

ABSTRACT

MANGUM, CHAD ROBERT. Representations of Twisted Toroidal Lie Algebras of Type A_{2n-1} .
(Under the direction of Kailash Misra and Naihuan Jing.)

In this thesis we study certain generalizations of Kac-Moody Lie algebras known as toroidal Lie algebras, and specifically twisted toroidal Lie algebras of type A_{2n-1} . Lie algebraic theory arose in the 1800s out of the study of certain geometric objects known as Lie groups, named in honor of mathematician Sophus Lie. The study of Lie algebras, and specifically their representation theory, has been significant in many areas of mathematics and physics since its inception, including the areas of combinatorics, geometry, vertex algebras, conformal field theory, quantum field theory, and string theory.

Lie algebras which are simple (that is, containing no nontrivial ideal) and finite dimensional over an algebraically closed field of characteristic zero were classified by 1900. These algebras were later generalized in the late 1960s to so-called Kac-Moody Lie algebras. The study of one class of Kac-Moody Lie algebras known as affine Lie algebras has proved particularly fruitful, and thus the advent of affine Lie algebras has only intensified the interest in Lie algebra theory. Affine Lie algebras come in two types, untwisted and twisted, and can be viewed as the Lie algebra of polynomial maps from the unit circle into a simple, finite dimensional Lie algebra. Drinfel'd gave an interesting presentation for these algebras in the 1980s which proved to be useful.

Study of a generalization of affine Lie algebras known as toroidal Lie algebras has also gained traction in recent decades; such algebras can be viewed as the Lie algebra of polynomial maps from the N -torus into a simple, finite dimensional Lie algebra. A Drinfel'd-type presentation of untwisted toroidal Lie algebras was given via generators and relations by Moody, Rao, and Yokonuma (MRY) in 1990, but to date, a similar presentation has not been given for twisted toroidal Lie algebras. The first main theorem in this thesis is an MRY-type presentation for twisted toroidal Lie algebras of type A_{2n-1} .

In the remainder of this thesis, we focus on using the MRY-type presentation to give two representations of the aforementioned algebra. In the 1980s, Feingold and Frenkel used Clifford (respectively, Weyl) algebras to give fermionic (respectively, bosonic) representations of affine Lie algebras. More recently, Jing, Misra, and Xu used the MRY presentation to construct fermionic and bosonic representations for classical untwisted toroidal Lie algebras. In this thesis, we use our MRY-type presentation for twisted toroidal Lie algebras of type A_{2n-1} to give fermionic and bosonic representations of these algebras.

© Copyright 2014 by Chad Robert Mangum

All Rights Reserved

Representations of Twisted Toroidal Lie Algebras of Type A_{2n-1}

by
Chad Robert Mangum

A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh, North Carolina

2014

APPROVED BY:

Kailash Misra
Co-chair of Advisory Committee

Naihuan Jing
Co-chair of Advisory Committee

Bojko Bakalov

Ernest Stitzinger

Ping-Tung Shaw

BIOGRAPHY

The author was born and raised in suburban Philadelphia, Pennsylvania. He graduated from Souderton Area High School in 2005. In 2009 he received a Bachelor of Science degree in Mathematics-Interdisciplinary from Taylor University in Upland, Indiana. He began graduate work at North Carolina State University in the fall of 2009 and received his Master of Science in Mathematics in December 2011. In the summer of 2013, he had the opportunity to augment his doctoral research by spending the summer working under Dr. Shaobin Tan at Xiamen University in Xiamen, Fujian, China. His Doctor of Philosophy degree in Mathematics was awarded the following year in the summer of 2014.

ACKNOWLEDGEMENTS

I am grateful to my advisers, Dr. Naihuan Jing and Dr. Kailash Misra, for their expertise and advice, and thankful to North Carolina State University for support. My committee members also deserve recognition for their sacrifice of time and effort: Dr. Bojko Bakalov, Dr. Ernest Stitzinger, and Dr. Ping-Tung Shaw. I wish also to thank Dr. Shaobin Tan and Xiamen University for their hospitality and fruitful mathematical discussions.

Heartfelt thanks are also due to my wife for her faithful support, my parents for their continued confidence and encouragement, and my brothers for their advice and friendship. I appreciate and rely on the prayers of family and friends (the names of whom would be too numerous to list), and most of all am thankful to God Himself for salvation, unwavering faithfulness, and unconditional love.

It is my hope that the results contained herein will be effective tools in the further study of twisted toroidal Lie algebras, Lie algebra representation theory more generally, and the ever-present links with mathematical physics.

SDG

TABLE OF CONTENTS

LIST OF FIGURES	v
Chapter 1 Introduction	1
Chapter 2 Background	4
2.1 Finite Dimensional	4
2.1.1 Scaling of Bilinear Form	7
2.1.2 Copies of \mathfrak{sl}_2	8
2.1.3 Example Calculations	9
2.1.4 Highest Root of $\dot{\Delta}$	9
2.1.5 Automorphism of A_{2n-1}	10
2.2 Toroidal Algebras	11
2.2.1 Central Extensions	11
2.2.2 Untwisted Toroidal	12
2.2.3 Twisted Toroidal	12
2.2.4 Module of Kähler differentials	13
2.3 Loop Algebra Realization of Toroidal Algebras	14
2.3.1 Untwisted Toroidal	14
2.3.2 Twisted Toroidal	14
Chapter 3 MRY Presentation of Toroidal Algebras	16
3.1 Untwisted Toroidal	16
3.2 Twisted Toroidal	17
3.3 First Main Theorem	21
Chapter 4 Fermionic Representation	57
4.1 Free Field Notation	57
4.2 Fermionic Representation of Twisted Toroidal A_{2n-1}	81
Chapter 5 Bosonic Representation	105
5.1 Free Field Notation	105
5.2 Bosonic Representation of Twisted Toroidal A_{2n-1}	127
Bibliography	149

LIST OF FIGURES

Figure 2.1	Automorphism of Dynkin diagram	10
Figure 2.2	Commutative diagram for a uce	11
Figure 2.3	The uce of a perfect Lie algebra is unique	11
Figure 3.1	Commutative diagram for the MRY presentation	22
Figure 3.2	Commutative diagram for the MRY presentation (duplicate)	56

Chapter 1

Introduction

The theory of Lie algebras arose in the 1800s from the study of infinitesimal transformations of geometric objects known as Lie groups, named for Norwegian mathematician Sophus Lie. By the end of that century, one of the most important classes of Lie algebras (simple, finite dimensional Lie algebras over an algebraically closed field of characteristic zero) had been classified by William Killing and Elie Cartan. Several decades later in the 1960s, a generalization of these algebras was defined independently by Victor G. Kac and Robert V. Moody. The algebras resulting from this generalization were appropriately called Kac-Moody (Lie) algebras.

Kac-Moody algebras come in three types: finite, affine, and indefinite. Finite type Kac-Moody algebras are (isomorphic to) the type classified by Killing and Cartan, while those of indefinite type are not yet classified. The third type, affine Kac-Moody algebras, or simply affine algebras (and specifically their representation theory), have been the subject of much study in the decades since their introduction. The utility of this theory has been now well-documented in many areas of mathematics and physics, including combinatorics, quantum groups, vertex algebras, and conformal field theory (as well as string theory more generally).

For each simple, finite dimensional Kac-Moody algebra \mathfrak{g} , one can associate an affine algebra which naturally contains \mathfrak{g} . The affine algebra can then be realized as the Lie algebra of polynomial maps from the unit circle S^1 into \mathfrak{g} . A natural generalization of affine algebras is a toroidal (Lie) algebra, the Lie algebra of polynomial maps from the N -torus $S^1 \times S^1 \times \cdots \times S^1$, into \mathfrak{g} . Toroidal algebras and their representations have been useful in many of the same contexts as affine algebras.

In this thesis, we meet two predominant goals: to describe a new presentation of a toroidal algebra of a particular type, which will be denoted $\bar{T}(A_{2n-1})$, and to construct new representations of this algebra. Two veins of theory have laid the groundwork for these results.

The first vein provides the basis for the new presentation of $\overline{T}(A_{2n-1})$. We trace the theory back to [G], in which Garland proved that one class of affine algebras, the untwisted affine algebras, are (one-dimensional) universal central extensions of loop algebras (Lie algebras of the form $\dot{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]$). In 1982, Wilson extended this result to twisted affine algebras; that is, he proved that twisted affine algebras are (one-dimensional) universal central extensions of twisted loop algebras (Lie algebras of the form $\bigoplus_{j \in \mathbb{Z}} \dot{\mathfrak{g}}_j \otimes t^j$ where $\dot{\mathfrak{g}}_j$ is the eigenspace of an automorphism σ of order r with eigenvalue ω^j , ω an r -th root of unity). Soon afterward, toward the end of the 1980s, V. G. Drinfel'd gave a different presentation of affine algebras in [D] which proved to be useful.

In 1990, Moody, Rao, and Yokonuma proved in [MRY] that an untwisted toroidal algebra is the universal central extension of an untwisted multi-loop algebra, that is, a loop algebra with multiple variables in the Laurent polynomials. In this thesis we deal only with the two variable case, making the untwisted multi-loop algebra of the form $\dot{\mathfrak{g}} \otimes \mathbb{C}[s, s^{-1}, t, t^{-1}]$. Their proof follows [W] closely. Berman and Krylyuk [BK] extended the result to the twisted case and showed that a twisted toroidal Lie algebra is the universal central extension of a twisted multi-loop algebra, and discussed the structure of the infinite dimensional central extension. The result in [BK] is for a more general setting than toroidal algebras; the structure for the specific case of the twisted toroidal Lie algebras as the direct sum of a twisted multi-loop algebra and Kähler differentials is summarized in [FJ].

In this thesis we consider the case of a twisted toroidal Lie algebra of type A_{2n-1} . We consider the order 2 automorphism σ of A_{2n-1} induced by the Dynkin diagram automorphism. We extend it to an automorphism $\bar{\sigma}$ of $A_{2n-1} \otimes \mathbb{C}[s, s^{-1}, t, t^{-1}]$ which acts as σ on A_{2n-1} and “twists” s . Our twisted toroidal Lie algebra of type A_{2n-1} is the universal central extension of the fixed point set of $\bar{\sigma}$.

In [MRY], a main result was to give a Drinfel'd type presentation of untwisted toroidal algebras. Specifically, a presentation of untwisted toroidal algebras was given via generators $\alpha_i(z), X(\pm\alpha_i, z), \not\in$ of an algebra t and relations similar to the defining relations of Kac-Moody algebras; we call this the MRY presentation. This presentation, like the Drinfel'd presentation of affine algebras, has proven to be useful. An MRY presentation has not yet been extended to the twisted toroidal case; the first primary result we give in this thesis is such a presentation of $\overline{T}(A_{2n-1})$.

In the final two chapters, our focus shifts to using this presentation of $\overline{T}(A_{2n-1})$ to construct representations of $\overline{T}(A_{2n-1})$. Another sequence of results in the literature supplies the background for the representations of interest, beginning with [FF] in 1985, which garnered inspiration from certain representations of affine algebras arising from mathematical physics

(indeed, they have uses in statistical mechanics, conformal field theory, and quantum field theory, for example), discussed in [F], [FK], [KP]. In [FF], Feingold and Frenkel used elements from Clifford and Weyl algebras, viewed as operators acting on a representation space, to give fermionic and bosonic representations of classical affine algebras; both untwisted and twisted affine are included.

In 2009 (respectively, 2010), Jing, Misra, and Xu in [JMX] (respectively, Jing and Misra in [JM]) used the [MRY] presentation and similar techniques to [FF] to construct bosonic (respectively, fermionic) representations of untwisted toroidal algebras. For the final two primary results in this thesis, we use the MRY presentation of $\bar{T}(A_{2n-1})$ and the techniques from [JMX], [JM] to give both fermionic and bosonic representations of $\bar{T}(A_{2n-1})$.

Many other results in the representation theory of affine and toroidal algebras, especially those making use of vertex algebras and vertex operator algebras, have appeared in recent years (see [JMT], [T1], [T2], [T3], [vdL], [FM], [L], [XH], [LT], [G], [B1], [B2], [BBS], [Li], [LTW], [K2] for examples). We hope this thesis will prove valuable in the ongoing study of the representation theory of Lie algebras and its pervasive connections with mathematical physics.

In this thesis we begin with brief background material about Lie algebras. We give special attention to the case of Lie algebras most relevant to the subsequent results, namely the simple, finite dimensional Lie algebras of type A_{2n-1} . For convenience, these algebras are defined over the field of complex numbers, \mathbb{C} , but any algebraically closed field of characteristic zero could be used instead. We then discuss the structure of the loop algebra realization of twisted toroidal Lie algebras of type A_{2n-1} . The final three chapters present the three primary results: the MRY presentation of twisted toroidal Lie algebras of type A_{2n-1} , and then a fermionic and bosonic representation of these algebras.

Chapter 2

Background

Throughout this thesis, the ground field will be \mathbb{C} unless otherwise stated. For example, all vector spaces, tensor products, and the linearity of operations will be implicitly understood to be over \mathbb{C} . The foundational background material of linear algebra, abstract algebra, finite dimensional Lie algebra theory, and representation theory can be found in such references as [E], [Gal], [HK], [H], [Me], and [Mi].

2.1 Finite Dimensional

We begin with a brief review of definitions and results for the simple, finite dimensional Lie algebra of type A_{2n-1} , $n \geq 3$ which are pertinent to our problem. A thorough coverage of the theory of finite dimensional Lie algebras can be found in [Mi], [H], [C], and the early chapters of [K] (since A_{2n-1} is a Kac-Moody algebra of finite type).

Definition 1. Here we collect some of the most important definitions for our description of A_{2n-1} .

- A *Lie algebra* L is a vector space endowed with a product $[\cdot, \cdot] : L \times L \rightarrow L$, called a *bracket product*, such that $[\cdot, \cdot]$ is bilinear, alternating (that is, $[x, y] = -[y, x] \forall x, y \in L$), and satisfies the Jacobi identity (that is, $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$). In general, L may be finite dimensional or infinite dimensional as a vector space.
- A subspace $M \subset L$ is an *ideal* of L if $[x, y] \in M$ whenever $x \in L, y \in M$.
- A Lie algebra L is *simple* if $[L, L] \neq \{0\}$ (L is not abelian) and if the only ideals of L are $\{0\}$ and L .

- Let $L^{(0)} \supset L^{(1)} \supset L^{(2)} \supset \dots$, where $L^{(m)} = [L^{(m-1)}, L^{(m-1)}]$ for $m = 1, 2, \dots$ and $L^{(0)} = L$, be called the *derived series*. L is *solvable* if $L^{(m)} = \{0\}$ for some $m \geq 0$.
- Let $L^0 \supset L^1 \supset L^2 \supset \dots$, where $L^m = [L, L^{m-1}]$ for $m = 1, 2, \dots$ and $L^0 = L$, be called the *lower central series*. L is *nilpotent* if $L^m = \{0\}$ for some $m \geq 0$.
- L is *semisimple* if the maximal solvable ideal (called the *radical of L*) is $\{0\}$.
- The notation $\text{adx}(y)$ (or $\text{ad}(x)(y)$ or $\text{ad}_x(y)$) will mean $[x, y]$. Notice that adx is thus a linear operator on L .

Remark. The axiom that a Lie algebra bracket be alternating is usually stated instead as $[x, x] = 0 \forall x \in L$, but the bracket being alternating is equivalent since \mathbb{C} does not have characteristic 2.

Example 2. If A is any associative algebra (a vector space with a bilinear, associative product $\cdot : A \times A \rightarrow A$), then A can be given the structure of a Lie algebra via the *commutator bracket*, $[a, b] = a \cdot b - b \cdot a \forall a, b \in A$.

Now we will give a realization of A_{2n-1} ; the bracket in A_{2n-1} will be denoted by $[\cdot, \cdot]'$ throughout.

Define formal generators of A_{2n-1} to be e'_i, f'_i, h'_i , with $i \in \{1, 2, \dots, 2n-1\}$, called *Chevalley generators*. Denote the subalgebra generated by $\{h'_i \mid i = 1, 2, \dots, 2n-1\}$ by $\dot{\mathfrak{h}}$. Define a matrix

$$A = (a_{ij})_{i,j=1}^{2n-1} = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & & -1 & 2 & -1 \\ 0 & 0 & \cdots & & -1 & 2 \end{bmatrix}$$

called the *Cartan matrix* for A_{2n-1} . Let $\alpha'_i \in \dot{\mathfrak{h}}^*$ for $i \in \{1, 2, \dots, 2n-1\}$, called *simple roots*, be determined by $\alpha'_j(h'_i) = a_{ij}$. Define a bracket structure among the Chevalley generators as follows; these relations appear in [H] Proposition 18.1.

1. $[e'_i, f'_j]' = \delta_{ij}h'_i$,
2. $[h'_i, h'_j]' = 0$,
3. $[h'_i, e'_j]' = \alpha'_j(h'_i)e'_j$,
4. $[h'_i, f'_j]' = -\alpha'_j(h'_i)f'_j$,

$$5. (\text{ad } e'_i)^{1-\alpha'_j(h'_i)}(e'_j) = 0 \text{ where } i \neq j,$$

$$6. (\text{ad } f'_i)^{1-\alpha'_j(h'_i)}(f'_j) = 0 \text{ where } i \neq j.$$

Define a bilinear form $\kappa(\cdot, \cdot) : A_{2n-1} \times A_{2n-1} \rightarrow \mathbb{C}$ by $\kappa(x, y) = \text{tr}(\text{ad}_x \text{ad}_y) \forall x, y \in A_{2n-1}$. This is known as the *Killing form*. Straightforward calculations (for some fixed n) show that the Killing form on A_{2n-1} is nondegenerate, that is, that $\{x \in A_{2n-1} \mid \kappa(x, y) = 0 \forall y \in A_{2n-1}\} = \{0\}$. By [Mi] Theorem 7.4, this implies that A_{2n-1} is semisimple. (This can also be seen by [Mi] Problem (2.18) (xv), which shows that A_{2n-1} is simple, combined with [Mi] Remark 4.11, which shows that simplicity implies semisimplicity.)

Since A_{2n-1} is semisimple, then thanks to the abstract Jordan decomposition ([H] §5.4), any element in $x \in A_{2n-1}$ can be decomposed into a sum of elements $x = x_s + x_n$ where x_s is ad-semisimple (that is, ad-diagonalizable) and x_n is ad-nilpotent. Further, again because A_{2n-1} is semisimple, A_{2n-1} is not solvable and hence not nilpotent. Thus, by Theorem 3.2 in [H], there must exist some elements in A_{2n-1} which are not ad-nilpotent, and hence ad-semisimple elements must exist. The span of such elements form, by definition, a *toral subalgebra*. By Lemma 8.1 in [H], such subalgebras are abelian. Of most interest in the theory is a *maximal toral subalgebra*, H .

By a common linear algebraic result, a commuting family of semisimple endomorphisms of a vector space is simultaneously diagonalizable (meaning that there exists a basis for which every element in H acts diagonally). Thus there exists a basis $\{x'_j\}$, with j lying in some index set (not $j = 1, \dots, 2n - 1$) so that $[h, x'_j]' = \alpha(h)x'_j \forall h \in H$ and some $\alpha \in \dot{\mathfrak{h}}^*$. Since these x'_j form a basis, A_{2n-1} is the direct sum of eigenspaces associated to the $\alpha \in \dot{\mathfrak{h}}^*$. Such a direct sum decomposition is known as a *root space decomposition*. Such nonzero α are called *roots*, and the corresponding *root system* (in the language of [H] §8.5 and §9) is denoted by $\dot{\Delta}$ for A_{2n-1} . We will call a nonzero set $\{x \in A_{2n-1} \mid [h, x]' = \alpha'(h)x \forall h \in \dot{\mathfrak{h}}\}$ a *root space* and denote it by $(A_{2n-1})_{\alpha'}$. In the construction of root systems in [H] §12.1, it is shown that the simple roots defined above indeed form a *base* for $\dot{\Delta}$, denoted by $\dot{\Pi}$ (straightforward computations show that the construction of simple roots therein coincides with our construction here). The lattice $\dot{Q} = \bigoplus_{i \in \{1, 2, \dots, 2n-1\}} \mathbb{Z}\alpha'_i$ is called the *root lattice*.

As can be seen from the relations above, $\dot{\mathfrak{h}}$ indeed forms a maximal toral subalgebra of A_{2n-1} . By [H] Corollary 15.3, $\dot{\mathfrak{h}}$ is then a *Cartan subalgebra (CSA)* of A_{2n-1} (a nilpotent, self-normalizing subalgebra).

Using the construction of A_{2n-1} in [K] §6.7, $\alpha' \in \dot{\Delta} \Rightarrow \alpha' = \alpha'_i + \alpha'_{i+1} + \dots + \alpha'_j$ for $1 \leq i \leq j \leq 2n - 1$. As in [K] (1.3.1) and §1.5, $A_{2n-1} = \bigoplus_{\alpha' \in \dot{Q}} (A_{2n-1})_{\alpha'}$ is a \dot{Q} -gradation of

A_{2n-1} . In particular, since $[(A_{2n-1})_{\beta_1}, (A_{2n-1})_{\beta_2}]' \subset (A_{2n-1})_{\beta_1+\beta_2}$, then for $\alpha' = \alpha'_i + \alpha'_{i+1} + \cdots + \alpha'_j \in \dot{\Delta} \subset \dot{Q}$, we can use the notation $e'_{\alpha'} = [e'_{\alpha'_i}, [e'_{\alpha'_{i+1}} [\cdots [e'_{\alpha'_{j-1}}, e'_{\alpha'_j}]']']'$. Similarly, $h'_{\alpha'} := h'_{\alpha'_i} + \cdots + h'_{\alpha'_j}$. Since the $f'_{\alpha'}$ are so-called “negative” roots, then the notation $f'_{\alpha'}$ means $[f'_{\alpha'_j}, [f'_{\alpha'_{j-1}} [\cdots [f'_{\alpha'_{i+1}}, f'_{\alpha'_i}]']']'$ (indices taken in reverse order).

2.1.1 Scaling of Bilinear Form

Here we define another bilinear form on A_{2n-1} which will be useful in what follows. On $\dot{\mathfrak{h}}^*$, the Killing form can be seen to be equal to $(\alpha'_i|\alpha'_j)' = a_{ij}, 1 \leq i \leq 2n-1$. By [K] Theorem 2.2, this form is bilinear, symmetric, invariant, and nondegenerate. However, in order to simplify relations in what follows (the MRY relations), we will double the form on $\dot{\mathfrak{h}}^*$ as follows.

Definition 3. Define $(\cdot|\cdot)': \dot{\mathfrak{h}} \times \dot{\mathfrak{h}} \rightarrow \mathbb{R}$ by $(\alpha'_i|\alpha'_j)' := 2a_{ij}$. $(\cdot|\cdot)'$ is also symmetric and bilinear by definition.

That this form is nondegenerate follows from the fact that the Killing form is; we will show below that this form is invariant as well; that is, that $([a, b]'|c)' = (a|[b, c]')' \forall a, b, c \in A_{2n-1}$. The spaces $\dot{\mathfrak{h}}^*$ and $\dot{\mathfrak{h}}$ are identified via the form; that is, the map

$$\nu: \dot{\mathfrak{h}}^* \rightarrow \dot{\mathfrak{h}}$$

is defined by $\alpha'_i \mapsto \nu(\alpha'_i)$ such that $\alpha'_j(\nu(\alpha'_i)) = (\alpha'_i|\alpha'_j)' = 2a_{ij} = 2\alpha'_j(h'_i) \Rightarrow \nu(\alpha'_i) = 2h'_i$. We can thus transfer the form to $\dot{\mathfrak{h}}$, using the same notation, via ν as follows:

$$(h'_i|h'_j)' = \left(\nu \left(\frac{\alpha'_i}{2} \right) \middle| \nu \left(\frac{\alpha'_j}{2} \right) \right)' = \frac{1}{4}(\alpha'_i|\alpha'_j)' = \frac{1}{2}a_{ij}.$$

Notice that, even with this scaling, we maintain the usual identity

$$\frac{2(\alpha'_i|\alpha'_j)'}{(\alpha'_i|\alpha'_i)'} = a_{ij} = \frac{2(h'_i|h'_j)'}{(h'_i|h'_i)'}.$$

Remark. To ensure invariance of $(\cdot|\cdot)'$, as in the proof of [K] Theorem 2.2, we must have:

$$\begin{aligned} ([e'_i, f'_j]'|h'_k)' &= (e'_i|[f'_j, h'_k]')' \Rightarrow (\delta_{ij}h'_i|h'_k)' = (e'_i|\alpha'_j(h'_k)f'_j)' \\ &\Rightarrow \frac{1}{2}\delta_{ij}a_{ik} = a_{kj}(e'_i|f'_j)' \Rightarrow \frac{1}{2}\delta_{ij} = (e'_i|f'_j)' \end{aligned}$$

which is indeed true. As in the referenced theorem, this is sufficient to establish invariance of this form (all other combinations $([a, b]'|c)' = (a|[b, c]')'$ are either trivial or can be written in terms of the case shown).

Proposition 4. *The bilinear form defined by*

1. $(e'_i|f'_j)' = \frac{1}{2}\delta_{ij}$,
2. $(h'_i|h'_j)' = \frac{1}{2}a_{ij}$,
3. $(h'_i|e'_j)' = (h'_i|f'_j)' = (e'_i|e'_j)' = (f'_i|f'_j)' = 0$,

for $1 \leq i \leq 2n - 1$ is a nondegenerate symmetric invariant bilinear form on A_{2n-1} . ■

2.1.2 Copies of \mathfrak{sl}_2

Since

- $[h'_i, e'_i]' = \alpha'_i(h'_i)e'_i = a_{ii}e'_i = 2e'_i$,
- $[h'_i, f'_i]' = -\alpha'_i(h'_i)f'_i = -a_{ii}e'_i = -2f'_i$, and
- $[e'_i, f'_i]' = h'_i$,

then $\{e'_i, f'_i, h'_i\}$ is an \mathfrak{sl}_2 triplet for each $i \in \{1, 2, \dots, 2n - 1\}$.

In fact for each $\alpha' \in \dot{\Delta}$, since $\alpha'(\nu(\frac{1}{2}\alpha')) = \alpha'(h'_{\alpha'}) = (\alpha'_i + \alpha'_{i+1} + \dots + \alpha'_j)(h'_i + h'_{i+1} + \dots + h'_j) = 2$, then we can choose $e'_{\alpha'} \in (A_{2n-1})_{\alpha'}$, $f'_{\alpha'} \in (A_{2n-1})_{-\alpha'}$ such that:

- $[e'_{\alpha'}, f'_{\alpha'}]' = h'_{\alpha'}$;
- $[h'_{\alpha'}, e'_{\alpha'}]' = \alpha'(h'_{\alpha'})e'_{\alpha'} = 2e'_{\alpha'}$;
- $[h'_{\alpha'}, f'_{\alpha'}]' = -\alpha'(h'_{\alpha'})f'_{\alpha'} = -2f'_{\alpha'}$.

In other words, $e'_{\alpha'}, f'_{\alpha'}, h'_{\alpha'}$ form an \mathfrak{sl}_2 triplet for any $\alpha' \in \dot{\Delta}$. Using our scaling of $(\cdot|\cdot)'$, we also have the identities:

- $(e'_{\alpha'}|f'_{\alpha'})' = \frac{2}{(\alpha'|\alpha')'}$,
- $\nu(\alpha') = \frac{(\alpha'|\alpha')'}{2}h'_{\alpha'}$, and
- $[e'_{\alpha'}, f'_{\alpha'}]' = (e'_{\alpha'}|f'_{\alpha'})'\nu(\alpha')$.

as in [MRY] (2.1). These identities follow from the fact all roots of A_{2n-1} have the same root length with respect to $(\cdot|\cdot)'$ since A_{2n-1} is simply-laced (its Cartan matrix is symmetric). Because of the scaling of $(\cdot|\cdot)'$, that root length is 4 because $(\alpha'|\alpha')' = 4 \forall \alpha' \in \dot{\Delta}$.

2.1.3 Example Calculations

It is instructive to see a concrete example of $(e'_{\alpha'}|f'_{\alpha'})' = \frac{2}{(\alpha'|\alpha')} = \frac{1}{2}$ to see how the form defined on Chevalley generators and $\dot{\mathfrak{h}}$ extends to arbitrary root spaces.

Example 5. Consider $\alpha = \alpha'_1 + \alpha'_2$. Then we want to show that $(e'_{\alpha}|f'_{\alpha})' = \frac{1}{2}$. We have, by invariance and the Jacobi identity (which makes the adjoint operator into a derivation) that: $(e'_{\alpha}|f'_{\alpha})' = ([e'_1, e'_2]'|[f'_2, f'_1]')' = (e'_1|[e'_2, [f'_2, f'_1]']')' = (e'_1|[e'_2, f'_2]', f'_1)' + [f'_2, [e'_2, f'_1]]')' = (e'_1|[h'_2, f'_1]' + 0)' = (e'_1 - \alpha'_1(h'_2)f'_1)' = (e'_1|f'_1)' = \frac{1}{2}$, as desired.

We also include a bracket calculation involving $\alpha = \alpha'_1 + \alpha'_2$.

Example 6. Since $[h'_{\alpha}, e'_{\alpha}]' = \alpha(h'_{\alpha})e'_{\alpha} = (\alpha'_1 + \alpha'_2)(h'_1 + h'_2)e'_{\alpha} = 2e'_{\alpha}$, we want to show that $[h'_{\alpha}, e'_{\alpha}]' = 2e'_{\alpha}$ using their definitions of brackets of CSA and Chevalley generators (respectively). Indeed, $[h'_{\alpha}, e'_{\alpha}]' = [h'_{\alpha'_1+\alpha'_2}, e'_{\alpha'_1+\alpha'_2}]' = [h'_1+h'_2, [e'_1, e'_2]']' = [h'_1, [e'_1, e'_2]']' + [h'_2, [e'_1, e'_2]']' = [[h'_1, e'_1], e'_2]' + [e'_1, [h'_1, e'_2]']' + [[h'_2, e'_1], e'_2]' + [e'_1, [h'_2, e'_2]']' = 2[e'_1, e'_2]' - [e'_1, e'_2]' - [e'_1, e'_2]' + 2[e'_1, e'_2]' = 2[e'_1, e'_2]' = 2e'_{\alpha}$.

2.1.4 Highest Root of $\dot{\Delta}$

As in the construction of A_{2n-1} in [K] §6.7, the *highest root* of $\dot{\Delta}$ is:

$$\theta = \alpha'_1 + \cdots + \alpha'_{2n-1}.$$

Then define h'_0, e'_0, f'_0 such that $e'_0 \in (A_{2n-1})_{\theta}, f'_0 \in (A_{2n-1})_{-\theta}, h'_0 := [e'_0, f'_0]'$ and the trio forms an \mathfrak{sl}_2 triplet as in the previous subsection. Since $[h'_0, e'_0]' = \theta(h'_0)e'_0 = 2e'_0$, we must have $\theta(h'_0) = 2 \Rightarrow h'_0 = h'_1 + \cdots + h'_{2n-1}$ as expected. Under our identification ν , we have $\nu(\theta) = \nu(\alpha'_1 + \cdots + \alpha'_{2n-1}) = 2h'_0$.

We record here some identities involving θ and 0-nodes for use in later calculations. Here $1 \leq p \leq 2n-1$.

- $\theta(h'_p) = (\alpha'_1 + \cdots + \alpha'_{2n-1})(h'_p) = \delta_{1p} + \delta_{2n-1,p}$
- $\theta(h'_0) = (\alpha'_1 + \cdots + \alpha'_{2n-1})(h'_1 + \cdots + h'_{2n-1}) = 2$
- $\alpha'_p(h'_0) = \alpha'_p(h'_1 + \cdots + h'_{2n-1}) = \delta_{1p} + \delta_{2n-1,p}$
- $(\theta|\theta)' = (\alpha'_1 + \cdots + \alpha'_{2n-1}|\alpha'_1)' + \cdots + (\alpha'_1 + \cdots + \alpha'_{2n-1}|\alpha'_{2n-1})' = 2(a_{11} + a_{12} + 0 + \cdots + 0 + a_{2n-2,2n-1} + a_{2n-1,2n-1}) = 4$
- $(h'_0|h'_0)' = (h'_1 + \cdots + h'_{2n-1}|h'_1)' + \cdots + (h'_1 + \cdots + h'_{2n-1}|h'_{2n-1})' = \frac{1}{2}(a_{11} + a_{12} + 0 + \cdots + 0 + a_{2n-2,2n-1} + a_{2n-1,2n-1}) = 1$

- $(h'_0|h'_p)' = (h'_1 + \dots + h'_{2n-1}|h'_p)' = \frac{1}{2}(\delta_{1p} + \delta_{2n-1,p})(a_{11} + a_{12}) = \frac{1}{2}(\delta_{1p} + \delta_{2n-1,p})$
- $(e'_0|f'_0)' = \frac{1}{2}$

2.1.5 Automorphism of A_{2n-1}

Fix an automorphism σ of A_{2n-1} of order 2. Then its minimal polynomial is $\sigma^2 - 1 = 0$ and thus the minimal polynomial splits into distinct linear factors $\sigma + 1$ and $\sigma - 1$, so σ is diagonalizable. Hence A_{2n-1} splits into a direct sum of the eigenspaces of σ :

$$A_{2n-1} = (A_{2n-1})_{\bar{0}} \oplus (A_{2n-1})_{\bar{1}}$$

where $(A_{2n-1})_{\bar{i}} = \{x \in A_{2n-1} \mid \sigma(x) = (-1)^i x, i = 0, 1\}$. In particular, $(A_{2n-1})_{\bar{0}}$ is the set of fixed points of σ .

We can define such a σ explicitly on the generators of A_{2n-1} as follows.

Definition 7. On A_{2n-1} , $\sigma(x'_i) = x'_{2n-i}$ for $x = e, f$, or h (notice that x'_n is fixed). We call σ defined this way the *diagram automorphism* of A_{2n-1} .

Remark. In [K] Proposition 8.3 it is proven that $(A_{2n-1})_{\bar{0}} \cong C_n$ and that $(A_{2n-1})_{\bar{1}}$ is an irreducible C_n -module.

Pictorially, for $n = 3$, σ acts on the Dynkin diagram as in the figure, and similarly for larger n . Indeed, this is the reason for the term “diagram automorphism.” To see that σ is indeed an

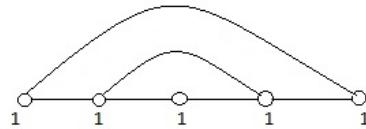


Figure 2.1 Automorphism of Dynkin diagram

automorphism as claimed, an argument is given in [K] §7.9.

2.2 Toroidal Algebras

2.2.1 Central Extensions

For any Lie algebra \mathfrak{g} , a *central extension* of \mathfrak{g} is a pair $(\hat{\mathfrak{g}}, \pi)$ where $\hat{\mathfrak{g}}$ is a Lie algebra and $\pi : \hat{\mathfrak{g}} \rightarrow \mathfrak{g}$ is a surjective homomorphism with $\ker(\pi) \subset Z(\hat{\mathfrak{g}})$, the center of $\hat{\mathfrak{g}}$.

A central extension $(\hat{\mathfrak{g}}, \pi)$ of \mathfrak{g} is the *universal central extension, uce* (or *universal covering algebra, uca*) if it satisfies the following universal property: for every central extension (\mathfrak{e}, φ) of \mathfrak{g} , $\exists!$ homomorphism $\psi : \hat{\mathfrak{g}} \rightarrow \mathfrak{e}$ such that $\varphi \cdot \psi = \pi$. That is, the diagram in Figure ?? commutes.

$$\begin{array}{ccc} & \mathfrak{e} & \\ \psi \uparrow & \swarrow \varphi & \\ \hat{\mathfrak{g}} & \xrightarrow{\pi} & \mathfrak{g} \end{array}$$

Figure 2.2 Commutative diagram for a uce

A Lie algebra is called *perfect* if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Note that A_{2n-1} is simple and hence perfect. As quoted in [BM] §2, Garland proves every perfect Lie algebra has a uce (that is, the uce exists) and that it is unique up to isomorphism. Uniqueness of the uce can be shown since the universal property holds for a uce; for, if A and B are both the uce of a Lie algebra \mathfrak{g} , then in particular they are each central extensions. Thus, by the discussion above, the following diagram commutes.

Thus $\rho\psi = \text{id}_B$ and $\psi\rho = \text{id}_A$, so ρ and ψ are invertible (they are inverses of each other), and

$$\begin{array}{ccc} & B & \\ \psi \uparrow \rho & \swarrow \varphi & \\ A & \xrightarrow{\pi} & \mathfrak{g} \end{array}$$

Figure 2.3 The uce of a perfect Lie algebra is unique

in particular bijective homomorphisms or isomorphisms.

2.2.2 Untwisted Toroidal

Define the multi-loop algebra:

$$L(A_{2n-1}) := A_{2n-1} \otimes \mathbb{C}[s, s^{-1}, t, t^{-1}]$$

Definition 8. The *untwisted (2-toroidal Lie algebra* is the use of the multi-loop algebra $L(A_{2n-1})$. We will denote this algebra by $T(A_{2n-1})$ (as in [JM]).

Remark. This definition could be given for $A_{2n-1} \otimes \mathbb{C}[t_1, t_1^{-1}, \dots, t_N, t_N^{-1}]$ for any integer $N \geq 2$, thus giving the definition of untwisted N -toroidal Lie algebras. However, in this paper, we will consider only the 2-toroidal case, and thus will call $T(A_{2n-1})$ simply a “toroidal” algebra with no modifier.

2.2.3 Twisted Toroidal

Extend σ of A_{2n-1} to an automorphism $\bar{\sigma}$ of $L(A_{2n-1})$ by:

$$\bar{\sigma}(x \otimes s^j t^m) := \sigma(x) \otimes ((-s)^j t^m)$$

for $j, m \in \mathbb{Z}, x \in (A_{2n-1})_{\bar{j}}$.

Remark. The “twist” is only on the variable s because only the exponent of that variable plays a nontrivial role in the definition of $\bar{\sigma}$.

Now define a twisted multi-loop algebra as follows:

$$L(A_{2n-1}, \sigma) = \bigoplus_{j \in \mathbb{Z}} L(A_{2n-1}, \sigma)_j$$

where $L(A_{2n-1}, \sigma)_j = (A_{2n-1})_{\bar{j}} \otimes \mathbb{C}[s, s^{-1}, t, t^{-1}]_j$ and $\mathbb{C}[s, s^{-1}, t, t^{-1}]_j = \text{span}\{s^j t^m | m \in \mathbb{Z}\}$. The $L(A_{2n-1}, \sigma)_j$ are thus graded subspaces of $L(A_{2n-1}, \sigma)$.

Remark. $L(A_{2n-1}, \sigma)$ is the subalgebra of $L(A_{2n-1})$ comprised of fixed points of $\bar{\sigma}$.

Definition 9. The *twisted toroidal Lie algebra*, denoted by $\overline{T}(A_{2n-1})$, is the use of the twisted multi-loop algebra $L(A_{2n-1}, \sigma)$.

2.2.4 Module of Kähler differentials

The structure of the central extension of toroidal algebras is more complicated than the one dimension needed in the affine case. This section describes the tools necessary for that structure.

For the preliminary definitions, it is most convenient to work in a more general algebraic setting. So, let A be any commutative associative algebra, and set $F = A \otimes A$. Make F into a two-sided A -module by the action $a \cdot (b_1 \otimes b_2) = ab_1 \otimes b_2 = (b_1 \otimes b_2) \cdot a$ for $a, b_1, b_2 \in A$. Set $K = \langle 1 \otimes ab - a \otimes b - b \otimes a \mid a, b \in A \rangle \subset F$ (that is, K is generated by all elements of such form). Define $\Omega_A = F/K$ and a map $d : A \rightarrow \Omega_A$ by $da = (1 \otimes a) + K$. Then Ω_A is called the A -module of Kähler differentials.

Remark. Notice that d is linear (since \otimes is), and $d(ab) = a(db) + (da)b$ for $a, b \in A$; thus the map d is a derivation. In [BK] it is remarked that the pair (Ω_A, d) is a sort of universal cover of the set D of all derivations from A to any A -module M .

Denote the canonical linear map $\Omega_A \rightarrow \frac{\Omega_A}{dA}$ by placing a bar over the elements from Ω_A . That is, $\overline{(1 \otimes a) + K} = 0 \in \frac{\Omega_A}{dA}$ for all $a \in A$. Thus, since $\overline{d(ab)} = 0$, we have $\overline{adb} = -\overline{(da)b} = -\overline{bda}$ for all $a, b \in A$. The elements of dA are called exact forms.

In [MRY] §2 it is shown that if $B = \mathbb{C}[s, s^{-1}, t, t^{-1}]$, a basis for $\frac{\Omega_B}{db}$ is $\left\{ \overline{s^{p-1}t^q ds}, \overline{s^p t^{-1} dt}, \overline{s^{-1} ds} \mid p \in \mathbb{Z}, q \in (\mathbb{Z} \setminus \{0\}) \right\}$.

2.2.4.1 Example Calculations in $\frac{\Omega_B}{db}$

We present two calculations to later use their results.

Example 10. First we show that

$$\overline{s^\ell ds^k} = \delta_{k,-\ell} ks^{\ell+k-1} ds.$$

By definition, $\overline{s^\ell ds^k} = \overline{(ds^k)s^\ell}$ since the module action is two-sided.

It is straightforward to show that the “bar” map is linear and, by a standard induction argument, $\overline{ds^m} = \overline{ms^{m-1}ds}$ (i.e. regular differentiation of polynomials holds). Hence, $\overline{(ds^k)s^\ell} = \overline{(ks^{k-1}ds)s^\ell} = \overline{ks^{k+\ell-1}ds}$. On the other hand, in $\frac{\Omega_B}{db}$ we have $\overline{(ds^k)s^\ell} = -\overline{s^k(ds^\ell)} = -\overline{\ell s^k(s^{\ell-1}ds)} = -\ell \overline{s^{k+\ell-1}ds} \Rightarrow \overline{ks^{k+\ell-1}ds} = -\ell \overline{s^{k+\ell-1}ds}$, and so $\overline{s^\ell ds^k} = \delta_{k,-\ell} ks^{\ell+k-1} ds$.

Example 11. Secondly, we show that

$$\overline{s^\ell t^{-1} d(s^k t)} = \delta_{k,-\ell} ks^{\ell+k-1} ds + \overline{s^{k+\ell} t^{-1} dt}.$$

Since d is a derivation, $\overline{s^\ell t^{-1} d(s^k t)} = \overline{s^\ell t^{-1} td(s^k)} + \overline{s^\ell t^{-1} s^k d(t)} = \overline{s^\ell ds^k} + \overline{s^{k+\ell} t^{-1} dt} = \delta_{k,-\ell} ks^{\ell+k-1} ds + \overline{s^{k+\ell} t^{-1} dt}$, where we use the result of the previous example.

2.3 Loop Algebra Realization of Toroidal Algebras

2.3.1 Untwisted Toroidal

In [MRY] Proposition 2.2, a realization of *untwisted* toroidal algebras is given. The definition is most conveniently given in a more general setting. Let the vector space $\mathfrak{u} = \dot{\mathfrak{g}} \otimes A \oplus \left(\frac{\Omega_A}{dA} \right)$, where $\dot{\mathfrak{g}}$ is any simple finite dimensional Lie algebra and A is any commutative algebra. Endow \mathfrak{u} with the Lie bracket:

- $[\dot{\mathfrak{g}} \otimes A, \frac{\Omega_A}{dA}] = 0$
- $[x \otimes a, y \otimes b] = [x, y]' \otimes ab + (x|y)' \overline{bda}$

for $a, b \in A, x, y \in \dot{\mathfrak{g}}$. Define the map $\omega : \mathfrak{u} \rightarrow \dot{\mathfrak{g}} \otimes A$ to be the projection with kernel $\frac{\Omega_A}{dA}$.

Proposition 12. [MRY] *Prop. 2.2. (\mathfrak{u}, ω) is the uce of $\dot{\mathfrak{g}} \otimes A$. ■*

Hence, when $\dot{\mathfrak{g}} = A_{2n-1}$ and $A = \mathbb{C}[s, s^{-1}, t, t^{-1}]$, the pair (\mathfrak{u}, ω) is the uce of $L(A_{2n-1})$, and thus is a realization of the untwisted toroidal algebra $T(A_{2n-1})$.

Remark. Notice that the central extension is one-dimensional for a loop algebra (the affine case), but is infinite-dimensional for a multi-loop algebra (the present toroidal case). This structure was investigated in [Kas] Thm. 2.1 and [BK].

2.3.2 Twisted Toroidal

Theorem 2.1 in [FJ] shows how this construction is modified for the twisted case (using Proposition 2.2 in [BK]).

Recall the grading on $B = \mathbb{C}[s, s^{-1}, t, t^{-1}]$, namely $B_j = \mathbb{C}[s, s^{-1}, t, t^{-1}]_j = \text{span}\{s^j t^m \mid m \in \mathbb{Z}\}$. Denote $\mathcal{K} = \frac{\Omega_B}{db} = \text{span}\{\overline{bda} \mid a, b \in B\}$ and the subalgebra $\mathcal{K}^\sigma = \text{span}\{\overline{bda} \mid a \in B_k, b \in B_l, k+l \equiv 0 \pmod{2}\}$. Now set

$$\bar{\tau} = L(A_{2n-1}, \sigma) \oplus \mathcal{K}^\sigma.$$

Define a bracket structure by

- $[L(A_{2n-1}, \sigma), \mathcal{K}^\sigma] = 0$
- $[x \otimes a, y \otimes b] = [x, y]' \otimes ab + (x|y)' \overline{bda}$ for $a \in B_k, b \in B_l, x \in (A_{2n-1})_{\bar{k}}, y \in (A_{2n-1})_{\bar{l}}$

Remark. Notice that if k is even and l is odd (without loss of generality), then it may appear at first that $[x \otimes a, y \otimes b]$ lies outside of $\bar{\tau}$ because $(x|y)' \overline{bda}$ term is not in \mathcal{K}^σ . However, in such cases, the bilinear form is defined in such a way that $(x|y)' = 0$. This follows from the fact that the form is a multiple of the Killing form. Thus, the form is invariant under any automorphism of A_{2n-1} , so $(x|y)' = (\sigma(x)|\sigma(y))' = (x|y)' = -(x|y)',$ hence $(x|y)' = 0$. So indeed $[x \otimes a, y \otimes b] \in \bar{\tau}$ and the above bracket is closed.

Define $\bar{\omega} : \bar{\tau} \rightarrow L(A_{2n-1}, \sigma)$ to be the natural projection map. We can now give the *loop algebra realization* of $\bar{T}(A_{2n-1})$.

Proposition 13. [FJ] Thm 2.1., [BK] Prop 2.2 *The pair $(\bar{\tau}, \bar{\omega})$ is the uce of $L(A_{2n-1}, \sigma)$, and hence is a realization of the twisted toroidal Lie algebra $\bar{T}(A_{2n-1})$. ■*

The quoted results [FJ] Thm 2.1., [BK] Prop 2.2 show that the uce of $L(A_{2n-1}, \sigma)$ does indeed exist, and the argument surrounding Figure 2.3 establishes uniqueness.

Extend $\bar{\sigma}$ to act on $\bar{\tau}$ by: $\bar{\sigma}(x \otimes s^j t^m + \overline{bda}) = \sigma(x) \otimes (-s)^j t^m + \overline{bda}$ where $x \in (A_{2n-1})_j, a = s^{a_1} t^{a_2}, b = s^{b_1} t^{b_2}$ such that $a_1, a_2, b_1, b_2, j, m \in \mathbb{Z}$ and $a_1 + b_1 \equiv 0 \pmod{2}$; in particular, $\bar{\sigma}$ acts as the identity on \mathcal{K}^σ . Notice that we keep the same notation for $\bar{\sigma}$ acting on $\bar{\tau}$.

Chapter 3

MRY Presentation of Toroidal Algebras

3.1 Untwisted Toroidal

In [MRY] §3, a different realization of $T(A_{2n-1})$ is given, which we summarize here.

Definition 14. Let t be the Lie algebra over \mathbb{C} with generators $\ell, \alpha'_i(k), X(\pm\alpha'_i, k)$ and satisfying relations:

- TA0: $[\ell, \alpha'_i(k)] = [\ell, X(\pm\alpha'_i, k)] = 0$
- TA1: $[\alpha'_i(k), \alpha'_j(m)] = k(\alpha'_i | \alpha'_j)' \delta_{k,-m} \ell$
- TA2: $[\alpha'_i(k), X(\pm\alpha'_j, m)] = \pm(\alpha'_i | \alpha'_j)' X(\pm\alpha'_j, m + k)$
- TA3: $[X(\alpha'_i, m), X(-\alpha'_j, l)] = -\delta_{ij} \left(\alpha'_i(m+l) + \frac{2l\delta_{m,-l}}{(\alpha'_i | \alpha'_j)'} \ell \right)$
- TA4: $[X(\alpha'_i, m), X(\alpha'_i, l)] = [X(-\alpha'_i, m), X(-\alpha'_i, m)] = 0$
- TA5: $\text{ad}(X(\alpha'_i, m))^{1-a_{ij}} X(\alpha'_j, l) = 0$, and $\text{ad}(X(-\alpha'_i, m))^{1-a_{ij}} X(-\alpha'_j, l) = 0$ for $i \neq j$

for $0 \leq i, j \leq 2n-1, k, m, l \in \mathbb{Z}$.

Also define a mapping $\pi : t \rightarrow L(A_{2n-1})$ by:

- $\ell \mapsto 0$
- $\alpha'_j(k) \mapsto h'_j \otimes s^k$

- $X(\alpha'_0, k) \mapsto f'_0 \otimes s^k t$
- $X(-\alpha'_0, k) \mapsto e'_0 \otimes s^k t^{-1}$
- $X(\alpha'_i, k) \mapsto e'_i \otimes s^k$
- $X(-\alpha'_i, k) \mapsto f'_i \otimes s^k$

for $0 \leq j \leq 2n - 1, 1 \leq i \leq 2n - 1, k \in \mathbb{Z}$, and extend by linearity.

Remark. We make some remarks about the definitions of t and π as compared to the definitions (3.1) and (3.2) in [MRY]. Each is inconsequential to the statement of the following theorem which follows because all pertinent properties are maintained.

1. $L(A_{2n-1})$ is written as $\mathbb{C}[s, s^{-1}, t, t^{-1}] \otimes A_{2n-1}$ in [MRY] instead of $A_{2n-1} \otimes \mathbb{C}[s, s^{-1}, t, t^{-1}]$ as we choose to do here.
2. In [MRY] it is chosen to write $\alpha_i^\vee(k)$ and $x_k(\alpha_i)$, whereas we choose the notations $\alpha'_i(k)$ and $X(\alpha'_i, k)$.
3. $\pi(X(-\alpha'_i, k)), 0 \leq i \leq 2n - 1$ is a different sign in [MRY].
4. The scaling on $(\alpha'_i | \alpha'_j)'$ is different than on the bilinear form used in [MRY].
5. Relation (TA5) is given in [MRY] (3.1) for only a few cases, and the remaining cases are proven in [MRY] Cor. 3.3. All cases have been listed in the definition here.

Theorem 15. [MRY] Prop 3.5 (t, π) is the uce of $L(A_{2n-1})$, and hence is a realization of $T(A_{2n-1})$. ■

3.2 Twisted Toroidal

We seek a similar realization for $\bar{\tau} = \bar{T}(A_{2n-1})$. To do so, we will define an algebra \bar{t} by generators and relations and prove that it is the uce of $L(A_{2n-1}, \sigma)$; hence (by uniqueness of the uce) it must also be a presentation of $\bar{T}(A_{2n-1})$.

Remark. Because of its similarity to the realization above, we will call this the MRY presentation of $\bar{T}(A_{2n-1})$.

Before defining \bar{t} , we must establish some more notation.

Definition 16. Define the following delta functions for formal variables z, w :

- $\delta(z - w) := \sum_{k \in \mathbb{Z}} w^k z^{-k-1} = \iota_{z,w} \frac{1}{z-w} + \iota_{w,z} \frac{1}{w-z}$
- $\delta(z + w) := \sum_{k \in \mathbb{Z}} (-w)^k z^{-k-1} = \iota_{z,w} \frac{1}{z+w} - \iota_{w,z} \frac{1}{w+z}$
- $\partial_w \delta(z - w) := \sum_{k \in \mathbb{Z}} k w^{k-1} z^{-k-1} = \iota_{z,w} \frac{1}{(z-w)^2} - \iota_{w,z} \frac{1}{(w-z)^2}$
- $\partial_w \delta(z + w) := \sum_{k \in \mathbb{Z}} -k(-w)^{k-1} z^{-k-1} = -\iota_{z,w} \frac{1}{(z+w)^2} + \iota_{w,z} \frac{1}{(w+z)^2}$

Here, $\iota_{z,w}$ indicates a power series expansion in the domain $|z| > |w|$.

Remark. In each of the four delta functions, the second equality follows from geometric series theory. Indeed, $\delta(z - w) = \sum_{k \in \mathbb{Z}} w^k z^{-k-1} = \sum_{k \in \mathbb{Z}} w^k z^{-k-1} = \sum_{k \in \mathbb{Z}_{\geq 0}} w^k z^{-k-1} + \sum_{k \in \mathbb{Z}_{< 0}} w^k z^{-k-1} = \sum_{k \in \mathbb{Z}_{\geq 0}} w^k z^{-k-1} + \sum_{k \in \mathbb{Z}_{\geq 0}} w^{-k-1} z^k = (z^{-1} + wz^{-2} + w^2 z^{-3} + \dots) + (w^{-1} + w^{-2} z + w^{-3} z^2 + \dots) = \iota_{z,w} \frac{\frac{1}{z}}{1 - \frac{w}{z}} + \iota_{w,z} \frac{\frac{1}{w}}{1 - \frac{z}{w}} = \iota_{z,w} \frac{1}{z-w} + \iota_{w,z} \frac{1}{w-z}$. The other delta function equalities can be shown to hold via similar computations.

We will also need the following Cartan matrix of type A_{2n-1} (the same Cartan matrix given previously):

$$A = (a_{ij})_{i,j=1}^{2n-1} = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \cdots & & -1 & 2 \end{bmatrix},$$

and that of untwisted affine type $C_n^{(1)}$:

$$C = (c_{ij})_{i,j=0}^n = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -2 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -2 \\ 0 & 0 & \cdots & & -1 & 2 \end{bmatrix}.$$

We now have all that is needed to define the algebra of interest.

Definition 17. Let \bar{t} be the Lie algebra over \mathbb{C} generated by components of the generating functions $\alpha_m^{\bar{\sigma}}(z) = \sum_{k \in \mathbb{Z}} \alpha_m^{\bar{\sigma}}(k) z^{-k-1}$, $X^{\bar{\sigma}}(\pm\alpha_m, z) = \sum_{k \in \mathbb{Z}} X^{\bar{\sigma}}(\pm\alpha_m, k) z^{-k-1}$, with $\not\epsilon$ central, and satisfying also the following relations. In the list of relations, $1 \leq i, j \leq n-1$, and $0 \leq m, p \leq n$.

1. $[\alpha_0^{\bar{\sigma}}(z), \alpha_m^{\bar{\sigma}}(w)] = (2\delta_{m0} - \delta_{m1})(\partial_w \delta(z-w) + \partial_w \delta(z+w))\not\epsilon$
2. $[\alpha_i^{\bar{\sigma}}(z), \alpha_j^{\bar{\sigma}}(w)] = a_{ij} \partial_w \delta(z-w)\not\epsilon$
3. $[\alpha_i^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(w)] = a_{in} (\partial_w \delta(z-w) + \partial_w \delta(z+w))\not\epsilon$
4. $[\alpha_n^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(w)] = a_{nn} (\partial_w \delta(z-w) + \partial_w \delta(z+w))\not\epsilon$
5. $[\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_m, w)] = \pm(2\delta_{m0} - \delta_{m1}) X^{\bar{\sigma}}(\pm\alpha_m, w) (\delta(z-w) + \delta(z+w))$
6. $[\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_0, w)] = \pm(-\delta_{i1}) X^{\bar{\sigma}}(\pm\alpha_0, w) (\delta(z-w) + \delta(z+w))$
7. $[\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_j, w)] = \pm a_{ij} X^{\bar{\sigma}}(\pm\alpha_j, w) \delta(z-w)$
8. $[\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_n, w)] = \pm a_{in} X^{\bar{\sigma}}(\pm\alpha_n, w) (\delta(z-w) + \delta(z+w))$
9. $[\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_0, w)] = 0$
10. $[\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_j, w)] = \pm a_{nj} X^{\bar{\sigma}}(\pm\alpha_j, w) (\delta(z-w) + \delta(z+w))$
11. $[\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_n, w)] = \pm a_{nn} X^{\bar{\sigma}}(\pm\alpha_n, w) (\delta(z-w) + \delta(z+w))$
12. $[X^{\bar{\sigma}}(\pm\alpha_m, z), X^{\bar{\sigma}}(\pm\alpha_m, w)] = 0$
13. $[X^{\bar{\sigma}}(\alpha_0, z), X^{\bar{\sigma}}(-\alpha_0, w)] = \alpha_0^{\bar{\sigma}}(w) (\delta(z-w) + \delta(z+w)) + (\partial_w \delta(z-w) + \partial_w \delta(z+w))\not\epsilon$
14. $[X^{\bar{\sigma}}(\alpha_i, z), X^{\bar{\sigma}}(-\alpha_i, w)] = \alpha_i^{\bar{\sigma}}(w) \delta(z-w) + \partial_w \delta(z-w)\not\epsilon$
15. $[X^{\bar{\sigma}}(\alpha_n, z), X^{\bar{\sigma}}(-\alpha_n, w)] = \alpha_n^{\bar{\sigma}}(w) (\delta(z-w) + \delta(z+w)) + (\partial_w \delta(z-w) + \partial_w \delta(z+w))\not\epsilon$
16. $[X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)] = 0$ for $p \neq m$
17. $\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_2) X^{\bar{\sigma}}(\pm\alpha_m, z_1) = 0$ if $c_{pm} = 0$
18. $\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_3) \text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_2) X^{\bar{\sigma}}(\pm\alpha_m, z_1) = 0$ if $c_{pm} = -1$
19. $\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_4) \text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_3) \text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_2) X^{\bar{\sigma}}(\pm\alpha_m, z_1) = 0$ if $c_{pm} = -2$

The generators will be referred to as *MRY generators*, the relations as *MRY relations*, and the algebra \bar{t} as the *MRY algebra*.

Remark. The elements $\{\alpha_i^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(z) \mid 1 \leq i \leq n-1\} \subset \bar{t}$ form a Heisenberg subalgebra.

Remark. As in [MRY] Remark 1 after (3.1), Relations 13-15 above show that the $X^{\bar{\sigma}}(\pm\alpha_m, z)$, $0 \leq m \leq n$ generate \bar{t} . In particular, since these generators can be recovered in the derived algebra $[\bar{t}, \bar{t}]$ (from relations 6-8, for example), then conclude that $\bar{t} \subset [\bar{t}, \bar{t}]$, and hence \bar{t} is perfect.

Remark. Notice that $\delta(z - w) = \sum_{k \in \mathbb{Z}} w^k z^{-k-1} = \sum_{\ell \in \mathbb{Z}} w^{-\ell-1} z^\ell = \delta(w - z)$. This observation motivates the following lemma.

Lemma 18. *Multiplication of a generating function in z by $\delta(z \pm w)$ allows z to be replaced by $\mp w$. That is, for any generating function $x(z)$, we have $x(z)\delta(z \pm w) = x(\mp w)\delta(z \pm w)$.*

Proof: $x(z)\delta(z \pm w) = \sum_{\ell \in \mathbb{Z}} x(k)z^{-k-1} \sum_{\ell \in \mathbb{Z}} (\mp w)^\ell z^{-\ell-1} = \sum_{k, \ell \in \mathbb{Z}} x(k)(\mp w)^\ell z^{-k-\ell-2}$. Setting $\ell' = k + \ell + 1$ and $k' = k$ gives $\sum_{k', \ell' \in \mathbb{Z}} x(k')(\mp w)^{\ell'-k'-1} z^{-\ell'-1} = \sum_{k' \in \mathbb{Z}} x(k')(\mp w)^{-k'-1} \cdot \sum_{\ell' \in \mathbb{Z}} (\mp w)^{\ell'} z^{-\ell'-1} = x(\mp w)\delta(z \pm w)$, as desired. ■

Define a map $\bar{\pi} : \bar{t} \rightarrow L(A_{2n-1}, \sigma)$ by:

- $\emptyset \mapsto 0$
- $\alpha_0^{\bar{\sigma}}(k) \mapsto -h'_0 \otimes s^k - h'_0 \otimes (-s)^k$
- $\alpha_i^{\bar{\sigma}}(k) \mapsto h'_i \otimes s^k + h'_{2n-i} \otimes (-s)^k$
- $X^{\bar{\sigma}}(\alpha_0, k) \mapsto f'_0 \otimes s^k t + f'_0 \otimes (-s)^k t$
- $X^{\bar{\sigma}}(-\alpha_0, k) \mapsto e'_0 \otimes s^k t^{-1} + e'_0 \otimes (-s)^k t^{-1}$
- $X^{\bar{\sigma}}(\alpha_i, k) \mapsto e'_i \otimes s^k + e'_{2n-i} \otimes (-s)^k$
- $X^{\bar{\sigma}}(-\alpha_i, k) \mapsto f'_i \otimes s^k + f'_{2n-i} \otimes (-s)^k$

for $1 \leq i \leq n$, and extend by linearity to the rest of the MRY algebra.

As noted in the definition of \bar{t} , we use generating functions to collect components of the MRY generators as above: $\alpha_j^{\bar{\sigma}}(z) = \sum_{k \in \mathbb{Z}} \alpha_j^{\bar{\sigma}}(k)z^{-k-1}$ and $X^{\bar{\sigma}}(\pm\alpha_j, z) = \sum_{k \in \mathbb{Z}} X^{\bar{\sigma}}(\pm\alpha_j, k)z^{-k-1}$ for $0 \leq j \leq n$ for a formal variable z . Then the $\bar{\pi}$ map on components can be written as power series identities; we use the same notation $\bar{\pi}$ for the map on power series. Thus the generating function version of $\bar{\pi}$ is:

- $\emptyset \mapsto 0$

- $\alpha_0^{\bar{\sigma}}(z) \mapsto \sum_{k \in \mathbb{Z}} (-h'_0 \otimes s^k - h'_0 \otimes (-s)^k) z^{-k-1}$
- $\alpha_i^{\bar{\sigma}}(z) \mapsto \sum_{k \in \mathbb{Z}} (h'_i \otimes s^k + h'_{2n-i} \otimes (-s)^k) z^{-k-1}$
- $X^{\bar{\sigma}}(\alpha_0, z) \mapsto \sum_{k \in \mathbb{Z}} (f'_0 \otimes s^k + f'_0 \otimes (-s)^k) t z^{-k-1}$
- $X^{\bar{\sigma}}(-\alpha_0, z) \mapsto \sum_{k \in \mathbb{Z}} (e'_0 \otimes s^k + e'_0 \otimes (-s)^k) t^{-1} z^{-k-1}$
- $X^{\bar{\sigma}}(\alpha_i, z) \mapsto \sum_{k \in \mathbb{Z}} (e'_i \otimes s^k + e'_{2n-i} \otimes (-s)^k) z^{-k-1}$
- $X^{\bar{\sigma}}(-\alpha_i, z) \mapsto \sum_{k \in \mathbb{Z}} (f'_i \otimes s^k + f'_{2n-i} \otimes (-s)^k) z^{-k-1}$

3.3 First Main Theorem

With this notation, we can state our first main theorem.

Theorem 19. *The pair $(\bar{t}, \bar{\pi})$ is the uce of $L(A_{2n-1}, \sigma)$ and hence is the MRY realization of $\bar{T}(A_{2n-1})$, the twisted toroidal Lie algebra of type A_{2n-1} .*

Proof: It is sufficient to show the following:

1. $(\bar{t}, \bar{\pi})$ is a central extension of $L(A_{2n-1}, \sigma)$ (that is, $\bar{\pi}$ is a surjective homomorphism and $\ker(\bar{\pi}) \subset Z(\bar{t})$, the center of \bar{t}).
2. There exists a homomorphism $\bar{\psi} : \bar{t} \rightarrow \bar{\pi}$.
3. $(\bar{t}, \bar{\pi})$ satisfies the universal property; that is, for an arbitrary central extension (\mathcal{V}, γ) of $L(A_{2n-1}, \sigma)$, the following diagram commutes:

This is sufficient because it shows that $(\bar{t}, \bar{\pi})$ is a uce of $L(A_{2n-1}, \sigma)$, which we know to be unique by the argument surrounding Figure 2.3.

Proof of (1): To see that $\bar{\pi}$ is surjective, we must show that $L(A_{2n-1}, \sigma) = \{x \otimes s^k t^m \mid k, m \in \mathbb{Z}, x \in (A_{2n-1})_{\bar{k}}\} \subset \text{im } \bar{\pi}$. It is convenient to use the component definition of $\bar{\pi}$. We split the proof into steps based on values of k and m .

By Proposition 8.3 (e) in [K], $(A_{2n-1})_{\bar{0}} \cong C_n$. We use the notation $\{\alpha_i \mid 1 \leq i \leq n\}$ for the roots of C_n . By Proposition 8.3 (b) in [K], the set $\{e'_i + e'_{2n-i}, f'_i + f'_{2n-i} \mid 1 \leq i \leq n\}$ generate C_n , and so $\alpha_i = \alpha'_i + \alpha'_{2n-i}$.

$$\begin{array}{ccc}
& \mathcal{V} & \\
\lambda \uparrow & \searrow \gamma & \\
& \overline{\omega} \rightarrow L(A_{2n-1}, \sigma) & \\
\overline{\tau} \uparrow & \nearrow \overline{\psi} & \\
\overline{t} & \nearrow \overline{\pi} &
\end{array}$$

Figure 3.1 Commutative diagram for the MRY presentation

Step 1: $k = 0, m = 0$.

$\{X^{\bar{\sigma}}(\pm\alpha_i, 0) | 1 \leq i \leq n\}$ generate all $\{x \otimes 1 | x \in C_n\}$ by Prop 8.3 (b) [K].

Step 2: $k = 0, m = \pm 1$.

Notice that $X^{\bar{\sigma}}(\alpha_0, 0) = 2f'_0 \otimes t$. Since $f'_0 \in (A_{2n-1})_{-\theta}$ and $\theta = \alpha'_1 + \dots + \alpha'_{2n-1} = \alpha_1 + \dots + \alpha_n$. Using the fact that $[(C_n)_\alpha, (C_n)_\beta] = (C_n)_{\alpha+\beta}$ for roots $\alpha, \beta, \alpha + \beta$ as in (7.8.5) in [K], and the root system of C_n (see §6.7 in [K]), we find that for each $1 \leq j \leq n$:

$$\begin{aligned}
& \left(\text{ad}X^{\bar{\sigma}}(\alpha_{j-1}, 0) \text{ad}X^{\bar{\sigma}}(\alpha_{j-2}, 0) \cdots \text{ad}X^{\bar{\sigma}}(\alpha_1, 0) \right. \\
& \cdot \left. \text{ad}X^{\bar{\sigma}}(\alpha_{j+1}, 0) \text{ad}X^{\bar{\sigma}}(\alpha_{j+2}, 0) \cdots \text{ad}X^{\bar{\sigma}}(\alpha_n, 0) \right) X^{\bar{\sigma}}(\alpha_0, 0) = (f'_j + f'_{2n-j}) \otimes t.
\end{aligned}$$

Then note $(\text{ad}X^{\bar{\sigma}}(\alpha_j, 0) \text{ad}X^{\bar{\sigma}}(\alpha_j, 0))(f'_j + f'_{2n-j}) \otimes t = -2(e'_j + e'_{2n-j}) \otimes t$. Similar computations using $X^{\bar{\sigma}}(-\alpha_i, 0), X^{\bar{\sigma}}(-\alpha_0, 0)$ yield $(f'_j + f'_{2n-j}) \otimes t^{-1}$ and $(e'_j + e'_{2n-j}) \otimes t^{-1}$.

So, if $y_1, \dots, y_p \in C_n$ are an indexed sequence of Chevalley generators such that

$[y_1, [y_2, [\dots, y_p]]] = x$, then in $L(A_{2n-1}, \sigma)$ the bracket $[y_1 \otimes 1, [y_2 \otimes 1, [\dots, y_p \otimes t^{\pm 1}]]] = x \otimes t^{\pm 1}$. Hence $C_n \otimes t^{\pm 1} \subset \text{im} \bar{\pi}$ since all generators are.

Step 3: $k = 0, m > 1$ or $m < -1$.

Use induction on m , the base case being $m = 1$ or $m = -1$, respectively. The induction step sets $m = m_0$, and repeats the same processes as in Step 2 to acquire $\pm m_0 \pm 1$. Hence, $\{C_n \otimes t^m | m \in \mathbb{Z}\} \subset \text{im} \bar{\pi}$.

Step 4: $k \in 2\mathbb{Z}, m \in \mathbb{Z}$.

For each such k , repeat Steps 1–3 beginning with $\{X^{\bar{\sigma}}(\pm\alpha_i, k) | 1 \leq i \leq n\}$. The argument shows that $\{x \otimes s^k t^m | k \in 2\mathbb{Z}, m \in \mathbb{Z}, x \in (A_{2n-1})_0\} \subset \text{im} \bar{\pi}$.

Step 5: $k = 1, m \in \mathbb{Z}$.

In contrast to Steps 1–4, for $x \otimes st^m$, we now have $x \in (A_{2n-1})_{\bar{1}}$. The element $X^{\bar{\sigma}}(\alpha_i, 1) = (e'_i - e'_{2n-i}) \otimes s$ is an example. Recall that, by Proposition 8.3 (d) [K], $(A_{2n-1})_{\bar{1}}$ is an irreducible

$(A_{2n-1})_{\bar{0}}$ -module. Thus, by applying some sequence of brackets of $X^{\bar{\sigma}}(\pm\alpha_i, 0)$, $1 \leq i \leq n$ to $X^{\bar{\sigma}}(\alpha_i, 1)$, all of $(A_{2n-1})_{\bar{1}} \otimes s$ can be retrieved.

For whatever sequence of brackets is used to retrieve $x \otimes s$ with $x \in (A_{2n-1})_{\bar{1}}$ we can replace one of the $X^{\bar{\sigma}}(\pm\alpha_i, 0) = (e'_i + e'_{2n-i}) \otimes 1$ with $(e'_i + e'_{2n-i}) \otimes t^m$ by the result of Steps 1–4. Hence, $\{x \otimes st^m \mid m \in \mathbb{Z}, x \in (A_{2n-1})_1\} \subset \text{im } \bar{\pi}$.

Step 6: $k \in 2\mathbb{Z} + 1, m \in \mathbb{Z}$.

Repeat Step 5 but beginning with $X^{\bar{\sigma}}(\pm\alpha_i, 2k+1)$, $1 \leq i \leq n$ instead of $X^{\bar{\sigma}}(\pm\alpha_i, 0)$. This shows that $\{x \otimes s^k t^m \mid k \in 2\mathbb{Z} + 1, m \in \mathbb{Z}, x \in (A_{2n-1})_1\} \subset \text{im } \bar{\pi}$.

Steps 1–6 prove that $\bar{\pi}$ is surjective.

Now we prove that $\ker(\bar{\pi}) \subset Z(\bar{t})$. This follows quickly when we note that \bar{t} can be viewed as a subalgebra of $t \cong A_{2n-1} \otimes B \oplus \frac{\Omega_B}{db} \cong T(A_{2n-1})$, the algebra defined in Definition 14. Recall also the map π from that section.

With this notation, $\bar{\pi}$ can be written in terms of π . For example, $\bar{\pi}(X^{\bar{\sigma}}(\pm\alpha_i, k)) = \pi(X(\pm\alpha'_i, k) + (-1)^k X(\pm\alpha'_{2n-i}, k))$, and similarly for the other generators. Thus, if the left-hand side of such an expression is 0, so must the right-hand side be. Therefore,

$$\ker \bar{\pi} \subset \ker \pi \subset Z(t) \subset Z(\bar{t}),$$

giving the desired inclusion. The middle inclusion is [MRY] Proposition 3.5, and the last inclusion follows from the fact that we can view \bar{t} as a subalgebra of t .

By the defining relations for \bar{t} , the $X^{\bar{\sigma}}(\pm\alpha_j, k)$, $0 \leq j \leq n$ generate \bar{t} , and since both maps are homomorphisms ($\bar{\pi}$ is proven to be a homomorphism in what follows, π is thanks to [MRY]), then for each $x \in \bar{t}$, there is some $y \in t$ such that $\bar{\pi}(x) = \pi(y)$ (in other words, though the notation is different in \bar{t} than in t , we can choose the respective elements so that their image under $\bar{\pi}$ or π , respectively, is equal). Hence $0 = \bar{\pi}(x) \Rightarrow 0 = \pi(y)$ for the x, y associated as above. So $x \in \ker \bar{\pi} \Rightarrow y \in \ker \pi$.

Since we can view $\bar{t} \subset t$ as a subalgebra (with different notation), we have shown $\ker \bar{\pi} \subset \ker \pi$. By [MRY] Proposition 3.5, $\ker \pi \subset Z(t)$, and since $\bar{t} \subset t$, then $Z(t) \subset Z(\bar{t})$. Therefore, $\ker(\bar{\pi}) \subset Z(\bar{t})$.

Notice that the elements x, y are the same in t , not just that their images are the same in the tensor product space. Our new notation for \bar{t} just renames certain elements from t for the purpose of simplifying the notation for the subalgebra \bar{t} . That is, we could define $X^{\bar{\sigma}}(\pm\alpha_i, k) = X(\pm\alpha_i, k) + (-1)^k X(\pm\alpha_{2n-i}, k)$.

Proof of (2): We begin by defining a map $\bar{\psi} : \bar{t} \rightarrow \bar{\tau}$ by: for $1 \leq i \leq n$,

$$\bullet \not\in \overline{s^{-1}ds}$$

- $\alpha_0^{\bar{\sigma}}(k) \mapsto -h'_0 \otimes s^k - h'_0 \otimes (-s)^k + \frac{1}{2} \left(\overline{s^k t^{-1} dt} + \overline{(-s)^k t^{-1} dt} \right)$
- $\alpha_i^{\bar{\sigma}}(k) \mapsto h'_i \otimes s^k + h'_{2n-i} \otimes (-s)^k$
- $X^{\bar{\sigma}}(\alpha_0, k) \mapsto f'_0 \otimes s^k t + f'_0 \otimes (-s)^k t$
- $X^{\bar{\sigma}}(-\alpha_0, k) \mapsto e'_0 \otimes s^k t^{-1} + e'_0 \otimes (-s)^k t^{-1}$
- $X^{\bar{\sigma}}(\alpha_i, k) \mapsto e'_i \otimes s^k + e'_{2n-i} \otimes (-s)^k$
- $X^{\bar{\sigma}}(-\alpha_i, k) \mapsto f'_i \otimes s^k + f'_{2n-i} \otimes (-s)^k$

The generating function version of $\bar{\psi}$ is:

- $\not\in \mapsto \overline{s^{-1} ds}$
- $\alpha_0^{\bar{\sigma}}(z) \mapsto \sum_{k \in \mathbb{Z}} \left(-h'_0 \otimes s^k - h'_0 \otimes (-s)^k + \frac{1}{2} \left(\overline{s^k t^{-1} dt} + \overline{(-s)^k t^{-1} dt} \right) \right) z^{-k-1}$
- $\alpha_i^{\bar{\sigma}}(z) \mapsto \sum_{k \in \mathbb{Z}} \left(h'_i \otimes s^k + h'_{2n-i} \otimes (-s)^k \right) z^{-k-1}$
- $X^{\bar{\sigma}}(\alpha_0, z) \mapsto \sum_{k \in \mathbb{Z}} \left(f'_0 \otimes s^k + f'_0 \otimes (-s)^k \right) t z^{-k-1}$
- $X^{\bar{\sigma}}(-\alpha_0, z) \mapsto \sum_{k \in \mathbb{Z}} \left(e'_0 \otimes s^k + e'_0 \otimes (-s)^k \right) t^{-1} z^{-k-1}$
- $X^{\bar{\sigma}}(\alpha_i, z) \mapsto \sum_{k \in \mathbb{Z}} \left(e'_i \otimes s^k + e'_{2n-i} \otimes (-s)^k \right) z^{-k-1}$
- $X^{\bar{\sigma}}(-\alpha_i, z) \mapsto \sum_{k \in \mathbb{Z}} \left(f'_i \otimes s^k + f'_{2n-i} \otimes (-s)^k \right) z^{-k-1}$

Notice that $\bar{\psi}$ and $\bar{\pi}$ agree except on $\not\in$ and $\alpha_0^{\bar{\sigma}}(z)$, and that the “new” terms are indeed elements of \mathcal{K}^{σ} because the sums of the exponents on s are even (in $\alpha_0^{\bar{\sigma}}(z)$, if k is odd, $\overline{s^k t^{-1} dt} + \overline{(-s)^k t^{-1} dt} = \overline{s^k t^{-1} dt} - \overline{s^k t^{-1} dt} = 0$.

It is clear from their definitions that $\bar{\pi} = \bar{\psi}\omega$. In particular, proving that $\bar{\psi}$ is a homomorphism is sufficient to prove that $\bar{\pi}$ is a homomorphism as well. We remark also that $\bar{\sigma}$, the automorphism on $L(A_{2n-1}, \sigma)$, does indeed fix the (components of the) images under $\bar{\psi}$ of $\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_i, z)$ with the “twist variable” being s . For example, $\bar{\sigma}(\alpha_i^{\bar{\sigma}}(k)) = \bar{\sigma}(h'_i \otimes s^k + h'_{2n-i} \otimes (-s)^k) = (-1)^k \sigma(h'_i) \otimes s^k + (-1)^k \sigma(h'_{2n-i}) \otimes (-s)^k = h'_{2n-i} \otimes (-s)^k + h'_i \otimes s^k = \alpha_i^{\bar{\sigma}}(k)$ where the latter equality follows since $(-1)^{-k} = (-1)^k \forall k \in \mathbb{Z}$. Thus, $\alpha_i^{\bar{\sigma}}(z)$ is fixed for $1 \leq i \leq n$. Similar calculations show that $\bar{\sigma}$ fixes $X^{\bar{\sigma}}(\pm\alpha_i, z)$ for $1 \leq i \leq n$ as well.

Proving $\bar{\psi}$ is a homomorphism requires that we show the MRY relations hold on either side of $\bar{\psi}$ being applied. For example, $[\bar{\psi}(\alpha_i^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(\pm\alpha_n, w))] = \bar{\psi}([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_n, w)])$, or rather, since we know the bracket $[\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_n, w)]$ by definition,

$$[\bar{\psi}(\alpha_i^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(\pm\alpha_n, w))] = \pm a_{in} \bar{\psi}(X^{\bar{\sigma}}(\pm\alpha_n, w)) (\delta(z-w) - \delta(z+w)).$$

We will show each relation according to its number in the list of MRY relations above.

Relation (1): $[\alpha_0^{\bar{\sigma}}(z), \alpha_m^{\bar{\sigma}}(w)] = (2\delta_{m0} - \delta_{m1})(\partial_w \delta(z-w) + \partial_w \delta(z+w))\phi$ for $0 \leq m \leq n$.

First consider the case $m = 0$.

$$\begin{aligned} & [\bar{\psi}(\alpha_0^{\bar{\sigma}}(z)), \bar{\psi}(\alpha_0^{\bar{\sigma}}(w))] \\ &= \left[\sum_{k \in \mathbb{Z}} \left(-h'_0 \otimes s^k - h'_0 \otimes (-s)^k + \frac{1}{2} \left(\overline{s^k t^{-1} dt} + \overline{(-s)^k t^{-1} dt} \right) \right) z^{-k-1}, \right. \\ & \quad \left. \sum_{\ell \in \mathbb{Z}} \left(-h'_0 \otimes s^\ell - h'_0 \otimes (-s)^\ell + \frac{1}{2} \left(\overline{s^\ell t^{-1} dt} + \overline{(-s)^\ell t^{-1} dt} \right) \right) w^{-\ell-1} \right] \\ &= \sum_{k, \ell \in \mathbb{Z}} \left([h'_0, h'_0]' \otimes s^{k+\ell} + (h'_0 | h'_0)' \overline{s^\ell ds^k} \right) z^{-k-1} w^{-\ell-1} \\ &+ \sum_{k, \ell \in \mathbb{Z}} \left([h'_0, h'_0]' \otimes s^{k+\ell} + (h'_0 | h'_0)' \overline{s^\ell ds^k} \right) (-1)^\ell z^{-k-1} w^{-\ell-1} \\ &+ \sum_{k, \ell \in \mathbb{Z}} \left([h'_0, h'_0]' \otimes s^{k+\ell} + (h'_0 | h'_0)' \overline{s^\ell ds^k} \right) (-1)^k z^{-k-1} w^{-\ell-1} \\ &+ \sum_{k, \ell \in \mathbb{Z}} \left([h'_0, h'_0]' \otimes s^{k+\ell} + (h'_0 | h'_0)' \overline{s^\ell ds^k} \right) (-1)^{k+\ell} z^{-k-1} w^{-\ell-1} \\ &= \sum_{k, \ell \in \mathbb{Z}} (h'_0 | h'_0)' \overline{s^\ell ds^k} (1 + (-1)^\ell + (-1)^k + (-1)^{\ell+k}) z^{-k-1} w^{-\ell-1}. \end{aligned}$$

As shown in Example 10, $\overline{s^\ell ds^k} = \delta_{k, -\ell} k \overline{s^{\ell+k-1} ds}$, so:

$$\begin{aligned} & (h'_0 | h'_0)' \sum_{k, \ell \in \mathbb{Z}} \delta_{k, -\ell} k \overline{s^{\ell+k-1} ds} (1 + (-1)^\ell + (-1)^k + (-1)^{\ell+k}) z^{-k-1} w^{-\ell-1} \\ &= (h'_0 | h'_0)' \sum_k k \overline{s^{-1} ds} (1 + (-1)^{-k} + (-1)^k + (-1)^0) z^{-k-1} w^{k-1} \\ &= 2(h'_0 | h'_0)' \left(\sum_k k z^{-k-1} w^{k-1} + \sum_k -k z^{-k-1} (-w)^{k-1} \right) \overline{s^{-1} ds} \\ &= 2 \frac{1}{2} a_{00} (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \bar{\psi}(\phi) \\ &= 2 (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \bar{\psi}(\phi) \\ &= \bar{\psi}([\alpha_0^{\bar{\sigma}}(z), \alpha_0^{\bar{\sigma}}(w)]). \end{aligned}$$

Now consider the case $1 \leq m \leq n$.

$$\begin{aligned} & [\bar{\psi}(\alpha_0^{\bar{\sigma}}(z)), \bar{\psi}(\alpha_m^{\bar{\sigma}}(w))] \\ &= \left[\sum_{k \in \mathbb{Z}} \left(-h'_0 \otimes s^k - h'_0 \otimes (-s)^k + \frac{1}{2} \left(\overline{s^k t^{-1} dt} + \overline{(-s)^k t^{-1} dt} \right) \right) z^{-k-1}, \right. \\ & \quad \left. \sum_{\ell \in \mathbb{Z}} (h'_m \otimes s^\ell + h'_{2n-m} \otimes (-s)^\ell) w^{-\ell-1} \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{k,\ell \in \mathbb{Z}} (-[h'_0, h'_m]' \otimes s^{k+\ell} - (h'_0|h'_m)' \overline{s^\ell ds^k}) z^{-k-1} w^{-\ell-1} \\
&+ \sum_{k,\ell \in \mathbb{Z}} (-[h'_0, h'_{2n-m}]' \otimes s^{k+\ell} - (h'_0|h'_{2n-m})' \overline{s^\ell ds^k}) (-1)^\ell z^{-k-1} w^{-\ell-1} \\
&+ \sum_{k,\ell \in \mathbb{Z}} (-[h'_0, h'_m]' \otimes s^{k+\ell} - (h'_0|h'_m)' \overline{s^\ell ds^k}) (-1)^k z^{-k-1} w^{-\ell-1} \\
&+ \sum_{k,\ell \in \mathbb{Z}} (-[h'_0, h'_{2n-m}]' \otimes s^{k+\ell} - (h'_0|h'_{2n-m})' \overline{s^\ell ds^k}) (-1)^{k+\ell} z^{-k-1} w^{-\ell-1} \\
&= - \sum_{k,\ell \in \mathbb{Z}} ((h'_0|h'_m)' \overline{s^\ell ds^k} (1 + (-1)^k) + (h'_0|h'_{2n-m})' \overline{s^\ell ds^k} ((-1)^\ell + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

As shown above, $(h'_0|h'_m)' = \frac{1}{2}\delta_{1m} = (h'_0|h'_{2n-m})'$, and from Example 10, $\overline{s^\ell ds^k} = \delta_{k,-\ell} k s^{\ell+k-1} ds$ so:

$$\begin{aligned}
&= -\frac{1}{2}\delta_{1m} \sum_{k,\ell \in \mathbb{Z}} k \delta_{k,-\ell} \overline{s^{-1} ds} (1 + (-1)^\ell + (-1)^k + (-1)^{\ell+k}) z^{-k-1} w^{-\ell-1} \\
&= -\frac{1}{2}\delta_{1m} \sum_{k \in \mathbb{Z}} k z^{-k-1} w^{k-1} (2 + 2(-1)^k) \overline{s^{-1} ds} \\
&= -\delta_{1m} \left(\sum_k k z^{-k-1} w^{k-1} + \sum_k -k z^{-k-1} (-w)^{k-1} \right) \overline{s^{-1} ds} \\
&= -\delta_{1m} (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \overline{\psi}(\phi) \\
&= -\delta_{1m} (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \overline{\psi}(\phi) \\
&= \overline{\psi}([\alpha_0^{\bar{\sigma}}(z), \alpha_m^{\bar{\sigma}}(w)]).
\end{aligned}$$

Relation (2): $[\alpha_i^{\bar{\sigma}}(z), \alpha_j^{\bar{\sigma}}(w)] = a_{ij} \partial_w \delta(z-w) \phi$ for $i \leq i, j \leq n-1$

$$\begin{aligned}
&[\overline{\psi}(\alpha_i^{\bar{\sigma}}(z)), \overline{\psi}(\alpha_j^{\bar{\sigma}}(w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (h'_i \otimes s^k + h'_{2n-i} \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (h'_j \otimes s^\ell + h'_{2n-j} \otimes (-s)^\ell) w^{-\ell-1} \right] \\
&= \sum_{k,\ell \in \mathbb{Z}} ([h'_i, h'_j]' \otimes s^{k+\ell} + (h'_i|h'_j)' \overline{s^\ell ds^k}) z^{-k-1} w^{-\ell-1} \\
&+ \sum_{k,\ell \in \mathbb{Z}} ([h'_i, h'_{2n-j}]' \otimes s^{k+\ell} + (h'_i|h'_{2n-j})' \overline{s^\ell ds^k}) (-1)^\ell z^{-k-1} w^{-\ell-1} \\
&+ \sum_{k,\ell \in \mathbb{Z}} ([h'_{2n-i}, h'_j]' \otimes s^{k+\ell} + (h'_{2n-i}|h'_j)' \overline{s^\ell ds^k}) (-1)^k z^{-k-1} w^{-\ell-1} \\
&+ \sum_{k,\ell \in \mathbb{Z}} ([h'_{2n-i}, h'_{2n-j}]' \otimes s^{k+\ell} + (h'_{2n-i}|h'_{2n-j})' \overline{s^\ell ds^k}) (-1)^{k+\ell} z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

We make a few remarks on the calculation:

1. All brackets of the form $[h'_p, h'_q]' = 0$
2. Since $1 \leq i, j \leq n-1$ and $(h'_p|h'_q)'$ is nonzero if and only if $j-1 \leq i \leq j+1$, then $(h'_i|h'_{2n-j})' = 0 = (h'_{2n-i}|h'_j)'$ (the i, j nodes and $2n-i, 2n-j$ nodes are “too far away.”)

$$3. (h'_i|h'_j)' = (h'_{2n-i}|h'_{2n-j})' = \frac{1}{2}a_{ij}$$

These remarks allow us to simplify the calculation as follows:

$$= \frac{1}{2}a_{ij} \sum_{k,\ell \in \mathbb{Z}} \overline{s^\ell ds^k} (1 + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1} = \frac{1}{2}a_{ij} \sum_{k,\ell \in \mathbb{Z}} \delta_{k,-\ell} k \overline{s^{\ell+k-1} ds} (1 + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}$$

since as shown in Example 10, $\overline{s^\ell ds^k} = \delta_{k,-\ell} k \overline{s^{\ell+k-1} ds}$.

Using the implications of $\delta_{k,-\ell}$ allows us to simplify as follows:

$$\begin{aligned} &= \frac{1}{2}a_{ij} \sum_{k \in \mathbb{Z}} k \overline{s^{-1} ds} (1 + (-1)^0) z^{-k-1} w^{k-1} = \frac{1}{2}a_{ij} \sum_{k \in \mathbb{Z}} 2k z^{-k-1} w^{k-1} \overline{s^{-1} ds} \\ &= a_{ij} \sum_{k \in \mathbb{Z}} k z^{-k-1} w^{k-1} \overline{s^{-1} ds} = a_{ij} \partial_w \delta(z - w) \overline{\psi}(\phi) \\ &= \overline{\psi}([\alpha_i^{\bar{\sigma}}(z), \alpha_j^{\bar{\sigma}}(w)]). \end{aligned}$$

$$\begin{aligned} \textbf{Relation (3): } & [\alpha_i^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(w)] = a_{in} (\partial_w \delta(z - w) + \partial_w \delta(z + w)) \phi \\ & [\overline{\psi}(\alpha_i^{\bar{\sigma}}(z)), \overline{\psi}(\alpha_n^{\bar{\sigma}}(w))] \\ &= \left[\sum_{k \in \mathbb{Z}} (h'_i \otimes s^k + h'_{2n-i} \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (h'_n \otimes s^\ell + h'_n \otimes (-s)^\ell) w^{-\ell-1} \right] \\ &= \sum_{k,\ell \in \mathbb{Z}} ([h'_i, h'_n]' \otimes s^{k+\ell} + (h'_i|h'_n)' \overline{s^\ell ds^k}) z^{-k-1} w^{-\ell-1} \\ &+ \sum_{k,\ell \in \mathbb{Z}} ([h'_i, h'_n]' \otimes s^{k+\ell} + (h'_i|h'_n)' \overline{s^\ell ds^k}) (-1)^\ell z^{-k-1} w^{-\ell-1} \\ &+ \sum_{k,\ell \in \mathbb{Z}} ([h'_{2n-i}, h'_n]' \otimes s^{k+\ell} + (h'_{2n-i}|h'_n)' \overline{s^\ell ds^k}) (-1)^k z^{-k-1} w^{-\ell-1} \\ &+ \sum_{k,\ell \in \mathbb{Z}} ([h'_{2n-i}, h'_n]' \otimes s^{k+\ell} + (h'_{2n-i}|h'_n)' \overline{s^\ell ds^k}) (-1)^{k+\ell} z^{-k-1} w^{-\ell-1}. \end{aligned}$$

As we know from the bracket definition and Definition 3, $[h'_i, h'_n]' = 0 = [h'_{2n-i}, h'_n]'$ and

$$(h'_i|h'_n)' = \frac{1}{2}a_{in} = \frac{1}{2}a_{2n-i,n} = (h'_{2n-i}|h'_n)'. \text{ Hence,}$$

$$\begin{aligned} &= \frac{1}{2}a_{in} \sum_{k,\ell \in \mathbb{Z}} \overline{s^\ell ds^k} (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1} \\ &= \frac{1}{2}a_{in} \sum_{k,\ell \in \mathbb{Z}} \delta_{k,-\ell} k \overline{s^{\ell+k-1} ds} (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1} \\ &= \frac{1}{2}a_{in} \sum_{k \in \mathbb{Z}} k \overline{s^{-1} ds} (1 + (-1)^{-k} + (-1)^k + (-1)^0) z^{-k-1} w^{k-1}. \end{aligned}$$

Since $(-1)^{-k} = (-1)^k = -(-1)^{k-1}$ for all $k \in \mathbb{Z}$, we have:

$$\begin{aligned} &= \frac{1}{2}a_{in} \sum_{k \in \mathbb{Z}} k z^{-k-1} w^{k-1} (2 + 2(-1)^k) \overline{s^{-1} ds} \\ &= 2 \frac{1}{2}a_{in} \left(\sum_{k \in \mathbb{Z}} k z^{-k-1} w^{k-1} + \sum_{k \in \mathbb{Z}} k z^{-k-1} w^{k-1} (-1)^k \right) \overline{s^{-1} ds} \end{aligned}$$

$$\begin{aligned}
&= 2 \frac{1}{2} a_{in} \left(\sum_{k \in \mathbb{Z}} k z^{-k-1} w^{k-1} + \sum_{k \in \mathbb{Z}} -k z^{-k-1} (-w)^{k-1} \right) \overline{s^{-1} ds} \\
&= a_{in} (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \overline{\psi(\phi)} \\
&= \overline{\psi}([\alpha_i^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(w)]).
\end{aligned}$$

$$\begin{aligned}
\textbf{Relation (4): } & [\alpha_n^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(w)] = a_{nn} (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \phi \\
& [\overline{\psi}(\alpha_n^{\bar{\sigma}}(z)), \overline{\psi}(\alpha_n^{\bar{\sigma}}(w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (h'_n \otimes s^k + h'_n \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (h'_n \otimes s^\ell + h'_n \otimes (-s)^\ell) w^{-\ell-1} \right] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([h'_n, h'_n]' \otimes s^{k+\ell} + (h'_n | h'_n)' \overline{s^\ell ds^k}) z^{-k-1} w^{-\ell-1} \\
&+ \sum_{k, \ell \in \mathbb{Z}} ([h'_n, h'_n]' \otimes s^{k+\ell} + (h'_n | h'_n)' \overline{s^\ell ds^k}) (-1)^\ell z^{-k-1} w^{-\ell-1} \\
&+ \sum_{k, \ell \in \mathbb{Z}} ([h'_n, h'_n]' \otimes s^{k+\ell} + (h'_n | h'_n)' \overline{s^\ell ds^k}) (-1)^k z^{-k-1} w^{-\ell-1} \\
&+ \sum_{k, \ell \in \mathbb{Z}} ([h'_n, h'_n]' \otimes s^{k+\ell} + (h'_n | h'_n)' \overline{s^\ell ds^k}) (-1)^{k+\ell} z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

As we know from the bracket definition and Definition 3, $[h'_n, h'_n]' = 0$ and $(h'_n | h'_n)'$

$$\begin{aligned}
&= \frac{1}{2} a_{nn} = 1. \text{ Hence,} \\
&= \sum_{k, \ell \in \mathbb{Z}} \overline{s^\ell ds^k} (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1} \\
&= \sum_{k, \ell \in \mathbb{Z}} \delta_{k, -\ell} k \overline{s^{\ell+k-1} ds} (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1} \\
&= \sum_{k \in \mathbb{Z}} k \overline{s^{-1} ds} (1 + (-1)^{-k} + (-1)^k + (-1)^0) z^{-k-1} w^{k-1}.
\end{aligned}$$

Since $(-1)^{-k} = (-1)^k = -(-1)^{k-1}$ for all $k \in \mathbb{Z}$, we have:

$$\begin{aligned}
&= \sum_{k \in \mathbb{Z}} k z^{-k-1} w^{k-1} (2 + 2(-1)^k) \overline{s^{-1} ds} = 2 \left(\sum_{k \in \mathbb{Z}} k z^{-k-1} w^{k-1} + \sum_{k \in \mathbb{Z}} k z^{-k-1} w^{k-1} (-1)^k \right) \overline{s^{-1} ds} \\
&= 2 \left(\sum_{k \in \mathbb{Z}} k z^{-k-1} w^{k-1} + \sum_{k \in \mathbb{Z}} -k z^{-k-1} (-w)^{k-1} \right) \overline{s^{-1} ds} \\
&= a_{nn} (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \overline{\psi(\phi)} \\
&= \overline{\psi}([\alpha_n^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(w)]).
\end{aligned}$$

Relation (5): $[\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm \alpha_m, w)] = \pm(2\delta_{m0} - \delta_{m1}) X^{\bar{\sigma}}(\pm \alpha_m, w) (\delta(z-w) + \delta(z+w))$

First we compute the case where $m = 0$ and for $X^{\bar{\sigma}}(\alpha_m, w)$.

$$\begin{aligned}
& [\overline{\psi}(\alpha_0^{\bar{\sigma}}(z)), \overline{\psi}(X^{\bar{\sigma}}(\alpha_0, w))] \\
&= \left[\sum_{k \in \mathbb{Z}} \left(-h'_0 \otimes s^k - h'_0 \otimes (-s)^k + \frac{1}{2} \left(\overline{s^k t^{-1} dt} + \overline{(-s)^k t^{-1} dt} \right) \right) z^{-k-1}, \right.
\end{aligned}$$

$$\sum_{\ell \in \mathbb{Z}} (f'_0 \otimes s^\ell + f'_0 \otimes (-s)^\ell) t w^{-\ell-1}.$$

Since the $\overline{s^k t^{-1} dt} + \overline{(-s)^k t^{-1} dt}$ terms commute with everything, we have:

$$\begin{aligned} &= - \sum_{k, \ell \in \mathbb{Z}} ([h'_0, f'_0]' \otimes s^{k+\ell} t (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1} \\ &= - \sum_{k, \ell \in \mathbb{Z}} (-\theta(h'_0) f'_0 \otimes s^{k+\ell} t (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1}. \end{aligned}$$

As shown above, $\theta(h'_0) = 2$. Also we replace the indices with $k' = k + \ell$ and $\ell' = k$ so that $\ell = k' - \ell'$ and use $(-1)^{k'-\ell'} = (-1)^{k'+\ell'}$ to get:

$$\begin{aligned} &= 2 \sum_{k', \ell' \in \mathbb{Z}} (f'_0 \otimes s^{k'} t (1 + (-1)^{k'-\ell'} + (-1)^{\ell'} + (-1)^{k'})) z^{-\ell'-1} w^{-k'+\ell'-1} \\ &= 2 \left\{ \sum_{k' \in \mathbb{Z}} (f'_0 \otimes s^{k'} t + f'_0 \otimes (-s)^{k'} t) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \right. \\ &\quad \left. + \sum_{k' \in \mathbb{Z}} (f'_0 \otimes s^{k'} t + f'_0 \otimes (-s)^{k'} t) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \right\} \\ &= 2\bar{\psi}(X^{\bar{\sigma}}(\alpha_0, w)) (\delta(z-w) + \delta(z+w)) \\ &= \bar{\psi}([\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_0, w)]). \end{aligned}$$

Similarly, $[\bar{\psi}(\alpha_0^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(-\alpha_0, w))]$

$$\begin{aligned} &= \left[\sum_{k \in \mathbb{Z}} (-h'_0 \otimes s^k - h'_0 \otimes (-s)^k + \frac{1}{2} (\overline{s^k t^{-1} dt} + \overline{(-s)^k t^{-1} dt})) z^{-k-1}, \right. \\ &\quad \left. \sum_{\ell \in \mathbb{Z}} (e'_0 \otimes s^\ell + e'_0 \otimes (-s)^\ell) t^{-1} w^{-\ell-1} \right]. \end{aligned}$$

Since the $\overline{s^k t^{-1} dt} + \overline{(-s)^k t^{-1} dt}$ terms commute with everything, we have:

$$\begin{aligned} &= - \sum_{k, \ell \in \mathbb{Z}} ([h'_0, e'_0]' \otimes s^{k+\ell} t^{-1} (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1} \\ &= - \sum_{k, \ell \in \mathbb{Z}} (\theta(h'_0) e'_0 \otimes s^{k+\ell} t^{-1} (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1}. \end{aligned}$$

As shown above, $\theta(h'_0) = 2$. Also we replace the indices with $k' = k + \ell$ and $\ell' = k$ so that $\ell = k' - \ell'$ and use $(-1)^{k'-\ell'} = (-1)^{k'+\ell'}$ to get:

$$\begin{aligned} &= -2 \sum_{k', \ell' \in \mathbb{Z}} (e'_0 \otimes s^{k'} t^{-1} (1 + (-1)^{k'-\ell'} + (-1)^{\ell'} + (-1)^{k'})) z^{-\ell'-1} w^{-k'+\ell'-1} \\ &= -2 \left\{ \sum_{k' \in \mathbb{Z}} (e'_0 \otimes s^{k'} t^{-1} + e'_0 \otimes (-s)^{k'} t^{-1}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \right. \\ &\quad \left. + \sum_{k' \in \mathbb{Z}} (e'_0 \otimes s^{k'} t^{-1} + e'_0 \otimes (-s)^{k'} t^{-1}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \right\} \\ &= -2\bar{\psi}(X^{\bar{\sigma}}(-\alpha_0, w)) (\delta(z-w) + \delta(z+w)) \\ &= \bar{\psi}([\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_0, w)]). \end{aligned}$$

For the remaining cases in this relation, let $1 \leq m \leq n$. First we compute for $X^{\bar{\sigma}}(\alpha_m, w)$.

$$\begin{aligned}
& [\bar{\psi}(\alpha_0^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(\alpha_m, w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (-h'_0 \otimes s^k - h'_0 \otimes (-s)^k + \frac{1}{2} (\overline{s^k t^{-1} dt} + \overline{(-s)^k t^{-1} dt}) \right) z^{-k-1}, \\
&\quad \sum_{\ell \in \mathbb{Z}} (e'_m \otimes s^\ell + e'_{2n-m} \otimes (-s)^\ell) w^{-\ell-1} \right] \\
&= - \sum_{k, \ell \in \mathbb{Z}} ([h'_0, e'_m]' \otimes s^{k+\ell} (1 + (-1)^k) + [h'_0, e'_{2n-m}]' \otimes s^{k+\ell} ((-1)^\ell + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1} \\
&= - \sum_{k, \ell \in \mathbb{Z}} (\alpha'_m(h'_0) e'_m \otimes s^{k+\ell} (1 + (-1)^k) + \alpha'_{2n-m}(h'_0) e'_{2n-m} \otimes s^{k+\ell} ((-1)^\ell + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

As shown above, $\alpha'_m(h'_0) = \delta_{m1} = \alpha'_{2n-m}(h'_0)$, so

$$= -\delta_{m1} \sum_{k, \ell \in \mathbb{Z}} (e'_m \otimes s^{k+\ell} (1 + (-1)^k) + e'_{2n-m} \otimes s^{k+\ell} ((-1)^\ell + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1}.$$

Now replace indices with $k' = k + \ell$ and $\ell' = k$ so that $\ell = k' - \ell'$ and:

$$= -\delta_{m1} \sum_{k', \ell' \in \mathbb{Z}} (e'_m \otimes s^{k'} (1 + (-1)^{\ell'}) + e'_{2n-m} \otimes s^{k'} ((-1)^{k'-\ell'} + (-1)^{k'})) z^{-\ell'-1} w^{-k'+\ell'-1}.$$

And since $(-1)^{k'-\ell'} = (-1)^{k'+\ell'}$,

$$\begin{aligned}
&= -\delta_{m1} \left\{ \sum_{k' \in \mathbb{Z}} (e'_m \otimes s^{k'} + e'_{2n-m} \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \right. \\
&\quad \left. + \sum_{k' \in \mathbb{Z}} (e'_m \otimes s^{k'} + e'_{2n-m} \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \right\} \\
&= -\delta_{m1} \bar{\psi}(X^{\bar{\sigma}}(\alpha_m, w)) (\delta(z-w) + \delta(z+w)) \\
&= \bar{\psi}([\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_m, w)]).
\end{aligned}$$

Similarly,

$$\begin{aligned}
& [\bar{\psi}(\alpha_0^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(-\alpha_m, w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (-h'_0 \otimes s^k - h'_0 \otimes (-s)^k + \frac{1}{2} (\overline{s^k t^{-1} dt} + \overline{(-s)^k t^{-1} dt}) \right) z^{-k-1}, \\
&\quad \sum_{\ell \in \mathbb{Z}} (f'_m \otimes s^\ell + f'_{2n-m} \otimes (-s)^\ell) w^{-\ell-1} \right] \\
&= - \sum_{k, \ell \in \mathbb{Z}} ([h'_0, f'_m]' \otimes s^{k+\ell} (1 + (-1)^k) + [h'_0, f'_{2n-m}]' \otimes s^{k+\ell} ((-1)^\ell + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1} \\
&= - \sum_{k, \ell \in \mathbb{Z}} (-\alpha'_m(h'_0) f'_m \otimes s^{k+\ell} (1 + (-1)^k) - \alpha'_{2n-m}(h'_0) f'_{2n-m} \otimes s^{k+\ell} ((-1)^\ell + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

As shown above, $\alpha'_m(h'_0) = \delta_{m1} = \alpha'_{2n-m}(h'_0)$, so

$$= \delta_{m1} \sum_{k, \ell \in \mathbb{Z}} (f'_m \otimes s^{k+\ell} (1 + (-1)^k) + f'_{2n-m} \otimes s^{k+\ell} ((-1)^\ell + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1}.$$

Now replace indices with $k' = k + \ell$ and $\ell' = k$ so that $\ell = k' - \ell'$ and:

$$= \delta_{m1} \sum_{k', \ell' \in \mathbb{Z}} (f'_m \otimes s^{k'} (1 + (-1)^{\ell'}) + f'_{2n-m} \otimes s^{k'} ((-1)^{k'-\ell'} + (-1)^{k'})) z^{-\ell'-1} w^{-k'+\ell'-1}.$$

And since $(-1)^{k'-\ell'} = (-1)^{k'+\ell'}$,

$$\begin{aligned}
&= \delta_{m1} \left\{ \sum_{k' \in \mathbb{Z}} (f'_m \otimes s^{k'} + f'_{2n-m} \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \right. \\
&\quad \left. + \sum_{k' \in \mathbb{Z}} (f'_m \otimes s^{k'} + f'_{2n-m} \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \right\} \\
&= \delta_{m1} \bar{\psi}(X^{\bar{\sigma}}(-\alpha_m, w)) (\delta(z-w) + \delta(z+w)) \\
&= \bar{\psi}([\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_m, w)]).
\end{aligned}$$

Relation (6): $[\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_0, w)] = \pm(-\delta_{i1}) X^{\bar{\sigma}}(\pm\alpha_0, w) (\delta(z-w) + \delta(z+w))$

First we compute for $X^{\bar{\sigma}}(\alpha_0, w)$.

$$\begin{aligned}
&[\bar{\psi}(\alpha_i^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(\alpha_0, w))] \\
&= [\sum_{k \in \mathbb{Z}} (h'_i \otimes s^k + h'_{2n-i} \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_0 \otimes s^\ell + f'_0 \otimes (-s)^\ell) t w^{-\ell-1}] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([h'_i, f'_0]' \otimes s^{k+\ell} (1 + (-1)^\ell) + [h'_{2n-i}, f'_0]' \otimes s^{k+\ell} ((-1)^k + (-1)^{k+\ell})) t z^{-k-1} w^{-\ell-1}. \\
\text{Since } &[h'_i, f'_0]' = -\theta(h'_i) f'_0 \text{ and } [h'_{2n-i}, f'_0]' = -\theta(h'_{2n-i}) f'_0 \text{ so we have:} \\
&= \sum_{k, \ell \in \mathbb{Z}} (-\theta(h'_i) f'_0 \otimes s^{k+\ell} (1 + (-1)^\ell) - \theta(h'_{2n-i}) f'_0 \otimes s^{k+\ell} ((-1)^k + (-1)^{k+\ell})) t z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

Also since $\theta(h'_i) = \delta_{i1} = \theta(h'_{2n-i})$, so the calculation above collapses to:

$$-\delta_{i1} \sum_{k, \ell \in \mathbb{Z}} (f'_0 \otimes s^{k+\ell} (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell})) t z^{-k-1} w^{-\ell-1}.$$

Now, replace indices $k' = k + \ell$ and $\ell' = k$ to get:

$$-\delta_{i1} \sum_{k', \ell' \in \mathbb{Z}} (f'_0 \otimes s^{k'} (1 + (-1)^{k'-\ell'} + (-1)^{\ell'} + (-1)^{k'})) t z^{-\ell'-1} w^{-k'+\ell'-1}.$$

Since $(-1)^{k'-\ell'} = (-1)^{k'+\ell'}$, we have:

$$\begin{aligned}
&-\delta_{i1} \sum_{k' \in \mathbb{Z}} (f'_0 \otimes s^{k'} + f'_0 \otimes (-s)^{k'}) t w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \\
&+ \sum_{k' \in \mathbb{Z}} (f'_0 \otimes s^{k'} + f'_0 \otimes (-s)^{k'}) t w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \\
&= -\delta_{i1} \bar{\psi}(X^{\bar{\sigma}}(\alpha_0, w)) (\delta(z-w) + \delta(z+w)) \\
&= \bar{\psi}([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_0, w)]).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&[\bar{\psi}(\alpha_i^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(-\alpha_0, w))] \\
&= [\sum_{k \in \mathbb{Z}} (h'_i \otimes s^k + h'_{2n-i} \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_0 \otimes s^\ell + e'_0 \otimes (-s)^\ell) t^{-1} w^{-\ell-1}] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([h'_i, e'_0]' \otimes s^{k+\ell} (1 + (-1)^\ell) + [h'_{2n-i}, e'_0]' \otimes s^{k+\ell} ((-1)^k + (-1)^{k+\ell})) t^{-1} z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

Since, $[h'_i, e'_0]' = \theta(h'_i) e'_0$ and $[h'_{2n-i}, e'_0]' = \theta(h'_{2n-i}) e'_0$ so we have:

$$\sum_{k, \ell \in \mathbb{Z}} (\theta(h'_i) e'_0 \otimes s^{k+\ell} (1 + (-1)^\ell) + \theta(h'_{2n-i}) e'_0 \otimes s^{k+\ell} ((-1)^k + (-1)^{k+\ell})) t^{-1} z^{-k-1} w^{-\ell-1}.$$

Also since $\theta(h'_i) = \delta_{i1} = \theta(h'_{2n-i})$, so the calculation above collapses to:

$$= \delta_{i1} \sum_{k,\ell \in \mathbb{Z}} (e'_0 \otimes s^{k+\ell} (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell})) t^{-1} z^{-k-1} w^{-\ell-1}.$$

Now, replace indices $k' = k + \ell$ and $\ell' = k$ to get:

$$= \delta_{i1} \sum_{k',\ell' \in \mathbb{Z}} (e'_0 \otimes s^{k'} (1 + (-1)^{k'-\ell'} + (-1)^{\ell'} + (-1)^{k'})) t^{-1} z^{-\ell'-1} w^{-k'+\ell'-1}.$$

Since $(-1)^{k'-\ell'} = (-1)^{k'+\ell'}$, we have:

$$\begin{aligned} &= \delta_{i1} \sum_{k' \in \mathbb{Z}} (e'_0 \otimes s^{k'} + e'_0 \otimes (-s)^{k'}) t^{-1} w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \\ &\quad + \sum_{k' \in \mathbb{Z}} (e'_0 \otimes s^{k'} + e'_0 \otimes (-s)^{k'}) t^{-1} w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \\ &= \delta_{i1} \bar{\psi}(X^{\bar{\sigma}}(-\alpha_0, w)) (\delta(z - w) + \delta(z + w)) \\ &= \bar{\psi}([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_0, w)]). \end{aligned}$$

Relation (7): $[\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_j, w)] = \pm a_{ij} X^{\bar{\sigma}}(\pm\alpha_j, w) \delta(z - w)$

First we compute for $X^{\bar{\sigma}}(\alpha_j, w)$.

$$\begin{aligned} &[\bar{\psi}(\alpha_i^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(\alpha_j, w))] \\ &= [\sum_{k \in \mathbb{Z}} (h'_i \otimes s^k + h'_{2n-j} \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_j \otimes s^\ell + e'_{2n-j} \otimes (-s)^\ell) w^{-\ell-1}] \\ &= \sum_{k,\ell \in \mathbb{Z}} ([h'_i, e'_j]' \otimes s^{k+\ell} + [h'_i, e'_{2n-j}]' \otimes s^{k+\ell} (-1)^\ell \\ &\quad + [h'_{2n-i}, e'_j]' \otimes s^{k+\ell} (-1)^k + [h'_{2n-i}, e'_{2n-j}]' \otimes s^{k+\ell} (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}. \end{aligned}$$

From the definition of the bracket, $[h'_i, e'_j]' = \alpha'_j(h'_i)e'_j$, $[h'_{2n-i}, e'_j]' = \alpha'_j(h'_{2n-i})e'_j$, $[h'_i, e'_{2n-j}]' = \alpha'_{2n-j}(h'_i)e'_{2n-j}$, and $[h'_{2n-i}, e'_{2n-j}]' = \alpha'_{2n-j}(h'_{2n-i})e'_{2n-j}$, so

$$\begin{aligned} &= \sum_{k,\ell \in \mathbb{Z}} (\alpha'_j(h'_i)e'_j \otimes s^{k+\ell} + \alpha'_{2n-j}(h'_i)e'_{2n-j} \otimes s^{k+\ell} (-1)^\ell \\ &\quad + \alpha'_j(h'_{2n-i})e'_j \otimes s^{k+\ell} (-1)^k + \alpha'_{2n-j}(h'_{2n-i})e'_{2n-j} \otimes s^{k+\ell} (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}. \end{aligned}$$

Now since $\alpha'_j(h'_i) = a_{ij} = a_{2n-i, 2n-j} = \alpha'_{2n-j}(h'_{2n-i})$ and $\alpha'_{2n-j}(h'_i) = a_{i, 2n-j} = a_{2n-i, j} = \alpha'_j(h'_{2n-i}) = 0$, we arrive at:

$$= a_{ij} \sum_{k,\ell \in \mathbb{Z}} (e'_j \otimes s^{k+\ell} + e'_{2n-j} \otimes (-s)^{k+\ell}) z^{-k-1} w^{-\ell-1}.$$

Set $k' = k + \ell$ and $\ell' = k$ so that $\ell = k' - \ell'$ and so:

$$\begin{aligned} &= a_{ij} \sum_{k',\ell' \in \mathbb{Z}} (e'_j \otimes s^{k'} + e'_{2n-j} \otimes (-s)^{k'}) z^{-\ell'-1} w^{-k'+\ell'-1} \\ &= a_{ij} \sum_{k' \in \mathbb{Z}} (e'_j \otimes s^{k'} + e'_{2n-j} \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \\ &= a_{ij} \bar{\psi}(X^{\bar{\sigma}}(\alpha_j, w)) \delta(z - w) \\ &= \bar{\psi}([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_j, w)]). \end{aligned}$$

Similarly,

$$[\bar{\psi}(\alpha_i^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(-\alpha_j, w))]$$

$$\begin{aligned}
&= \left[\sum_{k \in \mathbb{Z}} (h'_i \otimes s^k + h'_{2n-i} \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_j \otimes s^\ell + f'_{2n-j} \otimes (-s)^\ell) w^{-\ell-1} \right] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([h'_i, f'_j]' \otimes s^{k+\ell} + [h'_i, f'_{2n-j}]' \otimes s^{k+\ell}(-1)^\ell \\
&\quad + [h'_{2n-i}, f'_j]' \otimes s^{k+\ell}(-1)^k + [h'_{2n-i}, f'_{2n-j}]' \otimes s^{k+\ell}(-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

From the definition of the bracket, $[h'_i, f'_j]' = -\alpha'_j(h'_i)f'_j$, $[h'_{2n-i}, f'_j]' = -\alpha'_j(h'_{2n-i})f'_j$,

$$\begin{aligned}
&[h'_i, f'_{2n-j}]' = -\alpha'_{2n-j}(h'_i)f'_{2n-j}, \text{ and } [h'_{2n-i}, f'_{2n-j}]' = -\alpha'_{2n-j}(h'_{2n-i})f'_{2n-j}, \text{ so} \\
&= \sum_{k, \ell \in \mathbb{Z}} (-\alpha'_j(h'_i)f'_j \otimes s^{k+\ell} - \alpha'_{2n-j}(h'_i)f'_{2n-j} \otimes s^{k+\ell}(-1)^\ell \\
&\quad - \alpha'_j(h'_{2n-i})f'_j \otimes s^{k+\ell}(-1)^k - \alpha'_{2n-j}(h'_{2n-i})f'_{2n-j} \otimes s^{k+\ell}(-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

Now since $\alpha'_j(h'_i) = a_{ij} = a_{2n-i, 2n-j} = \alpha'_{2n-j}(h'_{2n-i})$ and $\alpha'_{2n-j}(h'_i) = a_{i, 2n-j}$

$= a_{2n-i, j} = \alpha'_j(h'_{2n-i}) = 0$, we arrive at:

$$= -a_{ij} \sum_{k, \ell \in \mathbb{Z}} (f'_j \otimes s^{k+\ell} + f'_{2n-j} \otimes (-s)^{k+\ell}) z^{-k-1} w^{-\ell-1}.$$

Set $k' = k + \ell$ and $\ell' = k$ so that $\ell = k' - \ell'$ and so:

$$\begin{aligned}
&= -a_{ij} \sum_{k', \ell' \in \mathbb{Z}} (f'_j \otimes s^{k'} + f'_{2n-j} \otimes (-s)^{k'}) z^{-\ell'-1} w^{-k'+\ell'-1} \\
&= -a_{ij} \sum_{k' \in \mathbb{Z}} (f'_j \otimes s^{k'} + f'_{2n-j} \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \\
&= -a_{ij} \bar{\psi}(X^{\bar{\sigma}}(-\alpha_j, w)) \delta(z - w) \\
&= \bar{\psi}([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_j, w)]).
\end{aligned}$$

Relation (8): $[\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm \alpha_n, w)] = \pm a_{in} X^{\bar{\sigma}}(\pm \alpha_n, w)(\delta(z - w) + \delta(z + w))$

First we compute for $X^{\bar{\sigma}}(\alpha_n, w)$.

$$\begin{aligned}
&[\bar{\psi}(\alpha_i^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(\alpha_n, w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (h'_i \otimes s^k + h'_{2n-i} \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_n \otimes s^\ell + e'_n \otimes (-s)^\ell) w^{-\ell-1} \right] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([h'_i, e'_n]' \otimes s^{k+\ell} (1 + (-1)^\ell) + [h'_{2n-i}, e'_n]' \otimes s^{k+\ell} ((-1)^k + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

From the bracket definition, $[h'_i, e'_n]' = \alpha'_n(h'_i)e'_n$ and $[h'_{2n-i}, e'_n]' = \alpha'_n(h'_{2n-i})e'_n$, so:

$$= \sum_{k, \ell \in \mathbb{Z}} (\alpha'_n(h'_i)e'_n \otimes s^{k+\ell} (1 + (-1)^\ell) + \alpha'_n(h'_{2n-i})e'_n \otimes s^{k+\ell} ((-1)^k + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1}.$$

We have $\alpha'_n(h'_i) = a_{in} = a_{2n-i, n} = \alpha'_n(h'_{2n-i})$. Also we replace indices with $k' = k + \ell$ and $\ell' = k$ so that $\ell = k' - \ell'$ and so:

$$= a_{in} \sum_{k', \ell' \in \mathbb{Z}} (e'_n \otimes s^{k'} (1 + (-1)^{k'-\ell'}) + e'_n \otimes s^{k'} ((-1)^{\ell'} + (-1)^{k'})) z^{-\ell'-1} w^{-k'+\ell'-1}.$$

Since $(-1)^{k'-\ell'} = (-1)^{k'+\ell'}$, we have

$$= a_{in} \sum_{k', \ell' \in \mathbb{Z}} (e'_n \otimes s^{k'} (1 + (-1)^{\ell'} + (-1)^{k'} + (-1)^{k'+\ell'})) z^{-\ell'-1} w^{-k'+\ell'-1}$$

$$\begin{aligned}
&= a_{in} \left\{ \sum_{k' \in \mathbb{Z}} (e'_n \otimes s^{k'} + e'_n \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \right. \\
&\quad \left. + \sum_{k' \in \mathbb{Z}} (e'_n \otimes s^{k'} + e'_n \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \right\} \\
&= a_{in} \bar{\psi}(X^{\bar{\sigma}}(\alpha_n, w)) (\delta(z - w) + \delta(z + w)) \\
&= \bar{\psi}([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_n, w)]).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&[\bar{\psi}(\alpha_i^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(-\alpha_n, w))] \\
&= [\sum_{k \in \mathbb{Z}} (h'_i \otimes s^k + h'_{2n-i} \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_n \otimes s^\ell + f'_n \otimes (-s)^\ell) w^{-\ell-1}] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([h'_i, f'_n]' \otimes s^{k+\ell} (1 + (-1)^\ell) + [h'_{2n-i}, f'_n]' \otimes s^{k+\ell} ((-1)^k + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

From the bracket definition, $[h'_i, f'_n]' = -\alpha'_n(h'_i)f'_n$ and $[h'_{2n-i}, f'_n]' = -\alpha'_n(h'_{2n-i})f'_n$, so:

$$=\sum_{k, \ell \in \mathbb{Z}} (-\alpha'_n(h'_i)f'_n \otimes s^{k+\ell} (1 + (-1)^\ell) - \alpha'_n(h'_{2n-i})f'_n \otimes s^{k+\ell} ((-1)^k + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1}.$$

We have $\alpha'_n(h'_i) = a_{in} = a_{2n-i, n} = \alpha'_n(h'_{2n-i})$. Also we replace indices with $k' = k + \ell$ and $\ell' = k$ so that $\ell = k' - \ell'$ and so:

$$=-a_{in} \sum_{k', \ell' \in \mathbb{Z}} (f'_n \otimes s^{k'} (1 + (-1)^{k'-\ell'}) + f'_n \otimes s^{k'} ((-1)^{\ell'} + (-1)^{k'})) z^{-\ell'-1} w^{-k'+\ell'-1}.$$

Since $(-1)^{k'-\ell'} = (-1)^{k'+\ell'}$, we have

$$\begin{aligned}
&=-a_{in} \sum_{k', \ell' \in \mathbb{Z}} (f'_n \otimes s^{k'} (1 + (-1)^{\ell'} + (-1)^{k'} + (-1)^{k'+\ell'})) z^{-\ell'-1} w^{-k'+\ell'-1} \\
&=-a_{in} \left\{ \sum_{k' \in \mathbb{Z}} (f'_n \otimes s^{k'} + f'_n \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \right. \\
&\quad \left. + \sum_{k' \in \mathbb{Z}} (f'_n \otimes s^{k'} + f'_n \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \right\} \\
&=-a_{in} \bar{\psi}(X^{\bar{\sigma}}(-\alpha_n, w)) (\delta(z - w) + \delta(z + w)) \\
&= \bar{\psi}([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_n, w)]).
\end{aligned}$$

Relation (9): $[\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_0, w)] = 0$

First we compute for $X^{\bar{\sigma}}(\alpha_0, w)$.

$$\begin{aligned}
&[\bar{\psi}(\alpha_n^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(\alpha_0, w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (h'_n \otimes s^k + h'_n \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_0 \otimes s^\ell + f'_0 \otimes (-s)^\ell) tw^{-\ell-1} \right] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([h'_n, f'_0]' \otimes s^{k+\ell} + (h'_n | f'_0)' \overline{s^\ell ds^k}) (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) tz^{-k-1} w^{-\ell-1} \\
&= 0 \\
&= \bar{\psi}([\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_0, w)]) \\
&\text{since } [h'_n, f'_0]' = 0 \text{ and } (h'_n | f'_0)' = 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& [\bar{\psi}(\alpha_n^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(-\alpha_0, w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (h'_n \otimes s^k + h'_n \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_0 \otimes s^\ell + e'_0 \otimes (-s)^\ell) t^{-1} w^{-\ell-1} \right] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([h'_n, e'_0]' \otimes s^{k+\ell} + (h'_n | e'_0)' \overline{s^\ell ds^k}) (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) t^{-1} z^{-k-1} w^{-\ell-1} \\
&= 0 \\
&= \bar{\psi}([\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_0, w)]) \\
&\text{since } [h'_n, e'_0]' = 0 \text{ and } (h'_n | e'_0)' = 0.
\end{aligned}$$

Relation (10): $[\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm \alpha_j, w)] = \pm a_{nj} X^{\bar{\sigma}}(\pm \alpha_j, w)(\delta(z-w) + \delta(z+w))$

First we compute for $X^{\bar{\sigma}}(\alpha_j, w)$.

$$\begin{aligned}
& [\bar{\psi}(\alpha_n^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(\alpha_j, w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (h'_n \otimes s^k + h'_n \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_j \otimes s^\ell + e'_{2n-j} \otimes (-s)^\ell) w^{-\ell-1} \right] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([h'_n, e'_j]' \otimes s^{k+\ell} + (h'_n | e'_j)' \overline{s^\ell ds^k}) (1 + (-1)^k) z^{-k-1} w^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} ([h'_n, e'_{2n-j}]' \otimes s^{k+\ell} + (h'_n | e'_{2n-j})' \overline{s^\ell ds^k}) ((-1)^\ell + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

We have that $[h'_n, e'_j]' = \alpha'_j(h'_n)e'_j$ and $[h'_n, e'_{2n-j}]' = \alpha'_{2n-j}(h'_n)e'_{2n-j}$, and $(h'_n | e'_j)' = 0 = (h'_n | e'_{2n-j})'$, so

$$= \sum_{k, \ell \in \mathbb{Z}} (\alpha'_j(h'_n)e'_j \otimes s^{k+\ell} (1 + (-1)^k) + \alpha'_{2n-j}(h'_n)e'_{2n-j} \otimes s^{k+\ell} ((-1)^\ell + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1}.$$

We also have that $\alpha'_j(h'_n) = a_{nj} = a_{2n-j, n} = \alpha'_{2n-j}(h'_n)$. We replace indices with $k' = k + \ell$ and $\ell' = k$ and use $(-1)^{k'-\ell'} = (-1)^{k'+\ell'}$ so that

$$\begin{aligned}
&= a_{nj} \sum_{k', \ell' \in \mathbb{Z}} (e'_j \otimes s^{k'} (1 + (-1)^{\ell'}) + e'_{2n-j} \otimes s^{k'} ((-1)^{k'-\ell'} + (-1)^{k'})) z^{-\ell'-1} w^{-k'+\ell'-1} \\
&= a_{nj} \left\{ \sum_{k' \in \mathbb{Z}} (e'_j \otimes s^{k'} + e'_{2n-j} \otimes s^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \right. \\
&\quad \left. + \sum_{k' \in \mathbb{Z}} (e'_j \otimes s^{k'} + e'_{2n-j} \otimes s^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \right\} \\
&= a_{nj} \bar{\psi}(X^{\bar{\sigma}}(\alpha_j, w)) (\delta(z-w) + \delta(z+w)) \\
&= \bar{\psi}([\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_j, w)]).
\end{aligned}$$

Similarly,

$$\begin{aligned}
& [\bar{\psi}(\alpha_n^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(-\alpha_j, w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (h'_n \otimes s^k + h'_n \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_j \otimes s^\ell + f'_{2n-j} \otimes (-s)^\ell) w^{-\ell-1} \right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k,\ell \in \mathbb{Z}} ([h'_n, f'_j]' \otimes s^{k+\ell} + (h'_n | f'_j)' \overline{s^\ell ds^k}) (1 + (-1)^k) z^{-k-1} w^{-\ell-1} \\
&+ \sum_{k,\ell \in \mathbb{Z}} ([h'_n, f'_{2n-j}]' \otimes s^{k+\ell} + (h'_n | f'_{2n-j})' \overline{s^\ell ds^k}) ((-1)^\ell + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

We have that $[h'_n, f'_j]' = -\alpha'_j(h'_n)f'_j$ and $[h'_n, f'_{2n-j}]' = -\alpha'_{2n-j}(h'_n)f'_{2n-j}$, and $(h'_n | f'_j)' = 0 = (h'_n | f'_{2n-j})'$, so

$$= \sum_{k,\ell \in \mathbb{Z}} (-\alpha'_j(h'_n)f'_j \otimes s^{k+\ell} (1 + (-1)^k) - \alpha'_{2n-j}(h'_n)f'_{2n-j} \otimes s^{k+\ell} ((-1)^\ell + (-1)^{k+\ell})) z^{-k-1} w^{-\ell-1}.$$

We also have that $\alpha'_j(h'_n) = a_{nj} = a_{2n-j,n} = \alpha'_{2n-j}(h'_n)$. We replace indices with $k' = k + \ell$ and $\ell' = k$ and use $(-1)^{k'-\ell'} = (-1)^{k'+\ell'}$ so that

$$\begin{aligned}
&= -a_{nj} \sum_{k',\ell' \in \mathbb{Z}} (f'_j \otimes s^{k'} (1 + (-1)^{\ell'}) + f'_{2n-j} \otimes s^{k'} ((-1)^{k'-\ell'} + (-1)^{k'})) z^{-\ell'-1} w^{-k'+\ell'-1} \\
&= -a_{nj} \left\{ \sum_{k' \in \mathbb{Z}} (f'_j \otimes s^{k'} + f'_{2n-j} \otimes s^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \right. \\
&\quad \left. + \sum_{k' \in \mathbb{Z}} (f'_j \otimes s^{k'} + f'_{2n-j} \otimes s^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \right\} \\
&= -a_{nj} \overline{\psi}(X^{\bar{\sigma}}(-\alpha_j, w)) (\delta(z-w) + \delta(z+w)) \\
&= \overline{\psi}([\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_j, w)]).
\end{aligned}$$

Relation (11): $[\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm \alpha_n, w)] = \pm a_{nn} X^{\bar{\sigma}}(\pm \alpha_n, w) (\delta(z-w) + \delta(z+w))$

First we compute for $X^{\bar{\sigma}}(\alpha_n, w)$.

$$\begin{aligned}
&[\overline{\psi}(\alpha_n^{\bar{\sigma}}(z)), \overline{\psi}(X^{\bar{\sigma}}(\alpha_n, w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (h'_n \otimes s^k + h'_n \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_n \otimes s^\ell + e'_n \otimes (-s)^\ell) w^{-\ell-1} \right] \\
&= \sum_{k,\ell \in \mathbb{Z}} ([h'_n, e'_n]' \otimes s^{k+\ell} + (h'_n | e'_n)' \overline{s^\ell ds^k}) (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

We have $[h'_n, e'_n]' = \alpha'_n(h'_n)e'_n = a_{nn}e'_n$ and $(h'_n | e'_n)' = 0$ so:

$$= a_{nn} \left\{ \sum_{k,\ell \in \mathbb{Z}} e'_n \otimes s^{k+\ell} (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1} \right\}.$$

This requires a change of indices to get in the desired form. Set $k' = k + \ell$ and $\ell' = k$ so that $\ell = k' - \ell'$ to get:

$$= a_{nn} \sum_{k',\ell' \in \mathbb{Z}} e'_n \otimes s^{k'} (1 + (-1)^{k'-\ell'} + (-1)^{\ell'} + (-1)^{k'}) z^{-\ell'-1} w^{-k'+\ell'-1}.$$

Since $(-1)^{k'-\ell'} = (-1)^{k'+\ell'}$, this sum breaks up into:

$$\begin{aligned}
&= a_{nn} \left\{ \sum_{k' \in \mathbb{Z}} (e'_n \otimes s^{k'} + e'_n \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \right. \\
&\quad \left. + \sum_{k' \in \mathbb{Z}} (e'_n \otimes s^{k'} + e'_n \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \right\} \\
&= a_{nn} \overline{\psi}(X^{\bar{\sigma}}(\alpha_n, w)) (\delta(z-w) + \delta(z+w))
\end{aligned}$$

$$= \bar{\psi}([\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_n, w)]).$$

Similarly,

$$\begin{aligned} & [\bar{\psi}(\alpha_n^{\bar{\sigma}}(z)), \bar{\psi}(X^{\bar{\sigma}}(-\alpha_n, w))] \\ &= \left[\sum_{k \in \mathbb{Z}} (h'_n \otimes s^k + h'_n \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_n \otimes s^\ell + f'_n \otimes (-s)^\ell) w^{-\ell-1} \right] \\ &= \sum_{k, \ell \in \mathbb{Z}} ([h'_n, f'_n]' \otimes s^{k+\ell} + (h'_n | f'_n)' \bar{s}^\ell ds^k) (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}. \end{aligned}$$

We have $[h'_n, f'_n]' = -\alpha'_n(h'_n)f'_n = -a_{nn}f'_n$ and $(h'_n | f'_n)' = 0$ so:

$$= -a_{nn} \left\{ \sum_{k, \ell \in \mathbb{Z}} f'_n \otimes s^{k+\ell} (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1} \right\}.$$

This requires a change of indices to get in the desired form. Set $k' = k + \ell$ and $\ell' = k$ so that $\ell = k' - \ell'$ to get:

$$= -a_{nn} \sum_{k', \ell' \in \mathbb{Z}} f'_n \otimes s^{k'} (1 + (-1)^{k'-\ell'} + (-1)^{\ell'} + (-1)^{k'}) z^{-\ell'-1} w^{-k'+\ell'-1}.$$

Since $(-1)^{k'-\ell'} = (-1)^{k'+\ell'}$, this sum breaks up into:

$$\begin{aligned} & -a_{nn} \left\{ \sum_{k' \in \mathbb{Z}} (f'_n \otimes s^{k'} + f'_n \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \right. \\ & \quad \left. + \sum_{k' \in \mathbb{Z}} (f'_n \otimes s^{k'} + f'_n \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \right\} \\ &= -a_{nn} \bar{\psi}(X^{\bar{\sigma}}(-\alpha_n, w)) (\delta(z-w) + \delta(z+w)) \\ &= \bar{\psi}([\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_n, w)]). \end{aligned}$$

Relation (12): $[X^{\bar{\sigma}}(\pm \alpha_m, z), X^{\bar{\sigma}}(\pm \alpha_m, w)] = 0$

First we examine the case $m = 0$ for α_m .

$$\begin{aligned} & [\bar{\psi}(X^{\bar{\sigma}}(\alpha_m, z)), \bar{\psi}(X^{\bar{\sigma}}(\alpha_m, w))] \\ &= \left[\sum_{k \in \mathbb{Z}} (f'_0 \otimes s^k + f'_0 \otimes (-s)^k) tz^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_0 \otimes s^\ell + f'_0 \otimes (-s)^\ell) tw^{-\ell-1} \right] \\ &= \sum_{k, \ell \in \mathbb{Z}} ([f'_0, f'_0]' \otimes s^{k+\ell} t^2 + (f'_0 | f'_0)') (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1} \\ &= 0 \\ &= \bar{\psi}([X^{\bar{\sigma}}(\alpha_m, z), X^{\bar{\sigma}}(\alpha_m, w)]) \end{aligned}$$

since $[f'_0, f'_0]' = 0$ and $(f'_0 | f'_0)' = 0$.

Similarly,

$$\begin{aligned} & [\bar{\psi}(X^{\bar{\sigma}}(-\alpha_m, z)), \bar{\psi}(X^{\bar{\sigma}}(-\alpha_m, w))] \\ &= \left[\sum_{k \in \mathbb{Z}} (e'_0 \otimes s^k + e'_0 \otimes (-s)^k) t^{-1} z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_0 \otimes s^\ell + e'_0 \otimes (-s)^\ell) t^{-1} w^{-\ell-1} \right] \\ &= \sum_{k, \ell \in \mathbb{Z}} ([e'_0, e'_0]' \otimes s^{k+\ell} t^{-2} + (e'_0 | e'_0)') (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1} \end{aligned}$$

$$\begin{aligned}
&= 0 \\
&= \bar{\psi}([X^{\bar{\sigma}}(-\alpha_m, z), X^{\bar{\sigma}}(-\alpha_m, w)]) \\
&\text{since } [e'_0, e'_0]' = 0 \text{ and } (e'_0, e'_0)' = 0.
\end{aligned}$$

Now we consider the case $1 \leq m \leq n$ for α_m .

$$\begin{aligned}
&[\bar{\psi}(X^{\bar{\sigma}}(\alpha_m, z)), \bar{\psi}(X^{\bar{\sigma}}(\alpha_m, w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (e'_m \otimes s^k + e'_{2n-m} \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_m \otimes s^\ell + e'_{2n-m} \otimes (-s)^\ell) w^{-\ell-1} \right] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([e'_m, e'_m]' \otimes s^{k+\ell} + (e'_m | e'_m)') z^{-k-1} w^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_m, e'_{2n-m}]' \otimes s^{k+\ell} + (e'_m | e'_{2n-m})') (-1)^\ell z^{-k-1} w^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_{2n-m}, e'_m]' \otimes s^{k+\ell} + (e'_{2n-m} | e'_m)') (-1)^k z^{-k-1} w^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_{2n-m}, e'_{2n-m}]' \otimes s^{k+\ell} + (e'_{2n-m} | e'_{2n-m})') (-1)^{k+\ell} z^{-k-1} w^{-\ell-1} \\
&= 0 \\
&= \bar{\psi}([X^{\bar{\sigma}}(\alpha_m, z), X^{\bar{\sigma}}(\alpha_m, w)])
\end{aligned}$$

since all brackets and bilinear forms involved are 0.

Similarly,

$$\begin{aligned}
&[\bar{\psi}(X^{\bar{\sigma}}(-\alpha_m, z)), \bar{\psi}(X^{\bar{\sigma}}(-\alpha_m, w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (f'_m \otimes s^k + f'_{2n-m} \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_m \otimes s^\ell + f'_{2n-m} \otimes (-s)^\ell) w^{-\ell-1} \right] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([f'_m, f'_m]' \otimes s^{k+\ell} + (f'_m | f'_m)') z^{-k-1} w^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} ([f'_m, f'_{2n-m}]' \otimes s^{k+\ell} + (f'_m | f'_{2n-m})') (-1)^\ell z^{-k-1} w^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} ([f'_{2n-m}, f'_m]' \otimes s^{k+\ell} + (f'_{2n-m} | f'_m)') (-1)^k z^{-k-1} w^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} ([f'_{2n-m}, f'_{2n-m}]' \otimes s^{k+\ell} + (f'_{2n-m} | f'_{2n-m})') (-1)^{k+\ell} z^{-k-1} w^{-\ell-1} \\
&= 0 \\
&= \bar{\psi}([X^{\bar{\sigma}}(-\alpha_m, z), X^{\bar{\sigma}}(-\alpha_m, w)])
\end{aligned}$$

since all brackets and bilinear forms involved are 0.

Relation (13): $[X^{\bar{\sigma}}(\alpha_0, z), X^{\bar{\sigma}}(-\alpha_0, w)] = \alpha_0^{\bar{\sigma}}(w)(\delta(z-w) + \delta(z+w)) + (\partial_w \delta(z-w) + \partial_w \delta(z+w))\phi$

$$[\bar{\psi}(X^{\bar{\sigma}}(\alpha_0, z)), \bar{\psi}(X^{\bar{\sigma}}(-\alpha_0, w))]$$

$$\begin{aligned}
&= \left[\sum_{k \in \mathbb{Z}} (f'_0 \otimes s^k + f'_0 \otimes (-s)^k) t z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_0 \otimes s^\ell + e'_0 \otimes (-s)^\ell) t^{-1} w^{-\ell-1} \right] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([f'_0, e'_0]' \otimes s^{k+\ell} + (f'_0 | e'_0)' \overline{s^\ell t^{-1} d(s^k t)}) (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

We know $[f'_0, e'_0]' = -h'_0$ and $(f'_0 | e'_0)' = \frac{1}{2}$. By Example 11, $\overline{s^\ell t^{-1} d(s^k t)} = \delta_{k, -\ell} k s^{\ell+k-1} ds + \overline{s^{k+\ell} t^{-1} dt}$, so we have

$$\begin{aligned}
&= \sum_{k, \ell \in \mathbb{Z}} \left(-h'_0 \otimes s^{k+\ell} + \frac{1}{2} \overline{s^{k+\ell} t^{-1} dt} \right) (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} \frac{1}{2} \delta_{k, -\ell} k s^{\ell+k-1} ds (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

Replace indices in the first sum with $k' = k + \ell$ and $\ell' = k$ to get:

$$\begin{aligned}
&= \sum_{k', \ell' \in \mathbb{Z}} \left(-h'_0 \otimes s^{k'} + \frac{1}{2} \overline{s^{k'} t^{-1} dt} \right) (1 + (-1)^{k'-\ell'} + (-1)^{\ell'} + (-1)^{k'}) z^{-\ell'-1} w^{-k'+\ell'-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} \frac{1}{2} \delta_{k, -\ell} k s^{\ell+k-1} ds (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

Now using $(-1)^{k'-\ell'} = (-1)^{k'+\ell'}$ gives

$$\begin{aligned}
&= \sum_{k' \in \mathbb{Z}} \left(-h'_0 \otimes s^{k'} - h'_0 \otimes (-s)^{k'} + \frac{1}{2} (\overline{s^{k'} t^{-1} dt} + \overline{(-s)^{k'} t^{-1} dt}) \right) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \\
&\quad + \sum_{k' \in \mathbb{Z}} \left(-h'_0 \otimes s^{k'} - h'_0 \otimes (-s)^{k'} + \frac{1}{2} (\overline{s^{k'} t^{-1} dt} + \overline{(-s)^{k'} t^{-1} dt}) \right) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \\
&\quad + \sum_{k \in \mathbb{Z}} \frac{1}{2} k s^{-1} ds (1 + (-1)^{-k} + (-1)^k + (-1)^0) z^{-k-1} w^{k-1} \\
&= \sum_{k' \in \mathbb{Z}} \left(-h'_0 \otimes s^{k'} - h'_0 \otimes (-s)^{k'} + \frac{1}{2} (\overline{s^{k'} t^{-1} dt} + \overline{(-s)^{k'} t^{-1} dt}) \right) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \\
&\quad + \sum_{k' \in \mathbb{Z}} \left(-h'_0 \otimes s^{k'} - h'_0 \otimes (-s)^{k'} + \frac{1}{2} (\overline{s^{k'} t^{-1} dt} + \overline{(-s)^{k'} t^{-1} dt}) \right) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \\
&\quad + \sum_{k \in \mathbb{Z}} \frac{1}{2} k s^{-1} ds (2 + 2(-1)^k) z^{-k-1} w^{k-1} \\
&= \sum_{k' \in \mathbb{Z}} \left(-h'_0 \otimes s^{k'} - h'_0 \otimes (-s)^{k'} + \frac{1}{2} (\overline{s^{k'} t^{-1} dt} + \overline{(-s)^{k'} t^{-1} dt}) \right) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \\
&\quad + \sum_{k' \in \mathbb{Z}} \left(-h'_0 \otimes s^{k'} - h'_0 \otimes (-s)^{k'} + \frac{1}{2} (\overline{s^{k'} t^{-1} dt} + \overline{(-s)^{k'} t^{-1} dt}) \right) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \\
&\quad + \left(\sum_{k \in \mathbb{Z}} k z^{-k-1} w^{k-1} + \sum_{k \in \mathbb{Z}} -k z^{-k-1} (-w)^{k-1} \right) \overline{s^{-1} ds}
\end{aligned}$$

since $(-1)^{-k} = (-1)^k = -(-1)^{k-1}$. Thus,

$$\begin{aligned}
&= \overline{\psi}(\alpha_0^\sigma(w)) (\delta(z-w) + \delta(z+w)) + (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \overline{\psi}(\phi) \\
&= \overline{\psi}([X^\sigma(\alpha_0, z), X^\sigma(-\alpha_0, w)]).
\end{aligned}$$

$$\begin{aligned}
& \text{Relation (14): } [X^{\bar{\sigma}}(\alpha_i, z), X^{\bar{\sigma}}(-\alpha_i, w)] = \alpha_i^{\bar{\sigma}}(w)\delta(z-w) + \partial_w\delta(z-w)\not{\epsilon} \\
& \quad [\bar{\psi}(X^{\bar{\sigma}}(\alpha_i, z)), \bar{\psi}(X^{\bar{\sigma}}(-\alpha_j, w))] \\
& = \left[\sum_{k \in \mathbb{Z}} (e'_i \otimes s^k + e'_{2n-i} \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_j \otimes s^\ell + f'_{2n-j} \otimes (-s)^\ell) w^{-\ell-1} \right] \\
& = \sum_{k, \ell \in \mathbb{Z}} ([e'_i, f'_j]' \otimes s^{k+\ell} + (e'_i | f'_j)' \overline{s^\ell ds^k}) z^{-k-1} w^{-\ell-1} \\
& \quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_i, f'_{2n-j}]' \otimes s^{k+\ell} + (e'_i | f'_{2n-j})' \overline{s^\ell ds^k}) (-1)^\ell z^{-k-1} w^{-\ell-1} \\
& \quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_{2n-i}, f'_j]' \otimes s^{k+\ell} + (e'_{2n-i} | f'_j)' \overline{s^\ell ds^k}) (-1)^k z^{-k-1} w^{-\ell-1} \\
& \quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_{2n-i}, f'_{2n-j}]' \otimes s^{k+\ell} + (e'_{2n-i} | f'_{2n-j})' \overline{s^\ell ds^k}) (-1)^{k+\ell} z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

We know that $[e'_i, f'_j]' = \delta_{ij} h'_i$ and $(e'_i | f'_j)' = \frac{1}{2}\delta_{ij}$. By the restrictions on i and j , $[e'_i, f'_{2n-j}]' = [e'_{2n-i}, f'_j]' = 0$ and $(e'_i | f'_{2n-j})' = (e'_{2n-i} | f'_j)' = 0$. Also, since $i = j$ iff $2n - i = 2n - j$, the calculation simplifies to:

$$= \delta_{ij} \sum_{k, \ell \in \mathbb{Z}} (h'_i \otimes s^{k+\ell} + \frac{1}{2} \overline{s^\ell ds^k}) z^{-k-1} w^{-\ell-1} + \delta_{ij} \sum_{k, \ell \in \mathbb{Z}} (h'_{2n-i} \otimes s^{k+\ell} + \frac{1}{2} \overline{s^\ell ds^k}) (-1)^{k+\ell} z^{-k-1} w^{-\ell-1}.$$

Now use $\overline{s^\ell ds^k} = \delta_{k, -\ell} k \overline{s^{\ell+k-1} ds}$ from Example 10 to get:

$$\begin{aligned}
& = \delta_{ij} \sum_{k, \ell \in \mathbb{Z}} (h'_i \otimes s^{k+\ell} + h'_{2n-i} \otimes (-s)^{k+\ell}) z^{-k-1} w^{-\ell-1} \\
& \quad + \frac{1}{2} \delta_{ij} \sum_{k, \ell \in \mathbb{Z}} \delta_{k, -\ell} \overline{s^{\ell+k-1} ds} (1 + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

In the first sum, use $k' = k + \ell$ and $\ell' = k$ and in the second sum, use the implications of $\delta_{k, -\ell}$ to reduce the number of indices. The calculation simplifies to:

$$\begin{aligned}
& = \delta_{ij} \sum_{k', \ell' \in \mathbb{Z}} (h'_i \otimes s^{k'} + h'_{2n-i} \otimes (-s)^{k'}) z^{-\ell'-1} w^{-k'+\ell'-1} \\
& \quad + \frac{1}{2} \delta_{ij} \sum_{k \in \mathbb{Z}} k \overline{s^{-1} ds} (1 + (-1)^0) z^{-k-1} w^{k-1} \\
& = \delta_{ij} \sum_{k' \in \mathbb{Z}} (h'_i \otimes s^{k'} + h'_{2n-i} \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \\
& \quad + \frac{1}{2} \cdot 2 \delta_{ij} \sum_{k \in \mathbb{Z}} k z^{-k-1} w^{k-1} \overline{s^{-1} ds} \\
& = \delta_{ij} \sum_{k' \in \mathbb{Z}} (h'_i \otimes s^{k'} + h'_{2n-i} \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} + \delta_{ij} \sum_{k \in \mathbb{Z}} k z^{-k-1} w^{k-1} \overline{s^{-1} ds} \\
& = \delta_{ij} \left(\bar{\psi}(\alpha_i^{\bar{\sigma}}(w)) \delta(z-w) + \partial_w \delta(z-w) \bar{\psi}(\not{\epsilon}) \right) \\
& = \bar{\psi}([X^{\bar{\sigma}}(\alpha_i, z), X^{\bar{\sigma}}(-\alpha_j, w)]).
\end{aligned}$$

Relation (15): $[X^{\bar{\sigma}}(\alpha_n, z), X^{\bar{\sigma}}(-\alpha_n, w)] = \alpha_n^{\bar{\sigma}}(w)(\delta(z-w) + \delta(z+w)) + (\partial_w \delta(z-w) + \partial_w \delta(z+w))\not{\epsilon}$

$$\begin{aligned}
& [\bar{\psi}(X^{\bar{\sigma}}(\alpha_n, z)), \bar{\psi}(X^{\bar{\sigma}}(-\alpha_n, w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (e'_n \otimes s^k + e'_n \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_n \otimes s^\ell + f'_n \otimes (-s)^\ell) w^{-\ell-1} \right] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([e'_n, f'_n]' \otimes s^{k+\ell} + (e'_n | f'_n)' \overline{s^\ell ds^k}) (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

We know $[e'_n, f'_n]' = h'_n$, and $(e'_n | f'_n)' = \frac{1}{2}$, and $\overline{s^\ell ds^k} = \delta_{k, -\ell} k \overline{s^{\ell+k-1} ds}$ from Example 10. Using these, we get:

$$\begin{aligned}
&= \sum_{k, \ell \in \mathbb{Z}} (h'_n \otimes s^{k+\ell} + \frac{1}{2} \delta_{k, -\ell} k \overline{s^{\ell+k-1} ds}) (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1} \\
&= \sum_{k, \ell \in \mathbb{Z}} h'_n \otimes s^{k+\ell} (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1} \\
&\quad + \frac{1}{2} \sum_{k, \ell \in \mathbb{Z}} \delta_{k, -\ell} k \overline{s^{\ell+k-1} ds} (1 + (-1)^\ell + (-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

In the first sum, use $k' = k + \ell$ and $\ell' = k$ and in the second sum, use the implications of $\delta_{k, -\ell}$ to reduce the number of indices. The calculation simplifies to:

$$\begin{aligned}
&= \sum_{k', \ell' \in \mathbb{Z}} h'_n \otimes s^{k'} (1 + (-1)^{k'-\ell'} + (-1)^{\ell'} + (-1)^{k'}) z^{-\ell'-1} w^{-k'+\ell'-1} \\
&\quad + \frac{1}{2} \sum_{k \in \mathbb{Z}} k \overline{s^{-1} ds} (1 + (-1)^{-k} + (-1)^k + (-1)^0) z^{-k-1} w^{k-1}.
\end{aligned}$$

Now use $(-1)^{k'-\ell'} = (-1)^{k'+\ell'}$ and $(-1)^{-k} = (-1)^k$ to get:

$$\begin{aligned}
&= \sum_{k' \in \mathbb{Z}} (h'_n \otimes s^{k'} + h'_n \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \\
&\quad + \sum_{k' \in \mathbb{Z}} (h'_n \otimes s^{k'} + h'_n \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \\
&\quad + \frac{1}{2} \sum_{k \in \mathbb{Z}} 2kz^{-k-1} w^{k-1} \overline{s^{-1} ds} + \frac{1}{2} \sum_{k \in \mathbb{Z}} 2kz^{-k-1} w^{k-1} (-1)^k \overline{s^{-1} ds}.
\end{aligned}$$

Now use $(-1)^k = -(-1)^{k-1}$ to simplify it to:

$$\begin{aligned}
&= \sum_{k' \in \mathbb{Z}} (h'_n \otimes s^{k'} + h'_n \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} w^{\ell'} \\
&\quad + \sum_{k' \in \mathbb{Z}} (h'_n \otimes s^{k'} + h'_n \otimes (-s)^{k'}) w^{-k'-1} \sum_{\ell' \in \mathbb{Z}} z^{-\ell'-1} (-w)^{\ell'} \\
&\quad + \left(\sum_{k \in \mathbb{Z}} kz^{-k-1} w^{k-1} + \sum_{k \in \mathbb{Z}} -kz^{-k-1} (-w)^{k-1} \right) \overline{s^{-1} ds} \\
&= \overline{\psi}(\alpha_n^{\bar{\sigma}}(z)) (\delta(z-w) + \delta(z+w)) + (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \overline{\psi}(\ell') \\
&= \overline{\psi}([X^{\bar{\sigma}}(\alpha_n, z), X^{\bar{\sigma}}(-\alpha_n, w)]).
\end{aligned}$$

Relation (16): $[X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)] = 0$ for $p \neq m$

We have several cases to check. First consider $p = 0$ and $1 \leq m \leq n$.

$$[\bar{\psi}(X^{\bar{\sigma}}(\alpha_p, z)), \bar{\psi}(X^{\bar{\sigma}}(-\alpha_m, w))]$$

$$\begin{aligned}
&= \left[\sum_{k \in \mathbb{Z}} (f'_0 \otimes s^k + f'_0 \otimes (-s)^k) t z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_m \otimes s^\ell + f'_{2n-m} \otimes (-s)^\ell) w^{-\ell-1} \right] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([f'_0, f'_m]' \otimes s^{k+\ell} t + (f'_0 | f'_m)' \overline{s^\ell ds^k t}) (1 + (-1)^k) z^{-k-1} w^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} ([f'_0, f'_{2n-m}]' \otimes s^{k+\ell} t + (f'_0 | f'_{2n-m})' \overline{s^\ell ds^k t}) ((-1)^\ell + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

We have $[f'_0, f'_m]' \in (A_{2n-1})_{-\theta - \alpha'_m}$ and $(f'_0 | f'_m)' = 0$. But since θ is the highest root of $\dot{\Delta}$, then $-\theta - \alpha'_m \notin \dot{\Delta}$, so $[f'_0, f'_m]' = 0$. For the same reasons, $[f'_0, f'_{2n-m}]' = 0$ and $(f'_0 | f'_{2n-m})' = 0$. Thus, this calculation is simply:

$$\begin{aligned}
&= 0 \\
&= \overline{\psi}([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]).
\end{aligned}$$

Now consider the case $1 \leq p \leq n$ and $m = 0$.

$$\begin{aligned}
&[\overline{\psi}(X^{\bar{\sigma}}(\alpha_p, z)), \overline{\psi}(X^{\bar{\sigma}}(-\alpha_m, w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (e'_p \otimes s^k + e'_{2n-p} \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_0 \otimes s^\ell + e'_0 \otimes (-s)^\ell) t^{-1} w^{-\ell-1} \right] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([e'_p, e'_0]' \otimes s^{k+\ell} t^{-1} + (e'_p | e'_0)' \overline{s^\ell t^{-1} ds^k}) (1 + (-1)^\ell) z^{-k-1} w^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_{2n-p}, e'_0]' \otimes s^{k+\ell} t^{-1} + (e'_{2n-p} | e'_0)' \overline{s^\ell t^{-1} ds^k}) ((-1)^k + (-1)^{k+\ell}) z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

We have $[e'_p, e'_0]' \in (A_{2n-1})_{\theta + \alpha'_p}$ and $(e'_p | e'_0)' = 0$. But since θ is the highest root of $\dot{\Delta}$, then $\theta + \alpha'_p \notin \dot{\Delta}$, so $[e'_p, e'_0]' = 0$. For the same reasons, $[e'_{2n-p}, e'_0]' = 0$ and $(e'_{2n-p} | e'_0)' = 0$. Thus, this calculation is simply:

$$\begin{aligned}
&= 0 \\
&= \overline{\psi}([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]).
\end{aligned}$$

Now consider the case $1 \leq p \neq m \leq n$.

$$\begin{aligned}
&[\overline{\psi}(X^{\bar{\sigma}}(\alpha_p, z)), \overline{\psi}(X^{\bar{\sigma}}(-\alpha_m, w))] \\
&= \left[\sum_{k \in \mathbb{Z}} (e'_p \otimes s^k + e'_{2n-p} \otimes (-s)^k) z^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_m \otimes s^\ell + f'_{2n-m} \otimes (-s)^\ell) w^{-\ell-1} \right] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([e'_p, f'_m]' \otimes s^{k+\ell} + (e'_p | f'_m)' \overline{s^\ell ds^k}) z^{-k-1} w^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_p, f'_{2n-m}]' \otimes s^{k+\ell} + (e'_p | f'_{2n-m})' \overline{s^\ell ds^k}) (-1)^\ell z^{-k-1} w^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_{2n-p}, f'_m]' \otimes s^{k+\ell} + (e'_{2n-p} | f'_m)' \overline{s^\ell ds^k}) (-1)^k z^{-k-1} w^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_{2n-p}, f'_{2n-m}]' \otimes s^{k+\ell} + (e'_{2n-p} | f'_{2n-m})' \overline{s^\ell ds^k}) (-1)^{k+\ell} z^{-k-1} w^{-\ell-1}.
\end{aligned}$$

Each bracket and bilinear form involved is 0 since $p \neq m$. Thus, this calculation is simply:

$$= 0$$

$$= \bar{\psi}([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]).$$

Relation (17): $\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_2)X^{\bar{\sigma}}(\pm\alpha_m, z_1) = 0$ if $c_{pm} = 0$

For the case $p = 0$, the matrix entry $c_{pm} = 0 \Leftrightarrow 2 \leq m \leq n$, so we consider this case first for positive α_p, α_m .

$$\begin{aligned} & [\bar{\psi}(X(\alpha_p, z_2)), \bar{\psi}(X(\alpha_m, z_1))] \\ &= \left[\sum_{k \in \mathbb{Z}} (f'_0 \otimes s^k + f'_0 \otimes (-s)^k) t z_2^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_m \otimes s^\ell + e'_{2n-m} \otimes (-s)^\ell) z_1^{-\ell-1} \right] \\ &= \sum_{k, \ell \in \mathbb{Z}} ([f'_0, e'_m]' \otimes s^{k+\ell} t + (f'_0 | e'_m)' \overline{s^\ell ds^k t}) (1 + (-1)^k) z_2^{-k-1} z_1^{-\ell-1} \\ &\quad + \sum_{k, \ell \in \mathbb{Z}} ([f'_0, e'_{2n-m}]' \otimes s^{k+\ell} t + (f'_0 | e'_{2n-m})' \overline{s^\ell ds^k t}) ((-1)^\ell + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1}. \end{aligned}$$

Now, $[f'_0, e'_m]' \in (A_{2n-1})_{-\theta+\alpha'_m}$. But by §6.7 in [K], $-\theta + \alpha'_m \notin \dot{\Delta}$ for these values of m . Hence, $[f'_0, e'_m]' = 0$, and by a similar argument, $[f'_0, e'_{2n-m}]' = 0$. Further, $(f'_0 | e'_m)' = 0 = (f'_0 | e'_{2n-m})'$, so the entire computation is simply:

$$\begin{aligned} &= 0 \\ &= \bar{\psi}([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

Similarly,

$$\begin{aligned} & [\bar{\psi}(X(-\alpha_p, z_2)), \bar{\psi}(X(-\alpha_m, z_1))] \\ &= \left[\sum_{k \in \mathbb{Z}} (e'_0 \otimes s^k + e'_0 \otimes (-s)^k) t^{-1} z_2^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_m \otimes s^\ell + f'_{2n-m} \otimes (-s)^\ell) z_1^{-\ell-1} \right] \\ &= \sum_{k, \ell \in \mathbb{Z}} ([e'_0, f'_m]' \otimes s^{k+\ell} t^{-1} + (e'_0 | f'_m)' \overline{s^\ell ds^k t^{-1}}) (1 + (-1)^k) z_2^{-k-1} z_1^{-\ell-1} \\ &\quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_0, f'_{2n-m}]' \otimes s^{k+\ell} t^{-1} + (e'_0 | f'_{2n-m})' \overline{s^\ell ds^k t^{-1}}) ((-1)^\ell + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1}. \end{aligned}$$

Now, $[e'_0, f'_m]' \in (A_{2n-1})_{\theta-\alpha'_m}$. But by §6.7 in [K], $\theta - \alpha'_m \notin \dot{\Delta}$ for these values of m . Hence, $[e'_0, f'_m]' = 0$, and by a similar argument, $[e'_0, f'_{2n-m}]' = 0$. Further, $(e'_0 | f'_m)' = 0 = (e'_0 | f'_{2n-m})'$, so the entire computation is simply:

$$\begin{aligned} &= 0 \\ &= \bar{\psi}([X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)]). \end{aligned}$$

Now if $m = 0$, the entry $c_{pm} = 0 \Leftrightarrow 2 \leq p \leq n$, so we consider this case first for positive α_p, α_m .

$$\begin{aligned} & [\bar{\psi}(X(\alpha_p, z_2)), \bar{\psi}(X(\alpha_m, z_1))] \\ &= \left[\sum_{k \in \mathbb{Z}} (e'_p \otimes s^k + e'_{2n-p} \otimes (-s)^k) z_2^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_0 \otimes s^\ell + f'_0 \otimes (-s)^\ell) t z_1^{-\ell-1} \right] \\ &= \sum_{k, \ell \in \mathbb{Z}} ([e'_p, f'_0]' \otimes s^{k+\ell} t + (e'_p | f'_0)' \overline{s^\ell t ds^k}) (1 + (-1)^\ell) z_2^{-k-1} z_1^{-\ell-1} \end{aligned}$$

$$+ \sum_{k,\ell \in \mathbb{Z}} ([e'_{2n-p}, f'_0]' \otimes s^{k+\ell} t + (e'_{2n-p}|f'_0)' \overline{s^\ell t ds^k}) ((-1)^k + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1}.$$

Now, $[e'_p, f'_0]' \in (A_{2n-1})_{-\theta+\alpha'_m}$. But by §6.7 in [K], $-\theta + \alpha'_m \notin \dot{\Delta}$ for these values of p . Hence, $[e'_p, f'_0]' = 0$, and by a similar argument, $[e'_{2n-p}, f'_0]' = 0$. Further, $(e'_p|f'_0)' = 0 = (e'_{2n-p}|f'_0)'$, so the entire computation is simply:

$$\begin{aligned} &= 0 \\ &= \overline{\psi}([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

Similarly,

$$\begin{aligned} &[\overline{\psi}(X(-\alpha_p, z_2)), \overline{\psi}(X(-\alpha_m, z_1))] \\ &= \left[\sum_{k \in \mathbb{Z}} (f'_p \otimes s^k + f'_{2n-p} \otimes (-s)^k) z_2^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_0 \otimes s^\ell + e'_0 \otimes (-s)^\ell) t^{-1} z_1^{-\ell-1} \right] \\ &= \sum_{k,\ell \in \mathbb{Z}} ([f'_p, e'_0]' \otimes s^{k+\ell} t^{-1} + (f'_p|e'_0)' \overline{s^\ell t^{-1} ds^k}) (1 + (-1)^\ell) z_2^{-k-1} z_1^{-\ell-1} \\ &\quad + \sum_{k,\ell \in \mathbb{Z}} ([f'_{2n-p}, e'_0]' \otimes s^{k+\ell} t^{-1} + (f'_{2n-p}|e'_0)' \overline{s^\ell t^{-1} ds^k}) ((-1)^k + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1}. \end{aligned}$$

Now, $[f'_p, e'_0]' \in (A_{2n-1})_{\theta-\alpha'_m}$. But by §6.7 in [K], $\theta - \alpha'_m \notin \dot{\Delta}$ for these values of p . Hence, $[f'_p, e'_0]' = 0$, and by a similar argument, $[f'_{2n-p}, e'_0]' = 0$. Further, $(f'_p|e'_0)' = 0 = (f'_{2n-p}|e'_0)'$, so the entire computation is simply:

$$\begin{aligned} &= 0 \\ &= \overline{\psi}([X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)]). \end{aligned}$$

For the remaining cases, we may assume that $p \neq 0$ and $m \neq 0$. For these values, $c_{pm} = 0$ when $|p - m| \geq 2$. With this in mind, we compute for α_m and α_p .

$$\begin{aligned} &[\overline{\psi}(X(\alpha_p, z_2)), \overline{\psi}(X(\alpha_m, z_1))] \\ &= \left[\sum_{k \in \mathbb{Z}} (e'_p \otimes s^k + e'_{2n-p} \otimes (-s)^k) z_2^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_m \otimes s^\ell + e'_{2n-m} \otimes (-s)^\ell) z_1^{-\ell-1} \right] \\ &= \sum_{k,\ell \in \mathbb{Z}} ([e'_p, e'_m]' \otimes s^{k+\ell} + (e'_p|e'_m)' \overline{s^\ell ds^k}) z_1^{-k-1} z_2^{-\ell-1} \\ &\quad + \sum_{k,\ell \in \mathbb{Z}} ([e'_p, e'_{2n-m}]' \otimes s^{k+\ell} + (e'_p|e'_{2n-m})' \overline{s^\ell ds^k}) (-1)^\ell z_1^{-k-1} z_2^{-\ell-1} \\ &\quad + \sum_{k,\ell \in \mathbb{Z}} ([e'_{2n-p}, e'_m]' \otimes s^{k+\ell} + (e'_{2n-p}|e'_m)' \overline{s^\ell ds^k}) (-1)^k z_1^{-k-1} z_2^{-\ell-1} \\ &\quad + \sum_{k,\ell \in \mathbb{Z}} ([e'_{2n-p}, e'_{2n-m}]' \otimes s^{k+\ell} + (e'_{2n-p}|e'_{2n-m})' \overline{s^\ell ds^k}) (-1)^{k+\ell} z_1^{-k-1} z_2^{-\ell-1}. \end{aligned}$$

Each of the bilinear forms is 0. Now, $[e'_p, e'_m]' \in (A_{2n-1})_{\alpha'_p+\alpha'_m}$. By §6.7 in [K], $\alpha'_p + \alpha'_m \notin \dot{\Delta}$ because $|p - m| \geq 2$. Hence $[e'_p, e'_m]' = 0$. The same argument applies to each of the other brackets since $|2n - p - (2n - m)| \geq 2$, $|2n - p - m| \geq 2$, and $|p - (2n - m)| \geq 2$. Therefore, the calculation is simply:

$$= 0 \\ = \bar{\psi}([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]).$$

Similarly,

$$\begin{aligned} & [\bar{\psi}(X(-\alpha_p, z_2)), \bar{\psi}(X(-\alpha_m, z_1))] \\ &= \left[\sum_{k \in \mathbb{Z}} (f'_p \otimes s^k + f'_{2n-p} \otimes (-s)^k) z_2^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_m \otimes s^\ell + f'_{2n-m} \otimes (-s)^\ell) z_1^{-\ell-1} \right] \\ &= \sum_{k, \ell \in \mathbb{Z}} ([f'_p, f'_m]' \otimes s^{k+\ell} + (f'_p | f'_m)' \overline{s^\ell ds^k}) z_1^{-k-1} z_2^{-\ell-1} \\ &+ \sum_{k, \ell \in \mathbb{Z}} ([f'_p, f'_{2n-m}]' \otimes s^{k+\ell} + (f'_p | f'_{2n-m})' \overline{s^\ell ds^k}) (-1)^\ell z_1^{-k-1} z_2^{-\ell-1} \\ &+ \sum_{k, \ell \in \mathbb{Z}} ([f'_{2n-p}, f'_m]' \otimes s^{k+\ell} + (f'_{2n-p} | f'_m)' \overline{s^\ell ds^k}) (-1)^k z_1^{-k-1} z_2^{-\ell-1} \\ &+ \sum_{k, \ell \in \mathbb{Z}} ([f'_{2n-p}, f'_{2n-m}]' \otimes s^{k+\ell} + (f'_{2n-p} | f'_{2n-m})' \overline{s^\ell ds^k}) (-1)^{k+\ell} z_1^{-k-1} z_2^{-\ell-1}. \end{aligned}$$

Each of the bilinear forms is 0. Now, $[f'_p, f'_m]' \in (A_{2n-1})_{-\alpha'_p - \alpha'_m}$. By §6.7 in [K], $-\alpha'_p - \alpha'_m \notin \dot{\Delta}$ because $|p - m| \geq 2$. Hence $[f'_p, f'_m]' = 0$. The same argument applies to each of the other brackets since $|2n - p - (2n - m)| \geq 2$, $|2n - p - m| \geq 2$, and $|p - (2n - m)| \geq 2$. Therefore, the calculation is simply:

$$= 0 \\ = \bar{\psi}([X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)]).$$

Relation (18): $\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_2)X^{\bar{\sigma}}(\pm\alpha_m, z_1) = 0$ if $c_{pm} = -1$

The condition $c_{pm} = -1$ occurs precisely when $|p - m| = 1$ except for the pairs $p = 1, m = 0$ and $p = n - 1, m = n$. First, we will compute when $p = 0, m = 1$ and for α_p, α_m .

$$\begin{aligned} & [\bar{\psi}(X(\alpha_p, z_2)), \bar{\psi}(X(\alpha_m, z_1))] \\ &= \left[\sum_{k \in \mathbb{Z}} (f'_0 \otimes s^k + f'_0 \otimes (-s)^k) t z_2^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_1 \otimes s^\ell + e'_{2n-1} \otimes (-s)^\ell) z_1^{-\ell-1} \right] \\ &= \sum_{k, \ell \in \mathbb{Z}} ([f'_0, e'_1]' \otimes s^{k+\ell} t + (f'_0 | e'_1)' \overline{s^\ell ds^k t}) (1 + (-1)^k) z_2^{-k-1} z_1^{-\ell-1} \\ &+ \sum_{k, \ell \in \mathbb{Z}} ([f'_0, e'_{2n-1}]' \otimes s^{k+\ell} t + (f'_0 | e'_{2n-1})' \overline{s^\ell ds^k t}) ((-1)^\ell + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1}. \end{aligned}$$

Each of the bilinear forms is 0. Now, $[f'_0, e'_1]' \in (A_{2n-1})_{-\theta + \alpha'_1}$. By §6.7 in [K], $-\theta + \alpha'_1 = -\alpha'_2 - \dots - \alpha'_{2n-1} \in \dot{\Delta}$, so $[f'_0, e'_1]'$ is nonzero. Similarly, $[f'_0, e'_{2n-1}]'$ is nonzero since $-\theta + \alpha'_{2n-1} = -\alpha'_1 - \dots - \alpha'_{2n-2} \in \dot{\Delta}$. Therefore, the calculation simplifies to:

$$= \sum_{k, \ell \in \mathbb{Z}} ([f'_0, e'_1]' \otimes s^{k+\ell} t (1 + (-1)^k) + [f'_0, e'_{2n-1}]' \otimes s^{k+\ell} t ((-1)^\ell + (-1)^{k+\ell})) z_2^{-k-1} z_1^{-\ell-1}.$$

We now must apply $X(\alpha_p, z_3) = \sum_{m \in \mathbb{Z}} (f'_0 \otimes s^m + f'_0 \otimes (-s)^m) t z_3^{-m-1}$ on the left.

$$\begin{aligned}
& \left[\sum_{m \in \mathbb{Z}} (f'_0 \otimes s^m + f'_0 \otimes (-s)^m) t z_3^{-m-1}, \right. \\
& \quad \left. \sum_{k, \ell \in \mathbb{Z}} ([f'_0, e'_1]' \otimes s^{k+\ell} t (1 + (-1)^k) + [f'_0, e'_{2n-1}]' \otimes s^{k+\ell} t ((-1)^\ell + (-1)^{k+\ell})) z_2^{-k-1} z_1^{-\ell-1} \right] \\
= & \sum_{\substack{k, \ell, m \in \mathbb{Z} \\ \cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}}} ([f'_0, [f'_0, e'_1]']' \otimes s^{k+\ell+m} t^2 + (f'_0 | [f'_0, e'_1]')' \overline{s^{k+\ell} t d s^m t}) (1 + (-1)^k) (1 + (-1)^m) \\
& + \sum_{\substack{k, \ell, m \in \mathbb{Z} \\ \cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}}} ([f'_0, [f'_0, e'_{2n-1}]']' \otimes s^{k+\ell+m} t^2 + (f'_0 | [f'_0, e'_{2n-1}]')' \overline{s^{k+\ell} t d s^m t}) ((-1)^\ell + (-1)^{k+\ell}) (1 + (-1)^m)
\end{aligned}$$

Each of the bilinear forms is 0. The bracket $[f'_0, [f'_0, e'_{2n-1}]']' \in (A_{2n-1})_{-2\theta + \alpha'_1}$, but by §6.7 in [K], $-2\theta + \alpha'_1 \notin \dot{\Delta}$, so $[f'_0, [f'_0, e'_{2n-1}]']' = 0$. By a similar argument, $[f'_0, [f'_0, e'_{2n-1}]']' = 0$. Hence, this entire calculation is:

$$\begin{aligned}
& = 0 \\
& = \overline{\psi}(\text{ad}X^{\bar{\sigma}}(\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_2)X^{\bar{\sigma}}(\alpha_m, z_1)).
\end{aligned}$$

Similarly, for $-\alpha_p, -\alpha_m$:

$$\begin{aligned}
& [\overline{\psi}(X(-\alpha_p, z_2)), \overline{\psi}(X(-\alpha_m, z_1))] \\
= & \left[\sum_{k \in \mathbb{Z}} (e'_0 \otimes s^k + e'_0 \otimes (-s)^k) t^{-1} z_2^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_1 \otimes s^\ell + f'_{2n-1} \otimes (-s)^\ell) z_1^{-\ell-1} \right] \\
= & \sum_{\substack{k, \ell \in \mathbb{Z} \\ \cdot z_2^{-k-1} z_1^{-\ell-1}}} ([e'_0, f'_1]' \otimes s^{k+\ell} t^{-1} + (e'_0 | f'_1)' \overline{s^\ell d s^k t^{-1}}) (1 + (-1)^k) z_2^{-k-1} z_1^{-\ell-1} \\
& + \sum_{\substack{k, \ell \in \mathbb{Z} \\ \cdot z_2^{-k-1} z_1^{-\ell-1}}} ([e'_0, f'_{2n-1}]' \otimes s^{k+\ell} t^{-1} + (e'_0 | f'_{2n-1})' \overline{s^\ell d s^k t^{-1}}) ((-1)^\ell + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1}.
\end{aligned}$$

Each of the bilinear forms is 0. Now, $[e'_0, f'_1]' \in (A_{2n-1})_{\theta - \alpha'_1}$. By §6.7 in [K], $\theta - \alpha'_1 = \alpha'_2 + \cdots + \alpha'_{2n-1} \in \dot{\Delta}$, so $[e'_0, f'_1]'$ is nonzero. Similarly, $[e'_0, f'_{2n-1}]'$ is nonzero since $\theta - \alpha'_{2n-1} = \alpha'_1 + \cdots + \alpha'_{2n-2} \in \dot{\Delta}$. Therefore, the calculation simplifies to:

$$= \sum_{k, \ell \in \mathbb{Z}} ([e'_0, f'_1]' \otimes s^{k+\ell} t^{-1} (1 + (-1)^k) + [e'_0, f'_{2n-1}]' \otimes s^{k+\ell} t^{-1} ((-1)^\ell + (-1)^{k+\ell})) z_2^{-k-1} z_1^{-\ell-1}.$$

We now must apply $X(-\alpha_p, z_3) = \sum_{m \in \mathbb{Z}} (e'_0 \otimes s^m + e'_0 \otimes (-s)^m) t^{-1} z_3^{-m-1}$ on the left.

$$\begin{aligned}
& \left[\sum_{m \in \mathbb{Z}} (e'_0 \otimes s^m + e'_0 \otimes (-s)^m) t^{-1} z_3^{-m-1}, \right. \\
& \quad \left. \sum_{k, \ell \in \mathbb{Z}} ([e'_0, f'_1]' \otimes s^{k+\ell} t^{-1} (1 + (-1)^k) + [e'_0, f'_{2n-1}]' \otimes s^{k+\ell} t^{-1} ((-1)^\ell + (-1)^{k+\ell})) z_2^{-k-1} z_1^{-\ell-1} \right] \\
= & \sum_{\substack{k, \ell, m \in \mathbb{Z} \\ \cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}}} ([e'_0, [e'_0, f'_1]']' \otimes s^{k+\ell+m} t^{-2} + (e'_0 | [e'_0, f'_1]')' \overline{s^{k+\ell} t^{-1} d s^m t^{-1}}) (1 + (-1)^k) (1 + (-1)^m) \\
& + \sum_{\substack{k, \ell, m \in \mathbb{Z} \\ \cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}}} ([e'_0, [e'_0, f'_{2n-1}]']' \otimes s^{k+\ell+m} t^{-2} + (e'_0 | [e'_0, f'_{2n-1}]')' \overline{s^{k+\ell} t^{-1} d s^m t^{-1}}) ((-1)^\ell + (-1)^{k+\ell})
\end{aligned}$$

$$(1 + (-1)^m)z_3^{-m-1}z_2^{-k-1}z_1^{-\ell-1}.$$

Each of the bilinear forms is 0. The bracket $[e'_0, [e'_0, f'_{2n-1}]']' \in (A_{2n-1})_{2\theta-\alpha'_1}$, but by §6.7 in [K], $2\theta - \alpha'_1 \notin \dot{\Delta}$, so $[e'_0, [e'_0, f'_{2n-1}]']' = 0$. By a similar argument, $[e'_0, [e'_0, f'_{2n-1}]']' = 0$. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \bar{\psi}(\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_2)X^{\bar{\sigma}}(-\alpha_m, z_1)). \end{aligned}$$

Now we compute for $1 \leq p, m \leq n$ where $|p - m| = 1$ and $m \neq n$.

$$\begin{aligned} &[\bar{\psi}(X(\alpha_p, z_2)), \bar{\psi}(X(\alpha_m, z_1))] \\ &= \left[\sum_{k \in \mathbb{Z}} (e'_p \otimes s^k + e'_{2n-p} \otimes (-s)^k) z_2^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_m \otimes s^\ell + e'_{2n-m} \otimes (-s)^\ell) z_1^{-\ell-1} \right] \\ &= \sum_{k, \ell \in \mathbb{Z}} ([e'_p, e'_m]' \otimes s^{k+\ell} + (e'_p | e'_m)' \overline{s^\ell ds^k}) z_1^{-k-1} z_2^{-\ell-1} \\ &\quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_p, e'_{2n-m}]' \otimes s^{k+\ell} + (e'_p | e'_{2n-m})' \overline{s^\ell ds^k}) (-1)^\ell z_1^{-k-1} z_2^{-\ell-1} \\ &\quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_{2n-p}, e'_m]' \otimes s^{k+\ell} + (e'_{2n-p} | e'_m)' \overline{s^\ell ds^k}) (-1)^k z_1^{-k-1} z_2^{-\ell-1} \\ &\quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_{2n-p}, e'_{2n-m}]' \otimes s^{k+\ell} + (e'_{2n-p} | e'_{2n-m})' \overline{s^\ell ds^k}) (-1)^{k+\ell} z_1^{-k-1} z_2^{-\ell-1}. \end{aligned}$$

Each of the bilinear forms is 0. The bracket $[e'_p, e'_m]' \in (A_{2n-1})_{\alpha'_p + \alpha'_m}$, and $\alpha'_p + \alpha'_m \in \dot{\Delta}$ by [K] §6.7 and the condition $|p - m| = 1$, so $[e'_p, e'_m]'$ is nonzero. Similarly, $[e'_{2n-p}, e'_{2n-m}]'$ is nonzero. However, since $\alpha'_p + \alpha'_{2n-m} \notin \dot{\Delta}$ and $\alpha'_{2n-p} + \alpha'_m \notin \dot{\Delta}$, so $[e'_p, e'_{2n-m}]' = 0$ and $[e'_{2n-p}, e'_m]' = 0$.

Thus the calculation simplifies to:

$$= \sum_{k, \ell \in \mathbb{Z}} ([e'_p, e'_m]' \otimes s^{k+\ell} + [e'_{2n-p}, e'_{2n-m}]' \otimes (-s)^{k+\ell}) z_1^{-k-1} z_2^{-\ell-1}.$$

We now must apply $X(\alpha_p, z_3) = \sum_{m \in \mathbb{Z}} (e'_p \otimes s^m + e'_{2n-p} \otimes (-s)^m) z_3^{-m-1}$ on the left.

$$\begin{aligned} &\left[\sum_{m \in \mathbb{Z}} (e'_p \otimes s^m + e'_{2n-p} \otimes (-s)^m) z_3^{-m-1}, \right. \\ &\quad \left. \sum_{k, \ell \in \mathbb{Z}} ([e'_p, e'_m]' \otimes s^{k+\ell} + [e'_{2n-p}, e'_{2n-m}]' \otimes (-s)^{k+\ell}) z_1^{-k-1} z_2^{-\ell-1} \right] \\ &= \sum_{k, \ell, m \in \mathbb{Z}} ([e'_p, [e'_p, e'_m]']' \otimes s^{k+\ell+m} + (e'_p | [e'_p, e'_m]')' \overline{s^{k+\ell} ds^m}) z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ &\quad + \sum_{k, \ell, m \in \mathbb{Z}} ([e'_p, [e'_{2n-p}, e'_{2n-m}]']' \otimes s^{k+\ell+m} + (e'_p | [e'_{2n-p}, e'_{2n-m}]')' \overline{s^{k+\ell} ds^m}) (-1)^{k+\ell} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ &\quad + \sum_{k, \ell, m \in \mathbb{Z}} ([e'_{2n-p}, [e'_p, e'_m]']' \otimes s^{k+\ell+m} + (e'_{2n-p} | [e'_p, e'_m]')' \overline{s^{k+\ell} ds^m}) (-1)^m z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ &\quad + \sum_{k, \ell, m \in \mathbb{Z}} ([e'_{2n-p}, [e'_{2n-p}, e'_{2n-m}]']' \otimes s^{k+\ell+m} + (e'_{2n-p} | [e'_{2n-p}, e'_{2n-m}]')' \overline{s^{k+\ell} ds^m}) (-1)^{k+\ell+m} \\ &\quad \cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}. \end{aligned}$$

Each of the bilinear forms is 0. The brackets are in A_{2n-1} root spaces of roots $2\alpha'_p + \alpha'_m$, $\alpha'_p + \alpha'_{2n-p} + \alpha'_{2n-m}$, $\alpha'_p + \alpha'_m + \alpha'_{2n-p}$, and $2\alpha'_{2n-p} + \alpha'_{2n-m}$, respectively, none of which are in $\dot{\Delta}$ by [K] §6.7 and the $|p - m| = 1, m \neq n$ conditions. Hence, this entire calculation is

$$= 0$$

$$= \bar{\psi}(\text{ad}X^{\bar{\sigma}}(\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_2)X^{\bar{\sigma}}(\alpha_m, z_1)).$$

Similarly, for $-\alpha_p, -\alpha_m$:

$$\begin{aligned} & [\bar{\psi}(X(-\alpha_p, z_2)), \bar{\psi}(X(-\alpha_m, z_1))] \\ &= \left[\sum_{k \in \mathbb{Z}} (f'_p \otimes s^k + f'_{2n-p} \otimes (-s)^k) z_2^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_m \otimes s^\ell + f'_{2n-m} \otimes (-s)^\ell) z_1^{-\ell-1} \right] \\ &= \sum_{k, \ell \in \mathbb{Z}} ([f'_p, f'_m]' \otimes s^{k+\ell} + (f'_p | f'_m)' \overline{s^\ell ds^k}) z_1^{-k-1} z_2^{-\ell-1} \\ &+ \sum_{k, \ell \in \mathbb{Z}} ([f'_p, f'_{2n-m}]' \otimes s^{k+\ell} + (f'_p | f'_{2n-m})' \overline{s^\ell ds^k}) (-1)^\ell z_1^{-k-1} z_2^{-\ell-1} \\ &+ \sum_{k, \ell \in \mathbb{Z}} ([f'_{2n-p}, f'_m]' \otimes s^{k+\ell} + (f'_{2n-p} | f'_m)' \overline{s^\ell ds^k}) (-1)^k z_1^{-k-1} z_2^{-\ell-1} \\ &+ \sum_{k, \ell \in \mathbb{Z}} ([f'_{2n-p}, f'_{2n-m}]' \otimes s^{k+\ell} + (f'_{2n-p} | f'_{2n-m})' \overline{s^\ell ds^k}) (-1)^{k+\ell} z_1^{-k-1} z_2^{-\ell-1}. \end{aligned}$$

Each of the bilinear forms is 0. The bracket $[f'_p, f'_m]' \in (A_{2n-1})_{-\alpha'_p - \alpha'_m}$, and $-\alpha'_p - \alpha'_m \in \dot{\Delta}$ by [K] §6.7 and the condition $|p - m| = 1$, so $[f'_p, f'_m]'$ is nonzero. Similarly, $[f'_{2n-p}, f'_{2n-m}]'$ is nonzero. However, since $-\alpha'_p - \alpha'_{2n-m} \notin \dot{\Delta}$ and $-\alpha'_{2n-p} - \alpha'_m \notin \dot{\Delta}$, so $[f'_p, f'_{2n-m}]' = 0$ and $[f'_{2n-p}, f'_m]' = 0$. Thus the calculation simplifies to:

$$= \sum_{k, \ell \in \mathbb{Z}} ([f'_p, f'_m]' \otimes s^{k+\ell} + [f'_{2n-p}, f'_{2n-m}]' \otimes (-s)^{k+\ell}) z_1^{-k-1} z_2^{-\ell-1}.$$

We now must apply $X(-\alpha_p, z_3) = \sum_{m \in \mathbb{Z}} (f'_p \otimes s^m + f'_{2n-p} \otimes (-s)^m) z_3^{-m-1}$ on the left.

$$\begin{aligned} & \left[\sum_{m \in \mathbb{Z}} (f'_p \otimes s^m + f'_{2n-p} \otimes (-s)^m) z_3^{-m-1}, \right. \\ & \quad \left. \sum_{k, \ell \in \mathbb{Z}} ([f'_p, f'_m]' \otimes s^{k+\ell} + [f'_{2n-p}, f'_{2n-m}]' \otimes (-s)^{k+\ell}) z_1^{-k-1} z_2^{-\ell-1} \right] \\ &= \sum_{k, \ell, m \in \mathbb{Z}} ([f'_p, [f'_p, f'_m]]' \otimes s^{k+\ell+m} + (f'_p | [f'_p, f'_m])' \overline{s^{k+\ell} ds^m}) z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ &+ \sum_{k, \ell, m \in \mathbb{Z}} ([f'_p, [f'_{2n-p}, f'_{2n-m}]]' \otimes s^{k+\ell+m} + (f'_p | [f'_{2n-p}, f'_{2n-m}])' \overline{s^{k+\ell} ds^m}) (-1)^{k+\ell} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ &+ \sum_{k, \ell, m \in \mathbb{Z}} ([f'_{2n-p}, [f'_p, f'_m]]' \otimes s^{k+\ell+m} + (f'_{2n-p} | [f'_p, f'_m])' \overline{s^{k+\ell} ds^m}) (-1)^m z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ &+ \sum_{k, \ell, m \in \mathbb{Z}} ([f'_{2n-p}, [f'_{2n-p}, f'_{2n-m}]]' \otimes s^{k+\ell+m} + (f'_{2n-p} | [f'_{2n-p}, f'_{2n-m}])' \overline{s^{k+\ell} ds^m}) (-1)^{k+\ell+m} \\ &\quad \cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}. \end{aligned}$$

Each of the bilinear forms is 0. The brackets are in A_{2n-1} root spaces of roots $-2\alpha'_p - \alpha'_m$, $-\alpha'_p -$

$\alpha'_{2n-p} - \alpha'_{2n-m}$, $-\alpha'_p - \alpha'_m - \alpha'_{2n-p}$, and $-2\alpha'_{2n-p} - \alpha'_{2n-m}$, respectively, none of which are in $\dot{\Delta}$ by [K] §6.7 and the $|p - m| = 1, m \neq n$ conditions. Hence, this entire calculation is

$$= 0$$

$$= \bar{\psi}(\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_2)X^{\bar{\sigma}}(-\alpha_m, z_1)).$$

Relation (19): $\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_4)\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_2)X^{\bar{\sigma}}(\pm\alpha_m, z_1) = 0$ if $c_{pm} = -2$

There are two cases to consider: $p = 1, m = 0$ and $p = n - 1, m = n$. First we compute for $p = 1, m = 0$ and α_p, α_m .

$$\begin{aligned} & [\bar{\psi}(X(\alpha_p, z_2)), \bar{\psi}(X(\alpha_m, z_1))] \\ &= \left[\sum_{k \in \mathbb{Z}} (e'_1 \otimes s^k + e'_{2n-1} \otimes (-s)^k) z_2^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_0 \otimes s^\ell + f'_0 \otimes (-s)^\ell) t z_1^{-\ell-1} \right] \\ &= \sum_{k, \ell \in \mathbb{Z}} ([e'_1, f'_0]' \otimes s^{k+\ell} t + (e'_1 | f'_0)' \overline{s^\ell t ds^k}) (1 + (-1)^\ell) z_2^{-k-1} z_1^{-\ell-1} \\ &\quad + \sum_{k, \ell \in \mathbb{Z}} ([e'_{2n-1}, f'_0]' \otimes s^{k+\ell} t + (e'_{2n-1} | f'_0)' \overline{s^\ell t ds^k}) ((-1)^k + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1}. \end{aligned}$$

We have $(e'_1 | f'_0)' = 0 = (e'_{2n-1} | f'_0)'$ and $[e'_1, f'_0]' \in (A_{2n-1})_{-\theta+\alpha'_1}$ and $[e'_{2n-1}, f'_0]' \in (A_{2n-1})_{-\theta+\alpha'_{2n-1}}$. Each of $-\theta + \alpha'_1$ and $-\theta + \alpha'_{2n-1}$ are elements of $\dot{\Delta}$ by [K] §6.7. Hence these brackets are nonzero

and the calculation becomes:

$$\begin{aligned} & \sum_{k, \ell \in \mathbb{Z}} ([e'_1, f'_0]' \otimes s^{k+\ell} t) (1 + (-1)^\ell) z_2^{-k-1} z_1^{-\ell-1} \\ &+ \sum_{k, \ell \in \mathbb{Z}} ([e'_{2n-1}, f'_0]' \otimes s^{k+\ell} t) ((-1)^k + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1}. \end{aligned}$$

We now must apply $X(\alpha_p, z_3) = \sum_{m \in \mathbb{Z}} (e'_1 \otimes s^m + e'_{2n-1} \otimes (-s)^m) z_3^{-m-1}$ on the left.

$$\begin{aligned} & \left[\sum_{m \in \mathbb{Z}} (e'_1 \otimes s^m + e'_{2n-1} \otimes (-s)^m) z_3^{-m-1}, \right. \\ & \sum_{k, \ell \in \mathbb{Z}} ([e'_1, f'_0]' \otimes s^{k+\ell} t) (1 + (-1)^\ell) z_2^{-k-1} z_1^{-\ell-1} \\ &+ \sum_{k, \ell \in \mathbb{Z}} ([e'_{2n-1}, f'_0]' \otimes s^{k+\ell} t) ((-1)^k + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1} \Big] \\ &= \sum_{k, \ell, m \in \mathbb{Z}} ([e'_1, [e'_1, f'_0]']' \otimes s^{k+\ell+m} t + (e'_1 | [e'_1, f'_0]') \overline{s^{k+\ell} t ds^m}) (1 + (-1)^\ell) z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ &+ \sum_{k, \ell, m \in \mathbb{Z}} ([e'_{2n-1}, [e'_1, f'_0]']' \otimes s^{k+\ell+m} t + (e'_{2n-1} | [e'_1, f'_0]') \overline{s^{k+\ell} t ds^m}) (1 + (-1)^\ell) (-1)^m \\ &\cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ &+ \sum_{k, \ell, m \in \mathbb{Z}} ([e'_1, [e'_{2n-1}, f'_0]']' \otimes s^{k+\ell+m} t + (e'_1 | [e'_{2n-1}, f'_0]') \overline{s^{k+\ell} t ds^m}) ((-1)^k + (-1)^{k+\ell}) \\ &\cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ &+ \sum_{k, \ell, m \in \mathbb{Z}} ([e'_{2n-1}, [e'_{2n-1}, f'_0]']' \otimes s^{k+\ell+m} t + (e'_{2n-1} | [e'_{2n-1}, f'_0]') \overline{s^{k+\ell} t ds^m}) ((-1)^k + (-1)^{k+\ell}) (-1)^m \end{aligned}$$

$$\cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}.$$

All bilinear forms involved are 0. The bracket $[e'_1, [e'_1, f'_0]']' \in (A_{2n-1})_{-\theta+2\alpha'_1}$, but $-\theta+2\alpha'_1 \notin \dot{\Delta}$ by [K] §6.7, so this bracket is 0. By a similar argument, $[e'_{2n-1}, [e'_{2n-1}, f'_0]']' = 0$. However, the brackets $[e'_{2n-1}, [e'_1, f'_0]']'$ and $[e'_1, [e'_{2n-1}, f'_0]']'$ are each in the $(A_{2n-1})_{-\theta+\alpha'_1+\alpha'_{2n-1}}$ root space, and this is nonzero since $-\theta+\alpha'_1+\alpha'_{2n-1} = -\alpha'_2 - \cdots - \alpha'_{2n-2} \in \dot{\Delta}$. Therefore, the calculation simplifies to:

$$= \sum_{k,\ell,m \in \mathbb{Z}} ([e'_{2n-1}, [e'_1, f'_0]']' \otimes s^{k+\ell+m} t) (1 + (-1)^\ell) (-1)^m z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ + \sum_{k,\ell,m \in \mathbb{Z}} ([e'_1, [e'_{2n-1}, f'_0]']' \otimes s^{k+\ell+m} t) ((-1)^k + (-1)^{k+\ell}) z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}.$$

We now apply $X(\alpha_p, z_4) = \sum_{q \in \mathbb{Z}} (e'_1 \otimes s^q + e'_{2n-1} \otimes (-s)^q) z_4^{-q-1}$ on the left.

$$\left[\sum_{q \in \mathbb{Z}} (e'_1 \otimes s^q + e'_{2n-1} \otimes (-s)^q) z_4^{-q-1}, \right. \\ = \sum_{k,\ell,m \in \mathbb{Z}} ([e'_{2n-1}, [e'_1, f'_0]']' \otimes s^{k+\ell+m} t) (1 + (-1)^\ell) (-1)^m z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ + \sum_{k,\ell,m \in \mathbb{Z}} ([e'_1, [e'_{2n-1}, f'_0]']' \otimes s^{k+\ell+m} t) ((-1)^k + (-1)^{k+\ell}) z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \Big] \\ = \sum_{k,\ell,m,q \in \mathbb{Z}} ([e'_1, [e'_{2n-1}, [e'_1, f'_0]']']' \otimes s^{k+\ell+m+q} t + (e'_1 | [e'_{2n-1}, [e'_1, f'_0]']')' \overline{s^{k+\ell+m} t d s^q}) (1 + (-1)^\ell) \\ \cdot (-1)^m z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ + \sum_{k,\ell,m,q \in \mathbb{Z}} ([e'_{2n-1}, [e'_{2n-1}, [e'_1, f'_0]']']' \otimes s^{k+\ell+m+q} t + (e'_{2n-1} | [e'_{2n-1}, [e'_1, f'_0]']')' \overline{s^{k+\ell+m} t d s^q}) \\ \cdot (1 + (-1)^\ell) (-1)^{m+q} z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ + \sum_{k,\ell,m,q \in \mathbb{Z}} ([e'_1, [e'_1, [e'_{2n-1}, f'_0]']']' \otimes s^{k+\ell+m+q} t + (e'_1 | [e'_1, [e'_{2n-1}, f'_0]']')' \overline{s^{k+\ell+m} t d s^q}) \\ \cdot ((-1)^k + (-1)^{k+\ell}) z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ + \sum_{k,\ell,m,q \in \mathbb{Z}} ([e'_{2n-1}, [e'_1, [e'_{2n-1}, f'_0]']']' \otimes s^{k+\ell+m+q} t + (e'_{2n-1} | [e'_1, [e'_{2n-1}, f'_0]']')' \overline{s^{k+\ell+m} t d s^q}) ((-1)^k + \\ \left. (-1)^{k+\ell}) (-1)^q z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}. \right]$$

As before, all bilinear forms involved are 0. Each bracket is either in the $(A_{2n-1})_{-\theta+\alpha'_1+2\alpha'_{2n-1}}$ or the $(A_{2n-1})_{-\theta+2\alpha'_1+\alpha'_{2n-1}}$ root space, both of which are 0 spaces since $-\theta+\alpha'_1+2\alpha'_{2n-1}$ and $-\theta+2\alpha'_1+\alpha'_{2n-1}$ are not roots. Therefore, this entire calculation is:

$$= 0 \\ = \overline{\psi}(\text{ad}X^{\bar{\sigma}}(\alpha_p, z_4)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_2)X^{\bar{\sigma}}(\alpha_m, z_1)).$$

We now calculate likewise for $-\alpha_p, -\alpha_m$.

$$[\overline{\psi}(X(-\alpha_p, z_2)), \overline{\psi}(X(-\alpha_m, z_1))]$$

$$\begin{aligned}
&= \left[\sum_{k \in \mathbb{Z}} (f'_1 \otimes s^k + f'_{2n-1} \otimes (-s)^k) z_2^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_0 \otimes s^\ell + e'_0 \otimes (-s)^\ell) t^{-1} z_1^{-\ell-1} \right] \\
&= \sum_{k, \ell \in \mathbb{Z}} ([f'_1, e'_0]' \otimes s^{k+\ell} t^{-1} + (f'_1 | e'_0)' \overline{s^\ell t^{-1} ds^k}) (1 + (-1)^\ell) z_2^{-k-1} z_1^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} ([f'_{2n-1}, e'_0]' \otimes s^{k+\ell} t^{-1} + (f'_{2n-1} | e'_0)' \overline{s^\ell t^{-1} ds^k}) ((-1)^k + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1}.
\end{aligned}$$

We have $(f'_1 | e'_0)' = 0 = (f'_{2n-1} | e'_0)', [f'_1, e'_0]' \in (A_{2n-1})_{\theta - \alpha'_1}$, and $[f'_{2n-1}, e'_0]' \in (A_{2n-1})_{\theta - \alpha'_{2n-1}}$. Each of $\theta - \alpha'_1$ and $\theta - \alpha'_{2n-1}$ are elements of $\dot{\Delta}$ by [K] §6.7. Hence these brackets are nonzero and the calculation becomes:

$$\begin{aligned}
&= \sum_{k, \ell \in \mathbb{Z}} ([f'_1, e'_0]' \otimes s^{k+\ell} t^{-1}) (1 + (-1)^\ell) z_2^{-k-1} z_1^{-\ell-1} \\
&\quad + \sum_{k, \ell \in \mathbb{Z}} ([f'_{2n-1}, e'_0]' \otimes s^{k+\ell} t^{-1}) ((-1)^k + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1}.
\end{aligned}$$

We now must apply $X(-\alpha_p, z_3) = \sum_{m \in \mathbb{Z}} (f'_1 \otimes s^m + f'_{2n-1} \otimes (-s)^m) z_3^{-m-1}$ on the left.

$$\begin{aligned}
&\left[\sum_{m \in \mathbb{Z}} (f'_1 \otimes s^m + f'_{2n-1} \otimes (-s)^m) z_3^{-m-1}, \right. \\
&\quad \sum_{k, \ell \in \mathbb{Z}} ([f'_1, e'_0]' \otimes s^{k+\ell} t^{-1}) (1 + (-1)^\ell) z_2^{-k-1} z_1^{-\ell-1} \\
&\quad \left. + \sum_{k, \ell \in \mathbb{Z}} ([f'_{2n-1}, e'_0]' \otimes s^{k+\ell} t^{-1}) ((-1)^k + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1} \right] \\
&= \sum_{k, \ell, m \in \mathbb{Z}} ([f'_1, [f'_1, e'_0]']' \otimes s^{k+\ell+m} t^{-1} + (f'_1 | [f'_1, e'_0]')' \overline{s^{k+\ell} t^{-1} ds^m}) (1 + (-1)^\ell) z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
&\quad + \sum_{k, \ell, m \in \mathbb{Z}} ([f'_{2n-1}, [f'_1, e'_0]']' \otimes s^{k+\ell+m} t^{-1} + (f'_{2n-1} | [f'_1, e'_0]')' \overline{s^{k+\ell} t^{-1} ds^m}) (1 + (-1)^\ell) (-1)^m \\
&\quad \cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
&\quad + \sum_{k, \ell, m \in \mathbb{Z}} ([f'_1, [f'_{2n-1}, e'_0]']' \otimes s^{k+\ell+m} t^{-1} + (f'_1 | [f'_{2n-1}, e'_0]')' \overline{s^{k+\ell} t^{-1} ds^m}) ((-1)^k + (-1)^{k+\ell}) \\
&\quad \cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
&\quad + \sum_{k, \ell, m \in \mathbb{Z}} ([f'_{2n-1}, [f'_{2n-1}, e'_0]']' \otimes s^{k+\ell+m} t^{-1} + (f'_{2n-1} | [f'_{2n-1}, e'_0]')' \overline{s^{k+\ell} t^{-1} ds^m}) ((-1)^k + (-1)^{k+\ell}) \\
&\quad \cdot (-1)^m z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}.
\end{aligned}$$

All bilinear forms involved are 0. The bracket $[f'_1, [f'_1, e'_0]']' \in (A_{2n-1})_{\theta - 2\alpha'_1}$, but $\theta - 2\alpha'_1 \notin \dot{\Delta}$ by [K] §6.7, so this bracket is 0. By a similar argument, $[f'_{2n-1}, [f'_{2n-1}, e'_0]']' = 0$. However, the brackets $[f'_{2n-1}, [f'_1, e'_0]']'$ and $[f'_1, [f'_{2n-1}, e'_0]']'$ are each in the $(A_{2n-1})_{\theta - \alpha'_1 - \alpha'_{2n-1}}$ root space, and this is nonzero since $\theta - \alpha'_1 - \alpha'_{2n-1} = \alpha'_2 + \dots + \alpha'_{2n-2} \in \dot{\Delta}$. Therefore, the calculation simplifies to:

$$= \sum_{k, \ell, m \in \mathbb{Z}} ([f'_{2n-1}, [f'_1, e'_0]']' \otimes s^{k+\ell+m} t^{-1}) (1 + (-1)^\ell) (-1)^m z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}$$

$$+ \sum_{k,\ell,m \in \mathbb{Z}} ([f'_1, [f'_{2n-1}, e'_0]']' \otimes s^{k+\ell+m} t^{-1}) ((-1)^k + (-1)^{k+\ell}) z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}.$$

We now apply $X(-\alpha_p, z_4) = \sum_{q \in \mathbb{Z}} (f'_1 \otimes s^q + f'_{2n-1} \otimes (-s)^q) z_4^{-q-1}$ on the left.

$$\begin{aligned} & \left[\sum_{q \in \mathbb{Z}} (f'_1 \otimes s^q + f'_{2n-1} \otimes (-s)^q) z_4^{-q-1}, \right. \\ &= \sum_{k,\ell,m \in \mathbb{Z}} ([f'_{2n-1}, [f'_1, e'_0]']' \otimes s^{k+\ell+m} t^{-1}) (1 + (-1)^\ell) (-1)^m z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ &+ \left. \sum_{k,\ell,m \in \mathbb{Z}} ([f'_1, [f'_{2n-1}, e'_0]']' \otimes s^{k+\ell+m} t^{-1}) ((-1)^k + (-1)^{k+\ell}) z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \right] \\ &= \sum_{k,\ell,m,q \in \mathbb{Z}} ([f'_1, [f'_{2n-1}, [f'_1, e'_0]']']' \otimes s^{k+\ell+m+q} t^{-1} + (f'_1|[f'_{2n-1}, [f'_1, e'_0]']')' \overline{s^{k+\ell+m} t^{-1} ds^q}) (1 + (-1)^\ell) \\ &\cdot (-1)^m z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ &+ \sum_{k,\ell,m,q \in \mathbb{Z}} ([f'_{2n-1}, [f'_{2n-1}, [f'_1, e'_0]']']' \otimes s^{k+\ell+m+q} t^{-1} + (f'_{2n-1}|[f'_{2n-1}, [f'_1, e'_0]']')' \overline{s^{k+\ell+m} t^{-1} ds^q}) (1 + (-1)^\ell) \\ &\cdot (-1)^{m+q} z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ &+ \sum_{k,\ell,m,q \in \mathbb{Z}} ([f'_1, [f'_1, [f'_{2n-1}, e'_0]']']' \otimes s^{k+\ell+m+q} t^{-1} + (f'_1|[f'_1, [f'_{2n-1}, e'_0]']')' \overline{s^{k+\ell+m} t^{-1} ds^q}) ((-1)^k + (-1)^{k+\ell}) z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\ &+ \sum_{k,\ell,m,q \in \mathbb{Z}} ([f'_{2n-1}, [f'_1, [f'_{2n-1}, e'_0]']']' \otimes s^{k+\ell+m+q} t^{-1} + (f'_{2n-1}|[f'_1, [f'_{2n-1}, e'_0]']')' \overline{s^{k+\ell+m} t^{-1} ds^q}) \\ &\cdot ((-1)^k + (-1)^{k+\ell}) (-1)^q z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}. \end{aligned}$$

As before, all bilinear forms involved are 0. Each bracket is either in the $(A_{2n-1})_{\theta - \alpha'_1 - 2\alpha'_{2n-1}}$ or the $(A_{2n-1})_{\theta - 2\alpha'_1 - \alpha'_{2n-1}}$ root space, both of which are 0 spaces since $\theta - \alpha'_1 - 2\alpha'_{2n-1}$ and $\theta - 2\alpha'_1 - \alpha'_{2n-1}$ are not roots. Therefore, this entire calculation is:

$$= 0$$

$$= \overline{\psi}(\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_4)\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_2)X^{\bar{\sigma}}(-\alpha_m, z_1)).$$

The final case to consider is $p = n - 1, m = n$. First we compute for α_p, α_m .

$$\begin{aligned} & [\overline{\psi}(X(\alpha_p, z_2)), \overline{\psi}(X(\alpha_m, z_1))] \\ &= \left[\sum_{k \in \mathbb{Z}} (e'_{n-1} \otimes s^k + e'_{n+1} \otimes (-s)^k) z_2^{-k-1}, \sum_{\ell \in \mathbb{Z}} (e'_n \otimes s^\ell + e'_n \otimes (-s)^\ell) z_1^{-\ell-1} \right] \\ &= \sum_{k,\ell \in \mathbb{Z}} ([e'_{n-1}, e'_n]' \otimes s^{k+\ell} + (e'_{n-1}|e'_n)' \overline{s^\ell ds^k}) (1 + (-1)^\ell) z_2^{-k-1} z_1^{-\ell-1} \\ &+ \sum_{k,\ell \in \mathbb{Z}} ([e'_{n+1}, e'_n]' \otimes s^{k+\ell} + (e'_{n+1}|e'_n)' \overline{s^\ell ds^k}) ((-1)^k + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1}. \end{aligned}$$

We have $(e'_{n-1}|e'_n)' = 0 = (e'_{n+1}|e'_n)', [e'_{n-1}, e'_n]' \in (A_{2n-1})_{\alpha'_{n-1} + \alpha'_n}$, and

$[e'_{n+1}, e'_n]' \in (A_{2n-1})_{\alpha'_n + \alpha'_{n+1}}$. Each of $\alpha'_{n-1} + \alpha'_n$ and $\alpha'_n + \alpha'_{n+1}$ are elements of $\dot{\Delta}$ by [K] §6.7.

Hence these brackets are nonzero and the calculation becomes:

$$\begin{aligned}
&= \sum_{k,\ell \in \mathbb{Z}} ([e'_{n-1}, e'_n]' \otimes s^{k+\ell}) (1 + (-1)^\ell) z_2^{-k-1} z_1^{-\ell-1} \\
&+ \sum_{k,\ell \in \mathbb{Z}} ([e'_{n+1}, e'_n]' \otimes s^{k+\ell}) ((-1)^k + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1}.
\end{aligned}$$

We now must apply $X(\alpha_p, z_3) = \sum_{m \in \mathbb{Z}} (e'_{n-1} \otimes s^m + e'_{n+1} \otimes (-s)^m) z_3^{-m-1}$ on the left.

$$\begin{aligned}
&\left[\sum_{m \in \mathbb{Z}} (e'_{n-1} \otimes s^m + e'_{n+1} \otimes (-s)^m) z_3^{-m-1}, \right. \\
&\sum_{k,\ell \in \mathbb{Z}} ([e'_{n-1}, e'_n]' \otimes s^{k+\ell}) (1 + (-1)^\ell) z_2^{-k-1} z_1^{-\ell-1} \\
&+ \sum_{k,\ell \in \mathbb{Z}} ([e'_{n+1}, e'_n]' \otimes s^{k+\ell}) ((-1)^k + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1} \Big] \\
&= \sum_{k,\ell,m \in \mathbb{Z}} ([e'_{n-1}, [e'_{n-1}, e'_n]']' \otimes s^{k+\ell+m} + (e'_{n-1}|[e'_{n-1}, e'_n]')' \overline{s^{k+\ell} ds^m}) (1 + (-1)^\ell) z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
&+ \sum_{k,\ell,m \in \mathbb{Z}} ([e'_{n+1}, [e'_{n-1}, e'_n]']' \otimes s^{k+\ell+m} + (e'_{n+1}|[e'_{n-1}, e'_n]')' \overline{s^{k+\ell} ds^m}) (1 + (-1)^\ell) (-1)^m \\
&\cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
&+ \sum_{k,\ell,m \in \mathbb{Z}} ([e'_{n-1}, [e'_{n+1}, e'_n]']' \otimes s^{k+\ell+m} + (e'_{n-1}|[e'_{n+1}, e'_n]')' \overline{s^{k+\ell} ds^m}) ((-1)^k + (-1)^{k+\ell}) \\
&\cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
&+ \sum_{k,\ell,m \in \mathbb{Z}} ([e'_{n+1}, [e'_{n+1}, e'_n]']' \otimes s^{k+\ell+m} + (e'_{n+1}|[e'_{n+1}, e'_n]')' \overline{s^{k+\ell} ds^m}) ((-1)^k + (-1)^{k+\ell}) (-1)^m \\
&\cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}.
\end{aligned}$$

All bilinear forms involved are 0. The bracket $[e'_{n-1}, [e'_{n-1}, e'_n]']' \in (A_{2n-1})_{2\alpha'_{n-1} + \alpha'_n}$, but $2\alpha'_{n-1} + \alpha'_n \notin \dot{\Delta}$ by [K] §6.7, so this bracket is 0. By a similar argument, $[e'_{n+1}, [e'_{n+1}, e'_n]']' = 0$. However, the brackets $[e'_{n+1}, [e'_{n-1}, e'_n]']'$ and $[e'_{n-1}, [e'_{n+1}, e'_n]']'$ are each in the $(A_{2n-1})_{\alpha'_{n-1} + \alpha'_n + \alpha'_{n+1}}$ root space, and this is nonzero since $\alpha'_{n-1} + \alpha'_n + \alpha'_{n+1} \in \dot{\Delta}$. Therefore, the calculation simplifies to:

$$\begin{aligned}
&= \sum_{k,\ell,m \in \mathbb{Z}} ([e'_{n+1}, [e'_{n-1}, e'_n]']' \otimes s^{k+\ell+m}) (1 + (-1)^\ell) (-1)^m z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
&+ \sum_{k,\ell,m \in \mathbb{Z}} ([e'_{n-1}, [e'_{n+1}, e'_n]']' \otimes s^{k+\ell+m}) ((-1)^k + (-1)^{k+\ell}) z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}.
\end{aligned}$$

We now apply $X(\alpha_p, z_4) = \sum_{q \in \mathbb{Z}} (e'_{n-1} \otimes s^q + e'_{n+1} \otimes (-s)^q) z_4^{-q-1}$ on the left.

$$\begin{aligned}
&\left[\sum_{q \in \mathbb{Z}} (e'_{n-1} \otimes s^q + e'_{n+1} \otimes (-s)^q) z_4^{-q-1}, \right. \\
&\sum_{k,\ell,m \in \mathbb{Z}} ([e'_{n+1}, [e'_{n-1}, e'_n]']' \otimes s^{k+\ell+m}) (1 + (-1)^\ell) (-1)^m z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
&+ \left. \sum_{k,\ell,m \in \mathbb{Z}} ([e'_{n-1}, [e'_{n+1}, e'_n]']' \otimes s^{k+\ell+m}) ((-1)^k + (-1)^{k+\ell}) z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k,\ell,m,q \in \mathbb{Z}} ([e'_{n-1}, [e'_{n+1}, [e'_{n-1}, e'_n]']']' \otimes s^{k+\ell+m+q} + (e'_{n-1}|[e'_{n+1}, [e'_{n-1}, e'_n]']')' \overline{s^{k+\ell+m} ds^q}) \\
&\quad \cdot (1 + (-1)^\ell)(-1)^m z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
&\quad + \sum_{k,\ell,m,q \in \mathbb{Z}} ([e'_{n+1}, [e'_{n+1}, [e'_{n-1}, e'_n]']']' \otimes s^{k+\ell+m+q} + (e'_{n+1}|[e'_{n+1}, [e'_{n-1}, e'_n]']')' \overline{s^{k+\ell+m} ds^q}) \\
&\quad \cdot (1 + (-1)^\ell)(-1)^{m+q} z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
&\quad + \sum_{k,\ell,m,q \in \mathbb{Z}} ([e'_{n-1}, [e'_{n-1}, [e'_{n+1}, e'_n]']']' \otimes s^{k+\ell+m+q} + (e'_{n-1}|[e'_{n-1}, [e'_{n+1}, e'_n]']')' \overline{s^{k+\ell+m} ds^q}) ((-1)^k + \\
&\quad (-1)^{k+\ell}) z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
&\quad + \sum_{k,\ell,m,q \in \mathbb{Z}} ([e'_{n+1}, [e'_{n-1}, [e'_{n+1}, e'_n]']']' \otimes s^{k+\ell+m+q} + (e'_{n+1}|[e'_{n-1}, [e'_{n+1}, e'_n]']')' \overline{s^{k+\ell+m} ds^q}) ((-1)^k + \\
&\quad (-1)^{k+\ell})(-1)^q z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}.
\end{aligned}$$

As before, all bilinear forms involved are 0. Each bracket is either in the $(A_{2n-1})_{\alpha'_{n-1} + \alpha'_n + 2\alpha'_{n+1}}$ or the $(A_{2n-1})_{2\alpha'_{n-1} + \alpha'_n + \alpha'_{n+1}}$ root space, both of which are 0 spaces since $\alpha'_{n-1} + \alpha'_n + 2\alpha'_{n+1}$ and $2\alpha'_{n-1} + \alpha'_n + \alpha'_{n+1}$ are not roots. Therefore, this entire calculation is:

$$\begin{aligned}
&= 0 \\
&= \overline{\psi}(\text{ad}X^{\bar{\sigma}}(\alpha_p, z_4)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_2)X^{\bar{\sigma}}(\alpha_m, z_1)).
\end{aligned}$$

Lastly, we repeat the calculation for $-\alpha_p, -\alpha_m$.

$$\begin{aligned}
&[\overline{\psi}(X(-\alpha_p, z_2)), \overline{\psi}(X(-\alpha_m, z_1))] \\
&= \left[\sum_{k \in \mathbb{Z}} (f'_{n-1} \otimes s^k + f'_{n+1} \otimes (-s)^k) z_2^{-k-1}, \sum_{\ell \in \mathbb{Z}} (f'_n \otimes s^\ell + f'_n \otimes (-s)^\ell) z_1^{-\ell-1} \right] \\
&= \sum_{k,\ell \in \mathbb{Z}} ([f'_{n-1}, f'_n]' \otimes s^{k+\ell} + (f'_{n-1}|f'_n)' \overline{s^\ell ds^k}) (1 + (-1)^\ell) z_2^{-k-1} z_1^{-\ell-1} \\
&\quad + \sum_{k,\ell \in \mathbb{Z}} ([f'_{n+1}, f'_n]' \otimes s^{k+\ell} + (f'_{n+1}|f'_n)' \overline{s^\ell ds^k}) ((-1)^k + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1}.
\end{aligned}$$

We have $(f'_{n-1}|f'_n)' = 0 = (f'_{n+1}|f'_n)', [f'_{n-1}, f'_n]' \in (A_{2n-1})_{-\alpha'_{n-1} - \alpha'_n}$, and

$[f'_{n+1}, f'_n]' \in (A_{2n-1})_{-\alpha'_n - \alpha'_{n+1}}$. Each of $-\alpha'_{n-1} - \alpha'_n$ and $-\alpha'_n - \alpha'_{n+1}$ are elements of Δ by [K] §6.7. Hence these brackets are nonzero and the calculation becomes:

$$\begin{aligned}
&= \sum_{k,\ell \in \mathbb{Z}} ([f'_{n-1}, f'_n]' \otimes s^{k+\ell}) (1 + (-1)^\ell) z_2^{-k-1} z_1^{-\ell-1} \\
&\quad + \sum_{k,\ell \in \mathbb{Z}} ([f'_{n+1}, f'_n]' \otimes s^{k+\ell}) ((-1)^k + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1}.
\end{aligned}$$

We now must apply $X(-\alpha_p, z_3) = \sum_{m \in \mathbb{Z}} (f'_{n-1} \otimes s^m + f'_{n+1} \otimes (-s)^m) z_3^{-m-1}$ on the left.

$$\begin{aligned}
&\left[\sum_{m \in \mathbb{Z}} (f'_{n-1} \otimes s^m + f'_{n+1} \otimes (-s)^m) z_3^{-m-1}, \right. \\
&\quad \left. \sum_{k,\ell \in \mathbb{Z}} ([f'_{n-1}, f'_n]' \otimes s^{k+\ell}) (1 + (-1)^\ell) z_2^{-k-1} z_1^{-\ell-1} \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{k,\ell \in \mathbb{Z}} ([f'_{n+1}, f'_n]' \otimes s^{k+\ell}) ((-1)^k + (-1)^{k+\ell}) z_2^{-k-1} z_1^{-\ell-1} \\
& = \sum_{k,\ell,m \in \mathbb{Z}} ([f'_{n-1}, [f'_{n-1}, f'_n]']' \otimes s^{k+\ell+m} + (f'_{n-1} | [f'_{n-1}, f'_n]')' \overline{s^{k+\ell} ds^m}) (1 + (-1)^\ell) z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
& + \sum_{k,\ell,m \in \mathbb{Z}} ([f'_{n+1}, [f'_{n-1}, f'_n]']' \otimes s^{k+\ell+m} + (f'_{n+1} | [f'_{n-1}, f'_n]')' \overline{s^{k+\ell} ds^m}) (1 + (-1)^\ell) (-1)^m \\
& \cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
& + \sum_{k,\ell,m \in \mathbb{Z}} ([f'_{n-1}, [f'_{n+1}, f'_n]']' \otimes s^{k+\ell+m} + (f'_{n-1} | [f'_{n+1}, f'_n]')' \overline{s^{k+\ell} ds^m}) ((-1)^k + (-1)^{k+\ell}) \\
& \cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
& + \sum_{k,\ell,m \in \mathbb{Z}} ([f'_{n+1}, [f'_{n+1}, f'_n]']' \otimes s^{k+\ell+m} + (f'_{n+1} | [f'_{n+1}, f'_n]')' \overline{s^{k+\ell} ds^m}) ((-1)^k + (-1)^{k+\ell}) (-1)^m \\
& \cdot z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}.
\end{aligned}$$

All bilinear forms involved are 0. The bracket $[f'_{n-1}, [f'_{n-1}, f'_n]']' \in (A_{2n-1})_{-2\alpha'_{n-1}-\alpha'_n}$, but $-2\alpha'_{n-1}-\alpha'_n \notin \dot{\Delta}$ by [K] §6.7, so this bracket is 0. By a similar argument, $[f'_{n+1}, [f'_{n+1}, f'_n]']' = 0$. However, the brackets $[f'_{n+1}, [f'_{n-1}, f'_n]']'$ and $[f'_{n-1}, [f'_{n+1}, f'_n]']'$ are each in the $(A_{2n-1})_{-\alpha'_{n-1}-\alpha'_n-\alpha'_{n+1}}$ root space, and this is nonzero since $-\alpha'_{n-1}-\alpha'_n-\alpha'_{n+1} \in \dot{\Delta}$. Therefore, the calculation simplifies to:

$$\begin{aligned}
& = \sum_{k,\ell,m \in \mathbb{Z}} ([f'_{n+1}, [f'_{n-1}, f'_n]']' \otimes s^{k+\ell+m}) (1 + (-1)^\ell) (-1)^m z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
& + \sum_{k,\ell,m \in \mathbb{Z}} ([f'_{n-1}, [f'_{n+1}, f'_n]']' \otimes s^{k+\ell+m}) ((-1)^k + (-1)^{k+\ell}) z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}.
\end{aligned}$$

We now apply $X(-\alpha_p, z_4) = \sum_{q \in \mathbb{Z}} (f'_{n-1} \otimes s^q + f'_{n+1} \otimes (-s)^q) z_4^{-q-1}$ on the left.

$$\begin{aligned}
& \left[\sum_{q \in \mathbb{Z}} (f'_{n-1} \otimes s^q + f'_{n+1} \otimes (-s)^q) z_4^{-q-1}, \right. \\
& = \sum_{k,\ell,m \in \mathbb{Z}} ([f'_{n+1}, [f'_{n-1}, f'_n]']' \otimes s^{k+\ell+m}) (1 + (-1)^\ell) (-1)^m z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
& + \sum_{k,\ell,m \in \mathbb{Z}} ([f'_{n-1}, [f'_{n+1}, f'_n]']' \otimes s^{k+\ell+m}) ((-1)^k + (-1)^{k+\ell}) z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
& = \sum_{k,\ell,m,q \in \mathbb{Z}} ([f'_{n-1}, [f'_{n+1}, [f'_{n-1}, f'_n]']]' \otimes s^{k+\ell+m+q} + (f'_{n-1} | [f'_{n+1}, [f'_{n-1}, f'_n]'])' \overline{s^{k+\ell+m} ds^q}) (1 + \\
& (-1)^\ell) (-1)^m z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
& + \sum_{k,\ell,m,q \in \mathbb{Z}} ([f'_{n+1}, [f'_{n+1}, [f'_{n-1}, f'_n]']]' \otimes s^{k+\ell+m+q} + (f'_{n+1} | [f'_{n+1}, [f'_{n-1}, f'_n]'])' \overline{s^{k+\ell+m} ds^q}) \\
& \cdot (1 + (-1)^\ell) (-1)^{m+q} z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1} \\
& + \sum_{k,\ell,m,q \in \mathbb{Z}} ([f'_{n-1}, [f'_{n-1}, [f'_{n+1}, f'_n]']]' \otimes s^{k+\ell+m+q} + (f'_{n-1} | [f'_{n-1}, [f'_{n+1}, f'_n]'])' \overline{s^{k+\ell+m} ds^q}) ((-1)^k + \\
& (-1)^{k+\ell}) z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}
\end{aligned}$$

$$+ \sum_{k,\ell,m,q \in \mathbb{Z}} ([f'_{n+1}, [f'_{n-1}, [f'_{n+1}, f'_n]']']' \otimes s^{k+\ell+m+q} + (f'_{n+1} | [f'_{n-1}, [f'_{n+1}, f'_n]']')' \overline{s^{k+\ell+m} ds^q}) ((-1)^k + (-1)^{k+\ell}) (-1)^q z_4^{-q-1} z_3^{-m-1} z_2^{-k-1} z_1^{-\ell-1}.$$

As before, all bilinear forms involved are 0. Each bracket is either in the $(A_{2n-1})_{-\alpha'_{n-1}-\alpha'_n-2\alpha'_{n+1}}$ or the $(A_{2n-1})_{-2\alpha'_{n-1}-\alpha'_n-\alpha'_{n+1}}$ root space, both of which are 0 spaces since $-\alpha'_{n-1}-\alpha'_n-2\alpha'_{n+1}$ and $-2\alpha'_{n-1}-\alpha'_n-\alpha'_{n+1}$ are not roots. Therefore, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \bar{\psi}(\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_4)\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_2)X^{\bar{\sigma}}(-\alpha_m, z_1)). \end{aligned}$$

Therefore, since $\bar{\psi}$ preserves the bracket in all of the defining relations for \bar{t} , we have shown that $\bar{\psi}$ is a homomorphism. This also proves in particular that $\bar{\pi} = \bar{\omega}\bar{\psi}$ is a homomorphism.

Proof of (3): Lastly, we must show that $(\bar{t}, \bar{\pi})$ satisfies the universal property. Let (\mathcal{V}, γ) be any central extension of $L(A_{2n-1}, \sigma)$. Recall the notation for the relevant homomorphisms:

$$\begin{array}{ccc} \mathcal{V} & & \\ \lambda \uparrow & \searrow \gamma & \\ \bar{\pi} & \xrightarrow{\bar{\omega}} & L(A_{2n-1}, \sigma) \\ \bar{\psi} \uparrow & \nearrow \bar{\pi} & \\ \bar{t} & & \end{array}$$

Figure 3.2 Commutative diagram for the MRY presentation (duplicate)

We must show that $\gamma\lambda\bar{\psi} = \bar{\pi}$.

As remarked earlier, the definitions of the maps on the generators of \bar{t} make it clear that $\bar{\omega}\bar{\psi} = \bar{\pi}$. Also, since \mathcal{V} is a central extension, and $(\bar{\pi}, \bar{\omega})$ is the uce of $L(A_{2n-1}, \sigma)$, then by definition $\gamma\lambda = \bar{\omega}$. Therefore, $\bar{\pi} = \bar{\omega}\bar{\psi} = \gamma\lambda\bar{\psi}$ and the universal property thus holds.

This completes the proof of the theorem. ■

Chapter 4

Fermionic Representation

In [FF], Feingold and Frenkel introduced an interesting representation of affine Lie algebras, both untwisted and twisted, using free fields. In the affine case, it has an advantage over the adjoint representation (using the realization as the central extension of a loop algebra) because the latter is not faithful since the center acts trivially. In the so-called FF representation, the center acts nontrivially, and so it is faithful. In the toroidal case, however, because there are infinitely many linearly independent central elements instead of just a one-dimensional central extension, we cannot prove faithfulness so easily. We do not claim that the following representation is faithful.

Two classes of these FF representations exist: fermionic and bosonic. In [JM], Jing and Misra use similar techniques as in [FF] to give a fermionic representation of *untwisted* toroidal Lie algebras (more about constructions of such operators, free fields, and their use in the theory of vertex algebras can be found throughout [K2]). The goal of this section is to use similar techniques, and the MRY presentation above, to give a fermionic representation of the twisted toroidal Lie algebra of type A_{2n-1} .

4.1 Free Field Notation

Let

$$\{\varepsilon_i \mid i = 0, 1, \dots, n+1\}$$

be a basis for \mathbb{C}^{n+2} as a vector space. Define an inner product $(\cdot|\cdot)$ by the condition:

$$(\varepsilon_i|\varepsilon_j) = \delta_{ij}.$$

The $\{\varepsilon_i\}$ basis is thus orthonormal with respect to the inner product $(\cdot|\cdot)$. Let also

$$\bar{c} = \frac{1}{\sqrt{2}}(\varepsilon_0 + i\varepsilon_{n+1}) \text{ and } \beta = -\bar{c} + \varepsilon_1$$

where i is the complex number such that $i^2 = -1$.

Similarly, define elements $\{\varepsilon_j^*, \bar{c}^* = \frac{1}{\sqrt{2}}(\varepsilon_0^* + i\varepsilon_{n+1}^*), \beta^* = -\bar{c}^* + \varepsilon_1^* | j = 0, 1, \dots, n+1\}$ such that $\{\varepsilon_i^*\}$ is a basis for another copy of \mathbb{C}^{n+2} as a vector space. Let $(\cdot|\cdot)$ be a bilinear form among these elements defined by $(\varepsilon_i^*|\varepsilon_j^*) = \delta_{ij}$ (the form uses the same notation as that among the $\{\varepsilon_i\}$ elements).

Form the \mathbb{C} -vector spaces P with basis $\{\varepsilon_i, \bar{c} | i = 1, \dots, n\}$, and P^* with basis $\{\varepsilon_i^*, \bar{c}^* | i = 1, \dots, n\}$.

Definition 20. Define the space $\mathcal{C} := P \oplus P^*$.

Remark. This decomposition of \mathcal{C} is a polarization into maximal isotropic subspaces with respect to the symmetric bilinear form $\langle \cdot, \cdot \rangle$ defined by:

$$\langle a, b \rangle = \langle a^*, b^* \rangle = 0 \text{ and } \langle a^*, b \rangle = \langle a, b^* \rangle = (a|b) \text{ for } a, b \in P.$$

Example 21. We record here some significant computations involving $\langle \cdot, \cdot \rangle$ among elements of \mathcal{C} .

$$\begin{aligned} \langle \bar{c}, \varepsilon_i \rangle &= \langle \bar{c}, \varepsilon_i^* \rangle = 0; \\ \langle \bar{c}^*, \varepsilon_i \rangle &= \langle \bar{c}^*, \varepsilon_i^* \rangle = 0; \\ \langle \bar{c}, \bar{c} \rangle &= \langle \bar{c}^*, \bar{c}^* \rangle = \langle \bar{c}^*, \bar{c} \rangle = 0; \\ \langle \beta, \varepsilon_i \rangle &= \langle \beta^*, \varepsilon_i^* \rangle = 0; \\ \langle \beta, \varepsilon_i^* \rangle &= \langle \beta^*, \varepsilon_i \rangle = \delta_{i1}; \\ \langle \beta, \beta \rangle &= \langle \beta^*, \beta^* \rangle = 0; \\ \langle \beta, \beta^* \rangle &= 1. \end{aligned}$$

Introduce an infinite dimensional Clifford algebra $Cl(P)$, an associative, unital algebra generated by an element \mathcal{K} and the set $\{a(k) | a \in \mathcal{C}, k \in \mathbb{Z} + \frac{1}{2}\}$ and subject to the relations that \mathcal{K} commutes with everything and:

$$\{a(k), b(l)\} = \langle a, b \rangle \delta_{k,-l} \mathcal{K};$$

where $\{a(k), b(l)\} := a(k)b(l) + b(l)a(k)$ and $a, b \in \mathcal{C}, k, l \in \mathbb{Z} + \frac{1}{2}$.

Definition 22. The representation space (or Fock space) is:

$$\overline{V} := \bigotimes_{a_i \in P} \left(\bigotimes_{k \in (\mathbb{Z} + \frac{1}{2})_{>0}} \mathbb{C}[a_i(-k)] \bigotimes_{k \in (\mathbb{Z} + \frac{1}{2})_{>0}} \mathbb{C}[a_i^*(-k)] \right).$$

The action of $Cl(P)$ on \overline{V} , written by juxtaposition, is described here. \overline{V} is a simple $Cl(P)$ -module with a distinguished vacuum vector $1 \in \overline{V}$. For $a \in \mathcal{C}, k \in (\mathbb{Z} + \frac{1}{2})_{>0}$ and any polynomial $v \in \overline{V}$, $a(-k)$ acts by left multiplication (as in $a(-k)v$) and $a(k)$ acts like $\frac{\partial}{\partial a(-k)}(v)$. Notice in particular that $a(k)1 = 0$. Hence, $a(k)$ is called an *annihilation operator* and $a(-k)$ is a *creation operator*. Also, \mathcal{K} acts as the identity.

For any $u \in \mathcal{C}$, we define a *generating function* or *free field operator* with a formal variable z using components and half-integers: $u(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} u(k)z^{-k - \frac{1}{2}}$. A generating function acts on $v \in \overline{V}$ by $u(z) \cdot v = \sum_{k \in \mathbb{Z} + \frac{1}{2}} (u(k) \cdot v)z^{-k - \frac{1}{2}}$. Hence the variable z acts only as a “bookkeeping” device and does not affect the action on \overline{V} .

Definition 23. The *normal ordering* of a quadratic expression $:u(z)v(w):$ is defined on its components by:

$$:u(m)v(n): = \begin{cases} u(m)v(n), & \text{if } m < 0; \\ -v(n)u(m), & \text{if } m > 0; \end{cases}$$

so that $:u(z)v(w): = \sum_{m,n \in \mathbb{Z} + \frac{1}{2}} :u(m)v(n): z^{-m - \frac{1}{2}} w^{-n - \frac{1}{2}}$.

Remark. Notice that $m \neq 0$ since $m \in \mathbb{Z} + \frac{1}{2}$.

Remark. The purpose of normal ordering is to put annihilation operators on the right to make the operators well-defined. The “unordered” operators could make an infinite (not well-defined) sequence since annihilation operators only necessarily kill the vacuum vector.

Remark. Since $:u(z)v(w): = -:v(w)u(z):$, these fields are indeed *fermionic fields*.

As before, the presence of $\delta(z \pm w)$ allows us to replace z with $\mp w$ in these generating functions; in particular, neither using half-integers in generating functions nor having multiple variables in a generating function disrupt this helpful property. We present it again here as a separate lemma since there are some differences in the proof.

Lemma 24. For $u \in \mathcal{C}$, and z, w formal variables, $:u(z)v(w):\delta(z \pm w) = :u(\mp w)v(w):\delta(z \pm w)$.

Proof: Note that $:u(z)v(w):\delta(z \pm w) = \sum_{m,n \in \mathbb{Z} + \frac{1}{2}} :u(m)v(n): z^{-m - \frac{1}{2}} w^{-n - \frac{1}{2}} \sum_{\ell \in \mathbb{Z}} (\mp w)^\ell z^{-\ell - 1}$

$$= \sum_{m,n \in \mathbb{Z} + \frac{1}{2}, \ell \in \mathbb{Z}} :u(m)v(n): z^{-m - \ell - \frac{3}{2}} w^{\ell - n - \frac{1}{2}} (\mp 1)^\ell.$$

Letting $m' = m + \ell + \frac{1}{2}$, $\ell' = m$, $n' = n$ makes this last summation equal to:

$\sum_{\ell', n' \in \mathbb{Z} + \frac{1}{2}, m' \in \mathbb{Z}} :u(\ell')v(n'): z^{-m'-1}w^{(m'-\ell'-\frac{1}{2})-n'-\frac{1}{2}}(\mp 1)^{m'-\ell'-\frac{1}{2}}$ since $\ell = m' - \ell' - \frac{1}{2}$. Thus, this

is equal to:

$$\begin{aligned} &= \sum_{\ell', n' \in \mathbb{Z} + \frac{1}{2}, m' \in \mathbb{Z}} :u(\ell')v(n'): z^{-m'-1}w^{m'-\ell'-n'-1}(\mp 1)^{m'-\ell'-\frac{1}{2}} \\ &= \sum_{\ell', n' \in \mathbb{Z} + \frac{1}{2}} :u(\ell')v(n'): w^{-\ell'-n'-1}(\mp 1)^{-\ell'-\frac{1}{2}} \sum_{m' \in \mathbb{Z}} (\mp w)^{m'} z^{-m'-1} \\ &= \sum_{\ell', n' \in \mathbb{Z} + \frac{1}{2}} :u(\ell')v(n'): (\mp w)^{-\ell'-\frac{1}{2}} w^{-n'-\frac{1}{2}} \sum_{m' \in \mathbb{Z}} (\mp w)^{m'} z^{-m'-1} = :u(\mp w)v(w): \delta(z \pm w). \blacksquare \end{aligned}$$

We can extend the definition of normal ordering to more than 2 fields inductively. For $u_1, u_2, u_3, \dots, u_m \in \mathcal{C}$ and $z_1, z_2, z_3, \dots, z_m$ formal variables,

$$\begin{aligned} &:u_1(z_1)u_2(z_2)\cdots u_m(z_m): = :u_1(z_1)(:u_2(z_2)\cdots u_m(z_m):): \\ &\quad = :u_1(z_1)(:u_2(z_2)(:u_3(z_3)\cdots u_m(z_m):):): \end{aligned}$$

and so on until the innermost parentheses contain only 2 fields. We also make the following notational definitions.

Definition 25. Define the *contraction* of two fields by

$$\underbrace{u(\pm z)v(\pm w)}_{=} = u(\pm z)v(\pm w) - :u(\pm z)v(\pm w):.$$

Definition 26. For $x_1, \dots, x_m \in \mathcal{C}$, define:

$$:x_1 \cdots \underbrace{x_i \cdots x_j}_{\phi} \cdots x_m: = \text{sgn}(\phi) \underbrace{x_i x_j}_{\phi} :x_1 \cdots x_{\hat{i}} \cdots x_{\hat{j}} \cdots x_m:$$

where $x_{\hat{i}}$ indicates that the index i is missing from the list, and $\text{sgn}(\phi)$ is the sign of the permutation $\phi = \begin{pmatrix} 1 & 2 & 3 & \cdots & m \\ i & j & 1 & \cdots & \hat{i} & \cdots & \hat{j} & \cdots & m \end{pmatrix}$, discussed in Chapter 5 of [Gal].

Wick's Theorem is crucial to our calculations. The following proposition and corollary are [FF] Theorem 5 and Corollary 6, respectively, and so are stated here without proof.

Proposition 27. Wick's Theorem for fermions. For $x_1, \dots, x_m, y_1, \dots, y_p \in \mathcal{C}$, we have:

1. $x_1 \cdots x_m = :x_1 \cdots x_m: + \sum :x_1 \cdots \underbrace{x_i \cdots x_j}_{\phi} \cdots x_m:, \text{ where the sum is taken over all combinations of sets of contractions (up to } \frac{m}{2} \text{ contractions if } m \text{ is even, or } \frac{m-1}{2} \text{ contractions if } m \text{ is odd).}$
2. $(:x_1 \cdots x_m:)(:y_1 \cdots y_p:) = :x_1 \cdots x_m y_1 \cdots y_p: + \sum :x_1 \cdots \underbrace{x_i \cdots x_m y_1 \cdots y_j}_{\phi} \cdots y_p:, \text{ where the sum is taken over all combinations of sets of contractions of some } x_i \text{'s with some } y_j \text{'s.}$

Equivalently, $(:x_1 \cdots x_m:)(:y_1 \cdots y_p:)$
 $= \sum_{s=0}^{\min(m,p)} \sum_{i_1 < \cdots < i_s, j_1 \neq \cdots \neq j_s} sgn(\phi) \underbrace{x_{i_1} y_{j_1}} \cdots \underbrace{x_{i_s} y_{j_s}} : x_1 \cdots x_m y_1 \cdots y_p :_{(i_1, \dots, i_s, j_1, \dots, j_s)}$ where
the subscript $(i_1, \dots, i_s, j_1, \dots, j_s)$ means that the elements $x_{i_1}, \dots, x_{i_s}, y_{j_1}, \dots, y_{j_s}$ are removed. ■

Corollary 28. $:x_1 \cdots x_m := (-1)^N :x_{\phi(1)} \cdots x_{\phi(m)}:$, where N is the number of transpositions of fermions in a decomposition of ϕ . ■

Corollary 29. For $u_1, u_2, u_3, u_4 \in \mathcal{C}$ and z, w formal variables, $:u_1(\pm z)u_2(\pm z)u_3(\pm w)u_4(\pm w): = :u_3(\pm w)u_4(\pm w)u_1(\pm z)u_2(\pm z):$

Proof: The permutation $\phi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ can be decomposed as (13)(24), so by the previous corollary, $:u_1u_2u_3u_4 := (-1)^2 :u_3u_4u_2u_1 := :u_3u_4u_1u_2:$. Thus,
 $:u_1(\pm z)u_2(\pm z)u_3(\pm w)u_4(\pm w): = \sum_{k,\ell \in \mathbb{Z} + \frac{1}{2}} :u_1u_2u_3u_4: (\pm z)^{-k-\frac{1}{2}} (\pm w)^{-\ell-\frac{1}{2}} = \sum_{k,\ell \in \mathbb{Z} + \frac{1}{2}} :u_3u_4u_1u_2:$
 $(\pm z)^{-k-\frac{1}{2}} (\pm w)^{-\ell-\frac{1}{2}} = :u_3(\pm z)u_4(\pm z)u_1(\pm w)u_2(\pm w):$, as desired. ■

Proposition 30. $\underbrace{a(\pm z)b(\pm w)}_{\pm z \mp w} = \iota_{z,w} \frac{\langle a,b \rangle}{\pm z \mp w}$ for $a, b \in \mathcal{C}$.

Proof: By definition we have $\underbrace{a(\pm z)b(\pm w)}_{\pm z \mp w} = a(\pm z)b(\pm w) - :a(\pm z)b(\pm w): =$
 $\sum_{k,l \in \mathbb{Z} + \frac{1}{2}} a(k)b(l)(\pm z)^{-k-\frac{1}{2}} (\pm w)^{-l-\frac{1}{2}} - \sum_{k,l \in \mathbb{Z} + \frac{1}{2}} :a(k)b(l): (\pm z)^{-k-\frac{1}{2}} (\pm w)^{-l-\frac{1}{2}}$
 $= \sum_{k,l \in \mathbb{Z} + \frac{1}{2}} a(k)b(l)(\pm z)^{-k-\frac{1}{2}} (\pm w)^{-l-\frac{1}{2}} - \sum_{k \in (\mathbb{Z} + \frac{1}{2})_{<0}, l \in \mathbb{Z} + \frac{1}{2}} a(k)b(l)(\pm z)^{-k-\frac{1}{2}} (\pm w)^{-l-\frac{1}{2}}$
 $+ \sum_{k \in (\mathbb{Z} + \frac{1}{2})_{>0}, l \in \mathbb{Z} + \frac{1}{2}} b(l)a(k)(\pm z)^{-k-\frac{1}{2}} (\pm w)^{-l-\frac{1}{2}}$
 $= \sum_{k \in (\mathbb{Z} + \frac{1}{2})_{>0}, l \in \mathbb{Z} + \frac{1}{2}} a(k)b(l)(\pm z)^{-k-\frac{1}{2}} (\pm w)^{-l-\frac{1}{2}} + \sum_{k \in (\mathbb{Z} + \frac{1}{2})_{>0}, l \in \mathbb{Z} + \frac{1}{2}} b(l)a(k)(\pm z)^{-k-\frac{1}{2}} (\pm w)^{-l-\frac{1}{2}}$
 $= \sum_{k \in (\mathbb{Z} + \frac{1}{2})_{>0}, l \in \mathbb{Z} + \frac{1}{2}} (a(k)b(l) + b(l)a(k))(\pm z)^{-k-\frac{1}{2}} (\pm w)^{-l-\frac{1}{2}}$
 $= \sum_{k \in (\mathbb{Z} + \frac{1}{2})_{>0}, l \in \mathbb{Z} + \frac{1}{2}} \{a(k), b(l)\}(\pm z)^{-k-\frac{1}{2}} (\pm w)^{-l-\frac{1}{2}}$
 $= \sum_{k \in (\mathbb{Z} + \frac{1}{2})_{>0}, l \in \mathbb{Z} + \frac{1}{2}} \langle a, b \rangle \delta_{k,-l} k'(\pm z)^{-k-\frac{1}{2}} (\pm w)^{-l-\frac{1}{2}}$
 $= \sum_{k \in (\mathbb{Z} + \frac{1}{2})_{>0}, l \in \mathbb{Z} + \frac{1}{2}} \langle a, b \rangle k'(\pm z)^{-k-\frac{1}{2}} (\pm w)^{k-\frac{1}{2}}$

This sum is a geometric series, and is equal to: $\iota_{z,w} \frac{\langle a, b \rangle \mathcal{K}(\pm z)^{-1}}{1 - \frac{\pm w}{\pm z}} = \iota_{z,w} \frac{\langle a, b \rangle \mathcal{K}}{\pm z \mp w}$. Since \mathcal{K} acts as the identity, as an operator this is equal to $\iota_{z,w} \frac{\langle a, b \rangle}{\pm z \mp w}$ as desired. ■

Remark. 4.1 As operators on \overline{V} , we choose \bar{c} so that $:a(z)\bar{c}(z):=0$ and $:a(z)\bar{c}^*(z):=0$ where $a \in \mathcal{C}$. Hence, in particular, $:a(z)\beta(z):=:a(z)\varepsilon_1(z):$ and $:a(z)\beta^*(z):=:a(z)\varepsilon_1^*(z):$.

The Lie algebra of operators in $Cl(P)$ is formed by endowing $Cl(P)$ with the commutator bracket. That is, the bracket among generating functions of operators is, by definition $[:a_1(z)b_1(z):, :a_2(w)b_2(w):] = :a_1(z)b_1(z)::a_2(w)b_2(w): - :a_2(w)b_2(w)::a_1(z)b_1(z):$.

Proposition 31. *For $a_1, b_1, a_2, b_2 \in \mathcal{C}$ and formal variables z, w , we have*

1. $[:a_1(z)b_1(z):, :a_2(w)b_2(w):]$

$$= \langle a_1, b_2 \rangle :b_1(w)a_2(w):\delta(z-w) - \langle a_1, a_2 \rangle :b_1(w)b_2(w):\delta(z-w)$$

$$+ \langle b_1, a_2 \rangle :a_1(w)b_2(w):\delta(z-w) - \langle b_1, b_2 \rangle :a_1(w)a_2(w):\delta(z-w)$$

$$+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w).$$
2. $[:a_1(-z)b_1(z):, :a_2(w)b_2(w):]$

$$= -\langle a_1, b_2 \rangle :b_1(-w)a_2(w):\delta(z+w) + \langle a_1, a_2 \rangle :b_1(-w)b_2(w):\delta(z+w)$$

$$+ \langle b_1, a_2 \rangle :a_1(-w)b_2(w):\delta(z-w) - \langle b_1, b_2 \rangle :a_1(-w)a_2(w):\delta(z-w)$$

$$- (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right).$$
3. $[:a_1(z)b_1(-z):, :a_2(w)b_2(w):]$

$$= \langle a_1, b_2 \rangle :b_1(-w)a_2(w):\delta(z-w) - \langle a_1, a_2 \rangle :b_1(-w)b_2(w):\delta(z-w)$$

$$- \langle b_1, a_2 \rangle :a_1(-w)b_2(w):\delta(z+w) + \langle b_1, b_2 \rangle :a_1(-w)a_2(w):\delta(z+w)$$

$$- (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right).$$
4. $[:a_1(z)b_1(z):, :a_2(-w)b_2(w):]$

$$= \langle a_1, b_2 \rangle :b_1(w)a_2(-w):\delta(z-w) - \langle a_1, a_2 \rangle :b_1(-w)b_2(w):\delta(z+w)$$

$$+ \langle b_1, a_2 \rangle :a_1(-w)b_2(w):\delta(z+w) - \langle b_1, b_2 \rangle :a_1(w)a_2(-w):\delta(z-w)$$

$$+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right).$$
5. $[:a_1(z)b_1(z):, :a_2(w)b_2(-w):]$

$$= \langle a_1, b_2 \rangle :b_1(-w)a_2(w):\delta(z+w) - \langle a_1, a_2 \rangle :b_1(w)b_2(-w):\delta(z-w)$$

$$+ \langle b_1, a_2 \rangle :a_1(w)b_2(-w):\delta(z-w) - \langle b_1, b_2 \rangle :a_1(-w)a_2(w):\delta(z+w)$$

$$+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right).$$

6. $[:a_1(-z)b_1(-z), :a_2(w)b_2(w):]$
 $= -\langle a_1, b_2 \rangle :b_1(w)a_2(w):\delta(z+w) + \langle a_1, a_2 \rangle :b_1(w)b_2(w):\delta(z+w)$
 $- \langle b_1, a_2 \rangle :a_1(w)b_2(w):\delta(z+w) + \langle b_1, b_2 \rangle :a_1(w)a_2(w):\delta(z+w)$
 $- (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z+w).$
7. $[:a_1(-z)b_1(z), :a_2(-w)b_2(w):]$
 $= -\langle a_1, b_2 \rangle :b_1(-w)a_2(-w):\delta(z+w) + \langle a_1, a_2 \rangle :b_1(w)b_2(w):\delta(z-w)$
 $+ \langle b_1, a_2 \rangle :a_1(w)b_2(w):\delta(z+w) - \langle b_1, b_2 \rangle :a_1(-w)a_2(-w):\delta(z-w)$
 $+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z+w) + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w).$
8. $[:a_1(-z)b_1(z), :a_2(w)b_2(-w):]$
 $= -\langle a_1, b_2 \rangle :b_1(w)a_2(w):\delta(z-w) + \langle a_1, a_2 \rangle :b_1(-w)b_2(-w):\delta(z+w)$
 $+ \langle b_1, a_2 \rangle :a_1(-w)b_2(-w):\delta(z-w) - \langle b_1, b_2 \rangle :a_1(w)a_2(w):\delta(z+w)$
 $- (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z-w) - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z+w).$
9. $[:a_1(z)b_1(-z), :a_2(-w)b_2(w):]$
 $= \langle a_1, b_2 \rangle :b_1(-w)a_2(-w):\delta(z-w) - \langle a_1, a_2 \rangle :b_1(w)b_2(w):\delta(z+w)$
 $- \langle b_1, a_2 \rangle :a_1(w)b_2(w):\delta(z-w) + \langle b_1, b_2 \rangle :a_1(-w)a_2(-w):\delta(z+w)$
 $- (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z-w) - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z+w).$
10. $[:a_1(z)b_1(-z), :a_2(w)b_2(-w):]$
 $= \langle a_1, b_2 \rangle :b_1(w)a_2(w):\delta(z+w) - \langle a_1, a_2 \rangle :b_1(-w)b_2(-w):\delta(z-w)$
 $- \langle b_1, a_2 \rangle :a_1(-w)b_2(-w):\delta(z+w) + \langle b_1, b_2 \rangle :a_1(w)a_2(w):\delta(z-w)$
 $+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z+w) + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w).$
11. $[:a_1(z)b_1(z), :a_2(-w)b_2(-w):]$
 $= \langle a_1, b_2 \rangle :b_1(-w)a_2(-w):\delta(z+w) - \langle a_1, a_2 \rangle :b_1(-w)b_2(-w):\delta(z+w)$
 $+ \langle b_1, a_2 \rangle :a_1(-w)b_2(-w):\delta(z+w) - \langle b_1, b_2 \rangle :a_1(-w)a_2(-w):\delta(z+w)$
 $+ (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle - \langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z+w).$
12. $[:a_1(-z)b_1(-z), :a_2(-w)b_2(w):]$
 $= -\langle a_1, b_2 \rangle :b_1(w)a_2(-w):\delta(z+w) + \langle a_1, a_2 \rangle :b_1(-w)b_2(w):\delta(z-w)$
 $- \langle b_1, a_2 \rangle :a_1(-w)b_2(w):\delta(z-w) + \langle b_1, b_2 \rangle :a_1(w)a_2(-w):\delta(z+w)$
 $+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right).$
13. $[:a_1(-z)b_1(-z), :a_2(w)b_2(-w):]$
 $= -\langle a_1, b_2 \rangle :b_1(-w)a_2(w):\delta(z-w) + \langle a_1, a_2 \rangle :b_1(w)b_2(-w):\delta(z+w)$

$$-\langle b_1, a_2 \rangle :a_1(w)b_2(-w):\delta(z+w) + \langle b_1, b_2 \rangle :a_1(-w)a_2(w):\delta(z-w) \\ + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right).$$

$$14. [:a_1(-z)b_1(z)::a_2(-w)b_2(-w):]$$

$$= -\langle a_1, b_2 \rangle :b_1(w)a_2(-w):\delta(z-w) + \langle a_1, a_2 \rangle :b_1(w)b_2(-w):\delta(z-w) \\ + \langle b_1, a_2 \rangle :a_1(w)b_2(-w):\delta(z+w) - \langle b_1, b_2 \rangle :a_1(w)a_2(-w):\delta(z+w) \\ - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right).$$

$$15. [:a_1(z)b_1(-z)::a_2(-w)b_2(-w):]$$

$$= \langle a_1, b_2 \rangle :b_1(w)a_2(-w):\delta(z+w) - \langle a_1, a_2 \rangle :b_1(w)b_2(-w):\delta(z+w) \\ - \langle b_1, a_2 \rangle :a_1(w)b_2(-w):\delta(z-w) + \langle b_1, b_2 \rangle :a_1(w)a_2(-w):\delta(z-w) \\ - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right).$$

$$16. [:a_1(-z)b_1(-z)::a_2(-w)b_2(-w):]$$

$$= -\langle a_1, b_2 \rangle :b_1(-w)a_2(-w):\delta(z-w) + \langle a_1, a_2 \rangle :b_1(-w)b_2(-w):\delta(z-w) \\ - \langle b_1, a_2 \rangle :a_1(-w)b_2(-w):\delta(z-w) + \langle b_1, b_2 \rangle :a_1(-w)a_2(-w):\delta(z-w) \\ + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w).$$

Proof: We will begin by dealing with all cases at once. Using Propositions 27 (the Wick Theorem for fermions) and 30, we have

$$:a_1(\pm z)b_1(\pm z)::a_2(\pm w)b_2(\pm w): =:a_1(\pm z)b_1(\pm z)a_2(\pm w)b_2(\pm w): \\ + \underbrace{a_1(\pm z)b_2(\pm w)}_{+} :b_1(\pm z)a_2(\pm w): - \underbrace{a_1(\pm z)a_2(\pm w)}_{+} :b_1(\pm z)b_2(\pm w): \\ + \underbrace{b_1(\pm z)a_2(\pm w)}_{+} :a_1(\pm z)b_2(\pm w): - \underbrace{b_1(\pm z)b_2(\pm w)}_{+} :a_1(\pm z)a_2(\pm w): \\ + \underbrace{a_1(\pm z)b_2(\pm w)}_{+} \underbrace{b_1(\pm z)a_2(\pm w)}_{+} - \underbrace{a_1(\pm z)a_2(\pm w)}_{+} \underbrace{b_1(\pm z)b_2(\pm w)}_{+} \\ =:a_1(\pm z)b_1(\pm z)a_2(\pm w)b_2(\pm w): \\ + \iota_{z,w} \frac{\langle a_1, b_2 \rangle \not k}{\pm z \mp w} :b_1(\pm z)a_2(\pm w): - \iota_{z,w} \frac{\langle a_1, a_2 \rangle \not k}{\pm z \mp w} :b_1(\pm z)b_2(\pm w): \\ + \iota_{z,w} \frac{\langle b_1, a_2 \rangle \not k}{\pm z \mp w} :a_1(\pm z)b_2(\pm w): - \iota_{z,w} \frac{\langle b_1, b_2 \rangle \not k}{\pm z \mp w} :a_1(\pm z)a_2(\pm w): \\ + \iota_{z,w} \frac{\langle a_1, b_2 \rangle \not k}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, a_2 \rangle \not k}{\pm z \mp w} - \iota_{z,w} \frac{\langle a_1, a_2 \rangle \not k}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, b_2 \rangle \not k}{\pm z \mp w}.$$

Recall from Corollary 28 that the sign of each term is $(-1)^N$, where N is the number of transpositions of fermions.

Similarly, $:a_2(\pm w)b_2(\pm w)::a_1(\pm z)b_1(\pm z): =:a_2(\pm w)b_2(\pm w)a_1(\pm z)b_1(\pm z):$

$$+ \underbrace{a_2(\pm w)b_1(\pm z)}_{+} :b_2(\pm w)a_1(\pm z): - \underbrace{a_2(\pm w)a_1(\pm z)}_{+} :b_2(\pm w)b_1(\pm z): \\ + \underbrace{b_2(\pm w)a_1(\pm z)}_{+} :a_2(\pm w)b_1(\pm z): - \underbrace{b_2(\pm w)b_1(\pm z)}_{+} :a_2(\pm w)a_1(\pm z):$$

$$\begin{aligned}
& + \underbrace{a_2(\pm w)b_1(\pm z)}_{=: a_2(\pm w)b_2(\pm w)a_1(\pm z)b_1(\pm z)} \underbrace{b_2(\pm w)a_1(\pm z)}_{+ \iota_{w,z} \frac{\langle a_2, b_1 \rangle \not k}{\pm w \mp z} : b_2(\pm w)a_1(\pm z)} - \underbrace{a_2(\pm w)a_1(\pm z)}_{+ \iota_{w,z} \frac{\langle a_2, b_1 \rangle \not k}{\pm w \mp z} : a_2(\pm w)b_1(\pm z)} \underbrace{b_2(\pm w)b_1(\pm z)}_{+ \iota_{w,z} \frac{\langle b_2, a_1 \rangle \not k}{\pm w \mp z} : a_2(\pm w)b_1(\pm z)} \\
& =: a_2(\pm w)b_2(\pm w)a_1(\pm z)b_1(\pm z) : \\
& + \iota_{w,z} \frac{\langle a_2, b_1 \rangle \not k}{\pm w \mp z} : b_2(\pm w)a_1(\pm z) : - \iota_{w,z} \frac{\langle a_2, a_1 \rangle \not k}{\pm w \mp z} : b_2(\pm w)b_1(\pm z) : \\
& + \iota_{w,z} \frac{\langle b_2, a_1 \rangle \not k}{\pm w \mp z} : a_2(\pm w)b_1(\pm z) : - \iota_{w,z} \frac{\langle b_2, b_1 \rangle \not k}{\pm w \mp z} : a_2(\pm w)a_1(\pm z) : \\
& + \iota_{w,z} \frac{\langle a_2, b_1 \rangle \not k}{\pm w \mp z} \iota_{w,z} \frac{\langle b_2, a_1 \rangle \not k}{\pm w \mp z} - \iota_{w,z} \frac{\langle a_2, a_1 \rangle \not k}{\pm w \mp z} \iota_{w,z} \frac{\langle b_2, b_1 \rangle \not k}{\pm w \mp z}.
\end{aligned}$$

Thus, compute the difference of these two halves, as prescribed by the commutator bracket as follows.

$$\begin{aligned}
& : a_1(\pm z)b_1(\pm z) :: a_2(\pm w)b_2(\pm w) : - : a_2(\pm w)b_2(\pm w) :: a_1(\pm z)b_1(\pm z) : \\
& =: a_1(\pm z)b_1(\pm z)a_2(\pm w)b_2(\pm w) : \\
& + \iota_{z,w} \frac{\langle a_1, b_2 \rangle \not k}{\pm z \mp w} : b_1(\pm z)a_2(\pm w) : - \iota_{z,w} \frac{\langle a_1, a_2 \rangle \not k}{\pm z \mp w} : b_1(\pm z)b_2(\pm w) : \\
& + \iota_{z,w} \frac{\langle b_1, a_2 \rangle \not k}{\pm z \mp w} : a_1(\pm z)b_2(\pm w) : - \iota_{z,w} \frac{\langle b_1, b_2 \rangle \not k}{\pm z \mp w} : a_1(\pm z)a_2(\pm w) : \\
& + \iota_{z,w} \frac{\langle a_1, b_2 \rangle \not k}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, a_2 \rangle \not k}{\pm z \mp w} - \iota_{z,w} \frac{\langle a_1, a_2 \rangle \not k}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, b_2 \rangle \not k}{\pm z \mp w} \\
& - : a_2(\pm w)b_2(\pm w)a_1(\pm z)b_1(\pm z) : \\
& - \iota_{w,z} \frac{\langle a_2, b_1 \rangle \not k}{\pm w \mp z} : b_2(\pm w)a_1(\pm z) : + \iota_{w,z} \frac{\langle a_2, a_1 \rangle \not k}{\pm w \mp z} : b_2(\pm w)b_1(\pm z) : \\
& - \iota_{w,z} \frac{\langle b_2, a_1 \rangle \not k}{\pm w \mp z} : a_2(\pm w)b_1(\pm z) : + \iota_{w,z} \frac{\langle b_2, b_1 \rangle \not k}{\pm w \mp z} : a_2(\pm w)a_1(\pm z) : \\
& - \iota_{w,z} \frac{\langle a_2, b_1 \rangle \not k}{\pm w \mp z} \iota_{w,z} \frac{\langle b_2, a_1 \rangle \not k}{\pm w \mp z} + \iota_{w,z} \frac{\langle a_2, a_1 \rangle \not k}{\pm w \mp z} \iota_{w,z} \frac{\langle b_2, b_1 \rangle \not k}{\pm w \mp z} \\
& =: a_1(\pm z)b_1(\pm z)a_2(\pm w)b_2(\pm w) : - : a_2(\pm w)b_2(\pm w)a_1(\pm z)b_1(\pm z) : \\
& + \iota_{z,w} \frac{\langle a_1, b_2 \rangle \not k}{\pm z \mp w} : b_1(\pm z)a_2(\pm w) : - \iota_{w,z} \frac{\langle b_2, a_1 \rangle \not k}{\pm w \mp z} : a_2(\pm w)b_1(\pm z) : \\
& - \iota_{z,w} \frac{\langle a_1, a_2 \rangle \not k}{\pm z \mp w} : b_1(\pm z)b_2(\pm w) : + \iota_{w,z} \frac{\langle a_2, a_1 \rangle \not k}{\pm w \mp z} : b_2(\pm w)b_1(\pm z) : \\
& + \iota_{z,w} \frac{\langle b_1, a_2 \rangle \not k}{\pm z \mp w} : a_1(\pm z)b_2(\pm w) : - \iota_{w,z} \frac{\langle a_2, b_1 \rangle \not k}{\pm w \mp z} : b_2(\pm w)a_1(\pm z) : \\
& - \iota_{z,w} \frac{\langle b_1, b_2 \rangle \not k}{\pm z \mp w} : a_1(\pm z)a_2(\pm w) : + \iota_{w,z} \frac{\langle b_2, b_1 \rangle \not k}{\pm w \mp z} : a_2(\pm w)a_1(\pm z) : \\
& + \iota_{z,w} \frac{\langle a_1, b_2 \rangle \not k}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, a_2 \rangle \not k}{\pm z \mp w} - \iota_{w,z} \frac{\langle a_2, b_1 \rangle \not k}{\pm w \mp z} \iota_{w,z} \frac{\langle b_2, a_1 \rangle \not k}{\pm w \mp z} \\
& - \iota_{z,w} \frac{\langle a_1, a_2 \rangle \not k}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, b_2 \rangle \not k}{\pm z \mp w} + \iota_{w,z} \frac{\langle a_2, a_1 \rangle \not k}{\pm w \mp z} \iota_{w,z} \frac{\langle b_2, b_1 \rangle \not k}{\pm w \mp z}.
\end{aligned}$$

Using Corollary 29, the quartic terms cancel. Also, by the symmetry of $\langle \cdot, \cdot \rangle$ and Corollary 28,

we have

$$\begin{aligned}
& = \iota_{z,w} \frac{\langle a_1, b_2 \rangle \not k}{\pm z \mp w} : b_1(\pm z)a_2(\pm w) : + \iota_{w,z} \frac{\langle a_1, b_2 \rangle \not k}{\pm w \mp z} : b_1(\pm z)a_2(\pm w) : \\
& - \iota_{z,w} \frac{\langle a_1, a_2 \rangle \not k}{\pm z \mp w} : b_1(\pm z)b_2(\pm w) : - \iota_{w,z} \frac{\langle a_1, a_2 \rangle \not k}{\pm w \mp z} : b_1(\pm z)b_2(\pm w) :
\end{aligned}$$

$$\begin{aligned}
& + \iota_{z,w} \frac{\langle b_1, a_2 \rangle \mathcal{K}}{\pm z \mp w} : a_1(\pm z) b_2(\pm w) : + \iota_{w,z} \frac{\langle b_1, a_2 \rangle \mathcal{K}}{\pm w \mp z} : a_1(\pm z) b_2(\pm w) : \\
& - \iota_{z,w} \frac{\langle b_1, b_2 \rangle \mathcal{K}}{\pm z \mp w} : a_1(\pm z) a_2(\pm w) : - \iota_{w,z} \frac{\langle b_2, b_1 \rangle \mathcal{K}}{\pm w \mp z} : a_1(\pm z) a_2(\pm w) : \\
& + \iota_{z,w} \frac{\langle a_1, b_2 \rangle \mathcal{K}}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, a_2 \rangle \mathcal{K}}{\pm z \mp w} - \iota_{w,z} \frac{\langle a_1, b_2 \rangle \mathcal{K}}{\pm w \mp z} \iota_{w,z} \frac{\langle b_1, a_2 \rangle \mathcal{K}}{\pm w \mp z} \\
& - \iota_{z,w} \frac{\langle a_1, a_2 \rangle \mathcal{K}}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, b_2 \rangle \mathcal{K}}{\pm z \mp w} + \iota_{w,z} \frac{\langle a_1, a_2 \rangle \mathcal{K}}{\pm w \mp z} \iota_{w,z} \frac{\langle b_1, b_2 \rangle \mathcal{K}}{\pm w \mp z} \\
& = \langle a_1, b_2 \rangle \mathcal{K} \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : b_1(\pm z) a_2(\pm w) : \\
& - \langle a_1, a_2 \rangle \mathcal{K} \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : b_1(\pm z) b_2(\pm w) : \\
& + \langle b_1, a_2 \rangle \mathcal{K} \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : a_1(\pm z) b_2(\pm w) : \\
& - \langle b_1, b_2 \rangle \mathcal{K} \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : a_1(\pm z) a_2(\pm w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle \mathcal{K}^2) \left(\iota_{z,w} \frac{1}{(\pm z \mp w)} \cdot \iota_{z,w} \frac{1}{(\pm z \mp w)} - \iota_{w,z} \frac{1}{(\pm w \mp z)} \cdot \iota_{w,z} \frac{1}{(\pm w \mp z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle \mathcal{K}^2) \left(\iota_{z,w} \frac{1}{(\pm z \mp w)} \cdot \iota_{z,w} \frac{1}{(\pm z \mp w)} - \iota_{w,z} \frac{1}{(\pm w \mp z)} \cdot \iota_{w,z} \frac{1}{(\pm w \mp z)} \right).
\end{aligned}$$

For the last two lines, notice that we cannot write $\frac{1}{(\pm z \mp w)^2}$ for $\frac{1}{(\pm z \mp w)} \cdot \frac{1}{(\pm z \mp w)}$ because the two factors have variables which come from different generating functions, and hence their signs may be different from each other. Also, since \mathcal{K} acts as the identity, those factors can be removed without changing the operators, so we have:

$$\begin{aligned}
& = \langle a_1, b_2 \rangle \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : b_1(\pm z) a_2(\pm w) : \\
& - \langle a_1, a_2 \rangle \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : b_1(\pm z) b_2(\pm w) : \\
& + \langle b_1, a_2 \rangle \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : a_1(\pm z) b_2(\pm w) : \\
& - \langle b_1, b_2 \rangle \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : a_1(\pm z) a_2(\pm w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\iota_{z,w} \frac{1}{(\pm z \mp w)} \cdot \iota_{z,w} \frac{1}{(\pm z \mp w)} - \iota_{w,z} \frac{1}{(\pm w \mp z)} \cdot \iota_{w,z} \frac{1}{(\pm w \mp z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\iota_{z,w} \frac{1}{(\pm z \mp w)} \cdot \iota_{z,w} \frac{1}{(\pm z \mp w)} - \iota_{w,z} \frac{1}{(\pm w \mp z)} \cdot \iota_{w,z} \frac{1}{(\pm w \mp z)} \right).
\end{aligned}$$

At this point, we can deduce each of the 16 cases stated in the proposition by choosing signs for each of the variables. We make use of the definitions of the various delta functions in Definition 16 and Lemma 24, and we continue the calculation above in each of the 16 cases. We proceed by using the same numbering of the cases as in the statement of the proposition.

To ease the notation, we will drop the leading $\iota_{z,w}$ and $\iota_{w,z}$ for the remainder of the proof.

$$\begin{aligned}
1. &= \langle a_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(z) a_2(w) : \\
&\quad - \langle a_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(z) b_2(w) : \\
&\quad + \langle b_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(z) b_2(w) : \\
&\quad - \langle b_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(z) a_2(w) : \\
&\quad + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w-z)} \right) \\
&\quad - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w-z)} \right) \\
&= \langle a_1, b_2 \rangle : b_1(z) a_2(w) : \delta(z-w) - \langle a_1, a_2 \rangle : b_1(z) b_2(w) : \delta(z-w) \\
&\quad + \langle b_1, a_2 \rangle : a_1(z) b_2(w) : \delta(z-w) - \langle b_1, b_2 \rangle : a_1(z) a_2(w) : \delta(z-w) \\
&\quad + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w) \\
&= \langle a_1, b_2 \rangle : b_1(w) a_2(w) : \delta(z-w) - \langle a_1, a_2 \rangle : b_1(w) b_2(w) : \delta(z-w) \\
&\quad + \langle b_1, a_2 \rangle : a_1(w) b_2(w) : \delta(z-w) - \langle b_1, b_2 \rangle : a_1(w) a_2(w) : \delta(z-w) \\
&\quad + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w). \\
2. &= \langle a_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : b_1(z) a_2(w) : \\
&\quad - \langle a_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : b_1(z) b_2(w) : \\
&\quad + \langle b_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(-z) b_2(w) : \\
&\quad - \langle b_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(-z) a_2(w) : \\
&\quad + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
&\quad - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
&= \langle a_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : b_1(z) a_2(w) : \\
&\quad - \langle a_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : b_1(z) b_2(w) : \\
&\quad + \langle b_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(-z) b_2(w) : \\
&\quad - \langle b_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(-z) a_2(w) : \\
&\quad + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right)
\end{aligned}$$

$$\begin{aligned}
& -(\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(z) a_2(w) : \delta(z+w) \\
& + \langle a_1, a_2 \rangle : b_1(z) b_2(w) : \delta(z+w) \\
& + \langle b_1, a_2 \rangle : a_1(-z) b_2(w) : \delta(z-w) \\
& - \langle b_1, b_2 \rangle : a_1(-z) a_2(w) : \delta(z-w) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle - \langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(-w) a_2(w) : \delta(z+w) \\
& + \langle a_1, a_2 \rangle : b_1(-w) b_2(w) : \delta(z+w) \\
& + \langle b_1, a_2 \rangle : a_1(-w) b_2(w) : \delta(z-w) \\
& - \langle b_1, b_2 \rangle : a_1(-w) a_2(w) : \delta(z-w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right).
\end{aligned}$$

$$\begin{aligned}
3. & = \langle a_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) a_2(w) : \\
& - \langle a_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(z) b_2(w) : \\
& - \langle b_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = \langle a_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) a_2(w) : \\
& - \langle a_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(z) b_2(w) : \\
& - \langle b_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = \langle a_1, b_2 \rangle : b_1(-z) a_2(w) : \delta(z-w) \\
& - \langle a_1, a_2 \rangle : b_1(-z) b_2(w) : \delta(z-w)
\end{aligned}$$

$$\begin{aligned}
& -\langle b_1, a_2 \rangle : a_1(z) b_2(w) : \delta(z + w) \\
& + \langle b_1, b_2 \rangle : a_1(z) a_2(w) : \delta(z + w) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle - \langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = \langle a_1, b_2 \rangle : b_1(-w) a_2(w) : \delta(z - w) \\
& - \langle a_1, a_2 \rangle : b_1(-w) b_2(w) : \delta(z - w) \\
& - \langle b_1, a_2 \rangle : a_1(-w) b_2(w) : \delta(z + w) \\
& + \langle b_1, b_2 \rangle : a_1(-w) a_2(w) : \delta(z + w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right).
\end{aligned}$$

$$\begin{aligned}
4. & = \langle a_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(z) a_2(-w) : \\
& - \langle a_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(z) b_2(w) : \\
& - \langle b_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(-w-z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(w-z)} \right). \\
& = \langle a_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(z) a_2(-w) : \\
& - \langle a_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : a_1(z) b_2(w) : \\
& - \langle b_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = \langle a_1, b_2 \rangle : b_1(z) a_2(-w) : \delta(z - w) \\
& - \langle a_1, a_2 \rangle : b_1(z) b_2(w) : \delta(z + w) \\
& + \langle b_1, a_2 \rangle : a_1(z) b_2(w) : \delta(z + w) \\
& - \langle b_1, b_2 \rangle : a_1(z) a_2(-w) : \delta(z - w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right)
\end{aligned}$$

$$\begin{aligned}
&= \langle a_1, b_2 \rangle : b_1(w) a_2(-w) : \delta(z - w) \\
&- \langle a_1, a_2 \rangle : b_1(-w) b_2(w) : \delta(z + w) \\
&+ \langle b_1, a_2 \rangle : a_1(-w) b_2(w) : \delta(z + w) \\
&- \langle b_1, b_2 \rangle : a_1(w) a_2(-w) : \delta(z - w) \\
&+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right).
\end{aligned}$$

$$\begin{aligned}
5. &= \langle a_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(z) a_2(w) : \\
&- \langle a_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(z) b_2(-w) : \\
&+ \langle b_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(z) b_2(-w) : \\
&- \langle b_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(z) a_2(w) : \\
&+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(w-z)} \right) \\
&- (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(-w-z)} \right) \\
&= \langle a_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(z) a_2(w) : \\
&- \langle a_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(z) b_2(-w) : \\
&+ \langle b_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(z) b_2(-w) : \\
&- \langle b_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : a_1(z) a_2(w) : \\
&+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
&- (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
&= \langle a_1, b_2 \rangle : b_1(z) a_2(w) : \delta(z+w) \\
&- \langle a_1, a_2 \rangle : b_1(z) b_2(-w) : \delta(z-w) \\
&+ \langle b_1, a_2 \rangle : a_1(z) b_2(-w) : \delta(z-w) \\
&- \langle b_1, b_2 \rangle : a_1(z) a_2(w) : \delta(z+w) \\
&+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
&= \langle a_1, b_2 \rangle : b_1(-w) a_2(w) : \delta(z+w) \\
&- \langle a_1, a_2 \rangle : b_1(w) b_2(-w) : \delta(z-w) \\
&+ \langle b_1, a_2 \rangle : a_1(w) b_2(-w) : \delta(z-w) \\
&- \langle b_1, b_2 \rangle : a_1(-w) a_2(w) : \delta(z+w)
\end{aligned}$$

$$+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right).$$

$$\begin{aligned}
6. &= \langle a_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : b_1(-z)a_2(w) : \\
&\quad - \langle a_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : b_1(-z)b_2(w) : \\
&\quad + \langle b_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(-z)b_2(w) : \\
&\quad - \langle b_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(-z)a_2(w) : \\
&\quad + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w+z)} \right) \\
&\quad - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w+z)} \right) \\
&= \langle a_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : b_1(-z)a_2(w) : \\
&\quad - \langle a_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : b_1(-z)b_2(w) : \\
&\quad + \langle b_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(-z)b_2(w) : \\
&\quad - \langle b_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(-z)a_2(w) : \\
&\quad + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)^2} - \frac{1}{(w+z)^2} \right) \\
&\quad - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)^2} - \frac{1}{(w+z)^2} \right) \\
&= -\langle a_1, b_2 \rangle : b_1(-z)a_2(w) : \delta(z+w) \\
&\quad + \langle a_1, a_2 \rangle : b_1(-z)b_2(w) : \delta(z+w) \\
&\quad - \langle b_1, a_2 \rangle : a_1(-z)b_2(w) : \delta(z+w) \\
&\quad + \langle b_1, b_2 \rangle : a_1(-z)a_2(w) : \delta(z+w) \\
&\quad - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z+w) \\
&= -\langle a_1, b_2 \rangle : b_1(w)a_2(w) : \delta(z+w) \\
&\quad + \langle a_1, a_2 \rangle : b_1(w)b_2(w) : \delta(z+w) \\
&\quad - \langle b_1, a_2 \rangle : a_1(w)b_2(w) : \delta(z+w) \\
&\quad + \langle b_1, b_2 \rangle : a_1(w)a_2(w) : \delta(z+w) \\
&\quad - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z+w).
\end{aligned}$$

$$7. = \langle a_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : b_1(z)a_2(-w) :$$

$$\begin{aligned}
& -\langle a_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : b_1(z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(-z) b_2(w) : \\
& - \langle b_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(-z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w+z)} \cdot \frac{1}{(-w-z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = \langle a_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : b_1(z) a_2(-w) : \\
& - \langle a_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : b_1(z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : a_1(-z) b_2(w) : \\
& - \langle b_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(-z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z+w)^2} + \frac{1}{(w+z)^2} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z-w)^2} + \frac{1}{(w-z)^2} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(z) a_2(-w) : \delta(z+w) \\
& + \langle a_1, a_2 \rangle : b_1(z) b_2(w) : \delta(z-w) \\
& + \langle b_1, a_2 \rangle : a_1(-z) b_2(w) : \delta(z+w) \\
& - \langle b_1, b_2 \rangle : a_1(-z) a_2(-w) : \delta(z-w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z+w) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w) \\
& = -\langle a_1, b_2 \rangle : b_1(-w) a_2(-w) : \delta(z+w) \\
& + \langle a_1, a_2 \rangle : b_1(w) b_2(w) : \delta(z-w) \\
& + \langle b_1, a_2 \rangle : a_1(w) b_2(w) : \delta(z+w) \\
& - \langle b_1, b_2 \rangle : a_1(-w) a_2(-w) : \delta(z-w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z+w) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w).
\end{aligned}$$

$$\begin{aligned}
8. & = \langle a_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : b_1(z) a_2(w) : \\
& - \langle a_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : b_1(z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(-z) b_2(-w) :
\end{aligned}$$

$$\begin{aligned}
& -\langle b_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(-z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(w-z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w+z)} \cdot \frac{1}{(-w-z)} \right) \\
& = \langle a_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : b_1(z) a_2(w) : \\
& - \langle a_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : b_1(z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(-z) b_2(-w) : \\
& - \langle b_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : a_1(-z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z-w)^2} + \frac{1}{(w-z)^2} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z+w)^2} + \frac{1}{(w+z)^2} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(z) a_2(w) : \delta(z-w) \\
& + \langle a_1, a_2 \rangle : b_1(z) b_2(-w) : \delta(z+w) \\
& + \langle b_1, a_2 \rangle : a_1(-z) b_2(-w) : \delta(z-w) \\
& - \langle b_1, b_2 \rangle : a_1(-z) a_2(w) : \delta(z+w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z-w) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z+w) \\
& = -\langle a_1, b_2 \rangle : b_1(w) a_2(w) : \delta(z-w) \\
& + \langle a_1, a_2 \rangle : b_1(-w) b_2(-w) : \delta(z+w) \\
& + \langle b_1, a_2 \rangle : a_1(-w) b_2(-w) : \delta(z-w) \\
& - \langle b_1, b_2 \rangle : a_1(w) a_2(w) : \delta(z+w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z-w) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z+w).
\end{aligned}$$

$$\begin{aligned}
9. & = \langle a_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) a_2(-w) : \\
& - \langle a_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(-z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(z) b_2(w) : \\
& - \langle b_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(-w+z)} \right)
\end{aligned}$$

$$\begin{aligned}
& -(\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = \langle a_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) a_2(-w) : \\
& - \langle a_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(-z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(z) b_2(w) : \\
& - \langle b_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z-w)^2} + \frac{1}{(w-z)^2} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z+w)^2} + \frac{1}{(w+z)^2} \right) \\
& = \langle a_1, b_2 \rangle : b_1(-z) a_2(-w) : \delta(z-w) \\
& - \langle a_1, a_2 \rangle : b_1(-z) b_2(w) : \delta(z+w) \\
& - \langle b_1, a_2 \rangle : a_1(z) b_2(w) : \delta(z-w) \\
& + \langle b_1, b_2 \rangle : a_1(z) a_2(-w) : \delta(z+w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z-w) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z+w). \\
& = \langle a_1, b_2 \rangle : b_1(-w) a_2(-w) : \delta(z-w) \\
& - \langle a_1, a_2 \rangle : b_1(w) b_2(w) : \delta(z+w) \\
& - \langle b_1, a_2 \rangle : a_1(w) b_2(w) : \delta(z-w) \\
& + \langle b_1, b_2 \rangle : a_1(-w) a_2(-w) : \delta(z+w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z-w) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z+w).
\end{aligned}$$

$$\begin{aligned}
10. & = \langle a_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(-z) a_2(w) : \\
& - \langle a_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(z) b_2(-w) : \\
& - \langle b_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(w+z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(-w+z)} \right) \\
& = \langle a_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(-z) a_2(w) :
\end{aligned}$$

$$\begin{aligned}
& -\langle a_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(z) b_2(-w) : \\
& - \langle b_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z+w)^2} + \frac{1}{(w+z)^2} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z-w)^2} + \frac{1}{(w-z)^2} \right) \\
& = \langle a_1, b_2 \rangle : b_1(-z) a_2(w) : \delta(z+w) \\
& - \langle a_1, a_2 \rangle : b_1(-z) b_2(-w) : \delta(z-w) \\
& - \langle b_1, a_2 \rangle : a_1(z) b_2(-w) : \delta(z+w) \\
& + \langle b_1, b_2 \rangle : a_1(z) a_2(w) : \delta(z-w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z+w) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w) \\
& = \langle a_1, b_2 \rangle : b_1(w) a_2(w) : \delta(z+w) \\
& - \langle a_1, a_2 \rangle : b_1(-w) b_2(-w) : \delta(z-w) \\
& - \langle b_1, a_2 \rangle : a_1(-w) b_2(-w) : \delta(z+w) \\
& + \langle b_1, b_2 \rangle : a_1(w) a_2(w) : \delta(z-w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z+w) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w).
\end{aligned}$$

$$\begin{aligned}
11. & = \langle a_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(z) a_2(-w) : \\
& - \langle a_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(z) b_2(-w) : \\
& - \langle b_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z+w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(-w-z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z+w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(-w-z)} \right) \\
& = \langle a_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(z) a_2(-w) : \\
& - \langle a_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : a_1(z) b_2(-w) :
\end{aligned}$$

$$\begin{aligned}
& -\langle b_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) :a_1(z)a_2(-w): \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)^2} - \frac{1}{(w+z)^2} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)^2} - \frac{1}{(w+z)^2} \right) \\
& = \langle a_1, b_2 \rangle :b_1(z)a_2(-w): \delta(z+w) \\
& - \langle a_1, a_2 \rangle :b_1(z)b_2(-w): \delta(z+w) \\
& + \langle b_1, a_2 \rangle :a_1(z)b_2(-w): \delta(z+w) \\
& - \langle b_1, b_2 \rangle :a_1(z)a_2(-w): \delta(z+w) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle - \langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z+w) \\
& = \langle a_1, b_2 \rangle :b_1(-w)a_2(-w): \delta(z+w) \\
& - \langle a_1, a_2 \rangle :b_1(-w)b_2(-w): \delta(z+w) \\
& + \langle b_1, a_2 \rangle :a_1(-w)b_2(-w): \delta(z+w) \\
& - \langle b_1, b_2 \rangle :a_1(-w)a_2(-w): \delta(z+w) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle - \langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z+w).
\end{aligned}$$

$$\begin{aligned}
12. & = \langle a_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) :b_1(-z)a_2(-w): \\
& - \langle a_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) :b_1(-z)b_2(w): \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) :a_1(-z)b_2(w): \\
& - \langle b_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) :a_1(-z)a_2(-w): \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(w+z)} \cdot \frac{1}{(-w+z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(w+z)} \right) \\
& = \langle a_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) :b_1(-z)a_2(-w): \\
& - \langle a_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) :b_1(-z)b_2(w): \\
& + \langle b_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) :a_1(-z)b_2(w): \\
& - \langle b_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) :a_1(-z)a_2(-w): \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right)
\end{aligned}$$

$$\begin{aligned}
&= -\langle a_1, b_2 \rangle : b_1(-z) a_2(-w) : \delta(z + w) \\
&+ \langle a_1, a_2 \rangle : b_1(-z) b_2(w) : \delta(z - w) \\
&- \langle b_1, a_2 \rangle : a_1(-z) b_2(w) : \delta(z - w) \\
&+ \langle b_1, b_2 \rangle : a_1(-z) a_2(-w) : \delta(z + w) \\
&+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
&= -\langle a_1, b_2 \rangle : b_1(w) a_2(-w) : \delta(z + w) \\
&+ \langle a_1, a_2 \rangle : b_1(-w) b_2(w) : \delta(z - w) \\
&- \langle b_1, a_2 \rangle : a_1(-w) b_2(w) : \delta(z - w) \\
&+ \langle b_1, b_2 \rangle : a_1(w) a_2(-w) : \delta(z + w) \\
&+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right).
\end{aligned}$$

$$\begin{aligned}
13. &= \langle a_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : b_1(-z) a_2(w) : \\
&- \langle a_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : b_1(-z) b_2(-w) : \\
&+ \langle b_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(-z) b_2(-w) : \\
&- \langle b_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(-z) a_2(w) : \\
&+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(w+z)} \right) \\
&- (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(w+z)} \cdot \frac{1}{(-w+z)} \right) \\
&= \langle a_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : b_1(-z) a_2(w) : \\
&- \langle a_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : b_1(-z) b_2(-w) : \\
&+ \langle b_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(-z) b_2(-w) : \\
&- \langle b_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(-z) a_2(w) : \\
&+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
&- (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
&= -\langle a_1, b_2 \rangle : b_1(-z) a_2(w) : \delta(z - w) \\
&+ \langle a_1, a_2 \rangle : b_1(-z) b_2(-w) : \delta(z + w) \\
&- \langle b_1, a_2 \rangle : a_1(-z) b_2(-w) : \delta(z + w) \\
&+ \langle b_1, b_2 \rangle : a_1(-z) a_2(w) : \delta(z - w)
\end{aligned}$$

$$\begin{aligned}
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(-w) a_2(w) : \delta(z-w) \\
& + \langle a_1, a_2 \rangle : b_1(w) b_2(-w) : \delta(z+w) \\
& - \langle b_1, a_2 \rangle : a_1(w) b_2(-w) : \delta(z+w) \\
& + \langle b_1, b_2 \rangle : a_1(-w) a_2(w) : \delta(z-w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right). \\
14. & = \langle a_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : b_1(z) a_2(-w) : \\
& - \langle a_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : b_1(z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(-z) b_2(-w) : \\
& - \langle b_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(-z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(z+w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(-w-z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(z+w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(-w-z)} \right) \\
& = \langle a_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : b_1(z) a_2(-w) : \\
& - \langle a_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : b_1(z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : a_1(-z) b_2(-w) : \\
& - \langle b_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : a_1(-z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(z) a_2(-w) : \delta(z-w) \\
& + \langle a_1, a_2 \rangle : b_1(z) b_2(-w) : \delta(z-w) \\
& + \langle b_1, a_2 \rangle : a_1(-z) b_2(-w) : \delta(z+w) \\
& - \langle b_1, b_2 \rangle : a_1(-z) a_2(-w) : \delta(z+w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(w) a_2(-w) : \delta(z-w) \\
& + \langle a_1, a_2 \rangle : b_1(w) b_2(-w) : \delta(z-w)
\end{aligned}$$

$$\begin{aligned}
& + \langle b_1, a_2 \rangle : a_1(w) b_2(-w) : \delta(z + w) \\
& - \langle b_1, b_2 \rangle : a_1(w) a_2(-w) : \delta(z + w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right). \\
15. & = \langle a_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(-z) a_2(-w) : \\
& - \langle a_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(-z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(z) b_2(-w) : \\
& - \langle b_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(-w+z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(-w+z)} \right) \\
& = \langle a_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(-z) a_2(-w) : \\
& - \langle a_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(-z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(z) b_2(-w) : \\
& - \langle b_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = \langle a_1, b_2 \rangle : b_1(-z) a_2(-w) : \delta(z + w) \\
& - \langle a_1, a_2 \rangle : b_1(-z) b_2(-w) : \delta(z + w) \\
& - \langle b_1, a_2 \rangle : a_1(z) b_2(-w) : \delta(z - w) \\
& + \langle b_1, b_2 \rangle : a_1(z) a_2(-w) : \delta(z - w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = \langle a_1, b_2 \rangle : b_1(w) a_2(-w) : \delta(z + w) \\
& - \langle a_1, a_2 \rangle : b_1(w) b_2(-w) : \delta(z + w) \\
& - \langle b_1, a_2 \rangle : a_1(w) b_2(-w) : \delta(z - w) \\
& + \langle b_1, b_2 \rangle : a_1(w) a_2(-w) : \delta(z - w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right).
\end{aligned}$$

$$\begin{aligned}
16. &= \langle a_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : b_1(-z) a_2(-w) : \\
&\quad - \langle a_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : b_1(-z) b_2(-w) : \\
&\quad + \langle b_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(-z) b_2(-w) : \\
&\quad - \langle b_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(-z) a_2(-w) : \\
&\quad + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(-w+z)} \right) \\
&\quad - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(-w+z)} \right) \\
&= \langle a_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : b_1(-z) a_2(-w) : \\
&\quad - \langle a_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : b_1(-z) b_2(-w) : \\
&\quad + \langle b_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(-z) b_2(-w) : \\
&\quad - \langle b_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(-z) a_2(-w) : \\
&\quad + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)^2} - \frac{1}{(w-z)^2} \right) \\
&\quad - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)^2} - \frac{1}{(w-z)^2} \right) \\
&= -\langle a_1, b_2 \rangle : b_1(-z) a_2(-w) : \delta(z-w) \\
&\quad + \langle a_1, a_2 \rangle : b_1(-z) b_2(-w) : \delta(z-w) \\
&\quad - \langle b_1, a_2 \rangle : a_1(-z) b_2(-w) : \delta(z-w) \\
&\quad + \langle b_1, b_2 \rangle : a_1(-z) a_2(-w) : \delta(z-w) \\
&\quad + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w) \\
&= -\langle a_1, b_2 \rangle : b_1(-w) a_2(-w) : \delta(z-w) \\
&\quad + \langle a_1, a_2 \rangle : b_1(-w) b_2(-w) : \delta(z-w) \\
&\quad - \langle b_1, a_2 \rangle : a_1(-w) b_2(-w) : \delta(z-w) \\
&\quad + \langle b_1, b_2 \rangle : a_1(-w) a_2(-w) : \delta(z-w) \\
&\quad + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle - \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w).
\end{aligned}$$

Each case is exactly as stated in the Proposition, proving it. ■

4.2 Fermionic Representation of Twisted Toroidal A_{2n-1}

We can use these fermionic fields to give a representation of the twisted toroidal Lie algebra of type A_{2n-1} as described in our second main theorem as follows.

Theorem 32. Define a map $\rho : \bar{t} \rightarrow \text{End}(\bar{V})$ by:

1. $\mathcal{K} \mapsto \mathcal{K} = \text{id}$
2. $\alpha_0^{\bar{\sigma}}(z) \mapsto : \varepsilon_1^*(z) \beta(z) : + : \varepsilon_1(-z) \beta^*(-z) :$
3. $\alpha_i^{\bar{\sigma}}(z) \mapsto : \varepsilon_i(z) \varepsilon_i^*(z) : - : \varepsilon_{i+1}(z) \varepsilon_{i+1}^*(z) :$
4. $\alpha_n^{\bar{\sigma}}(z) \mapsto : \varepsilon_n(z) \varepsilon_n^*(z) : - : \varepsilon_n(-z) \varepsilon_n^*(-z) :$
5. $X^{\bar{\sigma}}(\alpha_0, z) \mapsto : \beta^*(z) \varepsilon_1^*(-z) :$
6. $X^{\bar{\sigma}}(-\alpha_0, z) \mapsto : \beta(z) \varepsilon_1(-z) :$
7. $X^{\bar{\sigma}}(\alpha_i, z) \mapsto : \varepsilon_i(z) \varepsilon_{i+1}^*(z) :$
8. $X^{\bar{\sigma}}(-\alpha_i, z) \mapsto - : \varepsilon_i^*(z) \varepsilon_{i+1}(z) :$
9. $X^{\bar{\sigma}}(\alpha_n, z) \mapsto : \varepsilon_n(z) \varepsilon_n(-z) :$
10. $X^{\bar{\sigma}}(-\alpha_n, z) \mapsto - : \varepsilon_n^*(z) \varepsilon_n^*(-z) :$

for $1 \leq i \leq n-1$. Then ρ is an homomorphism, and hence a representation of the twisted toroidal Lie algebra of type A_{2n-1} on \bar{V} . Since \mathcal{K} acts as the identity, this representation is level 1.

Proof: By virtue of Theorem 19, we need only show that the images under ρ of the generators of \bar{t} satisfy the defining relations of \bar{t} . We will check each relation in turn. Since \mathcal{K} acts like a scalar, it commutes with all other operators (since they are linear), and thus we need not write down calculations showing that \mathcal{K} is central.

Relation (1): $[\alpha_0^{\bar{\sigma}}(z), \alpha_m^{\bar{\sigma}}(w)] = (2\delta_{m0} - \delta_{m1})(\partial_w \delta(z-w) + \partial_w \delta(z+w))\mathcal{K}$

First, consider the case in which $m = 0$.

$$\begin{aligned} & [\rho(\alpha_0^{\bar{\sigma}}(z)), \rho(\alpha_m^{\bar{\sigma}}(w))] \\ &= [: \varepsilon_1^*(z) \beta(z) : + : \varepsilon_1(-z) \beta^*(-z) :, : \varepsilon_1^*(w) \beta(w) : + : \varepsilon_1(-w) \beta^*(-w) :] \\ &= [: \varepsilon_1^*(z) \beta(z) :, : \varepsilon_1^*(w) \beta(w) :] + [: \varepsilon_1^*(z) \beta(z) :, : \varepsilon_1(-w) \beta^*(-w) :] \\ &\quad + [: \varepsilon_1(-z) \beta^*(-z) :, : \varepsilon_1^*(w) \beta(w) :] + [: \varepsilon_1(-z) \beta^*(-z) :, : \varepsilon_1(-w) \beta^*(-w) :]. \end{aligned}$$

We use Proposition 31 to compute these four brackets. The nonzero terms are:

$$\begin{aligned} &=: \beta(w) \varepsilon_1^*(w) : \delta(z - w) + : \varepsilon_1^*(w) \beta(w) : \delta(z - w) + \partial_w \delta(z - w) \\ &- : \beta(-w) \beta^*(-w) : \delta(z + w) - \varepsilon_1^*(-w) \varepsilon_1(-w) : \delta(z + w) + \partial_w \delta(z + w) \\ &+ : \beta^*(w) \beta(w) : \delta(z + w) + : \varepsilon_1(w) \varepsilon_1^*(w) : \delta(z + w) + \partial_w \delta(z + w) \\ &- : \beta^*(-w) \varepsilon_1(-w) : \delta(z - w) - : \varepsilon_1(-w) \beta^*(-w) : \delta(z - w) + \partial_w \delta(z - w). \end{aligned}$$

By Remark 4.1, we have

$$\begin{aligned} &=: \beta(w) \varepsilon_1^*(w) : \delta(z - w) + : \varepsilon_1^*(w) \beta(w) : \delta(z - w) + \partial_w \delta(z - w) \\ &- : \varepsilon_1(-w) \varepsilon_1^*(-w) : \delta(z + w) - \varepsilon_1^*(-w) \varepsilon_1(-w) : \delta(z + w) + \partial_w \delta(z + w) \\ &+ : \varepsilon_1^*(w) \varepsilon_1(w) : \delta(z + w) + : \varepsilon_1(w) \varepsilon_1^*(w) : \delta(z + w) + \partial_w \delta(z + w) \\ &- : \beta^*(-w) \varepsilon_1(-w) : \delta(z - w) - : \varepsilon_1(-w) \beta^*(-w) : \delta(z - w) + \partial_w \delta(z - w). \end{aligned}$$

By Corollary 28, we have

$$\begin{aligned} &=: \beta(w) \varepsilon_1^*(w) : \delta(z - w) - : \beta(w) \varepsilon_1^*(w) : \delta(z - w) + \partial_w \delta(z - w) \\ &- : \varepsilon_1(-w) \varepsilon_1^*(-w) : \delta(z + w) + \varepsilon_1(-w) \varepsilon_1^*(-w) : \delta(z + w) + \partial_w \delta(z + w) \\ &+ : \varepsilon_1^*(w) \varepsilon_1(w) : \delta(z + w) - : \varepsilon_1^*(w) \varepsilon_1(w) : \delta(z + w) + \partial_w \delta(z + w) \\ &- : \beta^*(-w) \varepsilon_1(-w) : \delta(z - w) + : \beta^*(-w) \varepsilon_1(-w) : \delta(z - w) + \partial_w \delta(z - w). \end{aligned}$$

We add \mathcal{K} without changing the operator since it acts as the identity:

$$\begin{aligned} &= 2(\partial_w \delta(z - w) + \partial_w \delta(z + w))\mathcal{K} \\ &= \rho([\alpha_0^{\bar{\sigma}}(z), \alpha_m^{\bar{\sigma}}(w)]). \end{aligned}$$

Next, consider the case $1 \leq m \leq n - 1$.

$$\begin{aligned} &[\rho(\alpha_0^{\bar{\sigma}}(z)), \rho(\alpha_m^{\bar{\sigma}}(w))] \\ &= [:\varepsilon_1^*(z) \beta(z) : + : \varepsilon_1(-z) \beta^*(-z) :, : \varepsilon_m(w) \varepsilon_m^*(w) : - : \varepsilon_{m+1}(w) \varepsilon_{m+1}^*(w) :] \\ &= [:\varepsilon_1^*(z) \beta(z) :, : \varepsilon_m(w) \varepsilon_m^*(w) :] + [:\varepsilon_1^*(z) \beta(z) :, : \varepsilon_{m+1}(w) \varepsilon_{m+1}^*(w) :] \\ &\quad + [:\varepsilon_1(-z) \beta^*(-z) :, : \varepsilon_m(w) \varepsilon_m^*(w) :] + [:\varepsilon_1(-z) \beta^*(-z) :, : \varepsilon_{m+1}(w) \varepsilon_{m+1}^*(w) :]. \end{aligned}$$

We use Proposition 31 to compute these four brackets. The nonzero terms are:

$$\begin{aligned} &-\delta_{m1} : \beta(w) \varepsilon_m^*(w) : \delta(z - w) - \delta_{m1} : \varepsilon_1^*(w) \varepsilon_m(w) : \delta(z - w) - \delta_{m1} \partial_w \delta(z - w) \\ &- \delta_{m+1,1} : \beta(w) \varepsilon_{m+1}^*(w) : \delta(z - w) - \delta_{m+1,1} : \varepsilon_1^*(w) \varepsilon_{m+1}(w) : \delta(z - w) - \delta_{m+1,1} \partial_w \delta(z - w) \\ &- \delta_{m1} : \beta^*(w) \varepsilon_m(w) : \delta(z + w) - \delta_{m1} : \varepsilon_1(w) \varepsilon_m^*(w) : \delta(z + w) - \delta_{m1} \partial_w \delta(z + w) \\ &- \delta_{m+1,1} : \beta^*(w) \varepsilon_{m+1}(w) : \delta(z + w) - \delta_{m+1,1} : \varepsilon_1(w) \varepsilon_{m+1}^*(w) : \delta(z + w) - \delta_{m+1,1} \partial_w \delta(z + w). \end{aligned}$$

Now notice that $\delta_{m+1,1} = 0$ by the restriction on values of m . Thus, by Remark 4.1, we have

$$\begin{aligned} &-\delta_{m1} (:\varepsilon_1(w) \varepsilon_m^*(w) : \delta(z - w) + : \varepsilon_1^*(w) \varepsilon_m(w) : \delta(z - w) + \partial_w \delta(z - w)) \\ &- \delta_{m1} (:\varepsilon_1^*(w) \varepsilon_m(w) : \delta(z + w) + : \varepsilon_1(w) \varepsilon_m^*(w) : \delta(z + w) + \partial_w \delta(z + w)). \end{aligned}$$

By Corollary 28, and since m must equal 1, we have

$$\begin{aligned} &-\delta_{m1} (:\varepsilon_1(w) \varepsilon_1^*(w) : \delta(z - w) - : \varepsilon_1(w) \varepsilon_1^*(w) : \delta(z - w) + \partial_w \delta(z - w)) \\ &- \delta_{m1} (:\varepsilon_1^*(w) \varepsilon_1(w) : \delta(z + w) - : \varepsilon_1^*(w) \varepsilon_1(w) : \delta(z + w) + \partial_w \delta(z + w)). \end{aligned}$$

We add \mathcal{K} without changing the operator since it acts as the identity:

$$\begin{aligned} &= -\delta_{m1}(\partial_w \delta(z-w) + \partial_w \delta(z+w))\mathcal{K} \\ &= \rho([\alpha_0^{\bar{\sigma}}(z), \alpha_m^{\bar{\sigma}}(w)]). \end{aligned}$$

The final case is $m = n$.

$$\begin{aligned} &[\rho(\alpha_0^{\bar{\sigma}}(z)), \rho(\alpha_m^{\bar{\sigma}}(w))] \\ &= [:\varepsilon_1^*(z)\beta(z): + :\varepsilon_1(-z)\beta^*(-z):, :\varepsilon_n(w)\varepsilon_n^*(w): - :\varepsilon_n(-w)\varepsilon_n^*(-w):] \\ &= [:\varepsilon_1^*(z)\beta(z):, :\varepsilon_n(w)\varepsilon_n^*(w):] + [:\varepsilon_1^*(z)\beta(z):, :\varepsilon_n(-w)\varepsilon_n^*(-w):] \\ &\quad + [:\varepsilon_1(-z)\beta^*(-z):, :\varepsilon_n(w)\varepsilon_n^*(w):] + [:\varepsilon_1(-z)\beta^*(-z):, :\varepsilon_n(-w)\varepsilon_n^*(-w):] \end{aligned}$$

All bilinear forms $\langle \cdot, \cdot \rangle$ (and hence contractions) are trivially 0, so this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([\alpha_0^{\bar{\sigma}}(z), \alpha_m^{\bar{\sigma}}(w)]). \end{aligned}$$

Relation (2): $[\alpha_i^{\bar{\sigma}}(z), \alpha_j^{\bar{\sigma}}(w)] = a_{ij}\partial_w \delta(z-w)\mathcal{K}$

$$\begin{aligned} &[\rho(\alpha_i^{\bar{\sigma}}(z)), \rho(\alpha_j^{\bar{\sigma}}(w))] \\ &= [:\varepsilon_i(z)\varepsilon_i^*(z): - :\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):, :\varepsilon_j(w)\varepsilon_j^*(w): - :\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w):] \\ &= [:\varepsilon_i(z)\varepsilon_i^*(z):, :\varepsilon_j(w)\varepsilon_j^*(w):] - [:\varepsilon_i(z)\varepsilon_i^*(z):, :\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w):] \\ &\quad - [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):, :\varepsilon_j(w)\varepsilon_j^*(w):] + [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):, :\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w):]. \end{aligned}$$

Here we use Proposition 31 which allows us to compute the brackets. The nonzero terms are:

$$\begin{aligned} &= \delta_{ij} :\varepsilon_i^*(w)\varepsilon_j(w): \delta(z-w) + \delta_{ij} :\varepsilon_i(w)\varepsilon_j^*(w): \delta(z-w) + \delta_{ij}\partial_w \delta(z-w) \\ &\quad - \delta_{i,j+1} :\varepsilon_i^*(w)\varepsilon_{j+1}(w): \delta(z-w) - \delta_{i,j+1} :\varepsilon_i(w)\varepsilon_{j+1}^*(w): \delta(z-w) - \delta_{i,j+1}\partial_w \delta(z-w) \\ &\quad - \delta_{i+1,j} :\varepsilon_{i+1}^*(w)\varepsilon_j(w): \delta(z-w) - \delta_{i+1,j} :\varepsilon_{i+1}(w)\varepsilon_j^*(w): \delta(z-w) - \delta_{i+1,j}\partial_w \delta(z-w) \\ &\quad + \delta_{ij} :\varepsilon_{i+1}^*(w)\varepsilon_{j+1}(w): \delta(z-w) + \delta_{ij} :\varepsilon_{i+1}(w)\varepsilon_{j+1}^*(w): \delta(z-w) + \delta_{ij}\partial_w \delta(z-w). \end{aligned}$$

By Corollary 28, we have

$$\begin{aligned} &= \delta_{ij} (: \varepsilon_i^*(w)\varepsilon_j(w) : - : \varepsilon_j^*(w)\varepsilon_i(w) :) \delta(z-w) + \delta_{ij}\partial_w \delta(z-w) \\ &\quad - \delta_{i,j+1} (: \varepsilon_i^*(w)\varepsilon_{j+1}(w) : - : \varepsilon_{j+1}^*(w)\varepsilon_i(w) :) \delta(z-w) - \delta_{i,j+1}\partial_w \delta(z-w) \\ &\quad - \delta_{i+1,j} (: \varepsilon_{i+1}^*(w)\varepsilon_j(w) : - : \varepsilon_j^*(w)\varepsilon_{i+1}(w) :) \delta(z-w) - \delta_{i+1,j}\partial_w \delta(z-w) \\ &\quad + \delta_{ij} (: \varepsilon_{i+1}^*(w)\varepsilon_{j+1}(w) : - : \varepsilon_{j+1}^*(w)\varepsilon_{i+1}(w) :) \delta(z-w) + \delta_{ij}\partial_w \delta(z-w). \end{aligned}$$

Because of the $\delta_{ij}, \delta_{i,j+1}, \delta_{i+1,j}$ coefficients, each pair of normally ordered products (that is, each pair inside of parentheses) cancels. All that remains is:

$$\begin{aligned} &= 2\delta_{ij}\partial_w \delta(z-w) - \delta_{i,j+1}\partial_w \delta(z-w) - \delta_{i+1,j}\partial_w \delta(z-w) \\ &= (2\delta_{ij} - \delta_{i,j+1} - \delta_{i+1,j})\partial_w \delta(z-w). \end{aligned}$$

Hence, for $i = j$, the coefficient is $2 = a_{ii}$. If $i = j \pm 1$, the coefficient is $-1 = a_{i,i\pm 1}$. In all other cases for i, j , the coefficient is $0 = a_{ij}$. Thus, the coefficient is a_{ij} in every case. Further, since \mathcal{K} acts as the identity, it can be added without changing the operator, so we can write:

$$\begin{aligned} &= a_{ij}\partial_w \delta(z-w)\mathcal{K} \\ &= \rho([\alpha_i^{\bar{\sigma}}(z), \alpha_j^{\bar{\sigma}}(w)]). \end{aligned}$$

Relation (3): $[\alpha_i^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(w)] = a_{in}(\partial_w \delta(z-w) + \partial_w \delta(z+w))\mathcal{K}$

$$\begin{aligned}
& [\rho(\alpha_i^{\bar{\sigma}}(z)), \rho(\alpha_n^{\bar{\sigma}}(w))] \\
&= [:\varepsilon_i(z)\varepsilon_i^*(z) : - :\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z) :, :\varepsilon_n(w)\varepsilon_n^*(w) : - :\varepsilon_n(-w)\varepsilon_n^*(-w) :] \\
&= [:\varepsilon_i(z)\varepsilon_i^*(z) :, :\varepsilon_n(w)\varepsilon_n^*(w) :] - [:\varepsilon_i(z)\varepsilon_i^*(z) :, :\varepsilon_n(-w)\varepsilon_n^*(-w) :] \\
&\quad - [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z) :, :\varepsilon_n(w)\varepsilon_n^*(w) :] + [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z) :, :\varepsilon_n(-w)\varepsilon_n^*(-w) :].
\end{aligned}$$

Notice that the first two brackets have contractions which are all trivially 0 because of the restriction on values of i . By Proposition 31, the remaining nonzero terms are:

$$\begin{aligned}
& -\delta_{i+1,n} :\varepsilon_{i+1}^*(w)\varepsilon_n(w) : \delta(z-w) - \delta_{i+1,n} :\varepsilon_{i+1}(w)\varepsilon_n^*(w) : \delta(z-w) - \delta_{i+1,n} \partial_w \delta(z-w) \\
& + \delta_{i+1,n} :\varepsilon_{i+1}^*(-w)\varepsilon_n(-w) : \delta(z+w) + \delta_{i+1,n} :\varepsilon_{i+1}(-w)\varepsilon_n^*(-w) : \delta(z+w) - \delta_{i+1,n} \partial_w \delta(z+w)
\end{aligned}$$

By Corollary 28, we have

$$\begin{aligned}
& -\delta_{i+1,n} (: \varepsilon_{i+1}^*(w)\varepsilon_n(w) : \delta(z-w) - : \varepsilon_n^*(w)\varepsilon_{i+1}(w) : \delta(z-w) + \partial_w \delta(z-w)) \\
& + \delta_{i+1,n} (: \varepsilon_{i+1}^*(-w)\varepsilon_n(-w) : \delta(z+w) - : \varepsilon_n^*(-w)\varepsilon_{i+1}(-w) : \delta(z+w) - \partial_w \delta(z+w))
\end{aligned}$$

Because of the $\delta_{i+1,n}$, each pair of normally ordered products cancels. All that remains is:

$$-\delta_{i+1,n} (\partial_w \delta(z-w) + \partial_w \delta(z+w)).$$

We add \mathcal{K} since it acts as the identity and hence does not change the operator to arrive at:

$$\begin{aligned}
& -\delta_{i+1,n} (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \mathcal{K} \\
& = a_{in} (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \mathcal{K} \\
& = \rho([\alpha_i^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(w)]).
\end{aligned}$$

Relation (4): $[\alpha_n^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(w)] = a_{nn} (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \mathcal{K}$

$$\begin{aligned}
& [\rho(\alpha_n^{\bar{\sigma}}(z)), \rho(\alpha_n^{\bar{\sigma}}(w))] \\
&= [:\varepsilon_n(z)\varepsilon_n^*(z) : - :\varepsilon_n(-z)\varepsilon_n^*(-z) :, :\varepsilon_n(w)\varepsilon_n^*(w) : - :\varepsilon_n(-w)\varepsilon_n^*(-w) :] \\
&= [:\varepsilon_n(z)\varepsilon_n^*(z) :, :\varepsilon_n(w)\varepsilon_n^*(w) :] - [:\varepsilon_n(z)\varepsilon_n^*(z) :, :\varepsilon_n(-w)\varepsilon_n^*(-w) :] \\
&\quad - [:\varepsilon_n(-z)\varepsilon_n^*(-z) :, :\varepsilon_n(w)\varepsilon_n^*(w) :] + [:\varepsilon_n(-z)\varepsilon_n^*(-z) :, :\varepsilon_n(-w)\varepsilon_n^*(-w) :].
\end{aligned}$$

We use Proposition 31 to compute the four brackets. The nonzero terms are:

$$\begin{aligned}
& = :\varepsilon_n^*(w)\varepsilon_n(w) : \delta(z-w) + :\varepsilon_n(w)\varepsilon_n^*(w) : \delta(z-w) + \partial_w \delta(z-w) \\
& - :\varepsilon_n^*(-w)\varepsilon_n(-w) : \delta(z+w) - :\varepsilon_n(-w)\varepsilon_n^*(-w) : \delta(z+w) + \partial_w \delta(z+w) \\
& + :\varepsilon_n^*(w)\varepsilon_n(w) : \delta(z+w) + :\varepsilon_n(w)\varepsilon_n^*(w) : \delta(z+w) + \partial_w \delta(z+w) \\
& - :\varepsilon_n^*(-w)\varepsilon_n(-w) : \delta(z-w) - :\varepsilon_n(-w)\varepsilon_n^*(-w) : \delta(z-w) + \partial_w \delta(z-w).
\end{aligned}$$

By Corollary 28, we have:

$$\begin{aligned}
& = (: \varepsilon_n^*(w)\varepsilon_n(w) : - : \varepsilon_n^*(w)\varepsilon_n(w) :) \delta(z-w) + \partial_w \delta(z-w) \\
& - (: \varepsilon_n^*(-w)\varepsilon_n(-w) : - : \varepsilon_n^*(-w)\varepsilon_n(-w) :) \delta(z+w) + \partial_w \delta(z+w) \\
& + (: \varepsilon_n^*(w)\varepsilon_n(w) : - : \varepsilon_n^*(w)\varepsilon_n(w) :) \delta(z+w) + \partial_w \delta(z+w) \\
& - (: \varepsilon_n^*(-w)\varepsilon_n(-w) : - : \varepsilon_n^*(-w)\varepsilon_n(-w) :) \delta(z-w) + \partial_w \delta(z-w) \\
& = 2(\partial_w \delta(z-w) + \partial_w \delta(z+w)).
\end{aligned}$$

We add \mathcal{K} without changing the operator, since it acts as the identity, to arrive at:

$$= a_{nn} (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \mathcal{K}$$

$$= \rho([\alpha_n^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(w)]).$$

Relation (5): $[\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_m, w)] = \pm(2\delta_{m0} - \delta_{m1})X^{\bar{\sigma}}(\pm\alpha_m, w)(\delta(z-w) + \delta(z+w))$

The first case to consider is $m = 0$. We calculate as follows, first for $X^{\bar{\sigma}}(\alpha_m, w)$:

$$\begin{aligned} & [\rho(\alpha_0^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(\alpha_m, w))] \\ &= [:\varepsilon_1^*(z)\beta(z): + :\varepsilon_1(-z)\beta^*(-z):, : \beta^*(w)\varepsilon_1^*(-w):] \\ &= [:\varepsilon_1^*(z)\beta(z):, : \beta^*(w)\varepsilon_1^*(-w):] + [:\varepsilon_1(-z)\beta^*(-z):, : \beta^*(w)\varepsilon_1^*(-w):] \end{aligned}$$

Using Proposition 31, the nonzero terms are:

$$\begin{aligned} & = :\varepsilon_1^*(w)\varepsilon_1^*(-w): \delta(z-w) - :\varepsilon_1^*(-w)\beta^*(w): \delta(z+w) \\ & - :\beta^*(-w)\beta^*(w): \delta(z-w) + :\beta^*(w)\varepsilon_1^*(-w): \delta(z+w). \end{aligned}$$

By Remark 4.1, we have

$$= (: \beta^*(w)\varepsilon_1^*(-w): - :\varepsilon_1^*(-w)\beta^*(w):) \delta(z-w) + (: \beta^*(w)\varepsilon_1^*(-w): - :\varepsilon_1^*(-w)\beta^*(w):) \delta(z+w)$$

By Corollary 28, we have

$$\begin{aligned} & = (: \beta^*(w)\varepsilon_1^*(-w): + :\beta^*(w)\varepsilon_1^*(-w):) \delta(z-w) + (: \beta^*(w)\varepsilon_1^*(-w): + :\beta^*(w)\varepsilon_1^*(-w):) \delta(z+w) \\ & = 2 :\beta^*(w)\varepsilon_1^*(-w): \delta(z-w) + 2 :\beta^*(w)\varepsilon_1^*(-w): \delta(z+w) \\ & = 2\rho(X^{\bar{\sigma}}(\alpha_m, w))(\delta(z-w) + \delta(z+w)) \\ & = \rho([\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_m, w)]). \end{aligned}$$

Similarly, for $X^{\bar{\sigma}}(-\alpha_m, w)$,

$$\begin{aligned} & [\rho(\alpha_0^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(-\alpha_m, w))] \\ &= [:\varepsilon_1^*(z)\beta(z): + :\varepsilon_1(-z)\beta^*(-z):, : \beta(w)\varepsilon_1(-w):] \\ &= [:\varepsilon_1^*(z)\beta(z):, : \beta(w)\varepsilon_1(-w):] + [:\varepsilon_1(-z)\beta^*(-z):, : \beta(w)\varepsilon_1(-w):] \end{aligned}$$

Using Proposition 31, the nonzero terms are:

$$\begin{aligned} & = :\beta(-w)\beta(w): \delta(z+w) - :\beta(w)\varepsilon_1(-w): \delta(z-w) \\ & - :\varepsilon_1(w)\varepsilon_1(-w): \delta(z+w) + :\varepsilon_1(-w)\beta(w): \delta(z-w). \end{aligned}$$

By Remark 4.1, we have

$$= -(: \beta(w)\varepsilon_1(-w): - :\varepsilon_1(-w)\beta(w):) \delta(z-w) - (: \beta(w)\varepsilon_1(-w): - :\varepsilon_1(-w)\beta(w):) \delta(z+w).$$

By Corollary 28, we have

$$\begin{aligned} & = -(: \beta(w)\varepsilon_1(-w): + :\beta(w)\varepsilon_1(-w):) \delta(z-w) - (: \beta(w)\varepsilon_1(-w): + :\beta(w)\varepsilon_1(-w):) \delta(z+w) \\ & = -2\rho(X^{\bar{\sigma}}(-\alpha_m, w))(\delta(z-w) + \delta(z+w)) \\ & = \rho([\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_m, w)]). \end{aligned}$$

Now consider the case for $1 \leq m \leq n-1$. For $X^{\bar{\sigma}}(\alpha_m, w)$,

$$\begin{aligned} & [\rho(\alpha_0^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(\alpha_m, w))] \\ &= [:\varepsilon_1^*(z)\beta(z): + :\varepsilon_1(-z)\beta^*(-z):, : \varepsilon_m(w)\varepsilon_{m+1}^*(w):] \\ &= [:\varepsilon_1^*(z)\beta(z):, : \varepsilon_m(w)\varepsilon_{m+1}^*(w):] + [:\varepsilon_1(-z)\beta^*(-z):, : \varepsilon_m(w)\varepsilon_{m+1}^*(w):] \end{aligned}$$

Using Proposition 31, the possibly nonzero terms are:

$$= -\delta_{m1} :\beta(w)\varepsilon_{m+1}^*(w): \delta(z-w) - \delta_{m+1,1} :\varepsilon_1^*(w)\varepsilon_m(w): \delta(z-w) - \delta_{m1}\delta_{m+1,1}\partial_w\delta(z-w)$$

$$= \delta_{m+1,1} : \beta^*(w) \varepsilon_m(w) : \delta(z + w) - \delta_{m1} : \varepsilon_1(w) \varepsilon_{m+1}^*(w) : \delta(z + w) - \delta_{m+1,1} \delta_{m1} \partial_w \delta(z + w).$$

By the restrictions on the values of m , in fact $\delta_{m+1,1} = 0$. Using that and Remark 4.1 gives us:

$$\begin{aligned} &= -\delta_{m1} : \beta(w) \varepsilon_{m+1}^*(w) : \delta(z - w) - \delta_{m1} : \varepsilon_1(w) \varepsilon_{m+1}^*(w) : \delta(z + w) \\ &= -\delta_{m1} (: \beta(w) \varepsilon_{m+1}^*(w) : \delta(z - w) + : \varepsilon_1(w) \varepsilon_{m+1}^*(w) : \delta(z + w)) \\ &= -\delta_{m1} (: \varepsilon_m(w) \varepsilon_{m+1}^*(w) : \delta(z - w) + : \varepsilon_m(w) \varepsilon_{m+1}^*(w) : \delta(z + w)) \\ &= -\delta_{m1} \rho(X^{\bar{\sigma}}(\alpha_m, w)) (\delta(z - w) + \delta(z + w)) \\ &= \rho([\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_m, w)]). \end{aligned}$$

Similarly, for $X^{\bar{\sigma}}(-\alpha_m, w)$,

$$\begin{aligned} &[\rho(\alpha_0^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(-\alpha_m, w))] \\ &= [: \varepsilon_1^*(z) \beta(z) : + : \varepsilon_1(-z) \beta^*(-z) :, - : \varepsilon_m^*(w) \varepsilon_{m+1}(w) :] \\ &= [: \varepsilon_1^*(z) \beta(z) :, - : \varepsilon_m^*(w) \varepsilon_{m+1}(w) :] + [: \varepsilon_1(-z) \beta^*(-z) :, - : \varepsilon_m^*(w) \varepsilon_{m+1}(w) :] \end{aligned}$$

Using Proposition 31, the possibly nonzero terms are:

$$\begin{aligned} &= -\delta_{m+1,1} : \beta(w) \varepsilon_m^*(w) : \delta(z - w) - \delta_{m1} : \varepsilon_1^*(w) \varepsilon_{m+1}(w) : \delta(z - w) - \delta_{m+1,1} \delta_{m1} \partial_w \delta(z - w) \\ &\quad - \delta_{m1} : \beta^*(w) \varepsilon_{m+1}(w) : \delta(z + w) - \delta_{m+1,1} : \varepsilon_1(w) \varepsilon_m^*(w) : \delta(z + w) - \delta_{m+1,1} \delta_{m1} \partial_w \delta(z + w). \end{aligned}$$

By the restrictions on the values of m , in fact $\delta_{m+1,1} = 0$. Using that and Remark 4.1 gives us:

$$\begin{aligned} &= -\delta_{m1} : \varepsilon_1^*(w) \varepsilon_{m+1}(w) : \delta(z - w) - \delta_{m1} : \beta^*(w) \varepsilon_{m+1}(w) : \delta(z + w) \\ &= \delta_{m1} (- : \varepsilon_1^*(w) \varepsilon_{m+1}(w) : \delta(z - w) - : \varepsilon_1^*(w) \varepsilon_{m+1}(w) : \delta(z + w)) \\ &= \delta_{m1} (- : \varepsilon_m^*(w) \varepsilon_{m+1}(w) : \delta(z - w) - : \varepsilon_m^*(w) \varepsilon_{m+1}(w) : \delta(z + w)) \\ &= \delta_{m1} \rho(X^{\bar{\sigma}}(-\alpha_m, w)) (\delta(z - w) + \delta(z + w)) \\ &= \rho([\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_m, w)]). \end{aligned}$$

The final case in this relation is $m = n$. For $X^{\bar{\sigma}}(\alpha_m, w)$,

$$\begin{aligned} &[\rho(\alpha_0^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(\alpha_m, w))] \\ &= [: \varepsilon_1^*(z) \beta(z) : + : \varepsilon_1(-z) \beta^*(-z) :, : \varepsilon_n(w) \varepsilon_n(-w) :] \\ &= [: \varepsilon_1^*(z) \beta(z) :, : \varepsilon_n(w) \varepsilon_n(-w) :] + [: \varepsilon_1(-z) \beta^*(-z) :, : \varepsilon_n(w) \varepsilon_n(-w) :] \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is simply:

$$\begin{aligned} &= 0 \\ &= \rho([\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_m, w)]). \end{aligned}$$

Similarly, for $X^{\bar{\sigma}}(-\alpha_m, w)$,

$$\begin{aligned} &[\rho(\alpha_0^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(-\alpha_m, w))] \\ &= [: \varepsilon_1^*(z) \beta(z) : + : \varepsilon_1(-z) \beta^*(-z) :, - : \varepsilon_n^*(-w) \varepsilon_n^*(w) :] \\ &= [: \varepsilon_1^*(z) \beta(z) :, - : \varepsilon_n^*(-w) \varepsilon_n^*(w) :] + [: \varepsilon_1(-z) \beta^*(-z) :, - : \varepsilon_n^*(-w) \varepsilon_n^*(w) :] \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is simply:

$$= 0$$

$$= \rho([\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_m, w)]).$$

Relation (6): $[\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_0, w)] = \pm(-\delta_{i1})X^{\bar{\sigma}}(\pm\alpha_0, w)(\delta(z-w) + \delta(z+w))$

We first calculate for $X^{\bar{\sigma}}(\alpha_0, w)$:

$$\begin{aligned} & [\rho(\alpha_i^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(\alpha_0, w))] \\ &= [:\varepsilon_i(z)\varepsilon_i^*(z): - :\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):, : \beta^*(w)\varepsilon_1^*(-w):] \\ &= [:\varepsilon_i(z)\varepsilon_i^*(z):, : \beta^*(w)\varepsilon_1^*(-w):] - [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):, : \beta^*(w)\varepsilon_1^*(-w):] \end{aligned}$$

Using Proposition 31, the possibly nonzero terms are:

$$\begin{aligned} &= \delta_{i1} :\varepsilon_i^*(-w)\beta^*(w):\delta(z+w) - \delta_{i1} :\varepsilon_i^*(w)\varepsilon_1^*(-w):\delta(z-w) \\ &\quad - \delta_{i+1,1} :\varepsilon_{i+1}^*(-w)\beta^*(w):\delta(z+w) + \delta_{i+1,1} :\varepsilon_{i+1}^*(w)\varepsilon_1^*(-w):\delta(z-w). \end{aligned}$$

By the restrictions on the values of i , $\delta_{i+1,1} = 0$. Using that and Remark 4.1 gives us:

$$\begin{aligned} &= \delta_{i1} :\varepsilon_i^*(-w)\beta^*(w):\delta(z+w) - \delta_{i1} :\varepsilon_i^*(w)\varepsilon_1^*(-w):\delta(z-w) \\ &= \delta_{i1} :\varepsilon_1^*(-w)\beta^*(w):\delta(z+w) - \delta_{i1} :\varepsilon_1^*(w)\varepsilon_1^*(-w):\delta(z-w) \\ &= \delta_{i1} :\varepsilon_1^*(-w)\beta^*(w):\delta(z+w) - \delta_{i1} :\beta^*(w)\varepsilon_1^*(-w):\delta(z-w). \end{aligned}$$

By Corollary 28, we have

$$\begin{aligned} &= -\delta_{i1} :\beta^*(w)\varepsilon_1^*(-w):\delta(z+w) - \delta_{i1} :\beta^*(w)\varepsilon_1^*(-w):\delta(z-w) \\ &= -\delta_{i1}\rho(X^{\bar{\sigma}}(\alpha_0, w))(\delta(z-w) + \delta(z+w)) \\ &= \rho([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_0, w)]). \end{aligned}$$

Now for $X^{\bar{\sigma}}(-\alpha_0, w)$:

$$\begin{aligned} & [\rho(\alpha_i^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(-\alpha_0, w))] \\ &= [:\varepsilon_i(z)\varepsilon_i^*(z): - :\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):, : \beta(w)\varepsilon_1(-w):] \\ &= [:\varepsilon_i(z)\varepsilon_i^*(z):, : \beta(w)\varepsilon_1(-w):] - [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):, : \beta(w)\varepsilon_1(-w):] \end{aligned}$$

Using Proposition 31, the possibly nonzero terms are:

$$\begin{aligned} &= \delta_{i1} :\varepsilon_i(w)\varepsilon_1(-w):\delta(z-w) - \delta_{i1} :\varepsilon_i(-w)\beta(w):\delta(z+w) \\ &\quad - \delta_{i+1,1} :\varepsilon_{i+1}(w)\varepsilon_1(-w):\delta(z-w) - \delta_{i+1,1} :\varepsilon_{i+1}(-w)\beta(w):\delta(z+w) \end{aligned}$$

By the restrictions on the values of i , $\delta_{i+1,1} = 0$. Using that and Remark 4.1 gives us:

$$\begin{aligned} &= \delta_{i1} :\varepsilon_i(w)\varepsilon_1(-w):\delta(z-w) - \delta_{i1} :\varepsilon_i(-w)\beta(w):\delta(z+w) \\ &= \delta_{i1} :\varepsilon_1(w)\varepsilon_1(-w):\delta(z-w) - \delta_{i1} :\varepsilon_1(-w)\beta(w):\delta(z+w) \\ &= \delta_{i1} :\beta(w)\varepsilon_1(-w):\delta(z-w) - \delta_{i1} :\varepsilon_1(-w)\beta(w):\delta(z+w). \end{aligned}$$

By Corollary 28, we have

$$\begin{aligned} &= \delta_{i1} :\beta(w)\varepsilon_1(-w):\delta(z-w) + \delta_{i1} :\beta(w)\varepsilon_1(-w):\delta(z+w) \\ &= \delta_{i1}\rho(X^{\bar{\sigma}}(-\alpha_0, w))(\delta(z-w) + \delta(z+w)) \\ &= \rho([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_0, w)]). \end{aligned}$$

Relation (7): $[\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_j, w)] = \pm a_{ij}X^{\bar{\sigma}}(\pm\alpha_j, w)\delta(z-w)$

We will first calculate for $X^{\bar{\sigma}}(\alpha_j, w)$.

$$[\rho(\alpha_i^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(\alpha_j, w))]$$

$$\begin{aligned}
&= [:\varepsilon_i(z)\varepsilon_i^*(z):-:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):,: \varepsilon_j(w)\varepsilon_{j+1}^*(w):] \\
&= [:\varepsilon_i(z)\varepsilon_i^*(z):,: \varepsilon_j(w)\varepsilon_{j+1}^*(w):] - [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):,: \varepsilon_j(w)\varepsilon_{j+1}^*(w):].
\end{aligned}$$

We use Proposition 31 to compute these two brackets, showing only the terms that are (possibly) nonzero.

$$\begin{aligned}
&= \delta_{i,j+1} :\varepsilon_i^*(w)\varepsilon_j(w):\delta(z-w) + \delta_{ij} :\varepsilon_i(w)\varepsilon_{j+1}^*(w):\delta(z-w) + \delta_{i,j+1}\delta_{ij}\partial_w\delta(z-w) \\
&\quad - \delta_{ij} :\varepsilon_{i+1}^*(w)\varepsilon_j(w):\delta(z-w) - \delta_{i+1,j} :\varepsilon_{i+1}(w)\varepsilon_{j+1}^*(w):\delta(z-w) - \delta_{ij}\delta_{i+1,j}\partial_w\delta(z-w) \\
&= \delta_{i,j+1} :\varepsilon_{j+1}^*(w)\varepsilon_j(w):\delta(z-w) + \delta_{ij} :\varepsilon_j(w)\varepsilon_{j+1}^*(w):\delta(z-w) + \delta_{i,j+1}\delta_{ij}\partial_w\delta(z-w) \\
&\quad - \delta_{ij} :\varepsilon_{j+1}^*(w)\varepsilon_j(w):\delta(z-w) - \delta_{i+1,j} :\varepsilon_j(w)\varepsilon_{j+1}^*(w):\delta(z-w) - \delta_{ij}\delta_{i+1,j}\partial_w\delta(z-w).
\end{aligned}$$

Clearly, $\delta_{ij}\delta_{i,j+1} = \delta_{ij}\delta_{i+1,j} = 0$. Also, by Corollary 28, we have

$$\begin{aligned}
&= -\delta_{i,j+1} :\varepsilon_j(w)\varepsilon_{j+1}^*(w):\delta(z-w) + \delta_{ij} :\varepsilon_j(w)\varepsilon_{j+1}^*(w):\delta(z-w) \\
&\quad + \delta_{ij} :\varepsilon_j(w)\varepsilon_{j+1}^*(w):\delta(z-w) - \delta_{i+1,j} :\varepsilon_j(w)\varepsilon_{j+1}^*(w):\delta(z-w). \\
&= (-\delta_{i,j+1} :\varepsilon_j(w)\varepsilon_{j+1}^*(w): + 2\delta_{ij} :\varepsilon_j(w)\varepsilon_{j+1}^*(w): - \delta_{i+1,j} :\varepsilon_j(w)\varepsilon_{j+1}^*(w):)\delta(z-w) \\
&= (-\delta_{i,j+1} + 2\delta_{ij} - \delta_{i+1,j})\rho(X^{\bar{\sigma}}(\alpha_j, w))\delta(z-w) \\
&= a_{ij}\rho(X^{\bar{\sigma}}(\alpha_j, w))\delta(z-w) \\
&= \rho([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_j, w)]).
\end{aligned}$$

We now repeat the calculation for $X^{\bar{\sigma}}(-\alpha_j, w)$.

$$\begin{aligned}
&[\rho(\alpha_i^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(-\alpha_j, w))] \\
&= [:\varepsilon_i(z)\varepsilon_i^*(z):-:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):-:\varepsilon_j^*(w)\varepsilon_{j+1}(w):] \\
&= [:\varepsilon_i(z)\varepsilon_i^*(z):-:\varepsilon_j^*(w)\varepsilon_{j+1}(w):] - [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):-:\varepsilon_j^*(w)\varepsilon_{j+1}(w):].
\end{aligned}$$

We use Proposition 31 to compute these two brackets, showing only the terms that are (possibly) nonzero.

$$\begin{aligned}
&= \delta_{ij} :\varepsilon_i^*(w)\varepsilon_{j+1}(w):\delta(z-w) + \delta_{i,j+1} :\varepsilon_i(w)\varepsilon_j^*(w):\delta(z-w) + \delta_{ij}\delta_{i,j+1}\partial_w\delta(z-w) \\
&\quad - \delta_{i+1,j} :\varepsilon_{i+1}^*(w)\varepsilon_{j+1}(w):\delta(z-w) - \delta_{ij} :\varepsilon_{i+1}(w)\varepsilon_j^*(w):\delta(z-w) - \delta_{i+1,j}\delta_{ij}\partial_w\delta(z-w) \\
&= \delta_{ij} :\varepsilon_j^*(w)\varepsilon_{j+1}(w):\delta(z-w) + \delta_{i,j+1} :\varepsilon_{j+1}(w)\varepsilon_j^*(w):\delta(z-w) + \delta_{ij}\delta_{i,j+1}\partial_w\delta(z-w) \\
&\quad - \delta_{i+1,j} :\varepsilon_j^*(w)\varepsilon_{j+1}(w):\delta(z-w) - \delta_{ij} :\varepsilon_{j+1}(w)\varepsilon_j^*(w):\delta(z-w) - \delta_{i+1,j}\delta_{ij}\partial_w\delta(z-w).
\end{aligned}$$

Clearly, $\delta_{ij}\delta_{i,j+1} = \delta_{ij}\delta_{i+1,j} = 0$. Also, by Corollary 28, we have

$$\begin{aligned}
&= \delta_{ij} :\varepsilon_j^*(w)\varepsilon_{j+1}(w):\delta(z-w) - \delta_{i,j+1} :\varepsilon_j^*(w)\varepsilon_{j+1}(w):\delta(z-w) \\
&\quad - \delta_{i+1,j} :\varepsilon_j^*(w)\varepsilon_{j+1}(w):\delta(z-w) + \delta_{ij} :\varepsilon_j^*(w)\varepsilon_{j+1}(w):\delta(z-w) \\
&= (-\delta_{i,j+1} + 2\delta_{ij} - \delta_{i+1,j}) :\varepsilon_j^*(w)\varepsilon_{j+1}(w):\delta(z-w) \\
&= -(-\delta_{i,j+1} + 2\delta_{ij} - \delta_{i+1,j})(-\varepsilon_j^*(w)\varepsilon_{j+1}(w):\delta(z-w)) \\
&= -a_{ij}\rho(X^{\bar{\sigma}}(-\alpha_j, w))\delta(z-w) \\
&= \rho([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_j, w)]).
\end{aligned}$$

Relation (8): $[\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_n, w)] = \pm a_{in}X^{\bar{\sigma}}(\pm\alpha_n, w)(\delta(z-w) + \delta(z+w))$

We will first calculate for $X^{\bar{\sigma}}(\alpha_n, w)$.

$$[\rho(\alpha_i^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(\alpha_n, w))]$$

$$\begin{aligned}
&= [:\varepsilon_i(z)\varepsilon_i^*(z):-:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):,: \varepsilon_n(w)\varepsilon_n(-w):] \\
&= [:\varepsilon_i(z)\varepsilon_i^*(z):,: \varepsilon_n(w)\varepsilon_n(-w):] - [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):,: \varepsilon_n(w)\varepsilon_n(-w):].
\end{aligned}$$

Using Proposition 31, the possibly nonzero terms are:

$$\begin{aligned}
&= \delta_{in} :\varepsilon_i(w)\varepsilon_n(-w):\delta(z-w) - \delta_{in} :\varepsilon_i(-w)\varepsilon_n(w):\delta(z+w) \\
&\quad - \delta_{i+1,n} :\varepsilon_{i+1}(w)\varepsilon_n(-w):\delta(z-w) + \delta_{i+1,n} :\varepsilon_{i+1}(-w)\varepsilon_n(w):\delta(z+w).
\end{aligned}$$

By the restrictions on the values of i , $\delta_{in} = 0$. Using that and Corollary 28, we have:

$$\begin{aligned}
&= -\delta_{i+1,n} :\varepsilon_{i+1}(w)\varepsilon_n(-w):\delta(z-w) + \delta_{i+1,n} :\varepsilon_{i+1}(-w)\varepsilon_n(w):\delta(z+w) \\
&= -\delta_{i+1,n} :\varepsilon_{i+1}(w)\varepsilon_n(-w):\delta(z-w) - \delta_{i+1,n} :\varepsilon_n(w)\varepsilon_{i+1}(-w):\delta(z+w) \\
&= -\delta_{i+1,n} (:\varepsilon_n(w)\varepsilon_n(-w):\delta(z-w) + : \varepsilon_n(w)\varepsilon_n(-w):\delta(z+w)) \\
&= a_{in}\rho((X^{\bar{\sigma}}(\alpha_n, w))(\delta(z-w) + \delta(z+w))) \\
&= \rho([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_n, w)]).
\end{aligned}$$

We now repeat the calculation for $X^{\bar{\sigma}}(-\alpha_n, w)$.

$$\begin{aligned}
&[\rho(\alpha_i^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(-\alpha_n, w))] \\
&= [:\varepsilon_i(z)\varepsilon_i^*(z):-:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):-:\varepsilon_n^*(-w)\varepsilon_n^*(w):] \\
&= [:\varepsilon_i(z)\varepsilon_i^*(z):-:\varepsilon_n^*(-w)\varepsilon_n^*(w):] - [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):-:\varepsilon_n^*(-w)\varepsilon_n^*(w):].
\end{aligned}$$

Using Proposition 31, the possibly nonzero terms are:

$$\begin{aligned}
&= -\delta_{in} :\varepsilon_i^*(w)\varepsilon_n^*(-w):\delta(z+w) + \delta_{in} :\varepsilon_i^*(-w)\varepsilon_n^*(w):\delta(z-w) \\
&\quad + \delta_{i+1,n} :\varepsilon_{i+1}^*(w)\varepsilon_n^*(-w):\delta(z+w) - \delta_{i+1,n} :\varepsilon_{i+1}^*(-w)\varepsilon_n^*(w):\delta(z-w)
\end{aligned}$$

By the restrictions on the values of i , $\delta_{in} = 0$. Using that and Corollary 28, we have:

$$\begin{aligned}
&= \delta_{i+1,n} :\varepsilon_{i+1}^*(w)\varepsilon_n^*(-w):\delta(z+w) - \delta_{i+1,n} :\varepsilon_{i+1}^*(-w)\varepsilon_n^*(w):\delta(z-w) \\
&= -\delta_{i+1,n} :\varepsilon_n^*(-w)\varepsilon_{i+1}^*(w):\delta(z+w) - \delta_{i+1,n} :\varepsilon_{i+1}^*(-w)\varepsilon_n^*(w):\delta(z-w) \\
&= \delta_{i+1,n} (-:\varepsilon_n^*(-w)\varepsilon_n^*(w):\delta(z+w) - : \varepsilon_n^*(-w)\varepsilon_n^*(w):\delta(z-w)) \\
&= -a_{in}\rho((X^{\bar{\sigma}}(-\alpha_n, w))(\delta(z-w) + \delta(z+w))) \\
&= \rho([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_n, w)]).
\end{aligned}$$

Relation (9): $[\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_0, w)] = 0$

We will first calculate for $X^{\bar{\sigma}}(\alpha_0, w)$.

$$\begin{aligned}
&[\rho(\alpha_n^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(\alpha_0, w))] \\
&= [:\varepsilon_n(z)\varepsilon_n^*(z):-:\varepsilon_n(-z)\varepsilon_n^*(-z):,: \beta^*(w)\varepsilon_1^*(-w):] \\
&= [:\varepsilon_n(z)\varepsilon_n^*(z):,: \beta^*(w)\varepsilon_1^*(-w):] - [:\varepsilon_n(-z)\varepsilon_n^*(-z):,: \beta^*(w)\varepsilon_1^*(-w):].
\end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is simply:

$$\begin{aligned}
&= 0 \\
&= \rho([\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_0, w)]).
\end{aligned}$$

We now repeat the calculation for $X^{\bar{\sigma}}(-\alpha_0, w)$.

$$[\rho(\alpha_n^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(-\alpha_0, w))]$$

$$\begin{aligned}
&= [:\varepsilon_n(z)\varepsilon_n^*(z):-:\varepsilon_n(-z)\varepsilon_n^*(-z):,: \beta(w)\varepsilon_1(-w):] \\
&= [:\varepsilon_n(z)\varepsilon_n^*(z):,: \beta(w)\varepsilon_1(-w):] - [:\varepsilon_n(-z)\varepsilon_n^*(-z):,: \beta(w)\varepsilon_1(-w):].
\end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is simply:

$$\begin{aligned}
&= 0 \\
&= \rho([\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_0, w)]).
\end{aligned}$$

Relation (10): $[\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_j, w)] = \pm a_{nj} X^{\bar{\sigma}}(\pm\alpha_j, w)(\delta(z-w) + \delta(z+w))$

We will first calculate for $X^{\bar{\sigma}}(\alpha_j, w)$.

$$\begin{aligned}
&[\rho(\alpha_n^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(\alpha_j, w))] \\
&= [:\varepsilon_n(z)\varepsilon_n^*(z):-:\varepsilon_n(-z)\varepsilon_n^*(-z):,: \varepsilon_j(w)\varepsilon_{j+1}^*(w):] \\
&= [:\varepsilon_n(z)\varepsilon_n^*(z):,: \varepsilon_j(w)\varepsilon_{j+1}^*(w):] - [:\varepsilon_n(-z)\varepsilon_n^*(-z):,: \varepsilon_j(w)\varepsilon_{j+1}^*(w):].
\end{aligned}$$

Using Proposition 31, the possibly nonzero terms are:

$$\begin{aligned}
&= \delta_{n,j+1} :\varepsilon_n^*(w)\varepsilon_j(w): \delta(z-w) + \delta_{nj} :\varepsilon_n(w)\varepsilon_{j+1}^*(w): \delta(z-w) + \delta_{n,j+1}\delta_{nj}\partial_w\delta(z-w) \\
&\quad + \delta_{n,j+1} :\varepsilon_n^*(w)\varepsilon_j(w): \delta(z+w) + \delta_{nj} :\varepsilon_n(w)\varepsilon_{j+1}^*(w): \delta(z+w) + \delta_{n,j+1}\delta_{nj}\partial_w\delta(z+w).
\end{aligned}$$

By the restrictions on the values of j , $\delta_{nj} = 0$. Using that and Corollary 28 gives us:

$$\begin{aligned}
&= \delta_{n,j+1} :\varepsilon_n^*(w)\varepsilon_j(w): \delta(z-w) + \delta_{n,j+1} :\varepsilon_n^*(w)\varepsilon_j(w): \delta(z+w) \\
&= -\delta_{n,j+1} (-:\varepsilon_j(w)\varepsilon_n^*(w): \delta(z-w) + : \varepsilon_j(w)\varepsilon_n^*(w): \delta(z+w)) \\
&= -\delta_{n,j+1} (-:\varepsilon_j(w)\varepsilon_{j+1}^*(w): \delta(z-w) + : \varepsilon_j(w)\varepsilon_{j+1}^*(w): \delta(z+w)) \\
&= a_{nj}\rho(X^{\bar{\sigma}}(\alpha_j, w))(\delta(z-w) + \delta(z+w)) \\
&= \rho([\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_j, w)]).
\end{aligned}$$

We now repeat the calculation for $X^{\bar{\sigma}}(-\alpha_j, w)$.

$$\begin{aligned}
&[\rho(\alpha_n^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(-\alpha_j, w))] \\
&= [:\varepsilon_n(z)\varepsilon_n^*(z):-:\varepsilon_n(-z)\varepsilon_n^*(-z):,-:\varepsilon_j^*(w)\varepsilon_{j+1}(w):] \\
&= [:\varepsilon_n(z)\varepsilon_n^*(z):,-:\varepsilon_j^*(w)\varepsilon_{j+1}(w):] - [:\varepsilon_n(-z)\varepsilon_n^*(-z):,-:\varepsilon_j^*(w)\varepsilon_{j+1}(w):].
\end{aligned}$$

Using Proposition 31, the possibly nonzero terms are:

$$\begin{aligned}
&= \delta_{nj} :\varepsilon_n^*(w)\varepsilon_{j+1}(w): \delta(z-w) + \delta_{n,j+1} :\varepsilon_n(w)\varepsilon_j^*(w): \delta(z-w) + \delta_{nj}\delta_{n,j+1}\partial_w\delta(z-w) \\
&\quad + \delta_{nj} :\varepsilon_n^*(w)\varepsilon_{j+1}(w): \delta(z+w) + \delta_{n,j+1} :\varepsilon_n(w)\varepsilon_j^*(w): \delta(z+w) + \delta_{nj}\delta_{n,j+1}\partial_w\delta(z+w).
\end{aligned}$$

By the restrictions on the values of j , $\delta_{nj} = 0$. Using that and Corollary 28 gives us:

$$\begin{aligned}
&= \delta_{n,j+1} :\varepsilon_n(w)\varepsilon_j^*(w): \delta(z-w) + \delta_{n,j+1} :\varepsilon_n(w)\varepsilon_j^*(w): \delta(z+w) \\
&= \delta_{n,j+1} (-:\varepsilon_j^*(w)\varepsilon_n(w): \delta(z-w) - : \varepsilon_j^*(w)\varepsilon_n(w): \delta(z+w)) \\
&= \delta_{n,j+1} (-:\varepsilon_j^*(w)\varepsilon_{j+1}(w): \delta(z-w) - : \varepsilon_j^*(w)\varepsilon_{j+1}(w): \delta(z+w)) \\
&= -a_{nj}\rho(X^{\bar{\sigma}}(-\alpha_j, w))(\delta(z-w) + \delta(z+w)) \\
&= \rho([\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_j, w)]).
\end{aligned}$$

Relation (11): $[\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_n, w)] = \pm a_{nn} X^{\bar{\sigma}}(\pm\alpha_n, w)(\delta(z-w) + \delta(z+w))$

We will first calculate for $X^{\bar{\sigma}}(\alpha_n, w)$.

$$\begin{aligned}
& [\rho(\alpha_n^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(\alpha_n, w))] \\
&= [:\varepsilon_n(z)\varepsilon_n^*(z):-:\varepsilon_n(-z)\varepsilon_n^*(-z):, :\varepsilon_n(w)\varepsilon_n(-w):] \\
&= [:\varepsilon_n(z)\varepsilon_n^*(z):, :\varepsilon_n(w)\varepsilon_n(-w):] - [:\varepsilon_n(-z)\varepsilon_n^*(-z):, :\varepsilon_n(w)\varepsilon_n(-w):].
\end{aligned}$$

Using Proposition 31, the nonzero terms are:

$$\begin{aligned}
& = :\varepsilon_n(w)\varepsilon_n(-w):\delta(z-w) - :\varepsilon_n(-w)\varepsilon_n(w):\delta(z+w) \\
& + :\varepsilon_n(w)\varepsilon_n(-w):\delta(z+w) - :\varepsilon_n(-w)\varepsilon_n(w):\delta(z-w).
\end{aligned}$$

By Corollary 28, we have:

$$\begin{aligned}
& = :\varepsilon_n(w)\varepsilon_n(-w):\delta(z-w) + :\varepsilon_n(w)\varepsilon_n(-w):\delta(z+w) \\
& + :\varepsilon_n(w)\varepsilon_n(-w):\delta(z+w) + :\varepsilon_n(w)\varepsilon_n(-w):\delta(z-w) \\
& = 2:\varepsilon_n(w)\varepsilon_n(-w):\delta(z-w) + 2:\varepsilon_n(w)\varepsilon_n(-w):\delta(z+w) \\
& = a_{nn}\rho(X^{\bar{\sigma}}(\alpha_n, w))(\delta(z-w) + \delta(z+w)) \\
& = \rho([\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_n, w)]).
\end{aligned}$$

We now repeat the calculation for $X^{\bar{\sigma}}(-\alpha_n, w)$.

$$\begin{aligned}
& [\rho(\alpha_n^{\bar{\sigma}}(z)), \rho(X^{\bar{\sigma}}(-\alpha_n, w))] \\
&= [:\varepsilon_n(z)\varepsilon_n^*(z):-:\varepsilon_n(-z)\varepsilon_n^*(-z):-:\varepsilon_n^*(-w)\varepsilon_n^*(w):] \\
&= [:\varepsilon_n(z)\varepsilon_n^*(z):-:\varepsilon_n^*(-w)\varepsilon_n^*(w):] - [:\varepsilon_n(-z)\varepsilon_n^*(-z):-:\varepsilon_n^*(-w)\varepsilon_n^*(w):].
\end{aligned}$$

Using Proposition 31, the nonzero terms are:

$$\begin{aligned}
& = -:\varepsilon_n^*(w)\varepsilon_n^*(-w):\delta(z-w) + :\varepsilon_n^*(-w)\varepsilon_n^*(w):\delta(z+w) \\
& - :\varepsilon_n^*(w)\varepsilon_n^*(-w):\delta(z+w) + :\varepsilon_n^*(-w)\varepsilon_n^*(w):\delta(z-w).
\end{aligned}$$

By Corollary 28, we have:

$$\begin{aligned}
& = +:\varepsilon_n^*(-w)\varepsilon_n^*(w):\delta(z-w) + :\varepsilon_n^*(-w)\varepsilon_n^*(w):\delta(z+w) \\
& + :\varepsilon_n^*(-w)\varepsilon_n^*(w):\delta(z+w) + :\varepsilon_n^*(-w)\varepsilon_n^*(w):\delta(z-w) \\
& = -2(-:\varepsilon_n^*(-w)\varepsilon_n^*(w):)\delta(z-w) - 2(-:\varepsilon_n^*(-w)\varepsilon_n^*(w):)\delta(z+w) \\
& = -a_{nn}\rho(X^{\bar{\sigma}}(-\alpha_n, w))(\delta(z-w) + \delta(z+w)) \\
& = \rho([\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(-\alpha_n, w)]).
\end{aligned}$$

Relation (12): $[X^{\bar{\sigma}}(\pm\alpha_m, z), X^{\bar{\sigma}}(\pm\alpha_m, w)] = 0$

First let $m = 0$. We will calculate for $X^{\bar{\sigma}}(\alpha_m, z), X^{\bar{\sigma}}(\alpha_m, w)$.

$$\begin{aligned}
& [\rho(X^{\bar{\sigma}}(\alpha_m, z)), \rho(X^{\bar{\sigma}}(\alpha_m, w))] \\
&= [:\beta^*(z)\varepsilon_1^*(-z):, :\beta^*(w)\varepsilon_1^*(-w):].
\end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is simply:

$$\begin{aligned}
& = 0 \\
& = \rho([X^{\bar{\sigma}}(\alpha_m, z), X^{\bar{\sigma}}(\alpha_m, w)]).
\end{aligned}$$

Now we repeat the calculation for $X^{\bar{\sigma}}(-\alpha_m, z), X^{\bar{\sigma}}(-\alpha_m, w)$.

$$[\rho(X^{\bar{\sigma}}(-\alpha_m, z)), \rho(X^{\bar{\sigma}}(-\alpha_m, w))]$$

$$= [:\beta(z)\varepsilon_1(-z), : \beta(w)\varepsilon_1(-w):].$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is simply:

$$= 0$$

$$= \rho([X^{\bar{\sigma}}(-\alpha_m, z), X^{\bar{\sigma}}(-\alpha_m, w)]).$$

The next case is for $1 \leq m \leq n - 1$. Calculating for $X^{\bar{\sigma}}(\alpha_m, z), X^{\bar{\sigma}}(\alpha_m, w)$ gives:

$$[\rho(X^{\bar{\sigma}}(\alpha_m, z)), \rho(X^{\bar{\sigma}}(\alpha_m, w))]$$

$$= [:\varepsilon_i(z)\varepsilon_{i+1}^*(z), : \varepsilon_i(w)\varepsilon_{i+1}^*(w):].$$

Using Proposition 31, every contraction (and hence bilinear form $\langle \cdot, \cdot \rangle$) is either trivially 0, or is equal to $\delta_{i,i+1}$, which is itself trivially 0. Hence, this calculation is simply:

$$= 0$$

$$= \rho([X^{\bar{\sigma}}(\alpha_m, z), X^{\bar{\sigma}}(\alpha_m, w)]).$$

Repeating the calculation for $X^{\bar{\sigma}}(-\alpha_m, z), X^{\bar{\sigma}}(-\alpha_m, w)$ gives:

$$[\rho(X^{\bar{\sigma}}(-\alpha_m, z)), \rho(X^{\bar{\sigma}}(-\alpha_m, w))]$$

$$= [- :\varepsilon_i^*(z)\varepsilon_{i+1}(z), - :\varepsilon_i^*(w)\varepsilon_{i+1}(w):].$$

Using Proposition 31, every contraction (and hence bilinear form $\langle \cdot, \cdot \rangle$) is either trivially 0, or is equal to $\delta_{i,i+1}$, which is itself trivially 0. Hence, this calculation is simply:

$$= 0$$

$$= \rho([X^{\bar{\sigma}}(-\alpha_m, z), X^{\bar{\sigma}}(-\alpha_m, w)]).$$

Finally, let $m = n$. Calculating for $X^{\bar{\sigma}}(\alpha_m, z), X^{\bar{\sigma}}(\alpha_m, w)$ gives:

$$[\rho(X^{\bar{\sigma}}(\alpha_m, z)), \rho(X^{\bar{\sigma}}(\alpha_m, w))]$$

$$= [:\varepsilon_n(z)\varepsilon_n(-z), : \varepsilon_n(w)\varepsilon_n(-w):].$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is simply:

$$= 0$$

$$= \rho([X^{\bar{\sigma}}(\alpha_m, z), X^{\bar{\sigma}}(\alpha_m, w)]).$$

Repeating the calculation for $X^{\bar{\sigma}}(-\alpha_m, z), X^{\bar{\sigma}}(-\alpha_m, w)$ gives:

$$[\rho(X^{\bar{\sigma}}(-\alpha_m, z)), \rho(X^{\bar{\sigma}}(-\alpha_m, w))]$$

$$= [- :\varepsilon_n^*(-z)\varepsilon_n^*(z), - :\varepsilon_n^*(-w)\varepsilon_n^*(w):].$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is simply:

$$= 0$$

$$= \rho([X^{\bar{\sigma}}(-\alpha_m, z), X^{\bar{\sigma}}(-\alpha_m, w)]).$$

Relation (13): $[X^{\bar{\sigma}}(\alpha_0, z), X^{\bar{\sigma}}(-\alpha_0, w)] = \alpha_0^{\bar{\sigma}}(w)(\delta(z - w) + \delta(z + w)) + (\partial_w \delta(z - w) + \partial_w \delta(z + w))k$

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_0, z)), \rho(X^{\bar{\sigma}}(-\alpha_0, w))] \\ &= [:\beta^*(z)\varepsilon_1^*(-z):, :\beta(w)\varepsilon_1(-w):]. \end{aligned}$$

Using Proposition 31, we have

$$\begin{aligned} & = :\varepsilon_1^*(w)\beta(w):\delta(z+w) - :\varepsilon_1^*(-w)\varepsilon_1(-w):\delta(z-w) \\ & - :\beta^*(-w)\varepsilon_1(-w):\delta(z+w) + :\beta^*(w)\beta(w):\delta(z-w) + \partial_w\delta(z-w) + \partial_w\delta(z+w). \end{aligned}$$

By Remark 4.1, we have

$$\begin{aligned} & = :\varepsilon_1^*(w)\beta(w):\delta(z+w) - :\beta^*(-w)\varepsilon_1(-w):\delta(z-w) \\ & - :\beta^*(-w)\varepsilon_1(-w):\delta(z+w) + :\varepsilon_1^*(w)\beta(w):\delta(z-w) + \partial_w\delta(z-w) + \partial_w\delta(z+w) \\ & = :\varepsilon_1^*(w)\beta(w):\delta(z-w) - :\beta^*(-w)\varepsilon_1(-w):\delta(z-w) \\ & + :\varepsilon_1^*(w)\beta(w):\delta(z+w) - :\beta^*(-w)\varepsilon_1(-w):\delta(z+w) + \partial_w\delta(z-w) + \partial_w\delta(z+w). \end{aligned}$$

By Corollary 28, we have

$$\begin{aligned} & = :\varepsilon_1^*(w)\beta(w):\delta(z-w) + :\varepsilon_1(-w)\beta^*(-w):\delta(z-w) \\ & + :\varepsilon_1^*(w)\beta(w):\delta(z+w) + :\varepsilon_1(-w)\beta^*(-w):\delta(z+w) + \partial_w\delta(z-w) + \partial_w\delta(z+w). \end{aligned}$$

Since \mathcal{K} acts as the identity, we add it without changing the operator to arrive at:

$$\begin{aligned} & = :\varepsilon_1^*(w)\beta(w):\delta(z-w) + :\varepsilon_1(-w)\beta^*(-w):\delta(z-w) \\ & + :\varepsilon_1^*(w)\beta(w):\delta(z+w) + :\varepsilon_1(-w)\beta^*(-w):\delta(z+w) + (\partial_w\delta(z-w) + \partial_w\delta(z+w))\mathcal{K} \\ & = \rho(\alpha_0^{\bar{\sigma}}(z))(\delta(z-w) + \delta(z+w)) + (\partial_w\delta(z-w) + \partial_w\delta(z+w))\mathcal{K} \\ & = \rho([X^{\bar{\sigma}}(\alpha_0, z), X^{\bar{\sigma}}(-\alpha_0, w)]). \end{aligned}$$

Relation (14): $[X^{\bar{\sigma}}(\alpha_i, z), X^{\bar{\sigma}}(-\alpha_i, w)] = \alpha_i^{\bar{\sigma}}(w)\delta(z-w) + \partial_w\delta(z-w)\mathcal{K}$

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_i, z)), \rho(X^{\bar{\sigma}}(-\alpha_i, w))] \\ &= [:\varepsilon_i(z)\varepsilon_{i+1}^*(z):, - :\varepsilon_i^*(w)\varepsilon_{i+1}(w):]. \end{aligned}$$

Using Proposition 31, the nonzero terms are:

$$= :\varepsilon_{i+1}^*(w)\varepsilon_{i+1}(w):\delta(z-w) + :\varepsilon_i(w)\varepsilon_i^*(w):\delta(z-w) + \partial_w\delta(z-w).$$

By Corollary 28, we have

$$\begin{aligned} & = - :\varepsilon_{i+1}(w)\varepsilon_{i+1}^*(w):\delta(z-w) + :\varepsilon_i(w)\varepsilon_i^*(w):\delta(z-w) + \partial_w\delta(z-w) \\ & = (:\varepsilon_i(w)\varepsilon_i^*(w): - :\varepsilon_{i+1}(w)\varepsilon_{i+1}^*(w):)\delta(z-w) + \partial_w\delta(z-w). \end{aligned}$$

We add \mathcal{K} since it acts as the identity and thus does not change the operator to arrive at:

$$\begin{aligned} & = \rho(\alpha_i^{\bar{\sigma}}(w))\delta(z-w) + \partial_w\delta(z-w)\mathcal{K} \\ & = \rho([X^{\bar{\sigma}}(\alpha_i, z), X^{\bar{\sigma}}(-\alpha_i, w)]). \end{aligned}$$

Relation (15): $[X^{\bar{\sigma}}(\alpha_n, z), X^{\bar{\sigma}}(-\alpha_n, w)] = \alpha_n^{\bar{\sigma}}(w)(\delta(z-w) + \delta(z+w)) + (\partial_w\delta(z-w) + \partial_w\delta(z+w))\mathcal{K}$

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_n, z)), \rho(X^{\bar{\sigma}}(-\alpha_n, w))] \\ &= [:\varepsilon_n(z)\varepsilon_n(-z):, - :\varepsilon_n^*(-w)\varepsilon_n^*(w):]. \end{aligned}$$

Using Proposition 31, we have

$$\begin{aligned}
&= - : \varepsilon_n(-w) \varepsilon_n^*(-w) : \delta(z-w) + : \varepsilon_n(w) \varepsilon_n^*(w) : \delta(z+w) \\
&\quad + : \varepsilon_n(w) \varepsilon_n^*(w) : \delta(z-w) - : \varepsilon_n(-w) \varepsilon_n^*(-w) : \delta(z+w) + \partial_w \delta(z-w) + \partial_w \delta(z+w) \\
&= - : \varepsilon_n(-w) \varepsilon_n^*(-w) : \delta(z-w) + : \varepsilon_n(w) \varepsilon_n^*(w) : \delta(z-w) \\
&\quad - : \varepsilon_n(-w) \varepsilon_n^*(-w) : \delta(z+w) + : \varepsilon_n(w) \varepsilon_n^*(w) : \delta(z+w) + \partial_w \delta(z-w) + \partial_w \delta(z+w).
\end{aligned}$$

We add \mathcal{K} since it acts as the identity and thus does not change the operator to arrive at:

$$\begin{aligned}
&= (: \varepsilon_n(w) \varepsilon_n^*(w) : - : \varepsilon_n(-w) \varepsilon_n^*(-w) :) \delta(z-w) \\
&\quad + (: \varepsilon_n(w) \varepsilon_n^*(w) : - : \varepsilon_n(-w) \varepsilon_n^*(-w) :) \delta(z+w) + (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \mathcal{K} \\
&= \rho(\alpha_n^{\bar{\sigma}}(w)) (\delta(z-w) + \delta(z+w)) + (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \mathcal{K} \\
&= \rho([X^{\bar{\sigma}}(\alpha_n, z), X^{\bar{\sigma}}(-\alpha_n, w)]).
\end{aligned}$$

Relation (16): $[X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)] = 0$ for $p \neq m$

We have several cases to check. First consider $p = 0$ and $1 \leq m \leq n-1$.

$$\begin{aligned}
&[\rho(X^{\bar{\sigma}}(\alpha_p, z)), \rho(X^{\bar{\sigma}}(-\alpha_m, w))] \\
&= [: \beta^*(z) \varepsilon_1^*(-z) : , - : \varepsilon_m^*(w) \varepsilon_{m+1}(w) :].
\end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are either trivially 0, or are equal to $\delta_{1,m+1}$, which is itself 0 by the restrictions on the values of m . Hence, this entire calculation is:

$$\begin{aligned}
&= 0 \\
&= \rho([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]).
\end{aligned}$$

Now consider $p = 0$ and $m = n$.

$$\begin{aligned}
&[\rho(X^{\bar{\sigma}}(\alpha_p, z)), \rho(X^{\bar{\sigma}}(-\alpha_m, w))] \\
&= [: \beta^*(z) \varepsilon_1^*(-z) : , - : \varepsilon_n^*(-w) \varepsilon_n^*(w) :].
\end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence, this entire calculation is:

$$\begin{aligned}
&= 0 \\
&= \rho([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]).
\end{aligned}$$

Now consider the case $1 \leq p \leq n-1$ and $m = 0$.

$$\begin{aligned}
&[\rho(X^{\bar{\sigma}}(\alpha_p, z)), \rho(X^{\bar{\sigma}}(-\alpha_m, w))] \\
&= [: \varepsilon_p(z) \varepsilon_{p+1}^*(z) : , : \beta(w) \varepsilon_1(-w) :].
\end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are either trivially 0, or are equal to $\delta_{p+1,1}$, which is itself 0 by the restrictions on the values of p . Hence, this entire calculation is:

$$= 0$$

$$= \rho([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]).$$

Now consider the case $p = n$ and $m = 0$.

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z)), \rho(X^{\bar{\sigma}}(-\alpha_m, w))] \\ &= [:\varepsilon_n(z)\varepsilon_n(-z):, :\beta(w)\varepsilon_1(-w):]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]). \end{aligned}$$

Now consider the case $1 \leq p \neq m \leq n - 1$.

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z)), \rho(X^{\bar{\sigma}}(-\alpha_m, w))] \\ &= [:\varepsilon_p(z)\varepsilon_{p+1}^*(z):, -:\varepsilon_m^*(w)\varepsilon_{m+1}(w):]. \end{aligned}$$

Using Proposition 31, we have:

$$= -\delta_{pm} : \varepsilon_{p+1}^*(w) \varepsilon_{m+1}(w) : \delta(z-w) - \delta_{pm} : \varepsilon_p(w) \varepsilon_m^*(w) : \delta(z-w) - \delta_{pm} \partial_w \delta(z-w).$$

Since by assumption $p \neq m$, $\delta_{pm} = 0$, so this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]). \end{aligned}$$

Now consider the case $1 \leq p \leq n - 1, m = n$.

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z)), \rho(X^{\bar{\sigma}}(-\alpha_m, w))] \\ &= [:\varepsilon_p(z)\varepsilon_{p+1}^*(z):, -:\varepsilon_n^*(-w)\varepsilon_n^*(w):]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are either trivially 0, or are equal to δ_{pn} , which is itself 0 by the restrictions on the values of p . Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]). \end{aligned}$$

Now consider the case $p = n, 1 \leq m \leq n - 1$.

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z)), \rho(X^{\bar{\sigma}}(-\alpha_m, w))] \\ &= [:\varepsilon_n(z)\varepsilon_n(-z):, -:\varepsilon_m^*(w)\varepsilon_{m+1}(w):]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are either trivially 0, or are equal to δ_{nm} , which is itself 0 by the restrictions on the values of m . Hence, this entire calculation is:

$$= 0$$

$$= \rho([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]).$$

Relation (17): $\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_2)X^{\bar{\sigma}}(\pm\alpha_m, z_1) = 0$ if $c_{pm} = 0$

For the case $p = 0$, the matrix entry $c_{pm} = 0 \Leftrightarrow 2 \leq m \leq n$. We must split this into two cases of the value of m , namely $2 \leq m \leq n - 1$ and $m = n$. First we consider $2 \leq m \leq n - 1$ for positive α_p, α_m .

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\beta^*(z_2)\varepsilon_1^*(-z_2):\, :\varepsilon_m(z_1)\varepsilon_{m+1}^*(z_1):]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are either trivially 0, or are equal to δ_{1m} , which is itself 0 by the restrictions on the values of m . Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

Similarly for $-\alpha_p, -\alpha_m$ we calculate

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(-\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(-\alpha_m, z_1))] \\ &= [:\beta(z_2)\varepsilon_1(-z_2):\, - :\varepsilon_m^*(z_1)\varepsilon_{m+1}(z_1):]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are either trivially 0, or are equal to δ_{1m} , which is itself 0 by the restrictions on the values of m . Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)]). \end{aligned}$$

When $p = 0$ and $m = n$, we have for α_p, α_m :

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\beta^*(z_2)\varepsilon_1^*(-z_2):\, :\varepsilon_n(z_1)\varepsilon_n(-z_1):]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are equal to $\delta_{1n} = 0$. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

Similarly for $-\alpha_p, -\alpha_m$ we calculate

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(-\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(-\alpha_m, z_1))] \\ &= [:\beta(z_2)\varepsilon_1(-z_2):\, - :\varepsilon_n^*(-z_1)\varepsilon_n^*(z_1):]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are equal to $\delta_{1n} = 0$. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)]). \end{aligned}$$

Now if $m = 0$, the entry $c_{pm} = 0 \Leftrightarrow 2 \leq p \leq n$. We must split this into two cases of the value of p , namely $2 \leq p \leq n - 1$ and $p = n$. First we consider $2 \leq p \leq n - 1$ for positive α_p, α_m .

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\varepsilon_p(z_2)\varepsilon_{p+1}^*(z_2):\, :\beta^*(z_1)\varepsilon_1^*(-z_1):\]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are either trivially 0, or are equal to δ_{p1} , which is itself 0 by the restrictions on the values of p . Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

Similarly, for $-\alpha_p, -\alpha_m$ we calculate

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(-\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(-\alpha_m, z_1))] \\ &= [-:\varepsilon_p^*(z_2)\varepsilon_{p+1}(z_2):\, :\beta(z_1)\varepsilon_1(-z_1):\]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are either trivially 0, or are equal to δ_{p1} , which is itself 0 by the restrictions on the values of p . Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)]). \end{aligned}$$

When $m = 0$ and $p = n$, we have, for α_p, α_m

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\varepsilon_n(z_2)\varepsilon_n(-z_2):\, :\beta^*(z_1)\varepsilon_1^*(-z_1):\]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are equal to $\delta_{n1} = 0$. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

Similarly, for $-\alpha_p, -\alpha_m$ we calculate

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(-\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(-\alpha_m, z_1))] \\ &= [-:\varepsilon_n^*(-z_2)\varepsilon_n^*(z_2):\, :\beta(z_1)\varepsilon_1(-z_1):\]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are equal to $\delta_{n1} = 0$. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)]). \end{aligned}$$

For the remaining cases, we may assume that $p \neq 0$ and $m \neq 0$. For these values, $c_{pm} = 0$ when $|p - m| \geq 2$. We compute for the case $1 \leq p, m \leq n - 1$ and α_p, α_m .

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\varepsilon_p(z_2)\varepsilon_{p+1}^*(z_2):\, :\varepsilon_m(z_1)\varepsilon_{m+1}^*(z_1):\]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are either trivially 0, or are equal to $\delta_{p,m+1}$ or $\delta_{p+1,m}$, which are both 0 by the restrictions on the values of p, m . Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

Similarly, for $-\alpha_p, -\alpha_m$,

$$\begin{aligned} &[\rho(X^{\bar{\sigma}}(-\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(-\alpha_m, z_1))] \\ &= [- : \varepsilon_p^*(z_2) \varepsilon_{p+1}(z_2) :, - : \varepsilon_m^*(z_1) \varepsilon_{m+1}(z_1) :]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are either trivially 0, or are equal to $\delta_{p,m+1}$ or $\delta_{p+1,m}$, which are both 0 by the restrictions on the values of p, m . Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)]). \end{aligned}$$

Now consider the case $p = n$ with $|p - m| \geq 2$. We calculate for α_p, α_m :

$$\begin{aligned} &[\rho(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [: \varepsilon_n(z_2) \varepsilon_n(-z_2) :, : \varepsilon_m(z_1) \varepsilon_{m+1}^*(z_1) :]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are either trivially 0, or are equal to $\delta_{n,m+1}$, which is itself 0 by the restrictions on the values of m . Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

Similarly, for $-\alpha_p, -\alpha_m$,

$$\begin{aligned} &[\rho(X^{\bar{\sigma}}(-\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(-\alpha_m, z_1))] \\ &= [- : \varepsilon_n^*(-z_2) \varepsilon_n^*(z_2) :, - : \varepsilon_m^*(z_1) \varepsilon_{m+1}(z_1) :]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are either trivially 0, or are equal to $\delta_{n,m+1}$, which is itself 0 by the restrictions on the values of m . Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho([X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)]). \end{aligned}$$

Finally, consider the case $m = n$ with $|p - m| \geq 2$. We calculate for α_p, α_m :

$$\begin{aligned} &[\rho(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [: \varepsilon_p(z_2) \varepsilon_{p+1}^*(z_2) :, : \varepsilon_n(z_1) \varepsilon_n(-z_1) :]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are either trivially 0, or are equal to $\delta_{p+1,n}$, which is itself 0 by the restrictions on the values of m . Hence, this entire calculation is:

$$= 0 \\ = \rho([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]).$$

Similarly, for $-\alpha_p, -\alpha_m$,

$$[\rho(X^{\bar{\sigma}}(-\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(-\alpha_m, z_1))] \\ = [- : \varepsilon_p^*(z_2) \varepsilon_{p+1}(z_2) :, - : \varepsilon_n^*(-z_1) \varepsilon_n^*(z_1) :].$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are either trivially 0, or are equal to $\delta_{p+1,n}$, which is itself 0 by the restrictions on the values of m . Hence, this entire calculation is:

$$= 0 \\ = \rho([X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)]).$$

Relation (18): $\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_2)X^{\bar{\sigma}}(\pm\alpha_m, z_1) = 0$ if $c_{pm} = -1$

The condition $c_{pm} = -1$ occurs precisely when $|p - m| = 1$ except for the pairs $p = 1, m = 0$ and $p = n - 1, m = n$. First, we will compute when $p = 0, m = 1$ and for α_p, α_m .

$$[\rho(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ = [:\beta^*(z_2) \varepsilon_1^*(-z_2) :, : \varepsilon_1(z_1) \varepsilon_2^*(z_1) :].$$

Using Proposition 31, we have:

$$= - : \varepsilon_1^*(-z_1) \varepsilon_2^*(z_1) : \delta(z_2 - z_1) - : \beta^*(-z_1) \varepsilon_2^*(z_1) : \delta(z_2 + z_1).$$

Using Remark 4.1, we have

$$= - : \varepsilon_1^*(-z_1) \varepsilon_2^*(z_1) : \delta(z_2 - z_1) - : \varepsilon_1^*(-z_1) \varepsilon_2^*(z_1) : \delta(z_2 + z_1) \\ = - : \varepsilon_1^*(-z_1) \varepsilon_2^*(z_1) : (\delta(z_2 - z_1) + \delta(z_2 + z_1)).$$

We now apply $X^{\bar{\sigma}}(\alpha_p, z_3)$ on the left.

$$[\rho(X^{\bar{\sigma}}(\alpha_p, z_3)), - : \varepsilon_1^*(-z_1) \varepsilon_2^*(z_1) : (\delta(z_2 - z_1) + \delta(z_2 + z_1))] \\ = [:\beta^*(z_3) \varepsilon_1^*(-z_3) :, - : \varepsilon_1^*(-z_1) \varepsilon_2^*(z_1) : (\delta(z_2 - z_1) + \delta(z_2 + z_1))].$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is:

$$= 0 \\ = \rho(\text{ad}X^{\bar{\sigma}}(\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_2)X^{\bar{\sigma}}(\alpha_m, z_1)).$$

Similarly, for $-\alpha_p, -\alpha_m$:

$$[\rho(X^{\bar{\sigma}}(-\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(-\alpha_m, z_1))] \\ = [:\beta(z_2) \varepsilon_1(-z_2) :, - : \varepsilon_1^*(z_1) \varepsilon_2(z_1) :].$$

Using Proposition 31, we have:

$$=: \varepsilon_1(-z_1) \varepsilon_2(z_1) : \delta(z_2 - z_1) + : \beta(-z_1) \varepsilon_2(z_1) : \delta(z_2 + z_1).$$

Using Remark 4.1, we have

$$=: \varepsilon_1(-z_1) \varepsilon_2(z_1) : \delta(z_2 - z_1) + : \varepsilon_1(-z_1) \varepsilon_2(z_1) : \delta(z_2 + z_1)$$

$$=: \varepsilon_1(-z_1)\varepsilon_2(z_1) : (\delta(z_2 - z_1) + \delta(z_2 + z_1)).$$

We now apply $X^{\bar{\sigma}}(-\alpha_p, z_3)$ on the left.

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(-\alpha_p, z_3)), : \varepsilon_1(-z_1)\varepsilon_2(z_1) : (\delta(z_2 - z_1) + \delta(z_2 + z_1))] \\ & = [:\beta(z_3)\varepsilon_1(-z_3) :, : \varepsilon_1(-z_1)\varepsilon_2(z_1) : (\delta(z_2 - z_1) + \delta(z_2 + z_1))]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is:

$$\begin{aligned} & = 0 \\ & = \rho(\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_2)X^{\bar{\sigma}}(-\alpha_m, z_1)). \end{aligned}$$

Now we compute for $1 \leq p, m \leq n - 1$ where $|p - m| = 1$. First we calculate for α_p, α_m .

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ & = [:\varepsilon_p(z_2)\varepsilon_{p+1}^*(z_2) :, : \varepsilon_m(z_1)\varepsilon_{m+1}^*(z_1) :]. \end{aligned}$$

Using Proposition 31, we have:

$$= \delta_{p,m+1} : \varepsilon_{p+1}^*(z_1)\varepsilon_m(z_1) : \delta(z_2 - z_1) + \delta_{p+1,m} : \varepsilon_p(z_1)\varepsilon_{m+1}^*(z_1) : \delta(z_2 - z_1) + \delta_{p,m+1}\delta_{p+1,m}\partial_{z_1}\delta(z_2 - z_1).$$

The coefficient $\delta_{p,m+1}\delta_{p+1,m} = 0$, so we have

$$= (\delta_{p,m+1} : \varepsilon_{p+1}^*(z_1)\varepsilon_m(z_1) : + \delta_{p+1,m} : \varepsilon_p(z_1)\varepsilon_{m+1}^*(z_1) :)\delta(z_2 - z_1).$$

We now apply $X^{\bar{\sigma}}(\alpha_p, z_3)$ on the left.

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z_3)), (\delta_{p,m+1} : \varepsilon_{p+1}^*(z_1)\varepsilon_m(z_1) : + \delta_{p+1,m} : \varepsilon_p(z_1)\varepsilon_{m+1}^*(z_1) :)\delta(z_2 - z_1)] \\ & = [:\varepsilon_p(z_3)\varepsilon_{p+1}^*(z_3) :, (\delta_{p,m+1} : \varepsilon_{p+1}^*(z_1)\varepsilon_m(z_1) : + \delta_{p+1,m} : \varepsilon_p(z_1)\varepsilon_{m+1}^*(z_1) :)\delta(z_2 - z_1)] \\ & = [:\varepsilon_p(z_3)\varepsilon_{p+1}^*(z_3) :, \delta_{p,m+1} : \varepsilon_{p+1}^*(z_1)\varepsilon_m(z_1) : \delta(z_2 - z_1)] \\ & \quad + [:\varepsilon_p(z_3)\varepsilon_{p+1}^*(z_3) :, \delta_{p+1,m} : \varepsilon_p(z_1)\varepsilon_{m+1}^*(z_1) : \delta(z_2 - z_1)]. \end{aligned}$$

Using Proposition 31, we have

$$\begin{aligned} & = \delta_{p,m+1} (-\delta_{p,p+1} : \varepsilon_{p+1}^*(z_1)\varepsilon_m(z_1) : \delta(z_3 - z_1) - \delta_{p+1,m} : \varepsilon_p(z_1)\varepsilon_{p+1}^*(z_1) : \delta(z_3 - z_1) \\ & \quad - \delta_{p,p+1}\delta_{p+1,m}\partial_{z_1}\delta(z_3 - z_1))\delta(z_2 - z_1) \\ & \quad + \delta_{p+1,m}(\delta_{p,m+1} : \varepsilon_{p+1}^*(z_1)\varepsilon_p(z_1) : \delta(z_3 - z_1) + \delta_{p+1,p} : \varepsilon_p(z_1)\varepsilon_{m+1}^*(z_1) : \delta(z_3 - z_1) \\ & \quad + \delta_{p,m+1}\delta_{p+1,p}\partial_{z_1}\delta(z_3 - z_1))\delta(z_2 - z_1). \end{aligned}$$

The coefficient $\delta_{p,p+1} = 0$, and each other term has coefficient $\pm\delta_{p,m+1}\delta_{p+1,m}$, which is 0. Hence, this calculation is:

$$\begin{aligned} & = 0 \\ & = \rho(\text{ad}X^{\bar{\sigma}}(\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_2)X^{\bar{\sigma}}(\alpha_m, z_1)). \end{aligned}$$

Similarly, for $-\alpha_p, -\alpha_m$:

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(-\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(-\alpha_m, z_1))] \\ & = [- : \varepsilon_p^*(z_2)\varepsilon_{p+1}(z_2) :, - : \varepsilon_m^*(z_1)\varepsilon_{m+1}(z_1) :]. \end{aligned}$$

Using Proposition 31, we have:

$$= \delta_{p,m+1} : \varepsilon_{p+1}(z_1) \varepsilon_m^*(z_1) : \delta(z_2 - z_1) + \delta_{p+1,m} : \varepsilon_p^*(z_1) \varepsilon_{m+1}(z_1) : \delta(z_2 - z_1) + \delta_{p,m+1} \delta_{p+1,m} \partial_{z_1} \delta(z_2 - z_1).$$

The coefficient $\delta_{p,m+1} \delta_{p+1,m} = 0$, so we have

$$= (\delta_{p,m+1} : \varepsilon_{p+1}(z_1) \varepsilon_m^*(z_1) : + \delta_{p+1,m} : \varepsilon_p^*(z_1) \varepsilon_{m+1}(z_1) :) \delta(z_2 - z_1).$$

We now apply $X^{\bar{\sigma}}(-\alpha_p, z_3)$ on the left.

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(-\alpha_p, z_3)), (\delta_{p,m+1} : \varepsilon_{p+1}(z_1) \varepsilon_m^*(z_1) : + \delta_{p+1,m} : \varepsilon_p^*(z_1) \varepsilon_{m+1}(z_1) :) \delta(z_2 - z_1)] \\ &= [- : \varepsilon_p^*(z_3) \varepsilon_{p+1}(z_3) :, (\delta_{p,m+1} : \varepsilon_{p+1}(z_1) \varepsilon_m^*(z_1) : + \delta_{p+1,m} : \varepsilon_p^*(z_1) \varepsilon_{m+1}(z_1) :) \delta(z_2 - z_1)] \\ &= [- : \varepsilon_p^*(z_3) \varepsilon_{p+1}(z_3) :, \delta_{p,m+1} : \varepsilon_{p+1}(z_1) \varepsilon_m^*(z_1) : \delta(z_2 - z_1)] \\ &\quad + [- : \varepsilon_p^*(z_3) \varepsilon_{p+1}(z_3) :, \delta_{p+1,m} : \varepsilon_p^*(z_1) \varepsilon_{m+1}(z_1) : \delta(z_2 - z_1)]. \end{aligned}$$

Using Proposition 31, we have

$$\begin{aligned} & = \delta_{p,m+1} (\delta_{p,p+1} : \varepsilon_{p+1}(z_1) \varepsilon_m^*(z_1) : \delta(z_3 - z_1) + \delta_{p+1,m} : \varepsilon_p^*(z_1) \varepsilon_{p+1}(z_1) : \delta(z_3 - z_1) \\ &\quad + \delta_{p,p+1} \delta_{p+1,m} \partial_{z_1} \delta(z_3 - z_1)) \delta(z_2 - z_1) \\ &\quad - \delta_{p+1,m} (\delta_{p,m+1} : \varepsilon_{p+1}(z_1) \varepsilon_p^*(z_1) : \delta(z_3 - z_1) + \delta_{p+1,p} : \varepsilon_p^*(z_1) \varepsilon_{m+1}(z_1) : \delta(z_3 - z_1) \\ &\quad + \delta_{p+1,m} \delta_{p+1,p} \partial_{z_1} \delta(z_3 - z_1)) \delta(z_2 - z_1). \end{aligned}$$

The coefficient $\delta_{p,p+1} = 0$, and each other term has coefficient $\pm \delta_{p,m+1} \delta_{p+1,m}$, which is 0. Hence, this calculation is:

$$\begin{aligned} &= 0 \\ &= \rho(\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_2)X^{\bar{\sigma}}(-\alpha_m, z_1)). \end{aligned}$$

The final case for this relation is $p = n$ and $m = n - 1$. We compute for α_p, α_m :

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\varepsilon_n(z_2) \varepsilon_n(-z_2) :, :\varepsilon_{n-1}(z_1) \varepsilon_n^*(z_1) :]. \end{aligned}$$

Using Proposition 31, we have:

$$\begin{aligned} & = :\varepsilon_n(-z_1) \varepsilon_{n-1}(z_1) : \delta(z_2 - z_1) + :\varepsilon_n(-z_1) \varepsilon_{n-1}(z_1) : \delta(z_2 + z_1) \\ &= :\varepsilon_n(-z_1) \varepsilon_{n-1}(z_1) : (\delta(z_2 - z_1) + \delta(z_2 + z_1)). \end{aligned}$$

We now apply $X^{\bar{\sigma}}(\alpha_p, z_3)$ on the left.

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z_3)), :\varepsilon_n(-z_1) \varepsilon_{n-1}(z_1) : (\delta(z_2 - z_1) + \delta(z_2 + z_1))] \\ &= [:\varepsilon_n(z_3) \varepsilon_n(-z_3) :, :\varepsilon_n(-z_1) \varepsilon_{n-1}(z_1) : (\delta(z_2 - z_1) + \delta(z_2 + z_1))]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is:

$$\begin{aligned} &= 0 \\ &= \rho(\text{ad}X^{\bar{\sigma}}(\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_2)X^{\bar{\sigma}}(\alpha_m, z_1)). \end{aligned}$$

Similarly, for $-\alpha_p, -\alpha_m$:

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(-\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(-\alpha_m, z_1))] \\ &= [- : \varepsilon_n^*(-z_2) \varepsilon_n^*(z_2) :, - : \varepsilon_{n-1}^*(z_1) \varepsilon_n(z_1) :]. \end{aligned}$$

Using Proposition 31, we have:

$$\begin{aligned} & = - : \varepsilon_n^*(-z_1) \varepsilon_{n-1}^*(z_1) : \delta(z_2 + z_1) - : \varepsilon_n^*(-z_1) \varepsilon_{n-1}^*(z_1) : \delta(z_2 - z_1) \\ &= - : \varepsilon_n^*(-z_1) \varepsilon_{n-1}^*(z_1) : (\delta(z_2 - z_1) + \delta(z_2 + z_1)). \end{aligned}$$

We now apply $X^{\bar{\sigma}}(-\alpha_p, z_3)$ on the left.

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(-\alpha_p, z_3)), - : \varepsilon_n^*(-z_1) \varepsilon_{n-1}^*(z_1) : (\delta(z_2 - z_1) + \delta(z_2 + z_1))] \\ &= [- : \varepsilon_n^*(-z_3) \varepsilon_n^*(z_3) :, - : \varepsilon_n^*(-z_1) \varepsilon_{n-1}^*(z_1) : (\delta(z_2 - z_1) + \delta(z_2 + z_1))]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is:

$$\begin{aligned} &= 0 \\ &= \rho(\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_2)X^{\bar{\sigma}}(-\alpha_m, z_1)). \end{aligned}$$

Relation (19): $\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_4)\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_2)X^{\bar{\sigma}}(\pm\alpha_m, z_1) = 0$ if $c_{pm} = -2$

There are two cases to consider: $p = 1, m = 0$ and $p = n - 1, m = n$. First we compute for $p = 1, m = 0$ and α_p, α_m .

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [: \varepsilon_1(z_2) \varepsilon_2^*(z_2) :, : \beta^*(z_1) \varepsilon_1^*(-z_1) :]. \end{aligned}$$

Using Proposition 31, we have

$$= : \varepsilon_2^*(-z_1) \beta^*(z_1) : \delta(z_2 + z_1) - : \varepsilon_2^*(z_1) \varepsilon_1^*(-z_1) : \delta(z_2 - z_1).$$

By Remark 4.1, we have

$$= : \varepsilon_2^*(-z_1) \varepsilon_1^*(z_1) : \delta(z_2 + z_1) - : \varepsilon_2^*(z_1) \varepsilon_1^*(-z_1) : \delta(z_2 - z_1).$$

We now apply $X^{\bar{\sigma}}(\alpha_p, z_3)$ on the left.

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z_3)), : \varepsilon_2^*(-z_1) \varepsilon_1^*(z_1) : \delta(z_2 + z_1) - : \varepsilon_2^*(z_1) \varepsilon_1^*(-z_1) : \delta(z_2 - z_1)] \\ &= [: \varepsilon_1(z_3) \varepsilon_2^*(z_3) :, : \varepsilon_2^*(-z_1) \varepsilon_1^*(z_1) : \delta(z_2 + z_1) - : \varepsilon_2^*(z_1) \varepsilon_1^*(-z_1) : \delta(z_2 - z_1)] \\ &= [: \varepsilon_1(z_3) \varepsilon_2^*(z_3) :, : \varepsilon_2^*(-z_1) \varepsilon_1^*(z_1) : \delta(z_2 + z_1)] - [: \varepsilon_1(z_3) \varepsilon_2^*(z_3) :, : \varepsilon_2^*(z_1) \varepsilon_1^*(-z_1) : \delta(z_2 - z_1)] \end{aligned}$$

Using Proposition 31, we have

$$= (: \varepsilon_2^*(z_1) \varepsilon_2^*(-z_1) : \delta(z_3 - z_1)) \delta(z_2 + z_1) - (: \varepsilon_2^*(-z_1) \varepsilon_2^*(z_1) : \delta(z_3 + z_1)) \delta(z_2 - z_1).$$

By Corollary 28,

$$= : \varepsilon_2^*(z_1) \varepsilon_2^*(-z_1) : (\delta(z_3 - z_1) \delta(z_2 + z_1) + \delta(z_3 + z_1) \delta(z_2 - z_1)).$$

We now apply $X^{\bar{\sigma}}(\alpha_p, z_4)$ on the left.

$$\begin{aligned} & [\rho(X^{\bar{\sigma}}(\alpha_p, z_4)), : \varepsilon_2^*(z_1) \varepsilon_2^*(-z_1) : (\delta(z_3 - z_1) \delta(z_2 + z_1) + \delta(z_3 + z_1) \delta(z_2 - z_1))] \\ &= [: \varepsilon_1(z_4) \varepsilon_2^*(z_4) :, : \varepsilon_2^*(z_1) \varepsilon_2^*(-z_1) : (\delta(z_3 - z_1) \delta(z_2 + z_1) + \delta(z_3 + z_1) \delta(z_2 - z_1))]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calcu-

lation is:

$$\begin{aligned} &= 0 \\ &= \rho(\text{ad}X^{\bar{\sigma}}(\alpha_p, z_4)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_2)X^{\bar{\sigma}}(\alpha_m, z_1)). \end{aligned}$$

Similarly, for $-\alpha_p, -\alpha_m$,

$$\begin{aligned} &[\rho(X^{\bar{\sigma}}(-\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(-\alpha_m, z_1))] \\ &= [- : \varepsilon_1^*(z_2)\varepsilon_2(z_2) :, : \beta(z_1)\varepsilon_1(-z_1) :]. \end{aligned}$$

Using Proposition 29, we have

$$= - : \varepsilon_2(-z_1)\beta(z_1) : \delta(z_2 + z_1) + : \varepsilon_2(z_1)\varepsilon_1(-z_1) : \delta(z_2 - z_1).$$

By Remark 4.1, we have

$$= - : \varepsilon_2(-z_1)\varepsilon_1(z_1) : \delta(z_2 + z_1) + : \varepsilon_2(z_1)\varepsilon_1(-z_1) : \delta(z_2 - z_1).$$

We now apply $X^{\bar{\sigma}}(-\alpha_p, z_3)$ on the left.

$$\begin{aligned} &[\rho(X^{\bar{\sigma}}(-\alpha_p, z_3)), - : \varepsilon_2(-z_1)\varepsilon_1(z_1) : \delta(z_2 + z_1) + : \varepsilon_2(z_1)\varepsilon_1(-z_1) : \delta(z_2 - z_1)] \\ &= [- : \varepsilon_1^*(z_3)\varepsilon_2(z_3) :, - : \varepsilon_2(-z_1)\varepsilon_1(z_1) : \delta(z_2 + z_1) + : \varepsilon_2(z_1)\varepsilon_1(-z_1) : \delta(z_2 - z_1)] \\ &= [- : \varepsilon_1^*(z_3)\varepsilon_2(z_3) :, - : \varepsilon_2(-z_1)\varepsilon_1(z_1) : \delta(z_2 + z_1)] + [- : \varepsilon_1^*(z_3)\varepsilon_2(z_3) :, : \varepsilon_2(z_1)\varepsilon_1(-z_1) : \delta(z_2 - z_1)]. \end{aligned}$$

Using Proposition 31, we have

$$= (: \varepsilon_2(z_1)\varepsilon_2(-z_1) : \delta(z_3 - z_1))\delta(z_2 + z_1) - (: \varepsilon_2(-z_1)\varepsilon_2(z_1) : \delta(z_3 + z_1))\delta(z_2 - z_1).$$

By Corollary 28,

$$= : \varepsilon_2(z_1)\varepsilon_2(-z_1) : (\delta(z_3 - z_1)\delta(z_2 + z_1) + \delta(z_3 + z_1)\delta(z_2 - z_1)).$$

We now apply $X^{\bar{\sigma}}(-\alpha_p, z_4)$ on the left.

$$\begin{aligned} &[\rho(X^{\bar{\sigma}}(-\alpha_p, z_4)), : \varepsilon_2(z_1)\varepsilon_2(-z_1) : (\delta(z_3 - z_1)\delta(z_2 + z_1) + \delta(z_3 + z_1)\delta(z_2 - z_1))] \\ &= [- : \varepsilon_1^*(z_4)\varepsilon_2(z_4) :, : \varepsilon_2(z_1)\varepsilon_2(-z_1) : (\delta(z_3 - z_1)\delta(z_2 + z_1) + \delta(z_3 + z_1)\delta(z_2 - z_1))]. \end{aligned}$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is:

$$\begin{aligned} &= 0 \\ &= \rho(\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_4)\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_2)X^{\bar{\sigma}}(-\alpha_m, z_1)). \end{aligned}$$

The final case to consider is $p = n - 1, m = n$. First we compute for α_p, α_m .

$$\begin{aligned} &[\rho(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [: \varepsilon_{n-1}(z_2)\varepsilon_n^*(z_2) :, : \varepsilon_n(z_1)\varepsilon_n(-z_1) :]. \end{aligned}$$

Using Proposition 31, we have

$$= : \varepsilon_{n-1}(z_1)\varepsilon_n(-z_1) : \delta(z_2 - z_1) - : \varepsilon_{n-1}(-z_1)\varepsilon_n(z_1) : \delta(z_2 + z_1).$$

We now apply $X^{\bar{\sigma}}(\alpha_p, z_3)$ on the left.

$$\begin{aligned} &[\rho(X^{\bar{\sigma}}(\alpha_p, z_3)), : \varepsilon_{n-1}(z_1)\varepsilon_n(-z_1) : \delta(z_2 - z_1) - : \varepsilon_{n-1}(-z_1)\varepsilon_n(z_1) : \delta(z_2 + z_1)] \\ &= [: \varepsilon_{n-1}(z_3)\varepsilon_n^*(z_3) :, : \varepsilon_{n-1}(z_1)\varepsilon_n(-z_1) : \delta(z_2 - z_1) - : \varepsilon_{n-1}(-z_1)\varepsilon_n(z_1) : \delta(z_2 + z_1)] \end{aligned}$$

$$= [: \varepsilon_{n-1}(z_3) \varepsilon_n^*(z_3) :, : \varepsilon_{n-1}(z_1) \varepsilon_n(-z_1) : \delta(z_2 - z_1)] \\ - [: \varepsilon_{n-1}(z_3) \varepsilon_n^*(z_3) :, : \varepsilon_{n-1}(-z_1) \varepsilon_n(z_1) : \delta(z_2 + z_1)].$$

Using Proposition 31, we have

$$= - : \varepsilon_{n-1}(-z_1) \varepsilon_{n-1}(z_1) : \delta(z_3 + z_1) \delta(z_2 - z_1) + : \varepsilon_{n-1}(z_1) \varepsilon_{n-1}(-z_1) : \delta(z_3 - z_1) \delta(z_2 + z_1).$$

By Corollary 28, we have

$$=: \varepsilon_{n-1}(z_1) \varepsilon_{n-1}(-z_1) : (\delta(z_3 + z_1) \delta(z_2 - z_1) + \delta(z_3 - z_1) \delta(z_2 + z_1))$$

We now apply $X^{\bar{\sigma}}(\alpha_p, z_4)$ on the left.

$$[\rho(X^{\bar{\sigma}}(\alpha_p, z_4)), : \varepsilon_{n-1}(z_1) \varepsilon_{n-1}(-z_1) : (\delta(z_3 + z_1) \delta(z_2 - z_1) + \delta(z_3 - z_1) \delta(z_2 + z_1))] \\ = [: \varepsilon_{n-1}(z_4) \varepsilon_n^*(z_4) :, : \varepsilon_{n-1}(z_1) \varepsilon_{n-1}(-z_1) : (\delta(z_3 + z_1) \delta(z_2 - z_1) + \delta(z_3 - z_1) \delta(z_2 + z_1))].$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is:

$$= 0$$

$$= \rho(\text{ad}X^{\bar{\sigma}}(\alpha_p, z_4)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_2)X^{\bar{\sigma}}(\alpha_m, z_1)).$$

Lastly, we repeat the calculation for $-\alpha_p, -\alpha_m$.

$$[\rho(X^{\bar{\sigma}}(-\alpha_p, z_2)), \rho(X^{\bar{\sigma}}(-\alpha_m, z_1))] \\ = [- : \varepsilon_{n-1}^*(z_2) \varepsilon_n(z_2) :, - : \varepsilon_n^*(-z_1) \varepsilon_n^*(z_1) :].$$

Using Proposition 31, we have

$$=: \varepsilon_{n-1}^*(-z_1) \varepsilon_n^*(z_1) : \delta(z_2 + z_1) - : \varepsilon_{n-1}^*(z_1) \varepsilon_n^*(-z_1) : \delta(z_2 - z_1).$$

We now apply $X^{\bar{\sigma}}(-\alpha_p, z_3)$ on the left.

$$[\rho(X^{\bar{\sigma}}(-\alpha_p, z_3)), : \varepsilon_{n-1}^*(-z_1) \varepsilon_n^*(z_1) : \delta(z_2 + z_1) - : \varepsilon_{n-1}^*(z_1) \varepsilon_n^*(-z_1) : \delta(z_2 - z_1)] \\ = [- : \varepsilon_{n-1}^*(z_3) \varepsilon_n(z_3) :, : \varepsilon_{n-1}^*(-z_1) \varepsilon_n^*(z_1) : \delta(z_2 + z_1) - : \varepsilon_{n-1}^*(z_1) \varepsilon_n^*(-z_1) : \delta(z_2 - z_1)] \\ = [- : \varepsilon_{n-1}^*(z_3) \varepsilon_n(z_3) :, : \varepsilon_{n-1}^*(-z_1) \varepsilon_n^*(z_1) : \delta(z_2 + z_1)] \\ + [- : \varepsilon_{n-1}^*(z_3) \varepsilon_n(z_3) :, - : \varepsilon_{n-1}^*(z_1) \varepsilon_n^*(-z_1) : \delta(z_2 - z_1)].$$

Using Proposition 31, we have

$$=: \varepsilon_{n-1}^*(z_1) \varepsilon_{n-1}^*(-z_1) : \delta(z_3 - z_1) \delta(z_2 + z_1) - : \varepsilon_{n-1}^*(-z_1) \varepsilon_{n-1}^*(z_1) : \delta(z_3 + z_1) \delta(z_2 - z_1).$$

By Corollary 28, we have

$$=: \varepsilon_{n-1}^*(z_1) \varepsilon_{n-1}^*(-z_1) : (\delta(z_3 - z_1) \delta(z_2 + z_1) + \delta(z_3 + z_1) \delta(z_2 - z_1)).$$

We now apply $X^{\bar{\sigma}}(-\alpha_p, z_4)$ on the left.

$$[\rho(X^{\bar{\sigma}}(-\alpha_p, z_4)), : \varepsilon_{n-1}^*(z_1) \varepsilon_{n-1}^*(-z_1) : (\delta(z_3 - z_1) \delta(z_2 + z_1) + \delta(z_3 + z_1) \delta(z_2 - z_1))] \\ = [- : \varepsilon_{n-1}^*(z_4) \varepsilon_n(z_4) :, : \varepsilon_{n-1}^*(z_1) \varepsilon_{n-1}^*(-z_1) : (\delta(z_3 - z_1) \delta(z_2 + z_1) + \delta(z_3 + z_1) \delta(z_2 - z_1))].$$

Using Proposition 31, all contractions (hence bilinear forms $\langle \cdot, \cdot \rangle$) are 0, so this calculation is:

$$= 0$$

$$= \rho(\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_4)\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(-\alpha_p, z_2)X^{\bar{\sigma}}(-\alpha_m, z_1)).$$

Hence, ρ is a homomorphism; thus we have constructed a representation of the twisted toroidal Lie algebra of type A_{2n-1} . ■

Chapter 5

Bosonic Representation

In this chapter, we give the other type of FF representation for the twisted toroidal Lie algebras of type A_{2n-1} , namely the bosonic representation.

5.1 Free Field Notation

Let $\{\varepsilon_i, \varepsilon_i^* \mid i = 0, 1, \dots, n+1\}, \bar{c}, \bar{c}^*, \beta, \beta^*$, the inner product $(\cdot|\cdot)$, and the vector spaces P, P^* exactly as in the fermionic representation.

Now introduce two copies of \mathbb{C}^n spanned by $\{\varepsilon_{\bar{i}}, \varepsilon_{\bar{i}}^* \mid i = 1, 2, \dots, n\}$, respectively. The $\varepsilon_{\bar{i}}$ are orthonormal with respect to the inner product $(\cdot|\cdot)$; that is, $(\varepsilon_{\bar{i}}|\varepsilon_{\bar{j}}) = \delta_{ij}$. Similarly, $(\varepsilon_{\bar{i}}^*|\varepsilon_{\bar{j}}^*) = \delta_{ij}$. Similarly define $\bar{\beta} = -\bar{c} + \varepsilon_{\bar{1}}$ and $\bar{\beta}^* = -\bar{c}^* + \varepsilon_{\bar{1}}^*$.

Form the vector spaces \bar{P} with basis $\{\varepsilon_{\bar{i}}, \bar{c} \mid i = 1, \dots, n\}$, and \bar{P}^* with basis $\{\varepsilon_{\bar{i}}^*, \bar{c}^* \mid i = 1, \dots, n\}$.

Definition 33. Define the space $\mathcal{C}' := \mathcal{C}_1 \oplus \mathcal{C}_1^*$ where $\mathcal{C}_1 = P \oplus \bar{P}$ and $\mathcal{C}_1^* = P^* \oplus \bar{P}^*$.

Remark. This decomposition of \mathcal{C}' is a polarization into maximal isotropic subspaces with respect to the antisymmetric bilinear form $\langle \cdot, \cdot \rangle$ defined by:

$$\langle a^*, b \rangle = -\langle a, b^* \rangle = \langle \bar{a}^*, \bar{b} \rangle = -\langle \bar{a}, \bar{b}^* \rangle = (a|b) \text{ for } a, b \in P, \text{ and all other combinations } 0.$$

We now introduce a Weyl algebra, $W(P)$. It is the associative, unital algebra generated by the set $\{a(k), k' \mid a \in \mathcal{C}', k \in \mathbb{Z} + \frac{1}{2}\}$ and subject to the relations

$$[a(k), b(l)] = \langle a, b \rangle \delta_{k,-l} k' \text{ for } a, b \in \mathcal{C}'.$$

Here $[\cdot, \cdot]$ denotes the usual commutator.

Remark. Note the differences between the fermionic case and bosonic; in the latter, we use an antisymmetric bilinear form (instead of a symmetric form), and the elements form a Weyl algebra under the commutator (instead of a Clifford algebra under the anticommutator).

Definition 34. The *representation space* (or *Fock space*) is:

$$\overline{V}' := \bigotimes_{a_i \in P, \overline{P}} \left(\bigotimes_{k \in (\mathbb{Z} + \frac{1}{2})_{>0}} \mathbb{C}[a_i(-k)] \bigotimes_{k \in (\mathbb{Z} + \frac{1}{2})_{>0}} \mathbb{C}[a_i^*(-k)] \right).$$

The action of $W(P)$ on \overline{V}' , written by juxtaposition, is described here. \overline{V}' is a simple $W(P)$ -module with a distinguished *vacuum vector* $1 \in \overline{V}'$. For $a \in \mathcal{C}'$, $k \in (\mathbb{Z} + \frac{1}{2})_{>0}$ and any polynomial $v \in \overline{V}'$, $a(-k)v$ acts by left multiplication (as in $a(-k)v$) and $a(k)$ acts like $\frac{\partial}{\partial a(-k)}(v)$. Notice in particular that $a(k)1 = 0$. Hence, $a(k)$ is called an *annihilation operator* and $a(-k)$ is a *creation operator*. Also, \mathcal{K}' acts as -2 .

For any $u \in \mathcal{C}'$, we define a *generating function* or *free field operator* with a formal variable z using components: $u(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} u(k)z^{-k-\frac{1}{2}}$. A generating function acts on $v \in \overline{V}'$ by $u(z) \cdot v = \sum_{k \in \mathbb{Z} + \frac{1}{2}} (u(k) \cdot v)z^{-k-\frac{1}{2}}$. Hence the variable z acts only as a “bookkeeping” device and does not affect the action on \overline{V}' .

Definition 35. The *normal ordering* of a quadratic expression $:u(z)v(w):$ is defined on its components by:

$$:u(m)v(n):= \begin{cases} u(m)v(n), & \text{if } m < 0; \\ v(n)u(m), & \text{if } m > 0; \end{cases}$$

$$\text{so that } :u(z)v(w):= \sum_{m,n \in \mathbb{Z} + \frac{1}{2}} :u(m)v(n): z^{-m-\frac{1}{2}} w^{-n-\frac{1}{2}}.$$

Remark. Since $:u(z)v(w):=:v(w)u(z):$, these fields are indeed *bosonic fields*.

We have also the following lemma, the proof of which is similar to the corresponding lemma in the previous section.

Lemma 36. For $u \in \mathcal{C}'$, and z, w formal variables, $:u(z)v(w):\delta(z \pm w) =:u(\mp w)v(w):\delta(z \pm w)$.

■

We can extend the definition of normal ordering to more than 2 fields inductively. For $u_1, u_2, u_3, \dots, u_m \in \mathcal{C}'$ and $z_1, z_2, z_3, \dots, z_m$ formal variables,

$$\begin{aligned} :u_1(z_1)u_2(z_2)\cdots u_m(z_m): &=:u_1(z_1)(:u_2(z_2)\cdots u_m(z_m):): \\ &=:u_1(z_1)(:u_2(z_2)(:u_3(z_3)\cdots u_m(z_m):):): \end{aligned}$$

and so on until the innermost parentheses contain only 2 fields. We also define the *contraction* of two bosonic fields by the same definition as previously.

Definition 37. For $x_1, \dots, x_m \in \mathcal{C}'$, define:

$$:x_1 \cdots \underbrace{x_i \cdots x_j}_{\text{}} \cdots x_m: = \underbrace{x_i x_j}_{\text{}} :x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_m:$$

where \hat{x}_i indicates that the index i is missing from the list.

Here we restate Wick's Theorem in the case of bosons. The only difference is the absence of the sign of a permutation in part 2. of the statement.

Proposition 38. Wick's Theorem for bosons. For $x_1, \dots, x_m, y_1, \dots, y_p \in \mathcal{C}'$, we have:

1. $x_1 \cdots x_m = :x_1 \cdots x_m: + \sum :x_1 \cdots \underbrace{x_i \cdots x_j}_{\text{}} \cdots x_m:, \text{ where the sum is taken over all combinations of sets of contractions (up to } \frac{m}{2} \text{ contractions if } m \text{ is even, or } \frac{m-1}{2} \text{ contractions if } m \text{ is odd).}$
2. $(:x_1 \cdots x_m:)(:y_1 \cdots y_p:) = :x_1 \cdots x_m y_1 \cdots y_p: + \sum :x_1 \cdots \underbrace{x_i \cdots x_m}_{\text{}} y_1 \cdots y_p:, \text{ where the sum is taken over all combinations of sets of contractions of some } x_i \text{'s with some } y_j \text{'s. Equivalently, } (:x_1 \cdots x_m:)(:y_1 \cdots y_p:) = \sum_{s=0}^{\min(m,p)} \sum_{i_1 < \dots < i_s, j_1 \neq \dots \neq j_s} \underbrace{x_{i_1} y_{j_1} \cdots x_{i_s} y_{j_s}}_{\text{}} :x_1 \cdots x_m y_1 \cdots y_p:_{(i_1, \dots, i_s, j_1, \dots, j_s)} \text{ where the subscript } (i_1, \dots, i_s, j_1, \dots, j_s) \text{ means that the elements } x_{i_1}, \dots, x_{i_s}, y_{j_1}, \dots, y_{j_s} \text{ are removed.}$

Wick's Theorem and the definition preceding it are simpler in the bosonic case because the presence of the sign of a permutation is not needed. Hence, the following corollary is immediate from the definition of normal ordering for bosons:

Corollary 39. For $u_1, u_2, u_3, u_4 \in \mathcal{C}'$ and z, w formal variables, $:u_1(\pm z)u_2(\pm z)u_3(\pm w)u_4(\pm w): =:u_3(\pm w)u_4(\pm w)u_1(\pm z)u_2(\pm z):$. ■

Proposition 40. $\underbrace{a(\pm z)b(\pm w)}_{\text{}} = \iota_{z,w} \frac{\langle a,b \rangle}{\pm z \mp w}$ for $a, b \in \mathcal{C}'$.

Proof: By definition we have $\underbrace{a(\pm z)b(\pm w)}_{\text{}} = a(\pm z)b(\pm w) - :a(\pm z)b(\pm w): = \sum_{k,l \in \mathbb{Z} + \frac{1}{2}} a(k)b(l)(\pm z)^{-k-\frac{1}{2}}(\pm w)^{-l-\frac{1}{2}} - \sum_{k,l \in \mathbb{Z} + \frac{1}{2}} :a(k)b(l): (\pm z)^{-k-\frac{1}{2}}(\pm w)^{-l-\frac{1}{2}}$

$$\begin{aligned}
&= \sum_{k,l \in \mathbb{Z} + \frac{1}{2}} a(k)b(l)(\pm z)^{-k-\frac{1}{2}}(\pm w)^{-l-\frac{1}{2}} - \sum_{k \in \mathbb{Z} + \frac{1}{2}, l \in \mathbb{Z} + \frac{1}{2}} a(k)b(l)(\pm z)^{-k-\frac{1}{2}}(\pm w)^{-l-\frac{1}{2}} \\
&- \sum_{k \in \mathbb{Z} + \frac{1}{2}, l > 0} b(l)a(k)(\pm z)^{-k-\frac{1}{2}}(\pm w)^{-l-\frac{1}{2}} \\
&= \sum_{k \in \mathbb{Z} + \frac{1}{2}, l > 0} a(k)b(l)(\pm z)^{-k-\frac{1}{2}}(\pm w)^{-l-\frac{1}{2}} - \sum_{k \in \mathbb{Z} + \frac{1}{2}, l > 0} b(l)a(k)(\pm z)^{-k-\frac{1}{2}}(\pm w)^{-l-\frac{1}{2}} \\
&= \sum_{k \in \mathbb{Z} + \frac{1}{2}, l > 0} (a(k)b(l) - b(l)a(k))(\pm z)^{-k-\frac{1}{2}}(\pm w)^{-l-\frac{1}{2}} \\
&= \sum_{k \in \mathbb{Z} + \frac{1}{2}, l > 0} [a(k), b(l)](\pm z)^{-k-\frac{1}{2}}(\pm w)^{-l-\frac{1}{2}} \\
&= \sum_{k \in \mathbb{Z} + \frac{1}{2}, l > 0} \langle a, b \rangle \delta_{k,-l} (\pm z)^{-k-\frac{1}{2}}(\pm w)^{-l-\frac{1}{2}} \\
&= \sum_{k \in \mathbb{Z} + \frac{1}{2}, l > 0} \langle a, b \rangle (\pm z)^{-k-\frac{1}{2}}(\pm w)^{k-\frac{1}{2}}
\end{aligned}$$

This sum is a geometric series, and is equal to: $\iota_{z,w} \frac{\langle a, b \rangle (\pm z)^{-1}}{1 - \frac{\pm w}{\pm z}} = \iota_{z,w} \frac{\langle a, b \rangle}{\pm z \mp w}$, which is as desired. ■

Remark. 5.1. As operators on \overline{V}' , we choose \bar{c} so that $:a(z)\bar{c}(z):=0$ and $:a(z)\bar{c}^*(z):=0$ where $a \in \mathcal{C}'$. Hence, in particular, $:a(z)\beta(z):=a(z)\varepsilon_1(z)$, $:a(z)\beta^*(z):=a(z)\varepsilon_1^*(z)$, $:a(z)\bar{\beta}^*(z):=a(z)\varepsilon_1^*(z)$, and $:a(z)\bar{\beta}(z):=a(z)\varepsilon_{\bar{1}}(z)$.

The Lie algebra of operators in $W(P)$ is formed by endowing $W(P)$ with the commutator bracket. That is, the bracket among generating functions of operators is, by definition $[:a_1(z)b_1(z), :a_2(w)b_2(w)]:=a_1(z)b_1(z)a_2(w)b_2(w)-a_2(w)b_2(w)a_1(z)b_1(z)$.

Proposition 41. For $a_1, b_1, a_2, b_2 \in \mathcal{C}'$ and formal variables z, w , we have

1. $[:a_1(z)b_1(z), :a_2(w)b_2(w)]:$
- $$\begin{aligned}
&= \langle a_1, b_2 \rangle :b_1(w)a_2(w):\delta(z-w) + \langle a_1, a_2 \rangle :b_1(w)b_2(w):\delta(z-w) \\
&+ \langle b_1, a_2 \rangle :a_1(w)b_2(w):\delta(z-w) + \langle b_1, b_2 \rangle :a_1(w)a_2(w):\delta(z-w) \\
&+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w).
\end{aligned}$$
2. $[:a_1(-z)b_1(z), :a_2(w)b_2(w)]:$
- $$\begin{aligned}
&= -\langle a_1, b_2 \rangle :b_1(-w)a_2(w):\delta(z+w) - \langle a_1, a_2 \rangle :b_1(-w)b_2(w):\delta(z+w) \\
&+ \langle b_1, a_2 \rangle :a_1(-w)b_2(w):\delta(z-w) + \langle b_1, b_2 \rangle :a_1(-w)a_2(w):\delta(z-w) \\
&- (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right).
\end{aligned}$$

3. $[:a_1(z)b_1(-z)::a_2(w)b_2(w):]$
 $= \langle a_1, b_2 \rangle :b_1(-w)a_2(w):\delta(z-w) + \langle a_1, a_2 \rangle :b_1(-w)b_2(w):\delta(z-w)$
 $- \langle b_1, a_2 \rangle :a_1(-w)b_2(w):\delta(z+w) - \langle b_1, b_2 \rangle :a_1(-w)a_2(w):\delta(z+w)$
 $- (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right).$
4. $[:a_1(z)b_1(z)::a_2(-w)b_2(w):]$
 $= \langle a_1, b_2 \rangle :b_1(w)a_2(-w):\delta(z-w) + \langle a_1, a_2 \rangle :b_1(-w)b_2(w):\delta(z+w)$
 $+ \langle b_1, a_2 \rangle :a_1(-w)b_2(w):\delta(z+w) + \langle b_1, b_2 \rangle :a_1(w)a_2(-w):\delta(z-w)$
 $+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right).$
5. $[:a_1(z)b_1(z)::a_2(w)b_2(-w):]$
 $= \langle a_1, b_2 \rangle :b_1(-w)a_2(w):\delta(z+w) + \langle a_1, a_2 \rangle :b_1(w)b_2(-w):\delta(z-w)$
 $+ \langle b_1, a_2 \rangle :a_1(w)b_2(-w):\delta(z-w) + \langle b_1, b_2 \rangle :a_1(-w)a_2(w):\delta(z+w)$
 $+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right).$
6. $[:a_1(-z)b_1(-z)::a_2(w)b_2(w):]$
 $= -\langle a_1, b_2 \rangle :b_1(w)a_2(w):\delta(z+w) - \langle a_1, a_2 \rangle :b_1(w)b_2(w):\delta(z+w)$
 $- \langle b_1, a_2 \rangle :a_1(w)b_2(w):\delta(z+w) - \langle b_1, b_2 \rangle :a_1(w)a_2(w):\delta(z+w)$
 $- (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z+w).$
7. $[:a_1(-z)b_1(z)::a_2(-w)b_2(w):]$
 $= -\langle a_1, b_2 \rangle :b_1(-w)a_2(-w):\delta(z+w) - \langle a_1, a_2 \rangle :b_1(w)b_2(w):\delta(z-w)$
 $+ \langle b_1, a_2 \rangle :a_1(w)b_2(w):\delta(z+w) + \langle b_1, b_2 \rangle :a_1(-w)a_2(-w):\delta(z-w)$
 $+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z+w) - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w).$
8. $[:a_1(-z)b_1(z)::a_2(w)b_2(-w):]$
 $= -\langle a_1, b_2 \rangle :b_1(w)a_2(w):\delta(z-w) - \langle a_1, a_2 \rangle :b_1(-w)b_2(-w):\delta(z+w)$
 $+ \langle b_1, a_2 \rangle :a_1(-w)b_2(-w):\delta(z-w) + \langle b_1, b_2 \rangle :a_1(w)a_2(w):\delta(z+w)$
 $- (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z-w) + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z+w).$
9. $[:a_1(z)b_1(-z)::a_2(-w)b_2(w):]$
 $= \langle a_1, b_2 \rangle :b_1(-w)a_2(-w):\delta(z-w) + \langle a_1, a_2 \rangle :b_1(w)b_2(w):\delta(z+w)$
 $- \langle b_1, a_2 \rangle :a_1(w)b_2(w):\delta(z-w) - \langle b_1, b_2 \rangle :a_1(-w)a_2(-w):\delta(z+w)$
 $- (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z-w) + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z+w).$
10. $[:a_1(z)b_1(-z)::a_2(w)b_2(-w):]$
 $= \langle a_1, b_2 \rangle :b_1(w)a_2(w):\delta(z+w) + \langle a_1, a_2 \rangle :b_1(-w)b_2(-w):\delta(z-w)$

$$\begin{aligned} & -\langle b_1, a_2 \rangle : a_1(-w) b_2(-w) : \delta(z+w) - \langle b_1, b_2 \rangle : a_1(w) a_2(w) : \delta(z-w) \\ & + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z+w) - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w). \end{aligned}$$

11. $[:a_1(z)b_1(z)::a_2(-w)b_2(-w):]$

$$\begin{aligned} & = \langle a_1, b_2 \rangle : b_1(-w) a_2(-w) : \delta(z+w) + \langle a_1, a_2 \rangle : b_1(-w) b_2(-w) : \delta(z+w) \\ & + \langle b_1, a_2 \rangle : a_1(-w) b_2(-w) : \delta(z+w) + \langle b_1, b_2 \rangle : a_1(-w) a_2(-w) : \delta(z+w) \\ & - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle + \langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z+w). \end{aligned}$$

12. $[:a_1(-z)b_1(-z)::a_2(-w)b_2(w):]$

$$\begin{aligned} & = -\langle a_1, b_2 \rangle : b_1(w) a_2(-w) : \delta(z+w) - \langle a_1, a_2 \rangle : b_1(-w) b_2(w) : \delta(z-w) \\ & - \langle b_1, a_2 \rangle : a_1(-w) b_2(w) : \delta(z-w) - \langle b_1, b_2 \rangle : a_1(w) a_2(-w) : \delta(z+w) \\ & + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right). \end{aligned}$$

13. $[:a_1(-z)b_1(-z)::a_2(w)b_2(-w):]$

$$\begin{aligned} & = -\langle a_1, b_2 \rangle : b_1(-w) a_2(w) : \delta(z-w) - \langle a_1, a_2 \rangle : b_1(w) b_2(-w) : \delta(z+w) \\ & - \langle b_1, a_2 \rangle : a_1(w) b_2(-w) : \delta(z+w) - \langle b_1, b_2 \rangle : a_1(-w) a_2(w) : \delta(z-w) \\ & + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right). \end{aligned}$$

14. $[:a_1(-z)b_1(z)::a_2(-w)b_2(-w):]$

$$\begin{aligned} & = -\langle a_1, b_2 \rangle : b_1(w) a_2(-w) : \delta(z-w) - \langle a_1, a_2 \rangle : b_1(w) b_2(-w) : \delta(z-w) \\ & + \langle b_1, a_2 \rangle : a_1(w) b_2(-w) : \delta(z+w) + \langle b_1, b_2 \rangle : a_1(w) a_2(-w) : \delta(z+w) \\ & - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right). \end{aligned}$$

15. $[:a_1(z)b_1(-z)::a_2(-w)b_2(-w):]$

$$\begin{aligned} & = \langle a_1, b_2 \rangle : b_1(w) a_2(-w) : \delta(z+w) + \langle a_1, a_2 \rangle : b_1(w) b_2(-w) : \delta(z+w) \\ & - \langle b_1, a_2 \rangle : a_1(w) b_2(-w) : \delta(z-w) - \langle b_1, b_2 \rangle : a_1(w) a_2(-w) : \delta(z-w) \\ & - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right). \end{aligned}$$

16. $[:a_1(-z)b_1(-z)::a_2(-w)b_2(-w):]$

$$\begin{aligned} & = -\langle a_1, b_2 \rangle : b_1(-w) a_2(-w) : \delta(z-w) - \langle a_1, a_2 \rangle : b_1(-w) b_2(-w) : \delta(z-w) \\ & - \langle b_1, a_2 \rangle : a_1(-w) b_2(-w) : \delta(z-w) - \langle b_1, b_2 \rangle : a_1(-w) a_2(-w) : \delta(z-w) \\ & + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w). \end{aligned}$$

Proof: We will begin by dealing with all cases at once. Using Propositions 38 and 40, we have

$$\begin{aligned}
& :a_1(\pm z)b_1(\pm z)::a_2(\pm w)b_2(\pm w): =:a_1(\pm z)b_1(\pm z)a_2(\pm w)b_2(\pm w): \\
& + \underbrace{a_1(\pm z)b_2(\pm w)}_{+} :b_1(\pm z)a_2(\pm w): + \underbrace{a_1(\pm z)a_2(\pm w)}_{+} :b_1(\pm z)b_2(\pm w): \\
& + \underbrace{b_1(\pm z)a_2(\pm w)}_{+} :a_1(\pm z)b_2(\pm w): + \underbrace{b_1(\pm z)b_2(\pm w)}_{+} :a_1(\pm z)a_2(\pm w): \\
& + \underbrace{a_1(\pm z)b_2(\pm w)}_{+} \underbrace{b_1(\pm z)a_2(\pm w)}_{+} + \underbrace{a_1(\pm z)a_2(\pm w)}_{+} \underbrace{b_1(\pm z)b_2(\pm w)}_{+} \\
& =:a_1(\pm z)b_1(\pm z)a_2(\pm w)b_2(\pm w): \\
& + \iota_{z,w} \frac{\langle a_1, b_2 \rangle}{\pm z \mp w} :b_1(\pm z)a_2(\pm w): + \iota_{z,w} \frac{\langle a_1, a_2 \rangle}{\pm z \mp w} :b_1(\pm z)b_2(\pm w): \\
& + \iota_{z,w} \frac{\langle b_1, a_2 \rangle}{\pm z \mp w} :a_1(\pm z)b_2(\pm w): + \iota_{z,w} \frac{\langle b_1, b_2 \rangle}{\pm z \mp w} :a_1(\pm z)a_2(\pm w): \\
& + \iota_{z,w} \frac{\langle a_1, b_2 \rangle}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, a_2 \rangle}{\pm z \mp w} + \iota_{z,w} \frac{\langle a_1, a_2 \rangle}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, b_2 \rangle}{\pm z \mp w}.
\end{aligned}$$

Similarly, $:a_2(\pm w)b_2(\pm w)::a_1(\pm z)b_1(\pm z): =:a_2(\pm w)b_2(\pm w)a_1(\pm z)b_1(\pm z):$

$$\begin{aligned}
& + \underbrace{a_2(\pm w)b_1(\pm z)}_{+} :b_2(\pm w)a_1(\pm z): + \underbrace{a_2(\pm w)a_1(\pm z)}_{+} :b_2(\pm w)b_1(\pm z): \\
& + \underbrace{b_2(\pm w)a_1(\pm z)}_{+} :a_2(\pm w)b_1(\pm z): + \underbrace{b_2(\pm w)b_1(\pm z)}_{+} :a_2(\pm w)a_1(\pm z): \\
& + \underbrace{a_2(\pm w)b_1(\pm z)}_{+} \underbrace{b_2(\pm w)a_1(\pm z)}_{+} + \underbrace{a_2(\pm w)a_1(\pm z)}_{+} \underbrace{b_2(\pm w)b_1(\pm z)}_{+} \\
& =:a_2(\pm w)b_2(\pm w)a_1(\pm z)b_1(\pm z): \\
& + \iota_{w,z} \frac{\langle a_2, b_1 \rangle}{\pm w \mp z} :b_2(\pm w)a_1(\pm z): + \iota_{w,z} \frac{\langle a_2, a_1 \rangle}{\pm w \mp z} :b_2(\pm w)b_1(\pm z): \\
& + \iota_{w,z} \frac{\langle b_2, a_1 \rangle}{\pm w \mp z} :a_2(\pm w)b_1(\pm z): + \iota_{w,z} \frac{\langle b_2, b_1 \rangle}{\pm w \mp z} :a_2(\pm w)a_1(\pm z): \\
& + \iota_{w,z} \frac{\langle a_2, b_1 \rangle}{\pm w \mp z} \iota_{w,z} \frac{\langle b_2, a_1 \rangle}{\pm w \mp z} + \iota_{w,z} \frac{\langle a_2, a_1 \rangle}{\pm w \mp z} \iota_{w,z} \frac{\langle b_2, b_1 \rangle}{\pm w \mp z}.
\end{aligned}$$

Thus, compute the difference of these two halves, as prescribed by the commutator bracket as follows.

$$\begin{aligned}
& :a_1(\pm z)b_1(\pm z)::a_2(\pm w)b_2(\pm w): - :a_2(\pm w)b_2(\pm w)::a_1(\pm z)b_1(\pm z):= \\
& :a_1(\pm z)b_1(\pm z)a_2(\pm w)b_2(\pm w): \\
& + \iota_{z,w} \frac{\langle a_1, b_2 \rangle}{\pm z \mp w} :b_1(\pm z)a_2(\pm w): + \iota_{z,w} \frac{\langle a_1, a_2 \rangle}{\pm z \mp w} :b_1(\pm z)b_2(\pm w): \\
& + \iota_{z,w} \frac{\langle b_1, a_2 \rangle}{\pm z \mp w} :a_1(\pm z)b_2(\pm w): + \iota_{z,w} \frac{\langle b_1, b_2 \rangle}{\pm z \mp w} :a_1(\pm z)a_2(\pm w): \\
& + \iota_{z,w} \frac{\langle a_1, b_2 \rangle}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, a_2 \rangle}{\pm z \mp w} + \iota_{z,w} \frac{\langle a_1, a_2 \rangle}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, b_2 \rangle}{\pm z \mp w} \\
& - :a_2(\pm w)b_2(\pm w)a_1(\pm z)b_1(\pm z): \\
& - \iota_{w,z} \frac{\langle a_2, b_1 \rangle}{\pm w \mp z} :b_2(\pm w)a_1(\pm z): - \iota_{w,z} \frac{\langle a_2, a_1 \rangle}{\pm w \mp z} :b_2(\pm w)b_1(\pm z): \\
& - \iota_{w,z} \frac{\langle b_2, a_1 \rangle}{\pm w \mp z} :a_2(\pm w)b_1(\pm z): - \iota_{w,z} \frac{\langle b_2, b_1 \rangle}{\pm w \mp z} :a_2(\pm w)a_1(\pm z): \\
& - \iota_{w,z} \frac{\langle a_2, b_1 \rangle}{\pm w \mp z} \iota_{w,z} \frac{\langle b_2, a_1 \rangle}{\pm w \mp z} - \iota_{w,z} \frac{\langle a_2, a_1 \rangle}{\pm w \mp z} \iota_{w,z} \frac{\langle b_2, b_1 \rangle}{\pm w \mp z} \\
& =:a_1(\pm z)b_1(\pm z)a_2(\pm w)b_2(\pm w): - :a_2(\pm w)b_2(\pm w)a_1(\pm z)b_1(\pm z):
\end{aligned}$$

$$\begin{aligned}
& + \iota_{z,w} \frac{\langle a_1, b_2 \rangle}{\pm z \mp w} : b_1(\pm z) a_2(\pm w) : - \iota_{w,z} \frac{\langle b_2, a_1 \rangle}{\pm w \mp z} : a_2(\pm w) b_1(\pm z) : \\
& + \iota_{z,w} \frac{\langle a_1, a_2 \rangle}{\pm z \mp w} : b_1(\pm z) b_2(\pm w) : - \iota_{w,z} \frac{\langle a_2, a_1 \rangle}{\pm w \mp z} : b_2(\pm w) b_1(\pm z) : \\
& + \iota_{z,w} \frac{\langle b_1, a_2 \rangle}{\pm z \mp w} : a_1(\pm z) b_2(\pm w) : - \iota_{w,z} \frac{\langle a_2, b_1 \rangle}{\pm w \mp z} : b_2(\pm w) a_1(\pm z) : \\
& + \iota_{z,w} \frac{\langle b_1, b_2 \rangle}{\pm z \mp w} : a_1(\pm z) a_2(\pm w) : - \iota_{w,z} \frac{\langle b_2, b_1 \rangle}{\pm w \mp z} : a_2(\pm w) a_1(\pm z) : \\
& + \iota_{z,w} \frac{\langle a_1, b_2 \rangle}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, a_2 \rangle}{\pm z \mp w} - \iota_{w,z} \frac{\langle a_2, b_1 \rangle}{\pm w \mp z} \iota_{w,z} \frac{\langle b_2, a_1 \rangle}{\pm w \mp z} \\
& + \iota_{z,w} \frac{\langle a_1, a_2 \rangle}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, b_2 \rangle}{\pm z \mp w} - \iota_{w,z} \frac{\langle a_2, a_1 \rangle}{\pm w \mp z} \iota_{w,z} \frac{\langle b_2, b_1 \rangle}{\pm w \mp z}.
\end{aligned}$$

Using Corollary 39, the quartic terms cancel. Also, by the anti-symmetry of $\langle \cdot, \cdot \rangle$ and Definition

35, we have

$$\begin{aligned}
& = \iota_{z,w} \frac{\langle a_1, b_2 \rangle}{\pm z \mp w} : b_1(\pm z) a_2(\pm w) : + \iota_{w,z} \frac{\langle a_1, b_2 \rangle}{\pm w \mp z} : b_1(\pm z) a_2(\pm w) : \\
& + \iota_{z,w} \frac{\langle a_1, a_2 \rangle}{\pm z \mp w} : b_1(\pm z) b_2(\pm w) : + \iota_{w,z} \frac{\langle a_1, a_2 \rangle}{\pm w \mp z} : b_1(\pm z) b_2(\pm w) : \\
& + \iota_{z,w} \frac{\langle b_1, a_2 \rangle}{\pm z \mp w} : a_1(\pm z) b_2(\pm w) : + \iota_{w,z} \frac{\langle b_1, a_2 \rangle}{\pm w \mp z} : a_1(\pm z) b_2(\pm w) : \\
& + \iota_{z,w} \frac{\langle b_1, b_2 \rangle}{\pm z \mp w} : a_1(\pm z) a_2(\pm w) : + \iota_{w,z} \frac{\langle b_1, b_2 \rangle}{\pm w \mp z} : a_1(\pm z) a_2(\pm w) : \\
& + \iota_{z,w} \frac{\langle a_1, b_2 \rangle}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, a_2 \rangle}{\pm z \mp w} - \iota_{w,z} \frac{\langle a_1, b_2 \rangle}{\pm w \mp z} \iota_{w,z} \frac{\langle b_1, a_2 \rangle}{\pm w \mp z} \\
& + \iota_{z,w} \frac{\langle a_1, a_2 \rangle}{\pm z \mp w} \iota_{z,w} \frac{\langle b_1, b_2 \rangle}{\pm z \mp w} - \iota_{w,z} \frac{\langle a_1, a_2 \rangle}{\pm w \mp z} \iota_{w,z} \frac{\langle b_1, b_2 \rangle}{\pm w \mp z} \\
& = \langle a_1, b_2 \rangle \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : b_1(\pm z) a_2(\pm w) : \\
& + \langle a_1, a_2 \rangle \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : b_1(\pm z) b_2(\pm w) : \\
& + \langle b_1, a_2 \rangle \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : a_1(\pm z) b_2(\pm w) : \\
& + \langle b_1, b_2 \rangle \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : a_1(\pm z) a_2(\pm w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\iota_{z,w} \frac{1}{(\pm z \mp w)} \cdot \iota_{z,w} \frac{1}{(\pm z \mp w)} - \iota_{w,z} \frac{1}{(\pm w \mp z)} \cdot \iota_{w,z} \frac{1}{(\pm w \mp z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\iota_{z,w} \frac{1}{(\pm z \mp w)} \cdot \iota_{z,w} \frac{1}{(\pm z \mp w)} - \iota_{w,z} \frac{1}{(\pm w \mp z)} \cdot \iota_{w,z} \frac{1}{(\pm w \mp z)} \right).
\end{aligned}$$

For the last two lines, notice that we cannot write $\frac{1}{(\pm z \mp w)^2}$ for $\frac{1}{(\pm z \mp w)} \cdot \frac{1}{(\pm z \mp w)}$ because the two factors have variables which come from different generating functions, and hence their signs may be different from each other. Thus we have:

$$= \langle a_1, b_2 \rangle \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : b_1(\pm z) a_2(\pm w) :$$

$$\begin{aligned}
& + \langle a_1, a_2 \rangle \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : b_1(\pm z) b_2(\pm w) : \\
& + \langle b_1, a_2 \rangle \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : a_1(\pm z) b_2(\pm w) : \\
& + \langle b_1, b_2 \rangle \left(\iota_{z,w} \frac{1}{\pm z \mp w} + \iota_{w,z} \frac{1}{\pm w \mp z} \right) : a_1(\pm z) a_2(\pm w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\iota_{z,w} \frac{1}{(\pm z \mp w)} \cdot \iota_{z,w} \frac{1}{(\pm z \mp w)} - \iota_{w,z} \frac{1}{(\pm w \mp z)} \cdot \iota_{w,z} \frac{1}{(\pm w \mp z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\iota_{z,w} \frac{1}{(\pm z \mp w)} \cdot \iota_{z,w} \frac{1}{(\pm z \mp w)} - \iota_{w,z} \frac{1}{(\pm w \mp z)} \cdot \iota_{w,z} \frac{1}{(\pm w \mp z)} \right).
\end{aligned}$$

At this point, we can deduce each of the 16 cases stated in the proposition by choosing signs for each of the variables. We make use of the definitions of the various delta functions in Definition 16 and Lemma 36, and we continue the calculation above in each of the 16 cases. We proceed by using the same numbering of the cases as in the statement of the proposition.

To ease the notation, we will drop the leading $\iota_{z,w}$ and $\iota_{w,z}$ for the remainder of the proof.

$$\begin{aligned}
1. & = \langle a_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(z) a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(z) b_2(w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w-z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w-z)} \right) \\
& = \langle a_1, b_2 \rangle : b_1(z) a_2(w) : \delta(z-w) + \langle a_1, a_2 \rangle : b_1(z) b_2(w) : \delta(z-w) \\
& + \langle b_1, a_2 \rangle : a_1(z) b_2(w) : \delta(z-w) + \langle b_1, b_2 \rangle : a_1(z) a_2(w) : \delta(z-w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w) \\
& = \langle a_1, b_2 \rangle : b_1(w) a_2(w) : \delta(z-w) + \langle a_1, a_2 \rangle : b_1(w) b_2(w) : \delta(z-w) \\
& + \langle b_1, a_2 \rangle : a_1(w) b_2(w) : \delta(z-w) + \langle b_1, b_2 \rangle : a_1(w) a_2(w) : \delta(z-w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w).
\end{aligned}$$

$$\begin{aligned}
2. & = \langle a_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : b_1(z) a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : b_1(z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(-z) b_2(w) :
\end{aligned}$$

$$\begin{aligned}
& + \langle b_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(-z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = \langle a_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : b_1(z) a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : b_1(z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(-z) b_2(w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(-z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(z) a_2(w) : \delta(z+w) \\
& - \langle a_1, a_2 \rangle : b_1(z) b_2(w) : \delta(z+w) \\
& + \langle b_1, a_2 \rangle : a_1(-z) b_2(w) : \delta(z-w) \\
& + \langle b_1, b_2 \rangle : a_1(-z) a_2(w) : \delta(z-w) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle + \langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(-w) a_2(w) : \delta(z+w) \\
& - \langle a_1, a_2 \rangle : b_1(-w) b_2(w) : \delta(z+w) \\
& + \langle b_1, a_2 \rangle : a_1(-w) b_2(w) : \delta(z-w) \\
& + \langle b_1, b_2 \rangle : a_1(-w) a_2(w) : \delta(z-w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right).
\end{aligned}$$

$$\begin{aligned}
3. & = \langle a_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(z) b_2(w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right)
\end{aligned}$$

$$\begin{aligned}
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = \langle a_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(z) b_2(w) : \\
& + \langle b_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = \langle a_1, b_2 \rangle : b_1(-z) a_2(w) : \delta(z-w) \\
& + \langle a_1, a_2 \rangle : b_1(-z) b_2(w) : \delta(z-w) \\
& - \langle b_1, a_2 \rangle : a_1(z) b_2(w) : \delta(z+w) \\
& - \langle b_1, b_2 \rangle : a_1(z) a_2(w) : \delta(z+w) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle + \langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = \langle a_1, b_2 \rangle : b_1(-w) a_2(w) : \delta(z-w) \\
& + \langle a_1, a_2 \rangle : b_1(-w) b_2(w) : \delta(z-w) \\
& - \langle b_1, a_2 \rangle : a_1(-w) b_2(w) : \delta(z+w) \\
& - \langle b_1, b_2 \rangle : a_1(-w) a_2(w) : \delta(z+w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right).
\end{aligned}$$

$$\begin{aligned}
4. & = \langle a_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(z) a_2(-w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(z) b_2(w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(-w-z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(w-z)} \right). \\
& = \langle a_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(z) a_2(-w) :
\end{aligned}$$

$$\begin{aligned}
& + \langle a_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : a_1(z) b_2(w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = \langle a_1, b_2 \rangle : b_1(z) a_2(-w) : \delta(z-w) \\
& + \langle a_1, a_2 \rangle : b_1(z) b_2(w) : \delta(z+w) \\
& + \langle b_1, a_2 \rangle : a_1(z) b_2(w) : \delta(z+w) \\
& + \langle b_1, b_2 \rangle : a_1(z) a_2(-w) : \delta(z-w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = \langle a_1, b_2 \rangle : b_1(w) a_2(-w) : \delta(z-w) \\
& + \langle a_1, a_2 \rangle : b_1(-w) b_2(w) : \delta(z+w) \\
& + \langle b_1, a_2 \rangle : a_1(-w) b_2(w) : \delta(z+w) \\
& + \langle b_1, b_2 \rangle : a_1(w) a_2(-w) : \delta(z-w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right).
\end{aligned}$$

$$\begin{aligned}
5. & = \langle a_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(z) a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(z) b_2(-w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(w-z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(-w-z)} \right) \\
& = \langle a_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(z) a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(z) b_2(-w) :
\end{aligned}$$

$$\begin{aligned}
& + \langle b_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : a_1(z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = \langle a_1, b_2 \rangle : b_1(z) a_2(w) : \delta(z+w) \\
& + \langle a_1, a_2 \rangle : b_1(z) b_2(-w) : \delta(z-w) \\
& + \langle b_1, a_2 \rangle : a_1(z) b_2(-w) : \delta(z-w) \\
& + \langle b_1, b_2 \rangle : a_1(z) a_2(w) : \delta(z+w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = \langle a_1, b_2 \rangle : b_1(-w) a_2(w) : \delta(z+w) \\
& + \langle a_1, a_2 \rangle : b_1(w) b_2(-w) : \delta(z-w) \\
& + \langle b_1, a_2 \rangle : a_1(w) b_2(-w) : \delta(z-w) \\
& + \langle b_1, b_2 \rangle : a_1(-w) a_2(w) : \delta(z+w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right).
\end{aligned}$$

$$\begin{aligned}
6. & = \langle a_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : b_1(-z) a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : b_1(-z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(-z) b_2(w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(-z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w+z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w+z)} \right) \\
& = \langle a_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : b_1(-z) a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : b_1(-z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(-z) b_2(w) : \\
& + \langle b_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(-z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)^2} - \frac{1}{(w+z)^2} \right)
\end{aligned}$$

$$\begin{aligned}
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)^2} - \frac{1}{(w+z)^2} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(-z) a_2(w) : \delta(z+w) \\
& - \langle a_1, a_2 \rangle : b_1(-z) b_2(w) : \delta(z+w) \\
& - \langle b_1, a_2 \rangle : a_1(-z) b_2(w) : \delta(z+w) \\
& - \langle b_1, b_2 \rangle : a_1(-z) a_2(w) : \delta(z+w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z+w) \\
& = -\langle a_1, b_2 \rangle : b_1(w) a_2(w) : \delta(z+w) \\
& - \langle a_1, a_2 \rangle : b_1(w) b_2(w) : \delta(z+w) \\
& - \langle b_1, a_2 \rangle : a_1(w) b_2(w) : \delta(z+w) \\
& - \langle b_1, b_2 \rangle : a_1(w) a_2(w) : \delta(z+w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z+w).
\end{aligned}$$

$$\begin{aligned}
7. & = \langle a_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : b_1(z) a_2(-w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : b_1(z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(-z) b_2(w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(-z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w+z)} \cdot \frac{1}{(-w-z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = \langle a_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : b_1(z) a_2(-w) : \\
& + \langle a_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : b_1(z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : a_1(-z) b_2(w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : a_1(-z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z+w)^2} + \frac{1}{(w+z)^2} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z-w)^2} + \frac{1}{(w-z)^2} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(z) a_2(-w) : \delta(z+w) \\
& - \langle a_1, a_2 \rangle : b_1(z) b_2(w) : \delta(z-w) \\
& + \langle b_1, a_2 \rangle : a_1(-z) b_2(w) : \delta(z+w)
\end{aligned}$$

$$\begin{aligned}
& + \langle b_1, b_2 \rangle : a_1(-z) a_2(-w) : \delta(z - w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z + w) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z - w) \\
& = - \langle a_1, b_2 \rangle : b_1(-w) a_2(-w) : \delta(z + w) \\
& - \langle a_1, a_2 \rangle : b_1(w) b_2(w) : \delta(z - w) \\
& + \langle b_1, a_2 \rangle : a_1(w) b_2(w) : \delta(z + w) \\
& + \langle b_1, b_2 \rangle : a_1(-w) a_2(-w) : \delta(z - w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z + w) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z - w). \\
8. & = \langle a_1, b_2 \rangle \left(\frac{1}{-z + w} + \frac{1}{-w + z} \right) : b_1(z) a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{-z - w} + \frac{1}{w + z} \right) : b_1(z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z - w} + \frac{1}{w - z} \right) : a_1(-z) b_2(-w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{z + w} + \frac{1}{-w - z} \right) : a_1(-z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z + w)} \cdot \frac{1}{(z - w)} - \frac{1}{(-w + z)} \cdot \frac{1}{(w - z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z - w)} \cdot \frac{1}{(z + w)} - \frac{1}{(w + z)} \cdot \frac{1}{(-w - z)} \right) \\
& = \langle a_1, b_2 \rangle \left(-\frac{1}{z - w} - \frac{1}{w - z} \right) : b_1(z) a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(-\frac{1}{z + w} + \frac{1}{w + z} \right) : b_1(z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z - w} + \frac{1}{w - z} \right) : a_1(-z) b_2(-w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{z + w} - \frac{1}{w + z} \right) : a_1(-z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z - w)^2} + \frac{1}{(w - z)^2} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z + w)^2} + \frac{1}{(w + z)^2} \right) \\
& = - \langle a_1, b_2 \rangle : b_1(z) a_2(w) : \delta(z - w) \\
& - \langle a_1, a_2 \rangle : b_1(z) b_2(-w) : \delta(z + w) \\
& + \langle b_1, a_2 \rangle : a_1(-z) b_2(-w) : \delta(z - w) \\
& + \langle b_1, b_2 \rangle : a_1(-z) a_2(w) : \delta(z + w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z - w) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z + w)
\end{aligned}$$

$$\begin{aligned}
&= -\langle a_1, b_2 \rangle : b_1(w) a_2(w) : \delta(z - w) \\
&- \langle a_1, a_2 \rangle : b_1(-w) b_2(-w) : \delta(z + w) \\
&+ \langle b_1, a_2 \rangle : a_1(-w) b_2(-w) : \delta(z - w) \\
&+ \langle b_1, b_2 \rangle : a_1(w) a_2(w) : \delta(z + w) \\
&- (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z - w) \\
&+ (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z + w).
\end{aligned}$$

$$\begin{aligned}
9. &= \langle a_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) a_2(-w) : \\
&+ \langle a_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(-z) b_2(w) : \\
&+ \langle b_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(z) b_2(w) : \\
&+ \langle b_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(z) a_2(-w) : \\
&+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(-w+z)} \right) \\
&+ (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(w+z)} \right) \\
&= \langle a_1, b_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) a_2(-w) : \\
&+ \langle a_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(-z) b_2(w) : \\
&+ \langle b_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(z) b_2(w) : \\
&+ \langle b_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(z) a_2(-w) : \\
&+ (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z-w)^2} + \frac{1}{(w-z)^2} \right) \\
&+ (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z+w)^2} + \frac{1}{(w+z)^2} \right) \\
&= \langle a_1, b_2 \rangle : b_1(-z) a_2(-w) : \delta(z - w) \\
&+ \langle a_1, a_2 \rangle : b_1(-z) b_2(w) : \delta(z + w) \\
&- \langle b_1, a_2 \rangle : a_1(z) b_2(w) : \delta(z - w) \\
&- \langle b_1, b_2 \rangle : a_1(z) a_2(-w) : \delta(z + w) \\
&- (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z - w) \\
&+ (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z + w). \\
&= \langle a_1, b_2 \rangle : b_1(-w) a_2(-w) : \delta(z - w) \\
&+ \langle a_1, a_2 \rangle : b_1(w) b_2(w) : \delta(z + w) \\
&- \langle b_1, a_2 \rangle : a_1(w) b_2(w) : \delta(z - w)
\end{aligned}$$

$$\begin{aligned}
& -\langle b_1, b_2 \rangle : a_1(-w) a_2(-w) : \delta(z + w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z - w) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z + w).
\end{aligned}$$

$$\begin{aligned}
10. &= \langle a_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(-z) a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(z) b_2(-w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(w+z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(-w+z)} \right) \\
& = \langle a_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(-z) a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{z-w} + \frac{1}{w-z} \right) : b_1(-z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(z) b_2(-w) : \\
& + \langle b_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z+w)^2} + \frac{1}{(w+z)^2} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z-w)^2} + \frac{1}{(w-z)^2} \right) \\
& = \langle a_1, b_2 \rangle : b_1(-z) a_2(w) : \delta(z + w) \\
& + \langle a_1, a_2 \rangle : b_1(-z) b_2(-w) : \delta(z - w) \\
& - \langle b_1, a_2 \rangle : a_1(z) b_2(-w) : \delta(z + w) \\
& - \langle b_1, b_2 \rangle : a_1(z) a_2(w) : \delta(z - w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z + w) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z - w) \\
& = \langle a_1, b_2 \rangle : b_1(w) a_2(w) : \delta(z + w) \\
& + \langle a_1, a_2 \rangle : b_1(-w) b_2(-w) : \delta(z - w) \\
& - \langle b_1, a_2 \rangle : a_1(-w) b_2(-w) : \delta(z + w) \\
& - \langle b_1, b_2 \rangle : a_1(w) a_2(w) : \delta(z - w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z + w) \\
& - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z - w).
\end{aligned}$$

$$\begin{aligned}
11. &= \langle a_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(z) a_2(-w) : \\
&\quad + \langle a_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(z) b_2(-w) : \\
&\quad + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(z) b_2(-w) : \\
&\quad + \langle b_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(z) a_2(-w) : \\
&\quad + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z+w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(-w-z)} \right) \\
&\quad + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z+w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(-w-z)} \right) \\
&= \langle a_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(z) a_2(-w) : \\
&\quad + \langle a_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(z) b_2(-w) : \\
&\quad + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : a_1(z) b_2(-w) : \\
&\quad + \langle b_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : a_1(z) a_2(-w) : \\
&\quad + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)^2} - \frac{1}{(w+z)^2} \right) \\
&\quad + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)^2} - \frac{1}{(w+z)^2} \right) \\
&= \langle a_1, b_2 \rangle : b_1(z) a_2(-w) : \delta(z+w) \\
&\quad + \langle a_1, a_2 \rangle : b_1(z) b_2(-w) : \delta(z+w) \\
&\quad + \langle b_1, a_2 \rangle : a_1(z) b_2(-w) : \delta(z+w) \\
&\quad + \langle b_1, b_2 \rangle : a_1(z) a_2(-w) : \delta(z+w) \\
&\quad - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle + \langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z+w) \\
&= \langle a_1, b_2 \rangle : b_1(-w) a_2(-w) : \delta(z+w) \\
&\quad + \langle a_1, a_2 \rangle : b_1(-w) b_2(-w) : \delta(z+w) \\
&\quad + \langle b_1, a_2 \rangle : a_1(-w) b_2(-w) : \delta(z+w) \\
&\quad + \langle b_1, b_2 \rangle : a_1(-w) a_2(-w) : \delta(z+w) \\
&\quad - (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle + \langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \partial_w \delta(z+w).
\end{aligned}$$

$$\begin{aligned}
12. &= \langle a_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : b_1(-z) a_2(-w) : \\
&\quad + \langle a_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : b_1(-z) b_2(w) : \\
&\quad + \langle b_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(-z) b_2(w) :
\end{aligned}$$

$$\begin{aligned}
& + \langle b_1, b_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(-z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(w+z)} \cdot \frac{1}{(-w+z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(w+z)} \right) \\
& = \langle a_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : b_1(-z) a_2(-w) : \\
& + \langle a_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : b_1(-z) b_2(w) : \\
& + \langle b_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(-z) b_2(w) : \\
& + \langle b_1, b_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(-z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(-z) a_2(-w) : \delta(z+w) \\
& - \langle a_1, a_2 \rangle : b_1(-z) b_2(w) : \delta(z-w) \\
& - \langle b_1, a_2 \rangle : a_1(-z) b_2(w) : \delta(z-w) \\
& - \langle b_1, b_2 \rangle : a_1(-z) a_2(-w) : \delta(z+w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(w) a_2(-w) : \delta(z+w) \\
& - \langle a_1, a_2 \rangle : b_1(-w) b_2(w) : \delta(z-w) \\
& - \langle b_1, a_2 \rangle : a_1(-w) b_2(w) : \delta(z-w) \\
& - \langle b_1, b_2 \rangle : a_1(w) a_2(-w) : \delta(z+w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right). \\
13. & = \langle a_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : b_1(-z) a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : b_1(-z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{-z-w} + \frac{1}{w+z} \right) : a_1(-z) b_2(-w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(-z) a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(-z-w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(w+z)} \right)
\end{aligned}$$

$$\begin{aligned}
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z-w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(w+z)} \cdot \frac{1}{(-w+z)} \right) \\
& = \langle a_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : b_1(-z)a_2(w) : \\
& + \langle a_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : b_1(-z)b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(-\frac{1}{z+w} + \frac{1}{w+z} \right) : a_1(-z)b_2(-w) : \\
& + \langle b_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(-z)a_2(w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(-z)a_2(w) : \delta(z-w) \\
& - \langle a_1, a_2 \rangle : b_1(-z)b_2(-w) : \delta(z+w) \\
& - \langle b_1, a_2 \rangle : a_1(-z)b_2(-w) : \delta(z+w) \\
& - \langle b_1, b_2 \rangle : a_1(-z)a_2(w) : \delta(z-w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(-w)a_2(w) : \delta(z-w) \\
& - \langle a_1, a_2 \rangle : b_1(w)b_2(-w) : \delta(z+w) \\
& - \langle b_1, a_2 \rangle : a_1(w)b_2(-w) : \delta(z+w) \\
& - \langle b_1, b_2 \rangle : a_1(-w)a_2(w) : \delta(z-w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right).
\end{aligned}$$

$$\begin{aligned}
14. & = \langle a_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : b_1(z)a_2(-w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : b_1(z)b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(-z)b_2(-w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : a_1(-z)a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(z+w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(-w-z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(z+w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(-w-z)} \right) \\
& = \langle a_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : b_1(z)a_2(-w) :
\end{aligned}$$

$$\begin{aligned}
& + \langle a_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : b_1(z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : a_1(-z) b_2(-w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : a_1(-z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} - \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(z) a_2(-w) : \delta(z-w) \\
& - \langle a_1, a_2 \rangle : b_1(z) b_2(-w) : \delta(z-w) \\
& + \langle b_1, a_2 \rangle : a_1(-z) b_2(-w) : \delta(z+w) \\
& + \langle b_1, b_2 \rangle : a_1(-z) a_2(-w) : \delta(z+w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right) \\
& = -\langle a_1, b_2 \rangle : b_1(w) a_2(-w) : \delta(z-w) \\
& - \langle a_1, a_2 \rangle : b_1(w) b_2(-w) : \delta(z-w) \\
& + \langle b_1, a_2 \rangle : a_1(w) b_2(-w) : \delta(z+w) \\
& + \langle b_1, b_2 \rangle : a_1(w) a_2(-w) : \delta(z+w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)} \cdot \frac{1}{(z+w)} + \frac{1}{(w-z)} \cdot \frac{1}{(w+z)} \right).
\end{aligned}$$

$$\begin{aligned}
15. & = \langle a_1, b_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(-z) a_2(-w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{z+w} + \frac{1}{-w-z} \right) : b_1(-z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(z) b_2(-w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(-w+z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(-w-z)} \cdot \frac{1}{(-w+z)} \right) \\
& = \langle a_1, b_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(-z) a_2(-w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{z+w} - \frac{1}{w+z} \right) : b_1(-z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(z) b_2(-w) :
\end{aligned}$$

$$\begin{aligned}
& + \langle b_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(-\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(-\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} - \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = \langle a_1, b_2 \rangle : b_1(-z) a_2(-w) : \delta(z+w) \\
& + \langle a_1, a_2 \rangle : b_1(-z) b_2(-w) : \delta(z+w) \\
& - \langle b_1, a_2 \rangle : a_1(z) b_2(-w) : \delta(z-w) \\
& - \langle b_1, b_2 \rangle : a_1(z) a_2(-w) : \delta(z-w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right) \\
& = \langle a_1, b_2 \rangle : b_1(w) a_2(-w) : \delta(z+w) \\
& + \langle a_1, a_2 \rangle : b_1(w) b_2(-w) : \delta(z+w) \\
& - \langle b_1, a_2 \rangle : a_1(w) b_2(-w) : \delta(z-w) \\
& - \langle b_1, b_2 \rangle : a_1(w) a_2(-w) : \delta(z-w) \\
& - (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z+w)} \cdot \frac{1}{(z-w)} + \frac{1}{(w+z)} \cdot \frac{1}{(w-z)} \right).
\end{aligned}$$

$$\begin{aligned}
16. & = \langle a_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : b_1(-z) a_2(-w) : \\
& + \langle a_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : b_1(-z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(-z) b_2(-w) : \\
& + \langle b_1, b_2 \rangle \left(\frac{1}{-z+w} + \frac{1}{-w+z} \right) : a_1(-z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(-w+z)} \right) \\
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(-z+w)} \cdot \frac{1}{(-z+w)} - \frac{1}{(-w+z)} \cdot \frac{1}{(-w+z)} \right) \\
& = \langle a_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : b_1(-z) a_2(-w) : \\
& + \langle a_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : b_1(-z) b_2(-w) : \\
& + \langle b_1, a_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(-z) b_2(-w) : \\
& + \langle b_1, b_2 \rangle \left(-\frac{1}{z-w} - \frac{1}{w-z} \right) : a_1(-z) a_2(-w) : \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle) \left(\frac{1}{(z-w)^2} - \frac{1}{(w-z)^2} \right)
\end{aligned}$$

$$\begin{aligned}
& + (\langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \left(\frac{1}{(z-w)^2} - \frac{1}{(w-z)^2} \right) \\
& = -\langle a_1, b_2 \rangle :b_1(-z)a_2(-w):\delta(z-w) \\
& - \langle a_1, a_2 \rangle :b_1(-z)b_2(-w):\delta(z-w) \\
& - \langle b_1, a_2 \rangle :a_1(-z)b_2(-w):\delta(z-w) \\
& - \langle b_1, b_2 \rangle :a_1(-z)a_2(-w):\delta(z-w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w) \\
& = -\langle a_1, b_2 \rangle :b_1(-w)a_2(-w):\delta(z-w) \\
& - \langle a_1, a_2 \rangle :b_1(-w)b_2(-w):\delta(z-w) \\
& - \langle b_1, a_2 \rangle :a_1(-w)b_2(-w):\delta(z-w) \\
& - \langle b_1, b_2 \rangle :a_1(-w)a_2(-w):\delta(z-w) \\
& + (\langle a_1, b_2 \rangle \langle b_1, a_2 \rangle + \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle) \partial_w \delta(z-w).
\end{aligned}$$

Each case is exactly as stated in the Proposition, proving it. ■

5.2 Bosonic Representation of Twisted Toroidal A_{2n-1}

We can now state and prove our third main theorem.

Theorem 42. Define a map $\rho' : \bar{t} \rightarrow \text{End}(\bar{V}')$ by:

1. $\not{\epsilon} \mapsto k' = -2$
2. $\alpha_0^{\bar{\sigma}}(z) \mapsto : \varepsilon_1^*(-z) \beta(-z) : - : \varepsilon_1(z) \beta^*(z) : - : \varepsilon_1^*(-z) \bar{\beta}(-z) : + : \varepsilon_1(z) \bar{\beta}^*(z) :$
3. $\alpha_i^{\bar{\sigma}}(z) \mapsto : \varepsilon_i(z) \varepsilon_i^*(z) : - : \varepsilon_{i+1}(z) \varepsilon_{i+1}^*(z) : - : \varepsilon_{\bar{i}}(z) \varepsilon_{\bar{i}}^*(z) : + : \varepsilon_{\bar{i+1}}(z) \varepsilon_{\bar{i+1}}^*(z) :$
4. $\alpha_n^{\bar{\sigma}}(z) \mapsto : \varepsilon_n(z) \varepsilon_n^*(z) : - : \varepsilon_n(-z) \varepsilon_n^*(-z) : - : \varepsilon_{\bar{n}}(z) \varepsilon_{\bar{n}}^*(z) : + : \varepsilon_{\bar{n}}(-z) \varepsilon_{\bar{n}}^*(-z) :$
5. $X^{\bar{\sigma}}(\alpha_0, z) \mapsto : \varepsilon_1^*(z) \bar{\beta}(-z) : - : \varepsilon_1^*(-z) \bar{\beta}(z) :$
6. $X^{\bar{\sigma}}(-\alpha_0, z) \mapsto : \varepsilon_1(-z) \bar{\beta}^*(z) : - : \varepsilon_1(z) \bar{\beta}^*(-z) :$
7. $X^{\bar{\sigma}}(\alpha_i, z) \mapsto : \varepsilon_i(z) \varepsilon_{i+1}^*(z) : - : \varepsilon_{\bar{i+1}}(z) \varepsilon_{\bar{i}}^*(z) :$
8. $X^{\bar{\sigma}}(-\alpha_i, z) \mapsto : \varepsilon_i^*(z) \varepsilon_{i+1}(z) : - : \varepsilon_{\bar{i+1}}^*(z) \varepsilon_{\bar{i}}(z) :$
9. $X^{\bar{\sigma}}(\alpha_n, z) \mapsto : \varepsilon_n(z) \varepsilon_{\bar{n}}^*(-z) : - : \varepsilon_n(-z) \varepsilon_{\bar{n}}^*(z) :$
10. $X^{\bar{\sigma}}(-\alpha_n, z) \mapsto : \varepsilon_n^*(-z) \varepsilon_{\bar{n}}(z) : - : \varepsilon_n^*(z) \varepsilon_{\bar{n}}(-z) :$

for $1 \leq i \leq n - 1$. Then ρ' is a homomorphism, and hence a representation of the twisted toroidal Lie algebra of type A_{2n-1} . Here \mathcal{K}' acts as -2 , so the representation is level -2 .

Proof: We will check each relation in turn.

$$\textbf{Relation (1): } [\alpha_0^{\bar{\sigma}}(z), \alpha_m^{\bar{\sigma}}(w)] = (2\delta_{m0} - \delta_{m1})(\partial_w \delta(z-w) + \partial_w \delta(z+w))\mathcal{K}'$$

First, consider the case in which $m = 0$.

$$\begin{aligned} & [\rho'(\alpha_0^{\bar{\sigma}}(z)), \rho'(\alpha_m^{\bar{\sigma}}(w))] \\ &= [:\varepsilon_1^*(-z)\beta(-z):-:\varepsilon_1(z)\beta^*(z):-:\varepsilon_1^*(-z)\bar{\beta}(-z):+:\varepsilon_1(z)\bar{\beta}^*(z):, \\ & :\varepsilon_1^*(-w)\beta(-w):-:\varepsilon_1(w)\beta^*(w):-:\varepsilon_1^*(-w)\bar{\beta}(-w):+:\varepsilon_1(w)\bar{\beta}^*(w):] \\ &= [:\varepsilon_1^*(-z)\beta(-z):,:\varepsilon_1^*(-w)\beta(-w):]-[:\varepsilon_1^*(-z)\beta(-z):,:\varepsilon_1(w)\beta^*(w):] \\ &\quad -[:\varepsilon_1^*(-z)\beta(-z):,:\varepsilon_1^*(-w)\bar{\beta}(-w):]+[:\varepsilon_1^*(-z)\beta(-z):,:\varepsilon_1(w)\bar{\beta}^*(w):] \\ &\quad -[:\varepsilon_1(z)\beta^*(z):,:\varepsilon_1^*(-w)\beta(-w):]+[:\varepsilon_1(z)\beta^*(z):,:\varepsilon_1(w)\beta^*(w):] \\ &\quad +[:\varepsilon_1(z)\beta^*(z):,:\varepsilon_1^*(-w)\bar{\beta}(-w):]-[:\varepsilon_1(z)\beta^*(z):,:\varepsilon_1(w)\bar{\beta}^*(w):] \\ &\quad -[:\varepsilon_1^*(-z)\bar{\beta}(-z):,:\varepsilon_1^*(-w)\beta(-w):]+[:\varepsilon_1^*(-z)\bar{\beta}(-z):,:\varepsilon_1(w)\beta^*(w):] \\ &\quad +[:\varepsilon_1^*(-z)\bar{\beta}(-z):,:\varepsilon_1^*(-w)\bar{\beta}(-w):]-[:\varepsilon_1^*(-z)\bar{\beta}(-z):,:\varepsilon_1(w)\bar{\beta}^*(w):] \\ &\quad +[:\varepsilon_1(z)\bar{\beta}^*(z):,:\varepsilon_1^*(-w)\beta(-w):]-[:\varepsilon_1(z)\bar{\beta}^*(z):,:\varepsilon_1(w)\beta^*(w):] \\ &\quad -[:\varepsilon_1(z)\bar{\beta}^*(z):,:\varepsilon_1^*(-w)\bar{\beta}(-w):]+[:\varepsilon_1(z)\bar{\beta}^*(z):,:\varepsilon_1(w)\bar{\beta}^*(w):]. \end{aligned}$$

We use Proposition 41 to compute these brackets. The nonzero terms are:

$$\begin{aligned} & = (-\langle \varepsilon_1^*, \beta \rangle : \beta(-w) \varepsilon_1^*(-w) : -\langle \beta, \varepsilon_1^* \rangle : \varepsilon_1^*(-w) \beta(-w) :) \delta(z-w) + \langle \varepsilon_1^*, \beta \rangle \langle \beta, \varepsilon_1^* \rangle \partial_w \delta(z-w) \\ & - \{ (-\langle \varepsilon_1^*, \varepsilon_1 \rangle : \beta(w) \beta^*(w) : -\langle \beta, \beta^* \rangle : \varepsilon_1^*(w) \varepsilon_1(w) :) \delta(z+w) - \langle \varepsilon_1^*, \varepsilon_1 \rangle \langle \beta, \beta^* \rangle \partial_w \delta(z+w) \} \\ & - \{ (\langle \beta^*, \beta \rangle : \varepsilon_1(-w) \varepsilon_1^*(-w) : +\langle \varepsilon_1, \varepsilon_1^* \rangle : \beta^*(-w) \beta(-w) :) \delta(z+w) - \langle \beta^*, \beta \rangle \langle \varepsilon_1, \varepsilon_1^* \rangle \partial_w \delta(z+w) \} \\ & + (\langle \beta^*, \varepsilon_1 \rangle : \varepsilon_1(w) \beta^*(w) : +\langle \varepsilon_1, \beta^* \rangle : \beta^*(w) \varepsilon_1(w) :) \delta(z-w) + \langle \beta^*, \varepsilon_1 \rangle \langle \varepsilon_1, \beta^* \rangle \partial_w \delta(z-w) \\ & + (-\langle \varepsilon_1^*, \bar{\beta} \rangle : \bar{\beta}(-w) \varepsilon_1^*(-w) : -\langle \bar{\beta}, \varepsilon_1^* \rangle : \varepsilon_1^*(-w) \bar{\beta}(-w) :) \delta(z-w) + \langle \varepsilon_1^*, \bar{\beta} \rangle \langle \bar{\beta}, \varepsilon_1^* \rangle \partial_w \delta(z-w) \\ & - \{ (-\langle \varepsilon_1^*, \varepsilon_{\bar{1}} \rangle : \bar{\beta}(w) \bar{\beta}^*(w) : -\langle \bar{\beta}, \bar{\beta}^* \rangle : \varepsilon_1^*(w) \varepsilon_{\bar{1}}(w) :) \delta(z+w) - \langle \varepsilon_1^*, \varepsilon_{\bar{1}} \rangle \langle \bar{\beta}, \bar{\beta}^* \rangle \partial_w \delta(z+w) \} \\ & - \{ (\langle \bar{\beta}^*, \bar{\beta} \rangle : \varepsilon_{\bar{1}}(-w) \varepsilon_1^*(-w) : +\langle \varepsilon_{\bar{1}}, \varepsilon_1^* \rangle : \bar{\beta}^*(-w) \bar{\beta}(-w) :) \delta(z+w) - \langle \bar{\beta}^*, \bar{\beta} \rangle \langle \varepsilon_{\bar{1}}, \varepsilon_1^* \rangle \partial_w \delta(z+w) \} \\ & + (\langle \bar{\beta}^*, \varepsilon_{\bar{1}} \rangle : \varepsilon_{\bar{1}}(w) \bar{\beta}^*(w) : +\langle \varepsilon_{\bar{1}}, \bar{\beta}^* \rangle : \bar{\beta}^*(w) \varepsilon_{\bar{1}}(w) :) \delta(z-w) + \langle \bar{\beta}^*, \varepsilon_{\bar{1}} \rangle \langle \varepsilon_{\bar{1}}, \bar{\beta}^* \rangle \partial_w \delta(z-w). \end{aligned}$$

By Remark 5.1, we see that every pair of normally ordered products cancels, and by antisymmetry of the form, we have

$$\begin{aligned} & = -4(\partial_w \delta(z-w) + \partial_w \delta(z+w)) \\ & = 2(\partial_w \delta(z-w) + \partial_w \delta(z+w))\mathcal{K}' \\ & = \rho'([\alpha_0^{\bar{\sigma}}(z), \alpha_m^{\bar{\sigma}}(w)]). \end{aligned}$$

Next, consider the case $1 \leq m \leq n - 1$.

$$\begin{aligned} & [\rho(\alpha_0^{\bar{\sigma}}(z)), \rho(\alpha_m^{\bar{\sigma}}(w))] \\ &= [:\varepsilon_1^*(-z)\beta(-z):-:\varepsilon_1(z)\beta^*(z):-:\varepsilon_1^*(-z)\bar{\beta}(-z):+:\varepsilon_1(z)\bar{\beta}^*(z):, \\ & :\varepsilon_j(w)\varepsilon_j^*(w):-:\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w):-:\varepsilon_{\bar{j}}(w)\varepsilon_{\bar{j}}^*(w):+:\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j+1}}^*(w):] \end{aligned}$$

$$\begin{aligned}
&= [:\varepsilon_1^*(-z)\beta(-z):, :\varepsilon_j(w)\varepsilon_j^*(w):] - [:\varepsilon_1^*(-z)\beta(-z):, :\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w):] \\
&\quad - [:\varepsilon_1^*(-z)\beta(-z):, :\varepsilon_{\bar{j}}(w)\varepsilon_{\bar{j}}^*(w):] + [:\varepsilon_1^*(-z)\beta(-z):, :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j+1}}^*(w):] \\
&\quad - [:\varepsilon_1(z)\beta^*(z):, :\varepsilon_j(w)\varepsilon_j^*(w):] + [:\varepsilon_1(z)\beta^*(z):, :\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w):] \\
&\quad + [:\varepsilon_1(z)\beta^*(z):, :\varepsilon_{\bar{j}}(w)\varepsilon_{\bar{j}}^*(w):] - [:\varepsilon_1(z)\beta^*(z):, :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j+1}}^*(w):] \\
&\quad - [:\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z):, :\varepsilon_j(w)\varepsilon_j^*(w):] + [:\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z):, :\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w):] \\
&\quad + [:\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z):, :\varepsilon_{\bar{j}}(w)\varepsilon_{\bar{j}}^*(w):] - [:\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z):, :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j+1}}^*(w):] \\
&\quad + [:\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, :\varepsilon_j(w)\varepsilon_j^*(w):] - [:\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, :\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w):] \\
&\quad - [:\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, :\varepsilon_{\bar{j}}(w)\varepsilon_{\bar{j}}^*(w):] + [:\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j+1}}^*(w):].
\end{aligned}$$

We use Proposition 41 to compute these brackets. The nonzero terms are:

$$\begin{aligned}
&= (-\langle \varepsilon_1^*, \varepsilon_j \rangle : \beta(w)\varepsilon_j^*(w) : - \langle \beta, \varepsilon_j^* \rangle : \varepsilon_1^*(w)\varepsilon_j(w) :) \delta(z+w) - \langle \varepsilon_1^*, \varepsilon_j \rangle \langle \beta, \varepsilon_j^* \rangle \partial_w \delta(z+w) \\
&\quad - \{ (\langle \beta^*, \varepsilon_j \rangle : \varepsilon_1(w)\varepsilon_j^*(w) : + \langle \varepsilon_1, \varepsilon_j^* \rangle : \beta^*(w)\varepsilon_j(w) :) \delta(z-w) + \langle \beta^*, \varepsilon_j \rangle \langle \varepsilon_1, \varepsilon_j^* \rangle \partial_w \delta(z-w) \} \\
&\quad + \left(-\langle \varepsilon_{\bar{1}}^*, \varepsilon_{\bar{j}} \rangle : \bar{\beta}(w)\varepsilon_{\bar{j}}^*(w) : - \langle \bar{\beta}, \varepsilon_{\bar{j}}^* \rangle : \varepsilon_{\bar{1}}^*(w)\varepsilon_{\bar{j}}(w) : \right) \delta(z+w) - \langle \varepsilon_{\bar{1}}^*, \varepsilon_{\bar{j}} \rangle \langle \bar{\beta}, \varepsilon_{\bar{j}}^* \rangle \partial_w \delta(z+w) \\
&\quad - \left\{ \left(\langle \bar{\beta}^*, \varepsilon_{\bar{j}} \rangle : \varepsilon_{\bar{1}}(w)\varepsilon_{\bar{j}}^*(w) : + \langle \varepsilon_{\bar{1}}, \varepsilon_{\bar{j}}^* \rangle : \bar{\beta}^*(w)\varepsilon_{\bar{j}}(w) : \right) \delta(z-w) + \langle \bar{\beta}^*, \varepsilon_{\bar{j}} \rangle \langle \varepsilon_{\bar{1}}, \varepsilon_{\bar{j}}^* \rangle \partial_w \delta(z-w) \right\}.
\end{aligned}$$

By Remark 5.1, we see that every pair of normally ordered products cancels, and by antisymmetry of the form, we have

$$\begin{aligned}
&= 2\delta_{j1}(\partial_w \delta(z-w) + \partial_w \delta(z+w)) \\
&= -\delta_{j1}(\partial_w \delta(z-w) + \partial_w \delta(z+w))k' \\
&= \rho'([\alpha_0^{\bar{\sigma}}(z), \alpha_m^{\bar{\sigma}}(w)]).
\end{aligned}$$

The final case is $m = n$.

$$\begin{aligned}
&[\rho(\alpha_0^{\bar{\sigma}}(z)), \rho(\alpha_m^{\bar{\sigma}}(w))] \\
&= [:\varepsilon_1^*(-z)\beta(-z):, - :\varepsilon_1(z)\beta^*(z):, - :\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z):, + :\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, \\
&\quad :\varepsilon_n(w)\varepsilon_n^*(w):, - :\varepsilon_n(-w)\varepsilon_n^*(-w):, - :\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w):, + :\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w):] \\
&= [:\varepsilon_1^*(-z)\beta(-z):, :\varepsilon_n(w)\varepsilon_n^*(w):] - [:\varepsilon_1^*(-z)\beta(-z):, :\varepsilon_n(-w)\varepsilon_n^*(-w):] \\
&\quad - [:\varepsilon_1^*(-z)\beta(-z):, :\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w):] + [:\varepsilon_1^*(-z)\beta(-z):, :\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w):] \\
&\quad - [:\varepsilon_1(z)\beta^*(z):, :\varepsilon_n(w)\varepsilon_n^*(w):] + [:\varepsilon_1(z)\beta^*(z):, :\varepsilon_n(-w)\varepsilon_n^*(-w):] \\
&\quad + [:\varepsilon_1(z)\beta^*(z):, :\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w):] - [:\varepsilon_1(z)\beta^*(z):, :\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w):] \\
&\quad - [:\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z):, :\varepsilon_n(w)\varepsilon_n^*(w):] + [:\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z):, :\varepsilon_n(-w)\varepsilon_n^*(-w):] \\
&\quad + [:\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z):, :\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w):] - [:\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z):, :\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w):] \\
&\quad + [:\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, :\varepsilon_n(w)\varepsilon_n^*(w):] - [:\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, :\varepsilon_n(-w)\varepsilon_n^*(-w):] \\
&\quad - [:\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, :\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w):] + [:\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, :\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w):].
\end{aligned}$$

All antisymmetric forms $\langle \cdot, \cdot \rangle$ (and hence contractions) are trivially 0, so this entire calculation is:

$$\begin{aligned}
&= 0 \\
&= \rho'([\alpha_0^{\bar{\sigma}}(z), \alpha_m^{\bar{\sigma}}(w)]).
\end{aligned}$$

$$\begin{aligned}
& \text{Relation (2): } [\alpha_i^{\bar{\sigma}}(z), \alpha_j^{\bar{\sigma}}(w)] = a_{ij} \partial_w \delta(z - w) k' \\
& [\rho'(\alpha_i^{\bar{\sigma}}(z)), \rho'(\alpha_j^{\bar{\sigma}}(w))] \\
& = [:\varepsilon_i(z)\varepsilon_i^*(z): - :\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z): - :\varepsilon_{\bar{i}}(z)\varepsilon_{\bar{i}}^*(z): + :\varepsilon_{\bar{i+1}}(z)\varepsilon_{\bar{i+1}}^*(z):, \\
& :\varepsilon_j(w)\varepsilon_j^*(w): - :\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w): - :\varepsilon_{\bar{j}}(w)\varepsilon_{\bar{j}}^*(w): + :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j+1}}^*(w):] \\
& = [:\varepsilon_i(z)\varepsilon_i^*(z):, :\varepsilon_j(w)\varepsilon_j^*(w):] - [:\varepsilon_i(z)\varepsilon_i^*(z):, :\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w):] \\
& - [:\varepsilon_i(z)\varepsilon_i^*(z):, :\varepsilon_{\bar{j}}(w)\varepsilon_{\bar{j}}^*(w):] + [:\varepsilon_i(z)\varepsilon_i^*(z):, :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j+1}}^*(w):] \\
& - [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):, :\varepsilon_j(w)\varepsilon_j^*(w):] + [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):, :\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w):] \\
& + [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):, :\varepsilon_{\bar{j}}(w)\varepsilon_{\bar{j}}^*(w):] - [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):, :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j+1}}^*(w):] \\
& - [:\varepsilon_{\bar{i}}(z)\varepsilon_{\bar{i}}^*(z):, :\varepsilon_j(w)\varepsilon_j^*(w):] + [:\varepsilon_{\bar{i}}(z)\varepsilon_{\bar{i}}^*(z):, :\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w):] \\
& + [:\varepsilon_{\bar{i}}(z)\varepsilon_{\bar{i}}^*(z):, :\varepsilon_{\bar{j}}(w)\varepsilon_{\bar{j}}^*(w):] - [:\varepsilon_{\bar{i}}(z)\varepsilon_{\bar{i}}^*(z):, :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j+1}}^*(w):] \\
& + [:\varepsilon_{\bar{i+1}}(z)\varepsilon_{\bar{i+1}}^*(z):, :\varepsilon_j(w)\varepsilon_j^*(w):] - [:\varepsilon_{\bar{i+1}}(z)\varepsilon_{\bar{i+1}}^*(z):, :\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w):] \\
& - [:\varepsilon_{\bar{i+1}}(z)\varepsilon_{\bar{i+1}}^*(z):, :\varepsilon_{\bar{j}}(w)\varepsilon_{\bar{j}}^*(w):] + [:\varepsilon_{\bar{i+1}}(z)\varepsilon_{\bar{i+1}}^*(z):, :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j+1}}^*(w):].
\end{aligned}$$

Here we use Proposition 41 which allows us to compute the brackets. The nonzero terms are:

$$\begin{aligned}
& (\langle \varepsilon_i, \varepsilon_j^* \rangle : \varepsilon_i^*(w)\varepsilon_j(w) : + \langle \varepsilon_i^*, \varepsilon_j \rangle : \varepsilon_i(w)\varepsilon_j^*(w) :) \delta(z - w) + \langle \varepsilon_i, \varepsilon_j^* \rangle \langle \varepsilon_i^*, \varepsilon_j \rangle \partial_w \delta(z - w) \\
& - \{ (\langle \varepsilon_i, \varepsilon_{j+1}^* \rangle : \varepsilon_i^*(w)\varepsilon_{j+1}(w) : + \langle \varepsilon_i^*, \varepsilon_{j+1} \rangle : \varepsilon_i(w)\varepsilon_{j+1}^*(w) :) \delta(z - w) \\
& + \langle \varepsilon_i, \varepsilon_{j+1}^* \rangle \langle \varepsilon_i^*, \varepsilon_{j+1} \rangle \partial_w \delta(z - w) \} \\
& - \{ (\langle \varepsilon_{i+1}, \varepsilon_j^* \rangle : \varepsilon_{i+1}^*(w)\varepsilon_j(w) : + \langle \varepsilon_{i+1}^*, \varepsilon_j \rangle : \varepsilon_{i+1}(w)\varepsilon_j^*(w) :) \delta(z - w) \\
& + \langle \varepsilon_{i+1}, \varepsilon_j^* \rangle \langle \varepsilon_{i+1}^*, \varepsilon_j \rangle \partial_w \delta(z - w) \} \\
& + (\langle \varepsilon_{i+1}, \varepsilon_{j+1}^* \rangle : \varepsilon_{i+1}^*(w)\varepsilon_{j+1}(w) : + \langle \varepsilon_{i+1}^*, \varepsilon_{j+1} \rangle : \varepsilon_{i+1}(w)\varepsilon_{j+1}^*(w) :) \delta(z - w) \\
& + \langle \varepsilon_{i+1}, \varepsilon_{j+1}^* \rangle \langle \varepsilon_{i+1}^*, \varepsilon_{j+1} \rangle \partial_w \delta(z - w) \\
& + \left(\langle \varepsilon_{\bar{i}}, \varepsilon_{\bar{j}}^* \rangle : \varepsilon_{\bar{i}}^*(w)\varepsilon_{\bar{j}}(w) : + \langle \varepsilon_{\bar{i}}^*, \varepsilon_{\bar{j}} \rangle : \varepsilon_{\bar{i}}(w)\varepsilon_{\bar{j}}^*(w) : \right) \delta(z - w) + \langle \varepsilon_{\bar{i}}, \varepsilon_{\bar{j}}^* \rangle \langle \varepsilon_{\bar{i}}^*, \varepsilon_{\bar{j}} \rangle \partial_w \delta(z - w) \\
& - \{ \left(\langle \varepsilon_{\bar{i}}, \varepsilon_{\bar{j+1}}^* \rangle : \varepsilon_{\bar{i}}^*(w)\varepsilon_{\bar{j+1}}(w) : + \langle \varepsilon_{\bar{i}}^*, \varepsilon_{\bar{j+1}} \rangle : \varepsilon_{\bar{i}}(w)\varepsilon_{\bar{j+1}}^*(w) : \right) \delta(z - w) \\
& + \langle \varepsilon_{\bar{i}}, \varepsilon_{\bar{j+1}}^* \rangle \langle \varepsilon_{\bar{i}}^*, \varepsilon_{\bar{j+1}} \rangle \partial_w \delta(z - w) \} \\
& - \{ \left(\langle \varepsilon_{\bar{i+1}}, \varepsilon_{\bar{j}}^* \rangle : \varepsilon_{\bar{i+1}}^*(w)\varepsilon_{\bar{j}}(w) : + \langle \varepsilon_{\bar{i+1}}^*, \varepsilon_{\bar{j}} \rangle : \varepsilon_{\bar{i+1}}(w)\varepsilon_{\bar{j}}^*(w) : \right) \delta(z - w) \\
& + \langle \varepsilon_{\bar{i+1}}, \varepsilon_{\bar{j}}^* \rangle \langle \varepsilon_{\bar{i+1}}^*, \varepsilon_{\bar{j}} \rangle \partial_w \delta(z - w) \} \\
& + \left(\langle \varepsilon_{\bar{i+1}}, \varepsilon_{\bar{j+1}}^* \rangle : \varepsilon_{\bar{i+1}}^*(w)\varepsilon_{\bar{j+1}}(w) : + \langle \varepsilon_{\bar{i+1}}^*, \varepsilon_{\bar{j+1}} \rangle : \varepsilon_{\bar{i+1}}(w)\varepsilon_{\bar{j+1}}^*(w) : \right) \delta(z - w) \\
& + \langle \varepsilon_{\bar{i+1}}, \varepsilon_{\bar{j+1}}^* \rangle \langle \varepsilon_{\bar{i+1}}^*, \varepsilon_{\bar{j+1}} \rangle \partial_w \delta(z - w).
\end{aligned}$$

By antisymmetry of the form, we have

$$\begin{aligned}
& = \delta_{ij} (- :\varepsilon_j^*(w)\varepsilon_j(w) : + :\varepsilon_j(w)\varepsilon_j^*(w) :) \delta(z - w) - \delta_{ij} \partial_w \delta(z - w) \\
& + \delta_{i,j+1} (:\varepsilon_{j+1}^*(w)\varepsilon_{j+1}(w) : - :\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w) :) \delta(z - w) + \delta_{i,j+1} \partial_w \delta(z - w) \\
& + \delta_{i+1,j} (:\varepsilon_j^*(w)\varepsilon_j(w) : - :\varepsilon_j^*(w)\varepsilon_j^*(w) :) \delta(z - w) + \delta_{i+1,j} \partial_w \delta(z - w) \\
& + \delta_{ij} (- :\varepsilon_{j+1}^*(w)\varepsilon_{j+1}(w) : + :\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w) :) \delta(z - w) - \delta_{ij} \partial_w \delta(z - w) \\
& + \delta_{ij} (- :\varepsilon_{\bar{j}}^*(w)\varepsilon_{\bar{j}}(w) : + :\varepsilon_{\bar{j}}(w)\varepsilon_{\bar{j}}^*(w) :) \delta(z - w) - \delta_{ij} \partial_w \delta(z - w) \\
& + \delta_{i,j+1} (:\varepsilon_{\bar{j+1}}^*(w)\varepsilon_{\bar{j+1}}(w) : - :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j+1}}^*(w) :) \delta(z - w) + \delta_{i,j+1} \partial_w \delta(z - w)
\end{aligned}$$

$$+\delta_{i+1,j}(:\varepsilon_{\bar{j}}^*(w)\varepsilon_{\bar{j}}(w):-:\varepsilon_{\bar{j}}^*(w)\varepsilon_{\bar{j}}^*(w):)\delta(z-w)+\delta_{i+1,j}\partial_w\delta(z-w) \\ +\delta_{ij}(-:\varepsilon_{j+1}^*(w)\varepsilon_{j+1}(w):+:\varepsilon_{j+1}(w)\varepsilon_{j+1}^*(w):)\delta(z-w)-\delta_{ij}\partial_w\delta(z-w).$$

Notice that we have used the $\delta_{ij}, \delta_{i,j+1}, \delta_{i+1,j}$ to put all indices in terms of j . Each pair of normally ordered products (that is, each pair inside of parentheses) cancels. All that remains is:

$$= -2(2\delta_{ij}\partial_w\delta(z-w)-\delta_{i,j+1}\partial_w\delta(z-w)-\delta_{i+1,j}\partial_w\delta(z-w)) \\ = (2\delta_{ij}-\delta_{i,j+1}-\delta_{i+1,j})\partial_w\delta(z-w)k'.$$

This coefficient is precisely a_{ij} . Thus we have:

$$= a_{ij}\partial_w\delta(z-w)k'. \\ = \rho'([\alpha_i^{\bar{\sigma}}(z), \alpha_j^{\bar{\sigma}}(w)]).$$

Relation (3): $[\alpha_i^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(w)] = a_{in}(\partial_w\delta(z-w) + \partial_w\delta(z+w))k'$

$$[\rho'(\alpha_i^{\bar{\sigma}}(z)), \rho'(\alpha_n^{\bar{\sigma}}(w))] \\ = [:\varepsilon_i(z)\varepsilon_i^*(z):-:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):-:\varepsilon_{\bar{i}}(z)\varepsilon_{\bar{i}}^*(z):+:\varepsilon_{\bar{i+1}}(z)\varepsilon_{\bar{i+1}}^*(z):, \\ :\varepsilon_n(w)\varepsilon_n^*(w):-:\varepsilon_n(-w)\varepsilon_n^*(-w):-:\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w):+:\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w):] \\ = [:\varepsilon_i(z)\varepsilon_i^*(z):-:\varepsilon_n(w)\varepsilon_n^*(w):]-[:\varepsilon_i(z)\varepsilon_i^*(z):,\varepsilon_n(-w)\varepsilon_n^*(-w):] \\ -[:\varepsilon_i(z)\varepsilon_i^*(z):,\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w):]+[:\varepsilon_i(z)\varepsilon_i^*(z):,\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w):] \\ -[:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):,\varepsilon_n(w)\varepsilon_n^*(w):]+[:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):,\varepsilon_n(-w)\varepsilon_n^*(-w):] \\ +[:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):,\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w):]-[:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):,\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w):] \\ -[:\varepsilon_{\bar{i}}(z)\varepsilon_{\bar{i}}^*(z):,\varepsilon_n(w)\varepsilon_n^*(w):]+[:\varepsilon_{\bar{i}}(z)\varepsilon_{\bar{i}}^*(z):,\varepsilon_n(-w)\varepsilon_n^*(-w):] \\ +[:\varepsilon_{\bar{i}}(z)\varepsilon_{\bar{i}}^*(z):,\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w):]-[:\varepsilon_{\bar{i}}(z)\varepsilon_{\bar{i}}^*(z):,\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w):] \\ +[:\varepsilon_{\bar{i+1}}(z)\varepsilon_{\bar{i+1}}^*(z):,\varepsilon_n(w)\varepsilon_n^*(w):]-[:\varepsilon_{\bar{i+1}}(z)\varepsilon_{\bar{i+1}}^*(z):,\varepsilon_n(-w)\varepsilon_n^*(-w):] \\ -[:\varepsilon_{\bar{i+1}}(z)\varepsilon_{\bar{i+1}}^*(z):,\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w):]+[:\varepsilon_{\bar{i+1}}(z)\varepsilon_{\bar{i+1}}^*(z):,\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w):].$$

By Proposition 41, the nonzero terms are:

$$-\{(\langle\varepsilon_{i+1}, \varepsilon_n^*\rangle : \varepsilon_{i+1}^*(w)\varepsilon_n(w): + \langle\varepsilon_{i+1}^*, \varepsilon_n\rangle : \varepsilon_{i+1}(w)\varepsilon_n^*(w):)\delta(z-w) \\ + \langle\varepsilon_{i+1}, \varepsilon_n^*\rangle \langle\varepsilon_{i+1}^*, \varepsilon_n\rangle \partial_w\delta(z-w)\} \\ + (\langle\varepsilon_{i+1}, \varepsilon_n^*\rangle : \varepsilon_{i+1}^*(-w)\varepsilon_n(-w): + \langle\varepsilon_{i+1}^*, \varepsilon_n\rangle : \varepsilon_{i+1}(-w)\varepsilon_n^*(-w):)\delta(z+w) \\ - \langle\varepsilon_{i+1}, \varepsilon_n^*\rangle \langle\varepsilon_{i+1}^*, \varepsilon_n\rangle \partial_w\delta(z+w) \\ - \{(\langle\varepsilon_{\bar{i+1}}, \varepsilon_{\bar{n}}^*\rangle : \varepsilon_{\bar{i+1}}^*(w)\varepsilon_{\bar{n}}(w): + \langle\varepsilon_{\bar{i+1}}^*, \varepsilon_{\bar{n}}\rangle : \varepsilon_{\bar{i+1}}(w)\varepsilon_{\bar{n}}^*(w):)\delta(z-w) \\ + \langle\varepsilon_{\bar{i+1}}, \varepsilon_{\bar{n}}^*\rangle \langle\varepsilon_{\bar{i+1}}^*, \varepsilon_{\bar{n}}\rangle \partial_w\delta(z-w)\} \\ + (\langle\varepsilon_{\bar{i+1}}, \varepsilon_{\bar{n}}^*\rangle : \varepsilon_{\bar{i+1}}^*(-w)\varepsilon_{\bar{n}}(-w): + \langle\varepsilon_{\bar{i+1}}^*, \varepsilon_{\bar{n}}\rangle : \varepsilon_{\bar{i+1}}(-w)\varepsilon_{\bar{n}}^*(-w):)\delta(z+w) \\ - \langle\varepsilon_{\bar{i+1}}, \varepsilon_{\bar{n}}^*\rangle \langle\varepsilon_{\bar{i+1}}^*, \varepsilon_{\bar{n}}\rangle \partial_w\delta(z+w).$$

Using antisymmetry of the form, we have

$$= \delta_{i+1,n} \{(:\varepsilon_n^*(w)\varepsilon_n(w):-:\varepsilon_n(w)\varepsilon_n^*(w):)\delta(z-w) + \partial_w\delta(z-w)\} \\ + \delta_{i+1,n} \{(-:\varepsilon_n^*(-w)\varepsilon_n(-w):+:\varepsilon_n(-w)\varepsilon_n^*(-w):)\delta(z+w) + \partial_w\delta(z+w)\} \\ + \delta_{i+1,n} \{(:\varepsilon_{\bar{n}}^*(w)\varepsilon_{\bar{n}}(w):-:\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w):)\delta(z-w) + \partial_w\delta(z-w)\}$$

$$+ \delta_{i+1,n} \{ (- : \varepsilon_n^*(-w) \varepsilon_{\bar{n}}(-w) : + : \varepsilon_{\bar{n}}(-w) \varepsilon_{\bar{n}}^*(-w) :) \delta(z+w) + \partial_w \delta(z+w) \}.$$

Notice that we have used the $\delta_{i+1,n}$ to put all indices in terms of j . Each pair of normally ordered products (that is, each pair inside of parentheses) cancels. All that remains is:

$$\begin{aligned} &= 2\delta_{i+1,n}(\partial_w \delta(z-w) + \partial_w \delta(z+w)) \\ &= -\delta_{i+1,n}(\partial_w \delta(z-w) + \partial_w \delta(z+w))k' \\ &= a_{in}(\partial_w \delta(z-w) + \partial_w \delta(z+w))k' \\ &= \rho'([\alpha_i^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(w)]). \end{aligned}$$

$$\begin{aligned} \text{Relation (4): } & [\alpha_n^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(w)] = a_{nn}(\partial_w \delta(z-w) + \partial_w \delta(z+w))k' \\ & [\rho'(\alpha_n^{\bar{\sigma}}(z)), \rho'(\alpha_n^{\bar{\sigma}}(w))] \\ &= [:\varepsilon_n(z)\varepsilon_n^*(z) : - :\varepsilon_n(-z)\varepsilon_n^*(-z) : - :\varepsilon_{\bar{n}}(z)\varepsilon_{\bar{n}}^*(z) : + :\varepsilon_{\bar{n}}(-z)\varepsilon_{\bar{n}}^*(-z) :] \\ & :\varepsilon_n(w)\varepsilon_n^*(w) : - :\varepsilon_n(-w)\varepsilon_n^*(-w) : - :\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w) : + :\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w) :] \\ &= [:\varepsilon_n(z)\varepsilon_n^*(z) :, :\varepsilon_n(w)\varepsilon_n^*(w) :] - [:\varepsilon_n(z)\varepsilon_n^*(z) :, :\varepsilon_n(-w)\varepsilon_n^*(-w) :] \\ & - [:\varepsilon_n(z)\varepsilon_n^*(z) :, :\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w) :] + [:\varepsilon_n(z)\varepsilon_n^*(z) :, :\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w) :] \\ & - [:\varepsilon_n(-z)\varepsilon_n^*(-z) :, :\varepsilon_n(w)\varepsilon_n^*(w) :] + [:\varepsilon_n(-z)\varepsilon_n^*(-z) :, :\varepsilon_n(-w)\varepsilon_n^*(-w) :] \\ & + [:\varepsilon_n(-z)\varepsilon_n^*(-z) :, :\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w) :] - [:\varepsilon_n(-z)\varepsilon_n^*(-z) :, :\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w) :] \\ & - [:\varepsilon_{\bar{n}}(z)\varepsilon_{\bar{n}}^*(z) :, :\varepsilon_n(w)\varepsilon_n^*(w) :] + [:\varepsilon_{\bar{n}}(z)\varepsilon_{\bar{n}}^*(z) :, :\varepsilon_n(-w)\varepsilon_n^*(-w) :] \\ & + [:\varepsilon_{\bar{n}}(z)\varepsilon_{\bar{n}}^*(z) :, :\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w) :] - [:\varepsilon_{\bar{n}}(z)\varepsilon_{\bar{n}}^*(z) :, :\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w) :] \\ & + [:\varepsilon_{\bar{n}}(-z)\varepsilon_{\bar{n}}^*(-z) :, :\varepsilon_n(w)\varepsilon_n^*(w) :] - [:\varepsilon_{\bar{n}}(-z)\varepsilon_{\bar{n}}^*(-z) :, :\varepsilon_n(-w)\varepsilon_n^*(-w) :] \\ & - [:\varepsilon_{\bar{n}}(-z)\varepsilon_{\bar{n}}^*(-z) :, :\varepsilon_{\bar{n}}(w)\varepsilon_{\bar{n}}^*(w) :] + [:\varepsilon_{\bar{n}}(-z)\varepsilon_{\bar{n}}^*(-z) :, :\varepsilon_{\bar{n}}(-w)\varepsilon_{\bar{n}}^*(-w)]. \end{aligned}$$

We use Proposition 41 to compute each bracket. The nonzero terms are:

$$\begin{aligned} & (\langle \varepsilon_n, \varepsilon_n^* \rangle : \varepsilon_n^*(w) \varepsilon_n(w) : + \langle \varepsilon_n^*, \varepsilon_n \rangle : \varepsilon_n(w) \varepsilon_n^*(w) :) \delta(z-w) + \langle \varepsilon_n, \varepsilon_n^* \rangle \langle \varepsilon_n^*, \varepsilon_n \rangle \partial_w \delta(z-w) \\ & - \{ (\langle \varepsilon_n, \varepsilon_n^* \rangle : \varepsilon_n^*(-w) \varepsilon_n(-w) : + \langle \varepsilon_n^*, \varepsilon_n \rangle : \varepsilon_n(-w) \varepsilon_n^*(-w) :) \delta(z+w) \\ & - \langle \varepsilon_n, \varepsilon_n^* \rangle \langle \varepsilon_n^*, \varepsilon_n \rangle \partial_w \delta(z+w) \} \\ & - \{ (-\langle \varepsilon_n, \varepsilon_n^* \rangle : \varepsilon_n^*(w) \varepsilon_n(w) : - \langle \varepsilon_n^*, \varepsilon_n \rangle : \varepsilon_n(w) \varepsilon_n^*(w) :) \delta(z+w) - \langle \varepsilon_n, \varepsilon_n^* \rangle \langle \varepsilon_n^*, \varepsilon_n \rangle \partial_w \delta(z+w) \} \\ & + (-\langle \varepsilon_n, \varepsilon_n^* \rangle : \varepsilon_n^*(-w) \varepsilon_n(-w) : - \langle \varepsilon_n^*, \varepsilon_n \rangle : \varepsilon_n(-w) \varepsilon_n^*(-w) :) \delta(z-w) \\ & + \langle \varepsilon_n, \varepsilon_n^* \rangle \langle \varepsilon_n^*, \varepsilon_n \rangle \partial_w \delta(z-w) \\ & + (\langle \varepsilon_{\bar{n}}, \varepsilon_{\bar{n}}^* \rangle : \varepsilon_{\bar{n}}^*(w) \varepsilon_{\bar{n}}(w) : + \langle \varepsilon_{\bar{n}}^*, \varepsilon_{\bar{n}} \rangle : \varepsilon_{\bar{n}}(w) \varepsilon_{\bar{n}}^*(w) :) \delta(z-w) + \langle \varepsilon_{\bar{n}}, \varepsilon_{\bar{n}}^* \rangle \langle \varepsilon_{\bar{n}}^*, \varepsilon_{\bar{n}} \rangle \partial_w \delta(z-w) \\ & - \{ (\langle \varepsilon_{\bar{n}}, \varepsilon_{\bar{n}}^* \rangle : \varepsilon_{\bar{n}}^*(-w) \varepsilon_{\bar{n}}(-w) : + \langle \varepsilon_{\bar{n}}^*, \varepsilon_{\bar{n}} \rangle : \varepsilon_{\bar{n}}(-w) \varepsilon_{\bar{n}}^*(-w) :) \delta(z+w) \\ & - \langle \varepsilon_{\bar{n}}, \varepsilon_{\bar{n}}^* \rangle \langle \varepsilon_{\bar{n}}^*, \varepsilon_{\bar{n}} \rangle \partial_w \delta(z+w) \} \\ & - \{ (-\langle \varepsilon_{\bar{n}}, \varepsilon_{\bar{n}}^* \rangle : \varepsilon_{\bar{n}}^*(w) \varepsilon_{\bar{n}}(w) : - \langle \varepsilon_{\bar{n}}^*, \varepsilon_{\bar{n}} \rangle : \varepsilon_{\bar{n}}(w) \varepsilon_{\bar{n}}^*(w) :) \delta(z+w) - \langle \varepsilon_{\bar{n}}, \varepsilon_{\bar{n}}^* \rangle \langle \varepsilon_{\bar{n}}^*, \varepsilon_{\bar{n}} \rangle \partial_w \delta(z+w) \} \\ & + (-\langle \varepsilon_{\bar{n}}, \varepsilon_{\bar{n}}^* \rangle : \varepsilon_{\bar{n}}^*(-w) \varepsilon_{\bar{n}}(-w) : - \langle \varepsilon_{\bar{n}}^*, \varepsilon_{\bar{n}} \rangle : \varepsilon_{\bar{n}}(-w) \varepsilon_{\bar{n}}^*(-w) :) \delta(z-w) + \langle \varepsilon_{\bar{n}}, \varepsilon_{\bar{n}}^* \rangle \langle \varepsilon_{\bar{n}}^*, \varepsilon_{\bar{n}} \rangle \partial_w \delta(z-w). \end{aligned}$$

By antisymmetry of the form, each pair of normally ordered products cancels. We are left with:

$$\begin{aligned} & = -4(\partial_w \delta(z-w) + \partial_w \delta(z+w)) \\ & = 2(\partial_w \delta(z-w) + \partial_w \delta(z+w))k' \\ & = a_{nn}(\partial_w \delta(z-w) + \partial_w \delta(z+w))k' \end{aligned}$$

$$= \rho'([\alpha_n^{\bar{\sigma}}(z), \alpha_n^{\bar{\sigma}}(w)]).$$

Relation (5): $[\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_m, w)] = \pm(2\delta_{m0} - \delta_{m1})X^{\bar{\sigma}}(\pm\alpha_m, w)(\delta(z-w) + \delta(z+w))$

The first case to consider is $m = 0$. We calculate as follows, first for $X^{\bar{\sigma}}(\alpha_m, w)$:

$$\begin{aligned} & [\rho'(\alpha_0^{\bar{\sigma}}(z)), \rho'(X^{\bar{\sigma}}(\alpha_m, w))] \\ &= [:\varepsilon_1^*(-z)\beta(-z): - :\varepsilon_1(z)\beta^*(z): - :\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z): + :\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, \\ & : \varepsilon_1^*(w)\bar{\beta}(-w): - :\varepsilon_1^*(-w)\bar{\beta}(w):] \\ &= [:\varepsilon_1^*(-z)\beta(-z):, : \varepsilon_1^*(w)\bar{\beta}(-w):] - [:\varepsilon_1^*(-z)\beta(-z):, : \varepsilon_1^*(-w)\bar{\beta}(w):] \\ &\quad - [:\varepsilon_1(z)\beta^*(z):, : \varepsilon_1^*(w)\bar{\beta}(-w):] + [:\varepsilon_1(z)\beta^*(z):, : \varepsilon_1^*(-w)\bar{\beta}(w):] \\ &\quad - [:\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z):, : \varepsilon_{\bar{1}}^*(w)\bar{\beta}(-w):] + [:\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z):, : \varepsilon_{\bar{1}}^*(-w)\bar{\beta}(w):] \\ &\quad + [:\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, : \varepsilon_{\bar{1}}^*(w)\bar{\beta}(-w):] - [:\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, : \varepsilon_{\bar{1}}^*(-w)\bar{\beta}(w):]. \end{aligned}$$

Using Proposition 41, the nonzero terms are:

$$\begin{aligned} & = -\langle \beta, \varepsilon_1^* \rangle : \varepsilon_1^*(w)\bar{\beta}(-w): \delta(z+w) - (-\langle \beta, \varepsilon_1^* \rangle) : \varepsilon_1^*(-w)\bar{\beta}(w): \delta(z-w) \\ & - \langle \varepsilon_1, \varepsilon_1^* \rangle : \beta^*(w)\bar{\beta}(-w): \delta(z-w) + \langle \varepsilon_1, \varepsilon_1^* \rangle : \beta^*(-w)\bar{\beta}(w): \delta(z+w) \\ & - (-\langle \varepsilon_{\bar{1}}^*, \bar{\beta} \rangle) : \bar{\beta}(-w)\varepsilon_1^*(w): \delta(z-w) - \langle \varepsilon_{\bar{1}}^*, \bar{\beta} \rangle : \bar{\beta}(w)\varepsilon_1^*(-w): \delta(z+w) \\ & + \langle \bar{\beta}^*, \bar{\beta} \rangle : \varepsilon_{\bar{1}}(-w)\varepsilon_1^*(w): \delta(z+w) - \langle \bar{\beta}^*, \bar{\beta} \rangle : \varepsilon_{\bar{1}}(w)\varepsilon_1^*(-w): \delta(z-w). \end{aligned}$$

By antisymmetry of the form, we have

$$\begin{aligned} & = : \varepsilon_1^*(w)\bar{\beta}(-w): \delta(z+w) - : \varepsilon_1^*(-w)\bar{\beta}(w): \delta(z-w) \\ & + : \beta^*(w)\bar{\beta}(-w): \delta(z-w) - : \beta^*(-w)\bar{\beta}(w): \delta(z+w) \\ & + : \bar{\beta}(-w)\varepsilon_1^*(w): \delta(z-w) - : \bar{\beta}(w)\varepsilon_1^*(-w): \delta(z+w) \\ & + : \varepsilon_{\bar{1}}(-w)\varepsilon_1^*(w): \delta(z+w) - : \varepsilon_{\bar{1}}(w)\varepsilon_1^*(-w): \delta(z-w). \end{aligned}$$

Using Remark 5.1 yields

$$\begin{aligned} & = 2(: \varepsilon_1^*(w)\bar{\beta}(-w): - : \varepsilon_1^*(-w)\bar{\beta}(w):)(\delta(z-w) + \delta(z+w)) \\ & = 2\rho'(X^{\bar{\sigma}}(\alpha_m, w))(\delta(z-w) + \delta(z+w)) \\ & = \rho'([\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_m, w)]). \end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_m, w)$.

Now consider the case for $1 \leq m \leq n-1$. For $X^{\bar{\sigma}}(\alpha_m, w)$,

$$\begin{aligned} & [\rho'(\alpha_0^{\bar{\sigma}}(z)), \rho'(X^{\bar{\sigma}}(\alpha_m, w))] \\ &= [:\varepsilon_1^*(-z)\beta(-z): - :\varepsilon_1(z)\beta^*(z): - :\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z): + :\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, \\ & : \varepsilon_m(w)\varepsilon_{m+1}^*(w): - : \varepsilon_{\bar{m}+1}(w)\varepsilon_{\bar{m}}^*(w):] \\ &= [:\varepsilon_1^*(-z)\beta(-z):, : \varepsilon_m(w)\varepsilon_{m+1}^*(w):] - [:\varepsilon_1^*(-z)\beta(-z):, : \varepsilon_{\bar{m}+1}(w)\varepsilon_{\bar{m}}^*(w):] \\ &\quad - [:\varepsilon_1(z)\beta^*(z):, : \varepsilon_m(w)\varepsilon_{m+1}^*(w):] + [:\varepsilon_1(z)\beta^*(z):, : \varepsilon_{\bar{m}+1}(w)\varepsilon_{\bar{m}}^*(w):] \\ &\quad - [:\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z):, : \varepsilon_m(w)\varepsilon_{m+1}^*(w):] + [:\varepsilon_{\bar{1}}^*(-z)\bar{\beta}(-z):, : \varepsilon_{\bar{m}+1}(w)\varepsilon_{\bar{m}}^*(w):] \\ &\quad + [:\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, : \varepsilon_m(w)\varepsilon_{m+1}^*(w):] - [:\varepsilon_{\bar{1}}(z)\bar{\beta}^*(z):, : \varepsilon_{\bar{m}+1}(w)\varepsilon_{\bar{m}}^*(w):]. \end{aligned}$$

Using Proposition 41, the possibly nonzero terms are:

$$= -\langle \varepsilon_1^*, \varepsilon_m \rangle : \beta(w)\varepsilon_{m+1}^*(w): \delta(z+w) - \langle \beta^*, \varepsilon_m \rangle : \varepsilon_1(w)\varepsilon_{m+1}^*(w): \delta(z-w)$$

$$-\langle \bar{\beta}, \varepsilon_{\bar{m}}^* \rangle : \varepsilon_{\bar{1}}^*(w) \varepsilon_{\bar{m+1}}(w) : \delta(z+w) - \langle \varepsilon_{\bar{1}}, \varepsilon_{\bar{m}}^* \rangle : \bar{\beta}^*(w) \varepsilon_{\bar{m+1}}(w) : \delta(z-w).$$

By the restrictions on the values of m , in fact $\delta_{m+1,1} = 0$. Using that and Remark 5.1 yields:

$$\begin{aligned} &= -\delta_{m1} : \beta(w) \varepsilon_{m+1}^*(w) : \delta(z+w) - \delta_{m1} : \beta(w) \varepsilon_{m+1}^*(w) : \delta(z-w) \\ &\quad + \delta_{m1} (: \varepsilon_{\bar{1}}^*(w) \varepsilon_{\bar{m+1}}(w) : \delta(z+w) + : \varepsilon_{\bar{1}}^*(w) \varepsilon_{\bar{m+1}}(w) : \delta(z-w)) \\ &= -\delta_{m1} (: \beta(w) \varepsilon_{m+1}^*(w) : - : \varepsilon_{\bar{1}}^*(w) \varepsilon_{\bar{m+1}}(w) :) (\delta(z-w) + \delta(z+w)) \\ &= -\delta_{m1} (: \varepsilon_m(w) \varepsilon_{m+1}^*(w) : - : \varepsilon_{\bar{m+1}}(w) \varepsilon_{\bar{m}}^*(w) :) (\delta(z-w) + \delta(z+w)) \\ &= -\delta_{m1} \rho'(X^{\bar{\sigma}}(\alpha_m, w)) (\delta(z-w) + \delta(z+w)) \\ &= \rho'([\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_m, w)]). \end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_m, w)$.

The final case in this relation is $m = n$. For $X^{\bar{\sigma}}(\alpha_m, w)$,

$$\begin{aligned} &[\rho'(\alpha_0^{\bar{\sigma}}(z)), \rho'(X^{\bar{\sigma}}(\alpha_m, w))] \\ &= [: \varepsilon_1^*(-z) \beta(-z) : - : \varepsilon_1(z) \beta^*(z) : - : \varepsilon_{\bar{1}}^*(-z) \bar{\beta}(-z) : + : \varepsilon_{\bar{1}}(z) \bar{\beta}^*(z) : , \\ &: \varepsilon_n(w) \varepsilon_{\bar{n}}^*(-w) : - : \varepsilon_n(-w) \varepsilon_{\bar{n}}^*(w) :]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is simply:

$$\begin{aligned} &= 0 \\ &= \rho'([\alpha_0^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_m, w)]). \end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_m, w)$.

Relation (6): $[\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm \alpha_0, w)] = \pm (-\delta_{i1}) X^{\bar{\sigma}}(\pm \alpha_0, w) (\delta(z-w) + \delta(z+w))$

We first calculate for $X^{\bar{\sigma}}(\alpha_0, w)$:

$$\begin{aligned} &[\rho'(\alpha_i^{\bar{\sigma}}(z)), \rho'(X^{\bar{\sigma}}(\alpha_0, w))] \\ &= [: \varepsilon_i(z) \varepsilon_i^*(z) : - : \varepsilon_{i+1}(z) \varepsilon_{i+1}^*(z) : - : \varepsilon_{\bar{i}}(z) \varepsilon_{\bar{i}}^*(z) : + : \varepsilon_{\bar{i+1}}(z) \varepsilon_{\bar{i+1}}^*(z) : , \\ &: \varepsilon_1^*(w) \bar{\beta}(-w) : - : \varepsilon_1^*(-w) \bar{\beta}(w) :] \\ &= [: \varepsilon_i(z) \varepsilon_i^*(z) : , : \varepsilon_1^*(w) \bar{\beta}(-w) :] - [: \varepsilon_i(z) \varepsilon_i^*(z) : , : \varepsilon_1^*(-w) \bar{\beta}(w) :] \\ &\quad - [: \varepsilon_{i+1}(z) \varepsilon_{i+1}^*(z) : , : \varepsilon_1^*(w) \bar{\beta}(-w) :] + [: \varepsilon_{i+1}(z) \varepsilon_{i+1}^*(z) : , : \varepsilon_1^*(-w) \bar{\beta}(w) :] \\ &\quad - [: \varepsilon_{\bar{i}}(z) \varepsilon_{\bar{i}}^*(z) : , : \varepsilon_1^*(w) \bar{\beta}(-w) :] + [: \varepsilon_{\bar{i}}(z) \varepsilon_{\bar{i}}^*(z) : , : \varepsilon_1^*(-w) \bar{\beta}(w) :] \\ &\quad + [: \varepsilon_{\bar{i+1}}(z) \varepsilon_{\bar{i+1}}^*(z) : , : \varepsilon_1^*(w) \bar{\beta}(-w) :] - [: \varepsilon_{\bar{i+1}}(z) \varepsilon_{\bar{i+1}}^*(z) : , : \varepsilon_1^*(-w) \bar{\beta}(w) :]. \end{aligned}$$

Using Proposition 41, the possibly nonzero terms are:

$$\begin{aligned} &= \langle \varepsilon_i, \varepsilon_1^* \rangle : \varepsilon_i^*(w) \bar{\beta}(-w) : \delta(z-w) - \langle \varepsilon_i, \varepsilon_1^* \rangle : \varepsilon_i^*(-w) \bar{\beta}(w) : \delta(z+w) \\ &\quad - \langle \varepsilon_{\bar{i}}^*, \bar{\beta} \rangle : \varepsilon_{\bar{i}}^*(-w) \varepsilon_1^*(w) : \delta(z+w) + \langle \varepsilon_{\bar{i}}^*, \bar{\beta} \rangle : \varepsilon_{\bar{i}}(w) \varepsilon_1^*(-w) : \delta(z-w). \end{aligned}$$

By antisymmetry of the form, we have:

$$\begin{aligned} &= -\delta_{i1} : \varepsilon_1^*(w) \bar{\beta}(-w) : \delta(z-w) + \delta_{i1} : \varepsilon_1^*(-w) \bar{\beta}(w) : \delta(z+w) \\ &\quad - \delta_{i1} : \varepsilon_{\bar{1}}(-w) \varepsilon_1^*(w) : \delta(z+w) + \delta_{i1} : \varepsilon_{\bar{1}}(w) \varepsilon_1^*(-w) : \delta(z-w) \\ &= -\delta_{i1} : \varepsilon_1^*(w) \bar{\beta}(-w) : \delta(z-w) + \delta_{i1} : \varepsilon_1^*(-w) \bar{\beta}(w) : \delta(z+w) \\ &\quad - \delta_{i1} : \varepsilon_1^*(w) \bar{\beta}(-w) : \delta(z+w) + \delta_{i1} : \varepsilon_1^*(-w) \bar{\beta}(w) : \delta(z-w) \end{aligned}$$

$$\begin{aligned}
&= -\delta_{i1} \left(: \varepsilon_1^*(w) \bar{\beta}(-w) : - : \varepsilon_1^*(-w) \bar{\beta}(w) : \right) (\delta(z-w) + \delta(z+w)) \\
&= -\delta_{i1} \rho' (X^{\bar{\sigma}}(\alpha_0, w)) (\delta(z-w) + \delta(z+w)) \\
&= \rho' ([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_0, w)]).
\end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_0, w)$.

Relation (7): $[\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_j, w)] = \pm a_{ij} X^{\bar{\sigma}}(\pm\alpha_j, w) \delta(z-w)$

We will first calculate for $X^{\bar{\sigma}}(\alpha_j, w)$.

$$\begin{aligned}
&[\rho'(\alpha_i^{\bar{\sigma}}(z)), \rho'(X^{\bar{\sigma}}(\alpha_j, w))] \\
&= [:\varepsilon_i(z)\varepsilon_i^*(z): - :\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z): - :\varepsilon_{\bar{i}}(z)\varepsilon_{\bar{i}}^*(z): + :\varepsilon_{\bar{i+1}}(z)\varepsilon_{\bar{i+1}}^*(z):, \\
&:\varepsilon_j(w)\varepsilon_{j+1}^*(w): - :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j}}^*(w):] \\
&= [:\varepsilon_i(z)\varepsilon_i^*(z):, :\varepsilon_j(w)\varepsilon_{j+1}^*(w):] - [:\varepsilon_i(z)\varepsilon_i^*(z):, :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j}}^*(w):] \\
&- [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):, :\varepsilon_j(w)\varepsilon_{j+1}^*(w):] + [:\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z):, :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j}}^*(w):] \\
&- [:\varepsilon_{\bar{i}}(z)\varepsilon_{\bar{i}}^*(z):, :\varepsilon_j(w)\varepsilon_{j+1}^*(w):] + [:\varepsilon_{\bar{i}}(z)\varepsilon_{\bar{i}}^*(z):, :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j}}^*(w):] \\
&+ [:\varepsilon_{\bar{i+1}}(z)\varepsilon_{\bar{i+1}}^*(z):, :\varepsilon_j(w)\varepsilon_{j+1}^*(w):] - [:\varepsilon_{\bar{i+1}}(z)\varepsilon_{\bar{i+1}}^*(z):, :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j}}^*(w):].
\end{aligned}$$

We use Proposition 41 to compute these brackets, showing only the terms that are (possibly) nonzero.

$$\begin{aligned}
&= (\langle \varepsilon_i, \varepsilon_{j+1}^* \rangle : \varepsilon_i^*(w) \varepsilon_j(w) : + \langle \varepsilon_i^*, \varepsilon_j \rangle : \varepsilon_i(w) \varepsilon_{j+1}^*(w) :) \delta(z-w) + \langle \varepsilon_i, \varepsilon_{j+1}^* \rangle \langle \varepsilon_i^*, \varepsilon_j \rangle \partial_w \delta(z-w) \\
&- \{ (\langle \varepsilon_{i+1}, \varepsilon_{j+1}^* \rangle : \varepsilon_{i+1}^*(w) \varepsilon_j(w) : + \langle \varepsilon_{i+1}^*, \varepsilon_j \rangle : \varepsilon_{i+1}(w) \varepsilon_{j+1}^*(w) :) \delta(z-w) \\
&+ \langle \varepsilon_{i+1}, \varepsilon_{j+1}^* \rangle \langle \varepsilon_{i+1}^*, \varepsilon_j \rangle \partial_w \delta(z-w) \} \\
&+ (\langle \varepsilon_{\bar{i}}, \varepsilon_{\bar{j}}^* \rangle : \varepsilon_{\bar{i}}^*(w) \varepsilon_{\bar{j+1}}(w) : + \langle \varepsilon_{\bar{i}}^*, \varepsilon_{\bar{j}} \rangle : \varepsilon_{\bar{i}}(w) \varepsilon_{\bar{j}}^*(w) :) \delta(z-w) + \langle \varepsilon_{\bar{i}}, \varepsilon_{\bar{j}}^* \rangle \langle \varepsilon_{\bar{i}}^*, \varepsilon_{\bar{j+1}} \rangle \partial_w \delta(z-w) \\
&- \{ (\langle \varepsilon_{\bar{i+1}}, \varepsilon_{\bar{j}}^* \rangle : \varepsilon_{\bar{i+1}}^*(w) \varepsilon_{\bar{j+1}}(w) : + \langle \varepsilon_{\bar{i+1}}^*, \varepsilon_{\bar{j+1}} \rangle : \varepsilon_{\bar{i+1}}(w) \varepsilon_{\bar{j}}^*(w) :) \delta(z-w) \\
&+ \langle \varepsilon_{\bar{i+1}}, \varepsilon_{\bar{j}}^* \rangle \langle \varepsilon_{\bar{i+1}}^*, \varepsilon_{\bar{j+1}} \rangle \partial_w \delta(z-w) \}.
\end{aligned}$$

Each $\partial_w \delta(z-w)$ terms has a multiple of $\delta_{ij} \delta_{i,j+1}$ or $\delta_{ij} \delta_{i+1,j}$ in front, each of which is 0. Also, by antisymmetry of the form, we have

$$\begin{aligned}
&= -\delta_{i,j+1} : \varepsilon_{j+1}^*(w) \varepsilon_j(w) : \delta(z-w) + \delta_{ij} : \varepsilon_j(w) \varepsilon_{j+1}^*(w) : \delta(z-w) \\
&+ \delta_{ij} : \varepsilon_{j+1}^*(w) \varepsilon_j(w) : \delta(z-w) - \delta_{i+1,j} : \varepsilon_j(w) \varepsilon_{j+1}^*(w) : \delta(z-w) \\
&- \delta_{ij} : \varepsilon_{\bar{j}}^*(w) \varepsilon_{\bar{j+1}}(w) : \delta(z-w) + \delta_{i,j+1} : \varepsilon_{\bar{j+1}}(w) \varepsilon_{\bar{j}}^*(w) : \delta(z-w) \\
&+ \delta_{i+1,j} : \varepsilon_{\bar{j}}^*(w) \varepsilon_{\bar{j+1}}(w) : \delta(z-w) - \delta_{ij} : \varepsilon_{\bar{j+1}}(w) \varepsilon_{\bar{j}}^*(w) : \delta(z-w) \\
&= (-\delta_{i,j+1} + 2\delta_{ij} - \delta_{i+1,j}) (:\varepsilon_j(w)\varepsilon_{j+1}^*(w): - :\varepsilon_{\bar{j+1}}(w)\varepsilon_{\bar{j}}^*(w):) \delta(z-w) \\
&= a_{ij} \rho' (X^{\bar{\sigma}}(\alpha_j, w)) \delta(z-w) \\
&= \rho' ([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_j, w)]).
\end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_j, w)$.

Relation (8): $[\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm\alpha_n, w)] = \pm a_{in} X^{\bar{\sigma}}(\pm\alpha_n, w) (\delta(z-w) + \delta(z+w))$

We will first calculate for $X^{\bar{\sigma}}(\alpha_n, w)$.

$$\begin{aligned}
&[\rho'(\alpha_i^{\bar{\sigma}}(z)), \rho'(X^{\bar{\sigma}}(\alpha_n, w))] \\
&= [:\varepsilon_i(z)\varepsilon_i^*(z): - :\varepsilon_{i+1}(z)\varepsilon_{i+1}^*(z): - :\varepsilon_{\bar{i}}(z)\varepsilon_{\bar{i}}^*(z): + :\varepsilon_{\bar{i+1}}(z)\varepsilon_{\bar{i+1}}^*(z):,
\end{aligned}$$

$$\begin{aligned}
& : \varepsilon_n(w) \varepsilon_n^*(-w) : - : \varepsilon_n(-w) \varepsilon_n^*(w) : \\
& = [: \varepsilon_i(z) \varepsilon_i^*(z) : , : \varepsilon_n(w) \varepsilon_n^*(-w) :] - [: \varepsilon_i(z) \varepsilon_i^*(z) : , : \varepsilon_n(-w) \varepsilon_n^*(w) :] \\
& - [: \varepsilon_{i+1}(z) \varepsilon_{i+1}^*(z) : , : \varepsilon_n(w) \varepsilon_n^*(-w) :] + [: \varepsilon_{i+1}(z) \varepsilon_{i+1}^*(z) : , : \varepsilon_n(-w) \varepsilon_n^*(w) :] \\
& - [: \varepsilon_i^*(z) : , : \varepsilon_n(w) \varepsilon_n^*(-w) :] + [: \varepsilon_i^*(z) : , : \varepsilon_n(-w) \varepsilon_n^*(w) :] \\
& + [: \varepsilon_{i+1}^*(z) : , : \varepsilon_n(w) \varepsilon_n^*(-w) :] - [: \varepsilon_{i+1}^*(z) : , : \varepsilon_n(-w) \varepsilon_n^*(w) :].
\end{aligned}$$

Using Proposition 41, the possibly nonzero terms are:

$$\begin{aligned}
& = -\langle \varepsilon_{i+1}^*, \varepsilon_n \rangle : \varepsilon_{i+1}(w) \varepsilon_n^*(-w) : \delta(z-w) + \langle \varepsilon_{i+1}^*, \varepsilon_n \rangle : \varepsilon_{i+1}(-w) \varepsilon_n^*(w) : \delta(z+w) \\
& + \langle \varepsilon_{i+1}^*, \varepsilon_n^* \rangle : \varepsilon_{i+1}^*(-w) \varepsilon_n(w) : \delta(z+w) - \langle \varepsilon_{i+1}^*, \varepsilon_n^* \rangle : \varepsilon_{i+1}^*(w) \varepsilon_n(-w) : \delta(z-w)
\end{aligned}$$

By antisymmetry of the form, we have:

$$\begin{aligned}
& = -\delta_{i+1,n} : \varepsilon_n(w) \varepsilon_n^*(-w) : \delta(z-w) + \delta_{i+1,n} : \varepsilon_n(-w) \varepsilon_n^*(w) : \delta(z+w) \\
& - \delta_{i+1,n} : \varepsilon_n^*(-w) \varepsilon_n(w) : \delta(z+w) + \delta_{i+1,n} : \varepsilon_n^*(w) \varepsilon_n(-w) : \delta(z-w) \\
& = -\delta_{i+1,n} (: \varepsilon_n(w) \varepsilon_n^*(-w) : - : \varepsilon_n(-w) \varepsilon_n^*(w) :) (\delta(z-w) + \delta(z+w)) \\
& = a_{in} \rho' (X^{\bar{\sigma}}(\alpha_n, w)) (\delta(z-w) + \delta(z+w)) \\
& = \rho' ([\alpha_i^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_n, w)]).
\end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_n, w)$.

Relation (9): $[\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm \alpha_0, w)] = 0$

We will first calculate for $X^{\bar{\sigma}}(\alpha_0, w)$.

$$\begin{aligned}
& [\rho'(\alpha_n^{\bar{\sigma}}(z)), \rho'(X^{\bar{\sigma}}(\alpha_0, w))] \\
& = [: \varepsilon_n(z) \varepsilon_n^*(z) : - : \varepsilon_n(-z) \varepsilon_n^*(-z) : - : \varepsilon_n(z) \varepsilon_n^*(z) : + : \varepsilon_n(-z) \varepsilon_n^*(-z) : , \\
& : \varepsilon_1^*(w) \bar{\beta}(-w) : - : \varepsilon_1^*(-w) \bar{\beta}(w) :].
\end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is simply:

$$\begin{aligned}
& = 0 \\
& = \rho' ([\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_0, w)]).
\end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_0, w)$.

Relation (10): $[\alpha_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm \alpha_j, w)] = \pm a_{nj} X^{\bar{\sigma}}(\pm \alpha_j, w) (\delta(z-w) + \delta(z+w))$

We will first calculate for $X^{\bar{\sigma}}(\alpha_j, w)$.

$$\begin{aligned}
& [\rho'(\alpha_n^{\bar{\sigma}}(z)), \rho'(X^{\bar{\sigma}}(\alpha_j, w))] \\
& = [: \varepsilon_n(z) \varepsilon_n^*(z) : - : \varepsilon_n(-z) \varepsilon_n^*(-z) : - : \varepsilon_n(z) \varepsilon_n^*(z) : + : \varepsilon_n(-z) \varepsilon_n^*(-z) : , \\
& : \varepsilon_j(w) \varepsilon_{j+1}^*(w) : - : \varepsilon_{j+1}(w) \varepsilon_j^*(w) :] \\
& = [: \varepsilon_n(z) \varepsilon_n^*(z) : , : \varepsilon_j(w) \varepsilon_{j+1}^*(w) :] - [: \varepsilon_n(z) \varepsilon_n^*(z) : , : \varepsilon_{j+1}(w) \varepsilon_j^*(w) :] \\
& - [: \varepsilon_n(-z) \varepsilon_n^*(-z) : , : \varepsilon_j(w) \varepsilon_{j+1}^*(w) :] + [: \varepsilon_n(-z) \varepsilon_n^*(-z) : , : \varepsilon_{j+1}(w) \varepsilon_j^*(w) :] \\
& - [: \varepsilon_n(z) \varepsilon_n^*(z) : , : \varepsilon_j(w) \varepsilon_{j+1}^*(w) :] + [: \varepsilon_n(z) \varepsilon_n^*(z) : , : \varepsilon_{j+1}(w) \varepsilon_j^*(w) :] \\
& + [: \varepsilon_n(-z) \varepsilon_n^*(-z) : , : \varepsilon_j(w) \varepsilon_{j+1}^*(w) :] - [: \varepsilon_n(-z) \varepsilon_n^*(-z) : , : \varepsilon_{j+1}(w) \varepsilon_j^*(w) :].
\end{aligned}$$

Using Proposition 41, the possibly nonzero terms are:

$$\begin{aligned}
&= \langle \varepsilon_n, \varepsilon_{j+1}^* \rangle : \varepsilon_n^*(w) \varepsilon_j(w) : \delta(z - w) - (-\langle \varepsilon_n, \varepsilon_{j+1}^* \rangle) : \varepsilon_n^*(w) \varepsilon_j(w) : \delta(z + w) \\
&\quad + \langle \varepsilon_n^*, \varepsilon_{\overline{j+1}} \rangle : \varepsilon_{\overline{n}}(w) \varepsilon_{\overline{j}}^*(w) : \delta(z - w) - (-\langle \varepsilon_{\overline{n}}, \varepsilon_{\overline{j+1}} \rangle) : \varepsilon_{\overline{n}}(w) \varepsilon_{\overline{j}}^*(w) : \delta(z + w) \\
&= -\delta_{n,j+1} : \varepsilon_{j+1}^*(w) \varepsilon_j(w) : \delta(z - w) - \delta_{n,j+1} : \varepsilon_{j+1}^*(w) \varepsilon_j(w) : \delta(z + w) \\
&\quad + \delta_{n,j+1} : \varepsilon_{\overline{j+1}}(w) \varepsilon_{\overline{j}}^*(w) : \delta(z - w) + \delta_{n,j+1} : \varepsilon_{\overline{j+1}}(w) \varepsilon_{\overline{j}}^*(w) : \delta(z + w) \\
&= -\delta_{n,j+1} (: \varepsilon_j(w) \varepsilon_{j+1}^*(w) : - : \varepsilon_{\overline{j+1}}(w) \varepsilon_{\overline{j}}^*(w) :) (\delta(z - w) + \delta(z + w)) \\
&= a_{nj} \rho'(X^{\bar{\sigma}}(\alpha_j, w)) (\delta(z - w) + \delta(z + w)) \\
&= \rho'([a_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_j, w)]).
\end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_j, w)$.

$$\textbf{Relation (11): } [a_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\pm \alpha_n, w)] = \pm a_{nn} X^{\bar{\sigma}}(\pm \alpha_n, w) (\delta(z - w) + \delta(z + w))$$

We will first calculate for $X^{\bar{\sigma}}(\alpha_n, w)$.

$$\begin{aligned}
&[\rho'(\alpha_n^{\bar{\sigma}}(z)), \rho'(X^{\bar{\sigma}}(\alpha_n, w))] \\
&= [: \varepsilon_n(z) \varepsilon_n^*(z) : - : \varepsilon_n(-z) \varepsilon_n^*(-z) : - : \varepsilon_{\overline{n}}(z) \varepsilon_{\overline{n}}^*(z) : + : \varepsilon_{\overline{n}}(-z) \varepsilon_{\overline{n}}^*(-z) : , \\
&\quad : \varepsilon_n(w) \varepsilon_{\overline{n}}^*(-w) : - : \varepsilon_n(-w) \varepsilon_{\overline{n}}^*(w) :] \\
&= [: \varepsilon_n(z) \varepsilon_n^*(z) : , : \varepsilon_n(w) \varepsilon_{\overline{n}}^*(-w) :] - [: \varepsilon_n(z) \varepsilon_n^*(z) : , : \varepsilon_n(-w) \varepsilon_{\overline{n}}^*(w) :] \\
&\quad - [: \varepsilon_n(-z) \varepsilon_n^*(-z) : , : \varepsilon_n(w) \varepsilon_{\overline{n}}^*(-w) :] + [: \varepsilon_n(-z) \varepsilon_n^*(-z) : , : \varepsilon_n(-w) \varepsilon_{\overline{n}}^*(w) :] \\
&\quad - [: \varepsilon_{\overline{n}}(z) \varepsilon_{\overline{n}}^*(z) : , : \varepsilon_n(w) \varepsilon_{\overline{n}}^*(-w) :] + [: \varepsilon_{\overline{n}}(z) \varepsilon_{\overline{n}}^*(z) : , : \varepsilon_n(-w) \varepsilon_{\overline{n}}^*(w) :] \\
&\quad + [: \varepsilon_{\overline{n}}(-z) \varepsilon_{\overline{n}}^*(-z) : , : \varepsilon_n(w) \varepsilon_{\overline{n}}^*(-w) :] - [: \varepsilon_{\overline{n}}(-z) \varepsilon_{\overline{n}}^*(-z) : , : \varepsilon_n(-w) \varepsilon_{\overline{n}}^*(w) :].
\end{aligned}$$

Using Proposition 41, the nonzero terms are:

$$\begin{aligned}
&= \langle \varepsilon_n^*, \varepsilon_n \rangle : \varepsilon_n(w) \varepsilon_{\overline{n}}^*(-w) : \delta(z - w) - \langle \varepsilon_n^*, \varepsilon_n \rangle : \varepsilon_n(-w) \varepsilon_{\overline{n}}^*(w) : \delta(z + w) \\
&\quad - (-\langle \varepsilon_n^*, \varepsilon_n \rangle) : \varepsilon_n(w) \varepsilon_{\overline{n}}^*(-w) : \delta(z + w) - \langle \varepsilon_n^*, \varepsilon_n \rangle : \varepsilon_n(-w) \varepsilon_{\overline{n}}^*(w) : \delta(z - w) \\
&\quad - \langle \varepsilon_{\overline{n}}, \varepsilon_{\overline{n}}^* \rangle : \varepsilon_{\overline{n}}^*(-w) \varepsilon_n(w) : \delta(z + w) + \langle \varepsilon_{\overline{n}}, \varepsilon_{\overline{n}}^* \rangle : \varepsilon_{\overline{n}}^*(w) \varepsilon_n(-w) : \delta(z - w) \\
&\quad - \langle \varepsilon_{\overline{n}}, \varepsilon_{\overline{n}}^* \rangle : \varepsilon_{\overline{n}}^*(-w) \varepsilon_n(w) : \delta(z - w) + \langle \varepsilon_{\overline{n}}, \varepsilon_{\overline{n}}^* \rangle : \varepsilon_{\overline{n}}^*(w) \varepsilon_n(-w) : \delta(z + w).
\end{aligned}$$

By antisymmetry of the form, we have:

$$\begin{aligned}
&=: \varepsilon_n(w) \varepsilon_{\overline{n}}^*(-w) : \delta(z - w) - : \varepsilon_n(-w) \varepsilon_{\overline{n}}^*(w) : \delta(z + w) \\
&\quad + : \varepsilon_n(w) \varepsilon_{\overline{n}}^*(-w) : \delta(z + w) - : \varepsilon_n(-w) \varepsilon_{\overline{n}}^*(w) : \delta(z - w) \\
&\quad + : \varepsilon_n(w) \varepsilon_{\overline{n}}^*(-w) : \delta(z + w) - : \varepsilon_n(-w) \varepsilon_{\overline{n}}^*(w) : \delta(z - w) \\
&\quad + : \varepsilon_n(w) \varepsilon_{\overline{n}}^*(-w) : \delta(z - w) - : \varepsilon_n(-w) \varepsilon_{\overline{n}}^*(w) : \delta(z + w). \\
&= 2 (: \varepsilon_n(w) \varepsilon_{\overline{n}}^*(-w) : - : \varepsilon_n(-w) \varepsilon_{\overline{n}}^*(w) :) (\delta(z - w) + \delta(z + w)) \\
&= a_{nn} \rho'(X^{\bar{\sigma}}(\alpha_n, w)) (\delta(z - w) + \delta(z + w)) \\
&= \rho'([a_n^{\bar{\sigma}}(z), X^{\bar{\sigma}}(\alpha_n, w)]).
\end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_n, w)$.

$$\textbf{Relation (12): } [X^{\bar{\sigma}}(\pm \alpha_m, z), X^{\bar{\sigma}}(\pm \alpha_m, w)] = 0$$

First let $m = 0$. We will calculate for $X^{\bar{\sigma}}(\alpha_m, z), X^{\bar{\sigma}}(\alpha_m, w)$.

$$\begin{aligned}
&[\rho'(X^{\bar{\sigma}}(\alpha_m, z)), \rho'(X^{\bar{\sigma}}(\alpha_m, w))] \\
&= [: \varepsilon_1^*(z) \bar{\beta}(-z) : - : \varepsilon_1^*(-z) \bar{\beta}(z) : , : \varepsilon_1^*(w) \bar{\beta}(-w) : - : \varepsilon_1^*(-w) \bar{\beta}(w) :].
\end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is simply:

$$\begin{aligned} &= 0 \\ &= \rho'([X^{\bar{\sigma}}(\alpha_m, z), X^{\bar{\sigma}}(\alpha_m, w)]). \end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_m, w)$.

The next case is for $1 \leq m \leq n - 1$. Calculating for $X^{\bar{\sigma}}(\alpha_m, z), X^{\bar{\sigma}}(\alpha_m, w)$ gives:

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_m, z)), \rho'(X^{\bar{\sigma}}(\alpha_m, w))] \\ &= [:\varepsilon_i(z)\varepsilon_{i+1}^*(z): - :\varepsilon_{i+1}(z)\varepsilon_i^*(z):, :\varepsilon_i(w)\varepsilon_{i+1}^*(w): - :\varepsilon_{i+1}(w)\varepsilon_i^*(w):]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric form $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is simply:

$$\begin{aligned} &= 0 \\ &= \rho'([X^{\bar{\sigma}}(\alpha_m, z), X^{\bar{\sigma}}(\alpha_m, w)]). \end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_m, w)$.

Finally, let $m = n$. Calculating for $X^{\bar{\sigma}}(\alpha_m, z), X^{\bar{\sigma}}(\alpha_m, w)$ gives:

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_n, z)), \rho'(X^{\bar{\sigma}}(\alpha_n, w))] \\ &= [:\varepsilon_n(z)\varepsilon_n^*(-z): - :\varepsilon_n(-z)\varepsilon_n^*(z):, :\varepsilon_n(w)\varepsilon_n^*(-w): - :\varepsilon_n(-w)\varepsilon_n^*(w):]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is simply:

$$\begin{aligned} &= 0 \\ &= \rho'([X^{\bar{\sigma}}(\alpha_n, z), X^{\bar{\sigma}}(\alpha_n, w)]). \end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_n, w)$.

$$\begin{aligned} \textbf{Relation (13): } &[X^{\bar{\sigma}}(\alpha_0, z), X^{\bar{\sigma}}(-\alpha_0, w)] = \alpha_0^{\bar{\sigma}}(w)(\delta(z - w) + \delta(z + w)) + (\partial_w \delta(z - w) + \partial_w \delta(z + w))k \\ &[\rho'(X^{\bar{\sigma}}(\alpha_0, z)), \rho'(X^{\bar{\sigma}}(-\alpha_0, w))] \\ &= [:\varepsilon_1^*(z)\bar{\beta}(-z): - :\varepsilon_1^*(-z)\bar{\beta}(z):, :\varepsilon_1(-w)\bar{\beta}^*(w): - :\varepsilon_1(w)\bar{\beta}^*(-w):] \\ &= [:\varepsilon_1^*(z)\bar{\beta}(-z):, :\varepsilon_1(-w)\bar{\beta}^*(w):] - [:\varepsilon_1^*(z)\bar{\beta}(-z):, :\varepsilon_1(w)\bar{\beta}^*(-w):] \\ &\quad - [:\varepsilon_1^*(-z)\bar{\beta}(z):, :\varepsilon_1(-w)\bar{\beta}^*(w):] + [:\varepsilon_1^*(-z)\bar{\beta}(z):, :\varepsilon_1(w)\bar{\beta}^*(-w):]. \end{aligned}$$

Using Proposition 41, we have

$$\begin{aligned} &= (\langle \varepsilon_1^*, \varepsilon_1 \rangle : \bar{\beta}(w)\bar{\beta}^*(w) : - \langle \bar{\beta}, \bar{\beta}^* \rangle : \varepsilon_1^*(-w)\varepsilon_1(-w) :)\delta(z + w) + \langle \varepsilon_1^*, \varepsilon_1 \rangle \langle \bar{\beta}, \bar{\beta}^* \rangle \partial_w \delta(z + w) \\ &\quad - \left\{ (\langle \varepsilon_1^*, \varepsilon_1 \rangle : \bar{\beta}(-w)\bar{\beta}^*(-w) : - \langle \bar{\beta}, \bar{\beta}^* \rangle : \varepsilon_1^*(w)\varepsilon_1(w) :)\delta(z - w) - \langle \varepsilon_1^*, \varepsilon_1 \rangle \langle \bar{\beta}, \bar{\beta}^* \rangle \partial_w \delta(z - w) \right\} \\ &\quad - \left\{ (- \langle \varepsilon_1^*, \varepsilon_1 \rangle : \bar{\beta}(w)\bar{\beta}^*(w) : + \langle \bar{\beta}, \bar{\beta}^* \rangle : \varepsilon_1^*(-w)\varepsilon_1(-w) :)\delta(z - w) - \langle \varepsilon_1^*, \varepsilon_1 \rangle \langle \bar{\beta}, \bar{\beta}^* \rangle \partial_w \delta(z - w) \right\} \\ &\quad + (- \langle \varepsilon_1^*, \varepsilon_1 \rangle : \bar{\beta}(-w)\bar{\beta}^*(-w) : + \langle \bar{\beta}, \bar{\beta}^* \rangle : \varepsilon_1^*(w)\varepsilon_1(w) :)\delta(z + w) + \langle \varepsilon_1^*, \varepsilon_1 \rangle \langle \bar{\beta}, \bar{\beta}^* \rangle \partial_w \delta(z + w). \end{aligned}$$

By antisymmetry of the form, we have

$$\begin{aligned} &= (:\bar{\beta}(w)\bar{\beta}^*(w): + :\varepsilon_1^*(-w)\varepsilon_1(-w): - :\bar{\beta}(-w)\bar{\beta}^*(-w): - :\varepsilon_1^*(w)\varepsilon_1(w):)\delta(z + w) \\ &\quad + (:\bar{\beta}(w)\bar{\beta}^*(w): + :\varepsilon_1^*(-w)\varepsilon_1(-w): - :\bar{\beta}(-w)\bar{\beta}^*(-w): - :\varepsilon_1^*(w)\varepsilon_1(w):)\delta(z - w) \end{aligned}$$

$$\begin{aligned}
& -2(\partial_w \delta(z-w) + \partial_w \delta(z+w)) \\
&= (: \varepsilon_1^*(-w) \beta(-w) : - : \varepsilon_1(w) \beta^*(w) : - : \varepsilon_{\bar{1}}^*(-w) \bar{\beta}(-w) : + : \varepsilon_{\bar{1}}(w) \bar{\beta}^*(w) :) (\delta(z-w) + \delta(z+w)) \\
&- 2(\partial_w \delta(z-w) + \partial_w \delta(z+w)) \\
&= \rho'(\alpha_0^{\bar{\sigma}}(z)) (\delta(z-w) + \delta(z+w)) + (\partial_w \delta(z-w) + \partial_w \delta(z+w)) k \\
&= \rho'([X^{\bar{\sigma}}(\alpha_0, z), X^{\bar{\sigma}}(-\alpha_0, w)]).
\end{aligned}$$

Relation (14): $[X^{\bar{\sigma}}(\alpha_i, z), X^{\bar{\sigma}}(-\alpha_i, w)] = \alpha_i^{\bar{\sigma}}(w) \delta(z-w) + \partial_w \delta(z-w) k$

$$\begin{aligned}
& [\rho'(X^{\bar{\sigma}}(\alpha_i, z)), \rho'(X^{\bar{\sigma}}(-\alpha_i, w))] \\
&= [: \varepsilon_i(z) \varepsilon_{i+1}^*(z) : - : \varepsilon_{\bar{i+1}}(z) \varepsilon_i^*(z) : , : \varepsilon_i^*(w) \varepsilon_{j+1}(w) : - : \varepsilon_{\bar{i+1}}^*(w) \varepsilon_{\bar{j}}(w) :] \\
&= [: \varepsilon_i(z) \varepsilon_{i+1}^*(z) : , : \varepsilon_i^*(w) \varepsilon_{i+1}(w) :] - [: \varepsilon_i(z) \varepsilon_{i+1}^*(z) : , : \varepsilon_{\bar{i+1}}^*(w) \varepsilon_{\bar{i}}(w) :] \\
&\quad - [: \varepsilon_{\bar{i+1}}(z) \varepsilon_i^*(z) : , : \varepsilon_i^*(w) \varepsilon_{i+1}(w) :] + [: \varepsilon_{\bar{i+1}}(z) \varepsilon_i^*(z) : , : \varepsilon_{\bar{i+1}}^*(w) \varepsilon_{\bar{i}}(w) :].
\end{aligned}$$

Using Proposition 41, the nonzero terms are:

$$\begin{aligned}
&= (\langle \varepsilon_i, \varepsilon_i^* \rangle : \varepsilon_{i+1}^*(w) \varepsilon_{i+1}(w) : + \langle \varepsilon_{i+1}, \varepsilon_{i+1} \rangle : \varepsilon_i(w) \varepsilon_i^*(w) :) \delta(z-w) + \langle \varepsilon_i, \varepsilon_i^* \rangle \langle \varepsilon_{i+1}^*, \varepsilon_{i+1} \rangle \partial_w \delta(z-w) \\
&+ (\langle \varepsilon_{\bar{i+1}}, \varepsilon_{\bar{i+1}}^* \rangle : \varepsilon_{\bar{i}}^*(w) \varepsilon_{\bar{i}}(w) : + \langle \varepsilon_{\bar{i}}, \varepsilon_{\bar{i}} \rangle : \varepsilon_{\bar{i+1}}(w) \varepsilon_{\bar{i+1}}^*(w) :) \delta(z-w) + \langle \varepsilon_{\bar{i+1}}, \varepsilon_{\bar{i+1}}^* \rangle \langle \varepsilon_{\bar{i}}^*, \varepsilon_{\bar{i}} \rangle \partial_w \delta(z-w).
\end{aligned}$$

By antisymmetry of the form, we have

$$\begin{aligned}
&= - : \varepsilon_{i+1}^*(w) \varepsilon_{i+1}(w) : \delta(z-w) + : \varepsilon_i(w) \varepsilon_i^*(w) : \delta(z-w) - \partial_w \delta(z-w). \\
&- : \varepsilon_{\bar{i}}^*(w) \varepsilon_{\bar{i}}(w) : + : \varepsilon_{\bar{i+1}}(w) \varepsilon_{\bar{i+1}}^*(w) : \delta(z-w) - \partial_w \delta(z-w) \\
&= (: \varepsilon_i(w) \varepsilon_i^*(w) : - : \varepsilon_{i+1}(w) \varepsilon_{i+1}^*(w) : - : \varepsilon_i(w) \varepsilon_i^*(w) : + : \varepsilon_{\bar{i+1}}(w) \varepsilon_{\bar{i+1}}^*(w) :) \delta(z-w) - 2 \partial_w \delta(z-w) \\
&= \rho'(\alpha_i^{\bar{\sigma}}(w)) \delta(z-w) + \partial_w \delta(z-w) k \\
&= \rho'([X^{\bar{\sigma}}(\alpha_i, z), X^{\bar{\sigma}}(-\alpha_i, w)]).
\end{aligned}$$

Relation (15): $[X^{\bar{\sigma}}(\alpha_n, z), X^{\bar{\sigma}}(-\alpha_n, w)] = \alpha_n^{\bar{\sigma}}(w) (\delta(z-w) + \delta(z+w)) + (\partial_w \delta(z-w) + \partial_w \delta(z+w)) k$

$$\begin{aligned}
& [\rho'(X^{\bar{\sigma}}(\alpha_n, z)), \rho'(X^{\bar{\sigma}}(-\alpha_n, w))] \\
&= [: \varepsilon_n(z) \varepsilon_{\bar{n}}^*(-z) : - : \varepsilon_n(-z) \varepsilon_{\bar{n}}^*(z) : , : \varepsilon_n^*(-w) \varepsilon_{\bar{n}}(w) : - : \varepsilon_n^*(w) \varepsilon_{\bar{n}}(-w) :] \\
&= [: \varepsilon_n(z) \varepsilon_{\bar{n}}^*(-z) : , : \varepsilon_n^*(-w) \varepsilon_{\bar{n}}(w) :] - [: \varepsilon_n(z) \varepsilon_{\bar{n}}^*(-z) : , : \varepsilon_n^*(w) \varepsilon_{\bar{n}}(-w) :] \\
&\quad - [: \varepsilon_n(-z) \varepsilon_{\bar{n}}^*(z) : , : \varepsilon_n^*(-w) \varepsilon_{\bar{n}}(w) :] + [: \varepsilon_n(-z) \varepsilon_{\bar{n}}^*(z) : , : \varepsilon_n^*(w) \varepsilon_{\bar{n}}(-w) :].
\end{aligned}$$

Using Proposition 41, we have

$$\begin{aligned}
&= (\langle \varepsilon_n, \varepsilon_n^* \rangle : \varepsilon_{\bar{n}}^*(w) \varepsilon_{\bar{n}}(w) : - \langle \varepsilon_{\bar{n}}, \varepsilon_{\bar{n}} \rangle : \varepsilon_n(-w) \varepsilon_n^*(-w) :) \delta(z+w) + \langle \varepsilon_n, \varepsilon_n^* \rangle \langle \varepsilon_{\bar{n}}^*, \varepsilon_{\bar{n}} \rangle \partial_w \delta(z+w) \\
&- \{ (\langle \varepsilon_n, \varepsilon_n^* \rangle : \varepsilon_{\bar{n}}^*(-w) \varepsilon_{\bar{n}}(-w) : - \langle \varepsilon_{\bar{n}}, \varepsilon_{\bar{n}} \rangle : \varepsilon_n(w) \varepsilon_n^*(w) :) \delta(z-w) - \langle \varepsilon_n, \varepsilon_n^* \rangle \langle \varepsilon_{\bar{n}}^*, \varepsilon_{\bar{n}} \rangle \partial_w \delta(z-w) \} \\
&- \{ (- \langle \varepsilon_n, \varepsilon_n^* \rangle : \varepsilon_{\bar{n}}^*(w) \varepsilon_{\bar{n}}(w) : + \langle \varepsilon_{\bar{n}}, \varepsilon_{\bar{n}} \rangle : \varepsilon_n(-w) \varepsilon_n^*(-w) :) \delta(z-w) \\
&- \langle \varepsilon_n, \varepsilon_n^* \rangle \langle \varepsilon_{\bar{n}}^*, \varepsilon_{\bar{n}} \rangle \partial_w \delta(z-w) \} \\
&+ (- \langle \varepsilon_n, \varepsilon_n^* \rangle : \varepsilon_{\bar{n}}^*(-w) \varepsilon_{\bar{n}}(-w) : + \langle \varepsilon_{\bar{n}}, \varepsilon_{\bar{n}} \rangle : \varepsilon_n(w) \varepsilon_n^*(w) :) \delta(z+w) + \langle \varepsilon_n, \varepsilon_n^* \rangle \langle \varepsilon_{\bar{n}}^*, \varepsilon_{\bar{n}} \rangle \partial_w \delta(z+w).
\end{aligned}$$

By antisymmetry of the form, we have:

$$\begin{aligned}
&= (- : \varepsilon_{\bar{n}}^*(w) \varepsilon_{\bar{n}}(w) : - : \varepsilon_n(-w) \varepsilon_n^*(-w) :) \delta(z+w) - \partial_w \delta(z+w) \\
& (: \varepsilon_n^*(-w) \varepsilon_{\bar{n}}(-w) : + : \varepsilon_n(w) \varepsilon_n^*(w) :) \delta(z-w) - \partial_w \delta(z-w)
\end{aligned}$$

$$\begin{aligned}
& (- : \varepsilon_n^*(w) \varepsilon_{\bar{n}}(w) : - : \varepsilon_n(-w) \varepsilon_n^*(-w) :) \delta(z-w) - \partial_w \delta(z-w) \\
& + (+ : \varepsilon_n^*(-w) \varepsilon_{\bar{n}}(-w) : + : \varepsilon_n(w) \varepsilon_n^*(w) :) \delta(z+w) - \partial_w \delta(z+w) \\
& = (: \varepsilon_n(w) \varepsilon_n^*(w) : - : \varepsilon_n(-w) \varepsilon_n^*(-w) : - : \varepsilon_{\bar{n}}(w) \varepsilon_{\bar{n}}^*(w) : + : \varepsilon_{\bar{n}}(-w) \varepsilon_{\bar{n}}^*(-w) :) (\delta(z-w) + \delta(z+w)) \\
& - 2(\partial_w \delta(z-w) + \partial_w \delta(z+w)) \\
& = \rho'(\alpha_n^{\bar{\sigma}}(w)) (\delta(z-w) + \delta(z+w)) + (\partial_w \delta(z-w) + \partial_w \delta(z+w)) \mathcal{K} \\
& = \rho'([X^{\bar{\sigma}}(\alpha_n, z), X^{\bar{\sigma}}(-\alpha_n, w)]).
\end{aligned}$$

Relation (16): $[X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)] = 0$ for $p \neq m$

We have several cases to check. First consider $p = 0$ and $1 \leq m \leq n-1$.

$$\begin{aligned}
& [\rho'(X^{\bar{\sigma}}(\alpha_p, z)), \rho'(X^{\bar{\sigma}}(-\alpha_m, w))] \\
& = [:\varepsilon_1^*(z)\bar{\beta}(-z) : - :\varepsilon_1^*(-z)\bar{\beta}(z) :, : \varepsilon_m^*(w)\varepsilon_{m+1}(w) : - :\varepsilon_{m+1}^*(w)\varepsilon_{\bar{m}}(w) :].
\end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence, this entire calculation is:

$$\begin{aligned}
& = 0 \\
& = \rho'([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]).
\end{aligned}$$

Now consider $p = 0$ and $m = n$.

$$\begin{aligned}
& [\rho'(X^{\bar{\sigma}}(\alpha_p, z)), \rho'(X^{\bar{\sigma}}(-\alpha_m, w))] \\
& = [:\varepsilon_1^*(z)\bar{\beta}(-z) : - :\varepsilon_1^*(-z)\bar{\beta}(z) :, : \varepsilon_n^*(-w)\varepsilon_{\bar{n}}(w) : - :\varepsilon_n^*(w)\varepsilon_{\bar{n}}(-w) :].
\end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence, this entire calculation is:

$$\begin{aligned}
& = 0 \\
& = \rho'([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]).
\end{aligned}$$

Now consider the case $1 \leq p \leq n-1$ and $m = 0$.

$$\begin{aligned}
& [\rho'(X^{\bar{\sigma}}(\alpha_p, z)), \rho'(X^{\bar{\sigma}}(-\alpha_m, w))] \\
& = [:\varepsilon_p(z)\varepsilon_{p+1}^*(z) : - :\varepsilon_{p+1}(z)\varepsilon_p^*(z) :, : \varepsilon_1(-w)\bar{\beta}^*(w) : - :\varepsilon_1(w)\bar{\beta}^*(-w) :].
\end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence, this entire calculation is:

$$\begin{aligned}
& = 0 \\
& = \rho'([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]).
\end{aligned}$$

Now consider the case $p = n$ and $m = 0$.

$$\begin{aligned}
& [\rho'(X^{\bar{\sigma}}(\alpha_p, z)), \rho'(X^{\bar{\sigma}}(-\alpha_m, w))] \\
& = [:\varepsilon_n(z)\varepsilon_n^*(-z) : - :\varepsilon_n(-z)\varepsilon_n^*(z) :, : \varepsilon_1(-w)\bar{\beta}^*(w) : - :\varepsilon_1(w)\bar{\beta}^*(-w) :].
\end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho'([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]). \end{aligned}$$

Now consider the case $1 \leq p \neq m \leq n - 1$.

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_p, z)), \rho'(X^{\bar{\sigma}}(-\alpha_m, w))] \\ &= [:\varepsilon_p(z)\varepsilon_{p+1}^*(z) : - :\varepsilon_{p+1}(z)\varepsilon_p^*(z) :, :\varepsilon_m^*(w)\varepsilon_{m+1}(w) : - :\varepsilon_{m+1}^*(w)\varepsilon_{m+1}(w) :]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0 or have coefficient δ_{pm} , which is 0 by assumption. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho'([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]). \end{aligned}$$

Now consider the case $1 \leq p \leq n - 1, m = n$.

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_p, z)), \rho'(X^{\bar{\sigma}}(-\alpha_m, w))] \\ &= [:\varepsilon_p(z)\varepsilon_{p+1}^*(z) : - :\varepsilon_{p+1}(z)\varepsilon_p^*(z) :, :\varepsilon_n^*(-w)\varepsilon_{\bar{n}}(w) : - :\varepsilon_n^*(w)\varepsilon_{\bar{n}}(-w) :]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho'([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]). \end{aligned}$$

Now consider the case $p = n, 1 \leq m \leq n - 1$.

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_p, z)), \rho'(X^{\bar{\sigma}}(-\alpha_m, w))] \\ &= [:\varepsilon_n(z)\varepsilon_{\bar{n}}^*(-z) : - :\varepsilon_n(-z)\varepsilon_{\bar{n}}^*(z) :, :\varepsilon_m^*(w)\varepsilon_{m+1}(w) : - :\varepsilon_{m+1}^*(w)\varepsilon_{\bar{m}}(w) :]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho'([X^{\bar{\sigma}}(\alpha_p, z), X^{\bar{\sigma}}(-\alpha_m, w)]). \end{aligned}$$

Relation (17): $\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_2)X^{\bar{\sigma}}(\pm\alpha_m, z_1) = 0$ if $c_{pm} = 0$

For the case $p = 0$, the matrix entry $c_{pm} = 0 \Leftrightarrow 2 \leq m \leq n$. We must split this into two cases of the value of m , namely $2 \leq m \leq n - 1$ and $m = n$. First we consider $2 \leq m \leq n - 1$ for positive α_p, α_m .

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho'(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\varepsilon_1^*(z_2)\bar{\beta}(-z_2) : - :\varepsilon_1^*(-z_2)\bar{\beta}(z_2) :, :\varepsilon_m(z_1)\varepsilon_{m+1}^*(z_1) : - :\varepsilon_{m+1}^*(z_1)\varepsilon_m^*(z_1) :]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho'([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)$.

When $p = 0$ and $m = n$, we have for α_p, α_m :

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho'(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\varepsilon_1^*(z_2)\bar{\beta}(-z_2): - :\varepsilon_1^*(-z_2)\bar{\beta}(z_2):, :\varepsilon_n(z_1)\varepsilon_n^*(-z_1): - :\varepsilon_n(-z_1)\varepsilon_n^*(z_1):]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho'([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)$.

Now if $m = 0$, the entry $c_{pm} = 0 \Leftrightarrow 2 \leq p \leq n$. We must split this into two cases of the value of p , namely $2 \leq p \leq n-1$ and $p = n$. First we consider $2 \leq p \leq n-1$ for positive α_p, α_m .

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho'(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\varepsilon_p(z_2)\varepsilon_{p+1}^*(z_2): - :\varepsilon_{p+1}^*(z_2)\varepsilon_p^*(z_2):, :\varepsilon_1^*(z_1)\bar{\beta}(-z_1): - :\varepsilon_1^*(-z_1)\bar{\beta}(z_1):]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho'([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)$.

When $m = 0$ and $p = n$, we have, for α_p, α_m

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho'(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\varepsilon_n(z_2)\varepsilon_n^*(-z_2): - :\varepsilon_n(-z_2)\varepsilon_n^*(z_2):, :\varepsilon_1^*(z_1)\bar{\beta}(-z_1): - :\varepsilon_1^*(-z_1)\bar{\beta}(z_1):]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho'([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)$.

For the remaining cases, we may assume that $p \neq 0$ and $m \neq 0$. For these values, $c_{pm} = 0$ when $|p - m| \geq 2$. We compute for the case $1 \leq p, m \leq n-1$ and α_p, α_m .

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho'(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\varepsilon_p(z_2)\varepsilon_{p+1}^*(z_2): - :\varepsilon_{p+1}^*(z_2)\varepsilon_p^*(z_2):, :\varepsilon_m(z_1)\varepsilon_{m+1}^*(z_1): - :\varepsilon_{m+1}^*(z_1)\varepsilon_m^*(z_1):]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence,

this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho'([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)$.

Now consider the case $p = n$ with $|p - m| \geq 2$. We calculate for α_p, α_m :

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho'(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\varepsilon_n(z_2)\varepsilon_n^*(-z_2) : - :\varepsilon_n(-z_2)\varepsilon_n^*(z_2) :, :\varepsilon_m(z_1)\varepsilon_{m+1}^*(z_1) : - :\varepsilon_{m+1}(z_1)\varepsilon_m^*(z_1) :]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho'([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)$.

Finally, consider the case $m = n$ with $|p - m| \geq 2$. We calculate for α_p, α_m :

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho'(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\varepsilon_p(z_2)\varepsilon_{p+1}^*(z_2) : - :\varepsilon_{p+1}(z_2)\varepsilon_p^*(z_2) :, :\varepsilon_n(z_1)\varepsilon_n^*(-z_1) : - :\varepsilon_n(-z_1)\varepsilon_n^*(z_1) :]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0. Hence, this entire calculation is:

$$\begin{aligned} &= 0 \\ &= \rho'([X^{\bar{\sigma}}(\alpha_p, z_2), X^{\bar{\sigma}}(\alpha_m, z_1)]). \end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)$.

Relation (18): $\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_2)X^{\bar{\sigma}}(\pm\alpha_m, z_1) = 0$ if $c_{pm} = -1$

The condition $c_{pm} = -1$ occurs precisely when $|p - m| = 1$ except for the pairs $p = 1, m = 0$ and $p = n - 1, m = n$. First, we will compute when $p = 0, m = 1$ and for α_p, α_m .

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho'(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\varepsilon_1^*(z_2)\bar{\beta}(-z_2) : - :\varepsilon_1^*(-z_2)\bar{\beta}(z_2) :, :\varepsilon_1(z_1)\varepsilon_2^*(z_1) : - :\varepsilon_2(z_1)\varepsilon_1^*(z_1) :] \\ &= [:\varepsilon_1^*(z_2)\bar{\beta}(-z_2) :, :\varepsilon_1(z_1)\varepsilon_2^*(z_1) :] - [:\varepsilon_1^*(z_2)\bar{\beta}(-z_2) :, :\varepsilon_2(z_1)\varepsilon_1^*(z_1) :] \\ &\quad - [:\varepsilon_1^*(-z_2)\bar{\beta}(z_2) :, :\varepsilon_1(z_1)\varepsilon_2^*(z_1) :] + [:\varepsilon_1^*(-z_2)\bar{\beta}(z_2) :, :\varepsilon_2(z_1)\varepsilon_1^*(z_1) :]. \end{aligned}$$

Using Proposition 41, we have:

$$\begin{aligned} &= \langle \varepsilon_1^*, \varepsilon_1 \rangle : \bar{\beta}(-z_1)\varepsilon_2^*(z_1) : \delta(z_2 - z_1) - (-\langle \bar{\beta}, \varepsilon_1^* \rangle) : \varepsilon_1^*(-z_1)\varepsilon_2(z_1) : \delta(z_2 + z_1) \\ &\quad - (-\langle \varepsilon_1^*, \varepsilon_1 \rangle) : \bar{\beta}(-z_1)\varepsilon_2^*(z_1) : \delta(z_2 + z_1) + \langle \bar{\beta}, \varepsilon_1^* \rangle : \varepsilon_1^*(-z_1)\varepsilon_2(z_1) : \delta(z_2 - z_1) \\ &= : \bar{\beta}(-z_1)\varepsilon_2^*(z_1) : \delta(z_2 - z_1) - : \varepsilon_1^*(-z_1)\varepsilon_2(z_1) : \delta(z_2 + z_1) \\ &\quad + : \bar{\beta}(-z_1)\varepsilon_2^*(z_1) : \delta(z_2 + z_1) - : \varepsilon_1^*(-z_1)\varepsilon_2(z_1) : \delta(z_2 - z_1) \\ &= (: \bar{\beta}(-z_1)\varepsilon_2^*(z_1) : - : \varepsilon_1^*(-z_1)\varepsilon_2(z_1) :) (\delta(z_2 - z_1) + \delta(z_2 + z_1)). \end{aligned}$$

We now apply $X^{\bar{\sigma}}(\alpha_p, z_3)$ on the left.

$$\begin{aligned} & [\rho'(X^{\bar{\sigma}}(\alpha_p, z_3)), (:\bar{\beta}(-z_1)\varepsilon_2^*(z_1):-:\varepsilon_1^*(-z_1)\varepsilon_{\bar{2}}(z_1):)(\delta(z_2 - z_1) + \delta(z_2 + z_1))] \\ &= [:\varepsilon_1^*(z_3)\bar{\beta}(-z_3):-:\varepsilon_1^*(-z_3)\bar{\beta}(z_3):, (:\bar{\beta}(-z_1)\varepsilon_2^*(z_1):-:\varepsilon_1^*(-z_1)\varepsilon_{\bar{2}}(z_1):)(\delta(z_2 - z_1) + \delta(z_2 + z_1))]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is:

$$\begin{aligned} &= 0 \\ &= \rho'(\text{ad}X^{\bar{\sigma}}(\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_2)X^{\bar{\sigma}}(\alpha_m, z_1)). \end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_p, z_3), X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)$.

Now we compute for $1 \leq p, m \leq n - 1$ where $|p - m| = 1$. First we calculate for α_p, α_m .

$$\begin{aligned} & [\rho'(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho'(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\varepsilon_p(z_2)\varepsilon_{p+1}^*(z_2):-:\varepsilon_{\bar{p}+1}(z_2)\varepsilon_{\bar{p}}^*(z_2):, :\varepsilon_m(z_1)\varepsilon_{m+1}^*(z_1):-:\varepsilon_{\bar{m}+1}(z_1)\varepsilon_{\bar{m}}^*(z_1):] \\ &= [:\varepsilon_p(z_2)\varepsilon_{p+1}^*(z_2):, :\varepsilon_m(z_1)\varepsilon_{m+1}^*(z_1):] - [:\varepsilon_p(z_2)\varepsilon_{p+1}^*(z_2):, :\varepsilon_{\bar{m}+1}(z_1)\varepsilon_{\bar{m}}^*(z_1):] \\ &\quad - [:\varepsilon_{\bar{p}+1}(z_2)\varepsilon_{\bar{p}}^*(z_2):, :\varepsilon_m(z_1)\varepsilon_{m+1}^*(z_1):] + [:\varepsilon_{\bar{p}+1}(z_2)\varepsilon_{\bar{p}}^*(z_2):, :\varepsilon_{\bar{m}+1}(z_1)\varepsilon_{\bar{m}}^*(z_1):]. \end{aligned}$$

Using Proposition 41, we have:

$$\begin{aligned} &= (\langle \varepsilon_p, \varepsilon_{m+1}^* \rangle : \varepsilon_{p+1}^*(z_1)\varepsilon_m(z_1) : + \langle \varepsilon_{p+1}^*, \varepsilon_m \rangle : \varepsilon_p(z_1)\varepsilon_{m+1}^*(z_1) :)\delta(z_2 - z_1) \\ &\quad + \langle \varepsilon_p, \varepsilon_{m+1}^* \rangle \langle \varepsilon_{p+1}^*, \varepsilon_m \rangle \partial_{z_1} \delta(z_2 - z_1) \\ &\quad + (\langle \varepsilon_{\bar{p}+1}, \varepsilon_{\bar{m}}^* \rangle : \varepsilon_{\bar{p}}^*(z_1)\varepsilon_{\bar{m}+1}(z_1) : + \langle \varepsilon_{\bar{p}}, \varepsilon_{\bar{m}+1} \rangle : \varepsilon_{\bar{p}+1}(z_1)\varepsilon_{\bar{m}}^*(z_1) :)\delta(z_2 - z_1) \\ &\quad + \langle \varepsilon_{\bar{p}+1}, \varepsilon_{\bar{m}}^* \rangle \langle \varepsilon_{\bar{p}}, \varepsilon_{\bar{m}+1} \rangle \partial_{z_1} \delta(z_2 + z_1) \end{aligned}$$

The coefficients $\langle \varepsilon_p, \varepsilon_{m+1}^* \rangle \langle \varepsilon_{p+1}^*, \varepsilon_m \rangle$ and $\langle \varepsilon_{\bar{p}+1}, \varepsilon_{\bar{m}}^* \rangle \langle \varepsilon_{\bar{p}}, \varepsilon_{\bar{m}+1} \rangle$ are 0, so we have

$$\begin{aligned} &= (-\delta_{p,m+1} : \varepsilon_{m+2}^*(z_1)\varepsilon_m(z_1) : + \delta_{p+1,m} : \varepsilon_{m-1}(z_1)\varepsilon_{m+1}^*(z_1) :)\delta(z_2 - z_1) \\ &\quad + (-\delta_{p+1,m} : \varepsilon_{m-1}^*(z_1)\varepsilon_{m+1}(z_1) : + \delta_{p,m+1} : \varepsilon_{m+2}(z_1)\varepsilon_{\bar{m}}^*(z_1) :)\delta(z_2 - z_1). \end{aligned}$$

We now apply $X^{\bar{\sigma}}(\alpha_p, z_3)$ on the left.

$$\begin{aligned} & [\rho'(X^{\bar{\sigma}}(\alpha_p, z_3)), (-\delta_{p,m+1} : \varepsilon_{m+2}^*(z_1)\varepsilon_m(z_1) : + \delta_{p+1,m} : \varepsilon_{m-1}(z_1)\varepsilon_{m+1}^*(z_1) :)\delta(z_2 - z_1) \\ &\quad + (-\delta_{p+1,m} : \varepsilon_{m-1}^*(z_1)\varepsilon_{m+1}(z_1) : + \delta_{p,m+1} : \varepsilon_{m+2}(z_1)\varepsilon_{\bar{m}}^*(z_1) :)\delta(z_2 - z_1)] \\ &= [:\varepsilon_p(z_3)\varepsilon_{p+1}^*(z_3) : - :\varepsilon_{\bar{p}+1}(z_3)\varepsilon_{\bar{p}}^*(z_3) :, \\ &\quad (-\delta_{p,m+1} : \varepsilon_{m+2}^*(z_1)\varepsilon_m(z_1) : + \delta_{p+1,m} : \varepsilon_{m-1}(z_1)\varepsilon_{m+1}^*(z_1) :)\delta(z_2 - z_1) \\ &\quad + (-\delta_{p+1,m} : \varepsilon_{m-1}^*(z_1)\varepsilon_{m+1}(z_1) : + \delta_{p,m+1} : \varepsilon_{m+2}(z_1)\varepsilon_{\bar{m}}^*(z_1) :)\delta(z_2 - z_1)] \\ &= [:\varepsilon_p(z_3)\varepsilon_{p+1}^*(z_3) :, -\delta_{p,m+1} : \varepsilon_{m+2}^*(z_1)\varepsilon_m(z_1) : \delta(z_2 - z_1)] \\ &\quad + [:\varepsilon_p(z_3)\varepsilon_{p+1}^*(z_3) :, \delta_{p+1,m} : \varepsilon_{m-1}(z_1)\varepsilon_{m+1}^*(z_1) : \delta(z_2 - z_1)] \\ &\quad - [:\varepsilon_{\bar{p}+1}(z_3)\varepsilon_{\bar{p}}^*(z_3) :, -\delta_{p+1,m} : \varepsilon_{m-1}^*(z_1)\varepsilon_{m+1}(z_1) : \delta(z_2 - z_1)] \\ &\quad - [:\varepsilon_{\bar{p}+1}(z_3)\varepsilon_{\bar{p}}^*(z_3) :, \delta_{p,m+1} : \varepsilon_{m+2}(z_1)\varepsilon_{\bar{m}}^*(z_1) : \delta(z_2 - z_1)]. \end{aligned}$$

Using Proposition 41, we have

$$\begin{aligned} &= -\delta_{p,m+1} \langle \varepsilon_{p+1}^*, \varepsilon_m \rangle : \varepsilon_p(z_1)\varepsilon_{m+2}^*(z_1) : \delta(z_2 - z_1)\delta(z_3 - z_1) \\ &\quad + \delta_{p+1,m} \langle \varepsilon_p, \varepsilon_{m+1}^* \rangle : \varepsilon_{p+1}^*(z_1)\varepsilon_{m-1}(z_1) : \delta(z_2 - z_1)\delta(z_3 - z_1) \end{aligned}$$

$$\begin{aligned}
& -(-\delta_{p+1,m}) \langle \varepsilon_{\bar{p}}^*, \varepsilon_{\bar{m+1}} \rangle : \varepsilon_{\bar{p+1}}(z_1) \varepsilon_{\bar{m-1}}^*(z_1) : \delta(z_2 - z_1) \delta(z_3 - z_1) \\
& - \delta_{p,m+1} \langle \varepsilon_{\bar{p+1}}^*, \varepsilon_{\bar{m}} \rangle : \varepsilon_{\bar{p}}^*(z_1) \varepsilon_{\bar{m+2}}(z_1) : \delta(z_2 - z_1) \delta(z_3 - z_1) \\
& = -\delta_{p,m+1} \delta_{p+1,m} : \varepsilon_p(z_1) \varepsilon_{m+2}^*(z_1) : \delta(z_2 - z_1) \delta(z_3 - z_1) \\
& - \delta_{p+1,m} \delta_{p,m+1} : \varepsilon_{p+1}^*(z_1) \varepsilon_{m-1}(z_1) : \delta(z_2 - z_1) \delta(z_3 - z_1) \\
& + \delta_{p+1,m} \delta_{p,m+1} : \varepsilon_{\bar{p+1}}(z_1) \varepsilon_{\bar{m-1}}^*(z_1) : \delta(z_2 - z_1) \delta(z_3 - z_1) \\
& + \delta_{p,m+1} \delta_{p+1,m} : \varepsilon_{\bar{p}}^*(z_1) \varepsilon_{\bar{m+2}}(z_1) : \delta(z_2 - z_1) \delta(z_3 - z_1)
\end{aligned}$$

The coefficient $\pm \delta_{p,m+1} \delta_{p+1,m} = 0$. Hence, this calculation is:

$$\begin{aligned}
& = 0 \\
& = \rho'(\text{ad}X^{\bar{\sigma}}(\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_2)X^{\bar{\sigma}}(\alpha_m, z_1)).
\end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_p, z_3), X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)$.

The final case for this relation is $p = n$ and $m = n - 1$. We compute for α_p, α_m :

$$\begin{aligned}
& [\rho'(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho'(X^{\bar{\sigma}}(\alpha_m, z_1))] \\
& = [:\varepsilon_n(z_2)\varepsilon_{\bar{n}}^*(-z_2) : - :\varepsilon_n(-z_2)\varepsilon_{\bar{n}}^*(z_2) :, :\varepsilon_{n-1}(z_1)\varepsilon_n^*(z_1) : - :\varepsilon_{\bar{n}}(z_1)\varepsilon_{\bar{n-1}}^*(z_1) :] \\
& = [:\varepsilon_n(z_2)\varepsilon_{\bar{n}}^*(-z_2) :, :\varepsilon_{n-1}(z_1)\varepsilon_n^*(z_1) :] - [:\varepsilon_n(z_2)\varepsilon_{\bar{n}}^*(-z_2) :, :\varepsilon_{\bar{n}}(z_1)\varepsilon_{\bar{n-1}}^*(z_1) :] \\
& - [:\varepsilon_n(-z_2)\varepsilon_{\bar{n}}^*(z_2) :, :\varepsilon_{n-1}(z_1)\varepsilon_n^*(z_1) :] + [:\varepsilon_n(-z_2)\varepsilon_{\bar{n}}^*(z_2) :, :\varepsilon_{\bar{n}}(z_1)\varepsilon_{\bar{n-1}}^*(z_1) :].
\end{aligned}$$

Using Proposition 41, we have:

$$\begin{aligned}
& = \langle \varepsilon_n, \varepsilon_n^* \rangle : \varepsilon_{\bar{n}}^*(-z_1)\varepsilon_{n-1}(z_1) : \delta(z_2 - z_1) + \langle \varepsilon_{\bar{n}}, \varepsilon_{\bar{n}} \rangle : \varepsilon_n(-z_1)\varepsilon_{\bar{n-1}}^*(z_1) : \delta(z_2 + z_1) \\
& + \langle \varepsilon_n, \varepsilon_n^* \rangle : \varepsilon_{\bar{n}}^*(-z_1)\varepsilon_{n-1}(z_1) : \delta(z_2 + z_1) + \langle \varepsilon_{\bar{n}}, \varepsilon_{\bar{n}} \rangle : \varepsilon_n(-z_1)\varepsilon_{\bar{n-1}}^*(z_1) : \delta(z_2 - z_1) \\
& = (:\varepsilon_n(-z_1)\varepsilon_{\bar{n-1}}^*(z_1) : - :\varepsilon_{\bar{n}}^*(-z_1)\varepsilon_{n-1}(z_1) :) (\delta(z_2 - z_1) + \delta(z_2 + z_1)).
\end{aligned}$$

We now apply $X^{\bar{\sigma}}(\alpha_p, z_3)$ on the left.

$$\begin{aligned}
& [\rho'(X^{\bar{\sigma}}(\alpha_p, z_3)), (:\varepsilon_n(-z_1)\varepsilon_{\bar{n-1}}^*(z_1) : - :\varepsilon_{\bar{n}}^*(-z_1)\varepsilon_{n-1}(z_1) :) (\delta(z_2 - z_1) + \delta(z_2 + z_1))] \\
& = [:\varepsilon_n(z_3)\varepsilon_{\bar{n}}^*(-z_3) : - :\varepsilon_n(-z_3)\varepsilon_{\bar{n}}^*(z_3) :, (:\varepsilon_n(-z_1)\varepsilon_{\bar{n-1}}^*(z_1) : - :\varepsilon_{\bar{n}}^*(-z_1)\varepsilon_{n-1}(z_1) :) \\
& \cdot (\delta(z_2 - z_1) + \delta(z_2 + z_1))].
\end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is:

$$\begin{aligned}
& = 0 \\
& = \rho'(\text{ad}X^{\bar{\sigma}}(\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_2)X^{\bar{\sigma}}(\alpha_m, z_1)).
\end{aligned}$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_p, z_3), X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)$.

Relation (19): $\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_4)\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\pm\alpha_p, z_2)X^{\bar{\sigma}}(\pm\alpha_m, z_1) = 0$ if $c_{pm} = -2$

There are two cases to consider: $p = 1, m = 0$ and $p = n - 1, m = n$. First we compute for $p = 1, m = 0$ and α_p, α_m .

$$\begin{aligned}
& [\rho'(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho'(X^{\bar{\sigma}}(\alpha_m, z_1))] \\
& = [:\varepsilon_1(z_2)\varepsilon_2^*(z_2) : - :\varepsilon_{\bar{2}}(z_2)\varepsilon_1^*(z_2) :, :\varepsilon_1^*(z_1)\bar{\beta}(-z_1) : - :\varepsilon_1^*(-z_1)\bar{\beta}(z_1) :] \\
& = [:\varepsilon_1(z_2)\varepsilon_2^*(z_2) :, :\varepsilon_1^*(z_1)\bar{\beta}(-z_1) :] - [:\varepsilon_1(z_2)\varepsilon_2^*(z_2) :, :\varepsilon_1^*(-z_1)\bar{\beta}(z_1) :]
\end{aligned}$$

$$-[:\varepsilon_{\bar{2}}(z_2)\varepsilon_{\bar{1}}^*(z_2):,::\varepsilon_1^*(z_1)\bar{\beta}(-z_1):]+[:\varepsilon_{\bar{2}}(z_2)\varepsilon_{\bar{1}}^*(z_2):,::\varepsilon_1^*(-z_1)\bar{\beta}(z_1):].$$

Using Proposition 41, we have

$$\begin{aligned} &= \langle \varepsilon_1, \varepsilon_1^* \rangle : \varepsilon_2^*(z_1) \bar{\beta}(-z_1) : \delta(z_2 - z_1) - \langle \varepsilon_1, \varepsilon_1^* \rangle : \varepsilon_2^*(-z_1) \bar{\beta}(z_1) : \delta(z_2 + z_1) \\ &- \langle \varepsilon_{\bar{1}}^*, \bar{\beta} \rangle : \varepsilon_{\bar{2}}(-z_1) \varepsilon_1^*(z_1) : \delta(z_2 + z_1) + \langle \varepsilon_{\bar{1}}^*, \bar{\beta} \rangle : \varepsilon_{\bar{2}}(z_1) \varepsilon_1^*(-z_1) : \delta(z_2 - z_1) \\ &= (:\varepsilon_{\bar{2}}(z_1) \varepsilon_1^*(-z_1) : - :\varepsilon_2^*(z_1) \bar{\beta}(-z_1) :) \delta(z_2 - z_1) \\ &+ (:\varepsilon_2^*(-z_1) \bar{\beta}(z_1) : - :\varepsilon_{\bar{2}}(-z_1) \varepsilon_1^*(z_1) :) \delta(z_2 + z_1). \end{aligned}$$

We now apply $X^{\bar{\sigma}}(\alpha_p, z_3)$ on the left.

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_p, z_3)), \\ &(:\varepsilon_{\bar{2}}(z_1) \varepsilon_1^*(-z_1) : - :\varepsilon_2^*(z_1) \bar{\beta}(-z_1) :) \delta(z_2 - z_1) + (:\varepsilon_2^*(-z_1) \bar{\beta}(z_1) : - :\varepsilon_{\bar{2}}(-z_1) \varepsilon_1^*(z_1) :) \delta(z_2 + z_1)] \\ &= [:\varepsilon_1(z_3) \varepsilon_2^*(z_3) : - :\varepsilon_{\bar{2}}(z_3) \varepsilon_{\bar{1}}^*(z_3) : , \\ &(:\varepsilon_{\bar{2}}(z_1) \varepsilon_1^*(-z_1) : - :\varepsilon_2^*(z_1) \bar{\beta}(-z_1) :) \delta(z_2 - z_1) + (:\varepsilon_2^*(-z_1) \bar{\beta}(z_1) : - :\varepsilon_{\bar{2}}(-z_1) \varepsilon_1^*(z_1) :) \delta(z_2 + z_1)] \\ &= [:\varepsilon_1(z_3) \varepsilon_2^*(z_3) : , (:\varepsilon_{\bar{2}}(z_1) \varepsilon_1^*(-z_1) : - :\varepsilon_2^*(z_1) \bar{\beta}(-z_1) :) \delta(z_2 - z_1)] \\ &+ [:\varepsilon_1(z_3) \varepsilon_2^*(z_3) : , (:\varepsilon_2^*(-z_1) \bar{\beta}(z_1) : - :\varepsilon_{\bar{2}}(-z_1) \varepsilon_1^*(z_1) :) \delta(z_2 + z_1)] \\ &- [:\varepsilon_{\bar{2}}(z_3) \varepsilon_{\bar{1}}^*(z_3) : , (:\varepsilon_{\bar{2}}(z_1) \varepsilon_1^*(-z_1) : - :\varepsilon_2^*(z_1) \bar{\beta}(-z_1) :) \delta(z_2 - z_1)] \\ &- [:\varepsilon_{\bar{2}}(z_3) \varepsilon_{\bar{1}}^*(z_3) : , (:\varepsilon_2^*(-z_1) \bar{\beta}(z_1) : - :\varepsilon_{\bar{2}}(-z_1) \varepsilon_1^*(z_1) :) \delta(z_2 + z_1)] \end{aligned}$$

Using Proposition 41, we have

$$\begin{aligned} &= \langle \varepsilon_1, \varepsilon_1^* \rangle : \varepsilon_2^*(-z_1) \varepsilon_{\bar{2}}(z_1) : \delta(z_3 + z_1) \delta(z_2 - z_1) - \langle \varepsilon_1, \varepsilon_1^* \rangle : \varepsilon_2^*(z_1) \varepsilon_{\bar{2}}(-z_1) : \delta(z_3 - z_1) \delta(z_2 + z_1) \\ &+ \langle \varepsilon_{\bar{1}}^*, \bar{\beta} \rangle : \varepsilon_{\bar{2}}(-z_1) \varepsilon_2^*(z_1) : \delta(z_3 + z_1) \delta(z_2 - z_1) - \langle \varepsilon_{\bar{1}}^*, \bar{\beta} \rangle : \varepsilon_{\bar{2}}(z_1) \varepsilon_2^*(-z_1) : \delta(z_3 - z_1) \delta(z_2 + z_1) \\ &= (:\varepsilon_2^*(z_1) \varepsilon_{\bar{2}}(-z_1) : - :\varepsilon_2^*(-z_1) \varepsilon_{\bar{2}}(z_1) :) (\delta(z_3 + z_1) \delta(z_2 - z_1) + \delta(z_3 - z_1) \delta(z_2 + z_1)). \end{aligned}$$

We now apply $X^{\bar{\sigma}}(\alpha_p, z_4)$ on the left.

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_p, z_4)), \\ &(:\varepsilon_2^*(z_1) \varepsilon_{\bar{2}}(-z_1) : - :\varepsilon_2^*(-z_1) \varepsilon_{\bar{2}}(z_1) :) (\delta(z_3 + z_1) \delta(z_2 - z_1) + \delta(z_3 - z_1) \delta(z_2 + z_1))] \\ &= [:\varepsilon_1(z_4) \varepsilon_2^*(z_4) : - :\varepsilon_{\bar{2}}(z_4) \varepsilon_{\bar{1}}^*(z_4) : , \\ &(:\varepsilon_2^*(z_1) \varepsilon_{\bar{2}}(-z_1) : - :\varepsilon_2^*(-z_1) \varepsilon_{\bar{2}}(z_1) :) (\delta(z_3 + z_1) \delta(z_2 - z_1) + \delta(z_3 - z_1) \delta(z_2 + z_1))]. \end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is:

$$= 0$$

$$= \rho'(\text{ad} X^{\bar{\sigma}}(\alpha_p, z_4) \text{ad} X^{\bar{\sigma}}(\alpha_p, z_3) \text{ad} X^{\bar{\sigma}}(\alpha_p, z_2) X^{\bar{\sigma}}(\alpha_m, z_1)).$$

The calculation is similar for $X^{\bar{\sigma}}(-\alpha_p, z_4), X^{\bar{\sigma}}(-\alpha_p, z_3), X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1)$.

The final case to consider is $p = n - 1, m = n$. First we compute for α_p, α_m .

$$\begin{aligned} &[\rho'(X^{\bar{\sigma}}(\alpha_p, z_2)), \rho'(X^{\bar{\sigma}}(\alpha_m, z_1))] \\ &= [:\varepsilon_{n-1}(z_2) \varepsilon_n^*(z_2) : - :\varepsilon_{\bar{n}}(z_2) \varepsilon_{\bar{n}-1}^*(z_2) : , : \varepsilon_n(z_1) \varepsilon_{\bar{n}}^*(-z_1) : - :\varepsilon_n(-z_1) \varepsilon_{\bar{n}}^*(z_1) :] \\ &= [:\varepsilon_{n-1}(z_2) \varepsilon_n^*(z_2) : , : \varepsilon_n(z_1) \varepsilon_{\bar{n}}^*(-z_1) :] - [:\varepsilon_{n-1}(z_2) \varepsilon_n^*(z_2) : , : \varepsilon_n(-z_1) \varepsilon_{\bar{n}}^*(z_1) :] \\ &- [:\varepsilon_{\bar{n}}(z_2) \varepsilon_{\bar{n}-1}^*(z_2) : , : \varepsilon_n(z_1) \varepsilon_n^*(-z_1) :] + [:\varepsilon_{\bar{n}}(z_2) \varepsilon_{\bar{n}-1}^*(z_2) : , : \varepsilon_n(-z_1) \varepsilon_n^*(z_1) :] \end{aligned}$$

Using Proposition 41, we have

$$\begin{aligned}
&= \langle \varepsilon_n^*, \varepsilon_n \rangle : \varepsilon_{n-1}(z_1) \varepsilon_{\overline{n}}^*(-z_1) : \delta(z_2 - z_1) - \langle \varepsilon_n^*, \varepsilon_n \rangle : \varepsilon_{n-1}(-z_1) \varepsilon_{\overline{n}}^*(z_1) : \delta(z_2 + z_1) \\
&- \langle \varepsilon_{\overline{n}}, \varepsilon_{\overline{n}}^* \rangle : \varepsilon_{\overline{n-1}}^*(-z_1) \varepsilon_n(z_1) : \delta(z_2 + z_1) + \langle \varepsilon_{\overline{n}}, \varepsilon_{\overline{n}}^* \rangle : \varepsilon_{\overline{n-1}}^*(z_1) \varepsilon_n(-z_1) : \delta(z_2 - z_1) \\
&=: \varepsilon_{n-1}(z_1) \varepsilon_{\overline{n}}^*(-z_1) : \delta(z_2 - z_1) - : \varepsilon_{n-1}(-z_1) \varepsilon_{\overline{n}}^*(z_1) : \delta(z_2 + z_1) \\
&+ : \varepsilon_{\overline{n-1}}^*(-z_1) \varepsilon_n(z_1) : \delta(z_2 + z_1) - : \varepsilon_{\overline{n-1}}^*(z_1) \varepsilon_n(-z_1) : \delta(z_2 - z_1).
\end{aligned}$$

We now apply $X^{\bar{\sigma}}(\alpha_p, z_3)$ on the left.

$$\begin{aligned}
&[\rho'(X^{\bar{\sigma}}(\alpha_p, z_3)), : \varepsilon_{n-1}(z_1) \varepsilon_{\overline{n}}^*(-z_1) : \delta(z_2 - z_1) - : \varepsilon_{n-1}(-z_1) \varepsilon_{\overline{n}}^*(z_1) : \delta(z_2 + z_1) \\
&+ : \varepsilon_{\overline{n-1}}^*(-z_1) \varepsilon_n(z_1) : \delta(z_2 + z_1) - : \varepsilon_{\overline{n-1}}^*(z_1) \varepsilon_n(-z_1) : \delta(z_2 - z_1)] \\
&= [: \varepsilon_{n-1}(z_3) \varepsilon_n^*(z_3) : - : \varepsilon_{\overline{n}}(z_3) \varepsilon_{\overline{n-1}}^*(z_3) : : \varepsilon_{n-1}(z_1) \varepsilon_{\overline{n}}^*(-z_1) : \delta(z_2 - z_1) - : \varepsilon_{n-1}(-z_1) \varepsilon_{\overline{n}}^*(z_1) : \\
&\delta(z_2 + z_1) \\
&+ : \varepsilon_{\overline{n-1}}^*(-z_1) \varepsilon_n(z_1) : \delta(z_2 + z_1) - : \varepsilon_{\overline{n-1}}^*(z_1) \varepsilon_n(-z_1) : \delta(z_2 - z_1)] \\
&= [: \varepsilon_{n-1}(z_3) \varepsilon_n^*(z_3) : : \varepsilon_{n-1}(z_1) \varepsilon_{\overline{n}}^*(-z_1) : \delta(z_2 - z_1)] - [: \varepsilon_{n-1}(z_3) \varepsilon_n^*(z_3) : : \varepsilon_{n-1}(-z_1) \varepsilon_{\overline{n}}^*(z_1) : \\
&\delta(z_2 + z_1)] \\
&+ [: \varepsilon_{n-1}(z_3) \varepsilon_n^*(z_3) : : \varepsilon_{\overline{n-1}}^*(-z_1) \varepsilon_n(z_1) : \delta(z_2 + z_1)] - [: \varepsilon_{n-1}(z_3) \varepsilon_n^*(z_3) : : \varepsilon_{\overline{n-1}}^*(z_1) \varepsilon_n(-z_1) : \delta(z_2 - \\
&z_1)] \\
&- [: \varepsilon_{\overline{n}}(z_3) \varepsilon_{\overline{n-1}}^*(z_3) : : \varepsilon_{n-1}(z_1) \varepsilon_{\overline{n}}^*(-z_1) : \delta(z_2 - z_1)] + [: \varepsilon_{\overline{n}}(z_3) \varepsilon_{\overline{n-1}}^*(z_3) : : \varepsilon_{n-1}(-z_1) \varepsilon_{\overline{n}}^*(z_1) : \delta(z_2 + \\
&z_1)] - [: \varepsilon_{\overline{n}}(z_3) \varepsilon_{\overline{n-1}}^*(z_3) : : \varepsilon_{\overline{n-1}}^*(-z_1) \varepsilon_n(z_1) : \delta(z_2 + z_1)] + [: \varepsilon_{\overline{n}}(z_3) \varepsilon_{\overline{n-1}}^*(z_3) : : \varepsilon_{\overline{n-1}}^*(z_1) \varepsilon_n(-z_1) : \\
&\delta(z_2 - z_1)].
\end{aligned}$$

Using Proposition 41, we have

$$\begin{aligned}
&= \langle \varepsilon_n^*, \varepsilon_n \rangle : \varepsilon_{n-1}(z_1) \varepsilon_{\overline{n-1}}^*(-z_1) : \delta(z_3 - z_1) \delta(z_2 + z_1) - \langle \varepsilon_n^*, \varepsilon_n \rangle : \varepsilon_{n-1}(-z_1) \varepsilon_{\overline{n-1}}^*(z_1) : \delta(z_3 + \\
&z_1) \delta(z_2 - z_1) \\
&- \langle \varepsilon_{\overline{n}}, \varepsilon_{\overline{n}}^* \rangle : \varepsilon_{\overline{n-1}}^*(-z_1) \varepsilon_{n-1}(z_1) : \delta(z_3 + z_1) \delta(z_2 - z_1) + \langle \varepsilon_{\overline{n}}, \varepsilon_{\overline{n}}^* \rangle : \varepsilon_{\overline{n-1}}^*(z_1) \varepsilon_{n-1}(-z_1) : \delta(z_3 - z_1) \delta(z_2 + \\
&z_1) \\
&= (: \varepsilon_{n-1}(z_1) \varepsilon_{\overline{n-1}}^*(-z_1) : - : \varepsilon_{n-1}(-z_1) \varepsilon_{\overline{n-1}}^*(z_1) :) (\delta(z_3 + z_1) \delta(z_2 - z_1) + \delta(z_3 - z_1) \delta(z_2 + z_1)).
\end{aligned}$$

We now apply $X^{\bar{\sigma}}(\alpha_p, z_4)$ on the left.

$$\begin{aligned}
&[\rho'(X^{\bar{\sigma}}(\alpha_p, z_4)), (: \varepsilon_{n-1}(z_1) \varepsilon_{\overline{n-1}}^*(-z_1) : - : \varepsilon_{n-1}(-z_1) \varepsilon_{\overline{n-1}}^*(z_1) :) (\delta(z_3 + z_1) \delta(z_2 - z_1) + \delta(z_3 - z_1) \delta(z_2 + z_1))] \\
&= [: \varepsilon_{n-1}(z_4) \varepsilon_n^*(z_4) : - : \varepsilon_{\overline{n}}(z_4) \varepsilon_{\overline{n-1}}^*(z_4) : \\
&(: \varepsilon_{n-1}(z_1) \varepsilon_{\overline{n-1}}^*(-z_1) : - : \varepsilon_{n-1}(-z_1) \varepsilon_{\overline{n-1}}^*(z_1) :) (\delta(z_3 + z_1) \delta(z_2 - z_1) + \delta(z_3 - z_1) \delta(z_2 + z_1))].
\end{aligned}$$

Using Proposition 41, all contractions (hence antisymmetric forms $\langle \cdot, \cdot \rangle$) are trivially 0, so this calculation is:

$$\begin{aligned}
&= 0 \\
&= \rho'(\text{ad}X^{\bar{\sigma}}(\alpha_p, z_4)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_3)\text{ad}X^{\bar{\sigma}}(\alpha_p, z_2)X^{\bar{\sigma}}(\alpha_m, z_1)). \\
\text{The calculation is similar for } X^{\bar{\sigma}}(-\alpha_p, z_4), X^{\bar{\sigma}}(-\alpha_p, z_3), X^{\bar{\sigma}}(-\alpha_p, z_2), X^{\bar{\sigma}}(-\alpha_m, z_1).
\end{aligned}$$

Hence, ρ' is a homomorphism; thus we have constructed a representation of the twisted

toroidal Lie algebra of type A_{2n-1} . ■

BIBLIOGRAPHY

- [1] [B1] Yuly Billig. *Principal Vertex Operator Representations for Toroidal Lie Algebras*, Journal of Mathematical Physics. **39** (1998), 3844-3864.
- [2] [B2] Yuly Billig. *Energy-Momentum Tensor for the Toroidal Lie Algebras*, Accessed May 12, 2014 at <http://arxiv.org/pdf/math/0201313v1.pdf>, January 2002.
- [3] [B3] Yuly Billig. *Representations of the Twisted Heisenberg-Virasoro Algebra at Level Zero*, Canad. Math. Bull. **46** (2003), 529-537.
- [4] [BB] Stephen Berman and Yuly Billig. *Irreducible Representations for Toroidal Lie Algebras*, Journal of Algebra. **221** (1999), 188-231.
- [5] [BBS] Stephen Berman, Yuly Billig and Jacek Szmigielski. *Vertex Operator Algebras and the Representation Theory of Toroidal Algebras*, Contemporary Mathematics. **297** (2002), 1-26.
- [6] [BM] G. M. Benkart and R. V. Moody. *Derivations, Central Extensions, and Affine Lie Algebras*, Algebras, Groups and Geometries. **3** (1986), 456-492.
- [7] [BK] S. Berman and Y. Krylyuk. *Universal Central Extensions of Twisted and Untwisted Lie Algebras Extended over Commutative Rings*, Journal of Algebra. **173** (1995), 302-347.
- [8] [C] Carter, Roger. Lie Algebras of Finite and Affine Type. Cambridge: Cambridge University Press, 2005. Print.
- [9] [D] V. G. Drinfel'd. *A New Realization of Yangians and Quantized Affine Algebras*, Soviet Math. Dokl. **36** (1988), 212-216.
- [10] [E] Etingof, Pavel et al. Introduction to Representation Theory. Providence, RI: American Mathematical Society, 2011. Print.

- [11] [EM] S. Eswara Rao and R. V. Moody. *Vertex Representations for N-Toroidal Lie Algebras and a Generalization of the Virasoro Algebra*, Comm. in Math. Phys. **159** (1994), 239-264.
- [12] [F] I. B. Frenkel. *Spinor Representations of Affine Lie Algebras*, Proc. Natl. Acad. Sci. USA. **77** (1980), 6303-6306.
- [13] [FF] A. J. Feingold and I. B. Frenkel. *Classical Affine Algebras*, Advances in Mathematics. **56** (1985), 117-172.
- [14] [FJ] J. Fu and C. Jiang. *Integrable Representations for the Twisted Full Toroidal Lie Algebras*, Journal of Algebra. **307** (2007), 769-794.
- [15] [FK] I. B. Frenkel and V. G. Kac. *Basic Representations of Affine Lie Algebras and Dual Resonance Models*, Invent. Math. **62** (1980), 23-66.
- [16] [FM] M. Fabbri and R. V. Moody. *Irreducible Representations of Virasoro-Toroidal Lie Algebras*, Commun. Math. Phys. **159** (1994), 1-13.
- [17] [G] H. Garland. *The arithmetic theory of loop groups*, Inst. Hautes Etudes Sci. Publ. Math. **52** (1980), 5-136.
- [18] [Gal] Gallian, Joseph A. Contemporary Abstract Algebra. Boston: Houghton Mifflin Company, 2006. Print.
- [19] [Gao] Y. Gao. *Fermionic and Bosonic Representations of the Extended Affine Lie Algebra $gl_N(\mathbb{C}_q)$* , Canad. Math. Bull. **45** (2002), 623-633.
- [20] [HK] Hoffman, Kenneth and Ray Kunze. Linear Algebra. Upper Saddle River, NJ: Prentice Hall, 1971. Print.
- [21] [H] Humphreys, James E. Introduction to Lie Algebras and Representation Theory. New York: Springer-Verlag, 1972. Print.

- [22] [J] N. Jing. *On Drinfel'd Realization of Quantum Affine Algebras*, Proc. Monster and Lie Algebras, Ohio Publ. Gruyter Verlag, 1998. 195-206.
- [23] [JM] N. Jing and K. C. Misra. *Fermionic Realization of Toroidal Lie Algebras of Classical Types*, Journal of Algebra. **324** (2010), 183-194.
- [24] [JM2] N. Jing and K. C. Misra. *Vertex Operators for Twisted Quantum Affine Algebras*, Transactions of the American Mathematical Society. **351** (1999), 1663-1690.
- [25] [JMT] Naihuan Jing, Kailash C. Misra, and Shaobin Tan. *Bosonic Realizations of Higher Level Toroidal Lie Algebras*, Pacific J. Math. **219** (2005), 183-194.
- [26] [JMX] Naihuan Jing, Kailash. C. Misra, and Chongbin Xu. *Bosonic Realization of Toroidal Lie Algebras of Classical Types*, Proceedings of the American Mathematical Society. **324** 11, (2009), 3609-3618.
- [27] [JZ] N. Jing and H. Zhang. *Braid group action and twisted quantum affine algebras*, Accessed on May 9, 2013 at <http://arxiv.org/pdf/1301.3550.pdf>. January 2013.
- [28] [K] Kac, Victor G. Infinite Dimensional Lie Algebras. Cambridge: Cambridge University Press, 1990. Print.
- [29] [K2] Kac, Victor Vertex Algebras for Beginners, 2nd Ed. Providence, RI: American Mathematical Society, 1998. Print.
- [30] [Kas] Kassel, Christian. *Kähler Differentials and Coverings of Complex Simple Lie Algebras Extended Over a Commutative Algebra*, Journal of Pure and Applied Algebra. **34** (1984), 265-275.
- [31] [KP] V. G. Kac and D. H. Peterson. *Spin and Wedge Representations of Infinite-Dimensional Lie Algebras and Groups*, Proc. Natl Acad. Sci. USA. **78** (1981), 3308-3312.

- [32] [L] M. Lau. *Bosonic and Fermionic Representations of Lie Algebra Central Extensions*, Adv. Math. **194** (2005), 225-245.
- [33] [Li] Haisheng Li. *A New Construction of Vertex Algebras and Quasi-Modules for Vertex Algebras*, Adv. Math. **202** (2006), 232-286.
- [34] [LT] Hai Feng Lian and Shaobin Tan. *Structure and Representation for a Class of Infinite-Dimensional Lie Algebras*, J. Algebra. **307** (2007), 804-828.
- [35] [LTW] Haisheng Li, Shaobin Tan, and Qing Wang. *Twisted Modules for Quantum Vertex Algebras*, Journal of Pure and Applied Algebra. **214** (2010), 201-220.
- [36] [Me] Meyer, Carl D. Matrix Analysis and Applied Linear Algebra. Philadelphia: Society for Industrial and Applied Mathematics, 2000. Print.
- [37] [Mi] Misra, Kailash C. Lecture Notes in Lie Algebra. Raleigh, NC: East Coast Digital Printing, 2010. Print.
- [38] [MRY] R. V. Moody and S. E. Rao and T. Yokonuma. *Toroidal Lie Algebras and Vertex Representations*, Geometriae Dedicata. **35** (1990), 283-307.
- [39] [MY] J. Morita and Y. Yoshii. *Universal Central Extensions of Chevalley Algebras Over Laurent Series Polynomial Rings and GIM Lie Algebras*, Proc. Japan Acad. Ser. A. **61** (1985), 179-181.
- [40] [R] S. E. Rao. *Irreducible Representations of the Lie Algebra of the Diffeomorphisms of a d-Dimensional Torus*, Journal of Algebra. **182** (1996), 401-421.
- [41] [T1] Shaobin Tan. *A Study of Vertex Operator Constructions for Some Infinite Dimensional Lie Algebras*, PhD Thesis. The University of Saskatchewan, 1998.
- [42] [T2] Shaobin Tan. *Principal Construction of the Toroidal Lie Algebra of Type A_1* , Mathematische Zeitschrift. **230** (1999), 621-657.

- [43] [T3] Shaobin Tan. *Vertex Operator Representations for Toroidal Lie Algebra of Type B_l* , Commun. in Algebra. **27** (1999), 3593-3618.
- [44] [vdL] Johan van de Leur. *Twisted Toroidal Lie Algebras*. Accessed August 13, 2012 at <http://arxiv.org/pdf/math/0106119.pdf>, February 2008.
- [45] [W] R. L. Wilson. *Euclidean Lie Algebras are Universal Central Extensions*, Lie Algebras and Related Topics, David Winter (ed.) New Brunswick, NJ: Springer Berlin Heidelberg. **933** (1982), 210-213.
- [46] [Wk] G. C. Wick, *Phys. Rev.*, **80** (1950), 268-272.
- [47] [XH] L. Xia and N. Hu. *Irreducible Representations for Virasoro-Toroidal Lie Algebras*, J. Pure Appl. Algebra. **194** (2004), 213-237.