</td>
<td>0.308 (0.062)</td>
<td>0.3100</td>
<td>0.2008</td>
<td>0.4551</td>
</tr>
<tr>
<td>$AC1[p - d]$</td>
<td>0.910 (0.306)</td>
<td>0.8502</td>
<td>0.7514</td>
<td>0.9263</td>
</tr>
<tr>
<td>$AC2[p - d]$</td>
<td>0.840 (0.290)</td>
<td>0.6759</td>
<td>0.4970</td>
<td>0.8306</td>
</tr>
<tr>
<td>$AC3[p - d]$</td>
<td>0.783 (0.269)</td>
<td>0.5272</td>
<td>0.2819</td>
<td>0.7469</td>
</tr>
</tbody>
</table>

3.7 Conclusion

In this chapter, instead of using calibration as in the previous chapters, we estimate the parameters. The estimation technique employed is the indirect inference of Gourieroux, Monfort, and Renault (1993). The auxiliary model consists of the moments used to assess the goodness of fit in Chapters 1 and 2. With estimated parameters, all our models are able to replicate most of the moments for the growth rates of durable consumption, nondurable consumption, and dividends.

For the estimated parameters we found the following. The risk aversion parameter $\gamma$ is estimated to be above one in the first, the second, and the adjusted first models and this is consistent with the value calibrated in Bansal and Yaron (2004) and Yang (2011). On the other hand, the third model generates an estimated value of $\gamma$ less than one. More importantly, the adjusted first model is the only model that provides an estimated $\gamma$ almost the same as the calibrated value of $\gamma$.

For the share of durable consumption $\alpha$, none of our models generate an estimate that
matches the calibrated value of 0.5. The first, the second, and the adjusted models generate smaller estimates of $\alpha$, whereas, the estimate of the third model is a bigger value. In comparison, the adjusted first model generates the closest estimate of $\alpha$ to the calibrated value of 0.5.

Moreover, the first, the second, and the third models fail to match, using estimated parameters, the mean of the return on the risk-free asset. Also, the first and the third models fail to match the mean of the excess return on the risky asset. The second model could generate a better match of the excess mean return on the risky asset than the first and third models. On the other hand, the adjusted first model can match successfully the mean return on both the risk-free asset and the risky asset.

Overall, we find that the adjusted first model is the best model among all of our models in matching moments of the empirical data whether we use estimated or calibrated parameters.
REFERENCES


APPENDICES
Appendix A

Let’s assume that utility $U_t$ is derived recursively over the (GES) utility function $\Omega_t$, which imposes non-separability between the surplus of nondurable and durable consumptions, and over a certainty equivalent of tomorrow’s utility $R_{g,t}(U_{t+1})$.

$$U_t = f(\Omega_t, R_{g,t}(U_{t+1})),$$

$$\Omega_t = C_t \left[ 1 - \alpha + \alpha \left( \frac{D_t}{C_t} \right)^{1-\frac{1}{\eta}} \right]^{\frac{1}{1-\frac{1}{\eta}}}$$

$$= C_t \nu \left( \frac{D_t}{C_t} \right),$$

$$R_{g,t}(U_{t+1}) = \left( E_t U_{t+1}^{1-\gamma} \right)^{\frac{1}{1-\gamma}},$$

$\nu \left( \frac{D_t}{C_t} \right)$ is defined right after equation (1.11).

The Bellman equation can be written as:

$$U_t = \left[ (1 - B) \Omega_t^{1-\frac{1}{\psi}} + B \left( E_t U_{t+1}^{1-\gamma} \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}$$

or equivalently as

$$U_t = \left[ (1 - B) \Omega_t^{1-\frac{1}{\psi}} + B R_{g,t}(U_{t+1})^{1-\frac{1}{\psi}} \right]^{\frac{1}{1-\frac{1}{\psi}}} \tag{3.8}$$

By definition the intertemporal marginal rate of substitution IMRS would be:

$$M_{t+1} = \frac{u_{c_{t+1}}}{u_{c_t}} \tag{3.9}$$

To know exactly what the IMRS of this economy we are going to take the F.O.C. of our utility function in equation (3.8) with respect to current nondurable consumption $C_t$ and with respect to tomorrow’s nondurable consumption $C_{t+1}$.
First, for the partial derivative with respect to $C_t$:

$$U_{c_t} = \frac{\partial U_t}{\partial \Omega_t} \cdot \frac{\partial \Omega_t}{\partial C_t} = f_1 (\Omega_t, R_{g,t}(U_{t+1})),$$

with

$$f_1 (\Omega_t, R_{g,t}(U_{t+1})) = (1 - B) \left[ (1 - B)\Omega_t^{1-\frac{1}{\psi}} + BR_{g,t}(U_{t+1})^{1-\frac{1}{\psi}} \right]^{\frac{1}{1-\frac{1}{\psi}}} \cdot (\mathcal{L}_1 - \mathcal{L}_2),$$

where

$$\mathcal{L}_1 = C_t^{-\frac{1}{\psi}} \left[ 1 - \alpha + \alpha (\frac{D_t}{C_t})^{1-\frac{1}{\rho}} \right]^{\frac{1}{1-\frac{1}{\rho}}} \frac{1}{\psi},$$

and

$$\mathcal{L}_2 = \alpha C_t^{1-\frac{1}{\psi}} \left[ 1 - \alpha + \alpha (\frac{D_t}{C_t})^{1-\frac{1}{\rho}} \right]^{\frac{1}{1-\frac{1}{\rho}}} \left( \frac{D_t}{C_t} \right)^{1-\frac{1}{\rho}} C_t^{-1}. $$

After some manipulations we get:

$$f_1 (\Omega_t, R_{g,t}(U_{t+1})) = (1 - B) U_t^{\frac{1}{\psi}} C_t^{-\frac{1}{\psi}} \left[ 1 - \alpha + \alpha (\frac{D_t}{C_t})^{1-\frac{1}{\rho}} \right]^{\frac{1}{1-\frac{1}{\rho}}} \times$$

$$\left( \left[ 1 - \alpha + \alpha (\frac{D_t}{C_t})^{1-\frac{1}{\rho}} \right] - \alpha (\frac{D_t}{C_t})^{1-\frac{1}{\rho}} \right),$$

$$= \left[ (1 - B)(1 - \alpha)U_t^{\frac{1}{\psi}} (C_t)^{-\frac{1}{\psi}} \nu \left( \frac{D_t}{C_t} \right)^{\frac{1}{\rho} - \frac{1}{\psi}} \right],$$

where

$$\nu \left( \frac{D_t}{C_t} \right) = \left[ 1 - \alpha + \alpha (\frac{D_t}{C_t})^{1-\frac{1}{\rho}} \right]^{\frac{1}{1-\frac{1}{\rho}}}. $$
Second, for the partial derivative with respect to $C_{t+1}$:

$$\frac{\partial U_{c_{t+1}}}{\partial C_{t+1}} = \frac{\partial R_{g,t}(U_{t+1})}{\partial U_{t+1}} \frac{\partial U_{t+1}}{\partial \Omega_{t+1}} \frac{\partial \Omega_{t+1}}{\partial C_{t+1}},$$

$$= f_2(\Omega_t, R_{g,t}(U_{t+1})) \left( E_t U_{t+1}^{1-\gamma} \right)^{\frac{1-\gamma}{\gamma}} f_1(\Omega_{t+1}, R_{g,t+1}(U_{t+2})), \quad \text{where}$$

$$f_2(\Omega_t, R_{g,t}(U_{t+1})) = B R_{g,t}(U_{t+1})^{-\frac{1}{\psi}} f(\Omega_t, R_{g,t}(U_{t+1}))^{\frac{1}{\psi}},$$

and

$$\frac{\partial R_{g,t}(U_{t+1})}{\partial U_{t+1}} = E_t \left( U_{t+1}^{1-\gamma} \right)^{\frac{1-\gamma}{\gamma}} U_{t+1}^{-\gamma}.$$

Substitute for $U_{c_t}$ and $U_{c_{t+1}}$ in equation (3.9) gives:

$$M_{t+1} = \frac{f_2(\Omega_t, R_{g,t}(U_{t+1})) \left( E_t U_{t+1}^{1-\gamma} \right)^{\frac{1-\gamma}{\gamma}} f_1(\Omega_{t+1}, R_{g,t+1}(U_{t+2}))}{f_1(\Omega_t, R_{g,t}(U_{t+1}))},$$

$$= B R_{g,t}(U_{t+1})^{-\frac{1}{\psi}} f(\Omega_t, R_{g,t}(U_{t+1}))^{\frac{1}{\psi}} \cdot L_3,$$

where

$$L_3 = \frac{\left( E_t U_{t+1}^{1-\gamma} \right)^{\frac{1-\gamma}{\gamma}} U_{t+1}^{-\gamma} (1 - B)(1 - \alpha) C_{t+1}^{-\frac{1}{\beta}} \nu \left( \frac{D_{t+1}}{C_{t+1}} \right)^{\left( \frac{1}{\beta} - \frac{1}{\beta} \right)} f(\Omega_{t+1}, R_{g,t+1}(U_{t+2}))^{\frac{1}{\psi}}}{(1 - B)(1 - \alpha) C_t^{-\frac{1}{\beta}} \nu \left( \frac{D_t}{C_t} \right)^{\left( \frac{1}{\beta} - \frac{1}{\beta} \right)} f(\Omega_t, R_{g,t}(U_{t+1}))^{\frac{1}{\psi}}}. $$
After some rearrangements we get:

\[
M_{t+1} = B \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left[ \frac{\nu' \left( \frac{D_{t+1}}{C_{t+1}} \right)}{\nu' \left( \frac{D_t}{C_t} \right)} \right]^{\left( \frac{1}{\psi} - \frac{1}{\gamma} \right)} R_{g,t}(U_{t+1})^{-\frac{1}{\psi}} \left( E_{t} U_{t+1} \right)^{\frac{1}{1-\gamma}} \times \\
U_{t+1}^{-\gamma} F \left( \Omega_{t+1}, R_{g,t+1}(U_{t+2}) \right)^{\frac{1}{\psi}}
\]

\[
= B \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left( \frac{\nu' \left( \frac{D_{t+1}}{C_{t+1}} \right)}{\nu' \left( \frac{D_t}{C_t} \right)} \right)^{\left( \frac{1}{\psi} - \frac{1}{\gamma} \right)} R_{g,t}(U_{t+1})^{-\frac{1}{\psi}} R_{g,t}(U_{t+1})^{\gamma} U_{t+1}^{-\frac{1}{\psi}} U_{t+1}^{\frac{1}{\psi}}
\]

\[
= B \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left( \frac{\nu' \left( \frac{D_{t+1}}{C_{t+1}} \right)}{\nu' \left( \frac{D_t}{C_t} \right)} \right)^{\left( \frac{1}{\psi} - \frac{1}{\gamma} \right)} \left( \frac{U_{t+1}}{R_{g,t}(U_{t+1})} \right)^{\frac{1}{\psi} - \gamma}.
\]

(3.10)

Guess and verify

let us guess that wealth \( W_t = G_t + E_t(M_{t+1}W_{t+1}) \) and the result is \( W_t = \frac{\tilde{V}_t}{\rho U_t R_{g,t}(U_{t+1})} \).

Once again \( G_t \) refers to total expenditures on consumption goods and it is equal to \( G_t = C_t + FD_t \). Once again the price for the nondurable goods is numeraire and \( F \) is the user cost of durable goods. The utility function, in this general case where \( \rho \neq 1 \), is:

\[
\Omega_t = \left[ (1 - \alpha)C_t^{\frac{1}{\phi - \rho}} + \alpha D_t^{\frac{1}{\phi - \rho}} \right]^{-\frac{1}{\phi - \rho}}.
\]

We established earlier that:

\[
F = \frac{\partial \Omega_t}{\partial D_t} = \frac{\partial \Omega_t}{\partial C_t} \frac{\partial D_t}{\partial C_t}.
\]

\[
= \frac{\alpha}{1 - \alpha} \left( \frac{D_t}{C_t} \right)^{-\frac{1}{\rho}}.
\]

and

\[
G_t = C_t + \frac{\alpha}{1 - \alpha} \left( \frac{D_t}{C_t} \right)^{-\frac{1}{\rho}} C_t.
\]
Since we know that \( \nu_t = \left[ 1 - \alpha + \alpha \left( \frac{D_t}{C_t} \right)^{1 - \frac{1}{\psi}} \right]^{\frac{1}{\psi}} \), the adjusted total expenditures on consumption can be rewritten as:

\[
G_t = \frac{C_t \nu \left( \frac{D_t}{C_t} \right)^{1 - \frac{1}{\psi}}}{(1 - \alpha)}.
\]

Now, we are going to use our guess to verify our result that \( W_t = \frac{U_t}{f_1(\Omega_t, R_{g,t}(U_{t+1}))} \). Let’s go back to the utility function and raise both sides by power \( \left( 1 - \frac{1}{\psi} \right) \):

\[
U_t^{1 - \frac{1}{\psi}} = (1 - B) \Omega_t^{1 - \frac{1}{\psi}} + B \left( E_t U_{t+1}^{1 - \gamma} \right)^{1 - \frac{1}{\psi}}. \tag{3.11}
\]

After substituting for \( \Omega_t \) and dividing both sides of equation (3.11) by \( f_1(\Omega_t, R_{g,t}(U_{t+1})) \), we have:

\[
\frac{U_t^{1 - \frac{1}{\psi}}}{f_1(\Omega_t, R_{g,t}(U_{t+1}))} = \frac{(1 - B) \Omega_t^{1 - \frac{1}{\psi}}}{f_1(\Omega_t, R_{g,t}(U_{t+1}))} \nu \left( \frac{D_t}{C_t} \right)^{1 - \frac{1}{\psi}} + \frac{B (E_t U_{t+1}^{1 - \gamma})^{1 - \frac{1}{\psi}}}{f_1(\Omega_t, R_{g,t}(U_{t+1}))}. \tag{3.12}
\]

Substituting for \( f_1(\Omega_t, R_{g,t}(U_{t+1})) \) in the right side of equation (3.12) and then multiplying both sides by \( U_t^{\frac{1}{\psi}} \) gives:

\[
\frac{U_t}{f_1(\Omega_t, R_{g,t}(U_{t+1}))} = \frac{(1 - B) \Omega_t^{1 - \frac{1}{\psi}}}{(1 - B)(1 - \alpha) C_t^{\frac{1}{\psi} - \frac{1}{\psi}}} \nu \left( \frac{D_t}{C_t} \right)^{1 - \frac{1}{\psi}} + \frac{B (E_t U_{t+1}^{1 - \gamma})^{1 - \frac{1}{\psi}}}{(1 - B)(1 - \alpha) C_t^{\frac{1}{\psi} - \frac{1}{\psi}}}. \tag{3.13}
\]

The certainty equivalent of next period’s utility \( \left( E_t U_{t+1}^{1 - \gamma} \right)^{1 - \frac{1}{\psi}} \) implies:

\[
\left( E_t U_{t+1}^{1 - \gamma} \right)^{1 - \frac{1}{\psi}} = \left( E_t U_{t+1}^{1 - \gamma} \right)^{1 - \frac{1}{\psi} - \frac{\gamma}{\psi}} = \frac{(E_t U_{t+1}^{1 - \gamma})^{1 - \frac{1}{\psi}}}{(E_t U_{t+1}^{1 - \gamma})^{\frac{1}{\psi} - \frac{1}{\psi}}} = \frac{(E_t U_{t+1}^{1 - \gamma})^{1 - \frac{1}{\psi}}}{R_{g,t}(U_{t+1})^{1 - \gamma} \psi} = \frac{(E_t U_{t+1}^{1 - \gamma})^{1 - \frac{1}{\psi}} (U_{t+1}^{1 - \frac{1}{\psi}})}{R_{g,t}(U_{t+1})^{1 - \gamma} \psi}.
\]

Plugging this back into equation (3.13) gives:

\[
\frac{U_t}{f_1(\Omega_t, R_{g,t}(U_{t+1}))} = \frac{C_t \nu \left( \frac{D_t}{C_t} \right)^{1 - \frac{1}{\psi}}}{(1 - \alpha)} + B E_t \left[ \frac{U_{t+1}^{1 - \frac{1}{\psi}}}{(1 - B)(1 - \alpha) C_t^{\frac{1}{\psi} - \frac{1}{\psi}} \nu \left( \frac{D_t}{C_t} \right)^{1 - \frac{1}{\psi}}} \left( \frac{R_{g,t}(U_{t+1})^{1 - \frac{1}{\psi}}}{\psi} \right) \right]. \tag{3.14}
\]
Multiply and divide the part under expectation in equation (3.14) by $C_{t+1}^{\frac{1}{\psi}} \nu \left( \frac{D_{t+1}}{C_{t+1}} \right)^{\frac{1}{\psi} - \frac{1}{\psi}}$ and we have:

$$\begin{bmatrix} \frac{U_t^{1-\frac{1}{\psi}}}{(1-B)(1-\alpha)C_t^{\frac{1}{\psi}} \nu \left( \frac{D_t}{C_t} \right)^{\frac{1}{\psi} - \frac{1}{\psi}}} & (U_{t+1})^{\frac{1}{\psi} - \gamma} \\ \frac{1}{C_{t+1}} \nu \left( \frac{D_{t+1}}{C_{t+1}} \right)^{\frac{1}{\psi} - \frac{1}{\psi}} & \frac{1}{C_t} \nu \left( \frac{D_t}{C_t} \right)^{\frac{1}{\psi} - \frac{1}{\psi}} \\ \end{bmatrix} \begin{bmatrix} C_{t+1}^{\frac{1}{\psi}} \nu \left( \frac{D_{t+1}}{C_{t+1}} \right)^{\frac{1}{\psi} - \frac{1}{\psi}} (U_{t+1})^{\frac{1}{\psi} - \gamma} \\ \frac{1}{C_{t+1}} \nu \left( \frac{D_{t+1}}{C_{t+1}} \right)^{\frac{1}{\psi} - \frac{1}{\psi}} (U_{t+1})^{\frac{1}{\psi} - \gamma} \\ \end{bmatrix} .$$

Plug this back into equation (3.14) gives:

$$\frac{U_t}{f_1(U_t, R_t, U_{t+1})} = G_t + B E_t \left[ \begin{bmatrix} C_{t+1}^{\frac{1}{\psi}} \nu \left( \frac{D_{t+1}}{C_{t+1}} \right)^{\frac{1}{\psi} - \frac{1}{\psi}} (U_{t+1})^{\frac{1}{\psi} - \gamma} \\ \frac{1}{C_{t+1}} \nu \left( \frac{D_{t+1}}{C_{t+1}} \right)^{\frac{1}{\psi} - \frac{1}{\psi}} (U_{t+1})^{\frac{1}{\psi} - \gamma} \\ \end{bmatrix} \right] .$$

\[
\therefore W_t = G_t + E_t (M_{t+1} W_{t+1}).
\]

which confirms our guess.

Now we are going to substitute for the part $\left( \frac{U_{t+1}}{R_{t+1}(U_{t+1})} \right)^{\frac{1}{\psi} - \gamma}$ in equation (3.10) and show that IMRS can be written as:

$$M_{t+1} = B^{\theta} \left( \frac{C_{t+1}}{C_t} \right)^{\frac{\theta}{\psi}} \left[ \frac{\nu_{t+1}}{\nu_t} \right]^{\theta - \frac{1}{\psi}} \int_{g^{s+1}}^{g_t} .$$
Based on our budget constraint, the return on consumption claim, or equivalently the return on the wealth portfolio can be written as:

\[
R_{g,t+1} = \frac{W_{t+1}}{W_t - G_t}
\]

\[
= \frac{W_{t+1}}{W_t - \frac{C_t \nu \left( \frac{D_t}{C_t} \right)^{1-\frac{1}{\alpha}}}{U_{t+1}}}
\]

\[
= \frac{f_1 \left( \Omega_{t+1}, R_{g,t+1}(U_{t+2}) \right)}{f_1(\Omega_t, R_{g,t}(U_{t+1}))} \cdot \frac{1}{U_{t+1}} \cdot \frac{U_t}{f_1(\Omega_t, R_{g,t}(U_{t+1})) - \frac{C_t \nu \left( \frac{D_t}{C_t} \right)^{1-\frac{1}{\alpha}}}{(1-\alpha)}}.
\]

Multiply and divide the right side ratio by \( f_1(\Omega_t, R_{g,t}(U_{t+1})) \) gives:

\[
R_{g,t+1} = \frac{U_{t+1}}{f_1(\Omega_{t+1}, R_{g,t+1}(U_{t+2}))} \cdot \frac{f_1(\Omega_t, R_{g,t}(U_{t+1}))}{U_t} \cdot \frac{1}{f_1(\Omega_t, R_{g,t}(U_{t+1})) - \frac{C_t \nu \left( \frac{D_t}{C_t} \right)^{1-\frac{1}{\alpha}}}{(1-\alpha)}}.
\]

(3.15)

Let’s go back to equation (3.8) and raise both sides by power \( (1 - \frac{1}{\psi}) \), then multiply both side by \( U_t^{\frac{1}{\psi}} \) to get the following:

\[
U_t = (1 - B)\Omega_t^{1-\frac{1}{\psi}} U_t^{\frac{1}{\psi}} + BR_{g,t}(U_{t+1})^{1-\frac{1}{\psi}} U_t^{\frac{1}{\psi}}.
\]

which gives:

\[
U_t - \frac{C_t \nu \left( \frac{D_t}{C_t} \right)^{1-\frac{1}{\alpha}}}{f_1(\Omega_t, R_{g,t}(U_{t+1}))} = BR_{g,t}(U_{t+1})^{1-\frac{1}{\psi}} f(\Omega_t, R_{g,t}(U_{t+1}))^{\frac{1}{\psi}}.
\]

(3.16)

Substituting equation (3.16) into equation (3.15) gives:

\[
R_{g,t+1} = \frac{U_{t+1}}{(1 - B)(1-\alpha)U_t^{\frac{1}{\psi}} C_{t+1}^{\frac{1}{\psi}} \nu \left( \frac{D_{t+1}}{C_{t+1}} \right)^{\left( \frac{1}{\psi} - \frac{1}{\phi} \right)} f(\Omega_t, R_{g,t}(U_{t+1}))^{\frac{1}{\psi}}} - \frac{B}{R_{g,t}(U_{t+1})^{1-\frac{1}{\psi}} f(\Omega_t, R_{g,t}(U_{t+1}))^{\frac{1}{\psi}}}
\]

\[
= \left[ B \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left( \frac{\nu \left( \frac{D_{t+1}}{C_{t+1}} \right)^{\left( \frac{1}{\psi} - \frac{1}{\phi} \right)} f(\Omega_t, R_{g,t}(U_{t+1}))^{\frac{1}{\psi}}}{U_{t+1}} \right) \right]^{-1}.
\]

(3.17)
After rearranging, (3.17) can be written as:

\[
\frac{R_{g,t+1}}{B\left(\frac{C_{t+1}}{C_t}\right)^{\frac{1}{\nu}} \left(\frac{D_{t+1}}{C_{t+1}}\right)^{\frac{1}{\nu} - \frac{1}{p}} \left(\frac{D_t}{C_t}\right)} = \frac{R_{g,t}(U_{t+1})}{U_{t+1}}.
\]

By plugging equation (3.18) into equation (3.10), we get the following

\[
M_{t+1} = B\left(\frac{C_{t+1}}{C_t}\right)^{\frac{1}{\nu}} \left(\frac{D_{t+1}}{C_{t+1}}\right)^{\frac{1}{\nu} - \frac{1}{p}} \left[ \frac{R_{g,t+1}}{B\left(\frac{C_{t+1}}{C_t}\right)^{\frac{1}{\nu}} \left(\frac{D_{t+1}}{C_{t+1}}\right)^{\frac{1}{\nu} - \frac{1}{p}}} \right]^{\frac{1}{p - \gamma}}
\]

\[
= B^{\left(1 + \frac{\frac{1}{\nu} \frac{1}{p - \gamma}}{1 - \frac{1}{\nu}}\right)} \left(\frac{C_{t+1}}{C_t}\right)^{\frac{1}{\nu} \left(1 + \frac{\frac{1}{\nu} \frac{1}{p - \gamma}}{1 - \frac{1}{\nu}}\right)} \left[ \frac{\nu\left(\frac{D_{t+1}}{C_{t+1}}\right)}{\nu\left(\frac{D_t}{C_t}\right)} \right]^{\frac{1}{p - \gamma}} \left(\frac{1}{p} \frac{1}{\nu} \frac{1}{p - \gamma}\right) R_{g,t+1}^{\frac{1}{p - \gamma}}
\]

\[
\therefore M_{t+1} = B^\theta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\nu}} \left(\frac{\nu\left(\frac{D_{t+1}}{C_{t+1}}s_{t+1}\right)}{\nu\left(\frac{D_t}{C_t}\right)}\right)^{\theta \left(\frac{1}{p} \frac{1}{\nu} \frac{1}{p - \gamma}\right)} R_{g,t+1}^{\theta - 1}.
\]
Appendix B

In the following, we provide an approximation for $\nu(\frac{D_t}{C_t})$ in the special case when $\rho = 1$

$$\nu(\frac{D_t}{C_t}) = \left[(1 - \alpha) + \alpha \left(\frac{D_t}{C_t}\right)^{1-\frac{1}{\rho}}\right]^{\frac{1}{1-\frac{1}{\rho}}}$$

$$= \exp\left(\frac{1}{1-\frac{1}{\rho}} \log \left(1 + (1 - \frac{1}{\rho}) \left(\frac{(\alpha-1)}{1-\frac{1}{\rho}} + \alpha \left(\frac{D_t}{C_t}\right)^{1-\frac{1}{\rho}}\right)^{-1}\right)\right)$$

$$\simeq \exp\left(\frac{(\alpha-1)}{1-\frac{1}{\rho}} + \alpha \left(\frac{D_t}{C_t}\right)^{1-\frac{1}{\rho}}\right) \quad \text{and when } \rho = 1$$

$$\simeq \exp\left(\alpha \ln \left(\frac{D_t}{C_t}\right)\right)$$

$$\simeq \left(\frac{D_t}{C_t}\right)^{\alpha}.$$
Appendix C

Here, we provide the first-order Taylor approximation for the return on consumption claim as reported in Campbell and Shiller (1988):

\[ R_{g,t+1} = \frac{P_{g,t+1} + G_{t+1}}{P_{g,t}} \]

Multiply and divide by \( \frac{G_{t+1}}{G_t} \) and \( \frac{G_t}{G_t} \)

\[ R_{g,t+1} = \frac{P_{g,t+1} + G_{t+1}}{P_{g,t}} \cdot \frac{G_{t+1}}{G_t} \cdot \frac{G_t}{G_t} \]

Rearrange gives:

\[ R_{g,t+1} = \frac{P_{g,t+1} + G_{t+1}}{G_{t+1}} \cdot \frac{G_{t+1}}{G_t} \cdot \frac{G_t}{P_{g,t}} \]

Since \( z_t = \log(P_{g,t}/G_t) \) and \( P_{g,t+1} \) is the price of future stream of total consumption.

\[ R_{g,t+1} = \left( \frac{P_{g,t+1} + G_{t+1}}{G_t} \right) + 1 \cdot \frac{G_{t+1}}{G_t} \cdot e^{-z_t} \]

Take logs gives:

\[ r_{g,t+1} = \log(e^{z_{t+1}} + 1) + \Delta g_{t+1} - z_t \] (3.19)

Let \( X_{t+1} = f(X_t) \), then the first order Taylor expansion around the steady state value \( \bar{X} \) is:

\[ X_{t+1} = f(\bar{X}) + f(\bar{X})(X_t - \bar{X}) \]

Therefore, the first term in equation (3.19) can be approximated as:

\[ \log(e^{z_{t+1}} + 1) \approx \log(e^{\bar{z}} + 1) + \left( \frac{e^{\bar{z}}}{e^{z_{t+1}}} \right) (z_{t+1} - \bar{z}) \approx A_0 + A_1 z_t, \]
where
\[ A_0 = \log(e^z + 1) - \left( \frac{e^z}{e^z + 1} \right) \bar{z}, \]
and
\[ A_1 = \left( \frac{e^z}{e^z + 1} \right). \]

Where \( \bar{z} \) is the average of the log of price-consumption ratio

\[ \therefore r_{g,t+1} = A_0 + A_1 z_{t+1} + \Delta g_{t+1} - z_t \quad (3.20) \]

Similarly the return on dividend claim is derived as follows:

\[ R_{m,t+1} = \frac{P_{m,t+1} + Y_{t+1}}{P_{m,t}}. \]

Multiply and divide by \( \frac{Y_{t+1}}{Y_t} \) and \( \frac{Y_t}{Y_{t+1}} \):

\[ R_{m,t+1} = \frac{P_{m,t+1} + Y_{t+1}}{P_{m,t}} \cdot \frac{Y_{t+1}}{Y_{t+1}} \cdot \frac{Y_t}{Y_t}. \]

Rearrange gives:

\[ R_{m,t+1} = \frac{P_{m,t+1} + Y_{t+1}}{Y_{t+1}} \cdot \frac{Y_{t+1}}{Y_t} \cdot \frac{Y_t}{Y_{m,t}}. \]

Since \( z_{m,t} = \log(P_{m,t}/Y_t) \) and \( P_{m,t+1} \) stands for the price of total future stream of dividends

\[ R_{m,t+1} = (\frac{P_{m,t+1} + Y_{t+1}}{Y_{t+1}} - 1) \cdot \frac{Y_{t+1}}{Y_t} \cdot e^{-z_{m,t}}, \]

Taking logs gives:

\[ r_{m,t+1} = \log(e^{z_{m,t+1}} + 1) + \Delta y_{t+1} - z_{m,t}, \]

or

\[ r_{m,t+1} = A_{m,0} + A_{m,1} z_{m,t+1} + \Delta y_{t+1} - z_{m,t}, \]
where
\[ A_{m,0} = \log(e^{\tilde{z}_m} + 1) - \left(\frac{e^{\tilde{z}_m}}{e^{\tilde{z}_m} + 1}\right) \tilde{z}_m, \]
and
\[ A_{m,1} = \left(\frac{e^{\tilde{z}_m}}{e^{\tilde{z}_m} + 1}\right), \]
\( \tilde{z}_m \) is the average of the log of price-dividend ratio.

Since the excess return can be expanded as the following:
\[
\begin{align*}
    r_{m,t+1} &= A_{m,0} + A_{m,1} \left[A_{y,0} + A_{y,1}x_{d,t+1} + A_{y,2}(s_{t+1} - \bar{s})\right] + \Delta y_{t+1} \\
    &\quad - (A_{y,0} + A_{y,1}x_{d,t} + A_{y,2}(s_{t} - \bar{s})).
\end{align*}
\]

and
\[
\begin{align*}
    r_{m,t+1} - E_t[r_{m,t+1}] &= A_{m,1}A_{y,1}\omega_{d,t}\sigma_{\epsilon_{s,t+1}} + A_{m,1}A_{y,2}\omega_{s,t}\sigma_{\epsilon_{s,t+1}} \\
    &\quad + \sigma_y \epsilon_{y,t+1}.
\end{align*}
\]

The innovation of IMRS is given as:
\[
\begin{align*}
    m_{t+1} - E_t[m_{t+1}] &= \lambda_{mc}\sigma_{\epsilon_{d,t+1}} - \lambda_{me}\sigma_{\epsilon_{s,t+1}} - \lambda_{ms}\sigma_{\epsilon_{s,t+1}} + \lambda_{mx}\sigma_{\epsilon_{s,t+1}},
\end{align*}
\]
where
\[
\begin{align*}
    \lambda_{mc} &= \theta(1 - \frac{1}{\psi})\alpha\omega_{d,t}, \\
    \lambda_{me} &= \left(\theta \left(\frac{1}{\psi} + (1 - \frac{1}{\psi})\alpha\right) + (1 - \theta)\right)\omega_{s,t}, \\
    \lambda_{ms} &= \theta \left[\frac{1}{\psi} + (1 - \frac{1}{\psi})\alpha\right] + (1 - \theta)(A_1A_{y,2} + 1), \\
    \lambda_{mx} &= (1 - \theta)A_1A_{y,1}\omega_{d,t}.
\end{align*}
\]
It follows that the mean of the equity premium is equal to:

\[
E_t(r_{m,t+1} - r_{f,t}) = -cov_t(r_{m,t+1} - E_t[r_{m,t+1}], m_{t+1} - E_t[m_{t+1}]) - 0.5\text{var}_t(r_{m,t+1})
\]

\[
= -(\gamma_x\omega_x^2\sigma_x^2 + \gamma_z\omega_z^2\sigma_z^2 + 0.5\text{var}_t(r_{m,t+1}))
\]

where

\[
\gamma_x = ((\theta - 1)(A_1A_{g,2} + 1) - \theta(\frac{1}{\psi} + (1 - \frac{1}{\psi})\alpha))A_{m,1}A_{y,2},
\]

\[
\gamma_z = (\theta - 1)A_1A_{g,1}A_{m,1}A_{y,1},
\]

\[
\text{var}_t(r_{m,t+1}) = [A_{m,1}A_{g,1}\omega_x^2\sigma_x^2]^2 + [A_{m,1}A_{g,2}\omega_z\sigma_z^2]^2 + [\sigma_y^2].
\]
Appendix D

We know that the standard asset pricing condition as

\[ E_t[M_{t+1}R_{t+1}] = 1, \quad (3.21) \]

or equivalently

\[ E_t[\exp(m_{t+1} + r_{t+1})] = 1, \quad (3.22) \]

By letting \( r_{t+1} = r_{g, t+1} \) and use equations (1.28) and (1.13), equation (3.22) can be written as:

\[ E_t[\exp(\theta \log(B) + \theta(1 - \frac{1}{\psi})\alpha(\Delta d_{t+1} - \Delta c_{t+1} - \Delta s_{t+1})) - \frac{\theta}{\psi}(\Delta c_{t+1} + \Delta s_{t+1}) + \theta r_{g, t+1}] = 1, \quad (3.23) \]

Taking logs of both sides and assuming log-normality between \( m_{t+1} \) and \( r_{t+1} \), we get

\[ E_t[m_{t+1} + r_{g, t+1}] = \frac{1}{2} \text{var}[m_{t+1} + r_{g, t+1}] = 0. \quad (3.24) \]

Concerning with terms linearly involved with \( x_t \) and \( (s_t - \bar{s}) \) and constants, we ignore the second terms of equation (3.24). Substituting equations (1.14), (1.15), (1.16), (1.18), (1.19), and (1.20) into equations (1.13) use that with equation (1.28) we can have:

\[ E_t[m_{t+1} + r_{g, t+1}] = \theta \log(B) + \theta(1 - \frac{1}{\psi})\alpha(\mu_c + x_t - \mu_c - (\phi - 1)(s_t - \bar{s})) \]

\[ + \frac{\theta}{\psi}(\mu_c + (\phi - 1)(s_t - \bar{s})) \]

\[ + \left[ A_0 + A_1 \left[ A_{g,0} + A_{g,1}\phi_c x_t + A_{g,2}\phi_c (s_t - \bar{s}) \right] \right] + \theta \left[ \frac{\Delta x_t = \Delta c_{t+1} + \Delta s_{t+1}}{\Delta g_{t+1} = \Delta c_{t+1} + \Delta s_{t+1}} \right] \approx 0. \quad (3.25) \]
Collecting constant terms:

\[ 0 = \theta \log(B) + \theta \left(1 - \frac{1}{\psi}\right) \alpha (\mu_d - \mu_c) - \frac{\theta}{\psi} \mu_c + \theta(A_0 + A_1 A_{g,0} + \mu_c - A_{g,0}) \]

\[ (1 - A_1)A_{g,0} = \log(B) + \left(1 - \frac{1}{\psi}\right) \alpha (\mu_d - \mu_c) - \frac{1}{\psi} \mu_c + A_0 + \mu_c \]

\[ A_{g,0} = \frac{\log(B) + \left(1 - \frac{1}{\psi}\right)(1 - \alpha)\mu_c + (1 - \frac{1}{\psi})\alpha \mu_d + A_0}{(1 - A_1)}. \]

Collecting linear terms in \( x_t \):

\[ 0 = \theta \left(1 - \frac{1}{\psi}\right) \alpha x_t + \theta(A_1 A_{g,1} \phi_x x_t - A_{g,1} x_t) \]

\[ \theta (1 - A_1 \phi_x) A_{g,1} x_t = \theta \left(1 - \frac{1}{\psi}\right) \alpha x_t \]

\[ A_{g,1} = \frac{\alpha \left(1 - \frac{1}{\psi}\right)}{(1 - A_1 \phi_x)}. \tag{3.26} \]

Collecting linear terms in \( s_t \):

\[ 0 = -\frac{\theta}{\psi} (\phi_x - 1) (s_t - \bar{s}) - \theta \left(1 - \frac{1}{\psi}\right) \alpha (\phi_x - 1)(s_t - \bar{s}) \]

\[ + \theta \left[A_1 A_{g,2} \phi_x (s_t - \bar{s}) + (\phi_x - 1)(s_t - \bar{s}) - A_{g,2} (s_t - \bar{s}) \right] \]

\[ [1 - A_1 \phi_x] A_{g,2} (s_t - \bar{s}) = \left[ (\phi_x - 1) - \frac{(\phi_x - 1)}{\psi} - (1 - \frac{1}{\psi}) \alpha (\phi_x - 1) \right] (s_t - \bar{s}) \]

\[ A_{g,2} = \frac{(\phi_x - 1) \left(1 - \frac{1}{\psi}\right)(1 - \alpha)}{(1 - A_1 \phi_x)}. \tag{3.27} \]

Along the same steps, for the stock market return we have:

\[ E_t [m_{t+1} + r_{m,t+1}] = \theta \log(B) + \theta \left(1 - \frac{1}{\psi}\right) \alpha (\mu_d + x_t - \mu_c - (\phi_x - 1)(s_t - \bar{s})) \]

\[ -\frac{\theta}{\psi} (\mu_c + (\phi_x - 1)(s_t - \bar{s})) \]

\[ + \theta \left[A_{m,0} + A_{m,1} A_{g,0} + A_{g,1} \phi_x x_t + A_{g,2} \phi_x (s_t - \bar{s}) \right] \]

\[ + \left[ (\mu_y + \lambda x_t + \lambda_z (s_t - \bar{s})) - (A_{y,0} + A_{y,1} x_t + A_{y,2} (s_t - \bar{s})) \right] \]

\[ \Delta y_{t+1} \approx 0. \]
Collecting all constant terms:

\[ 0 = \theta \log(B) + \theta (1 - \tfrac{1}{\psi}) \alpha (\mu_d - \mu_c) - \frac{\theta}{\psi} \mu_c + \theta (A_{m,0} + A_{m,1} A_{y,0} + \mu_y - A_{y,0}) \]

\[ (1 - A_{m,1}) A_{y,0} = \log(B) + (1 - \tfrac{1}{\psi}) \alpha (\mu_d - \mu_c) - \tfrac{1}{\psi} \mu_c + A_{m,0} + \mu_y \]

\[ A_{y,0} = \frac{\log(B) - \frac{1}{\psi} \mu_c + (1 - \tfrac{1}{\psi}) \alpha (\mu_d - \mu_c) + A_{m,0} + \mu_y}{(1 - A_{m,1})}. \]

Collecting all terms linear in \( x_t \):

\[ 0 = \left( (1 - \frac{1}{\psi}) \alpha + A_{m,1} A_{y,1} \phi_x - A_{y,1} + \lambda_x \right) x_t \]

\[ (1 - A_{m,1} \phi_x) A_{y,1} = (1 - \tfrac{1}{\psi}) \alpha + \lambda_x \]

\[ A_{y,1} = \frac{(1 - \tfrac{1}{\psi}) \alpha + \lambda_x}{(1 - A_{m,1} \phi_x)}. \]

Collecting all terms linear in \( (s_t - \bar{s}) \):

\[ 0 = -\frac{\theta}{\psi} (\phi_x - 1) (s_t - \bar{s}) - \theta (1 - \tfrac{1}{\psi}) \alpha (\phi_x - 1) (s_t - \bar{s}) \]

\[ + \theta [A_{m,1} A_{y,2} \phi_x (s_t - \bar{s}) + \lambda_x (s_t - \bar{s}) - A_{y,2} (s_t - \bar{s})] \]

\[ (1 - A_{m,1} \phi_x) A_{y,2} (s_t - \bar{s}) = [\lambda_x - \frac{(\phi_x - 1)}{\psi} - (1 - \tfrac{1}{\psi}) \alpha (\phi_x - 1)] (s_t - \bar{s}) \]

\[ A_{y,2} = \frac{[\lambda_x - \frac{(\phi_x - 1)}{\psi} - (1 - \tfrac{1}{\psi}) \alpha (\phi_x - 1)]}{(1 - A_{m,1} \phi_x)}. \]

(3.28)

(3.29)
Appendix E

Implementing the standard asset pricing condition

\[ E_t[\exp(m_{t+1} + r_{i,t+1})] = 1, \quad (3.30) \]

By letting \( r_{i,t+1} = r_{g,t+1} \) and use equations (2.8) and (1.13), Then, equation (3.30) can be written as:

\[ E_t[\exp(\theta \log(B) + (1 - \frac{1}{\psi})\alpha(\Delta d_{t+1} - \Delta c_{t+1}) - \frac{\theta}{\psi} \Delta c_{t+1} + \theta r_{g,t+1})] = 1, \]

Taking logs of both sides and assuming log-normality between \( m_{t+1} \) and \( r_{g,t+1} \):

\[ E_t[m_{t+1} + r_{g,t+1}] + \frac{1}{2} \text{var}[m_{t+1} + r_{g,t+1}] \approx 0. \quad (3.31) \]

Focusing on linear terms involved with \( x_{d,t} \) and \( x_{c,t} \) besides constant terms, we ignore the second terms of equation (3.31). Substituting equations (2.1a), (2.2), (2.3), (2.4a), (2.5), and (2.6), into the pricing kernel equation (1.13) and use that with equation (2.8) we can have:

\[
E_t[m_{t+1} + r_{g,t+1}] = \theta \log(B) + (1 - \frac{1}{\psi})\alpha(\mu_d + x_{d,t} - \mu_c - x_{c,t}) - \frac{\theta}{\psi} (\mu_c + x_{c,t})
\]

\[
+ \theta \left[ A_0 + A_1 \left[ A_{g,0} + A_{g,1} \phi_d x_{d,t} + A_{g,2} \phi_c x_{c,t} \right] \right]_{z_{t+1}} \approx 0.
\]

Collecting constant terms:

\[
0 = \theta \log(B) + (1 - \frac{1}{\psi})\alpha(\mu_d - \mu_c) - \frac{\theta}{\psi} \mu_c + \theta(A_0 + A_1 A_{g,0} + \mu_c - A_{g,0})
\]

\[
(1 - A_1)A_{g,0} = \log(B) + (1 - \frac{1}{\psi})\alpha(\mu_d - \mu_c) - \frac{1}{\psi} \mu_c + A_0 + \mu_c
\]

\[
A_{g,0} = \frac{\log(B) + (1 - \frac{1}{\psi})(1 - \alpha) \mu_c + (1 - \frac{1}{\psi})\alpha \mu_d + A_0}{(1 - A_1)}. \quad (3.32)
\]
Collecting linear terms in $x_{d,t}$:

$$0 = \theta (1 - \frac{1}{\psi}) \alpha x_{d,t} + \theta (A_1 A_{g,1} \phi_d x_{d,t} - A_{g,1} x_{d,t})$$

$$\theta (1 - A_1 \phi_d) A_{g,1} x_{d,t} = \theta (1 - \frac{1}{\psi}) \alpha x_{d,t}$$

$$A_{g,1} = \frac{\alpha (1 - \frac{1}{\psi})}{(1 - A_1 \phi_d)}.$$  \hfill (3.33)

Collecting linear terms in $x_{c,t}$:

$$0 = -\frac{\theta}{\psi} x_{c,t} - \theta (1 - \frac{1}{\psi}) \alpha x_{c,t} + \theta (A_1 A_{g,2} \phi_c x_{c,t} - A_{g,2} x_{c,t})$$

$$\theta (1 - A_1 \phi_c) A_{g,2} x_{c,t} = \theta (1 - \frac{1}{\psi})(1 - \alpha) x_{c,t}$$

$$A_{g,2} = \frac{(1 - \frac{1}{\psi})(1 - \alpha)}{(1 - A_1 \phi_c)}.$$  \hfill (3.34)

Since the excess return of the second model has the following expression:

$$r_{m,t+1} = A_{m,0} + A_{m,1} (A_{y,0} + A_{y,1} x_{d,t+1} + A_{y,2} x_{c,t+1}) + \Delta y_{t+1} - (A_{y,0} + A_{y,1} x_{d,t} + A_{y,2} x_{c,t}).$$

The innovation of the excess return can be derived as:

$$[r_{m,t+1} - E_t r_{m,t+1}] = A_{m,1} A_{y,1} \omega_{d,t} \sigma_{x_d} \varepsilon_{x_d,t+1} + A_{m,1} A_{y,2} \omega_{c,t} \sigma_{x_c} \varepsilon_{x_c,t+1} + \phi_{x} \omega_{c,t} \sigma_{x} \varepsilon_{x,t+1} + \sigma_{y} \varepsilon_{y,t+1}.$$  \hfill (3.35)

Using equation (3.35) and equation (2.10), the mean of the risk premium can be derived as the following:

$$E_t (r_{m,t+1} - r_{f,t}) = -cov([r_{m,t+1} - E_t r_{m,t+1}], [m_{t+1} - E_t m_{t+1}]) - 0.5 \text{var}(r_{m,t+1}) = - (Q_{c} \omega_{c,t}^2 \sigma_{c}^2 + Q_{x_d} \omega_{x_d,t}^2 \sigma_{x_d}^2 + Q_{x_c} \omega_{x_c,t}^2 \sigma_{x_c}^2 + 0.5 \text{var}(r_{m,t+1})).$$
where
\[ Q_c = \left[ -\theta \left( \frac{1}{\psi} - (1 - \frac{1}{\psi}) \alpha + (\theta - 1) \right) \phi_{cc} \right], \]
\[ Q_{xd} = (\theta - 1)A_1 A_{y,1} A_{m,1} A_{y,1}, \]
\[ Q_{xc} = (\theta - 1)A_1 A_{y,2} A_{m,1} A_{y,2}, \]
\[ \text{var}(r_{m,t+1}) = [A_{m,1} A_{y,1} \omega_{d,t} \sigma_{xd}]^2 + [A_{m,1} A_{y,2} \omega_{c,t} \sigma_{xc}]^2 + [\phi_{cc} \omega_{ct} \sigma_{c}]^2 + [\sigma_y]^2. \]

Letting \( r_{i,t+1} = r_{m,t+1} \) and using equations (2.9) and (1.13), equation (3.30) can be written as:
\[ E_t[\exp(\theta \log(B) + (1 - \frac{1}{\psi}) \alpha (\Delta d_{t+1} - \Delta c_{t+1}) - \frac{\theta}{\psi} \Delta c_{t+1} + \theta r_{m,t+1})] = 1 \]

Taking logs of both sides and assuming log-normality between \( m_{t+1} \) and \( r_{m,t+1} \):
\[ E_t[m_{t+1} + r_{m,t+1}] + \frac{1}{2} \text{var}[m_{t+1} + r_{m,t+1}] \approx 0. \tag{3.36} \]

Considering only linear terms involving the state variables with \( x_{d,t} \) and \( x_{c,t} \) and also constant terms, equation (3.36) can be approximately equal to zero such as:
\[ E_t[m_{t+1} + r_{m,t+1}] = \theta \left( \log(B) + (1 - \frac{1}{\psi}) \alpha (\mu_d + x_{d,t} - \mu_c - x_{c,t}) + \underbrace{\left( \frac{\mu_c + x_{c,t}}{\psi} + A_{m,0} + A_{m,1} \left( A_{y,0} + A_{y,1} \phi_{x} x_{d,t} + A_{y,2} \phi_{c} x_{c,t} \right) \right)}_{z_{m,t+1}} \right) \]
\[ + \theta \left( \underbrace{-\left( A_{y,0} + A_{y,1} x_{d,t} + A_{y,2} x_{c,t} \right)}_{z_{m,t}} \right) \approx 0. \]

Collecting constant terms:
\[ 0 = \theta \log(B) + \theta (1 - \frac{1}{\psi}) \alpha (\mu_d - \mu_c) - \frac{\theta}{\psi} \mu_c + \theta (A_{m,0} + A_{m,1} A_{y,0} + \mu_y - A_{y,0}) \]
\[ (1 - A_{m,1}) A_{y,0} = \log(B) + (1 - \frac{1}{\psi}) \alpha (\mu_d - \mu_c) - \frac{1}{\psi} \mu_c + A_{m,0} + \mu_y \]
\[ A_{y,0} = \frac{\log(B) - \frac{1}{\psi} \mu_c + (1 - \frac{1}{\psi}) \alpha (\mu_d - \mu_c) + A_{m,0} + \mu_y}{(1 - A_{m,1})}. \tag{3.37} \]
Collecting all terms linear in $x_{d,t}$:

$$0 = \theta (1 - \frac{1}{\psi}) \alpha x_{d,t} + \theta (A_{m,1} A_{y,1} \phi_d - A_{y,1} + \lambda_d) x_{d,t}$$

$$(1 - A_{m,1} \phi_d) A_{y,1} = (1 - \frac{1}{\psi}) \alpha + \lambda_d$$

$$A_{y,1} = \frac{(1 - \frac{1}{\psi}) \alpha + \lambda_d}{(1 - A_{m,1} \phi_d)}.$$

Collecting all terms linear in $x_{c,t}$:

$$0 = -\frac{\theta}{\psi} x_{c,t} - \theta (1 - \frac{1}{\psi}) \alpha x_{c,t} + \theta (A_{m,1} A_{y,2} \phi_c - A_{y,2} + \lambda_c) x_{c,t}$$

$$(1 - A_{m,1} \phi_c) A_{y,2} = (\frac{1}{\psi} - 1) \alpha + \lambda_c - \frac{1}{\psi}$$

$$A_{y,2} = \frac{(\frac{1}{\psi} - 1) \alpha + \lambda_c - \frac{1}{\psi}}{(1 - A_{m,1} \phi_c)}.$$

(3.38)
Appendix F

Using the standard asset pricing condition:

$$E_t[\exp(m_{t+1} + r_{t+1})] = 1,$$  \hfill (3.40)

and by letting $r_{i,t+1} = r_{g,t+1}$ and use equations (2.17) and (1.13), Then, equation (3.40) can be written as:

$$E_t[\exp(\theta \log(B) + \theta(1 - \frac{1}{\psi})\alpha(\Delta d_{t+1} - \Delta c_{t+1}) - \frac{\theta}{\psi}\Delta c_{t+1} + \theta r_{g,t+1})] = 1,$$

Taking logs of both sides and assuming log-normality between $m_{t+1}$ and $r_{g,t+1} \Rightarrow$

$$E_t[m_{t+1} + r_{g,t+1}] + \frac{1}{2} var[m_{t+1} + r_{g,t+1}] \approx 0. \hfill (3.41)$$

Substituting equations (2.11), (2.12), (2.13), (2.14), and (2.15) into the pricing kernel equation (1.13) and use that with equation (2.17), equation (3.41) can be expanded as the following:

$$\theta(\log(B) + (1 - \frac{1}{\psi})\alpha(\mu_d + x_{d,t} - \mu_c - \sigma_c \epsilon_{c,t+1}) - \frac{(\mu_c + \sigma_c \epsilon_{c,t+1})}{\psi\Delta_{c,t+1}} + \frac{\mu_c + \sigma_c \epsilon_{c,t+1})}{\Delta_{g,t+1}})$$

$$+ \theta \left[ A_0 + (A_{1,1} x_{d,t} + A_{2,1} \phi d x_{d,t} + A_{3,1} \sigma^2 + \phi d \sigma^2 - \phi d \sigma^2 + \sigma^2 \epsilon_{c,t+1}) \right]$$

$$\left[ \sigma^2 + \phi d \sigma^2 + \phi d \sigma^2 + \sigma^2 \epsilon_{c,t+1} \right]$$

$$- (A_{4,1} x_{d,t} + A_{4,2} \sigma^2)$$

$$+ \frac{1}{2} var \left[ (1 - \frac{1}{\psi})\alpha \omega_{d,t} \sigma_c \epsilon_{d,t+1} + \left( \theta - \frac{\theta}{\psi} \right) \sigma_c \epsilon_{d,t+1} \right]$$

$$\approx 0.$$
Collecting all constant terms:

\[ 0 = \theta \log(B) - \frac{\theta}{\psi} \mu_c + \theta (1 - \frac{1}{\psi}) \alpha (\mu_d - \mu_c) \]

\[ + \theta (A_0 + A_1 A_{g,0} + A_1 A_{g,2} (1 - \phi_c) \sigma^2_c + \mu_c - A_{g,0}) \]

\[ (1 - A_1) A_{g,0} = \log(B) - \frac{1}{\psi} \mu_c + (1 - \frac{1}{\psi}) \alpha (\mu_d - \mu_c) + A_0 + \mu_c + A_1 A_{g,2} (1 - \phi_c) \sigma^2_c \]

\[ A_{g,0} = \frac{\log(B) + (1 - \frac{1}{\psi})(1 - \alpha) \mu_c + (1 - \frac{1}{\psi}) \alpha \mu_d + A_0 + A_1 A_{g,2} (1 - \phi_c) \sigma^2_c}{(1 - A_1)}. \]  (3.42)

Collecting linear terms in \( x_{d,t} \):

\[ 0 = \theta (1 - \frac{1}{\psi}) \alpha x_{d,t} + \theta (A_1 A_{g,1} \phi_d x_{d,t} - A_{g,1} x_{d,t}) \]

\[ \theta (1 - A_1 \phi_d) A_{g,1} x_{d,t} = \theta (1 - \frac{1}{\psi}) \alpha x_{d,t} \]

\[ A_{g,1} = \frac{\alpha (1 - \frac{1}{\psi})}{(1 - A_1 \phi_d)}. \]  (3.43)

Collecting terms in \( \sigma^2_{\xi_t} \):

\[ 0 = \theta (A_1 A_{g,2} \phi_c - A_{g,2}) \sigma^2_{\xi_t} + \frac{1}{2} \left[ (\theta - \frac{\theta}{\psi}) - \theta (1 - \frac{1}{\psi}) \alpha \right]^2 \sigma^2_{\xi_t} \]

\[ \theta (1 - A_1 \phi_c) A_{g,2} \sigma^2_{\xi_t} = \frac{1}{2} \left[ (\theta - \frac{\theta}{\psi}) - \theta (1 - \frac{1}{\psi}) \alpha \right]^2 \sigma^2_{\xi_t} \]

\[ A_{g,2} = \frac{\frac{1}{2} \left[ (\theta - \frac{\theta}{\psi}) - \theta (1 - \frac{1}{\psi}) \alpha \right]^2}{\theta (1 - A_1 \phi_c)}. \]  (3.44)

Similarly, By letting \( r_{n,t+1} = r_{m,t+1} \) and use equations (2.18) and (1.13), Then, equation (3.40) can be written as:

\[ E_t[\exp(\theta \log(B) + \theta (1 - \frac{1}{\psi}) \alpha (\Delta d_{t+1} - \Delta c_{t+1}) - \frac{\theta}{\psi} \Delta c_{t+1} + \theta r_{m,t+1})] = 1, \]
Taking logs of both sides and assuming log-normality between $m_{t+1}$ and $r_{m,t+1}$

$$E_t[m_{t+1} + r_{m,t+1}] + \frac{1}{2} \text{var}[m_{t+1} + r_{m,t+1}] \approx 0.$$ 

$$\theta(\log(B) - \frac{(\mu_c + \sigma_c \epsilon_{c,t+1})}{\psi}) + (1 - \frac{1}{\psi}) \alpha (\mu_d + \epsilon_{d,t} - \mu_c - \sigma_c \epsilon_{c,t+1})$$

$$\theta(A_{m,0} + (A_{m,1} A_{y,0} + A_{y,1} \phi_d \epsilon_{d,t} + A_{y,2} \sigma_c^2 + \phi_c \sigma_c^2 - \sigma_c^2 + \sigma_c \epsilon_{c,t+1})))$$

$$+ \theta((\mu_y + \lambda_d \epsilon_d + \lambda_c \epsilon_{c,t+1} + \sigma_y \epsilon_{y,t+1}) - (A_{y,0} + A_{y,1} \epsilon_{d,t} + A_{y,2} \sigma_c^2))$$

$$+ \frac{1}{2} \text{var} \left[ \theta((1 - \frac{1}{\psi}) \alpha + \rho_d \omega_d \epsilon_{d,t} \sigma_d^2 \epsilon_{d,t+1} + \left( \theta \rho_c - \frac{\theta}{\psi} - (1 - \frac{1}{\psi}) \alpha \right) \sigma_c \epsilon_{c,t+1}$$

$$+ \theta(A_{m,1} A_{y,1} \omega_d \epsilon_{d,t} \sigma_d \epsilon_{d,t+1} + A_{m,1} A_{y,2} \sigma_c \epsilon_{c,t+1} + \sigma_y \epsilon_{y,t+1}) \right] \approx 0.$$ 

Collecting all constant terms:

$$0 = \theta \log(B) - \frac{\theta}{\psi} \mu_c + \theta(1 - \frac{1}{\psi}) \alpha (\mu_d - \mu_c) +$$

$$\theta(A_{m,0} + A_{m,1} A_{y,0} + A_{m,1} A_{y,2} (1 - \phi_c) \sigma_c^2 + \mu_y - A_{y,0})$$

$$(1 - A_{m,1}) A_{y,0} = \log(B) + (1 - \frac{1}{\psi}) \alpha (\mu_d - \mu_c) - \frac{1}{\psi} \mu_c + A_{m,0} + A_{m,1} A_{y,2} (1 - \phi_c) \sigma_c^2 + \mu_y$$

$$A_{y,0} = \frac{\log(B) - \frac{1}{\psi} \mu_c + (1 - \frac{1}{\psi}) \alpha (\mu_d - \mu_c) + A_{m,0} + A_{m,1} A_{y,2} (1 - \phi_c) \sigma_c^2 + \mu_y}{(1 - A_{m,1})}.$$

(3.45)

Collecting terms linear in $x_{d,t}$:

$$0 = \theta(1 - \frac{1}{\psi}) \alpha x_{d,t} + \theta(A_{m,1} A_{y,1} \phi_d - A_{y,1} + \lambda_d) x_{d,t}$$

$$(1 - A_{m,1} \phi_d) A_{y,1} = (1 - \frac{1}{\psi}) \alpha + \lambda_d$$

$$A_{y,1} = \frac{(1 - \frac{1}{\psi}) \alpha + \lambda_d}{(1 - A_{m,1} \phi_d)}.$$ 

(3.46)
Collecting terms in $\sigma_{c,t}^2$:

$$0 = \theta(A_{m,1}A_{y,2}\phi_e - A_{y,2})\sigma_{c,t}^2 + \frac{1}{2} \left[ (\theta \rho_e - \frac{\theta}{\psi})^2 + \left( \frac{1}{\psi} - 1 \right)^2 \alpha \right] \sigma_{c,t}^2$$

$$\theta(1 - A_{m,1}\phi_e)A_{y,2}\sigma_{c,t}^2 = \frac{1}{2} \left[ (\theta \rho_e - \frac{\theta}{\psi}) - \theta(1 - \frac{1}{\psi})\alpha \right]^2 \sigma_{c,t}^2$$

$$A_{y,2} = \frac{1}{\theta(1 - A_{1}\phi_e)}. \tag{3.47}$$

The excess return of the third model has the following form:

$$r_{m,t+1} = A_{m,0} + A_{m,1}(A_{y,0} + A_{y,1}x_{d,t+1} + A_{y,2}\sigma_{c,t+1}^2) + \Delta y_{t+1} - (A_{y,0} + A_{y,1}x_{d,t} + A_{y,2}\sigma_{c,t}^2),$$

and

$$[r_{m,t+1} - E_t r_{m,t+1}] = A_{m,1}A_{y,1}\omega_{d,t}\sigma_{\epsilon d,\epsilon d},t+1 + A_{m,1}A_{y,1}\sigma_{\epsilon \epsilon,\epsilon c,t+1}$$

$$+ \rho_c \sigma_{c,t}\epsilon_{c,t+1} + \rho_c \sigma_{c,t}\epsilon_{c,t+1} + \sigma_y \epsilon_{y,t+1}, \epsilon_{y,t+1}.$$

The mean of the risk premium of the third model derived as the following:

$$E_t(r_{m,t+1} - r_f,t) = -cov([r_{m,t+1} - E_t r_{m,t+1}, [m_{t+1} - E_t m_{t+1}]) - 0.5 \text{var}(r_{m,t+1}),$$

$$= - \begin{pmatrix} \theta(1 - \frac{1}{\psi})\alpha \rho_e \omega_{d,t}\sigma_{\epsilon d,\epsilon d}^2 + \left[ (\theta - 1) - \theta\left( \frac{1}{\psi} + (1 - \frac{1}{\psi})\alpha \right) \right] \rho_c \sigma_{c,t}^2 \\ \theta(1 - \frac{1}{\psi})\alpha \rho_e \omega_{d,t}\sigma_{\epsilon d,\epsilon d}^2 + \left[ (\theta - 1) - \theta\left( \frac{1}{\psi} + (1 - \frac{1}{\psi})\alpha \right) \right] \rho_c \sigma_{c,t}^2 \\ + (\theta - 1)A_{1}A_{g,2}A_{m,1}A_{y,2}\sigma_{\epsilon \epsilon,\epsilon c}^2 + (\theta - 1)A_{1}A_{g,1}A_{m,1}A_{y,1}\omega_{d,t}\sigma_{\epsilon d,\epsilon d}^2 \\ -0.5 \text{var}(r_{m,t+1}), \end{pmatrix}$$

where

$$\text{var}(r_{m,t+1}) = [A_{m,1}A_{y,1}\omega_{d,t}\sigma_{\epsilon d}]^2 + [A_{m,1}A_{y,2}\sigma_{\epsilon \epsilon}]^2 + [\rho_c \sigma_{c,t}]^2 + [\rho_c \omega_{d,t}\sigma_{\epsilon d}]^2 + [\sigma_y]^2.$$
Appendix G

Using the Euler equation condition:

\[ E_t[m_{t+1} + r_{g,t+1}] + \frac{1}{2} \text{var}[m_{t+1} + r_{g,t+1}] \approx 0. \]

As mentioned in the first model, we ignore the second part, so the solution to the coefficient \( A_{g,3} \) can be showed by collecting terms in \( x_{c,t} \) as:

\[ 0 = -\theta \frac{\psi}{\psi} x_{c,t} - \theta (1 - \frac{1}{\psi}) \alpha x_{c,t} + \theta (A_1 A_{g,3} \phi_c x_{c,t} - A_{g,3} x_{c,t}) \]

\[ \theta (1 - A_1 \phi_c) A_{g,3} x_{c,t} = \theta (1 - \frac{1}{\psi})(1 - \alpha) x_{c,t} \]

\[ A_{g,3} = \frac{(1 - \frac{1}{\psi})(1 - \alpha)}{(1 - A_1 \phi_c)}. \]  

(3.48)

Similarly, we the Euler equation condition:

\[ E_t[m_{t+1} + r_{m,t+1}] + \frac{1}{2} \text{var}[m_{t+1} + r_{m,t+1}] \approx 0. \]

The solution to the coefficient \( A_{y,3} \) can be derived by collecting terms in \( x_{c,t} \) as:

\[ 0 = -\theta \frac{\psi}{\psi} x_{c,t} - \theta (1 - \frac{1}{\psi}) \alpha x_{c,t} + \theta (A_{m,1} A_{y,3} \phi_c - A_{y,3} \lambda_c) x_{c,t} \]

\[ (1 - A_{m,1} \phi_c) A_{y,3} = \frac{(1 - \frac{1}{\psi}) \alpha + \lambda_c - \frac{1}{\psi}}{(1 - A_{m,1} \phi_c)}. \]

(3.49)

Also, the risk-free rate of the adjusted model takes the following form:

\[ r_{f,t} = -\theta \left( \log(B) - \frac{(\mu_c + x_{c,t} + (\phi_c - 1)(s_t - \bar{s}))}{\psi} \right) + (1 - \frac{1}{\psi}) \alpha (\mu_d + x_{d,t} - \mu_c - x_{c,t} - (\phi_c - 1)(s_t - \bar{s}))) \]

\[ - (\theta - 1) \left[ A_0 + A_1 [A_{g,0} + A_{g,1} \phi_d x_{d,t} + A_{g,2} \phi_d s_t - \bar{s}] + A_{g,3} \phi_c x_{c,t}] + (\mu_c + x_{c,t} + (\phi_c - 1)(s_t - \bar{s}) - [A_{g,0} + A_{g,1} x_t + A_{g,2} (s_t - \bar{s}) + A_{g,3} \phi_c x_{c,t}]} \right]. \]

\[ E_t[r_{g,t+1}] \]
with a volatility
\[
\text{var}(r_{f,t}) = \Gamma_d \text{var}(x_{d,t}) + \Gamma_c \text{var}(x_{c,t}) + \Gamma_s \text{var}((s_t - \bar{s}))
\]

where
\[
\begin{align*}
\Gamma_d &= \left( \theta \frac{1}{\psi} - 1 \right) \alpha + (1 - \theta)(A_1 A_{y,1} \phi_x - A_{y,1})^2, \\
\Gamma_c &= \left( \frac{\theta}{\psi} + \theta \left( 1 - \frac{1}{\psi} \right) \alpha \right) + (1 - \theta)(A_1 A_{y,3} \phi_x + 1 - A_{y,3})^2, \\
\Gamma_s &= \left( \phi_x - 1 \right) \left( \frac{\theta}{\psi} + \theta \left( 1 - \frac{1}{\psi} \right) \alpha \right) + (1 - \theta)(A_1 A_{y,2} \phi_x + (\phi_x - 1) - A_{y,2})^2.
\end{align*}
\]

The excess return of the adjusted model has the following form:
\[
\begin{align*}
\hat{r}_{m,t+1} &= A_{m,0} + A_{m,1} (A_{y,0} + A_{y,1} x_{d,t+1} + A_{y,2} (s_{t+1} - \bar{s}) + A_{y,3} x_{c,t+1}) + \Delta y_{t+1} \\
&\quad - (A_{y,0} + A_{y,1} x_{d,t} + A_{y,2} (s_{t} - \bar{s}) + A_{y,3} x_{c,t})
\end{align*}
\]

The innovation of the excess return for the adjusted model is:
\[
[r_{m,t+1} - \hat{E}_t r_{m,t+1}] = A_{m,1} A_{y,1} \omega_d \sigma_d \epsilon_d, t+1 + A_{m,1} A_{y,2} \omega_s \sigma_s \epsilon_s, t+1 + A_{m,1} A_{y,3} \omega_c \sigma_c \epsilon_c, t+1 + \rho_d \omega_d, t \sigma_d \epsilon_d, t+1 + \rho_s \omega_s, t \sigma_s \epsilon_s, t+1 + \rho_c \omega_c, t \sigma_c \epsilon_c, t+1.
\]

The equity premium of the adjusted model can be derived as the following:
\[
\begin{align*}
\hat{E}_t (r_{m,t+1} - r_{f,t}) &= -\text{cov}(r_{m,t+1} - \hat{E}_t r_{m,t+1}, [m_{t+1} - \hat{E}_t m_{t+1}]) - 0.5 \text{var}(r_{m,t+1}),
\end{align*}
\]

where
\[
\begin{align*}
\text{cov}(r_{m,t+1} - \hat{E}_t r_{m,t+1}, [m_{t+1} - \hat{E}_t m_{t+1}]) &= \gamma_{d,d,t} \omega_d^2 \sigma_d^2 + \gamma_{d,c,t} \omega_d \sigma_d \sigma_c + \gamma_{s,s,t} \omega_s^2 \sigma_s^2 \\
&\quad + \gamma_{c,c,t} \sigma_c^2 + \gamma_{d,c,t} \omega_d \sigma_c + \gamma_{c,s,t} \omega_s \sigma_c.
\end{align*}
\]
and

$$
\begin{align*}
\gamma_{\text{ad}} &= (\theta - 1)A_1 A_{y,1} A_{m,1} A_{y,1}, \\
\gamma_{\text{cd}} &= \theta(1 - \frac{1}{\psi})\alpha \rho_d, \\
\gamma_s &= ((\theta - 1)(A_1 A_{y,2} + 1) - \theta(\frac{1}{\psi} + (1 - \frac{1}{\psi})\alpha)A_{m,1} A_{y,2}, \\
\gamma_c &= \left[(\theta - 1) - \theta(\frac{1}{\psi} + (1 - \frac{1}{\psi})\alpha)\right] \rho_c, \\
\gamma_{xc} &= (\theta - 1)A_1 A_{y,3} A_{m,1} A_{y,3},
\end{align*}
$$

and

$$
\text{var}(r_{m,t+1}) = [A_{m,1} A_{y,1} \omega_{\text{st}} t \sigma_{\text{ad}}]^2 + [A_{m,1} A_{y,2} \omega_{\text{st}} t \sigma_s]^2 + [A_{m,1} A_{y,3} \omega_{\text{st}} t \sigma_{xc}]^2 \\
+ [\rho_d \omega_{\text{st}} t \sigma_c]^2 + [\rho_d \omega_{\text{st}} t \sigma_{xc}]^2.
$$