ABSTRACT

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Let M be an irreducible algebraic monoid with reductive unit group G. There exists an idempotent cross section Λ of $G \times G$ orbits that forms a lattice under the partial order $e \leq f \iff GeG \subseteq \overline{GfG}$, where the closure is in the Zariski topology. This cross section lattice is important in describing the structure of reductive monoids.

In this paper we study some properties of cross section lattices, particularly in the case where there are one or two minimal nonzero elements. We determine when these cross sections lattices are modular and distributive, and how distributive cross section lattices can be expressed as a product of chains. We also compute the zeta polynomials and characteristic polynomials of cross section lattices as well as describe the importance of corank in describing $\mathcal{J}$-irreducible cross section lattices.

On the Cross Section Lattice of Reductive Monoids

by
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BIOGRAPHY

Stephen Adams was born in Akron, Ohio to William and Vicki Adams on April 6, 1984. His family moved to Palm Harbor, Florida seven years later where he learned the importance of air conditioning. Mathematics was Stephen's favorite subject in elementary school and he had the opportunity to attend Safety Harbor Middle School where he participated in their Mathematics Education for Gifted Secondary School Students (MEGSSS) program and MathCounts. This helped transform mathematics from something Stephen was good at to something that he was passionate about. He then attended Palm Harbor University High School, was active in their Mu Alpha Theta chapter, and competed in regional, state, and national competitions. He has been writing tests for these competitions ever since he was in the tenth grade.

Stephen then applied to Florida State University as an engineering major. He decided to change his major to mathematics during freshman orientation and later added a second major in economics. Upon graduation, he attended the Ohio State University as a graduate student in economics. During this time Stephen realized that he didn't want to spend the rest of his life doing economics research, so he left the program after earning a Master's degree. He then entered the mathematics program at North Carolina State University in the fall of 2008. He conducted his research under Dr. Mohan Putcha and will defend his dissertation in June 2014. This August he will begin his job as an Assistant Professor at Cabrini College where he hopes to have a long career and a dog named Paws Scaggs.

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Chapter 1

Introduction

The study of reductive monoids began around 1980 and was developed independently by Mohan Putcha and Lex Renner. The theory is a rich blend of semigroup theory, algebraic groups, and torus embeddings. Monoids occur naturally in mathematics and the sciences since every linear algebraic monoid is isomorphic to a submonoid of the set of $n \times n$ matrices. Applications of monoids are abundant and include the areas of combinatorics and computer science.

A monoid is a semigroup with an identity element. As such it has a group of units G which is necessarily nonempty. A reductive monoid is a monoid whose unit group G is a reductive group. Reductive monoids are regular and are hence determined by the group of units and the set of idempotent elements. The structure of these reductive monoids is of particular interest. We can define a partial order on the set of $G \times G$ orbits of the monoid. These orbits form a lattice called the cross section lattice. Given this cross section lattice and a type map, we can construct the monoid M up to a central extension.

The purpose of this paper is to describe some properties of the cross section lattices of reductive monoids. Our results focus on two specific cases: the $\mathcal{J}$-irreducible case and the 2-reducible case where the cross section lattice has one and two minimal nonzero elements, respectively. In Chapter 2 we introduce the background material from algebraic geometry and algebraic groups that is required to define a reductive monoid. We then introduce all of the concepts from lattice theory that are used throughout the paper. Chapter 3 introduces the concept of the cross section lattice of a reductive monoid. This chapter contains the majority of the theory of cross section lattices that serves as the basis upon which the rest of the paper is built as well as several useful examples.

In Chapter 4 we determine when the cross section lattices of $\mathcal{J}$-irreducible and 2-reducible monoids are distributive. Chapter 5 investigates when the cross section lattices of distributive monoids can be expressed as a direct product of chains. The zeta polynomial is then calculated as an application. Chapter 6 details the Möbius function of a cross section lattice which is then used to calculate the lattice's characteristic polynomial. Chapter 7 briefly discusses the rank of cross section lattices and then investigates the consequences of corank in the $\mathcal{J}$-irreducible case. Finally, Chapter 8 summarizes possible directions of future research.

Chapter 2

Preliminaries

The study of reductive monoids is a rich blend of semigroup theory, algebraic geometry, and algebraic group theory. Our goal is to study the cross section lattice of a reductive monoid, which allows us to describe the structure of the monoid. The proofs of results in Chapters 4, 5, 6, and 7 are based entirely on lattice theory and the combinatorics of invariants of the monoid. As such, this chapter contains all of the algebraic background material necessary to introduce the notion of cross section lattices and nothing more. No knowledge of algebraic geometry, algebraic groups, or monoids is assumed. However, it will be assumed that the reader is familiar with the fundamentals of group theory and ring theory.

The material presented on algebraic geometry and algebraic groups comes from [4]. Additional material on root systems comes from [3]. For reductive monoids, we rely on [7], [14], and [16]. The majority of the lattice theory is from [17]. The reader who is interested in familiarizing themselves beyond the bare necessities is encouraged to peruse the relevant references.

2.1 Algebraic Geometry

Throughout we will assume that k is an algebraically closed field.

Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Since k is a field it is Noetherian and hence $k[x_1, \ldots, x_n]$ is Noetherian by the Hilbert Basis Theorem. I is therefore finitely generated, that is, $I = \langle f_1, \ldots, f_m \rangle$ for some polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$. The *zero set* of I is

$$\mathcal{V}(I) = \{(a_1, \ldots, a_n) \in k^n \mid f_i(a_1, \ldots, a_n) = 0,\ 1 \leq i \leq m\}.$$

This is the set of all points in k^n that vanish on every polynomial in I.

A set $X \subseteq k^n$ is an *affine variety* if it is the set of common zeros of a finite collection of polynomials. That is, an affine variety is of the form $X = \mathcal{V}(I)$ for some ideal I.

Example 2.1.1. Let $SL_n(k)$ be the group of $n \times n$ matrices of determinant 1 whose entries are elements of k. Since $SL_n(k)$ has n^2 entries, we can easily identify $SL_n(k)$ as a subset of k^{n^2}. If we view each of the entries x_{ij} of a matrix $A \in SL_n(k)$ as a variable, the determinant of A is a polynomial in these n^2 variables. $SL_n(k)$ therefore satisfies the polynomial equation $\det(A) = 1$, and hence $SL_n(k)$ is an affine variety.

Example 2.1.2. Let $GL_n(k) \subseteq k^{n^2}$ be the set of invertible $n \times n$ matrices whose entries are elements of k. Notice that $GL_n(k) \cong \left\{ \begin{pmatrix} A & 0 \\ 0 & x \end{pmatrix} \mid A \in GL_n(k),\, x\det(A) = 1 \right\} \subseteq k^{n^2+1}$. Therefore $GL_n(k)$ is an affine variety.

Example 2.1.3. Let $B_n(k) \subset GL_n(k)$ be the set of invertible upper triangular matrices. If $A = (a_{ij})$, then the entry $a_{ij} = 0$ for all $i > j$. $B_n(k)$ is therefore the zero set of finitely many polynomials and it is therefore an affine variety. Similarly, $D_n(k)$, the set of invertible diagonal matrices, and $U_n(k) = \{(a_{ij}) \in B_n(k) \mid a_{ii} = 1\}$, the set of unipotent upper triangular matrices, are affine varieties.

Let $X \subseteq k^n$. Let $\mathcal{I}(X)$ be the set of all polynomials that have X as a vanishing set. That is,

$$\mathcal{I}(X) = \{f \in k[x_1, \ldots, x_n] \mid f(x) = 0\ \forall x \in X\}.$$

It is easy to see that $\mathcal{I}(X)$ is an ideal. Furthermore

$$\begin{aligned} X &\subseteq \mathcal{V}(\mathcal{I}(X)) \\ I &\subseteq \mathcal{I}(\mathcal{V}(I)), \end{aligned}$$

however neither inclusion must hold with equality. The first will hold if X is an affine variety. The second will hold if I is a radical ideal, that is, $I = \{f \in k[x_1, \ldots, x_n] \mid f^i \in I \text{ for some } i \in \mathbb{N}\}$.

Let X and Y be affine varieties. A *morphism* is a mapping $\phi : X \to Y$ such that

$$\phi(x_1, \ldots, x_n) = (\psi_1(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n)),$$

where $\psi_i \in k[x_1, \ldots, x_n]/\mathcal{I}(x)$ for each i.

Affine varieties and morphisms are necessary to define algebraic groups in the next section.

It will be useful to topologize k^n by saying a set is *closed* if and only if it is an affine variety. The *closure* of a set A, denoted $\overline{A}$, is the smallest closed set containing A. It is not difficult to check that the axioms for a topology are met and the resulting topology is called the *Zariski topology*. Points in the Zariski topology are closed and every subcover has a finite subcover. However, open sets are dense and hence k^n is not a Hausdorff space. For this reason k^n is often said to be *quasicompact*, the term compact being reserved for a Hausdorff space. Despite this minor set back, k^n with the Zariski topology is a *noetherian* space, that is, closed sets satisfy the descending chain condition that if $X_1 \supseteq X_2 \supseteq \cdots$ is a sequence of closed subsets of k^n, then $X_i = X_{i+1} = \cdots$ for some integer i.

Example 2.1.4. Let $f(x) = x^2 - 1 \in \mathbb{C}[x]$. Then f has two zeros so the set $\{-1, 1\}$ is closed in the Zariski topology. In fact, any finite subset $\{c_1, \ldots, c_n\}$ of $\mathbb{C}$ is the set of zeros of the polynomial $\prod_{i=1}^{n}(x - c_i)$. All such sets are closed in the Zariski topology.

Example 2.1.5. Let $GL_n(k)$ be the group of $n \times n$ invertible matrices whose entries are elements of k. Notice that if X is a noninvertible $n \times n$ matrix, then $\det(X) = 0$. That is, the set of noninvertible matrices is closed in the Zariski topology. Therefore $GL_n(k)$ is an open subset of $M_n(k)$, the set of $n \times n$ matrices. Notice that $GL_n(k)$ is dense in $M_n(k)$, that is, $\overline{GL_n(k)} = M_n(k)$.

A topological space is *irreducible* if it cannot be written as the union of two proper nonempty closed sets. Equivalently, a topological space is irreducible if and only if the intersection of two nonempty open sets is nonempty. A variety is *irreducible* if it is nonempty and not the union of two proper subvarieties.

Example 2.1.6. k^n with the Zariski topology is irreducible since any open set is dense, and hence two nonempty open subsets have nonempty intersection. In particular, notice that if X is an open set in k^n, then $\overline{X} = k^n$.

Theorem 2.1.7. *Let X be a noetherian topological space. Then X has only finitely many maximal irreducible subspaces and their union is X.*

Theorem 2.1.7 allows us to express a noetherian space, such as k^n with the Zariski topology, as a union of maximal irreducible subspaces. These subspaces are called the *irreducible components* of X.

2.2 Algebraic Groups

Let G be an affine variety that satisfies the axioms of a group. If the two maps $\mu : G \times G \to G$, where $\mu(x, y) = xy$, and $\iota : G \to G$, where $\iota(x) = x^{-1}$, are morphisms of varieties, then G is called an *affine algebraic group*. A *linear algebraic group* is an affine algebraic group that is a subgroup of $GL_n(k)$. In view of Theorem 2.3.5 in the next section, we are primarily interested in linear algebraic groups.

Example 2.2.1. We have seen that $GL_n(k)$, $SL_n(k)$, $D_n(k)$, $B_n(k)$, and $U_n(k)$ are all affine varieties. Furthermore, they all satisfy the axioms of a group. Matrix multiplication and inversion are morphisms and therefore all five sets are linear algebraic groups.

The *irreducible components* of an affine algebraic group G are the irreducible components of G when considered as an affine variety. These irreducible components are called the *connected components* of G, as the term "irreducible" has a different meaning in regards to algebraic groups. There is a unique connected component, denoted G°, that contains the identity element 1 of G. This connected component is called the *identity component* of G. G° is a normal subgroup of G. G is said to be *connected* if $G = G^\circ$.

Example 2.2.2. $GL_n(k)$, $SL_n(k)$, $D_n(k)$, $B_n(k)$, and $U_n(k)$ are all connected linear algebraic groups.

Let G be a connected linear algebraic group. The *radical* of G, denoted $R(G)$, is the unique maximal connected normal solvable subgroup of G. An element $x \in G$ is *unipotent* if its only eigenvalue is 1. The *unipotent radical* of G, denoted $R_u(G)$, is the subgroup of $R(G)$ consisting of all the unipotent elements of G. If $G \neq \{1\}$, then G is called *semisimple* if $R(G) = \{1\}$ and *reductive* if $R_u(G) = \{1\}$. Notice that if G is semisimple, then it is reductive. The converse, however, it not necessarily true. G is a *simple group* if it has no closed connected normal subgroups other than itself and $\{1\}$.

Example 2.2.3. $GL_n(k)$ is reductive. $SL_n(k)$ is semisimple and hence also reductive. It is also simple. $B_n(k)$ is not reductive for $n \geq 2$ because its unipotent radical is $U_n(k)$, which is nontrivial.

An affine algebraic group is a *torus* if it is isomorphic to $k^* \times \cdots \times k^*$. Equivalently, a linear algebraic group is a torus if it is isomorphic to a subgroup of $D_n(k)$. A torus is a *maximal torus* if it is not properly contained in a larger torus.

A *Borel subgroup* of an affine algebraic group is a maximal closed connected solvable subgroup B of G. All Borel subgroups of G are conjugate to B. Furthermore, the maximal tori of G are those of the Borel subgroups of G. The maximal tori are also all conjugate. If B and B^- are Borel subgroups such that $B \cap B^- = T$ is a maximal torus, then B^- is called the *opposite Borel subgroup of B relative to T*. A proper subgroup P of G is *parabolic* if it contains a Borel subgroup as a subset. That is, the parabolic subgroups are the subgroups of G that are between B and G.

Let T be a maximal torus and $N = N_G(T) = \{x \in G \mid x^{-1}Tx = T\}$ be the normalizer of T in G. The *Weyl group* of G is $W = N/T$. Since all maximal tori are conjugate, the Weyl group is independent of the choice of T.

Example 2.2.4. Let $G = GL_n(k)$. Then the subgroup of invertible upper triangular matrices $B_n(k)$ is a Borel subgroup containing the maximal torus $D_n(k)$. The opposite Borel subgroup of $B_n(k)$ relative to $D_n(k)$ is $B_n^-(k)$, the subgroup of invertible lower triangular matrices. Notice that $B_n(k) \cap B_n^-(k) = D_n(k)$. The parabolic subgroups are the subgroups comprised of invertible upper block triangular matrices. Notice that these subgroups of G contain $B_n(k)$ as a subgroup. The Weyl group is isomorphic to S_n, the symmetric group on n elements.

Definition 2.2.5. A *root system* is a real vector space E together with a finite subset Φ such that

a) Φ spans E, and does not contain 0.

b) If $\alpha \in \Phi$, then the only other multiple of α in Φ is $-\alpha$.

c) If $\alpha \in \Phi$, then there is a reflection $s_\alpha : E \to E$ such that $s_\alpha(\alpha) = -\alpha$ and s_α leaves Φ stable.

d) If $\alpha, \beta \in \Phi$, then $s_\alpha(\beta) - \beta$ is an integral multiple of α.

The elements of Φ are called *roots*. A subset $\Delta = \{\alpha_1, \ldots, \alpha_k\}$ is called a *base* if Δ is a basis of E and each $\alpha \in \Phi$ can be expressed as a linear combination $\alpha = \sum c_i \alpha_i$, where the c_i are integers that are all either nonnegative or nonpositive. Bases exist and every root is an element of at least one base. The elements of Δ are called *simple roots*. A reflection s_α corresponding to the simple root $\alpha \in \Delta$ is called a *simple reflection*. The group W generated by the set of simple reflections $\{s_\alpha \mid \alpha \in \Delta\}$ is called the *Weyl group*. The Weyl group permutes the set of bases transitively. There is an inner product (α, β) on E relative to which W consists of orthogonal transformations. Furthermore, $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle$, where $\langle \beta, \alpha \rangle = 2(\beta, \alpha)/(\alpha, \alpha)$. Φ is said to be *irreducible* if it cannot be partitioned into a union of two mutually orthogonal proper subsets. Every root system is the disjoint union of irreducible root systems.

If G is a semisimple affine algebraic group with maximal torus T, then $E = X(T) \otimes \mathbb{R}$ with Φ as described above is a root system. The Weyl group is $W = N_G(T)/T$ and G is generated by the Borel subgroup B containing T and the normalizer $N_G(T)$.

A *graph* H is a set $V = \{v_1, \ldots, v_n\}$ of *vertices* with a set $E = \{e_1, \ldots, e_m\}$ of *edges*. The edges are two-element subsets of V. Each edge e_k therefore corresponds to an unordered pair (v_i, v_j) of elements of V. Graphs can be denoted pictorially by drawing points corresponding to each vertex of V and connecting two vertices v_i and v_j with a line segment if there exists $e_k \in E$ such that $e_k = (v_i, v_j)$. In this case we say that v_i and v_j are *adjacent*. A *path* is a sequence of edges that connect a sequence of vertices of a graph where no vertex is repeated. A graph is *connected* if any two vertices can be connected by a path. A *connected component* is a maximal connected subgraph. Notice that a graph is then connected if and only if it has one connected component.

The structure of affine algebraic groups can be described by the set of simple roots. We can create a graph called the *Dynkin diagram* as follows: Two nodes corresponding to the simple roots α and β are connected by $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle$ edges. $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle$ can be equal to 0, 1, 2, or 3. We will say that α and β are *adjacent* if they are connected by an edge in the Dynkin diagram of Δ. Two simple roots α and β are adjacent if and only if $s_\alpha s_\beta \neq s_\beta s_\alpha$. The Dynkin diagram is used to describe the structure of the algebraic group. The possible Dynkin diagrams for simple algebraic groups are listed in Figure 2.1. Notice that all of these Dynkin diagrams are connected graphs. If the algebraic group G is not simple, its Dynkin diagram is the disjoint union of Dynkin diagrams from Figure 2.1.

In Chapter 3 we will see how Dynkin diagrams can be used to help describe the structure of reductive monoids.

Let G be a reductive group and let B be a Borel subgroup of G containing a maximal torus T. Let S be the set of simple reflections corresponding to the base Δ determined by B and T. For $I \subset S$, let W_I be the subgroup of W generated by I. Let $P_I = BW_IB$.

Theorem 2.2.6.

a) *The only subgroups of G containing B are of the form P_I for $I \subset S$.*

b) *If P_I is conjugate to P_J, then $I = J$.*

c) *The following are equivalent.*

   i) $I = J$.

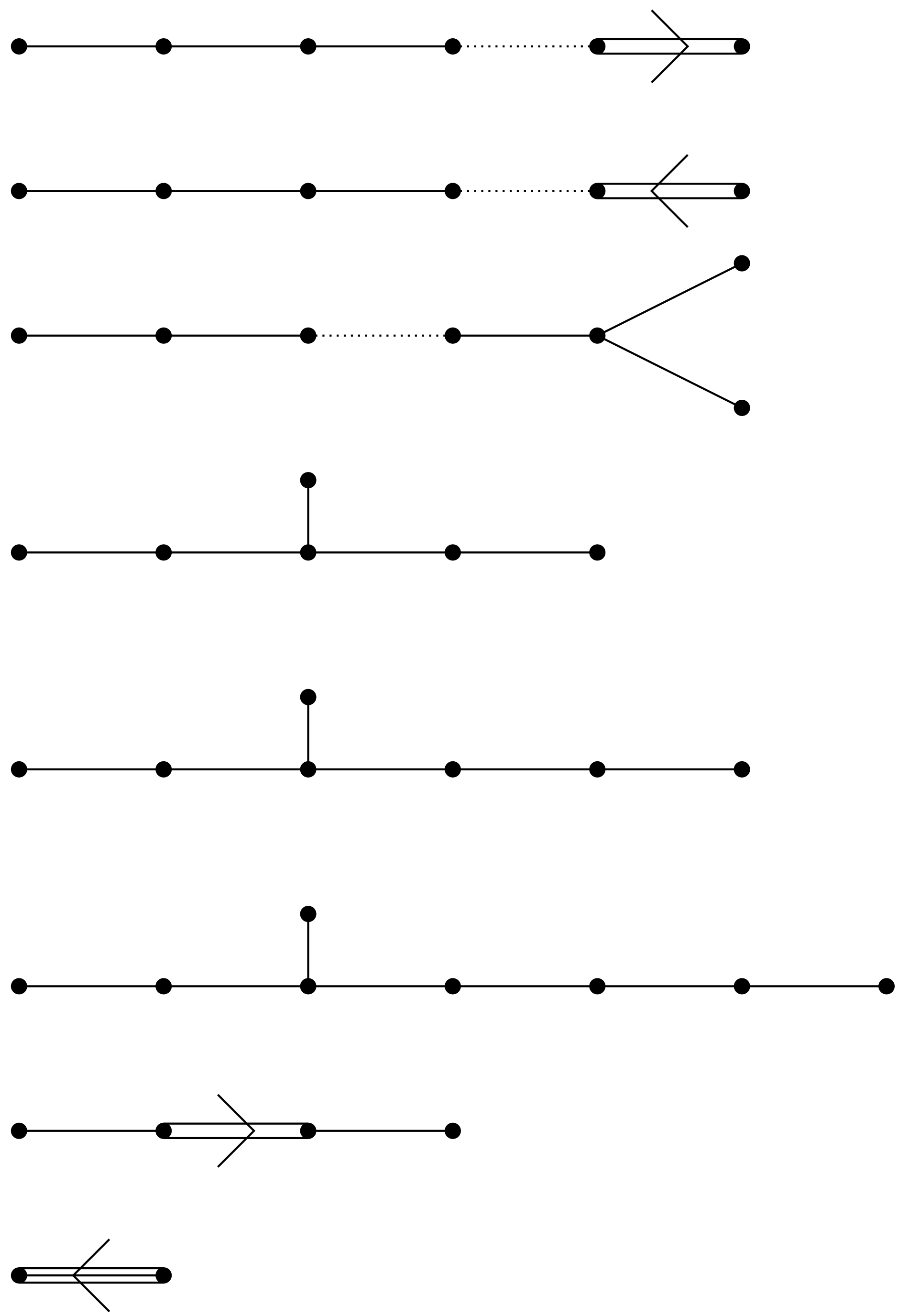
   ii) $W_I = W_J$.

   iii) $P_I = P_J$.

Notice that by Theorem 2.2.6 the P_I are the parabolic subgroups of G.

2.3 Reductive Monoids

Let S be a nonempty set with an associative binary operation $\cdot$. Then the set S is a called a *semigroup*. If there exists an element $1 \in S$ such that $1 \cdot x = x \cdot 1 = x$ for all $x \in S$, then 1 is called an *identity element* of S and S is called a *monoid*. An invertible element of a monoid M is called a *unit*. The set of units forms a group and this group of units is denoted by G. Notice that the identity element $1 \in M$ is a unit and therefore G is necessarily nonempty. If there exists an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ for all $x \in S$, then 0 is called a *zero element* of S and S is called a *semigroup with 0*. Any semigroup without an identity element can be turned into a monoid $S \cup \{1\}$ by adjoining an element 1 to S and defining $1 \cdot x = x \cdot 1 = x$ for all $x \in S$ and $1 \cdot 1 = 1$. The set $S \cup \{1\}$ is often denoted S^1 and $S^1 = S$ if S is a monoid. Similarly, if S is a semigroup without zero then we can adjoin a zero element 0 by defining $0 \cdot x = x \cdot 0 = 0$



$A_n (n \geq 1)$:

$B_n (n \geq 2)$:

$C_n (n \geq 3)$:

$D_n (n \geq 4)$:

E_6 :

E_7 :

E_8 :

F_4 :

G_2 :

Figure 2.1: Dynkin diagrams of simple algebraic groups

for all $x \in S$ and $0 \cdot 0 = 0$. Identity elements and zero elements of a semigroup S are unique and distinct provided $|S| > 1$.

We can think of a monoid as being like a group where not all of the elements are necessarily invertible. From this perspective we can consider the analogy that a monoid is to a group as a ring is to a field, and that, as is the case when comparing rings with fields, the structure of monoids differs greatly from that of groups.

Example 2.3.1. Let $2\mathbb{Z}$ be the set of even integers under multiplication. $2\mathbb{Z}$ is a semigroup with zero element 0 because $2\mathbb{Z}$ is closed under multiplication and the multiplication is associative. It is not a monoid because it does not have an identity element. The set $2\mathbb{Z} \cup \{1\}$, however, is a monoid under multiplication.

Example 2.3.2. The set $2\mathbb{Z}$ under addition is a monoid because the addition is associative and $0 \in 2\mathbb{Z}$ is an additive identity.

Notice that in the previous example the group of units is $G = 2\mathbb{Z}$. That is, the monoid is actually a group. In fact, all groups are monoids. However, no interesting insight can be gleaned from viewing a group as a monoid. It is therefore clear that we should focus our attention on monoids that are not actually groups.

Example 2.3.3. Consider $M_n(k)$, the set of $n \times n$ matrices whose entries are from the algebraically closed field k. Then $M_n(k)$ paired with matrix multiplication is a semigroup because matrix multiplication is associative. Notice that $M_n(k)$ contains I_n, the $n \times n$ identity matrix. Therefore $M_n(k)$ is actually a monoid. However, $M_n(k)$ is not a group because it contains matrices of determinant 0. The group of units is $G = GL_n(k)$, the set of invertible $n \times n$ matrices. Notice that $\overline{G} = M_n(k)$.

Let $x, y \in S$. *Green's Relations* are as follows:

a) $x\mathcal{R}y$ if $xS^1 = yS^1$.

b) $x\mathcal{L}y$ if $S^1x = S^1y$.

c) $x\mathcal{J}y$ if $S^1xS^1 = S^1yS^1$.

d) $x\mathcal{H}y$ if $x\mathcal{R}y$ and $x\mathcal{L}y$.

e) $x\mathcal{D}y$ if $x\mathcal{R}z$ and $z\mathcal{L}y$ for some $z \in S^1$.

Example 2.3.4. Let $a, b \in M = M_n(k)$. $a\mathcal{L}b$ if and only if a and b are row equivalent. $a\mathcal{R}b$ if and only if a and b are column equivalent. $a\mathcal{J}b$ if and only if $\text{rank}(a) = \text{rank}(b)$. Furthermore $\mathcal{J} = \mathcal{D}$.

Green's Relations are useful in describing the structure of semigroups. We will primarily be concerned with the $\mathcal{J}$-relation. We will see in Chapter 3 how this relation gives rise to the cross section lattice of a reductive monoid.

A *linear algebraic monoid* M is an affine variety with an associative morphism $\mu : M \times M \to M$ and an identity element $1 \in M$. M is *irreducible* if it cannot be expressed as the union of two proper closed subsets. The *irreducible components* of M are the maximal irreducible subsets of M. There is a unique irreducible component M° of M that contains the identity. In this case $M^\circ = \overline{G^\circ}$, the Zariski closure of the identity component of G. In particular if M is an irreducible linear algebraic monoid, then $M = \overline{G}$. It is therefore easy for us to generate monoids from their group of units. In fact, the structure of the monoid is determined in part by the structure of its group of units. This phenomenon is examined in more detail in Chapter 3.

Theorem 2.3.5. *Let M be a linear algebraic monoid. Then M is isomorphic to a closed submonoid of some $M_n(k)$.*

Theorem 2.3.5 tells us that we can view any linear algebraic monoid in terms of matrices. This is particularly convenient when constructing examples. It is for this reason that we think of the group of units G as a linear algebraic monoid rather than an affine algebraic monoid.

Let M be a linear algebraic monoid. The *set of idempotents* of M is $E(M) = \{e \in M \mid e^2 = e\}$. M is *regular* if for each $a \in M$ there exists $x \in M$ such that $axa = a$. M is *unit regular* if $M = GE(M) = E(M)G$. The property of being unit regular is quite desirable because it allows us to build the monoid given the group of units and the set of idempotents. If M is a regular irreducible linear algebraic monoid, then it is unit regular.

An irreducible linear algebraic monoid is *reductive* if its group of units G is a reductive algebraic group. If M has a zero element, then M is reductive if and only if M is regular. Reductive monoids are then determined by the group of units and the set of idempotents. As a consequence the structure of linear algebraic monoids is most interesting when the group of units G is a reductive algebraic group. We shall therefore only consider reductive monoids and it will be understood from this point forward that, unless otherwise stated, we mean "reductive monoid" whenever we use the term "monoid," whether the word "reductive" is omitted for the sake of terminological brevity or as a result of carelessness on the part of the author.

2.4 Lattices

In Chapter 3 we wish to describe the structure of the $G \times G$ orbits of a reductive monoid as a lattice. In this section we therefore collect all of the required notions from lattice theory that will be used not only in the next chapter, but throughout the remainder of the paper.

A *partially ordered set*, or *poset* for short, is a set P with a binary relation $\leq$ satisfying the following axioms:

a) *Reflexivity*: $x \leq x$ for all $x \in P$.

b) *Antisymmetry*: If $x \leq y$ and $y \leq x$, then $x = y$.

c) *Transitivity*: If $x \leq y$ and $y \leq z$, then $x \leq z$.

A poset is *finite* if it has finitely many elements. We will only consider finite posets. The *dual* of a poset P, denoted P^*, is the poset with the underlying set as P such that $x \leq y$ in P^* if and only if $y \leq x$ in P.

If $x \leq y$ but $x \neq y$, then we write $x < y$. To elements x and y of P are *comparable* if either $x \leq y$ or $y \leq x$; otherwise they are said to be *incomparable*. If $x < y$ and there is no $z \in P$ such that $x < z < y$, then y is said to *cover* x. P has a *least element*, denoted $\hat{0}$, if there exists an element $\hat{0} \in P$ such that $\hat{0} \leq x$ for all $x \in P$. Similarly, P has a *greatest element*, denoted $\hat{1}$, if there exists $\hat{1} \in P$ such that $x \leq \hat{1}$ for all $x \in P$. Two posets P and Q are *isomorphic* if there exists a bijection $\phi : P \to Q$ such that both ϕ and its inverse are order-preserving, that is, $x \leq y$ in P if and only if $\phi(x) \leq \phi(y)$ in Q.

Example 2.4.1. The set $[n] = \{1, 2, \ldots, n\}$ of the first n natural numbers forms a poset with order relation $\leq$. All elements of $[n]$ are comparable and y covers x if and only if $y = x + 1$. The least element of $[n]$ is $\hat{0} = 1$ and the greatest element is $\hat{1} = n$.

Given two elements x and y of a poset P, $z \in P$ is an *upper bound* if $x \leq z$ and $y \leq z$. z is a *least upper bound* if there does not exist an upper bound w such that $w < z$. If the least upper bound of x and y exists, it is denoted $x \vee y$ and is called the *join* of x and y. Lower bounds and greatest lower bounds can be defined similarly. If the greatest lower bound of x and y exists, it is denoted $x \wedge y$ and is called the *meet* of x and y. Joins and meets, if they exist, are unique. An element $x \in P$ is *join-irreducible* if x cannot be written as a join of two elements y and z where $y < x$ and $z < x$. A *lattice* is a poset L in which every pair of elements has a least upper bound and a greatest lower bound, both of which are in L. An *atom* of L is an element that covers $\hat{0}$. A *coatom* is an element that is covered by $\hat{1}$.

Example 2.4.2. Let S be a finite set of n elements. Let B_n be the set of subsets of S with the partial order given by $X \leq Y$ if and only if $X \subseteq Y$. Then B_n is a lattice called a *Boolean* lattice. Notice that as a set $B_n \cong 2^S$, the power set of S. The least element of B_n is $\hat{0} = \emptyset$ and the greatest element is $\hat{1} = S$. If X and Y are elements of B_n (and hence subsets of S), the join of X and Y is $X \vee Y = X \cup Y$ while the meet is $X \wedge Y = X \cap Y$.

An *(induced) subposet* of P is a subset Q of P with the same partial order as P. That is, Q is an induced subposet if $Q \subseteq P$ and if $x \leq y$ in Q, then $x \leq y$ in P. When discussing induced subposets, the word "induced" is often omitted. A *sublattice* M of a lattice L is a subposet that is closed under the operations of $\vee$ and $\wedge$. The operations $\vee$ and $\wedge$ are commutative, associative, and idempotent, that is, $x \wedge x = x \vee x = x$ for all $x \in L$. An important example of a subposet is the *interval* $[x, y] = \{z \in P \mid x \leq z \leq y\}$, defined whenever $x \leq y$.

Example 2.4.3. Let S be a finite set of n elements and $X \subset S$ be a proper, nonempty subset. Let $X^c = S \backslash X$ be the complement of X. Then $\{\emptyset, X, X^c, S\}$ is a sublattice of B_n since $X \vee X^c = X \cup X^c = S$ and $X \wedge X^c = X \cap X^c = \emptyset$. $[\hat{0}, X]$ is the sublattice of all subsets of X. Notice that $[\hat{0}, X] \cong B_{|X|}$.

We can represent the elements of a poset P along with the cover relations pictorially in a *Hasse diagram.* The Hasse diagram is a graph whose vertices are the elements of P. The vertices corresponding to two elements $x, y \in P$ are connected by an edge if and only if y covers x, in which case y is drawn "above" x. If they exist, $\hat{0}$ will be at the bottom of the Hasse diagram and $\hat{1}$ will be at the top.

Example 2.4.4. Let $S = \{x, y, z\}$ and $B_3 \cong 2^S$ be a Boolean lattice whose order relations are determined by set inclusion. The minimal element ofB_3 is $\hat{0} = \emptyset$, which will be at the bottom of the Hasse diagram. $\emptyset$ is covered by the one element subsets $\{x\}$, $\{y\}$, and $\{z\}$. These subsets all appear above $\emptyset$ in the Hasse diagram and all three are connected to $\emptyset$ by an edge. $\{x, y\}$ and $\{x, z\}$ cover $\{x\}$ so they are both above $\{x\}$ in the Hasse diagram and are connected to $\{x\}$ by an edge. This process is repeated until all cover relations are represented. The Hasse diagram is depicted in Figure 2.2. Notice that, for example, there is no edge connecting $\{x\}$ and $\{y, z\}$ because these two elements are incomparable. Additionally, $\{x\}$ and $\{x, y, z\}$ are comparable but not connected by an edge because $\{x, y, z\}$ does not cover $\{x\}$.

A *chain*, or *totally ordered set*, is a poset in which any two elements are comparable. A subset C of a poset P is called a *chain* if it is a chain when thought of as a subposet of P. The chain of n elements is denoted C_n. The *length* of a finite chain C_n, denoted $\ell(C_n)$, is defined by $\ell(C_n) = |C_n| - 1 = n - 1$. The *rank* of a finite poset P is the maximum of the length of all chains in P. If every maximal chain has the same length n, then P is said to be *graded of rank* n. In this case there exists a unique rank function $\rho : P \to \{0, 1, \ldots, n\}$ such that $\rho(x) = 0$ if x is a minimal element of P, and $\rho(y) = \rho(x) + 1$ is y covers x. If $\rho(x) = i$, then we say that the *rank* of x is i and the *corank* of x is $n - i$. Notice that the corank of x is the rank of x in the dual P^*. Let p_i be the number of elements of P of rank i. P is said to be *rank symmetric* if $p_i = p_{n-i}$ for all i. P is *locally rank symmetric* if every interval in P is rank symmetric.

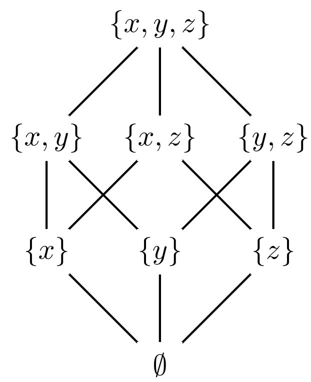


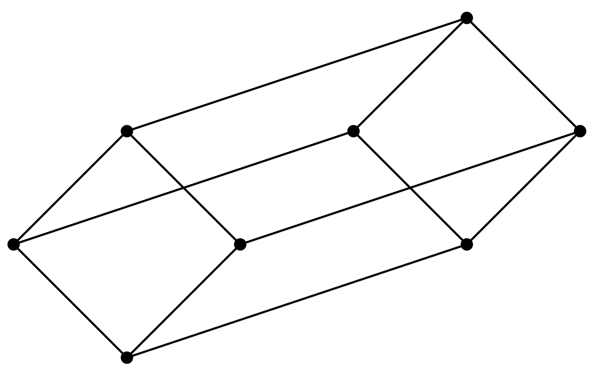
Figure 2.2: Hasse diagram of B_3

Example 2.4.5. $[n]$ is a chain because any two elements of $[n]$ are comparable. The length of a maximal chain is $n-1$, so $[n]$ is graded of rank $n-1$. For $i \in [n]$, $\rho(i) = i-1$. Notice that $[n] \cong C_n$.

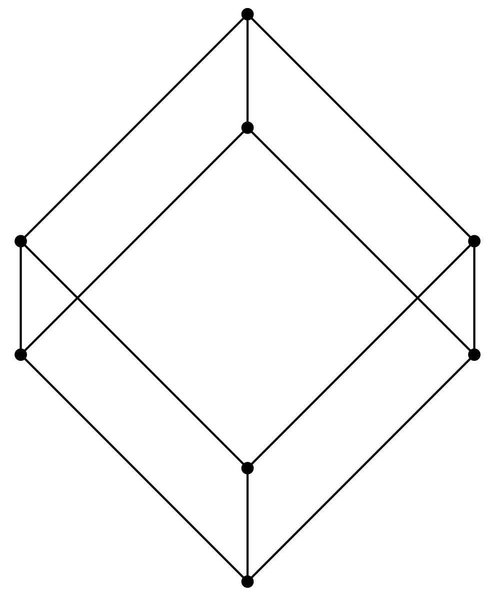
Example 2.4.6. The Boolean lattice $B_n \cong 2^{[n]}$ is not a chain. Consider, for example, $\{1\}, \{2\} \subseteq 2^{[n]}$. $\{1\} \not\subseteq \{2\}$ and $\{2\} \not\subseteq \{1\}$, and hence $\{1\}$ and $\{2\}$ are incomparable. The sublattice $\{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \ldots, [n]\}$ is a chain of length n. All maximal chains have length n, so $2^{[n]}$ is graded of rank n. If $X \subseteq 2^{[n]}$, then $\rho(X) = |X|$.

Given two posets P and Q, there are several ways we can build new posets, two of which will be of interest to us. The *direct product* of P and Q, denoted $P \times Q$, is the poset whose set of elements is $\{(x, y) \mid x \in P, y \in Q\}$ with order relation $(x, y) \leq (x', y')$ in $P \times Q$ if $x \leq x'$ in P and $y \leq y'$ in Q. The direct product of P with itself n times is denoted P^n. The Hasse diagram of $P \times Q$ can be drawn by placing a copy Q_x of Q at every vertex of P and then connecting the corresponding vertices of Q_x and Q_y if and only if y covers x in P. $P \times Q$ and $Q \times P$ are isomorphic, although it may not be immediately clear by looking at their respective Hasse diagrams.

Example 2.4.7. The Hasse diagram of $B_2 \times C_2$ is shown in Figure 2.3a while the Hasse diagram of $C_2 \times B_2$ is shown in Figure 2.3b. Comparing the Hasse diagram of $C_2 \times B_2$ with that of B_3 in Figure 2.2 makes it clear that $C_2 \times B_2 \cong B_3$. This is not as clear by comparing the Hasse diagrams of $B_2 \times C_2$ and B_3. This example shows that the Hasse diagrams of isomorphic posets may look different although they are isomorphic as graphs; this is particularly true as the number of elements of the posets increases. It will be advantageous for us to draw the Hasse diagram for this example to look like that of B_3 in Figure 2.2 as it emphasizes the rank of elements as we move from the bottom of the lattice to the top.



(a) $B_2 \times C_2$



(b) $C_2 \times B_2$

Figure 2.3: The direct product of lattices is commutative

Example 2.4.8. $B_n \cong C_2^n$.

The *ordinal sum* of two posets P and Q , denoted $P \oplus Q$ is the poset whose set of elements is $P \cup Q$ with order relation $x \leq y$ in $P \oplus Q$ if (a) $x, y \in P$ and $x \leq y$ in P, or (b) $x, y \in Q$ and $x \leq y$ in Q, or (c) $x \in P$ and $y \in Q$. In general, the ordinal sum of two posets is not commutative.

Example 2.4.9. The n element chain $C_n \cong C_1 \oplus \cdots \oplus C_1$ (n times).

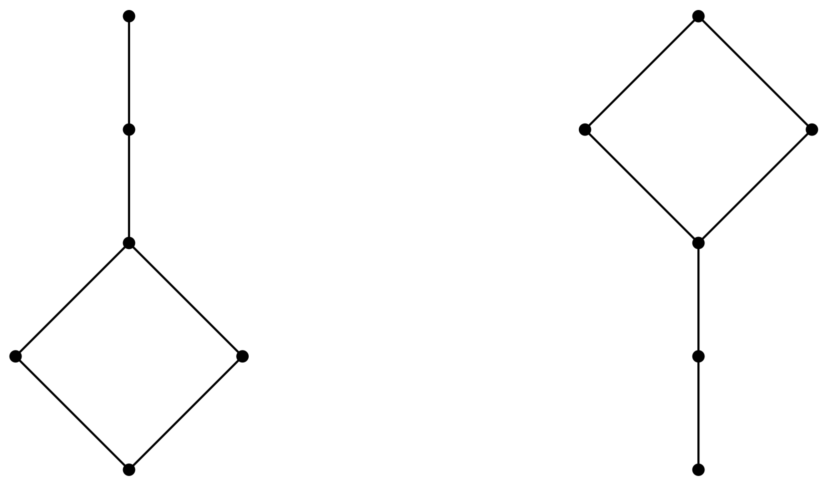
Example 2.4.10. The Hasse diagrams of $B_2 \oplus C_2$ and $C_2 \oplus B_2$ are shown in Figure 2.4. Notice that these Hasse diagrams are different and hence $B_2 \oplus C_2 \not\cong C_2 \oplus B_2$.

Theorem 2.4.11. *Let L be a finite lattice. The following are equivalent.*

a) L is graded, and the rank function ρ satisfies $\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y)$ for all $x, y \in L$.

b) If x and y both cover $x \wedge y$, then $x \vee y$ covers both x and y.

A finite lattice satisfying either of the properties of Theorem 2.4.11 is said to be *upper semimodular*. A finite lattice L is *lower semimodular* if its dual L^* is upper semimodular. L is *modular* if it is both upper semimodular and lower semimodular. That is, $\rho(x) + \rho(y) =$



(a) $B_2 \oplus C_2$ (b) $C_2 \oplus B_2$

Figure 2.4: The ordinal sum of lattices is not commutative

$\rho(x \wedge y) + \rho(x \vee y)$ for all $x, y \in L$. Equivalently, L is modular if and only if for all $x, y, z \in L$ such that $x \leq z$, we have $x \vee (y \wedge z) = (x \vee y) \wedge z$. This condition allows us to extend the idea of a modular lattice to lattices that are not graded.

A lattice L is *distributive* if the meet operation distributes over the join. That is, for all $x, y, z \in L$ we have $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. Equivalently, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Notice that if L is distributive and $x \leq z$, then

$$\begin{aligned} x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \\ &= (x \vee y) \wedge z. \end{aligned}$$

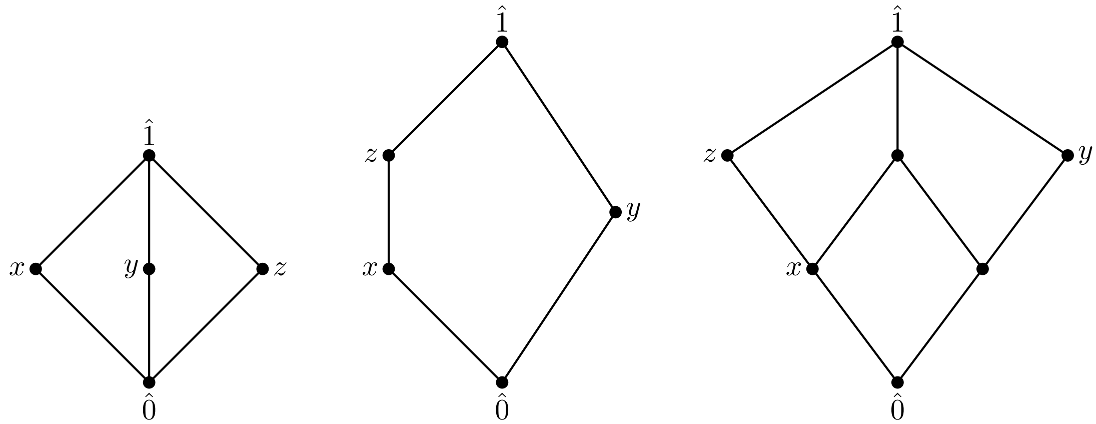
So all distributive lattices are modular. The converse, however, is not true.

Example 2.4.12. The Boolean lattice B_n is distributive and hence modular.

Example 2.4.13. The Hasse diagrams of three important lattices are show in Figure 2.5.

a) The lattice M_5 in Figure 2.5a is modular. Since M_5 is graded, this is easily seen by verifying that $\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y)$ for all $x, y \in L$. M_5, however, is not distributive. This is seen by noticing that $x \vee (y \wedge z) = x \neq \hat{1} = (x \vee y) \wedge (x \vee z)$.

b) The lattice N_5 in Figure 2.5b is not graded. Furthermore, $x \leq z$ and $x \vee (y \wedge z) = x \neq z = (x \vee y) \wedge z$. Therefore N_5 is not modular and hence it is not distributive. N_5 is the smallest non-modular lattice.



(a) M_5 (b) N_5 (c) N_7

Figure 2.5: Some important nondistributive lattices

c) The lattice N_7 in Figure 2.5c is graded and has $\{\hat{0}, x, y, z, \hat{1}\} \cong N_5$ as a sublattice. Therefore N_7 is neither modular nor distributive by the same argument as in part b).

It can often be tedious to check whether or not a lattice is modular or distributive by checking the respective conditions. This is particularly true when a lattice has many elements, only a few of which may not satisfy these conditions. The following theorem will give us a more efficient means to determine whether or not a lattice is modular or distributive:

Theorem 2.4.14. *Let L be a finite lattice.*

a) L is modular if and only if no sublattice is isomorphic to N_5.

b) L is distributive if and only if no sublattice is isomorphic to either M_5 or N_5.

Example 2.4.15. With Theorem 2.4.14 in tow it is now trivial that M_5 and N_5 are not distributive and that M_5 is modular while N_5 is not. Additionally, N_7 is neither modular nor distributive since it has N_5 as a sublattice.

Let L be a lattice with $\hat{0}$ and $\hat{1}$. The *complement* of $x \in L$, if it exists, is an element $y \in L$ such that $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. L is said to be *complemented* if every element of L has a complement. If every interval $[x, y]$ in L is complemented, then L is said to be *relatively complemented.*

Example 2.4.16. Let S be a set of n elements. Let $A \subseteq 2^S \cong B_n$. $A \wedge (S \backslash A) = \emptyset$ and $A \vee (S \backslash A) = S$ and hence B_n is complemented. Furthermore, any subinterval of B_n is isomorphic to a Boolean lattice. Therefore B_n is relatively complemented. In fact, a distributive lattice is complemented if and only if it is bounded and relatively complemented.

Table 2.1: Möbius function of N_5

	$\hat{0}$	x	y	z	$\hat{1}$
$\hat{0}$	1	-1	-1	0	1
x	-	1	-	-1	0
y	-	-	1	-	-1
z	-	-	-	1	-1
$\hat{1}$	-	-	-	-	1

Example 2.4.17. N_5 is complemented but it is not relatively complemented. To see this, consider the interval $[\hat{0}, z]$ of N_5 as in Figure 2.5b. The element $x \in [\hat{0}, z]$ does not have a complement so the interval $[\hat{0}, z]$ is not complemented.

The *Möbius function* of a poset P is defined by

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \leq z < y} \mu(x,z) & \text{for all } x < y \text{ in } P \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Example 2.4.18. Consider the lattice N_5 in Figure 2.5b. In order to calculate $\mu(\hat{0}, \hat{1})$ we can use equation (2.1) but we have to do so inductively:

$$\begin{aligned} \mu(\hat{0}, \hat{0}) &= 1 \\ \mu(\hat{0}, x) &= -\mu(\hat{0}, \hat{0}) = -1 \\ \mu(\hat{0}, y) &= -\mu(\hat{0}, \hat{0}) = -1 \\ \mu(\hat{0}, z) &= -(\mu(\hat{0}, \hat{0}) + \mu(\hat{0}, x)) = 0 \\ \mu(\hat{0}, \hat{1}) &= -(\mu(\hat{0}, \hat{0}) + \mu(\hat{0}, x) + \mu(\hat{0}, y) + \mu(\hat{0}, z)) = 1 \end{aligned}$$

All values of the Möbius function $\mu(a, b)$ are given by Table 2.1. The dashes in the table indicate that the Möbius function is not defined. For example, $\mu(x, y)$ is not defined because x and y are incomparable.

Let P be a graded poset with $\hat{0}$ of rank n. The *characteristic polynomial* of P is

$$\chi(P, x) = \sum_{y \in P} \mu(\hat{0}, y) x^{n-\rho(y)} = \sum_{i=0}^{n} w_i x^{n-i}.$$

The coefficient w_i is the *i-th Whitney number of P of the first kind*,

$$w_i = \sum_{\substack{y \in P \\ \rho(y)=i}} \mu(\hat{0}, y)$$

The characteristic polynomial is an important invariant of a poset that is used in the study of arrangements of hyperplanes in vector spaces.

Example 2.4.19. Let S be a set of n elements so $B_n \cong 2^S$. Let A and B be two comparable elements of B_n with $A \leq B$. Then $\mu(A, B) = (-1)^{|B|-|A|}$. In particular, $\mu(\hat{0}, A) = (-1)^{|A|}$. Since there are $\binom{n}{i}$ elements of B_n of rank i, $w_i = (-1)^i \binom{n}{i}$. Therefore

$$\chi(B_n, x) = \sum_{i=0}^{n} w_i x^{n-i} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^{n-i} = (x-1)^n$$

by the Binomial Theorem.

A multiset is a set-like object where the multiplicity of each element can be greater than one and is significant. For example, $\{1, 1, 2, 2, 3\}$ is a multiset which is different from the multiset $\{1, 1, 2, 3\}$. The number of multisets of k elements chosen from a set of n elements is denoted by $\left(\!\binom{n}{k}\!\right)$, read "$n$ multichoose k". The following formula can be used to count the number of multisets:

$$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k} \tag{2.2}$$

A *multichain* of a poset P is a chain with repeated elements. A multichain is then just a multiset of a chain. A *multichain of length* n is a sequence of elements $x_0 \leq x_1 \leq \cdots \leq x_n$ of P.

Let P be a finite poset. If $n \geq 2$, define $Z(P, n)$ to be the number of multichains $x_1 \leq x_2 \leq \cdots \leq x_{n-1}$. $Z(P, n)$ is a polynomial in n and is called the *zeta polynomial* of P. The zeta polynomial has the following properties: $Z(P, -1) = \mu(\hat{0}, \hat{1})$, $Z(P, 0) = 0$, $Z(P, 1) = 1$, $Z(P, 2)$ is the number of vertices of P, and $Z(P, 3)$ is the number of total relations in P.

Theorem 2.4.20. *Let P be a finite poset.*

a) Let b_i be the number of chains $x_1 < x_2 < \cdots < x_{i-1}$ in P. Then

$$Z(P, n) = \sum_{i \geq 2} b_i \binom{n-2}{i-2}. \tag{2.3}$$

b) $Z(P \times Q, n) \cong Z(P, n) Z(Q, n)$

Example 2.4.21. Consider $P = B_2$. There are two chains of length 1 and one chain of length 2. By equation 2.3,

$$Z(C_2, n) = 2\binom{n-2}{0} + \binom{n-2}{1} = 2 + n - 2 = n.$$

Since $B_k \cong C_2^k$, $Z(B_n) = n^k$ by Theorem 2.4.20.

Example 2.4.22. Let $P = N_5$. Then

$$Z(N_5, n) = 5\binom{n-2}{0} + 8\binom{n-2}{1} + 5\binom{n-2}{2} + \binom{n-2}{3} = \frac{1}{6}n^3 + n^2 - \frac{1}{6}n.$$

Notice that $\mu(\hat{0}, \hat{1}) = Z(N_5, -1) = 1$, which agrees with the value calculated in Example 2.4.18. $Z(N_5, 2) = 5$, the number of vertices of N_5. $Z(N_5, 3) = 13$, the total number of relations in N_3.

The following definitions are from Stanley [18]. A modular lattice L is said to be a *q-lattice* if every interval of rank two is either a chain or has $q + 1$ elements of rank one. A 0-lattice is a chain and a modular 1-lattice is the same thing as a distributive lattice. q-lattices can be defined for nonmodular lattices, but they will not be of interest to us in this paper; the interested reader should consult [18] for the definition. A lattice L is *semiprimary* if L is modular and whenever $x \in L$ is join-irreducible then the interval $[\hat{0}, x]$ is a chain. A semiprimary lattice is *primary* if every interval is either a chain or contains at least three atoms. L is *q-primary* if it is both a q-lattice and a primary lattice.

The following result is due to Regonati [12]:

Theorem 2.4.23. *Let L be a finite modular lattice. L is locally rank symmetric if and only if L can be written as a direct product of q-primary lattices.*

Example 2.4.24. The chain C_n is a 0-lattice. It is also a semiprimary lattice since C_n is modular and $[\hat{0}, x]$ is a chain for all $x \in C_n$. C_n is also primary since every interval is a chain. Therefore C_n is a 0-primary lattice.

Example 2.4.25. Consider the lattice L whose Hasse diagram is shown in Figure 2.6. L is modular by Theorem 2.4.14 and hence it is a 0-lattice. The join-irreducible elements of L are $\hat{0}$, x, y, and z. The intervals $[\hat{0}, \hat{0}]$, $[\hat{0}, x]$, $[\hat{0}, y]$, and $[\hat{0}, z]$ are all chains, so L is a semiprimary lattice. The interval $[x, \hat{1}]$, however, is not a chain and it has only two atoms, y and z. Therefore L is not primary and hence is not a q-primary lattice. Notice that L is not rank symmetric so it is not locally rank symmetric, and hence with Theorem 2.4.23 in mind our conclusion should not be terribly surprising.

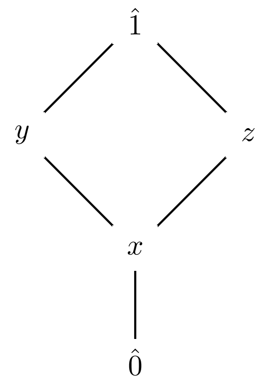


Figure 2.6: A lattice that is not q-primary

Chapter 3

Cross Section Lattices

With a modest knowledge of the theory of reductive monoids and lattices, we are now ready to introduce the idea of a cross section lattice. Whenever possible the original citations for all results have been included. The reader should be aware, however, that almost all results can be found in one or more of [7], [14], and [16].

3.1 Cross Section Lattices

Throughout we will assume that M is a reductive monoid with 0.

Let M be a reductive monoid with 0 with group of units G and set of idempotents $E(M)$. We saw in Chapter 2 that M is unit regular and hence $M = GE(M) = E(M)G$. That is, the structure of a reductive monoid is determined by the group of units and the idempotent elements. This is our first hint that the structure of reductive monoids is worth investigating in more detail. It turns out, however, that we can do more. Our goal is to describe the structure of the $G \times G$ orbits of M. From this we can create the type map from which M can be constructed up to a central extension.

Suppose $a, b \in M$. Then

$$a\mathcal{J}b \iff GaG = GbG \iff MaM = MbM.$$

That is, two elements of M are in the same $\mathcal{J}$-class if and only if they are in the same $G \times G$ orbit. Furthermore, we can define a partial order the $\mathcal{J}$-classes as follows:

$$\mathcal{J}_a \leq \mathcal{J}_b \iff GaG \subseteq \overline{GbG} \iff a \in MbM,$$

where $\mathcal{J}_a$ and $\mathcal{J}_b$ denote the $\mathcal{J}$-classes of two elements a and b of M, respectively, and the closure of GbG is in the Zariski topology. The following theorem is due to Putcha [7]:

Theorem 3.1.1. *Let M be a reductive monoid. Let $\mathcal{U}(M)$ denote the set of $\mathcal{J}$-classes of M. $\mathcal{U}(M)$ is a finite lattice with the partial order defined above.*

Fix a Borel subgroup B of G and a maximal torus T contained in B. We can define a partial order on $E(\overline{T})$, the set of idempotents of $\overline{T}$, as follows: Let $e, f \in E(\overline{T})$.

$$e \leq f \iff ef = fe = e.$$

We are now ready to define the cross section lattice of a reductive monoid. Cross section lattices were first introduced in 1983 by Putcha [6].

Definition 3.1.2. Let M be a reductive monoid with unit group G and maximal torus T contained in a Borel subgroup B of G. Then $\Lambda \subseteq E(\overline{T})$ is a *cross section lattice* of M relative to B and T if

a) $|\Lambda \cap J| = 1$ for all $J \in \mathcal{U}(M)$, and

b) If $e, f \in \Lambda$ then $J_e \leq J_f$ if and only if $e \leq f$.

The cross section lattice of M is then an order preserving cross section of the $\mathcal{J}$-classes of M where each $\mathcal{J}$-class is represented by an idempotent. Since the $G \times G$ orbit of 0 is of little interest, we will often be concerned with finding $\Lambda\backslash\{\hat{0}\}$ rather than Λ. Omitting $\hat{0}$ from our discussion of the cross section lattice will also often make the statements of theorems a little cleaner. The downside, of course, is that $\Lambda\backslash\{\hat{0}\}$ is not actually a lattice unless there is a minimal nonzero element of Λ. See Section 3.2 for details. We hope the reader will agree, however, that this does not raise any significant difficulties.

The following was first observed by Putcha in [5]:

Theorem 3.1.3. *Let M be a reductive monoid with maximal torus T contained in a Borel subgroup B of G. Let $W = N_G(T)/T$ be the Weyl group. Then*

a) Cross section lattices exist.

b) Any two cross section lattices are conjugate by an element of W.

c) There is a one-to-one correspondence between the cross section lattices and Borel subgroups of G containing T.

Example 3.1.4. Let $M = M_n(k)$. Then $G = GL_n(k)$. Choose a maximal torus $T = D_n(k)$ contained in the Borel subgroup $B = B_n(k)$. $\overline{T} = \overline{D_n(k)}$, the set of diagonal matrices. Then the set of idempotents is $E(\overline{T}) = \{(a_{ij}) | a_{ij} = 0 \text{ if } i \neq j \text{ and } a_{ij} = 0 \text{ or } 1 \text{ if } i = j\}$. By Example 2.3.4 two matrices are in the same $\mathcal{J}$-class if and only if they have the same rank. Let $e_k = I_k \oplus 0_{n-k}$, where I_k is the $k \times k$ identity matrix and 0_{n-k} is the $(n-k) \times (n-k)$ zero matrix. $\Lambda \backslash \{\hat{0}\} = \{e_1, \ldots, e_n\}$ is the cross section lattice of M relative to B and T. Notice that Λ contains exactly one element of rank i for $1 \leq i \leq n$. Furthermore $e_i e_j = e_j e_i = e_i$ for all $i \neq j$. Also notice that the lattice of $\mathcal{J}$-classes $\mathcal{U}$ is isomorphic to the chain C_n.

Example 3.1.5. Let $M = M_n(k)$. Then $G = GL_n(k)$. Choose a maximal torus $T = D_n(k)$ contained in the Borel subgroup $B_n^-(k)$, the set of invertible lower triangular matrices. Let $e'_k = 0_{n-k} \oplus I_k$. Then $\Lambda' \backslash \{\hat{0}\} = \{e'_1, \ldots, e'_n\}$ is the cross section lattice of M relative to B' and T. Notice that the Weyl group is $W = N_G(T)/T \cong S_n$. Let $\sigma \in W$ be given by the $n \times n$ matrix

$$\sigma = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}.$$

Then $\sigma^{-1} \Lambda' \sigma = \Lambda$.

Example 3.1.6. Let $M = M_n(k)$ and $T = D_n(k)$ as in Examples 3.1.4 and 3.1.5. Let $e = I_k \oplus 0_{n-k}$ and $f = 0_k \oplus I_{n-k}$ for some value of k such that $k > n-k$. Suppose $e, f \in \Lambda'' \subseteq E(\overline{T})$. Notice that $ef = fe \neq e$ and $ef = fe \neq f$. Therefore e and f are incomparable in the lattice $E(\overline{T})$. But e and f have different ranks and $\mathcal{U}(M)$ is a chain as seen in Example 3.1.4. Since $n > n-k$ it follows that $J_e > J_f$ and Λ'' cannot be the cross section lattice of M relative to any Borel subgroup B'' and T since Λ'' and $\mathcal{U}(M)$ have a different partial order.

Cross section lattices are not unique but they are related to each other in a very precise way. Since all cross section lattices are conjugate, we are not concerned so much with any particular cross section lattice. In particular, we will fix a Borel subgroup B and a maximal tours T and refer to the corresponding cross section lattice Λ as *the* cross section lattice of M if there is no chance of confusion.

Example 3.1.6 shows that both conditions of Definition 3.1.2 must hold in order for $\Lambda \subseteq E(\overline{T})$ to be a cross section lattice. That is, the cross section lattice is not just a cross section of the $\mathcal{J}$-classes, but rather it must *also* preserve the order of $\mathcal{UJ}(M)$. It should therefore not come as a surprise to the reader that the problem of coming up with the cross section lattice of M relative to a given Borel subgroup B and a maximal torus T merely through inspection can be a nontrivial task. Fortunately Putcha [8] noticed the following:

Theorem 3.1.7. *Let M be a reductive monoid with unit group G and maximal torus T contained in a Borel subgroup B of G.*

a) $\Lambda = \{e \in E(\overline{T}) \mid Be = eBe\} \cong G\backslash M/G$.

b) $M = \bigsqcup_{e \in \Lambda} GeG$

c) $P(e) = \{x \in G \mid xe = exe\}$ *is a parabolic subgroup of* G.

Definition 3.1.8. Let M be a reductive monoid with unit group G and maximal torus T contained in a Borel subgroup B. Let Δ be the set of simple roots of G relative to B and T and let S be the corresponding set of simple reflections. Let Λ be the cross section lattice of M.

a) The *type map* is the map $\lambda : \Lambda \to 2^{\Delta}$ where $\lambda(e) \subseteq \Delta$ is the unique subset such that $P(e) = P_{\lambda(e)}$.

b) $\lambda_*(e) = \bigcap_{f \leq e} \lambda(f)$

c) $\lambda^*(e) = \bigcap_{f \geq e} \lambda(f)$

Theorem 3.1.9. *Let* $e, f \in \Lambda$

a) $\lambda(e) = \lambda^*(e) \sqcup \lambda_*(e)$.

b) $\lambda(e) \cap \lambda(f) \subseteq \lambda(e \vee f) \cap \lambda(e \wedge f)$.

c) If $e \leq f$, *then* $\lambda_*(f) \subseteq \lambda_*(e)$ *and* $\lambda^*(e) \subseteq \lambda^*(f)$.

Example 3.1.10. Let $M = M_n(k)$, $B = B_n(k)$, and $T = D_n(k)$. We saw in Example 3.1.4 that the cross section lattice of M is $\Lambda\backslash\{0\} = \{e_1, \ldots, e_n\}$ where $e_k = I_k \oplus 0_{n-k}$ for $1 \leq k \leq n$. The partial order of Λ is $e_0 < e_1 < \cdots < e_n$. The set of simple roots is $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\}$ which is of the type A_{n-1}. The Weyl group is $W \cong S_n = \langle s_1, \ldots, s_{n-1}\rangle$ where $s_i = (s_i\, s_{i+1})$ corresponds to α_i for $1 \leq i \leq n-1$. Let $x = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ where X_1 is a $k \times k$ matrix, X_2 is a $k \times (n-k)$ matrix, X_3 is a $(n-k) \times k$ matrix, and X_4 is a $(n-k) \times (n-k)$ matrix. $xe_k = \begin{pmatrix} X_1 & 0 \\ X_3 & 0 \end{pmatrix}$ and $e_k x e_k = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore if $x \in P(e_k)$, then it must be that X_3 is the $(n-k) \times k$ zero matrix and hence $P(e_k)$ is the set of all matrices of the form $\begin{pmatrix} X_1 & X_2 \\ 0 & X_4 \end{pmatrix}$. Notice that this is a subgroup of upper block triangular matrices which is a parabolic subgroup of $GL_n(k)$ by Example 2.2.4. Therefore $\lambda(e)$ is the subset of S such that the parabolic subgroup $P_{\lambda(e_k)} = P(e_k)$. Recall that $P_{\lambda(e_k)} = BW_{\lambda(e_k)}B$, where $W_{\lambda(e_k)}$ is the subgroup of W generated by $\lambda(e_k)$. $\lambda(e_k)$ is then isomorphic to the set of permutation matrices that permute columns 1 through k as well as columns $k+2$ through n. That is, $\lambda(e_k) = \{\alpha_1, \ldots, \alpha_{k-1}\} \sqcup \{\alpha_{k+1}, \ldots, \alpha_{n-1}\}$. Then $\lambda^*(e_k) = \{\alpha_1, \ldots, \alpha_{k-1}\}$ and $\lambda_*(e_k) = \{\alpha_{k+1}, \ldots, \alpha_{n-1}\}$. Notice that $\lambda(e_k) = \lambda^*(e_k) \sqcup \lambda_*(e_k)$. Additionally, $\lambda^*(e_i) \subseteq \lambda^*(e_j)$ and $\lambda_*(e_j) \subseteq \lambda_*(e_i)$ if $i \leq j$.

The following is from [7].

Theorem 3.1.11. *Let M be a reductive monoid with unit group G and cross section lattice Λ and type map λ. Let $e \in \Lambda$.*

a) Let

$$eMe = \{x \in M \mid x = exe\}.$$

Then eMe is a reductive monoid with group of units $eC_G(e)$. A cross section lattice of eMe is $e\Lambda = \{f \in \Lambda \mid fe = f\}$. λ^* restricted to eMe is the λ^* of eMe.

b) Let

$$M_e = \overline{\{x \in G \mid ex = xe = e\}^0}.$$

Then M_e is a reductive monoid with group of units $\{x \in G \mid ex = xe = e\}^0$. A cross section lattice of M_e is $\Lambda_e = \{f \in \Lambda \mid fe = e\}$. λ_* restricted to M_e is the λ_* of M_e.

Given a reductive monoid M, Theorem 3.1.11 allows us to construct new reductive monoids whose cross section lattices are related to the cross section lattice Λ of M. In particular, the cross section lattice of eMe is isomorphic to the interval $[e, \hat{1}]$ of Λ and the cross section lattice of M_e is isomorphic to the interval $[\hat{0}, e]$ of Λ.

A reductive monoid M is said to be *semisimple* if $dim(Z(G)) = 1$. The following theorem is from [7].

Theorem 3.1.12. *Let M be a semisimple monoid with cross section lattice Λ and set of simple roots Δ relative to a maximal torus T and Borel subgroup B. Then there exists $e_\alpha \in \Lambda \backslash \{\hat{0}\}$ such that $\lambda(e_\alpha) = \Delta \backslash \{\alpha\}$. Moreover, e_α is unique.*

We will now introduce some more terminology concerning cross section lattices that will be useful in Section 3.3. The rest of the material in this section was first presented in [11].

Definition 3.1.13. Let M be a reductive monoid with cross section lattice Λ and let Λ_1 be the set of minimal nonzero elements of Λ.

a) The *core* C of Λ is

$$C = \{e \in \Lambda \mid e = e_1 \vee \cdots \vee c_k, \text{ for some } e_i \in \Lambda_1\}.$$

b) Define $\theta : \Lambda \backslash \{\hat{0}\} \to C$ by $\theta(e) = \vee\{f \in \Lambda_1 \mid f \leq e\}$.

c) Define $\Lambda_h = \theta^{-1}(h)$ for $h \in C$.

The core of Λ is then all of the elements of Λ that can be expressed as the join of one or more minimal nonzero elements of Λ. If $e \in \Lambda_h$ where $h \in C$, then $e \geq h$ and h is the maximal element of the core such that e and h are comparable. Clearly then $\Lambda \backslash \{\hat{0}\} = \bigsqcup_{h \in C} \Lambda_h$. We will see in Section 3.3 that the structure of each Λ_h can vary quite a bit depending upon $h \in C$.

Proposition 3.1.14. *Let M be a reductive monoid with k minimal nonzero elements $e_1, \ldots, e_k$. Suppose $\lambda^*(e_1 \vee \cdots \vee e_k) = \emptyset$. Then $\lambda_*(e_{i_1} \vee \cdots \vee e_{i_t}) = \lambda_*(e_{i_1}) \cap \cdots \cap \lambda_*(e_{i_t})$.*

Proof. Let Λ be the cross section lattice of M and let C be the core. If $h \in C$, then $h \leq e_1 \vee \cdots \vee e_k$. Therefore $\lambda^*(h) \subseteq \lambda^*(e_1 \vee \cdots \vee e_k) = \emptyset$ and hence $\lambda^*(h) = \emptyset$ for all $h \in C$. Then $\lambda(h) = \lambda_*(h)$. We proceed by induction. Suppose $\lambda_*(e_{i_1} \vee \cdots \vee e_{i_{t-1}}) = \lambda_*(e_{i_1}) \cap \cdots \cap \lambda_*(e_{i_{t-1}})$. By Theorem 3.1.9b, $\lambda_*(e_{i_1}) \cap \cdots \cap \lambda_*(e_{i_{t-1}}) \cap \lambda_*(e_{i_t}) = \lambda_*(e_{i_1} \vee \cdots \vee e_{i_{t-1}}) \cap \lambda_*(e_{i_t}) \subseteq \lambda_*(e_{i_1} \vee \cdots \vee e_{i_{t-1}} \vee e_{i_t})$. However, $e_{i_j} \leq e_{i_1} \vee \cdots \vee e_{i_t}$ so $\lambda_*(e_{i_1} \vee \cdots \vee e_{i_t}) \subseteq \lambda_*(e_{i_j})$ for each $1 \leq j \leq t$. Therefore $\lambda_*(e_{i_1} \vee \cdots \vee e_{i_t}) \subseteq \lambda_*(e_{i_1}) \cap \cdots \cap \lambda_*(e_{i_t})$. $\square$

Definition 3.1.15. Let M be a semisimple monoid with cross section lattice Λ.

a) Define $\pi : \Delta \to C$ by $\pi(\alpha) = \theta(e_\alpha)$.

b) Define $\Delta_h = \pi^{-1}(h)$ for $h \in C$.

We will be primarily interested in semisimple monoids as they have the most interesting structure. In this case we are able to partition the set of simple roots by $\Delta = \bigsqcup_{h \in C} \Delta_h$.

The type map is the ultimate combinatorial invariant of a reductive monoid. It allows us to determine the structure of the $G \times G$ orbits and how they can be "pieced together" to build the monoid. From this perspective the type map can be thought of as the monoid version of the Dynkin diagram that describes the structure of algebraic groups and Lie algebras. Our goal is therefore to determine the type map in terms of some minimal information about the monoid. The following theorem provides us with a good start.

Theorem 3.1.16.

a) If $e \in \Lambda_h$, then $\lambda_*(e) = \{\alpha \in \lambda_*(h) \mid s_\alpha s_\beta = s_\beta s_\alpha$ for all $\beta \in \lambda^*(e)\}$.

b) If $e \in \Lambda_h$ and $f \in \Lambda_k$, then $e \leq f$ if and only if $h \leq k$ and $\lambda^*(e) \subseteq \lambda^*(f)$.

Theorem 3.1.16 provides us with two important consequences. If we happen to know $\lambda^*(e)$, then we can determine $\lambda_*(e)$ and hence $\lambda(e) = \lambda^*(e) \sqcup \lambda_*(e)$. Furthermore, we can use λ^* to determine the partial order on Λ. Our main objective then is to determine λ^* for a given monoid. This is a very difficult task in general. In Sections 3.2 and 3.3 we will look at two special cases where we will be able to determine λ^* and hence the cross section lattice and the type map.

3.2 $\mathcal{J}$-irreducible Reductive Monoids

In this section we will examine a special class of monoids with a single minimal nonzero $G \times G$ orbit. The results are precise. Given some minimal invariants of the monoid we will be able to explicitly calculate the type map and cross section lattice. All of the results in the section are due to Putcha and Renner [10].

Definition 3.2.1.

a) A reductive monoid M is *$\mathcal{J}$-irreducible* if there is a unique minimal nonzero element e_0 of the cross section lattice Λ. The *type* of M is $I = \lambda_*(e_0)$.

b) A reductive monoid M is *$\mathcal{J}$-coirreducible* if there is a unique element e^0 of the cross section lattice Λ of corank 1. The *cotype* of M is $J = \lambda^*(e^0)$.

c) M is $\mathcal{J}$-linear if Λ is a chain.

A $\mathcal{J}$-irreducible monoid is a special case of Definition 3.1.13 where $\Lambda_1 = C = \{e_0\}$, $\Lambda_{e_0} = \Lambda \backslash \{0\}$, and $\Delta_{e_0} = \Delta$. Putcha showed in [7] that $\mathcal{J}$-irreducible and $\mathcal{J}$-coirreducible monoids are semisimple. Notice that, by definition, $\mathcal{J}$-linear monoids are $\mathcal{J}$-irreducible, although the converse is not necessarily true.

Notice that if e_0 is a minimal element of the cross section lattice, then by Definition 3.1.8

$$\lambda_*(e_0) = \bigcap_{f \leq e} \lambda(f) = \lambda(e_0).$$

Also, since $\lambda(e_0) = \lambda^*(e_0) \sqcup \lambda_*(e_0)$, it follows that $\lambda^*(e_0) = \emptyset$. Calculating λ^* for the rest of the cross section lattice, however, is not immediately clear. The following theorem allows us to do so in terms of the type I and the set of simple roots Δ.

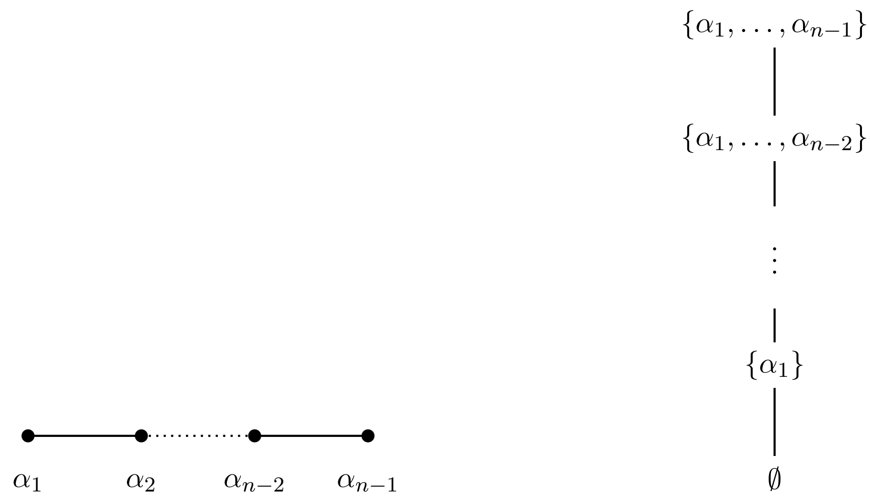
Theorem 3.2.2. *Let M be a $\mathcal{J}$-irreducible monoid of type I and set of simple roots Δ.*

a) Let $X \subseteq \Delta$. $X = \lambda^*(e)$ for some $e \in \Lambda \backslash \{\hat{0}\}$ if and only if no connected component of X lies entirely in I.

b) For any $e \in \Lambda \backslash \{\hat{0}\}$, $\lambda_*(e) = \{\alpha \in I \backslash \lambda^*(e) \mid s_\alpha s_\beta = s_\beta s_\alpha$ for all $\beta \in \lambda^*(e)\}$.

c) λ is injective.

Example 3.2.3. Let $M = M_n(k)$. Then $G = GL_n(k)$. Choose a maximal torus $T = D_n(k)$ contained in the Borel subgroup $B = B_n(k)$. Let $e_k = I_k \oplus 0_{n-k}$. The set of simple roots is $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\}$ and is of type A_{n-1}. We saw in Example 3.1.4 that $\Lambda \backslash \{\hat{0}\} = \{e_1, \ldots, e_n\}$ is the cross section lattice of M with partial order $e_1 < e_2 < \cdots < e_n$. The minimal nonzero element

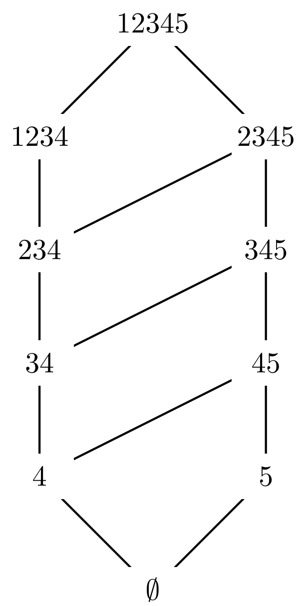


(a) Dynkin diagram of Δ (b) Cross section lattice $\Lambda\backslash\{\hat{0}\}$

Figure 3.1: $\mathcal{J}$-irreducible monoid with $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\}$ and $I = \{\alpha_2, \ldots, \alpha_{n-1}\}$

of Λ is e_1. From Example 3.1.10 we have the type of M is $\lambda_*(e_1) = \{\alpha_2, \ldots, \alpha_{n-1}\}$. To find the subsets $X \subseteq \Delta$ such that $X = \lambda^*(e)$ for some $e \in \Lambda\backslash\{\hat{0}\}$, we need to determine the subsets of Δ that have no connected component contained in $\{\alpha_2, \ldots, \alpha_{n-1}\}$. These are precisely $\emptyset$ and the connected subsets of Δ that contain α_1. That is, $\lambda^*(e_1) = \emptyset$ and $\lambda^*(e_i) = \{\alpha_1, \ldots, \alpha_{i-1}\}$ for $2 \leq i \leq i-1$. This agrees with our calculations from Example 3.1.10. The Dynkin diagram and cross section lattice are shown in Figure 3.1.

Comparing the Dynkin diagram with the cross section lattice in Figure 3.1 probably does not instill much enthusiasm in the reader. After all, they look very similar to each other, except that the cross section lattice is drawn vertically to emphasize the lattice structure. They are not identical, however, as the Dynkin diagram has $n-1$ nodes while the cross section lattice $\Lambda\backslash\{\hat{0}\}$ has n elements. Example 3.2.3 is not particularly interesting, however, and was chosen due to the simplicity of calculating the cross section using several different techniques and not for the aesthetics of the resulting figures. It may not be immediately clear to the reader that Dynkin diagrams and cross section lattices are different enough from each other to warrant further study; a simple realization, however, can do the trick without the need for any explicit calculations. This realization is that the cross section lattice Λ of a reductive monoid is precisely what the name indicates: it is a lattice. Even in the $\mathcal{J}$-irreducible case this lattice can be constructed using the Dynkin diagram of Δ and the type I of the monoid, and Λ will indeed be a lattice regardless of the number of connected components of Δ. That is, Λ is connected when viewed as a graph even when Δ is not. It is the author's hope that this realization along with the



(a) Cross section lattice $\Lambda\backslash\{\hat{0}\}$

$\lambda^*(e)$	$\lambda_*(e)$	$\lambda(e)$
$\emptyset$	123	123
4	12	412
5	123	**1235**
34	1	134
45	12	**1245**
234	$\emptyset$	234
345	1	**1345**
1234	$\emptyset$	**1234**
2345	$\emptyset$	**2345**
12345	$\emptyset$	12345

(b) Type map of M

Figure 3.2: $\mathcal{J}$-irreducible monoid with $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and $I = \{\alpha_1, \alpha_2, \alpha_3\}$

following examples will convince the reader that the topic is worthy of study.

Example 3.2.4. Let M be a $\mathcal{J}$-irreducible monoid of type $I = \{\alpha_1, \alpha_2, \alpha_3\}$ and set simple roots $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, where $s_{\alpha_i} s_{\alpha_j} \neq s_{\alpha_j} s_{\alpha_i}$ if $|i - j| = 1$. Δ is then of the type A_5. The subsets $X \subseteq \Delta$ that are in the image of λ^* are the subsets with no connected component contained in $I = \{\alpha_1, \alpha_2, \alpha_3\}$. These subsets are $\emptyset$, $\{\alpha_4\}$, $\{\alpha_5\}$, $\{\alpha_3, \alpha_4\}$, $\{\alpha_4, \alpha_5\}$, $\{\alpha_2, \alpha_3, \alpha_4\}$, $\{\alpha_3, \alpha_4, \alpha_5\}$, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, and $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. The cross section lattice $\Lambda\backslash\{\hat{0}\}$ is shown in Figure 3.2a, where the vertices of $\Lambda\backslash\{\hat{0}\}$ are labeled by the indices of the respective subsets of Δ.

We can also use Theorem 3.2.2 to determine the subsets $X \subseteq \Delta$ that are in the image of λ_*. For example, suppose $\lambda^*(e) = \{\alpha_4, \alpha_5\}$. Then $\lambda_*(e)$ is all of the elements of $I = \{\alpha_1, \alpha_2, \alpha_3\}$ that are not adjacent to either α_4 or α_5 in the Dynkin diagram of Δ. Therefore $\lambda_*(e) = \{\alpha_1, \alpha_2\}$. The table in Figure 3.2b shows the values of λ^*, λ_*, and λ for each element of $\Lambda\backslash\{\hat{0}\}$ labeled by their respective indices. The bolded entries represent the elements whose type is of the form $\Delta\backslash\{\alpha\}$ for some $\alpha \in \Delta$. Notice that M is semisimple. Also notice that λ is injective although λ_* is not. λ^* will be injective if and only if M is $\mathcal{J}$-irreducible.

Notice that we don't need an explicit description of the $\mathcal{J}$-irreducible monoid M in order to determine λ and Λ. We don't even need to specify the group of units G, maximal torus T, or Borel subgroup B. The only information we need is the type of the minimal idempotent e_0 and the graph structure of the Dynkin diagram of Δ.

3.3 2-reducible Reductive Monoids

As in the $\mathcal{J}$-irreducible case, we can explicitly calculate the type map and cross section lattice of a monoid with two minimal, nonzero elements. The results in this section were developed by Putcha and Renner [11], [14].

Definition 3.3.1. A reductive monoid M is *2-reducible* if there are exactly two minimal nonzero elements e_+ and e_- of the cross section lattice. $I_+ = \lambda_*(e_+)$ and $I_- = \lambda_*(e_-)$ are the *types* of M.

Let M be a 2-reducible monoid with minimal nonzero elements e_+ and e_-. The core of M is $C = \{e_+, e_-, e_0\}$, where $e_0 = e_+ \vee e_-$. The types of M are $I_+ = \lambda_*(e_+)$ and $I_- = \lambda_*(e_-)$. Let $I_0 = \lambda_*(e_0)$. Then $I_0 = I_+ \cap I_-$ by Proposition 3.1.14. Furthermore, $\Lambda = \Lambda_+ \sqcup \Lambda_- \sqcup \Lambda_0$ where

a) $\Lambda_+ = \Lambda_{e_+} = \{e \in \Lambda\backslash\{0\} \mid e \geq e_+, e \not\geq e_-\}$

b) $\Lambda_- = \Lambda_{e_-} = \{e \in \Lambda\backslash\{0\} \mid e \geq e_-, e \not\geq e_+\}$

c) $\Lambda_0 = \Lambda_{e_0} = \{e \in \Lambda\backslash\{0\} \mid e \geq e_0\}$

Theorem 3.3.2. *If M is not semisimple, then $\dim(Z(G)) = 2$. Additionally, λ^* is determined by*

a) $\lambda^*(\Lambda_+) = \{X \subseteq \Delta \mid$ no component of X is contained in $I_+\}$

b) $\lambda^*(\Lambda_-) = \{X \subseteq \Delta \mid$ no component of X is contained in $I_-\}$

c) $\lambda^*(\Lambda_0) = \{X \subseteq \Delta \mid$ no component of X is contained in $I_0\}$

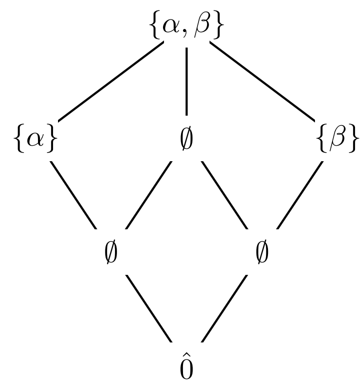
and λ_* is determined by Theorem 3.2.2.

The structure and connections with geometry, however, are more interesting when M is semisimple. We will therefore assume that all 2-reducible monoids are semisimple unless otherwise stated.

Let M be a semisimple 2-reducible monoid. We can then decompose the set of simple roots as $\Delta = \Delta_+ \sqcup \Delta_- \sqcup \Delta_0$. Let e_α be the unique element of $\Lambda\backslash\{\hat{0}\}$ such that $\lambda(e_\alpha) = \Delta\backslash\{\alpha\}$. Then

a) $\Delta_+ = \{\alpha \in \Delta \mid e_\alpha \in \Lambda_+\}$

b) $\Delta_- = \{\alpha \in \Delta \mid e_\alpha \in \Lambda_-\}$



(a) Cross section lattice Λ

	$\lambda^*(e)$	$\lambda_*(e)$	$\lambda(e)$
Λ_+	$\emptyset$	$\emptyset$	$\emptyset$
	$\{\alpha\}$	$\emptyset$	$\{\alpha\}$
Λ_-	$\emptyset$	$\emptyset$	$\emptyset$
	$\{\beta\}$	$\emptyset$	$\{\beta\}$
Λ_0	$\emptyset$	$\emptyset$	$\emptyset$
	$\{\alpha, \beta\}$	$\emptyset$	$\{\alpha, \beta\}$

(b) Type map of M

Figure 3.3: 2-reducible monoid with $\Delta = \{\alpha, \beta\}$, $I_+ = I_- = I_0 = \emptyset$, $\Delta_+ = \{\beta\}$, $\Delta_- = \{\alpha\}$

c) $\Delta_0 = \{\alpha \in \Delta \mid e_\alpha \in \Lambda_0\}$

The following theorem allows us to construct the cross section lattice of a semisimple 2-reducible monoid in terms of the invariants I_+, I_-, Δ_+, and Δ_-.

Theorem 3.3.3. *Let M be a semisimple 2-reducible monoid. Then*

a) $\lambda^*(\Lambda_+) = \{X \subseteq \Delta \mid$ *no component of* X *is contained in* I_+ *and* $\Delta_+ \not\subseteq X\}$

b) $\lambda^*(\Lambda_-) = \{X \subseteq \Delta \mid$ *no component of* X *is contained in* I_- *and* $\Delta_- \not\subseteq X\}$

c) $\lambda^*(\Lambda_0) = \{X \subseteq \Delta \mid$ *no component of* X *is contained in* I_0*, and either* $\Delta_+ \not\subseteq X, \Delta_- \not\subseteq X$ *or else* $\Delta_+ \sqcup \Delta_- \subseteq X\}$

Furthermore, λ^* *is injective on* Λ_+*,* Λ_-*, and* Λ_0*.*

Example 3.3.4. Let M be a semisimple 2-reducible monoid with connected set of simple roots $\Delta = \{\alpha, \beta\}$ and $I_+ = I_- = I_0 = \emptyset$, $\Delta_+ = \{\beta\}$, $\Delta_- = \{\alpha\}$. Notice that since I_+, I_-, and I_0 are all empty, no component of any subset of X can be contained in any of these sets. $\lambda^*(\Lambda_+)$ is then the set of all subsets of Δ that don't have Δ_+ as a subset. Therefore $\lambda^*(\Lambda_+) = \{\emptyset, \{\alpha\}\}$. Similarly, $\lambda^*(\Lambda_-) = \{\emptyset, \{\beta\}\}$. $\lambda^*(\Lambda_0)$ is the set of all subsets of Δ that either don't have Δ_+ and Δ_- as subsets, or have both Δ_+ and Δ_- as subsets. Therefore $\lambda^*(\Lambda_0) = \{\emptyset, \{\alpha, \beta\}\}$. The cross section lattice Λ and the type map are shown in Figure 3.3.

Notice that there exists $e_\beta \in \Lambda_+$ such that $\lambda^*(e_\beta) = \{\alpha\} = \Delta\backslash\{\beta\}$. Similarly there exists $e_\alpha \in \Lambda_+$ such that $\lambda^*(e_\alpha) = \{\beta\} = \Delta\backslash\{\alpha\}$. Also, λ^* is injective on Λ_+, Λ_-, and Λ_0. However, λ^* is not injective on all of $\Lambda\backslash\{0\}$. In fact, for any 2-reducible monoid $\lambda^*(e_+) = \lambda^*(e_-) = \lambda^*(e_0) = \emptyset$, so λ^* can never be injective on the entire cross section lattice for a 2-reducible monoid. Notice that since the types $I_+ = I_- = I_0 = \emptyset$, $\lambda^* = \lambda$ and hence λ is not injective on $\Lambda\backslash\{0\}$ either.

Example 3.3.5. Let M be a semisimple 2-reducible monoid with connected set of simple roots $\Delta = \{\alpha, \beta, \gamma\}$ where $s_\alpha s_\beta \neq s_\beta s_\alpha$ and $s_\beta s_\gamma \neq s_\gamma s_\beta$. Let $I_+ = \{\beta, \gamma\}$, $I_- = \{\alpha, \beta\}$, $\Delta_+ = \{\alpha\}$, $\Delta_- = \{\beta\}$. Then

$$\lambda(e_-) = \lambda^*(e_-) \sqcup \lambda_*(e_-) = \emptyset \sqcup I_- = \{\alpha, \beta\} \implies \gamma \in \Delta_-,$$

a contradiction. Therefore M is not semisimple and hence no semisimple 2-reducible monoid exists with the given choices of I_+, I_-, Δ_+, and Δ_-.

Example 3.3.5 brings us to the unfortunate realization that there may not exist a semisimple 2-reducible monoid for arbitrary I_+, I_-, Δ_+, and Δ_-. One question we may want to ask is which choices of these invariants will give rise to a semisimple 2-reducible monoid? A complete solution to this problem is not known. Some restrictions, however, are as follows:

Theorem 3.3.6. *a) $I_+, I_- \subset \Delta$ are the types for some 2-reducible semisimple monoid if and only if $I_+ \neq I_-$ or else $I_+ = I_-$ and $|\Delta \backslash I_+| \geq 2$.*

b) $\Delta_+ \neq \emptyset$ and $\Delta_- \neq \emptyset$.

c) There exists a 2-reducible semisimple monoid with $I_+ = I_- = \emptyset$ if and only if $\Delta_+ \neq \emptyset$, $\Delta_- \neq \emptyset$, and $\Delta_+ \cap \Delta_- = \emptyset$.

d) If Δ is of the type A_n with $I_+ = \Delta \backslash \{\alpha_1\}$ and $I_- = \Delta \backslash \{\alpha_i\}$, then either

i) $\Delta_+ = \{\alpha_1, \ldots, \alpha_j\}$ and $\Delta_- = \{\alpha_{j+2}, \ldots, \alpha_n\}$ for some $1 \leq j \leq i-1$, or

ii) $\Delta_+ = \{\alpha_1, \ldots, \alpha_{j+1}\}$ and $\Delta_- = \{\alpha_{j+2}, \ldots, \alpha_n\}$ for some $0 \leq j \leq i-1$

Notice that the 2-reducible monoid in Example 3.3.4 is permissible by Theorem 3.3.6. Example 3.3.5, however, is not. $I_+ = \Delta \backslash \{\alpha\}$ and $I_- = \Delta \backslash \{\gamma\}$, but $\Delta_- = \{\beta\}$ is not allowed since in either case we must have $\gamma \in \Delta_-$.

Chapter 4

Distributive Cross Section Lattices

In [15] Renner shows that the type of a $\mathcal{J}$-irreducible monoid is combinatorially smooth if the cross section lattice is distributive. This provides us with a motivation to study when a given monoid is distributive. In Section 4.1 we show that a $\mathcal{J}$-irreducible cross section lattice is distributive if and only if it is modular. We then devise a method to determine if the cross section lattice is distributive in terms of its type. In Section 4.2 we show that a 2-reducible cross section lattice is distributive if and only if it is modular.

4.1 Distributive $\mathcal{J}$-irreducible Cross Section Lattices

Before we try to describe when a given $\mathcal{J}$-irreducible cross section is distributive, we prove the following proposition which will come in handy throughout this section.

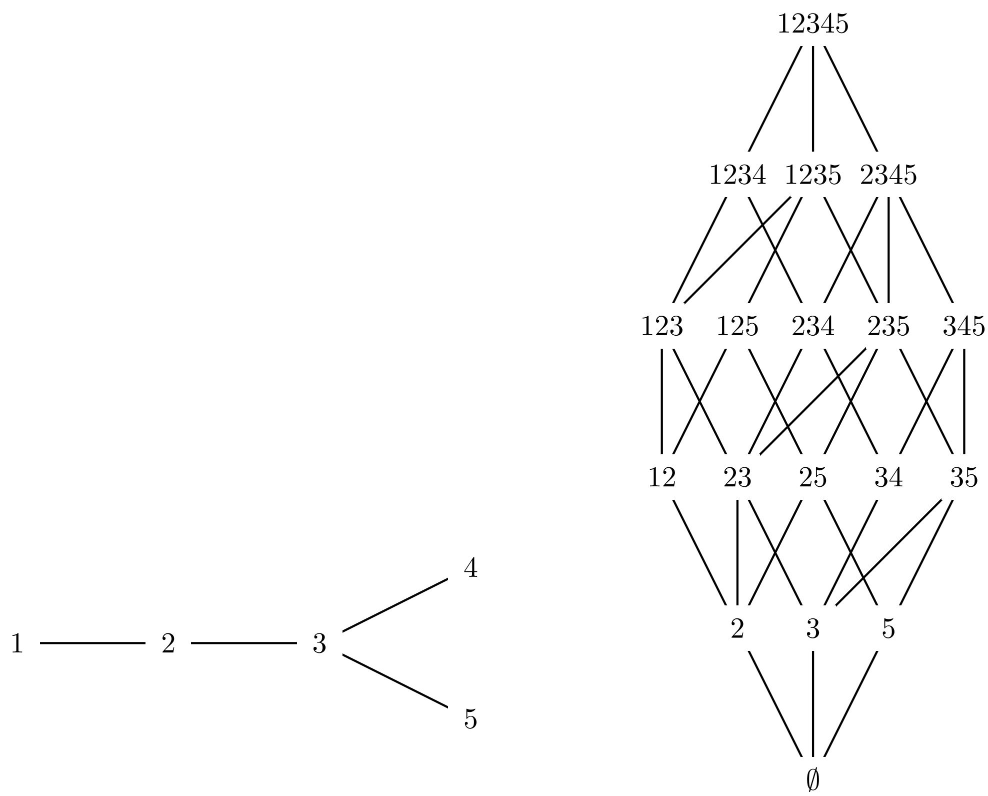
Proposition 4.1.1. *Let M be a $\mathcal{J}$-irreducible reductive monoid and let $\lambda^*(e) = U$ and $\lambda^*(f) = V$. The following are true.*

a) $\lambda^*(e) \cup \lambda^*(f)$ is in the image of λ^*.

b) $\lambda^*(e \vee f) = \lambda^*(e) \cup \lambda^*(f)$.

Proof.

a) Suppose not. Then there is a connected component C of $\lambda^*(e) \cup \lambda^*(f)$ that is contained in I. If $C \subseteq \lambda^*(f)$ then C would be connected in $\lambda^*(f)$. Then a connected component of $\lambda^*(f)$ would be contained in I, a contradiction. Therefore $C \nsubseteq \lambda^*(f)$ hence $C \cap \lambda^*(e) \neq \emptyset$. Similarly $C \cap \lambda^*(f) \neq \emptyset$. The connected components of $\lambda^*(e)$ will be contained in the connected components of $\lambda^*(e) \cup \lambda^*(f)$. Let C' be a connected component of $C \cap \lambda^*(e)$. Then $C' \subseteq C \subseteq I$, a contradiction.



(a) Dynkin diagram of type D_5 (b) Cross section lattice $\Lambda\backslash\{\hat{0}\}$

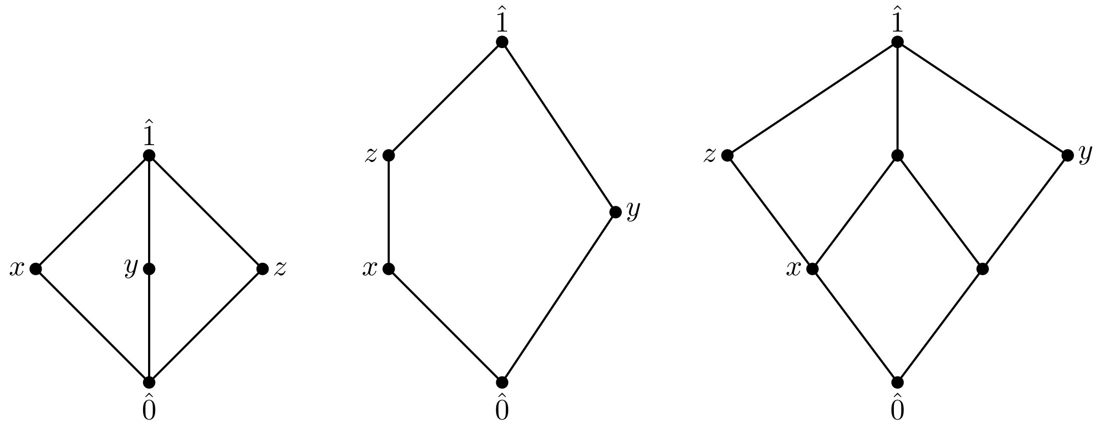
Figure 4.1: $\mathcal{J}$-irreducible monoid with $\Delta = \{\alpha_1, \ldots, \alpha_5\}$ and $I = \{\alpha_1, \alpha_4\}$

b) Let $h \in \Lambda\backslash\{\hat{0}\}$ such that $\lambda^*(h) = \lambda^*(e) \cup \lambda^*(f)$. Since $\lambda^*(e) \subseteq \lambda^*(h)$ and $\lambda^*(f) \subseteq \lambda^*(h)$, $h \geq e \vee f$. On the other hand, $\lambda^*(e) \subseteq \lambda^*(e \vee f)$ and $\lambda^*(f) \subseteq \lambda^*(e \vee f)$. So $\lambda^*(h) = \lambda^*(e) \cup \lambda^*(f) \subseteq \lambda^*(e \vee f)$. Therefore $h \leq e \vee f$ and hence $h = e \vee f$. The result follows since λ^* is injective.

□

Example 4.1.2. Let M be a $\mathcal{J}$-irreducible of type $I = \{\alpha_1, \alpha_4\}$ and set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_5\}$ of type D_5. The Dynkin diagram of Δ is shown in Figure 4.1a and the cross section lattice $\Lambda\backslash\{\hat{0}\}$ is shown in Figure 4.1b. Notice that the union of any two elements in the image of λ^* is in the image and this union is the join.

It should be pointed out that Proposition 4.1.1 is not necessarily true for 2-reducible monoids.



(a) M_5 (b) N_5 (c) N_7

Figure 4.2: Some important nondistributive lattices

Example 4.1.3. Let M be a 2-reducible semisimple monoid with set of simple roots $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$ with $\Delta_+ = \{\alpha_1\}$, $\Delta_- = \{\alpha_2, \alpha_3\}$, and $I_+ = I_- = I_0 = \emptyset$. Notice that such a monoid exists by Theorem 3.3.6. No component of $\{\alpha_2\}$ is contained in I_0 and neither Δ_+ nor Δ_- are subsets of $\{\alpha_2\}$. Therefore there exists $e \in \Lambda_0$ such that $\lambda^*(e) = \{\alpha_2\}$. Similarly there exists $f \in \Lambda_0$ such that $\lambda^*(f) = \{\alpha_3\}$. Although no component of $\{\alpha_2, \alpha_3\}$ is contained in I_0, Δ_- is a subset of $\{\alpha_2, \alpha_3\}$ while Δ_+ is not. Therefore there does not exist an element $h \in \Lambda_0$ such that $\lambda^*(h) = \{\alpha_2, \alpha_3\}$. Therefore $e \vee f = \hat{1}$ and hence $\lambda^*(e) \cup \lambda^*(f) \neq \lambda^*(e \vee f)$.

We saw in Section 2.4.14 how we can determine whether a given lattice is modular or distributive in terms of the special lattices M_5 and N_5. For convenience these lattices have been reproduced in Figure 4.2.

Proposition 4.1.4. *Let M be a $\mathcal{J}$-irreducible reductive monoid. $\Lambda \backslash \{\hat{0}\}$ does not contain M_5 as a sublattice.*

Proof. Consider $\{e, e_1, e_2, e_3, f\} \subseteq \Lambda \backslash \{\hat{0}\}$ such that $e < e_i < f$, $e_i \wedge e_j = e$, and $e_i \vee e_j = f$ for all $i \neq j$. Then $\{e, e_1, e_2, e_3, f\} \cong M_5$. $\lambda^*(e_1) \subseteq \lambda^*(f)$ and $\lambda^*(e_2) \subseteq \lambda^*(f)$, so $\lambda^*(e_1) \cup \lambda^*(e_2) \subseteq \lambda^*(f)$. $\lambda^*(e_1) \cup \lambda^*(e_2)$ is in the image of λ^* by Proposition 4.1.1, so $\lambda^*(h) = \lambda^*(e_1) \cup \lambda^*(e_2)$ for some $h \in \Lambda \backslash \{\hat{0}\}$. Therefore $e_1 \leq h$ and $e_2 \leq h$ and hence $e_1 \vee e_2 \leq h$. However, $e_1 \vee e_2 = f$, so $f \leq h$. On the other hand, $\lambda^*(h) = \lambda^*(e_1) \cup \lambda^*(e_2) \subseteq \lambda^*(f)$ so $h \leq f$. Therefore $h = f$ and hence $\lambda^*(e_1) \cup \lambda^*(e_2) = \lambda^*(f)$. Similarly $\lambda^*(e_1) \cup \lambda^*(e_3) = \lambda^*(e_2) \cup \lambda^*(e_3) = \lambda^*(f)$. Let $A_i = \lambda^*(f) \backslash \lambda^*(e_i)$ for $i = 1, 2, 3$. Then $A_i \subseteq \lambda^*(e_j)$ for all $i \neq j$. Let $e' \in [e, e_1]$ such that e' covers e. Since M is semisimple, there exists $\alpha \in \Delta$ such that $\lambda^*(e') = \lambda^*(e) \sqcup \{\alpha\}$. $\alpha \in \lambda^*(e_1)$ since $e' \leq e_1$. Suppose $\alpha \in \lambda^*(e_2)$. Then $\lambda^*(e') = \lambda^*(e) \sqcup \{\alpha\} \subseteq \lambda^*(e_2)$. Therefore $e_1 \wedge e_2 \geq e' > e$, a contradiction. So $\alpha \notin \lambda^*(e_2)$. That is, $\alpha \in A_2$. But $A_2 \subseteq \lambda^*(e_3)$. Therefore $\alpha \in \lambda^*(e_3)$ and $e_1 \wedge e_3 > e$, a contradiction. □

Corollary 4.1.5. *Let M be a $\mathcal{J}$-irreducible reductive monoid. $\Lambda\backslash\{\hat{0}\}$ is distributive if and only if it is modular.*

Proof. Suppose $\Lambda\backslash\{\hat{0}\}$ is distributive. Then by Theorem 2.4.14 $\Lambda\backslash\{\hat{0}\}$ does not have N_5 as a sublattice and hence $\Lambda\backslash\{\hat{0}\}$ is modular.

Now suppose $\Lambda\backslash\{\hat{0}\}$ is modular. Then by Theorem 2.4.14 $\Lambda\backslash\{\hat{0}\}$ does not have N_5 as a sublattice and by Proposition 4.1.4 $\Lambda\backslash\{\hat{0}\}$ does not have M_5 as a sublattice, and hence $\Lambda\backslash\{\hat{0}\}$ is distributive. □

In general, it is possible for a lattice to be modular but not distributive. The lattice M_5 in Figure 4.2a is an example. Corollary 4.1.5, however, shows that the concept of modular and distributive lattices are equivalent in $\mathcal{J}$-irreducible cross section lattices. With this knowledge in hand, we would like to be able to determine when a $\mathcal{J}$-irreducible cross section lattice is distributive/modular in terms of the set of simple roots and the type of the monoid. The following lemma will be useful in doing so.

Lemma 4.1.6. *Let M be a $\mathcal{J}$-irreducible reductive monoid and let $\{e_1, e_2, e_3, e_4, e_5\}$ be a sublattice isomorphic to N_5. Then there exist e_2' and e_4' in $\Lambda\backslash\{\hat{0}\}$ such that $\{e_1, e_2', e_3, e_4', e_5\}$ is a sublattice isomorphic to N_5 with $\lambda^*(e_2') \cap \lambda^*(e_4') = \lambda^*(e_1)$.*

Proof. Let $A = (\lambda^*(e_2) \cap \lambda^*(e_4))\backslash\lambda^*(e_1)$. If $A = \emptyset$, then $e_2 = e_2'$ and $e_4 = e_4'$ and the statement is trivial. So let $\alpha \in A$. $\lambda^*(e_1) \sqcup \{\alpha\}$ is not in the image of λ^* because if it were then $e_2 \wedge e_4 > e_1$. So $\alpha \in I$ and α is not adjacent to an element of $\lambda^*(e_1)$. Notice that $A \subseteq I$. We would like to be able to remove the elements of A from $\lambda^*(e_2)$, but doing so may leave a connected component that is contained in I. So we need to remove any extraneous elements of I as well. To do this let A' be the path (or paths) of elements of $(I\backslash A) \cap \lambda^*(e_2)$ with no elements adjacent to an element of $\lambda^*(e_1)$ but at least one element adjacent to an element of A. It is possible that A' could be empty. Let $B = A \sqcup A'$. Then by construction no component of $\lambda^*(e_2)\backslash B$ or $\lambda^*(e_4) \cup B$ is contained in I. So there exists e_2' and e_4' in $\Lambda\backslash\{\hat{0}\}$ such that $\lambda^*(e_2') = \lambda^*(e_2)\backslash B$ and $\lambda^*(e_4') = \lambda^*(e_4) \cup B$. Furthermore, $\lambda^*(e_2') \cap \lambda^*(e_4') = \lambda^*(e_1)$. Notice that since $B \subseteq \lambda^*(e_2)$ and no element of B is adjacent to an element of $\lambda^*(e_1)$, there is an element $\gamma \in \lambda^*(e_2') = \lambda^*(e_2)\backslash B$ that is not in $\lambda^*(e_1)$. That is, $e_2' \neq e_1$. Furthermore, $\gamma \in \lambda^*(e_5)$ but $\gamma \notin \lambda^*(e_4')$, so $e_4' \neq e_5$.

All that remains is to show that $\{e_1, e_2', e_3, e_4', e_5\}$ is a sublattice isomorphic to N_5. By construction, $e_1 \leq e_2' \leq e_3 \leq e_5$ and $e_1 \leq e_4' \leq e_5$. Since $\lambda^*(e_2') \cap \lambda^*(e_4') = \lambda^*(e_1)$, $e_2' \wedge e_4' = e_1$. By Proposition 4.1.1, $\lambda^*(e_2' \vee e_4') = \lambda^*(e_2') \cup \lambda^*(e_4') = \lambda^*(e_2) \cup \lambda^*(e_4) = \lambda^*(e_5)$, so $e_2' \vee e_4' = e_5$. Since $e_3 \vee e_4 = e_5$ and $e_4 \leq e_4' \leq e_5$, $e_3 \vee e_4' = e_5$. If $\beta \in B$, then $\beta \in I$ and β is not adjacent to an element of $\lambda^*(e_1)$. So $\lambda^*(e_1) \sqcup B'$ is not in the image of λ^* for any $\emptyset \neq B' \subseteq B$. Therefore $e_3 \wedge e_4' = e_3 \wedge e_4 = e_1$. $\square$

Example 4.1.7. Let M be a $\mathcal{J}$-irreducible monoid of type $I = \{\alpha_4, \alpha_5, \alpha_6\}$ with set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_7\}$ of type A_7. There exist elements $e_1, e_2, e_3, e_4, e_5 \in \Lambda \backslash \{\hat{0}\}$ such that $\lambda^*(e_1) = \{\alpha_1\}$, $\lambda^*(e_2) = \{\alpha_1, \alpha_2, \alpha_5, \alpha_6, \alpha_7\}$, $\lambda^*(e_3) = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$, $\lambda^*(e_4) = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$, and $\lambda^*(e_5) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ since no component of these sets is contained in I. $\lambda^*(e_2) \cap \lambda^*(e_4) = \{\alpha_1, \alpha_5\}$ but this is not in the image of λ^* since the component $\{\alpha_5\}$ is contained in I. Therefore $e_2 \wedge e_4 = e_1$. Similarly $e_3 \wedge e_4 = e_1$. $\lambda^*(e_2) \cup \lambda^*(e_4) = \lambda^*(e_3) \cup \lambda^*(e_4) = \lambda^*(e_5)$, so $e_2 \vee e_4 = e_3 \vee e_4 = e_5$ by Proposition 4.1.1. Therefore $\{e_1, e_2, e_3, e_4, e_5\}$ is a sublattice that is isomorphic to N_5.

Using the notation in the proof of Lemma 4.1.6, $A = (\lambda^*(e_2) \cap \lambda^*(e_4)) \backslash \lambda^*(e_1) = \{\alpha_5\}$. $A' = \{\alpha_6\}$ so $B = \{\alpha_5, \alpha_6\}$. Notice that no component of $\lambda^*(e_2) \backslash B = \{\alpha_1, \alpha_2, \alpha_7\}$ or $\lambda^*(e_4) \cup B = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ is contained in I. Therefore there exist $e_2', e_4' \in \Lambda \backslash \{\hat{0}\}$ such that $\lambda^*(e_2') = \{\alpha_1, \alpha_2, \alpha_7\}$ and $\lambda^*(e_4') = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. Then $e_2' \wedge e_4' = e_3 \wedge e_4' = e_1$ and $e_2' \vee e_4' = e_3 \vee e_4' = e_5$ and hence $\{e_1, e_2', e_3, e_4', e_5\}$ is isomorphic to N_5.

We are now ready to prove the main result of this section.

Theorem 4.1.8. *Let M be a $\mathcal{J}$-irreducible reductive monoid with set of simple roots Δ. Let $\Delta = \Delta_1 \sqcup \cdots \sqcup \Delta_k$, where the Δ_i's are the connected components of Δ and let $I_i = I \cap \Delta_i$ for $1 \leq i \leq k$. Then $\Lambda \backslash \{\hat{0}\}$ is distributive if and only if $\Delta_i \backslash I_i$ is connected for all i.*

Proof. Suppose $\Delta_i \backslash I_i$ is not connected for some i. We would like to show that $\Lambda \backslash \{\hat{0}\}$ is not distributive. Let I_i' be a connected component of I_i such that $\Delta_i \backslash I_i'$ is not connected. Such a connected component is guaranteed to exist since $\Delta_i \backslash I_i$ is not connected. We will construct a sublattice isomorphic to N_5 of rank 3 for the cases $|I_i'| = 1$, $|I_i'| = 2$, $|I_i'| > 2$.

Suppose $I_i' = \{\beta\}$. Since $\Delta_i \backslash \{\beta\}$ is not connected, there exist $\alpha, \gamma \in \Delta_i \backslash \{\beta\}$ such that $s_\alpha s_\beta \neq s_\beta s_\alpha$ and $s_\beta s_\gamma \neq s_\gamma s_\beta$. Clearly α and γ are in different connected components of $\Delta_i \backslash \{\beta\}$ and neither is contained in I. Let $\lambda^*(e_1) = \emptyset$, $\lambda^*(e_2) = \{\alpha\}$, $\lambda^*(e_3) = \{\alpha, \beta\}$, $\lambda^*(e_4) = \{\beta, \gamma\}$, and $\lambda^*(e_5) = \{\alpha, \beta, \gamma\}$. No connected component of these sets is contained in I so they are all in the image of λ^*. Clearly $e_2 \wedge e_4 = e_1$, $e_2 \vee e_4 = e_5$, and $e_3 \vee e_4 = e_5$. Since $\beta \in I_i' \subseteq I$, $\{\beta\}$ is not in the image of λ^*. Therefore $e_3 \wedge e_4 = e_1$ and $\{e_1, e_2, e_3, e_4, e_5\}$ is a sublattice isomorphic to N_5 of rank 3. The Hasse diagram of the interval $[e_1, e_5]$ is shown in Figure 4.3.

Now suppose $I_i' = \{\beta_1, \beta_2\}$. Since I_i' is a connected component of I_i, $s_{\beta_1} s_{\beta_2} \neq s_{\beta_2} s_{\beta_1}$. Since $\Delta_i \backslash \{\beta_1, \beta_2\}$ is not connected, there exist $\alpha, \gamma \in \Delta_i \backslash \{\beta_1, \beta_2\}$ such that $s_\alpha s_{\beta_1} \neq s_{\beta_1} s_\alpha$ and $s_{\beta_2} s_\gamma \neq s_\gamma s_{\beta_2}$. Clearly α and γ are in different connected components of $\Delta_i \backslash \{\beta_1, \beta_2\}$ and neither is in I. Let $\lambda^*(e_1) = \{\alpha\}$, $\lambda^*(e_2) = \{\alpha, \beta_1\}$, $\lambda^*(e_3) = \{\alpha, \beta_1, \beta_2\}$, $\lambda^*(e_4) = \{\alpha, \beta_2, \gamma\}$, and $\lambda^*(e_5) = \{\alpha, \beta_1, \beta_2, \gamma\}$. No connected component of these sets is contained in I so they are

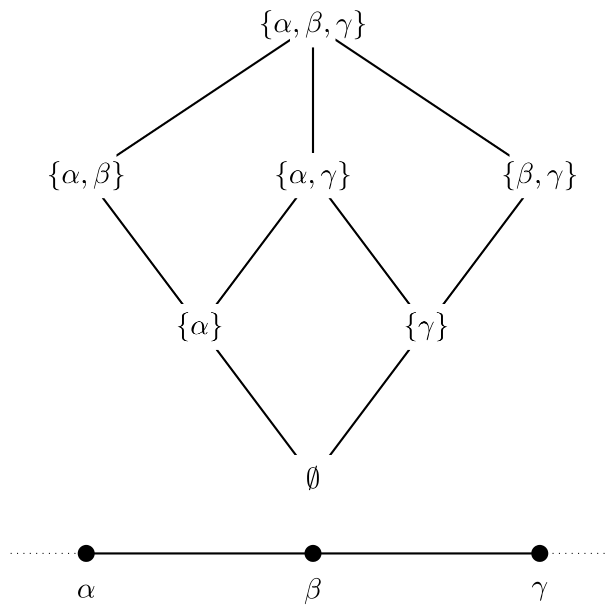


Figure 4.3: $|I_i'| = \{\beta\}$

all in the image of λ^*. Clearly $e_2 \wedge e_4 = e_1$, $e_2 \vee e_4 = e_5$, and $e_3 \vee e_4 = e_5$. Since $\beta_2 \in I_i' \subseteq I$ and $s_\alpha s_{\beta_2} \neq s_{\beta_2} s_\alpha$, $\{\alpha, \beta_2\}$ is not in the image of λ^*. Therefore $e_3 \wedge e_4 = e_1$ and $\{e_1, e_2, e_3, e_4, e_5\}$ is a sublattice isomorphic to N_5 of rank 3. The Hasse diagram of the interval $[e_1, e_5]$ is shown in Figure 4.4.

Finally, suppose $I_i' = \{\beta_1, \ldots, \beta_k\}$ where $k \geq 3$. We will assume that since I_i' is a connected component of I_i, I_i' is a path such that $s_{\beta_{j-1}} s_{\beta_j} \neq s_{\beta_j} s_{\beta_{j-1}}$ for $2 \leq j \leq k$. The case where I_i' contains an endpoint of Δ_i if Δ_i is of the type D_n, E_6, E_7, or E_8 is not interestingly different. Since $\Delta_i \backslash I_i'$ is not connected, there exist $\alpha, \gamma \in \Delta_i \backslash I_i'$ such that $s_\alpha s_{\beta_1} \neq s_{\beta_1} s_\alpha$ and $s_{\beta_k} s_\gamma \neq s_\gamma s_{\beta_k}$. Clearly α and γ are in different connected components of $\Delta_i \backslash I_i'$ and neither is in I. Let $\lambda^*(e_1) = \{\alpha, \beta_1, \ldots, \beta_{k-2}\}$, $\lambda^*(e_2) = \{\alpha, \beta_1, \ldots, \beta_{k-1}\}$, $\lambda^*(e_3) = \{\alpha, \beta_1, \ldots, \beta_k\}$, $\lambda^*(e_4) = \{\alpha, \beta_1, \ldots, \beta_{k-2}, \beta_k, \gamma\}$, and $\lambda^*(e_5) = \{\alpha, \beta_1, \ldots, \beta_k, \gamma\}$. No connected component of these sets is contained in I so they are all in the image of λ^*. Clearly $e_2 \wedge e_4 = e1$, $e_2 \vee e_4 = e_5$, and $e_3 \vee e_4 = e_5$. Since $\beta_k \in I_i' \subseteq I$ and $s_{\beta_{k-2}} s_{\beta_k} \neq s_{\beta_k} s_{\beta_{k-2}}$, $\{\alpha, \beta_1, \ldots, \beta_{k-2}, \beta_k\}$ is not in the image of λ^*. Therefore $e_3 \wedge e_4 = e_1$ and $\{e_1, e_2, e_3, e_4, e_5\}$ is a sublattice isomorphic to N_5 of rank 3. The Hasse diagram of the interval $[e_1, e_5]$ is shown in Figure 4.5.

Now suppose $\{e_1, e_2, e_3, e_4, e_5\}$ is a sublattice of $\Lambda \backslash \{\hat{0}\}$ that is isomorphic to N_5. We need to show that $\Delta_i \backslash I_i$ is not connected for some i. If $\lambda^*(e_2) \cap \lambda^*(e_4) \neq \lambda^*(e_1)$, then by Lemma 4.1.6 there exist $e_2', e_4' \in \Lambda \backslash \{\hat{0}\}$ such that $\{e_1, e_2', e_3, e_4', e_5\}$ is a sublattice isomorphic to N_5 with $\lambda^*(e_2') \cap \lambda^*(e_4') = \lambda^*(e_1)$. So let us assume, without loss of generality, that $\lambda^*(e_2) \cap \lambda^*(e_4) =$

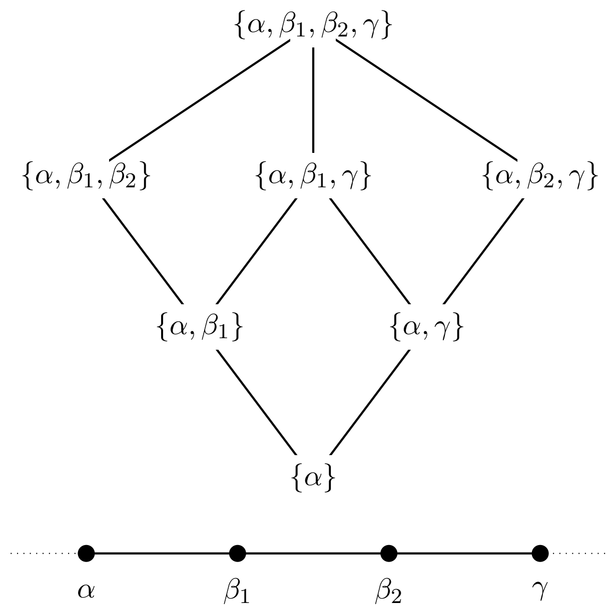


Figure 4.4: $|I_i'| = \{\beta_1, \beta_2\}$

$\lambda^*(e_1)$. Since M is semisimple, there exists $\beta \in \lambda^*(e_3)\backslash\lambda^*(e_2)$ such that $\lambda^*(e_2) \sqcup \{\beta\}$ is in the image of λ^*. Since $\beta \notin \lambda^*(e_2)$, $\lambda^*(e_2) \cap \lambda^*(e_4) = \lambda^*(e_1)$, and $\lambda^*(e_2) \cup \lambda^*(e_4) = \lambda^*(e_5)$ it must be that $\beta \in \lambda^*(e_4)$. However, $\lambda^*(e_1) \sqcup \{\beta\}$ is not in the image of λ^*, for if it were then $e_3 \wedge e_4 \neq e_1$. Therefore $\beta \in I$ and β is not adjacent to any elements of $\lambda^*(e_1)$. Since $\lambda^*(e_2) \sqcup \{\beta\}$ is in the image of λ^*, β is an element of a path of elements of I with at least one element adjacent to some $\alpha \in \lambda^*(e_2)$ such that $\alpha \notin I$. Similarly, since $\lambda^*(e_4)$ is in the image of λ^* and $\beta \in \lambda^*(e_4)$, β is an element of a path of elements of I with at least one element adjacent to some $\gamma \in \lambda^*(e_4)$ such that $\gamma \notin I$. β is therefore an element of a path of elements of I at least one of which is adjacent to $\alpha \in \Delta_i\backslash I_i$ and another is adjacent to $\gamma \in \Delta_i\backslash I_i$ for some i. Therefore $\Delta_i\backslash I_i$ is not connected. $\square$

The following corollary was first observed, without proof, by Putcha and Renner in [10].

Corollary 4.1.9. *Let M be a $\mathcal{J}$-irreducible reductive monoid with connected set of simple roots Δ and type I. Then $\Lambda\backslash\{\hat{0}\}$ is distributive if and only if $\Delta\backslash I$ is connected.*

Example 4.1.10. Consider the $\mathcal{J}$-irreducible monoid of type $I = \{\alpha_1, \alpha_4\}$ and set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_5\}$ of type D_5 from Example 4.1.2. $\Delta\backslash I = \{\alpha_2, \alpha_3, \alpha_5\}$ is connected so the cross section lattice $\Lambda\backslash\{\hat{0}\}$ is distributive by Corollary 4.1.9. This can be verified by noticing that no sublattice of $\Lambda\backslash\{\hat{0}\}$ in Figure 4.1b is isomorphic to N_5.

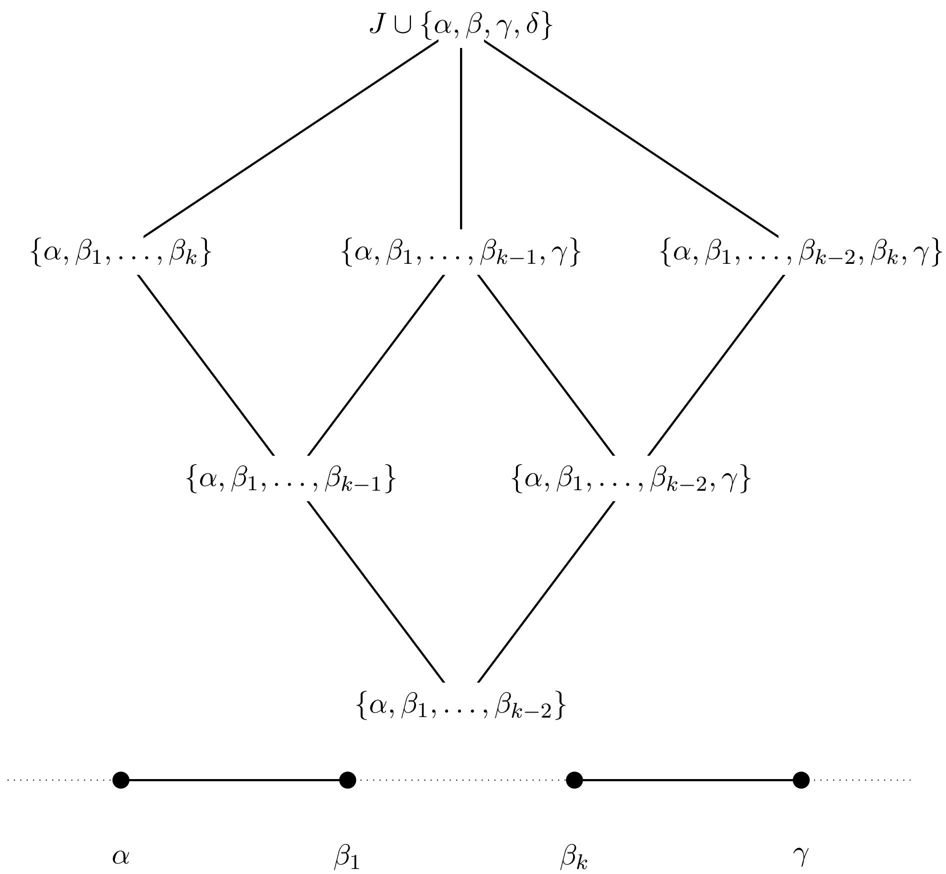


Figure 4.5: $|I_i'| = \{\beta_1, \ldots, \beta_k\}, k \geq 3$

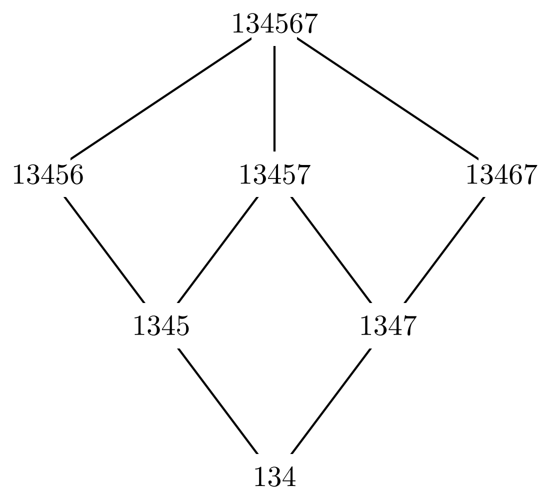


Figure 4.6: A sublattice of a nondistributive cross section lattice that is isomorphic to N_7

Corollary 4.1.11. *Let M be a $\mathcal{J}$-irreducible reductive monoid with cross section lattice $\Lambda\backslash\{\hat{0}\}$. If $\Lambda\backslash\{\hat{0}\}$ has a sublattice isomorphic to N_5, then $\Lambda\backslash\{\hat{0}\}$ has a sublattice isomorphic to N_5 of rank 3.*

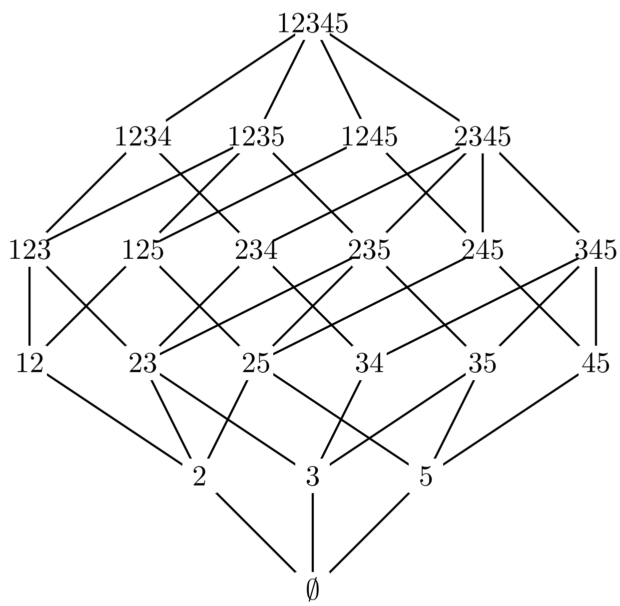
Proof. If $\Lambda\backslash\{0\}$ has a sublattice isomorphic to N_5, then $\Lambda\backslash\{\hat{0}\}$ is not distributive and hence $\Delta_i\backslash I_i$ is not connected for some i. A method for constructing a sublattice of $\Lambda\backslash\{\hat{0}\}$ isomorphic to N_5 of rank 3 is described in the proof of Theorem 4.1.8.

□

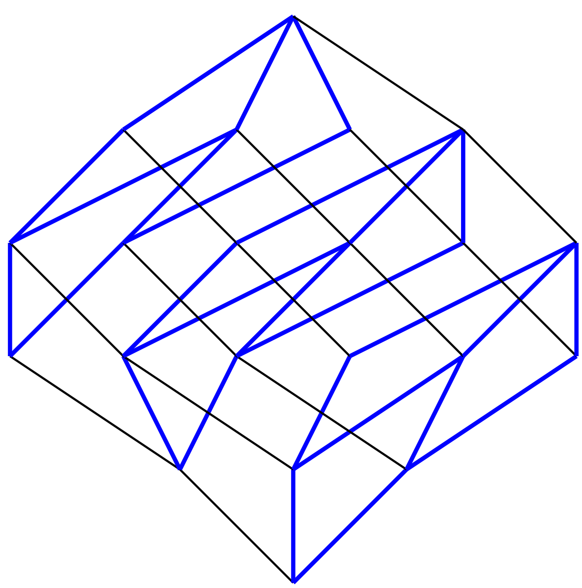
Corollary 4.1.12. *Let M be a $\mathcal{J}$-irreducible reductive monoid with nondistributive cross section lattice $\Lambda\backslash\{\hat{0}\}$. Then $\Lambda\backslash\{\hat{0}\}$ has a sublattice isomorphic to N_5 of rank 3.*

Notice that if M is a nondistributive $\mathcal{J}$-irreducible monoid, then its cross section lattice has a *sublattice* $\{e_1, e_2, e_3, e_4, e_5\}$ of rank 3 that is isomorphic to N_5. The *interval* $[e_1, e_5]$ will be isomorphic to the lattice N_7 in Figure 4.2c.

Example 4.1.13. Consider the $\mathcal{J}$-irreducible monoid from Example 4.1.7. The sublattice $\{e_1, e_2, e_3, e_4, e_5\}$ is isomorphic to N_5 where $\lambda^*(e_1) = \{\alpha_1\}$, $\lambda^*(e_2) = \{\alpha_1, \alpha_2, \alpha_5, \alpha_6, \alpha_7\}$, $\lambda^*(e_3) = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$, $\lambda^*(e_4) = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$, and
$\lambda^*(e_5) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$. There exists a sublattice $\{e'_1, e'_2, e'_3, e'_4, e'_5\}$ that is isomorphic to N_5 where $\lambda^*(e'_1) = \{\alpha_1, \alpha_3, \alpha_4\}$, $\lambda^*(e'_2) = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$, $\lambda^*(e'_3) = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$, $\lambda^*(e'_4) = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6, \alpha_7\}$, and $\lambda^*(e'_5) = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$. The interval $[e'_1, e'_5]$ is shown in Figure 4.6. Notice that this interval is isomorphic to N_7.



(a) $\Lambda\backslash\{\hat{0}\}$ with type $I = \{\alpha_1, \alpha_4\}$ and set of simple roots Δ of type $A_2 \oplus A_3$



(b) Intervals of length 3 that are isomorphic to N_7

Figure 4.7: A nondistributive cross section lattice

Example 4.1.14. Let M be a $\mathcal{J}$-irreducible of type $I = \{\alpha_1, \alpha_4\}$ and set of simple roots $\Delta = \{\alpha_1, \alpha_2\} \sqcup \{\alpha_3, \alpha_4, \alpha_5\}$ of type $A_2 \oplus A_3$. Let $I_1 = \{\alpha_1\}$, $I_2 = \{\alpha_4\}$, $\Delta_1 = \{\alpha_1, \alpha_2\}$ and $\Delta_2 = \{\alpha_3, \alpha_4, \alpha_5\}$. The cross section lattice $\Lambda\backslash\{\hat{0}\}$ is shown in Figure 4.7a. Even though $\Delta_1\backslash I_1 = \{\alpha_2\}$ is connected, $\Lambda\backslash\{\hat{0}\}$ is not distributive because $\Delta_2\backslash I_2 = \{\alpha_3, \alpha_5\}$ is not connected. Notice that it only takes $\Delta_i\backslash I_i$ to be not connected for a single value of i in order for the cross section lattice to be nondistributive. There are three intervals of $\Lambda\backslash\{\hat{0}\}$ of length 3 that are isomorphic to N_7: $[\emptyset, 345]$, $[2, 2345]$, and $[12, 12345]$. These intervals are highlighted in Figure 4.7b.

4.2 Distributive 2-reducible Cross Section Lattices

In Section 4.1 we showed that the cross section lattice of a $\mathcal{J}$-irreducible monoid is distributive if and only if it is modular. We were able to due so by showing that a $\mathcal{J}$-irreducible cross section lattice could not contain a sublattice isomorphic to M_5. We would like to prove an analogous result of Corollary 4.1.5 for 2-reducible semisimple monoids. This case, however, is more complicated as the following example illustrates.

Example 4.2.1. Let $\Delta = \{\alpha_1, \cdots, \alpha_{10}\}$ with corresponding set of simple reflections $S = \{s_1, \cdots, s_{10}\}$. Let Δ be of type A_{10} so that $s_i s_{i+1} \neq s_{i+1} s_i$ for $i = 1, \cdots, n-1$. Let $I_+ = \{\alpha_2, \alpha_3, \alpha_6, \alpha_9\}$, $I_- = \{\alpha_3, \alpha_4, \alpha_7\}$, $\Delta_+ = \{\alpha_3, \alpha_6, \alpha_9\}$, and $\Delta_- = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_7, \alpha_8, \alpha_{10}\}$. Then $I_0 = I_+ \cap I_- = \{\alpha_3\}$ and $\Delta_0 = \Delta\backslash(\Delta_+ \sqcup \Delta_-) = \emptyset$. No connected component of $\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6\}$ is contained in I_+ and $\Delta_+ \not\subseteq \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6\}$. Therefore there exists $e_1 \in \Lambda_+$ such that $\lambda^*(e_1) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6\}$. Similarly there exist $e_2, e_3 \in \Lambda_+$ such that $\lambda^*(e_2) = \{\alpha_1, \alpha_3, \alpha_4, \alpha_8, \alpha_9\}$ and $\lambda^*(e_3) = \{\alpha_1, \alpha_6, \alpha_7, \alpha_9, \alpha_{10}\}$. Notice that $\lambda^*(e_1) \cap \lambda^*(e_2) = \{\alpha_1, \alpha_3\}$. However, there is no element $e \in \Lambda_+$ such that $\lambda^*(e) = \{\alpha_1, \alpha_3\}$ because $\{\alpha_3\}$ is a component of $\{\alpha_1, \alpha_3\}$ that is contained in I_+. However, there exists $e \in \Lambda_+$ such that $\lambda^*(e) = \{\alpha_1\}$ and hence $e_1 \wedge e_2 = e$. Similarly $e_1 \wedge e_3 = e_2 \wedge e_3 = e$.

On the other hand, $\Delta_+ \subseteq \lambda^*(e_1) \cup \lambda^*(e_2)$, so $f = e_1 \vee e_2 \notin \Lambda_+$. Since $\lambda^*(e_1) \cup \lambda^*(e_2) \subseteq \lambda^*(f)$, it follows that $f \in \Lambda_0$ and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(f)$. Since $\Delta_+ \sqcup \Delta_- = \Delta$, $\lambda^*(f) = \Delta$. Similarly $e_1 \vee e_3 = e_2 \vee e_3 = f$.

Example 4.2.1 provides several insights into the structure of cross section lattices of 2-reducible reductive monoids. First of all, Proposition 4.1.1 does not hold in the 2-reducible case. Additionally $\{e, e_1, e_2, e_3, f\}$ is a sublattice isomorphic to M_5, which was not possible in the $\mathcal{J}$-irreducible case. This suggests that the problem of determining the relationship between modular and distributive cross section lattices in the 2-reducible case may be more complicated than the $\mathcal{J}$-irreducible case. The following example provides a hopeful insight.

Example 4.2.2. Let Δ, I_+, I_-, Δ_+, and Δ_- be as in Example 4.2.1. No component of $\{\alpha_1, \alpha_2, \alpha_5, \alpha_6\}$ is contained in I_+ and $\Delta_+ \not\subseteq \{\alpha_1, \alpha_2, \alpha_5, \alpha_6\}$. Therefore there exists $e' \in \Lambda_+$

such that $\lambda^*(e') = \{\alpha_1, \alpha_2, \alpha_5, \alpha_6\}$. Since $e_1, e' \in \Lambda_+$ and $\lambda^*(e') \subseteq \lambda^*(e_1)$, it follows by Theorem 3.1.16 that $e' \leq e_1$. Since $\lambda^*(e') \cap \lambda^*(e_2) = \{\alpha_1\} = \lambda^*(e)$, it follows that $e' \wedge e_2 = e$. Furthermore, $\lambda^*(e') \cup \lambda^*(e_2) = \lambda^*(e_1) \cup \lambda^*(e_2)$, so $e' \vee e_2 = e_1 \vee e_2 = f$. Therefore $\{e, e', e_1, e_2, f\}$ is a sublattice that is isomorphic to N_5. $\square$

Example 4.2.2 shows that even though a cross section lattice can have a sublattice isomorphic to M_5, it may be possible to find a sublattice that is isomorphic is N_5. The following lemmas will allow us to prove that this must be the case.

Throughout the remainder of the section let $\{e, e_1, e_2, e_3, f\}$ be a sublattice of Λ that is isomorphic to M_5 where $e_i \wedge e_j = e$ and $e_i \vee e_j = f$ for $i, j = 1, 2, 3$ and $i \neq j$. The following notation will be useful in the following lemmas.

Definition 4.2.3. Let e, e_1, e_2, and e_3 be as above. Define $A_{ij} \subseteq \Delta$ by

$$A_{ij} = (\lambda^*(e_i) \cap \lambda^*(e_j)) \backslash \lambda^*(e)$$

for $i \neq j$.

Lemma 4.2.4.

a) If $\{e, e_1, e_2, e_3\} \subseteq \Lambda_+$ and $f \in \Lambda_0$, then A_{ij} is nonempty for all $i \neq j$.

b) If $\{e, e_1, e_2\} \subseteq \Lambda_+$, $\{e_3, f\} \subseteq \Lambda_0$, and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e_3)$, then A_{ij} is nonempty for all $i \neq j$.

c) If $\{e, e_1, e_2\} \subseteq \Lambda_+$, $\{e_3, f\} \subseteq \Lambda_0$, and $\Delta_+, \Delta_- \nsubseteq \lambda^*(e_3)$, then A_{ij} is nonempty for some $i \neq j$.

d) If $\{e, e_1, e_2, e_3, f\} \subseteq \Lambda_0$ and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e)$, then A_{ij} is nonempty for all $i \neq j$.

e) If $\{e, e_1, e_2, e_3, f\} \subseteq \Lambda_0$ and $\Delta_+, \Delta_- \nsubseteq \lambda^*(f)$, then A_{ij} is nonempty for all $i \neq j$.

f) If $\{e, e_1, e_2, e_3, f\} \subseteq \Lambda_0$, $\Delta_+, \Delta_- \nsubseteq \lambda^*(e), \lambda^*(e_i)$, and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(f)$, then A_{ij} is nonempty for some $i \neq j$.

Proof.

a) Let $\{e_1, e_2, e_3\} \subseteq \Lambda_+$ and $f \in \Lambda_0$. No component of $\lambda^*(e_i)$ is contained in I_+ for $i = 1, 2, 3$ and hence no component of $\lambda^*(e_i) \cup \lambda^*(e_j)$ is contained in I_+. Since $e_i \vee e_j \notin \Lambda_+$ for $i \neq j$, by Theorem 3.3.2 it must be that $\Delta_+ \subseteq \lambda^*(e_i) \cup \lambda^*(e_j)$ for $i \neq j$. Since, for example, $\Delta_+ \nsubseteq \lambda^*(e_2)$, there exists $\alpha \in \Delta_+$ such that $\alpha \in \lambda^*(e_1)$ and $\alpha \notin \lambda^*(e_2)$. Since $\alpha \notin \lambda^*(e_2)$ and $\Delta_+ \subseteq \lambda^*(e_2) \cup \lambda^*(e_3)$, $\alpha \in \lambda^*(e_3)$. That is, $A_{13} \neq \emptyset$. A similar argument shows that A_{12} and A_{23} are nonempty as well.

b) Let $\{e_1, e_2\} \subseteq \Lambda_+$, $\{e_3, f\} \subseteq \Lambda_0$, and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e_3)$. No component of $\lambda^*(e_1) \cup \lambda^*(e_2)$ is contained in I_+. Since $e_1 \vee e_2 \in \Lambda_0$, $\Delta_+ \subseteq \lambda^*(e_1) \cup \lambda^*(e_2)$. Since $\Delta_+ \not\subseteq \lambda^*(e_2)$, there exists $\alpha \in \Delta_+$ such that $\alpha \in \lambda^*(e_1)$ and $\alpha \notin \lambda^*(e_2)$. Since $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e_3)$, $\alpha \in \lambda^*(e_3)$ and hence $\alpha \in A_{13}$. Similarly A_{23} is nonempty.

No component of $\lambda^*(e_1)$ is contained in I_+ and hence no component of $\lambda^*(e_1)$ is contained in I_0. Since no component of $\lambda^*(e_3)$ is contained in I_0, it follows that no component of $\lambda^*(e_1) \cup \lambda^*(e_3)$ is contained in I_0. Since $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e_3)$, $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e_1) \cup \lambda^*(e_3)$. Therefore $\lambda^*(e_1) \cup \lambda^*(e_3) = \lambda^*(f)$. Since $\lambda^*(e_1) \not\subseteq \lambda^*(e_3)$, there exists $\beta \in \lambda^*(e_1)$ such that $\beta \notin \lambda^*(e_3)$. Similarly $\lambda^*(e_2) \cup \lambda^*(e_3) = \lambda^*(f)$, so $\beta \in \lambda^*(e_2)$ and hence $\beta \in A_{12}$.

c) Let $\{e_1, e_2\} \subseteq \Lambda_+$, $\{e_3, f\} \subseteq \Lambda_0$, and $\Delta_+, \Delta_- \not\subseteq \lambda^*(e_3)$. Since $e_1 \vee e_2 = f \in \Lambda_0$, $\Delta_+ \subseteq \lambda^*(e_1) \cup \lambda^*(e_2)$ and hence $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(f)$.

First suppose that no component of $\lambda^*(e_3)$ is contained in I_+. Then since $\Delta_+ \not\subseteq \lambda^*(e_3)$, there exists $e_3' \in \Lambda_+$ such that $\lambda^*(e_3') = \lambda^*(e_3)$. If $e_1 \vee e_3' = e_2 \vee e_3' = f$, then we are in the case of part a). So suppose, without loss of generality, $e_1 \vee e_3' \neq f$. No component of $\lambda^*(e_1)$ is contained in I_+ and hence no component of $\lambda^*(e_1)$ is contained in I_0. Additionally no component of $\lambda^*(e_3) = \lambda^*(e_3')$ is contained in I_0. Therefore no component of $\lambda^*(e_1) \cup \lambda^*(e_3)$ is contained in I_0. Since $e_1 \vee e_3 = f$ and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(f)$, either $\Delta_+ \subseteq \lambda^*(e_1) \cup \lambda^*(e_3) = \lambda^*(e_1) \cup \lambda^*(e_3')$ or $\Delta_- \subseteq \lambda^*(e_1) \cup \lambda^*(e_3) = \lambda^*(e_1) \cup \lambda^*(e_3')$. Therefore if $e_1 \vee e_3' \in \Lambda_0$ it must be that $e_1 \vee e_3' = f$. So $e_1 \vee e_3' \in \Lambda_+$. Then $\Delta_+ \not\subseteq \lambda^*(e_1) \cup \lambda^*(e_3') = \lambda^*(e_1) \cup \lambda^*(e_3)$. So $\Delta_- \subseteq \lambda^*(e_1) \cup \lambda^*(e_3)$. Since $\Delta_- \not\subseteq \lambda^*(e_3)$, there exists $\alpha \in \Delta_-$ such that $\alpha \in \lambda^*(e_1)$ and $\alpha \notin \lambda^*(e_3)$. $\Delta_+ \subseteq \lambda^*(e_1) \cup \lambda^*(e_2)$, there exists $\beta \in \Delta_+$ such that $\beta \in \lambda^*(e_1)$ and $\beta \notin \lambda^*(e_2)$.

If $\Delta_+ \subseteq \lambda^*(e_2) \cup \lambda^*(e_3)$, then $\beta \in \lambda^*(e_3)$ and hence $\beta \in A_{13}$. If, on the other hand, $\Delta_+ \not\subseteq \lambda^*(e_2) \cup \lambda^*(e_3)$ then $\Delta_+ \not\subseteq \lambda^*(e_2) \cup \lambda^*(e_3')$ and hence $e_2 \vee e_3' \in \Lambda_+$. Since $e_2 \vee e_3 = f$ and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(f)$ it must be that $\Delta_- \subseteq \lambda^*(e_2) \cup \lambda^*(e_3)$. So there exists $\gamma \in \Delta_-$ such that $\gamma \in \lambda^*(e_2)$ but $\gamma \notin \lambda^*(e_3)$. But $\Delta_- \subseteq \lambda^*(e_1) \cup \lambda^*(e_3)$, so $\gamma \in \lambda^*(e_1)$ and hence $\gamma \in A_{12}$.

Now suppose a component of $\lambda^*(e_3)$ is contained in I_+. So there does not exist an element $e_3' \in \Lambda_+$ such that $\lambda^*(e_3') = \lambda^*(e_3)$. As before $\Delta_+ \subseteq \lambda^*(e_1) \cup \lambda^*(e_2) \subseteq \lambda^*(f)$. No component of $\lambda^*(e_1) \cup \lambda^*(e_3)$ is contained in I_0. If $\Delta_+, \Delta_- \not\subseteq \lambda^*(e_1) \cup \lambda^*(e_3)$, then $e_1 \vee e_3 < f$. So either $\Delta_+ \subseteq \lambda^*(e_1) \cup \lambda^*(e_3)$ or $\Delta_- \subseteq \lambda^*(e_1) \cup \lambda^*(e_3)$. Suppose $\Delta_+ \subseteq \lambda^*(e_1) \cup \lambda^*(e_3)$. Then there exists $\alpha \in \Delta_+$ such that $\alpha \in \lambda^*(e_3)$ and $\alpha \notin \lambda^*(e_1)$. Since $\Delta_+ \subseteq \lambda^*(e_1) \cup \lambda^*(e_2)$, $\alpha \in \lambda^*(e_2)$ and hence $\alpha \in A_{23}$. Suppose instead $\Delta_- \subseteq \lambda^*(e_1) \cup \lambda^*(e_3)$. By a similar argument to the above either $\Delta_+ \subseteq \lambda^*(e_2) \cup \lambda^*(e_3)$ or $\Delta_- \subseteq \lambda^*(e_2) \cup \lambda^*(e_3)$. If $\Delta_+ \subseteq \lambda^*(e_2) \cup \lambda^*(e_3)$ then there exists $\beta \in \Delta_+$ such that $\beta \in \lambda^*(e_3)$ and $\beta \notin \lambda^*(e_2)$.

Since $\Delta_+ \subseteq \lambda^*(e_1) \cup \lambda^*(e_2)$, $\beta \in \lambda^*(e_1)$ and hence $\beta \in A_{13}$. If $\Delta_- \subseteq \lambda^*(e_2) \cup \lambda^*(e_3)$ then there exists $\gamma \in \Delta_-$ such that $\gamma \in \lambda^*(e_2)$ and $\gamma \notin \lambda^*(e_3)$. Since $\Delta_- \subseteq \lambda^*(e_1) \cup \lambda^*(e_3)$, $\gamma \in \lambda^*(e_1)$ and hence $\gamma \in A_{23}$.

In any case at least one of A_{12}, A_{13} or A_{23} is nonempty.

d) Let $\{e, e_1, e_2, e_3, f\} \subseteq \Lambda_0$ and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e)$. Then $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e_i)$ for all i. No component of $\lambda^*(e_i)$ or $\lambda^*(e_j)$ is contained in I_0, so no component of $\lambda^*(e_i) \cup \lambda^*(e_j)$ is contained in I_0. Therefore $\lambda^*(e_i) \cup \lambda^*(e_j) = \lambda^*(f)$. Since $\lambda^*(e_1) \not\subseteq \lambda^*(e_3)$, there exists $\alpha \in \lambda^*(e_1)$ such that $\alpha \notin \lambda^*(e_3)$. So then $\alpha \in \lambda^*(f) = \lambda^*(e_2) \cup \lambda^*(e_3)$. Therefore $\alpha \in \lambda^*(e_2)$ and hence $\alpha \in A_{12}$. Similarly A_{13} and A_{23} are nonempty.

e) Let $\{e, e_1, e_2, e_3, f\} \subseteq \Lambda_0$ and $\Delta_+, \Delta_- \not\subseteq \lambda^*(f)$. Then $\Delta_+, \Delta_- \not\subseteq \lambda^*(e_i)$ for all i. No component of $\lambda^*(e_i)$ or $\lambda^*(e_j)$ is contained in I_0, so no component of $\lambda^*(e_i) \cup \lambda^*(e_j)$ is contained in I_0. Therefore $\lambda^*(e_i) \cup \lambda^*(e_j) = \lambda^*(f)$. Since $\lambda^*(e_1) \not\subseteq \lambda^*(e_3)$, there exists $\alpha \in \lambda^*(e_1)$ such that $\alpha \notin \lambda^*(e_3)$. So then $\alpha \in \lambda^*(f) = \lambda^*(e_2) \cup \lambda^*(e_3)$. Therefore $\alpha \in \lambda^*(e_2)$ and hence $\alpha \in A_{12}$. Similarly A_{13} and A_{23} are nonempty.

f) Let $\{e, e_1, e_2, e_3, f\} \subseteq \Lambda_0$, $\Delta_+, \Delta_- \not\subseteq \lambda^*(e)$, and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(f)$. If $\Delta_+, \Delta_- \not\subseteq \lambda^*(e_1), \lambda^*(e_2)$ and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e_3)$ the proof that A_{ij} is nonempty for all $i \neq j$ is similar to the proof of b). If $\Delta_+, \Delta_- \not\subseteq \lambda^*(e_1), \lambda^*(e_2), \lambda^*(e_3)$ the proof that A_{ij} is nonempty for some $i \neq j$ is similar to the proof of c).

Finally, suppose $\Delta_+, \Delta_- \not\subseteq \lambda^*(e_1)$ and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e_2), \lambda^*(e_3)$. No component of $\lambda^*(e_i)$ or $\lambda^*(e_j)$ is contained in I_0, so no component of $\lambda^*(e_i) \cup \lambda^*(e_j)$ is contained in I_0. Since $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e_2), \lambda^*(e_3)$, $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e_i) \cup \lambda^*(e_j)$ for all $i \neq j$ and hence $\lambda^*(e_i) \cup \lambda^*(e_j) = \lambda^*(f)$. Since $\lambda^*(e_2) \not\subseteq \lambda^*(e_3)$, there exists $\beta \in \lambda^*(e_2)$ such that $\beta \notin \lambda^*(e_3)$. Since $\lambda^*(e_2) \subseteq \lambda^*(f) = \lambda^*(e_1) \cup \lambda^*(e_3)$, $\beta \in \lambda^*(e_1)$ and hence $\beta \in A_{12}$.

$\square$

Lemma 4.2.5.

a) Let $\{e, e_1, e_2\} \subseteq \Lambda_+$ and $f \in \Lambda_0$. If $\alpha \in A_{ij}$, then $\alpha \in I_+$.

b) Let $\{e, e_i, e_j, f\} \subseteq \Lambda_0$ and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e)$. If $\alpha \in A_{ij}$, then $\alpha \in I_0$.

c) Let $\{e, e_i, e_j, f\} \subseteq \Lambda_0$ and $\Delta_+, \Delta_- \not\subseteq \lambda^*(f)$. If $\alpha \in A_{ij}$, then $\alpha \in I_0$.

d) Let $\{e, e_i, e_j, f\} \subseteq \Lambda_0$, $\Delta_+, \Delta_- \not\subseteq \lambda^*(e), \lambda^*(e_i)$, and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(f)$. If $\alpha \in A_{ij}$, then $\alpha \in I_0$.

Proof. We will prove part a). The proofs of the other parts are similar.

Let $\{e, e_1, e_2\} \subseteq \Lambda_+$ and $f \in \Lambda_0$. Let $\alpha \in A_{ij}$. $\lambda^*(e) \sqcup \{\alpha\}$ cannot be in $\lambda^*(\Lambda_+)$ for if it were then $e_i \wedge e_j > e$. Since $\Delta_+ \not\subseteq \lambda^*(e_i)$, $\Delta_+ \not\subseteq \lambda^*(e) \sqcup \{\alpha\}$. Therefore a component of $\lambda^*(e) \sqcup \{\alpha\}$ is contained in I_+. That is, $\alpha \in I_+$. $\square$

Lemma 4.2.6.

a) Let $\{e, e_i, e_j\} \subseteq \Lambda_+$, $f \in \Lambda_0$, and A_{ij} be nonempty. Then there exists $e' \in \Lambda_+$ and $\alpha \in A_{ij}$ such that $\lambda^*(e') = \lambda^*(e_i)\backslash\{\alpha\}$.

b) Let $\{e, e_i, e_j, f\} \subseteq \Lambda_0$, $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e)$, and A_{ij} be nonempty. Then there exists $e' \in \Lambda_0$ and $\alpha \in A_{ij}$ such that $\lambda^*(e') = \lambda^*(e_i)\backslash\{\alpha\}$.

c) Let $\{e, e_i, e_j, f\} \subseteq \Lambda_0$, $\Delta_+, \Delta_- \not\subseteq \lambda^*(f)$, and A_{ij} be nonempty. Then there exists $e' \in \Lambda_0$ and $\alpha \in A_{ij}$ such that $\lambda^*(e') = \lambda^*(e_i)\backslash\{\alpha\}$.

d) Let $\{e, e_i, e_j, f\} \subseteq \Lambda_0$, $\Delta_+, \Delta_- \not\subseteq \lambda^*(e), \lambda^*(e_i)$, $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(f)$, and A_{ij} be nonempty. Then there exists $e' \in \Lambda_0$ and $\alpha \in A_{ij}$ such that $\lambda^*(e') = \lambda^*(e_i)\backslash\{\alpha\}$.

Proof. We will prove part a). The proofs of the other three parts are similar.

Let $\alpha \in A_{ij}$. By Lemma 4.2.5 $\alpha \in I_+$ and α is not adjacent to an element of $\lambda^*(e)$. Since $\alpha \in \lambda^*(e_i)$, α is adjacent to some $\delta_1 \in \lambda^*(e_i)\backslash(\lambda^*(e) \sqcup \{\alpha\})$ that itself is in a chain of elements of $\lambda^*(e_i)$ that are adjacent to an element of $\lambda^*(e)$. Similarly, since $\alpha \in \lambda^*(e_j)$, α is adjacent to some $\delta_2 \in \lambda^*(e_j)\backslash(\lambda^*(e) \sqcup \{\alpha\})$ that itself is in a chain of elements of $\lambda^*(e_j)$ that are adjacent to an element of $\lambda^*(e)$. If α is an endpoint of $\lambda^*(e_i)$, then $\lambda^*(e_i)\backslash\{\alpha\}$ is in Λ_+. So suppose α is not an endpoint of $\lambda^*(e_i)$. If we cannot pick δ_1 and δ_2 in such a way that $\delta_1 \neq \delta_2$, then α is an endpoint of a path of elements $A \subseteq A_{ij}$ such that $A\backslash\{\alpha\} \subseteq I_+$ is a connected component of $\lambda^*(e_i)\backslash\{\alpha\}$. Let α' be another endpoint of this path. Then $\lambda^*(e_i)\backslash\{\alpha'\}$ and $\lambda^*(e_j)\backslash\{\alpha'\}$ are in $\lambda^*(\Lambda_+)$. So suppose $\delta_1 \neq \delta_2$. If α is adjacent to only two elements of Δ, then it must be that $\delta_2 \in \lambda^*(e_i)$ since α is not an endpoint of $\lambda^*(e_i)$. Furthermore, the path of $\lambda^*(e_i)\backslash\{\alpha\}$ containing δ_2 must be a subset of I_+. Notice that $\delta_2 \in A_{ij}$. We will then choose δ_2 to take the place of α and repeat this process until we find an $\alpha' \in A_{ij}$ such that α' is adjacent to some $\delta \in \lambda^*(e_j)$ such that $\delta \notin I_+$. Such an α' and δ are guaranteed to exist since the original δ_2 was an element of a path of elements of $\lambda^*(e_j)$ that was adjacent to an element of $\lambda^*(e)$. Then no component of $\lambda^*(e_i)\backslash\{\alpha'\}$ is contained in I_+ and hence there exists $e' \in \Lambda_+$ such that $\lambda^*(e') = \lambda^*(e_i)\backslash\{\alpha'\}$. On the other hand, suppose α is adjacent to three elements of Δ: $\delta_1 \in \lambda^*(e_i)$, $\delta_2 \in \lambda^*(e_j)$, and δ_3, no two of which are equal. δ_3 is an endpoint of Δ by the choice of δ_1 and δ_2. If either $\delta_3 \notin \lambda^*(e_i)$ or $\delta_3 \notin I_+$, then δ_3 has no effect on whether or not a component of $\lambda^*(e_i)\backslash\{\alpha\}$ is contained in I_+. So suppose $\delta_3 \in \lambda^*(e_i) \cap I_+$. If $\delta_3 \notin \lambda^*(e_j)$, then choose δ_2 to take the place of α and proceed as before. If $\delta_3 \in \lambda^*(e_j)$, then $\delta_3 \in A_{ij}$ and both $\lambda^*(e_i)\backslash\{\delta_3\}$ and $\lambda^*(e_i)\backslash\{\delta_3\}$

are in $\lambda^*(\Lambda_+)$ since δ_3 is an endpoint of both $\lambda^*(e_i)$ and $\lambda^*(e_j)$. In any case, there exists some $\alpha' \in A_{ij}$ and $e' \in \Lambda_+$ such that $\lambda^*(e') = \lambda^*(e_i)\backslash\{\alpha'\}$. $\square$

Proposition 4.2.7. *Let M be a 2-reducible semisimple reductive monoid with cross section lattice Λ. If Λ has a sublattice isomorphic to M_5, then it has a sublattice isomorphic to N_5.*

Proof. First consider the case where one element of $\{e_1, e_2, e_3\}$ is in Λ_+ and another is in Λ_-. Without loss of generality, suppose $e_1 \in \Lambda_+$ and $e_2 \in \Lambda_-$. Then $e_1 \wedge e_2 = e = \hat{0}$. If $e_3 \in \Lambda_+$, then $e_1 \wedge e_3 = e_+ > \hat{0}$, a contradiction. Similarly if $e_3 \in \Lambda_-$, then $e_2 \wedge e_3 = e_- > 0$. If $e_3 \in \Lambda_0$, then $e_1 \wedge e_3 = e_+ > \hat{0}$. In any case elements of $\{e_1, e_2, e_3\}$ cannot be in both Λ_+ and Λ_-.

Suppose $\{e, e_1\} \subseteq \Lambda_+$ and $\{e_2, e_3, f\} \subseteq \Lambda_0$. Then $e_1 \wedge e_2 \in \Lambda_+$ and $e_2 \wedge e_3 \in \Lambda_0$, so $e_1 \wedge e_2 \neq e_2 \wedge e_3$. That is, $\{e, e_1, e_2, e_3, f\}$ cannot be isomorphic to M_5.

Now suppose that $\{e, e_1, e_2, e_3\} \subseteq \Lambda_+$. and $f \in \Lambda_+$. Then fMf is a $\mathcal{J}$-irreducible monoid with $\{e, e_1, e_2, e_3, f\}$ as a sublattice isomorphic to M_5. This is not possible by Proposition 4.1.4.

Suppose $\{e, e_1, e_2, e_3\} \subseteq \Lambda_+$ and $\{f\} \subseteq \Lambda_0$. By Lemma 4.2.6 there exists $\alpha \in A_{12}$ such that $\lambda^*(e_1)\backslash\{\alpha\}$ is in $\lambda^*(\Lambda_+)$. Notice that since $\alpha \in (\lambda^*(e_1) \cap \lambda^*(e_2))\backslash\lambda^*(e)$ and $\lambda^*(e_1) \neq \lambda^*(e_2)$, $|\lambda^*(e_1)\backslash\lambda^*(e)| \geq 2$. Therefore $e_1 > e' > e$. Since e' and e_2 are both in Λ_+ and $\lambda^*(e') \cup \lambda^*(e_2) = \lambda^*(e_1) \cup \lambda^*(e_2)$, it follows that $e' \vee e_2 = e_1 \vee e_2 = f$. Furthermore, $e' \wedge e_2 = e_1 \wedge e_2 = e$ since e_1 covers e'. Therefore $\{e, e', e_1, e_2, f\}$ is a sublattice of Λ that is isomorphic to N_5. Similarly if $\{e_1, e_2\} \subseteq \Lambda_+$ and $\{e_3, f\} \subseteq \Lambda_0$ then there is a sublattice $\{e, e', e_1, e_2, f\}$ of Λ that is isomorphic to N_5.

Similarly suppose $\{e, e_1, e_2, e_3, f\} \subseteq \Lambda_0$ and either $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e)$; $\Delta_+, \Delta_- \nsubseteq \lambda^*(f)$; or $\Delta_+, \Delta_- \nsubseteq \lambda^*(e), \lambda^*(e_i)$ and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(f)$. Then a similar argument to the above shows that there exists a sublattice $\{e, e', e_i, e_j, f\}$ of Λ that is isomorphic to N_5.

Suppose $\{e, e_1, e_2, e_3, f\} \subseteq \Lambda_0$, $\Delta_+, \Delta_- \nsubseteq \lambda^*(e)$, and $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(e_i)$ for all i. Then $\Delta_+ \sqcup \Delta_- \subseteq \lambda^*(f)$. Notice that $A = \lambda^*(e) \cup \Delta_+ \cup \Delta_- \notin \lambda^*(\Lambda_0)$ for if it were then $e_i \wedge e_j > e$ for all $i \neq j$. So a component of $\lambda^*(e) \cup \Delta_+ \cup \Delta_-$ is contained in I_0. Call this component I'. There are at most two endpoints of I' that are not endpoints of Δ. Let these endpoints be α and β. The case where there is only one endpoint is similar. Notice that $\lambda^*(e_i) \cup \lambda^*(e_j) = \lambda^*(f)$ for $i \neq j$, so if $\gamma \in \lambda^*(e_1)\backslash A$ then γ is an element of either $\lambda^*(e_2)\backslash A$ or $\lambda^*(e_3)\backslash A$. So then $\lambda^*(e) \sqcup \{\gamma\}$ is not in $\lambda^*(\Lambda_0)$. Since $\gamma \notin \Delta_+ \sqcup \Delta_-$, it must be that $\Delta_+, \Delta_- \nsubseteq \lambda^*(e) \sqcup \{\gamma\}$ and hence a component of $\lambda^*(e) \sqcup \{\gamma\}$ is contained in I_0. That is, $\gamma \in I_0$ and hence $\lambda^*(e_i)\backslash A \subseteq I_0$ for all i. $\{\alpha, \beta\} \subseteq I' \subseteq \lambda^*(e_i)$ for all i. Therefore either α or β is connected to an element of $\Delta\backslash I_0$ by a path of elements of $\lambda^*(e_i) \cap I_0$. That is, there exist elements $\{\delta_1, \ldots \delta_k\} \subset (\lambda^*(e_i) \cap \lambda^*(e_j)\backslash A$ for some $i \neq j$ such that $A \sqcup \{\delta_1, \ldots, \delta_k\} \in \lambda^*(\Lambda_0)$. But then $e_i \wedge e_j > e$. This is a contradiction and therefore this case is not possible.

We have now exhausted all cases. In every case the sublattice isomorphic to M_5 was either not possible or we were able to find a sublattice isomorphic to N_5.

□

Corollary 4.2.8. *Let M be a 2-reducible semisimple reductive monoid with cross section lattice Λ. Λ is distributive if and only if it is modular.*

Proof. Suppose Λ is distributive. Then by Theorem 2.4.14 Λ does not have N_5 as a sublattice and hence Λ is modular.

Suppose Λ is not distributive. Then by Theorem 2.4.14 Λ has a sublattice isomorphic to either M_5 or N_5. If Λ has a sublattice isomorphic to M_5, then by Proposition 4.2.7 Λ has a sublattice isomorphic to N_5. Therefore if Λ is not distributive then Λ has a sublattice isomorphic to N_5 and hence is not modular. □

Corollary 4.2.8 tells us that the cross section lattice of a 2-reducible semisimple monoid is distributive if and only if it is modular. We had a similar result for the cross section lattices of $\mathcal{J}$-irreducible monoids. It is then natural to make the following conjecture:

Conjecture 4.2.9. Let M be a semisimple k-reducible reductive monoid with cross section lattice Λ. Λ is distributive if and only if it is modular.

There is an obvious difficulty in proving such a conjecture. In proving the result for the $\mathcal{J}$-irreducible and 2-reducible cases we relied heavily upon Theorem 3.2.2 and Theorem 3.3.2 which allowed us to describe the image of λ^* in terms of the set of simple roots and the type(s) of the monoid. In order to generalize our results we will either need to develop a characterization of the cross section lattices of k-reducible monoids when $k > 2$, or we will have to devise another method to prove our conjecture.

We know that the cross section lattice of a 2-reducible semisimple monoid is distributive if and only if it is modular. Following the blueprint laid out in studying distributive $\mathcal{J}$-irreducible cross section lattices, a natural question to ask is under what conditions will a 2-reducible cross section lattice be distributive/modular. We do not have an answer to this problem. One major difficulty was realized in Example 3.3.5: there may not exist a 2-reducible semisimple reductive monoid corresponding to arbitrarily chosen Δ_+, Δ_-, I_+, and I_-. It is therefore difficult to create examples (or non-examples) that actually exist. The reader should notice that the question of the equivalence of distributive and modular cross section lattices in this chapter was approached purely from a combinatorial viewpoint. That is, aside from the characterization of the cross section lattice in terms of the set of simple roots and the type(s) of the monoid, no algebraic techniques were used in our proofs. In particular, the results in Section 4.2 concerning the cross section lattices of 2-reducible monoids is valid for *any* lattice described by Theorem 3.3.2 regardless of whether or not it corresponds to an actual monoid. It seems possible that a classification of distributive 2-reducible monoids in terms of Δ_+, Δ_-, I_+, and I_- may be different than a classification of distributive lattices that are described by Theorem 3.3.2. It may therefore be advantageous to try to extend the list of possible 2-reducible monoids that is given by Theorem 3.3.6.

The difficulties described above arise when trying to generalize most results concerning $\mathcal{J}$-irreducible monoids to the 2-reducible case. The reader should keep this in mind as the majority of the remainder of this paper will focus on the $\mathcal{J}$-irreducible case.

Chapter 5

Direct Products of $\mathcal{J}$-Irreducible Reductive Monoids

Given a reductive monoid M, our goal is to understand its structure. This structure is encoded by the type map, which we think of as the monoid equivalent of the Dynkin diagram that is so important in describing the structure of semisimple Lie algebras. The type map in turn can be described by the cross section lattice Λ of M. The structure of the cross section lattice can be extremely complicated in general. In Chapter 3 we saw that the structure of the cross section lattice of a $\mathcal{J}$-irreducible reductive monoid can be described very precisely. This structure, however, can still be complicated. In Chapter 4 we were able to determine precisely when such a monoid is distributive. In this special case the structure is a little more tractable.

In Section 5.1 we study how distributive $\mathcal{J}$-irreducible monoids can be expressed as direct products of chains. We then use these results in Section 5.2 to compute zeta polynomial of these cross section lattices.

5.1 Direct Products

We begin by showing how the cross section lattice of a $\mathcal{J}$-irreducible monoid with disconnected set of simple roots can be written as a product of cross section lattices.

Proposition 5.1.1. *Let M be a $\mathcal{J}$-irreducible monoid of type I with set of simple roots $\Delta = \Delta_1 \sqcup \Delta_2$, where Δ_1 and Δ_2 are the connected components of Δ. Let $\Lambda' = \Lambda \backslash \{\hat{0}\}$ be the cross section lattice of M. Let $\Lambda_i' = \Lambda_i \backslash \{\hat{0}\}$ be the cross section lattice of the $\mathcal{J}$-irreducible monoid M_i of type $I_i = I \cap \Delta_i$ for $i = 1, 2$. Then $\Lambda' \cong \Lambda_1' \times \Lambda_2'$.*

Proof. Let $e \in \Lambda'$. There exists $X \subseteq \Delta$ such that $\lambda^*(e) = X$. Let $X_1 = X \cap \Delta_1$ and $X_2 = X \cap \Delta_2$. Since no component of X is contained in I, no component of X_1 is contained in I_1. So there exists $e_1 \in \Lambda_1'$ such that $\lambda_1^*(e_1) = X_1$, where λ_1 is the type map of M_1. Similarly there exists $e_2 \in \Lambda_2'$ such that $\lambda_2^*(e_2) = X_2$.

Let $\varphi : \Lambda' \to \Lambda_1' \times \Lambda_2'$ be given by $\varphi(e) = e_1 \times e_2$. φ is injective because the type maps of $\mathcal{J}$-irreducible monoids are injective. Let $e_1 \times e_2 \in \Lambda_1' \times \Lambda_2'$. Then there exist $X_1, X_2 \subseteq \Delta$ such that $X_1 = \lambda_1^*(e_1)$ and $X_2 = \lambda_2^*(e_2)$. No component of $X_1 \sqcup X_2$ is contained in I, so there exists $e \in \Lambda'$ such that $\lambda^*(e) = X_1 \sqcup X_2$. So $\varphi(e) = e_1 \times e_2$. Therefore φ is surjective and hence bijective.

Let $e, f \in \Lambda'$ such that $e \leq f$. Let $\lambda^*(e) = X$ and $\lambda^*(f) = Y$. Then $X \subseteq Y$. Let $\lambda^*(e) = e_1 \times e_2$ and $\lambda^*(f) = f_1 \times f_2$. Let $\lambda^*(e_i) = X_i = X \cap \Delta_i$ and $\lambda^*(f_i) = Y_i = Y \cap \Delta_i$ for $i = 1, 2$. Since $X \subseteq Y$, $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$. So $e_1 \leq f_1$ in Λ_1' and $e_2 \leq f_2$ in Λ_2'. Therefore $e_1 \times e_2 \leq f_1 \times f_2$ in $\Lambda_1' \times \Lambda_2'$ and hence $\varphi(e) \leq \varphi(f)$.

Since φ is bijective and order preserving, $\Lambda' \cong \Lambda_1' \times \Lambda_2'$. $\square$

Corollary 5.1.2. *Let M be a $\mathcal{J}$-irreducible monoid of type I with set of simple roots $\Delta = \Delta_1 \sqcup \cdots \sqcup \Delta_k$, where Δ_i is a connected component of Δ for $i = 1, \ldots, k$. Let $\Lambda' = \Lambda \backslash \{\hat{0}\}$ be the cross section lattice of M. Let $\Lambda_i' = \Lambda_i \backslash \{\hat{0}\}$ be the cross section lattice of the $\mathcal{J}$-irreducible monoid M_i of type $I_i = I \cap \Delta_i$ for $i = 1, \ldots, k$. Then $\Lambda' \cong \Lambda_1' \times \cdots \times \Lambda_k'$.*

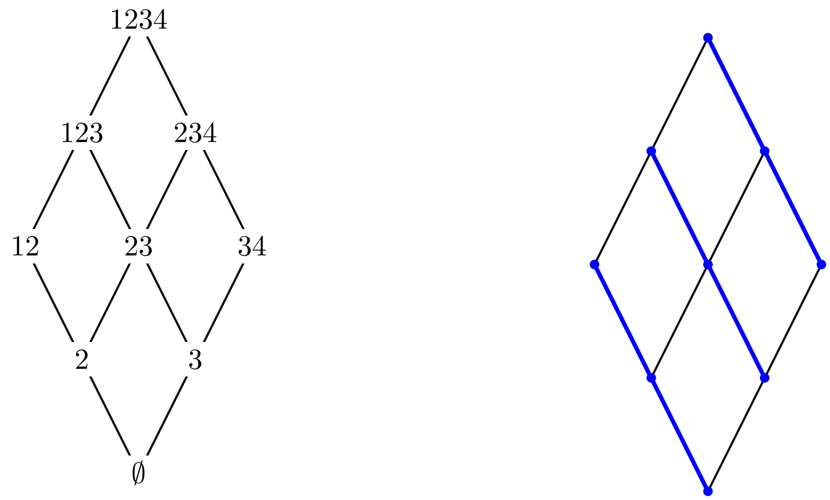
The following theorem is due to Can [1]:

Theorem 5.1.3. *Let M be a distributive $\mathcal{J}$-irreducible monoid of type I with minimal nonzero element e_0. Let the set of simple roots Δ be connected and of type A_n, B_n, C_n, F_4, or G_2. If $|\Delta \backslash I| > 1$ then $\Lambda \backslash \{\hat{0}\}$ is isomorphic to a product of chains. If $|\Delta \backslash I| = 1$ then $\Lambda \backslash \{\hat{0}, e_0\}$ is isomorphic to a product of chains.*

Example 5.1.4. Let M be a $\mathcal{J}$-irreducible monoid of type $I = \{\alpha_1, \alpha_4\}$ and set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_4\}$ of type A_4. The cross section lattice $\Lambda \backslash \{\hat{0}\}$ is shown in Figure 5.1a. $\Delta \backslash I = \{\alpha_2, \alpha_3\}$ is connected, so $\Lambda \backslash \{\hat{0}\}$ is distributive by Corollary 4.1.9. Therefore $\Lambda \backslash \{\hat{0}\}$ is isomorphic to a product of chains by Theorem 5.1.3. It is easy to see by inspection that $\Lambda \backslash \{\hat{0}\} \cong C_3^2$. This product is shown in Figure 5.1b with some edges colored blue to emphasize the product.

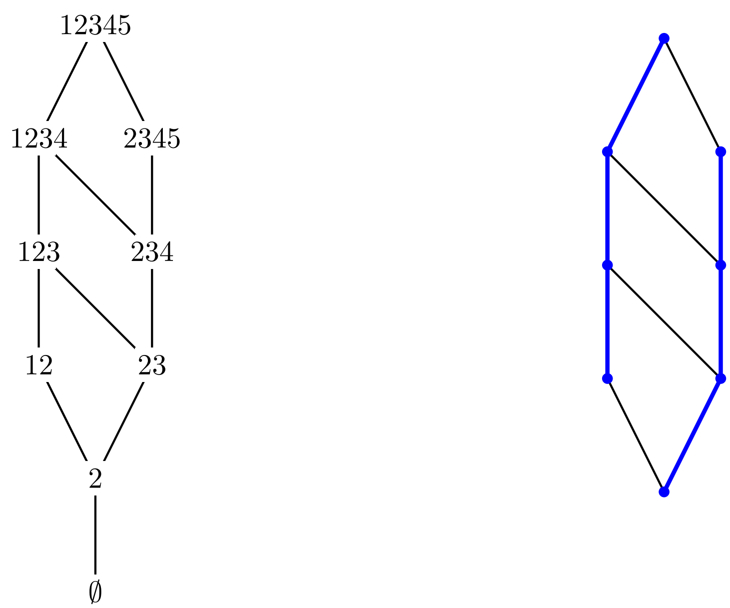
Example 5.1.5. Let M be a $\mathcal{J}$-irreducible monoid of type $I = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$ and set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_5\}$ of type A_5. The cross section lattice $\Lambda \backslash \{\hat{0}\}$ is shown in Figure 5.2a. $\Delta \backslash I = \{\alpha_2\}$ is connected, so $\Lambda \backslash \{\hat{0}\}$ is distributive by Corollary 4.1.9. Therefore $\Lambda \backslash \{\hat{0}, e_0\}$ is isomorphic to a product of chains by Theorem 5.1.3. It is easy to see by inspection that $\Lambda \backslash \{\hat{0}, e_0\} \cong C_4 \times C_2$. This product is shown in Figure 5.2b.

Notice that $\Lambda \backslash \{\hat{0}\} \cong e_0 \oplus (C_4 \times C_2)$. In general if $\Delta \backslash I = \{\alpha\}$, then $\{\alpha\}$ is the only singleton subset of Δ with no component contained in I. That is, Λ only has one element of rank 1. Then $\Lambda \backslash \{\hat{0}\}$ is of the form $e_0 \oplus \Lambda'$, where Λ' is isomorphic to a product of chains.



(a) $\Lambda \backslash \{\hat{0}\}$ with type $I = \{\alpha_1, \alpha_4\}$ (b) $C_3 \times C_3$

Figure 5.1: A distributive cross section lattice as a product of chains



(a) $\Lambda \backslash \{\hat{0}\}$ with type $I = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$ (b) $C_4 \times C_2$

Figure 5.2: A distributive cross section lattice as a product of chains

Theorem 5.1.3 tells us that distributive cross section lattices can be represented as a product of chains if the set of simple roots is connected and is of the type A_n, B_n, C_n, F_4, or G_2. It does not, however, tell us what that product is. In Example 5.1.4 and Example 5.1.5 we were able to determine what this product was by looking at the cross section lattice. This has two drawbacks. We were only able to recognize how to express the cross section lattices in the previous examples as products of chains because the examples were specifically chosen because of their simplicity. As the number of elements of the cross section lattice increases it becomes much more difficult to express it as a product of chains merely through inspection. More importantly, we would like to be able to determine the cross section lattice without explicitly computing λ^* for each element of $\Lambda\backslash\{\hat{0}\}$. The following two propositions tell us how we can do so when the set of simple roots is connected and of the type A_n, B_n, C_n, F_4, or G_2.

Proposition 5.1.6. *Let M be a distributive $\mathcal{J}$-irreducible monoid of type I and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be connected and of type A_n, B_n, C_n, F_4, or G_2.*

a) If $I = \{\alpha_1, \ldots, \alpha_k\}$ and $|\Delta\backslash I| \geq 2$, then $\Lambda\backslash\{\hat{0}\} \cong C_{k+2} \times C_2^{n-k-1}$.

b) If $I = \{\alpha_1, \ldots, \alpha_{n-1}\}$, then $\Lambda\backslash\{\hat{0}, e_0\} \cong C_n$.

Proof.

a) If $I = \{\alpha_1, \ldots, \alpha_k\}$, then $\Delta\backslash I$ is connected and hence $\Lambda\backslash\{\hat{0}\}$ is distributive by Corollary 4.1.9 and isomorphic to a product of chains by Theorem 5.1.3.

We will first find the number of elements of $\Lambda\backslash\{\hat{0}\}$. This is equal to the number of subsets of Δ with no component contained in $I = \{\alpha_1, \ldots, \alpha_k\}$. Since $|\Delta\backslash I| = n-k$, there are 2^{n-k} subsets X of Δ such that $X \cap I = \emptyset$. If X is in the image of λ^* and $X \cap I \neq \emptyset$, then $\{\alpha_k, \alpha_{k+1}\} \subseteq X$. There are k possibilities of $X \cap I$: $\{\alpha_k\}, \{\alpha_{k-1}, \alpha_k\}, \ldots, \{\alpha_1, \ldots, \alpha_k\}$. There are 2^{n-k-1} possibilities of $X \cap \{\alpha_{k+2}, \ldots, \alpha_n\}$. There are therefore

$$2^{n-k} + k \cdot 2^{n-k-1} = (k+2) \cdot 2^{n-k-1}$$

possible subsets of X that are in the image of λ^*.

Notice that C_n is a chain of n elements, so its length is $n-1$. Furthermore, $C_m \times C_n$ has mn elements and the length of a maximal chain is $(m-1)+(n-1) = m+n-2$ elements. Since $\Lambda\backslash\{\hat{0}\}$ is a product of chains, $\Lambda\backslash\{\hat{0}\} \cong C_{i_1} \times C_{i_2} \times \cdots C_{i_l}$ for some l. Since $\Lambda\backslash\{\hat{0}\}$ has $(k+2) \cdot 2^{n-k-1}$ elements and the length of a maximal chain is n, it follows that $i_1 \cdots i_l = (k+2) \cdot 2^{n-k-1}$ and $i_1 + \cdots + i_l - l = n$. Notice that if $i_1 = k+2$ and $i_j = 2$ for $2 \leq j \leq n-k$, then $i_1 \cdots i_{n-k} = (k+2) \cdot 2^{n-k-1}$ and $i_1 + \cdots + i_{n-k} - (n-k) = k+2+(n-k-1) \cdot 2 - n + k = n$. Furthermore, this is the only solution. Therefore $\Lambda\backslash\{\hat{0}\} \cong C_{k+2} \times C_2^{n-k-1}$.

b) If $I = \{\alpha_1, \ldots, \alpha_{n-1}\}$, then the only subsets of Δ with no component contained in I are $\emptyset, \{\alpha_n\}, \{\alpha_{n-1}, \alpha_n\}, \ldots, \{\alpha_1, \ldots, \alpha_n\}$. Then $\Lambda \backslash \{\hat{0}\}$ is a chain of $n+1$ elements, and hence $\Lambda \backslash \{\hat{0}\} \cong C_{n+1}$.

□

Proposition 5.1.7. *Let M be a distributive $\mathcal{J}$-irreducible monoid of type I and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be connected and of type A_n, B_n, C_n, F_4, or G_2.*

a) If $I = \{\alpha_1, \ldots, \alpha_k\} \sqcup \{\alpha_l, \ldots, \alpha_n\}$ and $|\Delta \backslash I| \geq 2$, then $\Lambda \backslash \{\hat{0}\} \cong C_{k+2} \times C_{n-l+3} \times C_2^{l-k-3}$.

b) If $I = \{\alpha_1, \ldots, \alpha_k\} \sqcup \{\alpha_{k+2}, \ldots, \alpha_n\}$, then $\Lambda \backslash \{\hat{0}, e_0\} \cong C_{k+1} \times C_{n-k}$.

Proof.

a) Let $I = I_1 \sqcup I_2 = \{\alpha_1, \ldots, \alpha_k\} \sqcup \{\alpha_l, \ldots, \alpha_n\}$. Then $\Delta \backslash I = \{\alpha_{k+1}, \ldots, \alpha_{l-1}\}$ is connected and hence $\Lambda \backslash \{\hat{0}\}$ is distributive by Corollary 4.1.9 and isomorphic to a product of chains by Theorem 5.1.3. Additionally, since $|\Delta \backslash I| = l - k - 1 \geq 2$, it follows that $l - k - 3 \geq 0$.

The number of elements of λ^* that are subsets of $\Delta \backslash I$ is 2^{l-k-1}. If $X \subseteq \Delta$ is in the image of λ^* such that $X \cap I_1 \neq \emptyset$ and $X \cap I_2 = \emptyset$, then $\{\alpha_k, \alpha_{k+1}\} \subseteq X$. There are then $k \cdot 2^{l-k-2}$ such subsets X. Similarly, if $X \subseteq \Delta$ is in the image of λ^* such that $X \cap I_2 \neq \emptyset$ and $X \cap I_1 = \emptyset$, then there are $(n - l + 1) \cdot 2^{l-k-2}$ such subsets X. If $X \subseteq \Delta$ is in the image of λ^* such that $X \cap I_1 \neq \emptyset$ and $X \cap I_2 \neq \emptyset$, then there are $k(n-l+1) \cdot 2^{l-k-3}$ such subsets X. The total number of elements in $\Lambda \backslash \{\hat{0}\}$ is therefore

$$
\begin{aligned}
|\Lambda \backslash \{\hat{0}\}| &= 2^{l-k-1} + k \cdot 2^{l-k-2} + (n-l+1) \cdot 2^{l-k-2} + k \cdot (n-l+1) \cdot 2^{l-k-3} \\
&= [4 + 2k + 2(n-l+1) + k(n-l+1)] \cdot 2^{l-k-3} \\
&= (k+2)(n-l+3) \cdot 2^{l-k-3}.
\end{aligned}
$$

Since $\Lambda \backslash \{\hat{0}\}$ is a product of chains, $\Lambda \backslash \{\hat{0}\} \cong C_{i_1} \times C_{i_2} \times \cdots C_{i_m}$, where $i_1 \cdots i_m = (k+2)(n-l+3) \cdot 2^{l-k-3}$ and $i_1 + \cdots + i_m - m = n$. Notice that if $i_1 = k+2$, $i_2 = n-l+3$, and $i_3 = \cdots = i_{l-k-1} = 2$, then $i_1 \cdots i_{l-k-1} = (k+2)(n-l+3) \cdot 2^{l-k-3}$ and $i_1 + \cdots i_{l-k1} - (l-k-1) = (k+2) + (n-l+3) + 2(l-k-3) - (l-k-1) = n$. Furthermore, this is the only possibility. Therefore $\Lambda \backslash \{\hat{0}\} \cong C_{k+2} \times C_{n-l+3} \times C_2^{l-k-3}$.

b) Let $I = I_1 \sqcup I_2 = \{\alpha_1, \ldots, \alpha_k\} \sqcup \{\alpha_{k+2}, \ldots, \alpha_n\}$. Then $\Delta \backslash I = \{\alpha_{k+1}\}$ is connected and hence $\Lambda \backslash \{\hat{0}, e_0\}$ is distributive by Corollary 4.1.9 and isomorphic to a product of chains by Theorem 5.1.3.

The number of elements of λ^* that are subsets of $\Delta \backslash I = \{\alpha_{k+1}\}$ is 2. However, we disregard the empty set because $\lambda^*(e_0) = \emptyset$ and we are only concerned with counting the elements of $\Lambda\backslash\{\hat{0}, e_0\}$. If $X \subseteq \Delta$ is in the image of λ^* such that $X \cap I_1 \neq \emptyset$ and $X \cap I_2 = \emptyset$, then there are k such subsets X. If $X \subseteq \Delta$ is in the image of λ^* such that $X \cap I_2 \neq \emptyset$ and $X \cap I_1 = \emptyset$, then there are $n-k-1$ such subsets X. If $X \subseteq \Delta$ is in the image of λ^* such that $X \cap I_1 \neq \emptyset$ and $X \cap I_2 \neq \emptyset$, then there are $k(n-k-1)$ such subsets X. The total number of elements in $\Lambda\backslash\{\hat{0}, e_0\}$ is therefore

$$\begin{aligned} |\Lambda\backslash\{\hat{0}, e_0\}| &= 1 + k + n - k - 1 + k(n-k-1) \\ &= n + kn - k^2 - k \\ &= (n-k)(k+1). \end{aligned}$$

Since $\Lambda\backslash\{\hat{0}, e_0\}$ is a product of chains, $\Lambda\backslash\{\hat{0}, e_0\} \cong C_{i_1} \times C_{i_2} \times \cdots C_{i_m}$, where $i_l \cdots i_m = (n-k)(k+1)$ and $i_1 + \cdots + i_m - m = n-1$, the length of a maximal chain of $\Lambda\backslash\{\hat{0}, e_0\}$. Notice that if $i_1 = n-k$, and $i_2 = k+1$, then $i_1 \cdot i_2 = (n-k)(k+1)$ and $i_1 + i_2 - 2 = (n-k) + (k+1) - 2 = n-1$. Furthermore, this is the only possibility. Therefore $\Lambda\backslash\{\hat{0}, e_0\} \cong C_{n-k} \times C_{k+1}$.

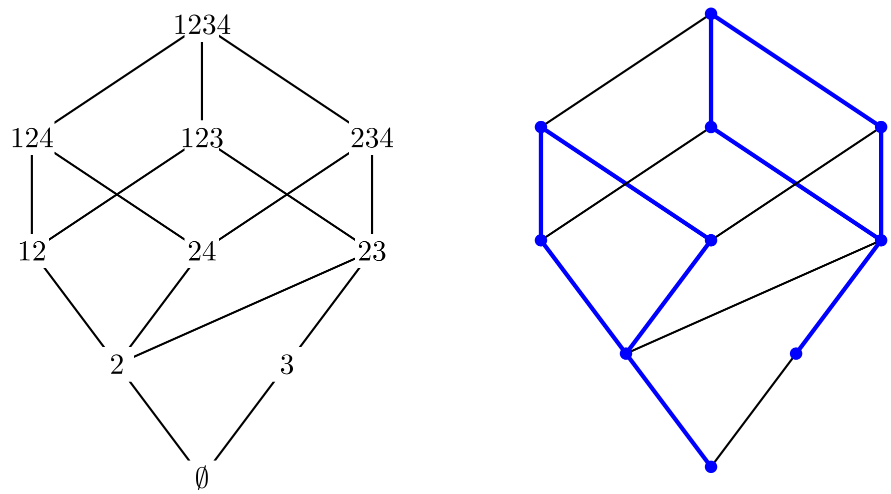
□

Example 5.1.8. Consider the monoid from Example 5.1.4 where $I = \{\alpha_1, \alpha_4\}$ and $\Delta = \{\alpha_1, \ldots, \alpha_4\}$. $n = 4$, $k = 1$, and $l = 4$ so by Proposition 5.1.7 $\Lambda\backslash\{\hat{0}\} \cong C_3 \times C_3 \times C_2^0 \cong C_3^2$.

Similarly, consider the monoid from Example 5.1.5 where $I = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$ and $\Delta = \{\alpha_1, \ldots, \alpha_5\}$. $n = 5$ and $k = 1$ so $\Lambda\backslash\{\hat{0}, e_0\} \cong C_2 \times C_4$.

Example 5.1.9. Let $\Delta = D_4 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where $s_{\alpha_1}s_{\alpha_2} \neq s_{\alpha_2}s_{\alpha_1}$, $s_{\alpha_2}s_{\alpha_3} \neq s_{\alpha_3}s_{\alpha_2}$, and $s_{\alpha_2}s_{\alpha_4} \neq s_{\alpha_4}s_{\alpha_2}$. Let $I = \{\alpha_1, \alpha_4\}$. Notice that $\Delta\backslash I$ is connected and therefore $\Lambda\backslash\{\hat{0}\}$ is distributive. However, $\Lambda\backslash\{\hat{0}\}$ is not rank symmetric and hence not locally rank symmetric. Therefore by Theorem 2.4.23, $\Lambda\backslash\{\hat{0}\}$ cannot be expressed as a direct product of chains or even as a direct product of primary q-lattices. However, it can easily be seen that $\Lambda\backslash\{\hat{0}\} \cong C_2 \times (C_1 \oplus (C_2 \times C_2))$. Figure 5.3a shows the cross section lattice $\Lambda\backslash\{\hat{0}\}$. Figure 5.3b shows the two copies of $C_1 \oplus +(C_2 \times C_2)$ that appear in the cross section lattice.

Example 5.1.9 shows that the structure of distributive cross section lattices can be much more complicated when Δ is of the type D_n. The cross section lattice may still be a product of chains, as is seen in the trivial case when $I = \emptyset$. In this case $\Lambda\backslash\{\hat{0}\}$ is isomorphic to the Boolean lattice $C_2^m \cong 2^{|\Delta|}$. However, Example 5.1.9 shows that this is not necessarily the case. It should be pointed out, however, that the cross section lattice is "almost" isomorphic to a product of chains in the sense that $C_1 \oplus (C_2 \times C_2)$ is a product of two chains with a new $\hat{0}$ adjoined to the bottom of the lattice. Although the structure of the cross section lattice in this case is difficult to describe in general, it appears that $\Lambda\backslash\{\hat{0}\}$ can have one of two forms, described as follows:



(a) $\Lambda\backslash\{\hat{0}\}$ (b) $C_2 \times (C_1 \oplus (C_2 \times C_2))$

Figure 5.3: Δ of type D_4, $I = \{\alpha_1, \alpha_4\}$

Conjecture 5.1.10. Let M be a distributive $\mathcal{J}$-irreducible monoid of type I and let Δ be connected of type D_n.

a) If $|\Delta\backslash I| \geq 2$, then $\Lambda\backslash\{\hat{0}\}$ is either a product of chains or the product of a product of chains with a lattice of the form $C_i \oplus (C_j \times C_k)$.

b) If $|\Delta\backslash I| = 1$, then $\Lambda\backslash\{\hat{0}, e_0\}$ is the product of a product of chains with a lattice of the form $C_i \oplus (C_j \times C_k)$.

Let M be a distributive $\mathcal{J}$-irreducible monoid of type I where $\Delta = \Delta_1 \sqcup \cdots \sqcup \Delta_k$ is not connected. Let the Δ_i be the connected components of Δ for $1 \leq i \leq k$. If we are able to express the distributive $\mathcal{J}$-irreducible monoid M_i of type $I_i = I \cap \Delta_i$ as a product of chains using Proposition 5.1.6, Proposition 5.1.7, or Conjecture 5.1.10, then we can use Corollary 5.1.2 to express the cross section lattice of M as a product of lattices. In particular, if Δ_i is of type A_n, B_n, C_n, F_4, or G_2 and $|\Delta_i\backslash I_i| \geq 2$ for all i, then the cross section lattice $\Lambda\backslash\{\hat{0}\}$ of M will be a product of chains.

Example 5.1.11. Let M be a $\mathcal{J}$-irreducible monoid with set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_4\} \sqcup \{\alpha_5, \ldots, \alpha_9\} \sqcup \{\alpha_{10}, \alpha_{11}\}$ of type $A_4 \oplus A_5 \oplus A_2$. Let the type of M be $I = \{\alpha_3, \alpha_4, \alpha_5, \alpha_9\}$. Then $\Lambda\backslash\{\hat{0}\} \cong C_4 \times C_2 \times C_3 \times C_3 \times C_2 \times C_2^2 \cong C_2^4 \times C_3^2 \times C_4$. Notice that $\Lambda\backslash\{\hat{0}\}$ has $2^4 \cdot 3^2 \cdot 4 = 576$ elements. The preceding propositions allow us to calculate the cross section lattice in just a few minutes. Calculating it explicitly would take an inordinately long time.

Example 5.1.12. Let M be a $\mathcal{J}$-irreducible monoid with $\Delta = \Delta_1 \sqcup \Delta_2 = \{\alpha_1, \alpha_2\} \sqcup \{\alpha_3, \alpha_4, \alpha_5\}$ where Δ_1 is of type A_2 and Δ_2 is of type A_3. Let $I = \{\alpha_1, \alpha_3, \alpha_5\}$. Then $I_1 = \{\alpha_1\}$ and $I_2 = \{\alpha_3, \alpha_5\}$. Notice that $\Delta_1 \backslash I_1 = \{\alpha_2\}$ and $\Delta_2 \backslash I_2 = \{\alpha_4\}$ are connected, and hence $\Lambda \backslash \{\hat{0}\}$ is distributive.

Let M_i be the $\mathcal{J}$-irreducible monoid with set of simple roots Δ_i and let I_i be the type for $i = 1, 2$. Let Λ_i be the cross section lattice. $\Lambda_1 \backslash \{\hat{0}\} \cong C_3$. By Proposition 5.1.7 $\Lambda_2 \backslash \{\hat{0}, e_0\} \cong C_2^2$, and hence $\Lambda_2 \backslash \{\hat{0}\} \cong C_1 \oplus C_2^2$. Then by Corollary 5.1.2, $\Lambda \backslash \{\hat{0}\} \cong \Lambda_1 \backslash \{\hat{0}\} \times \Lambda_2 \backslash \{\hat{0}\} \cong C_3 \times (C_1 \oplus C_2^2)$. Notice that even though the connected components of Δ are of type A_2 and A_3, the cross section lattice $\Lambda \backslash \{\hat{0}\}$ is not a product of chains since $|\Delta_2 \backslash I_2| = 1$. Figure 5.4a shows the cross section lattice $\Lambda \backslash \{\hat{0}\}$. Figure 5.4b shows the cross section lattice as the product $C_3 \times (C_1 \oplus C_2^2)$.

5.2 Zeta Polynomials

Calculating the zeta polynomial of a lattice is difficult in general because it requires us to be able to count all of the multichains of the lattice. When we are able to express a cross section lattice as a product of chains, however, we can avoid this issue by instead only calculating the number of multichains for the respective chains in the product. The zeta polynomial has some interesting combinatorial properties which are listed in Chapter 2. At the very least, it provides us with an interesting application of the results from the previous section.

Proposition 5.2.1. *Let M be a distributive $\mathcal{J}$-irreducible monoid of type I and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be connected and of type A_n, B_n, C_n, F_4, or G_2.*

a) If $I = \{\alpha_1, \ldots, \alpha_k\}$ and $|\Delta \backslash I| \geq 2$, then $Z(\Lambda \backslash \{\hat{0}\}, x) = x^{n-k-1} \binom{x+k}{k+1}$.

b) If $I = \{\alpha_1, \ldots, \alpha_{n-1}\}$, then $Z(\Lambda \backslash \{\hat{0}\}, x) = \binom{x+k-2}{k-1}$.

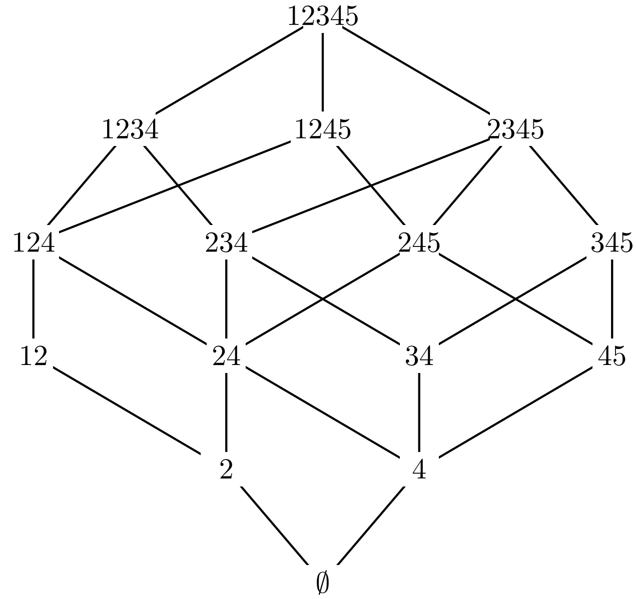
c) If $I = \{\alpha_1, \ldots, \alpha_k\} \sqcup \{\alpha_l, \ldots, \alpha_n\}$ and $|\Delta \backslash I| \geq 2$, then

$$Z(\Lambda \backslash \{\hat{0}\}, x) = x^{l-k-3} \binom{x+k}{k+1} \binom{x+n-l+1}{n-l+2}.$$

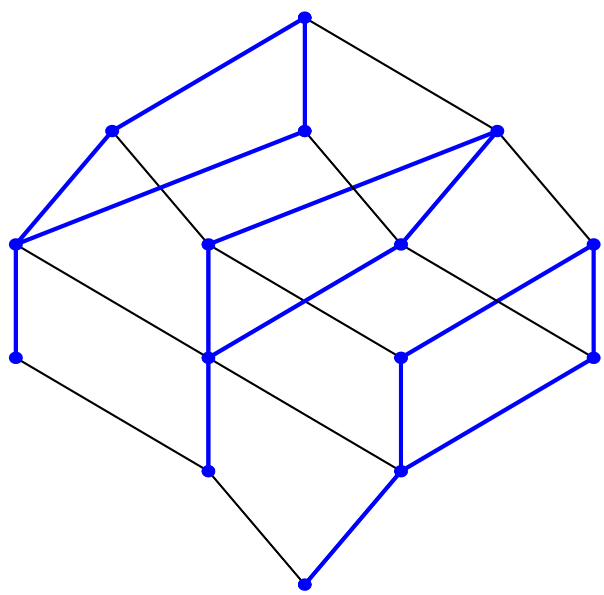
d) If $I = \{\alpha_1, \ldots, \alpha_k\} \sqcup \{\alpha_{k+2}, \ldots, \alpha_n\}$, then

$$Z(\Lambda \backslash \{\hat{0}, e_0\}, x) = \binom{x+k-1}{k} \binom{x+n-k-2}{n-k-1}.$$

Proof. $Z(\Lambda \backslash \{\hat{0}\}, x)$ is the number of multichains $e_1 \leq e_2 \leq \cdots \leq e_{x-1}$, where $x \geq 2$ is an



(a) $\Lambda \backslash \{\hat{0}\}$



(b) $C_3 \times (C_1 \oplus C_2^2)$

Figure 5.4: Δ of type $A_2 \cup A_3$, $I = \{\alpha_1, \alpha_3, \alpha_5\}$

integer. The number of multichains of length $x-1$ in the chain C_k can be calculated as

$$Z(C_k, x) = \left(\!\!\binom{k}{x-1}\!\!\right) = \binom{x+k-2}{k-1}.$$

Additionally, $Z(P \times Q, x) = Z(P, x)Z(Q, x)$ for any two posets P and Q. The results then follow from Propositions 5.1.6 and 5.1.7. $\square$

Chapter 6

Möbius Functions and Characteristic Polynomials

The Möbius function is an important invariant of a lattice that helps describe its structure. Its generating function is the characteristic polynomial. In Section 6.1 we see how the Möbius function of a cross section lattice can be computed in terms of relatively complemented intervals. In Section 6.2 we calculate the characteristic polynomial of a certain class of cross section lattices.

6.1 Möbius Functions of Cross Section Lattices

In [9] Putcha calculated the Möbius function of an arbitrary cross section lattice:

Theorem 6.1.1. *Let M be a reductive monoid with cross section lattice Λ.*

a) Let $e, f \in \Lambda$, $e \leq f$. Then

$$\mu(e,f) = \begin{cases} (-1)^{rk(e)+rk(f)} & \textit{if } [e,f] \textit{ is relatively complemented,} \\ 0 & \textit{otherwise.} \end{cases}$$

b) Let $e, f \in \Lambda$, $e \leq f$. Then $[e,f]$ is relatively complemented if and only if $\lambda_*(e) \cap \lambda^*(f) = \emptyset$.

Theorem 6.1.1 holds for any reductive monoid M, not just $\mathcal{J}$-irreducible or 2-reducible monoids. Therefore calculating the Möbius function of a cross section lattice is an issue of determining when an interval is relatively complemented. This problem was solved by Can in [1] for the $\mathcal{J}$-irreducible case:

Theorem 6.1.2. *Let M be a $\mathcal{J}$-irreducible monoid with cross section lattice Λ. An interval of $\Lambda \backslash \{\hat{0}\}$ is relatively complemented if and only if it is isomorphic to a Boolean lattice.*

Example 6.1.3. Let M be the $\mathcal{J}$-irreducible monoid of type $I = \{\alpha_1, \alpha_2, \alpha_3\}$ and set simple roots $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. We calculated the cross section lattice and type map in Example 3.2.4. The cross section lattice $\Lambda \backslash \{\hat{0}\}$ and type map have been reproduced in in Figure 6.1a and Figure 6.1b, respectively. Using Theorem 6.1.2 we can easily tell which intervals are relatively complemented. The Möbius function on $\Lambda \backslash \{\hat{0}\}$ is shown in Figure 6.1c.

One question we may ask is if the result of Theorem 6.1.2 generalizes to k-reducible monoids. The answer is no.

Example 6.1.4. Let M be a semisimple 2-redicuble monoid with types $I_+ = I_- = \emptyset$ and set of simple roots $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$ where $s_{\alpha_i} s_{\alpha_j} \neq s_{\alpha_j} s_{\alpha_i}$ if $|i - j| = 1$. Let $\Delta_+ = \{\alpha_1, \alpha_2\}$ and $\Delta_- = \{\alpha_3\}$. Notice that by Theorem 3.3.6 such a monoid exists. The cross section lattice Λ is shown in Figure 6.2a, where the vertices are labeled by the indices of the α_i. The interval $[\emptyset, 123]$ is shown in Figure 6.2b, where $\emptyset \in \Lambda_+$. This interval is relatively complemented but it is not a Boolean lattice.

6.2 Characteristic Polynomials of Cross Section Lattices

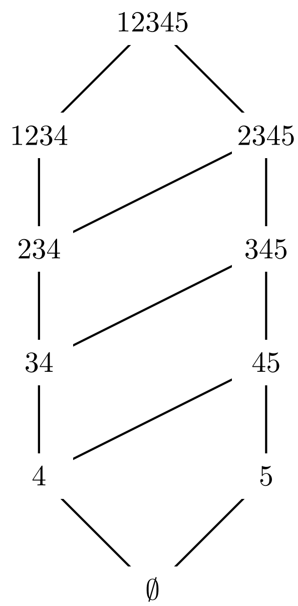
Proposition 6.2.1. *Let M be a $\mathcal{J}$-irreducible reductive monoid of type I, where $|I| = k$. The characteristic polynomial of $\Lambda \backslash \{\hat{0}\}$ is $\chi(\Lambda \backslash \{\hat{0}\}, x) = x^k (x-1)^{n-k}$.*

Proof. $\chi(\Lambda \backslash \{\hat{0}\}, x) = \sum_{i=0}^{n} w_i x^{n-i}$, where $w_i = \sum_{\substack{e \in \Lambda \backslash \{\hat{0}\} \\ rk(e) = i}}^{n} \mu(e_0, e)$ is the ith Whitney number of $\Lambda \backslash \{\hat{0}\}$ of the first kind. Since $rk(e_0) = 0$, by Theorem 6.1.1 we have

$$\mu(e_0, e) \quad = \quad \begin{cases} (-1)^{rk(e)} & \text{if } [e_0, e] \text{ is relatively complemented,} \\ 0 & \text{otherwise.} \end{cases}$$

$[e_0, e]$ is relatively complemented if and only if $\lambda_*(e_0) \cap \lambda^*(e) = I \cap \lambda^*(e) = \emptyset$. To find w_i we therefore need to find the number of subsets of $\Delta \backslash I$ with i elements. This is $\binom{|\Delta| - |I|}{i} = \binom{n-k}{i}$ and w_i is positive if i is even and negative if i is odd. Additionally $w_i = 0$ if $i > n - k$. Notice that by the Binomial Theorem

$$(x-1)^{n-k} = \sum_{i=0}^{n-k} \binom{n-k}{i} (-1)^i x^{n-k-i}.$$



(a) Cross section lattice $\Lambda \backslash \{\hat{0}\}$

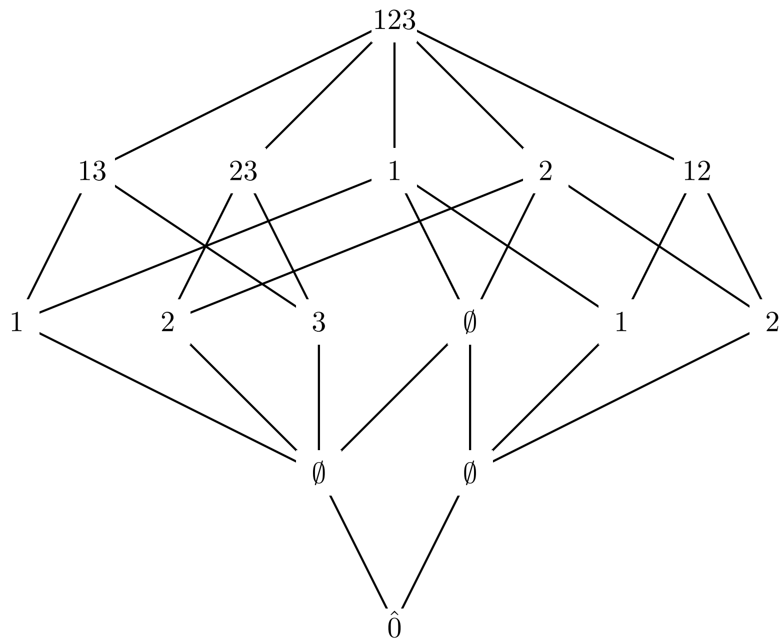
$\lambda^*(e)$	$\lambda_*(e)$	$\lambda(e)$
$\emptyset$	123	123
4	12	412
5	123	**1235**
34	1	134
45	12	**1245**
234	$\emptyset$	234
345	1	**1345**
1234	$\emptyset$	**1234**
2345	$\emptyset$	**2345**
12345	$\emptyset$	12345

(b) Type map of M

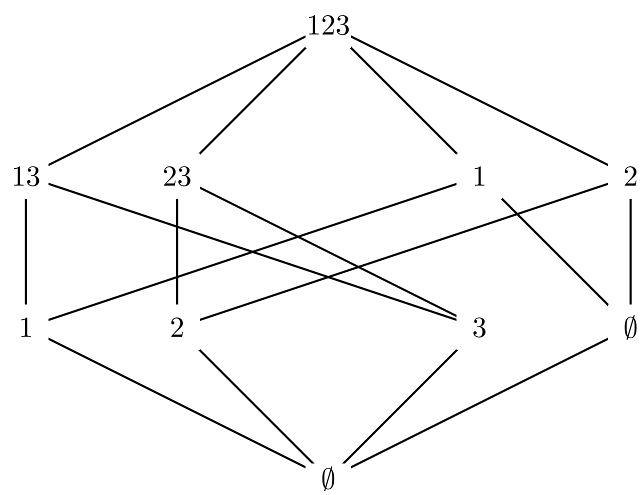
	$\emptyset$	4	5	34	45	234	345	1234	2345	12345
$\emptyset$	1	-1	-1	0	1	0	0	0	0	0
4	-	1	-	-1	-1	0	1	0	0	0
5	-	-	1	-	-1	-	0	-	0	0
34	-	-	-	1	-	-1	-1	0	1	0
45	-	-	-	-	1	-	-1	-	0	0
234	-	-	-	-	-	1	-	-1	-1	1
345	-	-	-	-	-	-	1	-	-1	0
1234	-	-	-	-	-	-	-	1	-	-1
2345	-	-	-	-	-	-	-	-	1	-1
12345	-	-	-	-	-	-	-	-	-	1

(c) Möbius function

Figure 6.1: $\mathscr{J}$-irreducible monoid with $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and $I = \{\alpha_1, \alpha_2, \alpha_3\}$



(a) Cross section lattice Λ



(b) Relatively complemented interval $[\emptyset, 123]$, where $\emptyset \in \Lambda_+$

Figure 6.2: 2-reducible monoid with $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$, $I_+ = I_- = \emptyset$, $\Delta_+ = \{\alpha_1, \alpha_2\}$ and $\Delta_- = \{\alpha_3\}$

Therefore

$$\begin{aligned} x^k(x-1)^{n-k} &= \sum_{i=0}^{n-k} \binom{n-k}{i} (-1)^i x^{n-i} \\ &= \sum_{i=0}^{n} w_i x^{n-i} \\ &= \chi(\Lambda \backslash \{\hat{0}\}, x). \end{aligned}$$

□

Notice that Proposition 6.2.1 does not depend upon whether or not Δ is connected since Theorem 6.1.1 does not depend upon whether or not Δ is connected.

Example 6.2.2. Consider the $\mathcal{J}$-irreducible monoid from Example 6.1.3 whose Möbius function is shown in Figure 6.2b. The ith Whitney numbers of $\Lambda\backslash\{\hat{0}\}$ of the first kind $w_i = \displaystyle\sum_{\substack{e\in\Lambda\backslash\{\hat{0}\} \\ rk(e)=i}}^{n} \mu(e_0, e)$ are $w_0 = 1$, $w_1 = -2$, $w_2 = 1$, $w_3 = w_4 = w_5 = 0$. The characteristic polynomial is then $\chi(\Lambda\backslash\{\hat{0}\}, x) = \displaystyle\sum_{i=0}^{n} w_i x^{n-i} = x^5 - 2x^4 + x^3 = x^3(x-1)^2$. Notice that in this example $k = |I| = 3$, $n = 5$, and $\chi(\Lambda\backslash\{\hat{0}\}, x) = x^k(x-1)^{n-k}$.

Recall that the core of a monoid M is the set $C = \vee\{h \in \Lambda_1\}$, where Λ_1 is the set of minimal nonzero elements of the cross section lattice. We will say that the core is *full* if $\{\hat{0}\} \sqcup C \cong 2^{\Lambda_1}$. Notice that $\mathcal{J}$-irreducible and 2-reducible monoids have full core.

Proposition 6.2.3. *Let M is a semisimple k-reducible monoid with full core and cross section lattice Λ. Then $\chi(\Lambda, x) = x^{n-k+1}(x-1)^k$.*

Proof. Let $\Lambda_1 = \{e_1, \ldots, e_k\}$ be the minimal nonzero elements of Λ and let $C = \vee\{h \in \Lambda_1\}$ be the core of Λ. Notice that $\{\hat{0}\} \sqcup C \cong 2^{\Lambda_1}$, so $[0, h]$ is relatively complemented for all $h \in C$. Let $e \in \Lambda\backslash(\{\hat{0} \sqcup C)\}$. Then $e \in \Lambda_h$ for some $h \in C$. Suppose $[0, e]$ is relatively complemented. Then there exists $x \in [0, e]$ such that $x \wedge h = 0$ and $x \vee h = e$. Clearly $x \neq h$. If $x \in C$, then $x \vee h = h \neq e$. If $x \in \Lambda_h$, then $x \wedge h = h \neq 0$. Therefore $[0, e]$ cannot be relatively complemented unless $e \in C$. The only elements $e \in \Lambda$ of rank i such that $[0, e]$ is relatively complemented are the elements of C of rank i. There are $\binom{k}{i}$ such elements so $w_i = (-1)^i \binom{k}{i}$ if $i \leq k$ and 0 otherwise. Notice that by the Binomial Theorem

$$(x-1)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^i x^{k-i}.$$

Therefore

$$\begin{aligned} x^{n-k+1}(x-1)^k &= \sum_{i=0}^{k} \binom{k}{i} (-1)^i x^{n-i+1} \\ &= \sum_{i=0}^{n+1} w_i x^{n-i+1} \\ &= \chi(\Lambda, x). \end{aligned}$$

$\square$

Corollary 6.2.4.

a) Let M be a $\mathcal{J}$-irreducible monoid. Then $\chi(\Lambda, x) = x^n(x-1)$.

b) Let M be a 2-reducible semisimple monoid. Then $\chi(\Lambda, x) = x^{n-1}(x-1)^2$.

Example 6.2.5. Consider the $\mathcal{J}$-irreducible monoid from Example 6.1.3 whose cross section lattice $\Lambda \backslash \{\hat{0}\}$ is in Figure 6.1a. The Hasse diagram of Λ is $\hat{0} \oplus \Lambda \backslash \{\hat{0}\}$. The only interval of Λ that is relatively complemented is $[0, \emptyset]$. The characteristic polynomial is therefore $\chi(\Lambda, x) = x^6 - x^5 = x^5(x-1) = x^n(x-1)$.

Chapter 7

Rank and Corank in Cross Section Lattices

In Chapter 3 we saw how the cross section lattice and hence the structure of a $\mathcal{J}$-irreducible reductive monoid M can be described in terms of the minimal nonzero $G \times G$ orbit of the monoid. In such a case we can define a rank function on the lattice $\Lambda \backslash \{\hat{0}\}$ and there is a unique element of the cross section lattice of rank 0. Do the maximal $G \times G$ orbits that are not equal to M tell us anything about the monoid? That is, do the elements of the cross section lattice of corank 1 help describe the structure of the monoid? The answer to this question is yes, although the information they provide is not as rich as the type of M.

Section 7.1 contains some general remarks about rank in cross section lattices. The main result of this chapter is in Section 7.2 where we find some descriptions of a $\mathcal{J}$-irreducible cross section lattice in terms of the elements of corank 1.

7.1 Rank of Cross Section Lattices

The following definition introduces some notation that will be convenient throughout this chapter.

Definition 7.1.1. Let M be a reductive monoid with cross section lattice Λ.

a) Let Λ_i be the set of elements of Λ of rank i.

b) Let Λ^j be the set of elements of Λ of corank j.

Let Λ be the cross section lattice of a semisimple reductive monoid M. Recall that $\Lambda \backslash \{\hat{0}\}$ is a ranked poset. That is, the length of any maximal chain is the same and is equal to the rank of the maximal element of $\Lambda \backslash \{\hat{0}\}$. If the set of simple roots is Δ such that $|\Delta| = n$, then the

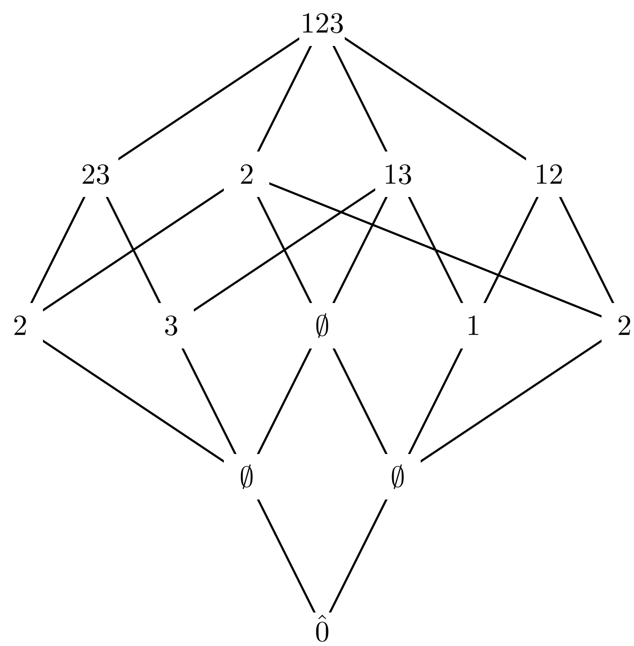


Figure 7.1: 2-reducible monoid with $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$, $I_+ = I_- = \emptyset$, $\Delta_+ = \{\alpha_1\}$ and $\Delta_- = \{\alpha_3\}$

length of any maximal chain is n. In terms of Definition 7.1.1, M is $\mathcal{J}$-irreducible if and only if $|\Lambda_0| = 1$ and M is $\mathcal{J}$-coirreducible if and only if $|\Lambda^{n-1}| = 1$. If M is $\mathcal{J}$-irreducible, then it is easy to define the rank function: $\rho(e) = |\lambda^*(e)|$ for all $e \in \Lambda \backslash \{\hat{0}\}$. This is a little harder to do in the 2-reducible case.

Example 7.1.2. Let M be a 2-reducible reductive monoid with $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$, $I_+ = I_- = \emptyset$, $\Delta_+ = \{\alpha_1\}$, and $\Delta_- = \{\alpha_3\}$. The cross section lattice Λ is shown in Figure 7.1.

Notice that it is possible for two elements e and f of the cross section lattice to have the same rank while $\lambda^*(e)$ and $\lambda^*(f)$ have different cardinalities. Furthermore, two elements of the same partition of Λ (in this case Λ_0) can have the same rank but their image under λ^* can have different cardinalities.

Even though the rank of elements of a $\mathcal{J}$-irreducible cross section lattice are easy to describe in terms of the type map, Example 7.1.2 shows that describing the rank of an arbitrary cross section lattice is more difficult. This is particularly true when M is semisimple. Notice that if M is a 2-reducible reductive monoid that is not semisimple, then by Theorem 3.3.2 the length of a maximal chain will be $n + 1$.

It should be pointed out that we are considering the rank of the poset $\Lambda \backslash \{\hat{0}\}$ rather than the lattice Λ. No difficulties arise and we could just as easily discuss the rank of Λ instead but we choose not to do so due to the nice symmetry of the length of maximal chains and the number of simple roots in the $\mathcal{J}$-irreducible case.

7.2 Corank of $\mathcal{J}$-irreducible Cross Section Lattices

We saw in Chapter 3 how we can construct the cross section lattice (and hence the type map) of a $\mathcal{J}$-irreducible monoid in terms of the minimal nonzero element e_0. See Theorem 3.2.2 for details. Our goal in this section is to describe the cross section lattice of M in terms of the elements of Λ of corank 1. Throughout let M be a $\mathcal{J}$-irreducible reductive monoid of type $I = \lambda_*(e_0)$. Let $\Lambda\backslash\{\hat{0}\}$ be a cross section lattice of M and let Δ be the set of simple roots.

We first look at $\mathcal{J}$-irreducible monoids that are also $\mathcal{J}$-coirreducible. The results are surprisingly precise.

Proposition 7.2.1. *Let M be a $\mathcal{J}$-irreducible monoid that is also $\mathcal{J}$-coirreducible of cotype $J = \Delta\backslash\{\alpha\}$. The following are true:*

a) Δ is connected.

b) The Dynkin diagram of Δ is of the type A_n, B_n, C_n, F_4, or G_2.

c) M is $\mathcal{J}$-linear.

d) α is an endpoint of the Dynkin diagram of Δ.

e) The type of M is $I = \Delta\backslash\{\beta\}$, where β is the other endpoint of the Dynkin diagram of Δ.

Proof. We induct on the dimension of M. If $dim(M) = 2$ then $\Delta = \{\alpha, \beta\}$ and e^0 will cover e_0. Let $\lambda^*(e^0) = \Delta\backslash\{\alpha\} = \{\beta\}$. Since M is $\mathcal{J}$-coirreducible, $\{\alpha\}$ is not in the image of λ^*. Therefore $\alpha \in I$ and hence $I = \Delta\backslash\{\beta\} = \{\alpha\}$. Additionally notice that Δ is connected for if it weren't a component, namely $\{\alpha\}$, would be contained in I, a contradiction since $\lambda^*(1) = \Delta$.

Now suppose $dim(M) = n + 1$ and let $\lambda^*(e^0) = \Delta\backslash\{\alpha\}$ so no component of $\Delta\backslash\{\alpha\}$ is contained in I. Let $f \in \Lambda\backslash\{\hat{0}\}$ be covered by e^0. Then there exists $\gamma \in \Delta\backslash\{\alpha\}$ such that $\lambda^*(f) = \Delta\backslash\{\alpha, \gamma\}$. No component of $\Delta\backslash\{\alpha, \gamma\}$ is contained in I. However, since M is $\mathcal{J}$-coirreducible, a component of $\Delta\backslash\{\gamma\}$ is contained in I. Therefore $\{\alpha\}$ is a connected component of $\Delta\backslash\{\gamma\}$ and $\alpha \in I$. Then $s_\alpha s_\delta = s_\delta s_\alpha$ for all $\delta \in \Delta\backslash\{\gamma\}$. There are two possibilities: $\{\alpha\}$ is a connected component of Δ, or $s_\alpha s_\gamma \neq s_\gamma s_\alpha$. Suppose $\{\alpha\}$ is a connected component of Δ. Since $\alpha \in I$, Δ is not in the image of λ^*, a contradiction since $\lambda^*(1) = \Delta$. Therefore $s_\alpha s_\gamma \neq s_\gamma s_\alpha$ and hence α is an endpoint of the Dynkin diagram of Δ. Suppose there exists $f' \in \Lambda\backslash\{\hat{0}\}$ such that $f' \neq f$ and $\lambda^*(f') = \Delta\backslash\{\alpha, \delta\}$ for some $\delta \in \Delta\backslash\{\alpha, \gamma\}$. Then by the above reasoning $s_\alpha s_\delta \neq s_\delta s_\alpha$ but this isn't possible since $s_\alpha s_\delta = s_\delta s_\alpha$ for all $\delta \in \Delta\backslash\{\gamma\}$. Therefore f is the unique element of $\Lambda\backslash\{\hat{0}\}$ of corank 2.

Let $\Delta' = \Delta\backslash\{\alpha\}$. Then e^0Me^0 is $\mathcal{J}$-coirreducible of type $J' = \Delta'\backslash\{\gamma\}$. By the induction hypothesis e^0Me^0 is $\mathcal{J}$-irreducible of type $I' = \Delta'\backslash\{\beta\} = \Delta\backslash\{\alpha,\beta\}$. Additionally Δ' is connected and the Dynkin diagram is of the type A_n, B_n, C_n, F_4, or G_2 with endpoints γ and β. Since $s_\alpha s_\gamma \neq s_\gamma s_\alpha$ and $s_\alpha s_\delta = s_\delta s_\alpha$ for all $\delta \in \Delta\backslash\{\gamma\}$, Δ is connected and the Dynkin diagram is of the type A_n, B_n, C_n, F_4, or G_2 with endpoints α and β. e^0Me^0 is $\mathcal{J}$-irreducible of type $I' = I \cap \lambda^*(e^0)$, so $\Delta\backslash\{\alpha,\beta\} \subseteq I$. Since $\alpha \in I$ and $I \neq \Delta$, $I = \Delta\backslash\{\beta\}$. Finally since e^0Me^0 is $\mathcal{J}$-linear and f is the unique element of $\Lambda\backslash\{\hat{0}\}$ of corank 2, M is $\mathcal{J}$-linear. □

Corollary 7.2.2. *Let M be a $\mathcal{J}$-irreducible reductive monoid with cross section lattice $\Lambda\backslash\{\hat{0}\}$. If $|\Lambda_i| = 1$ for some $i < n$, then $|\Lambda_k| = 1$ for $k = 0,\ldots,i-1$. Furthermore, Δ is connected.*

Proof. Let $|\Lambda_i| = 1$ where $1 \leq i < n$. Let e_i be the unique element of $\Lambda\backslash\{\hat{0}\}$ of rank i. Let f cover e_i. Then fMf is $\mathcal{J}$-coirreducible. Since $f \geq e$ for all $e \in \Lambda\backslash\{\hat{0}\}$ of rank less than or equal to i, fMf is $\mathcal{J}$-linear. Notice that if $|\Lambda_1| = 1$, there is only one singleton subset of Δ with no component contained in I. Therefore $I = \Delta\backslash\{\alpha\}$ for some $\alpha \in \Delta$. If Δ is not connected, then a connected component of Δ must be contained in I, which is not possible.

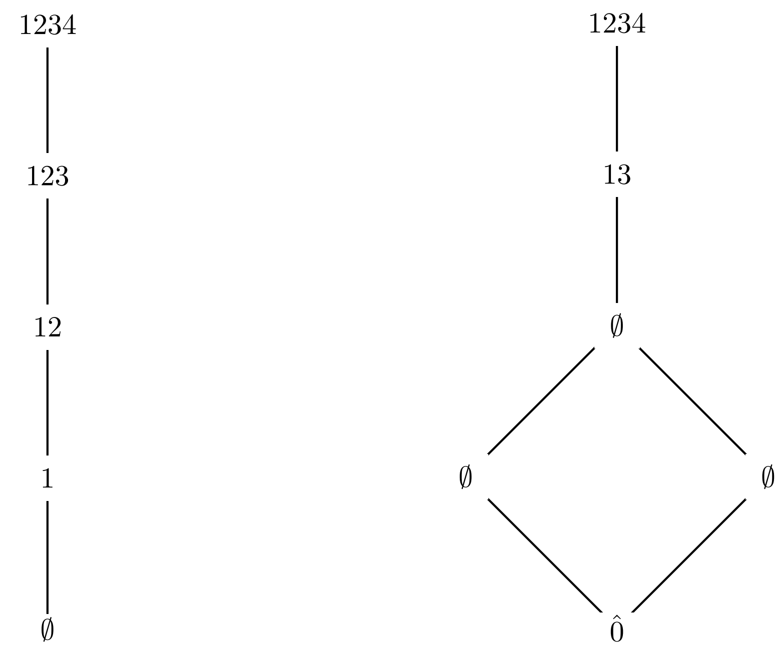
□

Example 7.2.3. Let M be a $\mathcal{J}$-coirreducible monoid with set of simple roots $\Delta = \{\alpha_1,\ldots,\alpha_4\}$ of type A_4 and cotype $J = \{\alpha_1,\alpha_2,\alpha_3\}$. Then the type of M is $I = \{\alpha_2,\alpha_3,\alpha_4\}$. Notice that the image of λ^* contains the empty set as well as the connected subsets of Δ that contain α_1 and hence M is $\mathcal{J}$-linear. In particular J is the only subset of three elements with no component contained in I. The cross section lattice is shown in Figure 7.2a.

Example 7.2.4. Let M be a 2-reducible monoid with set of simple roots $\Delta = \{\alpha_1,\alpha_2,\alpha_3\}$, $I_+ = \{\alpha_2,\alpha_3\}$, $I_- = \{\alpha_1,\alpha_2\}$, $\Delta_+ = \{\alpha_1\}$, and $\Delta_- = \{\alpha_3\}$. Such a monoid exists by Theorem 3.3.6. The cross section lattice is shown in Figure 7.2b. Notice that M is $\mathcal{J}$-coirreducible but not $\mathcal{J}$-linear.

Example 7.2.4 shows that the results of Theorem 7.2.1 and Corollary 7.2.2 do not generalize to k-reducible monoids. The reader should notice by comparing Theorem 3.2.2 with Theorem 3.3.3 that the structure of semisimple monoids can be quite different when there are one or two minimal nonzero elements of the cross section lattice. It should therefore not be terribly surprising that many results about $\mathcal{J}$-irreducible monoids do not generalize.

As mentioned above the structure of a semisimple reductive monoid can be quite different depending upon whether there are one or two minimal nonzero elements. It turns out that $\mathcal{J}$-irreducible monoids with two elements of corank 1 are much easier to describe. The trade-off, however, is that our results are not quite as descriptive.



(a) $\mathcal{J}$-irreducible cross section lattice of cotype $J = \{\alpha_1, \alpha_2, \alpha_3\}$

(b) $I_+ = \{\alpha_2, \alpha_3\}$, $I_- = \{\alpha_1, \alpha_2\}$, $\Delta_+ = \{\alpha_1\}$, and $\Delta_- = \{\alpha_3\}$

Figure 7.2: Examples of $\mathcal{J}$-coirreducible cross section lattices

Proposition 7.2.5. *Let M be a $\mathcal{J}$-irreducible monoid of type $\lambda_*(e_0) = I$ with connected set of simple roots Δ. Let $\lambda^*(\Lambda^1) = \{\Delta\backslash\{\alpha\}, \Delta\backslash\{\beta\}\}$. Then the following are true:*

a) α and β are endpoints of the Dynkin diagram of Δ.

b) If the Dynkin diagram of Δ is of type D_n, E_6, E_7, or E_8, then $I = \Delta\backslash\{\gamma\}$ where γ is the third endpoint of the Dynkin diagram of Δ.

c) If the Dynkin diagram of Δ is of type A_n, B_n, C_n, F_4, or G_2 then $I = \Delta\backslash\{\gamma, \delta\}$ where $s_\gamma s_\delta \neq s_\delta s_\gamma$ or else $I = \Delta\backslash\{\gamma\}$ where γ is not an endpoint of the Dynkin diagram of Δ.

Proof.

a) First notice that if $|\Delta| = 2$, then $\Delta = \{\alpha, \beta\}$ and $\lambda^*(\Lambda^1) = \{\{\alpha\}, \{\beta\}\}$ where α and β are endpoints of the Dynkin diagram of Δ, which is of type A_2, B_2, C_2, or G_2. This is only possible if $I = \emptyset = \Delta\backslash\{\alpha, \beta\}$ where $s_\alpha s_\beta \neq s_\beta s_\alpha$.

Let $|\Delta| > 2$ and suppose that α is not an endpoint of the Dynkin diagram of Δ. Then there exists $\gamma \in \Delta\backslash\{\alpha, \beta\}$ that is an endpoint of the Dynkin diagram of Δ. $\Delta\backslash\{\gamma\}$ is not in the image of λ^*, so a connected component must be contained in I. Since γ is an endpoint, $\Delta\backslash\{\gamma\}$ is connected so $\Delta\backslash\{\gamma\} \subseteq I$. Since $\Delta \neq I$ it must be that $I = \Delta\backslash\{\gamma\}$. Since α is not an endpoint, $\Delta\backslash\{\alpha\}$ will have at least two connected components. One will contain γ and the other(s) will not. So then at least one connected component of $\Delta\backslash\{\alpha\}$ is contained in I, a contradiction. Therefore α (and similarly β) must be an endpoint.

b) If the Dynkin diagram of Δ is of type D_n, E_6, E_7, or E_8 then there is a third endpoint $\gamma \in \Delta\backslash\{\alpha, \beta\}$. By the above reasoning $I = \Delta\backslash\{\gamma\}$.

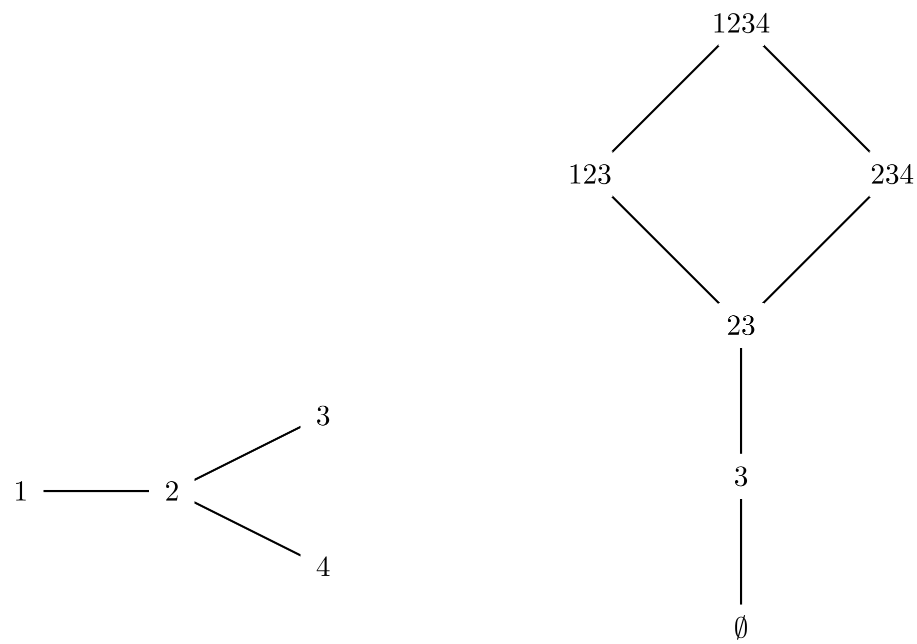
c) Let Δ be of type A_n, B_n, C_n, F_4, or G_2 and $|\Delta| > 2$. Our goal is to find all possible subsets $A \subseteq \Delta$ such that $I = \Delta\backslash A$. So let $\gamma \in A$, or equivalently $\gamma \notin I$. First suppose that γ is an endpoint of Δ. From the above we know that no component of $\Delta\backslash\{\gamma\}$ is contained in I. Let $\delta \in \Delta\backslash\{\gamma\}$ such that $s_\gamma s_\delta \neq s_\delta s_\gamma$. Since $|\Delta| > 2$, δ is not an endpoint of Δ and hence a component of $\Delta\backslash\{\delta\}$ is contained in I. The components of $\Delta\backslash\{\delta\}$ are $\{\gamma\}$ and $\Delta\backslash\{\gamma, \delta\}$. Since $\gamma \notin I$ it must be that $\Delta\backslash\{\gamma, \delta\} \subseteq I$. Notice that $\delta \notin I$ for if it were then $\Delta\backslash\{\gamma\} \subseteq I$ which is not possible. So $I = \Delta\backslash\{\gamma, \delta\}$ where $s_\gamma s_\delta \neq s_\delta s_\gamma$.

   So suppose γ is not an endpoint. Then there exists $\delta, \epsilon \in \Delta\backslash\{\gamma\}$ such that $s_\gamma s_\delta \neq s_\delta s_\gamma$ and $s_\gamma s_\epsilon \neq s_\epsilon s_\gamma$. Since γ is not an endpoint at least one component of $\Delta\backslash\{\gamma\}$ must be contained in I. Since one of these components contains δ and the other contains ϵ it follows that either $\delta \in I$, $\epsilon \in I$, or both. First suppose that $\delta \in I$ and $\epsilon \notin I$. Then the connected component of $\Delta\backslash\{\gamma\}$ containing δ must be contained in I. If ϵ is an endpoint, then $I = \Delta\backslash\{\gamma, \epsilon\}$, the component of $\Delta\backslash\{\gamma\}$ containing δ. If ϵ is not an endpoint, then $\Delta\backslash\{\epsilon\}$ will have two connected components. Let these two components be E_1 and E_2 with $\gamma \in E_1$. Since ϵ is not an endpoint at least one of E_1 or E_2 is contained in I. Since $\gamma \notin I$ it must be that $E_2 \subseteq I$ and hence $I = \Delta\backslash\{\gamma, \epsilon\}$. The case where $\epsilon \in I$ and $\delta \notin I$ is similar. Now suppose that $\delta \in I$ and $\epsilon \in I$. If δ and ϵ are both endpoints, then $|\Delta| = 3$ and $I = \{\delta, \epsilon\} = \Delta\backslash\{\gamma\}$. If exactly one of δ or ϵ is an endpoint, then by the reasoning above $I = \Delta\backslash\{\gamma\}$. If neither δ nor ϵ are endpoints, then the components of $\Delta\backslash\{\delta\}$ and $\Delta\backslash\{\epsilon\}$ not containing γ will be in I. Then since $\{\delta, \epsilon\} \subseteq I$, $I = \Delta\backslash\{\gamma\}$.

   In any case either $I = \Delta\backslash\{\gamma, \delta\}$ where $s_\gamma s_\delta \neq s_\delta s_\gamma$ or else $I = \Delta\backslash\{\gamma\}$ where γ is not an endpoint of the Dynkin diagram of Δ.

□

Example 7.2.6. Let M be $\mathcal{J}$-irreducible with set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_4\}$ of type D_4. The Dynkin diagram of Δ is shown in Figure 7.3a. Let the type of M be $I = \{\alpha_1, \alpha_2, \alpha_4\} = \Delta\backslash\{\alpha_3\}$. The cross section lattice $\Lambda\backslash\{\hat{0}\}$ is shown in Figure 7.3b. Notice that there are two elements of the cross section lattice of corank 1. Also notice that λ^* of these two elements are of the form $\Delta\backslash\{\alpha_1\}$ and $\Delta\backslash\{\alpha_4\}$ and that α_1 and α_4 are the two endpoints of Δ that are elements of I.



(a) Dynkin of type D_4 (b) Cross section lattice $\Lambda\backslash\{\hat{0}\}$

Figure 7.3: $\mathcal{J}$-irreducible monoid with $\Delta = \{\alpha_1, \dots, \alpha_4\}$ and $I = \{\alpha_1, \alpha_2, \alpha_4\}$

Proposition 7.2.5 only considers the case of two elements of corank 1 when the set of simple roots Δ is connected. The following proposition considers the possibility that Δ may not be connected.

Proposition 7.2.7. *Let M be a $\mathcal{J}$-irreducible monoid of type $\lambda_*(e_0) = I$. Let $\lambda^*(\Lambda^1) = \{\Delta\backslash\{\alpha\}, \Delta\backslash\{\beta\}\}$. Then either*

a) Δ is connected and is as described in Proposition 7.2.5, or

b) Δ has two connected components Δ_1 and Δ_2 of type A_n, B_n, C_n, F_4, or G_2 where α is an endpoint of Δ_1 and β is an endpoint of Δ_2, and

i) If $|\Delta_1| \geq 2$ and $|\Delta_2| \geq 2$, then $I = \Delta\backslash\{\gamma, \delta\}$ where γ and δ are the other endpoints of Δ_1 and Δ_2.

ii) If $|\Delta_1| \geq 2$ and $|\Delta_2| = 1$, then $I = \Delta\backslash\{\gamma\}$ where γ is the other endpoint of Δ_1

iii) If $|\Delta_1| = 1$ and $|\Delta_2| \geq 2$, then $I = \Delta\backslash\{\delta\}$ where δ is the other endpoint of Δ_2.

Proof. Let $\lambda^*(\Lambda') = \{\Delta\backslash\{\alpha\}, \Delta\backslash\{\beta\}\}$ and let the connected components of Δ be $\Delta_1, \dots, \Delta_k$. Clearly $\Delta_i \not\subseteq I$ for all $1 \leq i \leq k$. Without loss of generality assume $\alpha \in \Delta_1$. Suppose that α is not an endpoint of Δ_1. Then there exists $\gamma \in \Delta\backslash\{\alpha\}$ such that $\gamma \neq \beta$ and γ is an endpoint of Δ_1. Since $\Delta\backslash\{\gamma\}$ is not in the image of λ^*, a component must be contained in I. The connected components of $\Delta\backslash\{\gamma\}$ are $\Delta_1\backslash\{\gamma\}$, $\Delta_2, \ldots, \Delta_k$ so $\Delta_1\backslash\{\gamma\} \subseteq I$. Since no component of $\Delta\backslash\{\alpha\}$ is contained in I, no component of $\Delta_1\backslash\{\alpha\}$ is contained in I. Let A be a component of $\Delta_1\backslash\{\alpha\}$ that does not contain γ. Then $A \subseteq \Delta_1\backslash\{\gamma\} \subseteq I$, a contradiction. Therefore α is an endpoint of Δ_1. Similarly β is an endpoint of a connected component of Δ.

Suppose Δ has at least two connected components and $\{\alpha, \beta\} \subseteq \Delta_1$. Let $\gamma \in \Delta$ be an endpoint of Δ_2 such that $\Delta_2\backslash\{\gamma\} \nsubseteq I$ (such an endpoint exists since $\Delta_2 \nsubseteq I$). Then no component of $\Delta\backslash\{\gamma\}$ is contained in I, and hence $|\Lambda^1| \geq 3$, a contradiction. Therefore Δ must be connected if α and β are in the same connect component. Clearly then if Δ is not connected it has exactly two connected components. Let $\alpha \in \Delta_1$ and $\beta \in \Delta_2$. If $|\Delta_1| \geq 2$ and $|\Delta_2| \geq 2$, then by Proposition 7.2.1 both Δ_1 and Δ_2 are of the type A_n, B_n, C_n, E_4, or G_2 and $I = \Delta\backslash\{\gamma, \delta\}$ where γ and δ are the other endpoints of Δ_1 and Δ_2. If $|\Delta_1| \geq 2$ and $|\Delta_2| = 1$, then Δ_1 is of the type A_n, B_n, C_n, E_4, or G_2 and $I = \Delta\backslash\{\gamma\}$ where γ is the other endpoint of Δ_1. If $|\Delta_1| = 1$ and $|\Delta_2| \geq 2$, then Δ_2 is of the type A_n, B_n, C_n, E_4, or G_2 and $I = \Delta\backslash\{\delta\}$ where δ is the other endpoint of Δ_2.

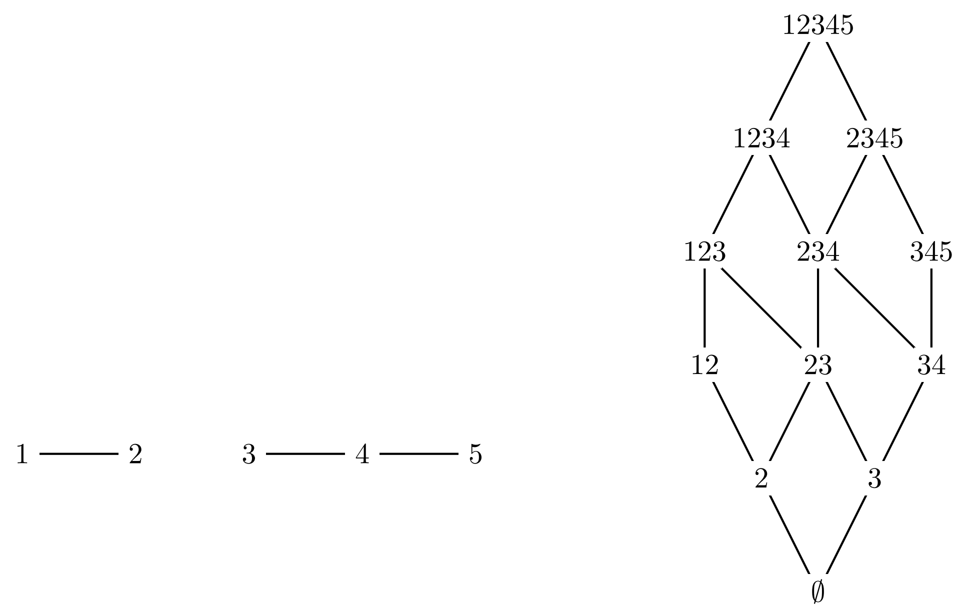
□

Example 7.2.8. Let M be $\mathcal{J}$-irreducible with set of simple roots $\Delta = \Delta_1 \sqcup \Delta_2 = \{\alpha_1, \alpha_2\} \sqcup \{\alpha_3, \alpha_4, \alpha_5\}$ of type $A_2 \oplus A_3$. The Dynkin diagram of Δ is shown in Figure 7.4a. Let the type of M be $I = \{\alpha_1, \alpha_4, \alpha_5\} = \Delta\backslash\{\alpha_2, \alpha_3\}$ where α_2 and α_3 are endpoints of Δ_1 and Δ_2, respectively. The cross section lattice $\Lambda\backslash\{\hat{0}\}$ is shown in Figure 7.4b. Notice that there are two elements of the cross section lattice of corank 1. Also notice that λ^* of these two elements are of the form $\Delta\backslash\{\alpha_1\}$ and $\Delta\backslash\{\alpha_5\}$ and that α_1 and α_5 are the two endpoints of Δ that are elements of I.

Comparing Proposition 7.2.1 and Proposition 7.2.5 we see that as the number of elements of corank 1 increases, the number of possibilities of the type of M increases and these possibilities become more complicated. We will therefore only consider one more case: the maximal case where all n subsets of the form $\Delta\backslash\{\alpha\}$ are in the image of λ^*.

Proposition 7.2.9. *Let M be a $\mathcal{J}$-irreducible monoid of type $\lambda_*(e_0) = I$. Then $|\Lambda^1| = n$ if and only if no endpoint of the Dynkin diagram of Δ is contained in I.*

Proof. Let $|\Lambda^1| = n$. Suppose $\alpha \in \Delta$ is an endpoint of the Dynkin diagram of Δ such that $\alpha \in I$. Then there exists a unique $\beta \in \Delta$ such that $s_\alpha s_\beta \neq s_\beta s_\alpha$. Since $|\Lambda^1| = n$, $\Delta\backslash\{\beta\}$ is in the image of λ^*. Since $\{\alpha\}$ is a connected component of $\Delta\backslash\{\beta\}$ and no component of $\Delta\backslash\{\beta\}$ is contained in I, $\alpha \notin I$. This is a contradiction and hence no endpoint of the Dynkin diagram of Δ is contained in I.



(a) Dynkin diagram of type $A_2 \oplus A_3$ (b) Cross section lattice $\Lambda\backslash\{\hat{0}\}$

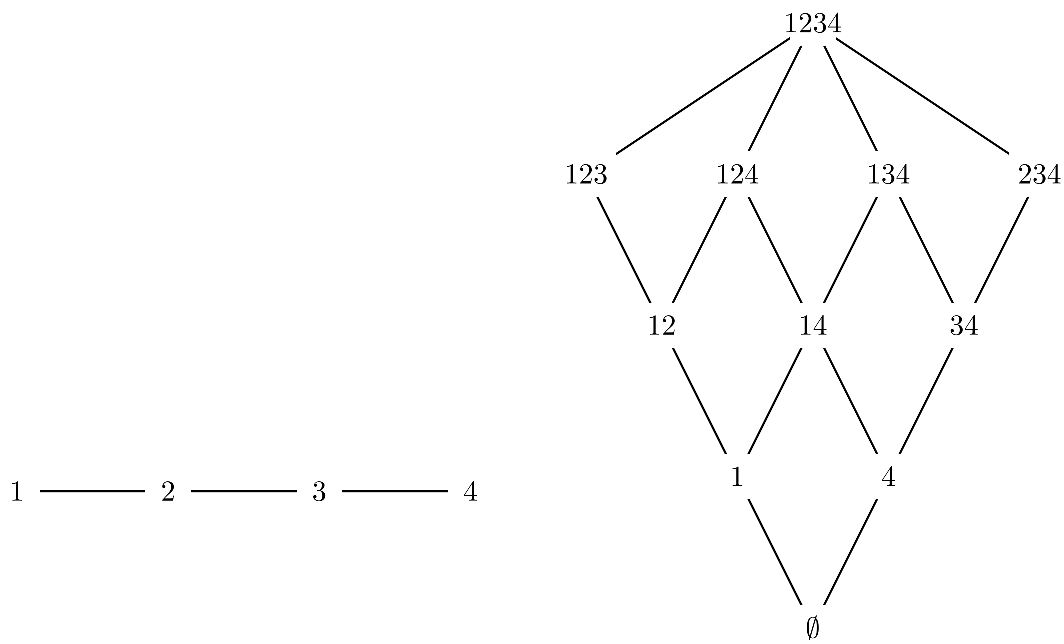
Figure 7.4: $\mathcal{J}$-irreducible monoid with $\Delta = \{\alpha_1, \alpha_2\} \sqcup \{\alpha_3, \alpha_4, \alpha_5\}$ and $I = \{\alpha_1, \alpha_2, \alpha_4\}$

Now suppose no endpoint of the Dynkin diagram of Δ is contained in I. If $I = \emptyset$, then the image of λ^* is 2^Δ, the power set of Δ, so $|\Lambda^1| = n$. Let $\alpha \in \Delta$. If $|\Delta| = 1$, then $I \neq \Delta$ otherwise Δ would not be in the image of λ^*. So it must be that $I = \emptyset$. If $|\Delta| > 1$, then any component of $\Delta\backslash\{\alpha\}$ will contain an endpoint of the Dynkin diagram, and hence no component of $\Delta\backslash\{\alpha\}$ can be contained in I. Therefore $\Delta\backslash\{\alpha\}$ is in the image of λ^* for all $\alpha \in \Delta$ and hence $|\Lambda^1| = n$. $\square$

Example 7.2.10. Let M be $\mathcal{J}$-irreducible with set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_4\}$ of type A_4. The Dynkin diagram of Δ is shown in Figure 7.5a. Let the type of M be $I = \{\alpha_2, \alpha_3\}$. Notice that I does not contain any endpoints of Δ. The cross section lattice $\Lambda\backslash\{\hat{0}\}$ is shown in Figure 7.4b. Notice that every three element subset of Δ is in the image of λ^*.

Example 7.2.11. Consider the monoid in Example 7.2.10. $\Delta\backslash I = \{\alpha_1, \alpha_4\}$ is not connected, so by Corollary 4.1.9 $\Lambda\backslash\{\hat{0}\}$ is not distributive. More generally, let M is a $\mathcal{J}$-irreducible monoid of type I with set of simple roots Δ. Suppose $|\Lambda^1| = |\Delta|$. By Proposition 7.2.9, no endpoint of Δ is contained in I. Therefore such a monoid will be distributive if and only if $I = \emptyset$. In this case $\Lambda\backslash\{\hat{0}\} \cong 2^\Delta$, the Boolean lattice of rank $|\Delta|$.

Notice that the previous proposition does not require that Δ be connected. In general Δ can have up to n connected components while $|\Lambda^1| = n$. The maximal case occurs when $I = \emptyset$. This is a special case of the following:



(a) Dynkin diagram of type A_4 (b) Cross section lattice $\Lambda \backslash \{\hat{0}\}$

Figure 7.5: $\mathscr{J}$-irreducible monoid lattice with $\Delta = \{\alpha_1, \ldots, \alpha_4\}$ and $I = \{\alpha_1, \alpha_2, \alpha_4\}$

Proposition 7.2.12. *Let M be a $\mathscr{J}$-irreducible reductive monoid of type $I = \lambda_*(e_0)$.*

a) If Δ has k connected components, then $|\Delta^1| \geq k$.

b) If $|\Delta^1| = k$, then Δ has at most k connected components.

Proof.

a) Let the connected components of Δ be Δ_j for $1 \leq j \leq k$. For any Δ_j either $|\Delta_j| = 1$ or $|\Delta_j| > 1$. If $|\Delta_j| = 1$, then $\Delta_j \not\subseteq I$ or else Δ wouldn't be in the image of λ^*. If $|\Delta_j| > 1$, then Δ_j has at least two endpoints. Let two of these endpoints be α_j and β_j. We claim that either $\Delta_j \backslash \{\alpha_j\}$ is contained in I or no component of $\Delta_j \backslash \{\beta_j\}$ is contained in I. So suppose a component of $\Delta_j \backslash \{\alpha_j\}$ is contained in I. Let $I_j = \Delta_j \cap I$. Then a component of $\Delta_j \backslash \{\alpha_j\}$ is contained in I_j. Since $\Delta_j \backslash \{\alpha_j\}$ is connected and $\Delta_j \not\subseteq I$ (and hence $\Delta_j \not\subseteq I_j$), it follows that $I_j = \Delta_j \backslash \{\alpha_j\}$. Since $\alpha_j \in \Delta_j \backslash \{\beta_j\}$ and $\Delta_j \backslash \{\beta_j\}$ is connected, no component of $\Delta_j \backslash \{\beta_j\}$ is contained in $I_j \subseteq I$. Then no component of $\Delta_j \backslash \{\beta_j\}$ is contained in I. Since $\Delta_i \not\subseteq I$ for any $1 \leq i \leq n$ it follows that no component of $\Delta \backslash \{\beta_j\}$ is contained in I. That is, for every connected component of Δ there is at least one element of $|\Lambda^1|$, so $|\Lambda^1| \geq k$.

b) Let $|\Delta^1| = k$ and suppose Δ has more than k connected components. Then there exists a component Δ_i such that $\Delta \backslash \{\alpha\}$ is not in the image of λ^* for any $\alpha \in \Delta_i$. Let α and

β be two endpoints of Δ_i. Then $\Delta_i \backslash \{\alpha\} \subseteq I$ and $\Delta_i \backslash \{\beta\} \subseteq I$. Therefore $\Delta_i \subseteq I$, a contradiction.

$\square$

Chapter 8

Conclusion

In this paper we have focused mainly on the cross section lattices of $\mathcal{J}$-irreducible reductive monoids. The reason for this is not surprising. These monoids are the best understood and their respective cross section lattices are the easiest to describe. However, that does not mean there is nothing left to study. In Chapter 5 we saw how $\mathcal{J}$-irreducible cross section lattices could be expressed as a product of chains (and in some cases lattices that are "almost" chains) when the set of simple roots is connected and of the type A_n, B_n, C_n, F_4, or G_2. We were then left with a conjecture as to how the cross section lattice can be factored when the set of simple roots is of the type D_n. Studying the case for E_6, E_7, and E_8 would then be a natural conclusion to this intriguing problem.

We also spent time studying the structure of 2-reducible semisimple monoids. While the structure of these monoids is more complicated than that of the $\mathcal{J}$-irreducible monoids, we are still able to describe the cross section lattice (and hence the type map) precisely provided that we have the invariant sets Δ_+, Δ_-, I_+, and I_- in hand. We saw in Example 3.3.5 that there may not exist a corresponding 2-reducible semisimple monoid when these invariants are chosen randomly. This is a large deterrent to our ability to study these monoids and their respective cross section lattices. It is not impossible, however, as we were able to prove Corollary 4.2.8 which stated that such a cross section lattice is modular if and only if it is distributive. We were able to do so by thinking the problem as an exercise in dealing with the combinatorics of sets and removing all connections to the algebra behind these sets. This approached worked, although the resulting proof may be more complicated than is necessary. This realization in itself is motivation enough to try to understand which choices of these invariants are permissible. Theorem 3.3.6 gives us some possibilities but the results are incomplete. A better understanding of this problem will also help in generalizing many of the results of this paper concerning $\mathcal{J}$-irreducible monoids to the 2-reducible case.

Another potential area of work is to come up with a theorem to construct the cross section lattice of k-reducible semisimple monoids when $k > 2$. The ability to do so would provide a great deal of insight into the structure of reductive monoids in general.

The field of reductive monoids is still relatively young, especially when compared to other areas of algebra. The study of cross section lattices is but one small area in this field. This paper is meant to shed some light on the structure of these lattices. It is the author's hope that, at the very least, the reader has developed an appreciation for the simultaneous beauty and complexity of these lattices and that their study is only just beginning.

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