ABSTRACT

LI, YANMING. Geometric Phase Holograms for Light Orbital Angular Momentum Control and Wavefront Shaping. (Under the direction of Michael Escuti and Keith Weninger.)

Effective and efficient control of light is central in nearly all optics and photonics systems. One important part of the control is on the wavefront. Generally, there are always demands on simple and efficient optical elements. Specifically, to control novel wavefronts such as helical wavefronts, novel optics are greatly needed. Although many beam shaping techniques have been developed, most of them are based on dynamic phase shift. Geometric phase, as a separate phase shift, has not been systematically studied for wavefront manipulating. One main reason is that there has not been a general practical implementation available.

Consequently, this dissertation is guided by two tasks; 1) Building theory and classification of optical elements and systems based on geometric phase (GPHs), 2) verifying and applying the theory for high efficient wavefront manipulation, by developing and experimenting on novel GPHs that manipulate helical wavefronts.

We develop a general theory for Geometric Phase Holograms (GPHs) and identify the efficiency dependences. Based on these analysis, we conclude that GPHs are highly potential wavefront manipulation elements. To verify and apply this, we propose and demonstrate a compact, low cost, light weight, highly efficient GPH, Forked Polarization Grating (FPG), for controlling helical wavefront. We verify for the first time that FPG can generate and modify orbital angular momentum of light with very high efficiency (> 95%) at visible wavelengths and the output wavefront mode can be controlled by incident polarization. We then extend the GPH theory from single element to a stack system. We propose a ternary design of GPH relays that can actively manipulate multiple wavefronts. We verify and apply this design to develop the $Q$-stack, which controls multiple helical wavefront by fast electrical switching. We demonstrate 27 helical mode controlling with a 3-stage $Q$-stack, with high efficiency (> 80%) and mode purity (85%). Finally, we propose an implement method of GPH based on multi-twist liquid crystals that allows us to modify the efficiency dependence over wavelength. To achieve this, we identify the parameters in a general GPH that are functions of wavelength and proof that these parameters can be modified over a wavelength range by numerical designed
multi-twist structures. As a result, we developed broadband GPHs that control helical wavefronts with high efficiency (> 97%) over the whole visible wavelength range.

The significant of this work on the theory and implementation of GPHs and systems based on GPHs is that it enables high efficient general wavefront manipulation with simple, compact, low cost, and light weight elements, which is not commonly possible with conventional optics. We conclude with a summary of results and suggested directions for future research.
Geometric Phase Holograms for Light Orbital Angular Momentum Control
and Wavefront Shaping

by
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A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Physics

Raleigh, North Carolina
2014

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DEDICATION

To my parents.
BIOGRAPHY

Yanming Li was born and brought up in Tianjin, China. She received her Bachelor of Science degree (2008) in Physics from Nankai University in China and the Master of Science degree in Physics (2010) and Electrical Engineering (2013) from North Carolina State University, where she has been a Ph.D. student in Physics since 2008. Yanming’s main research interest is in optics and photonics, for which she conducted the dissertation research with the Opto-electronics and Lightwave Engineering Group (OLEG).
ACKNOWLEDGEMENTS

I would like to thank Dr. Michael Escuti for providing valuable insight, motivation, and guidance at every stage of this thesis, and whose enthusiasm about science and technology were a constant inspiration. I also wish to thank all my committee members, Dr. Harald Ade, Dr. David Aspnes, Dr. Keith Weninger, and the Physics Department at NCSU for supporting me, both financially and spiritually, in my cross-disciplinary research.

Many thanks to my colleagues at OLEG and ImagineOptix Inc. for their help and insightful discussions. I would like to convey my deepest gratitude to all my colleagues, Ravi Komanduri, Jihwan Kim, Matthew Miskiewicz, Kathryn Hornburg, Anirudh Venugopal, Erin Clark, and Jason Kekas.

A special thanks to all my friends, Qianru, Sihui, Tian, Xin, Jianbang, Xiaomeng, and (especially) Guojing for all the good foods and good laughs, for companying me through good and bad times, and for making my stay at NCSU a memorable one.
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<tbody>
<tr>
<td>δ</td>
<td>Phase</td>
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<tr>
<td>ϕ</td>
<td>Azimuthal angle coordinate in cylindrical coordinate system</td>
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<tr>
<td>Φ</td>
<td>Azimuthal angle of the optical axis</td>
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<tr>
<td>l</td>
<td>Topological charge, mode number, and order number in helical beams, OAM state number</td>
</tr>
<tr>
<td>n</td>
<td>Index of refraction</td>
</tr>
<tr>
<td>λ</td>
<td>Wavelength</td>
</tr>
<tr>
<td>Ω</td>
<td>Solid angle</td>
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<tr>
<td>Γ</td>
<td>Phase retardation</td>
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Symbols in embedded manuscripts may have different meanings.
Chapter 1

Introduction

This dissertation is concerned primarily with the development of a category of optical elements based on geometric phase for wavefront shaping, with the emphasis on helical wavefront shaping. To date, there has not been systematical research on geometric phase in wavefront shaping. This provides the motivation for this work as we investigate the theoretical gap and develop a feasible implement for this theory.

1.1 Optical Wavefront Shaping

Light as electromagnetic wave can be described by a vectorial wave function with three key attributes: amplitude, phase, and polarization. Wavefront is the equal-phase surface of a light wave. Optical wavefront shaping is the art of controlling the phase profile of a beam. Wavefront shaping is nested in a larger topic of beam shaping, where one modify one or more of the three attributes to shape a beam.

Many research and applications require or can benefit from the use of light beams with a specified shape. The demand ranges across academic, industrial, medical, and military areas. Various exotic shaped beams are of great interest for basic research by themselves as well. Optical beam shaping, in modern days mainly laser beam shaping, is an important science that enables other researches while developing itself.

People have had the need and wisdom of controlling light since ancient times. Long before the laser was invented, the ancient philosophers knew about shaping light in their favor. In the west, there is the dramatic event involving the use of mirror arrays by the Greeks to set fire to Roman ships in the harbor during the siege of Syracuse (214–212
BC). This purported weapon is sometimes called the “Archimedes burning glasses”. In the east, ancient Chinese had recorded since the Western Zhou dynasty (1046-771 BC) of using concave mirrors to gather light from the sun in order to ignite. This tool was called “solar-flint”.

After the laser was invented, laser beam shaping became a large and significant part in the optical research. Generally, any modification on light waves is essentially beam shaping. This includes the simple cases of focusing and defocusing with lens and blocking light with apertures. More commonly, laser beam shaping refers to engineering more complex or special beam shapes. A very common laser beam shaping objective is the flat-top beam, where the intensity profile is uniform over some cross-section.

1.1.1 Helical Beams and Orbital Angular Momentum

Helical beams are a category of special laser beams that interests many researchers recently. It will also be an emphasis in this dissertation. As how it is called, a helical beam has wavefront in shape of a helix. The axis of the helix is coincident with the center of the beam and follows the beam propagation direction. Mathematically, the phase profile of the helical beams is a linear function of the azimuthal angle in the cylindrical coordinates with z axis aligned with the beam axis, the wave function thus has this signature term $e^{il\phi}$, where $\phi$ is azimuthal angle and $l$ is called topological charge that can take any integer. A common example of helical beam is Laguerre-Gaussian (LG) beam. The different modes of LG beams are described by different orders of Laguerre polynomials with two integer parameters $l$ and $p$. Since the azimuthal order $l$ is the same $l$ in the phase term, when refer to the $l$ parameter, people use these three terminologies interchangeable: (topological) charge, mode, and order.

Several helical beam examples taking different $l$ values are illustrated in Fig. 1.1. Note that here we only take account of the helical phase term. For realistic beams such as LG beams, there could exist additional radius dependent phase term that deforms the wavefront from these modeled in the figure. For $l = 0$, beam reduces to non-helical shape. The wavefront is perpendicular to the beam axis. The phase profile on a beam cross-section takes a constant value. For higher order $l = 1$ and $l = 2$, wavefront is twisted with respect to beam axis due to the azimuthal phase delay. At a local point on radius, the wavefront is tilted from the $l = 0$ case with an angle that increases with $l$. If the $l = 1$ wavefront is seen as a helix, then the $l = 2$ wavefront is a double-helix. The phase profile
Figure 1.1: Illustration of helical beams of order 0, ±1, and ±2. Left column: wavefronts. Right column: phase profile at beam cross-section. The color represent the phase value from 0 to $2\pi$. 
on a beam cross-section gain an azimuthal slope. The phase change in a loop is $2l\pi$ for an order $l$ beam. For negative orders $l = -1$ and $l = -2$, the phase profiles resemble their positive counterparts. The only difference is that the handedness changes for both the wavefront twisting and the cross-section phase ramp.

The helical beams are of great interest mainly because of their two key features: doughnut shape transverse intensity profile and Orbital Angular Momentum (OAM) carried by the beam. The helical wavefront by necessity creates a phase singularity at its axis. As the same kind of research in optical vortex shows, the phase singularity lead to a zero intensity. Dark center surrounded by high intensity ring, this kind of intensity profile is a good gradient trap for small particles. Thus helical beams are used in many optical trapping and transporting works. The size of the dark center increases with the helical charge but is obviously always smaller than the whole beam spot. By pairing with tailored dye and light, the helical beams act as a key role in stimulated emission depletion microscopy. The doughnut shape intensity ring deactivates the fluorescence on the outside of a spot and leaves the very small center to be detected by the other light. This scheme much enhances the resolution compared to the traditional confocal microscopy which is limited by the diffraction limit.

An order $l$ helical beam carries OAM of $lh$ per photon. This can be easily understood within the ray optical picture. The Poynting vector determines the direction of a ray emitting from the local point on the wavefront. The rays are presentations that carry linear momentum. Helical wavefront leads to a twist of these local rays so that each linear momentum has an transverse component. When integrated over the whole beam, the total effect is an angular momentum in the direction of beam propagation. Unlike the Spin Angular Momentum (SAM) associated with photon polarization, this OAM is unbounded. SAM has the value of $\pm \hbar$ per photon when spin takes the eigenstates of up and down. OAM can take a unbounded number of values since $l$ can be any integer. Here parameter $l$ gains its fourth meaning, the quantum number that identifies the OAM states.

A direct use of OAM is to manipulate particles as it can transfer to matter as SAM does. Microscopic particles are demonstrated to spin or orbit about the beam center depending on the relative size and particle properties. The second use is based on the discrete and unbounded nature of OAM states. As the two states of SAM can form a qubit system, the OAM states could support a qudit system, where the level $d$ is
theoretically unbounded as $l$. Several demonstrations with OAM have already shown increased date rate of the optical communication links. Some great effort has also been taken into utilizing OAM in enhanced cryptography.

1.2 Research Motivation: Lack of Efficient Wavefront Manipulating Elements

The motivation for this dissertation is the lack of efficient wavefront manipulating element. The term efficiency in concern has a broad meaning. It includes aspects like energy efficiency, cost efficiency, versatility, and spatial efficiency, among others. In this section, we will briefly review the conventional wavefront shaping elements and identify the limitation of each element.

From the well-known equation for phase calculation $\delta = 2\pi (nd/\lambda)$, to modify the phase distribution across a plane one can modify the physical path length $d$ for different part of the wave or modify the refractive index $n$ spatially. Both routes lead to spatially varying optical path length and spatially varying phase at output.

Reflective elements, like concave and convex mirrors, deformable mirrors, mirror arrays, and reflective gratings, change the physical paths of the local rays in a light beam. The reflective surface is made by depositing a metallic coating or multiple thin layers of dielectric material on a substrate of glass or other optical materials. The reflectance can be very high (> 95%) and most reflective elements have very high energy efficiency. However, these elements generally reverse the beam propagation direction, thus not ideal for many applications. Archimedes mirror array and the modern expensive deformable mirrors have great use in adaptive optics because of the versatility. However, they are limited on the range of controllable wavefronts by the actuator stroke and resolution, the number and distribution of actuators. The attempts to improve these parameters often result in bulkier and more expensive products. The complexity and cost also make it hard to scale to large sizes.

Refractive elements, like lens, prisms, and refractive gratings, change the refractive index therefore change the optical path of the light beam. Most elements are made of glass or crystals. They take up space and need compatible mounting. The fabrication is not flexible. For example, large numerical number lens is hard to make. Complex designs are also not easy. Recently, one category of adjustable refractive elements are widely used.
in optical research areas, they are called Spatial Light Modulator (SLM). A SLM is an electrically programmable device that modulates light on a pixel-by-pixel basis. They have the ability to shape intensity, phase, or both. For the work in this dissertation, we only discuss the SLM in phase mode. SLM has two kinds, LCOS (Liquid Crystal on Silicon) which is reflective and LCD (Liquid Crystal Display) which is transmissive. Both of them work by locally tilting the index ellipsoid of liquid crystal. SLMs are commonly used in research because of their versatility of displaying various phase pattern through easy control. However, SLMs are like the deformable mirrors, expensive, unpractical for commercial applications. In many cases, SLMs don’t have high energy efficiency in wavefront shaping due to the pixelated nature. This is especially problematic with transmissive SLMs since they often have a small pixel fill factor (around 50%), which is the pixels percentage of light sensitive material. The relatively large amount of optically inactive space often causes extra, undesirable diffraction orders to appear in the output. Filtering optics like aperture is commonly needed. The portion of the wave with the desired wavefront contains a small part of the input energy.

Another common problem for both reflective and refractive elements is wavelength sensitivity. The wavefront shaping is based on precise phase calculation, which means the target optical path length variation across the beam front is about the scale of optical wavelength. This also means a particular elements designed for a wavefront shape for one wavelength will have obvious distortion if used for another wavelength. This either is not applicable for wavelengths outside specification or need calibrate before each new usage. Most is even not possible to use with broadband light source.

After reviewing the above conventional wavefront shaping elements, we ask the question "Can we find and realize new optical elements for wavefront shaping which is efficient in light energy and cost, compact, lightweight, and also versatile?"

### 1.3 Research Objective and Scope

We start from the physics of phase change. We notice that there are two essentially different mechanisms to modify wavefront: through dynamic phase and through geometric phase. Most of the conventional wavefront shaping as discussed above alter the optical path length, which affects the dynamic phase. We find that geometric phase has its unique potential for highly effective wavefront shaping. Wavefront shaping through
geometric phase is naturally wavelength insensitive. The phase change is not originated from optical path length, instead, it is pure geometric effect due to the vectorial nature of light. However, geometric phase for wavefront shaping has not been widely and systematically studied. One main reason is that there has not been a practical implementation available.

The objective of this dissertation is to propose the concept and develop a implementation of Geometric Phase Hologram (GPH). We define GPH as a category of optical elements that achieve wavefront modification based on the geometric phase.

Our analysis on the theory of GPH for wavefront shaping is aimed for arbitrary wavefront shaping. We also try to generalize the theory for the broad types of optical anisotropies, including birefringence and dichroism. However, our experimental work will be mainly focused on shaping helical wavefronts and the birefringence anisotropy in liquid crystal materials.

Our implement of GPH is based on liquid crystal materials. There are two main properties of liquid crystal materials that make them suitable for GPH. The first property is the anisotropies due to their partial crystalline nature. The optical anisotropy such as birefringence is especially useful in manipulate light. The second property is the flexibility due to the partial liquid nature. This provides the possibility of adjustable control of light.

Our proposed GPH made by liquid crystal is easy to fabricate, compact in size, versatile to implement in application, and most importantly highly efficient in shaping wavefront without losing energy. The potential highest efficiency is 100% and the experimental efficiency is expected to reach 99% for most wavefront shapes.

1.4 Research Approach and Contributions

The main theme of this dissertation is to study and implement GPH as simple, compact, low-cost, lightweight, highly efficient wavefront method, with emphasis on helical wavefront shaping. We approach this goal with the structured chapters as follows.

Chapter 2 contains the relevant background topics necessary to understand this research, including a review of conventional optical wavefront shaping methods, an introduction to helical beams and orbital angular momentum, a description on geometric phase with comparison with dynamic phase, and a summary on the liquid crystal materials and
their properties.

Chapter 3 presents our theoretical study on how the GPHs can control wavefront by the optical anisotropy patterns on GPHs. We utilize Jones calculus to derive the transfer matrix of a general GPHs with an arbitrary transverse anisotropy pattern and the output field from such a GPH. We show that the explicit relation between the element pattern and the wavefront change is the base of GPH wavefront shaping. We discuss the properties of the GPH induced wave change including polarization sensitivity and develop the efficiency equations of GPHs.

Chapter 4 demonstrates a GPH example for helical wavefront shaping. This novel element is called Forked Polarization Grating (FPG). This work verifies the theory and derived the properties about GPH through both numerical simulation and experiments. We demonstrate that FPGs are compact orbital angular momentum controller and helical mode transformer with high efficiency.

Chapter 5 expand the research from single GPH elements to GPH relays and from the ability to shape single wavefront to a series of wavefronts. We propose and demonstrate a design consisting of a stack of active GPHs and wave plates. The system can shape the input light into a category of many wavefronts. It has the ability to freely distribute the input energy into these wavefront shapes as desired by pure electrical control. Particularly, the input light beam can be transformed to any of the possible wavefronts with near-zero energy loss. Based on this design, we experimentally demonstrated an agile OAM control system consisting of a stack of azimuthal GPHs, which can switch the output wave among 27 helical modes. The output wavefronts and the transform efficiencies is also examined.

Chapter 6 investigates in modifying the wavelength dependence of GPH wavefront shaping efficiency. For the GPHs demonstrated in Chapter 4 and Chapter 5, since the optical birefringence from nematic LC is homogenous in the element depth, GPH efficiencies are dependent on operation wavelength in a natural dispersive manner. We propose a method that can suppress or modify this dependence without losing wavefront shaping quality or increasing fabrication difficulty. We approach this by designing a structured birefringence in depth, which is adapted from the multilayer twist nematic LC idea in broadband retarder. We prove that the same multilayer structure can be applied to a general GPH to change its spectral response and meanwhile keep the same wavefront shaping properties. Experimentally, two GPHs for broadband helical wavefront shaping
are fabricated and characterized.

1.5 Related Publications

The research presented in this dissertation and its applications has been published in 4 peer-reviewed journals, 5 peer-reviewed conference papers with 4 additional journal submissions in preparation:


Chapter 2

Background

In this chapter, we provide the background information of several different areas that are related to the following chapters. We begin with a review of optical wavefront shaping methods in Section 2.1. Concept and prior implements of light orbital angular momentum follows in Section 2.2, which is commonly possessed by the beams with helical shaped wavefronts. In Section 2.3, the background on geometric phase is provided, as it is a less utilized phase shift compared to dynamics phase which most prior wavefront shaping methods are based on. The basics of LC materials and the related alignment methods are provided in Section 2.4 ending with a discussion on their potential in implementing optical wavefront shaping elements.

2.1 Optical Beam Shaping

The control and modification of the light field is a very broad area in science and engineering. Fully control over light field includes the control on amplitude, phase, and polarization. However, conventional light wave shaping usually limits the concern to the scaler part of the filed, thus targets at a particular intensity distribution or phase profile.

We note a concept clarification here between the target and the method of optical beam shaping. The target of optical beam shaping is a desired light field at some plane or range after the beam shaping element or system. In some cases, only the intensity of the field is specified, such as focused or uniform illumination[190, 92, 126, 164] and flat-top beams[62, 86]. In other cases, only the phase is specified, such as helical beams[3] as well as the various adaptive wavefront compensation and correction applications[11,
There are also cases where both intensity and phase are required to be in a particular form, such as vortex Bessel beams. The method of optical beam shaping is the field change directly from the optical shaping element or system, in other words, the change in the near field exiting the element or system compared to the incident. Some methods modify only the amplitude of the incident field, such as apertures, zone plates, amplitude gratings and holograms (including computer generated holograms). Other methods modify only the phase of the incident field, such as refractive optics (lens, axicon, prism), phase gratings, and phase-only Spatial Light Modulators (SLMs). Due to the diffraction nature of a light field, the quantity changed in a method may not be identical to the target. Actually in many cases, phase-modifying method is used for intensity redistribution in far field and amplitude-modifying method is used for changing phase profile.

Wavefront shaping in this dissertation refers to the optical beam shaping method that directly modifies the phase profile of the light field. The far field result could be a structured wavefront or an intensity distribution. In practice, due to the simplicity and high efficiency, phase-only methods are commonly used to approximate complex fields. The elements for wavefront shaping can also be combined with other elements or systems to construct a full field controller.

2.2 Helical Beams and Orbital Angular Momentum

Among various wavefronts of interest, we especially focus on the helical wavefront in this dissertation. The Poynting vector in the beams with helical wavefront follows a spiral path and the beams carry an angular momentum regardless of its polarization. Study related to helical beams has become a rapidly growing area, in both basic wave optics and quantum optics, as well as in engineering applications.

2.2.1 Orbital Angular Momentum of Light

The Orbital Angular Momentum (OAM) of a light beam was first been investigated by Allen et al. in 1992 with the paraxial approximation. For the research and work with practical lasers, the paraxial approximation is normally sufficient to describe beam propagation properties. It is recognized in this paper that all beams with an helical phase structure carry an OAM of $\hbar l$ per photon. OAM is an independent charac-
teristic to the Spin Angular Momentum (SAM), which is originated from the rotation of electromagnetic field vector. In optics, this rotation is noted as circular polarization.

The beams carry OAM have helical wavefront, therefore they are called helical beams. For helical beams propagating along +z-axis we can write the complex amplitude of the field in cylindrical coordinates as

$$\psi(r, \phi, z) = A(r, z) e^{ikz} e^{il\phi}$$

(2.1)

where \(r\) is the radial distance, \(\phi\) is the azimuthal angle, \(z\) is the longitudinal distance from the beam waist, \(A\) is the amplitude, \(l\) is an integer, and \(k\) is the wave vector. The choices of signs follow the convention we note in Appendix A.1.

From Maxwell’s equations, one physically realizable solution that satisfies the paraxial approximation is the family of Laguerre-Gaussian (LG) modes. The complex expression of these modes is[4]

$$LG_p^l(r, \phi, z) = C_{p}^{LG} \left(\frac{\sqrt{2}r}{w(z)}\right)^l \exp \left[ -\frac{r^2}{w^2(z)} \right] L_p^l \left(\frac{2r^2}{w^2(z)}\right)$$

$$\cdot \exp \left[ -\frac{i kr^2 z}{2(2z^2 + z_R^2)} \right] \exp (il\phi) \exp \left[ i(2p + l + 1) \arctan \left(\frac{z}{z_R}\right) \right]$$

(2.2)

where \(C_{p}^{LG}\) is the normalization constant, \(L_p^l \left(\frac{2r^2}{w^2(z)}\right)\) is a generalized Laguerre polynomial, the width of the beam at position \(z\) is

$$w(z) = w(0) \sqrt{\frac{z^2 + z_R^2}{z^2}}$$

(2.3)

where \(w(0)\) is the beam waist, and \(z_R\) is the Rayleigh range.

Based on these experimentally realizable modes, researchers have derived and confirmed many properties of helical beams, such as the signature doughnut shape intensity distribution. The radius corresponding to maximum amplitude increases with the order number \(l\),

$$r(z)_{\text{max}} = \frac{\sqrt{2w(z)}}{2} \sqrt{l}$$

(2.4)

Later experiments further consider the diffraction factor in the practical laser beams and propose a different equation with high power to the order number, \(r \propto l\) [48].
Nevertheless, this cross-sectional doughnut shaped intensity profile with zero intensity at its center is the most simple and direct phenomenon one could observe from helical beams, in contrast to the Gaussian distribution from a Gaussian wave and the uniform distribution from a plan wave.

Although the LG modes are the earliest and widely studied family of helical waves, they are not the only solutions. Other beams such as high order Bessel beams[10], high order Mathieu beams[38], and Ince-Gaussian beams[12] also have helical wavefront. Each of them has unique properties other than carrying OAM, such as non-diffraction in Bessel beams, which lead to interesting research in diverse areas.

After the first OAM study in the 1992 paper[3], researchers investigate the similarities and differences between OAM and SAM, as well as the total optical angular momentum in terms of spin and orbital components. Both OAM and SAM can be derived from Maxwells equations, which give an angular momentum to energy ratio of $l/\omega$ for a helical wave, and $\sigma/\omega$ for a circularly polarized wave, where $\sigma = 0, \pm 1$ for linearly, left and right circularly polarized light, respectively. Since the energy of each photon is $h\omega$, the OAM and SAM are simply $lh$ and $\sigma h$, respectively. These expressions state that the two angular momenta are separated and independent. Sequently, several theoretical discussion[182] and experimental investigations[3, 167] established that in the paraxial approximation, the SAM can add to, or subtract from, the OAM to give a total angular momentum and reversely, the total angular momentum can be separated into spin and orbital components and each can be observed and measured.

The interaction between light with OAM and matter is studied at the same time. Similar to circularly polarized light rotating birefringence plate, OAM can be transferred to matter and set micron size absorptive particles into rotation[81]. Moreover, OAM is shown to be mechanically equivalent to SAM[167]. These early experiments confirm that the total light angular momentum is $(l + \sigma)\hbar$.

We should note that this clear addition/separation rule of OAM and SAM components in a light wave is only valid under paraxial approximation. In the case of tight focused beam, however, the polarization state is not well defined, the total angular momentum takes a more complicated form[14, 198, 136]. An analyze in terms of angular momentum flux was proposed for this situation[13]. In most cases including the work in this dissertation, however, considering the OAM and SAM as distinct and independent quantities is sufficient and helpful.
2.2.2 Applications of Helical Beams

The research and applications of helical beams and OAM can be broadly categorized into two groups based on their two main properties. The first group is optical manipulation such as in “optical tweezer” and “optical trap”, which is based on the unique momentum or energy distribution of the helical beams. The focus is on the interaction of helical beams with matter. The first subgroup utilize the large angular momentum and study the transferring of momentum between light and matter[81, 64, 167, 151, 69, 68, 110]. Compared to SAM, which takes value between +¯\hbar and −¯\hbar, OAM can have many times larger magnitude since the l in its expression ℓ\hbar can be any integer. The utilizing of OAM in addition to SAM not only increases usable angular momentum, it also provides an extra freedom in optical manipulation. For example, the particles can be set to spinning about its own axis by SAM and at the same time orbiting about the light beam axis by OAM. The second subgroup utilize the intensity gradient at the beam center to trap particles[82, 104, 55, 49]. Momentum transferring usually also happens complimentarily in these cases. Overview the related applications, optical manipulation by helical beams has promising utilizations in the context of cell biology, to manipulate living cells and chromosomes[99, 151]; microrheology[9, 26]; microfluidics, as micro-optomechanical pumps[104, 109]; and micromechanics[63, 65].

The second group of applications of helical beams is in optical communications and information, which is based on the unbounded OAM states they can possess. The OAM states are discrete and orthogonal to each other. The state number l can theoretically take any integer value. These properties make transmitting larger amount of data possible. Systems that based on SAM, which has +1 or −1 two eigenstate, work on bits. The data is coded on a base of 2. However, in systems that based on OAM, which has unbounded eigenstates, the information can be coded on a much bigger base, whose size is only limited by the physical realization. Great interest has been aroused to research on high capacity optical communication links based on OAM and many schemes have been investigated[71, 193, 120, 174, 187]. In the subcategory of free-space optical links, extra consideration on the aberrations from atmospheric turbulence and schemes to overcome them is also an active research area[150, 181, 152, 156]. The utilization of OAM can also be combined with other quantities of light to form multiple dimension multiplexing, such as with polarization[71, 187], wavelength[60], and both[89]. The latest one reports a 100Tbit/s free-space data link enabled by three-dimensional multiplexing of OAM, polarization, and
wavelength. Besides higher transmission rate, researchers have also investigated OAM as a way to make the optical communication more secure. The larger state base of OAM provides higher coding density in quantum information systems, such as in quantum cryptography[119]. Research has demonstrate an entangled trinary quantum systems (qutrits) for quantum key distribution (QKD), where the qutrits are encoded into the OAM of photons, carried by three LaguerreGaussian modes[73]. Similar to the free-space data links, the influence of atmospheric turbulence on quantum cryptography systems is also studied[120].

2.2.3 Generation and Measurement methods

There are several existing methods to create helical beams that carry OAM. Depending on whether the generation methods involve amplifying media, these methods can be categorized as active or passive[197]. The active methods modify the laser cavity to select the helical mode Laguerre-Gaussian $LG_{1}^{0}$ as the lasing mode instead of Gaussian mode ($LG_{0}^{0}$). Two same frequency Hermite-Gaussian modes $HG_{1,0}$ and $HG_{0,1}$ are phase locked[175, 78] to form the $LG_{1}^{0}$ mode[155].

Passive methods are more frequently used compared to active methods, because they generate helical mode in the free space outside laser cavity and can work with any conventional laser. The most direct method is spiral phase plate, which is a refractive plate with a helical thickness[18, 180, 141, 172]. This is a straightforward method but needs precise fabrication of the helical pitch to match the wavelength and the materials.

Another passive methods which was originally used by Allen in 1992 is also based on the fact that LG modes are superposition of HG modes. A pair of cylindrical lens converts a $HG_{m,n}$ mode to a $LG_{m-n}^{m-n}$ mode[3]. Later work on mode converter between HG and LG modes expand this method to multiple HG modes[19]. This method is nearly lossless, however, did not gain much usage due to two reasons. The first reason is the setup of two cylindrical lenses takes space and time. The second reason is that one need to generate HG modes to begin with, while most laser resonators in laboratories emit Gaussian mode.

The third passive method is to use conventional diffractive optics including Computer Generated Holograms (CGHs) and Spatial Light Modulators (SLMs). These elements are isotropic, they modulate the amplitude or phase of the lightwave, and diffract it into multiple orders. The pattern is either a spiral Fresnel lens[83], which is a Fresnel lens with
an azimuthal ramp, or a “forked grating”[16, 17, 84], which is a 1D grating with a fork shape singularity. These diffractive elements are easy to set up and the patterns can be changed for different modes. However, the big problem with these diffractive elements is low efficiency; most power goes to the 0th order which is unmodified. SLMs are also very expensive.

The fourth passive method is a special type of axial wave plates ($q$-plates)[122, 127, 41, 131], the optical axis of which is axially symmetric in-plane. At half wave retardation, a circularly polarized incident beam goes through an axially symmetric polarization rotation, which results in an azimuthal phase. In such way, these axial wave plates can convert a Gaussian beam or plan wave to a helical beam. This element is essential different from the conventional diffractive optics or refractive optics, since it realized the phase modification by polarization control. We will discussion in the following content that the key is geometric phase other than dynamic phase. The advance of this elements lays in high efficiency and simple structure. The geometric nature also leads to unique features such as polarization sensitivity. To obtain a pure helical beam, the incident polarization needs to be perfect circular and the wave plate should be at exact half wave retardation for the incident wavelength. Tunable $q$-plate was developed later to simplify the manufacturing procedure and to rely on electrical control to reach high efficiency for different wavelengths[168].

Other methods includes the using of multi-mode fiber[128, 50, 2, 192, 32] and special coupling systems[35]. So far, the smallest generation device is a silicon-integrated emitter using angular gratings to extract light confined in whispering gallery modes with high OAM into free-space beams with well-controlled amounts of OAM[35].

To observe a helical beam, the two most easy phenomena are the dark centered intensity profile and the characteristic interference patterns with either a oblique plan wave or coaxial spherical wave. Some good example figures are in Ref. [146]. We will make these observations in the later chapters as well. The interference patterns are good indication of helical mode number[15, 78, 170] since the strength of the singularity in the beam shows as the number of fringes in the interference pattern. Other experiments interferes a helical beam with its mirror image which produces an interference pattern with doubled number of fringes[78, 145, 108]. Also based on diffraction, researchers found that OAM can be detected by shining the beam through apertures[20, 85, 116, 130]. The output fields have special form, the number of intensity maxima corresponds to the
OAM mode number. These interference techniques are good for identifying the existence of helical mode or the dominant OAM state in a beam. However, it cannot show if the mode is pure or is a superposition of many modes, since one dominant mode will create the most strong fringes. Moreover, these methods involves pattern recognition, which is hard to automate.

To quantitively measure the OAM carried by helical beams, more complex schemes are developed. Most schemes involves forked diffractive gratings as mode filters which convert the matching helical modes back to Gaussian mode. The results from these gratings are then compared to a Gaussian mode to get the original mode information. One scheme uses computer generated holograms as reverse mode converter and measure the result through a single mode fiber[119]. Another scheme uses forked grating pattern on SLM as mode filter and measure the result through an on-axis spatial filter[115, 57]. These measurements are good for arbitrary OAM mode and superpositions. The drawback is the complex setup and alignment, and the need of exchanging the filter holograms to extract information on the different incident modes.

Most recently, Lavery and coworkers developed an OAM mode sorter based on log-polar field transform[105, 106]. This system transforms the azimuthal phase ramp in helical beams into linear phase profile, which results in lateral intensity shift in the far field. The linear phase slope and the lateral intensity shift is simply proportional to the OAM charge in the incident beam. This system is a very efficient OAM measurement because it consists only of refractive optics, with no mechanical or electrical moving part. It converts all the azimuthal frequencies in helical beam to linear frequencies, thus gives the full OAM spectrum at the output. This could be especially useful in detecting complex OAM states in a timely manner.

2.3 Geometric Phase

Phase is an important quantity in the description of a wave. It is a function of location at a given time, which takes different profiles for different beams. Wavefront is the equal-phase surface of a light wave. Optical wavefront shaping is the art of controlling the phase profile of a light beam.
2.3.1 Dynamic Phase and Geometric Phase

There are two types of phase shift that are different by origins in physics and are named by their nature. They are called geometric phase and dynamic phase. The dynamic phase is usually just referred to “phase” since it is the common phase that happens in most optical systems. Dynamic phase origins from the wave propagation. It is a function of the optical path length

$$\delta_D = 2\pi \frac{nd}{\lambda} \quad (2.5)$$

where \(n\) is the refractive index of the medium, \(d\) is the distance light travels in said medium, and \(\lambda\) is the wavelength of the light in vacuum. This phase shift is independent of the polarization state of the light. Most conventional optics are based on creating optical path length difference to modify the dynamic phase across a light beam, such as prisms, lenses, phase gratings, and phase-mode SLMs.

The geometric phase has a very different origin. The wave function of a quantum system can gain a phase shift when the parameters of the system undergo a cyclic change, which is extended from the adiabatic theorem of quantum mechanics by Berry in 1984[21]. Examples in physical systems include the Foucault pendulum[185], the precession of a neutron in a magnetic field[97], and many others in the references[188, 121, 22, 6, 7, 43]. The mathematical interpretation was found by Simon[166] in terms of anholonomy resulting from parallel transporting. Although Berry originally discussed the effect from cycled evolution in adiabatic system, the geometric phase theory has been generalized later to non-adiabatic systems and partial cycles[1, 158, 93]. The geometric phase is a pure geometric effect which is reflected in its general expression

$$\delta_G = -m_s \Omega \quad (2.6)$$

where \(\Omega\) is the solid angle subtended by the trace of the change of a parameter on a sphere of radius equal to the said parameter module in parameter space. The sign of \(\Omega\) is decided by the direction of the trajectory. \(m_s\) is the component of spin along the said parameter direction.

We summarize and compare the properties of dynamic phase and geometric phase in the following table (Table. 2.1).
### Table 2.1: Comparison of Dynamic Phase and Geometric Phase

<table>
<thead>
<tr>
<th>Origin</th>
<th>Dynamic Phase</th>
<th>Geometric Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Wave propagation</td>
<td>System change in parameter space</td>
</tr>
<tr>
<td>Expression</td>
<td>$\delta_D = 2\pi \frac{nd}{\lambda}$</td>
<td>$\delta_G = -m_s \Omega$</td>
</tr>
<tr>
<td>Wavelength dependence</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Measurable by interference</td>
<td>Yes</td>
<td>Yes[24, 186]</td>
</tr>
<tr>
<td>Measurable at single-photon level</td>
<td>Yes</td>
<td>Yes[103, 76]</td>
</tr>
</tbody>
</table>

### 2.3.2 Geometric Phase in Optics

The geometric phase in optics was first observed by Pancharatnam[149] in an even earlier time than the theory of Berry’s phase. Berry rediscovered this work in 1987[23] and connect the Pancharatnam phase with Berry phase. So this type of geometric phase in optics is also called Pancharatnam-Berry phase. This is the most observed manifestation of the geometric phase occur in optics that results from a cycle of change in polarization state of a beam. Many other manifestations are also studies, as summarized below[75]:

1. Pancharatnam-Berry phase: phase shift from a cycle of change in polarization state of a beam[149, 23];

2. Spin-redirection phase: phase shift from a cycle change in the direction of the propagation of a beam[40, 179];

3. Phase shift from a cycle change in squeezed states of light[39];

4. Phase shift from light reflected at optical multilayers[184];

5. Phase shift from light transmitted through a smoothly inhomogeneous isotropic medium[27];

6. Phase shift from a cycle of beam transverse mode transform[66]...

The first two manifestations have been studied since the early years of discover of Berry phase and are good representations of geometric phase in two different parameter spaces. We review the geometric phase in this two manifestations in parallel to gain a better
Table 2.2: Two manifestations of Geometric Phase in Optics: Spin-redirection Phase and Pancharatnam-Berry Phase

<table>
<thead>
<tr>
<th>Parameter Space</th>
<th>Spin-redirection Phase</th>
<th>Pancharatnam-Berry Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter Sphere</td>
<td>Momentum space</td>
<td>Polarization state space</td>
</tr>
<tr>
<td>$m_s$ Value</td>
<td>Unit sphere of directions</td>
<td>Poincare sphere</td>
</tr>
<tr>
<td>Geometric Phase Value</td>
<td>$\delta_G = \pm \Omega$</td>
<td>$\delta_G = \pm \frac{1}{2} \Omega$</td>
</tr>
</tbody>
</table>

understanding of geometric phase. A summary is in Table 2.2. Photons are spin-1 particles, however, the coefficients for geometric phase calculation are different. In the case of spin-direction phase, because the wave vector frame makes an angle with the spinor, the path on the unit sphere of direction represents the rotation of a full three-component spinor reflecting the unit spin of the photon[23]. Therefore, geometric phase is $\delta_G = \pm \Omega$. In the case of Pancharatnam-Berry phase, the Poincare sphere represents a two-state system, where photons are considered to have only two states (spin-1/2). This is based on the transverse nature of the plan waves, in which the spin takes either parallel or antiparallel direction to the propagation direction. Therefore for Pancharatnam-Berry phase, the relation to the solid angle is $\delta_G = \pm \frac{1}{2} \Omega$.

Several features of geometric phase allow them to be a useful alternative of dynamic phase in wavefront shaping. First, the geometric phase in optics is a topological property of the system which is not based on optical path difference effect as the dynamic phase does. Therefore, it is intrinsically achromatic. We will discuss this more in Chapter 6. Second, the geometric phase is not $2\pi$ modulo but unbound[123]. Therefore, perfect
Table 2.3: Geometric Phase Optical Elements

<table>
<thead>
<tr>
<th>General Category</th>
<th>Specific Element</th>
</tr>
</thead>
<tbody>
<tr>
<td>Space variant geometric Pancharatnam-Berry phase</td>
<td>Polarization diffraction gratings (via sub-wavelength structures)[80]</td>
</tr>
<tr>
<td>elements (SVGPBPE)</td>
<td></td>
</tr>
<tr>
<td>(Quantized) Pancharatnam-Berry phase diffractive optics (QPBOEs)</td>
<td>Blazed polarization diffraction grating, Diffractive focusing lens, Spiral retarder[138, 122, 41]</td>
</tr>
<tr>
<td>Spatially variable anisotropic transparencies</td>
<td>Polarization grating (dichroic)[72]</td>
</tr>
<tr>
<td>Polarization kinoform</td>
<td>Polarization gratings[94]</td>
</tr>
<tr>
<td>Diffractive waveplates (DW)</td>
<td>Cycloidal diffractive waveplates, Optical axis gratings, Axial waveplates[131, 132, 133]</td>
</tr>
<tr>
<td>Polarization holograms</td>
<td>Polarization gratings[178, 137, 45, 154, 58, 143]</td>
</tr>
</tbody>
</table>

Continuous phase profiles and large phase shifts are easily realizable.

2.3.3 Geometric Phase Optical Elements

The early studies on the nature of geometric phase were mainly conducted with homogeneous optics, such as wave plates. Although explicit use of geometric phase for wavefront shaping has not been an active area, there exist several special optical elements that are essentially working on geometric phase. Most of the elements were developed for an individual propose with exceptional unique features, such as extreme high efficiency. However, there has not been an effort to summarize these elements and look for insight into the common origin of these features. Here we try to identify the geometric nature of these elements and the various names they have been given in Table. 2.3. Our aim is to summarize these elements as special cases in a large category of optical elements, called Geometric Phase Holograms (GPHs).
2.4 Liquid Crystals and Alignment Methods

We choose liquid crystal materials to implement our proposed wavefront manipulating elements. The optical properties of liquid crystals make them capable of interacting with light waves. Some other physical properties like the electro-optical properties make them easy to control. We will review these properties of liquid crystal materials in the first section. The key of creating spatially varying optical anisotropy is to align the liquid crystals in a desired manner. We will review the aligning technologies in the second section in this chapter.

2.4.1 Liquid Crystal Materials

A liquid crystal (LC)\cite{52} is a phase of soft condensed matter that simultaneously exhibits characteristics of both an isotropic liquid and a crystalline solid. It flows and takes the shape of the container like a liquid. The molecules do not occupy specific average positions. However, the molecules do not orient in a totally random way like in a liquid. They possess a partial order where some preferred local orientation presents, which results in a variety of anisotropies like in a crystalline solid.

The first discovery of LCs is done by Friedrich Reinitzer in 1888 on a cholesteric LCs. It is then followed by the first systematical study on LCs by Otto Lehmann in the late 19th century. In the early 20th century, Georges Friedel\cite{144} first suggested that liquid crystals are mesomorphic states that constitute of states of matter where molecular properties are intermediary between those of crystals and those of ordinary liquids. After almost half a century of inactivity, Pierre-Gilles de Gennes rekindle the research on LCs in the early 1970s. LCs and related technologies have become of expanding interest to communities in both science and engineering to the present day.

Liquid crystals can be broadly classified based on the intermolecular structure or the nature of the order that they possess. The two classes of materials are called nematics and smectics. The nematic phase LCs have only orientational long-range order and no positional order. Cholesteric LCs, or chiral nematic LCs, are a special class of the nematic phase where in addition to the orientational order, the average molecular orientation rotates about a helical axis.

In the smectic phase LCs, there exist both the orientational order and a small amount of positional order. The molecules are still free to move randomly like in liquid, but they
tend to orient along a preferred direction and arrange themselves in layers. There are different smectic phases can be further distinguished by the in-layer molecular orientation related to the layer normal and to that of the different layers.

LC molecules are usually illustrated by elliptical rod-like or disc-like shapes, which represent the anisotropy they possess at molecular level. However, the overall anisotropy in a LC medium is also dependent on the long-range order. We can consider the anisotropy of such a medium as a combination result of the molecular anisotropy and the intermolecular order. At a spatial point in the media, on an average, the main axes of the LC molecules point in a particular direction. This average direction is referred to as the nematic director, \( \mathbf{n} \). The nematic director can be used to characterize the anisotropic properties of a liquid crystal at macroscopic level since it represents the average direction of the principle axis of index ellipsoid, thus the optical axis. The LC medium exhibit dielectric anisotropy described by \( \Delta \epsilon = \epsilon_\parallel - \epsilon_\perp \). At optical frequencies, the LCs show the optical anisotropy, conveniently described by birefringence \( \Delta n = n_e - n_o \).

When an external electric field is applied to the LC, it tends to polarize the free charges within the molecules and leads to a dipole moment. Since the LC molecules have the freedom, they will react to this induced dipole moment and reorient either parallel or perpendicular to the applied electric field. More commonly, without external electrical field, the orientation of the nematic director in a LC medium is decided by boundary conditions of a surface due to chemical or microscopic structural interactions\cite{173}. The methods of engineering these conditions and therefore controlling the orientation of LC direction are discussed in the next section.

Among different liquid crystal materials, a range of liquid crystal mixtures with polymerizable end groups are of special interest\cite{33, 95}. These liquid crystal molecules contain reactive groups that can be polymerized and cross-linked to form a anisotropic network as a thin, solid film. The polymerization can be photochemical or thermal. The resultant densely cross-linked networks contain the liquid crystal that is permanently fixed at the patterned director and order. The resultant film is then inactive to external modifications such as electrical fields. We call the optical elements made from these materials the “Polymer LC elements”. These elements have fixed optical axis orientations. They are relatively thermal and electrical stable, which is much desired in applications such as optical compensation film in displays.

In the contrary, the optical elements made from non-polymerized liquid crystals that
retain the freedom of being modified by external controls such as electrical fields are called “Active LC elements”. The LC materials are sandwiched by two conducting substrates and follow the alignment pattern from the surface condition of the substrates. When an external electrical field is applied to the LCs through the conducting substrates, the LC molecules reorient according to the field and lose the patterned alignment. When the external field disappear, the LCs can realign to the substrate surface thus re-form the desired optical axis pattern. These active elements are more versatile since they offer more than one optical configuration but also less stable than the polymer elements at the same time. The degradation after many times of switching should also be considered in applications.

2.4.2 Alignment Methods

Most optical elements made of LC consist a command layer that provide the LC orientation pattern. Due to long range orientational interaction, the alignment condition for liquid crystal given by this layer extends into the liquid crystal bulk on a macroscopic scale. The material used in this command layer is usually a thin polymer film, mostly polyimide, among others.

Rubbing is the most conventional alignment process. When the polymer is rubbed with a cloth, LCs tend to be aligned by surface interactions. Rubbing method is simple and easy to applied to large area, therefore widely used in industry. However, the coarse mechanical nature of this method have several problems that can make the alignment layer imperfect. It is also impossible to create a high resolution alignment pattern using rubbing method. Therefore, for our propose of high efficient wavefront shaping elements, we need a more advanced LC alignment methods.

A number of non-rubbing alignment techniques have been developed, including photoalignment, chemically treated surfaces, and oblique evaporation. Among these methods, photoalignment appears to be the most versatile and promising one.

In photoalignment, the photosensitive alignment materials is exposed to a linearly polarized light. The molecules absorb light differently based upon their relative orientation with respect to the polarization direction of the incident light. An orientationally dependent reaction of the molecules of the aligning material leads to a surface orientational ordering and an anisotropic surface condition. There are a variety of photoalignment materials that contain photosensitive species with angularly dependent absorption. Ac-
cordingly, there are also different photochemistry that the materials undergo during the photoalignment process, including reversible trans-cis photoisomerization, irreversible photo-destruction, and cycloaddition type photo-crosslinking[194]. Each of these materials has advantages and disadvantages. We prefer high stability against light and heat and large anchoring strengths. So far our best choice is linear photopolymerizable polymers (LPPs). LPPs are one kind of photo-crosslinking materials, in which process called linear photopolymerization presents[159]. The abbreviation LPP sometimes stands for this process as well as the material. A critical property of LPPs is that they support high resolution spatially varying LC alignments. The LPP film provides planar uniaxial alignment on single substrates, for both adjacent monomeric liquid crystals and liquid crystals polymers (LCPs). The LCP alignment induced by LPP is preserved during the subsequent cross-linking of the LCP layers. This allows us to build bulk aligned LCs up to several micrometers based on a single LPP alignment layer. These important properties of LPPs make them the optimal alignment materials for our research and the essential role in creating the spatial configurations in our elements.
Chapter 3

Wavefront Shaping via Geometric Phase Holograms

The phase change in optical waves has two possible mechanisms: dynamical or geometrical, thus called dynamic phase or geometric phase. This has been discovered and studied since Pancharatnam (1956) as summarized in Section 2.3. However, the research on geometric phase has mostly been focused on the physics with homogenous beams and optics. Meanwhile, the large research area of wavefront shaping or beam shaping has been mainly utilizing the various controls over dynamic phase, sometimes along with amplitude. Controlling wavefront through geometric phase has not been widely or systematically studied. There are a number of works on individual inhomogenous anisotropic elements such as PGs[80, 153, 59] and q-plates[122]. These were introduced as novel elements without revealing the geometric nature of their function. Among these, Hasman and coworkers summarized their series of work on sub-wavelength grating structures in 2002[30]. In this article, they propose the concept of Pancharatnam-Berry phase Optical Elements (PBOEs) as birefringent plates with a constant retardation and a spatial varying fast axis.

Our idea is to propose the concept and theory of Geometric Phase Hologram (GPH), which includes Hasman’s elements and the other inhomogenous anisotropic elements previously studied. GPH is a broad category of optical elements that achieve wavefront modification based on the geometric phase.
3.1 Lightwaves through spatially varying anisotropy

We limit the discussion to the geometric phase associated with the change in light polarization state, which is commonly referred to Pancharatnam-Berry phase to honor the great physicists for their discovery and rediscovery of this phase. The polarization state of light may alter due to birefringence or dichroism, two classes of anisotropy in the complex refractive index of optical medium. This complex refractive index is a constant number in isotropic medium and a tensor in anisotropic medium. In birefringent media, the real part of the complex refractive index depends on the polarization and propagation direction of the light. In dichroic (biattenuation) medium, the imaginary part of the complex refractive index (i.e., corresponding to absorption) varies for different polarization. Each class of the anisotropy can be further divided into two kinds, in one the refractive index take the extremes at two orthogonal linear polarizations and the other at two orthogonal circular polarizations. Therefore, here we will study all four kinds of optical anisotropy, Linear Birefringence (LB), Linear Dichroism (LD), Circular Birefringence (CB) (optical rotation/activity), and Circular Dichroism (CD).

For a light beam to gain non-uniform phase shift, the polarization state across the wavefront need to go through different changes, which translate to different paths on Poincare sphere. The optical element that can implement such polarization change will need to have spatially varying birefringence or dichroism.

In order to realize spatially variant anisotropy, we could use one of these two types of designs:

**Type A** Inhomogeneous optical axis orientation and constant amplitude of anisotropy

**Type B** Inhomogeneous amplitude of anisotropy and constant optical axis orientation

Since circular anisotropic material is rotational invariant, CB and CD materials can only form Type B elements. For linear anisotropy LB and LD, both Type A and Type B elements are possible (see Table 3.1). We investigate each combination as following.

### 3.1.1 Linear Birefringence, Type A and B

The Jones transfer matrix of a waveplate is

\[ T_{LB}(\Gamma, \Phi) = \cos \left( \frac{\Gamma}{2} \right) I + i \sin \left( \frac{\Gamma}{2} \right) \begin{bmatrix} \cos(2\Phi) & \sin(2\Phi) \\ \sin(2\Phi) & -\cos(2\Phi) \end{bmatrix} \]  

(3.1)
Table 3.1: Spatial Variant Anisotropy Combinations

<table>
<thead>
<tr>
<th>Anisotropy</th>
<th>Variables</th>
<th>Types possible</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear Birefringence (LB)</td>
<td>$\Phi$, optical axis orientation $\Gamma$, retardation</td>
<td>A, B, A+B</td>
</tr>
<tr>
<td>Linear Dichroism (LD)</td>
<td>$\Phi$, transmission axis orientation $k_H, k_V$, dichroism coefficients</td>
<td>A, B, A+B</td>
</tr>
<tr>
<td>Circular Birefringence (CB)</td>
<td>$\Gamma$, retardation</td>
<td>B</td>
</tr>
<tr>
<td>Circular Dichroism (CD)</td>
<td>$k_L, k_R$, dichroism coefficients</td>
<td>B</td>
</tr>
</tbody>
</table>

where $\Gamma$ is the phase retardation in radians. $\Phi$ is the angle that optical axis makes from $+x$ axis. For Type A elements, the retardation is a constant and the local optical axis angle is a function of location, $\Phi = \Phi(x, y)$. For Type B elements, the retardation is a function of location $\Gamma = \Gamma(x, y)$ and the local optical axis angle is a function of location. We include the both types here and rewrite the transfer matrix as operators in SU(2) parameter space\[42\]

$$T_{LB}(x, y) = \cos \left( \frac{\Gamma(x, y)}{2} \right) I + i \sin \left( \frac{\Gamma(x, y)}{2} \right) (e^{i2\Phi(x, y)}S_- + e^{-i2\Phi(x, y)}S_+)$$  \hspace{1cm} (3.2)

$S_{\pm} = \hbar \left( \begin{array}{cc} 1 & \pm i \\ \pm i & -1 \end{array} \right)$ are the raising and lowering operators in quantum mechanics. We normalize them to get unitary matrices (operators) so that they act on the polarization state without adding amplitude factors.

$$S'_{\pm} = \frac{1}{2\hbar} S_{\pm} = \frac{1}{2} \left( \begin{array}{cc} 1 & \pm i \\ \pm i & -1 \end{array} \right)$$  \hspace{1cm} (3.3)

By writing the transfer matrix in this form we can easily identity the parts that effect different attributes of a lightwave: the matrices affect the polarization, the exponential terms affect the phase, and the trigonometric functions affect amplitude.

We first look at the phase related terms. The exponential terms $e^{\pm i2\Phi(x, y)}$ indicates two opposite phase profiles. Thus from the transfer matrix we can deduce that the output field is consist of three waves, one with the same wavefront as input and two with shaped wavefront. The deviation of the two shaped waves from the input wave are opposite in
sign and equal in magnitude. The magnitude has a very simple relation to the pattern on the element, \( \Delta \delta(x, y) = 2\Phi(x, y) \), which is dependent on the optical axis pattern only and independent on the retardation.

The amplitude terms decide the component weighting among these three waves. \( \cos \left( \frac{\Gamma(x, y)}{2} \right) \) is the amplitude for the unchanged wave and \( \sin \left( \frac{\Gamma(x, y)}{2} \right) \) is the amplitude for the shaped wave. We can see that the power distribution among the three output waves are solely dependent on the retardation pattern and independent on the optical axis. The squares of the two orthogonal trigonometric functions nicely sum up to unity, which means that the LB elements are intrinsic lossless.

For the polarization parts, we introduce \( S'_\pm \) operators due to the reason that they are ladder operators that can help easily monitoring the change on spin states. We note the two spin eigenstates as \( \chi^{(\pm)} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ \pm 1 \end{array} \right) \), corresponding to left/right circular polarization states. And the unitary ladder operators function as follows

\[
\begin{align*}
S'_{\pm} \chi^{(\mp)} &= \chi^{(\pm)} \\
S'_{\pm} \chi^{(\pm)} &= 0
\end{align*}
\]

This pair of equations indicate a polarization conversion and selection. Looking again at the transfer matrix we can deduce another property of the output waves which is polarization sensitivity. The two circular polarized components in the input wave are converted to the orthogonal states and applied opposite phase. The unchanged, leakage wave has the same polarization as the input wave.

Summarizing these properties, we can see that a pure Type A LB element can control the input wavefront in a direct and efficient manner. By selecting the input polarization and the homogenous retardation, one can modify wavefront directly to the desired profile imbedded in the optical axis pattern. On contrary, a pure Type B LB element cannot directly change phase. Instead, it is possible to create an amplitude modification to the input wave, which is possible to create different wavefronts during wave propagation. The combination of the two types in one element provides a method to fully control the scaler lightwave, phase and amplitude independently.
3.1.2 Linear Dichroism, Type A and B

Similar to the first case, the transfer matrix is

$$T_{LD}(a,b,\Phi) = \begin{bmatrix} k_H \cos^2 \Phi + k_V \sin^2 \Phi & (k_H - k_V) \cos \Phi \sin \Phi \\ (k_H - k_V) \cos \Phi \sin \Phi & k_H \sin^2 \Phi + k_V \cos^2 \Phi \end{bmatrix}$$

$$= \frac{k_H + k_V}{2} I + \frac{k_H - k_V}{2} \begin{bmatrix} \cos(2\Phi) & \sin(2\Phi) \\ \sin(2\Phi) & -\cos(2\Phi) \end{bmatrix} \tag{3.5}$$

$k_H$ and $k_V$ are the transmission coefficients of two axes. $\Phi$ is the angle that one axis ($k_H$) makes from $+x$ axis. For Type A and B elements, these parameters are functions of locations. Rewrite as ladder operator in SU(2) parameter space

$$T_{LD}(x,y) = \frac{k_H + k_V}{2} I + \frac{k_H - k_V}{2} (e^{i2\Phi} S'_- + e^{-i2\Phi} S'_+) \tag{3.6}$$

The output field is consist of three waves, one with the same wavefront as input and two with shaped wavefront. The component weighting among waves are dependent on the two transmission coefficients. The composition of the output wave is very similar to the first case.

Note that for dichroism, $k_H$ and $k_V$ cannot be 1 at the same time. Therefore, the sum of squares of the amplitude coefficients are always less than 1. The element is intrinsic lossy. This is actually necessary since dichroism is an absorption phenomenon.

A spatially varying ideal linear polarizer is a special case of Type A element when $k_H = 1$ and $k_V = 0$

$$T_{Pol}(\Phi) = \begin{bmatrix} \cos^2 \Phi & \cos \Phi \sin \Phi \\ \cos \Phi \sin \Phi & \sin^2 \Phi \end{bmatrix}$$

$$= \frac{1}{2} I + \frac{1}{2} \begin{bmatrix} \cos(2\Phi) & \sin(2\Phi) \\ \sin(2\Phi) & -\cos(2\Phi) \end{bmatrix} \tag{3.7}$$

$\Phi = \Phi(x,y)$ is the angle that the transmission axis makes from $+x$ axis. Rewrite as ladder operator in SU(2) parameter space

$$T_{Pol}(x,y) = \frac{1}{2} I + \frac{1}{2} (e^{i2\Phi(x,y)} S'_- + e^{-i2\Phi(x,y)} S'_+) \tag{3.8}$$
3.1.3 Linear Birefringence and Linear Dichroism

When both linear birefringence and linear dichroism exist in the same medium, their principal axes are normally coincidental. The transfer matrix can be represented as

\[
T_{LB+LD}(x, y) = \frac{1}{2} e^{-i(\Gamma n'_x - \frac{\Gamma}{2})} \left\{ \left( (a + b) \cos \frac{\Gamma}{2} + i(a - b) \sin \frac{\Gamma}{2} \right) I^+ \right. \\
\left. \left( (a - b) \cos \frac{\Gamma}{2} + i(a + b) \sin \frac{\Gamma}{2} \right) \left( e^{i2\Phi(x,y)} S'_- + e^{-i2\Phi(x,y)} S'_+ \right) \right\}
\]  

(3.9)

\(a\) and \(b\) are the dichroism coefficients. \(\Gamma = \Gamma' \Delta n = (2\pi d/\lambda) \Delta n\). \(n'_x\) is one of the indices of refraction. This exponential term is just a trivial overall phase. There are always three possible waves. One with positive phase, one with negative phase, and the other with unaltered phase.

3.1.4 Circular Birefringence (Optical Rotation), Type B

The transfer matrix of an optical active medium is simply

\[
T_{CB} = \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix}
\]  

(3.10)

where \(\Theta\) is the rotation angle \(\Theta = \frac{\pi d}{\lambda_0} (n_L - n_R)\), \(n_L\) and \(n_R\) being the refractive index for left and right circular polarization, respectively. Calculating the rotated version of this matrix, one will find

\[
T_{CB}(\Phi) = T_{CB}(0)
\]  

(3.11)

So the matrix is invariant under rotation. If \(\Theta\) is a spatial varying function \(\Theta(x, y)\)

\[
T_{CB}(x, y) = e^{i\Theta(x,y)} S'_+ S'_+ + e^{-i\Theta(x,y)} S'_+ S'_-
\]  

(3.12)

This case is different from the previous three. There are only two output waves and the operators are not simple raise/lowering operators, instead, \(S'_+ S'_+ = \frac{1}{2} \left( \begin{array}{cc} 1 & \pm i \\ \mp i & 1 \end{array} \right)\). Let us
examine their functions on spin states

\[
\begin{align*}
S'_\mp S'_\pm \chi(\mp) &= \chi(\mp) \\
S'_\mp S'_\pm \chi(\pm) &= 0
\end{align*}
\]

Interestingly, these operators work as spin state filters. In the first term, \(S_- S_+\) blocks the \(\chi^+(+)\) state can let only the \(\chi^-(+)\) state transmit with a phase term \(e^{+i\Theta(x,y)}\). In the second term, \(S_+ S_-\) blocks the opposite \(\chi^-(+)\) state can let only the \(\chi^+(+)\) state transmit with an opposite phase term \(e^{-i\Theta(x,y)}\). Compare to the previous linear anisotropies cases, although the polarization states go through different changes in CB, the overall effect is the same: Wavefront is modified in two possible ways that are selected by input polarization. The difference is that there is no amplitude function due to the one less variable in CB elements. On one hand, this appears to be less control over the input lightwave. On the other hand, it is even a better wavefront shaping method than linear anisotropies since one does not need to worry about leakage wave. The theoretical 100\% conversion efficiency is easier to achieve, given the rotation angle pattern can by accurately engineered.

### 3.1.5 Circular Dichroism, Type B

\[
T_{\text{CD}} = \frac{1}{2} \begin{bmatrix}
  k_L + k_R & -i(k_L - k_R) \\
  i(k_L - k_R) & k_L + k_R 
\end{bmatrix}
\]

\[
= k_L S_+ S_- + k_R S_- S_+
\]

\(k_L\) and \(k_R\) are the circular dichroism coefficients. It is obvious that even the dichroism can be controlled to be a function of location, the transfer matrix indicates no phase modification. The coefficients affect the amplitude of two orthogonal polarized components.

### 3.1.6 General anisotropy, Type A

After examining the four kinds of optical anisotropies individually, we further derive a general formula for Type A elements, which has spatially varying optical axis and constant anisotropy value. We assume the matrix for a local point can be written as the following Jones matrix

\[
T_0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]
where \(a, b, c, d\) are complex numbers. Then the spatial varying anisotropy over the whole area is this local point rotated by an location dependent angle \(\Phi(x,y)\). The rotation matrix is \(R(\Phi) = \begin{bmatrix} \cos \Phi & \sin \Phi \\ -\sin \Phi & \cos \Phi \end{bmatrix}\). The matrix for the element at \((x,y)\) point is

\[
T(x,y) = R(-\Phi) T_0 R(\Phi) = \frac{1}{2} \left\{ (a - d) \begin{bmatrix} \cos(2\Phi) & \sin(2\Phi) \\ \sin(2\Phi) & -\cos(2\Phi) \end{bmatrix} + (a + d)I \\
+ (b + c) \begin{bmatrix} -\sin(2\Phi) & \cos(2\Phi) \\ \cos(2\Phi) & \sin(2\Phi) \end{bmatrix} + (b - c) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}
\]

(3.16)

where the coefficients are

\[
A = \frac{1}{2} (a - d + i(b + c)) \\
B = \frac{1}{2} (a - d - i(b + c)) \\
C = \frac{1}{2} (a + d) \\
D = \frac{1}{2} (b - c)
\]

(3.17)

For the subcategory of lossless cases, which includes birefringence but not dichroism, the matrix \(T_0\) is a unitary matrix and these relations must be satisfied: \(d = a^*\) and \(c = -b^*\). Then the coefficients can be simplified to

\[
A = \text{Im}(a) + i\text{Im}(b) \\
B = \text{Im}(a) - i\text{Im}(b) \\
C = \text{Re}(a) \\
D = \text{Re}(b)
\]

(3.18)

The first two terms in Eq. 3.16 are modified waves with opposite phase derivation and the last two terms are leakage waves with the same phase profile as the input wave. As discussed in Section 3.1.1, the first(second) wave is left(right) circularly polarized, has a phase change of \(\pm 2\Phi(x,y)\) from the input, its amplitude is decided by \(A(B)\). The third
term is the transmitted leakage wave where both polarization and phase are unchanged. The last term is another leakage wave but its polarization is converted to the state orthogonal to input polarization.

3.2 Geometric Phase Holograms

After discussing the possibilities of spatially varying polarization change, let us review our objective and evaluate the possibilities towards it. Our aim is thin, light weight, efficient optical elements that can perform wavefront shaping. An element in this category should work via geometric phase and possess the whole phase information of the target light beam at the output plane. We will use the name “Geometric Phase Holograms (GPH)” for these elements. The word “hologram” emphasizes the free-form wavefront shaping ability based on compact, thin elements. It is also appropriate due to the properties we will discuss below.

First, we examine if the light phase profile can be easily controlled. All the spatially varying anisotropies discussed in Section 3.1 except Circular Dichroism (Section 3.1.5) provides easy control over incident light phase. The phase formula is simply doubling the optical axis angle in linear anisotropies or doubling the optical rotation angle in optical activity. In all cases, a possible output wave with positive phase profile (noted by “+”) is always companied by a possible output wave with negative phase profile (noted by “−”). There is also sometimes a leakage wave (noted by “0”) and a modified leakage wave (noted by “0⊥”). This property resembles the phenomena when a traditional hologram is replayed. The two possible phase-modified output waves are like the reconstructed objective wave and its conjugate wave from a hologram (illustrated in Fig. 3.1).

Second, we make an observation on the polarization state of the output waves. For all the cases, the wave with positive phase change (has the term $e^{-i2\Phi(x,y)}$ or $e^{-i2\Theta(x,y)}$), called the “primary” wave, is left circularly polarized and the wave with negative phase change (has the term $e^{i2\Phi(x,y)}$ or $e^{i2\Theta(x,y)}$), called the “conjugate” wave, is right circularly polarized. If there is a leakage wave, the polarization is the same as the input wave. This is very different compared to conventional holograms, from which all output waves have the same polarization state as the input wave. Polarization plays a critical role in the GPH function. One of the reasons is that the shaped wave always has a defined and pure circular polarization. The second reason lays in the next discussion on efficiency.
In the third place, we examine the theoretical wavefront shaping efficiency in each case. The efficiency distribution among the different output waves is dependent on input polarization state, which we will use a Jones vector $\chi^{in} = \begin{bmatrix} \chi_x^{in} \\ \chi_y^{in} \end{bmatrix}$ to represent. For all the cases, including LB and LD in both Type A and B, and CB in Type B, the efficiencies for each of the output waves can be derived as

$$
\eta_+ = \alpha^2 |\langle \chi^- | \chi^{in} \rangle|^2 \\
\eta_- = \alpha^2 |\langle \chi^+ | \chi^{in} \rangle|^2 \\
\eta_0 = \beta^2 \\
\eta_{0\perp} = \gamma^2
$$

(3.19)

where $\langle \chi^{\pm} | \chi^{in} \rangle$ is the inner product of the input polarization with circular polarization state. It calculates the coefficients of the circular polarization components when the input state is expressed by these two states. Relate to Stokes description of polarization, $|\langle \chi^- | \chi^{in} \rangle|^2 - |\langle \chi^+ | \chi^{in} \rangle|^2 = S_3^2$, which is the normalized last Stoke parameter corresponding to the preponderance of right circularly polarized over left circularly polarized components.
So expressed by Stokes parameters, the efficiencies are

\[
\eta_+ = \frac{\alpha^2}{2} (1 + S_3')
\]
\[
\eta_- = \frac{\alpha^2}{2} (1 - S_3')
\]
\[
\eta_0 = \beta^2
\]
\[
\eta_{\perp} = \gamma^2
\]

(3.20)

We point out a very important note here. The wavefront shaping efficiency is defined as the efficiency of the shaped waves \((\eta_{\pm})\). Therefore, out of the three parameters, \(\alpha\) is the most important one when we examine the theoretical maximum wavefront shaping efficiency. Assuming that we can always control the polarization state of the input wave, the key to achieve 100% wavefront shaping efficiency is to have \(\alpha = 1\).

The weighting factors between the shaped waves and the leakage waves \(\alpha\) and \(\beta, \gamma\) take different forms in each case. For example in LB (Section 3.1.1), they are functions of retardation,

\[
\alpha = \sin \frac{\Gamma}{2}
\]
\[
\beta = \cos \frac{\Gamma}{2}
\]
\[
\gamma = 0
\]

(3.21)

Thus under half-wave retardation \(\Gamma = \pi\) condition the theoretical maximum wavefront shaping efficiency is 100%.

In LD (Section 3.1.2) they are functions of the transmission coefficients

\[
\alpha = \frac{k_H - k_V}{2}
\]
\[
\beta = \frac{k_H + k_V}{2}
\]
\[
\gamma = 0
\]

(3.22)

Under the condition \(k_H = 1\) and \(k_V = 0\) the theoretical maximum wavefront shaping efficiency is 50%.
In CB (Section 3.1.4) they are constants

\[
\begin{align*}
\alpha &= 1 \\
\beta &= 0 \\
\gamma &= 0
\end{align*}
\]

(3.23)

The theoretical maximum wavefront shaping efficiency is 100% unconditionally.

In the case of general anisotropy in Type A (Section 3.1.6)

\[
\begin{align*}
\alpha &= \sqrt{A^2} = \sqrt{B^2} \\
\beta &= C \\
\gamma &= D
\end{align*}
\]

(3.24)

where the expression of \(A\), \(B\), \(C\), and \(D\) are listed in Eq. (3.17) and Eq. (3.18). The maximum wavefront shaping efficiency is 100% when the modulus of \(A\) and \(B\) is 1.

Summarizing the above discussions, we can conclude the wavefront shaping function of a general Geometric Phase Hologram by this formula

\[
e^{i\delta_n}|\chi^{in}\rangle \rightarrow_{\text{GPH}} \sqrt{\eta_-} e^{i(\delta_n + 2\Phi)} |\chi^-\rangle + \sqrt{\eta_+} e^{i(\delta_n - 2\Phi)} |\chi^+\rangle + \sqrt{\eta_0} |\chi^{in}\rangle + \sqrt{\eta_{0\perp}} |\chi^{in}_{\perp}\rangle
\]

(3.25)

where the coefficients are found in Eq. (3.19)–(3.24). \(|\chi^{in}\rangle\) represents the orthogonal state of \(|\chi^{in}\rangle\). \(\Phi = \Phi(x, y)\) is the spatially varying pattern of optical anisotropy. It is optical axis for Type A GPHs and is replaced by optical rotation angle \((\Theta = \Theta(x, y))\) in optical active GPH.

To achieve a potential 100% wavefront shaping efficiency, one should choose from (a) CB in Type B configuration and (b) LB or other lossless anisotropy in Type A configuration. Between (a) and (b), (a) seems a better option since there is no leakage term. However, we suggest that (b) is advantageous due to the following reasons. First, the phase control term in (b) is the optical axis orientation \(\Phi\), which is simply a geometric variable and independent on the input light wavelength. This ensures the output wavefront has the desired phase profile regardless of input light. In contrary, in (a) the phase control term is the optical rotation angle \(\Theta\), which is a function of the operating wavelength. The circular refractive indices has chromatic dispersion which also contributes to the wavelength dependence of \(\Theta\). As a result, the phase profile of the output wave
will distort if the incident wave is at a different frequency than designed or incident is broadband light. The second reason is that in the case of (b), the GPH function contains more controllable variables which generally provides broader control over the light field. The efficiency factor is adjustable by functions of retardation. One could utilize this feature to engineer the spectral response of a GPH in a way that no other method could. This part of work will be shown in Chapter 6.

Therefore, in the work of this dissertation, we will choose to use linear birefringence and its variation in Type A configuration, where the optical axis spatially varies across the element.

3.3 Implementation of GPH

The implementation of GPH could use many different types of materials and the corresponding patterning methods, such as sub-wavelength structures created by lithography and liquid crystal layer aligned by rubbed patterns. We choose to use liquid crystal materials and photo-alignment technique due to the features discussed in Section 2.4. The liquid crystal molecules react to UV light and align themselves to the light polarization orientation in plane. The unit vector parameter “director” represents the direction of the average orientation of molecules in the neighborhood of any local point. Due to the geometric and optical anisotropies in liquid crystal molecules, the director orientation is also the optical axis orientation of the birefringence formed by the aligned liquid crystal.

To create an optical axis pattern, the principle is to create different UV light field for the liquid crystal molecules at different locations. We call this process the patterning. The patterning methods can be put into two categories: holography and direct writing.

First let us look at the holography patterning method. GPHs as holograms naturally lead us to think that they can be made by holography. However, GPHs are not like conventional intensity holograms, thus the holography to made GPH is not the conventional intensity holography. We will use polarization holography that was initially introduced to produce polarization gratings[178, 72, 177, 45, 47, 58, 153]. Two orthogonal circularly polarized plane waves forms a 1D periodic polarization pattern, which is recorded to a polarization grating.

Based on the polarization holography used for polarization grating, we can extend this method to record GPHs. Instead of two plane waves, we modify one or two of the
waves by optical elements such as lens or axicon. By the principle of holography, the result interference pattern at the sample plan will have the information of the optics. Assume the two coherent waves are

\[ \Psi_1 = E e^{-i\delta_1} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \]

\[ \Psi_2 = E e^{-i\delta_2} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \]

Then the total field is

\[ \Psi_1 + \Psi_2 = \frac{E}{\sqrt{2}} \begin{bmatrix} e^{-i\delta_1} + e^{i\delta_1} \\ i(e^{-i\delta_1} - e^{i\delta_1}) \end{bmatrix} \]

\[ = \sqrt{2}E e^{-i\delta_+} \begin{bmatrix} \cos \delta_- \\ \sin \delta_- \end{bmatrix} \]

where \( \delta_+ = \frac{\delta_1 + \delta_2}{2} \) and \( \delta_- = \frac{\delta_1 - \delta_2}{2} \). This is a linearly polarized field. The orientation of the linear polarization is decided by \( \delta_- \). Therefore, if the phase of the recording wave is a function of location, then the polarization of the total field is also a function of location. The two functions simply have linear relation. Once the liquid crystal is aligned, the director follows the UV light polarization orientation. The optical axis of resultant birefringence will be identical or perpendicular to the director depending on the type of liquid crystal, \( \Phi(x,y) = \delta_-(x,y) \) or \( \delta_-(x,y) + \pi/2 \).

This holographic method is good for creating most of the regular patterns that generate regular wavefronts. For example, a lens can be used to create a spherical recording wave. The result GPH is essentially a thin film lens.

The second category of GPH patterning method is direct writing. A UV laser is focused into a small spot on the sample with controllable polarization. The relative position of the laser spot on the sample is also controllable. The polarization pattern is directly written onto the sample in a pixel-by-pixel manner. This method enormously extends the experimentally realizable types and applications of GPHs, because of its freedom of creating almost arbitrary pattern.

In the experimental part of this dissertation, we will use both methods and also other patterning setups customized for particular patterns. We will demonstrate with OAM controlling GPHs that the fabricated GPHs can achieve wavefront shaping with high efficiency as the theory predicts.
Chapter 4

OAM control via Geometric Phase Holograms – Forked Polarization Gratings

The first GPH element we created is Forked Polarization Grating (FPG). Its optical axis has a complex 2D profile with point singularities. The function of this element is to manipulate helical wavefronts, thus to generate, control, and measure optical OAM.

This chapter is reformatted from the published manuscript: Yanming Li, Jihwan Kim, and Michael J. Escuti, ”Orbital angular momentum generation and mode transformation with high efficiency using forked polarization gratings”, Applied Optics, vol. 51, no. 34, pp. 8236-8245, 2012.

The major content in this chapter is finished by the dissertation author, who thank journal manuscript co-author Jihwan Kim for the helpful discussions and collaborations on the experiment settings and manuscript drafting and for making the sample mount for $q$-plate exposure setup.

Original article abstract: We present a novel optical element that efficiently generates orbital angular momentum (OAM) of light and transforms light between OAM modes based on a polarization grating with a fork-shaped singularity. This forked polarization grating (FPG) is composed of liquid crystalline materials and can be made either static or switchable with high diffraction efficiency (i.e., 100% theoretically) into a single order. By spatially varying the Pancharatnam-Berry phase, FPGs shape the wavefront and thus control the OAM mode. We demonstrate theoretically and empirically that a charge $l_g$
FPG creates helical modes with OAM charge \( \pm l_g \) when a Gaussian beam is input, and more generally, transforms the incident helical mode with OAM charge \( l_{in} \) into output modes with OAM charge \( l_{in} \pm l_g \). We also show for the first time that this conversion into a single mode can be very efficient (\( i.e., \sim 95\% \) experimentally) at visible wavelengths, and the relative power between the two possible output modes is polarization-controllable from 0% to \( \sim 100\% \). We developed a fabrication method that substantially improves FPG quality and efficiency over prior work. We also successfully fabricated switchable FPGs, which can be electrically switched between an OAM generating/transforming state and a transmissive state. Our experimental results showed \( \sim 92\% \) conversion efficiency for both configurations at 633 nm. These holographically fabricated elements are compact (\( i.e., \) thin glass plates), lightweight, and easily optimized for nearly any wavelength from ultraviolet to infrared, for a wide range of OAM charge, and for large or small clear apertures. They are ideal elements for enhanced control of OAM, \( e.g., \) in optical trapping and high capacity information.


4.1 Introduction

It has long been recognized that photons carry spin angular momentum that corresponds to polarization. During the last two decades, research has revealed that lightwaves can carry orbital angular momentum (OAM) as well, which is associated with the helical characteristic of the phase front [3]. It was first suggested and experimentally demonstrated by Allen’s group that lightwaves with an azimuthal angle-dependent phase term \( \exp(-i l \phi) \) carry OAM of \( l \hbar \) per photon, where \( l \) can take any integer value. Since this discovery, OAM has attracted intense attention and has been studied in various applications, including particle manipulation[81], micro-fabrication[74], and quantum information sciences[71, 70]. However, the generation, manipulation, and detection of OAM is still challenging; in particular, very few methods have yet been found that can generate and transform OAM states efficiently, compactly, and with generous tolerance to input polarization, incidence, and OAM purity.

Here we present a highly efficient OAM generator and controller based on polarization gratings (PGs). PGs are a category of diffraction gratings that are formed in anisotropic materials, and function by affecting the polarization of incident light in such a way as to control the Pancharatnam-Berry phase [178, 142, 23]. This in-plane wavefront shaping occurs within a thin anisotropic layer and leads to unique behavior: 100% diffraction into a single order for wide angular acceptance and wide range of periods. In one sub-class, researchers suggested to embody the birefringence profile by spatially aligning Liquid Crystal (LC) materials[47], which was later achieved with \( \sim 100\% \) efficiency by others[59, 153, 132]. These PGs are thin and lightweight, operate with extremely high efficiency, and can be made either static or switchable. These attractive properties of PGs have been used in several applications including laser beam steering[96], optical filters[134], attenuators[135], polarization imaging[102], and displays[100, 162]. Traditional PGs have a one-dimensional spatially-varying optical axis that follows \( \Phi(x) = \pi x/\Lambda \). Our new OAM element introduced here is a two-dimensional variation \( \Phi(x, y) \) that deftly manipulates OAM while otherwise preserving the advantageous features of traditional PGs.

Choi and coworkers have investigated two modified static PGs as OAM beam generators and have shown mixed experiment results[42]. The first is a modified reflective PG that diffracts always into multiple orders, wherein which they noticed the first-order diffraction manifests OAM. Because of the undesired zero and high order diffraction, this
structure has very limited efficiency and utility. The second structure is a modified transmissive PG made by two adjacent diffraction orders from a forked Computer Generated Hologram (CGH). They demonstrate that their samples can convert a Gaussian beam into an OAM beam, and that the OAM mode is sensitive to the polarization handedness. However, their preliminary report suggests very low diffraction efficiency and high scattering losses. Furthermore, they only discuss OAM generation from a Gaussian wave, and ignore the potential for conversion between OAM modes in general. All these issues make their structures insufficient for a practical mode converter.

Here we report on forked polarization gratings (FPGs), which can not only generate OAM from a Gaussian wave (i.e., with $l = 0$) input, but can also convert incoming OAM modes into higher or lower modes (Fig. 4.1) with high efficiency. Some of our preliminary results have been presented on these two conferences[111, 112]. In this work, we go substantially beyond our initial reports of the basic concept by providing a more clear and concise theoretical analysis, more complete experimental results, and much more comprehensive characterization. In particular, high quality polymer FPGs for visible wavelength are demonstrated for the first time. We also discuss the issues with current fabrication process and directions of the future improvement.

4.2 Forked Polarization Grating (FPG)

FPGs are a two-dimensional variation of the traditional one-dimensional PG, with a spatially-varying birefringence profile whose optical axis follows

$$\Phi(x, y) = \frac{1}{2} l_g \phi(x, y) - \pi x / \Lambda + \Phi_0$$

(4.1)

in the $xy$ plane and is homogeneous in the third ($z$) dimension. In the equation, $\phi(x, y) = \tan^{-1}(y/x)$ is the azimuthal angle of all positions within the $xy$ plane and $l_g$ denotes the topological charge of the FPG, sometimes called its order. $\Lambda$ is a design constant called grating pitch. For FPG analysis, it is convenient to assume the constant phase factor $\Phi_0 = 0$, as it does not affect the resulting OAM, polarization, or propagation direction.

An FPG is essentially an inhomogenous waveplate, with its optical axis spatially-varying in the $xy$ plane; it reshapes the wavefront by locally modifying the Pancharatnam-Berry phase of the incoming light. This modulation can be achieved by spatially varying birefringence and/or dichroism and is polarization sensitive as shown in Fig. 4.1. Among
Figure 4.1: Operation of our FPG concept as an OAM mode generator and transformer. The input beam has OAM charge $l_0$, which is diffracted by the FPG (with charge $l_g$) into (a-b): circularly polarized beam with charge $l_0 - l_g$ or $l_0 + l_g$ for a right or left circularly polarized input. (c): If the input beam is linearly polarized or unpolarized, both outputs exist. In a more general case, there is a zero-order leakage that has the same polarization and OAM charge as the input. Li01-intro.eps

many possible types and creation methods, in this work, we utilize the “circular PGs” [142] created by polarization holography and recorded on LC materials with positive birefringence. Fig. 4.2 shows FPG structures and two examples of FPGs. Here we only show the center areas since they are of the most interest. The arrows indicate the local optical axis, which is also the local nematic director in LC-based FPGs. Colored background is also used as an indication of the optical axis orientation, where purple(red) corresponds to optical axis parallel to $x(y)$ axis. It clearly shows the “forks” at the FPG center. It also predict the throughput distribution of FPGs when viewed under polarizing microscope.

4.2.1 Theoretical Behavior of FPGs

The well-known Jones matrix of a waveplate can be directly written for the FPG using Eq. (4.1), which describes the spatially-varying orientation $\Phi(x, y)$ of the local optical axis:

$$
T(x, y) = \cos \zeta I + i \sin \zeta \begin{bmatrix}
\cos 2\Phi(x, y) & \sin 2\Phi(x, y) \\
\sin 2\Phi(x, y) & -\cos 2\Phi(x, y)
\end{bmatrix}
$$

(4.2)
where $\mathbf{I}$ is the identity matrix, $\zeta = \pi \Delta n d / \lambda$ is a normalized retardation (i.e., half of the phase retardation), $\Delta n$ and $d$ are the birefringence and thickness of the waveplate, and $\lambda$ is the wavelength. This can be conveniently expressed as the ladder operators in SU(2) parameter space as

$$
T(x, y) = \cos \zeta \mathbf{I} + i \sin \zeta \frac{1}{2\hbar} \left( e^{-i2\Phi(x,y)} \mathbf{S}_+ + e^{i2\Phi(x,y)} \mathbf{S}_- \right) \tag{4.3}
$$

where $\mathbf{S}_\pm = \hbar \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}$ are the raising/lowering operators. Here, $\mathbf{S}_\pm \chi^{(\mp)} = 2\hbar \chi^{(\pm)}$, where $\chi^{(\pm)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$ are the two spin eigenstates, corresponding to right (+) and left (−) circular polarization states (i.e., RCP and LCP, respectively).

Note that $\pm 2\Phi(x, y) = \pm (l_g \phi(x, y) - 2\pi x / \Lambda)$ is the net phase change (via Pancharatnam-Berry effect) caused by the FPG, for the LCP and RCP components (the second and the third terms in Eq. (4.3)), respectively. Therefore, an FPG with charge $l_g$ should add or subtract an azimuthal phase variation of $2\pi l_g$ about the singularity, and thus transform light OAM to a state $l_g$ higher or lower.

To further observe this transformation, consider a circularly polarized incident beam with electric field $\mathbf{E}_{in}(x, y) = e^{i\phi(x,y)} \chi^{(\pm)}$, which has OAM charge $l_{in}$, RCP/LCP
polarization, and unity magnitude. The near-field electric field \( \mathbf{D}(x, y) \) immediately after the FPG is

\[
\mathbf{D}(x, y) = \mathbf{T}(x, y) \mathbf{E}_m(x, y) = \cos \zeta e^{i l_m \phi(x,y)} \chi^{(\pm)} + \sin \zeta e^{i (l_m \pm l_g) \phi(x,y) + i 2\pi x / \Lambda} \chi^{(\mp)}. \tag{4.4}
\]

Three waves can be identified in this output, by inspection. The linear dependence on \( x \) in the second term results in a diffracted wave (symmetric directions for the two circular polarizations), whereas the first term is the directly transmitted wave. The azimuthal dependence in both terms suggests helical wavefronts, which persist in each wave under paraxial conditions. The far-field electric field in each diffraction order \( m = 0, \pm 1 \) can be summarized as:

\[
\mathbf{D}_m = \cos \zeta \delta_m e^{i l_m \phi(x,y)} \chi^{(\pm)} + \sin \zeta \delta_{m \pm 1} e^{i (l_m \pm l_g) \phi(x,y)} \chi^{(\mp)}. \tag{4.5}
\]

Note that for a circularly polarized input there are only two diffraction orders, \( m = 0 \) and \( m = \pm 1 \). For RCP/LCP input, the output takes the top/bottom set of signs in Eq. (4.5). However, if the input is polarized in any other way, the diffraction will be the superposition of the circular cases, containing all the three orders, \( m = 0, m = +1, \) and \( m = -1 \).

The net power-transfer to any of these diffraction orders is most easily described by a ratio of output to input intensity \( \eta_m = |\mathbf{D}_m|^2 / |\mathbf{E}_m|^2 \), called the diffraction efficiency:

\[
\eta_0 = \cos^2 \zeta \tag{4.6a}
\]
\[
\eta_{\pm 1} = |\langle \chi^{(\mp)} | \Psi \rangle|^2 \sin^2 \zeta \tag{4.6b}
\]

where \( |\langle \chi^{(\mp)} | \Psi \rangle|^2 \) is the fraction of the input power in each \( \chi^{(\mp)} \) spin eigenstate, \( \Psi \) being the normalized input wave function. Note that \( |\langle \chi^{(+) | \Psi} \rangle|^2 - |\langle \chi^{(-)} | \Psi \rangle|^2 = S'_3 \), the normalized Stokes parameter corresponding to the fraction of incident circular polarization.

We can now summarize several notable features of ideal FPGs that are packed into Eqs. (4.5) and (4.6):

1. The output of an FPG has only three possible diffraction orders: \( m = 0 \) and \( \pm 1 \). The relative power balance between the zero and the first-orders depends only on the retardation \( \zeta \). The first-orders reach 100\% efficiency when the retardation is
halfwave \((i.e., \zeta = \pi/2)\), and all of this light is directed into a single first-order for circularly polarized input. This behavior is the same as traditional one-dimensional PGs.

2. The first-order outputs \((m = \pm 1)\) have their OAM charge changed by \(l_g\), such that \(l_{m=+1} = l_{in} - l_g\) and \(l_{m=-1} = l_{in} + l_g\), as can be observed in Eq. (4.5). The linear momentum is also affected, as these propagate along first-order diffraction directions set by the classic grating equation \((\sin \theta_{\pm 1} = \pm \lambda/\Lambda)\), and will include the normal grating dispersion. The first-orders are always circularly polarized, orthogonal with each other, and orthogonal to the circular polarization present in the input.

3. The zero-order output \((m = 0)\) has the same OAM, polarization, and direction as the input beam. Note that both the output OAM and polarization behavior of the FPG output are independent of the retardation.

It is worthwhile to note that the first-order output of the FPG transforms all three types of momentum simultaneously: OAM, spin angular momentum \((i.e., \text{polarization})\), and linear momentum \((i.e., \text{wave vector} \vec{k})\). In the case of halfwave retardation \((\zeta = \pi/2)\) and normal incident, the transform can be concluded as

\[
|l_{in}\rangle^{(+)} , \vec{k} \cdot \hat{x} = 0 \rightarrow |l_{in} + l_g\rangle^{(-)} , \vec{k} \cdot \hat{x} < 0
\]

\[
|l_{in}\rangle^{(-)} , \vec{k} \cdot \hat{x} = 0 \rightarrow |l_{in} - l_g\rangle^{(+)} , \vec{k} \cdot \hat{x} > 0
\]

(4.7)

This is very rare to occur in a single element, and can be preferable in some applications.

4.2.2 Numerical simulation of FPGs

As a first step to empirically validate the theoretical behavior of FPGs, we used numerical simulation of the Jones calculus above. For various inputs and FPG characteristics, we calculated the near-field output electric field via Eq. (4.4) on a two-dimensional \((x, y)\) grid. We then used Fourier transforms to simulate beam propagation and examined the resulting far-field electric field. Because the three possible diffraction orders of the FPG spatially separate in the far-field, we can observe amplitude and phase information of each separately. The result from several simulations where the FPG had halfwave retardation are shown in Fig. 4.3, with both linear and circular polarization inputs. The
doughnut-shaped intensity is a characteristic of an OAM beam, with the dark center radius increasing with its charge. Phase plots in the bottom rows show the OAM charge of each output beam. Each light to dark transition presents a $|2\pi|$ phase difference. In the phase plots of the cross-sections normal to the diffraction wave vectors, the direction of this transition going about the center decides the sign of phase change and therefore the sign of OAM charge. In the bottom phase plots, which are views at an oblique cross-section, OAM charge can be observed through the number of the fork-shaped branches. The results confirm the theory above — most notably, the OAM charge of the first-orders is changed by the charge of the FPG, and that 100% of the input is diffracted into a single first-order when the input is circularly polarized.

Figure 4.3: Numerical simulation of two cases of FPG diffraction, showing far field intensity (a–c) and phase at diffracted beam cross-sections (d–e). The FPG charge $l_g$ and input $l_0$ for each simulation is shown at the top of each column. Diffraction order $m$ is shown at the bottom of each column. The input polarization is (a)linear, (b)right circular, and (c)left circular. The phase distributions of diffracted beams are calculated at (d)on-axis cross-section and (e)off-axis cross-section of each beam. Li03-fpgFun2.eps
4.3 Fabrication Methods

The fabrication of FPGs involves polarization holography and, in our approach, axial wave plates called \( q \)-plates \([122, 127]\) acting as helical wave generators. We will discuss these both in turn, and then discuss the detailed fabrication of the FPG itself.

4.3.1 Polarization holography

We fabricated the FPGs by writing a polarization hologram into photo-alignment materials. Polarization holography records the polarization standing wave of the interference of two beams, rather than the intensity (bright/dark) fringes. Two coherent beams with orthogonal circular polarization are superimposed with a small angle between them, which creates a spatially varying standing wave in the polarization pattern, with constant intensity everywhere. For two equal power plane waves, the resulting field is linearly polarized everywhere, with a one dimensional periodic linear modulation of its polarization orientation \([178, 47]\).

4.3.2 Liquid crystal and photo-alignment principles

While polarization holograms can be recorded and developed in many polarization-sensitive materials, we use a thin layer of photo-sensitive polymer called Linear Photopolymerizable Polymer (LPP) \([160]\) on a glass substrate. This establishes a spatially-varying alignment condition in the LPP that follows the local linear polarization direction of the standing wave. When LC encounters this LPP profile, the latter aligns the former to achieve the desired optical axis orientation. For a static (i.e., passive) configuration, we spin-coat a reactive mesogen mixture, and after it aligns to the LPP we photo-polymerize it into a solid cross-linked film. This processing is based on the procedure detailed in Ref. \([59]\). For a switchable (i.e., active) configuration, the fabrication process is as follows: first, ITO-coated substrates are spin-coated with the LPP and then assembled into a cell with a fixed thickness set by glass bead spacers; second, the cell is exposed to the polarization hologram; third, this cell is heated and filled with nematic LC in its isotropic phase and then cooled down to room temperature.
4.3.3 \textit{q}-plate fabrication

To record FPGs, we need one or two of the writing beams to carry helical wavefront(s). In this work, we realized this by inserting inhomogeneous wave plates called \textit{q}-plate(s) in the optical path(s)\cite{111}. A \textit{q}-plate is essentially a radially symmetric half wave plate. It has azimuthal varying optical axis in the transverse plane which follows $\Phi(\phi) = q\phi + \Phi_0$\cite{122}. In our project, we realized the azimuthal-varying photo-alignment by relative rotation between the sample and the light polarization orientation, as a revised setup in Ref. \cite{127}. Specifically, we focused the linearly polarized light from a HeCd laser using a cylindrical lens into a slim strip on the sample and simultaneously rotated the cylindrical lens and the substrate at respective speeds during the exposure. Since we will use them in the UV polarization holography, we optimized their retardation for halfwave condition at 325 nm wavelength, with the anisotropy recorded by the same materials that we used for the FPGs.

4.3.4 FPG fabrication

We adapted the conventional PG fabrication setup for FPG fabrication by inserting a \textit{q}-plate into the holography optical path (see Fig. 4.4(a)). To avoid the low intensity center caused by the OAM singularity during light propagation, we minimized the distance between the \textit{q}-plate and the sample. This resulted in a uniform intensity distribution to align the photo-sensitive polymer, leading to good alignment over the whole sample. This is crucial to FPG quality, since the center area is where the fork-shape singularity will be formed. In order to keep the \textit{q}-plate close to the sample, it has to be in the path of both writing beams, as shown in Fig. 4.4(a). As a result, two singularities will be recorded on the sample. That is, for every singular point we record, there will be a doubled singularity side by side on the same sample (Fig. 4.4(b)). While this may be a concern for some contexts, for this work it was not, since the \textit{q}-plates we used were of low charges and the non-center part of them resembled a normal halfwave plate. We therefore considered the result as two FPGs on one sample, with singularities located several hundred $\mu$m or several mm apart. Since our area of interest is confined to the region around one singularity, it is equivalent to polarization holography with a \textit{q}-plate and a half wave plate in the two writing arms respectively, as shown in Fig. 4.4(c).

The fabrication recipe was tailored as following: we exposed LPP (DIC: LIA-C001)
under polarization holography by HeCd (325 nm) laser with 5 J/cm$^2$. The LC layer was always arranged into a half wave thickness (i.e., here for 633 nm), depending on the configuration as follows. For the polymer FPGs, we used the reactive mesogen mixture from Merck (RMS10-025) with $\Delta n = 0.16$ birefringence. We coated a first layer with diluted LC (25%) at 1500 rpm for 30 seconds, then a second layer with 100% LC at 630 rpm for 45 seconds. For the switchable FPGs, we filled LC material MDA-06-177 ($\Delta n = 0.135$ at 589 nm, EMD Chemicals) into a $\sim 2.2 \mu m$ gap cell.

Figure 4.4: Fabrication of FPG: (a) polarized holography setup; (b) detailed view of the interference, where the $q$-plate singularity is projected twice onto the sample plane (yellow arrows); (c) equivalent holographic interference for each singularity in (b). M:Mirror, BS:Beam Splitter, QWP:Quarter-Wave Plate, HWP:Half-Wave Plate. Li04-setupNew.eps

4.4 Experimental Results

4.4.1 LC alignment fidelity

We first verified that our samples have the desired optical axis alignment. Fig. 4.5 shows the polarized optical microscope pictures of the FPGs fabricated as described in the previous section. The two polymer FPGs are of charge $l_g = +1$ (Fig. 4.5(a)) and $l_g = +2$ (Fig. 4.5(b)). In both cases, the pictures are centered at the singularity to show the variation. In sample areas other than the singularities, the alignment is uniform in one dimension with a period of 16 $\mu m$ in the other dimension, just like a PG (shown in
Figure 4.5: Polarizing optical microscope pictures of FPGs between crossed polarizers: (a) at the singularity of a polymer FPG \((l_g = +1)\); (b) at the singularity of another polymer FPG \((l_g = +2)\); (c) a view well away from the singularity of the FPG in (a); (d) low-mag view of both singularities of the FPG in part (a); A switchable FPG \((l_g = +1)\) at (e) 0 V and (f) 15 V. In (a)-(d), a fullwave retarder was inserted in the microscope to emphasize the forked alignment pattern; in (e) and (f) this was removed.

Li05-fpgMicroSFPG.eps

Fig. 4.5(c)). Comparing to Fig. 4.2(c–d), we see the alignment is a good match with both theory and simulation. Our fabrication method is valid as discussed in Section 4.3.4. For the charge \(l_g = +1\) FPG, we also show its location relative to the doubled singularity in Fig. 4.5(d). The two singularities are measured \(~667\) nm apart on this sample. A \(l_g = +1\) switchable sample is shown in Fig. 4.5(e) and (f). The same overall periodic pattern and a bifurcation are present. Some of the point defects are the spacers in the cell gap. With a voltage of 15 V \((\gg\) threshold voltage) applied, the grating structure fades out and disclination lines appear over large length scales (Fig. 4.5(f)). This phenomenon suggests the LC molecules are mostly aligned perpendicular to the substrates as expected. This state of a switchable FPG is defined as “on” state and the FPG at this state is addressed to be on its transmissive mode. Correspondingly, the state with zero applied voltage is defined as “off” state and FPG is addressed to be on generating/transforming mode.
4.4.2 Far-field spectral and diffraction behavior

We then investigated the far-field spectral and diffractive behavior. To measure the spectra, we used a collimated, unpolarized broadband light source with small beam size along with an integrating sphere connected to a fiber spectrometer (Ocean Optics) to isolate individual diffraction orders. The ratio of the power measured within a diffraction order to the input power is the transmittance, and corresponds to the total net throughput into that order. In all our samples, this varied from 95% to 99% in the visible range. However, as with many diffraction grating analyses, it is more useful to consider the ratio of the power measured within a diffraction order to the total power transmitted into the output hemisphere, a quantity called the diffraction efficiency. This isolates the diffractive behavior of the PG itself in the experimental measurements and allows direct comparison to Eq. (4.6) by normalizing out the minor effect of the substrate, interface reflections, and absorption, but preserving the effect of any scattering.

The diffraction efficiency spectra of zero-order, first-order, and total first-orders of a polymer FPG are shown in Fig. 4.6. Two aspects are particularly important. First, as predicted in PG theory (Eq. (4.6)), the zero-order and first-order efficiency vary with wavelength. Their valley and peak locations verify that the birefringence of our sample gives halfwave retardation at wavelength of 633 nm, which was our target. Second, where the normalized retardation $\zeta = \pi/2$ at 633 nm, most light is diffracted into the two first-orders. Very little power is leaked into zero-order or high orders. In this case, the input light is unpolarized, each of the two first-orders equally obtain half of the total power. This is very close to the theoretical values, $\eta_0 = 0$ and $\eta_{\pm 1} = 1/2$. The small difference is due to scattering and very dim higher orders, which may be caused by material point defects or slight alignment mismatch.

Switchable FPGs on generating/transforming mode were measured to have the same zero-order and first-order efficiency spectra as the polymer FPGs. The same switchable FPGs on transmissive mode become nearly vertical-aligned and non-diffractive, thus a ~ 100% zero-order efficiency across the measured wavelength range was obtained as expected[112].

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4.4.3 Polarization response

Here we describe the polarization behavior of FPGs at their optimized wavelength. We used a collimated beam from a HeNe laser (633 nm) as the normally incident light, and received the diffraction from the FPG by a photo-detector with integrating sphere in the far field region. The laser beam has a waist of 0.5 mm, so that we can shine it at one FPG on our sample without involving the other singularity. As shown in Table 4.1, most of the transmitted light ($\sim 96\%$) is diffracted to the lowest three orders, among which the zero-order was much weaker than the first-orders. Around $2\%$ of the light goes to higher orders, the remaining power ($\sim 2\%$) is lost by scattering. The power distribution between the two first-orders is dependent on the polarization of the incident light as predicted by Eq. 4.6(b). Left circularly polarized light ($|\langle \chi^(-) | \Psi \rangle|^2 = 1, |\langle \chi^{(+)} | \Psi \rangle|^2 = 0$) is mostly diffracted to the first-order ($\eta_{+1} = 1, \eta_{-1} = 0$), whereas right circularly polarized light ($|\langle \chi^(-) | \Psi \rangle|^2 = 0, |\langle \chi^{(+)} | \Psi \rangle|^2 = 1$) is diffracted to the negative-first-order ($\eta_{+1} = 0, \eta_{-1} = 1$). When the incident light is linearly polarized, which is equivalent to equally composed right and left circular polarization ($|\langle \chi^{(+)} | \Psi \rangle|^2 = 1/2$), the total power is equally distributed into the two first-orders ($\eta_{\pm 1} = 1/2$). Unpolarized input works the same way, as the case in Fig. 4.6 showed. Thus, the FPGs are demonstrated to have the polarization sensitivity predicted by theory and simulation.

We also graphed these data points in Fig. 4.6 (black squares) to compare them with
Table 4.1: Measured FPG diffraction efficiency (%).

<table>
<thead>
<tr>
<th>Input Polarization</th>
<th>$m = -1$</th>
<th>$m = 0$</th>
<th>$m = +1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>45.4</td>
<td>2.1</td>
<td>48.6</td>
</tr>
<tr>
<td>Left circular</td>
<td>0.4</td>
<td>1.2</td>
<td>95.5</td>
</tr>
<tr>
<td>Right circular</td>
<td>94.8</td>
<td>2.0</td>
<td>0.3</td>
</tr>
</tbody>
</table>

the data obtained by the spectrometer. The first-order diffraction efficiencies appear slightly lower and the zero-order leakage higher for the HeNe laser than those from the broadband light source. We think the reason for the difference is beam size related. The laser beam was much narrower than the white light and covered a smaller area of the FPG. Thus the singularity on the FPG took a higher percentage of the whole effective area. Since the singularity is essentially a kind of defect, we expect more loss from the laser.

4.4.4 OAM mode transformation behavior

To examine the OAM charge of each individual diffracted beam, we set up a Mach-Zehnder interferometer with the HeNe laser (633 nm) and captured the far field interference pattern with a CCD camera. The objective beam is each order of the diffraction from FPG, the reference beam is an expanded Gaussian wave. We used the direct output mode from the laser, which was Gaussian with waist of 0.5 mm as input for the FPG. This beam size ensured that only one singularity on each sample was lit.

Fig. 4.7 (a-d) shows the results for a linearly polarized Gaussian input beam and a charge $l_g = +1$ FPG. The interference patterns show that the three orders of diffraction have different OAM charges, which are indicated by the different fork-shape topological charges. Thus, OAM modes of $l = \pm 1$ are generated at the $m = \pm 1$ diffraction. Meanwhile, the very weak $m = 0$ order leakage remains Gaussian. Fig. 4.7 (e-h) shows the results for a OAM charge $l = -2$ input light and the same $l_g = +1$ FPG. The $m = +1$ diffracted light transforms to OAM mode of charge $l = -3$, which is the input mode lowered by 1; $m = -1$ diffracted light transforms to OAM mode of charge $l = -1$, which is the input mode raised by 1; and $m = 0$ leakage remains the same mode, $l = -2$. Note the topological resemblance between these patterns and Fig. 4.3(e). Easily from interference principle, the result matches our FPG formula and simulation, aside from
Table 4.2: Measured switchable FPG diffraction efficiency (%).

<table>
<thead>
<tr>
<th>Input polarization</th>
<th>Linear</th>
<th>Right circular</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$m = -1$</td>
<td>$m = 0$</td>
</tr>
<tr>
<td>Applied voltage</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 V</td>
<td>45.8</td>
<td>2.8</td>
</tr>
<tr>
<td>5.5 V</td>
<td>26.1</td>
<td>39.8</td>
</tr>
<tr>
<td>15 V</td>
<td>1.9</td>
<td>93.3</td>
</tr>
</tbody>
</table>

some distortion that we will discuss in later section.

Figure 4.7: Interference pattern of input light and each diffraction order from a $l_g = +1$ FPG. (a–d): Gaussian input; (e–h): OAM charge $l = -2$ input. (a,e): input beam; (b,f): $m = +1$ diffraction; (c,g): $m = 0$ diffraction; (d,h): $m = -1$ diffraction. Li07-1129611297.eps

4.4.5 Voltage response of switchable FPGs

For the switchable configuration, the samples work the same way when no external voltage is applied. To verify the diffraction properties when switching, we did the same characterization with several applied voltages using a HeNe laser which has the targeted wavelength 633 nm. Table 4.2 and Fig. 4.8 show the diffraction efficiency and OAM analysis of a $l_g = +1$ switchable FPG for three states: “off”, “on”, and an intermediate state (Details in Ref. [112]).

The result agrees very well with our theory in Sec. 4.2.1. First of all, the OAM of all
three diffraction orders is polarization insensitive, just as the polymer FPGs. Secondly and the most significantly, we testify that the output OAM change is unrelated to applied voltage either. The proof is that the interference patterns at different applied voltages are topologically identical (Fig. 4.8, bottom row).

We then did comparative measurements of the voltage response of the diffraction efficiency at two wavelengths: the optimized wavelength of the sample, 633 nm and a control wavelength 532 nm. The zero-order and total first-orders diffraction efficiencies at 633 nm is shown in Fig. 4.9, as the solid lines with blue and red, respectively. The squares and circles are actual measured data points. In the same manner, cyan and magenta dash lines with diamond and circle data points are the zero-order and total first-orders diffraction efficiencies at 532 nm. As expected for most LC materials, all these diffraction efficiencies start to flip at a threshold voltage $V_{th} \sim 2.5$ V and reach saturation at some high voltage. An interesting observation is that for the non-optimal wavelength 532 nm, its diffraction efficiencies reach maximum and minimum at a non-zero applied voltage $V_{th} \sim 2.5$ V. The values are $\sim 3\%$ and $\sim 92\%$ for zero-order and total first orders, which are very close to the values of 633 nm with zero applied voltage. This result suggests an useful feature of switchable FPGs: a sample that is optimized for a certain wavelength at its “off” state can be tuned for other wavelengths by simply applying a small external voltage. In other words, switchable FPGs can achieve broadband high efficiencies through electrical tuning.
Figure 4.9: Zero order \((m = 0)\) and first-orders \((\Sigma m = \pm 1)\) diffraction efficiency responses to applied voltage. Solid lines are measured using 633 nm laser. Dashed lines are measured using 532 nm laser. Li09-TvsV-633n532.eps

The raise and the fall switching times of the switchable FPGs were measured with a HeNe laser and an oscillating square voltage. The zero order intensity was measured during switching. Fig. 4.10(a) shows the LC response to a 15V (rms) square wave. The 10%-90% rise and fall times versus applied voltage is shown in Fig. 4.10(b). As expected, the rise time is sensitive to applied voltage, whereas the fall time is mainly material dependent and tends to be longer. Applied > 5V voltage, the total switching times of all samples are on the order of 3 ms, similar to the expected value for a homogeneous half wave plate configured with this liquid crystal. Here we highlight the fast total switching time of all samples, which is on the order of 3 ms. This is about the average value of a conventional homogeneous halfwave plate configured with the same type LC materials. Therefore, our FPGs preserve the good electro-optical switching fashion of LC elements.

4.5 Discussion

FPGs with good OAM mode transformation and efficiency have been fabricated. The key enabling our high quality result is utilizing a classic polarization holography method, which resulted in excellent LC alignment without defects and minimal scattering. The resulting FPGs exhibit all the properties predicted by theory and simulation. Power
measurements of the diffraction orders show our current FPGs have very good diffraction efficiencies (92% – 95%). We believe by adjusting the recording setup and fine-tuning the processing, it is possible to achieve diffraction efficiency > 99%. A layer of anti-reflection coating on the substrate will help improve the absolute transmission as well. This means that these FPGs could modulate the OAM state of light with very little power loss.

For switchable FPGs, the applied voltage controls the efficiency ratio between zero-order and the total first-order diffraction, which is identical to the power ratio between the unchanged and changed OAM. Thus, by adjusting applied voltage, we can control the percentage of light being transformed, from entirely transformed (100%) to not transformed at all (0%). And for every voltage (percentage of transformation), the incident polarization determines the efficiency ratio between $m = -1$ and $m = +1$ diffraction, that is, how much of the transformed light goes to a upper OAM state and how much goes to a lower OAM state. As a result, by combining voltage and polarization controls, a switchable FPG can amazingly control these three output OAM states: the original OAM, the OAM state lowered by $l_g$, and the OAM state raised by $l_g$, with arbitrary power ratio from 0% to ∼100%. This is very unique among all the current OAM controlling techniques.
The dynamic response of a switchable FPG is fast, with a total switching time of \( \sim 3 \) ms, which is on the order to a conventional nematic LC halfwave switch. For certain applications, switchable FPGs can make low-cost, flexible and compact substitution for SLM systems.

The special diffractive property of FPG makes it a significant better OAM mode transformer in some circumstances, compared to its close kin \( q \)-plate. Unlike \( q \)-plates, which only work well with circularly polarized light, FPGs work well with arbitrary polarization. The “fork” shaped birefringence pattern modulates the wavefront through Pancharatnam-Berry phase and results in both linear momentum and OAM change. A pure helical input will always be transformed to pure helical beams. For instance, if an OAM eigenstate is needed from a Gaussian source, we need to make sure the input light is perfectly circularly polarized before sending it through a \( q \)-plate. Otherwise, the output will be a superposition of \( +2q \) and \( -2q \) modes. Additionally, the \( q \)-plate should be exact half-wave, or the leakage wave (with unchanged OAM) will add to the output as well. Although supplementary polarization filters could be used, it adds to the complexity of the system. Alternatively, with FPGs, even if the input is not perfectly circular or is not the optimal wavelength, we will always get a pure OAM eigenstate at the first-order diffraction. The undesired modes are automatically filtered \( (i.e., \text{ diffracted into other directions}) \). This feature makes them superior in applications where purity of a single eigenstate is preferred.

Challenges remain as well. One might notice the little distortion in some of the interference patterns of the high charge beams, such as in Fig. 4.7(g). The distortion indicates that the charge \( l = 2 \) singularity tends to split into two unit charge ones. We know this can happen when the beam is not perfectly symmetric [61], due to perturbation or ellipticity of the beam itself. In our case, the imperfect alignment of the FPG may have caused this problem. A close look at Fig. 4.5(b) suggests that the anisotropy is slightly different from the vector plot in Fig. 4.2. One possible reason relates to the optical axis patterning process. For a high charge OAM beam, the center area carries very high topological strength, which means the photo-alignment material we exposed to the beam will experience more difficulties in anchoring accordingly. This happens for both \( q \)-plates and FPGs. This could be improved by increasing exposure fluence or adjusting the processing methods. We will leave the comprehensive diagnosis and improvement for our further work.
With high quality FPGs now realized, many applications of FPGs can be considered; instead of spiral phase plates or CGHs, one can use FPGs to easily generate helical light modes, which can be used in particle manipulation or micro-fabrication. We can expect easier fabrication and simpler control in these fields with the use of FPGs.

4.6 Conclusion

We have proposed a polarization-controlled OAM mode generator and transformer called a Forked Polarization Grating and have successful realized it. We demonstrated that with our polarization hologram setup and photo-alignment technique we can make high-quality FPGs that modulate light’s OAM highly efficiently. By adjusting the input polarization and FPG retardation (by applied voltage on switchable FPGs), we are able to achieve a $\sim 100\%$ OAM conversion of one beam, or generate three beams with different OAM at the same time and freely control their ratio. Moreover, FPGs modulate the wavefront through Pancharatnam-Berry phase and can spatially separate the outgoing beams with different OAM states. With these competitive advantages, including simple processing, compact size and light weight, FPGs will greatly promote the development of applications that utilize the OAM of light, especially in high-capacity information sciences.

4.7 Acknowledgment

The authors gratefully acknowledge the support of the National Science Foundation (NSF grant ECCS-0955127).
Chapter 5

Geometric Phase Holograms Relay – Agile Orbital Angular Momentum Control and Vector Beam Generator

In this chapter, we try to answer the question if geometric phase holograms can be used to actively generate multiple wavefronts without manually exchange the phase element, like an electrical spatial light modulator dose. To answer this question, we expand the theoretical analysis from a single GPH element to GPH relays and propose a system design with the ability to shape single wavefront to a series of wavefronts. We propose and demonstrate a design consisting of a stack of active GPHs and wave plates. The system can shape the input light into a category of many wavefronts. The number of controllable wavefronts expands exponentially with the system size. It has the ability to freely distribute the input energy into these wavefront shapes as desired by pure electrical control. Particularly, the input light beam can be transformed to any of the possible wavefronts with near-zero energy loss.

This system is adaptable for a wide range of wavefront groups that share topological commons. We set our experimental demonstration of this design in this chapter as the multiple helical wavefront modes in singular beams. We experimentally demonstrated an agile singular beam control system consisting of a stack of azimuthal GPHs, which can switch the output wave among 27 helical modes. The output wavefronts and the transform efficiencies is also examined.

The major content in this chapter is finished by the dissertation author. The ternary
arrangement was inspired by journal manuscript co-author Jihwan Kims prior work on laser beam steering. The application of the ternary arrangement in OAM control was proposed by both co-authors.

5.1 Overview on Singular Beam Mode Control

Singular beams are light beams that process local singularities in one or more of their field parameters, such as polarization or phase. Due to the special properties associated with these field singularities, singular beams is becoming a research subject of growing interest. One kind of the most studied polarization singularities is in cylindrical vector beams, where the polarization has cylindrical symmetry\[197\]. Their properties such as tight focusing has led to various application in optical imaging\[28\] and manipulation\[196\]. The singularities in phase has been studied in optical vortices and helical beams that carry Orbital Angular Momentum (OAM). These helical modes as high angular momentum carriers and a complete orthogonal base set, are gaining rapid development in optical tweezing\[68\], microfabrication\[74\], and quantum information applications\[71, 70\].

Among a variety of methods to generate and manipulation single singular beams\[3, 15, 18, 80, 122, 113, 36\], most are only capable for a single or a limited number of modes while the ability to easily generate singular beams of multiple orders are often desirable in applications. Spatial light modulator as the most commonly used method in laboratories to generate various helical modes is expensive and inefficient. Many alternatives have been studies with \(\varphi\)-plates and other elements. Stalder et al. discussed the possibility of cascading elements to generate any polarization order and demonstrated a charge 3 field in their work\[171\]. Similarly, O’Dwyer et al. reported switching between three states by cascading conical refraction from biaxial crystals\[140\]. However, their schemes have low mode-per-element efficiency and become hard to implement for high orders due to the limited element type they are based on.

5.2 Geometric Phase Hologram Relay System Design

In this work, we present a ternary system for high efficient singular beam generation called \(Q\)-stack based on a stack of electrical controlled Geometric Phase Holograms (GPH). We
formulate the effective topological charge for the whole system over each element status and experimentally demonstrate a working device generating multiple polarization and phase singular modes.

\[
\text{(Circular Input)}
\]

\[
\text{l in}
\]

\[
\text{L/RCP}
\]

\[
\text{QP, V = 0}
\]

\[
\text{lin +2q (RCP)}
\]

\[
\text{lin (LCP/RCP)}
\]

\[
\text{lin -2q (LCP)}
\]

\[
\frac{M = 3^N}{\text{L/RCP}}
\]

\[
\text{QP, V > V th}
\]

\[
\text{n_1 n_2 n_3}
\]

\[
\text{M = 3^N}
\]

Figure 5.1: Ternary design of the Q-stack. (a) An active \( q \)-plate with circular polarized input. (b) A \( q \)-plate with applied voltage. (c) A stage that consists of a half-wave plate and a \( q \)-plate. (d) A ternary \( Q \)-stack with three \( (N = 3) \) stages \( n_1, n_2, \) and \( n_3 \), which has \( M = 3^3 = 27 \) possible outputs. WP–wave plate, QP–\( q \)-plate. L/RCP–left/right circular polarization.

Geometric Phase Holograms (GPH) works on the geometric relation in vector optical fields[147, 23], which bridge the polarization and the phase. In general, an active GPH has three possible output waves depending on the external voltage and incident polarization.
With zero external voltage, a half-wave GPH add a spatially varying geometric phase to the incident wave for one circular polarization and add a conjugate phase for the orthogonal circular polarization. The phase profile is directly related to the optical axis profile of the GPH $\delta = \pm 2\Phi$. With an external voltage much greater that the material threshold ($V \gg V_{th}$), the optical anisotropic profile can be effectively erased and any incident wave will pass through the element unchanged.

The key of our design to multiple output modes is to relay GPHs with proper polarization and retardation control, commonly with switchable wave plates. There are many designs of some stacked GPHs that could achieve this kind of control, such as the the one dimensional gratings used in non-mechanical beam-steering[96]. A unit of an active wave plate and an active GPH can electrically switch among three outputs. In a GPH stack, we call this unit a stage. An independent external voltage can be applied on each individual element. All elements are half wave retardation without external voltage.

The Jones matrix of the whole stack of N stages can by expressed as follows, assuming the birefringent elements are close to each other:

$$T^{(N)} = T^G_N T^W_N \cdots T^G_1 T^W_1 = \prod_{i=1}^{N} T^G_i T^W_i$$  \hspace{1cm} (5.1)

where $T^G_i$ and $T^W_i$ are the Jones matrices of the GPH and the half-wave plates in the $i^{th}$ stage, respectively:

$$T^G_i = \begin{bmatrix} \cos 2\Phi_i & \sin 2\Phi_i \\ \sin 2\Phi_i & -\cos 2\Phi_i \end{bmatrix}$$  \hspace{1cm} (5.2a) \\
$$T^W_i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$  \hspace{1cm} (5.2b)

After some matrix algebra of Eq. 5.1 and 5.2, the final Jones matrix for the GPH stack can be reduced to:

$$T^{(N)} = \begin{bmatrix} \cos 2\Phi^{(N)} & \sin 2\Phi^{(N)} \\ \sin 2\Phi^{(N)} & -\cos 2\Phi^{(N)} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{\beta}$$  \hspace{1cm} (5.3)

$$\Phi^{(N)} = \sum_{i=1}^{N} (-1)^{\alpha_i} V^G_i \Phi_i$$  \hspace{1cm} (5.4)
This interesting result shows that the total effect of the stack is the same as several half-wave plates followed by a GPH with an effective optical axis $\Phi^{(N)}$.

The parameters in Eq. (5.3) and Eq. (5.4) are:

$$\alpha_i = \begin{cases} 
\sum_{j=i+1}^{N} (V^W_j - 1) & \text{if } i < N \\
0 & \text{if } i = N 
\end{cases} \quad (5.5a)$$

$$\beta = N - \sum_{i=1}^{N} V^W_i \quad (5.5b)$$

and $V^W_i$ and $V^G_i$ are the states of the wave plate and the GPH in the $i^{th}$ stage, respectively. They take the value of 1 or 0 when zero external voltage or a higher-than-threshold voltage is applied on the cells.

So far the expression is valid for arbitrary GPH with optical axis pattern $\Phi_i$. In this work, we choose azimuthal varying GPHs in the relay. This category of GPHs have been used for both cylindrical vector beam\cite{171, 29, 127, 37} and OAM beam generation\cite{25, 122}. In many cases, they are referred to $q$-plates\cite{122}. The optical axis follows $\Phi(r, \phi) = \Phi(\phi) = q\phi + \Phi_0$ in polar coordinates, where $q$ is the topological charge, $\Phi_0$ is a constant. As a GPH, $q$-plate is polarization sensitive. For linearly polarized incident light, the output is cylindrical vector beam of order $+2q$. With an additional half-wave plate field of the negative order $-2q$ can be generated as well.\cite{171} For circularly polarized input, helical beams of order $+2q$ or $-2q$ is generated at output depending on the input polarization handedness (Fig. 5.1(a)).\cite{122} When the $q$-plate is switched to isotropic in plane, then the incident beam just transmits without gaining singularity (Fig. 5.1(b)). Mathematically, by plugging in optical axis function $\Phi_i(\phi) = q_i\phi$ and replacing the notation $V^G_i$ by specific $V^Q_i$ in Eq. 5.4, we get the absolute value of the output singularity charge:

$$L = 2 \sum_{i=1}^{N} (-1)^{\alpha_i} V^Q_i q_i \quad (5.6)$$

For circularly polarized input, we can get either charge $+L$ or $-L$ OAM modes by switching the first wave plate. For linearly polarized input, we will get an axial polarized output of charge $+L$. If we add an additional half-wave plate to the end of the stack, then we will also be able to get $-L$ axial polarized modes by switching this last wave.
plate.

Use the idea of ternary design[96] where stages are arranged in the way that the $q$-plate in the following stage has triple the singularity charge of that in the previous stage ($q_{i+1} = 3q_i$), we can achieve $M$ controllable singular modes at the final output of a $N$ stage $Q$-stack with a uniform step of $\Delta$:

$$M = 3^N \quad \text{(5.7a)}$$

$$\Delta = 2q_1 \quad \text{(5.7b)}$$

where $q_1$ is the charge of the first $q$-plate.

In this work, we have implemented three stages, step one ($N = 3$, $\Delta = 1$) stack and showed 27 different output singular modes (Figure. 5.1(d)).

Similar to Kim’s reasoning, the overall system transmittance can be estimated as

$$T = (1 - D)^N (1 - R)^{2N} (1 - A)^{2N} \quad \text{(5.8)}$$

$D$ is the diffuse scattering of each $q$-plate, $R$ and $A$ are the Fresnel reflectance and absorption losses, respectively, of each LC cell.

However, the intrinsic efficiency of each stage is not included in Eq. 5.8. Because neither $q$-plates nor wave plates change the beam propagation direction, all modified beams and leakage beams are collected at output. Therefore, it is more useful to investigate the mode purity $P$, which is the weighting of the theoretically predicted mode in the overall transmitted output. We can estimate the mode purity as

$$P = \sum_{i=1}^{N} \eta_i \quad \text{(5.9)}$$

where $\eta_i$ is the intrinsic efficiency of each stage. $\eta = (1/2)(1\pm S_3')\sin^2\frac{\Gamma}{2}$, where $S_3' = S_3/S_0$ is the normalized Stoke parameter corresponding to the ellipticity of the incident light and $\Gamma$ is the normalized retardation of the $q$-plate. For perfect wave plates and $q$-plates, $\eta_i = 1$ thus $P = 1$. In practice, we will expect a few percentage lower before further improvement on fabrication.
5.3 Active Geometric Phase Hologram System Fabrication

All elements in the $Q$-stack are electrical controlled birefringent cells with liquid crystal. Two indium-tin-oxide (ITO) glass substrates coated with LPP (Linearly Photo-polymerizable Polymer, Rolic ROP-108) are separated by 2.2 $\mu$m spacers to form a thin cell. The azimuthal profiles for $q$-plates or the uniform profile for wave plates is written into the LPP using UV light. A nematic liquid crystal material (Merck, MDA-06-177) is then injected into the cell and it automatically aligns to the exposed LPP pattern. Then we seal and wire the cells. All cells are designed to have half wave retardation at 633 nm.

We use photo-alignment to pattern each cell, which is a major improvement over Stalder’s rubbing method. It allows more kinds of structures to be written, including the topological charge 0.5, 1.5 and 4.5 $q$-plates in this demonstration stack. We also engineer a high orientation accuracy across the elements till a very small center. The size of the center defect is $\sim 2 \mu$m for the charge 0.5 $q$-plate and $< 20 \mu$m for charge 4.5.

Finally, all cells are aligned in the designed order and glued as a stack by optical glue. This process is monitored under a polarizing microscope to make sure that the lateral offset of the singularities is less than 5 $\mu$m.

Our demonstration $q$-stack contains three stages, five cells. The complete wired sample is shown in Fig. 5.2(a, left). The first wave plate in the design is left out in this first demonstration for convenience. Instead, its function of polarization control is done by a rotatable quarter-wave plate. Each cell is $\sim 2.2$ mm thick, of which 2 mm is the thickness of the substrates, $\sim 2.2 \mu$m is the cell gap. Therefore, the active thickness is 11 $\mu$m. The stack is examined between crossed polarizers with several combinations of voltages on each cell. As expected, the total effective optical axis has different orientations which lead to different transmittance patterns between crossed polarizers as shown in Fig. 5.2(a, right).

The measured transmittance of different operation modes of the $q$-stack varies between 83% and 87%. The reflectance and absorption of the substrates with ITO electrodes are $R = 1.2\%$ and $A = 1.0\%$, respectively. This leads to a theoretical maximum transmittance of $\sim 87\%$ according to Eq. 5.8. Therefore, the cells are shown to have low scattering thus good quality and the rest power is mainly lost as absorption and reflection from the
ITO coatings and substrate surfaces.

## 5.4 Experimental Results of Singular Beams

The \( q \)-stack is examined in three experiments. First, to verify that the \( Q \)-stack converts a homogenous linearly polarized beam to cylindrical vector beams, we used a HeNe linear polarized laser as the incident beam and a CCD camera to capture the output intensity from the \( Q \)-stack, as shown in Fig. 5.2(b). An additional analyzer is inserted before the camera when capturing the spatial polarization variation within the beam. The example output of three modes (charge 13, 7, and 2) are shown in Fig. 5.3. The output beams shown in the first row have uniform intensity in the azimuthal direction, whereas in the
second row, the same beams captured with the analyzer show intensity fringes. These fringes rotate with the analyzer angle and indicate the polarization orientation change. These two sets of results verify that the polarization of the output beams are angular dependent and azimuthal symmetric. The among of variation is revealed by the number of bright or dark segments through the analyzer.

The second experiment demonstrates that the $Q$-stack generates single OAM modes for circularly polarized input. The HeNe laser was split by a non-polarizing beam splitter to form a Mach-Zehnder interferometer (Fig. 5.2(c)). One arm is the output from the $Q$-stack. The other arm is a reference plane wave. The phase profile of the output beams is visualized by the interference pattern captured by the camera. In the results, we can see the signature fork-shaped patterns for helical beams as shown in the bottom two rows.
in Fig. 5.3. We show the same three examples of OAM charge ±13, ±7, and ±2 here. For all the output modes in each column, the states of the cells in the $Q$-stack are the same. The incident polarization is set to RCP or LCP in the bottom two rows. The number of branches of the fork corresponds to the topological charge of the spiral phase profile. For a pair of modes with opposite charges, the “fork openings” point to opposite directions (up and down).

The third experiment is an azimuthal mode decomposition using a set of diffractive masks (Fig. 5.2(d)). A forked grating is known to be able to select the matched helical beam[119, 71, 115]. Only if the input beam charge matches the mask charge, the first order diffraction from the mask becomes a Gaussian beam, whose cross-section intensity peaks at the beam center. In any other cases, the first order diffraction is still a helical beam which is dark-centered. In our experiment, a series of binary forked masks of different charges from 0 to 14 is used to create a OAM component base from $-14$ to $+14$. A CCD camera is used to capture the diffracted beam intensity profile. The images are then processed numerically to gather the intensity value at the beam center. An example set of CCD images and decomposition result is shown in Fig. 5.4. The target beam under investigation is one of the possible helical modes from the $Q$-stack with an expected OAM charge of $l = 7$. The filter masks have singularity charges from $-14$ to $14$. The target beam goes through these 29 filters consecutively and results in 29 different intensity profiles on the CCD camera. As examples, 15 of them are shown as the insets in Fig. 5.4. The processed decomposition result is shown in the bar chart. As expected, the mode weighting peaks at $l = 7$ component and falls significantly on other components. A very low crosstalk around 3% can be seen at $l = -2, 6, 8$.

A full decomposition result of all 27 output modes is shown in Fig. 5.5. In the grid, each horizontal row is the decomposition result of an output mode, whose expected charge is marked on the $y$-axis. The result value on each OAM component ($x$-axis) is represented by different colors. The row of target output mode $l = 7$ in white dashed box corresponds to the result in Fig. 5.4. For all output modes, the matched component weighting is 71–94%. The average value is about 85%. The crosstalk is less than 0.4% at most points on the grid. Even the highest value did not exceed 10%. We believe that the crosstalk is mainly caused by some retardation offset of one or more cells. Whenever a cell in a stage is not giving exact half wave retardation or not getting to zero retardation with applied voltage, the stage will let through leakage mode and the intrinsic efficiency
Figure 5.4: Azimuthal mode decomposition result of one (OAM $l = 7$) of the output modes on a base of OAM $-14$ to $+14$. The inset images are the results of the beam going through different filter masks and captured by the CCD camera.

of the desired mode decreases. Naturally, as predicted in Eq. 5.9, the final mode purity also decreases.

5.5 Scalability and Wavefront Mode Purity

This $Q$-stack design can be easily expanded to higher mode capacity with additional stages. The exponentially increasing number of modes versus number of stages is shown in Fig. 5.6. Note that more than two hundreds of controllable states can be generated by a mere $N = 5$ stack.

A challenge for high $N$ $Q$-stacks is the mode purity. Since the intrinsic efficiency of each stage is practically less than unity in Eq. 5.9, the stack mode purity will decrease as the number of stages increases. The degree of this effect is obviously sensitively depends on individual stage efficiency. In Fig. 5.6 we show several mode purity curves assuming different levels of single stage intrinsic efficiency. Take $N = 5$ for example, output mode purity can be increased by $\sim 10\%$ with each 2% improvement on single stage intrinsic efficiency. A high single stage intrinsic efficiency relies on the accurate retardation of each cell. This can be improved in the future by either precise controlling over the cell gap during fabrication or adjusting applied voltages according to the retardation-voltage curve of each cell.
Figure 5.5: Decomposition results of the 27 output modes on a base of OAM $-14$ to $+14$.

Figure 5.6: Modes number (bars) and estimated purity (lines) for $N = 1–5$ $Q$-stacks. Purity estimations are based on individual stage intrinsic efficiencies $\eta_i = 95\%$, 97\%, and 99\%.
The high transmittance thus low power loss of a $Q$-stack is a big advantage over other systems. To compare, most SLMs have less than 50% efficiency when configured for singular beams. Our preliminary $Q$-stack transmits more than 80% power and it can be further improved in the future by choosing thinner glass and adding anti-reflection coatings to the first and last surfaces of the stack. Different liquid crystal and cell gap combinations can also be tested to reduce absorption.

In another perspective, we can utilize the continuous control over retardation of the cells to intentionally create superposition modes from the $Q$-stack. In Fig. 5.7, we show one such example (illustrated by the hollow bars) together with a pure mode for comparison (illustrated by the solid bars). We applied zero voltage on the two $q$-plates in stage 1 and 2, full voltage on the wave plate in stage 2, and an intermediate voltage on both the wave plate and the $q$-plate in stage 3. This would make the third stage generate all the three possible outcome at the same time. In the experimental decomposition result shown in Fig. 5.7 (hollow bars), we see three high weighting components at $-7$, $2$, and $11$, which match the theoretical prediction. Naturally, the qualitatively controlled superpositions can only be achieved based on a experimental measurement and calibration of the electrical response of each cell.

In conclusion, we have demonstrated an $q$-plate based, wide range singular beam generating device. The ternary design offers exponential scaling ($3^N$) of the number of output modes though easy electrical control. The system optical properties of this device is described in simple effective Jones matrices. The whole device is a solid, compact stack of glass cells with no moving, bulky, or expensive optics. The fabrication is relatively easier and more cost-effective compared to other conventional approaches such as spatial light modulator. Note that while the result and discussion here are limited to generate singular beams from homogeneously incident beam, the device could also be used as a mode transformer, which changes the charge of the incident singular beam in $3^N$ ways. Specially it could be a mode detector since the singular beam of charge that matching to the operating mode would be selected and transformed to non-singular beam.
Figure 5.7: Decomposition results of a pure mode (solid bars) and a superposition (hollow bars) created by the $Q$-stack. Bar labels are the charges of azimuthal mode components. Bar height is the mode weighting in log scale.
Chapter 6

Multi-Twist Geometric Phase Holograms

Geometric Phase Holograms (GPHs) are thin film optics that can accomplish arbitrary phase modulation and beam shaping via the geometric phase effect, e.g., the Pancharatnam-Berry phase. Such wavefront shaping is extremely important for many researches and applications involving beam mode transfer, divergence and direction control, where often cumbersome combinations of refractive optics such as lenses, phase plates, etc. Instead, flat GPH optics offer a convenient alternative to write and reproduce arbitrary phase profiles into monolithic elements avoiding the complexity with all the prior approaches, and excel at many applications including radial/axial symmetric retarders[25, 29, 122, 41, 127], arithmetically designed phase mask[169], etc.

Many wavefront shaping tasks highly prefer broadband operation, such as the vortex phase shaping in coronagraph[125, 124]. Geometric phase is intrinsically achromatic. However, the efficiency of conventional GPH is dependent upon the retardation, which is wavelength dependent. As a result, their operation is optimal at a specific wavelength, thus narrowband.

Methods to compensate retardation dispersion have existed for a long time, some of which are commercially used in achromatic wave plates, such as utilizing the reversed dispersions of two materials (in many cases, crystallized quartz and MgF) in achromatic zero order wave plates[46, 77, 161, 31]. Another method employs multiple homogenous plates of the same material, where their optical axis and retardation is carefully arranged. Many designs are belong to this method, including the early two plates design[53], the
three plates QHQ design[147, 148], and more complicated cases[129, 161, 157]. The use of LC materials extend this method to homogenous or inhomogenous plates made by LC cells[199, 191, 107, 165]. Sub-wavelength gratings have been designed with compensating dispersion to achieve broadband performance[139, 195]. Special nanostructures are also investigated[67, 114]. Unfortunately, these methods cannot easily be applied to the fabrication of GPHs with complex spatially varying optical axis. They either are only able to be tailored for simple patterns or require extreme fine alignment between multiple layers, which is unpractical for arbitrary pattern. In this paper, we will introduce a category of MTGPHs that work with optimal efficiency over a broad spectral band.

6.1 Generalized GPH theory for complex anisotropy

We recall the formulas for GPHs composed by a lossless general anisotropy from Chapter 3. The transfer matrix for GPH is Eq. 3.16 and the coefficients are in Eq. 3.18. The corresponding wavefront shaping efficiencies are calculated by Eq. 3.20 and Eq. 3.24. These formulas are valid for a general anisotropy that can represent a element that is either homogenous or inhomogenous in-depth (along z direction). Surely, a stack of such elements also follows the same theory. For such complex anisotropy, there is not a simple true optical axis, a direction along which polarization is preserved. Therefore, we extend the definition of the function $\Phi(x, y)$ used in Chapter 3 from the angle of optical axis to the angle of relative rotation. Assuming the local anisotropy structure at the location $(0, 0)$ is expressed by matrix $T_0$ and at zero angle, then at any other point $(x, y)$ of the GPH the local anisotropy is a rotated version of that at point $(0, 0)$, the angle of which is $\Phi(x, y)$. An alternative view is that the function $\Phi(x, y)$ still describes the optical axis orientation. However, it is not the optical axis of the full thickness but the local optical axis on the $z = 0$ plane. $\Phi(x, y)$ is a short-handed note for $\Phi(x, y, 0)$. These two descriptions are equivalent except a possible constant difference.

As discussed in Section 3.1.6, the core matrix takes the unitary form

$$T_0 = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$$

(6.1)

After some more matrix algebra started from Eq. 3.16, the total matrix of GPH can be rewritten in three terms that are identified with different phase and polarization
modulations 

\[ T(x, y) = A \cdot \mathbf{R}(-\alpha) + iB \left( e^{i(2\Phi + \beta)}S'_- + e^{-i(2\Phi + \beta)}S'_+ \right) \] (6.2)

where

\[ A = \sqrt{\text{Re}(a)^2 + \text{Re}(b)^2} \] (6.3a)
\[ B = \sqrt{\text{Im}(a)^2 + \text{Im}(b)^2} \] (6.3b)
\[ \alpha = \arctan \frac{\text{Re}(b)}{\text{Re}(a)} \] (6.3c)
\[ \beta = \arctan \frac{\text{Im}(b)}{\text{Im}(a)} \] (6.3d)

To show how this general GPH effects a light wave, we express an unit input field by 

\[ E_i = e^{-i\delta}\chi^\text{in}, \] where \( E \) is the amplitude, \( \delta \) is the phase, and \( \chi^\text{in} \) is the Jones vector representing the polarization state. The output field at the back surface of the GPH is then

\[ E_{\text{out}} = T \cdot E_i = A e^{-i\delta}\mathbf{R}(-\alpha)\chi^\text{in} + iB \]
\[ \cdot \left( \langle \chi^{(+)} | \chi^\text{in} \rangle e^{-i(\delta - 2\Phi - \beta)}\chi^{(-)} + \langle \chi^{(-)} | \chi^\text{in} \rangle e^{-i(\delta + 2\Phi + \beta)}\chi^{(+)} \right) \] (6.4)

The GPH adds a geometric phase factor to the circularly polarized components of the input wave, which equals \( \pm(2\Phi(x, y) + \beta) \). This is intrinsically achromatic and guarantee the same phase modulation can be applied to all wavelength. The weighting of this part of output is defined as the phase modulation efficiency of the GPH. A leakage wave with no phase modulation exist with a weighting coefficient \( A \). We define the diffraction efficiency as the ratio of output to input intensity (\( \eta_m = |E_m|^2/|E_i|^2 \)), then for the three waves

\[ \eta_0 = A^2 \] (6.5a)
\[ \eta_{\pm} = B^2 \left| \langle \chi^{(+)} | \chi^\text{in} \rangle \right|^2 = B^2 \frac{1 \pm S'_3}{2} \] (6.5b)

where \( S'_3 = S_3/S_0 \) is the normalized Stokes parameter corresponding to ellipticity of the incident light.
For a narrow band GPH, which is homogenous wave plates at local points, the core matrix reduce to

$$T_0(\lambda) = \begin{bmatrix} e^{i\frac{\Gamma}{2}} & 0 \\ 0 & e^{-i\frac{\Gamma}{2}} \end{bmatrix}$$ \hspace{1cm} (6.6)

Then in the GPH matrix, $A = \cos \frac{\Gamma}{2}$, $B = \sin \frac{\Gamma}{2}$, and the first matrix is the identity matrix. Note that the normalized retardation $\Gamma = \frac{2\pi \Delta nd}{\lambda}$ ($\Delta n$ and $d$ are the birefringence and thickness of the waveplate) is chromatic, which results in an efficiency dispersion. The phase modulation efficiency reaches the maximum only for several discrete wavelengths.

### 6.2 Spectral Response in Multi-Twist GPH

Our idea of GPH with tailored spectral response is to design a compact, easy fabricated birefringence which matches a desired efficiency curve in a given wavelength range. A practical example is broadband GPH, where the in-depth structure is designed to keep the core matrix $T_0$ resembling a half-wave plate in a wide wavelength range. We use multiple twist nematic liquid crystal layers to realize such complex birefringence. Therefore, the resultant GPHs are called Multi-Twist GPHs (MTGPHs). This structure has been investigated and utilized for broadband wave plates, called Multi-Twist Retarders (MTRs)[101]. A numerical design method was introduced. The mode suggests that broad bandwidth can be realized by as less as two or three layers. The fabricated MTRs match or exceed all traditional approaches using multiple homogenous retarders in bandwidth and general behavior. Before MTRs, a special case of two symmetric twist liquid crystal layers has also been demonstrated to create broadband response in visible range[143]. A PG consist of two twist layers shows more than a fourfold increase of bandwidth over conventional PG which is homogenous in-depth.

The MT structure consists of multiple twisted Liquid Crystal (LC) layers on a single alignment layer and substrate. Its most important feature is the simple alignment of complex layers. The first LC layer is aligned to the alignment layer just as the conventional polymer LC element. However, the subsequent LC layers do not need new alignment layers, instead, they can be aligned to the previous layer and clone the pattern. In this way, the initial alignment pattern can propagate through all LC layers. Depending on the type of chiral dopants and coating recipe, each layer has controlled thickness and the LC molecules go through a in-layer twist, the angle of which is decided by the dopant
concentration of the chiral material of that layer. The final result of the multiple LC layers is a monolithic film that is capable of retardation control for nearly arbitrary bandwidths and shapes. In practice, the MTGPHs are very easy and cheap to fabricate due to the self-aligned nature. The fact that no external alignment is required between layers also ensures a high structure precision and almost ideal performance predicted in theory, especially for larger number of layers.

The MTGPH structure is illustrated in Fig. 6.1. The single initial alignment layer (LPP, in our case) on a single substrate is shown by the pattern on the glass slab. Fig. 6.1(a) shows an example of three layer MTGPH structure by displaying the starting and ending orientation pattern of each layer. There are four patterns in the figure because the ending pattern of a layer is also the starting pattern of the layer above it. We can see how the layers resemble in the in-plane alignment and the alignment pattern be passed onto the top. The relative LC orientation angle at any point \((x, y)\) with respect to that at \((0, 0)\) on the plane is the same for all layers. However, the absolute orientation angles between adjacent layers differ by an angle that equals to the in-layer twist angle. A column is highlighted to show the whole twist that started at a particular location on the alignment layer. The whole twist is overall continuous, but it has three segments that corresponds to different chiral conditions in the three layers. Fig. 6.1(b) illustrates the continuous twist within one layer. Fig. 6.1(c) is the view from the top of a MTGPH.

The transfer matrix for an \(N\)-twist-layer retarder is:

\[
T_0(\lambda) = \prod_{i=1}^{N} T_i(\theta_i, d_i, \lambda) \tag{6.7}
\]

\(\theta_i\) and \(d_i\) are the twist angle and thickness for the \(i\)th layer. There are \(2N\) degrees of freedom for a \(N\)-twist-layer retarder. Therefore, in theory, we can specify the form of the transfer matrix at any wavelength given a large enough \(N\). For example, an element can be design to have half wave retardation at telecommunication wavelength in infrared but zero or full wave retardation at visible range. A second example is that elements can be designed to behave as certain constant retardation across a broad spectral range, making them achromatic within the wavelength concerned.

For a general MTGPH, the numerical design target function is the terms that related to the output efficiencies \(A\) and \(B\), the result of which could be a very complex forms of anisotropy. Due to the inhomogenous anisotropy, there does not always exist a true opti-
Figure 6.1: MTGPH structure. (a) Layer view with a highlighted column showing the overall twist started at a particular location. (b) Twist view within one layer. (c) Top view.

cal axis for the whole element thickness. This does no effect the wavefront manipulation and efficiency as we have demonstrated theoretically and will experimentally. However, if a true optical axis is required in any case, or the zero order polarization is required to be the same as the input, we can easily adjust to it. In additional to the target function related to the output efficiencies $A$ and $B$, in the case of designing true optical axis MTGPH, we just have to include an extra constrain from Eq. (6.3c)

$$\alpha = \arctan \frac{\text{Re}(b)}{\text{Re}(a)} = 0 \quad (6.8)$$

Here we examine the design of broadband GPH as an example. The target is low zero order leakage across a wide wavelength range. We construct the twists so that the
resultant effective matrix \( T_0(\lambda) \) satisfies the condition \( A \approx 0 \) according to Eq. (6.3a) and Eq. (6.5) across the desired spectral range by numerical optimization. Therefore, the leakage of a GPH reduces to nearly zero for a continuous spectral band. MTGPH became broadband in efficiency. The effective matrix for the MTGPH

\[
T(\lambda) = R(-\Phi)T_0(\lambda)R(\Phi) \\
\rightarrow e^{i2\Phi(x,y)} \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} + e^{-i2\Phi(x,y)} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}
\]

(6.9)
is valid for the whole designed spectral range. The simulation output is an array of the twist layer parameters, which contains \( N \) thickness values and \( N \) twist angles.

We emphasis a very important insight here that once the design of MT layers is fixed, the efficiency distribution among the three output waves is fixed, which is independent to the base alignment pattern. Mathematically, the explicit expression of the surface optical axis pattern \( \Phi(x, y) \) was not part of the derivation and does not appear in the efficiency equations Eq. (6.5). This is very useful because it indicates that we can focus on designing the spectral bandwidth and shapes without taking alignment pattern into consideration. More importantly, once a MT structure design is optimized for a particular spectral shape, it is ready to be used upon any GPH pattern.

### 6.3 Fabrication of MTGPH

Fabrication of broadband GPH requires no alignment between layers and can be apply to arbitrary optics axis pattern. The same photo-alignment technique used for narrowband GPH creates the alignment layer with the desired pattern. The photo-alignment could be done with polarization holography or direct laser scanning. Twist nematic LC materials are then applied on the alignment layer sequently. Twist angle and thickness of each layer is tuned according to the simulation result. In practice, we can choose appropriate LC materials, control the chiral doping, and adjust the spin-casting parameters.

### 6.4 MTGPHs for broadband OAM manipulation

In this section, we apply the general MTGPH theory and method to a category of elements for broadband helical wavefront shaping and OAM control of various light sources:
a broadband $q$-plate and a broadband Forked Polarization Grating (FPG). We discuss the design principles involved and experimentally demonstrate broadband $q$-plates and FPGs that are highly efficient (> 90%) in the visible wavelength range. These thin film elements enable easy integration into various optical systems requiring broadband OAM manipulation such as optical trapping and high capacity information.

6.4.1 Broadband OAM introduction

The momentum of a propagating light wave has both linear and angular contributions. The linear momentum is associated with the wave vector. The angular momentum can be further broken down into two more parts: Spin Angular Momentum (SAM) that is associated with polarization, and Orbital Angular Momentum (OAM) that is associated with spatial distribution of the phase front. Unlike the first two, current study on OAM of light is fairly recent. OAM is a new degree of freedom of light that we can utilize. Moreover, comparing to SAM which can only be $\pm \hbar$ per photon, OAM could offer larger momentum to exchange when interact with matter, or wider state set when encode information. An increasingly intense set of work has recently been conducted on the applications of light OAM, including optical trapping[68] and information science and technology[98, 71]. While several OAM manipulation methods (described in more detail below) have been suggested for these applications, none of them has the capability to function over a wide spectral band with good power efficiency and flexibility. Highly efficient broadband OAM will be particularly useful for applications such as OAM-based wavelength-division multiplexing.

Current methods to generate, manipulate, and detect OAM states include the use of spiral phase plate[18], cylindrical lens pair[3], computer generated hologram (CGH), and spatial light modulator (SLM)[17, 90]. In general, these conventional approaches result in bulky and expensive devices limiting the optical performance. In addition approaches using CGHs usually result in low power efficiency while SLMs suffer from resolution and wavelength limits. Most recently, two novel liquid crystal elements were introduced to manipulate the OAM of light. The first is an axially varying half-wave plate called $q$-plate[122], where $q$ is the singularity charge. A $q$-plate converts incoming circularly polarized light to OAM $l = \pm 2q$ states, and generally arbitrary polarization to a superposition of these two OAM states. Both static and tunable versions of $q$-plate have been reported[168]. The second element, introduced by our group[111, 112, 113],

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is a Forked Polarization Grating (FPG), which converts incoming light with OAM \( l_0 \) to one or both of the OAM \( l = l_0 \pm l_g \) eigenstates, where \( l_g \) is the singularity charge of the FPG. As a diffractive optical element, an FPG also changes the linear momentum. Light that goes through different OAM changes (raising or lowering, depending on input polarization), will be diffracted to different direction as well. We recently introduced this approach with both static and switchable FPG elements formed in liquid crystals, where high OAM conversion efficiencies of 96% were demonstrated for UV and visible light sources.

Both \( q \)-plates and FPGs can be tailored for nearly any wavelength from ultraviolet to infrared using commercially available nematic LCs. However, due to the dispersion in the LC birefringence, the high diffraction efficiency occurs only over a modest bandwidth centered around a single optimized wavelength. Here we report on broadband FPGs and \( q \)-plates that accomplish highly efficient (> 97%) OAM manipulation over a much wider spectral range in the visible region (500–700 nm).

### 6.4.2 Broadband \( q \)-plate and broadband FPG design and fabrication

Our target is to develop broadband helical wavefront shaping and OAM control elements for the broad visible band. A two twist layer design \( (d_1 = d_2, \theta_1 = 70^\circ, \theta_2 = -70^\circ) \) has been demonstrated to work for this bandwidth with a one dimensional grating alignment pattern[142, 143]. We adapt this design to create broadband FPGs and \( q \)-plates for the same bandwidth, since we have proved that the twist structure is independent to and can be build on arbitrary alignment pattern.

We experimentally realized the broadband FPGs and \( q \)-plates formed as liquid crystal thin films fabricated by polarization holography and photoalignment techniques. We utilized a linear photo-polymerizable polymer (LPP) as the photoalignment material (ROP108, Rolic Ltd). The surface alignment pattern was recorded in the LPP layer using an HeCd laser (325 nm). For \( q \)-plates, this pattern was created by a rotating line beam with appropriately controlled linear polarization[111, 127]. For FPGs, we utilize polarization holography with a \( q \)-plate as helical beam generator[113]. After UV exposure, liquid crystal polymer (LCP) was spin-coated onto the patterned LPP-coated substrate. We utilized RMS03-001C (EMD Chemicals, \( \Delta n = 0.16 \) @ 589 nm) to create this birefringent LCP film, according to the method previously reported[143]. A diluted
mixture (1:3 of RMS03-001C:PGMEA) layer was first coated to improve later coating quality. The next layer, the first chiral layer, was composed of the LCP doped with a small amount (approximately 0.3%) of the chiral molecule CB15 (EMD Chemicals). The last layer, the second chiral layer, was composed of the LCP doped with a small amount (approximately 0.3%) of the chiral molecule MLC-6742 (EMD Chemicals). The thickness of layers were tuned so that the half-wave retardation \( nd = \lambda_0 / 2 \) (at \( \lambda_0 = 550 \text{ nm} \)) and a twist angle \( \theta = 70^\circ \) occurred simultaneously. Spin-coating speeds were 2000 rpm for the diluted LC layer, 580 rpm and 530 rpm for the first and second chiral layers, respectively.

Polarizing microscope pictures of the samples are shown in Fig. 6.2 which indicate good LC alignment producing the required spatial pattern in both cases. Both elements are made on one inch square substrates. The FPG has a period of 10\( \mu \text{m} \), picture is zoomed in to show the fork-shaped singularity. These results are comparable to prior work on such elements. Next, we attempt to further characterize the optical properties of both the broadband FPG and \( q \)-plate.

### 6.4.3 Experimental results

The characterization of the OAM conversion is conducted with a Mach-Zender interferometer. With Gaussian mode laser beam \( (l = l_0) \) incident, the output beams are examined by interfering with a tilted plan wave. To demonstrate broadband ability, we repeated this examination using two lasers at 633 nm and 532 nm, respectively. Captured interference patterns are shown in Fig. 6.3 with optical path illustrations with them.
The result perfectly agrees with the theory. Predicted patterns are observed at both wavelengths. For FPGs, both circularly and linearly polarized input were tested. Circular input will diffract to a single first order and linear input will diffract to both first orders. At the same time, all polarizations have a very weak zero order leakage. As shown in Fig. 6.3(a), the zero order diffraction is always unchanged, with \( l = l_0 = 0 \), and the first orders are always modulated: OAM of \( m = -1 \) diffraction is increased by \( l_g = 1 \), which in this case, equals to 1, and for \( m = +1 \) diffraction, OAM is decreased by \( l_g = 1 \), to \( l = -1 \). For \( q \)-plates, circular polarized input is also converted to helical beam (Fig. 6.3(b)). Opposite polarization handedness result in the same OAM value but opposite helical handedness. Linear polarized input is converted to a superposition of helical modes, which is essentially an “axial polarized beam”. This spatial varying linear polarization can be examined by observing its intensity distribution through an analyzer (Fig. 6.3(c)).

To measure the conversion efficiency over the visible range we took the transmission spectra with an unpolarized broadband light source. Due to the diffractive feature of a broadband FPG, its useful output (first orders) can be easily selected by a spatial filter. We measured the zero order transmission by a spectrometer and normalized it by the reference total transmission of a bare glass substrate to get normalized zero order diffraction efficiency \( \eta_0 \). This normalization excludes the effect of substrate reflection and absorption, which could be prominently reduced by proper coating. Then we estimated the normalized total first order diffraction efficiency \( \sum \eta_{\pm} = 100\% - \eta_0 \), shown in Fig. 6.4. We did this estimation because zero order is non-dispersive thus can be measured most accurately. Clearly, the broadband FPG manifests high diffraction efficiency 97% across almost all visible wavelengths (483–720 nm), which is a substantial improvement over the narrowband FPG (580–675 nm), almost 2.5 times wider. A spectrum of a sample optimized for 633 nm is shown in the figure as reference. Direct first order efficiency measurements using lasers are also shown here as markers. Diffraction efficiency is calculated by single order intensity over total transmitted intensity. A 2% difference from estimated values is due to high order diffraction, scattering, and some material absorption. This should be further optimized by improving fabrication details as we will discuss in a later section. \( q \)-plates fabricated using the same recipe should have the same theoretical efficiency as FPGs. However, because output from \( q \)-plates is not spatially separated, we used a different approach to measure the efficiency by filtering out the useful fraction by
Figure 6.3: Pictures of output from (a) FPG ($l_g = 1$) and (b-c) $q$-plate ($q = 1$). Top: HeNe (633 nm) red laser; bottom: NdYAG (532 nm) green laser. First row: far field intensity. Second row: (a-b) interference with tilted plane wave or (c) intensity through an analyzer.
polarization. With circularly polarized input, the modulated fraction reverts its handedness, whereas the unmodulated leakage fraction remains the same. Thus, by measuring the transmission through circular polarizer, we verified the \( q \)-plates conversion efficiency of \( > 97.6\% \) at both 532 nm and 633 nm by lasers.

So far, we have demonstrated that we fabricated broadband FPGs and broadband \( q \)-plates that extent the working bandwidth significantly comparing to their narrowband versions. These elements have two main advantages over others. First, they can be used to control OAM of a broadband light with high efficiency, which could not be achieved by narrowband elements. For example, a narrowband FPG will have significant higher zero-order leakage at off-center wavelengths, making it low efficient for overall bandwidth. A narrowband \( q \)-plate has even more trouble with broadband light, since the zero-order leakage will superimpose with first-orders and make the output a complex OAM state that varies across the spectral band. The essential that makes our new thin films broadband is the material. Because of the two opposite chiral doping, there are more parameters we could adjust in the LC coating process in order to adjust both the shape and the position of the spectrum. Although our current recipe is optimized to cover the visible range, these elements are capable of serving other spectral ranges as well. This feature makes the simultaneous OAM controlling of a broadband light field easily achievable and could be particularly useful in wavelength-division multiplexing in OAM-based systems.

Second, these elements are easy to fabricate. Because the broadband versions have the
same surface alignment patterns as conventional narrowband versions, they can be easily fabricated by photo-alignment techniques that are currently available. Generally, for most arbitrary two-dimensional GPH, we can keep its surface alignment and introduce this double twist structure to make it broadband. As a result, many nice features of narrowband FPGs and $q$-plates still apply to our broadband elements. For instance, they have good diffraction efficiency, polarization-controlled conversion, they are light weight and flexible. Combining with their wide working wavelength range, this elements are extremely unique and excellent OAM controllers.

6.5 Discussion

MTGPHs represents a new family of wavefront shaping elements that can be designed for arbitrary phase profile and arbitrary spectral band, easy to fabricate, compact, and high efficient.

The thickness and twist parameters of the layers in MTGPH provide the extra freedom for shaping the efficiency dispersion curve. The general trend is the more layers being used, the more complicated band shape can be achieved. For instance in the case of MTGPH designed for broadband applications, the normalized bandwidth increases with the number of layers as shown in Fig. 6.5 (data from Ref. [101]). The normalized bandwidth $\Delta \lambda/\lambda_0$ is defined as the ratio of the spectral range $\Delta \lambda$ (over which high wavefront shaping efficiency $\eta_{\pm 1} \geq 99\%$ occurs) to the center wavelength $\lambda_0$. However, in practice, none of the LC layer is perfect. The tradeoff of using more twist layers is normally a lower quality. A big problem is point defects which is common in LC coating. In these point defects the LC molecules are not aligned to the desired pattern, which usually leads to scattering or zero leakage. We also expect the material of each layer absorb a very low precent of light. To estimate the general trend, we assume each layer has a transmittance of $T_i$ and the overall transmittance is the product of $T_i$ in all layers. The estimation is drawn in Fig. 6.5 with three $T_i$ values: 99.5%, 99.0%, and 98.5%. We can see that the element transmittance naturally decreases with the number of layers. However, if the quality of each single LC layer is relatively high (99.5%), the overall result is still satisfactory for many layers (> 97.5% for $N = 5$). This is the case in our fabrication; 99.5% normally can be achieved for a single LC layer.

In designing of an MT structure, the numerical calculation usually gives multiple solu-
Figure 6.5: Normalized bandwidth of broadband MTGPHs (bars) and estimated overall transmittance of such MTGPHs (lines).

tions that correspond to different local minima and lead to similar performance. However, to choose from these solutions we need to take other limitations into consideration. The true optical axis condition as discussed above is one when the target is such MTGPH. More commonly, the consideration is on the practical implementation convenience and confidence level of the LC layers. Signs of an easy-to-fabricate layer are small thickness and small twist angle, especially zero twist in some layers. Signs of a hard-to-fabricate layer are very large thickness and large twist angles. Above all, fewer layers is always easier than more layers.

In summary, the MTGPH massively expands the usage of GPH by providing an easy-to-fabricate and low-cost method to reform efficiency dispersion in arbitrary wave-front shaping. This is especially beneficial for the applications in astronomy, such as coronagraphs.[125, 124].
Chapter 7

Discussion and Conclusion

The main motivation of the work in this dissertation was to develop simple and efficient elements for wavefront manipulation. The objective was thus to investigate the possibilities of Geometric Phase Holograms (GPHs) and apply them as a category of simple, compact, low-cost, light-weight, and highly efficient wavefront shaping elements. We approached this goal with both theoretical analysis and experimental verifications. The wavefront shaping functionality and properties of GPH was demonstrated with an emphasis on helical wavefront shaping and OAM controlling.

7.1 Summary

In Chapter 3, we tried to verify if GPHs have the potential for highly efficient wavefront manipulation. We first investigated the unique mechanism of wavefront shaping through geometric phase compared to the conventional dynamic phase. We derived the theoretical maximum wavefront shaping efficiency of different types of GPHs using Jones calculus. We found that among them the linear birefringent GPHs are both high efficient and flexible. We developed a general theory for GPHs for the first time and identified the efficiency dependences, including the anisotropy structure of the element and the polarization state of the incident wave. Our analysis predicts that GPHs can provide theoretically 100% wavefront shaping efficiency and easily adjustable spectral range. Therefore, they are ideal elements for high efficiency wavefront manipulating.

In Chapter 4, we tried to verify if we can create a GPH that controls helical wavefronts with high efficiency. We applied the GPHs theory in helical wavefront shaping and Orbital
Angular Momentum (OAM) generation and mode transformation. We proposed and numerically simulated the design and fabrication method of Forked Polarization Grating (FPG) and verified the wavefront shaping theory of GPH. We also derived the special function in the case of OAM states related to the topological charge on the FPG. We demonstrated theoretically and empirically that a charge \( l_g \) FPG creates helical modes with OAM charge \( \pm l_g \) when a Gaussian beam is incident, and more generally, transforms the incident helical mode with OAM charge \( l_{in} \) into output modes with OAM charge \( l_{in} \pm l_g \). We developed a fabrication method involving polarization holography and \( q \)-plate that substantially improves FPG quality and efficiency over prior work. We also successfully fabricated switchable FPGs, which can be electrically switched between an OAM generating/transforming state and a transmissive state. Our experimental results showed > 92% conversion efficiency for both configurations at 633 nm. As a contribution, we develop the novel optical element FPG that efficiently generates and modifies OAM of light and verified for the first time that this conversion into a single mode can be very efficient (i.e., \( \sim 95\% \) experimentally) at visible wavelengths, and the relative power between the two possible output modes is polarization-controllable from 0\% to \( \sim 100\% \). The FPGs are compact (i.e., thin glass plates), lightweight, and easily optimized for nearly any wavelength from ultraviolet to infrared, for a wide range of OAM charge, and for large or small clear apertures. Therefore, they are ideal elements for enhanced control of helical beams and OAM, e.g., in optical trapping and high-capacity information.

Chapter 5 addressed the following question: Can we efficiently manipulate multiple wavefronts using a single GPH or GPH system? We proposed a ternary design and derived the general formula of the output wavefront profile in terms of the individual GPH patterns and the applied voltage on each stage. Compared to a single GPH that controls three possible wavefronts, our design of the active GPH relay is able to control a series of \( 3^N \) possible wavefronts, where \( N \) is the number of stages and each stage is composed by two GPHs. This design requires no spacing between elements; all GPHs are stacked back to back. Therefore, the overall system is still compact and light-weight compared to the conventional bulk wavefront shaping optics and systems. We also form the particular theory for a \( Q \)-stack that controls OAM states. We derived the output OAM state as a function of the charges of each composing \( q \)-plate and external voltages. Experimentally we fabricated a 3-stage \( Q \)-stack that can control 27 OAM modes and their superpositions with 71–94\% purity for visible light via simple electrical switches. As our
second contribution, the development of active GPH relays exponentially extends the range of controllable wavefront modes. The significance of the development of general GPH relay is that it provides a simple, compact, and high efficient solution to many situations where a series of wavefronts are of interest and need to be switched quickly. Compared to other reconfigurable wavefront shaping methods, such as Spatial Light Modulators (SLMs) and deformable mirrors, these GPH relays are not only more efficient but also more compact and affordable.

In Chapter 6, we tried to find solution to the following question Can we overcome and even control the wavelength dependence of GPH efficiency? To tackle this problem we investigated the key wavelength dependent factors that decide the GPH efficiency in Jones calculus and demonstrated that the Multi-Twist design can be used with arbitrary GPH without affecting its wavefront shaping function. In another word, we found the MTGPH has two dimension of design freedom at the same time: the freedom of arbitrary wavefront modification and the freedom of modified efficiency dispersion over wavelength. We applied this MTGPH to helical wavefront shaping and OAM control and developed the broadband \( q \)-plate and broadband FPG. The experimental efficiency is >97% across almost all visible wavelengths (483–720 nm), which is a substantial improvement over the narrowband GPH (580–675 nm), almost 2.5 times wider. Our contribution in this chapter is that we developed an special implementation of GPH with layered liquid crystal design called Multi-Twist GPH (MTGPH), which achieves adjustable wavefront shaping efficiency over a broad spectral range by design. We believe the MTGPHs can enable many novel research and applications that require accurate wavefront shaping at specific spectral range.

7.2 Future Work Directions

We now look at the directions of future work on GPH. The first direction is to continue the research on GPHs for helical wavefront shaping. Specifically, ever better elements and systems for orbital angular momentum detection are still of great demand. Systems based on GPHs is a very promising option due to the high efficiency and high configurability packed in thin films. An example that we have considered is GPH full angular momentum sorter, which consists of a log-polar transforming function overlapped with an orthogonal grating. This will be able to sort OAM and SAM on the two orthogonal axes of the sorting
plane. All the combinations of light OAM and SAM will correspond to a map on the output. One will be able to easily detect the OAM and SAM carried by the incident beam by observing or measuring the intensity distribution on the sorting map.

The second direction is to develop GPHs with new combination of birefringence patterns and multi-twist designs for various research and applications. The pattern decides the output phase profile and the MT design decides the spectral response. The freedom of independently controlling these two features will enable great amount of ideas. For instance in Stimulated Emission Depletion (STED) microscopy, assume one wants an element that creates a focused IR beam to excite the target sample, a separate shorter IR beam to deplete the surrounding area, and be transparent to visible light at the same time. This can be achieved by using GPH. We could first make a MTGPH lens which has half wave retardation at the exciting wavelength and full wave retardation across the visible range. We make a second MTGPH $q$-plate which has half wave retardation at the depleting wavelength and full wave retardation across the visible range. By stacking these two GPHs back to back, we will get a thin film GPH that does said function.

The third direction is to expand the present GPHs with in-plane birefringence from nematic liquid crystal to other anisotropies. With complex in-plane and in-depth anisotropy, the advanced GPHs will have the potential to fully control the incident light wave, including its phase, amplitude, and polarization. This direction will need the investigation on new materials and new alignment techniques.

The fourth direction is based on a limitation of current GPHs. Although the pattern on each GPH can be arbitrary, once the pattern is created, it cannot be altered on each single element. We partially overcame this disadvantage by the GPH array design shown in Chapter 5. However, the output is still restricted among limited numbers of wavefronts with similar topological patterns. Thus the present GPHs are not ideal for research or applications that need repeatedly changing to rather different new wavefronts or for freeform wavefront correcting in closed loop adaptive optics systems. In the future, however, a reconfigurable GPH might be possible with LC in-plane reorientation technologies, like the IPS (in-plane switching) technology that is widely used in some liquid crystal displays.

To conclude, we investigated the wavefront shaping method based on geometric phase change and developed the concept and generalized theory of Geometric Phase Holograms (GPHs). The GPHs provide high efficient, low cost, and light weight solutions to many
wavefront shaping applications. They have unique properties such as intrinsic achromaticity and polarization selection. To enable active switching among a series of wavefronts, we developed the theory and design of GPH relay, which allows fast electrical control over many wavefront modifications. To reform the efficiency-wavelength function of GPHs, we developed the theory of MTGPH, which achieves such target with simple and easy fabrication based on liquid crystal layers.

The above work on the GPH is general, for arbitrary wavefront shaping. We applied the general theory to a special wavefront, helical wavefront, in each part. As a result, we successfully developed four novel elements for helical wavefront shaping and the accompanied Orbital Angular Momentum (OAM) control: Forked Polarization Grating (FPG), Q-stack, broadband $q$-plate, and broadband FPG. These elements modify helical wavefronts with high efficiency, low cost, and high adaptability in fabrication. They provide ideal solutions to the problem of efficient OAM control in various science and engineering areas, including optical manipulation and optical communications.
REFERENCES


99


[83] NR Heckenberg and R McDuff. Generation of optical phase singularities by


[88] Lifa Hu, Li Xuan, Yongjun Liu, Zhaogliang Cao, Dayu Li, and QuanQuan Mu. Phase-only liquid crystal spatial light modulator for wavefront correction with high precision. *Optics Express*, 12(26):6403–6409, 2004.


2008.


[146] Miles Padgett, Johannes Courtial, and Les Allen. Light’s orbital angular moment-


[184] Jean-Marie Vigoureux and Daniel Van Labeke. A geometric phase in optical mul-

[185] Jens von Bergmann and HsingChi von Bergmann. Foucault pendulum through 

[186] Apoorva G Wagh and Veer Chand Rakhecha. On measuring the Pancharatnam 

[187] Jian Wang, Jeng-Yuan Yang, Irfan M. Fazal, Nisar Ahmed, Yan Yan, Hao Huang, 
Yongxiong Ren, Yang Yue, Samuel Dolinar, Moshe Tur, and Alan E. Willner. 
Terabit free-space data transmission employing orbital angular momentum multi-


[189] Dominik Wildanger, Eva Rittweger, Lars Kastrup, and Stefan W Hell. STED 
microscopy with a supercontinuum laser source. _Optics Express_, 16(13):9614–21, 
June 2008.

16(2):89–95, 1974.

[191] Thomas X Wu, Yuhua Huang, and Shin-Tson Wu. Design optimization of broad-
band linear polarization converter using twisted nematic liquid crystal. _Japanese 


Appendix A

Mathematical convention for vector light waves

In the mathematical descriptions of light wave and its polarization, the forms are usually not exclusive, especially in the cases of representing the handedness of elliptical polarization states. People have adopted and been using different conventions in science and engineering, in earlier and modern days. These choices are mathematical rather than physical. One just should be wary when consulting references.

The conventions used in this dissertation is listed as follows.

A.1 Wave functions

The complex representation of an electromagnetic wave, including optical wave is

\[ \psi(\vec{r}, t) = A(\vec{r}, t)e^{i(\vec{k} \cdot \vec{r} - \omega t)} = A(\vec{r}, t)e^{-i\delta(\vec{r}, t)} \]  \hspace{1cm} (A.1)

where \( A \) is the amplitude and \( \delta = \omega t - \vec{k} \cdot \vec{r} \) is the phase of the wave.

For a plane wave propagating along the +z direction, phase \( \delta = \omega t - kz \).

A helical wave with topological charge \( l \) has the phase term \( \delta = \omega t - kz - l\phi \), where \( \phi \) is the azimuthal angle in cylindrical coordinates. Thus the exponential term \( e^{il\phi} \) in a wave function is usually a signature of the wavefront being helical.
A.2 Polarization states

All polarization states can be represented as superposition of a pair of orthogonal polarization states. This pair is usually chosen to be two orthogonal linear polarization states or the two circular polarization states. Most theoretical analysis in the dissertation is conducted on the two orthogonal circular polarization basis. The mathematical expressions for circular polarization states in this dissertation are chosen to follow these conventions in Tab. A.1.
Table A.1: Expressions for circular polarization states

<table>
<thead>
<tr>
<th>State of polarization</th>
<th>Eigenspinor</th>
<th>Jones vector</th>
<th>Stokes vector</th>
<th>Dirac notation</th>
<th>Spin angular momentum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Right circular</td>
<td>$\chi^\left(-\right)$</td>
<td>$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \ 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 \ 0 \ 0 \ 1 \end{pmatrix}$</td>
<td>$</td>
<td>\left(-\right&gt;\rangle$</td>
</tr>
<tr>
<td>Left circular</td>
<td>$\chi^\left(+\right)$</td>
<td>$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \ i \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 \ 0 \ 0 \ -1 \end{pmatrix}$</td>
<td>$</td>
<td>\left(+\right&gt;\rangle$</td>
</tr>
</tbody>
</table>