ABSTRACT

KHUHIRUN, BORWORN. Classification of Nilpotent Lie Algebras with Small Breadth. (Under the direction of Ernest L. Stitzinger and Kailash C. Misra.)

A Lie algebra, L is said to be of breadth k if the maximal dimension of the images of left multiplication by elements of the algebra is k, introduced by Leedham-Green, Neumann and Wiegold. Inspired by the work of Parmeggani and Stellmacher on finite p-groups, we characterize nilpotent Lie algebras of breadth 1 and 2. We show that a nilpotent Lie algebra L has breadth 1 if and only if the derived algebra of L has dimension 1 which is equivalent to L being a Heisenberg Lie algebra with possible abelian direct summands. The nilpotent Lie algebra L has breadth 2 if and only if either the derived algebra of L has dimension 2 or the derived algebra and central quotient both have dimension 3. These results parallel results in finite p-groups. Unlike its group theory counter part, we use our characterization to determine the isomorphism classes of nilpotent Lie algebras of breadth 1 and 2. In this classification we focus on Lie algebras with no abelian direct summand, algebras which we call pure. So our classification results are always for pure nilpotent Lie algebras. One can harmlessly add abelian direct summands to these algebras to get further examples. For breadth 2, we determine the isomorphism classes of all Lie algebras with three dimensional derived algebra and all Lie algebras with two dimensional derived algebra and one dimensional center. For the only remaining case, where the derived algebra and center both have dimension two, we classify the algebras up to dimension six.

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Classification of Nilpotent Lie Algebras with Small Breadth

by
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BIOGRAPHY

Early in the morning of September 21, 1984, Borworn Khuhirun was born in Lopburi, Thailand. Two months later, he moved to Bangkok, Thailand, raised by his two aunts. He graduated from Horwang School before studying Mathematics at Chulalongkorn University for undergraduate and master degrees. Later in 2009, he got a scholarship for studying abroad and then chose to attend doctoral program at North Carolina State University .

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Chapter 1

Introduction

Classifying algebraic objects is a central theme of mathematical research. A paramount example is the classification of finite dimensional complex simple Lie algebras due to Killing and Cartan. Less progress has been made for other classes of Lie algebras. In particular, the vast number of nilpotent Lie algebras has made the classification problem formidable for this class. As a result, authors have made progress by classifying nilpotent Lie algebras satisfying certain conditions. Research in finite group theory has followed a similar path. Simple groups have been classified, but the large number of p-groups has led researchers to investigate p-groups with added conditions. An example is the work of Parmeggiani and Stellmacher where the concept of breadth is used. In particular, they have given characterizations of finite p-groups of breadth 1 and 2. However, so far there does not exist a classification of these finite p-groups.

The analogous concept of breadth for Lie algebra has been introduced by Leedham-Green, Neumann and Wiegold. They define the breadth of a Lie algebra to be the maximum of the dimensions of the images of ad_x where x runs over the algebra. We consider this concept for finite dimensional nilpotent Lie algebras and give a characterization for breadth 1 and 2. In particular, we show that a finite dimensional nilpotent Lie algebra is of breadth 1 if and only if its derived algebra is one dimensional. We also show that a finite dimensional nilpotent Lie algebra L has breadth 2 if and only if either the derived algebra of L has dimension 2 or the derived algebra and the central quotient both have dimension 3. These results parallel results in finite p-groups.

Finally we use our characterizations to classify finite dimensional nilpotent Lie algebras of breadth 1 and 2. We define a nilpotent Lie algebra to be *pure* if it does not have abelian direct summands. Then we classify finite dimensional pure nilpotent Lie algebras of breadth one and two since abelian summands can be added harmlessly. In particular, we show that a finite dimensional pure nilpotent Lie algebra of breadth 1 is isomorphic to a Heisenberg Lie algebra. For a finite dimensional pure nilpotent Lie algebras L, the center is contained in the derived algebra. By our characterization result, the dimension of the derived algebra of a finite dimensional pure nilpotent Lie algebra L of breadth 2 is either 2 or 3. We determine the isomorphism classes of finite dimensional pure nilpotent Lie algebras of breadth two with three dimensional derived algebra. We also determine the isomorphism classes of finite dimensional pure nilpotent Lie algebras of breadth two with two dimensional derived algebra and one dimensional center. For the remaining case where the derived algebra and center coincide with dimension 2, we determine their isomorphism classes up to dimension 6. We hope these classification results will lead to corresponding classification results in finite p-groups.

Chapter 2

Preliminaries

We begin this chapter by introducing some basic definitions and notations we use throughout this paper. All of these following definitions and notations can be found in Humphreys and lecture notes in Lie algebra. We consider finite dimensional Lie algebra together with underlying field $\mathbb{F}$ such that $\mathrm{char}(\mathbb{F}) \neq 2$ for the most of the first half. Meanwhile, we develop our focus to finite dimensional nilpotent Lie algebra over $\mathbb{F}$ in the second half. Let L be a Lie algebra. Define a sequence of ideals of L called *lower central series* or *decending central series* by

$$L^0 \supseteq L^1 \supseteq L^2 \supseteq L^3 \supseteq \ldots \supseteq L^m \supseteq \ldots$$

where $L^0 = L, L^1 = [L,L], L^2 = [L,[L,L]] = [L,L^1], L^3 = [L,L^2], \ldots, L^m = [L,L^{m-1}], \ldots$. L is said to be *nilpotent* if $L^m = \{0\}$ for some $m \in \mathbb{Z}_{\geq 0}$.

During classification process, nilpotency of Lie algebra and its center play important roles, so we would like to provide some facts about them. Every nilpotent Lie algebra has nontrivial center. Furthermore, homomorphic image and quotient of nilpoent Lie algebra are nilpotent. Note that when we consider a Lie algebra L as a finite dimensional vector space, we could apply rank-nullity theorem in order to get

$$\dim L = \mathrm{nullity}\, \varphi + \mathrm{rank}\, \varphi$$

where $\varphi : L \to L$ is a Lie algebra homomorphism.

Chapter 3

Nilpotent Lie Algebras of Breadth 1

3.1 Basic Definitions and Properties of Breadth

We start this section with definitions and properties of breadth on Lie algebra developed from group theory.

Definition 3.1.1. Let L be a finite dimensional Lie algebra. For any $x \in L$, *breadth* of x, denoted by $b(x)$ is

$$\begin{aligned} b(x) &= \dim(L/\ker ad_x) \\ &= \dim L - \text{nullity}\, ad_x \\ &= \text{rank}\, ad_x. \end{aligned}$$

More generally, for any ideal A of L, we define

$$\begin{aligned} b_A(x) &= \dim(A/\ker ad_x|_A) \\ &= \dim A - \text{nullity}\, ad_x|_A \\ &= \text{rank}\, ad_x|_A. \end{aligned}$$

Definition 3.1.2. Let L be a finite dimensional Lie algebra. We define *breadth* of L, denoted by $b(L)$ to be

$$b(L) = \max\{b(x) \mid x \in L\}.$$

Moreover, for any ideal A of L, we have

$$b_A(L) = \max\{b_A(x) \mid x \in L\}.$$

Remark. *Let L be a finite dimensional Lie algebra. Then $Z(L) = \{x \in L \mid b(x) = 0\}$.*

Remark. *Let L be a finite dimensional Lie algebra and A an ideal of L. Then the following hold for any $x \in L$:*

1. $b_A(x) \leq b(x)$.
2. $b_A(L) \leq b(L)$.

Definition 3.1.3. Let L be a finite dimensional Lie algebra and A an ideal of L. We define

$$\begin{aligned} B &= \{x \in L \mid b(x) = b(L)\}, \\ B_A &= \{x \in L \mid b_A(x) = b_A(L)\}, \\ T_A &= \{x \in L \mid b_A(x) = 1\}. \end{aligned}$$

Next proposition shows that Lie algebra which has breadth equal to zero is equivalent to that Lie algebra is abelian.

Proposition 3.1.4. *Let L be a finite dimensional Lie algebra. Then $b(L) = 0$ if and only if L is abelian.*

Proof. Let L be a finite dimensional Lie algebra. Then it is easy to see that

$$\begin{aligned} b(L) = \max\{b(x) \mid x \in L\} = 0 &\Longleftrightarrow b(x) = \operatorname{rank} ad_x = 0 \quad \forall x \in L \\ &\Longleftrightarrow ad_x = 0 \quad \forall x \in L \\ &\Longleftrightarrow [L, L] = \{0\} \\ &\Longleftrightarrow L \text{ is abelian.} \end{aligned}$$

□

Lemma 3.1.5. *Let L be a finite dimensional Lie algebra and A an ideal of L. Then $b(L) \leq \dim[L, L]$ and $b_A(L) \leq \dim[A, L]$.*

Proof. For any $x \in L$, we have $ad_x : L \to [L, L]$. Then $\operatorname{im} ad_x \subseteq [L, L]$, so we obtain $b(x) = \operatorname{rank} ad_x \leq \dim[L, L]$. Since $x \in L$ is arbitrary, $b(L) \leq \dim[L, L]$. Similarly, we also have $ad_x|_A : L \to [A, L]$, so $\operatorname{im} ad_x|_A \subseteq [A, L]$. Thus $b_A(x) = \operatorname{rank} ad_x|_A \leq \dim[A, L]$. Because $x \in L$ is arbitrary, $b_A(L) \leq \dim[A, L]$. □

Corollary 3.1.6. *Let L be a finite dimensional Lie algebra. Suppose that there exists $x \in L$ such that $b(x) = \dim[L, L]$. Then $b(L) = \dim[L, L]$. In particular, let A be an ideal of L. Suppose that there exists $x \in L$ such that $b_A(x) = \dim[A, L]$. Then $b_A(L) = \dim[A, L]$.*

Proof. Let L be a finite dimensional Lie algebra. Suppose that there exists $x \in L$ such that $b(x) = \dim[L, L]$. By Lemma 3.1.5, we know that $b(L) \leq \dim[L, L]$. Therefore we have

$$\dim[L, L] = b(x) \leq b(L) \leq \dim[L, L],$$

so $b(L) = \dim[L, L]$. On the other hand, if we let A be an ideal of L and assume that there exists $x \in L$ such that $b_A(x) = \dim[A, L]$. By Lemma 3.1.5, we have $b_A(L) \leq \dim[A, L]$. Hence

$$\dim[A, L] = b_A(x) \leq b_A(L) \leq \dim[A, L],$$

so $b_A(L) = \dim[A, L]$. □

Even though we define breadth of Lie algebra to be maximum value of breadth of all elements and Lie algebra can be considered as a vector space spanned by a basis, we cannot determine breadth of Lie algebra form its basis.

Example 3.1.7. (Breadth of a Lie algebra cannot be determined from its basis)
Let $L = H_1 \oplus H_2$ where H_1 and H_2 are Heisenberg Lie algebra. Then

$$L = \text{span}\{x_1, y_1, z_1\} \oplus \text{span}\{x_2, y_2, z_2\}$$

where $[x_1, y_1] = z_1$ and $[x_2, y_2] = z_2$. Note that L is a six dimensional nilpotent Lie algebra because $[L, L] = \text{span}\{z_1, z_2\} = Z(L)$ and $[L, [L, L]] = [L, Z(L)] = \{0\}$. Observe that

$$b(x_i) = b(y_i) = 1 \quad \text{and} \quad b(z_i) = 0 \quad \text{for all} \quad i = 1, 2,$$

but $b(x_1 + x_2) = 2$ since $[x_1 + x_2, y_i] = z_i$ for all $i = 1, 2$. By Corollary 3.1.6, we have $b(L) = 2$ since $b(x_1 + x_2) = 2 = \dim[L, L]$.

Example 3.1.8. Let $L = \text{span}\{x_1, x_2, \ldots, x_n\}$ together with bracket relations defined by

$$[x_1, x_n] = 0 \quad \text{and} \quad [x_1, x_i] = x_{i+1}$$

where $i = 2, 3, \ldots, n-1$. Then L is an n-dimensional nilpotent Lie algebra of breadth $n-2$.

First, we need to show that Jacobi identity holds. Let $x, y, z \in L$. Then there exist $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{C}$ such that

$$x = a_1 x_1 + \ldots + a_n x_n,$$
$$y = b_1 x_1 + \ldots + b_n x_n,$$
$$z = c_1 x_1 + \ldots + c_n x_n.$$

Then we have

$$\begin{aligned}[x,[y,z]] &= [a_1x_1+\ldots+a_nx_n,[b_1x_1+\ldots+b_nx_n,c_1x_1+\ldots+c_nx_n]]\\ &= [a_1x_1+\ldots+a_nx_n,(b_1c_2-b_2c_1)x_3+(b_1c_3-b_3c_1)x_4+\ldots+(b_1c_{n-1}-b_{n-1}c_1)x_n]\\ &= a_1(b_1c_2-b_2c_1)x_4+a_1(b_1c_3-b_3c_1)x_5+\ldots+a_1(b_1c_{n-2}-b_{n-2}c_1)x_n,\end{aligned}$$

$$\begin{aligned}[[x,y],z] &= [[a_1x_1+\ldots+a_nx_n,b_1x_1+\ldots+b_nx_n],c_1x_1+\ldots+c_nx_n]\\ &= [(a_1b_2-a_2b_1)x_3+(a_1b_3-a_3b_1)x_4+\ldots+(a_1b_{n-1}-a_{n-1}b_1)x_n,c_1x_1+\ldots+c_nx_n]\\ &= -c_1(a_1b_2-a_2b_1)x_4-c_1(a_1b_3-a_3b_1)x_5-\ldots-c_1(a_1b_{n-2}-a_{n-2}b_1)x_n,\end{aligned}$$

$$\begin{aligned}[y,[x,z]] &= [b_1x_1+\ldots+b_nx_n,[a_1x_1+\ldots+a_nx_n,c_1x_1+\ldots+c_nx_n]]\\ &= [b_1x_1+\ldots+b_nx_n,(a_1c_2-a_2c_1)x_3+(a_1c_3-a_3c_1)x_4+\ldots+(a_1c_{n-1}-a_{n-1}c_1)x_n]\\ &= b_1(a_1c_2-a_2c_1)x_4+b_1(a_1c_3-a_3c_1)x_5+\ldots+b_1(a_1c_{n-2}-a_{n-2}c_1)x_n.\end{aligned}$$

Therefore we have

$$\begin{aligned}[[x,y],z]+[y,[x,z]] =&(-c_1(a_1b_2-a_2b_1)x_4-c_1(a_1b_3-a_3b_1)x_5-\ldots-c_1(a_1b_{n-2}-a_{n-2}b_1)x_n)\\ &+(b_1(a_1c_2-a_2c_1)x_4+b_1(a_1c_3-a_3c_1)x_5+\ldots+b_1(a_1c_{n-2}-a_{n-2}c_1)x_n)\\ =&(-a_1b_2c_1+a_2b_1c_1+a_1b_1c_2-a_2b_1c_1)x_4+(-a_1b_3c_1+a_3b_1c_1+a_1b_1c_3\\ &-a_3b_1c_1)x_5+\ldots+(-a_1b_{n-2}c_1+a_{n-2}b_1c_1+a_1b_1c_{n-2}-a_{n-2}b_1c_1)x_n\\ =&(-a_1b_2c_1+a_1b_1c_2)x_4+(-a_1b_3c_1+a_1b_1c_3)x_5+\ldots+(-a_1b_{n-2}c_1\\ &+a_1b_1c_{n-2})x_n\\ =&a_1(b_1c_2-b_2c_1)x_4+a_1(b_1c_3-b_3c_1)x_5+\ldots+a_1(b_1c_{n-2}-b_{n-2}c_1)x_n\\ =&[x,[y,z]].\end{aligned}$$

As a result, the Jacobi identity holds, so L is an n-dimensional Lie algebra.

Observe that $[L,L]=\text{span}\{x_3,x_4,\ldots,x_n\}$ which is $(n-2)$-dimensional and $L^n=\{0\}$, so L is nilpotent. Moreover, by Corollary 3.1.6, we have $b(L)=n-2$ since $b(x_1)=n-2=\dim[L,L]$. Hence L is an n-dimensional nilpotent Lie algebra of breadth $n-2$.

Theorem 3.1.9. *Let L be a finite dimensional Lie algebra such that $b(L)=n\in\mathbb{Z}_{>0}$. Then $\dim(L/Z(L))\geq n+1$.*

Proof. Let L be a finite dimensional Lie algebra such that $b(L)=n\in\mathbb{Z}_{>0}$. Let $x\in B$. Then $b(x)=b(L)=n$, so there exist $y_1,y_2,\ldots,y_n\in L$ and $z_1,z_2,\ldots,z_n\in[L,L]$ such that $[x,y_i]=z_i$

for all $i = 1, 2, \ldots, n$ and $\{z_1, z_2, \ldots, z_n\}$ is linearly independent. Note that $\alpha \in Z(L)$ if and only if $b(\alpha) = 0$. Therefore $y_1, y_2, \ldots, y_n \notin Z(L)$ because $b(y_i) \geq 1$ for all $i = 1, 2, \ldots, n$. Thus $y_1 + Z(L), y_2 + Z(L), \ldots, y_n + Z(L) \neq Z(L)$. Similarly, we also have $x \notin Z(L)$ because $b(x) = n > 0$. Therefore $x + Z(L) \neq Z(L)$.

Next, we claim that $\{x + Z(L), y_1 + Z(L), y_2 + Z(L), \ldots, y_n + Z(L)\} \subseteq L/Z(L)$ is linearly independent, let $a_0, a_1, a_2, \ldots, a_n \in \mathbb{F}$ be such that

$$a_0(x + Z(L)) + a_1(y_1 + Z(L)) + a_2(y_2 + Z(L)) + \ldots + a_n(y_n + Z(L)) = Z(L).$$

Then $a_0x + a_1y_1 + a_2y_2 + \ldots + a_ny_n \in Z(L)$, so we have

$$\begin{aligned} 0 &= [x, a_0x + a_1y_1 + a_2y_2 + \ldots + a_ny_n] \\ &= a_0[x, x] + a_1[x, y_1] + a_2[x, y_2] + \ldots + a_n[x, y_n] \\ &= a_1z_1 + a_2z_2 + \ldots + a_nz_n. \end{aligned}$$

Since $\{z_1, z_2, \ldots, z_n\}$ is linearly independent, $a_1, a_2, \ldots, a_n = 0$. By assumption, we also get $a_0(x + Z(L)) = Z(L)$. Since $x + Z(L) \neq Z(L)$, $a_0 = 0$. Consequently, $a_0, a_1, a_2, \ldots, a_n = 0$, so $\{x + Z(L), y_1 + Z(L), y_2 + Z(L), \ldots, y_n + Z(L)\}$ is a linearly independent subset of $L/Z(L)$. Hence $\dim(L/Z(L)) \geq n + 1$. □

Lemma 3.1.10. *Let L be a finite dimensional Lie algebra. Suppose that* $\dim(L/Z(L)) = n \in \mathbb{Z}_{>0}$. *Then* $\dim[L, L] \leq \binom{n}{2}$.

Proof. Let L be a finite dimensional Lie algebra. Then there exists $m \in \mathbb{Z}_{\geq 0}$ such that $Z(L) = \text{span}\{x_1, x_2, \ldots, x_m\}$. Then we extend this basis to $L = \text{span}\{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\}$. Since $x_1, x_2, \ldots, x_r \in Z(L)$, we have

$$\begin{aligned} [L, L] = \text{span}\{&[y_1, y_2], [y_1, y_3], \ldots, [y_1, y_n], \\ &[y_2, y_3], \ldots, [y_2, y_n], \\ &\ldots, [y_{n-1}, y_n]\} \end{aligned}$$

and then

$$\begin{aligned} \dim[L, L] &= \dim \text{span}\{[y_1, y_2], [y_1, y_3], \ldots, [y_{n-1}, y_n]\} \\ &\leq n + (n-1) + (n-2) + \ldots + 1 \\ &= \frac{n}{2}(n-1) \\ &= \binom{n}{2}. \end{aligned}$$

Hence $\dim[L, L] \leq \binom{n}{2}$ as we wanted. $\square$

Finally, we are going to show that breadth of the direct sum of finite dimensional Lie algebras is equal to sum of their breadthes.

Lemma 3.1.11. *Let L_1 and L_2 be finite dimensional Lie algebras. Then $b_{L_1 \oplus L_2}(x_1 + x_2) = b_{L_1}(x_1) + b_{L_2}(x_2)$ for any $x_1 \in L_1$ and $x_2 \in L_2$.*

Proof. Let L_1 and L_2 be finite dimensional Lie algebras and $L = L_1 \oplus L_2$. Let $x_1 \in L_1$ and $x_2 \in L_2$. Since $L = L_1 \oplus L_2$, we know that $[L_1, L_2] = L_1 \cap L_2 = \{0\}$, so L_1 and L_2 can be considered as ideals of L. Then we have

$$\begin{aligned} \operatorname{im} ad_{x_1+x_2}|_{L_1} &= [x_1 + x_2, L_1] = [x_1, L_1] + [x_2, L_1] = \operatorname{im} ad_{x_1}|_{L_1}, \\ \operatorname{im} ad_{x_1+x_2}|_{L_2} &= [x_1 + x_2, L_2] = [x_1, L_2] + [x_2, L_2] = \operatorname{im} ad_{x_2}|_{L_2}, \end{aligned}$$

so we get

$$\begin{aligned} \operatorname{im} ad_{x_1+x_2}|_{L_1 \oplus L_2} &= [x_1 + x_2, L_1 \oplus L_2] \\ &= [x_1 + x_2, L_1] \oplus [x_1 + x_2, L_2] \\ &= \operatorname{im} ad_{x_1+x_2}|_{L_1} \oplus \operatorname{im} ad_{x_1+x_2}|_{L_2} \\ &= \operatorname{im} ad_{x_1}|_{L_1} \oplus \operatorname{im} ad_{x_2}|_{L_2}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} b_{L_1 \oplus L_2}(x_1 + x_2) &= \operatorname{rank} ad_{x_1+x_2}|_{L_1 \oplus L_2} \\ &= \dim \operatorname{im} ad_{x_1+x_2}|_{L_1 \oplus L_2} \\ &= \dim(\operatorname{im} ad_{x_1}|_{L_1} \oplus \operatorname{im} ad_{x_2}|_{L_2}) \\ &= \dim \operatorname{im} ad_{x_1}|_{L_1} + \dim \operatorname{im} ad_{x_2}|_{L_2} \\ &= \operatorname{rank} ad_{x_1}|_{L_1} + \operatorname{rank} ad_{x_1}|_{L_2} \\ &= b_{L_1}(x_1) + b_{L_2}(x_2). \end{aligned}$$

Hence we get $b_L(x_1 + x_2) = b_{L_1}(x_1) + b_{L_2}(x_2)$ for any $x_1 \in L_1$ and $x_2 \in L_2$ as we wanted. $\square$

Theorem 3.1.12. *Let L_1 and L_2 be finite dimensional Lie algebras. Then $b(L_1 \oplus L_2) = b(L_1) + b(L_2)$.*

Proof. Let L_1 and L_2 be finite dimensional Lie algebras. Then there exist $x_1 \in L_1$ and $x_2 \in L_2$ such that $b_{L_1}(x_1) = b(L_1)$ and $b_{L_2}(x_2) = b(L_2)$, respectively. By using Lemma 3.1.11, we have

$$b(L_1) + b(L_2) = b_{L_1}(x_1) + b_{L_2}(x_2) = b_{L_1 \oplus L_2}(x_1 + x_2) \leq b(L_1 \oplus L_2).$$

On the other hand, we let $y \in L_1 \oplus L_2$. Then $y = y_1 + y_2$ for some $y_1 \in L_1$ and $y_2 \in L_2$. Note that $b_{L_1}(y_1) \leq b(L_1)$ and $b_{L_2}(y_2) \leq b(L_2)$, so we get

$$b_{L_1 \oplus L_2}(y) = b_{L_1 \oplus L_2}(y_1 + y_2) = b_{L_1}(y_1) + b_{L_2}(y_2) \leq b(L_1) + b(L_2)$$

by Lemma 3.1.11. Since $y \in L_1 \oplus L_2$ is arbitrary, we have $b(L_1 \oplus L_2) \leq b(L_1) + b(L_2)$. Hence $b(L_1 \oplus L_2) = b(L_1) + b(L_2)$. □

Corollary 3.1.13. *Let $L_1, L_2, \ldots, L_n$ be finite dimensional Lie algebras for some $n \in \mathbb{Z}_{>0}$. Then $b(L_1 \oplus L_2 \oplus \ldots \oplus L_n) = b(L_1) + b(L_2) + \ldots + b(L_n)$.*

3.2 Classification of Nilpotent Lie Algebras of Breadth 1

For Lie algebra of breadth 1, we get a result analogous to the result in group theory provided by [2].

Theorem 3.2.1. *Let L be a finite dimensional Lie algebra. Then $b(L) = 1$ if and only if* $\dim[L, L] = 1$.

Proof. Let L be a finite dimensional Lie algebra such that $\dim[L, L] = 1$. Then by Lemma 3.1.5, we have $b(L) \leq 1$. Since L is not abelian, $b(L) = 1$ by Proposition 3.1.4.

On the other hand, assume that $b(L) = 1$ and $\dim[L, L] \neq 1$. By Proposition 3.1.4, since $b(L) \neq 0$, $\dim[L, L] \neq 0$. Thus $\dim[L, L] \geq 2$. Let $z_1, z_2 \in [L, L]$ be such that $\{z_1, z_2\}$ is linearly indenpendent. Then there exist $x_1, x_2, y_1, y_2 \in L$ such that $[x_1, y_1] = z_1$ and $[x_2, y_2] = z_2$. Note that

$$ad_{x_1}(y_1) = z_1, \quad ad_{y_1}(x_1) = -z_1,$$
$$ad_{x_2}(y_2) = z_2, \quad ad_{y_2}(x_2) = -z_2,$$

and $b(L) = 1$, so $b(x_1) = b(x_2) = b(y_1) = b(y_2) = 1$. Therefore we have

$$ad_{x_1}, ad_{y_1} : L \to \operatorname{span}\{z_1\},$$
$$ad_{x_2}, ad_{y_2} : L \to \operatorname{span}\{z_2\}.$$

Next, we consider $[x_1, x_2] = ad_{x_1}(x_2) \in \operatorname{span}\{z_1\}$. On the other hand, $[x_1, x_2] = -ad_{x_2}(x_1) \in \operatorname{span}\{z_2\}$. Thus $[x_1, x_2] \in \operatorname{span}\{z_1\} \cap \operatorname{span}\{z_2\} = \{0\}$, so $[x_1, x_2] = 0$. Similarly, we also get $[x_1, y_2] = [y_1, x_2] = [y_1, y_2] = 0$. Consequently, we obtain

$$[x_1 + x_2, y_1] = [x_1, y_1] + [x_2, y_1] = z_1,$$

$$[x_1 + x_2, y_2] = [x_1, y_2] + [x_2, y_2] = z_2.$$

Since $\{z_1, z_2\}$ is linearly indenpendent, $b(x_1+x_2) = \text{rank}\, ad_{x_1+x_2} \geq 2$, which contradicts $b(L) = 1$. Hence $\dim[L, L] = 1$ by contradiction. □

Lemma 3.2.2. *Let L be a finite dimensional Lie algebra of breadth 1. Then L is nilpotent if and only if $[L, L] \subseteq Z(L)$.*

Proof. Let L be a finite dimensional Lie algebra of breadth 1. Then by Theorem 3.2.1, we know that $\dim[L, L] = 1$. Suppose that L is nilpotent and $[L, L] \nsubseteq Z(L)$. Thus $[L, L] \cap Z(L) = \{0\}$ because $\dim[L, L] = 1$. Let $x \in [L, L] - \{0\}$. Then $x \notin Z(L)$, so there exists $y \in L$ such that $[y, x] \neq 0$. Therefore $[y, x] = \alpha x$ for some $\alpha \neq 0$. Consequently, we have $ad_y^N(x) = \alpha^N x \neq 0$ for all $N \in \mathbb{Z}_{>0}$ which is a contradiction. Hence $[L, L] \subseteq Z(L)$.

Conversely, suppose that $[L, L] \subseteq Z(L)$. Then we have $L^3 = [L, [L, L]] \subseteq [L, Z(L)] = \{0\}$, so $L^3 = \{0\}$. Hence L is nilpotent. □

In order to classify Lie algebra of breadth 1, we use the concept of alternate bilinear form and its application that could be found in [5] as the following:

Definition 3.2.3. Let V be a finite dimensional vector space and $\varphi(\ ,\ ) : V \times V \to \mathbb{F}$ a bilinear form on V. Then φ is called *alternate* if $\varphi(v, v) = 0$ for all $v \in V$.

Theorem 3.2.4 (cf. [6], Theorem 6.3)**.** *Let V be a finite dimensional vector space such that $\dim V = n \in \mathbb{Z}_{>0}$. Let $\varphi(\ ,\ ) : V \times V \to \mathbb{F}$ be an alternate bilinear form on V. Then there exists a basis*

$$S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z_1, \ldots, z_{n-2r}\}$$

for V such that the matrix of B relative to this basis has the form

$$B_\varphi = \text{diag}\{S_1, S_2, \ldots, S_r, 0, \ldots, 0\}$$

where $r \in \mathbb{Z}_{>0}$ such that $r \leq \frac{n}{2}$ and $S_1 = S_2 = \ldots = S_r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Finally, we classify finite dimensional Lie algebras of breadth 1 as the following theorem. Note that if we consider finite dimensional nilpotent Lie algebras, then they are actually direct sums of Heisenberg Lie algebra and abelian Lie algebra.

Theorem 3.2.5. *Let L be a finite dimensional Lie algebra of breadth 1 such that $\dim L = n \in \mathbb{Z}_{>0}$. Let $0 \neq z \in [L, L]$. Then there exists a basis*

$$S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z_1, \ldots, z_{n-2r}\}$$

for L such that

$$[v_i, v_j] = \begin{cases} z & \textit{if } \ i = -j > 0, \\ -z & \textit{if } \ i = -j < 0, \\ 0 & \textit{otherwise}, \end{cases}$$

for every $i, j \in \{\pm 1, \pm 2, \ldots, \pm r\}$ and $Z(L) = \operatorname{span}\{z_1 \ldots, z_{n-2r}\}$.

Proof. Let L be a finite dimensional Lie algebra of breadth 1 such that $\dim L = n \in \mathbb{Z}_{>0}$. Then $b(L) = 1$. By Theorem 3.2.1, $\dim[L, L] = 1$. Let $0 \neq z \in [L, L]$. Then we have $[L, L] = \operatorname{span}\{z\}$. For any $x, y \in L$, we have $[x, y] = \alpha z$ for some $\alpha \in \mathbb{F}$. Define a bilinear form $\varphi : L \times L \to \mathbb{F}$ to be $\varphi(x, y) = \alpha$. Note that φ is bilinear since bracket is bilinear. Moreover, this is an alternate form since $[x, x] = 0$ for all $x \in L$. By Theorem 3.2.4, there exists a basis

$$S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z_1, \ldots, z_{n-2r}\}$$

for L such that the matrix of φ relative to S has the form

$$B_\varphi = \operatorname{diag}\{S_1, S_2, \ldots, S_r, 0, \ldots, 0\}$$

where $r \in \mathbb{N}$ such that $r \leq \frac{n}{2}$ and $S_1 = S_2 = \ldots = S_r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
As a result, for every $i, j \in \{1, 2, \ldots r\}$,

$$\varphi(v_i, v_j) = \begin{cases} 1 & \text{if } \ i = -j > 0, \\ -1 & \text{if } \ i = -j < 0, \\ 0 & \text{otherwise}, \end{cases}$$

and $z_1, \ldots, z_{n-2r} \in Z(L)$. Hence we have

$$[v_i, v_j] = \begin{cases} z & \text{if } \ i = -j > 0, \\ -z & \text{if } \ i = -j < 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $i, j \in \{1, 2, \ldots r\}$ and $z_1, \ldots, z_{n-2r} \in Z(L)$.

Next, we will claim that $Z(L) = \operatorname{span}\{z_1 \ldots, z_{n-2r}\}$. Since $z_1, \ldots, z_{n-2r} \in Z(L)$, we get $\operatorname{span}\{z_1 \ldots, z_{n-2r}\} \subseteq Z(L)$. Conversely, without loss of generality, we let $a_1, a_{-1}, \ldots, a_r, a_{-r} \in \mathbb{F}$ such that

$$a_1 v_1 + a_{-1} v_{-1} + a_2 v_2 + a_{-2} v_{-2} + \ldots + a_r v_r + a_{-r} v_{-r} = 0.$$

Let $k \in \{1, 2, \ldots, r\}$. Then we have

$$0 = [v_k, a_1 v_1 + a_{-1} v_{-1} + \ldots + a_r v_r + a_{-r} v_{-r}] = a_{-k}[v_k, v_{-k}] = a_{-k} z,$$
$$0 = [v_{-k}, a_1 v_1 + a_{-1} v_{-1} + \ldots + a_r v_r + a_{-r} v_{-r}] = a_k [v_{-k}, v_k] = a_k(-z) = -a_k z,$$

so $a_k = a_{-k} = 0$ for all $k = 1, 2, \ldots, r$. Therefore we obtain $Z(L) \subseteq \text{span}\{z_1 \ldots, z_{n-2r}\}$. Consequently, $Z(L) = \text{span}\{z_1 \ldots, z_{n-2r}\}$. In summary, there exists a basis

$$S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z_1, \ldots, z_{n-2r}\}$$

for L such that

$$[v_i, v_j] = \begin{cases} z & \text{if } i = -j > 0, \\ -z & \text{if } i = -j < 0, \\ 0 & \text{otherwise,} \end{cases}$$

for every $i, j \in \{1, 2, \ldots r\}$ and $Z(L) = \text{span}\{z_1 \ldots, z_{n-2r}\}$. □

Theorem 3.2.6. *Let L be a finite dimensional nilpotent Lie algebra of breadth 1 such that* $\dim L = n \in \mathbb{Z}_{>0}$. *Let $0 \neq z \in [L, L]$. Then there exists a basis*

$$S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z, w_1, \ldots, w_{n-2r-1}\}$$

for L such that

$$[v_i, v_j] = \begin{cases} z & \textit{if } i = -j > 0, \\ -z & \textit{if } i = -j < 0, \\ 0 & \textit{otherwise,} \end{cases}$$

for every $i, j \in \{\pm 1, \pm 2, \ldots, \pm r\}$ and $Z(L) = \text{span}\{z, w_1 \ldots, w_{n-2r-1}\}$.

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 1 such that $\dim L = n \in \mathbb{Z}_{>0}$. Let $0 \neq z \in [L, L]$. By Theorem 3.2.5, there exists a basis

$$S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z_1, \ldots, z_{n-2r}\}$$

for L such that

$$[v_i, v_j] = \begin{cases} z & \text{if } i = -j > 0, \\ -z & \text{if } i = -j < 0, \\ 0 & \text{otherwise,} \end{cases}$$

for every $i, j \in \{1, 2, \ldots r\}$ and $Z(L) = \text{span}\{z_1 \ldots, z_{n-2r}\}$. Since L is nilpotent, we have

$[L, L] \subseteq Z(L)$ by Lemma 3.2.2. Thus we have $z \in Z(L)$, so we can pick z as a basis element so that $Z(L) = \text{span}\{z, w_1 \ldots, w_{n-2r-1}\}$ and $S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z, w_1, \ldots, w_{n-2r-1}\}$. As a result, there exists a basis

$$S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z, w_1, \ldots, w_{n-2r-1}\}$$

for L such that

$$[v_i, v_j] = \begin{cases} z & \text{if } \; i = -j > 0, \\ -z & \text{if } \; i = -j < 0, \\ 0 & \text{otherwise,} \end{cases}$$

for every $i, j \in \{\pm 1, \pm 2, \ldots, \pm r\}$ and $Z(L) = \text{span}\{z, w_1 \ldots, w_{n-2r-1}\}$. □

Chapter 4

Nilpotent Lie Algebras of Breadth 2

4.1 Properties and Lemmas

In this section, we prove several theorems that we need to use in our main theorem in the next section. We begin this part with a defintion which is derived from centralizer in group theory.

Definition 4.1.1. Let L be a finite dimensional Lie algebra, A an ideal of L and $S \subseteq L$. We define

$$\begin{aligned} C_A(S) &= \{\alpha \in A \mid ad_\alpha(x) = 0 \text{ for all } x \in S\} \\ &= \{\alpha \in A \mid ad_x(\alpha) = 0 \text{ for all } x \in S\} \\ &= \bigcap_{x \in S} \{\alpha \in A \mid ad_x(\alpha) = 0\} \\ &= \bigcap_{x \in S} \ker ad_x|_A. \end{aligned}$$

In particular, we have

$$C_L(A) = \bigcap_{a \in A} \ker ad_a$$

and if $S = \{x\}$ for some $x \in L$, then

$$C_A(S) = C_A(\{x\}) = \ker ad_x|_A.$$

Remark. *Let L be a finite dimensional Lie algebra and A an ideal of L. Then the following holds:*

1. *If $S_1 \subseteq S_2 \subseteq L$, then $C_A(S_2) \subseteq C_A(S_1)$.*

2. *If* $S = \operatorname{span}\{x_1, x_2, \ldots, x_n\} \subseteq L$ *for some* $n \in \mathbb{Z}_{>0}$, *then*

$$C_A(S) = \bigcap_{x \in S} \ker ad_x|_A = \bigcap_{i=1}^{n} \ker ad_{x_i}|_A.$$

3. *For any* $x \in L$, $x \in C_L(A)$ *if and only if* $b_A(x) = 0$.

Lemma 4.1.2. *Let* L *be a finite dimensional Lie algebra and* A *an ideal of* L. *Let* $x, y \in L$ *be such that* $\operatorname{im} ad_x|_A \cap \operatorname{im} ad_y|_A = \{0\}$. *Then* $\ker ad_x|_A \subseteq \ker ad_y|_A$ *or* $b_A(x) < b_A(x+y)$.

Proof. Let L be a finite dimensional Lie algebra and A an ideal of L. Let $x, y \in L$ be such that $\operatorname{im} ad_x|_A \cap \operatorname{im} ad_y|_A = \{0\}$. Suppose that there exists $\alpha \in A$ such that $\alpha \in \ker ad_x|_A - \ker ad_y|_A$. Then we have $[x, \alpha] = 0$ but $[y, \alpha] \neq 0$. We observe that

$$[x+y, \alpha] = [x, \alpha] + [y, \alpha] = 0 + [y, \alpha] = [y, \alpha] \neq 0,$$

so $\operatorname{im} ad_{x+y}|_A \cap \operatorname{im} ad_y|_A$ is not trivial. Note that we have

$$\dim(\operatorname{im} ad_{x+y}|_A + \operatorname{im} ad_y|_A) = \dim \operatorname{im} ad_{x+y}|_A + \dim \operatorname{im} ad_y|_A - \dim(\operatorname{im} ad_{x+y}|_A \cap \operatorname{im} ad_y|_A)$$

As a result,

$$\dim(\operatorname{im} ad_{x+y}|_A + \operatorname{im} ad_y|_A) < \dim \operatorname{im} ad_{x+y}|_A + \dim \operatorname{im} ad_y|_A.$$

We define a map $\varphi : \operatorname{im} ad_x|_A \times \operatorname{im} ad_y|_A \to \operatorname{im} ad_{x+y}|_A + \operatorname{im} ad_y|_A$ by $\varphi(x_1, y_1) = x_1 + y_1$ where $x_1 \in \operatorname{im} ad_x|_A$ and $y_1 \in \operatorname{im} ad_y|_A$. We will show that φ is an isomorphism. Let $(x_1, y_1), (x_2, y_2) \in \operatorname{im} ad_x|_A \times \operatorname{im} ad_y|_A$. Then we have $x_1, x_2 \in \operatorname{im} ad_x|_A$ and $y_1, y_2 \in \operatorname{im} ad_y|_A$, so there exist $a_1, a_2, b_1, b_2 \in A$ such that $[x, a_i] = x_i$ and $[y, b_i] = y_i$ for all $i = 1, 2$. We will verify that $\operatorname{im} \varphi \subseteq \operatorname{im} ad_{x+y}|_A + \operatorname{im} ad_y|_A$. Observe that

$$\varphi(x_1, y_1) = x_1 + y_1 = [x, a_1] + [y, b_1] = [x+y, a_1] + [y, b_1 - a_1] \in \operatorname{im} ad_{x+y}|_A + \operatorname{im} ad_y|_A.$$

Thus $\operatorname{im} \varphi \subseteq \operatorname{im} ad_{x+y}|_A + \operatorname{im} ad_y|_A$. Moreover, it is easy to see that φ is linear. In order to show that φ is injective, we suppose that $\varphi(x_1, y_1) = 0$. Then we get $[x, a_1] + [y, b_1] = x_1 + y_1 = \varphi(x_1, y_1) = 0$, so $[x, a_1] = [y, -b_1] \in \operatorname{im} ad_x|_A \cap \operatorname{im} ad_y|_A = \{0\}$. Therefore $x_1 = [x, a_1] = 0$ and $y_1 = [y, b_1] = 0$ which implies $(x_1, y_1) = 0$. Thus φ is injective. To claim that φ is surjective, let $z \in \operatorname{im} ad_{x+y}|_A + \operatorname{im} ad_y|_A$. Then $z = z_1 + z_2$ where $z_1 \in \operatorname{im} ad_{x+y}|_A$ and $z_2 \in \operatorname{im} ad_y|_A$. There exists $c_1, c_2 \in A$ such that $[x+y, c_1] = z_1$ and $[y, c_2] = z_2$. Note that we have

$$\varphi([x, c_1], [y, c_1 + c_2]) = [x, c_1] + [y, c_1 + c_2] = [x+y, c_1] + [y, c_2] = z_1 + z_2 = z.$$

Therefore φ is surjective. Hence $\varphi : \operatorname{im} ad_x|_A \times \operatorname{im} ad_y|_A \to \operatorname{im} ad_{x+y}|_A + \operatorname{im} ad_y|_A$ is an isomorphism which implies

$$\dim(\operatorname{im} ad_x|_A \times \operatorname{im} ad_y|_A) = \dim(\operatorname{im} ad_{x+y}|_A + \operatorname{im} ad_y|_A).$$

As a consequence, we have

$$\begin{aligned} b_A(x) + b_A(y) &= \operatorname{rank} ad_x|_A + \operatorname{rank} ad_y|_A \\ &= \dim \operatorname{im} ad_x|_A + \dim \operatorname{im} ad_y|_A \\ &= \dim(\operatorname{im} ad_x|_A \times \operatorname{im} ad_y|_A) \\ &= \dim(\operatorname{im} ad_{x+y}|_A + \operatorname{im} ad_y|_A) \\ &< \dim \operatorname{im} ad_{x+y}|_A + \dim \operatorname{im} ad_y|_A \\ &= \operatorname{rank} ad_{x+y}|_A + \operatorname{rank} ad_y|_A \\ &= b_A(x+y) + b_A(y). \end{aligned}$$

Hence $b_A(x) < b_A(x+y)$. □

Theorem 4.1.3. *Let L be a finite dimensional Lie algebra and A an ideal of L. Let $x, y \in T_A$, $T = \operatorname{span}\{x, y\}$ and*

$$\bar{T} = (T + C_L(A))/C_L(A) \cong T/(T \cap C_L(A)).$$

Then we have the following two cases:

1. *If $\dim \bar{T} = 1$, then $\operatorname{im} ad_x|_A = \operatorname{im} ad_y|_A$ and $\ker ad_x|_A = \ker ad_y|_A$.*
2. *If $\dim \bar{T} = 2$, then $\operatorname{im} ad_x|_A \neq \operatorname{im} ad_y|_A$ or $\ker ad_x|_A \neq \ker ad_y|_A$.*

However, if $x + y \in T_A$, then the consequence of the second case is "either or".

Proof. Note that $\bar{T} = (T + C_L(A))/C_L(A) \cong T/(T \cap C_L(A))$ comes from second isomorphism theorem ($(S + I)/I \cong S/(S \cap I)$). For the first case, we assume that $\dim \bar{T} = 1$. Then we have $\dim(T \cap C_L(A)) = \dim T - \dim \bar{T} = 2 - 1 = 1$. Since $x, y \in T_A$, $b_A(x) = b_A(y) = 1$. Therefore $x, y \notin C_L(A)$, so $x, y \notin T \cap C_L(A)$. By considering $T/(T \cap C_L(A))$ as a 1-dimensional quotient space, there exists $\alpha \in \mathbb{F} - \{0\}$ such that $x + (T \cap C_L(A)) = -\alpha y + (T \cap C_L(A))$, so $x + \alpha y \in (T \cap C_L(A))$. Since $\dim(T \cap C_L(A)) = 1$ and $x + \alpha y \neq 0$, we have $T \cap C_L(A) = \operatorname{span}\{x + \alpha y\}$. First, we will claim that $\operatorname{im} ad_x|_A = \operatorname{im} ad_y|_A$. To show that $\operatorname{im} ad_x|_A \subseteq \operatorname{im} ad_y|_A$, let $z \in \operatorname{im} ad_x|_A$. Then there exists $a \in A$ such that $[x, a] = z$. Because $x + \alpha y \in T \cap C_L(A)$, $[x + \alpha y, a] = 0$. Therefore we get

$$0 = [x + \alpha y, a] = [x, a] + \alpha[y, a] = z - [y, -\alpha a],$$

so $z = [y, -\alpha a]$, that is $z \in \operatorname{im} ad_y|_A$. Conversely, let $z \in \operatorname{im} ad_y|_A$. Then there exists $a \in A$ such that $[y, a] = z$. Again, we get

$$0 = [x + \alpha y, a] = [x, a] + \alpha[y, a] = [x, a] + \alpha z,$$

so $z = \frac{-1}{\alpha}[x, a] = [x, \frac{-a}{\alpha}]$. Thus $z \in \operatorname{im} ad_x|_A$. Hence $\operatorname{im} ad_x|_A = \operatorname{im} ad_y|_A$. Next, we will show that $\ker ad_x|_A = \ker ad_y|_A$. Suppose $\ker ad_x|_A \neq \ker ad_y|_A$. Without loss of generality, there exists $a \in A$ such that $[y, a] = 0$ but $[x, a] \neq 0$. Then we have

$$[x + \alpha y, a] = [x, a] + \alpha[y, a] = [x, a] + 0 = [x, a] \neq 0$$

which contradicts $x + \alpha y \in T \cap C_L(A)$. Hence $\ker ad_x|_A = \ker ad_y|_A$.

For the second case, suppose that $\dim \bar{T} = 2$. Then we have $\dim(T \cap C_L(A)) = \dim T - \dim \bar{T} = 2 - 2 = 0$, so $T \cap C_L(A) = \{0\}$. Suppose that $\ker ad_x|_A = \ker ad_y|_A$. Then we have to show that $\operatorname{im} ad_x|_A \neq \operatorname{im} ad_y|_A$. Let $\alpha \in \mathbb{F}$. Then $x + \alpha y \in T - \{0\}$, so $x + \alpha y \notin C_L(A)$. Thus there exists $a \in A$ such that

$$0 \neq [x + \alpha y, a] = [x, a] + \alpha[y, a].$$

Note that $a \notin \ker ad_x|_A = \ker ad_y|_A$ since $[x, a] + \alpha[y, a] \neq 0$. Therefore $[x, a] \neq 0$ and $[y, a] \neq 0$, but we have

$$[x, a] \neq -\alpha[y, a] \quad \text{for all} \quad \alpha \in \mathbb{F}.$$

Since $\operatorname{rank} ad_x|_A = \operatorname{rank} ad_y|_A = 1$, $\operatorname{im} ad_x|_A \neq \operatorname{im} ad_y|_A$. Hence we proved the second case.

Finally, we will show that the consequence of the second case is "either or" if $x + y \in T_A$. Suppose additionally to the second case that $x + y \in T_A$. Then we have $b_A(x + y) = 1$. Assume that $\operatorname{im} ad_x|_A \neq \operatorname{im} ad_y|_A$. Since $x, y \in T_A$, $b_A(x) = b_A(y) = 1$, so $\operatorname{im} ad_x|_A \cap \operatorname{im} ad_y|_A = \{0\}$. By Lemma 4.1.2, we get

$$\ker ad_x|_A \subseteq \ker ad_y|_A \quad \text{or} \quad b_A(x) < b_A(x + y).$$

Because $b_A(x) = 1 = b_A(x + y)$, we have $\ker ad_x|_A \subseteq \ker ad_y|_A$. Similarly, we also have $\ker ad_y|_A \subseteq \ker ad_x|_A$ since $b_A(y) = 1 = b_A(x + y)$. Therefore $\ker ad_x|_A = \ker ad_y|_A$. On the other hand, we suppose that $\ker ad_x|_A \neq \ker ad_y|_A$. Without loss of generality, assume that $\ker ad_x|_A \nsubseteq \ker ad_y|_A$. Then $\operatorname{im} ad_x|_A \cap \operatorname{im} ad_y|_A \neq \{0\}$ by contrapositive of Lemma 4.1.2. Since $b_A(x) = b_A(y) = 1$, we get $\operatorname{im} ad_x|_A = \operatorname{im} ad_y|_A$ as desired. □

Proposition 4.1.4. *Let L be a finite dimensional Lie algebra and A an ideal of L. Let $x, y, z \in L$ be such that $y - z \notin C_L(A)$. Suppose that $b_A(x) > 1$. Then at least one of the elements*

$$y,\ z,\ y+z,\ x+y,\ x+z,\ x+y+z$$

is not in T_A.

Proof. Let L be a finite dimensional Lie algebra and A an ideal of L. Let $x, y, z \in L$ be such that $b_A(x) > 1$ and $y - z \notin C_L(A)$. Suppose that $y, z, y+z, x+y, x+z, x+y+z \in T_A$. Additionally, we let

$$\begin{aligned} T_1 &= \text{span}\{y, y-z\} = \text{span}\{y, z-y\} = \text{span}\{y, z\}, \\ T_2 &= \text{span}\{y-z, x+z\} = \text{span}\{x+y, x+z\}, \\ T_3 &= \text{span}\{y, x+z\}. \end{aligned}$$

For $i = 1, 2, 3$, T_i is not contained in $C_L(A)$, so $T_i \cap C_L(A)$ is zero or 1-dimensional which implies $\bar{T_i} = (T_i + C_L(A))/C_L(A) \cong T_i/(T_i \cap C_L(A))$ is 1 or 2-dimensional. Next, we consider

$$\begin{aligned} \text{im}\, ad_{T_1}|_A &:= \text{span}\{\text{im}\, ad_y|_A, \text{im}\, ad_z|_A\}, \\ \text{im}\, ad_{T_2}|_A &:= \text{span}\{\text{im}\, ad_{x+y}|_A, \text{im}\, ad_{x+z}|_A\}, \\ \text{im}\, ad_{T_3}|_A &:= \text{span}\{\text{im}\, ad_y|_A, \text{im}\, ad_{x+z}|_A\}. \end{aligned}$$

By Theorem 4.1.3, we know that for each $i = 1, 2, 3$,

1. If $\dim T_i = 1$, then we have $\text{im}\, ad_{T_1}|_A$ is 1-dimensional.

2. If $\dim T_i = 2$, then we have $\text{im}\, ad_{T_1}|_A$ is 2-dimensional.

Hence $\text{im}\, ad_{T_1}|_A, \text{im}\, ad_{T_2}|_A$ and $\text{im}\, ad_{T_3}|_A$ are 1 or 2-dimensional. By pigeonhole principle, there exist $\alpha, \beta \in \{1, 2, 3\}$ such that $\alpha \neq \beta$ and $\dim(\text{im}\, ad_{T_\alpha}|_A) = \dim(\text{im}\, ad_{T_\beta}|_A)$. Let $n := \dim(\text{im}\, ad_{T_\alpha}|_A) = \dim(\text{im}\, ad_{T_\beta}|_A)$. Then we consider the following two cases:

For $n = 1$, we consider

$$\begin{aligned} T_1 &= \text{span}\{y, y-z\}, \\ T_2 &= \text{span}\{y-z, x+z\}, \\ T_3 &= \text{span}\{x+z, y\}. \end{aligned}$$

Then we have $N := \text{im}\, ad_y|_A = \text{im}\, ad_{y-z}|_A = \text{im}\, ad_{x+z}|_A$ is 1-dimensional. Next, we will claim that $\operatorname{im} ad_x|_A \subseteq N$. Let $a \in A$. Since $x = (x+z) + (y-z) - y$, we get

$$[x, a] = [x+z, a] + [y-z, a] - [y, a] \in N.$$

Thus $\operatorname{im} ad_x|_A \subseteq N$, so $b_A(x) \leq 1$ which contradicts the assumption $b_A(x) > 1$.

For $n = 2$, we consider

$$\begin{aligned} T_1 &= \operatorname{span}\{y, z-y\}, \\ T_2 &= \operatorname{span}\{x+z, y-z\}, \\ T_3 &= \operatorname{span}\{y, x+z\}, \\ T_\alpha &= \operatorname{span}\{\alpha_1, \alpha_2\}, \\ T_\beta &= \operatorname{span}\{\beta_1, \beta_2\}, \end{aligned}$$

so that

$$\begin{aligned} \operatorname{im} ad_{T_\alpha}|_A &:= \operatorname{span}\{\operatorname{im} ad_{\alpha_1}|_A, \operatorname{im} ad_{\alpha_2}|_A\}, \\ \operatorname{im} ad_{T_\beta}|_A &:= \operatorname{span}\{\operatorname{im} ad_{\beta_1}|_A, \operatorname{im} ad_{\beta_2}|_A\}. \end{aligned}$$

Since $n = 2$, we obtain $\operatorname{im} ad_{\alpha_1}|_A \neq \operatorname{im} ad_{\alpha_2}|_A$ and $\operatorname{im} ad_{\beta_1}|_A \neq \operatorname{im} ad_{\beta_2}|_A$. Note that $\dim \bar{T}_\alpha = \dim \bar{T}_\beta = 2$ and for T_1, T_2, T_3, we have

$$\begin{aligned} b_A(y) = 1 &= b_A(z) = b_A(y + (z-y)), \\ b_A(x+z) = 1 &= b_A(x+y) = b_A((x+z) + (y-z)), \\ b_A(y) = 1 &= b_A(x+y+z) = b_A(y + (x+z)). \end{aligned}$$

Thus $b_A(\alpha_1 + \alpha_2) = b_A(\alpha_1) = 1$ and $b_A(\beta_1 + \beta_2) = b_A(\beta_1) = 1$, so $\alpha_1 + \alpha_2, \beta_1 + \beta_2 \in T_A$. By Theorem 4.1.3, (2) with “either or”, we have

$$\ker ad_{\alpha_1}|_A = \ker ad_{\alpha_2}|_A \quad \text{and} \quad \ker ad_{\beta_1}|_A = \ker ad_{\beta_2}|_A.$$

It is easy to see that $\ker ad_{y-z}|_A = \ker ad_{z-y}|_A$, so we have $M := \ker ad_y|_A = \ker ad_{y-z}|_A = \ker ad_{x+z}|_A$. Next we will prove that $M \subseteq \ker ad_x|_A$. Let $m \in M$. Since $x = (x+z)+(y-z)-y$, we get

$$[x, m] = [x+z, m] + [y-z, m] - [y, m] = 0.$$

Thus $M \subseteq \ker ad_x|_A$, that means $\operatorname{nullity} ad_y|_A \leq \operatorname{nullity} ad_x|_A$. Hence we have

$$\operatorname{rank} ad_x|_A = \dim A - \operatorname{nullity} ad_x|_A$$

$$
\begin{aligned}
&= (\text{nullity}\, ad_y|_A + \text{rank}\, ad_y|_A) - \text{nullity}\, ad_x|_A \\
&= (\text{nullity}\, ad_y|_A - \text{nullity}\, ad_x|_A) + \text{rank}\, ad_y|_A \\
&\leq \text{rank}\, ad_y|_A \\
&\leq 1,
\end{aligned}
$$

so $b_A(x) \leq 1$ which is a contradiction. In consequence, from the two cases above, at least one of the elements $y,\ z,\ y+z,\ x+y,\ x+z,\ x+y+z$ is not in T_A. □

From now on, we begin to consider finite dimensional nilpotent Lie algebra in order to guarantee that it has an abelian ideal.

Definition 4.1.5. Let L be a finite dimensional nilpotent Lie algebra and A an abelian ideal of L. For any $x \in L$, we define

$$
\begin{aligned}
M_x &= A + \ker ad_x, \\
L_x &= \text{span}\{M_{a+x} \mid a \in A\}, \\
D_x &= \bigcap_{a \in A} M_{a+x}.
\end{aligned}
$$

Proposition 4.1.6. *Let L be a finite dimensional nilpotent Lie algebra and A an abelian ideal of L. Let $x \in L$ and $a \in A$. Then the following hold:*

1. $A \subseteq D_x \subseteq M_{a+x}$.
2. $[A, x] = [A, a+x] = [M_{a+x}, a+x] = [D_x, a+x]$.
3. $[L_x, x] \subseteq [A, L]$.
4. $[a, M_{a+x} \cap M_x] \subseteq [A, x]$.
5. *If $L = M_x + U = M_{a+x} + U$ for some $a \in A$ and subspace U of L, then*

$$[a, L] \subseteq [U, x] + [A, x] + [A, U].$$

6. $[A, D_x] \subseteq [A, x]$.
7. *If $b_A(L) = b(L)$, then $\dim[A, L] = b_A(L)$ and $L = D_z$ for all $z \in B_A$.*

Proof. Let L be a finite dimensional nilpotent Lie algebra and A an abelian ideal of L. Let $x \in L$ and $a \in A$.

1. Since $A \subseteq A + \ker ad_{a'+x} = M_{a'+x}$ for all $a' \in A$, $A \subseteq D_x$. It is clear that $D_x \subseteq M_{a+x}$ because $0 \in A$. Hence $A \subseteq D_x \subseteq M_{a+x}$.

2. First, it is obvious that $[A, x] = [A, a + x]$ since A is abelian. Because $M_{a+x} = A + \ker ad_{a+x}$, we have

$$\begin{aligned}
[M_{a+x}, a + x] &= [A + \ker ad_{a+x}, a + x] \\
&= [A, a] + [A, x] + [\ker ad_{a+x}, a + x] \\
&= 0 + [A, x] + 0 \\
&= [A, x].
\end{aligned}$$

   Thus $[A, x] = [M_{a+x}, a+x]$. Next, we will show that $[A, x] = [D_x, a+x]$. Since $D_x \subseteq M_{a+x}$, we get $[D_x, a + x] \subseteq [M_{a+x}, a + x] = [A, x]$. Conversely, we have $[A, x] = [A, a + x] \subseteq [D_x, a + x]$ since $A \subseteq D_x$. Therefore $[A, x] = [D_x, a + x]$. Hence $[A, x] = [A, a + x] = [M_{a+x}, a + x] = [D_x, a + x]$.

3. Let $a \in A$ be arbitrary. Let $y \in M_{a+x}$. Then $y = a_y + c_y$ where $a_y \in A$ and $c_y \in \ker ad_{a+x}$. Thus $[c_y, a + x] = 0$, so $[c_y, x] = [a, c_y]$. Therefore

$$[y, x] = [a_y + c_y, x] = [a_y, x] + [c_y, x] = [a_y, x] + [a, c_y] \in [A, L].$$

   Since $y \in M_{a+x}$ and $a \in A$ is arbitrary, $[L_x, x] \subseteq [A, L]$.

4. Let $y \in M_{a+x} \cap M_x$. Since $y \in M_{a+x}$, $y = a_y + c_y$ where $a_y \in A$ and $c_y \in \ker ad_{a+x}$. Thus $[c_y, a+x] = 0$, so $[c_y, x] = [a, c_y]$. On the other hand, $y = a'_y + c'_y$ where $a'_y \in A$ and $c'_y \in \ker ad_x$ because $y \in M_x$. Therefore $[c'_y, x] = 0$, so we have

$$[y, x] = [a'_y + c'_y, x] = [a'_y, x] + [c'_y, x] = [a'_y, x].$$

   Consequently,

$$\begin{aligned}
[a, y] &= [a, a_y + c_y] \\
&= [a, a_y] + [a, c_y] \\
&= 0 + [a, c_y] \\
&= [c_y, x] \\
&= [y - a_y, x] \\
&= [y, x] - [a_y, x] \\
&= [a'_y, x] - [a_y, x]
\end{aligned}$$

$$\in [A, x].$$

Hence $[a, M_{a+x} \cap M_x] \subseteq [A, x]$.

5. Suppose that $L = M_x + U = M_{a+x} + U$ for some $a \in A$ and subspace U of L. Since $L = M_{a+x} + U$, we have

$$\begin{aligned}
[a, L] &= [a, M_{a+x} + U] \\
&= [a, A + \ker ad_{a+x} + U] \\
&= [a, A] + [a, \ker ad_{a+x}] + [a, U] \\
&= 0 + [a, \ker ad_{a+x}] + [a, U] \\
&= [a, \ker ad_{a+x}] + [a, U].
\end{aligned}$$

On the other hand, since $L = M_x + U$ and $[a + x, \ker ad_{a+x}] = 0$, we obtain

$$\begin{aligned}
[a, \ker ad_{a+x}] &= [\ker ad_{a+x}, x] \\
&\subseteq [L, x] \\
&= [M_x + U, x] \\
&= [A + \ker ad_x + U, x] \\
&= [A, x] + [\ker ad_x, x] + [U, x] \\
&= [A, x] + 0 + [U, x] \\
&= [A, x] + [U, x],
\end{aligned}$$

so $[a, \ker ad_{a+x}] \subseteq [A, x] + [U, x]$. As a result, we get

$$\begin{aligned}
[a, L] &= [a, \ker ad_{a+x}] + [a, U] \\
&\subseteq [A, x] + [U, x] + [a, U] \\
&\subseteq [A, x] + [U, x] + [A, U].
\end{aligned}$$

Hence $[a, L] \subseteq [U, x] + [A, x] + [A, U]$.

6. Let $a' \in A$. By using part (4) of this proposition, we have

$$[a, D_x] = [a, \bigcap_{a \in A} M_{a+x}] \subseteq [a, M_{a+x} \cap M_x] \subseteq [A, x].$$

Hence $[a, D_x] \subseteq [A, x]$.

7. Suppose that $b_A(L) = b(L)$. We will prove that $L = D_z$ for all $z \in B_A$. Let $z \in B_A$ and $a' \in A$. Then $b_A(z) = b_A(L)$. Since A is abelian, we have

$$\operatorname{rank} ad_{a'+z}|_A = b_A(a'+z) = \operatorname{rank} ad_{a'+z}|_A = \operatorname{rank} ad_z|_A = b_A(z) = b_A(L) = b(L).$$

Then $\operatorname{rank} ad_{a'+z} = \operatorname{rank} ad_{a'+z}|_A$. Thus $\operatorname{im} ad_{a'+z} = \operatorname{im} ad_{a'+z}|_A$, which means, for any $\alpha \in L - A$ there exists $\beta \in A$ such that $ad_{a'+z}(\alpha) = ad_{a'+z}(\beta)$. Therefore $ad_{a'+z}(\alpha - \beta) = 0$, so $\alpha - \beta \in \ker ad_{a'+z}$. Hence we obtain

$$\alpha = \beta + (\alpha - \beta) \in A + \ker ad_{a'+z} = M_{a'+z}.$$

Since $\alpha \in L - A$ is arbitrary, $L - A \subseteq M_{a'+z}$. Moreover, it is obvious that $A \subseteq A + \ker ad_{a'+z} = M_{a'+z}$. As a result, $L = (L - A) + A \subseteq M_{a'+z}$, so $L = M_{a'+z}$. Since $a' \in A$ is arbitrary, we get $L = \bigcap_{a' \in A} L = \bigcap_{a' \in A} M_{a'+z} = D_z$ as desired. To show that $\dim[A, L] = b(L)$, we fix $z \in B_A$. By using part (6) of this proposition and $L = D_z$, we obtain $[A, L] = [A, D_z] \subseteq [A, z] = \operatorname{im} ad_z|_A$. In consequence, we have

$$\dim[A, L] = \dim \operatorname{im} ad_z|_A = b_A(L) = b(L).$$

□

Lemma 4.1.7. *Let L be a finite dimensional nilpotent Lie algebra, A an abelian ideal of L and $x \in B \cap B_A$. Suppose that $b(L) = b_A(L) + 1$ and $L \neq L_x$. Then $\dim L/M_x = 1$ and $M_x = M_{a+x}$ for every $a \in A$.*

Proof. Let L be a finite dimensional nilpotent Lie algebra, A an abelian ideal of L and $x \in B \cap B_A$. Then we have $b(x) = b(L)$ and $b_A(x) = b_A(L)$. Suppose that $b(L) = b_A(L) + 1$ and $L \neq L_x$. Thus $b(x) = b_A(x) + 1$, so $\operatorname{rank} ad_x = \operatorname{rank} ad_x|_A + 1$. Note that for any $x \in L$, we know that $\ker ad_x|_A = A \cap \ker ad_x \subseteq \ker ad_x$. Define

$$D := \operatorname{span}\{\alpha \mid \alpha \in \ker ad_x - \ker ad_x|_A\}.$$

Next, we will show that $A \cap \ker ad_x = \ker ad_x|_A = A \cap \ker ad_{a+x}$. It is clear that $A \cap \ker ad_x = \ker ad_x|_A$. To show that $\ker ad_x|_A = A \cap \ker ad_{a+x}$, let $a \in A$. Let $y \in \ker ad_x|_A$. Then $[x, y] = 0$ and $y \in A$, so $[a, y] = 0$. Thus $[a + x, y] = [a, y] + [x, y] = 0$. Therefore $y \in A \cap \ker ad_{a+x}$, so $\ker ad_x|_A \subseteq A \cap \ker ad_{a+x}$. Conversely, let $y \in A \cap \ker ad_{a+x}$. Then $[a + x, y] = 0$ and $y \in A$, so $[a, y] = 0$. Thus $[x, y] = [a, y] + [x, y] = [a + x, y] = 0$. Hence $y \in \ker ad_x|_A$, so $\ker ad_x|_A \supseteq A \cap \ker ad_{a+x}$. Consequently, we have $A \cap \ker ad_x = \ker ad_x|_A = A \cap \ker ad_{a+x}$.

Moreover, since $\ker ad_x|_A \subseteq \ker ad_x$, we get

$$\dim D = \text{nullity}\, ad_x - \text{nullity}\, ad_x|_A.$$

On the other hand, because $A \cap \ker ad_x = \ker ad_x|_A$, we have $M_x = A + \ker ad_x = A \oplus D$. Therefore $\dim M_x = \dim A + \dim D$. Additionally, we know that

$$\dim L = \text{nullity}\, ad_x + \text{rank}\, ad_x,$$
$$\dim A = \text{nullity}\, ad_x|_A + \text{rank}\, ad_x|_A.$$

Therefore we have

$$\dim L - \dim A = (\text{nullity}\, ad_x - \text{nullity}\, ad_x|_A) + (\text{rank}\, ad_x - \text{rank}\, ad_x|_A) = \dim D + 1,$$

so we get

$$\dim L/M_x = \dim L - \dim M_x = \dim L - (\dim A + \dim D) = 1.$$

In order to prove that $M_x = M_{a+x}$ for every $a \in A$, assume that there exists $a \in A$ such that M_{a+x} is not contained in M_x. Since $\dim L = \dim M_x + 1$, we have $L_x = \text{span}\{M_{a+x} \mid a \in A\} = L$, which contradicts the assumption $L \neq L_x$. Hence $M_{a+x} \subseteq M_x$ for every $a \in A$. Conversely, suppose that there exists $a \in A$ such that $M_{a+x} \not\supseteq M_x$. Then $A + \ker ad_{a+x} \not\supseteq A + \ker ad_x$. Let V_{a+x} and V_x be complementary subspaces of A in M_{a+x} and M_x, respectively. Then $M_{a+x} = A \oplus V_{a+x}$ and $M_x = A \oplus V_x$. Because $M_{a+x} \not\supseteq M_x$, we know that $V_{a+x} \not\supseteq V_x$. Since $A \cap \ker ad_x = A \cap \ker ad_{a+x}$, we have

$$\ker ad_{a+x} = (A \cap \ker ad_{a+x}) \oplus V_{a+x} = (A \cap \ker ad_x) \oplus V_{a+x} \not\supseteq (A \cap \ker ad_x) \oplus V_x = \ker ad_x.$$

Therefore $\text{nullity}\, ad_{a+x} < \text{nullity}\, ad_x$, so we obtain

$$b(x) = \text{rank}\, ad_x = \dim L - \text{nullity}\, ad_x < \dim L - \text{nullity}\, ad_{a+x} = \text{rank}\, ad_{a+x} = b(a+x).$$

Thus $b(x) < b(a+x)$. Since $x \in B$, $b(x) = b(L)$, so we have $b(a+x) > b(L)$, which is a contradiction. Consequently, $M_x = M_{a+x}$ for every $a \in A$. □

Theorem 4.1.8. *Let L be a finite dimensional nilpotent Lie algebra and A an abelian ideal of L. Suppose that $b(L) \leq b_A(L) + 1$. Then $\text{im}\, ad_x|_A$ is an ideal of L for every $x \in B_A$.*

Proof. Let L be a finite dimensional nilpotent Lie algebra and A an abelian ideal of L. Suppose that $b(L) \leq b_A(L)+1$. Let $x \in B_A$. Since $b_A(L) \leq b(L)$, we have $b_A(L) \leq b(L) \leq b_A(L)+1$, so

$$b(L) = b_A(L) \quad \text{or} \quad b(L) = b_A(L) + 1.$$

First, we will claim that $\operatorname{im} ad_{a+x}|_A$ is an ideal of M_{a+x} for every $a \in A$. Let $a \in A$. Then it is obvious that $\operatorname{im} ad_{a+x}|_A \subseteq A \subseteq A + \ker ad_{a+x} = M_{a+x}$. Let $a' \in A$ and $y \in M_{a+x}$. Then $[a+x, a'] \in \operatorname{im} ad_{a+x}|_A$ and $y = a_y + c_y$ where $a_y \in A$ and $c_y \in \ker ad_{a+x}$. Thus $[c_y, a+x] = 0$. Note that $[[a_y, a+x], a'] = 0$ because A is an abelian ideal. Therefore we get

$$\begin{aligned}
[y,[a+x,a']] &= [[y,a+x],a'] + [a+x,[y,a']] \\
&= [[a_y+c_y,a+x],a'] + [a+x,[y,a']] \\
&= [[a_y,a+x],a'] + [[c_y,a+x],a'] + [a+x,[y,a']] \\
&= 0 + [0,a'] + [a+x,[y,a']] \\
&\in [a+x, A] \\
&= \operatorname{im} ad_{a+x}|_A.
\end{aligned}$$

Hence $\operatorname{im} ad_{a+x}|_A$ is an ideal of M_{a+x} for every $a \in A$.

If $L = L_x = \operatorname{span}\{M_{a+x} | a \in A\}$, then $\operatorname{im} ad_x|_A = \operatorname{im} ad_{a+x}|_A$ is an ideal of L by previous claim. Suppose that $L \neq L_x$. If $b(L) = b_A(L)$, then by Proposition 4.1.6 (7), we have $L = D_x$. Thus $L = D_x \subseteq M_x \subseteq L_x$. It is clear that $L_x \subseteq L$, so $L = L_x$, which contradicts $L \neq L_x$. Therefore $b(L) = b_A(L) + 1$. Next, we need to show that $x \in B$. Assume $x \notin B$. Then $b(x) \neq b(L)$, so we have

$$b_A(x) \leq b(x) \leq b(L) - 1 = b_A(L) = b_A(x),$$

which implies $b(x) = b_A(x)$, so $\operatorname{rank} ad_x = \operatorname{rank} ad_x|_A$. Thus $\operatorname{im} ad_x = \operatorname{im} ad_x|_A$, which means, for any $\alpha \in L - A$ there exists $\beta \in A$ such that $ad_x(\alpha) = ad_x(\beta)$. Therefore $ad_x(\alpha - \beta) = 0$, so $\alpha - \beta \in \ker ad_x$. As a result,

$$\alpha = \beta + (\alpha - \beta) \in A + \ker ad_x = M_x.$$

Since $\alpha \in L - A$ is arbitrary, $L - A \subseteq M_x$. Moreover, it is easy to see that $A \subseteq A + \ker ad_x = M_x$. Consequently, $L = (L-A) + A \subseteq M_x$. Because $M_x \subseteq L_x$, $L \subseteq L_x$. Therefore $L = L_x$, which again contradicts $L \neq L_x$. Hence $x \in B$. In conclusion, we know that $x \in B \cap B_A$. By Lemma 4.1.7, $\dim L/M_x = 1$ and $M_x = M_{a+x}$ for every $a \in A$, so $D_x = \bigcap_{a \in A} M_{a+x} = M_x$. Next, we will show that $[A, x] = [A, M_x]$. By Proposition 4.1.6 (6), $[A, D_x] \subseteq [A, x]$. Since $D_x = M_x$, we get $[A, M_x] = [A, D_x] \subseteq [A, x]$. On the other hand, we let $a \in A$. Then $[a, x] \in [A, x]$. Because

$x \in \ker ad_x$, we obtain

$$[a,x] \in [A, \ker ad_x] \subseteq [A, A + \ker ad_x] = [A, M_x].$$

Thus $[A,x] \subseteq [A, M_x]$. Hence $[A,x] = [A, M_x]$. Next, we will claim that $M_x = A + \ker ad_x$ is a Lie subalgebra of L. It is clear that M_x is a subspace of L. Let $m_1, m_2 \in M_x$. Then there exist $a_1, a_2 \in A$ and $c_1, c_2 \in \ker ad_x$ such that $m_i = a_i + c_i$ for $i = 1, 2$. We observe that $ad_x([c_1, c_2]) = [x, [c_1, c_2]] = [[x, c_1], c_2] + [c_1, [x, c_2]] = 0$, so $[c_1, c_2] \in \ker ad_x$. Therefore we have

$$[m_1, m_2] = [a_1 + c_1, a_2 + c_2] = [a_1, a_2] + [a_1, c_2] + [c_1, a_2] + [c_1, c_2] \in A + \ker ad_x = M_x.$$

Since $m_1, m_2 \in M_x$ are arbitrary, we get $[M_x, M_x] \subseteq M_x$, which means M_x is closed under bracket. Thus M_x is a Lie subalgebra of L. Since $\dim L/M_x = 1$, there exists $y \in L - M_x$ such that $L = M_x \oplus \operatorname{span}\{y\}$, which means $\operatorname{span}\{y\}$ is the complementary subspace of M_x in L. To show that M_x is an ideal of L, we suppose that M_x is not an ideal of L. Then there exist $m \in M_x$ and $z \in L$ such that $[m, z] \notin M_x$. Since $L = M_x \oplus \operatorname{span}\{y\}$, there exist $m' \in M_x$ and $\alpha \in \mathbb{F}$ such that $z = m' + \alpha y$. We observe that

$$[m, m'] + \alpha[m, y] = [m, m' + \alpha y] = [m, z] \notin M_x.$$

Since $[m, m'] \in M_x$, we obtain $\alpha \neq 0$ and $[m, y] \notin M_x$, so $[m, y] = m'' + \beta y$ where $m'' \in M_x$ and $\beta \neq 0$. Consequently, we get $ad_m^N(y) \neq 0$ for any $N \in \mathbb{Z}_{>0}$ which contradicts nilpotency of L. Hence M_x is an ideal of L. Because we know that $\operatorname{im} ad_x|_A = [A, x] = [A, M_x]$ and A, M_x are ideals of L, $\operatorname{im} ad_x|_A$ is also an ideal of L. Since $x \in B_A$ is arbitrary, $\operatorname{im} ad_x|_A$ is an ideal of L for every $x \in B_A$. □

Lemma 4.1.9. *Let L be a finite dimensional Lie algebra and A an ideal of L such that $b_A(L) > 1$. Let $x \in B_A$ be given. Suppose that there exists $y \in L$ such that y and $x + y$ do not satisfy*

$$b_A(z) > 1 \quad \textit{and} \quad 2b_A(z) \geq b_A(L). \tag{4.1}$$

Then $b_A(y) = b_A(x + y) = 1$ and $b_A(L) = 2$.

Proof. Let L be a finite dimensional Lie algebra and A an ideal of L such that $b_A(L) > 1$. Let $x \in B_A$ be given. Then $b_A(x) = b_A(L)$. Suppose that there exists $y \in L$ such that y and $x + y$ do not satisfy

$$b_A(z) > 1 \quad \text{and} \quad 2b_A(z) \geq b_A(L). \tag{4.1}$$

First, we claim that $b_A(L) = 2$ by using contrapositive. Assume that $b_A(L) \neq 2$. Then we have $b_A(L) \geq 3$ since $b_A(L) > 1$. In order to show that for all $y \in L$, y or $x + y$ satisfies (4.1), we let $y \in L$ be such that y does not satisfy (4.1). Then we get $b_A(y) \leq 1$ or $2b_A(y) < b_A(L)$. Note that if $b_A(y) \leq 1$, then we also get $2b_A(y) \leq 2 < 3 \leq b_A(L)$. Thus we can assume that $2b_A(y) < b_A(L)$. Then we apply the fact that

$$\operatorname{rank}(A + B) \leq \operatorname{rank} A + \operatorname{rank} B$$

where A and B are linear transformations, so we get

$$\operatorname{rank} ad_x|_A \leq \operatorname{rank}(ad_x|_A + ad_y|_A) + \operatorname{rank}(-ad_y|_A).$$

Because $ad_x|_A + ad_y|_A = ad_{x+y}|_A$ and $\operatorname{rank}(-ad_y|_A) = \operatorname{rank} ad_y|_A$, we have

$$\operatorname{rank} ad_x|_A \leq \operatorname{rank} ad_{x+y}|_A + \operatorname{rank} ad_y|_A. \tag{4.2}$$

Thus $\operatorname{rank} ad_{x+y}|_A \geq \operatorname{rank} ad_x|_A - \operatorname{rank} ad_y|_A$, which also means $b_A(x + y) \geq b_A(x) - b_A(y)$. Since $2b_A(y) < b_A(L)$ and $b_A(x) = b_A(L)$, we obtain

$$b_A(x + y) \geq b_A(x) - b_A(y) > b_A(L) - \frac{b_A(L)}{2} > \frac{b_A(L)}{2}.$$

Therefore $2b_A(x + y) > b_A(L)$. We also get $b_A(x + y) > \frac{b_A(L)}{2} \geq \frac{3}{2} > 1$. Hence $x + y$ satisfies (4.1). Consequently, $b_A(L) = 2$ as we claimed.

Next, we will prove that $b_A(x) = b_A(x + y) = 1$. Since y and $x + y$ do not satisfy (4.1) and $b_A(L) = 2$, we have

$$b_A(y) \leq 1 \quad \text{or} \quad 2b_A(y) < b_A(L) = 2$$

and

$$b_A(x + y) \leq 1 \quad \text{or} \quad 2b_A(x + y) < b_A(L) = 2,$$

which can be reduced to $b_A(y) \leq 1$ and $b_A(x + y) \leq 1$. Suppose that $b_A(y) = 0$. This means $\operatorname{im} ad_y|_A = \{0\}$, so we have

$$b_A(x + y) = \operatorname{rank} ad_{x+y}|A = \operatorname{rank} ad_x|_A = b_A(x) = b_A(L) = 2$$

which contradicts $b_A(x + y) \leq 1$. Hence $b_A(y) = 1$. Next, we assume that $b_A(x + y) = 0$. Then $\operatorname{rank} ad_{x+y}|_A = 0$. By applying this to (4.2), we get $\operatorname{rank} ad_x|_A \leq \operatorname{rank} ad_y|_A$. Therefore

$$b_A(y) = \operatorname{rank} ad_y|_A \geq \operatorname{rank} ad_x|_A = b_A(x) = b_A(L) = 2,$$

which contradicts $b_A(y) = 1$. Hence $b_A(x+y) = 1$. As a result, we obtain $b_A(y) = b_A(x+y) = 1$ and $b_A(L) = 2$ as desired. □

Theorem 4.1.10. *Let L be a finite dimensional nilpotent Lie algebra and A an abelian ideal of L such that $b_A(L) > 1$. Suppose that $[L, z] \subseteq [A, L]$ for all $z \in L$ satisfying*

$$b_A(z) > 1 \quad \textit{and} \quad 2b_A(z) \geq b_A(L). \tag{4.1}$$

Then $[L, L] = [C_L(A), L]$. In addition, if $A = C_L(A)$ and $b(L) = b_A(L)$, then $\dim[L, L] = b_A(L)$.

Proof. Let L be a finite dimensional nilpotent Lie algebra and A an abelian ideal of L. Let $b_A(L) > 1$ and suppose that $[L, z] \subseteq [A, L]$ for all $z \in L$ satisfying

$$b_A(z) > 1 \quad \text{and} \quad 2b_A(z) \geq b_A(L). \tag{4.1}$$

Fix an $x \in B_A$. Then we have the following two cases to consider.

The first case is y or $x + y$ satisfy (4.1) for every $y \in L$. Then by assumption, $[L, y] \subseteq [A, L]$ or $[L, x + y] \subseteq [A, L]$. Next, we will show that $[L, x + y] \subseteq [A, L]$ also implies $[L, y] \subseteq [A, L]$. Suppose that $[L, x + y] \subseteq [A, L]$. Since $x \in B_A$, $b_A(x) = b_A(L) > 1$. It is clear that $2b_A(x) > b_A(x) = b_A(L)$, so x also satisfies (4.1). Thus $[L, x] \subseteq [A, L]$. To show $[L, y] \subseteq [A, L]$, we let $z \in L$. Then $[z, x + y] \in [L, x + y] \subseteq [A, L]$ and $[z, x] \in [L, x] \subseteq [A, L]$, so we get $[z, y] = [z, x + y] - [z, x] \in [A, L]$. Since $z \in L$ is arbitrary, $[L, y] \in [A, L]$. Hence we have $[L, y] \in [A, L]$ for any $y \in L$. Because A is abelian, $A \subseteq C_L(A)$, so $[A, L] \subseteq [C_L(A), L]$. Thus $[L, y] \subseteq [A, L] \subseteq [C_L(A), L]$ for every $y \in L$. Since $y \in L$ is arbitrary, $[L, L] \subseteq [C_L(A), L]$. Conversely, it is easy to see that $[C_L(A), L] \subseteq [L, L]$. Hence $[L, L] = [C_L(A), L]$.

For the second case, assume that there exists $y \in L$ such that y and $x + y$ do not satisfy (4.1). By Lemma 4.1.9, we have

$$b_A(y) = b_A(x + y) = 1 \quad \text{and} \quad b_A(L) = 2.$$

We have to claim that $[L, z] \subseteq [C_L(A), L]$ for all $z \in L$ such that $b_A(z) \neq 1$. Let $z \in L$ be such that $b_A(z) \neq 1$. If $b_A(z) = 0$, then $z \in C_L(A)$, so $[L, z] \subseteq [C_L(A), L]$. If $b_A(z) = 2 = b_A(L)$, then z satisfies (4.1), so $[L, z] \subseteq [A, L]$ by assumption. Since A is abelian, we have $A \subseteq C_L(A)$. Therefore $[L, z] \subseteq [A, L] \subseteq [C_L(A), L]$. Hence $[L, z] \subseteq [C_L(A), L]$ for all $z \in L$ such that $b_A(z) \neq 1$. In order to show that $[L, L] \subseteq [C_L(A), L]$, suppose that $[L, L] \nsubseteq [C_L(A), L]$. Then there exist $u, v \in L$ such that $[u, v] \notin [C_L(A), L]$. By previous claim, we have $b_A(u) = b_A(v) = 1$. Note that

$$[u + v, v] = [u - v, v] = [u, v] \notin [C_L(A), L],$$

so we also have $b_A(u+v) = b_A(u-v) = 1$. Since $x \in B_A$, $b_A(x) = b_A(L) = 2$. Therefore $[x,v], [u,x] \in [C_L(A), L]$ by previous claim. Thus we have

$$[x+u, v] = [x,v] + [u,v] \notin [C_L(A), L],$$
$$[u, x+v] = [u,x] + [u,v] \notin [C_L(A), L],$$
$$[x+u+v, v] = [x,v] + [u+v,v] \notin [C_L(A), L],$$

so $b_A(x+u) = b_A(x+v) = b_A(x+u+v) = 1$. Note that $u - v \notin C_L(A)$ because $b_A(u-v) = 1$. In summary, we have

$$b_A(u) = b_A(v) = b_A(u+v) = b_A(u-v) = b_A(x+u) = b_A(x+v) = b_A(x+u+v) = 1.$$

Hence we have $u, v, u+v, x+u, x+v, x+u+v \in T_A$, which contradicts Proposition 4.1.4. Therefore $[L,L] \subseteq [C_L(A), L]$. Conversely, it is clear that $[C_L(A), L] \subseteq [L,L]$. Hence $[L,L] = [C_L(A), L]$.

In addition, assume that we also have $A = C_L(A)$ and $b(L) = b_A(L)$. Then $[L,L] = [C_L(A), L] = [A, L]$. Consequently, $\dim[L,L] = \dim[A,L] = b_A(L)$ by Proposition 4.1.6(7). □

Notice that there are relations between set of elements of breadth 0 and 1, as we show in the next lemma.

Lemma 4.1.11. *Let L be a finite dimensional Lie algebra and A an ideal of L. Suppose that $b_A(L) = 1$. Then the following hold:*

1. *$L = T_A \cup C_L(A)$.*
2. *$T_A \cap C_L(A) = \emptyset$.*
3. *$T_A \cup \{0\}$ is a subspace of L.*

Proof. Let L be a finite dimensional Lie algebra and A an ideal of L. Suppose that $b_A(L) = 1$.

1. Let $x \in L$. Then we have $b_A(x) = 0$ or $b_A(x) = 1$, that is $x \in C_L(A)$ or $x \in T_A$. Therefore $L \subseteq T_A \cup C_L(A)$. Conversely, we know that $T_A \subseteq L$ and $C_L(A) \subseteq L$, so $T_A \cup C_L(A) \subseteq L$. Hence $L = T_A \cup C_L(A)$.
2. It is clear that $T_A \cap C_L(A) = \emptyset$ by their definitions.
3. Note that $C_L(A) = \bigcap_{a \in A} \ker ad_a$. Since $\ker ad_a$ is a subspace of L for all $a \in A$, $C_L(A)$ is also a subspace of L. Because L is finite dimensional, so is $C_L(A)$. Therefore we write $C_L(A) = \text{span}\{c_1, c_2, \ldots, c_n\}$ and extend this basis to $L = \text{span}\{c_1, c_2, \ldots, c_n, t_1, t_2, \ldots, t_m\}$. Then we have $T_A \cup \{0\} = \text{span}\{t_1, t_2, \ldots, t_m\}$ and $L = C_L(A) \oplus (T_A \cup \{0\})$. Hence $T_A \cup \{0\}$ is a subspace of L.

□

In next lemma, we prove a few properties of maximal abelian ideal which we are going to use in our main theorem.

Lemma 4.1.12. *Let L be a finite dimensional nilpotent Lie algebra and A a maximal abelian ideal of L. Then the following hold:*

1. $\ker ad_a = A$ *for all* $a \in A - Z(L)$.

2. $C_L(A) = A$.

3. $Z(L) \subseteq A$.

Proof. Let L be a finite dimensional nilpotent Lie algebra and A a maximal abelian ideal of L.

1. Let $a \in A - Z(L)$. Then $\ker ad_a \neq L$ because $a \notin Z(L)$. Since $a \in A$ and A is abelian, we have $A \subseteq \ker ad_a$. Next, we will show that $\ker ad_a$ is an ideal of L. Let $x \in \ker ad_a$ and $y \in L$. Then $[a, x] = 0$, so we have

$$ad_a([x, y]) = [a, [x, y]] = [[a, x], y] + [x, [a, y]] = [x, [a, y]] \in A \subseteq \ker ad_a.$$

   Thus $\ker ad_a$ is an ideal of L that contains A. Since A is maximal and $\ker ad_a \neq L$, $\ker ad_a = A$. Hence $A = \ker ad_a$ for all $a \in A - Z(L)$.

2. Observe that if $a \in Z(L)$, then we have $\ker ad_a = L$. Consequently, we obtain

$$\begin{aligned} C_L(A) &= \bigcap_{a \in A} \ker ad_a \\ &= \Big(\bigcap_{a \in A - Z(L)} \ker ad_a\Big) \cap \Big(\bigcap_{a \in A \cap Z(L)} \ker ad_a\Big) \\ &= \Big(\bigcap_{a \in A - Z(L)} A\Big) \cap \Big(\bigcap_{a \in A \cap Z(L)} L\Big) \\ &= A \cap L \\ &= A. \end{aligned}$$

   Hence $C_L(A) = A$ as we want.

3. Let $x \in Z(L)$. Then we get $ad_x = 0$, so $b(x) = 0$. Since $b_A(x) \leq b(x)$, we have $b_A(x) = 0$. Thus $x \in C_L(A)$. By part two of this lemma, we know that $C_L(A) = A$, so $x \in C_L(A) = A$. Hence $Z(L) \subseteq A$.

□

Theorem 4.1.13. *Let L be a finite dimensional nilpotent Lie algebra and A an abelian ideal of L. Suppose that $b_A(L) = 1$. Then*

1. $\dim(A/(A \cap Z(L))) = 1$ *and* $\ker ad_x|_A = \ker ad_y|_A$ *for all* $x, y \in L - C_L(A)$.
   In addition, if A is a maximal abelian ideal of L, then $\dim(L/Z(L)) = b(L) + 1$.

or

2. $\dim[A, L] = 1$. *In addition, if A is a maximal abelian ideal of L, then* $b(L/[A, L]) < b(L)$.

Proof. Let L be a finite dimensional nilpotent Lie algebra and A an abelian ideal of L. Suppose that $b_A(L) = 1$. By Lemma 4.1.11, we know that $L = T_A \cup C_L(A)$, $T_A \cap C_L(A) = \emptyset$ and $T_A \cup \{0\}$ is a subspace of L. As a result, we have $T_A = L - C_L(A)$. Define $T = \text{span}\{x, y\}$ and

$$\bar{T} = (T + C_L(A))/C_L(A) \cong T/(T \cap C_L(A))$$

where $x, y \in T_A$, as defined in Theorem 4.1.3. Since $T_A \cap C_L(A) = \emptyset$ and $T \subseteq T_A \cup \{0\}$, we have

$$\begin{aligned} T \cap C_L(A) &\subseteq (T_A \cup \{0\}) \cap C_L(A) \\ &= (T_A \cap C_L(A)) \cup (\{0\} \cap C_L(A)) \\ &= \emptyset \cup \{0\} \\ &= \{0\}, \end{aligned}$$

so $T \cap C_L(A) = \{0\}$. Hence $\bar{T} \cong T/(T \cap C_L(A)) \cong T$.

Suppose that $T_A \cup \{0\}$ is not 1-dimensional. Then we get $\dim T_A \cup \{0\} \geq 2$, so there exist $x, y \in T_A$ such that $T = \text{span}\{x, y\}$ is 2-dimensional. Thus $\dim \bar{T} = \dim T = 2$ because $\bar{T} \cong T$. Since $\{x, y\}$ is linearly independent and $T_A \cup \{0\}$ is a subspace of L, we have $x + y \in T_A$. By Theorem 4.1.3 (2), we know that

$$\text{either} \quad \text{im}\, ad_x|_A \neq \text{im}\, ad_y|_A \quad \text{or} \quad \ker ad_x|_A \neq \ker ad_y|_A,$$

so we consider the following two cases:

1. $\text{im}\, ad_x|_A = \text{im}\, ad_y|_A =: K$ and $\ker ad_x|_A \neq \ker ad_y|_A$. Let $z \in T_A$. Then

   $$\ker ad_z|_A \neq \ker ad_x|_A \quad \text{or} \quad \ker ad_z|_A \neq \ker ad_y|_A.$$

   Without loss of generality, we suppose that $\ker ad_z|_A \neq \ker ad_x|_A$. Next we will show that $x + z \in T_A$. Since $x, z \in T_A$ and $T_A \cup \{0\}$ is a subspace of L, we have $x + z \in T_A \cup \{0\}$. If $x + z = 0$, then $x = -z$, so we get $\ker ad_x|_A = \ker ad_{-z}|_A = \ker ad_z|_A$, which is a contradiction. Therefore $x + z \in T_A$. By Theorem 4.1.3 (2) with "either or", we have

$\operatorname{im} ad_z|_A = \operatorname{im} ad_x|_A$, so $[A, z] = \operatorname{im} ad_z|_A = \operatorname{im} ad_x|_A = K$. Since $z \in T_A$ is arbitrary, $[A, T_A] = K$. On the other hand, if $z \in C_L(A)$, then $[A, z] = \{0\}$. Because $z \in C_L(A)$ is arbitrary, $[A, C_L(A)] = 0$. Consequently, we obtain

$$[A, L] = [A, T_A \cup C_L(A)] = \operatorname{span}\{[A, T_A], [A, C_L(A)]\} = \operatorname{span}\{K, 0\} = K.$$

Hence $\dim[A, L] = \dim K = b_A(x) = 1$.

2. $\operatorname{im} ad_x|_A \neq \operatorname{im} ad_y|_A$ and $\ker ad_x|_A = \ker ad_y|_A =: K'$. Let $z \in T_A$. Then

$$\operatorname{im} ad_z|_A \neq \operatorname{im} ad_x|_A \quad \text{or} \quad \operatorname{im} ad_z|_A \neq \operatorname{im} ad_y|_A.$$

Without loss of generality, we assume that $\operatorname{im} ad_z|_A \neq \operatorname{im} ad_x|_A$. Next we will show that $x + z \in T_A$. Since $x, z \in T_A$ and $T_A \cup \{0\}$ is a subspace of L, we have $x + z \in T_A \cup \{0\}$. If $x + z = 0$, then $x = -z$, so we get $\operatorname{im} ad_x|_A = \operatorname{im} ad_{-z}|_A = \operatorname{im} ad_z|_A$, which is a contradiction. Therefore $x + z \in T_A$. By Theorem 4.1.3 (2) with "either or", we have $\ker ad_z|_A = \ker ad_x|_A = K'$. Since $z \in T_A$ is arbitrary, we get $\ker ad_z|_A = K'$ for any $z \in T_A = L - C_L(A)$. Hence $\ker ad_x|_A = \ker ad_y|_A$ for all $x, y \in L - C_L(A)$.

On the other hand, we assume that $T_A \cup \{0\}$ is 1-dimensional. Then $T_A \cup \{0\} = T \cong \bar{T}$, so T and $\bar{T}$ are also 1-dimensional for every $x, y \in T_A$. By Theorem 4.1.3 (1), we have

$$\operatorname{im} ad_x|_A = \operatorname{im} ad_y|_A \quad \text{and} \quad \ker ad_x|_A = \ker ad_y|_A$$

for every $x, y \in T_A$. Note that if $z \in C_L(A)$, then $[A, z] = \{0\}$. Therefore $[A, L] = \operatorname{im} ad_x|_A$ for some $x \in T_A$, so $\dim[A, L] = b_A(x) = 1$. Hence $\dim[A, L] = 1$ and $\ker ad_x|_A = \ker ad_y|_A$ for every $x, y \in L - C_L(A)$.

Next, we will claim that $\ker ad_x|_A = \ker ad_y|_A$ for all $x, y \in L - C_L(A)$ implies $\dim(A/(A \cap Z(L))) = 1$. Suppose that $\ker ad_x|_A = \ker ad_y|_A$ for all $x, y \in L - C_L(A)$. Note that for every $z \in C_L(A)$, we have $\ker ad_z|_A = A$. We also know that $Z(L) = \bigcap_{x \in L} \ker ad_x$, so

$$\begin{aligned} A \cap Z(L) &= A \cap (\bigcap_{x \in L} \ker ad_x) \\ &= \bigcap_{x \in L} (A \cap \ker ad_x) \\ &= \bigcap_{x \in L} \ker ad_x|_A \\ &= A \cap \ker ad_\alpha|_A \\ &= \ker ad_\alpha|_A \end{aligned}$$

for some $\alpha \in L - C_L(A) = T_A$. Consequently, we obtain

$$\begin{aligned}\dim(A/(A\cap Z(L))) &= \dim(A/\ker ad_\alpha|_A)\\ &= \dim A - \text{nullity}\, ad_\alpha|_A\\ &= \text{rank}\, ad_\alpha|_A\\ &= b_A(\alpha)\\ &= 1.\end{aligned}$$

Hence $\dim(A/(A\cap Z(L))) = 1$.

Finally, we suppose that A is a maximal abelian ideal of L. For the first result, we have $\dim(A/(A\cap Z(L))) = 1$. By Lemma 4.1.12 (1) & (3), we get $A = \ker ad_a$ for every $a \in A - Z(L)$ and $Z(L) \subseteq A$, respectively. Then we have $A \cap Z(L) = Z(L)$. Fix $\alpha \in A - Z(L)$, so

$$\begin{aligned}\dim(L/Z(L)) &= \dim L - \dim Z(L)\\ &= (\dim L - \dim A) + (\dim A - \dim Z(L))\\ &= (\dim L - \dim\ker ad_\alpha) + (\dim A - \dim(A\cap Z(L)))\\ &= (\dim L - \text{nullity}\, ad_\alpha) + \dim(A/(A\cap Z(L)))\\ &= \text{rank}\, ad_\alpha + 1\\ &= b(\alpha) + 1\\ &\le b(L) + 1.\end{aligned}$$

On the other hand, we know that $\dim(L/Z(L)) \ge b(L) + 1$ by Theorem 3.1.9. Consequently, $\dim(L/Z(L)) = b(L) + 1$.

For the second result, we have $\dim[A, L] = 1$. Note that $[A, L]$ is an ideal of L because both A and L are ideals of L. Thus we can consider the quotient Lie algebra $L/[A, L]$. Let $x \in L$. Then $x + [A, L] \in L/[A, L]$ and $ad_{x+[A,L]} : L/[A, L] \to L/[A, L]$ is given by

$$y + [A, L] \mapsto [x, y] + [A, L]$$

where $y + [A, L] \in L/[A, L]$. By Lemma 4.1.11 (1) & (2), we know that $L = T_A \cup C_L(A)$ and $T_A \cap C_L(A) = \emptyset$, respectively. Moreover, by Lemma 4.1.12 (2), we also know that $C_L(A) = A$. Therefore x must be contained in A or T_A. Then we consider the following two cases:

1. If $x \in A$, then $[x, y] \in [A, L]$ for any $y \in L$. Thus $\text{im}\, ad_{x+[A,L]} = \{[A, L]\}$, so we have $b(x + [A, L]) = 0$. Note that $b(L) \ge b_A(L) = 1$. Hence $b(x + [A, L]) < b(L)$ for any $x \in A$.

2. Assume that $x \in T_A$. Then $b_A(x) = 1$, which imples $[A, x] = \text{im}\, ad_x|_A$ is 1-dimensional.

Since $[A, x] \subseteq [A, L]$, $\operatorname{im} ad_x \cap [A, L] \neq \{0\}$. Hence $b(x + [A, L]) < b(x) \leq b(L)$ for any $x \in T_A$.

Hence $b(x + [A, L]) < b(L)$ for any $x \in L$. Since $x \in L$ is arbitrary, $b(L/[A, L]) < b(L)$. □

4.2 Main Theorem

Our main theorem shows necessary and sufficient conditions for finite dimensional nilpotent Lie algebra of breadth 2. Observe that dimension of finite dimensional nilpotent Lie algebra of breadth 2 is not bounded above but it is relatively small when we consider its square.

Theorem 4.2.1. *Let L be a finite dimensional nilpotent Lie algebra. Then $b(L) = 2$ if and only if one of the following holds:*

1. $\dim[L, L] = 2$

or

2. $\dim[L, L] = 3$ *and* $\dim(L/Z(L)) = 3$.

Proof. Let L be a finite dimensional nilpotent Lie algebra. Suppose $b(L) = 2$. Since $b(L) \neq 0$, by Proposition 3.1.4, L is not abelian, so $Z(L) \subsetneq L$. On the other hand, $Z(L) \neq \{0\}$ because L is nilpotent. Therefore $\{0\} \neq Z(L) \subsetneq L$, which guarantee that L has a maximal abelian ideal. Then we consider the following two cases:

First, there exists a maximal abelian ideal A of L such that $b_A(L) = 2$. Let $x \in L$ be such that $b_A(x) > 1$. Since $b_A(L) = 2$, we get $b_A(x) = 2$. Then

$$b_A(x) = 2 > 1 \quad \text{and} \quad 2b_A(x) = 4 \geq 2 = b_A(L).$$

Thus x satisfies (4.1). Since $b_A(x) = 2 = b(L)$, $\operatorname{rank} ad_x|_A = \operatorname{rank} ad_x = 2$. Therefore we have

$$[L, x] = \operatorname{im} ad_x = \operatorname{im} ad_x|_A = [A, x] \subseteq [A, L].$$

As a result, A meets all requirements in Theorem 4.1.10. In addition, we have $b_A(L) = 2 = b(L)$ and $C_L(A) = A$ by Lemma 4.1.12 (2). Hence $\dim[L, L] = b_A(L) = 2$ by Theorem 4.1.10.

The complementary case of the previous one is $b_A(L) \leq 1$ for every maximal abelian ideal A of L. We will show that for every maximal abelian ideal A of L, $b_A(L) \neq 0$. Suppose that there exists a maximal abelian ideal A of L such that $b_A(L) = 0$. Then for every $x \in L$, $b_A(x) = 0$. Thus $\operatorname{rank} ad_x|_A = 0$, which implies $[A, x] = \operatorname{im} ad_x|_A = \{0\}$. Since $x \in L$ is arbitrary, $[A, L] = \{0\}$, so $C_L(A) = L$ which contradicts $C_L(A) = A$ by Lemma 4.1.12 (2). As a result, this case turns into $b_A(L) = 1$ for every maximal abelian ideal A of L. Next, we apply Theorem 4.1.13 to this case, so we have two following subcases to consider:

1. $\dim(A/(A\cap Z(L))) = 1$, $\ker ad_x|_A = \ker ad_y|_A$ for all $x, y \in L - C_L(A)$ and $\dim(L/Z(L)) = b(L) + 1 = 2 + 1 = 3$. In addition, we have $\dim[L, L] \leq \binom{3}{2} = 3$ by Lemma 3.1.10. Since $b(L) = 2 \neq 0, 1$, we have $\dim[L, L] \neq 0, 1$ by Proposition 3.1.4 and Theorem 3.2.1, respectively. Hence $\dim[L, L] = 2, 3$ and $\dim(L/Z(L)) = 3$.

2. $\dim[A, L] = 1$ and $b(L/[A, L]) < b(L) = 2$. Then we get $b(L/[A, L]) = 0, 1$. Next, we will claim that $b(L/[A, L]) \neq 0$. Assume that $b(L/[A, L]) = 0$. Then $[L/[A, L], L/[A, L]] = \{0\}$ by Proposition 3.1.4. Therefore

$$\dim[L, L]/[A, L] = \dim[L/[A, L], L/[A, L]] = 0,$$

so $\dim[L, L] = \dim[A, L] = 1$. By Theorem 3.2.1, $b(L) = 1$, which contradicts $b(L) = 2$. Hence $b(L/[A, L]) \neq 0$, which implies $b(L/[A, L]) = 1$. As a result, we have

$$\dim[L, L]/[A, L] = \dim[L/[A, L], L/[A, L]] = 1$$

by Theorem 3.2.1. Since $\dim[A, L] = 1$, $\dim[L, L] = \dim[A, L] + 1 = 2$ in this case.

Conversely, if $\dim[L, L] = 2$, then $b(L) \leq \dim[L, L] = 2$ by Lemma 3.1.5. Since $\dim[L, L] \neq 0$, L is not abelian, so $b(L) \neq 0$ by Proposition 3.1.4. Similarly, we have $b(L) \neq 1$ by Theorem 3.2.1. Hence $b(L) = 2$ in this case.

On the other hand, if $\dim[L, L] = 3$ and $\dim(L/Z(L)) = 3$, then $b(L) \leq \dim[L, L] = 3$ by Lemma 3.1.5. Similar to the case $\dim[L, L] = 2$, we get $b(L) \neq 0, 1$ by Proposition 3.1.4 and Theorem 3.2.1, respectively. If $b(L) = 3$, then by Theorem 3.1.9, $\dim(L/Z(L)) \geq b(L) + 1 = 4$ which contradicts $\dim(L/Z(L)) = 3$. Hence $b(L) = 2$. □

Corollary 4.2.2. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 and A a maximal abelian ideal of L. Suppose that* $\dim[L, L] = 3$. *Then* $\dim(A/Z(L)) = 1$.

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 and A a maximal abelian ideal of L. Suppose that $\dim[L, L] = 3$. Then $b(L) = 2$. By the proof of Theorem 4.2.1, this must be the result of the case $b_{A'}(L) = 1$ for every maximal abelian ideal A' of L together with the first result of Theorem 4.1.13. Consequently, $\dim(A/(A \cap Z(L))) = 1$ by Theorem 4.1.13 (1). Similar to Theorem 4.2.1, we have $\{0\} \neq Z(L) \not\subseteq L$, which is an abelian ideal of L. By Lemma 4.1.12 (3), we know that $Z(L) \subseteq A$, so $A \cap Z(L) = Z(L)$. Hence $\dim(A/Z(L)) = \dim(A/(A \cap Z(L))) = 1$. □

Chapter 5

Classification of Nilpotent Lie Algebras of Breadth 2

5.1 Structure of Nilpotent Lie Algebras of Breadth 2

We begin this section by showing that any finite dimensional nilpotent Lie algebra of breadth 2 has dimension greater than 3. Thus we know our starting dimension of the classification process.

Lemma 5.1.1. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that* $\dim L \in \mathbb{Z}_{>0}$. *Then*

$$1 \leq \dim Z(L) \leq \dim L - 3.$$

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $\dim L := n \in \mathbb{Z}_{>0}$. Then $b(L) = 2$. By Theorem 3.1.9, we know that $\dim(L/Z(L)) \geq b(L) + 1$, so we have $\dim L - \dim Z(L) \geq 3$. Thus $\dim Z(L) \leq \dim L - 3$. On the other hand, L has nontrivial center since L is nilpotent. Therefore $\dim Z(L) \geq 1$. Hence $1 \leq \dim Z(L) \leq \dim L - 3$. $\square$

Corollary 5.1.2. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2. Then* $\dim L \geq 4$.

Definition 5.1.3. A Lie algebra L is called *pure* if it does not have an abelian ideal as a direct summand.

Lemma 5.1.4. *Let L be a finite dimensional nilpotent Lie algebra. Then L is pure if and only if* $Z(L) \subseteq [L, L]$.

Proof. Let L be a finite dimensional nilpotent Lie algebra. Suppose that $Z(L)$ is not contained in $[L, L]$. Then there exists $x \in Z(L) - [L, L]$. Note that $x \neq 0$ since $x \notin [L, L]$. Let

$I = \text{span}\{x\}$. Then I is a nonzero ideal of L contained in $Z(L)$. Next, we extend I to a basis of L, say $L = \text{span}\{x, y_1, y_2, \ldots, y_n\}$ for some $n \in \mathbb{Z}_{>0}$. Let $J = \text{span}\{y_1, y_2, \ldots, y_n\}$. Then we see that $L = I \oplus J$, so we need to show that J is also an ideal of L. Since $x \in Z(L) - [L, L]$, $I \cap [L, L] = \{0\}$, so $[L, L] \subseteq J$. Thus J is an ideal of L. Hence $L = I \oplus J$ where I and J are ideals of L and $I \subseteq Z(L)$. Consequently, L is not pure.

Conversely, assume that L is not pure. Then $L = I \oplus J$ where I and J are ideals of L and $I \subseteq Z(L)$. Let $x \in I - \{0\} \subseteq Z(L) - \{0\}$. Next we will claim that $[L, L] \subseteq J$. Let $a, b \in L$. Then a and b can be written as $a = a_I + a_J$ and $b = b_I + b_J$ where $a_I, b_I \in I \subseteq Z(L)$ and $a_J, b_J \in J$. Therefore we have

$$[a, b] = [a_I + a_J, b_I + b_J] = [a_I, b_I] + [a_I, b_J] + [a_J, b_I] + [a_J, b_J] = [a_J, b_J] \in J.$$

Since a and b are arbitrary, $[L, L] \subseteq J$. Because $x \neq 0$, $x \notin J$ which also implies $x \notin [L, L]$. Hence $x \in Z(L) - [L, L]$, so $Z(L)$ is not contained in $[L, L]$.

□

Therefore we get a condition that is equivalent to purity of Lie algebras. In general, we consider only pure Lie algebras, so we will include the condition $Z(L) \subseteq [L, L]$ in our classification process. Note that in order to obtain a Lie algebra which is not pure, we begin with a pure Lie algebra with smaller dimension and add an abelian part to it.

Theorem 5.1.5. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) \subseteq [L, L]$. Then L is a direct sum of smaller Lie algebras if and only if L is a direct sum of two Heisenberg Lie algebras.*

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) \subseteq [L, L]$. Suppose that L is a direct sum of smaller Lie algebras, says $L = L_1 \oplus L_2 \oplus \ldots \oplus L_n$ for some $n \in \mathbb{Z}_{>0}$. Since $Z(L) \subseteq [L, L]$, we know that L is pure, so each summand is not abelian. Thus $b(L_i) \neq 0$ for all $i = 1, 2, \ldots, n$. By Corollary 3.1.13, we have

$$b(L_1) + b(L_2) + \ldots + b(L_n) = b(L_1 \oplus L_2 \oplus \ldots \oplus L_n) = b(L) = 2,$$

which leave us only one choice, $n = 2$ and $b(L_1) = b(L_2) = 1$. Hence L is a direct sum of two Heisenberg Lie algebras by Theorem 3.2.6. The converse implication is clear. Consequently, L is a direct sum of smaller Lie algebras if and only if L is a direct sum of two Heisenberg Lie algebras. □

Corollary 5.1.6. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) \subseteq [L, L]$. Suppose that L is a direct sum of smaller Lie algebras. Then* $\dim L$ *is even.*

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) \subseteq [L, L]$. Suppose that L is a direct sum of smaller Lie algebras. Then L is a direct sum of two Heisenberg Lie algebras by Theorem 5.1.5, says $L = H_1 \oplus H_2$. Since $Z(L) \subseteq [L, L]$, L is pure which implies H_1 and H_2 are also pure. Thus both $\dim H_1$ and $\dim H_2$ are odd. Hence $\dim L = \dim(H_1 \oplus H_2) = \dim H_1 + \dim H_2$ is even. □

5.2 Nilpotent Lie Algebras of Breadth 2 with $\dim[L, L] = 3$ and $\dim(L/Z(L)) = 3$

As we have already seen in Theorem 4.2.1, nilpotent Lie algebra of breadth 2 has two equivalent conditions. In this part, we consider the second condition and classify it as stated in the next theorem. From now on, we may not write all bracket relations of L. We assume that all of the bracket relations are equal to zero, unless we state otherwise.

Theorem 5.2.1. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) \subseteq [L, L]$. Suppose that $\dim[L, L] = 3$ and $\dim L/Z(L) = 3$. Then L is isomorphic to either*

1. $L = \text{span}\{x, y, v, w_1, w_2\}$ *where* $[x, y] = v, [x, v] = w_1$ *and* $[y, v] = w_2$

or

2. $L = \text{span}\{x, y, z, w_1, w_2, w_3\}$ *where* $[x, y] = w_1, [x, z] = w_2$ *and* $[y, z] = w_3$.

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) \subseteq [L, L]$. Then $b(L) = 2$. Suppose that $\dim[L, L] = 3$ and $\dim L/Z(L) = 3$. Because $Z(L) \subseteq [L, L]$ and $\dim[L, L] = 3$, we have $\dim Z(L) = 0, 1, 2$ or 3. Since L is nilpotent, L has nontrivial center, so $\dim Z(L) \neq 0$. Therefore we have 3 cases to consider:

1. Case I : $\dim Z(L) = 1$. Then $3 = \dim L/Z(L) = \dim L - \dim Z(L) = \dim L - 1$, so $\dim L = 4$. Let $Z(L) = \text{span}\{z\}$. Then extend it to $[L, L] = \text{span}\{u, v, z\}$ and then $L = \text{span}\{x, u, v, z\}$. Note that the bracket relations on L are defined by $[x, u], [x, v]$ and $[u, v]$. Since $[L, L] = \text{span}\{u, v, z\}$, we say that

$$[x, u] = \alpha_1 u + \alpha_2 v + \alpha_3 z,$$
$$[x, v] = \beta_1 u + \beta_2 v + \beta_3 z,$$
$$[u, v] = \gamma_1 u + \gamma_2 v + \gamma_3 z$$

for some $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}$ and $i = 1, 2, 3$. Because of the nilpotency of L, we have $\alpha_1 = \beta_2 = \gamma_1 = \gamma_2 = 0$. Thus we obtain

$$[x, u] = \alpha_2 v + \alpha_3 z,$$
$$[x, v] = \beta_1 u + \beta_3 z,$$
$$[u, v] = \gamma_3 z.$$

Note that $\alpha_2, \beta_1, \gamma_3 \neq 0$ because $\dim[L, L] = 3$. Then we have $(ad_x)^N(u) \neq 0$ for any $N \in \mathbb{Z}_{>0}$ which contradicts the nilpotency of L.

2. Case II : $\dim Z(L) = 2$. Then $3 = \dim L/Z(L) = \dim L - \dim Z(L) = \dim L - 2$, so $\dim L = 5$. Let $Z(L) = \text{span}\{z_1, z_2\}$. Then extend it to $[L, L] = \text{span}\{u, z_1, z_2\}$ and then $L = \text{span}\{x, y, u, z_1, z_2\}$. The bracket relations on L are defined by $[x, y], [x, u]$ and $[y, u]$. Since $[L, L] = \text{span}\{u, z_1, z_2\}$, we say that

$$[x, y] = \alpha_1 u + \alpha_2 z_1 + \alpha_3 z_2,$$
$$[x, u] = \beta_1 u + \beta_2 z_1 + \beta_3 z_2,$$
$$[y, u] = \gamma_1 u + \gamma_2 z_1 + \gamma_3 z_2$$

for some $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}$ and $i = 1, 2, 3$. Since L is nilpotent, $\beta_1 = \gamma_1 = 0$. Then we get

$$[x, y] = \alpha_1 u + \alpha_2 z_1 + \alpha_3 z_2 =: v,$$
$$[x, u] = \beta_2 z_1 + \beta_3 z_2 =: w_1,$$
$$[y, u] = \gamma_2 z_1 + \gamma_3 z_2 =: w_2.$$

Since $\dim[L, L] = 3$, we get $Z(L) = \text{span}\{w_1, w_2\}$ and $[L, L] = \text{span}\{v, w_1, w_2\}$, so $\alpha_1 \neq 0$. Let $w_1' = \alpha_1 w_1$ and $w_2' = \alpha_1 w_2$. Hence $Z(L) = \text{span}\{w_1', w_2'\}, [L, L] = \text{span}\{v, w_1', w_2'\}$ and $L = \text{span}\{x, y, v, w_1', w_2'\}$ where

$$[x, y] = v,$$
$$[x, v] = [x, \alpha_1 u + \alpha_2 z_1 + \alpha_3 z_2] = \alpha_1 w_1 = w_1',$$
$$[y, v] = [y, \alpha_1 u + \alpha_2 z_1 + \alpha_3 z_2] = \alpha_1 w_2 = w_2'.$$

3. Case III : $\dim Z(L) = 3$. Then $3 = \dim L/Z(L) = \dim L - \dim Z(L) = \dim L - 3$, so $\dim L = 6$. Thus $Z(L) = [L, L]$, says $Z(L) = \text{span}\{w_1, w_2, w_3\}$. Next, we extend this basis to $L = \text{span}\{x, y, z, w_1, w_2, w_3\}$. Note that the bracket relations on L are defined by $[x, y], [x, z]$ and $[y, z]$. Since $\dim[L, L] = 3$, $[L, L] = Z(L) = \text{span}\{[x, y], [x, z], [y, z]\}$.

Let $[x, y] = w'_1, [x, z] = w'_2$ and $[y, z] = w'_3$. Hence $[L, L] = Z(L) = \text{span}\{w'_1, w'_2, w'_3\}$ and $L = \text{span}\{x, y, z, w'_1, w'_2, w'_3\}$ where $[x, y] = w'_1, [x, z] = w'_2$ and $[y, z] = w'_3$.

In conclusion, L is isomorphic to either

1. $L = \text{span}\{x, y, v, w_1, w_2\}$ where $[x, y] = v, [x, v] = w_1$ and $[y, v] = w_2$

or

2. $L = \text{span}\{x, y, z, w_1, w_2, w_3\}$ where $[x, y] = w_1, [x, z] = w_2$ and $[y, z] = w_3$.

□

5.3 Nilpotent Lie Algebras of Breadth 2 with $\dim[L, L] = 2$ and $\dim Z(L) = 1$

As stated in the first condition of Theorem 4.2.1, we now consider finite dimensional nilpotent Lie algebra L such that $\dim[L, L] = 2$. Since we also consider the condition $Z(L) \subseteq [L, L]$, $Z(L)$ could be 1 or 2-dimensional. In this section, we classify one with $\dim Z(L) = 1$ and leave the case $\dim Z(L) = 2$ to the next section.

Proposition 5.3.1. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $\dim L =: n \geq 4$. Suppose that $\dim[L, L] = 2$, $\dim Z(L) = 1$ and $Z(L) \subseteq [L, L]$. Then $L/Z(L)$ is isomorphic to $\text{span}\{x + Z(L), y + Z(L), v + Z(L), w_1 + Z(L), w_2 + Z(L), \ldots, w_{n-4} + Z(L)\}$ such that $[x + Z(L), y + Z(L)] = v + Z(L)$ and $v + Z(L), w_i + Z(L) \in Z(L/Z(L))$ for all $i = 1, 2, \ldots, n - 4$.*

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $\dim L =: n \geq 4$. Suppose that $\dim[L, L] = 2$, $\dim Z(L) = 1$ and $Z(L) \subseteq [L, L]$. Let $Z(L) = \text{span}\{z\}$. Then we extend $Z(L)$ to $[L, L] = \text{span}\{v, z\}$. Thus we have $z \neq 0$ and $v \in [L, L] - Z(L)$. Next, we consider $L/Z(L)$. Since $L/Z(L)$ is a homomorphic image of L which is nilpotent, $L/Z(L)$ is also nilpotent. In addition, $b(L/Z(L)) = 1$ because

$$[L/Z(L), L/Z(L)] = [L, L]/Z(L) = \text{span}\{v + Z(L)\}$$

is 1-dimensional by Theorem 3.2.1. As a result, by Theorem 3.2.6, $L/Z(L)$ is isomorphic to

$$\begin{aligned}&\text{span}\{x_1 + Z(L), y_1 + Z(L), x_2 + Z(L), y_2 + Z(L), \ldots, x_m + Z(L), y_m + Z(L),\\ &\quad v + Z(L), w_1 + Z(L), w_2 + Z(L), \ldots, w_{n-2m-2} + Z(L)\}\end{aligned}$$

such that $[x_i + Z(L), y_i + Z(L)] = v + Z(L)$ and $v + Z(L), w_1 + Z(L), \ldots, w_{n-2m-2} + Z(L) \in Z(L/Z(L))$ for some $m \in \{1, 2, \ldots, \lfloor \frac{n-2}{2} \rfloor\}$ and for all $i = 1, 2, \ldots, m$. Next, we will claim that $m = 1$. Suppose that $m \geq 1$. Then we consider L from $L/Z(L)$, so L is isomorphic to

$$\text{span}\{x_1, y_1, x_2, y_2, \ldots, x_m, y_m, v, w_1, w_2, \ldots, w_{n-2m-2}, z\}$$

such that $[x_i, y_i] = v + \alpha_i z$ and the rest of the bracket relations lie in $Z(L) = \text{span}\{z\}$ for some $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ and for all $i = 1, 2, \ldots, m$. Since $m \geq 1$, we can choose $i \neq j \in \{1, 2, \ldots, m\}$. Note that $[x_i, x_j], [x_i, y_j] \in Z(L)$, so

$$[x_i, v] = [x_i, v + \alpha_j z] = [x_i, [x_j, y_j]] = [[x_i, x_j], y_j] + [x_j, [x_i, y_j]] = 0.$$

Similarly, we also have

$$[y_i, v] = [y_i, v + \alpha_j z] = [y_i, [x_j, y_j]] = [[y_i, x_j], y_j] + [x_j, [y_i, y_j]] = 0$$

because $[y_i, x_j], [y_i, y_j] \in Z(L)$. Moreover, for $k = 1, 2, \ldots, n - 2m - 2$, we get

$$[w_k, v] = [w_k, v + \alpha_1 z] = [w_k, [x_1, y_1]] = [[w_k, x_1], y_1] + [x_1, [w_k, y_1]] = 0$$

since $[w_k, x_1], [w_k, y_1] \in Z(L)$. We also know that $[v, v] = [v, z] = 0$. Thus $v \in Z(L)$ which is a contradiction. Hence $m = 1$, so $L/Z(L)$ is isomorphic to $\text{span}\{x + Z(L), y + Z(L), v + Z(L), w_1 + Z(L), w_2 + Z(L), \ldots, w_{n-4} + Z(L)\}$ such that $[x + Z(L), y + Z(L)] = v + Z(L)$ and $v + Z(L), w_i + Z(L) \in Z(L/Z(L))$ for all $i = 1, 2, \ldots, n - 4$. □

Lemma 5.3.2. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that* $\dim L =: n \geq 4$. *Suppose that* $\dim[L, L] = 2$ *and* $Z(L) = \text{span}\{z\} \subseteq [L, L]$ *is 1-dimensional. Then L is isomorphic to* $\text{span}\{x, y, z, v, w_1, w_2, \ldots, w_{n-4}\}$ *such that* $[x, y] = v, [x, v] = z$ *and* $[y, v] = [x, w_i] = [v, w_i] = 0$ *for all* $i = 1, 2, \ldots, n - 4$ *and the rest of the bracket relations lie in* $Z(L)$.

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $\dim L =: n \geq 4$. Suppose that $\dim[L, L] = 2$ and $Z(L) = \text{span}\{z\} \subseteq [L, L]$ is 1-dimensional. By Proposition 5.3.1, $L/Z(L)$ is isomorphic to $\text{span}\{x + Z(L), y + Z(L), v + Z(L), w_1 + Z(L), w_2 + Z(L), \ldots, w_{n-4} + Z(L)\}$ such that $[x + Z(L), y + Z(L)] = v + Z(L)$ and $v + Z(L), w_i + Z(L) \in Z(L/Z(L))$ for all $i = 1, 2, \ldots, n - 4$.

Next, we pull this back so $L \cong \text{span}\{x, y, z, v, w_1, w_2, \ldots, w_{n-4}\}$ such that $[x, y] = v + \alpha z$ for some $\alpha \in \mathbb{F}$ and the rest of the bracket relations lie in $Z(L)$. Take $v' = v + \alpha z = [x, y]$. Then $L \cong \text{span}\{x, y, z, v', w_1, w_2, \ldots, w_{n-4}\}$ such that $[x, y] = v'$ and the rest of the bracket relations lie in $Z(L)$. Note that

$$[w_i, v'] = [w_i, [x, y]] = [[w_i, x], y] + [x, [w_i, y]] = 0$$

for all $i = 1, 2, \ldots, n-4$ because $[w_i, x], [w_i, y] \in Z(L)$. Consequently, $[w_i, v'] = 0$ for all $i = 1, 2, \ldots, n-4$. Since $v' \notin Z(L)$, we have $[x, v'] \neq 0$ or $[y, v'] \neq 0$. Without loss of generality, we assume that $[x, v'] \neq 0$, says $[x, v'] = \beta z =: z'$ for some $\beta \in \mathbb{F} - \{0\}$. Then we take $Z(L) = \text{span}\{z'\}$. Let $[x, w_i] = \gamma_i z$ for some $\gamma_i \in \mathbb{F}$ and for all $i = 1, 2, \ldots, n-4$. Then we take $w'_i = \beta w_i - \gamma_i v'$. As a consequence, we have

$$\begin{aligned} [x, w'_i] &= [x, \beta w_i - \gamma_i v'] = \beta[x, w_i] - \gamma_i[x, v'] = \beta\gamma_i z - \gamma_i \beta z = 0, \\ [v', w'_i] &= [v', \beta w_i - \gamma_i v'] = \beta[v', w_i] - \gamma_i[v', v'] = 0. \end{aligned}$$

Finally, observe that $[y, v'] = \delta z'$ for some $\delta \in \mathbb{F}$. By taking $y' = y - \delta x$, we have

$$\begin{aligned} [x, y'] &= [x, y - \delta x] = [x, y] - \delta[x, x] = v', \\ [y', v'] &= [y - \delta x, v'] = [y, v'] - \delta[x, v'] = \delta z' - \delta z' = 0. \end{aligned}$$

Hence L is isomorphic to $\text{span}\{x, y', z', v', w'_1, w'_2, \ldots, w'_{n-4}\}$ such that $[x, y] = v, [x, v] = z$ and $[y, v] = [x, w_i] = [v, w_i] = 0$ for all $i = 1, 2, \ldots, n-4$ and the rest of the bracket relations lie in $Z(L)$. □

Theorem 5.3.3. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that* $\dim L =: n \geq 4$. *Suppose that* $\dim[L, L] = 2$ *and* $Z(L) = \text{span}\{z\} \subseteq [L, L]$ *is 1-dimensional. Then the following holds:*

1. *If n is even, then L is isomorphic to* $\text{span}\{x, y, z, v, w_1, w_2, \ldots, w_{n-4}\}$ *such that* $[x, y] = v$, $[x, v] = z$ *and* $[w_i, w_{i+1}] = z$ *for all* $i = 1, 3, 5, \ldots, n-5$.
2. *If n is odd, then L is isomorphic to* $\text{span}\{x, y, z, v, w_1, w_2, \ldots, w_{n-4}\}$ *such that* $[x, y] = v$, $[x, v] = z$, $[y, w_1] = z$ *and* $[w_i, w_{i+1}] = z$ *for all* $i = 2, 4, 6, \ldots, n-5$.

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $\dim L =: n \geq 4$. Suppose that $\dim[L, L] = 2$ and $Z(L) = \text{span}\{z\} \subseteq [L, L]$ is 1-dimensional. It is clear that, by Lemma 5.3.2, L is isomorphic to $\text{span}\{x, y, z, v\}$ such that $[x, y] = v$, $[x, v] = z$ and $[y, v] = 0$ if $\dim L = 4$. Moreover, if $\dim L = 5$, then by Lemma 5.3.2, L is isomorphic to $\text{span}\{x, y, z, v, w\}$ such that $[x, y] = v$, $[x, v] = z$ and $[y, v] = [x, w] = [v, w] = 0$ and the rest of the bracket relations lie in $Z(L)$. Since $w \notin Z(L)$, we have $[y, w] = \alpha z$ for some $\alpha \neq 0$. By taking $w' = \frac{w}{\alpha}$, we have

$$[y,w'] = [y, \frac{w}{\alpha}] = \frac{1}{\alpha}[y,w] = \frac{1}{\alpha}\alpha z = z \quad \text{and} \quad [x,w'] = [v,w'] = 0.$$

Hence L is isomorphic to $\operatorname{span}\{x,y,z,v,w'\}$ such that $[x,y] = v$, $[x,v] = [y,w'] = z$ and $[y,v] = [x,w'] = [v,w'] = 0$.

Assume that $\dim L = n \geq 6$. By Lemma 5.3.2, L is isomorphic to $\operatorname{span}\{x,y,z,v,w_1,w_2,\ldots,w_{n-4}\}$ such that $[x,y] = v$, $[x,v] = z$ and $[y,v] = [x,w_i] = [v,w_i] = 0$ for all $i = 1,2,\ldots,n-4$ and the rest of the bracket relations lie in $Z(L)$. Let $W := \operatorname{span}\{z,w_1,w_2,\ldots,w_{n-4}\}$. Then $\operatorname{im} ad_x|_W = [x,W] = \{0\}$ and $\operatorname{im} ad_v|_W = [v,W] = \{0\}$. Observe that $[W,W] \subseteq \operatorname{span}\{z\}$. Suppose that $[W,W] = \{0\}$. Then we get $[y,w_i] \neq 0$ for all $i = 1,2,\ldots,n-4$. Thus $[y,w_1] = a_1 z$ and $[y,w_2] = a_2 z$ where $a_1, a_2 \in \mathbb{F} - \{0\}$. Therefore we have

$$[y, a_2 w_1 - a_1 w_2] = a_2[y,w_1] - a_1[y,w_2] = a_2 a_1 z - a_1 a_2 z = 0,$$

so $a_2 w_1 - a_1 w_2 \in Z(L)$, which is a contradiction. Consequently, $[W,W] = \operatorname{span}\{z\}$ which is 1-dimensional, so W is a nilpotent Lie subalgebra of L such that $b(W) = 1$ by Theorem 3.2.1. By Theorem 3.2.6, $W = \operatorname{span}\{z, w'_1, w'_2, \ldots, w'_{2k}, \ldots, w'_{n-4}\}$ such that $[w'_i, w'_{i+1}] = z$ for all $i = 1,3,\ldots,2k-1$ and $Z(W) = \{z, w'_{2k+1}, \ldots, w'_{n-4}\}$ where $2k \leq n-4$. Observe that for $i = 1,2,\ldots,2k$, we have $[y,w'_i] = \alpha_i z$ where $\alpha_i \in \mathbb{F}$. Let

$$y' = y + \sum_{\substack{i=1 \\ i \text{ is odd}}}^{2k-1} \alpha_i w'_{i+1} - \sum_{\substack{i=2 \\ i \text{ is even}}}^{2k} \alpha_i w'_{i-1}.$$

As a result, we have

$$[y', w'_i] = \begin{cases} [y,w'_i] + [\alpha_i w'_{i+1}, w'_i] = \alpha_i z - \alpha_i z = 0 & \text{if } i \text{ is odd,} \\ [y,w'_i] - [\alpha_i w'_{i-1}, w'_i] = \alpha_i z - \alpha_i z = 0 & \text{if } i \text{ is even.} \end{cases}$$

Therefore $[y', w'_i] = 0$ for all $i = 1,2,\ldots,2k$. Observe that

$$[x,y'] = [x,y] = v \quad \text{and} \quad [y',v] = [y,v] = 0$$

because $[x,W] = [v,W] = \{0\}$. Notice that $[y',W] \subseteq Z(L)$ since $[y,W],[W,W] \subseteq Z(L)$. By considering $L = \operatorname{span}\{x,y',z,v,w'_1,w'_2,\ldots,w'_{2k},\ldots,w'_{n-4}\}$, we know that $[x,w'_j] = [v,w'_j] = [w'_i,w'_j] = 0$ for all $i = 1,2,\ldots,n-4$ and $j = 2k+1,\ldots,n-4$. Since $w'_{2k+1},\ldots,w'_{n-4} \notin Z(L)$, we get $[y',w'_j] = \beta_j z$ where $\beta_j \in \mathbb{F} - \{0\}$ for all $j = 2k+1,\ldots,n-4$. If $2k+2 \leq n-4$, then

$[y', w'_{2k+1}] = \beta_{2k+1}z$ and $[y', w'_{2k+2}] = \beta_{2k+2}z$. Consequently, we obtain

$$\begin{aligned}[y', \beta_{2k+2}w'_{2k+1} - \beta_{2k+1}w'_{2k+2}] &= \beta_{2k+2}[y', w'_{2k+1}] - \beta_{2k+1}[y', w'_{2k+2}] \\ &= \beta_{2k+2}\beta_{2k+1}z - \beta_{2k+1}\beta_{2k+2}z \\ &= 0,\end{aligned}$$

so $\beta_{2k+2}w'_{2k+1} - \beta_{2k+1}w'_{2k+2} \in Z(L)$ which is a contradiction. Hence we get $2k+2 > n-4$, which implies $2k = n-4$ or $2k+1 = n-4$. Then we consider the following two cases:

1. Case I : $2k = n-4$. Then n is even and L is isomorphic to $\operatorname{span}\{x, y', z, v, w'_1, w'_2, \ldots, w'_{n-4}\}$ such that $[x, y'] = v$, $[x, v] = z$ and $[w'_i, w'_{i+1}] = z$ for all $i = 1, 3, 5, \ldots, n-5$.

2. Case II : $2k+1 = n-4$. Then n is odd. Since $w'_{n-4} \notin Z(L)$, we get $[y', w'_{n-4}] = \beta_{n-4}z$ where $\beta_{n-4} \in \mathbb{F} - \{0\}$. Let $\bar{w}_{n-4} = \frac{w'_{n-4}}{\beta_{n-4}}$. Then we have

$$[y', \bar{w}_{n-4}] = [y', \frac{w'_{n-4}}{\beta_{n-4}}] = \frac{1}{\beta_{n-4}}[y', w'_{n-4}] = \frac{1}{\beta_{n-4}}\beta_{n-4}z = z.$$

   Hence L is isomorphic to $\operatorname{span}\{x, y', z, v, w'_1, w'_2, \ldots, w'_{n-5}, \bar{w}_{n-4}\}$ such that $[x, y'] = v$, $[x, v] = z$, $[y', \bar{w}_{n-4}] = z$ and $[w'_i, w'_{i+1}] = z$ for all $i = 1, 3, 5, \ldots, n-6$.

Since the result from the two cases above are complement to each other, we can modify our result as follows:

1. If n is even, then L is isomorphic to $\operatorname{span}\{x, y, z, v, w_1, w_2, \ldots, w_{n-4}\}$ such that $[x, y] = v$, $[x, v] = z$ and $[w_i, w_{i+1}] = z$ for all $i = 1, 3, 5, \ldots, n-5$.

2. If n is odd, then L is isomorphic to $\operatorname{span}\{x, y, z, v, w_1, w_2, \ldots, w_{n-4}\}$ such that $[x, y] = v$, $[x, v] = z$, $[y, w_1] = z$ and $[w_i, w_{i+1}] = z$ for all $i = 2, 4, 6, \ldots, n-5$.

□

5.4 Nilpotent Lie Algebras of Breadth 2 with $\dim[L, L] = 2$ and $\dim Z(L) = 2$

To begin this section, we introduce the concept of component of a Lie algebra which we use throughout our classification process. For any finite dimensional Lie algebra L, its center can be written as $Z(L) = \operatorname{span}\{z_1, z_2, \ldots, z_m\}$ for some $m \in \mathbb{Z}_{\geq 0}$. Then we extend this basis to $L = \operatorname{span}\{x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_m\}$ where $n \in \mathbb{Z}_{\geq 0}$. Thus $\dim L = n + m$. We denote a subspace $L' := \operatorname{span}\{x_1, x_2, \ldots, x_n\} \subseteq L$.

Definition 5.4.1. Let L be a finite dimensional Lie algebra. A subspace $M = \text{span}\{y_1, \ldots, y_k\} \subseteq L'$ is a *component* of L of dimension k if $M + C_L(M) = L$.

Proposition 5.4.2. *Let L be a finite dimensional Lie algebra and M be a component of L. Then $b(x) \geq 1$ for all $x \in M - \{0\}$.*

Proof. Let L be a finite dimensional Lie algebra and M be a component of L. Since $Z(L) = \{x \in L \mid b(x) = 0\}$, we know that $b(x) \geq 1$ for all $x \in L' - \{0\}$. Hence $b(x) \geq 1$ for all $x \in M - \{0\}$ because $M - \{0\} \subseteq L' - \{0\}$. □

Lemma 5.4.3. *Let L be a finite dimensional Lie algebra and M be a subspace of L' such that $M + C_L(M) = L$. Then $M \cap C_L(M) = \{0\}$.*

Proof. Let L be a finite dimensional Lie algebra and M be a subspace of L' such that $M + C_L(M) = L$. We write $M = \text{span}\{u_1, u_2, \ldots, u_k\}$ for some $k \in \mathbb{Z}_{\geq 0}$ and extend it to a basis $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_{n-k}, z_1, z_2, \ldots, z_m\}$ of L where $Z(L) = \text{span}\{z_1, z_2, \ldots, z_m\}$. Let $x \in M \cap C_L(M)$. Then x can be written as $x = a_1u_1 + a_2u_2 + \ldots + a_ku_k$ where $a_i \in \mathbb{F}$ for all $i = 1, 2, \ldots, k$. Next we will claim that $x \in Z(L)$. Let $y \in L$. Since $L = M + C_L(M)$, y can be written as $y = y_M + c_M$ where $y_M \in M$ and $c_M \in C_L(M)$. Because $x \in M \cap C_L(M)$, we have $[x, y] = [x, y_M + c_M] = [x, y_M] + [x, c_M] = 0$. Since $y \in L$ is arbitrary, we obtain $x \in Z(L)$. Therefore x can also be written as $x = b_1z_1 + b_2z_2 + \ldots + b_mz_m$ where $b_j \in \mathbb{F}$ for all $j = 1, 2, \ldots, m$. Consequently, we obtain

$$
\begin{aligned}
0 &= x - x \\
&= (a_1u_1 + a_2u_2 + \ldots + a_ku_k) - (b_1z_1 + b_2z_2 + \ldots + b_mz_m) \\
&= a_1u_1 + a_2u_2 + \ldots + a_ku_k - b_1z_1 - b_2z_2 - \ldots - b_mz_m.
\end{aligned}
$$

Since $\{u_1, u_2, \ldots, u_k, z_1, z_2, \ldots, z_m\}$ is linearly independent, $a_i = b_j = 0$ for all $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$. Hence $x = a_1u_1 + a_2u_2 + \ldots + a_ku_k = 0$, so $M \cap C_L(M) = \{0\}$. □

By using previous lemma, we can develop our definition of component to be direct summand instead of normal summand.

Definition 5.4.4. Let L be a finite dimensional Lie algebra. A subspace $M = \text{span}\{y_1, \ldots, y_k\} \subseteq L'$ is a *component* of L of dimension k if $M \oplus C_L(M) = L$.

Remark. *Let L be a finite dimensional Lie algebra. Then L' is the largest component of L.*

Proposition 5.4.5. *Let L be a finite dimensional Lie algebra and M be a component of L of dimension k. Then $k \geq 2$.*

Proof. Let L be a finite dimensional Lie algebra and M be a component of L of dimension k. Let $x \in M - \{0\} \subseteq L' - \{0\}$. Then $b(x) \geq 1$ by Proposition 5.4.2. Thus there exists $y \in L - \{0\}$ such that $[x, y] \neq 0$, so $y \notin C_L(M)$. Since $L = M \oplus C_L(M)$, we write $y = y_M + c_M$ where $y_M \in M$ and $c_M \in C_L(M)$. Note that $y_M \neq 0$ because $y \notin C_L(M)$. Moreover, we have

$$[x, y_M] = [x, y - c_M] = [x, y] - [x, c_M] = [x, y] \neq 0.$$

As a result, we consider $y_M \in M$ and notice that $\{x, y_M\} \subseteq M$ is linearly independent because $[x, y_M] \neq 0$. Hence we have $k \geq 2$. □

Next, we define the reducibility of component. Note that a component is called *irreducible* if it is not reducible.

Definition 5.4.6. Let L be a finite dimensional Lie algebra and M be a component of L. Then M is said to be *reducible* if there exist components M_1 and M_2 such that $M = M_1 \oplus M_2$.

Remark. *Let L be a finite dimensional Lie algebra and $M = M_1 \oplus M_2$ be a reducible component of L. Then $M_1 \subseteq C_L(M_2)$ and $M_2 \subseteq C_L(M_1)$.*

By Proposition 5.4.5, the smallest component is 2-dimensional. Thus we easily get the following corollary.

Corollary 5.4.7. *Let L be a finite dimensional Lie algebra and M be a component of L of dimension 2 or 3. Then M is irreducible.*

Theorem 5.4.8. *Let L be a finite dimensional Lie algebra and M be a component of L of dimension $k \geq 2$. Then for any $x \in M - \{0\}$, $1 \leq b(x) \leq k - 1$.*

Proof. Let L be a finite dimensional Lie algebra and M be a component of L of dimension $k \geq 2$. By Proposition 5.4.2, $b(x) \geq 1$ for all $x \in M - \{0\}$. Suppose that there exists $x \in M - \{0\}$ such that $b(x) \geq k$. Since $M \oplus C_L(M) = L$, without loss of generality, there exist $y_1, y_2, \ldots, y_k \in M - \{0\}$ such that $[x, y_i] = z_i$ for all $i = 1, 2, \ldots, k$ where $\{z_1, z_2, \ldots, z_k\}$ is linearly independent. Next, we will show that $\{x, y_1, y_2, \ldots, y_k\} \subseteq M$ is linearly independent. Let $a, a_1, a_2, \ldots, a_k \in \mathbb{F}$ be such that $ax + a_1y_1 + a_2y_2 + \ldots + a_ky_k = 0$. Then we have

$$\begin{aligned} 0 &= [x, ax + a_1y_1 + a_2y_2 + \ldots + a_ky_k] \\ &= a[x, x] + a_1[x, y_1] + a_2[x, y_2] + \ldots + a_k[x, y_k] \\ &= a_1z_1 + a_2z_2 + \ldots + a_kz_k. \end{aligned}$$

Since $\{z_1, z_2, \ldots, z_k\}$ is linearly independent, $a_1 = a_2 = \ldots = a_k = 0$, which also implies $a = 0$. Hence $\{x, y_1, y_2, \ldots, y_k\} \subseteq M$ is linearly independent, which is a contradiction. Consequently, $1 \leq b(x) \leq k - 1$ for any $x \in M - \{0\}$. □

Corollary 5.4.9. *Let L be a finite dimensional Lie algebra and M be a component of L of dimension 2. Then $b(x) = 1$ for all $x \in M - \{0\}$.*

By using previous corollary, we can identify the structure of component of dimension 2 as we prove in next theorem.

Theorem 5.4.10. *Let L be a finite dimensional Lie algebra and M be a component of L of dimension 2. Then $M = \text{span}\{x_1, x_2\}$ such that $[x_1, x_2] \neq 0$. In particular, $[M, L] = \text{span}\{[x_1, x_2]\}$.*

Proof. Let L be a finite dimensional Lie algebra and M be a component of L of dimension 2. Let $x_1 \in M - \{0\}$. By Corollary 5.4.9, $b(x_1) = 1$. Since $L = M \oplus C_L(M)$, without loss of generality, there exists $x_2 \in M - \{0\}$ such that $[x_1, x_2] \neq 0$. We know that $x_2 \notin \text{span}\{x_1\}$, so $M = \text{span}\{x_1, x_2\}$. To show that $[M, L]$ is 1-dimensional, let $x \in M$ and $y \in L$. Since $L = M \oplus C_L(M)$, x and y can be written as $x = a_1x_1 + a_2x_2$ and $y = b_1x_1 + b_2x_2 + c$ where $a_i, b_i \in \mathbb{F}$ for $i = 1, 2$ and $c \in C_L(M)$. Note that $[x_1, c] = [x_2, c] = 0$ because $c \in C_L(M)$. Then we obtain

$$\begin{aligned}
[x, y] &= [a_1x_1 + a_2x_2, b_1x_1 + b_2x_2 + c] \\
&= a_1b_1[x_1, x_1] + a_1b_2[x_1, x_2] + a_1[x_1, c] + a_2b_1[x_2, x_1] + a_2b_2[x_2, x_2] + a_2[x_2, c] \\
&= a_1b_2[x_1, x_2] + a_2b_1[x_2, x_1] \\
&= (a_1b_2 - a_2b_1)[x_1, x_2] \\
&\in \text{span}\{[x_1, x_2]\}.
\end{aligned}$$

Since $x \in M$ and $y \in L$ are arbitrary, we have $[M, L] = \text{span}\{[x_1, x_2]\}$. □

Next theorem clarify the picture of reducible component. We simply need to find a smaller part of component in order to tell that it is reducible.

Theorem 5.4.11. *Let L be a finite dimensional Lie algebra and M be a component of L of dimension $k \geq 4$. Suppose that there is a proper subspace $M_1 \subseteq M$ such that $M_1 + C_L(M_1) = L$. Then M is reducible. In particular, $M = M_1 \oplus M_2$ where $M_2 \subseteq C_L(M_1)$ is a component spanned by basis of M extended from M_1.*

Proof. Let L be a finite dimensional Lie algebra and M be a component of L of dimension $k \geq 4$. Suppose that there is a proper subspace $M_1 \subseteq M$ such that $M_1 + C_L(M_1) = L$. By Lemma 5.4.3, we have $M_1 \cap C_L(M_1) = \{0\}$, so $M_1 \oplus C_L(M_1) = L$. Thus M_1 is a component of L. Assume that $M_1 = \text{span}\{x_1, x_2, \ldots, x_t\}$ for some $t < k$. Since $L = M_1 \oplus C_L(M_1)$, we extend this basis to $M = \text{span}\{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_s\}$ such that $y_1, y_2, \ldots, y_s \in C_L(M_1)$ where $s + t = k$. Let $M_2 = \text{span}\{y_1, y_2, \ldots, y_s\} \subseteq C_L(M_1)$. Then $M = M_1 \oplus M_2$. Since $M_2 \subseteq C_L(M_1)$, we have

$[M_1, M_2] = \{0\}$. Thus $M_1 \subseteq C_L(M_2)$. On the other hand, we also have $C_L(M) \subseteq C_L(M_2)$ because $M_2 \subseteq M$. Similarly, we get $C_L(M) \subseteq C_L(M_1)$, so $M_1 \cap C_L(M) \subseteq M_1 \cap C_L(M_1) = \{0\}$. Consequently, we have $M_1 \oplus C_L(M) \subseteq C_L(M_2)$ and

$$L = M \oplus C_L(M) = (M_1 \oplus M_2) \oplus C_L(M) = M_2 \oplus (M_1 \oplus C_L(M)) \subseteq M_2 + C_L(M_2).$$

Thus $M_2 + C_L(M_2) = L$. Again, by Lemma 5.4.3, $M_2 \cap C_L(M_2) = \{0\}$, so $M_2 \oplus C_L(M_2) = L$. As a result, M_2 is a component of L of dimension s. Hence $M = M_1 \oplus M_2$ is reducible. □

Corollary 5.4.12. *Let L be a finite dimensional Lie algebra and M be an irreducible component of L of dimension $k \geq 4$. Then for any proper subspace $M' \subseteq M$, $M' + C_L(M')$ is a proper subspace of L.*

Theorem 5.4.13. *Let L be a finite dimensional Lie algebra and M be an irreducible component of L. Then for any proper subspace $M' \subseteq M$, there exists $x \in M - M'$ such that $x \notin C_L(M')$.*

Proof. Let L be a finite dimensional Lie algebra and M be an irreducible component of L. Let M' be a proper subspace of M. Suppose that for any $x \in M - M'$, $x \in C_L(M')$. Then we have $M - M' \subseteq C_L(M')$. Since $M' \subseteq M$, $C_L(M) \subseteq C_L(M')$. Next, we will show that $M' + C_L(M') = L$. Suppose that $M' = \text{span}\{x_1, x_2, \ldots, x_t\}$ for some $t \geq 1$. Then we extend this basis to $M = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_s\}$ for some $s \geq 1$. Let $y \in L$. Since $M + C_L(M) = L$, y can be written as $y = a_1x_1 + a_2x_2 + \ldots + a_tx_t + b_1y_1 + b_2y_2 + \ldots + b_sy_s + c$ where $a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_s \in \mathbb{F}$ and $c \in C_L(M)$. We observe that

$$\begin{aligned} y &= a_1x_1 + a_2x_2 + \ldots + a_tx_t + b_1y_1 + b_2y_2 + \ldots + b_sy_s + c \\ &= (a_1x_1 + a_2x_2 + \ldots + a_tx_t) + (b_1y_1 + b_2y_2 + \ldots + b_sy_s) + c \\ &\in M' + (M - M') + C_L(M) \\ &\in M' + C_L(M'). \end{aligned}$$

Therefore $M' + C_L(M') = L$. By Theorem 5.4.11, M is reducible, which is a contradiction. Hence there exists $x \in M - M'$ such that $x \notin C_L(M')$. □

Theorem 5.4.14. *Let L be a finite dimensional Lie algebra and M be an irreducible component of L of dimension $k \geq 3$. Then there exist $x \in M - \{0\}$ such that $b(x) > 1$.*

Proof. Let L be a finite dimensional Lie algebra and M be an irreducible component of L of dimension $k \geq 3$. Suppose that $b(x) = 1$ for every $x \in M - \{0\}$. Let $x_1 \in M - \{0\}$. Since $L = M \oplus C_L(M)$, without loss of generality, there exists $x_2 \in M - \{0\}$ such that $[x_1, x_2] \neq 0$. Because $x_2 \in M - \{0\}$, $b(x_2) = 1$. Note that $x_2 \notin \text{span}\{x_1\}$, so $\{x_1, x_2\}$ is linearly independent. Let $M_1 := \text{span}\{x_1, x_2\}$. Since $b(x_1) = b(x_2) = 1$, by rank-nullity theorem, we know that nullity $ad_{x_i} = \dim L - \operatorname{rank} ad_{x_i} = \dim L - 1$ for $i = 1, 2$. Therefore $\ker ad_{x_1} \neq \ker ad_{x_2}$ but they are both $(\dim L - 1)$-dimensional. Thus $C_L(M_1) = \ker ad_{x_1} \cap \ker ad_{x_2}$ is $(\dim L - 2)$-dimensional. Next we will claim that $M_1 \cap C_L(M_1) = \{0\}$. Let $x \in M_1 \cap C_L(M_1)$. Then we write $x = a_1x_1 + a_2x_2$ for some $a_1, a_2 \in \mathbb{F}$. Since $x \in C_L(M_1)$, $[x_i, x] = 0$ for $i = 1, 2$. Consequently, we have

$$0 = [x_1, x] = [x_1, a_1x_1 + a_2x_2] = a_1[x_1, x_1] + a_2[x_1, x_2] = a_2[x_1, x_2],$$
$$0 = [x_2, x] = [x_2, a_1x_1 + a_2x_2] = a_1[x_2, x_1] + a_2[x_2, x_2] = -a_1[x_1, x_2],$$

so $a_1 = a_2 = 0$. Thus $M_1 \cap C_L(M_1) = \{0\}$. By counting dimension, $M_1 \oplus C_L(M_1) = L$. Hence $M_1 \subseteq M$ is a component of L of dimension 2. If $k \geq 4$, then by Theorem 5.4.11, M is reducible, which is a contradiction. Next, we assume that $k = 3$. Since $L = M_1 \oplus C_L(M_1)$, we let $0 \neq y \in M \cap C_L(M_1)$. Then $M = \operatorname{span}\{x_1, x_2, y\}$. We will claim that $y \in Z(L)$. Let $x \in L$. Since $L = M + C_L(M)$, x can be written as $x = a_1x_1 + a_2x_2 + by + c$ where $a_1, a_2, b \in \mathbb{F}$ and $c \in C_L(M)$. Then $[y, c] = 0$ because $y \in M$. Moreover, $[y, x_1] = [y, x_2] = 0$ since $y \in C_L(M_1)$. Therefore we have

$$[y, x] = [y, a_1x_1 + a_2x_2 + by + c] = a_1[y, x_1] + a_2[y, x_2] + b[y, y] + [y, c] = 0.$$

Thus $y \in Z(L)$, which is a contradiction. Hence there exist $x \in M - \{0\}$ such that $b(x) > 1$. □

Next theorem gives us the structure of component of dimension 3. Furthermore, we also obtain the classification of nilpotent Lie algebras L of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and $\dim L = 5$ as the upcoming corollary.

Theorem 5.4.15. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Let M be a component of L of dimension 3. Then $M = \operatorname{span}\{x_1, x_2, x_3\}$ such that $[x_1, x_2] = z_1$, $[x_1, x_3] = z_2$ and $[x_1, x_3] = 0$ where $Z(L) = \operatorname{span}\{z_1, z_2\}$.*

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Let M be a component of L of dimension 3. By Theorem 5.4.14, there exists $x_1 \in M - \{0\}$ such that $b(x_1) > 1$. By Theorem 5.4.8, we have $1 < b(x_1) \leq 3 - 1 = 2$, so $b(x_1) = 2$. Since $M \oplus C_L(M) = L$, without loss of generality, there exist $x_2', x_3' \in M - \{0\}$ such that $[x_1, x_2'] = z_1$ and $[x_1, x_3'] = z_2$ where $\{z_1, z_2\}$ is linearly independent. Since $Z(L) = [L, L]$ are 2-dimensional, we get $Z(L) = \operatorname{span}\{z_1, z_2\}$. Next, we observe $[x_2', x_3'] \in [L, L] = Z(L)$. Then there exist $a_1, a_2 \in \mathbb{F}$ such that $[x_2', x_3'] = a_1z_1 + a_2z_2$. Let $x_2 = x_2' - a_2x_1$ and $x_3 = x_3' + a_1x_1$. Then we have

$$[x_1, x_2] = [x_1, x_2' - a_2x_1] = [x_1, x_2'] - a_2[x_1, x_1] = z_1,$$

$$\begin{aligned}[x_1, x_3] &= [x_1, x'_3 + a_1x_1] = [x_1, x'_3] + a_1[x_1, x_1] = z_2,\\ [x_2, x_3] &= [x'_2 - a_2x_1, x'_3 + a_1x_1]\\ &= [x'_2, x'_3] + a_1[x'_2, x_1] - a_2[x_1, x'_3] - a_2a_3[x_1, x_1]\\ &= (a_1z_1 + a_2z_2) + a_1(-z_1) - a_2z_2\\ &= 0.\end{aligned}$$

Note that $\{x_1, x'_2, x'_3\}$ is linearly independent and so is $\{x_1, x_2, x_3\}$. Hence $M = \operatorname{span}\{x_1, x_2, x_3\}$ such that $[x_1, x_2] = z_1$, $[x_1, x_3] = z_2$ and $[x_1, x_3] = 0$ where $Z(L) = \operatorname{span}\{z_1, z_2\}$. □

Corollary 5.4.16. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and $\dim L = 5$. Then $L = \operatorname{span}\{x_1, x_2, x_3, z_1, z_2\}$ such that $[x_1, x_2] = z_1$, $[x_1, x_3] = z_2$ and $[x_1, x_3] = 0$ where $Z(L) = \operatorname{span}\{z_1, z_2\}$.*

Next, we provide a definition of standard n-dimensional subspace of a component and its properties.

Definition 5.4.17. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and M be an irreducible component of L. For $n \geq 2$, define an n-dimensional subspace $M_n := \operatorname{span}\{x_1, x_2, \ldots, x_n\} \subseteq M$ such that

$$b(x_1) = 1, \quad b(x_2) = b(x_3) = \ldots = b(x_{n-1}) = 2, \quad b(x_n) \geq 1$$

and

$$[x_i, x_{i+1}] = \begin{cases} z_1 & \text{if } i \text{ is odd} \\ z_2 & \text{if } i \text{ is even} \end{cases}$$

where $i \in \{1, 2, \ldots, n-1\}$ and $Z(L) = \operatorname{span}\{z_1, z_2\}$.

Proposition 5.4.18. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and M be an irreducible component of L. Then $M_n \cap C_L(M_{n-1}) = \{0\}$ for all $n \geq 3$.*

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and M be an irreducible component of L. Let $n \geq 3$ and $a_1, a_2, \ldots, a_n \in \mathbb{F}$ be such that

$$x = a_1x_1 + a_2x_2 + \ldots + a_nx_n \in M_n \cap C_L(M_{n-1}) = \operatorname{span}\{x_1, x_2, \ldots, x_n\} \cap C_L(M_{n-1}).$$

Since $x \in C_L(M_{n-1}) = \bigcap_{i=1}^{n-1} \ker ad_{x_i}$, we have $[x_i, x] = 0$ for all $i = 1, 2, \ldots, n-1$. As a result, for any $i \in \{2, 3, \ldots, n-1\}$, we have

$$\begin{aligned} 0 &= [x_i, x] \\ &= [x_i, a_1x_1 + a_2x_2 + \ldots + a_nx_n] \\ &= a_1[x_i, x_1] + a_2[x_i, x_2] + \ldots + a_n[x_i, x_n] \\ &= a_{i-1}[x_i, x_{i-1}] + a_{i+1}[x_i, x_{i+1}] \\ &= \begin{cases} -a_{i-1}z_2 + a_{i+1}z_1 & \text{if } i \text{ is odd} \\ -a_{i-1}z_1 + a_{i+1}z_2 & \text{if } i \text{ is even} \end{cases} \end{aligned}$$

Since $Z(L) = \{z_1, z_2\}$ is linearly independent, $a_{i-1} = a_{i+1} = 0$ for all $i = 2, 3, \ldots, n-1$. Thus $a_i = 0$ for all $i = 1, 2, \ldots, n$. Hence $x = 0$, so $M_n \cap C_L(M_{n-1}) = \{0\}$ for all $n \geq 3$. □

Corollary 5.4.19. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and M be an irreducible component of L. Then $M_n \cap C_L(M_n) = \{0\}$ for all $n \geq 2$.*

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and M be an irreducible component of L. Let $n \geq 2$. For $n = 2$, we let $a_1, a_2 \in \mathbb{F}$ be such that

$$x = a_1x_1 + a_2x_2 \in M_2 \cap C_L(M_2) = \text{span}\{x_1, x_2\} \cap C_L(M_2).$$

Since $x \in C_L(M_2) = \ker ad_{x_1} \cap \ker ad_{x_2}$, we have

$$\begin{aligned} 0 &= [x_1, x] = [x_1, a_1x_1 + a_2x_2] = a_1[x_1, x_1] + a_2[x_1, x_2] = a_2z_1, \\ 0 &= [x_2, x] = [x_2, a_1x_1 + a_2x_2] = a_1[x_2, x_1] + a_2[x_2, x_2] = a_1(-z_1). \end{aligned}$$

Thus $a_1 = a_2 = 0$, so $x = 0$. Hence $M_2 \cap C_L(M_2) = \{0\}$. Suppose that $n \geq 3$. By Proposition 5.4.18, $M_n \cap C_L(M_{n-1}) = \{0\}$, so we have

$$\begin{aligned} M_n \cap C_L(M_n) &= M_n \cap (\ker ad_{x_n} \cap C_L(M_{n-1})) \\ &= \ker ad_{x_n} \cap (M_n \cap C_L(M_{n-1})) \\ &= \ker ad_{x_n} \cap \{0\} \\ &= \{0\}. \end{aligned}$$

Therefore $M_n \cap C_L(M_n) = \{0\}$ for all $n \geq 3$. Hence $M_n \cap C_L(M_n) = \{0\}$ for all $n \geq 2$. □

Theorem 5.4.20. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Let M be an irreducible component of L and M_n be a subspace of M for $n \geq 2$. Then*

1. $L = \ker ad_{x_1} \oplus \operatorname{span}\{x_2\}$

2. $L = \ker ad_{x_i} \oplus \operatorname{span}\{x_{i-1}, x_{i+1}\}$ *for all* $i = 2, 3, \ldots, n-1$

3. $L = \begin{cases} \ker ad_{x_n} \oplus \operatorname{span}\{x_{n-1}\} & \textit{if } \; b(x_n) = 1 \\ \ker ad_{x_n} \oplus \operatorname{span}\{x_{n-1}, x\} & \textit{if } \; b(x_n) = 2 \end{cases}$

   where $Z(L) = \operatorname{span}\{[x_{n-1}, x_n], [x_n, x]\}$.

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Let M be an irreducible component of L and M_n be a subspace of M for $n \geq 2$.

1. First, we observe $x_1 \in M_n$. Note that $b(x_1) = 1$ and $[x_1, x_2] = z_1 \neq 0$. By rank-nullity theorem, we get nullity $ad_{x_1} = \dim L - \operatorname{rank} ad_{x_1} = \dim L - 1$. Since $x_2 \notin \ker ad_{x_1}$, we have $L = \ker ad_{x_1} \oplus \operatorname{span}\{x_2\}$.

2. Let $i \in \{2, 3, \ldots, n-1\}$. Then we have $b(x_i) = 2$. Without loss of generality, we suppose that i is even so that $[x_{i-1}, x_i] = z_1$ and $[x_i, x_{i+1}] = z_2$. To show that $L = \ker ad_{x_i} + \operatorname{span}\{x_{i-1}, x_{i+1}\}$, let $y \in L$. If $y \in \ker ad_{x_i}$, then $y = y + 0 \in \ker ad_{x_i} + \operatorname{span}\{x_{i-1}, x_{i+1}\}$. Assume that $y \notin \ker ad_{x_i}$. Then $[x_i, y] = a_1 z_1 + a_2 z_2$ for some $a_1, a_2 \in \mathbb{F}$, so we get

$$\begin{aligned} [x_i, y + a_1 x_{i-1} - a_2 x_{i+1}] &= [x_i, y] + a_1[x_i, x_{i-1}] - a_2[x_i, x_{i+1}] \\ &= (a_1 z_1 + a_2 z_2) + a_1(-z_1) - a_2 z_2 \\ &= 0, \end{aligned}$$

   so $c := y + a_1 x_{i-1} - a_2 x_{i+1} \in \ker ad_{x_i}$. As a result, we have

$$y = c - a_1 x_{i-1} + a_2 x_{i+1} \in \ker ad_{x_i} + \operatorname{span}\{x_{i-1}, x_{i+1}\}.$$

   Hence $L = \ker ad_{x_i} + \operatorname{span}\{x_{i-1}, x_{i+1}\}$. Since nullity $ad_{x_i} = \dim L - \operatorname{rank} ad_{x_i} = \dim L - 2$, by counting dimension, we also know that $\ker ad_{x_i} \cap \operatorname{span}\{x_{i-1}, x_{i+1}\} = \{0\}$. Hence $L = \ker ad_{x_i} \oplus \operatorname{span}\{x_{i-1}, x_{i+1}\}$.

3. First, we observe $x_n \in M_n$. Without loss of generality, we assume that n is even so that $[x_{n-1}, x_n] = z_1$. Suppose that $b(x_n) = 1$. By rank-nullity theorem, nullity $ad_{x_n} =$

$\dim L - \operatorname{rank} ad_{x_n} = \dim L - 1$. Since $x_{n-1} \notin \ker ad_{x_n}$, we have $L = \ker ad_{x_n} \oplus \operatorname{span}\{x_{n-1}\}$. Next, we assume that $b(x_n) = 2$. Then there exists $x \in L$ such that $[x_n, x] = a_1 z_1 + a_2 z_2$ where $a_1, a_2 \in \mathbb{F}$ and $a_2 \neq 0$. Next, we will claim that $L = \ker ad_{x_n} + \operatorname{span}\{x_{n-1}, x\}$. Let $y \in L$. If $y \in \ker ad_{x_n}$, then we get $y = y + 0 \in \ker ad_{x_n} + \operatorname{span}\{x_{n-1}, x\}$. Suppose that $y \notin \ker ad_{x_n}$. Then $[x_n, y] = b_1 z_1 + b_2 z_2$ for some $b_1, b_2 \in \mathbb{F}$. Therefore we have

$$\begin{aligned}
[x_n, y + (b_1 - \frac{b_2}{a_2}a_1)x_{n-1} - \frac{b_2}{a_2}x] &= [x_n, y] + (b_1 - \frac{b_2}{a_2}a_1)[x_n, x_{n-1}] - \frac{b_2}{a_2}[x_n, x] \\
&= (b_1 z_1 + b_2 z_2) + (b_1 - \frac{b_2}{a_2}a_1)(-z_1) - \frac{b_2}{a_2}(a_1 z_1 + a_2 z_2) \\
&= (b_1 - b_1 + \frac{b_2}{a_2}a_1 - \frac{b_2}{a_2}a_1)z_1 + (b_2 - \frac{b_2}{a_2}a_2)z_2 \\
&= 0.
\end{aligned}$$

Thus $c := y + (b_1 - \frac{b_2}{a_2}a_1)x_{n-1} - \frac{b_2}{a_2}x \in \ker ad_{x_n}$, so we have

$$y = c - (b_1 - \frac{b_2}{a_2}a_1)x_{n-1} + \frac{b_2}{a_2}x \in \ker ad_{x_n} + \operatorname{span}\{x_{n-1}, x\}.$$

Hence $L = \ker ad_{x_n} + \operatorname{span}\{x_{n-1}, x\}$. Since nullity $ad_{x_n} = \dim L - \operatorname{rank} ad_{x_n} = \dim L - 2$, $\ker ad_{x_n} \cap \operatorname{span}\{x_{n-1}, x\} = \{0\}$ by counting dimension. Consequently, $L = \ker ad_{x_n} \oplus \operatorname{span}\{x_{n-1}, x\}$. Additionally, since $[x_n, x] = a_1 z_1 + a_2 z_2 \notin \operatorname{span}\{z_1\} = \operatorname{span}\{[x_{n-1}, x_n]\}$, we get $Z(L) = \operatorname{span}\{[x_{n-1}, x_n], [x_n, x]\}$.

□

From now on, we are going to identify the structure of component of dimension 4 by constructing a standard subspace inside it as the following 2 theorems.

Theorem 5.4.21. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Let M be an irreducible component of L of dimension 4. Suppose that $M_3 \subseteq M$. Then $b(x_3) = 2$.*

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Let M be an irreducible component of L of dimension 4. Suppose that $M_3 \subseteq M$ and $b(x_3) = 1$. Then we have $M_3 = \operatorname{span}\{x_1, x_2, x_3\}$ such that $b(x_1) = 1 = b(x_3)$, $b(x_2) = 2$ and $[x_1, x_2] = z_1$, $[x_2, x_3] = z_2$, $[x_1, x_3] = 0$ where $Z(L) = \operatorname{span}\{z_1, z_2\}$. Therefore $\operatorname{im} ad_{x_1} = \operatorname{span}\{z_1\}$, $\operatorname{im} ad_{x_2} = Z(L)$ and $\operatorname{im} ad_{x_3} = \operatorname{span}\{z_2\}$.

Next, we will show that M_3 is a component of L. Let $x \in L$. If $x \in C_L(M_3)$, then $x = 0 + x \in M_3 + C_L(M_3)$. Assume that $x \notin C_L(M_3)$. Then we have $[x_1, x] = az_1$, $[x_2, x] = b_1 z_1 + b_2 z_3$ and

$[x_3, x] = cz_2$ where $a, b_1, b_2, c \in \mathbb{F}$. Let $y = x + b_1x_1 - ax_2 - b_2x_3$. Therefore we get

$$\begin{aligned}
[x_1, y] &= [x_1, x + b_1x_1 - ax_2 - b_2x_3] \\
&= [x_1, x] + b_1[x_1, x_1] - a[x_1, x_2] - b_2[x_1, x_3] \\
&= az_1 - az_1 \\
&= 0, \\
[x_2, y] &= [x_2, x + b_1x_1 - ax_2 - b_2x_3] \\
&= [x_2, x] + b_1[x_2, x_1] - a[x_2, x_2] - b_2[x_2, x_3] \\
&= (b_1z_1 + b_2z_2) + b_1(-z_1) - b_2z_2 \\
&= 0, \\
[x_3, y] &= [x_3, x + b_1x_1 - ax_2 - b_2x_3] \\
&= [x_3, x] + b_1[x_3, x_1] - a[x_3, x_2] - b_2[x_3, x_3] \\
&= cz_2 - a(-z_2) \\
&= (c + a)z_2 \\
&=: c'z_2
\end{aligned}$$

where $c' = c + a$. If $c' = 0$, then $y \in C_L(M_3)$, so we have $x = (-b_1x_1 + ax_2 + b_2x_3) + y \in M_3 + C_L(M_3)$. Suppose that $c' \neq 0$. Since M is a component of L, we write $y \in L = M \oplus C_L(M)$ as $y = y_M + c_M$ where $y_M \in M$ and $c_M \in C_L(M)$. Next, we will claim that $y_M \notin M_3$. Suppose that $y_M \in M_3$. Then $y_M = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3$ where $\alpha_i \in \mathbb{F}$ for $i = 1, 2, 3$. Therefore we have

$$\begin{aligned}
x &= y - b_1x_1 + ax_2 + b_2x_3 \\
&= (y_M + c_M) - b_1x_1 + ax_2 + b_2x_3 \\
&= (\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + c_M) - b_1x_1 + ax_2 + b_2x_3 \\
&= (\alpha_1 - b_1)x_1 + (\alpha_2 + a)x_2 + (\alpha_3 + b_2)x_3 + c_M.
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}
az_1 &= [x_1, x] \\
&= [x_1, (\alpha_1 - b_1)x_1 + (\alpha_2 + a)x_2 + (\alpha_3 + b_2)x_3 + c_M] \\
&= (\alpha_1 - b_1)[x_1, x_1] + (\alpha_2 + a)[x_1, x_2] + (\alpha_3 + b_2)[x_1, x_3] + [x_1, c_M] \\
&= (\alpha_2 + a)z_1, \\
cz_2 &= [x_3, x] \\
&= [x_3, (\alpha_1 - b_1)x_1 + (\alpha_2 + a)x_2 + (\alpha_3 + b_2)x_3 + c_M]
\end{aligned}$$

$$
\begin{aligned}
&= (\alpha_1 - b_1)[x_3, x_1] + (\alpha_2 + a)[x_3, x_2] + (\alpha_3 + b_2)[x_3, x_3] + [x_3, c_M] \\
&= (\alpha_2 + a)(-z_2) \\
&= -(\alpha_2 + a)z_2.
\end{aligned}
$$

Hence $a = \alpha_2 + a$ and $c = -(\alpha_2 + a)$, so $\alpha_2 = 0$ and $c = -a$. Thus $c' = c + a = 0$ which is a contradiction. Consequently, $y_M \notin M_3$, so $y_M \in M - M_3$. Let $y'_M = \frac{y_M}{c'}$. Then $y'_M \in M - M_3$, so $M = \text{span}\{x_1, x_2, x_3, y'_M\}$. Moreover, we observe that

$$
\begin{aligned}
[x_1, y_M] &= [x_1, y_M + c_M] = [x_1, y] = 0, \\
[x_2, y_M] &= [x_2, y_M + c_M] = [x_2, y] = 0, \\
[x_3, y_M] &= [x_3, y_M + c_M] = [x_3, y] = c' z_2.
\end{aligned}
$$

Hence we have $[x_1, y'_M] = 0 = [x_2, y'_M]$ and $[x_3, y'_M] = z_2$. Observe that

$$
\begin{aligned}
[x_1, x_2 + y'_M] &= [x_1, x_2] + [x_1, y'_M] = z_1, \\
[x_2, x_2 + y'_M] &= [x_2, x_2] + [x_2, y'_M] = 0, \\
[x_3, x_2 + y'_M] &= [x_3, x_2] + [x_3, y'_M] = -z_2 + z_2 = 0, \\
[y'_M, x_2 + y'_M] &= [y'_M, x_2] + [y'_M, y'_M] = 0.
\end{aligned}
$$

Since $L = M \oplus C_L(M) = \text{span}\{x_1, x_2, x_3, y'_M\} \oplus C_L(M)$, we get $\text{im}\, ad_{x_2 + y'_M} = \text{span}\{z_1\}$, so $b(x_2 + y'_M) = 1$. Additionally, we have

$$
\begin{aligned}
\ker ad_{x_1} &= \text{span}\{x_1, x_3, y'_M\} \oplus C_L(M), \\
\ker ad_{x_2 + y'_M} &= \text{span}\{x_2, x_3, y'_M\} \oplus C_L(M).
\end{aligned}
$$

We consider $M = \text{span}\{x_1, x_2 + y'_M, x_3, y'_M\}$ and let $M' = \text{span}\{x_1, x_2 + y'_M\} \subseteq M$. Then

$$
C_L(M') = \ker ad_{x_1} \cap \ker ad_{x_2 + y'_M} = \text{span}\{x_3, y'_M\} \oplus C_L(M).
$$

Consequently, we have

$$
\begin{aligned}
L &= M \oplus C_L(M) \\
&= \text{span}\{x_1, x_2 + y'_M, x_3, y'_M\} \oplus C_L(M) \\
&= \text{span}\{x_1, x_2 + y'_M\} \oplus \text{span}\{x_3, y'_M\} \oplus C_L(M) \\
&= M' \oplus C_L(M'),
\end{aligned}
$$

so $M' \subseteq M$ is a component of L which contradicts irreducibility of M. Hence $x \in M_3 + C_L(M_3)$.

Since $x \in L$ is arbitrary, $L = M_3 + C_L(M_3)$. By Corollary 5.4.19, we know that $M_3 \cap C_L(M_3) = \{0\}$, so $L = M_3 \oplus C_L(M_3)$. Thus $M_3 \subseteq M$ is a component of L which is also a contradiction. Hence $b(x_3) = 2$. □

Theorem 5.4.22. *Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Let M be an irreducible component of L of dimension 4. Suppose that M contains an element of breadth 1. Then $M = \text{span}\{x_1, x_2, x_3, x_4\}$ such that $[x_1, x_2] = z_1$, $[x_1, x_3] = z_2$ and $[x_3, x_4] = z_1$ where $Z(L) = \text{span}\{z_1, z_2\}$.*

Proof. Let L be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Let M be an irreducible component of L of dimension 4. Suppose that M contains an element of breadth 1, says $x \in M$ such that $b(x) = 1$. Let $u_1 = x$ and $M_1 = \text{span}\{u_1\} \subseteq M$. By Theorem 5.4.13, there exists $u_2 \in M - M_1$ such that $u_2 \notin C_L(M_1)$. Thus $[u_1, u_2] = z_1 \in Z(L) - \{0\}$.

Next, we let $M_2 = \text{span}\{u_1, u_2\} \subseteq M$. We will show that $b(u_2) = 2$. Suppose that $b(u_2) = 1$. By Theorem 5.4.20, we know that $L = \text{span}\{u_1\} \oplus \ker ad_{u_2} = \text{span}\{u_2\} \oplus \ker ad_{u_1}$. Since $u_2 \in \ker ad_{u_2}$, $L = \ker ad_{u_1} + \ker ad_{u_2}$, so we have

$$\begin{aligned}
\dim L &= \dim(\ker ad_{u_1} + \ker ad_{u_2}) \\
&= \dim \ker ad_{u_1} + \dim \ker ad_{u_2} - \dim(\ker ad_{u_1} \cap \ker ad_{u_2}) \\
&= \text{nullity}\, ad_{u_1} + \text{nullity}\, ad_{u_2} - \dim C_L(M_2) \\
&= (\dim L - \text{rank}\, ad_{u_1}) + (\dim L - \text{rank}\, ad_{u_2}) - \dim C_L(M_2) \\
&= (\dim L - b(u_1)) + (\dim L - b(u_2)) - \dim C_L(M_2) \\
&= (\dim L - 1) + (\dim L - 1) - \dim C_L(M_2) \\
&= 2 \dim L - 2 - \dim C_L(M_2).
\end{aligned}$$

Thus $\dim C_L(M_2) = \dim L - 2$. By Corollary 5.4.19, we get $M_2 \cap C_L(M_2) = \{0\}$. Since $\dim M_2 = 2$ we have $L = M_2 \oplus C_L(M_2)$, so $M_2 \subseteq M$ is a component of L which is a contradiction. Therefore $b(u_2) = 2$.

Because $b(u_2) = 2$, there exists $u_3 \in M$ such that $[u_2, u_3] = z_2$ where $Z(L) = \text{span}\{z_1, z_2\}$. Note that $u_3 \notin M_2$ because $[u_2, M_2] = \text{span}\{z_1\}$. Since $b(u_1) = b(x) = 1$ and $\text{im}\, ad_{u_1} = \text{span}\{z_1\}$, we suppose that $[u_1, u_3] = az_1$ where $a \in \mathbb{F}$. Let $v_1 = u_1$, $v_2 = u_2$ and $v_3 = u_3 - au_2$. Then we have

$$\begin{aligned}
&[v_1, v_2] = [u_1, u_2] = z_1, \\
&[v_2, v_3] = [u_2, u_3 - au_2] = [u_2, u_3] - a[u_2, u_2] = z_2, \\
&[v_1, v_3] = [u_1, u_3 - au_2] = [u_1, u_3] - a[u_1, u_2] = az_1 - az_1 = 0.
\end{aligned}$$

Let $M_3 = \text{span}\{v_1, v_2, v_3\}$. Then $b(v_1) = b(x) = 1, b(v_2) = 2$ and $b(v_3) \geq 1$. By Theorem 5.4.21, we have $b(v_3) = 2$, so there exists $v_4 \in M$ such that $[v_3, v_4] = c_1 z_1 + c_2 z_2$ where $c_1, c_2 \in \mathbb{F}$ and $c_1 \neq 0$. Note that $v_4 \notin M_3$ because $[v_3, M_3] = \text{span}\{z_2\}$. Thus $M = \text{span}\{v_1, v_2, v_3, v_4\}$. Since $b(v_1) = 1$ and $b(v_2) = 2$, we assume that $[v_1, v_4] = a_1 z_1$ and $[v_2, v_4] = b_1 z_1 + b_2 z_2$ where $a_1, b_1, b_2 \in \mathbb{F}$. Let $y = v_4 + b_1 v_1 - a_1 v_2 - b_2 v_3$. Then $M = \text{span}\{v_1, v_2, v_3, y\}$ such that

$$\begin{aligned}
[v_1, y] &= [v_1, v_4 + b_1 v_1 - a_1 v_2 - b_2 v_3] \\
&= [v_1, v_4] + b_1[v_1, v_1] - a_1[v_1, v_2] - b_2[v_1, v_3] \\
&= a_1 z_1 - a_1 z_1 \\
&= 0, \\
[v_2, y] &= [v_2, v_4 + b_1 v_1 - a_1 v_2 - b_2 v_3] \\
&= [v_2, v_4] + b_1[v_2, v_1] - a_1[v_2, v_2] - b_2[v_2, v_3] \\
&= (b_1 z_1 + b_2 z_2) + b_1(-z_1) - b_2 z_2 \\
&= 0, \\
[v_3, y] &= [v_3, v_4 + b_1 v_1 - a_1 v_2 - b_2 v_3] \\
&= [v_3, v_4] + b_1[v_3, v_1] - a_1[v_3, v_2] - b_2[v_3, v_3] \\
&= (c_1 z_1 + c_2 z_2) - a_1(-z_2) \\
&= c_1 z_1 + (c_2 + a_1) z_2.
\end{aligned}$$

Let $x_1 = v_1$, $x_2 = v_2$, $x_3 = v_3$ and $x_4 = \frac{y}{c_1}$. Hence $M = \text{span}\{x_1, x_2, x_3, x_4\}$ such that

$$\begin{aligned}
&[x_1, x_2] = [v_1, v_2] = z_1, \\
&[x_2, x_3] = [v_2, v_3] = z_2, \\
&[x_3, x_4] = [v_3, \frac{y}{c_1}] = \frac{1}{c_1}[v_3, y] = \frac{1}{c_1}(c_1 z_1 + (c_2 + a_1) z_2) = z_1 + (\frac{c_2 + a_1}{c_1}) z_2 =: z_1 + \alpha z_2, \\
&[x_1, x_3] = [v_1, v_3] = 0, \\
&[x_1, x_4] = [v_1, \frac{y}{c_1}] = \frac{1}{c_1}[v_1, y] = 0, \\
&[x_2, x_4] = [v_2, \frac{y}{c_1}] = \frac{1}{c_1}[v_2, y] = 0.
\end{aligned}$$

where $\alpha := \frac{c_2 + a_1}{c_1} \in \mathbb{F}$ and $Z(L) = \text{span}\{z_1, z_2\}$. To show that $\alpha = 0$, we suppose that $\alpha \neq 0$. Let $s_1 = x_1$, $s_2 = \alpha x_2 + x_4$, $s_3 = \alpha x_3 - x_1$ and $s_4 = x_4$. Then we have $M = \text{span}\{s_1, s_2, s_3, s_4\}$ such that

$$[s_1, s_2] = [x_1, \alpha x_2 + x_4] = \alpha[x_1, x_2] + [x_1, x_4] = \alpha z_1,$$

$$
\begin{aligned}
[s_3, s_4] &= [\alpha x_3 - x_1, x_4] = \alpha[x_3, x_4] - [x_1, x_4] = \alpha(z_1 + \alpha z_2),\\
[s_1, s_3] &= [x_1, \alpha x_3 - x_1] = \alpha[x_1, x_3] - [x_1, x_1] = 0,\\
[s_1, s_4] &= [x_1, x_4] = 0,\\
[s_2, s_3] &= [\alpha x_2 + x_4, \alpha x_3 - x_1]\\
&= \alpha^2[x_2, x_3] - \alpha[x_2, x_1] + \alpha[x_4, x_3] - [x_4, x_1]\\
&= \alpha^2 z_2 + \alpha z_1 - \alpha(z_1 + \alpha z_2)\\
&= 0,\\
[s_2, s_4] &= [\alpha x_2 + x_4, x_4] = \alpha[x_2, x_4] + [x_4, x_4] = 0
\end{aligned}
$$

where $Z(L) = \text{span}\{z_1, z_2\} = \text{span}\{\alpha z_1, \alpha(z_1 + \alpha z_2)\}$. Thus $M = M' \oplus M''$ where $M' = \text{span}\{s_1, s_2\}$ and $M'' = \text{span}\{s_3, s_4\}$ are components of L. Hence M is reducible which is a contradiction. Consequently, we obtain $\alpha = 0$, so we get $M = \text{span}\{x_1, x_2, x_3, x_4\}$ such that $[x_1, x_2] = z_1$, $[x_1, x_3] = z_2$ and $[x_3, x_4] = z_1$ where $Z(L) = \text{span}\{z_1, z_2\}$. □

At this point, we can classify nilpotent Lie algebra L of breadth 2 such that $Z(L) = [L, L]$ and $\dim L = 6$ under the assumption that L has an element of breadth 1 as we show in the following corollary.

Corollary 5.4.23. *Let L be a 6-dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Suppose that L contains an element of breadth 1. Then $L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\}$ such that*

1. $[x_1, x_2] = z_1$, $[x_2, x_3] = z_2$ *and* $[x_3, x_4] = z_1$

or

2. $[x_1, x_2] = z_1$ *and* $[x_3, x_4] = z_2$

where $Z(L) = \text{span}\{z_1, z_2\}$.

Proof. Let L be a 6-dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Suppose that L contains an element of breadth 1. Since $Z(L)$ is 2-dimensional, we have $\dim L' = \dim L - \dim Z(L) = 6 - 2 = 4$. Then we have L' is a component of L of dimension 4. Suppose that L' is irreducible. By Theorem 5.4.22, $L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\}$ such that $[x_1, x_2] = z_1$, $[x_1, x_3] = z_2$ and $[x_3, x_4] = z_1$ where $Z(L) = \text{span}\{z_1, z_2\}$.

On the other hand, we assume that L' is reducible. Then L' must be composed of 2 irreducible components of dimension 2. Since $Z(L) = [L, L]$ are 2-dimensional, we obtain $L' = \text{span}\{x_1, x_2\} \oplus \text{span}\{x_3, x_4\}$ such that $[x_1, x_2] = z_1$ and $[x_3, x_4] = z_2$ where $Z(L) =$

$[L, L] = \text{span}\{z_1, z_2\}$. We also note that

$$b(\alpha x_1 + \beta x_3) = b(\alpha x_1 + \beta x_4) = b(\alpha x_2 + \beta x_3) = b(\alpha x_2 + \beta x_4) = 2$$

for any $\alpha, \beta \in \mathbb{F}$. In this case, it is not isomorphic to previous case because of the component property. Hence $L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\}$ such that $[x_1, x_2] = z_1$ and $[x_3, x_4] = z_2$ where $Z(L) = [L, L] = \text{span}\{z_1, z_2\}$. □

In the next part, we prove that 6-dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional contains an element of breadth 1 if the underlying field is algebraically closed. Thus we get a complete classification if we consider Lie algebras over algebraically closed field.

Lemma 5.4.24. *Let L be a 6-dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Suppose that for any $x \in L - Z(L)$, $b(x) = 2$. Then $L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\}$ such that $[x_1, x_2] = [x_3, x_4] = z_1$, $[x_2, x_3] = z_2$ and $[x_1, x_4] = \alpha z_2$ where $\alpha \neq 0$.*

Proof. Let L be a 6-dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Since $Z(L)$ is 2-dimensional, we have $\dim L' = \dim L - \dim Z(L) = 6 - 2 = 4$. Then we have L' is a component of L of dimension 4. Suppose that for any $x \in L - Z(L)$, $b(x) = 2$. Note that $L' \subseteq L - Z(L)$, so $b(x) = 2$ for any $x \in L'$ Let $u_2 \in L'$. Then $b(u_2) = 2$, so there exist $u_1, u_3 \in L'$ such that $[u_1, u_2] = w_1$, $[u_2, u_3] = w_2$ and $[u_1, u_3] = a_1 w_1 + a_2 w_2$ where $a_1, a_2 \in \mathbb{F}$ and $Z(L) = \text{span}\{w_1, w_2\}$. Let $v_1 = u_1 - a_2 u_2$, $v_2 = u_2$, $v_3 = u_3 - a_1 u_2$ and $M = \text{span}\{v_1, v_2, v_3\} \subseteq L'$. Then we have

$$\begin{aligned}
[v_1, v_2] &= [u_1 - a_2 u_2, u_2] = [u_1, u_2] - a_2 [u_2, u_2] = w_1, \\
[v_2, v_3] &= [u_2, u_3 - a_1 u_2] = [u_2, u_3] - a_1 [u_2, u_2] = w_2, \\
[v_1, v_3] &= [u_1 - a_2 u_2, u_3 - a_1 u_2] \\
&= [u_1, u_3] - a_1 [u_1, u_2] - a_2 [u_2, u_3] + a_2 a_1 [u_2, u_2] \\
&= (a_1 w_1 + a_2 w_2) - a_1 w_1 - a_2 w_2 \\
&= 0.
\end{aligned}$$

Since L' is 4-dimensional, there exists $v_4 \in L'$. Then we write

$$\begin{aligned}
[v_1, v_4] &= \alpha_1 w_1 + \alpha_2 w_2, \\
[v_2, v_4] &= \beta_1 w_1 + \beta_2 w_2, \\
[v_3, v_4] &= \gamma_1 w_1 + \gamma_2 w_2
\end{aligned}$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}$ for $i = 1, 2$. Since $b(v_1) = b(v_3) = 2$, we have $\alpha_2, \gamma_1 \neq 0$. Without loss of generality, we assume that $\gamma_1 = 1$. Then we have $L' = \text{span}\{v_1, v_2, v_3, v_4\}$ such that

$$\begin{aligned}
[v_1, v_4] &= \alpha_1 w_1 + \alpha_2 w_2, \\
[v_2, v_4] &= \beta_1 w_1 + \beta_2 w_2, \\
[v_3, v_4] &= w_1 + \gamma_2 w_2
\end{aligned}$$

Let $y_1 = v_1,\ y_2 = v_2,\ y_3 = v_3$ and $y_4 = v_4 + \beta_1 v_1 + \gamma_2 v_2 - \beta_2 v_3$. Then we have $L' = \text{span}\{y_1, y_2, y_3, y_4\}$ such that

$$\begin{aligned}
[y_1, y_2] &= [v_1, v_2] = w_1, \\
[y_1, y_3] &= [v_1, v_3] = 0, \\
[y_2, y_3] &= [v_2, v_3] = w_2, \\
[y_1, y_4] &= [v_1, v_4 + \beta_1 v_1 + \gamma_2 v_2 - \beta_2 v_3] \\
&= [v_1, v_4] + \beta_1 [v_1, v_1] + \gamma_2 [v_1, v_2] - \beta_2 [v_1, v_3] \\
&= (\alpha_1 w_1 + \alpha_2 w_2) + \gamma_2 w_1 \\
&= (\alpha_1 + \gamma_2) w_1 + \alpha_2 w_2 \\
&=: \delta w_1 + \alpha_2 w_2, \\
[y_2, y_4] &= [v_2, v_4 + \beta_1 v_1 + \gamma_2 v_2 - \beta_2 v_3] \\
&= [v_2, v_4] + \beta_1 [v_2, v_1] + \gamma_2 [v_2, v_2] - \beta_2 [v_2, v_3] \\
&= (\beta_1 w_1 + \beta_2 w_2) + \beta_1 (-w_1) - \beta_2 w_2 \\
&= 0, \\
[y_3, y_4] &= [v_3, v_4 + \beta_1 v_1 + \gamma_2 v_2 - \beta_2 v_3] \\
&= [v_3, v_4] + \beta_1 [v_3, v_1] + \gamma_2 [v_3, v_2] - \beta_2 [v_3, v_3] \\
&= (w_1 + \gamma_2 w_2) + \gamma_2 (-w_2) \\
&= w_1
\end{aligned}$$

where $\delta = \alpha_1 + \gamma_2$. Next, we let $x_1 = y_1 - \frac{\delta}{2} y_3,\ x_2 = y_2,\ x_3 = y_3,\ x_4 = y_4 - \frac{\delta}{2} y_2,\ z_1 = w_1 + \frac{\delta}{2} w_2,\ z_2 = w_2$ and $\alpha = \alpha_2 - \frac{\delta^2}{4}$. Then $L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\}$ such that

$$\begin{aligned}
[x_1, x_2] &= [y_1 - \frac{\delta}{2} y_3, y_2] = [y_1, y_2] - \frac{\delta}{2} [y_3, y_2] = w_1 + \frac{\delta}{2} w_2 = z_1, \\
[x_1, x_3] &= [y_1 - \frac{\delta}{2} y_3, y_3] = [y_1, y_3] - \frac{\delta}{2} [y_3, y_3] = 0, \\
[x_2, x_3] &= [y_2, y_3] = w_2 = z_2,
\end{aligned}$$

$$
\begin{aligned}
[x_1, x_4] &= [y_1 - \frac{\delta}{2}y_3, y_4 - \frac{\delta}{2}y_2] \\
&= [y_1, y_4] - \frac{\delta}{2}[y_1, y_2] - \frac{\delta}{2}[y_3, y_4] + \frac{\delta^2}{4}[y_3, y_2] \\
&= (\delta w_1 + \alpha_2 w_2) - \frac{\delta}{2}w_1 - \frac{\delta}{2}w_1 + \frac{\delta^2}{4}(-w_2) \\
&= (\alpha_2 - \frac{\delta^2}{4})w_2 \\
&= \alpha z_2, \\
[x_2, x_4] &= [y_2, y_4 - \frac{\delta}{2}y_2] = [y_2, y_4] - \frac{\delta}{2}[y_2, y_2] = 0, \\
[x_3, x_4] &= [y_3, y_4 - \frac{\delta}{2}y_2] = [y_3, y_4] - \frac{\delta}{2}[y_3, y_2] = w_1 - \frac{\delta}{2}(-w_2) = w_1 + \frac{\delta}{2}w_2 = z_1.
\end{aligned}
$$

Moreover, if $\alpha = 0$, then we have $b(x_1) = b(x_4) = 1$ which is a contradiction. Thus $\alpha \neq 0$. Hence $L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\}$ such that $[x_1, x_2] = [x_3, x_4] = z_1$, $[x_2, x_3] = z_2$ and $[x_1, x_4] = \alpha z_2$ where $\alpha \neq 0$. □

Theorem 5.4.25. *Let L be a 6-dimensional nilpotent Lie algebra of breadth 2 over an algebraically closed field such that $Z(L) = [L, L]$ are 2-dimensional. Then L contains an element of breadth 1.*

Proof. Let L be a 6-dimensional nilpotent Lie algebra of breadth 2 over an algebraically closed field such that $Z(L) = [L, L]$ are 2-dimensional. Suppose that L does not contain an element of breadth 1. Then $b(x) = 2$ for any $x \in L - Z(L)$. By Lemma 5.4.24, $L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\}$ such that $[x_1, x_2] = [x_3, x_4] = z_1$, $[x_2, x_3] = z_2$ and $[x_1, x_4] = \alpha z_2$ where $\alpha \neq 0$. Since we consider L over algebraically closed field, $\sqrt{-\alpha}$ exists. Consider $y = \sqrt{-\alpha}x_2 + x_4$. Then we have

$$
\begin{aligned}
[y, x_1] &= [\sqrt{-\alpha}x_2 + x_4, x_1] = \sqrt{-\alpha}[x_2, x_1] + [x_4, x_1] = \sqrt{-\alpha}(-z_1) - \alpha z_2 = -\sqrt{-\alpha}(z_1 - \sqrt{-\alpha}z_2), \\
[y, x_2] &= [\sqrt{-\alpha}x_2 + x_4, x_2] = \sqrt{-\alpha}[x_2, x_2] + [x_4, x_2] = 0, \\
[y, x_3] &= [\sqrt{-\alpha}x_2 + x_4, x_3] = \sqrt{-\alpha}[x_2, x_3] + [x_4, x_3] = \sqrt{-\alpha}(z_2) - z_1 = -(z_1 - \sqrt{-\alpha}z_2), \\
[y, x_4] &= [\sqrt{-\alpha}x_2 + x_4, x_4] = \sqrt{-\alpha}[x_2, x_4] + [x_4, x_4] = 0.
\end{aligned}
$$

Therefore $\operatorname{im} ad_y = \text{span}\{z_1 - \sqrt{-\alpha}z_2\}$, so $b(y) = 1$ which is a contradiction. Hence L contains an element of breadth 1. □

Corollary 5.4.26. *Let L be a 6-dimensional nilpotent Lie algebra of breadth 2 over an algebraically closed field such that $Z(L) = [L, L]$ are 2-dimensional. Then $L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\}$ such that*

1. $[x_1, x_2] = z_1$*,* $[x_2, x_3] = z_2$ *and* $[x_3, x_4] = z_1$

or

2. $[x_1, x_2] = z_1$ *and* $[x_3, x_4] = z_2$

where $\alpha \neq 0$ *and* $Z(L) = \operatorname{span}\{z_1, z_2\}$*.*

Note that for any odd dimensional nilpotent Lie algebra L of breadth 2 such that $Z(L) = [L, L]$ is 2-dimensional, we do not need algebraically closed field to find an element of breadth 1.

Proposition 5.4.27. *Let* L *be an odd dimensional nilpotent Lie algebra of breadth 2 such that* $Z(L) = [L, L]$ *is 2-dimensional. Then there exists* $x \in L$ *such that* $b(x) = 1$ *and* $\operatorname{im} ad_x = \operatorname{span}\{z\}$ *for any* $z \in Z(L)$*.*

Proof. Let L be an odd dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ is 2-dimensional. Then $\dim L = n \in \mathbb{Z}_{>0}$ which is odd. Let $z \in L$ and $I = \operatorname{span}\{z\}$. Then I is an ideal of L since $z \in Z(L)$. Then we observe that $[L/I, L/I] = [L, L]/I$, so we have $\dim[L/I, L/I] = \dim[L, L] - \dim I = 2 - 1 = 1$. By Theorem 3.2.1, we have $b(L/I) = 1$. Note that $\dim L/I = \dim L - \dim I = n - 1$, so by Theorem 3.2.6, there exists a basis

$$S = \{v_1 + I, v_{-1} + I, v_2 + I, v_{-2} + I, \ldots, v_r + I, v_{-r} + I, z' + I, w_1 + I \ldots, w_{(n-1)-2r-1} + I\}$$

for L/I such that

$$[v_i + I, v_j + I] = \begin{cases} z' + I & \text{if } \ i = -j > 0, \\ -z' + I & \text{if } \ i = -j < 0, \\ I & \text{otherwise} \end{cases}$$

for every $i, j \in \{\pm 1, \pm 2, \ldots, \pm r\}$ and $z' + I, w_1 + I \ldots, w_{(n-1)-2r-1} + I \in Z(L/I)$ where $r \in \mathbb{Z}_{>0}$. Observe that $(n-1) - 2r - 1 = n - 2(r+1) > 0$ because n is odd. Thus there exists $w_1 + I \in Z(L/I)$, so $\operatorname{im} ad_{w_1+I} = \operatorname{span}\{I\}$. Therefore for any $y \in L$, we have $[w_1, y] + I = [w_1 + I, y + I] = I$, so $[w_1, y] \in I$. Since $y \in L$ is arbitrary, $\operatorname{im} ad_{w_1} \subseteq I = \operatorname{span}\{z\}$. Next, we will claim that $w_1 \notin Z(L)$. Suppost that $w_1 \in Z(L)$. Then we obtain

$$w_1 + I \in Z(L)/I = [L, L]/I = [L/I, L/I] = \operatorname{span}\{z' + I\}.$$

Thus $w_1 + I = az' + I$ for some $a \in \mathbb{F}$, so $\{w_1 + I, z' + I\}$ is not linearly independent which is a contradiction. Hence $w_1 \notin Z(L)$. Consequently, $b(w_1) = 1$ and $\operatorname{im} ad_{w_1} = I = \operatorname{span}\{z\}$. □

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