ABSTRACT

KHUHIRUN, BORWORN. Classification of Nilpotent Lie Algebras with Small Breadth. (Under the direction of Ernest L. Stitzinger and Kailash C. Misra.)

A Lie algebra, $L$ is said to be of breadth $k$ if the maximal dimension of the images of left multiplication by elements of the algebra is $k$, introduced by Leedham-Green, Neumann and Wiegold. Inspired by the work of Parmeggani and Stellmacher on finite $p$-groups, we characterize nilpotent Lie algebras of breadth 1 and 2. We show that a nilpotent Lie algebra $L$ has breadth 1 if and only if the derived algebra of $L$ has dimension 1 which is equivalent to $L$ being a Heisenberg Lie algebra with possible abelian direct summands. The nilpotent Lie algebra $L$ has breadth 2 if and only if either the derived algebra of $L$ has dimension 2 or the derived algebra and central quotient both have dimension 3. These results parallel results in finite $p$-groups. Unlike its group theory counter part, we use our characterization to determine the isomorphism classes of nilpotent Lie algebras of breadth 1 and 2. In this classification we focus on Lie algebras with no abelian direct summand, algebras which we call pure. So our classification results are always for pure nilpotent Lie algebras. One can harmlessly add abelian direct summands to these algebras to get further examples. For breadth 2, we determine the isomorphism classes of all Lie algebras with three dimensional derived algebra and all Lie algebras with two dimensional derived algebra and one dimensional center. For the only remaining case, where the derived algebra and center both have dimension two, we classify the algebras up to dimension six.
Classification of Nilpotent Lie Algebras with Small Breadth

by

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BIOGRAPHY

Early in the morning of September 21, 1984, Borworn Khuhirun was born in Lopburi, Thailand. Two months later, he moved to Bangkok, Thailand, raised by his two aunts. He graduated from Horwang School before studying Mathematics at Chulalongkorn University for undergraduate and master degrees. Later in 2009, he got a scholarship for studying abroad and then chose to attend doctoral program at North Carolina State University.
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Chapter 1

Introduction

Classifying algebraic objects is a central theme of mathematical research. A paramount example is the classification of finite dimensional complex simple Lie algebras due to Killing and Cartan. Less progress has been made for other classes of Lie algebras. In particular, the vast number of nilpotent Lie algebras has made the classification problem formidable for this class. As a result, authors have made progress by classifying nilpotent Lie algebras satisfying certain conditions. Research in finite group theory has followed a similar path. Simple groups have been classified, but the large number of $p$-groups has led researchers to investigate $p$-groups with added conditions. An example is the work of Parmeggiani and Stellmacher where the concept of breadth is used. In particular, they have given characterizations of finite $p$-groups of breadth 1 and 2. However, so far there does not exist a classification of these finite $p$-groups.

The analogous concept of breadth for Lie algebra has been introduced by Leedham-Green, Neumann and Wiegold. They define the breadth of a Lie algebra to be the maximum of the dimensions of the images of $\text{ad}_x$ where $x$ runs over the algebra. We consider this concept for finite dimensional nilpotent Lie algebras and give a characterization for breadth 1 and 2. In particular, we show that a finite dimensional nilpotent Lie algebra is of breadth 1 if and only if its derived algebra is one dimensional. We also show that a finite dimensional nilpotent Lie algebra $L$ has breadth 2 if and only if either the derived algebra of $L$ has dimension 2 or the derived algebra and the central quotient both have dimension 3. These results parallel results in finite $p$-groups.

Finally we use our characterizations to classify finite dimensional nilpotent Lie algebras of breadth 1 and 2. We define a nilpotent Lie algebra to be pure if it does not have abelian direct summands. Then we classify finite dimensional pure nilpotent Lie algebras of breadth one and two since abelian summands can be added harmlessly. In particular, we show that a finite dimensional pure nilpotent Lie algebra of breadth 1 is isomorphic to a Heisenberg Lie algebra. For a finite dimensional pure nilpotent Lie algebras $L$, the center is contained in the derived
algebra. By our characterization result, the dimension of the derived algebra of a finite dimensional pure nilpotent Lie algebra $L$ of breadth 2 is either 2 or 3. We determine the isomorphism classes of finite dimensional pure nilpotent Lie algebras of breadth two with three dimensional derived algebra. We also determine the isomorphism classes of finite dimensional pure nilpotent Lie algebras of breadth two with two dimensional derived algebra and one dimensional center. For the remaining case where the derived algebra and center coincide with dimension 2, we determine their isomorphism classes up to dimension 6. We hope these classification results will lead to corresponding classification results in finite $p$-groups.
Chapter 2

Preliminaries

We begin this chapter by introducing some basic definitions and notations we use throughout this paper. All of these following definitions and notations can be found in Humphreys and lecture notes in Lie algebra. We consider finite dimensional Lie algebra together with underlying field $\mathbb{F}$ such that char$(\mathbb{F}) \neq 2$ for the most of the first half. Meanwhile, we develop our focus to finite dimensional nilpotent Lie algebra over $\mathbb{F}$ in the second half. Let $L$ be a Lie algebra. Define a sequence of ideals of $L$ called lower central series or descending central series by

\[ L^0 \supseteq L^1 \supseteq L^2 \supseteq L^3 \supseteq \ldots \supseteq L^m \supseteq \ldots \]

where $L^0 = L, L^1 = [L, L], L^2 = [L, [L, L]] = [L, L^1], L^3 = [L, L^2], \ldots, L^m = [L, L^{m-1}], \ldots$

$L$ is said to be nilpotent if $L^m = \{0\}$ for some $m \in \mathbb{Z}_{\geq 0}$.

During classification process, nilpotency of Lie algebra and its center play important roles, so we would like to provide some facts about them. Every nilpotent Lie algebra has nontrivial center. Furthermore, homomorphic image and quotient of nilpotent Lie algebra are nilpotent. Note that when we consider a Lie algebra $L$ as a finite dimensional vector space, we could apply rank-nullity theorem in order to get

\[ \dim L = \text{nullity } \varphi + \text{rank } \varphi \]

where $\varphi : L \to L$ is a Lie algebra homomorphism.
Chapter 3

Nilpotent Lie Algebras of Breadth 1

3.1 Basic Definitions and Properties of Breadth

We start this section with definitions and properties of breadth on Lie algebra developed from group theory.

**Definition 3.1.1.** Let $L$ be a finite dimensional Lie algebra. For any $x \in L$, breadth of $x$, denoted by $b(x)$ is

$$
b(x) = \dim(L/\ker ad_x) = \dim L - \text{nullity } ad_x = \text{rank } ad_x.
$$

More generally, for any ideal $A$ of $L$, we define

$$
b_A(x) = \dim(A/\ker ad_x|_A) = \dim A - \text{nullity } ad_x|_A = \text{rank } ad_x|_A.
$$

**Definition 3.1.2.** Let $L$ be a finite dimensional Lie algebra. We define breadth of $L$, denoted by $b(L)$ to be

$$
b(L) = \max\{b(x) \mid x \in L\}.
$$

Moreover, for any ideal $A$ of $L$, we have

$$
b_A(L) = \max\{b_A(x) \mid x \in L\}.
$$
Remark. Let $L$ be a finite dimensional Lie algebra. Then $Z(L) = \{ x \in L \mid b(x) = 0 \}$.

Remark. Let $L$ be a finite dimensional Lie algebra and $A$ an ideal of $L$. Then the following hold for any $x \in L$:

1. $b_A(x) \leq b(x)$.
2. $b_A(L) \leq b(L)$.

Definition 3.1.3. Let $L$ be a finite dimensional Lie algebra and $A$ an ideal of $L$. We define

$$B = \{ x \in L \mid b(x) = b(L) \},$$

$$B_A = \{ x \in L \mid b_A(x) = b_A(L) \},$$

$$T_A = \{ x \in L \mid b_A(x) = 1 \}.$$ 

Next proposition shows that Lie algebra which has breadth equal to zero is equivalent to that Lie algebra is abelian.

Proposition 3.1.4. Let $L$ be a finite dimensional Lie algebra. Then $b(L) = 0$ if and only if $L$ is abelian.

Proof. Let $L$ be a finite dimensional Lie algebra. Then it is easy to see that

$$b(L) = \max \{ b(x) \mid x \in L \} = 0 \iff b(x) = \text{rank } ad_x = 0 \quad \forall x \in L$$

$$\iff ad_x = 0 \quad \forall x \in L$$

$$\iff [L, L] = \{0\}$$

$$\iff L \text{ is abelian}.$$ 

\[\square\]

Lemma 3.1.5. Let $L$ be a finite dimensional Lie algebra and $A$ an ideal of $L$. Then $b(L) \leq \dim[L, L]$ and $b_A(L) \leq \dim[A, L]$.

Proof. For any $x \in L$, we have $ad_x : L \to [L, L]$. Then $\text{im } ad_x \subseteq [L, L]$, so we obtain $b(x) = \text{rank } ad_x \leq \dim[L, L]$. Since $x \in L$ is arbitrary, $b(L) \leq \dim[L, L]$. Similarly, we also have $ad_x|_A : L \to [A, L]$, so $\text{im } ad_x|_A \subseteq [A, L]$. Thus $b_A(x) = \text{rank } ad_x|_A \leq \dim[A, L]$. Because $x \in L$ is arbitrary, $b_A(L) \leq \dim[A, L]$. 

\[\square\]

Corollary 3.1.6. Let $L$ be a finite dimensional Lie algebra. Suppose that there exists $x \in L$ such that $b(x) = \dim[L, L]$. Then $b(L) = \dim[L, L]$. In particular, let $A$ be an ideal of $L$. Suppose that there exists $x \in L$ such that $b_A(x) = \dim[A, L]$. Then $b_A(L) = \dim[A, L]$. 

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Proof. Let $L$ be a finite dimensional Lie algebra. Suppose that there exists $x \in L$ such that $b(x) = \dim[L, L]$. By Lemma 3.1.5, we know that $b(L) \leq \dim[L, L]$. Therefore we have

$$\dim[L, L] = b(x) \leq b(L) \leq \dim[L, L],$$

so $b(L) = \dim[L, L]$. On the other hand, if we let $A$ be an ideal of $L$ and assume that there exists $x \in L$ such that $b_A(x) = \dim[A, L]$. By Lemma 3.1.5, we have $b_A(L) \leq \dim[A, L]$. Hence

$$\dim[A, L] = b_A(x) \leq b_A(L) \leq \dim[A, L],$$

so $b_A(L) = \dim[A, L].$

Even though we define breadth of Lie algebra to be maximum value of breadth of all elements and Lie algebra can be considered as a vector space spanned by a basis, we cannot determine breadth of Lie algebra from its basis.

**Example 3.1.7.** (Breadth of a Lie algebra cannot be determined from its basis)

Let $L = H_1 \oplus H_2$ where $H_1$ and $H_2$ are Heisenberg Lie algebra. Then

$$L = \text{span}\{x_1, y_1, z_1\} \oplus \text{span}\{x_2, y_2, z_2\}$$

where $[x_1, y_1] = z_1$ and $[x_2, y_2] = z_2$. Note that $L$ is a six dimensional nilpotent Lie algebra because $[L, L] = \text{span}\{z_1, z_2\} = Z(L)$ and $[L, [L, L]] = [L, Z(L)] = \{0\}$. Observe that

$$b(x_i) = b(y_i) = 1 \quad \text{and} \quad b(z_i) = 0 \quad \text{for all} \quad i = 1, 2,$$

but $b(x_1 + x_2) = 2$ since $[x_1 + x_2, y_i] = z_i$ for all $i = 1, 2$. By Corollary 3.1.6, we have $b(L) = 2$ since $b(x_1 + x_2) = 2 = \dim[L, L].$

**Example 3.1.8.** Let $L = \text{span}\{x_1, x_2, \ldots, x_n\}$ together with bracket relations defined by

$$[x_1, x_n] = 0 \quad \text{and} \quad [x_1, x_i] = x_{i+1}$$

where $i = 2, 3, \ldots, n - 1$. Then $L$ is an $n$-dimensional nilpotent Lie algebra of breadth $n - 2$.

First, we need to show that Jacobi identity holds. Let $x, y, z \in L$. Then there exist $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{C}$ such that

$$x = a_1 x_1 + \ldots + a_n x_n,$$
$$y = b_1 x_1 + \ldots + b_n x_n,$$
$$z = c_1 x_1 + \ldots + c_n x_n.$$
Then we have

\[[x, y] = [a_1 x_1 + \ldots + a_n x_n, [b_1 x_1 + \ldots + b_n x_n, c_1 x_1 + \ldots + c_n x_n]]
= [a_1 x_1 + \ldots + a_n x_n, (b_1 c_2 - b_2 c_1)x_3 + (b_1 c_3 - b_3 c_1)x_4 + \ldots + (b_1 c_{n-1} - b_{n-1} c_1)x_n]
= a_1(b_1 c_2 - b_2 c_1)x_4 + a_1(b_1 c_3 - b_3 c_1)x_5 + \ldots + a_1(b_1 c_{n-2} - b_{n-2} c_1)x_n.

\[[x, y], z = [[a_1 x_1 + \ldots + a_n x_n, b_1 x_1 + \ldots + b_n x_n], c_1 x_1 + \ldots + c_n x_n]
= [(a_1 b_2 - a_2 b_1)x_3 + (a_1 b_3 - a_3 b_1)x_4 + \ldots + (a_1 b_{n-1} - a_{n-1} b_1)x_n, c_1 x_1 + \ldots + c_n x_n]
= -c_1(a_1 b_2 - a_2 b_1)x_4 - c_1(a_1 b_3 - a_3 b_1)x_5 - \ldots - c_1(a_1 b_{n-2} - a_{n-2} b_1)x_n.

\[[y, x], z = [b_1 x_1 + \ldots + b_n x_n, [a_1 x_1 + \ldots + a_n x_n, c_1 x_1 + \ldots + c_n x_n]]
= [b_1 x_1 + \ldots + b_n x_n, (a_1 c_2 - a_2 c_1)x_3 + (a_1 c_3 - a_3 c_1)x_4 + \ldots + (a_1 c_{n-1} - a_{n-1} c_1)x_n]
= b_1(a_1 c_2 - a_2 c_1)x_4 + b_1(a_1 c_3 - a_3 c_1)x_5 + \ldots + b_1(a_1 c_{n-2} - a_{n-2} c_1)x_n.

Therefore we have

\[[[x, y], z] + [y, [x, z]] = (-c_1(a_1 b_2 - a_2 b_1)x_4 - c_1(a_1 b_3 - a_3 b_1)x_5 - \ldots - c_1(a_1 b_{n-2} - a_{n-2} b_1)x_n)
+ (b_1(a_1 c_2 - a_2 c_1)x_4 + b_1(a_1 c_3 - a_3 c_1)x_5 + \ldots + b_1(a_1 c_{n-2} - a_{n-2} c_1)x_n)
= (-a_1 b_2 c_1 + a_2 b_1 c_1 + a_1 b_3 c_1 - a_3 b_1 c_1)x_4 + (-a_1 b_3 c_1 + a_3 b_1 c_1 + a_1 b_1 c_3
- a_3 b_1 c_1)x_5 + \ldots + (-a_1 b_{n-2} c_1 + a_{n-2} b_1 c_1 + a_1 b_1 c_{n-2} - a_{n-2} b_1 c_1)x_n
= (-a_1 b_2 c_1 + a_1 b_2 c_1)x_4 + (-a_1 b_3 c_1 + a_1 b_3 c_1)x_5 + \ldots + (-a_1 b_{n-2} c_1
+ a_1 b_1 c_{n-2})x_n
= a_1(b_1 c_2 - b_2 c_1)x_4 + a_1(b_1 c_3 - b_3 c_1)x_5 + \ldots + a_1(b_1 c_{n-2} - b_{n-2} c_1)x_n
= [x, [y, z]].

As a result, the Jacobi identity holds, so \(L\) is an \(n\)-dimensional Lie algebra.

Observe that \([L, L] = \text{span}\{x_3, x_4, \ldots, x_n\}\) which is \((n - 2)\)-dimensional and \(L^n = \{0\}\), so \(L\) is nilpotent. Moreover, by Corollary 3.1.6, we have \(b(L) = n - 2\) since \(b(x_1) = n - 2 = \dim[L, L]\). Hence \(L\) is an \(n\)-dimensional nilpotent Lie algebra of breadth \(n - 2\).

**Theorem 3.1.9.** Let \(L\) be a finite dimensional Lie algebra such that \(b(L) = n \in \Z_{>0}\). Then \(\dim(L/Z(L)) \geq n + 1\).

**Proof.** Let \(L\) be a finite dimensional Lie algebra such that \(b(L) = n \in \Z_{>0}\). Let \(x \in B\). Then \(b(x) = b(L) = n\), so there exist \(y_1, y_2, \ldots, y_n \in L\) and \(z_1, z_2, \ldots, z_n \in [L, L]\) such that \([x, y_i] = z_i\).
for all \( i = 1, 2, \ldots, n \) and \( \{z_1, z_2, \ldots, z_n\} \) is linearly independent. Note that \( \alpha \in Z(L) \) if and only if \( b(\alpha) = 0 \). Therefore \( y_1, y_2, \ldots, y_n \notin Z(L) \) because \( b(y_i) \geq 1 \) for all \( i = 1, 2, \ldots, n \).

Thus \( y_1 + Z(L), y_2 + Z(L), \ldots, y_n + Z(L) \neq Z(L) \). Similarly, we also have \( x \notin Z(L) \) because \( b(x) = n > 0 \).

Next, we claim that \( \{x + Z(L), y_1 + Z(L), y_2 + Z(L), \ldots, y_n + Z(L)\} \subseteq L/Z(L) \) is linearly independent, let \( a_0, a_1, a_2, \ldots, a_n \in F \) be such that

\[
a_0(x + Z(L)) + a_1(y_1 + Z(L)) + a_2(y_2 + Z(L)) + \ldots + a_n(y_n + Z(L)) = Z(L).
\]

Then \( a_0x + a_1y_1 + a_2y_2 + \ldots + a_ny_n \in Z(L) \), so we have

\[
0 = [x, a_0x + a_1y_1 + a_2y_2 + \ldots + a_ny_n]
= a_0[x, x] + a_1[x, y_1] + a_2[x, y_2] + \ldots + a_n[x, y_n]
= a_1z_1 + a_2z_2 + \ldots + a_nz_n.
\]

Since \( \{z_1, z_2, \ldots, z_n\} \) is linearly independent, \( a_1, a_2, \ldots, a_n = 0 \). By assumption, we also get \( a_0(x + Z(L)) = Z(L) \). Since \( x + Z(L) \neq Z(L) \), \( a_0 = 0 \). Consequently, \( a_0, a_1, a_2, \ldots, a_n = 0 \), so \( \{x + Z(L), y_1 + Z(L), y_2 + Z(L), \ldots, y_n + Z(L)\} \) is a linearly independent subset of \( L/Z(L) \).

Hence \( \dim(L/Z(L)) \geq n + 1 \).

**Lemma 3.1.10.** Let \( L \) be a finite dimensional Lie algebra. Suppose that \( \dim(L/Z(L)) = n \in \mathbb{Z}_{>0} \). Then \( \dim[L, L] \leq \binom{n}{2} \).

**Proof.** Let \( L \) be a finite dimensional Lie algebra. Then there exists \( m \in \mathbb{Z}_{>0} \) such that \( Z(L) = \operatorname{span}\{x_1, x_2, \ldots, x_m\} \). Then we extend this basis to \( L = \operatorname{span}\{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\} \).

Since \( x_1, x_2, \ldots, x_r \in Z(L) \), we have

\[
[L, L] = \operatorname{span}\{[y_1, y_2], [y_1, y_3], \ldots, [y_1, y_n],
[y_2, y_3], \ldots, [y_2, y_n],
\ldots, [y_{n-1}, y_n]\}
\]

and then

\[
\dim[L, L] = \dim \operatorname{span}\{[y_1, y_2], [y_1, y_3], \ldots, [y_{n-1}, y_n]\}
\leq n + (n - 1) + (n - 2) + \ldots + 1
= \frac{n}{2}(n - 1)
= \binom{n}{2}.
\]
Hence \( \dim[L, L] \leq \binom{n}{2} \) as we wanted.

Finally, we are going to show that breadth of the direct sum of finite dimensional Lie algebras is equal to sum of their breadths.

**Lemma 3.1.11.** Let \( L_1 \) and \( L_2 \) be finite dimensional Lie algebras. Then \( b_{L_1 \oplus L_2}(x_1 + x_2) = b_{L_1}(x_1) + b_{L_2}(x_2) \) for any \( x_1 \in L_1 \) and \( x_2 \in L_2 \).

**Proof.** Let \( L_1 \) and \( L_2 \) be finite dimensional Lie algebras and \( L = L_1 \oplus L_2 \). Let \( x_1 \in L_1 \) and \( x_2 \in L_2 \). Since \( L = L_1 \oplus L_2 \), we know that \([L_1, L_2] = L_1 \cap L_2 = \{0\}\), so \( L_1 \) and \( L_2 \) can be considered as ideals of \( L \). Then we have

\[
\begin{align*}
\text{im } ad_{x_1+x_2}|_{L_1} &= [x_1, x_2, L_1] = [x_1, L_1] + [x_2, L_1] = \text{im } ad_{x_1}|_{L_1}, \\
\text{im } ad_{x_1+x_2}|_{L_2} &= [x_1, x_2, L_2] = [x_1, L_2] + [x_2, L_2] = \text{im } ad_{x_2}|_{L_2},
\end{align*}
\]

so we get

\[
\text{im } ad_{x_1+x_2}|_{L_1 \oplus L_2} = [x_1 + x_2, L_1 \oplus L_2] = [x_1 + x_2, L_1] \oplus [x_1 + x_2, L_2] = \text{im } ad_{x_1}|_{L_1} \oplus \text{im } ad_{x_2}|_{L_2} = \text{im } ad_{x_1}|_{L_1} \oplus \text{im } ad_{x_2}|_{L_2}.
\]

Therefore we obtain

\[
b_{L_1 \oplus L_2}(x_1 + x_2) = \text{rank } ad_{x_1+x_2}|_{L_1 \oplus L_2} = \dim \text{im } ad_{x_1+x_2}|_{L_1 \oplus L_2} = \dim(\text{im } ad_{x_1}|_{L_1} \oplus \text{im } ad_{x_2}|_{L_2}) = \dim \text{im } ad_{x_1}|_{L_1} + \dim \text{im } ad_{x_2}|_{L_2} = \text{rank } ad_{x_1}|_{L_1} + \text{rank } ad_{x_2}|_{L_2} = b_{L_1}(x_1) + b_{L_2}(x_2).
\]

Hence we get \( b_L(x_1 + x_2) = b_{L_1}(x_1) + b_{L_2}(x_2) \) for any \( x_1 \in L_1 \) and \( x_2 \in L_2 \) as we wanted. \( \square \)

**Theorem 3.1.12.** Let \( L_1 \) and \( L_2 \) be finite dimensional Lie algebras. Then \( b(L_1 \oplus L_2) = b(L_1) + b(L_2) \).

**Proof.** Let \( L_1 \) and \( L_2 \) be finite dimensional Lie algebras. Then there exist \( x_1 \in L_1 \) and \( x_2 \in L_2 \) such that \( b_{L_1}(x_1) = b(L_1) \) and \( b_{L_2}(x_2) = b(L_2) \), respectively. By using Lemma 3.1.11, we have

\[
b(L_1) + b(L_2) = b_{L_1}(x_1) + b_{L_2}(x_2) = b_{L_1 \oplus L_2}(x_1 + x_2) \leq b(L_1 \oplus L_2).
\]
On the other hand, we let \( y \in L_1 \oplus L_2 \). Then \( y = y_1 + y_2 \) for some \( y_1 \in L_1 \) and \( y_2 \in L_2 \). Note that \( b_{L_1}(y_1) \leq b(L_1) \) and \( b_{L_2}(y_2) \leq b(L_2) \), so we get

\[
b_{L_1 \oplus L_2}(y) = b_{L_1 \oplus L_2}(y_1 + y_2) = b_{L_1}(y_1) + b_{L_2}(y_2) \leq b(L_1) + b(L_2)
\]

by Lemma 3.1.11. Since \( y \in L_1 \oplus L_2 \) is arbitrary, we have \( b(L_1 \oplus L_2) \leq b(L_1) + b(L_2) \). Hence \( b(L_1 \oplus L_2) = b(L_1) + b(L_2) \).

**Corollary 3.1.13.** Let \( L_1, L_2, \ldots, L_n \) be finite dimensional Lie algebras for some \( n \in \mathbb{Z}_{>0} \). Then \( b(L_1 \oplus L_2 \oplus \cdots \oplus L_n) = b(L_1) + b(L_2) + \cdots + b(L_n) \).

### 3.2 Classification of Nilpotent Lie Algebras of Breadth 1

For Lie algebra of breadth 1, we get a result analogous to the result in group theory provided by [2].

**Theorem 3.2.1.** Let \( L \) be a finite dimensional Lie algebra. Then \( b(L) = 1 \) if and only if \( \dim[L, L] = 1 \).

**Proof.** Let \( L \) be a finite dimensional Lie algebra such that \( \dim[L, L] = 1 \). Then by Lemma 3.1.5, we have \( b(L) \leq 1 \). Since \( L \) is not abelian, \( b(L) = 1 \) by Proposition 3.1.4.

On the other hand, assume that \( b(L) = 1 \) and \( \dim[L, L] \neq 1 \). By Proposition 3.1.4, since \( b(L) \neq 0 \), \( \dim[L, L] \neq 0 \). Thus \( \dim[L, L] \geq 2 \). Let \( z_1, z_2 \in [L, L] \) be such that \( \{z_1, z_2\} \) is linearly independent. Then there exist \( x_1, x_2, y_1, y_2 \in L \) such that \( [x_1, y_1] = z_1 \) and \( [x_2, y_2] = z_2 \). Note that

\[
ad_{x_1}(y_1) = z_1, \quad ad_{y_1}(x_1) = -z_1,
\]

\[
ad_{x_2}(y_2) = z_2, \quad ad_{y_2}(x_2) = -z_2,
\]

and \( b(L) = 1 \), so \( b(x_1) = b(x_2) = b(y_1) = b(y_2) = 1 \). Therefore we have

\[
ad_{x_1}, ad_{y_1} : L \to \text{span}\{z_1\},
\]

\[
ad_{x_2}, ad_{y_2} : L \to \text{span}\{z_2\}.
\]

Next, we consider \( [x_1, x_2] = ad_{x_1}(x_2) \in \text{span}\{z_1\} \). On the other hand, \( [x_1, x_2] = -ad_{x_2}(x_1) \in \text{span}\{z_2\} \). Thus \( [x_1, x_2] \in \text{span}\{z_1\} \cap \text{span}\{z_2\} = \{0\} \), so \( [x_1, x_2] = 0 \). Similarly, we also get \( [x_1, y_2] = [y_1, x_2] = [y_1, y_2] = 0 \). Consequently, we obtain

\[
[x_1 + x_2, y_1] = [x_1, y_1] + [x_2, y_1] = z_1,
\]
\[ [x_1 + x_2, y_2] = [x_1, y_2] + [x_2, y_2] = z_2. \]

Since \( \{z_1, z_2\} \) is linearly independent, \( b(x_1 + x_2) = \text{rank} \; ad_{x_1 + x_2} \geq 2 \), which contradicts \( b(L) = 1 \). Hence \( \dim[L, L] = 1 \) by contradiction.

**Lemma 3.2.2.** Let \( L \) be a finite dimensional Lie algebra of breadth 1. Then \( L \) is nilpotent if and only if \( [L, L] \subseteq Z(L) \).

**Proof.** Let \( L \) be a finite dimensional Lie algebra of breadth 1. Then by Theorem 3.2.1, we know that \( \dim[L, L] = 1 \). Suppose that \( L \) is nilpotent and \( [L, L] \not\subseteq Z(L) \). Thus \( [L, L] \cap Z(L) = \{0\} \) because \( \dim[L, L] = 1 \). Let \( x \in [L, L] - \{0\} \). Then \( x \notin Z(L) \), so there exists \( y \in L \) such that \( [y, x] \neq 0 \). Therefore \( [y, x] = \alpha x \) for some \( \alpha \neq 0 \). Consequently, we have \( ad_{y}^{N}(x) = \alpha^{N}x \neq 0 \) for all \( N \in \mathbb{Z}_{>0} \) which is a contradiction. Hence \( [L, L] \subseteq Z(L) \).

Conversely, suppose that \( [L, L] \subseteq Z(L) \). Then we have \( L^{3} = [L, [L, L]] \subseteq [L, Z(L)] = \{0\} \), so \( L^{3} = \{0\} \). Hence \( L \) is nilpotent.

In order to classify Lie algebra of breadth 1, we use the concept of alternate bilinear form and its application that could be found in [5] as the following:

**Definition 3.2.3.** Let \( V \) be a finite dimensional vector space and \( \varphi(\ , \) : \( V \times V \rightarrow \mathbb{F} \) a bilinear form on \( V \). Then \( \varphi \) is called alternate if \( \varphi(v, v) = 0 \) for all \( v \in V \).

**Theorem 3.2.4** (cf. [6], Theorem 6.3). Let \( V \) be a finite dimensional vector space such that \( \dim V = n \in \mathbb{Z}_{>0} \). Let \( \varphi(\ , \) : \( V \times V \rightarrow \mathbb{F} \) be an alternate bilinear form on \( V \). Then there exists a basis

\[
S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z_1, \ldots, z_{n-2r}\}
\]

for \( V \) such that the matrix of \( B \) relative to this basis has the form

\[
B_{\varphi} = \text{diag}\{S_1, S_2, \ldots, S_r, 0, \ldots, 0\}
\]

where \( r \in \mathbb{Z}_{>0} \) such that \( r \leq \frac{n}{2} \) and \( S_1 = S_2 = \ldots = S_r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

Finally, we classify finite dimensional Lie algebras of breadth 1 as the following theorem. Note that if we consider finite dimensional nilpotent Lie algebras, then they are actually direct sums of Heisenberg Lie algebra and abelian Lie algebra.

**Theorem 3.2.5.** Let \( L \) be a finite dimensional Lie algebra of breadth 1 such that \( \dim L = n \in \mathbb{Z}_{>0} \). Let \( 0 \neq z \in [L, L] \). Then there exists a basis

\[
S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z_1, \ldots, z_{n-2r}\}
\]
for \( L \) such that

\[
[v_i, v_j] = \begin{cases} 
  z & \text{if } i = -j > 0, \\
  -z & \text{if } i = -j < 0, \\
  0 & \text{otherwise},
\end{cases}
\]

for every \( i, j \in \{\pm 1, \pm 2, \ldots, \pm r\} \) and \( Z(L) = \text{span}\{z_1, \ldots, z_{n-2r}\} \).

Proof. Let \( L \) be a finite dimensional Lie algebra of breadth 1 such that \( \dim L = n \in \mathbb{Z}_{>0} \). Then \( b(L) = 1 \). By Theorem 3.2.1, \( \dim[L, L] = 1 \). Let \( 0 \neq z \in [L, L] \). Then we have \([L, L] = \text{span}\{z\}\).

For any \( x, y \in L \), we have \([x, y] = \alpha z \) for some \( \alpha \in \mathbb{F} \). Define a bilinear form \( \varphi : L \times L \to \mathbb{F} \) to be \( \varphi(x, y) = \alpha \). Note that \( \varphi \) is bilinear since \([x, x] = 0 \) for all \( x \in L \). By Theorem 3.2.4, there exists a basis

\[
S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z_1, \ldots, z_{n-2r}\}
\]

for \( L \) such that the matrix of \( \varphi \) relative to \( S \) has the form

\[
B_\varphi = \text{diag}\{S_1, S_2, \ldots, S_r, 0, \ldots, 0\}
\]

where \( r \in \mathbb{N} \) such that \( r \leq \frac{n}{2} \) and \( S_1 = S_2 = \ldots = S_r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

As a result, for every \( i, j \in \{1, 2, \ldots r\} \),

\[
\varphi(v_i, v_j) = \begin{cases} 
  1 & \text{if } i = -j > 0, \\
  -1 & \text{if } i = -j < 0, \\
  0 & \text{otherwise},
\end{cases}
\]

and \( z_1, \ldots, z_{n-2r} \in Z(L) \). Hence we have

\[
[v_i, v_j] = \begin{cases} 
  z & \text{if } i = -j > 0, \\
  -z & \text{if } i = -j < 0, \\
  0 & \text{otherwise},
\end{cases}
\]

where \( i, j \in \{1, 2, \ldots r\} \) and \( z_1, \ldots, z_{n-2r} \in Z(L) \).

Next, we will claim that \( Z(L) = \text{span}\{z_1, \ldots, z_{n-2r}\} \). Since \( z_1, \ldots, z_{n-2r} \in Z(L) \), we get \( \text{span}\{z_1, \ldots, z_{n-2r}\} \subseteq Z(L) \). Conversely, without loss of generality, we let \( a_1, a_{-1}, \ldots, a_r, a_{-r} \in \mathbb{F} \) such that

\[
a_1 v_1 + a_{-1} v_{-1} + a_2 v_2 + a_{-2} v_{-2} + \ldots + a_r v_r + a_{-r} v_{-r} = 0.
\]
Let $k \in \{1, 2, \ldots, r\}$. Then we have
\[
0 = [v_k, a_1 v_1 + a_{-1} v_{-1} + \ldots + a_r v_r + a_{-r} v_{-r}] = a_{-k} [v_k, v_{-k}] = a_{-k} z,
\]
\[
0 = [v_{-k}, a_1 v_1 + a_{-1} v_{-1} + \ldots + a_r v_r + a_{-r} v_{-r}] = a_k [v_{-k}, v_k] = a_k (-z) = -a_k z,
\]
so $a_k = a_{-k} = 0$ for all $k = 1, 2, \ldots, r$. Therefore we obtain $Z(L) \subseteq \text{span}\{z_1, \ldots, z_{n-2r}\}$.

Consequently, $Z(L) = \text{span}\{z_1, \ldots, z_{n-2r}\}$. In summary, there exists a basis $S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z_1, \ldots, z_{n-2r}\}$ for $L$ such that
\[
[v_i, v_j] = \begin{cases} 
  z & \text{if } i = -j > 0, \\
  -z & \text{if } i = -j < 0, \\
  0 & \text{otherwise},
\end{cases}
\]
for every $i, j \in \{1, 2, \ldots, r\}$ and $Z(L) = \text{span}\{z_1, \ldots, z_{n-2r}\}$. $\square$

**Theorem 3.2.6.** Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 1 such that $\dim L = n \in \mathbb{Z}_{>0}$. Let $0 \neq z \in [L, L]$. Then there exists a basis $S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z, w_1, \ldots, w_{n-2r-1}\}$ for $L$ such that
\[
[v_i, v_j] = \begin{cases} 
  z & \text{if } i = -j > 0, \\
  -z & \text{if } i = -j < 0, \\
  0 & \text{otherwise},
\end{cases}
\]
for every $i, j \in \{\pm 1, \pm 2, \ldots, \pm r\}$ and $Z(L) = \text{span}\{z, w_1, \ldots, w_{n-2r-1}\}$.

**Proof.** Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 1 such that $\dim L = n \in \mathbb{Z}_{>0}$. Let $0 \neq z \in [L, L]$. By Theorem 3.2.5, there exists a basis $S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z_1, \ldots, z_{n-2r}\}$ for $L$ such that
\[
[v_i, v_j] = \begin{cases} 
  z & \text{if } i = -j > 0, \\
  -z & \text{if } i = -j < 0, \\
  0 & \text{otherwise},
\end{cases}
\]
for every $i, j \in \{1, 2, \ldots, r\}$ and $Z(L) = \text{span}\{z_1, \ldots, z_{n-2r}\}$. Since $L$ is nilpotent, we have
$[L, L] \subseteq Z(L)$ by Lemma 3.2.2. Thus we have $z \in Z(L)$, so we can pick $z$ as a basis element so that $Z(L) = \text{span}\{z, w_1, \ldots, w_{n-2r-1}\}$ and $S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z, w_1, \ldots, w_{n-2r-1}\}$.

As a result, there exists a basis

$$S = \{v_1, v_{-1}, v_2, v_{-2}, \ldots, v_r, v_{-r}, z, w_1, \ldots, w_{n-2r-1}\}$$

for $L$ such that

$$[v_i, v_j] = \begin{cases} 
  z & \text{if } i = -j > 0, \\
  -z & \text{if } i = -j < 0, \\
  0 & \text{otherwise}, 
\end{cases}$$

for every $i, j \in \{\pm 1, \pm 2, \ldots, \pm r\}$ and $Z(L) = \text{span}\{z, w_1, \ldots, w_{n-2r-1}\}$. \hfill \Box
Chapter 4

Nilpotent Lie Algebras of Breadth 2

4.1 Properties and Lemmas

In this section, we prove several theorems that we need to use in our main theorem in the next section. We begin this part with a definition which is derived from centralizer in group theory.

**Definition 4.1.1.** Let \( L \) be a finite dimensional Lie algebra, \( A \) an ideal of \( L \) and \( S \subseteq L \). We define

\[
C_A(S) = \{ \alpha \in A \mid ad_\alpha(x) = 0 \text{ for all } x \in S \} = \{ \alpha \in A \mid ad_x(\alpha) = 0 \text{ for all } x \in S \} = \bigcap_{x \in S} \{ \alpha \in A \mid ad_x(\alpha) = 0 \} = \bigcap_{x \in S} \ker ad_x|_A.
\]

In particular, we have

\[
C_L(A) = \bigcap_{a \in A} \ker ad_a
\]

and if \( S = \{x\} \) for some \( x \in L \), then

\[
C_A(S) = C_A(\{x\}) = \ker ad_x|_A.
\]

**Remark.** Let \( L \) be a finite dimensional Lie algebra and \( A \) an ideal of \( L \). Then the following holds:

1. If \( S_1 \subseteq S_2 \subseteq L \), then \( C_A(S_2) \subseteq C_A(S_1) \).
2. If $S = \text{span}\{x_1, x_2, \ldots, x_n\} \subseteq L$ for some $n \in \mathbb{Z}_{>0}$, then

$$C_A(S) = \bigcap_{x \in S} \ker ad_x|_A = \bigcap_{i=1}^{n} \ker ad_{x_i}|_A.$$ 

3. For any $x \in L$, $x \in C_L(A)$ if and only if $b_A(x) = 0$.

**Lemma 4.1.2.** Let $L$ be a finite dimensional Lie algebra and $A$ an ideal of $L$. Let $x, y \in L$ be such that $\text{im} \ ad_x|_A \cap \text{im} \ ad_y|_A = \{0\}$. Then $\ker ad_x|_A \subseteq \ker ad_y|_A$ or $b_A(x) < b_A(x + y)$.

**Proof.** Let $L$ be a finite dimensional Lie algebra and $A$ an ideal of $L$. Let $x, y \in L$ be such that $\text{im} \ ad_x|_A \cap \text{im} \ ad_y|_A = \{0\}$. Suppose that there exists $\alpha \in A$ such that $\alpha \in \ker ad_x|_A - \ker ad_y|_A$. Then we have $[x, \alpha] = 0$ but $[y, \alpha] \neq 0$. We observe that

$$[x + y, \alpha] = [x, \alpha] + [y, \alpha] = 0 + [y, \alpha] = [y, \alpha] \neq 0,$$

so $\text{im} \ ad_{x+y}|_A \cap \text{im} \ ad_y|_A$ is not trivial. Note that we have

$$\dim(\text{im} \ ad_{x+y}|_A + \text{im} \ ad_y|_A) = \dim \text{im} \ ad_{x+y}|_A + \dim \text{im} \ ad_y|_A - \dim(\text{im} \ ad_{x+y}|_A \cap \text{im} \ ad_y|_A)$$

As a result,

$$\dim(\text{im} \ ad_{x+y}|_A + \text{im} \ ad_y|_A) < \dim \text{im} \ ad_{x+y}|_A + \dim \text{im} \ ad_y|_A.$$

We define a map $\varphi : \text{im} \ ad_x|_A \times \text{im} \ ad_y|_A \to \text{im} \ ad_{x+y}|_A + \text{im} \ ad_y|_A$ by $\varphi(x_1, y_1) = x_1 + y_1$ where $x_1 \in \text{im} \ ad_x|_A$ and $y_1 \in \text{im} \ ad_y|_A$. We will show that $\varphi$ is an isomorphism. Let $(x_1, y_1), (x_2, y_2) \in \text{im} \ ad_x|_A \times \text{im} \ ad_y|_A$. Then we have $x_1, x_2 \in \text{im} \ ad_x|_A$ and $y_1, y_2 \in \text{im} \ ad_y|_A$, so there exist $a_1, a_2, b_1, b_2 \in A$ such that $[x, a_i] = x_i$ and $[y, b_i] = y_i$ for all $i = 1, 2$. We will verify that $\text{im} \ \varphi \subseteq \text{im} \ ad_{x+y}|_A + \text{im} \ ad_y|_A$. Observe that

$$\varphi(x_1, y_1) = x_1 + y_1 = [x, a_1] + [y, b_1] = [x + y, a_1] + [y, b_1 - a_1] \in \text{im} \ ad_{x+y}|_A + \text{im} \ ad_y|_A.$$ 

Thus $\text{im} \ \varphi \subseteq \text{im} \ ad_{x+y}|_A + \text{im} \ ad_y|_A$. Moreover, it is easy to see that $\varphi$ is linear. In order to show that $\varphi$ is injective, we suppose that $\varphi(x_1, y_1) = 0$. Then we get $[x, a_1] + [y, b_1] = x_1 + y_1 = \varphi(x_1, y_1) = 0$, so $[x, a_1] = [y, -b_1] \in \text{im} \ ad_x|_A \cap \text{im} \ ad_y|_A = \{0\}$. Therefore $x_1 = [x, a_1] = 0$ and $y_1 = [y, b_1] = 0$ which implies $(x_1, y_1) = 0$. Thus $\varphi$ is injective. To claim that $\varphi$ is surjective, let $z \in \text{im} \ ad_{x+y}|_A + \text{im} \ ad_y|_A$. Then $z = z_1 + z_2$ where $z_1 \in \text{im} \ ad_{x+y}|_A$ and $z_2 \in \text{im} \ ad_y|_A$. There exists $c_1, c_2 \in A$ such that $[x + y, c_1] = z_1$ and $[y, c_2] = z_2$. Note that we have

$$\varphi([x, c_1], [y, c_1 + c_2]) = [x, c_1] + [y, c_1 + c_2] = [x + y, c_1] + [y, c_2] = z_1 + z_2 = z.$$ 

Therefore $\varphi$ is surjective. Hence $\varphi : \text{im} \ ad_x|_A \times \text{im} \ ad_y|_A \to \text{im} \ ad_{x+y}|_A + \text{im} \ ad_y|_A$ is an isomor-
Therefore, \(\dim(\im\ad_x|_A \times \im\ad_y|_A) = \dim(\im\ad_{x+y}|_A + \im\ad_y|_A)\).

As a consequence, we have

\[
b_A(x) + b_A(y) = \text{rank } \ad_x|_A + \text{rank } \ad_y|_A \\
= \dim \im\ad_x|_A + \dim \im\ad_y|_A \\
= \dim(\im\ad_x|_A \times \im\ad_y|_A) \\
= \dim(\im\ad_{x+y}|_A + \im\ad_y|_A) \\
< \dim \im\ad_{x+y}|_A + \dim \im\ad_y|_A \\
= \text{rank } \ad_{x+y}|_A + \text{rank } \ad_y|_A \\
= b_A(x + y) + b_A(y).
\]

Hence \(b_A(x) < b_A(x + y)\).

\[\square\]

**Theorem 4.1.3.** Let \(L\) be a finite dimensional Lie algebra and \(A\) an ideal of \(L\). Let \(x, y \in T_A\), \(T = \text{span}\{x, y\}\) and

\[
\bar{T} = (T + C_L(A))/C_L(A) \cong T/(T \cap C_L(A)).
\]

Then we have the following two cases:

1. If \(\dim \bar{T} = 1\), then \(\im\ad_x|_A = \im\ad_y|_A\) and \(\ker\ad_x|_A = \ker\ad_y|_A\).
2. If \(\dim \bar{T} = 2\), then \(\im\ad_x|_A \neq \im\ad_y|_A\) or \(\ker\ad_x|_A \neq \ker\ad_y|_A\).

However, if \(x + y \in T_A\), then the consequence of the second case is “either or”.

**Proof.** Note that \(\bar{T} = (T + C_L(A))/C_L(A) \cong T/(T \cap C_L(A))\) comes from second isomorphism theorem ((\(S + I)/I \cong S/(S \cap I)\)). For the first case, we assume that \(\dim \bar{T} = 1\). Then we have \(\dim(T \cap C_L(A)) = \dim T - \dim\bar{T} = 2 - 1 = 1\). Since \(x, y \in T_A\), \(b_A(x) = b_A(y) = 1\). Therefore \(x, y \notin C_L(A)\), so \(x, y \notin T \cap C_L(A)\). By considering \(T/(T \cap C_L(A))\) as a 1-dimensional quotient space, there exists \(\alpha \in \mathbb{F} - \{0\}\) such that \(x + (T \cap C_L(A)) = -\alpha y + (T \cap C_L(A))\), so \(x + \alpha y \in (T \cap C_L(A))\). Since \(\dim(T \cap C_L(A)) = 1\) and \(x + \alpha y \neq 0\), we have \(T \cap C_L(A) = \text{span}\{x + \alpha y\}\). First, we will claim that \(\im\ad_x|_A = \im\ad_y|_A\). To show that \(\im\ad_x|_A \subseteq \im\ad_y|_A\), let \(z \in \im\ad_x|_A\). Then there exists \(a \in A\) such that \([x, a] = z\). Because \(x + \alpha y \in T \cap C_L(A)\), \([x + \alpha y, a] = 0\). Therefore we get

\[
0 = [x + \alpha y, a] = [x, a] + \alpha[y, a] = z - [y, -\alpha a],
\]
so \( z = [y, -\alpha a] \), that is \( z \in \text{im} \, ad_y|_A \). Conversely, let \( z \in \text{im} \, ad_y|_A \). Then there exists \( a \in A \) such that \( [y, a] = z \). Again, we get

\[
0 = [x + \alpha y, a] = [x, a] + \alpha [y, a] = [x, a] + \alpha z,
\]

so \( z = \frac{1}{\alpha} [x, a] = \left[ x, \frac{-a}{\alpha} \right] \). Thus \( z \in \text{im} \, ad_x|_A \). Hence \( \text{im} \, ad_x|_A = \text{im} \, ad_y|_A \). Next, we will show that \( \ker \, ad_x|_A = \ker \, ad_y|_A \). Suppose \( \ker \, ad_x|_A \neq \ker \, ad_y|_A \). Without loss of generality, there exists \( a \in A \) such that \( [y, a] = 0 \) but \( [x, a] \neq 0 \). Then we have

\[
[x + \alpha y, a] = [x, a] + \alpha [y, a] = [x, a] + 0 = [x, a] \neq 0
\]

which contradicts \( x + \alpha y \in T \cap C_L(A) \). Hence \( \ker \, ad_x|_A = \ker \, ad_y|_A \).

For the second case, suppose that \( \dim T = 2 \). Then we have \( \dim(T \cap C_L(A)) = \dim T - \dim \bar{T} = 2 - 2 = 0 \), so \( T \cap C_L(A) = \{0\} \). Suppose that \( \ker \, ad_x|_A = \ker \, ad_y|_A \). Then we have to show that \( \text{im} \, ad_x|_A \neq \text{im} \, ad_y|_A \). Let \( \alpha \in \mathbb{F} \). Then \( x + \alpha y \in T - \{0\} \), so \( x + \alpha y \notin C_L(A) \). Thus there exists \( a \in A \) such that

\[
0 \neq [x + \alpha y, a] = [x, a] + \alpha [y, a].
\]

Note that \( a \notin \ker \, ad_x|_A = \ker \, ad_y|_A \) since \( [x, a] + \alpha [y, a] \neq 0 \). Therefore \( [x, a] \neq 0 \) and \( [y, a] \neq 0 \), but we have

\[
[x, a] \neq -\alpha [y, a] \quad \text{for all} \quad \alpha \in \mathbb{F}.
\]

Since \( \dim \, ad_x|_A = \dim \, ad_y|_A = 1 \), \( \text{im} \, ad_x|_A \neq \text{im} \, ad_y|_A \). Hence we proved the second case.

Finally, we will show that the consequence of the second case is “either or” if \( x + y \in T_A \). Suppose additionally to the second case that \( x + y \in T_A \). Then we have \( b_A(x + y) = 1 \). Assume that \( \text{im} \, ad_x|_A \neq \text{im} \, ad_y|_A \). Since \( x, y \in T_A \), \( b_A(x) = b_A(y) = 1 \), so \( \text{im} \, ad_x|_A \cap \text{im} \, ad_y|_A = \{0\} \). By Lemma 4.1.2, we get

\[
\ker \, ad_x|_A \subseteq \ker \, ad_y|_A \quad \text{or} \quad b_A(x) < b_A(x + y).
\]

Because \( b_A(x) = 1 = b_A(x + y) \), we have \( \ker \, ad_x|_A \subseteq \ker \, ad_y|_A \). Similarly, we also have \( \ker \, ad_y|_A \subseteq \ker \, ad_x|_A \) since \( b_A(y) = 1 = b_A(x + y) \). Therefore \( \ker \, ad_x|_A = \ker \, ad_y|_A \). On the other hand, we suppose that \( \ker \, ad_x|_A \neq \ker \, ad_y|_A \). Without loss of generality, assume that \( \ker \, ad_x|_A \nsubseteq \ker \, ad_y|_A \). Then \( \text{im} \, ad_x|_A \cap \text{im} \, ad_y|_A \neq \{0\} \) by contrapositive of Lemma 4.1.2. Since \( b_A(x) = b_A(y) = 1 \), we get \( \text{im} \, ad_x|_A = \text{im} \, ad_y|_A \) as desired. \( \square \)
Proposition 4.1.4. Let \( L \) be a finite dimensional Lie algebra and \( A \) an ideal of \( L \). Let \( x, y, z \in L \) be such that \( y - z \notin C_L(A) \). Suppose that \( b_A(x) > 1 \). Then at least one of the elements

\[
y, z, y + z, x + y, x + z, x + y + z
\]

is not in \( T_A \).

Proof. Let \( L \) be a finite dimensional Lie algebra and \( A \) an ideal of \( L \). Let \( x, y, z \in L \) be such that \( y - z \notin C_L(A) \). Suppose that \( y, z, y + z, x + y, x + z, x + y + z \in T_A \). Additionally, we let

\[
T_1 = \text{span}\{y, y - z\} = \text{span}\{y, z - y\} = \text{span}\{y, z\},
T_2 = \text{span}\{y - z, x + z\} = \text{span}\{x + y, x + z\},
T_3 = \text{span}\{y, x + z\}.
\]

For \( i = 1, 2, 3 \), \( T_i \) is not contained in \( C_L(A) \), so \( T_i \cap C_L(A) \) is zero or 1-dimensional which implies \( T_i = (T_i + C_L(A))/C_L(A) \cong T_i/(T_i \cap C_L(A)) \) is 1 or 2-dimensional. Next, we consider

\[
\begin{align*}
im ad_{T_1}|_A &:= \text{span}\{im ad_y|_A, im ad_z|_A\}, \\
im ad_{T_2}|_A &:= \text{span}\{im ad_{x+y}|_A, im ad_{x+z}|_A\}, \\
im ad_{T_3}|_A &:= \text{span}\{im ad_y|_A, im ad_{x+z}|_A\}.
\end{align*}
\]

By Theorem 4.1.3, we know that for each \( i = 1, 2, 3 \),

1. If \( \dim T_i = 1 \), then we have \( \text{im } ad_{T_i}|_A \) is 1-dimensional.
2. If \( \dim T_i = 2 \), then we have \( \text{im } ad_{T_i}|_A \) is 2-dimensional.

Hence \( \text{im } ad_{T_1}|_A, \text{im } ad_{T_2}|_A \) and \( \text{im } ad_{T_3}|_A \) are 1 or 2-dimensional. By pigeonhole principle, there exist \( \alpha, \beta \in \{1, 2, 3\} \) such that \( \alpha \neq \beta \) and \( \dim(\text{im } ad_{T_\alpha}|_A) = \dim(\text{im } ad_{T_\beta}|_A) \). Let \( n := \dim(\text{im } ad_{T_\alpha}|_A) = \dim(\text{im } ad_{T_\beta}|_A) \). Then we consider the following two cases:

For \( n = 1 \), we consider

\[
\begin{align*}
T_1 &= \text{span}\{y, y - z\}, \\
T_2 &= \text{span}\{y - z, x + z\}, \\
T_3 &= \text{span}\{x + z, y\}.
\end{align*}
\]

Then we have \( N := \text{im } ad_y|_A = \text{im } ad_{y-z}|_A = \text{im } ad_{x+z}|_A \) is 1-dimensional. Next, we will claim
that im $ad_x|A \subseteq N$. Let $a \in A$. Since $x = (x + z) + (y - z) - y$, we get

$$[x, a] = [x + z, a] + [y - z, a] - [y, a] \in N.$$  

Thus im $ad_x|A \subseteq N$, so $b_A(x) \leq 1$ which contradicts the assumption $b_A(x) > 1$.

For $n = 2$, we consider

$$T_1 = \text{span}\{ y, z - y \},$$
$$T_2 = \text{span}\{ x + z, y - z \},$$
$$T_3 = \text{span}\{ y, x + z \},$$
$$T_\alpha = \text{span}\{ \alpha_1, \alpha_2 \},$$
$$T_\beta = \text{span}\{ \beta_1, \beta_2 \},$$

so that

$$\text{im } ad_{T_\alpha}|A := \text{span}\{ \text{im } ad_{\alpha_1}|A, \text{im } ad_{\alpha_2}|A \},$$
$$\text{im } ad_{T_\beta}|A := \text{span}\{ \text{im } ad_{\beta_1}|A, \text{im } ad_{\beta_2}|A \}.$$  

Since $n = 2$, we obtain $\text{im } ad_{\alpha_1}|A \neq \text{im } ad_{\alpha_2}|A$ and $\text{im } ad_{\beta_1}|A \neq \text{im } ad_{\beta_2}|A$. Note that dim $\bar{T}_\alpha = \dim \bar{T}_\beta = 2$ and for $T_1, T_2, T_3$, we have

$$b_A(y) = 1 = b_A(z) = b_A(y + (z - y)),$$
$$b_A(x + z) = 1 = b_A(x + y) = b_A((x + z) + (y - z)),$$
$$b_A(y) = 1 = b_A(x + y + z) = b_A(y + (x + z)).$$  

Thus $b_A(\alpha_1 + \alpha_2) = b_A(\alpha_1) = 1$ and $b_A(\beta_1 + \beta_2) = b_A(\beta_1) = 1$, so $\alpha_1 + \alpha_2, \beta_1 + \beta_2 \in T_A$. By Theorem 4.1.3, (2) with “either or”, we have

$$\ker ad_{\alpha_1}|A = \ker ad_{\alpha_2}|A \quad \text{and} \quad \ker ad_{\beta_1}|A = \ker ad_{\beta_2}|A.$$  

It is easy to see that $\ker ad_{y - z}|A = \ker ad_{z - y}|A$, so we have $M := \ker ad_y|A = \ker ad_{y - z}|A = \ker ad_{x + z}|A$. Next we will prove that $M \subseteq \ker ad_x|A$. Let $m \in M$. Since $x = (x + z) + (y - z) - y$, we get

$$[x, m] = [x + z, m] + [y - z, m] - [y, m] = 0.$$  

Thus $M \subseteq \ker ad_x|A$, that means nullity $ad_y|A \leq \text{nullity } ad_x|A$. Hence we have

$$\text{rank } ad_x|A = \dim A - \text{nullity } ad_x|A$$

20
\[
= (\text{nullity } \text{ad}_y|_A + \text{rank } \text{ad}_y|_A) - \text{nullity } \text{ad}_x|_A \\
= (\text{nullity } \text{ad}_y|_A - \text{nullity } \text{ad}_x|_A) + \text{rank } \text{ad}_y|_A \\
\leq \text{rank } \text{ad}_y|_A \\
\leq 1,
\]
so \(b_A(x) \leq 1\) which is a contradiction. In consequence, from the two cases above, at least one of the elements \(y, z, y + z, x + y, x + z, x + y + z\) is not in \(T_A\).

From now on, we begin to consider finite dimensional nilpotent Lie algebra in order to guarantee that it has an abelian ideal.

**Definition 4.1.5.** Let \(L\) be a finite dimensional nilpotent Lie algebra and \(A\) an abelian ideal of \(L\). For any \(x \in L\), we define

\[
M_x = A + \ker \text{ad}_x, \\
L_x = \text{span}\{M_{a+x} \mid a \in A\}, \\
D_x = \bigcap_{a \in A} M_a + x.
\]

**Proposition 4.1.6.** Let \(L\) be a finite dimensional nilpotent Lie algebra and \(A\) an abelian ideal of \(L\). Let \(x \in L\) and \(a \in A\). Then the following hold:

1. \(A \subseteq D_x \subseteq M_{a+x}\).
2. \([A, x] = [A, a + x] = [M_{a+x}, a + x] = [D_x, a + x]\).
3. \([L_x, x] \subseteq [A, L]\).
4. \([a, M_{a+x} \cap M_x] \subseteq [A, x]\).
5. If \(L = M_x + U = M_{a+x} + U\) for some \(a \in A\) and subspace \(U\) of \(L\), then
   \([a, L] \subseteq [U, x] + [A, x] + [A, U]\).
6. \([A, D_x] \subseteq [A, x]\).
7. If \(b_A(L) = b(L)\), then \(\dim [A, L] = b_A(L)\) and \(L = D_z\) for all \(z \in B_A\).

**Proof.** Let \(L\) be a finite dimensional nilpotent Lie algebra and \(A\) an abelian ideal of \(L\). Let \(x \in L\) and \(a \in A\).
1. Since $A \subseteq A + \ker ad_{a'+x} = M_{a'+x}$ for all $a' \in A$, $A \subseteq D_x$. It is clear that $D_x \subseteq M_{a+x}$ because $0 \in A$. Hence $A \subseteq D_x \subseteq M_{a+x}$.

2. First, it is obvious that $[A, x] = [A, a + x]$ since $A$ is abelian. Because $M_{a+x} = A + \ker ad_{a+x}$, we have

$$[M_{a+x}, a + x] = [A + \ker ad_{a+x}, a + x] = [A, a] + [A, x] + [\ker ad_{a+x}, a + x] = 0 + [A, x] + 0 = [A, x].$$

Thus $[A, x] = [M_{a+x}, a+x]$. Next, we will show that $[A, x] = [D_x, a+x]$. Since $D_x \subseteq M_{a+x}$, we get $[D_x, a + x] \subseteq [M_{a+x}, a + x] = [A, x]$. Conversely, we have $[A, x] = [A, a + x] \subseteq [D_x, a + x]$ since $A \subseteq D_x$. Therefore $[A, x] = [D_x, a + x]$. Hence $[A, x] = [A, a + x] = [M_{a+x}, a + x] = [D_x, a + x]$. 

3. Let $a \in A$ be arbitrary. Let $y \in M_{a+x}$. Then $y = a_y + c_y$ where $a_y \in A$ and $c_y \in \ker ad_{a+x}$. Thus $[c_y, a + x] = 0$, so $[c_y, x] = [a, c_y]$. Therefore

$$[y, x] = [a_y + c_y, x] = [a_y, x] + [c_y, x] = [a_y, x] + [a, c_y] \in [A, L].$$

Since $y \in M_{a+x}$ and $a \in A$ is arbitrary, $[L_x, x] \subseteq [A, L]$.

4. Let $y \in M_{a+x} \cap M_x$. Since $y \in M_{a+x}$, $y = a_y + c_y$ where $a_y \in A$ and $c_y \in \ker ad_{a+x}$. Thus $[c_y, a + x] = 0$, so $[c_y, x] = [a, c_y]$. On the other hand, $y = a_y' + c_y'$ where $a_y' \in A$ and $c_y' \in \ker ad_x$ because $y \in M_x$. Therefore $[c_y', x] = 0$, so we have

$$[y, x] = [a_y' + c_y', x] = [a_y', x] + [c_y', x] = [a_y', x].$$

Consequently,

$$[a, y] = [a, a_y + c_y] = [a, a_y] + [a, c_y] = 0 + [a, c_y] = [c_y, x] = [y - a_y, x] = [y, x] - [a_y, x] = [a_y', x] - [a_y, x]$$
\[ \in [A, x]. \]

Hence \([a, M_{a+x} \cap M_x] \subseteq [A, x]\).

5. Suppose that \(L = M_x + U = M_{a+x} + U\) for some \(a \in A\) and subspace \(U\) of \(L\). Since \(L = M_{a+x} + U\), we have

\[
[a, L] = [a, M_{a+x} + U] \\
= [a, A + \ker \text{ad}_{a+x} + U] \\
= [a, A] + [a, \ker \text{ad}_{a+x}] + [a, U] \\
= 0 + [a, \ker \text{ad}_{a+x}] + [a, U] \\
= [a, \ker \text{ad}_{a+x}] + [a, U].
\]

On the other hand, since \(L = M_x + U\) and \([a + x, \ker \text{ad}_{a+x}] = 0\), we obtain

\[
[a, \ker \text{ad}_{a+x}] = [\ker \text{ad}_{a+x}, x] \\
\subseteq [L, x] \\
= [M_x + U, x] \\
= [A + \ker \text{ad}_x + U, x] \\
= [A, x] + [\ker \text{ad}_x, x] + [U, x] \\
= [A, x] + 0 + [U, x] \\
= [A, x] + [U, x],
\]

so \([a, \ker \text{ad}_{a+x}] \subseteq [A, x] + [U, x]\). As a result, we get

\[
[a, L] = [a, \ker \text{ad}_{a+x}] + [a, U] \\
\subseteq [A, x] + [U, x] + [a, U] \\
\subseteq [A, x] + [U, x] + [A, U].
\]

Hence \([a, L] \subseteq [U, x] + [A, x] + [A, U]\).

6. Let \(a' \in A\). By using part (4) of this proposition, we have

\[
[a, D_x] = [a, \bigcap_{a \in A} M_{a+x}] \subseteq [a, M_{a+x} \cap M_x] \subseteq [A, x].
\]

Hence \([a, D_x] \subseteq [A, x]\).
Lemma 4.1.7. Let \( L \) be a finite dimensional nilpotent Lie algebra, \( A \) an abelian ideal of \( L \) and \( x \in B \cap B_A \). Suppose that \( b(L) = b_A(L) + 1 \) and \( L \neq L_x \). Then \( \dim L/M_x = 1 \) and \( M_x = M_{x+1} \) for every \( a \in A \).

Proof. Let \( L \) be a finite dimensional nilpotent Lie algebra, \( A \) an abelian ideal of \( L \) and \( x \in B \cap B_A \). Then we have \( b(x) = b(L) \) and \( b_A(x) = b_A(L) \). Suppose that \( b(L) = b_A(L) + 1 \) and \( L \neq L_x \). Thus \( b(x) = b_A(x) + 1 \), so \( \text{rank } ad_x = \text{rank } ad_x|_A + 1 \). Note that for any \( x \in L \), we know that \( \ker ad_x|_A = A \cap \ker ad_x \subset \ker ad_x \). Define

\[
D := \operatorname{span}\{\alpha \mid \alpha \in \ker ad_x - \ker ad_x|_A \}.
\]

Next, we will show that \( A \cap \ker ad_x = \ker ad_x|_A = A \cap \ker ad_{a+1} \). It is clear that \( A \cap \ker ad_x = \ker ad_x|_A \). To show that \( \ker ad_x|_A = A \cap \ker ad_{a+1} \), let \( a \in A \). Let \( y \in \ker ad_x|_A \). Then \([x,y] = 0\) and \( y \in A \), so \([a,y] = 0\). Thus \([a,x+y] = [a,y] + [x,y] = 0\). Therefore \( y \in A \cap \ker ad_{a+1} \), so \( \ker ad_x|_A \subset A \cap \ker ad_{a+1} \). Conversely, let \( y \in A \cap \ker ad_{a+1} \). Then \([a+x,y] = 0\) and \( y \in A \), so \([a,y] = 0\). Thus \([x,y] = [a,y] + [x,y] = [a+x,y] = 0\). Hence \( y \in \ker ad_x|_A \), so \( \ker ad_x|_A \supset A \cap \ker ad_{a+1} \). Consequently, we have \( A \cap \ker ad_x = \ker ad_x|_A = A \cap \ker ad_{a+1} \).
Moreover, since \( \ker \text{ad}_x|A \subseteq \ker \text{ad}_x \), we get
\[
dim D = \text{nullity} \text{ad}_x - \text{nullity} \text{ad}_x|A.
\]
On the other hand, because \( A \cap \ker \text{ad}_x = \ker \text{ad}_x|A \), we have \( M_x = A + \ker \text{ad}_x = A \oplus D \). Therefore \( \dim M_x = \dim A + \dim D \). Additionally, we know that
\[
\dim L = \text{nullity} \text{ad}_x + \text{rank} \text{ad}_x,
\]
\[
\dim A = \text{nullity} \text{ad}_x|A + \text{rank} \text{ad}_x|A.
\]
Therefore we have
\[
\dim L - \dim A = (\text{nullity} \text{ad}_x - \text{nullity} \text{ad}_x|A) + (\text{rank} \text{ad}_x - \text{rank} \text{ad}_x|A) = \dim D + 1,
\]
so we get
\[
\dim L/M_x = \dim L - \dim M_x = \dim L - (\dim A + \dim D) = 1.
\]
In order to prove that \( M_x = M_{a+x} \) for every \( a \in A \), assume that there exists \( a \in A \) such that \( M_{a+x} \) is not contained in \( M_x \). Since \( \dim L = \dim M_x + 1 \), we have \( L_x = \text{span}\{M_{a+x} \mid a \in A\} = L \), which contradicts the assumption \( L \neq L_x \). Hence \( M_{a+x} \subseteq M_x \) for every \( a \in A \). Conversely, suppose that there exists \( a \in A \) such that \( M_{a+x} \nsubseteq M_x \). Then \( A + \ker \text{ad}_{a+x} \nsubseteq A + \ker \text{ad}_x \).

Let \( V_{a+x} \) and \( V_x \) be complementary subspaces of \( A \) in \( M_{a+x} \) and \( M_x \), respectively. Then \( M_{a+x} = A \oplus V_{a+x} \) and \( M_x = A \oplus V_x \). Because \( M_{a+x} \nsubseteq M_x \), we know that \( V_{a+x} \nsubseteq V_x \). Since \( A \cap \ker \text{ad}_x = A \cap \ker \text{ad}_{a+x} \), we have
\[
\ker \text{ad}_{a+x} = (A \cap \ker \text{ad}_{a+x}) \oplus V_{a+x} = (A \cap \ker \text{ad}_x) \oplus V_{a+x} \nsubseteq (A \cap \ker \text{ad}_x) \oplus V_x = \ker \text{ad}_x.
\]
Therefore nullity \( \text{ad}_{a+x} < \text{nullity} \text{ad}_x \), so we obtain
\[
b(x) = \text{rank} \text{ad}_x = \dim L - \text{nullity} \text{ad}_x < \dim L - \text{nullity} \text{ad}_{a+x} = \text{rank} \text{ad}_{a+x} = b(a + x).
\]
Thus \( b(x) < b(a + x) \). Since \( x \in B \), \( b(x) = b(L) \), so we have \( b(a + x) > b(L) \), which is a contradiction. Consequently, \( M_x = M_{a+x} \) for every \( a \in A \). \(\square\)

**Theorem 4.1.8.** Let \( L \) be a finite dimensional nilpotent Lie algebra and \( A \) an abelian ideal of \( L \). Suppose that \( b(L) \leq b_A(L) + 1 \). Then \( \text{im} \text{ad}_x|A \) is an ideal of \( L \) for every \( x \in B_A \).

**Proof.** Let \( L \) be a finite dimensional nilpotent Lie algebra and \( A \) an abelian ideal of \( L \). Suppose
that \(b(L) \leq b_A(L) + 1\). Let \(x \in B_A\). Since \(b_A(L) \leq b(L)\), we have \(b_A(L) \leq b(L) \leq b_A(L) + 1\), so
\[
b(L) = b_A(L) \quad \text{or} \quad b(L) = b_A(L) + 1.
\]

First, we will claim that \(\text{im} \ ad_{a+x}|_A\) is an ideal of \(M_{a+x}\) for every \(a \in A\). Let \(a \in A\). Then it is obvious that \(\text{im} \ ad_{a+x}|_A \subseteq A \subseteq A + \ker ad_{a+x} = M_{a+x}\). Let \(a' \in A\) and \(y \in M_{a+x}\). Then \([a + x, a'] \in \text{im} \ ad_{a+x}|_A\) and \(y = a_y + c_y\) where \(a_y \in A\) and \(c_y \in \ker ad_{a+x}\). Thus \([c_y, a + x] = 0\). Note that \([a_y, a + x], a'] = 0\) because \(A\) is an abelian ideal. Therefore we get
\[
[y, [a + x, a']] = [[y, a + x], a'] + [a + x, [y, a']] \\
= [[a_y + c_y, a + x], a'] + [a + x, [y, a']] \\
= [[[a_y, a + x], a'], [c_y, a + x], a'] + [a + x, [y, a']] \\
= 0 + [0, a'] + [a + x, [y, a']] \\
\in [a + x, A] \\
= \text{im} \ ad_{a+x}|_A.
\]

Hence \(\text{im} \ ad_{a+x}|_A\) is an ideal of \(M_{a+x}\) for every \(a \in A\).

If \(L = L_x = \text{span}\{M_{a+x}|_A | a \in A\}\), then \(\text{im} \ ad_x|_A = \text{im} \ ad_{a+x}|_A\) is an ideal of \(L\) by previous claim. Suppose that \(L \neq L_x\). If \(b(L) = b_A(L)\), then by Proposition 4.1.6 (7), we have \(L = D_x\). Thus \(L = D_x \subseteq M_x \subseteq L_x\). It is clear that \(L_x \subseteq L\), so \(L = L_x\), which contradicts \(L \neq L_x\). Therefore \(b(L) = b_A(L) + 1\). Next, we need to show that \(x \in B\). Assume \(x \notin B\). Then \(b(x) \neq b(L)\), so we have
\[
b_A(x) \leq b(x) \leq b(L) - 1 = b_A(L) = b_A(x),
\]
which implies \(b(x) = b_A(x)\), so \(\text{rank} \ ad_x = \text{rank} \ ad_{a+x}|_A\). Thus \(\text{im} \ ad_x = \text{im} \ ad_{a+x}|_A\), which means, for any \(\alpha \in L - A\) there exists \(\beta \in A\) such that \(ad_x(\alpha) = ad_x(\beta)\). Therefore \(ad_x(\alpha - \beta) = 0\), so \(\alpha - \beta \in \ker ad_x\). As a result,
\[
\alpha = \beta + (\alpha - \beta) \in A + \ker ad_x = M_x.
\]

Since \(\alpha \in L - A\) is arbitrary, \(L - A \subseteq M_x\). Moreover, it is easy to see that \(A \subseteq A + \ker ad_x = M_x\). Consequently, \(L = (L - A) + A \subseteq M_x\). Because \(M_x \subseteq L_x, L \subseteq L_x\). Therefore \(L = L_x\), which again contradicts \(L \neq L_x\). Hence \(x \in B\). In conclusion, we know that \(x \in B \cap B_A\). By Lemma 4.1.7, \(\dim L/M_x = 1\) and \(M_x = M_{a+x}\) for every \(a \in A\), so \(D_x = \bigcap_{a \in A} M_{a+x} = M_x\). Next, we will show that \([A, x] = [A, M_x]\). By Proposition 4.1.6 (6), \([A, D_x] \subseteq [A, x]\). Since \(D_x = M_x\), we get \([A, M_x] = [A, D_x] \subseteq [A, x]\). On the other hand, we let \(a \in A\). Then \([a, x] \in [A, x]\). Because
$x \in \ker \text{ad}_x$, we obtain

$$[a, x] \in [A, \ker \text{ad}_x] \subseteq [A, A + \ker \text{ad}_x] = [A, M_x].$$

Thus $[A, x] \subseteq [A, M_x]$. Hence $[A, x] = [A, M_x]$. Next, we will claim that $M_x = A + \ker \text{ad}_x$ is a Lie subalgebra of $L$. It is clear that $M_x$ is a subspace of $L$. Let $m_1, m_2 \in M_x$. Then there exist $a_1, a_2 \in A$ and $c_1, c_2 \in \ker \text{ad}_x$ such that $m_i = a_i + c_i$ for $i = 1, 2$. We observe that $\text{ad}_x([c_1, c_2]) = [x, [c_1, c_2]] = [[x, c_1], c_2] + [c_1, [x, c_2]] = 0$, so $[c_1, c_2] \in \ker \text{ad}_x$. Therefore we have

$$[m_1, m_2] = [a_1 + c_1, a_2 + c_2] = [a_1, a_2] + [a_1, c_2] + [a_1, c_2] + [c_1, c_2] \in A + \ker \text{ad}_x = M_x.$$

Since $m_1, m_2 \in M_x$ are arbitrary, we get $[M_x, M_x] \subseteq M_x$, which means $M_x$ is closed under bracket. That $M_x$ is a Lie subalgebra of $L$. Since $\dim L/M_x = 1$, there exists $y \in L - M_x$ such that $L = M_x \oplus \text{span}\{y\}$, which means $\text{span}\{y\}$ is the complementary subspace of $M_x$ in $L$. To show that $M_x$ is an ideal of $L$, we suppose that $M_x$ is not an ideal of $L$. Then there exist $m \in M_x$ and $z \in L$ such that $[m, z] \notin M_x$. Since $L = M_x \oplus \text{span}\{y\}$, there exist $m' \in M_x$ and $\alpha \in \mathbb{F}$ such that $z = m' + \alpha y$. We observe that

$$[m, m'] + \alpha[m, y] = [m, m' + \alpha y] = [m, z] \notin M_x.$$

Since $[m, m'] \in M_x$, we obtain $\alpha \neq 0$ and $[m, y] \notin M_x$, so $[m, y] = m'' + \beta y$ where $m'' \in M_x$ and $\beta \neq 0$. Consequently, we get $\text{ad}_m^N(y) \neq 0$ for any $N \in \mathbb{Z}_{>0}$ which contradicts nilpotency of $L$. Hence $M_x$ is an ideal of $L$. Because we know that $\text{im} \text{ad}_x|_A = [A, x] = [A, M_x]$ and $A, M_x$ are ideals of $L$, $\text{im} \text{ad}_x|_A$ is also an ideal of $L$. Since $x \in B_A$ is arbitrary, $\text{im} \text{ad}_x|_A$ is an ideal of $L$ for every $x \in B_A$. \hfill \Box

**Lemma 4.1.9.** Let $L$ be a finite dimensional Lie algebra and $A$ an ideal of $L$ such that $b_A(L) > 1$. Let $x \in B_A$ be given. Suppose that there exists $y \in L$ such that $y$ and $x + y$ do not satisfy

$$b_A(z) > 1 \quad \text{and} \quad 2b_A(z) \geq b_A(L). \quad (4.1)$$

Then $b_A(y) = b_A(x + y) = 1$ and $b_A(L) = 2$.

**Proof.** Let $L$ be a finite dimensional Lie algebra and $A$ an ideal of $L$ such that $b_A(L) > 1$. Let $x \in B_A$ be given. Then $b_A(x) = b_A(L)$. Suppose that there exists $y \in L$ such that $y$ and $x + y$ do not satisfy

$$b_A(z) > 1 \quad \text{and} \quad 2b_A(z) \geq b_A(L). \quad (4.1)$$

Then $b_A(y) = b_A(x + y) = 1$ and $b_A(L) = 2$. 

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First, we claim that \( b_A(L) = 2 \) by using contrapositive. Assume that \( b_A(L) \neq 2 \). Then we have \( b_A(L) \geq 3 \) since \( b_A(L) > 1 \). In order to show that for all \( y \in L \), \( y \) or \( x + y \) satisfies (4.1), we let \( y \in L \) be such that \( y \) does not satisfy (4.1). Then we get \( b_A(y) \leq 1 \) or \( 2b_A(y) < b_A(L) \).

Note that if \( b_A(y) \leq 1 \), then we also get \( 2b_A(y) \leq 2 < 3 \leq b_A(L) \). Thus we can assume that \( 2b_A(y) < b_A(L) \). Then we apply the fact that

\[
\text{rank}(A + B) \leq \text{rank} A + \text{rank} B
\]

where \( A \) and \( B \) are linear transformations, so we get

\[
\text{rank ad}_x|_A \leq \text{rank}(ad_x|_A + ad_y|_A) + \text{rank}(-ad_y|_A).
\]

Because \( ad_x|_A + ad_y|_A = ad_{x+y}|_A \) and \( \text{rank}(-ad_y|_A) = \text{rank} ad_y|_A \), we have

\[
\text{rank ad}_x|_A \leq \text{rank ad}_{x+y}|_A + \text{rank} ad_y|_A. \quad (4.2)
\]

Thus \( \text{rank ad}_{x+y}|_A \geq \text{rank ad}_x|_A - \text{rank ad}_y|_A \), which also means \( b_A(x + y) \geq b_A(x) - b_A(y) \).

Since \( 2b_A(y) < b_A(L) \) and \( b_A(x) = b_A(L) \), we obtain

\[
b_A(x + y) \geq b_A(x) - b_A(y) > b_A(L) - \frac{b_A(L)}{2} > \frac{b_A(L)}{2}.
\]

Therefore \( 2b_A(x + y) > b_A(L) \). We also get \( b_A(x + y) > \frac{b_A(L)}{2} \geq \frac{3}{2} > 1 \). Hence \( x + y \) satisfies (4.1). Consequently, \( b_A(L) = 2 \) as we claimed.

Next, we will prove that \( b_A(x) = b_A(x + y) = 1 \). Since \( y \) and \( x + y \) do not satisfy (4.1) and \( b_A(L) = 2 \), we have

\[
b_A(y) \leq 1 \quad \text{or} \quad 2b_A(y) < b_A(L) = 2
\]

and

\[
b_A(x + y) \leq 1 \quad \text{or} \quad 2b_A(x + y) < b_A(L) = 2,
\]

which can be reduced to \( b_A(y) \leq 1 \) and \( b_A(x + y) \leq 1 \). Suppose that \( b_A(y) = 0 \). This means \( \text{im ad}_y|_A = \{0\} \), so we have

\[
b_A(x + y) = \text{rank ad}_{x+y}|_A = \text{rank ad}_x|_A = b_A(x) = b_A(L) = 2
\]

which contradicts \( b_A(x + y) \leq 1 \). Hence \( b_A(y) = 1 \). Next, we assume that \( b_A(x + y) = 0 \). Then \( \text{rank ad}_{x+y}|_A = 0 \). By applying this to (4.2), we get \( \text{rank ad}_x|_A \leq \text{rank ad}_y|_A \). Therefore

\[
b_A(y) = \text{rank ad}_y|_A \geq \text{rank ad}_x|_A = b_A(x) = b_A(L) = 2,
\]
which contradicts $b_A(y) = 1$. Hence $b_A(x+y) = 1$. As a result, we obtain $b_A(y) = b_A(x+y) = 1$ and $b_A(L) = 2$ as desired. \hfill \Box

**Theorem 4.1.10.** Let $L$ be a finite dimensional nilpotent Lie algebra and $A$ an abelian ideal of $L$ such that $b_A(L) > 1$. Suppose that $[L, z] \subseteq [A, L]$ for all $z \in L$ satisfying

$$b_A(z) > 1 \quad \text{and} \quad 2b_A(z) \geq b_A(L).$$

Then $[L, L] = [C_L(A), L]$. In addition, if $A = C_L(A)$ and $b(L) = b_A(L)$, then $\dim[L, L] = b_A(L)$.

**Proof.** Let $L$ be a finite dimensional nilpotent Lie algebra and $A$ an abelian ideal of $L$. Let $b_A(L) > 1$ and suppose that $[L, z] \subseteq [A, L]$ for all $z \in L$ satisfying

$$b_A(z) > 1 \quad \text{and} \quad 2b_A(z) \geq b_A(L).$$

Fix an $x \in B_A$. Then we have the following two cases to consider.

The first case is $y$ or $x + y$ satisfy (4.1) for every $y \in L$. Then by assumption, $[L, y] \subseteq [A, L]$ or $[L, x + y] \subseteq [A, L]$. Next, we will show that $[L, x + y] \subseteq [A, L]$ also implies $[L, y] \subseteq [A, L]$. Suppose that $[L, x + y] \subseteq [A, L]$. Since $x \in B_A$, $b_A(x) = b_A(L) > 1$. It is clear that

$$2b_A(x) > b_A(L) = 2,$$  

so $x$ also satisfies (4.1). Thus $[L, x] \subseteq [A, L]$. To show $[L, y] \subseteq [A, L]$, we let $z \in L$. Then $[z, x + y] \subseteq [L, x + y] \subseteq [A, L]$ and $[z, x] \subseteq [L, x] \subseteq [A, L]$, so we get

$$[z, y] = [z, x + y] - [z, x] \in [A, L].$$

Since $z \in L$ is arbitrary, $[L, y] \subseteq [A, L]$. Hence we have $[L, y] \subseteq [A, L]$ for any $y \in L$. Because $A$ is abelian, $A \subseteq C_L(A)$, so $[A, L] \subseteq [C_L(A), L]$. Thus $[L, y] \subseteq [A, L] \subseteq [C_L(A), L]$ for every $y \in L$. Since $y \in L$ is arbitrary, $[L, L] \subseteq [C_L(A), L]$. Conversely, it is easy to see that $[C_L(A), L] \subseteq [L, L]$. Hence $[L, L] = [C_L(A), L]$.

For the second case, assume that there exists $y \in L$ such that $y$ and $x + y$ do not satisfy (4.1). By Lemma 4.1.9, we have

$$b_A(y) = b_A(x + y) = 1 \quad \text{and} \quad b_A(L) = 2.$$
so we also have \( b_A(u + v) = b_A(u - v) = 1 \). Since \( x \in B_A \), \( b_A(x) = b_A(L) = 2 \). Therefore \([x, v], [u, x] \in [C_L(A), L] \) by previous claim. Thus we have

\[
[x + u, v] = [x, v] + [u, v] \notin [C_L(A), L],
\]

\[
[u, x + v] = [u, x] + [u, v] \notin [C_L(A), L],
\]

\[
[x + u + v, v] = [x, v] + [u + v, v] \notin [C_L(A), L],
\]

so \( b_A(x + u) = b_A(x + v) = b_A(x + u + v) = 1 \). Note that \( u - v \notin C_L(A) \) because \( b_A(u - v) = 1 \). In summary, we have

\[
b_A(u) = b_A(v) = b_A(u + v) = b_A(u - v) = b_A(x + u) = b_A(x + v) = b_A(x + u + v) = 1.
\]

Hence we have \( u, v, u + v, x + u, x + v, x + u + v \in T_A \), which contradicts Proposition 4.1.4. Therefore \([L, L] \subseteq [C_L(A), L] \). Conversely, it is clear that \([C_L(A), L] \subseteq [L, L] \). Hence \([L, L] = [C_L(A), L] \).

In addition, assume that we also have \( A = C_L(A) \) and \( b(L) = b_A(L) \). Then \([L, L] = [C_L(A), L] = [A, L] \). Consequently, \( \dim[L, L] = \dim[A, L] = b_A(L) \) by Proposition 4.1.6(7). □

Notice that there are relations between set of elements of breadth 0 and 1, as we show in the next lemma.

**Lemma 4.1.11.** Let \( L \) be a finite dimensional Lie algebra and \( A \) an ideal of \( L \). Suppose that \( b_A(L) = 1 \). Then the following hold:

1. \( L = T_A \cup C_L(A) \).
2. \( T_A \cap C_L(A) = \emptyset \).
3. \( T_A \cup \{0\} \) is a subspace of \( L \).

**Proof.** Let \( L \) be a finite dimensional Lie algebra and \( A \) an ideal of \( L \). Suppose that \( b_A(L) = 1 \).

1. Let \( x \in L \). Then we have \( b_A(x) = 0 \) or \( b_A(x) = 1 \), that is \( x \in C_L(A) \) or \( x \in T_A \). Therefore \( L \subseteq T_A \cup C_L(A) \). Conversely, we know that \( T_A \subseteq L \) and \( C_L(A) \subseteq L \), so \( T_A \cup C_L(A) \subseteq L \). Hence \( L = T_A \cup C_L(A) \).

2. It is clear that \( T_A \cap C_L(A) = \emptyset \) by their definitions.

3. Note that \( C_L(A) = \bigcap_{a \in A} \ker ad_a \). Since \( \ker ad_a \) is a subspace of \( L \) for all \( a \in A \), \( C_L(A) \) is also a subspace of \( L \). Because \( L \) is finite dimensional, so is \( C_L(A) \). Therefore we write \( C_L(A) = \text{span}\{c_1, c_2, \ldots, c_n\} \) and extend this basis to \( L = \text{span}\{c_1, c_2, \ldots, c_n, t_1, t_2, \ldots, t_m\} \). Then we have \( T_A \cup \{0\} = \text{span}\{t_1, t_2, \ldots, t_m\} \) and \( L = C_L(A) \oplus (T_A \cup \{0\}) \). Hence \( T_A \cup \{0\} \) is a subspace of \( L \).
In next lemma, we prove a few properties of maximal abelian ideal which we are going to use in our main theorem.

**Lemma 4.1.12.** Let $L$ be a finite dimensional nilpotent Lie algebra and $A$ a maximal abelian ideal of $L$. Then the following hold:

1. $\ker \text{ad}_a = A$ for all $a \in A - Z(L)$.
2. $C_L(A) = A$.
3. $Z(L) \subseteq A$.

**Proof.** Let $L$ be a finite dimensional nilpotent Lie algebra and $A$ a maximal abelian ideal of $L$.

1. Let $a \in A - Z(L)$. Then $\ker \text{ad}_a \neq L$ because $a \notin Z(L)$ since $a \in A$ and $A$ is abelian, we have $A \subseteq \ker \text{ad}_a$. Next, we will show that $\ker \text{ad}_a$ is an ideal of $L$. Let $x \in \ker \text{ad}_a$ and $y \in L$. Then $[a, x] = 0$, so we have

$$\text{ad}_a([x, y]) = [a, [x, y]] = [[a, x], y] + [x, [a, y]] = [x, [a, y]] \in A \subseteq \ker \text{ad}_a.$$ 

Thus $\ker \text{ad}_a$ is an ideal of $L$ that contains $A$. Since $A$ is maximal and $\ker \text{ad}_a \neq L$, $\ker \text{ad}_a = A$. Hence $A = \ker \text{ad}_a$ for all $a \in A - Z(L)$.

2. Observe that if $a \in Z(L)$, then we have $\ker \text{ad}_a = L$. Consequently, we obtain

$$C_L(A) = \bigcap_{a \in A} \ker \text{ad}_a = \bigcap_{a \in A - Z(L)} \ker \text{ad}_a \cap \bigcap_{a \in A \cap Z(L)} \ker \text{ad}_a = \bigcap_{a \in A - Z(L)} A \cap \bigcap_{a \in A \cap Z(L)} L = A \cap L = A.$$

Hence $C_L(A) = A$ as we want.

3. Let $x \in Z(L)$. Then we get $\text{ad}_x = 0$, so $b(x) = 0$. Since $b_A(x) \leq b(x)$, we have $b_A(x) = 0$. Thus $x \in C_L(A)$. By part two of this lemma, we know that $C_L(A) = A$, so $x \in C_L(A) = A$. Hence $Z(L) \subseteq A$.

$\square$
**Theorem 4.1.13.** Let $L$ be a finite dimensional nilpotent Lie algebra and $A$ an abelian ideal of $L$. Suppose that $b_A(L) = 1$. Then

1. $\dim(A/(A \cap Z(L))) = 1$ and $\ker ad_x|_A = \ker ad_y|_A$ for all $x, y \in L - C_L(A)$.

In addition, if $A$ is a maximal abelian ideal of $L$, then $\dim(L/Z(L)) = b(L) + 1$.

or

2. $\dim[A, L] = 1$. In addition, if $A$ is a maximal abelian ideal of $L$, then $b(L/[A, L]) < b(L)$.

**Proof.** Let $L$ be a finite dimensional nilpotent Lie algebra and $A$ an abelian ideal of $L$. Suppose that $b_A(L) = 1$. By Lemma 4.1.11, we know that $L = T_A \cup C_L(A), T_A \cap C_L(A) = \emptyset$ and $T_A \cup \{0\}$ is a subspace of $L$. As a result, we have $T_A = L - C_L(A)$. Define $T = \text{span}\{x, y\}$ and

$$T = (T + C_L(A))/C_L(A) \cong T/(T \cap C_L(A))$$

where $x, y \in T_A$, as defined in Theorem 4.1.3. Since $T_A \cap C_L(A) = \emptyset$ and $T \subseteq T_A \cup \{0\}$, we have

$$T \cap C_L(A) \subseteq (T_A \cup \{0\}) \cap C_L(A)$$

$$= (T_A \cap C_L(A)) \cup \{0\} \cap C_L(A))$$

$$= \emptyset \cup \{0\}$$

$$= \{0\},$$

so $T \cap C_L(A) = \{0\}$. Hence $T \cong T/(T \cap C_L(A)) \cong T$.

Suppose that $T_A \cup \{0\}$ is not 1-dimensional. Then we get $\dim T_A \cup \{0\} \geq 2$, so there exist $x, y \in T_A$ such that $T = \text{span}\{x, y\}$ is 2-dimensional. Thus $\dim T = \dim T = 2$ because $T \cong T$. Since $\{x, y\}$ is linearly independent and $T_A \cup \{0\}$ is a subspace of $L$, we have $x + y \in T_A$. By Theorem 4.1.3 (2), we know that

$$\text{either } \ker ad_x|_A \neq \ker ad_y|_A \text{ or } \ker ad_x|_A \neq \ker ad_y|_A,$$

so we consider the following two cases:

1. $\ker ad_x|_A = \ker ad_y|_A =: K$ and $\ker ad_x|_A \neq \ker ad_y|_A$. Let $z \in T_A$. Then

$$\ker ad_z|_A \neq \ker ad_x|_A \text{ or } \ker ad_z|_A \neq \ker ad_y|_A.$$

Without loss of generality, we suppose that $\ker ad_z|_A \neq \ker ad_x|_A$. Next we will show that $x + z \in T_A$. Since $x, z \in T_A$ and $T_A \cup \{0\}$ is a subspace of $L$, we have $x + z \in T_A \cup \{0\}$. If $x + z = 0$, then $x = -z$, so we get $\ker ad_x|_A = \ker ad_{-z}|_A = \ker ad_z|_A$, which is a contradiction. Therefore $x + z \in T_A$. By Theorem 4.1.3 (2) with “either or”, we have
im\(ad_z|_A\) = im\(ad_x|_A\), so \([A, z] = im\(ad_z|_A\) = im\(ad_x|_A\) = K\). Since \(z \in T_A\) is arbitrary, \([A, T_A] = K\). On the other hand, if \(z \in C_L(A)\), then \([A, z] = \{0\}\). Because \(z \in C_L(A)\) is arbitrary, \([A, C_L(A)] = 0\). Consequently, we obtain

\[[A, L] = [A, T_A \cup C_L(A)] = \text{span}\{[A, T_A], [A, C_L(A)]\} = \text{span}\{K, 0\} = K\]

Hence \(dim[A, L] = \dim K = b_A(x) = 1\).

2. \(im\(ad_x|_A\) \neq im\(ad_y|_A\) and \(ker\(ad_x|_A\) = ker\(ad_y|_A\) =: K’\). Let \(z \in T_A\). Then

\[im\(ad_x|_A\) \neq im\(ad_y|_A\) \quad \text{or} \quad im\(ad_z|_A\) \neq im\(ad_y|_A\).

Without loss of generality, we assume that \(im\(ad_z|_A\) \neq im\(ad_x|_A\)\). Next we will show that \(x + z \in T_A\). Since \(x, z \in T_A\) and \(T_A \cup \{0\}\) is a subspace of \(L\), we have \(x + z \in T_A \cup \{0\}\). If \(x + z = 0\), then \(x = -z\), so we get \(im\(ad_x|_A\) = im\(ad_z|_A\) = im\(ad_x|_A\), which is a contradiction. Therefore \(x + z \in T_A\). By Theorem 4.1.3 (2) with “either or”, we have \(ker\(ad_z|_A\) = ker\(ad_x|_A\) = K’\). Since \(z \in T_A\) is arbitrary, we get \(ker\(ad_z|_A\) = K’\) for any \(z \in T_A = L - C_L(A)\). Hence \(ker\(ad_x|_A\) = ker\(ad_y|_A\) for all \(x, y \in L - C_L(A)\).

On the other hand, we assume that \(T_A \cup \{0\}\) is 1-dimensional. Then \(T_A \cup \{0\} = T \cong \bar{T}\), so \(T\) and \(\bar{T}\) are also 1-dimensional for every \(x, y \in T_A\). By Theorem 4.1.3 (1), we have

\[im\(ad_x|_A\) = im\(ad_y|_A\) \quad \text{and} \quad ker\(ad_x|_A\) = ker\(ad_y|_A\)

for every \(x, y \in T_A\). Note that if \(z \in C_L(A)\), then \([A, z] = \{0\}\). Therefore \([A, L] = im\(ad_x|_A\)\) for some \(x \in T_A\), so \(dim[A, L] = b_A(x) = 1\). Hence \(dim[A, L] = 1\) and \(ker\(ad_x|_A\) = ker\(ad_y|_A\) for every \(x, y \in L - C_L(A)\).

Next, we will claim that \(ker\(ad_x|_A\) = ker\(ad_y|_A\) for all \(x, y \in L - C_L(A)\) implies \(dim(A/(A \cap Z(L))) = 1\). Suppose that \(ker\(ad_x|_A\) = ker\(ad_y|_A\) for all \(x, y \in L - C_L(A)\). Note that for every \(z \in C_L(A)\), we have \(ker\(ad_z|_A\) = A\). We also know that \(Z(L) = \bigcap_{x \in L} ker\(ad_x\)\), so

\[A \cap Z(L) = A \cap (\bigcap_{x \in L} ker\(ad_x\)) = (A \cap ker\(ad_x\)) \cap ker\(ad_x\) = A \cap ker\(ad\alpha|_A\) = ker\(ad\alpha|_A\)\]
for some $\alpha \in L - C_L(A) = T_A$. Consequently, we obtain

\[
\dim(A/(A \cap Z(L))) = \dim(A/\ker ad_\alpha|_A) = \dim A - \text{nullity } ad_\alpha|_A = \text{rank } ad_\alpha|_A = b_A(\alpha) = 1.
\]

Hence $\dim(A/(A \cap Z(L))) = 1$.

Finally, we suppose that $A$ is a maximal abelian ideal of $L$. For the first result, we have $\dim(A/(A \cap Z(L))) = 1$. By Lemma 4.1.12 (1) & (3), we get $A = \ker ad_a$ for every $a \in A - Z(L)$ and $Z(L) \subseteq A$, respectively. Then we have $A \cap Z(L) = Z(L)$. Fix $\alpha \in A - Z(L)$, so

\[
\dim(L/Z(L)) = \dim L - \dim Z(L) = (\dim L - \dim A) + (\dim A - \dim Z(L)) = (\dim L - \dim \ker ad_a) + (\dim A - \dim(A \cap Z(L))) = (\dim L - \text{nullity } ad_\alpha) + \dim(A/(A \cap Z(L))) = \text{rank } ad_\alpha + 1 = b(\alpha) + 1 \leq b(L) + 1.
\]

On the other hand, we know that $\dim(L/Z(L)) \geq b(L) + 1$ by Theorem 3.1.9. Consequently, $\dim(L/Z(L)) = b(L) + 1$.

For the second result, we have $\dim[A, L] = 1$. Note that $[A, L]$ is an ideal of $L$ because both $A$ and $L$ are ideals of $L$. Thus we can consider the quotient Lie algebra $L/[A, L]$. Let $x \in L$. Then $x + [A, L] \in L/[A, L]$ and $ad_{x+[A,L]} : L/[A, L] \to L/[A, L]$ is given by

\[
y + [A, L] \mapsto [x,y] + [A, L]
\]

where $y + [A, L] \in L/[A, L]$. By Lemma 4.1.11 (1) & (2), we know that $L = T_A \cup C_L(A)$ and $T_A \cap C_L(A) = \emptyset$, respectively. Moreover, by Lemma 4.1.12 (2), we also know that $C_L(A) = A$. Therefore $x$ must be contained in $A$ or $T_A$. Then we consider the following two cases:

1. If $x \in A$, then $[x,y] \in [A, L]$ for any $y \in L$. Thus $\text{im } ad_{x+[A,L]} = \{[A, L]\}$, so we have $b(x + [A, L]) = 0$. Note that $b(L) \geq b_A(L) = 1$. Hence $b(x + [A, L]) < b(L)$ for any $x \in A$.

2. Assume that $x \in T_A$. Then $b_A(x) = 1$, which implies $[A, x] = \text{im } ad_x|_A$ is 1-dimensional.
Since \([A, x] \subseteq [A, L]\), \(\text{im} \, \text{ad}_x \cap [A, L] \neq \{0\}\). Hence \(b(x + [A, L]) < b(x) \leq b(L)\) for any \(x \in T_A\).

Hence \(b(x + [A, L]) < b(L)\) for any \(x \in L\). Since \(x \in L\) is arbitrary, \(b(L/[A, L]) < b(L)\).

\[\square\]

### 4.2 Main Theorem

Our main theorem shows necessary and sufficient conditions for finite dimensional nilpotent Lie algebra of breadth 2. Observe that dimension of finite dimensional nilpotent Lie algebra of breadth 2 is not bounded above but it is relatively small when we consider its square.

**Theorem 4.2.1.** Let \(L\) be a finite dimensional nilpotent Lie algebra. Then \(b(L) = 2\) if and only if one of the following holds:

1. \(\dim[L, L] = 2\)

or

2. \(\dim[L, L] = 3\) and \(\dim(L/Z(L)) = 3\).

**Proof.** Let \(L\) be a finite dimensional nilpotent Lie algebra. Suppose \(b(L) = 2\). Since \(b(L) \neq 0\), by Proposition 3.1.4, \(L\) is not abelian, so \(Z(L) \subsetneq L\). On the other hand, \(Z(L) \neq \{0\}\) because \(L\) is nilpotent. Therefore \(\{0\} \neq Z(L) \subsetneq L\), which guarantees that \(L\) has a maximal abelian ideal. Then we consider the following two cases:

First, there exists a maximal abelian ideal \(A\) of \(L\) such that \(b_A(L) = 2\). Let \(x \in L\) be such that \(b_A(x) > 1\). Since \(b_A(L) = 2\), we get \(b_A(x) = 2\). Then

\[b_A(x) = 2 > 1 \quad \text{and} \quad 2b_A(x) = 4 \geq 2 = b_A(L).\]

Thus \(x\) satisfies (4.1). Since \(b_A(x) = 2 = b(L)\), \(\text{rank} \, \text{ad}_x|_A = \text{rank} \, \text{ad}_x = 2\). Therefore we have

\([L, x] = \text{im} \, \text{ad}_x = \text{im} \, \text{ad}_x|_A = [A, x] \subseteq [A, L].\)

As a result, \(A\) meets all requirements in Theorem 4.1.10. In addition, we have \(b_A(L) = 2 = b(L)\) and \(C_L(A) = A\) by Lemma 4.1.12 (2). Hence \(\dim[L, L] = b_A(L) = 2\) by Theorem 4.1.10.

The complementary case of the previous one is \(b_A(L) \leq 1\) for every maximal abelian ideal \(A\) of \(L\). We will show that for every maximal abelian ideal \(A\) of \(L\), \(b_A(L) \neq 0\). Suppose that there exists a maximal abelian ideal \(A\) of \(L\) such that \(b_A(L) = 0\). Then for every \(x \in L\), \(b_A(x) = 0\). Thus \(\text{rank} \, \text{ad}_x|_A = 0\), which implies \([A, x] = \text{im} \, \text{ad}_x|_A = \{0\}\). Since \(x \in L\) is arbitrary, \([A, L] = \{0\}\), so \(C_L(A) = L\) which contradicts \(C_L(A) = A\) by Lemma 4.1.12 (2). As a result, this case turns into \(b_A(L) = 1\) for every maximal abelian ideal \(A\) of \(L\). Next, we apply Theorem 4.1.13 to this case, so we have two following subcases to consider:
1. \( \dim(A/(A \cap Z(L))) = 1, \ker ad_x |_A = \ker ad_y |_A \) for all \( x, y \in L - C_L(A) \) and \( \dim(L/Z(L)) = b(L) + 1 = 2 + 1 = 3 \). In addition, we have \( \dim[L, L] \leq \left( \binom{3}{2} \right) = 3 \) by Lemma 3.1.10. Since \( b(L) = 2 \neq 0,1 \), we have \( \dim[L, L] \neq 0,1 \) by Proposition 3.1.4 and Theorem 3.2.1, respectively. Hence \( \dim[L, L] = 2,3 \) and \( \dim(L/Z(L)) = 3 \).

2. \( \dim[A, L] = 1 \) and \( b(L/[A, L]) < b(L) = 2 \). Then we get \( b(L/[A, L]) = 0,1 \). Next, we will claim that \( b(L/[A, L]) \neq 0 \). Assume that \( b(L/[A, L]) = 0 \). Then \( [L/[A, L], L/[A, L]] = \{0\} \) by Proposition 3.1.4. Therefore

\[
\dim[L, L]/[A, L] = \dim[L/[A, L], L/[A, L]] = 0,
\]

so \( \dim[L, L] = \dim[A, L] = 1 \). By Theorem 3.2.1, \( b(L) = 1 \), which contradicts \( b(L) = 2 \). Hence \( b(L/[A, L]) \neq 0 \), which implies \( b(L/[A, L]) = 1 \). As a result, we have

\[
\dim[L, L]/[A, L] = \dim[L/[A, L], L/[A, L]] = 1
\]


Conversely, if \( \dim[L, L] = 2 \), then \( b(L) \leq \dim[L, L] = 2 \) by Lemma 3.1.5. Since \( \dim[L, L] \neq 0 \), \( L \) is not abelian, so \( b(L) \neq 0 \) by Proposition 3.1.4. Similarly, we have \( b(L) \neq 1 \) by Theorem 3.2.1. Hence \( b(L) = 2 \) in this case.

On the other hand, if \( \dim[L, L] = 3 \) and \( \dim(L/Z(L)) = 3 \), then \( b(L) \leq \dim[L, L] = 3 \) by Lemma 3.1.5. Similar to the case \( \dim[L, L] = 2 \), we get \( b(L) \neq 0,1 \) by Proposition 3.1.4 and Theorem 3.2.1, respectively. If \( b(L) = 3 \), then by Theorem 3.1.9, \( \dim(L/Z(L)) \geq b(L) + 1 = 4 \) which contradicts \( \dim(L/Z(L)) = 3 \). Hence \( b(L) = 2 \).

**Corollary 4.2.2.** Let \( L \) be a finite dimensional nilpotent Lie algebra of breadth 2 and \( A \) a maximal abelian ideal of \( L \). Suppose that \( \dim[L, L] = 3 \). Then \( \dim(A/Z(L)) = 1 \).

**Proof.** Let \( L \) be a finite dimensional nilpotent Lie algebra of breadth 2 and \( A \) a maximal abelian ideal of \( L \). Suppose that \( \dim[L, L] = 3 \). Then \( b(L) = 2 \). By the proof of Theorem 4.2.1, this must be the result of the case \( b_{A'}(L) = 1 \) for every maximal abelian ideal \( A' \) of \( L \) together with the first result of Theorem 4.1.13. Consequently, \( \dim(A/(A \cap Z(L))) = 1 \) by Theorem 4.1.13 (1). Similar to Theorem 4.2.1, we have \( \{0\} \neq Z(L) \subseteq L \), which is an abelian ideal of \( L \). By Lemma 4.1.12 (3), we know that \( Z(L) \subseteq A \), so \( A \cap Z(L) = Z(L) \). Hence \( \dim(A/Z(L)) = \dim(A/(A \cap Z(L))) = 1 \). \( \square \)
Chapter 5

Classification of Nilpotent Lie Algebras of Breadth 2

5.1 Structure of Nilpotent Lie Algebras of Breadth 2

We begin this section by showing that any finite dimensional nilpotent Lie algebra of breadth 2 has dimension greater than 3. Thus we know our starting dimension of the classification process.

Lemma 5.1.1. Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $\dim L \in \mathbb{Z}_{>0}$. Then

$$1 \leq \dim Z(L) \leq \dim L - 3.$$

Proof. Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $\dim L := n \in \mathbb{Z}_{>0}$. Then $b(L) = 2$. By Theorem 3.1.9, we know that $\dim(L/Z(L)) \geq b(L) + 1$, so we have $\dim L - \dim Z(L) \geq 3$. Thus $\dim Z(L) \leq \dim L - 3$. On the other hand, $L$ has nontrivial center since $L$ is nilpotent. Therefore $\dim Z(L) \geq 1$. Hence $1 \leq \dim Z(L) \leq \dim L - 3$.

Corollary 5.1.2. Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2. Then $\dim L \geq 4$.

Definition 5.1.3. A Lie algebra $L$ is called pure if it does not have an abelian ideal as a direct summand.

Lemma 5.1.4. Let $L$ be a finite dimensional nilpotent Lie algebra. Then $L$ is pure if and only if $Z(L) \subseteq [L, L]$.

Proof. Let $L$ be a finite dimensional nilpotent Lie algebra. Suppose that $Z(L)$ is not contained in $[L, L]$. Then there exists $x \in Z(L) - [L, L]$. Note that $x \neq 0$ since $x \notin [L, L]$. Let
\( I = \text{span}\{x\} \). Then \( I \) is a nonzero ideal of \( L \) contained in \( Z(L) \). Next, we extend \( I \) to a basis of \( L \), say \( L = \text{span}\{x, y_1, y_2, \ldots, y_n\} \) for some \( n \in \mathbb{Z}_{>0} \). Let \( J = \text{span}\{y_1, y_2, \ldots, y_n\} \). Then we see that \( L = I \oplus J \), so we need to show that \( J \) is also an ideal of \( L \). Since \( x \in Z(L) - [L, L] \), \( I \cap [L, L] = \{0\} \), so \([L, L] \subseteq J \). Thus \( J \) is an ideal of \( L \). Hence \( L = I \oplus J \) where \( I \) and \( J \) are ideals of \( L \) and \( I \subseteq Z(L) \). Consequently, \( L \) is not pure.

Conversely, assume that \( L \) is not pure. Then \( L = I \oplus J \) where \( I \) and \( J \) are ideals of \( L \) and \( I \subseteq Z(L) \). Let \( x \in I - \{0\} \subseteq Z(L) - \{0\} \). Next we will claim that \([L, L] \subseteq J \). Let \( a, b \in L \). Then \( a \) and \( b \) can be written as \( a = a_I + a_J \) and \( b = b_I + b_J \) where \( a_I, b_I \in I \subseteq Z(L) \) and \( a_J, b_J \in J \). Therefore we have

\[
[a, b] = [a_I + a_J, b_I + b_J] = [a_I, b_I] + [a_I, b_J] + [a_J, b_I] + [a_J, b_J] = [a_J, b_J] \in J.
\]

Since \( a \) and \( b \) are arbitrary, \([L, L] \subseteq J \). Because \( x \neq 0 \), \( x \not\in J \) which also implies \( x \notin [L, L] \). Hence \( x \in Z(L) - [L, L] \), so \( Z(L) \) is not contained in \([L, L] \).

Therefore we get a condition that is equivalent to purity of Lie algebras. In general, we consider only pure Lie algebras, so we will include the condition \( Z(L) \subseteq [L, L] \) in our classification process. Note that in order to obtain a Lie algebra which is not pure, we begin with a pure Lie algebra with smaller dimension and add an abelian part to it.

**Theorem 5.1.5.** Let \( L \) be a finite dimensional nilpotent Lie algebra of breadth 2 such that \( Z(L) \subseteq [L, L] \). Then \( L \) is a direct sum of smaller Lie algebras if and only if \( L \) is a direct sum of two Heisenberg Lie algebras.

**Proof.** Let \( L \) be a finite dimensional nilpotent Lie algebra of breadth 2 such that \( Z(L) \subseteq [L, L] \). Suppose that \( L \) is a direct sum of smaller Lie algebras, says \( L = L_1 \oplus L_2 \oplus \ldots \oplus L_n \) for some \( n \in \mathbb{Z}_{>0} \). Since \( Z(L) \subseteq [L, L] \), we know that \( L \) is pure, so each summand is not abelian. Thus \( b(L_i) \neq 0 \) for all \( i = 1, 2, \ldots, n \). By Corollary 3.1.13, we have

\[
b(b_1) + b(b_2) + \ldots + b(b_n) = b(L_1 \oplus L_2 \oplus \ldots \oplus L_n) = b(L) = 2,
\]

which leave us only one choice, \( n = 2 \) and \( b(L_1) = b(L_2) = 1 \). Hence \( L \) is a direct sum of two Heisenberg Lie algebras by Theorem 3.2.6. The converse implication is clear. Consequently, \( L \) is a direct sum of smaller Lie algebras if and only if \( L \) is a direct sum of two Heisenberg Lie algebras. 

**Corollary 5.1.6.** Let \( L \) be a finite dimensional nilpotent Lie algebra of breadth 2 such that \( Z(L) \subseteq [L, L] \). Suppose that \( L \) is a direct sum of smaller Lie algebras. Then \( \dim L \) is even.

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Proof. Let \( L \) be a finite dimensional nilpotent Lie algebra of breadth 2 such that \( Z(L) \subseteq [L, L] \). Suppose that \( L \) is a direct sum of smaller Lie algebras. Then \( L \) is a direct sum of two Heisenberg Lie algebras by Theorem 5.1.5, says \( L = H_1 \oplus H_2 \). Since \( Z(L) \subseteq [L, L] \), \( L \) is pure which implies \( H_1 \) and \( H_2 \) are also pure. Thus both \( \dim H_1 \) and \( \dim H_2 \) are odd. Hence \( \dim L = \dim (H_1 \oplus H_2) = \dim H_1 + \dim H_2 \) is even.

5.2 Nilpotent Lie Algebras of Breadth 2 with \( \dim[L, L] = 3 \) and \( \dim(L/Z(L)) = 3 \)

As we have already seen in Theorem 4.2.1, nilpotent Lie algebra of breadth 2 has two equivalent conditions. In this part, we consider the second condition and classify it as stated in the next theorem. From now on, we may not write all bracket relations of \( L \). We assume that all of the bracket relations are equal to zero, unless we state otherwise.

**Theorem 5.2.1.** Let \( L \) be a finite dimensional nilpotent Lie algebra of breadth 2 such that \( Z(L) \subseteq [L, L] \). Suppose that \( \dim[L, L] = 3 \) and \( \dim L/Z(L) = 3 \). Then \( L \) is isomorphic to either

1. \( L = \text{span}\{x, y, v, w_1, w_2\} \) where \([x, y] = v, [x, v] = w_1 \) and \([y, v] = w_2\)

or

2. \( L = \text{span}\{x, y, z, w_1, w_2, w_3\} \) where \([x, y] = w_1, [x, z] = w_2 \) and \([y, z] = w_3\).

**Proof.** Let \( L \) be a finite dimensional nilpotent Lie algebra of breadth 2 such that \( Z(L) \subseteq [L, L] \). Then \( b(L) = 2 \). Suppose that \( \dim[L, L] = 3 \) and \( \dim L/Z(L) = 3 \). Because \( Z(L) \subseteq [L, L] \) and \( \dim[L, L] = 3 \), we have \( \dim Z(L) = 0, 1, 2 \) or 3. Since \( L \) is nilpotent, \( L \) has nontrivial center, so \( \dim(Z(L)) \neq 0 \). Therefore we have 3 cases to consider:

1. Case I : \( \dim Z(L) = 1 \). Then \( 3 = \dim L/Z(L) = \dim L - \dim Z(L) = \dim L - 1 \), so \( \dim L = 4 \). Let \( Z(L) = \text{span}\{z\} \). Then extend it to \( [L, L] = \text{span}\{u, v, z\} \) and then \( L = \text{span}\{x, u, v, z\} \). Note that the bracket relations on \( L \) are defined by \([x, u], [x, v] \) and \([u, v] \). Since \([L, L] = \text{span}\{u, v, z\} \), we say that

\[
\begin{align*}
[x, u] &= \alpha_1 u + \alpha_2 v + \alpha_3 z, \\
[x, v] &= \beta_1 u + \beta_2 v + \beta_3 z, \\
[u, v] &= \gamma_1 u + \gamma_2 v + \gamma_3 z
\end{align*}
\]
for some $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}$ and $i = 1, 2, 3$. Because of the nilpotency of $L$, we have $\alpha_1 = \beta_2 = \gamma_1 = \gamma_2 = 0$. Thus we obtain

\[
\begin{align*}
[x, u] &= \alpha_2 v + \alpha_3 z, \\
[x, v] &= \beta_1 u + \beta_3 z, \\
[u, v] &= \gamma_3 z.
\end{align*}
\]

Note that $\alpha_2, \beta_1, \gamma_3 \neq 0$ because $\dim L/Z(L) = 3$. Then we have $(ad_x)^N(u) \neq 0$ for any $N \in \mathbb{Z}_{>0}$ which contradicts the nilpotency of $L$.

2. Case II : $\dim Z(L) = 2$. Then $3 = \dim L/Z(L) = \dim L - \dim Z(L) = \dim L - 2$, so $\dim L = 5$. Let $Z(L) = \operatorname{span}\{z_1, z_2\}$. Then extend it to $[L, L] = \operatorname{span}\{u, z_1, z_2\}$ and then $L = \operatorname{span}\{x, y, u, z_1, z_2\}$. The bracket relations on $L$ are defined by $[x, y], [x, u]$ and $[y, u]$.

Since $[L, L] = \operatorname{span}\{u, z_1, z_2\}$, we say that

\[
\begin{align*}
[x, y] &= \alpha_1 u + \alpha_2 z_1 + \alpha_3 z_2, \\
[x, u] &= \beta_1 u + \beta_2 z_1 + \beta_3 z_2, \\
[y, u] &= \gamma_1 u + \gamma_2 z_1 + \gamma_3 z_2
\end{align*}
\]

for some $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}$ and $i = 1, 2, 3$. Since $L$ is nilpotent, $\beta_1 = \gamma_1 = 0$. Then we get

\[
\begin{align*}
[x, y] &= \alpha_1 u + \alpha_2 z_1 + \alpha_3 z_2 =: v, \\
[x, u] &= \beta_2 z_1 + \beta_3 z_2 =: w_1, \\
[y, u] &= \gamma_2 z_1 + \gamma_3 z_2 =: w_2.
\end{align*}
\]

Since $\dim L/Z(L) = 3$, we get $Z(L) = \operatorname{span}\{w_1, w_2\}$ and $[L, L] = \operatorname{span}\{v, w_1, w_2\}$, so $\alpha_1 \neq 0$. Let $w'_1 = \alpha_1 w_1$ and $w'_2 = \alpha_1 w_2$. Hence $Z(L) = \operatorname{span}\{w'_1, w'_2\}, [L, L] = \operatorname{span}\{v, w'_1, w'_2\}$ and $L = \operatorname{span}\{x, y, v, w'_1, w'_2\}$ where

\[
\begin{align*}
[x, y] &= v, \\
[x, v] &= [x, \alpha_1 u + \alpha_2 z_1 + \alpha_3 z_2] = \alpha_1 w_1 = w'_1, \\
[y, v] &= [y, \alpha_1 u + \alpha_2 z_1 + \alpha_3 z_2] = \alpha_1 w_2 = w'_2.
\end{align*}
\]

3. Case III : $\dim Z(L) = 3$. Then $3 = \dim L/Z(L) = \dim L - \dim Z(L) = \dim L - 3$, so $\dim L = 6$. Thus $Z(L) = [L, L]$, says $Z(L) = \operatorname{span}\{w_1, w_2, w_3\}$. Next, we extend this basis to $L = \operatorname{span}\{x, y, z, w_1, w_2, w_3\}$. Note that the bracket relations on $L$ are defined by $[x, y], [x, z]$ and $[y, z]$. Since $\dim L/Z(L) = 3$, $[L, L] = Z(L) = \operatorname{span}\{[x, y], [x, z], [y, z]\}$. 

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Let \([x, y] = w'_1, [x, z] = w'_2\) and \([y, z] = w'_3\). Hence \([L, L] = Z(L) = \text{span}\{w'_1, w'_2, w'_3\}\) and \(L = \text{span}\{x, y, z, w'_1, w'_2, w'_3\}\) where \([x, y] = w'_1, [x, z] = w'_2\) and \([y, z] = w'_3\).

In conclusion, \(L\) is isomorphic to either

1. \(L = \text{span}\{x, y, v, w_1, w_2\}\) where \([x, y] = v, [x, v] = w_1\) and \([y, v] = w_2\)

or

2. \(L = \text{span}\{x, y, z, w_1, w_2, w_3\}\) where \([x, y] = w_1, [x, z] = w_2\) and \([y, z] = w_3\).

\[\square\]

### 5.3 Nilpotent Lie Algebras of Breadth 2 with \(\dim[L, L] = 2\) and \(\dim Z(L) = 1\)

As stated in the first condition of Theorem 4.2.1, we now consider finite dimensional nilpotent Lie algebra \(L\) such that \(\dim[L, L] = 2\). Since we also consider the condition \(Z(L) \subseteq [L, L]\), \(Z(L)\) could be 1 or 2-dimensional. In this section, we classify one with \(\dim Z(L) = 1\) and leave the case \(\dim Z(L) = 2\) to the next section.

**Proposition 5.3.1.** Let \(L\) be a finite dimensional nilpotent Lie algebra of breadth 2 such that \(\dim L =: n \geq 4\). Suppose that \(\dim[L, L] = 2, \dim Z(L) = 1\) and \(Z(L) \subseteq [L, L]\). Then \(L/Z(L)\) is isomorphic to \(\text{span}\{x + Z(L), y + Z(L), v + Z(L), w_1 + Z(L), w_2 + Z(L), \ldots, w_{n-4} + Z(L)\}\) such that \([x + Z(L), y + Z(L)] = v + Z(L)\) and \(v + Z(L), w_i + Z(L) \in Z(L/Z(L))\) for all \(i = 1, 2, \ldots, n - 4\).

**Proof.** Let \(L\) be a finite dimensional nilpotent Lie algebra of breadth 2 such that \(\dim L =: n \geq 4\). Suppose that \(\dim[L, L] = 2, \dim Z(L) = 1\) and \(Z(L) \subseteq [L, L]\). Let \(Z(L) = \text{span}\{z\}\). Then we extend \(Z(L)\) to \([L, L] = \text{span}\{v, z\}\). Thus we have \(z \neq 0\) and \(v \in [L, L] - Z(L)\). Next, we consider \(L/Z(L)\). Since \(L/Z(L)\) is a homomorphic image of \(L\) which is nilpotent, \(L/Z(L)\) is also nilpotent. In addition, \(b(L/Z(L)) = 1\) because

\([L/Z(L), L/Z(L)] = [L, L]/Z(L) = \text{span}\{v + Z(L)\}\)

is 1-dimensional by Theorem 3.2.1. As a result, by Theorem 3.2.6, \(L/Z(L)\) is isomorphic to

\[\text{span}\{x_1 + Z(L), y_1 + Z(L), x_2 + Z(L), y_2 + Z(L), \ldots, x_m + Z(L), y_m + Z(L), v + Z(L), w_1 + Z(L), w_2 + Z(L), \ldots, w_{n-2m-2} + Z(L)\}\]

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such that \([x_i + Z(L), y_i + Z(L)] = v + Z(L)\) and \(v + Z(L), w_1 + Z(L), \ldots, w_{n-2m-2} + Z(L) \in Z(L/Z(L))\) for some \(m \in \{1, 2, \ldots, \lfloor \frac{n-2}{2} \rfloor \}\) and for all \(i = 1, 2, \ldots, m\). Next, we will claim that \(m = 1\). Suppose that \(m \geq 1\). Then we consider \(L\) from \(L/Z(L)\), so \(L\) is isomorphic to

\[
\text{span}\{x_1, y_1, x_2, y_2, \ldots, x_m, y_m, v, w_1, w_2, \ldots, w_{n-2m-2}, z\}
\]

such that \([x_i, y_i] = v + \alpha_i z\) and the rest of the bracket relations lie in \(Z(L) = \text{span}\{z\}\) for some \(\alpha_1, \ldots, \alpha_m \in \mathbb{F}\) and for all \(i = 1, 2, \ldots, m\). Since \(m \geq 1\), we can choose \(i \neq j \in \{1, 2, \ldots, m\}\). Noting that \([x_i, x_j], [x_i, y_j] \in Z(L)\), so

\[
[x_i, v] = [x_i, v + \alpha_j z] = [x_i, [x_j, y_j]] = [[x_i, x_j], y_j] + [x_j, [x_i, y_j]] = 0.
\]

Similarly, we also have

\[
y_i, v = [y_i, v + \alpha_j z] = [y_i, [x_j, y_j]] = [[y_i, x_j], y_j] + [x_j, [y_i, y_j]] = 0
\]

because \([y_i, x_j], [y_i, y_j] \in Z(L)\). Moreover, for \(k = 1, 2, \ldots, n-2m-2\), we get

\[
[w_k, v] = [w_k, v + \alpha_1 z] = [w_k, [x_1, y_1]] = [[w_k, x_1], y_1] + [x_1, [w_k, y_1]] = 0
\]

since \([w_k, x_1], [w_k, y_1] \in Z(L)\). We also know that \([v, v] = [v, z] = 0\). Thus \(v \in Z(L)\) which is a contradiction. Hence \(m = 1\), so \(L/Z(L)\) is isomorphic to \(\text{span}\{x + Z(L), y + Z(L), v + Z(L), w_1 + Z(L), w_2 + Z(L), \ldots, w_{n-4} + Z(L)\}\) such that \([x + Z(L), y + Z(L)] = v + Z(L)\) and \(v + Z(L), w_1 + Z(L) \in Z(L/Z(L))\) for all \(i = 1, 2, \ldots, n-4\). □

**Lemma 5.3.2.** Let \(L\) be a finite dimensional nilpotent Lie algebra of breadth 2 such that \(\dim L = n \geq 4\). Suppose that \(\dim [L, L] = 2\) and \(Z(L) = \text{span}\{z\} \subseteq [L, L]\) is 1-dimensional. Then \(L\) is isomorphic to \(\text{span}\{x, y, z, v, w_1, w_2, \ldots, w_{n-4}\}\) such that \([x, y] = v, [x, v] = z\) and \([y, v] = [x, w_1] = [v, w_i] = 0\) for all \(i = 1, 2, \ldots, n-4\) and the rest of the bracket relations lie in \(Z(L)\).

**Proof.** Let \(L\) be a finite dimensional nilpotent Lie algebra of breadth 2 such that \(\dim L = n \geq 4\). Suppose that \(\dim [L, L] = 2\) and \(Z(L) = \text{span}\{z\} \subseteq [L, L]\) is 1-dimensional. By Proposition 5.3.1, \(L/Z(L)\) is isomorphic to \(\text{span}\{x + Z(L), y + Z(L), v + Z(L), w_1 + Z(L), w_2 + Z(L), \ldots, w_{n-4} + Z(L)\}\) such that \([x + Z(L), y + Z(L)] = v + Z(L)\) and \(v + Z(L), w_1 + Z(L) \in Z(L/Z(L))\) for all \(i = 1, 2, \ldots, n-4\).

Next, we pull this back so \(L \cong \text{span}\{x, y, z, v, w_1, w_2, \ldots, w_{n-4}\}\) such that \([x, y] = v + \alpha z\) for some \(\alpha \in \mathbb{F}\) and the rest of the bracket relations lie in \(Z(L)\). Take \(v' = v + \alpha z = [x, y]\). Then \(L \cong \text{span}\{x, y, z, v', w_1, w_2, \ldots, w_{n-4}\}\) such that \([x, y] = v'\) and the rest of the bracket relations
lie in $Z(L)$. Note that

$$[w_i, v'] = [w_i, [x, y]] = [[w_i, x], y] + [x, [w_i, y]] = 0$$

for all $i = 1, 2, \ldots, n - 4$ because $[w_i, x], [w_i, y] \in Z(L)$. Consequently, $[w_i, v'] = 0$ for all $i = 1, 2, \ldots, n - 4$. Since $v' \notin Z(L)$, we have $[x, v'] \neq 0$ or $[y, v'] \neq 0$. Without loss of generality, we assume that $[x, v'] \neq 0$, says $[x, v'] = \beta z =: z'$ for some $\beta \in \mathbb{F} - \{0\}$. Then we take $Z(L) = \text{span}\{z'\}$. Let $[x, w_i] = \gamma_i z$ for some $\gamma_i \in \mathbb{F}$ and for all $i = 1, 2, \ldots, n - 4$. Then we take $w'_i = \beta w_i - \gamma_i v'$. As a consequence, we have

$$[x, w'_i] = [x, \beta w_i - \gamma_i v'] = \beta [x, w_i] - \gamma_i [x, v'] = \beta \gamma_i z - \gamma_i \beta z = 0,$$

$$[v', w'_i] = [v', \beta w_i - \gamma_i v'] = \beta [v', w_i] - \gamma_i [v', v'] = 0.$$

Finally, observe that $[y, v'] = \delta z'$ for some $\delta \in \mathbb{F}$. By taking $y' = y - \delta x$, we have

$$[x, y'] = [x, y - \delta x] = [x, y] - \delta [x, x] = v',$$

$$[y', v'] = [y - \delta x, v'] = [y, v'] - \delta [x, v'] = \delta z' - \delta z' = 0.$$

Hence $L$ is isomorphic to $\text{span}\{x, y', z', v', w'_1, w'_2, \ldots, w'_{n-4}\}$ such that $[x, y] = v, [x, v] = z$ and $[y, v] = [x, w_i] = [v, w_i] = 0$ for all $i = 1, 2, \ldots, n - 4$ and the rest of the bracket relations lie in $Z(L)$.

\[\square\]

**Theorem 5.3.3.** Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $\dim L =: n \geq 4$. Suppose that $\dim[L, L] = 2$ and $Z(L) = \text{span}\{z\} \subseteq [L, L]$ is 1-dimensional. Then the following holds:

1. If $n$ is even, then $L$ is isomorphic to $\text{span}\{x, y, z, v, w_1, w_2, \ldots, w_{n-4}\}$ such that $[x, y] = v, [x, v] = z$ and $[w_i, w_{i+1}] = z$ for all $i = 1, 3, 5, \ldots, n - 5$. 

2. If $n$ is odd, then $L$ is isomorphic to $\text{span}\{x, y, z, v, w_1, w_2, \ldots, w_{n-4}\}$ such that $[x, y] = v, [x, v] = z, [y, w_1] = z$ and $[w_i, w_{i+1}] = z$ for all $i = 2, 4, 6, \ldots, n - 5$.

**Proof.** Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $\dim L =: n \geq 4$. Suppose that $\dim[L, L] = 2$ and $Z(L) = \text{span}\{z\} \subseteq [L, L]$ is 1-dimensional. It is clear that, by Lemma 5.3.2, $L$ is isomorphic to $\text{span}\{x, y, z, v\}$ such that $[x, y] = v, [x, v] = z$ and $[y, v] = 0$ if $\dim L = 4$. Moreover, if $\dim L = 5$, then by Lemma 5.3.2, $L$ is isomorphic to $\text{span}\{x, y, z, v, w\}$ such that $[x, y] = v, [x, v] = z$ and $[y, v] = [x, w] = [v, w] = 0$ and the rest of the bracket relations lie in $Z(L)$. Since $w \notin Z(L)$, we have $[y, w] = \alpha z$ for some $\alpha \neq 0$. By taking $w' = \frac{w}{\alpha}$,
we have

\[ [y, w'] = [y, \frac{w}{\alpha}] = \frac{1}{\alpha} [y, w] = \frac{1}{\alpha} \alpha z = z \quad \text{and} \quad [x, w'] = [v, w'] = 0. \]

Hence \( L \) is isomorphic to \( \text{span}\{x, y, z, v, w'\} \) such that \([x, y] = v, [x, v] = [y, w'] = z \) and \([y, v] = [x, w'] = [v, w'] = 0\).

Assume that \( \dim L = n \geq 6 \). By Lemma 5.3.2, \( L \) is isomorphic to \( \text{span}\{x, y, z, v, w_1, w_2, \ldots, w_{n-4}\} \) such that \([x, y] = v, [x, v] = [x, w_i] = [v, w_i] = 0 \) for all \( i = 1, 2, \ldots, n - 4 \) and the rest of the bracket relations lie in \( Z(L) \). Let \( W := \text{span}\{z, w_1, w_2, \ldots, w_{n-4}\} \). Then \( \text{im} \, \text{ad}_x |_W = [x, W] = \{0\} \) and \( \text{im} \, \text{ad}_v |_W = [v, W] = \{0\} \). Observe that \( [W, W] \subseteq \text{span}\{z\} \).

Suppose that \([W, W] = \{0\}\). Then we get \([y, w_i] \neq 0 \) for all \( i = 1, 2, \ldots, n-4 \). Thus \([y, w_1] = a_1 z \) and \([y, w_2] = a_2 z \) where \( a_1, a_2 \in \mathbb{F} - \{0\} \). Therefore we have

\[ [y, a_2 w_1 - a_1 w_2] = a_2 [y, w_1] - a_1 [y, w_2] = a_2 a_1 z - a_1 a_2 z = 0, \]

so \( a_2 w_1 - a_1 w_2 \in Z(L) \), which is a contradiction. Consequently, \([W, W] = \text{span}\{z\}\) which is 1-dimensional, so \( W \) is a nilpotent Lie subalgebra of \( L \) such that \( b(W) = 1 \) by Theorem 3.2.1. By Theorem 3.2.6, \( W = \text{span}\{z, w_1, w_2', \ldots, w_{2k}', \ldots, w_{n-4}'\} \) such that \([w_i', w_j'] = z \) for all \( i = 1, 3, \ldots, 2k - 1 \) and \( Z(W) = \{z, w_{2k+1}', \ldots, w_{n-4}'\} \) where \( 2k \leq n - 4 \). Observe that for \( i = 1, 2, \ldots, 2k \), we have \([y, w_i'] = \alpha_i z \) where \( \alpha_i \in \mathbb{F} \). Let

\[ y' = y + \sum_{i=1}^{2k-1} \alpha_i w'_{i+1} - \sum_{i=2}^{2k} \alpha_i w'_{i-1}. \]

As a result, we have

\[ [y', w_i'] = \begin{cases} [y, w_i'] + [\alpha_i w'_{i+1}, w_i'] = \alpha_i z - \alpha_i z = 0 & \text{if } i \text{ is odd}, \\ [y, w_i'] - [\alpha_i w'_{i-1}, w_i'] = \alpha_i z - \alpha_i z = 0 & \text{if } i \text{ is even}. \end{cases} \]

Therefore \([y', w_i'] = 0\) for all \( i = 1, 2, \ldots, 2k \). Observe that

\[ [x, y'] = [x, y] = v \quad \text{and} \quad [y', v] = [y, v] = 0 \]

because \([x, W] = [v, W] = \{0\}\). Notice that \([y', W] \subseteq Z(L)\) since \([y, W], [W, W] \subseteq Z(L)\). By considering \( L = \text{span}\{x, y', z, v, w_1', w_2', \ldots, w_{2k}', \ldots, w_{n-4}'\} \), we know that \([x, w_j'] = [v, w_j'] = [w_i', w_j'] = 0 \) for all \( i = 1, 2, \ldots, n - 4 \) and \( j = 2k + 1, \ldots, n - 4 \). Since \( w_{2k+1}', \ldots, w_{n-4}' \notin Z(L) \), we get \([y', w_j'] = \beta_j z \) where \( \beta_j \in \mathbb{F} - \{0\} \) for all \( j = 2k + 1, \ldots, n - 4 \). If \( 2k + 2 \leq n - 4 \), then
\[ [y', w_{2k+1}] = \beta_{2k+1}z \text{ and } [y', w_{2k+2}] = \beta_{2k+2}z. \] Consequently, we obtain

\[
[y', \beta_{2k+2}w_{2k+1} - \beta_{2k+1}w_{2k+2}] = \beta_{2k+2}[y', w_{2k+1}] - \beta_{2k+1}[y', w_{2k+2}]
= \beta_{2k+2}\beta_{2k+1}z - \beta_{2k+1}\beta_{2k+2}z
= 0,
\]

so \( \beta_{2k+2}w_{2k+1} - \beta_{2k+1}w_{2k+2} \in Z(L) \) which is a contradiction. Hence we get \( 2k + 2 > n - 4 \), which implies \( 2k = n - 4 \) or \( 2k + 1 = n - 4 \). Then we consider the following two cases:

1. **Case I** : \( 2k = n - 4 \). Then \( n \) is even and \( L \) is isomorphic to \( \text{span}\{x, y', z, v, w_1', w_2', \ldots, w_{n-4}'\} \) such that \([x, y'] = v, [x, v] = z \) and \([w_i', w_{i+1}'] = z \) for all \( i = 1, 3, 5, \ldots, n - 5 \).

2. **Case II** : \( 2k + 1 = n - 4 \). Then \( n \) is odd. Since \( w_{n-4}' \notin Z(L) \), we get \([y', w_{n-4}'] = \beta_{n-4}z \) where \( \beta_{n-4} \in \mathbb{F} - \{0\} \). Let \( \bar{w}_{n-4} = \frac{w_{n-4}'}{\beta_{n-4}} \). Then we have

\[
[y', \bar{w}_{n-4}] = [y', \frac{w_{n-4}'}{\beta_{n-4}}] = \frac{1}{\beta_{n-4}}[y', w_{n-4}'] = \frac{1}{\beta_{n-4}}\beta_{n-4}z = z.
\]

Hence \( L \) is isomorphic to \( \text{span}\{x, y', z, v, w_1', w_2', \ldots, w_{n-5}', \bar{w}_{n-4}\} \) such that \([x, y'] = v, [x, v] = z, [y', \bar{w}_{n-4}] = z \) and \([w_i', w_{i+1}'] = z \) for all \( i = 1, 3, 5, \ldots, n - 6 \).

Since the result from the two cases above are complement to each other, we can modify our result as follows:

1. If \( n \) is even, then \( L \) is isomorphic to \( \text{span}\{x, y, z, v, w_1, w_2, \ldots, w_{n-4}\} \) such that \([x, y] = v, [x, v] = z \) and \([w_i, w_{i+1}] = z \) for all \( i = 1, 3, 5, \ldots, n - 5 \).

2. If \( n \) is odd, then \( L \) is isomorphic to \( \text{span}\{x, y, z, v, w_1, w_2, \ldots, w_{n-4}\} \) such that \([x, y] = v, [x, v] = z, [y, w_1] = z \) and \([w_i, w_{i+1}] = z \) for all \( i = 2, 4, 6, \ldots, n - 5 \).

\[ \square \]

### 5.4 Nilpotent Lie Algebras of Breadth 2 with \( \dim[L, L] = 2 \) and \( \dim Z(L) = 2 \)

To begin this section, we introduce the concept of component of a Lie algebra which we use throughout our classification process. For any finite dimensional Lie algebra \( L \), its center can be written as \( Z(L) = \text{span}\{z_1, z_2, \ldots, z_m\} \) for some \( m \in \mathbb{Z}_{\geq 0} \). Then we extend this basis to \( L = \text{span}\{x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_m\} \) where \( n \in \mathbb{Z}_{\geq 0} \). Thus \( \dim L = n + m \). We denote a subspace \( L' := \text{span}\{x_1, x_2, \ldots, x_n\} \subseteq L \).
**Definition 5.4.1.** Let \( L \) be a finite dimensional Lie algebra. A subspace \( M = \text{span}\{y_1, \ldots, y_k\} \subseteq L' \) is a component of \( L \) of dimension \( k \) if \( M + C_L(M) = L \).

**Proposition 5.4.2.** Let \( L \) be a finite dimensional Lie algebra and \( M \) be a component of \( L \). Then \( b(x) \geq 1 \) for all \( x \in M - \{0\} \).

**Proof.** Let \( L \) be a finite dimensional Lie algebra and \( M \) be a component of \( L \). Since \( Z(L) = \{x \in L \mid b(x) = 0\} \), we know that \( b(x) \geq 1 \) for all \( x \in L' - \{0\} \). Hence \( b(x) \geq 1 \) for all \( x \in M - \{0\} \) because \( M - \{0\} \subseteq L' - \{0\} \).

**Lemma 5.4.3.** Let \( L \) be a finite dimensional Lie algebra and \( M \) be a subspace of \( L' \) such that \( M + C_L(M) = L \). Then \( M \cap C_L(M) = \{0\} \).

**Proof.** Let \( L \) be a finite dimensional Lie algebra and \( M \) be a subspace of \( L' \) such that \( M + C_L(M) = L \). We write \( M = \text{span}\{u_1, u_2, \ldots, u_k\} \) for some \( k \in \mathbb{Z}_{\geq 0} \) and extend it to a basis \( \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_{n-k}, z_1, z_2, \ldots, z_m\} \) of \( L \) where \( Z(L) = \text{span}\{z_1, z_2, \ldots, z_m\} \). Let \( x \in M \cap C_L(M) \). Then \( x \) can be written as \( x = a_1u_1 + a_2u_2 + \ldots + a_ku_k \) where \( a_i \in \mathbb{F} \) for all \( i = 1, 2, \ldots, k \). Next we will claim that \( x \in Z(L) \). Let \( y \in L \). Since \( L = M + C_L(M) \), \( y \) can be written as \( y = y_M + c_M \) where \( y_M \in M \) and \( c_M \in C_L(M) \). Because \( x \in M \cap C_L(M) \), we have \( [x, y] = [x, y_M + c_M] = [x, y_M] + [x, c_M] = 0 \). Since \( y \in L \) is arbitrary, we obtain \( x \in Z(L) \). Therefore \( x \) can also be written as \( x = b_1z_1 + b_2z_2 + \ldots + b mz_m \) where \( b_j \in \mathbb{F} \) for all \( j = 1, 2, \ldots, m \). Consequently, we obtain

\[
0 = x - x = (a_1u_1 + a_2u_2 + \ldots + a_ku_k) - (b_1z_1 + b_2z_2 + \ldots + b mz_m) = a_1u_1 + a_2u_2 + \ldots + a_ku_k - b_1z_1 - b_2z_2 - \ldots - b mz_m.
\]

Since \( \{u_1, u_2, \ldots, u_k, z_1, z_2, \ldots, z_m\} \) is linearly independent, \( a_i = b_j = 0 \) for all \( i = 1, 2, \ldots, k \) and \( j = 1, 2, \ldots, m \). Hence \( x = a_1u_1 + a_2u_2 + \ldots + a_ku_k = 0 \), so \( M \cap C_L(M) = \{0\} \).

By using previous lemma, we can develop our definition of component to be direct summand instead of normal summand.

**Definition 5.4.4.** Let \( L \) be a finite dimensional Lie algebra. A subspace \( M = \text{span}\{y_1, \ldots, y_k\} \subseteq L' \) is a component of \( L \) of dimension \( k \) if \( M \oplus C_L(M) = L \).

**Remark.** Let \( L \) be a finite dimensional Lie algebra. Then \( L' \) is the largest component of \( L \).

**Proposition 5.4.5.** Let \( L \) be a finite dimensional Lie algebra and \( M \) be a component of \( L \) of dimension \( k \). Then \( k \geq 2 \).
Proof. Let \( L \) be a finite dimensional Lie algebra and \( M \) be a component of \( L \) of dimension \( k \). Let \( x \in M - \{0\} \subseteq L - \{0\} \). Then \( b(x) \geq 1 \) by Proposition 5.4.2. Thus there exists \( y \in L - \{0\} \) such that \([x, y] \neq 0\), so \( y \notin C_L(M) \). Since \( L = M \oplus C_L(M) \), we write \( y = y_M + c_M \) where \( y_M \in M \) and \( c_M \in C_L(M) \). Note that \( y_M \neq 0 \) because \( y \notin C_L(M) \). Moreover, we have

\[
[x, y_M] = [x, y] = [x, c_M] = [x, y] \neq 0.
\]

As a result, we consider \( y_M \in M \) and notice that \( \{x, y_M\} \subseteq M \) is linearly independent because \([x, y_M] \neq 0\). Hence we have \( k \geq 2 \).

Next, we define the reducibility of component. Note that a component is called \textit{irreducible} if it is not reducible.

**Definition 5.4.6.** Let \( L \) be a finite dimensional Lie algebra and \( M \) be a component of \( L \). Then \( M \) is said to be \textit{reducible} if there exist components \( M_1 \) and \( M_2 \) such that \( M = M_1 \oplus M_2 \).

**Remark.** Let \( L \) be a finite dimensional Lie algebra and \( M = M_1 \oplus M_2 \) be a reducible component of \( L \). Then \( M_1 \subseteq C_L(M_2) \) and \( M_2 \subseteq C_L(M_1) \).

By Proposition 5.4.5, the smallest component is 2-dimensional. Thus we easily get the following corollary.

**Corollary 5.4.7.** Let \( L \) be a finite dimensional Lie algebra and \( M \) be a component of \( L \) of dimension 2 or 3. Then \( M \) is irreducible.

**Theorem 5.4.8.** Let \( L \) be a finite dimensional Lie algebra and \( M \) be a component of \( L \) of dimension \( k \geq 2 \). Then for any \( x \in M - \{0\}, 1 \leq b(x) \leq k - 1 \).

Proof. Let \( L \) be a finite dimensional Lie algebra and \( M \) be a component of \( L \) of dimension \( k \geq 2 \). By Proposition 5.4.2, \( b(x) \geq 1 \) for all \( x \in M - \{0\} \). Suppose that there exists \( x \in M - \{0\} \) such that \( b(x) \geq k \). Since \( M \oplus C_L(M) = L \), without loss of generality, there exist \( y_1, y_2, \ldots, y_k \in M - \{0\} \) such that \([x, y_i] = z_i \) for all \( i = 1, 2, \ldots, k \) where \( \{z_1, z_2, \ldots, z_k\} \) is linearly independent. Next, we will show that \( \{x, y_1, y_2, \ldots, y_k\} \subseteq M \) is linearly independent. Let \( a, a_1, a_2, \ldots, a_k \in \mathbb{F} \) be such that \( ax + a_1y_1 + a_2y_2 + \ldots + a_ky_k = 0 \). Then we have

\[
0 = [x, ax + a_1y_1 + a_2y_2 + \ldots + a_ky_k] = a[x, x] + a_1[x, y_1] + a_2[x, y_2] + \ldots + a_k[x, y_k] = a_1z_1 + a_2z_2 + \ldots + a_kz_k.
\]

Since \( \{z_1, z_2, \ldots, z_k\} \) is linearly independent, \( a_1 = a_2 = \ldots = a_k = 0 \), which also implies \( a = 0 \). Hence \( \{x, y_1, y_2, \ldots, y_k\} \subseteq M \) is linearly independent, which is a contradiction. Consequently, \( 1 \leq b(x) \leq k - 1 \) for any \( x \in M - \{0\} \).
Corollary 5.4.9. Let $L$ be a finite dimensional Lie algebra and $M$ be a component of $L$ of dimension 2. Then $b(x) = 1$ for all $x \in M - \{0\}$.

By using previous corollary, we can identify the structure of component of dimension 2 as we prove in next theorem.

Theorem 5.4.10. Let $L$ be a finite dimensional Lie algebra and $M$ be a component of $L$ of dimension 2. Then $M = \text{span}\{x_1, x_2\}$ such that $[x_1, x_2] \neq 0$. In particular, $[M, L] = \text{span}\{[x_1, x_2]\}$.

Proof. Let $L$ be a finite dimensional Lie algebra and $M$ be a component of $L$ of dimension 2. Let $x_1 \in M - \{0\}$. By Corollary 5.4.9, $b(x_1) = 1$. Since $L = M \oplus C_L(M)$, without loss of generality, there exists $x_2 \in M - \{0\}$ such that $[x_1, x_2] \neq 0$. We know that $x_2 \notin \text{span}\{x_1\}$, so $M = \text{span}\{x_1, x_2\}$. To show that $[M, L]$ is 1-dimensional, let $x \in M$ and $y \in L$. Since $L = M \oplus C_L(M)$, $x$ and $y$ can be written as $x = a_1x_1 + a_2x_2$ and $y = b_1x_1 + b_2x_2 + c$ where $a_i, b_i \in \mathbb{F}$ for $i = 1, 2$ and $c \in C_L(M)$. Note that $[x_1, c] = [x_2, c] = 0$ because $c \in C_L(M)$. Then we obtain

$$
[x, y] = [a_1x_1 + a_2x_2, b_1x_1 + b_2x_2 + c] \\
= a_1b_1[x_1, x_1] + a_1b_2[x_1, x_2] + a_1[x_1, c] + a_2b_1[x_2, x_1] + a_2b_2[x_2, x_2] + a_2[x_2, c] \\
= a_1b_2[x_1, x_2] + a_2b_1[x_2, x_1] \\
= (a_1b_2 - a_2b_1)[x_1, x_2] \\
\in \text{span}\{[x_1, x_2]\}.
$$

Since $x \in M$ and $y \in L$ are arbitrary, we have $[M, L] = \text{span}\{[x_1, x_2]\}$.

Next theorem clarify the picture of reducible component. We simply need to find a smaller part of component in order to tell that it is reducible.

Theorem 5.4.11. Let $L$ be a finite dimensional Lie algebra and $M$ be a component of $L$ of dimension $k \geq 4$. Suppose that there is a proper subspace $M_1 \subseteq M$ such that $M_1 + C_L(M_1) = L$. Then $M$ is reducible. In particular, $M = M_1 \oplus M_2$ where $M_2 \subseteq C_L(M_1)$ is a component spanned by basis of $M$ extended from $M_1$.

Proof. Let $L$ be a finite dimensional Lie algebra and $M$ be a component of $L$ of dimension $k \geq 4$. Suppose that there is a proper subspace $M_1 \subseteq M$ such that $M_1 + C_L(M_1) = L$. By Lemma 5.4.3, we have $M_1 \cap C_L(M_1) = \{0\}$, so $M_1 \oplus C_L(M_1) = L$. Thus $M_1$ is a component of $L$. Assume that $M_1 = \text{span}\{x_1, x_2, \ldots, x_t\}$ for some $t < k$. Since $L = M_1 \oplus C_L(M_1)$, we extend this basis to $M = \text{span}\{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_s\}$ such that $y_1, y_2, \ldots, y_s \in C_L(M_1)$ where $s + t = k$. Let $M_2 = \text{span}\{y_1, y_2, \ldots, y_s\} \subseteq C_L(M_1)$. Then $M = M_1 \oplus M_2$. Since $M_2 \subseteq C_L(M_1)$, we have
Thus $M_1 \subseteq C_L(M_2)$. On the other hand, we also have $C_L(M) \subseteq C_L(M_2)$ because $M_2 \subseteq M$. Similarly, we get $C_L(M) \subseteq C_L(M_1)$, so $M_1 \cap C_L(M) \subseteq M_1 \cap C_L(M_1) = \{0\}$. Consequently, we have $M_1 \oplus C_L(M) \subseteq C_L(M_2)$ and

$$L = M \oplus C_L(M) = (M_1 \oplus M_2) \oplus C_L(M) = M_2 \oplus (M_1 \oplus C_L(M)) \subseteq M_2 + C_L(M_2).$$

Thus $M_2 + C_L(M_2) = L$. Again, by Lemma 5.4.3, $M_2 \cap C_L(M_2) = \{0\}$, so $M_2 \oplus C_L(M_2) = L$. As a result, $M_2$ is a component of $L$ of dimension $s$. Hence $M = M_1 \oplus M_2$ is reducible. \hfill \Box

**Corollary 5.4.12.** Let $L$ be a finite dimensional Lie algebra and $M$ be an irreducible component of $L$ of dimension $k \geq 4$. Then for any proper subspace $M' \subseteq M$, $M' + C_L(M')$ is a proper subspace of $L$.

**Theorem 5.4.13.** Let $L$ be a finite dimensional Lie algebra and $M$ be an irreducible component of $L$. Then for any proper subspace $M' \subseteq M$, there exists $x \in M - M'$ such that $x \notin C_L(M')$.

**Proof.** Let $L$ be a finite dimensional Lie algebra and $M$ be an irreducible component of $L$. Let $M'$ be a proper subspace of $M$. Suppose that for any $x \in M - M'$, $x \in C_L(M')$. Then we have $M - M' \subseteq C_L(M')$. Since $M' \subseteq M$, $C_L(M) \subseteq C_L(M')$. Next, we will show that $M' + C_L(M') = L$. Suppose that $M' = \text{span}\{x_1, x_2, \ldots, x_t\}$ for some $t \geq 1$. Then we extend this basis to $M = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_s\}$ for some $s \geq 1$. Let $y \in L$. Since $M + C_L(M) = L$, $y$ can be written as $y = a_1x_1 + a_2x_2 + \ldots + a_t x_t + b_1 y_1 + b_2 y_2 + \ldots + b_s y_s + c$ where $a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_s \in \mathbb{F}$ and $c \in C_L(M)$. We observe that

$$y = a_1x_1 + a_2x_2 + \ldots + a_t x_t + b_1 y_1 + b_2 y_2 + \ldots + b_s y_s + c$$

$$= (a_1x_1 + a_2x_2 + \ldots + a_t x_t) + (b_1 y_1 + b_2 y_2 + \ldots + b_s y_s) + c$$

$$\in M' + (M - M') + C_L(M)$$

$$\in M' + C_L(M').$$

Therefore $M' + C_L(M') = L$. By Theorem 5.4.11, $M$ is reducible, which is a contradiction. Hence there exists $x \in M - M'$ such that $x \notin C_L(M')$. \hfill \Box

**Theorem 5.4.14.** Let $L$ be a finite dimensional Lie algebra and $M$ be an irreducible component of $L$ of dimension $k \geq 3$. Then there exists $x \in M - \{0\}$ such that $b(x) > 1$.

**Proof.** Let $L$ be a finite dimensional Lie algebra and $M$ be an irreducible component of $L$ of dimension $k \geq 3$. Suppose that $b(x) = 1$ for every $x \in M - \{0\}$. Let $x_1 \in M - \{0\}$. Since $L = M \oplus C_L(M)$, without loss of generality, there exists $x_2 \in M - \{0\}$ such that $[x_1, x_2] \neq 0$. Because $x_2 \in M - \{0\}$, $b(x_2) = 1$. Note that $x_2 \notin \text{span}\{x_1\}$, so $\{x_1, x_2\}$ is linearly independent. Let $M_1 := \text{span}\{x_1, x_2\}$. Since $b(x_1) = b(x_2) = 1$, by rank-nullity theorem, we know that
nullity $ad_{x_i} = \dim L - \text{rank } ad_{x_i} = \dim L - 1$ for $i = 1, 2$. Therefore $\ker ad_{x_1} \neq \ker ad_{x_2}$ but they are both $(\dim L - 1)$-dimensional. Thus $C_L(M_1) = \ker ad_{x_1} \cap \ker ad_{x_2}$ is $(\dim L - 2)$-dimensional. Next we will claim that $M_1 \cap C_L(M_1) = \{0\}$. Let $x \in M_1 \cap C_L(M_1)$. Then we have $x = a_1x_1 + a_2x_2$ for some $a_1, a_2 \in \mathbb{F}$. Since $x \in C_L(M_1)$, $[x_i, x] = 0$ for $i = 1, 2$. Consequently, we have

$$0 = [x_1, x] = [x_1, a_1x_1 + a_2x_2] = a_1[x_1, x_1] + a_2[x_1, x_2] = a_2[x_1, x_2],$$
$$0 = [x_2, x] = [x_2, a_1x_1 + a_2x_2] = a_1[x_2, x_1] + a_2[x_2, x_2] = -a_1[x_1, x_2],$$

so $a_1 = a_2 = 0$. Thus $M_1 \cap C_L(M_1) = \{0\}$. By counting dimension, $M_1 \oplus C_L(M_1) = L$. Hence $M_1 \subseteq M$ is a component of $L$ of dimension 2. If $k \geq 4$, then by Theorem 5.4.11, $M$ is reducible, which is a contradiction. Next, we assume that $k = 3$. Since $L = M_1 \oplus C_L(M_1)$, we let $0 \neq y \in M \cap C_L(M_1)$. Then $M = \text{span}\{x_1, x_2, y\}$. We will claim that $y \in Z(L)$. Let $x \in L$. Since $L = M + C_L(M)$, $x$ can be written as $x = a_1x_1 + a_2x_2 + by + c$ where $a_1, a_2, b \in \mathbb{F}$ and $c \in C_L(M)$. Then $[y, c] = 0$ because $y \in M$. Moreover, $[y, x_1] = [y, x_2] = 0$ since $y \in C_L(M_1)$. Therefore we have

$$[y, x] = [y, a_1x_1 + a_2x_2 + by + c] = a_1[y, x_1] + a_2[y, x_2] + b[y, y] + [y, c] = 0.$$

Thus $y \in Z(L)$, which is a contradiction. Hence there exist $x \in M - \{0\}$ such that $b(x) > 1$. \hfill \Box

Next theorem gives us the structure of component of dimension 3. Furthermore, we also obtain the classification of nilpotent Lie algebras $L$ of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and $\dim L = 5$ as the upcoming corollary.

**Theorem 5.4.15.** Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Let $M$ be a component of $L$ of dimension 3. Then $M = \text{span}\{x_1, x_2, x_3\}$ such that $[x_1, x_2] = z_1$, $[x_1, x_3] = z_2$ and $[x_1, x_3] = 0$ where $Z(L) = \text{span}\{z_1, z_2\}$.

**Proof.** Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Let $M$ be a component of $L$ of dimension 3. By Theorem 5.4.14, there exists $x_1 \in M - \{0\}$ such that $b(x_1) > 1$. By Theorem 5.4.8, we have $1 < b(x_1) \leq 3 - 1 = 2$, so $b(x_1) = 2$. Since $M \oplus C_L(M) = L$, without loss of generality, there exist $x'_2, x'_3 \in M - \{0\}$ such that $[x_1, x'_2] = z_1$ and $[x_1, x'_3] = z_2$ where $\{z_1, z_2\}$ is linearly independent. Since $Z(L) = [L, L]$ are 2-dimensional, we get $Z(L) = \text{span}\{z_1, z_2\}$. Next, we observe $[x'_2, x'_3] \in [L, L] = Z(L)$. Then there exist $a_1, a_2 \in \mathbb{F}$ such that $[x'_2, x'_3] = a_1z_1 + a_2z_2$. Let $x_2 = x'_2 - a_2x_1$ and $x_3 = x'_3 + a_1x_1$. Then we have

$$[x_1, x_2] = [x_1, x'_2 - a_2x_1] = [x_1, x'_2] - a_2[x_1, x_1] = z_1,$$

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Let $Z$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and $\dim L = 5$. Then $L = \text{span}\{x_1, x_2, x_3, z_1, z_2\}$ such that $\{x_1, x_2\} = z_1$, $\{x_1, x_3\} = z_2$ and $\{x_1, x_3\} = 0$ where $Z(L) = \text{span}\{z_1, z_2\}$. \hfill $\square$

**Corollary 5.4.16.** Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and $\dim L = 5$. Then $L = \text{span}\{x_1, x_2, x_3, z_1, z_2\}$ such that $\{x_1, x_2\} = z_1$, $\{x_1, x_3\} = z_2$ and $\{x_1, x_3\} = 0$ where $Z(L) = \text{span}\{z_1, z_2\}$.

Next, we provide a definition of standard $n$-dimensional subspace of a component and its properties.

**Definition 5.4.17.** Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and $M$ be an irreducible component of $L$. For $n \geq 2$, define an $n$-dimensional subspace $M_n := \text{span}\{x_1, x_2, \ldots, x_n\} \subseteq M$ such that

$$b(x_1) = 1, \quad b(x_2) = b(x_3) = \ldots = b(x_{n-1}) = 2, \quad b(x_n) \geq 1$$

and

$$[x_i, x_{i+1}] = \begin{cases} 
    z_1 & \text{if } i \text{ is odd} \\
    z_2 & \text{if } i \text{ is even}
\end{cases}$$

where $i \in \{1, 2, \ldots, n - 1\}$ and $Z(L) = \text{span}\{z_1, z_2\}$.

**Proposition 5.4.18.** Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and $M$ be an irreducible component of $L$. Then $M_n \cap C_L(M_{n-1}) = \{0\}$ for all $n \geq 3$.

**Proof.** Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and $M$ be an irreducible component of $L$. Let $n \geq 3$ and $a_1, a_2, \ldots, a_n \in \mathbb{F}$ be such that

$$x = a_1x_1 + a_2x_2 + \ldots + a_nx_n \in M_n \cap C_L(M_{n-1}) = \text{span}\{x_1, x_2, \ldots, x_n\} \cap C_L(M_{n-1}).$$
Since $x \in C_L(M_{n-1}) = \bigcap_{i=1}^{n-1} \ker ad_{x_i}$, we have $[x_i, x] = 0$ for all $i = 1, 2, \ldots, n-1$. As a result, for any $i \in \{2, 3, \ldots, n-1\}$, we have

$$0 = [x_i, x] = [x_i, a_1x_1 + a_2x_2 + \ldots + a_nx_n] = a_1[x_i, x_1] + a_2[x_i, x_2] + \ldots + a_n[x_i, x_n] = a_{i-1}[x_i, x_{i-1}] + a_{i+1}[x_i, x_{i+1}]
$$

Thus $a_i = 0$ for all $i = 1, 2, \ldots, n$. Hence $x = 0$, so $M_n \cap C_L(M_{n-1}) = \{0\}$ for all $n \geq 3$.

**Corollary 5.4.19.** Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and $M$ be an irreducible component of $L$. Then $M_n \cap C_L(M_n) = \{0\}$ for all $n \geq 2$.

**Proof.** Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional and $M$ be an irreducible component of $L$. Let $n \geq 2$. For $n = 2$, we let $a_1, a_2 \in \mathbb{F}$ be such that

$$x = a_1x_1 + a_2x_2 \in M_2 \cap C_L(M_2) = \text{span}\{x_1, x_2\} \cap C_L(M_2).$$

Since $x \in C_L(M_2) = \ker ad_{x_1} \cap \ker ad_{x_2}$, we have

$$0 = [x_1, x] = [x_1, a_1x_1 + a_2x_2] = a_1[x_1, x_1] + a_2[x_1, x_2] = a_2z_1,$$

$$0 = [x_2, x] = [x_2, a_1x_1 + a_2x_2] = a_1[x_2, x_1] + a_2[x_2, x_2] = a_1(-z_1).$$

Thus $a_1 = a_2 = 0$, so $x = 0$. Hence $M_2 \cap C_L(M_2) = \{0\}$. Suppose that $n \geq 3$. By Proposition 5.4.18, $M_n \cap C_L(M_{n-1}) = \{0\}$, so we have

$$M_n \cap C_L(M_n) = M_n \cap (\ker ad_{x_n} \cap C_L(M_{n-1}))$$

$$= \ker ad_{x_n} \cap (M_n \cap C_L(M_{n-1}))$$

$$= \ker ad_{x_n} \cap \{0\}$$

$$= \{0\}.$$

Therefore $M_n \cap C_L(M_n) = \{0\}$ for all $n \geq 3$. Hence $M_n \cap C_L(M_n) = \{0\}$ for all $n \geq 2$. 

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Theorem 5.4.20. Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Let $M$ be an irreducible component of $L$ and $M_n$ be a subspace of $M$ for $n \geq 2$. Then

1. $L = \ker ad_{x_1} \oplus \text{span}\{x_2\}$

2. $L = \ker ad_{x_i} \oplus \text{span}\{x_{i-1}, x_{i+1}\}$ for all $i = 2, 3, \ldots, n - 1$

3. $L = \begin{cases} 
\ker ad_{x_n} \oplus \text{span}\{x_{n-1}\} & \text{if } b(x_n) = 1 \\
\ker ad_{x_n} \oplus \text{span}\{x_{n-1}, x\} & \text{if } b(x_n) = 2
\end{cases}$

where $Z(L) = \text{span}\{[x_{n-1}, x_n], [x_n, x]\}$.

Proof. Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Let $M$ be an irreducible component of $L$ and $M_n$ be a subspace of $M$ for $n \geq 2$.

1. First, we observe $x_1 \in M_n$. Note that $b(x_1) = 1$ and $[x_1, x_2] = z_1 \neq 0$. By rank-nullity theorem, we get nullity $ad_{x_1} = \dim L - \rank ad_{x_1} = \dim L - 1$. Since $x_2 \notin \ker ad_{x_1}$, we have $L = \ker ad_{x_1} \oplus \text{span}\{x_2\}$.

2. Let $i \in \{2, 3, \ldots, n - 1\}$. Then we have $b(x_i) = 2$. Without loss of generality, we suppose that $i$ is even so that $[x_{i-1}, x_i] = z_1$ and $[x_i, x_{i+1}] = z_2$. To show that $L = \ker ad_{x_i} + \text{span}\{x_{i-1}, x_{i+1}\}$, let $y \in L$. If $y \in \ker ad_{x_i}$, then $y = y + 0 \in \ker ad_{x_i} + \text{span}\{x_{i-1}, x_{i+1}\}$. Assume that $y \notin \ker ad_{x_i}$. Then $[x_i, y] = a_1z_1 + a_2z_2$ for some $a_1, a_2 \in \mathbb{F}$, so we get

$$[x_i, y + a_1x_{i-1} - a_2x_{i+1}] = [x_i, y] + a_1[x_i, x_{i-1}] - a_2[x_i, x_{i+1}] = (a_1z_1 + a_2z_2) + a_1(-z_1) - a_2z_2 = 0,$$

so $c := y + a_1x_{i-1} - a_2x_{i+1} \in \ker ad_{x_i}$. As a result, we have

$$y = c - a_1x_{i-1} + a_2x_{i+1} \in \ker ad_{x_i} + \text{span}\{x_{i-1}, x_{i+1}\}.$$

Hence $L = \ker ad_{x_i} + \text{span}\{x_{i-1}, x_{i+1}\}$. Since nullity $ad_{x_i} = \dim L - \rank ad_{x_i} = \dim L - 2$, by counting dimension, we also know that $\ker ad_{x_i} \cap \text{span}\{x_{i-1}, x_{i+1}\} = \{0\}$. Hence $L = \ker ad_{x_i} \oplus \text{span}\{x_{i-1}, x_{i+1}\}$.

3. First, we observe $x_n \in M_n$. Without loss of generality, we assume that $n$ is even so that $[x_{n-1}, x_n] = z_1$. Suppose that $b(x_n) = 1$. By rank-nullity theorem, nullity $ad_{x_n} = \ldots$
Let \( \text{dim } L - \text{rank } \text{ad} x_n = \text{dim } L - 1 \). Since \( x_{n-1} \notin \ker \text{ad} x_n \), we have \( L = \ker \text{ad} x_n \oplus \text{span}\{x_{n-1}\} \).

Next, we assume that \( b(x_n) = 2 \). Then there exists \( x \in L \) such that \([x_n, x] = a_1z_1 + a_2z_2 \) where \( a_1, a_2 \in F \) and \( a_2 \neq 0 \). Next, we will claim that \( L = \ker \text{ad} x_n + \text{span}\{x_{n-1}, x\} \). Let \( y \in L \). If \( y \in \ker \text{ad} x_n \), then we get \( y = y + 0 \in \ker \text{ad} x_n + \text{span}\{x_{n-1}, x\} \). Suppose that \( y \notin \ker \text{ad} x_n \). Then \([x, y] = b_1z_1 + b_2z_2 \) for some \( b_1, b_2 \in F \). Therefore we have

\[
[x, y + (b_1 - \frac{b_2}{a_2}a_1)x_{n-1} - \frac{b_2}{a_2}x] = [x, y] + (b_1 - \frac{b_2}{a_2}a_1)[x_n, x_{n-1}] - \frac{b_2}{a_2}[x_n, x]
\]

\[
= (b_1z_1 + b_2z_2) + (b_1 - \frac{b_2}{a_2}a_1)(-z_1) - \frac{b_2}{a_2}(a_1z_1 + a_2z_2)
\]

\[
= (b_1 - b_1 + \frac{b_2}{a_2}a_1 - \frac{b_2}{a_2}a_1)z_1 + (b_2 - \frac{b_2}{a_2}a_2)z_2
\]

\[
= 0.
\]

Thus \( c := y + (b_1 - \frac{b_2}{a_2}a_1)x_{n-1} - \frac{b_2}{a_2}x \in \ker \text{ad} x_n \), so we have

\[
y = c - (b_1 - \frac{b_2}{a_2}a_1)x_{n-1} + \frac{b_2}{a_2}x \in \ker \text{ad} x_n + \text{span}\{x_{n-1}, x\}.
\]

Hence \( L = \ker \text{ad} x_n + \text{span}\{x_{n-1}, x\} \). Since nullity \( \text{ad} x_n = \text{dim } L - \text{rank } \text{ad} x_n = \text{dim } L - 2 \), \( \ker \text{ad} x_n \cap \text{span}\{x_{n-1}, x\} = \{0\} \) by counting dimension. Consequently, \( L = \ker \text{ad} x_n \oplus \text{span}\{x_{n-1}, x\} \). Additionally, since \([x_n, x] = a_1z_1 + a_2z_2 \notin \text{span}\{z_1\} = \text{span}\{[x_{n-1}, x_n]\}, \) we get \( Z(L) = \text{span}\{[x_{n-1}, x_n], [x_n, x]\} \).

\[\square\]

From now on, we are going to identify the structure of component of dimension 4 by constructing a standard subspace inside it as the following 2 theorems.

**Theorem 5.4.21.** Let \( L \) be a finite dimensional nilpotent Lie algebra of breadth 2 such that \( Z(L) = [L, L] \) are 2-dimensional. Let \( M \) be an irreducible component of \( L \) of dimension 4. Suppose that \( M_3 \subseteq M \). Then \( b(x_3) = 2 \).

**Proof.** Let \( L \) be a finite dimensional nilpotent Lie algebra of breadth 2 such that \( Z(L) = [L, L] \) are 2-dimensional. Let \( M \) be an irreducible component of \( L \) of dimension 4. Suppose that \( M_3 \subseteq M \) and \( b(x_3) = 1 \). Then we have \( M_3 = \text{span}\{x_1, x_2, x_3\} \) such that \( b(x_1) = 1 = b(x_3) \), \( b(x_2) = 2 \) and \([x_1, x_2] = z_1, [x_2, x_3] = z_2, [x_1, x_3] = 0 \) where \( Z(L) = \text{span}\{z_1, z_2\} \). Therefore \( \text{im } \text{ad} x_1 = \text{span}\{z_1\}, \text{im } \text{ad} x_2 = Z(L) \) and \( \text{im } \text{ad} x_3 = \text{span}\{z_2\} \).

Next, we will show that \( M_3 \) is a component of \( L \). Let \( x \in L \). If \( x \in C_L(M_3) \), then \( x = 0 + x \in M_3 + C_L(M_3) \). Assume that \( x \notin C_L(M_3) \). Then we have \([x_1, x] = az_1, [x_2, x] = b_1z_1 + b_2z_2 \) and
\( [x_3, x] = cz_2 \) where \( a, b_1, b_2, c \in \mathbb{F} \). Let \( y = x + b_1x_1 - ax_2 - b_2x_3 \). Therefore we get

\[
[x_1, y] = [x_1, x + b_1x_1 - ax_2 - b_2x_3] \\
= [x_1, x] + b_1[x_1, x_1] - a[x_1, x_2] - b_2[x_1, x_3] \\
= az_1 - az_1 \\
= 0,
\]

\[
[x_2, y] = [x_2, x + b_1x_1 - ax_2 - b_2x_3] \\
= [x_2, x] + b_1[x_2, x_1] - a[x_2, x_2] - b_2[x_2, x_3] \\
= (b_1z_1 + b_2z_2) + b_1(-z_1) - b_2z_2 \\
= 0,
\]

\[
[x_3, y] = [x_3, x + b_1x_1 - ax_2 - b_2x_3] \\
= [x_3, x] + b_1[x_3, x_1] - a[x_3, x_2] - b_2[x_3, x_3] \\
= cz_2 - a(-z_2) \\
= (c + a)z_2 \\
=: c'z_2
\]

where \( c' = c + a \). If \( c' = 0 \), then \( y \in C_L(M_3) \), so we have \( x = (-b_1x_1 + ax_2 + b_2x_3) + y \in M_3 + C_L(M_3) \). Suppose that \( c' \neq 0 \). Since \( M \) is a component of \( L \), we write \( y \in L = M \oplus C_L(M) \) as \( y = y_M + c_M \) where \( y_M \in M \) and \( c_M \in C_L(M) \). Next, we will claim that \( y_M \notin M_3 \). Suppose that \( y_M \in M_3 \). Then \( y_M = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 \) where \( \alpha_i \in \mathbb{F} \) for \( i = 1, 2, 3 \). Therefore we have

\[
x = y - b_1x_1 + ax_2 + b_2x_3 \\
= (y_M + c_M) - b_1x_1 + ax_2 + b_2x_3 \\
= (\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + c_M) - b_1x_1 + ax_2 + b_2x_3 \\
= (\alpha_1 - b_1)x_1 + (\alpha_2 + a)x_2 + (\alpha_3 + b_2)x_3 + c_M.
\]

As a result, we obtain

\[
az_1 = [x_1, x] \\
= [x_1, (\alpha_1 - b_1)x_1 + (\alpha_2 + a)x_2 + (\alpha_3 + b_2)x_3 + c_M] \\
= (\alpha_1 - b_1)[x_1, x_1] + (\alpha_2 + a)[x_1, x_2] + (\alpha_3 + b_2)[x_1, x_3] + [x_1, c_M] \\
= (\alpha_2 + a)z_1,
\]

\[
cz_2 = [x_3, x] \\
= [x_3, (\alpha_1 - b_1)x_1 + (\alpha_2 + a)x_2 + (\alpha_3 + b_2)x_3 + c_M]
\]

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Hence \( a = \alpha_2 + a \) and \( c = -(\alpha_2 + a) \), so \( \alpha_2 = 0 \) and \( c = -a \). Thus \( c' = c + a = 0 \) which is a contradiction. Consequently, \( y_M \notin M_3 \), so \( y_M \in M - M_3 \). Let \( \frac{y_M}{y_M} = \frac{y_M}{y_M} \). Then \( y_M' \in M - M_3 \), so \( M = \text{span}\{x_1, x_2, x_3, y_M'\} \). Moreover, we observe that

\[
\begin{align*}
[x_1, y_M] &= [x_1, y_M + c_M] = [x_1, y] = 0, \\
[x_2, y_M] &= [x_2, y_M + c_M] = [x_2, y] = 0, \\
[x_3, y_M] &= [x_3, y_M + c_M] = [x_3, y] = c'z_2.
\end{align*}
\]

Hence we have \([x_1, y_M'] = 0 = [x_2, y_M']\) and \([x_3, y_M'] = z_2\). Observe that

\[
\begin{align*}
[x_1, x_2 + y_M'] &= [x_1, x_2] + [x_1, y_M'] = z_1, \\
[x_2, x_2 + y_M'] &= [x_2, x_2] + [x_2, y_M'] = 0, \\
[x_3, x_2 + y_M'] &= [x_3, x_2] + [x_3, y_M'] = -z_2 + z_2 = 0, \\
[y_M', x_2 + y_M'] &= [y_M', x_2] + [y_M', y_M'] = 0.
\end{align*}
\]

Since \( L = M \oplus C_L(M) = \text{span}\{x_1, x_2, x_3, y_M'\} \oplus C_L(M) \), we get \( \text{im} ad_{x_2+y_M'} = \text{span}\{z_1\} \), so \( b(x_2 + y_M') = 1 \). Additionally, we have

\[
\begin{align*}
\ker ad_{x_1} &= \text{span}\{x_1, x_3, y_M'\} \oplus C_L(M), \\
\ker ad_{x_2+y_M'} &= \text{span}\{x_2, x_3, y_M'\} \oplus C_L(M).
\end{align*}
\]

We consider \( M = \text{span}\{x_1, x_2 + y_M', x_3, y_M'\} \) and let \( M' = \text{span}\{x_1, x_2 + y_M'\} \subseteq M \). Then

\[
C_L(M') = \ker ad_{x_1} \cap \ker ad_{x_2+y_M'} = \text{span}\{x_3, y_M'\} \oplus C_L(M).
\]

Consequently, we have

\[
L = M \oplus C_L(M) = \text{span}\{x_1, x_2 + y_M', x_3, y_M'\} \oplus C_L(M) = \text{span}\{x_1, x_2 + y_M'\} \oplus \text{span}\{x_3, y_M'\} \oplus C_L(M) = M' \oplus C_L(M'),
\]

so \( M' \subseteq M \) is a component of \( L \) which contradicts irreducibility of \( M \). Hence \( x \in M_3 + C_L(M_3) \).
Since $x \in L$ is arbitrary, $L = M_3 + C_L(M_3)$. By Corollary 5.4.19, we know that $M_3 \cap C_L(M_3) = \{0\}$, so $L = M_3 \oplus C_L(M_3)$. Thus $M_3 \subseteq M$ is a component of $L$ which is also a contradiction. Hence $b(x_3) = 2$. 

**Theorem 5.4.22.** Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Let $M$ be an irreducible component of $L$ of dimension 4. Suppose that $M$ contains an element of breadth 1. Then $M = \text{span}\{x_1, x_2, x_3, x_4\}$ such that $[x_1, x_2] = z_1$, $[x_1, x_3] = z_2$ and $[x_3, x_4] = z_1$ where $Z(L) = \text{span}\{z_1, z_2\}$.

**Proof.** Let $L$ be a finite dimensional nilpotent Lie algebra of breadth 2 such that $Z(L) = [L, L]$ are 2-dimensional. Let $M$ be an irreducible component of $L$ of dimension 4. Suppose that $M$ contains an element of breadth 1, says $x \in M$ such that $b(x) = 1$. Let $u_1 = x$ and $M_1 = \text{span}\{u_1\} \subseteq M$. By Theorem 5.4.13, there exists $u_2 \in M - M_1$ such that $u_2 \notin C_L(M_1)$. Thus $[u_1, u_2] = z_1 \in Z(L) - \{0\}$.

Next, we let $M_2 = \text{span}\{u_1, u_2\} \subseteq M$. We will show that $b(u_2) = 2$. Suppose that $b(u_2) = 1$. By Theorem 5.4.20, we know that $L = \text{span}\{u_1\} \oplus \ker ad_{u_2} = \text{span}\{u_2\} \oplus \ker ad_{u_1}$. Since $u_2 \in \ker ad_{u_2}$, $L = \ker ad_{u_1} + \ker ad_{u_2}$, so we have

$$
\dim L = \dim(\ker ad_{u_1} + \ker ad_{u_2}) = \dim \ker ad_{u_1} + \dim \ker ad_{u_2} - \dim(\ker ad_{u_1} \cap \ker ad_{u_2}) = \text{nullity } ad_{u_1} + \text{nullity } ad_{u_2} - \dim C_L(M_2)
$$

$$
= (\dim L - \text{rank } ad_{u_1}) + (\dim L - \text{rank } ad_{u_2}) - \dim C_L(M_2)
$$

$$
= (\dim L - b(u_1)) + (\dim L - b(u_2)) - \dim C_L(M_2)
$$

$$
= (\dim L - 1) + (\dim L - 1) - \dim C_L(M_2)
$$

$$
= 2 \dim L - 2 - \dim C_L(M_2).
$$

Thus $\dim C_L(M_2) = \dim L - 2$. By Corollary 5.4.19, we get $M_2 \cap C_L(M_2) = \{0\}$. Since $\dim M_2 = 2$ we have $L = M_2 \oplus C_L(M_2)$, so $M_2 \subseteq M$ is a component of $L$ which is a contradiction. Therefore $b(u_2) = 2$.

Because $b(u_2) = 2$, there exists $u_3 \in M$ such that $[u_2, u_3] = z_2$ where $Z(L) = \text{span}\{z_1, z_2\}$. Note that $u_3 \notin M_2$ because $[u_2, M_2] = \text{span}\{z_1\}$. Since $b(u_1) = b(x) = 1$ and $\text{im } ad_{u_1} = \text{span}\{z_1\}$, we suppose that $[u_1, u_3] = az_1$ where $a \in F$. Let $v_1 = u_1$, $v_2 = u_2$ and $v_3 = u_3 - au_2$. Then we have

$$
[v_1, v_2] = [u_1, u_2] = z_1,
$$

$$
[v_2, v_3] = [u_2, u_3 - au_2] = [u_2, u_3] - a[u_2, u_2] = z_2,
$$

$$
[v_1, v_3] = [u_1, u_3 - au_2] = [u_1, u_3] - a[u_1, u_2] = az_1 - az_1 = 0.
$$

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Let $M_3 = \text{span}\{v_1, v_2, v_3\}$. Then $b(v_1) = b(x) = 1, b(v_2) = 2$ and $b(v_3) \geq 1$. By Theorem 5.4.21, we have $b(v_3) = 2$, so there exists $v_4 \in M$ such that $[v_3, v_4] = c_1 z_1 + c_2 z_2$ where $c_1, c_2 \in \mathbb{F}$ and $c_1 \neq 0$. Note that $v_4 \notin M_3$ because $[v_3, M_3] = \text{span}\{z_2\}$. Thus $M = \text{span}\{v_1, v_2, v_3, v_4\}$. Since $b(v_1) = 1$ and $b(v_2) = 2$, we assume that $[v_1, v_4] = a_1 z_1$ and $[v_2, v_4] = b_1 z_1 + b_2 z_2$ where $a_1, b_1, b_2 \in \mathbb{F}$. Let $y = v_4 + b_1 v_1 - a_1 v_2 - b_2 v_3$. Then $M = \text{span}\{v_1, v_2, v_3, y\}$ such that

\[
[v_1, y] = [v_1, v_4 + b_1 v_1 - a_1 v_2 - b_2 v_3] \\
\quad = [v_1, v_4] + b_1 [v_1, v_1] - a_1 [v_1, v_2] - b_2 [v_1, v_3] \\
\quad = a_1 z_1 - a_1 z_1 \\
\quad = 0,
\]

\[
[v_2, y] = [v_2, v_4 + b_1 v_1 - a_1 v_2 - b_2 v_3] \\
\quad = [v_2, v_4] + b_1 [v_2, v_1] - a_1 [v_2, v_2] - b_2 [v_2, v_3] \\
\quad = (b_1 z_1 + b_2 z_2) + b_1 (-z_1) - b_2 z_2 \\
\quad = 0,
\]

\[
[v_3, y] = [v_3, v_4 + b_1 v_1 - a_1 v_2 - b_2 v_3] \\
\quad = (c_1 z_1 + c_2 z_2) - a_1 (-z_2) \\
\quad = c_1 z_1 + (c_2 + a_1) z_2.
\]

Let $x_1 = v_1, x_2 = v_2, x_3 = v_3$ and $x_4 = \frac{y}{c_1}$. Hence $M = \text{span}\{x_1, x_2, x_3, x_4\}$ such that

\[
[x_1, x_2] = [v_1, v_2] = z_1, \\
[x_2, x_3] = [v_2, v_3] = z_2, \\
[x_3, x_4] = [v_3, \frac{y}{c_1}] = \frac{1}{c_1} [v_3, y] = \frac{1}{c_1} (c_1 z_1 + (c_2 + a_1) z_2) = z_1 + \frac{(c_2 + a_1)}{c_1} z_2 =: z_1 + \alpha z_2, \\
[x_1, x_3] = [v_1, v_3] = 0, \\
[x_1, x_4] = [v_1, \frac{y}{c_1}] = \frac{1}{c_1} [v_1, y] = 0, \\
[x_2, x_4] = [v_2, \frac{y}{c_1}] = \frac{1}{c_1} [v_2, y] = 0.
\]

where $\alpha := \frac{c_2 + a_1}{c_1} \in \mathbb{F}$ and $Z(L) = \text{span}\{z_1, z_2\}$. To show that $\alpha = 0$, we suppose that $\alpha \neq 0$. Let $s_1 = x_1, s_2 = \alpha x_2 + x_4, s_3 = \alpha x_3 - x_1$ and $s_4 = x_4$. Then we have $M = \text{span}\{s_1, s_2, s_3, s_4\}$ such that

\[
[s_1, s_2] = [x_1, \alpha x_2 + x_4] = \alpha [x_1, x_2] + [x_1, x_4] = \alpha z_1,
\]

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\[ s_3, s_4 = [ax_3 - x_1, x_4] = \alpha [x_3, x_4] - [x_1, x_4] = \alpha (z_1 + \alpha z_2), \]
\[ [s_1, s_3] = [x_1, \alpha x_3 - x_1] = \alpha [x_1, x_3] - [x_1, x_1] = 0, \]
\[ [s_1, s_4] = [x_1, x_4] = 0, \]
\[ [s_2, s_3] = [\alpha x_2 + x_4, \alpha x_3 - x_1] \]
\[ = \alpha^2 [x_2, x_3] - \alpha [x_2, x_1] + \alpha [x_4, x_3] - [x_4, x_1] \]
\[ = \alpha^2 z_2 + \alpha z_1 - \alpha (z_1 + \alpha z_2) \]
\[ = 0, \]
\[ [s_2, s_4] = [\alpha x_2 + x_4, x_4] = \alpha [x_2, x_4] + [x_4, x_4] = 0 \]

where \( Z(L) = \text{span}\{z_1, z_2\} = \text{span}\{\alpha z_1, \alpha (z_1 + \alpha z_2)\} \). Thus \( M = M' \oplus M'' \) where \( M' = \text{span}\{s_1, s_2\} \) and \( M'' = \text{span}\{s_3, s_4\} \) are components of \( L \). Hence \( M \) is reducible which is a contradiction. Consequently, we obtain \( \alpha = 0 \), so we get \( M = \text{span}\{x_1, x_2, x_3, x_4\} \) such that \( [x_1, x_2] = z_1, [x_1, x_3] = z_2 \) and \( [x_3, x_4] = z_1 \) where \( Z(L) = \text{span}\{z_1, z_2\} \).

At this point, we can classify nilpotent Lie algebra \( L \) of breadth 2 such that \( Z(L) = [L, L] \) and \( \dim L = 6 \) under the assumption that \( L \) has an element of breadth 1 as we show in the following corollary.

**Corollary 5.4.23.** Let \( L \) be a 6-dimensional nilpotent Lie algebra of breadth 2 such that \( Z(L) = [L, L] \) are 2-dimensional. Suppose that \( L \) contains an element of breadth 1. Then \( L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\} \) such that

1. \([x_1, x_2] = z_1, [x_2, x_3] = z_2 \) and \([x_3, x_4] = z_1 \)

or

2. \([x_1, x_2] = z_1 \) and \([x_3, x_4] = z_2 \)

where \( Z(L) = \text{span}\{z_1, z_2\} \).

**Proof.** Let \( L \) be a 6-dimensional nilpotent Lie algebra of breadth 2 such that \( Z(L) = [L, L] \) are 2-dimensional. Suppose that \( L \) contains an element of breadth 1. Since \( Z(L) \) is 2-dimensional, we have \( \dim L' = \dim L - \dim Z(L) = 6 - 2 = 4 \). Then we have \( L' \) is a component of \( L \) of dimension 4. Suppose that \( L' \) is irreducible. By Theorem 5.4.22, \( L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\} \) such that \([x_1, x_2] = z_1, [x_1, x_3] = z_2 \) and \([x_3, x_4] = z_1 \) where \( Z(L) = \text{span}\{z_1, z_2\} \).

On the other hand, we assume that \( L' \) is reducible. Then \( L' \) must be composed of 2 irreducible components of dimension 2. Since \( Z(L) = [L, L] \) are 2-dimensional, we obtain \( L' = \text{span}\{x_1, x_2\} \oplus \text{span}\{x_3, x_4\} \) such that \([x_1, x_2] = z_1 \) and \([x_3, x_4] = z_2 \) where \( Z(L) = \text{span}\{z_1, z_2\} \).
\[ [L, L] = \text{span}\{z_1, z_2\}. \] We also note that
\[
b(\alpha x_1 + \beta x_3) = b(\alpha x_1 + \beta x_4) = b(\alpha x_2 + \beta x_3) = b(\alpha x_2 + \beta x_4) = 2
\]
for any \( \alpha, \beta \in F \). In this case, it is not isomorphic to previous case because of the component property. Hence \( L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\} \) such that \([x_1, x_2] = z_1 \) and \([x_3, x_4] = z_2 \) where \( Z(L) = [L, L] = \text{span}\{z_1, z_2\}. \) \( \Box \)

In the next part, we prove that 6-dimensional nilpotent Lie algebra of breadth 2 such that \( Z(L) = [L, L] \) are 2-dimensional contains an element of breadth 1 if the underlying field is algebraically closed. Thus we get a complete classification if we consider Lie algebras over algebraically closed field.

**Lemma 5.4.24.** Let \( L \) be a 6-dimensional nilpotent Lie algebra of breadth 2 such that \( Z(L) = [L, L] \) are 2-dimensional. Suppose that for any \( x \in L - Z(L) \), \( b(x) = 2. \) Then \( L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\} \) such that \([x_1, x_2] = [x_3, x_4] = z_1 \), \([x_2, x_3] = z_2 \) and \([x_1, x_4] = \alpha z_2 \) where \( \alpha \neq 0. \)

**Proof.** Let \( L \) be a 6-dimensional nilpotent Lie algebra of breadth 2 such that \( Z(L) = [L, L] \) are 2-dimensional. Since \( Z(L) \) is 2-dimensional, we have \( \dim L' = \dim L - \dim Z(L) = 6 - 2 = 4. \) Then we have \( L' \) is a component of \( L \) of dimension 4. Suppose that for any \( x \in L - Z(L), \) \( b(x) = 2. \) Note that \( L' \subseteq L - Z(L), \) so \( b(x) = 2 \) for any \( x \in L' \) Let \( u_2 \in L' \). Then \( b(u_2) = 2, \) so there exist \( u_1, u_3 \in L' \) such that \([u_1, u_2] = w_1, \) \([u_2, u_3] = w_2 \) and \([u_1, u_3] = a_1 w_1 + a_2 u_2 \) where \( a_1, a_2 \in F \) and \( Z(L) = \text{span}\{w_1, w_2\}. \) Let \( u_1 = u_1 - a_2 u_2, \) \( u_2 = u_2, \) \( u_3 = u_3 - a_1 u_2 \) and \( M = \text{span}\{v_1, v_2, v_3\} \subseteq L'. \) Then we have
\[
[v_1, v_2] = [u_1 - a_2 u_2, u_2] = [u_1, u_2] - a_2[u_2, u_2] = w_1,
\]
\[
[v_2, v_3] = [u_2, u_3 - a_1 u_2] = [u_2, u_3] - a_1[u_2, u_2] = w_2,
\]
\[
[v_1, v_3] = [u_1 - a_2 u_2, u_3 - a_1 u_2]
\]
\[
= [u_1, u_3] - a_1[u_1, u_2] - a_2[u_2, u_3] + a_2a_1[u_2, u_2]
\]
\[
= (a_1 w_1 + a_2 w_2) - a_1 w_1 - a_2 w_2
\]
\[
= 0.
\]

Since \( L' \) is 4-dimensional, there exists \( v_4 \in L'. \) Then we write
\[
[v_1, v_4] = \alpha w_1 + \alpha_2 w_2,
\]
\[
[v_2, v_4] = \beta w_1 + \beta_2 w_2,
\]
\[
[v_3, v_4] = \gamma w_1 + \gamma_2 w_2
\]
where \( \alpha_i, \beta_i, \gamma_i \in \mathbb{F} \) for \( i = 1, 2 \). Since \( b(v_1) = b(v_3) = 2 \), we have \( \alpha_2, \gamma_1 \neq 0 \). Without loss of generality, we assume that \( \gamma_1 = 1 \). Then we have \( L' = \text{span}\{v_1, v_2, v_3, v_4\} \) such that

\[
\begin{align*}
[v_1, v_4] &= \alpha_1 w_1 + \alpha_2 w_2, \\
[v_2, v_4] &= \beta_1 w_1 + \beta_2 w_2, \\
[v_3, v_4] &= w_1 + \gamma_2 w_2
\end{align*}
\]

Let \( y_1 = v_1, y_2 = v_2, y_3 = v_3 \) and \( y_4 = v_4 + \beta_1 v_1 + \gamma_2 v_2 - \beta_2 v_3 \). Then we have \( L' = \text{span}\{y_1, y_2, y_3, y_4\} \) such that

\[
\begin{align*}
[y_1, y_2] &= [v_1, v_2] = w_1, \\
[y_1, y_3] &= [v_1, v_3] = 0, \\
[y_2, y_3] &= [v_2, v_3] = w_2, \\
[y_1, y_4] &= [v_1, v_4 + \beta_1 v_1 + \gamma_2 v_2 - \beta_2 v_3] \\
&= [v_1, v_4] + \beta_1[v_1, v_1] + \gamma_2[v_1, v_2] - \beta_2[v_1, v_3] \\
&= (\alpha_1 w_1 + \alpha_2 w_2) + \gamma_2 w_1 \\
&= (\alpha_1 + \gamma_2) w_1 + \alpha_2 w_2 \\
&= \delta w_1 + \alpha_2 w_2, \\
[y_2, y_4] &= [v_2, v_4 + \beta_1 v_1 + \gamma_2 v_2 - \beta_2 v_3] \\
&= [v_2, v_4] + \beta_1[v_2, v_1] + \gamma_2[v_2, v_2] - \beta_2[v_2, v_3] \\
&= (\beta_1 w_1 + \beta_2 w_2) + \beta_1(-w_1) - \beta_2 w_2 \\
&= 0, \\
[y_3, y_4] &= [v_3, v_4 + \beta_1 v_1 + \gamma_2 v_2 - \beta_2 v_3] \\
&= [v_3, v_4] + \beta_1[v_3, v_1] + \gamma_2[v_3, v_2] - \beta_2[v_3, v_3] \\
&= (w_1 + \gamma_2 w_2) + \gamma_2(-w_2) \\
&= w_1
\end{align*}
\]

where \( \delta = \alpha_1 + \gamma_2 \). Next, we let \( x_1 = y_1 - \frac{\delta}{2} y_3, x_2 = y_2, x_3 = y_3, x_4 = y_4 - \frac{\delta}{2} y_2, z_1 = w_1 + \frac{\delta}{2} w_2, z_2 = w_2 \) and \( \alpha = \alpha_2 - \frac{\delta^2}{4} \). Then \( L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\} \) such that

\[
\begin{align*}
[x_1, x_2] &= [y_1 - \frac{\delta}{2} y_3, y_2] = [y_1, y_2] - \frac{\delta}{2}[y_3, y_2] = w_1 + \frac{\delta}{2} w_2 = z_1, \\
[x_1, x_3] &= [y_1 - \frac{\delta}{2} y_3, y_3] = [y_1, y_3] - \frac{\delta}{2}[y_3, y_3] = 0, \\
[x_2, x_3] &= [y_2, y_3] = w_2 = z_2
\end{align*}
\]
Moreover, if \( \alpha = 0 \), then we have \( b(x_1) = b(x_4) = 1 \) which is a contradiction. Thus \( \alpha \neq 0 \). Hence

\[
L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\}
\]

such that \( [x_1, x_2] = [x_3, x_4] = z_1 \), \( [x_2, x_3] = z_2 \) and \( [x_1, x_4] = \alpha z_2 \)

where \( \alpha \neq 0 \).

**Theorem 5.4.25.** Let \( L \) be a 6-dimensional nilpotent Lie algebra of breadth 2 over an algebraically closed field such that \( Z(L) = [L, L] \) are 2-dimensional. Then \( L \) contains an element of breadth 1.

**Proof.** Let \( L \) be a 6-dimensional nilpotent Lie algebra of breadth 2 over an algebraically closed field such that \( Z(L) = [L, L] \) are 2-dimensional. Suppose that \( L \) does not contain an element of breadth 1. Then \( b(x) = 2 \) for any \( x \in L - Z(L) \). By Lemma 5.4.24, \( L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\} \) such that \( [x_1, x_2] = [x_3, x_4] = z_1 \), \( [x_2, x_3] = z_2 \) and \( [x_1, x_4] = \alpha z_2 \) where \( \alpha \neq 0 \). Since we consider \( L \) over algebraically closed field, \( \sqrt{-\alpha} \) exists. Consider \( y = \sqrt{-\alpha} x_2 + x_4 \). Then we have

\[
[y, x_1] = [\sqrt{-\alpha} x_2 + x_4, x_1] = \sqrt{-\alpha} [x_2, x_1] + [x_4, x_1] = \sqrt{-\alpha} (-z_1) = \sqrt{-\alpha} z_2 = -\sqrt{-\alpha} (z_1 - \sqrt{-\alpha} z_2),
\]
\[
[y, x_2] = [\sqrt{-\alpha} x_2 + x_4, x_2] = \sqrt{-\alpha} [x_2, x_2] + [x_4, x_2] = 0,
\]
\[
[y, x_3] = [\sqrt{-\alpha} x_2 + x_4, x_3] = \sqrt{-\alpha} [x_2, x_3] + [x_4, x_3] = \sqrt{-\alpha} (z_2) = z_1 = - (z_1 - \sqrt{-\alpha} z_2),
\]
\[
[y, x_4] = [\sqrt{-\alpha} x_2 + x_4, x_4] = -\sqrt{-\alpha} (x_2, x_4) + [x_4, x_4] = 0.
\]

Therefore \( \text{im } \text{ad}_y = \text{span}\{z_1 - \sqrt{-\alpha} z_2\} \), so \( b(y) = 1 \) which is a contradiction. Hence \( L \) contains an element of breadth 1.

**Corollary 5.4.26.** Let \( L \) be a 6-dimensional nilpotent Lie algebra of breadth 2 over an algebraically closed field such that \( Z(L) = [L, L] \) are 2-dimensional. Then \( L = \text{span}\{x_1, x_2, x_3, x_4, z_1, z_2\} \) such that
1. \([x_1, x_2] = z_1, [x_2, x_3] = z_2\) and \([x_3, x_4] = z_1\)

or

2. \([x_1, x_2] = z_1\) and \([x_3, x_4] = z_2\)

where \(\alpha \neq 0\) and \(Z(L) = \text{span}\{z_1, z_2\}\).

Note that for any odd dimensional nilpotent Lie algebra \(L\) of breadth 2 such that \(Z(L) = [L, L]\) is 2-dimensional, we do not need algebraically closed field to find an element of breadth 1.

**Proposition 5.4.27.** Let \(L\) be an odd dimensional nilpotent Lie algebra of breadth 2 such that \(Z(L) = [L, L]\) is 2-dimensional. Then there exists \(x \in L\) such that \(b(x) = 1\) and \(\text{im} \ ad_x = \text{span}\{z\}\) for any \(z \in Z(L)\).

**Proof.** Let \(L\) be an odd dimensional nilpotent Lie algebra of breadth 2 such that \(Z(L) = [L, L]\) is 2-dimensional. Then \(\dim L = n \in \mathbb{Z}_{>0}\) which is odd. Let \(z \in L\) and \(I = \text{span}\{z\}\). Then \(I\) is an ideal of \(L\) since \(z \in Z(L)\). Then we observe that \([L/I, L/I] = [L, L]/I\), so we have \(\dim[L/I, L/I] = \dim[L, L] - \dim I = 2 - 1 = 1\). By Theorem 3.2.1, we have \(b(L/I) = 1\). Note that \(\dim L/I = \dim L - \dim I = n - 1\), so by Theorem 3.2.6, there exists a basis

\[
S = \{v_1 + I, v_{-1} + I, v_2 + I, v_{-2} + I, \ldots, v_r + I, v_{-r} + I, z' + I, w'_1 + I, \ldots, w_{(n-1)-2r-1} + I\}
\]

for \(L/I\) such that

\[
[v_i + I, v_j + I] = \begin{cases} 
z' + I & \text{if } i = -j > 0, \\
-z' + I & \text{if } i = -j < 0, \\
I & \text{otherwise}
\end{cases}
\]

for every \(i, j \in \{\pm 1, \pm 2, \ldots, \pm r\}\) and \(z' + I, w'_1 + I, \ldots, w_{(n-1)-2r-1} + I \in Z(L/I)\) where \(r \in \mathbb{Z}_{>0}\). Observe that \((n - 1) - 2r - 1 = n - 2(r + 1) > 0\) because \(n\) is odd. Thus there exists \(w_1 + I \in Z(L/I)\), so \(\text{im} \ ad_{w_1 + I} = \text{span}\{I\}\). Therefore for any \(y \in L\), we have \([w_1, y] + I = [w_1 + I, y + I] = I\), so \([w_1, y] \in I\). Since \(y \in L\) is arbitrary, \(\text{im} \ ad_{w_1} \subseteq I = \text{span}\{z\}\). Next, we will claim that \(w_1 \notin Z(L)\). Suppose that \(w_1 \in Z(L)\). Then we obtain

\[
w_1 + I \in Z(L)/I = [L, L]/I = [L/I, L/I] = \text{span}\{z' + I\}.
\]

Thus \(w_1 + I = az' + I\) for some \(a \in \mathbb{F}\), so \(\{w_1 + I, z' + I\}\) is not linearly independent which is a contradiction. Hence \(w_1 \notin Z(L)\). Consequently, \(b(w_1) = 1\) and \(\text{im} \ ad_{w_1} = I = \text{span}\{z\}\). \(\square\)
REFERENCES


