ABSTRACT

YAN, XIANG. Conversion of Evanescent into Propagating Guided Waves in Plates. (Under the direction of Dr. Fuh-Gwo Yuan).

Current guided wave-based damage detection methods using advanced signal/image processing techniques can only process the damage information contained in the propagating guided waves since evanescent guided waves scattered from the damage decay exponentially away from it. The valuable subwavelength damage information concealed in the evanescent waves is therefore permanently lost in the damage image which may prevent the detection of the damage at the very early stage. Motivated by the possibility of achieving a ‘super resolution’ damage image by re-capturing subwavelength damage information contained in evanescent guided waves, this dissertation presents a feasibility study of converting evanescent into propagating guided waves in isotropic plates so that the far-field sensors can retrieve such valuable localized (subwavelength) damage information.

The properties of evanescent and propagating guided waves, i.e., Lamb waves and shear horizontal waves (SH waves), are first investigated by revisiting the formation of guided waves in isotropic plates and are characterized by the complex-valued dispersive curves. The important phase information and mode shapes for the propagating and evanescent guided waves are uncovered.

The differential and integral form complex reciprocity relations are derived for guided waves in isotropic plates as a theoretical base for solving waveguide mode analysis under either displacement or traction excitations in plates. Using the integral form reciprocity relations, the orthogonality relations for the guided wave modes are obtained and are discussed for propagating and evanescent modes, respectively. The power flow delivered to
an elastic plate can be completely described by a complex power flow, with real part representing the radiative power and imaginary part describing the reactive power. This type of separation of complex power flow is then proved to be related to the usual separation made on the basis of propagating and evanescent guided waves as propagating guided waves carry pure real power flow and the power flow for evanescent waves turns out to be pure imaginary.

The conversion of evanescent into propagating Lamb waves is substantiated by prescribing Lamb evanescent time-harmonic displacements or tractions distributed through a narrow aperture at the edge of a semi-infinite plate. First, a purely Lamb evanescent field can be generated when Lamb evanescent displacements or tractions are incident upon the entire edge of the plate. The amplitude coefficient of the propagating Lamb waves being converted from evanescent is determined through a theoretical formulation based on the complex reciprocity theorem via the finite element analysis (FEA) and is verified through the validation of a complex power conservation relation. Power conversion efficiency analysis shows that the propagating power converted from evanescent excitation is strongly frequency dependent and can be significant.

The conversion of evanescent into propagating SH waves is demonstrated by prescribing SH evanescent time-harmonic displacement excitations through a narrow aperture at the edge of a semi-infinite plate. The formula derivation of the SH wave is much simpler than the Lamb wave counterpart. A purely SH evanescent field can be generated when SH evanescent displacement is incident. The conversion process is then demonstrated by solving the SH wave governing equation using a finite element method. A theoretical model based on the complex reciprocity theorem with the aid of FEA is proposed to quantify
the amplitude coefficient of the converted propagating SH mode from evanescent and the amplitude coefficient is verified by proving a complex power conservation relation. Power conversion analysis shows that the conversion efficiency for propagating SH modes converted from evanescent varies dramatically as excitation frequency changes and can be significant.

The demonstration of converting evanescent into propagating guided waves in isotropic plates provides a theoretical foundation for subwavelength damage detection and imaging and may have potential application in structural health monitoring.
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Conversion of Evanescent into Propagating Guided Waves in Plates

by

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DEDICATION

To my dearest wife, Ni Sui,

my lovely daughter, Annie Yan,

and my devoting parents and parents-in-law.
BIOGRAPHY

Xiang Yan was born on July 27, 1986 in Guannan, Jiangsu Province, China. He graduated from Xinhai High School of Lianyungang in 2005 and in the same year, he was admitted to Jiangsu University, Zhenjiang, China, studying Civil Engineering. In 2009, he received his Bachelor degree at Jiangsu University. After graduation, he was admitted without examination to study in the Solid Mechanics PhD program and studied one-year in the same university.

In 2010, he was enrolled as a doctoral student in the Department of Mechanical and Aerospace Engineering at North Carolina State University under the direction of Dr. Fuh-Gwo Yuan. He conducted research on flexoelectric sensing and acoustic/elastic metamaterials for structural health monitoring (SHM) between 2011 and 2013. He started this fundamental dissertation work on the conversion of evanescent into propagating guided waves in plates towards the goal of achieving subwavelength damage detection and imaging for SHM in the summer of 2011 and was awarded a M.S. degree in 2012. He is being supported by the China Scholarship Council (CSC) and North Carolina State University.

The author married Ni Sui in 2010 and his daughter Annie Yan was born in 2012.
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**LIST OF SYMBOLS AND ABBREVIATIONS**

**Coordinate Systems and Locations**

\[ \mathbf{x} = (x_1, x_2, x_3) = (x, y, z) \]  
Rectangular coordinate system variables

**Greek Symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>(\xi)</td>
<td>Power conversion efficiency</td>
</tr>
<tr>
<td>(\lambda)</td>
<td>Wave length</td>
</tr>
<tr>
<td>(\mu)</td>
<td>Shear modulus</td>
</tr>
<tr>
<td>(\nu)</td>
<td>Poisson’s ratio</td>
</tr>
<tr>
<td>(\omega)</td>
<td>Angular frequency</td>
</tr>
<tr>
<td>(\omega_c)</td>
<td>Cut-off angular frequency</td>
</tr>
<tr>
<td>(\rho)</td>
<td>Density</td>
</tr>
<tr>
<td>(\phi)</td>
<td>Scalar potential</td>
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<td>(\psi)</td>
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<td>(\theta)</td>
<td>Phase surface</td>
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<tr>
<td>(\sigma)</td>
<td>Stress tensor</td>
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**Roman Symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>(A)</td>
<td>Amplitude coefficient</td>
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<td>(c_p)</td>
<td>Phase velocity</td>
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<td>Symbol</td>
<td>Description</td>
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<td>----------------------------------</td>
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<tr>
<td>$c_g$</td>
<td>Group velocity</td>
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<tr>
<td>$c_T$</td>
<td>Transverse velocity</td>
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<td>$E$</td>
<td>Young’s modulus</td>
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<tr>
<td>$f$</td>
<td>Body force</td>
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<tr>
<td>$f$</td>
<td>Frequency</td>
</tr>
<tr>
<td>$h$</td>
<td>Plate thickness</td>
</tr>
<tr>
<td>$\text{Im}$</td>
<td>Imaginary part of complex variables</td>
</tr>
<tr>
<td>$k$</td>
<td>Wavenumber</td>
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<tr>
<td>$L$</td>
<td>Length of the plate</td>
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<tr>
<td>$n$</td>
<td>Surface normal unit vector</td>
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<tr>
<td>$P$</td>
<td>Complex acoustic Poynting vector</td>
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<td>$EW$</td>
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### Modifiers

<table>
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<tr>
<td>*</td>
<td>Complex conjugate</td>
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<tr>
<td>\nabla</td>
<td>Del operator</td>
</tr>
<tr>
<td>T</td>
<td>Matrix transpose</td>
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If we knew what it was we were doing, it would not be called research, would it?

--- Albert Einstein
Chapter 1

Introduction
1.1 Research Motivation

Guided wave-based non-destructive inspection (NDI) and structural health monitoring (SHM) methods for damage detection have been attracting much attention mainly due to their capability of long-range and through-the-thickness interrogation of structures. The goal of NDI and SHM for damage detection is to acquire the location/size as well as the severity of the damage using ultrasonic waves, and thus the needs for accurate sizing and imaging of the damage has brought continuous interest for researchers. However, even with a number of well-developed advanced imaging processing techniques, obtaining a damage image that includes as much as details to be seen in a photograph-like manner is still an extravagant demand, and this is because the image resolution is always limited by the wavelength due to the well-known ‘diffraction limit’ (Abbe, 1873) that hinders the subwavelength imaging. Therefore, while the use of guided ultrasonic waves provides better penetration depth, the diffraction limit excludes the possibility of detecting subwavelength features of the damages. If there is a means to break the diffraction limit, it will be possible to resolve subwavelength characteristics of the damage and a super resolution damage image could be achieved.

In a common guided wave-based SHM setup for damage detection, omnidirectional circular piezoelectric wafers are often used to excite and sense the wave signals (as shown in Figure 1.1). Far-field sensors receive the signals scattered from damage or geometrical features and such scattered signals are usually in the form of propagating waves (PW) since the non-propagating or evanescent waves (EW) have already diminished. The scattered signals are then post processed using advanced signal processing techniques to obtain the
damage image which only includes the information of propagating waves and thus the image resolution is limited by the wavelength of the probing propagating waves so that subwavelength information about the damage cannot be resolved from the propagating waves. The reason that causes the loss of subwavelength information is because evanescent waves, which carry subwavelength information about the damage and only exist in the immediate vicinity of the damage, are produced when the waves scattered from the damage along with the propagating waves and these evanescent waves decay exponentially away from the damage. The underlying cause that prevents the damage image to be ‘super-resolved’ is the loss of the subwavelength information processed in the evanescent waves, and therefore, collecting or recapturing evanescent waves that can be included in the sensor signals is promising to get a ‘super resolution’ damage image.
As plate-like structures are often seen in a number of mechanical, aerospace and civil applications, detecting the damage at an early stage is critical for safe operation of these structures. A ‘super-resolved’ image can provide much more details about the damages so that they can be diagnosed at the incipient state. Motivated by the possibility of achieving a ‘super resolution’ damage image by re-attaining subwavelength damage information concealed in the evanescent waves, the goal of the dissertation work is to investigate the feasibility of converting evanescent into propagating guided waves in plate-like structures so that the sensors located in the far-field can retrieve such valuable localized (subwavelength) damage information. The work presented in this dissertation provides a theoretical foundation for the study of converting evanescent into propagating guided waves (including both Lamb waves and SH waves) in plates and may holds promise for the subwavelength damage imaging for plate-like structures.

1.2 Literature Review

1.2.1 Non-Propagation (Evanescent) Waves in Plates

The wave motions in the plate can be classified as two types of wave motions: plane strain and antiplane shear motions, where the plane strain case corresponding to the famous Lamb waves (L-SV waves) which are coupled longitudinal and shear vertical waves and the antiplane shear represents for SH waves. The propagation of elastic waves traveling in plate-like structures has been extensively studied since Horace Lamb published his pioneer work in (1917). It was Lamb who first initiated the study of the lowest propagating symmetric and
antisymmetric Lamb modes (real roots for Rayleigh Lamb equation). Following Lamb in the next half century, most of the efforts had been attempted on the characterization of higher order modes of propagating waves. It was not until (1955), the purely imaginary roots of the dispersion equation were obtained by Lyon. The presence of complex roots of the dispersion equation was demonstrated in the remarkable work done by Mindlin (1957). Shortly after that, a complete dispersion curves including real, imaginary, and complex wavenumbers is elaborated by Mindlin (1960). The amplitudes of Lamb waves with purely imaginary or complex wavenumbers exponentially decay away from the source, and these waves are usually referred to as non-propagating or evanescent Lamb waves. The propagation of SH waves is much simpler, since they are decoupled from the L-SV (Lamb) waves and have simple dispersion characteristics. A complete dispersion curves including pure imaginary and real wavenumbers of SH waves in plates can be found in many books (Achenbach, 1975; Graff, 1975; Miklowitz, 1978; Auld, 1990). The SH waves with pure imaginary wavenumbers which exhibit exponentially decay and thus are called non-propagating or evanescent SH waves.

It is well known that the reflection of guided waves from the free edge of a plate usually accompanied by the generation of evanescent guided waves to satisfy the traction-free boundary conditions. These localized non-propagating modes can solely exist in the close proximity of the plate edge upon reflection, and have been studied and reported quite extensively in the literature e.g., (Torvik, 1967; Gregory et al., 1983; Diligent et al., 2003; Morvan et al., 2003; Gunawan et al., 2004; Santhanam et al., 2013), where the work by
Torvik (1967) is highly recommended. The purpose of studying these localized modes is to obtain an accurate field distribution around the structural discontinuities when applying normal mode superstition techniques (e.g., Auld, 1976; Gunawan et al., 2004, Moreau, 2006). In addition, as guided waves are scattered from damage/cracks (scatters), the superposition of incident and scattered guided waves needs to satisfy the boundary condition on the top and bottom of the plate as well as the surface of the scatter. The existence of non-propagating modes (evanescent waves) in the scattered waves is a necessary condition to fulfill this requirement (Simonetti et al., 2005; An et al., 2013). After considering the non-propagating modes, a complete description of wave fields around the scatters can be obtained. Apart from the research related to the evanescent guided waves mentioned above, as far as we know, very few works (Diligent et al., 2003, Predoi, 2004; Simonetti et al., 2005) in the literature solely dedicated to study the property or gave some insights on the physical meaning of evanescent guided waves.

1.2.2 Retrieving Evanescent Waves Information for Subwavelength Imaging

Making a flawless image has been the dream of researchers for centuries. It is already known that evanescent waves that carry subwavelength features of the scatter decaying exponentially away from the scatter are permanently lost in the image, resulting in a ‘diffraction limited’ image. Synge (1928) proposed that by collecting the evanescent waves produced as waves scattered from the specimen, subwavelength information about the specimen could be resolved and thus can break the Rayleigh diffraction limit and achieve super-resolution image. This method requires a subwavelength aperture to scan within one
wavelength of the surface of the specimen. It was not until (1972) Ash and Nicholls experimentally proved Synge’s idea. In their experiment, two ‘object’ separated by $\lambda/60$ which broke the diffraction limit in the microwave regime. Since then this near-field scanning technique became the well-established field called ‘Near-field Scanning Optical Microscopy’ (NSOM) (Betzig et al., 1991; Lewis et al., 2001; Courjon, 2003) and has been widely used. Generally, there are three approaches (Courjon, 2003) that are available in NSOM: illumination mode, collection mode and apertureless mode. The physics behind all of the three approaches is that the evanescent waves produced at the surface of the specimen are converted to propagating waves due to the tunneling effect. With the converted propagating waves from evanescent, the localized information about the specimen is recovered and enables the realization of subwavelength resolution. Since NSOM uses point-by-point scanning, the collected signals still need to be post-processed in order to be imaged and thus it is time-consuming.

In the late 1990s, Pendry (2000) proposed a class of ‘superlenses’ to substantially amplify evanescent waves through a slab of negative refractive index material (NIM) (Veselago, 1968), compensating the loss of evanescent waves and thus capturing the information contained in the evanescent waves to restoring an image below the diffraction limit. The NIM is also termed as metamaterials (Wikipedia, 2014), a manmade composite through artificially design which exhibits unique properties that cannot be found in nature. The implementations of the superlenses idea have been realized in both electromagnetic field (e.g., Smith et al., 2000; Shelby et al., 2001) (the realization of negative permittivity $\varepsilon$ and/or
negative permeability $\mu$) and acoustic fields (e.g., Ambati et al., 2007; Jia et al., 2010; Park et al., 2011) (negative density $\rho$ and/or negative modulus $E$). The amplification of evanescent waves relies on the fact the NIM supports resonant surface waves in which evanescent waves can be effectively coupled into these surface modes and enhanced by their resonant nature when their wave vectors are matched (Zhang et al., 2008). As the resonance based NIM ‘superlenses’ is constructed using realistic materials, the intrinsic energy dissipation or loss hinders the resolution of a ‘perfect image’. Another drawback of ‘superlenses’ is that the distance between the slab and both the image and object, as well as the thickness of the slab is strictly restricted so as to obtain an efficient resonant enhancement of evanescent waves. Another type of superlens using photonic crystals (e.g. Luo et al., 2003) and phononic crystals (Sukhovich et al., 2009) which can exhibit negative refraction were proposed for subwavelength imaging. The coupling between incident evanescent waves and the slab mode of photonic crystals or phononic crystals leads to the amplification of evanescent waves. Those superlenses are only capable of projecting subwavelength image in the near field, as the evanescent waves will decay again away from the lenses.

In order to overcome the major limitation that a simple slab superlens can only be used at near field, several approaches were proposed after the concept of superlenses to achieve the subwavelength in far-field. A far-field superlens (FSL) (Durant et al., 2006) based on a silver slab was first proposed to project subwavelength image into the far-field. The principle of the FSL using a two-stage process: evanescent wave amplification via surface resonance, and, subsequently, conversion to propagating waves via coupling elements. Jacob (2006) and
Salandrino (2006) independently proposed that a cylindrical geometry anisotropic metamaterials with hyperbolic dispersion, termed as ‘hyperlens’, can allow image magnification for subwavelength imaging. When evanescent waves enter the anisotropic medium, the evanescent waves become propagating and can propagate through the hyperlens. The working mechanism (Lu et al., 2012) is essentially because the anisotropic medium with hyperbolic dispersion relation continuously compresses the large wave vectors of the original evanescent wave vectors in the hyperlens region. Since the transverse wave vectors are compressed to be sufficiently small, the propagating waves remain propagating as they propagate outside the anisotropic region and can propagate to the far-field. The concept of hyperlens and subwavelength imaging were successfully verified by experiments in the electromagnetic (e.g., Lee et al., 2007; Liu et al., 2007) and acoustic fields (e.g., Ao et al., 2008; Torrent et al., 2008; Li et al., 2009). Among these work, Lee (2011) proposed an elastic hyperlens using a near flat dispersion relation that can resolve two sources in a plate which is 0.45λ apart.

Simonetti (2006) from the NDE group at Imperial College, London proposed that by considering the multiple scattering (or higher-order) effects in the scattering problem two point-like scatters that are 1/3 λ in distance can be resolved and can lead to super resolution. Simonetti stated by accounting for the multiple scattering, evanescent waves with higher spatial frequency (greater than 2k) are encoded in the far-field and thus subwavelength (<1/2 λ) information can be retrieved. However, this approach does not rely on any imaging system being used in the near field as discussed above and they attribute the mechanism as the
interaction between the scatters that cause the evanescent waves to be encoded. Since the higher order effects in the scattering fields are difficult to be described, in other words, not all the evanescent modes can be encoded, this prevents a better subwavelength image to be obtained.

1.3 Objective and Outline

In view of the motivation and the literature review for the dissertation, conversion of evanescent into propagating guided waves in plates is probably the best way to achieve far-field subwavelength damage detection and imaging in plates. As the study on conversion of evanescent into propagating guided waves in plates is absent in literature, the objective of the dissertation is to give a comprehensive study on the evanescent guided waves and quantitatively investigate the conversion process for converting evanescent into propagating guided waves in isotropic plates:

- To characterize the properties of evanescent guided waves in isotropic plates;
- To separate the complex power flow into the plate on the basis of propagating and evanescent guided waves.
- To provide analytical model for the generation of evanescent guided waves in isotropic plates and verify using FEA.
- To propose an analytical model based on the complex reciprocity theorem via FEA to quantify the amplitude coefficient and the power conversion efficiency for the converted propagating guided waves from evanescent.
This study would provide a theoretical foundation for the future study of far-field subwavelength damage detection and imaging in plates.

In order to explore the feasibility of converting evanescent into propagating guided waves in plates, this dissertation is organized as follows:

Chapter 1 presents the motivation and problem statement, the literature review, objective and outlines of the dissertation work.

Chapter 2 introduces a complete theoretical description of guided waves propagation in plates. The bulk wave propagation in an infinite medium is first studied followed by the study of the two types of guided waves in plates: Lamb waves and SH waves. The properties of propagating and evanescent guided Lamb and SH waves are studied, respectively. The propagating and evanescent guided waves are first characterized from the complex dispersion curves. The phase information of propagating and evanescent guided waves is then investigated from the displacements and stresses distributions.

Chapter 3 describes the reciprocity, orthogonality relations and power flow for guided waves in plates. The generic reciprocity relation is first derived and is discussed for Lamb and SH waves, respectively. The orthogonality relations for Lamb waves and SH waves are then derived from the reciprocity relations. Finally, the power flows for propagating and evanescent guided waves in plates are discussed in detail.

Chapter 4 quantitatively investigates the conversion process of evanescent into propagating Lamb waves. Generation of evanescent Lamb waves is first studied both theoretically and numerically. The evanescent Lamb waves are generated by prescribing
evanescent displacement or tractions at the edge of a 2-D isotropic semi-infinite plate. A theoretical model based on the reciprocity theorem with the aid of finite element analysis (FEA) model is then used to investigate the conversion of evanescent into propagating Lamb waves. The conversion process is studied by imposing evanescent excitations through a narrow aperture at the edge of the 2-D isotropic semi-infinite plate. The amplitude coefficient of the converted propagating waves is determined from the theoretical model with the aid of FEA is validated by proving a complex power conservation relation. Finally, the propagating power conversion efficiency for Lamb waves is studied.

Chapter 5 quantitatively investigates the conversion process of evanescent into propagating SH waves. To model the SH waves propagation in two-dimensional, the commercially available software COMSOL is proposed to solve the governing equation for SH waves in plates by a finite element method. The generation of evanescent SH waves is demonstrated by prescribing evanescent SH displacement at the edge of a 2-D isotropic semi-infinite plate and the conversion process is studied by prescribing evanescent SH displacement through a narrow aperture at the edge. The amplitude coefficient of the converted propagating mode is determined from a theoretical model based on the complex reciprocity theorem via FEA and is verified by a complex power conservation. Finally, the propagating power conversion efficiency for SH waves is defined.

Chapter 6 summarizes the main results and contributions of the dissertation work and recommends some future research topics.
Chapter 2

Guided Waves in Plates
2.1 Introduction

The sensors are used to receive wave signals from the actuators from which the wave disturbance is generated. Such signals are then analyzed to extract information of the medium through which the wave propagates and reflected or scattered from the damages. It is therefore necessary that some background in different types of waves for different media must be acquired for complete understanding of various signal processing techniques.

If an elastic medium is under the action of an excitation varying in time, different types of traveling waves can arise in the medium. Each type of wave can transfer disturbances from one part of the medium to the other at a finite speed. The theory of stress waves of elastic media shows that in an infinite isotropic elastic solid an arbitrary disturbance is propagated by means of two types of bulk waves (Achenbach, 1975), longitudinal (P) and transverse, or shear (S), waves, each traveling with its own constant velocity. The transverse waves can be further classified as shear vertical (SV) and shear horizontal (SH) waves. Conventional ultrasonic nondestructive methods based on these waves have been used to inspect flaws with some success (Krautkramer et al., 1990 and Bray et al., 1992). Most inspection techniques, such as conventional ultrasonics or eddy currents, require a transducer to be scanned over each point of the structure which is to be inspected. This is a time-consuming process and cannot inspect in inaccessible areas.

The elastic waves can propagate only in a medium. A guided wave is one whose propagation is guided by a structural form or boundary. For thin plate-like structures, the longitudinal and vertical shear waves experience repeated reflections at the upper and lower
surfaces alternately and the resulting disturbance propagation from their mutual interference is guided by the plate surfaces and is directed along the plate. The plate-like structures can serve as waveguides. The guided wave can be modeled by imposing surface boundary conditions on the equations of motion and can effectively describe the wave behavior. However, this approach introduces dispersion phenomenon; that is, the velocity of propagation of a disturbance along the plate being a function of frequency or, equivalently, wavelength. Therefore the dispersion in the case of an elastic medium is simply an interference phenomenon, rather than a physical property of the material. Consequently, the shape of the signal of a wave packet (a short-time wave train) may vary with the distance and time of propagation.

An important feature of guided wave propagation is that there exist generally an infinite number of branches, each corresponding to a particular mode of wave propagation. This is in contrast to bulk waves which exhibit primarily three types of waves. In addition, at least two modes can co-exist at any given frequency. Mode conversion can in general occur at boundaries and at any other discontinuities such as defects, thus multi-mode waves need to be interpreted. Another key characteristic of guided waves is that they can propagate over very long distances because the structure, or waveguide, retains the energy by its surface boundaries and a large region can therefore be interrogated for each transducer position efficiently. Consequently, the use of guided waves in the ultrasonic nondestructive evaluation of structural components such as plate-like structures has received considerable attention over the past two decades (Rose, 1999). More recently, active diagnostic techniques are
being used by exciting ultrasonic signals from smart materials mounted on or embedded in the structure for detecting localized damage in plate-like structures. The development of diagnostic techniques require the study of complicated wave propagation phenomena and relies strongly on the use of predictive modeling tools to enable the best structural health monitoring strategies to be identified and their sensitivities to be evaluated.

In this Chapter, the wave propagation in an infinite elastic medium is first studied. Two bulk wave speeds are defined and three types of waves are identified. Then two types of guided waves, Lamb waves and shear horizontal waves, are discussed. The wave solutions satisfy the equations of motions and boundary conditions related to traction-free surfaces of the two parallel flat surfaces. The concept of a dispersion relation, giving \( \omega \) as a function \( k \) will be introduced. Since the propagating and non-propagating (evanescent) guided waves exhibit different behaviors, the properties of propagating and evanescent guided Lamb waves and SH waves are studied, respectively. The propagating and evanescent guided waves are first characterized from the complex dispersion curves. The phase information for propagating and evanescent guided waves is then investigated from the displacements and stresses distributions.

### 2.2 Bulk Waves

In the absence of body forces the components of the displacement vector in a homogeneous, isotropic, linearly elastic medium are governing by the following equation of motion that can be expressed by
\[ \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} = \rho \ddot{\mathbf{u}} \] 

(2.1)

where \( \mathbf{u} = (u_1, u_2, u_3) \) is the displacement in the medium at point \( \mathbf{x} = (x_1, x_2, x_3) \) and time \( t \). \( \rho \) is the density, \( \lambda \) and \( \mu \) are the Lame’s constant and shear modulus, respectively. In the Helmholtz decomposition, the displacement vector is expressed by the sum of a scalar potential \( \phi \) and a vector potential \( \psi \)

\[ \mathbf{u} = \nabla \phi + \nabla \times \psi \] 

(2.2)

Note that Eq. (2.2) relates three components of the displacement vector to four other functions: the scalar potential and the three components of the vector potential. This indicates that \( \phi \) and the components of \( \psi \) should be under an additional constraint condition. The condition \( \nabla \cdot \psi = 0 \) can provide the sufficient additional condition to uniquely determine the three components of \( \mathbf{u} \) from the four components of \( \phi \) and \( \psi \), but it is not a necessary condition. Some other relation between \( \phi \) and \( \psi \) must be specified if \( \nabla \cdot \psi = 0 \) is not used.

Substituting Eq. (2.2) into (2.1) yields

\[ \mu \nabla^2 [\nabla \phi + \nabla \times \psi] + (\lambda + \mu) \nabla \nabla \cdot [\nabla \phi + \nabla \times \psi] = \rho [\nabla \dot{\phi} + \nabla \times \dot{\psi}] \] 

(2.3)

Since \( \nabla \cdot \psi = \nabla^2 \phi \) and \( \nabla \cdot \psi = 0 \), the above equation upon rearranging terms leads to

\[ \nabla [(\lambda + 2 \mu) \nabla^2 \phi - \rho \ddot{\phi}] + \nabla \times [\mu \nabla^2 \psi - \rho \ddot{\psi}] = 0 \] 

(2.4)

Clearly the equation of motion is satisfied if the potentials displacement potentials \( \phi \) and \( \psi \) satisfy the uncoupled wave equations

\[ \nabla^2 \phi - \frac{1}{c_L^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \] 

(2.5a)
and

\[ \nabla^2 \psi - \frac{1}{c_T^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \]  

(2.5b)

where \( c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{E(1-v)}{[(1+v)(1-2v)\rho]}}, \) \( c_T = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E}{2(1+v)\rho}} \), and its ratio squared is defined as \( \alpha = c_L/c_T = \sqrt{\frac{\lambda + 2\mu}{\mu}} = \sqrt{\frac{2(1-v)}{(1-2v)}} \).

The scalar potential \( \phi \) represents longitudinal waves traveling with a wave speed \( c_L \) while the vector potential \( \psi \) denotes transverse waves traveling with a wave speed \( c_T \). Since Eq. (2.5) are simpler in form than the displacements equations of motion, elastic wave propagation problems in isotropic solids are usually approached by first expressing the displacement field in terms of these two potentials.

Both longitudinal and transverse waves are referred to as bulk waves since they propagate in the volume (bulk) of a solid. With the identity \( \nabla \cdot (\nabla \phi) = 0 \) and \( \nabla \cdot \nabla \times \psi = 0 \), the longitudinal (L) bulk waves are also called P-waves, pressure waves, compressional waves, dilatational waves, or irrotational waves. Transverse waves are often called shear (S) waves, distortional waves, equivoluminal waves, or rotational waves.

Since both bulk wave speeds in the equations are independent of frequency, harmonic wave solutions of the type given below represent non-dispersive waves. This is strictly true, however, if the solid is perfectly elastic so that no frequency dependent energy attenuation or dissipation mechanisms are present.
The stress wave field at large distances (compared to the dominant wavelength or smallest characteristic dimension in the medium) from an excitation source can, for many purposes, be quite accurately represented by a plane wave in a Cartesian coordinate system

\[ \mathbf{u} = \mathbf{a} f(\hat{k} \cdot \mathbf{x} / c_p - t) \]  

(2.6)

where \( \mathbf{a} \) is defined by the vector of the direction of motion (displacement) and \( \hat{k} \) is the unit vector (direction cosines) of the direction of wave propagation \( \mathbf{k}, \hat{k} = k \hat{k}, k = |\mathbf{k}| \) respectively, \( k \) is often called wavenumber, \( \mathbf{k} \) wavevector.

It is worth noting that \( \hat{k} \cdot \mathbf{x} = \text{constant} \) describes a plane normal to the unit propagation wave vector \( \hat{k} \) and the constant represents the distance of the plane from the origin (along the normal). If the position of the plane is associated with \( t = 0 \), then for \( t > 0 \), the plane moves with speed \( c_p \) in the direction of \( \hat{k} \). Hence the equation of the plane at time \( t \) is described by \( \hat{k} \cdot \mathbf{x} - c_p t = \text{constant} \), which is called a wavefront. Since at any instant of time the wave crests lie in parallel planes, the motion represented by Eq. (2.6) is called a train of plane waves. Substituting Eq. (2.6) into Eq. (2.1) yields

\[ \left[ \mu \mathbf{a} + (\lambda + \mu)(\mathbf{k} \cdot \mathbf{a})\mathbf{k} - \rho c_p^2 \mathbf{a} \right] f^{*}(\hat{k} \cdot \mathbf{x} / c_p - t) = 0 \]

Or

\[ \left[ \mu - \rho c_p^2 \right] \mathbf{a} + (\lambda + \mu)(\mathbf{k} \cdot \mathbf{a})\hat{k} = 0 \]  

(2.7)

Since \( \hat{k} \) and \( \mathbf{a} \) are two different vectors, Eq. (2.7) can be satisfied in two ways only

\[ \mathbf{a} \parallel \hat{k} \text{ or } \mathbf{a} \cdot \hat{k} = 0 \]  

(2.8)

(a) \( \mathbf{a} \parallel \hat{k} \)
Eq. (2.7) yields
\[ c_p = c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} \] (2.9)

In this case, the time-varying displacement is parallel to the direction of propagation and the wave is therefore called **bulk longitudinal wave**.

(b) \( \mathbf{a} \cdot \hat{k} = 0 \)

\[ c_p = c_T = \sqrt{\frac{\mu}{\rho}} \] (2.10)

In this case, the time-varying displacement is normal to the direction of propagation and the wave is therefore called **bulk transverse (shear) wave**. The displacement can have any direction in a plane normal to the direction of propagation. When the \( (x_1, x_2) \)-plane is normally chosen to contain the vector \( \hat{k} \), motions can be either in the \( (x_1, x_2) \)-plane or normal to the \( (x_1, x_2) \)-plane, i.e., along the \( x_3 \) direction. These transverse motions propagating at the same speed are called shear vertical (SV) and shear horizontal (SH) polarized waves, respectively. Note that to describe each type of waves, two constants (such as amplitude and phase) are needed to specify each wave type, in addition to the direction of propagation.

Harmonic waves are steady-state waves; waves which are not steady-state are said to be transient waves (pulses). However, the transient waves in linear elastic materials can be obtained by superimposing harmonic waves in Fourier integrals. A plane harmonic wave propagating with phase velocity \( c_p \) in a direction of wave vector \( \hat{k} \) is often convenient to represent the wave in a complex form

\[ \mathbf{u} = \mathbf{a} e^{i(\omega \hat{k} \cdot \mathbf{x} / c_p - \omega t)} = \mathbf{a} e^{i(k \cdot \mathbf{x} / c_p - \omega t)} \] (2.11)
where \( \mathbf{a} \) is a complex vector consisting of amplitude and phase angle, and \( c_p = \omega / k \). \( \omega \) is the angular (circular) frequency (radians per second) and \( k \) is the wavenumber.

In all linear operations on the complex wave form Eq. (2.11), the real part of the derived wave solutions is equal to the solutions of the same operations applied to the real part of the original wave form. The actual solution of Eq. (2.11) is

\[
\text{Re}(u) = \text{Re}\left\{ a \ e^{i[kx/c_p, - \omega t]} \right\} = |a| \cos[(k \cdot x - \omega t + \eta], \ \eta = \arg \mathbf{a}
\]

(2.12)

The quantity

\[
\theta(x,t) = k \cdot x - \omega t
\]

(2.13)
gives the relationship between \( x \) and \( t \), and is generally called the phase of the waves; it determines the position on the cycle between a crest, where \( u \) is maximum, and a trough, where \( u \) is a minimum. In this phase wave equation, phase surfaces \( \theta = \text{constant} \) are parallel planes. Hence, Eq. (2.12) represents a plane wave whose planes of constant phases are normal to the \( k \). The gradient of \( -\theta \) in space is the wavenumber \( k \), whose direction is normal to the planes and whose magnitude is the average number of crests per \( 2\pi \) units of distance in that direction. Similarly, \( \theta \) is the frequency \( \omega \), the average number of crests per \( 2\pi \) units of time.

\[
\frac{\partial \theta}{\partial x} = k, \ \ \frac{\partial \theta}{\partial t} = \omega
\]

(2.14)

where \( \omega \) and \( k \) are related to the period \( T \), the frequency (cycles per second or \( Hz \)), and the wavelength \( \lambda \) by \( \omega = 2\pi / T \), \( \omega = 2\pi f \), and \( k = 2\pi / \lambda \). From Eq. (2.14), wavenumber \( k \) which
is the $-\theta$ gradient in space can also be called *spatial frequency* which is analogy to the frequency which is the $-\theta$ gradient in time.

The wave motion is recognized from Eq. (2.12). Any particular phase surface is moving with normal velocity $\omega k$ in the direction of $k$. To emphasize the direction of the phase velocity, a phase velocity vector can be defined as

$$c_p = \frac{\omega}{k} \hat{k} \quad (2.15)$$

In terms of potentials, the harmonic wave solutions can be represented by

$$\phi = Ae^{i(kx-\omega t)} \quad (2.16)$$
$$\varphi = Be^{i(\kappa x-\omega t)} \quad (2.17)$$

where $A$ and $B$ are arbitrarily complex constants; $k$ and $\kappa$ are wave vectors; $k = |k|$, $\kappa = |\kappa|$, $c_L = \omega / k$, $c_T = \omega / \kappa$.

**2.3 Lamb Waves**

**2.3.1 Basic Equations**

For a plate bounded by the surfaces $z = \pm h/2$ and is of infinite extent in the $x$ and $y$ directions ($x = (x, y, z)$ is equivalent to $x = (x_1, x_2, x_3)$), the harmonic wave motion can be divided into two classes of wave motions: plane strain and antiplane shear (or shear horizontal) motions.
A shear horizontal (SH) wave can propagate alone in the $y$ direction and its polarization is unchanged on reflection or refraction. This is not the case for a longitudinal (L) or shear vertical (SV) wave – these waves are coupled at a surface. Thus, reflection at a free surface is associated with partial conversion, so that an incident L (SV) wave gives a reflected L (SV) wave and a converted SV (L) wave (Figure 2.1). The wave vectors of the SV and L partial waves must all have the same components $k$ along the $x$ direction, and the S and L partial waves consequently propagate at different angles to the $x$ direction for these modes, which are called *Lamb waves*. The analysis of wave propagation in a plate is thus more complicated for the polarized wave than for a SH wave.

![Figure 2.1 Lamb wave formation in an isotropic plate with free boundaries- pressure wave and shear vertical wave incident and reflection.](image)

For harmonic wave motion in plane strain in the ($x, z$)-plane of an elastic plate, the guided wave field can be represented by a standing wave in the $z$ direction and a propagating
wave in the \( x \) direction. Using the Helmholtz decomposition in plane strain in the \( y \) direction 
\((u_y=0, \ \partial / \partial x = 0)\), the displacement can be written as
\[
\begin{align*}
    u_x &= \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z} \\
    u_y &= \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial x}
\end{align*}
\]
(2.18)
where the subscript 2 from the function \( \psi_2 \) has been omitted for simplicity.

The in-plane stress components can be expressed by the potentials from Hooke’s law as
\[
\begin{align*}
    \sigma_x &= \lambda \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right) + 2\mu \frac{\partial u_z}{\partial x} = \lambda \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + 2\mu \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial z} \right) \\
    \sigma_z &= \lambda \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right) + 2\mu \frac{\partial u_z}{\partial z} = \lambda \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + 2\mu \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right) \\
    \tau_{xz} &= \mu \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right) = \mu \left( 2 \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \psi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right)
\end{align*}
\]
(2.19)

As discussed before, the potentials \( \phi \) and \( \psi \) satisfy wave equations
\[
\begin{align*}
    \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} &= \frac{1}{c_L^2} \frac{\partial^2 \phi}{\partial t^2} \\
    \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} &= \frac{1}{c_T^2} \frac{\partial^2 \psi}{\partial t^2}
\end{align*}
\]
(2.20)

The wave solution can be considered in the following complex form
\[
\begin{align*}
    \phi &= \Phi(z)e^{i(kx-\omega t)} \\
    \psi &= \Psi(z)e^{i(kx-\omega t)}
\end{align*}
\]
(2.21)
These functions represent waves traveling coherently in the $x_1$ direction with the same angular frequency $\omega$ and the same wavenumber $k$. Substituting Eq. (2.21) into (2.20) leads to

$$
\Phi(z) = A_1 \sin(pz) + A_2 \cos(pz) \\
\Psi(z) = B_1 \sin(qz) + B_2 \cos(qz)
$$

(2.22)

where

$$
p^2 = \frac{\omega^2}{c_L^2} - k^2, \quad q^2 = \frac{\omega^2}{c_T^2} - k^2
$$

(2.23)

$p$ and $q$ stands for the transverse wavenumber of the bulk pressure and shear wave, respectively. Inspection of the displacements from Eq. (2.18) by using Eq. (2.21) and (2.22) indicates the motion can be separated into symmetric and antisymmetric modes

(a) Symmetric modes, in which the longitudinal component is an even function of $z$ and the transverse component is an odd function of $z$.

(b) Antisymmetric modes, in which the longitudinal component is an odd function of $z$ and the transverse component is an even function of $z$.
Figure 2.2 Lamb waves. (a) Symmetric—the longitudinal displacement is symmetric and the transverse displacement is antisymmetric with respect to the mid-plane of the plate are opposite. (b) Antisymmetric—the transverse displacement is symmetric and the longitudinal displacement is antisymmetric with respect to the mid-plane of the plate.

The displacements associated with symmetric and antisymmetric Lamb waves are shown in Figure 2.2. The functions associated with the symmetric and antisymmetric modes are listed in the following

(a) Symmetric modes

\[ \Phi = A_2 \cos(pz) \]
\[ \Psi = B_1 \sin(qz) \]
\[ u_x = ikA_2 \cos(pz) + qB_1 \cos(qz) \]
\[ u_z = -pA_2 \sin(pz) - ikB_1 \sin(qz) \]
\[ \sigma_x = \mu[(2p^2 - k^2 - q^2)A_2 \cos(pz) + 2ikqB_1 \cos(qz)] \]
\[ \sigma_z = \mu[(k^2 - q^2)A_2 \cos(pz) - 2ikqB_1 \cos(qz)] \]
\[ \tau_{xz} = \mu[-2ikpA_2 \sin(pz) + (k^2 - q^2)B_1 \sin(qz)] \]
(b) Antisymmetric modes

\[ \Phi = A_1 \sin(pz) \]
\[ \Psi = B_2 \cos(qz) \]
\[ u_x = ikA_1 \sin(pz) - qB_2 \sin(qz) \]
\[ u_z = pA_1 \cos(pz) - ikB_2 \cos(qz) \]
\[ \sigma_z = \mu[(2p^2 - k^2 - q^2)A_1 \sin(pz) - 2ikB_2 \sin(qz)] \]
\[ \tau_{xz} = \mu[2ikpA_1 \cos(pz) + (k^2 - q^2)B_1 \cos(qz)] \]

where the term \( \exp[i(\kappa - \omega t)] \) has been dropped in the expression of displacements and stresses.

### 2.3.2 Lamb Wave Dispersion Relation

The dispersion relation can be obtained by imposing the traction-free boundary conditions at \( z = \pm h/2 \)

\[ \sigma_z = \tau_{xz} = 0 \]  

Substituting Eq. (2.26) into symmetric and antisymmetric modes in Eq. (2.24) and (2.25) gives two homogeneous equations for coefficients \( A \) and \( B \). A necessary and sufficient condition for the existence of a solution to these equations is that the determinant of the coefficients is zero. The implicit equation between \( k \) and \( \omega \) formed by setting the determinant to zero is usually called as dispersion relation. The dispersion relation for symmetric modes are given by

\[ (k^2 - q^2)^2 \cos(ph/2)\sin(qh/2) + 4k^2 pqs\sin(ph/2)\cos(qh/2) = 0 \]  

or

\[ (k^2 - q^2)^2 \cos(ph/2)\sin(qh/2) + 4k^2 pqs\sin(ph/2)\cos(qh/2) = 0 \]
\[
\frac{\tan(qh / 2)}{\tan(ph / 2)} = -\frac{4k^2 pq}{(q^2 - k^2)^2}
\] (2.27b)

For antisymmetric modes

\[
(k^2 - q^2)^2 \sin(ph / 2) \cos(qh / 2) + 4k^2 pq \cos(ph / 2) \sin(qh / 2) = 0
\] (2.28a)

or

\[
\frac{\tan(qh / 2)}{\tan(ph / 2)} = -\frac{(q^2 - k^2)^2}{4k^2 pq}
\] (2.28b)

The above two equations can be combined into one equation

\[
\frac{\omega^4}{c^4} = 4k^2 q^2 \left[ 1 - \frac{p}{q} \frac{\tan(ph / 2 + \gamma)}{\tan(qh / 2 + \gamma)} \right]
\] (2.29)

where \(\gamma = 0\) and \(\pi/2\) represent symmetric and antisymmetric Lamb wave modes, respectively. Eq. (2.29) is commonly known as Rayleigh-Lamb dispersion relation. The dispersion relation results in an infinite number of wave modes and Lamb waves dispersion curves will be plotted in next section.

### 2.3.3 Propagating and Evanescent Lamb Waves

The displacements for Lamb wave modes can be readily derived from Section 2.3.1 as

\[
u_x = AU(z) \ e^{i(kz-\omega t)}
\] (2.30a)

\[
u_z = AW(z) \ e^{i(kz-\omega t)}
\] (2.30b)

where \(A\) is amplitude coefficient and

\[
U(z) = q \left[ -\frac{2k^2}{k^2 - q^2} \frac{\cos(qh / 2 + \gamma)}{\cos(ph / 2 + \gamma)} \cos(pz + \gamma) + \cos(qz + \gamma) \right]
\] (2.31a)
The stress components can be obtained as

\[ \sigma_x = A t_x(z) e^{j(kx-\omega t)} \]  

(2.32a)

\[ \sigma_z = A t_z(z) e^{j(kz-\omega t)} \]  

(2.32b)

\[ \tau_{xz} = A t_{xz}(z) e^{j(kx-\omega t)} \]  

(2.32c)

where

\[
t_x(z) = 2i\mu k q \left[ \frac{2pq - q^2 - k^2}{k^2 - q^2} \frac{\cos(qh / 2 + \gamma)}{\cos(ph / 2 + \gamma)} \cos(pz + \gamma) + \cos(qz + \gamma) \right]
\]  

(2.33a)

\[
t_z(z) = 2i\mu k q \left[ \frac{\cos(qh / 2 + \gamma)}{\cos(ph / 2 + \gamma)} \cos(pz + \gamma) - \cos(qz + \gamma) \right]
\]  

(2.33b)

\[
t_{xz}(z) = \mu \left[ \frac{4k^2 pq}{k^2 - q^2} \frac{\cos(qh / 2 + \gamma)}{\cos(ph / 2 + \gamma)} \sin(pz + \gamma) + (k^2 - q^2) \sin(qz + \gamma) \right]
\]  

(2.33c)

From the displacements and stresses distributions of Lamb waves, i.e., Eq. (2.30) and Eq. (2.32), the value of wavenumber \( k \) (pure real, pure imaginary or complex) can determine a Lamb wave mode is \textit{propagating} or \textit{evanescent}. As the Lamb wave dispersion relation has been given in Eq. (2.29), the dispersion curves which give the relationship between frequency and wavenumber \( k \) can be obtained. To plot the dispersion curves, it is convenient to introduce the non-dimensional variables. The non-dimensional form of the dispersion relation gives
\[
\bar{\omega}^4 = 4k^2q^2 \left[ 1 - \frac{\bar{p} \tan(\bar{p}h/2 + \gamma)}{\bar{q} \tan(\bar{q}h/2 + \gamma)} \right]
\]  
(2.34)

where the non-dimensional variables are defined by

\[
\bar{\omega} = \frac{\omega h}{c_T}, \quad \bar{k} = kh, \quad \bar{p}^2 = (ph)^2 = \alpha^2 \bar{\omega}^2 - \bar{k}^2, \quad \bar{q}^2 = (qh)^2 = \bar{\omega}^2 \bar{k}^2
\]  
(2.35)

Recall \(\alpha = \frac{c_L}{c_T} = \sqrt{\frac{2(1-\nu)}{1-2\nu}}\). Clearly the non-dimensional dispersion curves in linear elastic isotropic medium are only functions of Poisson’s ratio. Since Eq. (2.34) is a transcendental equation and no analytical solution is available, a Muller numerical algorithm (Stoer, 1993) is used to solve the dispersion relation. A non-dimensional complex dispersion curves for an isotropic plate Poisson’s ratio 0.33 obtained from Eq. (2.34) is shown in Figure 2.3. For clarity, only lower- order wave modes are plotted. For completeness, the Lamb wave phase and group velocity dispersive curves are shown in Figure 2.4 and Figure 2.5.
Figure 2.3 Lamb waves dispersion curves including both propagating and evanescent wave modes for an isotropic plate with Poisson’s ratio 0.33.
Figure 2.4 Lamb waves phase velocity dispersive curves for propagating wave modes with Poisson’s ratio 0.33.
Figure 2.5 Lamb waves group velocity dispersive curves for propagating wave modes with Poisson’s ratio 0.33.

From the complex dispersion curve, the wavenumber $k$ may be real, pure imaginary or complex values.

Case 1: $k$ is pure real

From Eq. (2.30), the displacements for Lamb waves can be rewritten as $u = AUe^{(kx-\omega t)}$, where $u=(u_x, u_z)$ and $U = (U(z), W(z))$. When $k$ is pure real, the displacements are sinusoidal function for both space and time and thus Lamb waves with pure real wavenumbers are
propagating Lamb waves. From Figure 2.3, at any given frequency, there is a finite number of propagating Lamb modes.

Case 2: $k$ is pure imaginary and $k = ik_I (k_I > 0)$

$$u = A U e^{-k_I x} e^{-i \omega t}$$

(2.36)

It is seen from Eq. (2.36), the displacements exhibit exponentially decay from positive $x$ direction and the values of the wavenumbers determine the decay rate. The Lamb wave modes with pure imaginary wavenumbers are non-propagating (evanescent) Lamb waves. From Figure 2.3, at any given frequency, there is a finite number evanescent Lamb wave modes with pure imaginary wavenumbers.

Case 3: $k$ is complex and $k = k_R + ik_I (k_I > 0)$

$$u = A U e^{-k_I x} e^{(k_R x - i \omega t)}$$

(2.37)

Eq. (2.37) can be characterized as waves propagating with a sinusoidal variation described by the real part of the wavenumbers, modulated by an exponential decaying function controlled by the imaginary part of the wavenumbers. Therefore, the Lamb wave modes with complex wavenumbers are non-propagating (evanescent) Lamb waves as well. From Figure 2.3, at any given frequency, there are an infinite number of evanescent Lamb modes with complex wavenumbers. The non-propagating $A_1$ mode has the purely imaginary wavenumber which means that below the $A_1$ cutoff, the mode is characterized by the exponential decay. In addition, among all the non-propagating modes, the magnitude of the $A_1$ imaginary part is usually much smaller implying the slower decay rate.
From the dispersion curves, the Lamb waves are labeled as $S_0$, $S_1$, $S_2$... for symmetric modes and $A_0$, $A_1$, $A_2$... for antisymmetric modes. The transition frequency that connects propagating and evanescent Lamb modes are called a cut-off frequency. The cut-off frequency is associated with the lowest frequency at which propagating modes can exist in a mode; that is, the lowest frequency at which a given mode exist with a real-valued wavenumber. Below the cut-off frequencies, the Lamb modes are evanescent and thus evanescent fields are exponentially decaying fields that do not possess real power. The exponential decay associated with the mode is not associated with losses in the medium. Rather, the energy in the mode is stored in a region rather than being propagated freely. The power for propagating and evanescent Lamb waves in plates will be discussed in detail in the next Chapter.

Among all the Lamb wave modes, there exist two modes $S_0$ and $A_0$ that do not have a cut-off frequency. For these fundamental modes, $\omega$ approaches zero as $k$ approaches zero, and the waves are all propagating across the entire frequency range. For Lamb waves with cut-off frequencies, the cut-off frequency $\omega_c$ can be obtained by letting $k = 0$. For the symmetric modes, the traction-free boundary conditions, Eq. (2.27a), give

$$q^4 \cos(p h / 2) \sin(q h / 2) = 0$$  \hspace{1cm} (2.38)

For $q = 0$, $\omega_c = 0$. This is the fundamental $S_0$ mode. In the case of non-zero cut-off frequencies, two cases can be considered

(a) $\cos(p h / 2) = 0$

The solutions lead to
\[ \frac{ph}{2} = \frac{(2m+1)\pi}{2}, \quad m = 0, 1, 2, \ldots \]

or

\[ \frac{ph}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \]

Since \( p = \frac{\omega_c}{c_L}, \frac{\omega_h}{2} = \frac{\pi c_L}{2}, \frac{3\pi c_L}{2}, \frac{5\pi c_L}{2}, \ldots \)

Thus the symmetric modes may be named as \( S_{2m+1}, \quad m = 0, 1, 2 \ldots \) The first few nondimensional cut-off frequencies associated with the symmetric modes are listed as follows

<table>
<thead>
<tr>
<th>Mode</th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_{ch}/c_T )</td>
<td>( \alpha \pi )</td>
<td>( 3\alpha \pi )</td>
<td>( 5\alpha \pi )</td>
</tr>
</tbody>
</table>

(b) \( \sin(\frac{qh}{2}) = 0 \)

The solutions yield

\[ \frac{qh}{2} = n\pi, \quad n = 1, 2, 3, \ldots \]

Since \( q = \frac{\omega_c}{c_T}, \frac{\omega_h}{2} = \frac{\pi c_T}{2}, \frac{2\pi c_T}{2}, \frac{3\pi c_T}{2}, \ldots \) The symmetric modes may be named as \( S_{2n}, \quad n = 1, 2, 3, \ldots \) The first few non-dimensional cut-off frequencies associated with the symmetric modes are listed as follows

<table>
<thead>
<tr>
<th>Mode</th>
<th>( S_2 )</th>
<th>( S_4 )</th>
<th>( S_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_{ch}/c_T )</td>
<td>( 2\pi )</td>
<td>( 4\pi )</td>
<td>( 6\pi )</td>
</tr>
</tbody>
</table>

Following a similar procedure for antisymmetric modes, the traction free boundary conditions, Eq. (2.28a), give

\[ q^4 \sin(\frac{ph}{2})\cos(\frac{qh}{2}) = 0 \quad (2.39) \]
For \( q = 0 \), \( \omega_c = 0 \). This is the fundamental \( A_0 \) mode. In the case of non-zero cut-off frequencies, by letting \( \sin(\phi h / 2) = 0 \) or \( \cos(\phi h / 2) = 0 \), the first few non-dimensional cut-off frequencies associated with the antisymmetric modes are listed as follows

<table>
<thead>
<tr>
<th>Mode</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( A_3 )</th>
<th>( A_4 )</th>
<th>( A_5 )</th>
<th>( A_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_c )</td>
<td>( \pi )</td>
<td>( 2\pi )</td>
<td>( 3\pi )</td>
<td>( 4\pi )</td>
<td>( 5\pi )</td>
<td>( 6\pi )</td>
</tr>
</tbody>
</table>

Since the displacements and stresses distributions for Lamb waves in the plate are critical, it is necessary to obtain the phase information of displacements and stresses components for propagating and evanescent Lamb waves, respectively. The phase information of displacements and stresses components for propagating Lamb waves are first discussed. From the definition of \( \bar{p} \) and \( \bar{q} \) given by Eq. (2.35), the slowness curve for Lamb waves in an isotropic plate with thickness \( h \) can be obtained and are shown in Figure 2.6. From the slowness curves, the wavenumber \( \vec{k} = \vec{k}_x \) is always real for propagating modes, and thus the transverse wavenumber \( \vec{k}_z \), which is either \( \bar{p} \) or \( \bar{q} \) for P or SV waves can be pure real, and/or pure imaginary depending on whether the wave vectors for L and SV waves can locate on the slowness curve. There are three possible cases for \( \bar{p} \) and \( \bar{q} \) being pure real, and/or pure imaginary as shown in Figure 2.6 (a), (b) and (c). Therefore, the real (\( \omega \), \( \vec{k} \)) plane of the dispersion curves can be divided into three regions as shown in Figure 2.7 in which the displacements and stresses distributions for propagating waves may be different.
Figure 2.6 Slowness curves for Lamb waves in an isotropic plate of thickness $h$. (a) Both the wave vectors for S and L waves locate on the slowness curve, i.e., $\vec{p}$ and $\vec{q}$ are both real, (b) S wave’s wave vector locate on the slowness curve and L wave’s wave vector fails to locate on the slowness curve, i.e., $\vec{q}$ is real and $\vec{p}$ is imaginary, (c) both the wave vectors for S and L waves fail to locate on the slowness curve, i.e., $\vec{p}$ and $\vec{q}$ are both imaginary.
\( \vec{k}_x / \vec{\omega} \)

(a) \( \vec{k}_x < \vec{\omega} / \alpha \)
(\( \vec{p} \) and \( \vec{q} \) are real)

(b) \( \vec{\omega} / \alpha < \vec{k}_x < \vec{\omega} \)
(\( \vec{q} \) is real and \( \vec{p} \) is imaginary)

(c) \( \vec{k}_x > \vec{\omega} \)
(\( \vec{q} \) and \( \vec{p} \) are imaginary)

\( (a) \) \( (b) \) \( (c) \)
The non-dimensional displacements and stresses can be obtained from Eq. (2.30)- Eq. (2.33) as

$$\bar{u}_x = A\bar{U}(z) = Aq\left[-\frac{2k^2}{k^2 - q^2} \cos\left(\frac{q}{2 + \gamma}\right) \cos\left(\frac{p}{2 + \gamma}\right) + \cos\left(\frac{q}{2 + \gamma}\right)\right]$$  \hspace{1cm} (2.40)

$$\bar{u}_z = A\bar{W}(z) = -ikA\left[\frac{2pq}{k^2 - q^2} \cos\left(\frac{q}{2 + \gamma}\right) \sin\left(\frac{p}{2 + \gamma}\right) + \sin\left(\frac{q}{2 + \gamma}\right)\right]$$  \hspace{1cm} (2.41)

$$\bar{\sigma}_x(z) = A\bar{\tau}_x(z) = -2iA\bar{k}q\left[\frac{2p^2 - q^2 - k^2}{k^2 - q^2} \cos\left(\frac{q}{2 + \gamma}\right) \cos\left(\frac{p}{2 + \gamma}\right) + \cos\left(\frac{q}{2 + \gamma}\right)\right]$$  \hspace{1cm} (2.42)

$$\bar{\sigma}_z(z) = A\bar{\tau}_z(z) = 2A\bar{k}q\left[\frac{\cos\left(\frac{q}{2 + \gamma}\right)}{\cos\left(\frac{p}{2 + \gamma}\right)} \cos\left(\frac{p}{2 + \gamma}\right) - \cos\left(\frac{q}{2 + \gamma}\right)\right]$$  \hspace{1cm} (2.43)
\[
\tau_{xc}(z) = A\bar{\tau}_{xc}(z) = -A \left[ \frac{4k^2pq}{k^2-q^2} \frac{\cos(\bar{q} / 2 + \gamma)}{\cos(\bar{p} / 2 + \gamma)} \sin(\bar{p} z + \gamma) + (k^2-q^2) \sin(\bar{q} z + \gamma) \right] \tag{2.44}
\]

Now, the phase relations between displacements and stresses components are discussed in the three regions for the propagating Lamb waves by substituting \( \bar{p}, \bar{q} \) and real \( \bar{k} \) into Eq. (2.40)- Eq. (2.44), respectively.

(a) \( \bar{q} \) and \( \bar{p} \) are real- region I in Figure 2.7

For both symmetric and antisymmetric Lamb wave modes, \( \bar{u}_x \) is pure real and \( \bar{u}_z \) is pure imaginary. Thus, \( \bar{u}_x \) always lags 90 degree in phase behind \( \bar{u}_z \). \( \bar{\sigma}_x \) and \( \bar{\sigma}_z \) are pure imaginary and \( \tau_{xc} \) is pure real. Thus, \( \bar{\sigma}_x \) and \( \bar{\sigma}_z \) are in phase with each other and lead 90 degree in phase ahead of \( \tau_{xc} \). Figure 2.8 shows the mode shapes for \( A_1 \) mode propagating Lamb waves at \( \omega h / c_T = 3.56 \ (f h = 1.8 \text{ MHz}\cdot\text{mm}) \) located in region I.
Figure 2.8 Mode shapes for $A_1$ mode propagating Lamb waves at $\omega h/c_T = 3.56$ ($f_h = 1.8$ MHz·mm) located in region I of Figure 2.7. (a) Displacement mode shapes; (b) Stress mode shapes.

For both symmetric and antisymmetric Lamb wave modes, $\bar{u}_x$ is pure real and $\bar{u}_z$ is pure imaginary. Thus, $\bar{u}_z$ always lags 90 degree in phase behind $\bar{u}_z$. $\bar{\sigma}_x$ and $\bar{\sigma}_z$ are pure imaginary and $\bar{\tau}_{xz}$ is pure real. Thus, $\bar{\sigma}_x$ and $\bar{\sigma}_z$ are in phase with each other and lead 90 degree in phase over $\bar{\tau}_{xz}$. Figure 2.9 shows the mode shapes for $S_0$ mode propagating Lamb waves at $\omega h/c_T = 3.56$ ($f_h = 1.8$ MHz·mm) located in region II.
Figure 2.9 Mode shapes for $S_0$ mode propagating Lamb waves at $\omega h/c_T = 3.56$ ($f_h = 1.8 \text{ MHz/mm}$) located in region II of Figure 2.7. (a) Displacement mode shapes; (b) Stress mode shapes.

(c) $\tilde{q}$ and $\tilde{p}$ are imaginary- region III in Figure 2.7

For symmetric Lamb wave modes, $\tilde{u}_x$ is pure imaginary and $\tilde{u}_z$ is pure real. $\tilde{\sigma}_x$ and $\tilde{\sigma}_z$ are pure real and $\tilde{\sigma}_{xz}$ is pure imaginary. Therefore, $\tilde{u}_z$ always lags 90 degree in phase behind $\tilde{u}_x$, $\tilde{\sigma}_x$ and $\tilde{\sigma}_z$ are in phase with each other and lag 90 degree in phase behind $\tilde{\sigma}_{xz}$. For antisymmetric Lamb wave modes, $\tilde{u}_z$ is pure real and $\tilde{u}_z$ is pure imaginary. $\tilde{\sigma}_x$ and $\tilde{\sigma}_z$ are pure imaginary and $\tilde{\sigma}_{xz}$ is pure real. Therefore, $\tilde{u}_z$ always leads 90 degree in phase over $\tilde{u}_x$, $\tilde{\sigma}_x$ and $\tilde{\sigma}_z$ are in phase with each other and leads 90 degree in phase ahead of $\tilde{\sigma}_{xz}$. Figure 2.10 shows the mode shapes for $A_0$ mode propagating Lamb waves at $\omega h/c_T = 3.56$ ($f_h = 1.8 \text{ MHz/mm}$).
MHz·mm) located in region III. Figure 2.11 shows the mode shapes for \( S_0 \) mode propagating Lamb waves at \( \omega h/c_T = 7.4 \) \( (f_h = 3.74 \text{ MHz·mm}) \) located in region III.

![Figure 2.11 Mode shapes for \( S_0 \) mode propagating Lamb waves at \( \omega h/c_T = 7.4 \) \( (f_h = 3.74 \text{ MHz·mm}) \) located in region III.](image)

Figure 2.10 Mode shapes for \( A_0 \) mode propagating Lamb waves at \( \omega h/c_T = 3.56 \) \( (f_h = 1.8 \text{ MHz·mm}) \) located in region III of Figure 2.7. (a) Displacement mode shapes; (b) Stress mode shapes.
The phase relations between displacements and stresses components for *evanescent Lamb modes* need to be considered for evanescent modes with pure imaginary wavenumbers and evanescent modes with complex wavenumbers, respectively.

(a) \( \vec{k} \) is pure imaginary

From Eq. (2.35), if \( \vec{k} \) is pure imaginary, \( \vec{q} \) and \( \vec{p} \) are both real values. Therefore, after substituting \( \vec{p} \), \( \vec{q} \) and imaginary \( \vec{k} \) into Eq. (2.40)- Eq. (2.44), it is found that the displacements components \( \vec{u}_x \) and \( \vec{u}_z \) are in phase with each other and the stresses components \( \vec{\sigma}_x \), \( \vec{\sigma}_z \), and \( \vec{\tau}_{xz} \) are in phase with each other as well. Figure 2.12 shows the mode shapes for \( A_1 \) mode evanescent Lamb waves at \( \omega h/c_T = 1.8 \) (\( fh = 0.9 \, MHz.mm \)).

Figure 2.11 Mode shapes for \( S_0 \) mode propagating Lamb waves at \( \omega h/c_T = 7.4 \) (\( fh = 3.74 \, MHz.mm \)) located in region III of Figure 2.7. (a) Displacement mode shapes; (b) Stress mode shapes.
Figure 2.12 Mode shapes for $A_1$ mode evanescent Lamb waves at $\omega h/c_T = 1.8$ ($fh = 0.9 \text{ MHz} \cdot \text{mm}$). (a) Displacement mode shapes; (b) Stress mode shapes.

(b) $k$ is complex

If $k$ becomes complex, as defined in Eq. (2.35), $q$ and $p$ are always complex valued and all the displacements and stresses are determined to be complex valued from Eq. (2.40)- Eq. (2.44). For this case, the phase relations between the displacements and stresses components cannot be easily obtained as the phase information are included in the complex valued displacements and stresses components. However, the complex valued displacements and stresses components indicate that each of these components can be treated as a linear combination of two time-harmonic responses which are different in amplitude and out-of-phase with each other. Figure 2.13 shows the mode shapes for $S_1/S_2$ mode evanescent Lamb waves at $\omega h/c_T = 1.8$ ($fh = 0.9 \text{ MHz} \cdot \text{mm}$).
2.4 SH Waves

2.4.1 Basic Equations and SH Waves Dispersion Relation

Different from Lamb waves in a plate, another type of wave motion in the plate involves antiplane shear motion. In a plate with surfaces normal to \( z \) direction, the wave polarized in the \((x, z)\) plane with components \( u_x \) and \( u_z \) is decoupled from the SH wave polarized along \( y \) direction with displacement \( u_y \) alone, and the propagation direction is here defined as \( x \). The governing equation for the SH wave is

\[
\mu \left( \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2} \right) = \rho \frac{\partial^2 u_y}{\partial t^2}
\]

(2.45)

where \( \rho \) is density. The displacement is assumed in the form

\[
u_y = f(z)e^{i(kx - \omega t)}
\]

(2.46)

Figure 2.13 Mode shapes for \( S_1/ S_2 \) mode evanescent Lamb waves at \( \omega h/c_T = 1.8 \) (\( fh = 0.9 \, MHz \cdot mm \)). (a) Displacement mode shapes; (b) Stress mode shapes.
Substituting Eq. (2.46) into Eq. (2.45) and solving for \( f(z) \) gives

\[
f(z) = B_1 \sin(qz) + B_2 \cos(qz) \tag{2.47}
\]

Given that the traction-free boundary conditions on the top and bottom surfaces \( z = \pm h/2 \) of the plate

\[
\tau_{yz} = \mu \frac{\partial u_y}{\partial z} = 0 \tag{2.48}
\]

and solving this boundary value problem yields

\[
B_1 \cos(qh/2) + B_2 \cos(qh/2) = 0 \tag{2.49}
\]

where

\[
q^2 = \frac{\omega^2}{c_T^2} - k^2 \tag{2.50}
\]

Eq. (2.49) can be satisfied in two ways

\[
B_1 = 0 \quad \text{and} \quad \sin(qh/2) = 0 \tag{2.51}
\]

\[
B_2 = 0 \quad \text{and} \quad \cos(qh/2) = 0 \tag{2.52}
\]

in which \( B_1 = 0 \) gives that the displacement is symmetric with respect to the mid-plane of the plate and the displacement is antisymmetric if \( B_2 = 0 \). In both cases, the following relation must be satisfied

\[
qh = m\pi \tag{2.53}
\]

where \( m = 0, 2, 4, \ldots \) for symmetric modes, and \( m = 1, 3, 5, \ldots \) for antisymmetric modes.

Combining Eq. (2.50) and Eq. (2.53), the dispersion relation can be written in an analytical form as
\[
\left(\frac{\omega}{c_T}\right)^2 = k^2 + \left(\frac{m\pi}{h}\right)^2
\]  
(2.54)

The hyperbolic dispersion relation can be normalized and is given by

\[
\bar{\omega}^2 = \bar{k}^2 + (m\pi)^2
\]  
(2.55)

where \(\bar{\omega} = \omega h / c_T\), \(\bar{k} = kh\). The displacements for SH waves can be derived as

\[
u_y = A_m V(z)e^{i(k,x-\omega t)}
\]  
(2.56)

where \(A_m\) is amplitude coefficient and

\[
V(z) = \begin{cases} 
\cos\left(\frac{m\pi}{h} z\right) & \text{(symmetric modes)} \\
\sin\left(\frac{m\pi}{h} z\right) & \text{(antisymmetric modes)} 
\end{cases}
\]  
(2.57)

The stresses components can be obtained as

\[
\tau_{xy} = A_m t_{xy}(z)e^{i(k,x-\omega t)}
\]  
(2.58a)

\[
\tau_{yz} = A_m t_{yz}(z)e^{i(k,x-\omega t)}
\]  
(2.58b)

where

\[
t_{xy}(z) = \begin{cases} 
-iG_m A_m \cos\left(\frac{m\pi}{h} z\right) & \text{(symmetric modes)} \\
-iG_m A_m \sin\left(\frac{m\pi}{h} z\right) & \text{(antisymmetric modes)} 
\end{cases}
\]  
(2.59a)

\[
t_{yz}(z) = \begin{cases} 
-G_m A_m \cos\left(\frac{m\pi}{h} z\right) & \text{(symmetric modes)} \\
G_m A_m \sin\left(\frac{m\pi}{h} z\right) & \text{(antisymmetric modes)} 
\end{cases}
\]  
(2.59b)
2.4.2 Propagating and Evanescent SH Waves

In order to characterize the propagating and evanescent SH waves, the formation of SH guided waves is investigated first. From the SH dispersion relation Eq. (2.54), SH guided waves propagation can be treated as waves propagating transversely upward possess proper angles and wave vectors so that they can reconstruct themselves after reflection from the top and bottom surfaces of the plate, as shown in Figure 2.14. The constant wavenumber for the bulk wave along $x$ direction is $k$ and the transverse wavenumber must be $q = m\pi/h$. The physical meaning of the dispersion relation can be easily interpreted by using the slowness curve. A normalized slowness curve which is a circle with radius equals to 1 is sketched to unveil some general features for guided SH waves, as shown in Figure 2.15, where the normalized dispersion relation $k^2 + q^2 = \omega^2$ is used, and $q = qh$. From the slowness curve, the incident and reflected bulk waves have the transverse wave vector components of $m\pi/\bar{\omega}$ and $-m\pi/\bar{\omega}$, and incident and reflected angle $\phi$, respectively. As the normalized frequency increases, the incident wave angle $\phi$ increases and the transverse wavenumber is compressed, and the waves propagate entirely along the $z$ direction as a very high frequency is reached. However, the incident wave angle $\phi$ decreases as the normalized frequency decreases, and zero incident angle $\phi$ appears as $\bar{\omega} = \bar{\omega}_c = m\pi$. For $\bar{\omega} < \bar{\omega}_c$, the incident waves fail to lie on the slowness curve and the transverse wavenumber $q = m\pi$ has to be always satisfied, the propagation wavenumber $\bar{k}$ must be pure imaginary. Therefore, $\bar{\omega} = \bar{\omega}_c = m\pi$ is a cut-off frequency that can determine a SH wave to be either propagating (for $\bar{\omega} > \bar{\omega}_c$), where $e^{i\bar{k}x}$ is a
harmonic function) or evanescent (for $\bar{\omega} < \bar{\omega}_c$, where $e^{-\bar{k}_x}$ is an exponential decaying function), and it is obvious that the evanescent wave does not transport energy along the plate.

Figure 2.14 SH guided wave formation in an isotropic plate with free boundaries-SH wave incident and reflection.

Figure 2.15 Slowness curve for the SH wave in an isotropic plate with thickness $h$. 
Furthermore, a non-dimensional complex dispersion curves for an aluminum plate obtained from Eq. (2.55) are shown in Figure 2.16 to further explore the properties of propagating and evanescent waves. For completeness, the SH wave phase and group velocity dispersive curves are shown in Figure 2.17 and Figure 2.18. From the complex dispersion curves, the wavenumber may be real or pure imaginary values; the real wavenumber imply propagating waves and pure imaginary wavenumbers representing non-propagating (or evanescent) waves. At any given frequency, there is a finite number of propagating modes and an infinite number of non-propagating (evanescent modes). It is noticed from the SH dispersion curves that except the first non-dispersive mode, all other dispersive modes are continuous across the entire frequency spectrum with cutoff frequency \( m\pi \) as the transition point which distinguish the propagating and non-propagating modes clearly. From Eq. (2.46), the displacement field for the propagating SH waves can be written as

\[
\mathbf{u}_y = A_m \ V(z)e^{i(k_x x - \omega t)}
\]  

(2.60)

where \( k_R \) stands for real wavenumber, and it is clear that the propagating wave is a sinusoidal wave. The displacement field for the evanescent SH waves is given by

\[
\mathbf{u}_y = A_m \ V(z)e^{-k_I x}e^{-i\omega t}
\]  

(2.61)

where \( k_I \) represents imaginary part of wavenumbers, and this is a spatially varying disturbances which has a decreasing exponential curve. Therefore, the evanescent waves with pure imaginary wavenumbers exhibit exponentially decay away from the location where evanescent wave is excited. From the dispersion relation given in Eq. (2.55), the magnitude of the pure imaginary wavenumbers for a given evanescent mode can be obtained as
\[ \bar{k}_I = \sqrt{(m\pi)^2 - \bar{\omega}^2} \]  

(2.62)

It is seen from Eq. (2.62), the values of the imaginary wavenumbers determine the decaying rate, and Eq. (2.62) indicates that the value of \( \bar{k}_I \) decreases as the frequency increases for every evanescent \( SH \) mode. In addition, among all the non-propagating modes, the magnitude of the \( SH_1 \) imaginary part is the smallest implying the lowest decaying rate.

Figure 2.16 SH waves dispersion curves including both propagating and evanescent wave modes for an isotropic plate.
Figure 2.17 Propagating SH wave phase velocity dispersive curves.

Figure 2.18 Propagating SH wave group velocity dispersive curves.
Since different SH modes exist in the SH waves, each of the mode has a characteristic variation of field quantities like displacements, velocities and stresses through the waveguide, called ‘mode shape’. It is seen from Eq. (2.56), the propagating and evanescent SH modes bear the same displacement mode shape, since the displacement amplitude is independent of frequency and wavenumber. The stresses components, i.e., Eq. (2.58) and Eq. (2.59), for SH waves have a much simpler form than those of Lamb wave modes, thus the phase relations between stresses components can be easily obtained. The shear stress $\tau_{xy}$ is always lags 90 degree in phase behind $\tau_{yz}$ for propagating SH modes while for evanescent SH modes there is no phase difference between stresses components. Figure 2.19 (a) and (b) show the stress mode shapes at $\omega h/c_T = 3.972$ ($f_h = 2 \text{ MHz}\cdot\text{mm}$) for the propagating $SH_0$ and $SH_1$ modes, respectively. The stress mode shapes at $\omega h/c_T = 1.986$ ($f_h = 1 \text{ MHz}\cdot\text{mm}$) for evanescent $SH_2$ and $SH_1$ modes are plotted in Figure 2.20(a) and (b).
Figure 2.19 Stress mode shapes for propagating SH modes at $\omega h/c_T = 3.972$ ($fh = 2$ MHz·mm) (a) $SH_0$ mode, (b) $SH_1$ mode (The normalized stresses

$\bar{\tau}_{xy} = \tau_{xy} / G, \quad \bar{\tau}_{yz} = \tau_{yz} / G$ are used).

Figure 2.20 Stress mode shapes for evanescent SH modes at $\omega h/c_T = 1.986$ ($fh = 1$ MHz·mm) (a) $SH_2$ mode, (b) $SH_1$ mode (The normalized stresses

$\bar{\tau}_{xy} = \tau_{xy} / G, \quad \bar{\tau}_{yz} = \tau_{yz} / G$ are used).
2.5 Summary

In this Chapter, the basic theory for bulk wave propagation in a homogeneous, isotropic, linearly elastic medium is introduced first. Two types of bulk waves: longitudinal and shear bulk waves which satisfy the uncoupled bulk wave governing equations are defined. From the bulk wave governing equation, and assuming plane strain in the y direction, two types of guided waves, Lamb waves and SH waves that can propagate in a plate are presented. After applying the traction free boundary condition on the two flat surfaces, the important dispersion relations for Lamb waves and SH waves are derived. Using the dispersion relations, the properties of propagating and evanescent Lamb and SH waves are characterized. From the displacements and stresses responses, the mode shapes and the important phase information for propagating and evanescent guided waves are uncovered, respectively.
Chapter 3

Reciprocity, Orthogonality Relations and Power Flow for Guided Waves in Plates
3.1 Introduction

Reciprocity is like in daily life, you give someone something and you will get something else, equally or more valuable returned. The reciprocity for states of deformation in elastic bodies is implemented: For a given known solution under one loading case (virtual fields), some important features of another loading case (real fields) can be more valuable given. Although the actual fields may sometimes incomplete, at least it generates an equation to solve such field. A formal definition of elastodynamic reciprocity theorem is given by Achenbach in his book (2003): ‘A reciprocity theorem relates, in a specific manner, two admissible elastodynamic states that can occur in the same time-invariant linearly elastic material parameters and its own set of loading conditions. The domain to apply the reciprocity theorem may be bounded or unbounded.’ In elasticity theory, the reciprocity theorem provides a relation between displacements/velocity, traction components and body forces for two different loading states of a single body or two bodies which have the same geometry. The elastostatic reciprocity theorem was first formulated by Betti (1872) and a more general theorem which includes the elastodynamic case is given by Raleigh (1873). The statements of reciprocity relations can be found, among other, in books by Achenbach (1973), Auld (1973), Achenbach et al., (1982) and de Hoop (1995). Figure 3.1 shows the static reciprocity relation. In the figure, the generic body volume is \( V \) and boundary \( S \) subjected to body forces \( \mathbf{f}_1 \) and \( \mathbf{f}_2 \) at points \( P_1 \) and \( P_2 \), respectively. The static reciprocity relation states that the work done at point \( P_1 \) by force \( \mathbf{f}_1 \) upon the displacement \( \mathbf{u}_{12} \) produced
by the force $\mathbf{f}_2$ is the same as the work done at point $P_2$ by force $\mathbf{f}_2$ upon the displacement $\mathbf{u}_{21}$ produced by the force $\mathbf{f}_1$, i.e.,

$$\mathbf{f}_1 \cdot \mathbf{u}_{12} = \mathbf{f}_2 \cdot \mathbf{u}_{21}$$  \hspace{1cm} (3.1)

![Figure 3.1 Static reciprocity relations (Elastic body with volume $V$ and boundary $S$ subjected to concentrated load $\mathbf{f}_1$ and $\mathbf{f}_2$).](image)

Mode analysis is a powerful tool to analyze the waveguide excitation and scattering for the elastic waves propagating in elastic plates. To apply this technique, any arbitrary elastic wave field distributions must be expanded in terms of a superposition of waveguide modes which require the proof that the field distributions of individual waveguide mode are orthogonal. Before proving the orthogonality relation between waveguide modes, it is necessary to derive the reciprocity relations for elastic waves propagation in plates. Both real
and complex reciprocity relations are available depending on different applications (Auld 1990).

The real reciprocity relation can be applied to both lossy and lossless medium with some restrictions when used in the lossless medium, while the complex reciprocity relation can only be employed for a lossless media which requires real constitutive matrices. In addition, the real reciprocity relation is more suitable for scattering analysis while the complex reciprocity is adequate when solving the waveguide mode analysis and velocity/displacement or traction disturbance problems (Auld 1990). In this dissertation, all the media is assumed lossless and the study of evanescent waves gets complex wavenumbers involved. Moreover, the displacement/traction excitation will be used to excite the waveguide with the aid of waveguide modal analysis. Therefore, the complex reciprocity relation is selected for the study of conversion of evanescent into propagating guided waves in plates. In order to quantitatively characterize the conversion of evanescent into propagating guided waves in plates, knowing the amount of propagating power that can be converted from evanescent excitation is crucial to calculate the power conversion efficiency during the conversion. As a result, it is important to understand the power flow for propagating and evanescent guided waves and the physical meaning of the power flow associated with these waves, respectively.

In this Chapter, the reciprocity relation will be first derived and will be discussed for Lamb and SH waves, respectively. The orthogonality relations which discuss the orthogonal relations for Lamb waves and SH waves are then derived from the reciprocity relations. In
the last part of this Chapter, the power flow for propagating and evanescent guided waves in plates are discussed in detail.

3.2 Reciprocity Relations

The equation of motion and the strain-displacements relations for elastic solid subjected to a body force, in the tensor notation are given by

\[
\begin{align*}
\nabla \cdot \sigma &= \rho \frac{\partial^2 u}{\partial t^2} - f \\
\nabla \cdot u &= \varepsilon
\end{align*}
\]

where \( \rho \) is volume density, \( \sigma \) and \( \varepsilon \) are the second rank stress and strain tensor, \( u \) is the displacement vector, and \( f \) is the body force. The Hook’s law in tensor notation is given by

\[
\sigma = C : \varepsilon \quad \text{or} \quad \varepsilon = S : \sigma
\]

where \( C \) is the fourth rank stiffness tensor, \( S \) is the fourth rank compliance tensor.

Consider two body force sources \( f_1 \) and \( f_2 \) that can produce two wave fields 1 and 2, with displacement, velocity and stress fields \([u_1, v_1, \sigma_1]\) and \([u_2, v_2, \sigma_2]\). In order to derive the complex reciprocity relation, the wave field 2 is taken to be induced by the body force source \( f_2^* \), where \(^*\) denotes the complex conjugate. Using the equation of motion and strain-displacement relations shown in Eq. (3.2), the following equations can be written for wave field 1 and 2.
\[
\begin{align*}
\nabla \sigma_1 &= \rho \frac{\partial v_1}{\partial t} - f_1 \quad (3.4) \\
\nabla \cdot v_1 &= \frac{\partial \varepsilon_1}{\partial t}
\end{align*}
\]

\[
\begin{align*}
\nabla \sigma_2^* &= \rho \frac{\partial v_2^*}{\partial t} - f_2^* \quad (3.5) \\
\nabla \cdot v_2^* &= \frac{\partial \varepsilon_2^*}{\partial t}
\end{align*}
\]

After multiplying Eq. (3.4) with wave field 2, e.g., \( v_2^* \) and \( \sigma_2^* \), and Eq. (3.5) with wave field 1, e.g., \( v_1 \) and \( \sigma_1 \), one can obtain

\[
\begin{align*}
\nabla \cdot (v_2^* \cdot \sigma_1) &= \rho \frac{\partial v_2^* \cdot \hat{v}_1}{\partial t} - v_2^* \cdot f_1 \\
\sigma_2^* : \nabla \cdot v_1 &= \sigma_2^* : \frac{\partial \varepsilon_1}{\partial t}
\end{align*}
\]

\[
\begin{align*}
\nabla \cdot (v_1 \cdot \sigma_2^*) &= \rho \frac{\partial v_1 \cdot \hat{v}_2^*}{\partial t} - v_1 \cdot f_2^* \\
\sigma_1 : \nabla \cdot v_2^* &= \sigma_1 : \frac{\partial \varepsilon_2^*}{\partial t}
\end{align*}
\]

Combining Eq. (3.6) and Eq. (3.7), the following relation can be obtained

\[
\begin{align*}
\nabla \cdot (v_2^* \cdot \sigma_1) + \sigma_1 : \nabla \cdot v_2^* + v_1 \cdot (\nabla \cdot \sigma_2^*) + \sigma_2^* : \nabla \cdot v_1 \\
= \rho \frac{\partial v_2^* \cdot \hat{v}_1}{\partial t} + \rho \frac{\partial v_1 \cdot \hat{v}_2^*}{\partial t} + \sigma_2^* : \frac{\partial \varepsilon_1}{\partial t} + \sigma_1 : \frac{\partial \varepsilon_2^*}{\partial t} - v_2^* \cdot f_1 - v_1 \cdot f_2^*
\end{align*}
\]

(3.8)

Recall the identity that

\[
\begin{align*}
\nabla \cdot (v_2^* \cdot \sigma_1) &= v_2^* \cdot (\nabla \cdot \sigma_1) + \sigma_1 : \nabla v_2^* \\
\nabla \cdot (v_1 \cdot \sigma_2^*) &= v_1 \cdot (\nabla \cdot \sigma_2^*) + \sigma_2^* : \nabla v_1
\end{align*}
\]

(3.9)

Substituting Eq. (3.9) into Eq. (3.8) yields
\[ \nabla \cdot (v_2^* \cdot \sigma_1 + v_1^* \cdot \sigma_2^*) = \rho v_2^* \cdot \frac{\partial v_1^*}{\partial t} + \rho v_1^* \cdot \frac{\partial v_2^*}{\partial t} + \sigma_2^* : \frac{\partial \epsilon_1^*}{\partial t} + \sigma_1^* : \frac{\partial \epsilon_2^*}{\partial t} - v_2^* \cdot f_1 - v_1^* \cdot f_2^* \] (3.10)

Through the use of the constitutive relation Eq. (3.2), one can obtain

\[ \sigma_1^* : \frac{\partial \epsilon_2^*}{\partial t} + \sigma_2^* : \frac{\partial \epsilon_1^*}{\partial t} = \sigma_1 : S : \frac{\partial \sigma_2^*}{\partial t} + \sigma_2 : S : \frac{\partial \sigma_1^*}{\partial t} \] (3.11)

The compliance matrix \( s \) is symmetric and is assumed to be real, thus

\[ \sigma_1 : S : \frac{\partial \sigma_2^*}{\partial t} = \frac{\partial \sigma_2}{\partial t} : S : \sigma_1 \] (3.12)

Substituting Eq. (3.12) into Eq. (3.11) and subsequently into Eq. (3.10), the complex reciprocity relation takes the form as

\[ \nabla \cdot (v_2^* \cdot \sigma_1 + v_1^* \cdot \sigma_2^*) = \frac{\partial}{\partial t} (\rho v_2^* \cdot \nabla + \sigma_2^* : \epsilon_1^*) - v_2^* \cdot f_1 - v_1^* \cdot f_2^* \] (3.13)

For steady-state time-harmonic fields, i.e.,

\[ f_1 = f_1(x, y, z)e^{-i\omega t}, \quad v_1 = v_1(x, y, z)e^{-i\omega t}, \quad \sigma_1 = \sigma_1(x, y, z)e^{-i\omega t}, \quad \epsilon_1 = \epsilon_1(x, y, z)e^{-i\omega t} \]
\[ f_2 = f_2(x, y, z)e^{-i\omega t}, \quad v_2 = v_2(x, y, z)e^{-i\omega t}, \quad \sigma_2 = \sigma_2(x, y, z)e^{-i\omega t}, \quad \epsilon_2 = \epsilon_2(x, y, z)e^{-i\omega t} \] (3.14)

Taking conjugate of Eq. (3.14) yields

\[ f_1^* = f_1^*(x, y, z)e^{i\omega t}, \quad v_1^* = v_1^*(x, y, z)e^{i\omega t}, \quad \sigma_1^* = \sigma_1^*(x, y, z)e^{i\omega t}, \quad \epsilon_1^* = \epsilon_1^*(x, y, z)e^{i\omega t} \]
\[ f_2^* = f_2^*(x, y, z)e^{i\omega t}, \quad v_2^* = v_2^*(x, y, z)e^{i\omega t}, \quad \sigma_2^* = \sigma_2^*(x, y, z)e^{i\omega t}, \quad \epsilon_2^* = \epsilon_2^*(x, y, z)e^{i\omega t} \] (3.15)

Substituting Eq. (3.14) and Eq. (3.15) into Eq. (3.13), the complex reciprocity relation for steady-state time-harmonic variation functions follows a simpler form

\[ \nabla \cdot (v_2^* \cdot \sigma_1 + v_1^* \cdot \sigma_2^*) = - (v_2^* \cdot f_1 + v_1^* \cdot f_2^*) \] (3.16a)

or
\[ i \omega \nabla \cdot (u_2^* \cdot \sigma_i - u_i \cdot \sigma_2^*) = -i \omega (u_2^* \cdot f_1 - u_1 \cdot f_2^*) \] (3.16b)

and this is the differential form reciprocity theorem. The integral form complex reciprocity relation can be obtained by integrating Eq. (3.16a) and Eq. (3.16b) over a region \( V \) with boundary \( S \). The volume integration on the left-hand side is converted into a surface integral by using Gauss’s theorem and is given by

\[
\int_s \left( v_2^* \cdot \sigma_i + v_1 \cdot \sigma_2^* \right) \cdot n \, dS = -\int_v \left( v_2^* \cdot f_1 + v_1 \cdot f_2^* \right) \, dV 
\] (3.17a)

or

\[
i \omega \int_s \left( u_2^* \cdot \sigma_i - u_i \cdot \sigma_2^* \right) \cdot n \, dS = -i \omega \int_v \left( u_2^* \cdot f_1 - u_1 \cdot f_2^* \right) \, dV
\] (3.17b)

where \( n \) is the surface outward normal.

### 3.2.1 Reciprocity Relations for Lamb Waves

For Lamb waves propagating in an elastic plate, the velocity vector for Lamb waves is

\[
v = (v_x \quad 0 \quad v_z)
\] (3.18)

The stress matrix is given by

\[
\sigma = \begin{pmatrix}
\sigma_x & \tau_{xz} \\
0 & 0 & 0 \\
\tau_{xz} & 0 & \sigma_z
\end{pmatrix}
\] (3.19)

Rewriting Eq. (3.6) and Eq. (3.7) for Lamb waves
\[
\begin{align*}
{v}_x^2 \frac{\partial \sigma_x^3}{\partial x} + {v}_x^2 \frac{\partial \tau_{xz}}{\partial x} &= \rho \frac{\partial v_x^2}{\partial t} - {v}_x^2 f_x^1 \\
{v}_z^2 \frac{\partial \tau_{xz}}{\partial x} + {v}_z^2 \frac{\partial \sigma_z^3}{\partial z} &= \rho \frac{\partial v_z^2}{\partial t} - {v}_z^2 f_z^1 \\
\sigma_x^3 \frac{\partial v_x^1}{\partial x} &= \sigma_x^1 \frac{\partial \omega_x^1}{\partial t} \\
\sigma_z^3 \frac{\partial v_z^1}{\partial z} &= \sigma_z^1 \frac{\partial \omega_z^1}{\partial t} \\
\tau_{xz}^1 \left( \frac{\partial v_x^2}{\partial x} + \frac{\partial v_z^2}{\partial z} \right) &= \tau_{xz}^1 \frac{\partial \gamma_{xz}^2}{\partial t}
\end{align*}
\]

(3.20a)

\[
\begin{align*}
{v}_x^3 \frac{\partial \sigma_x^4}{\partial x} + {v}_x^3 \frac{\partial \tau_{xz}}{\partial x} &= \rho \frac{\partial v_x^3}{\partial t} - {v}_x^3 f_x^3 \\
{v}_z^3 \frac{\partial \tau_{xz}}{\partial x} + {v}_z^3 \frac{\partial \sigma_z^4}{\partial z} &= \rho \frac{\partial v_z^3}{\partial t} - {v}_z^3 f_z^3 \\
\sigma_x^4 \frac{\partial v_x^2}{\partial x} &= \sigma_x^1 \frac{\partial \omega_x^2}{\partial t} \\
\sigma_z^4 \frac{\partial v_z^2}{\partial z} &= \sigma_z^1 \frac{\partial \omega_z^2}{\partial t} \\
\tau_{xz}^2 \left( \frac{\partial v_x^3}{\partial x} + \frac{\partial v_z^3}{\partial z} \right) &= \tau_{xz}^1 \frac{\partial \gamma_{xz}^2}{\partial t}
\end{align*}
\]

(3.20b)

Adding Eq. (3.20a) into Eq. (3.20b) and upon rearrangement, the differential form complex reciprocity relation for Lamb waves can be obtained as

\[
\frac{\partial}{\partial x} \left( \sigma_x^1 \sigma_x^3 + \sigma_z^1 \sigma_z^3 + v_x^2 \tau_{xz}^1 + v_z^2 \tau_{xz}^3 \right) + \frac{\partial}{\partial z} \left( v_x^2 \tau_{xz}^1 + v_z^2 \tau_{xz}^3 + v_x^2 \sigma_x^1 + v_z^2 \sigma_z^1 \right) + \frac{\partial}{\partial t} \left( v_x^2 \sigma_x^1 + v_z^2 \sigma_z^1 \right)
\]

\[
= \rho \frac{\partial}{\partial t} \left( v_x^1 v_x^2 + v_z^1 v_z^2 \right) + \frac{\partial}{\partial x} \left( \sigma_x^1 \sigma_x^3 + \sigma_x^1 \sigma_z^3 + \tau_{xz}^1 \gamma_{xz}^2 \right) - \left( v_x^2 f_x^1 + v_z^2 f_z^1 \right)
\]

(3.21)

For the steady-state time-harmonic fields, Eq. (3.19) can be simplified as
\[
\frac{\partial}{\partial x}\left(v^2_x \sigma^1_{xx} + v^1_x \sigma^2_{xx} + v^2_z \tau^1_{xz} + v^1_z \sigma^2_{xz}\right) + \frac{\partial}{\partial z}\left(v^2_x \tau^1_{xz} + v^1_z \tau^2_{xz} + v^2^*_z \sigma^1_{z} + v^1_z \sigma^2_{z}\right) \\
= -\left(v^2_x f^1_x + v^1_x f^2_x + v^2_z f^1_z + v^1_z f^2_z\right)
\]

(3.22a)

Or

\[
\quad i \omega \frac{\partial}{\partial x}\left(u^2_x \sigma^1_{xx} - u^1_x \sigma^2_{xx} + u^2_z \tau^1_{xz} - u^1_z \tau^2_{xz}\right) + i \omega \frac{\partial}{\partial z}\left(u^2_x \tau^1_{xz} - u^1_z \tau^2_{xz} + u^2^*_z \sigma^1_{z} - u^1_z \sigma^2_{z}\right) \\
= -i \omega \left(u^2_x f^1_x - u^1_x f^2_x + u^2_z f^1_z - u^1_z f^2_z\right)
\]

(3.22b)

The steady-state time-harmonic fields integral form complex reciprocity relations for Lamb waves can be easily obtained by substituting Eq. (3.18) and Eq. (3.19) into Eq. (3.17a) or Eq. (3.17b) as

\[
\int_S \left(v^2_x \sigma^1_{xx} + v^1_x \sigma^2_{xx} + v^2_z \tau^1_{xz} + v^1_z \sigma^2_{xz} + v^2 \tau^1_{xz} + v^1_x \sigma^2_{xz} + v^2 \sigma^1_{z} + v^1_x \sigma^2_{z}\right) \cdot \mathbf{n} \, dS \\
= -\int_V \left(v^2_x f^1_x + v^1_x f^2_x + v^2_z f^1_z + v^1_z f^2_z\right) \, dV
\]

(3.23a)

Or

\[
\quad i \omega \int_S \left(u^2_x \sigma^1_{xx} - u^1_x \sigma^2_{xx} + u^2_z \tau^1_{xz} - u^1_z \tau^2_{xz}\right) + i \omega \int_S \left(u^2_x \tau^1_{xz} - u^1_z \tau^2_{xz} + u^2 \sigma^1_{z} - u^1_z \sigma^2_{z}\right) \cdot \mathbf{n} \, dS \\
= -i \omega \int_V \left(u^2_x f^1_x - u^1_x f^2_x + u^2_z f^1_z - u^1_z f^2_z\right) \, dV
\]

(3.23b)

### 3.2.2 Reciprocity Relations for SH Waves

For SH waves propagating in elastic plates, the velocity vector for SH waves is

\[
\mathbf{v} = (0 \quad v_y \quad 0)
\]

(3.24)

The stress matrix is given by

\[
\mathbf{\sigma} = \begin{pmatrix}
0 & \tau_{xy} & 0 \\
\tau_{xy} & 0 & \tau_{yz} \\
0 & \tau_{yz} & 0
\end{pmatrix}
\]

(3.25)
Rewriting Eq. (3.6) and Eq. (3.7) for SH waves

\[
\begin{align*}
& v_y^{2*} \frac{\partial \tau_{xy}^{1}}{\partial x} + v_y^{2*} \frac{\partial \tau_{xy}^{1}}{\partial z} = \rho v_y^{2*} \frac{\partial v_{y}^{1}}{\partial t} - v_y^{2*} f_y^{1} \\
& \tau_{xy}^{2*} \frac{\partial v_{y}^{1}}{\partial x} = \tau_{xy}^{2*} \frac{\partial \gamma_{xy}^{1}}{\partial t} \\
& \tau_{xy}^{2*} \frac{\partial v_{y}^{1}}{\partial z} = \tau_{xy}^{2*} \frac{\partial \gamma_{xy}^{1}}{\partial t} \\
\end{align*}
\]

(3.26a)

\[
\begin{align*}
& v_y^{1} \frac{\partial \tau_{xy}^{2*}}{\partial x} + v_y^{1} \frac{\partial \tau_{xy}^{2*}}{\partial z} = \rho v_y^{1} \frac{\partial v_{y}^{2*}}{\partial t} - v_y^{1} f_y^{2*} \\
& \tau_{xy}^{1} \frac{\partial v_{y}^{2*}}{\partial x} = \tau_{xy}^{1} \frac{\partial \gamma_{xy}^{2*}}{\partial t} \\
& \tau_{xy}^{1} \frac{\partial v_{y}^{2*}}{\partial z} = \tau_{xy}^{1} \frac{\partial \gamma_{xy}^{2*}}{\partial t} \\
\end{align*}
\]

(3.26b)

Adding Eq. (3.26a) into Eq. (3.26b) and upon rearrangement, the differential form complex reciprocity relations for SH waves is given as

\[
\begin{align*}
& \frac{\partial}{\partial x} \left( v_y^{2*} \tau_{xy}^{1} + v_y^{1} \tau_{xy}^{2*} \right) + \frac{\partial}{\partial z} \left( v_y^{2*} \tau_{yz}^{1} + v_y^{1} \tau_{yz}^{2*} \right) \\
& = \frac{\partial}{\partial t} \left( \rho v_y^{1} v_y^{2*} + \tau_{yz}^{2*} \gamma_{yz}^{1} + \tau_{xy}^{2*} \gamma_{xy}^{1} \right) - \left( v_y^{2*} f_y^{1} + v_y^{1} f_y^{2*} \right)
\end{align*}
\]

(3.27)

For the steady-state time-harmonic fields, Eq. (3.27) can be simplified as

\[
\begin{align*}
& \frac{\partial}{\partial x} \left( v_y^{2*} \tau_{xy}^{1} + v_y^{1} \tau_{xy}^{2*} \right) + \frac{\partial}{\partial z} \left( v_y^{2*} \tau_{yz}^{1} + v_y^{1} \tau_{yz}^{2*} \right) = - \left( v_y^{2*} f_y^{1} + v_y^{1} f_y^{2*} \right)
\end{align*}
\]

(3.28a)

Or

\[
\begin{align*}
& i\omega \frac{\partial}{\partial x} \left( u_y^{2*} \tau_{xy}^{1} - u_y^{1} \tau_{xy}^{2*} \right) + i\omega \frac{\partial}{\partial z} \left( u_y^{2*} \tau_{yz}^{1} - u_y^{1} \tau_{yz}^{2*} \right) = -i\omega \left( u_y^{2*} f_y^{1} - u_y^{1} f_y^{2*} \right)
\end{align*}
\]

(3.28b)
The steady-state time-harmonic fields integral form complex reciprocity relations for SH waves can be easily obtained by substituting Eq. (3.23) and Eq. (3.24) into Eq. (3.17a) or Eq. (3.17b) as

\[
\int_S \left( v_y^{2*} \tau_{xy}^1 + v_y^1 \tau_{xy}^{2*} + v_y^{2*} \tau_{xz}^1 + v_y^1 \tau_{xz}^{2*} \right) \cdot \mathbf{n} \, dS = -\int_v \left( v_y^{2*} f_x^1 + v_y^1 f_x^{2*} \right) \, dV
\]  

(3.29a)

Or

\[
i\omega \int_S \left( u_y^{2*} \tau_{xy}^1 - u_y^1 \tau_{xy}^{2*} + u_y^{2*} \tau_{xz}^1 - u_y^1 \tau_{xz}^{2*} \right) \cdot \mathbf{n} \, dS = -i\omega \int_v \left( u_y^{2*} f_x^1 - u_y^1 f_x^{2*} \right) \, dV
\]  

(3.29a)

### 3.3 Orthogonality Relations

In this Section, the orthogonality relation for wave propagation in elastic plates is derived with the assumptions that both wave fields 1 and 2 are time-harmonic and space-harmonic in \( x \) (the time-harmonic term \( e^{i\omega t} \) is omitted in the context herein). The wave field 1 is in the form

\[
v_1(x, z) = v_m(z) \, e^{ik_x x}
\]

\[
\sigma_1(x, z) = \sigma_m(z) \, e^{ik_x x}
\]  

(3.30)

The wave field 2 is taken to be

\[
v_2^*(x, z) = v_n^*(z) \, e^{-ik_x x}
\]

\[
\sigma_2^*(x, z) = \sigma_n^*(z) \, e^{-ik_x x}
\]  

(3.31)

where \( m, n \) may be positive or negative integers, respectively. The domain for applying the integral form complex reciprocity relations (Eq. (3.17a) or Eq. (3.17b)) for both wave fields
is considered between \( x = 0 \) and a given arbitrary \( x \), \(-h/2 \leq z \leq h/2\), as shown in Figure 3.2.

Since the top and bottom surfaces \((z = \pm h/2)\) of the plate are traction-free, the surface integrations at \( z = \pm h/2 \) in the integral form complex reciprocity relation disappear. At \( x = x \), the surface normal \( \hat{n}_x = (1 \ 0 \ 0)^T \) and at \( x = 0 \), the surface normal is \( \hat{n}_{x=0} = (-1 \ 0 \ 0)^T \). Therefore, applying the integral form complex reciprocity relation Eq. (3.17a) or Eq. (3.17b) on both the wave fields and set the body force term \( f_1 = f_2 = 0 \) yields

\[
\int_{-h/2}^{h/2} F_{mn} \bigg|_{x=x} dz - \int_{-h/2}^{h/2} F_{mn} \bigg|_{x=0} dz = 0
\]

(3.32)

where

\[
F_{mn} = \left[V^*_n(z) \cdot \sigma_m(z) + V_m(z) \cdot \sigma^*_n(z)\right] \cdot n \ e^{i(k_n - k_m)x}
\]

(3.33a)
Or

\[ F_{mn} = i\omega [u_m^*(z) \cdot \sigma_m(z) - u_n(z) \cdot \sigma_n^*(z)] \cdot n_x e^{i(k_n - k_n^*)x} \]  

(3.33b)

Substituting Eq. (3.33a) and Eq. (3.33b) into Eq. (3.32) leads to the following relation

\[ 4P_{mn}[e^{i(k_n - k_n^*)x} - 1] = 0 \]  

(3.34)

where

\[ P_{mn} = -\frac{1}{4} \int_{-h/2}^{h/2} [v_m^*(z) \cdot \sigma_m(z) + v_m(z) \cdot \sigma_m^*(z)] \cdot n_x \, dz \]  

(3.35)

or alternatively

\[ P_{mn} = -\frac{i\omega}{4} \int_{-h/2}^{h/2} [u_m^*(z) \cdot \sigma_m(z) - u_n(z) \cdot \sigma_n^*(z)] \cdot n_x \, dz \]  

(3.36)

If \( k_m \neq k_n^* \), then Eq. (3.34) becomes an generic orthogonality relation for guided wave propagation in plates, i.e.,

\[ P_{mn} = 0 \quad \text{if} \quad k_m \neq k_n^* \]  

(3.37)

In the following, the orthogonality relations for the propagating wave modes with real wavenumbers, the evanescent waves with pure imaginary wavenumbers or complex wavenumber are described individually.

(a) For propagating modes, the wavenumbers \( k_m \) and \( k_n \) are real. The orthogonality relation among propagating wave modes holds if

\[ m \neq n \]  

(3.38)
(b) For evanescent modes with pure imaginary wavenumbers \( k_m \) and \( k_n \), the orthogonality relation among evanescent wave modes holds if

\[
m \neq -n
\]

(c) For evanescent modes with complex wavenumbers \( k_m \) and \( k_n \), the orthogonality relation among evanescent wave modes holds if

\[
m \neq n, -n
\]

(d) The orthogonality condition always holds between propagating modes and evanescent modes, and between evanescent modes with pure imaginary wavenumbers and evanescent modes with complex wave numbers, respectively. Since \( k_m \neq k_n^* \) occurs naturally and Eq. (3.34) is satisfied automatically.

A special case where \( m = n \) is considered to study the physical meaning of \( P_{mm} \). Let \( m = n, P_{mm} \) is given by

\[
P_{mm} = -\frac{1}{4} \int_{-h/2}^{h/2} \left( v_m^+ \cdot \sigma_m^+ + v_m^- \cdot \sigma_m^- \right) \cdot n_x dz
\]

\[
= Re \left( \int_{-h/2}^{h/2} -\frac{v_m^+ \cdot \sigma_m^+}{2} n_x dz \right)
\]

\[
= Re \left( \int_{-h/2}^{h/2} P \cdot n_x dz \right) = Re(P_m)
\]

where \( Re \) denotes the real part and

\[
P = -\frac{v_m^+ \cdot \sigma_m}{2}
\]

is defined as the complex acoustic Poynting vector, and
\[ P_m = \int_{-h/2}^{h/2} \mathbf{P} \cdot \mathbf{n}_z dz \]  

(3.43)

is the *complex power flow* for guided wave modes in plates. The *complex acoustic Poynting vector* and *complex power flow* will be discussed in detail in Section 3.4. If \( m \) is a *propagating mode*, since \( m = n \), then \( k_m = k_n^* \) and Eq. (3.37) no longer holds and \( P_{mm} \) has a nonzero value. Thus, from Eq. (3.41), the real part of the complex power flow for propagating guide waves through the plate has a nonzero real value and can propagate power out, i.e.,

\[ P_{mm} = \text{Re}(P_m) \neq 0 \]  

(3.44)

If \( m \) is a *non-propagating (evanescent) mode*, \( k_m - k_m^* \neq 0 \), then the orthogonality relations still applies and \( P_{nm} = 0 \). Thus, from Eq. (3.41), it is found that the real part of complex power flow for evanescent guided waves is zero.

\[ P_{mm} = \text{Re} (P_m) = 0 \]  

(3.45)

Therefore, \( P_{mm} \) is a measure of real power flow for guided waves propagating through the plate.

### 3.3.1 Orthogonality Relations for Lamb Waves

Recall Eq. (3.30) and Eq. (3.31), the solutions for wave field 1 and 2 for Lamb waves are

\[ \mathbf{v}_1(x,z) = [v_x^m(z) \mathbf{i} + v_z^m(z) \mathbf{k}] e^{ik_m z} \]  

(3.46a)
\[ \mathbf{\sigma}_i(x, z) = \begin{pmatrix} \sigma^m_x(z) & 0 & \tau^m_{xz}(z) \\ 0 & 0 & 0 \\ \tau^m_{xz}(z) & 0 & \sigma^m_z(z) \end{pmatrix} e^{ik_x x} \]  
(3.46b)

\[ \mathbf{v}^*(x, z) = \left[ v^m_x(z) \mathbf{i} + v^n_z(z) k \right] e^{-ik_x x} \]  
(3.47a)

\[ \mathbf{\sigma}_2(x, z) = \begin{pmatrix} \sigma^m_x(z) & 0 & \tau^m_{xz}(z) \\ 0 & 0 & 0 \\ \tau^m_{xz}(z) & 0 & \sigma^m_z(z) \end{pmatrix} e^{-ik_x x} \]  
(3.47b)

Substituting Eq. (3.46) and Eq. (3.47) into Eq. (3.32) to obtain

\[ 4[e^{i(k_n-k_n) x} - 1]P_{mn} = 0 \]  
(3.48)

where

\[ P_{mn} = -\frac{1}{4} \int_{-h/2}^{h/2} \left[ v^m_x(z)\sigma_x^m(z) + v^n_x(z)\tau_{xz}^m(z) + v^m_x(z)\sigma_x^m(z) + v^n_x(z)\tau_{xz}^m(z) \right] dz \]  
(3.49a)

or

\[ P_{mn} = -\frac{i\omega}{4} \int_{-h/2}^{h/2} \left[ u^m_x(z)\sigma_x^m(z) + u^n_x(z)\tau_{xz}^m(z) - u^m_x(z)\sigma_x^m(z) - u^n_x(z)\tau_{xz}^m(z) \right] dz \]  
(3.49b)

Eq. (3.48) defines the orthogonality relation for Lamb waves. The orthogonality relation for propagating and evanescent Lamb wave modes follows the same discussions as the generic orthogonality relations for propagating and evanescent guided wave modes.

(I) For propagating Lamb waves, if two modes are identical, i.e., \( m = n \), \( P_{mn} \) can be obtained from Eq. (3.46a) or Eq. (3.46b) as

\[ P_{mn} = \text{Re} \left[ \int_{-h/2}^{h/2} -\frac{v_x^m(z)\sigma_x^m(z) + v_z^m(z)\tau_{xz}^m(z)}{2} \right] = \text{Re}(P_m) \]  
(3.50a)
Or

\[ P_{mn} = \text{Re} \left[ -i \omega \int_{-h/2}^{h/2} \frac{u_x^m(z)\sigma_x^m(z) - u_z^m(z)\tau_x^m(z)}{2} dz \right] = \text{Re} (P_m) \]  

(3.50b)

Therefore, for propagating Lamb wave, \( P_{mn} \) represents for the real power flow carried by propagating Lamb wave modes.

(II) For evanescent Lamb waves, if two modes are identical, i.e., \( m = n \) and \( k_m = k_n^* \), the orthogonality relation requires

\[ P_{mn} = \text{Re} \left[ \int_{-h/2}^{h/2} \frac{v_x^m(z)\sigma_x^m(z) + v_z^m(z)\tau_x^m(z)}{2} dz \right] = \text{Re} (P_m) = 0 \]  

(3.51)

Thus, evanescent Lamb wave modes cannot carry real energy and propagate energy out.

3.3.2 Orthogonality Relation for SH Waves

Recall Eq. (3.30) and Eq. (3.31), the solutions for wave field 1 and 2 for SH waves are

\[ \mathbf{v}_1(x, z) = v_y^m(z) \mathbf{j} e^{ik_m x} \]  

(3.52a)

\[ \mathbf{\sigma}_1(x, z) = \begin{pmatrix} 0 & \tau_x^m(z) & 0 \\ \tau_{xy}^m(z) & 0 & \tau_y^m(z) \\ 0 & \tau_{yz}^m(z) & 0 \end{pmatrix} e^{ik_m x} \]  

(3.52b)

\[ \mathbf{v}_2(x, z) = v_y^{n*}(z) \mathbf{j} e^{-ik_m x} \]  

(3.53a)

\[ \mathbf{\sigma}_2(x, z) = \begin{pmatrix} 0 & \tau_x^{n*}(z) & 0 \\ \tau_{xy}^{n*}(z) & 0 & \tau_y^{n*}(z) \\ 0 & \tau_{yz}^{n*}(z) & 0 \end{pmatrix} e^{-ik_m x} \]  

(3.53b)

Substituting Eq. (3.52) and Eq. (3.53) into Eq. (3.32) to obtain
\[ 4[e^{i(k_m-k_n)x} - 1]P_{mn} = 0 \] (3.54)

where

\[ P_{mn} = \frac{1}{4} \int_{-h/2}^{h/2} [v_y^m(z)v_{y}^{*n}(z) + v_y^n(z)v_{y}^{*m}(z)] \, dz \] (3.55a)

Or

\[ P_{mn} = -\frac{i\omega}{4} \int_{-h/2}^{h/2} [u_y^m(z)\tau_{xy}^m(z) - u_y^n(z)\tau_{xy}^n(z)] \, dz \] (3.55b)

Eq. (3.54) defines the orthogonality relation for SH wave modes in plates. The orthogonality relation for propagating and evanescent SH modes follows the same discussions as the generic orthogonality relations for propagating and evanescent guided wave modes in plates.

(I) For propagating SH waves, if two modes are identical, i.e., \( m = n \), \( P_{mn} \) can be obtained from Eq. (3.55a) or Eq. (3.55b) as

\[ P_{mn} = \text{Re} \left[ \frac{1}{4} \int_{-h/2}^{h/2} \frac{v_y^m(z)v_{y}^{*n}(z)}{2} \, dz \right] = \text{Re}(P_m) \] (3.56a)

or

\[ P_{mn} = -\frac{i\omega}{4} \int_{-h/2}^{h/2} \frac{u_y^m(z)\tau_{xy}^m(z)}{2} \, dz = \text{Re}(P_m) \] (3.56b)

Therefore, for propagating SH modes, \( P_{mn} \) represents for the real power flow carried by propagating SH modes.

(I) For evanescent SH waves, if two modes are the same, i.e., \( k_m = k_n^{*} \), \( P_{mn} \) has to be zero to satisfy the orthogonality relations and
Thus, no real power flow exists in the evanescent SH modes and the power for evanescent SH modes is stored.

### 3.4 Power Flow

In a number of elastic wave propagation applications, it is very useful to define the power flow in an elastic medium. As a force \( \mathbf{F} \) acts on a material particle moving with velocity \( \mathbf{v} \) (\( \mathbf{v} = \mathbf{\hat{u}} \)), the power delivered by the force can be defined by \( \mathbf{F} \cdot \mathbf{v} \). If a surface element \( dS \) is considered, the force acting on the surface is then \( \mathbf{T} dS = \mathbf{\sigma} \cdot \mathbf{\hat{n}} dS = \mathbf{\sigma} \cdot dS \), where \( \mathbf{T} \) is the traction, \( \mathbf{\hat{n}} \) is the normal direction to the surface \( S \), and the vector \( dS \) points outward normal to the surface. The power transmitted at the surface is thereby \( \mathbf{v} \cdot \mathbf{\sigma} \cdot dS \). The power flow into the volume can be obtained by adding a minus sign on the power transmitted at the surface. The power flow density is therefore given by

\[
\mathbf{P} = -\mathbf{\sigma} \cdot \mathbf{v} = -\mathbf{\sigma} \cdot \mathbf{\hat{u}}
\]  

Eq. (3.58) is the elastodynamic equivalent of the Poynting vector in electromagnetics and is called the acoustic Poynting vector. The power that flows outward through a surface is thus

\[
P = \int_{S} -\mathbf{\sigma} \cdot \mathbf{\hat{n}} \ dS = \int_{S} -\mathbf{\sigma} \cdot \mathbf{\hat{u}} \cdot \mathbf{\hat{n}} \ dS
\]  

The instantaneous power flow density into the elastic medium is given by

\[
P(t) = -\mathbf{\sigma}(t) \cdot \mathbf{v}(t)
\]
Eq. (3.60) applies to stresses and velocities with arbitrary functions. For time-harmonic stresses and velocities

\[
\mathbf{\sigma}(t) = |\mathbf{\sigma}_0|\cos(\omega t + \phi_\mathbf{\sigma}) = \frac{\mathbf{\sigma}_0 e^{i\omega t} + \mathbf{\sigma}_0^* e^{-i\omega t}}{2} = \text{Re} (\mathbf{\sigma}_0 e^{i\omega t})
\]

(3.61)

\[
\mathbf{v}(t) = |\mathbf{v}_0|\cos(\omega t + \phi_\mathbf{v}) = \frac{\mathbf{v}_0 e^{i\omega t} + \mathbf{v}_0^* e^{-i\omega t}}{2} = \text{Re} (\mathbf{v}_0 e^{i\omega t})
\]

where the complex numbers

\[
\mathbf{\sigma}_0 = |\mathbf{\sigma}_0|e^{i\phi_\mathbf{\sigma}}
\]

(3.62)

\[
\mathbf{v}_0 = |\mathbf{v}_0|e^{i\phi_\mathbf{v}}
\]

define both the amplitudes and phases of the stress and velocity, respectively.

After substituting Eq. (3.61) into Eq. (3.60), the instantaneous power flow density can be written as

\[
P(t) = \text{Re} \left( -\frac{\mathbf{v}_0^* \cdot \mathbf{\sigma}_0}{2} - \frac{\mathbf{v}_0 \cdot \mathbf{\sigma}_0 e^{2i\omega t} + \mathbf{v}_0^* \cdot \mathbf{\sigma}_0 e^{-2i\omega t}}{4} \right)
\]

(3.63)

\[
= \text{Re} \left( -\frac{\mathbf{v}_0^* \cdot \mathbf{\sigma}_0}{2}(1 + \cos 2\omega t) + \text{Im} \left( -\frac{\mathbf{v}_0^* \cdot \mathbf{\sigma}_0}{2} \right) \sin(2\omega t) \right)
\]

For a source such as traction or displacement that can excite the elastic wave field, the first term in Eq. (3.63) represents the power irretrievably propagated to infinity and it has a time-averaged value

\[
P_{av} = \frac{1}{T} \int_0^T P(t) \, dt = \text{Re} \left( -\frac{\mathbf{v}_0^* \cdot \mathbf{\sigma}_0}{2} \right) = P_R
\]

(3.64)

where \( T \) is the period of excitation. The second term stands for the reactive power that oscillates between the source and the field. The time average of the reactive power is zero but
it has a peak value which is the amplitude of the reversible or reactive power and is defined as

\[ P_I = \text{Im} \left( -\frac{v_0^* \cdot \sigma_0}{2} \right) \]  \hspace{1cm} (3.65)

At every instant, the summation of the two terms in Eq. (3.63) is equal to the total instantaneous power flow. The instantaneous power form (Eq. (3.63)) is completely described by the average radiative power and the peak reactive power as

\[ \mathbf{P} = P_R + iP_I = -\frac{v_0^* \cdot \sigma_0}{2} \]  \hspace{1cm} (3.66)

And this defines the complex Poynting vector, and the complex power flow outward through a surface is

\[ P_c = \int_S -\frac{(v_0^* \cdot \sigma_0) \cdot \hat{n}}{2} dS \]  \hspace{1cm} (3.67)

The physical basis for the separation of Eq. (3.66) is similar to that in circuit theory (Seshadri, 1971) where the first term would represent the instantaneous power delivered to the resistor and the second term would represent the power associated with the discharging and charging of a capacitor. A similar statement (Seshadri, 2008) was made in electromagnetic field that if a source generates complex power, the real part of the complex power will become the radiative power through the propagation of propagating wave while the imaginary part of the complex power is the reactive one that oscillates between the source and the field in the form of evanescent waves.
If a 2-D elastic plate is excited by traction or displacement at the left edge \((x = 0)\), as shown in Figure 3.3, from Eq. (3.63), the instantaneous complex power flow into the plate due to source excitation through the surface at \(x = 0\) can then be calculated as

\[
P^0_0(t) = \int_{-h/2}^{h/2} \left[ \text{Re} \left( i\omega \frac{u_0^* \cdot T_0}{2} \right) (1 + \cos 2\omega t) + \text{Im} \left( i\omega \frac{u_0^* \cdot T_0}{2} \right) \sin(2\omega t) \right] \, dz
\]

The complex power flow into plate is then obtained from Eq. (3.67) as

\[
P^0_c = i\omega \int_{-h/2}^{h/2} \frac{(u_0^*(z) \cdot T_0(z))}{2} \, dz = P^0_R + P^0_I
\]

where \(P^0_R\) represents average-power flow into the plate, and \(P^0_I\) denotes the peak reactive power that is provided to the plate. Both the real power and reactive power may be delivered into the plate. As the guided waves are excited in the plate, the guided waves propagate in the plate in the forms of propagating and/or evanescent waves, thus the complex power delivered into the plate are related to the propagating and/or evanescent waves. Therefore, the complex power delivered into the plate must be associated with the propagating and evanescent waves. In the following, the complex power flow passing through any vertical cross section of the plate for propagating/evanescent Lamb and SH waves are discussed, respectively.
3.4.1 Power Flow for Lamb Waves in Plates

Consider a plate for Lamb waves propagation shown in Figure 3.4, in the figure, at any arbitrary cross section $x$, the velocity and stress corresponding to Lamb waves are denoted as $v_x$, $v_z$ and $\sigma_x$, $\tau_{xz}$, and $\sigma_z$. The complex acoustic Poynting vector for Lamb waves is then half of the scalar product between velocity and stress matrix

$$P_c = \frac{1}{2} \left( v^* \cdot \sigma \right) = \frac{1}{2} \left( v^*_x \sigma_x + v^*_z \tau_{xz} + v^*_z \sigma_z \right)$$  \hspace{1cm} (3.70)

As Lamb waves passing through any arbitrary position $x$, the power flow through the surface $dydz$ with surface normal $\hat{n}_x$ (shown in Figure 3.4) are given from Eq. (3.67) as

$$P_c = -\frac{1}{2} \left( v^* \cdot \sigma \right) \cdot \hat{n}_x dydz$$  \hspace{1cm} (3.71)

The surface normal $\hat{n}_x$ is defined as

$$\hat{n}_x = (1 \ 0 \ 0)^T$$  \hspace{1cm} (3.72)
Substituting Eq. (3.70) and Eq. (3.72) into Eq. (3.71) yields

$$ P_c = -\frac{1}{2} \int_{-y-h/2}^{y-h/2} \left( v_y^* \sigma_x + v_z^* \tau_{xz} \right) dy dz $$

(3.73)

As the plate is assumed to be plane-strain, velocity and shear stress components are $y$ independent and Lamb wave propagation is $y$-invariant. Therefore, the power flow per unit front length is considered as

$$ P_c = -\frac{1}{2} \int_{-h/2}^{h/2} \left( v_x^* \sigma_x + v_z^* \tau_{xz} \right) dz = -\frac{1}{2} i \omega \int_{-h/2}^{h/2} \left( u_x^* \sigma_x + u_z^* \tau_{xz} \right) dz $$

(3.74)

Eq. (3.74) defines the complex power flow per unit length through any arbitrary position $x$ of Lamb waves propagation in $x$ direction. Now let’s consider the power flow outward through any arbitrary position $x$ for propagating and evanescent Lamb waves, respectively.

![Diagram showing the velocity and stresses for Lamb waves at any arbitrary cross section in a plate.](image-url)
(I) For propagating Lamb waves, as phase relations between Lamb waves displacements and stresses components have been discussed in section 3.2.2, since \( u_x \) is out-of phase with \( \sigma_x \) and \( u_z \) is out-of phase with \( \tau_{xz} \), the complex power flow through any arbitrary position \( x \) is determined from Eq. (3.74) as pure real, thus

\[
P^{\text{PW}} = -\frac{1}{2} i \omega |A_m|^2 \int_{-h/2}^{h/2} [U^{m*}(z) \tau_x^m(z) + W^{m*}(z) \tau_{xz}^m(z)] \, dz
\]  

(3.75)

Therefore, the complex power flow for propagating Lamb waves is pure real, which means the propagating Lamb waves have time-averaged power and can carry energy out. It is also noticed that the real power flow is \( x \) location independent which implies that propagating Lamb waves can propagate energy continuously throughout the plate and this part of power is irreversible.

(II) For evanescent Lamb waves, since \( u_x \) is in phase with \( \sigma_x \) and \( u_z \) is in phase with \( \tau_{xz} \), from Eq. (3.74), the complex power flow through any arbitrary position \( x \) is determined from Eq. (3.74) as pure imaginary, thus

\[
P^{\text{EW}} = iP_i
\]

(3.76)

where

\[
P_i = -\frac{1}{2} \omega |A_m|^2 e^{-2i k_{im} x} \int_{-h/2}^{h/2} [U^{m*}(z) \tau_x^m(z) + W^{m*}(z) \tau_{xz}^m(z)] \, dz
\]  

(3.77)

where \( k_{im} \) represents imaginary part of wavenumber for a given evanescent mode. From Eq. (3.77), it is obvious that the amplitude of the reactive power flow for evanescent Lamb waves is \( x \) position dependent and appears to be an exponentially decaying function (decays
exponentially as propagation distance increases), and this explains why evanescent Lamb waves are exponentially decaying and can only exist in the region near the source. The reactive power flow which cannot be consumed oscillates between the source and the near field.

If Lamb waves are excited by a boundary traction/or displacement applied at the edge, the power flow into the plate due to the source excitation is given by from Eq. (3.67) as

\[
P^0_C = -i\omega \int_{-h/2}^{h/2} \frac{[u^*_0(z) \cdot T_0(z)]}{2} dz
\]

\[
= \frac{1}{2} i\omega \int_{-h/2}^{h/2} (u_0^{0*} \sigma_{xx}^0 + u_0^{0*} \tau_{xz}^0) dz
\]

\[
= P^0_R + iP^0_I
\]

In order to calculate the complex power flow into the plate, either the applied traction \( T_0 \), or displacement \( u_0 \) needs to be known as well as the displacement \( u_0 \) or stress \( \sigma_0 \) corresponding to different excitation. Since the propagating Lamb waves carry pure real power which is position independent, the propagating Lamb waves contribute to the real part of the complex power provided by the source and radiate it to infinity, and

\[
P^{pw} = P^0_R
\]

As the complex power flow for evanescent waves is determined as pure imaginary from Eq. (3.76), the evanescent Lamb waves contribute to the imaginary part of the complex power input to the plate from the source.
3.4.2 Power Flow for SH Waves in Plates

Consider a plate for SH waves propagation shown in Figure 3.5, in the figure, at any arbitrary cross section $x$, the velocity and shear stress corresponding to SH waves are denoted as $v_y$ and $\tau_{xy}$ and $\tau_{yz}$. The complex acoustic Poynting vector for SH waves is then half of the scalar product between velocity and stress matrix

$$\frac{-1}{2}(v^* \cdot \sigma) = \frac{-1}{2}(v_y^* \tau_{xy}, 0, v_y^* \tau_{yz})$$

(3.80)

As SH waves passing through any arbitrary position $x$, the power flow through the surface $dydz$ with surface normal $\hat{n}_y$ (shown in Figure 3.5) are given from Eq. (3.67) as

$$P_c = -\frac{1}{2} \int_{-y-h/2}^{y} \int_{-h/2}^{h/2} (v^* \cdot \sigma) \cdot \hat{n}_y dydz$$

(3.81)

The surface normal $\hat{n}_y$ is defined as

$$\hat{n}_y = (1 \ 0 \ 0)^T$$

(3.82)

Substituting Eq. (3.80) and Eq. (3.82) into Eq. (3.81) yields

$$P_c = -\frac{1}{2} \int_{-y-h/2}^{y} \int_{-h/2}^{h/2} v_y^* \tau_{xy} dydz$$

(3.83)

Since both the velocity and shear stress components are $y$ independent, SH wave propagation is $y$-invariant. Therefore, the power flow per unit front length is considered as

$$P_c = -\frac{1}{2} \int_{-h/2}^{h/2} v_y^* \tau_{xy} dz = -\frac{1}{2} \int_{-h/2}^{h/2} u_y^* \tau_{xy} dz$$

(3.84)
Eq. (3.84) defines the power flow per unit length through any arbitrary position $x$ of SH waves propagation in $x$ direction. Now the power flow outward through any arbitrary position $x$ for propagating and evanescent SH waves is considered, respectively. As the displacement and stresses responses for SH waves are much simpler than Lamb waves, analytical solution of the power flow for SH waves can be obtained easily.

Figure 3.5 The velocity and shear stresses for SH waves at any arbitrary cross section in a plate.

(I) For propagating SH waves, after substituting Eq. (2.56) and Eq. (2.58a) (wavenumber $k$ is real) into Eq. (3.84), the complex power flow through any arbitrary position $x$ is determined as
\[ P_c = \frac{1}{2} G\omega |A_m|^2 k_m^R h \int_{-h/2}^{h/2} \cos^2 \left( \frac{m\pi z}{h} \right) dz \]

(3.85a)

\[ = \frac{1}{4} G\omega |A_m|^2 h k_m^R \]

for symmetric modes, and

\[ P_c = \frac{1}{2} G\omega |A_m|^2 k_m^R h \int_{-h/2}^{h/2} \sin^2 \left( \frac{m\pi z}{h} \right) dz \]

(3.85b)

\[ = \frac{1}{4} G\omega |A_m|^2 h k_m^R \]

Hence, the power flow for propagating SH modes can be written as

\[ P^{PW} = \frac{1}{4} G\omega |A_m|^2 h k_m^R \]

(3.86)

where \( m = 0, 2, 4 \ldots \) for symmetric modes, \( m = 1, 3, 5 \) for antisymmetric modes and \( k_m^R \) denotes real wavenumber. As already discussed above, the pure real power flow means the propagating wave power can be radiated to infinity and is irreversible.

(II) Similarly, the complex power flow for evanescent SH modes passing through any arbitrary position \( x \) is obtained by substituting pure imaginary wavenumber \( k \) into Eq. (2.56) and Eq. (2.58a) and subsequently into Eq. (3.84) to get

\[ P_c = \frac{1}{2} i G\omega |A_m|^2 k_m^I e^{-2k_m^I x} h \int_{-h/2}^{h/2} \cos^2 \left( \frac{m\pi z}{h} \right) dz \]

(3.87a)

\[ = \frac{1}{4} i G\omega h |A_m|^2 k_m^I e^{-2k_m^I x} \]

for symmetric modes, and
\[ P_c = \frac{1}{2} i G \omega |A_m|^2 k^t_m e^{-2i\mu_k x} \int_{-h/2}^{h/2} \sin^2 \left( \frac{m\pi z}{h} \right) dz \]

\[ = \frac{1}{4} i G \omega h |A_m|^2 k^t_m e^{-2i\mu_k x} \]

Therefore,

\[ P^{EW} = iP_I \] \hspace{1cm} (3.88)

where

\[ P_I = \frac{1}{4} G \omega h |A_m|^2 k^t_m e^{-2i\mu_k x} \] \hspace{1cm} (3.89)

where \( k^t_m \) represents imaginary part of the wavenumber. From Eq. (3.89), it is obvious that the complex power flow for evanescent SH modes is \( x \) location dependent and exponentially decays as the propagation direction increases. The pure imaginary power flow stands for the reactive power which cannot be consumed and is stored in the plate.

If SH waves are excited by a boundary traction/or displacement applied at the edge, the complex power flow into the plate due to the source excitation is given by from Eq. (3.67) as

\[ P_c^0 = i \omega \int_{-h/2}^{h/2} \frac{u_y(z) \cdot T_3(z)}{2} dz \]

\[ = \frac{1}{2} i \omega \int_{-h/2}^{h/2} u_y^0 \tau_{xy}^0 dz \]

\[ = P_R^0 + iP_I^0 \] \hspace{1cm} (3.90)

Similar to the analysis for Lamb waves, the real part of the complex power input is attributed from the propagating SH waves which have pure real power flow and

\[ P^{pw} = P_R^0 \] \hspace{1cm} (3.91)
Since evanescent SH waves carry pure imaginary power flow, evanescent SH modes contribute to the imaginary part of the complex power input.

3.5 Summary

In this Chapter, the generic differential form complex reciprocity relation for guided waves in plates is first derived using the equation of motion and the constitutive relation (strain-stress relation) for elastic solids. After applying the Gauss’s theorem on the differential form complex reciprocity relation, the integral form complex reciprocity relation is obtained. The differential form reciprocity relation is convenient for studying the waveguide excited by boundary excitations, i.e., traction or displacement excitations. The complex reciprocity relation is then derived for Lamb and SH waves, respectively. From the complex reciprocity relation, the orthogonality relations between waveguide modes are proved and are discussed individually for propagating, evanescent modes. The orthogonality relation for Lamb and SH waves is then readily obtained. After revisiting the expression for $P_{mm}$, it is found that $P_{mm}$ stands for the real power flow for any guided modes propagating in plates. The real power flow for propagating guided modes has nonzero real value while for evanescent guided modes the real power flow is zero. In Section 3.4, the power flow in an elastic medium is first defined, and after calculating the instantaneous power flow density into an elastic medium, it is found that the instantaneous power flow density is completely described by the real power flow density (average) and the imaginary part of the power flow density and thus a complex power flow density for an elastic medium is defined. The real
power flow represents the radiative power and imaginary power flow is the reactive power that cannot be consumed. The complex power flow for guided waves in plates is then discussed in detail and the expressions for propagating and evanescent guided waves are derived, respectively. It should be addressed that the complex power flow for propagating guided waves is pure real and is position independent and the complex power flow for evanescent guided waves is pure imaginary which exhibits exponentially decaying as propagation distance increases. The propagating guided waves are thereby contribute to the real part of the complex power delivered to the plate and the imaginary part of the complex power input to the plate is attributed to the evanescent guided waves. The propagating guided waves can carry energy to the far-field of the plate and evanescent waves localize the reactive power which oscillates between the source and the near fields.
Chapter 4

Conversion of Evanescent into Propagating Lamb Waves in Plates
4.1 Introduction

In this Chapter, in order to investigate the conversion of evanescent into propagating Lamb waves, a means of generating a pure evanescent Lamb wave field is required. Imposing time-harmonic Lamb evanescent displacements/tractions at the edge of a 2-D semi-infinite plate is proposed to generate evanescent Lamb wave field. The evanescent Lamb wave generation is first theoretically studied by using the complex reciprocity theorem and is then verified by the FEA. The conversion of evanescent into propagating Lamb waves is then demonstrated by prescribing the evanescent displacements/stresses distributions through a narrow aperture at the edge of a 2-D semi-infinite plate. The amplitude coefficient of the converted propagating wave is determined by using a theoretical model based on the complex reciprocity theorem with the aid of the FEA and is validated by proving a complex power conservation relation. Finally, the power analysis is performed to calculate the power conversion efficiency for propagating Lamb waves from the evanescent excitations.

4.2 Generation of Evanescent Lamb Waves

As discussed in Section 3.2, from Eq. (3.17b) the waveguide can be excited by prescribed tractions $T = \sigma \cdot n$ or prescribed displacement $u$ at the plate boundaries. A time-harmonic evanescent displacement or traction excitation prescribed at the left boundary ($x = 0$) of a 2-D semi-infinite plate, as shown in Figure 4.1, is proposed to generate a purely Lamb evanescent field. For the semi-infinite plate under time-harmonic motion represented by $e^{-i \omega t}$,
and \( \text{Im}(k) > 0 \) represent a forward decaying evanescent wave away from the boundary in the forward \( x \) direction.

![Schematic view of a semi-infinite plate of thickness \( h \) where Lamb evanescent displacement or traction excitations are prescribed at the edge across the thickness to generate a purely Lamb evanescent field.](image)

**Figure 4.1**

4.2.1 Through Edge Displacement Excitations

4.2.1.1 Theoretical Formulation

Two wave fields 1 and 2 are required to use the complex reciprocity relation for Lamb waves. The solutions for wave field 1 (real field) is the field generated from an actual excitation, i.e., evanescent displacement excitation in this case, and the excited field in the waveguide can be represented by the mode expansion as

\[
\mathbf{u}_1(x, z) = \sum_{m=0}^{\infty} [u_x^m(z) \mathbf{i} + u_z^m(z) \mathbf{k}] e^{ik_m x} \tag{4.1}
\]

\[
\mathbf{\sigma}_1(x, z) = \sum_{m=0}^{\infty} \begin{pmatrix}
\sigma_x^m(z) & 0 & \tau_x^m(z) \\
0 & 0 & 0 \\
\tau_x^m(z) & 0 & \sigma_z^m(z)
\end{pmatrix} e^{ik_m x} \tag{4.2}
\]
The wave field 2 is chosen to be a specific virtual mode.

\[
\mathbf{u}_x^*(x, z) = [u_x^n(z) \mathbf{i} + u_z^n(z) \mathbf{k}] e^{-ik_x z} \tag{4.3}
\]

\[
\mathbf{\sigma}_x^*(x, z) = \begin{pmatrix} \sigma_x^n(z) & 0 & \tau_{xz}^n(z) \\ 0 & 0 & 0 \\ \tau_{xz}^n(z) & 0 & \sigma_z^n(z) \end{pmatrix} e^{-ik_x z} \tag{4.4}
\]

Substituting Eq. (4.1) - Eq. (4.4) into Eq. (3.30), the complex reciprocity relation at \( x = 0 \) gives

\[
\int_{-h/2}^{h/2} F_{mn} \bigg|_{x=0} dz = -\sum_{m=0}^{\infty} i\omega \int_{-h/2}^{h/2} [u_x^n(z)\sigma_x^n(z) + u_z^n(z)\tau_{xz}^n(z) - u_x^m(z)\sigma_x^m(z) - u_z^m(z)\tau_{xz}^m(z)] dz \tag{4.5}
\]

For the Lamb evanescent edge displacement excitations, the Lamb evanescent displacements corresponding to an evanescent Lamb mode applied at \( x = 0 \) can be obtained from Eq. (2.30) as

\[
u_x^{EW}(z) = A^{EW} U_x^{EW}(z) = \sum_{m=0}^{\infty} u_x^m(z) \tag{4.6a}
\]

\[
u_z^{EW}(z) = A^{EW} W_z^{EW}(z) = \sum_{m=0}^{\infty} u_z^m(z) \tag{4.6b}
\]

The corresponding stresses at \( x = 0 \) can be expressed via Eq. (2.31)

\[
\sigma_x^0(z) = A^{EW} \sigma_x^0(z) = \sum_{m=0}^{\infty} \sigma_x^m(z) \tag{4.7a}
\]

\[
\tau_{xz}^0(z) = A^{EW} \tau_{xz}^0(z) = \sum_{m=0}^{\infty} \tau_{xz}^m(z) \tag{4.7b}
\]
where $A_{EW}$ is the amplitude coefficient for the applied Lamb evanescent displacement, $U_{EW}(z)$, $W_{EW}(z)$, $t_x^0(z)$, $t_{xz}^0(z)$ can be obtained by substituting wavenumber $k$ which is either complex or purely imaginary into Eq. (2.31) and Eq. (2.33). Substituting Eq. (4.6) and Eq. (4.7) into Eq. (4.5), the complex reciprocity at $x = 0$ can be given by

$$
\int_{-h/2}^{h/2} F_{mn} \bigg|_{x=0} \, dz = -i\omega A_{EW}^{\text{EF}} \int_{-h/2}^{h/2} \left[ u_x^n(z) t_x^0(z) + u_z^n(z) t_{xz}^0(z) \right] \left( -U_{EW}^n(z) \sigma_x^n(z) - W_{EW}^n(z) \tau_{xz}^n(z) \right) \, dz \quad (4.8)
$$

The complex reciprocity relation at $x = x$ becomes

$$
\int_{-h/2}^{h/2} F_{mn} \bigg|_{x=x} \, dz = -\sum_{m=0}^{\infty} i\omega A_m \int_{-h/2}^{h/2} \left[ u_x^n(z) t_x^m(z) + u_z^n(z) t_{xz}^m(z) \right] e^{i(k_x - k_x')x} \bigg|_{x=x} \, dz \quad (4.9)
$$

Substituting Eq. (4.8) and Eq. (4.9) into Eq. (3.30), the complex reciprocity relation for the waveguide excited by Lamb evanescent displacement prescribed at the plate edge can be simplified as

$$
4\sum_{m=0}^{\infty} A_m P_{mn} e^{i(k_x - k_x')x} = i\omega A_{EW}^{\text{EF}} \int_{-h/2}^{h/2} \left[ u_x^n(z) t_x^0(z) + u_z^n(z) t_{xz}^0(z) \right] \left( -U_{EW}^n(z) \sigma_x^n(z) - W_{EW}^n(z) \tau_{xz}^n(z) \right) \, dz \quad (4.10)
$$

where $P_{mn}$ is defined in Eq. (3.46b). If the wave field 2 is take to be a propagating Lamb mode, since the orthogonality condition is automatically satisfied between evanescent modes and propagating Lamb modes, as discussed in Section 3.3, the right-hand side of Eq. (4.10) is always zero. Then using orthogonality condition on the left-hand side of Eq. (47), it follows that

$$
4A_m P_{mn} e^{i(k_x - k_x')x} = 0 \quad (4.11)
$$

and thus
which means there are no propagating Lamb waves survived in the mode expansion of the excited field at \( x = x \) if the plate is excited by the time-harmonic displacement excitation and only purely evanescent Lamb modes can exist in the excited field. Therefore, applying evanescent displacements distributions at the edge of the plate can be used to generate purely evanescent Lamb wave modes.

**4.2.1.2 Finite Element Analysis Verification**

A frequency domain finite element analysis (Structural Mechanics Module, COMSOL Multiphysics) is used to verify the generation of Lamb evanescent field under edge displacement excitation. The governing equation for the frequency domain study is given by

\[
\nabla \cdot \sigma = -\rho \omega^2 \mathbf{u},
\]

(4.13)

For this study, the plate is chosen to be an aluminum plate with thickness \( h = 3 \text{ mm} \), density \( \rho = 2713 \text{ kg/m}^3 \), Young’s modulus \( E = 72.6 \text{ GPa} \) and Poisson’s ratio \( \nu = 0.33 \). Free triangular elements are used and the size of the elements is 0.4 mm, which is less than 1/20th of the wavelength of the \( A_0 \) propagating mode. Generation of \( A_1, S_1/S_2 \) and \( A_2/A_3 \) Lamb evanescent modes at \( \omega h/c_T = 0.0946 \) (\( f_h = 0.3 \text{ MHz}\cdot\text{mm} \)) is investigated. Figures 4.2(a), (b), (c) show the normalized displacement distribution at \( x=0 \) for the generation of Lamb evanescent \( A_1, S_1/S_2 \) and \( A_2/A_3 \) modes, respectively with unit amplitude coefficient, \( A = 1 \). The normalized displacements are given from Eq. (2.40) - Eq. (2.44).
Figure 4.2 Time-harmonic normalized displacement distributions applied at $x = 0$ for the generation of (a) $A_1$ evanescent mode; (b) $S_1/S_2$ evanescent mode; (c) $A_2/A_3$ evanescent mode at $\omega h/c_T = 0.0946$ ($fh = 0.3 \text{ MHz mm}$).
It is seen from Figure 4.2, the displacements are complex values and thus the excitations can be treated as a linear combination of a time-harmonic excitation with amplitude \( \text{Re}(\bar{u}_x)/\text{Re}(\bar{u}_z) \) and another out-of-phase time-harmonic excitation with amplitude \( \text{Im}(\bar{u}_x)/\text{Im}(\bar{u}_z) \).

Therefore, the displacements prescribed at the edge can be written as

\[
\bar{u}_x = \text{Re}(\bar{u}_x)e^{-i\omega t} + \text{Im}(\bar{u}_x)e^{i(\pi/2 - \omega t)}
\]

\[
\bar{u}_z = \text{Re}(\bar{u}_z)e^{-i\omega t} + \text{Im}(\bar{u}_z)e^{i(\pi/2 - \omega t)}
\]

Figures 4.3(a), (b) and (c) show the normalized evanescent displacement fields for \( A_1, S_1/S_2, \) and \( A_2/A_3 \) evanescent modes under evanescent edge displacement excitations at \( \omega h/c_T = 0.0946 (\omega h = 0.3 \text{ MHz:mm}) \), respectively. A perfect matched layer is adopted on the far right region to eliminate any reflection that is not considered in this dissertation. From Figures 4.3(a) and (c), it is shown that the displacement \( u_x \) is antisymmetric and the displacement \( u_z \) is symmetric with respect to the neural axis, respectively, which verifies the displacements distribution of antisymmetric Lamb modes. The displacement \( u_x \) is symmetric and the displacement \( u_z \) is antisymmetric with respect to the neural axis, respectively, as shown in Figure 4.3(b) which represent for a symmetric Lamb modes. The displacement responses of both \( u_x \) and \( u_z \) shown in Figure 4.3(a)-(c) decay exponentially away from the excitation source, which demonstrate the evidence of Lamb evanescent field. Therefore, from the FEA it is confirmed that the time-harmonic edge displacement excitations prescribed at the left boundary can be used to generate Lamb evanescent field which is decayed exponentially away from the boundary. This result agrees with the theoretical formulations discussed in
Section 4.2.1.1 that there are no propagating Lamb waves survived in the mode expansion of the excited field by time-harmonic displacement excitation. Consequently, combining the results obtained from FEA and theoretical formulation, the time-harmonic edge displacement excitations can be used to generate a purely Lamb evanescent field.
Figure 4.3 Evanescent Lamb wave displacement field generated by edge displacement excitations at \( \frac{\omega h}{c_T} = 0.0946 \) (\( f h = 0.3 \text{ MHz·mm} \)) (a) \( A_1 \) evanescent mode; (b) \( S_1/S_2 \) evanescent mode; (c) \( A_2/A_3 \) evanescent mode.
4.2.2 Through Edge Traction Excitations

4.2.2.1 Theoretical Formulation

For the Lamb evanescent edge traction excitations, the Lamb evanescent tractions corresponding to an evanescent Lamb mode applied at $x = 0$ can be obtained from Eq. (2.33) as

$$T^E_x(z) = -A^E t^E_x(z)$$  \hfill (4.16a)

$$T^E_z(z) = -A^E t^E_z(z)$$  \hfill (4.16b)

The corresponding displacements at $x=0$ can be expressed via Eq. (2.30)

$$u^0_x(z) = A^E U^0(z)$$  \hfill (4.17a)

$$u^0_z(z) = A^E W^0(z)$$  \hfill (4.17b)

where $t^E_x(z)$, $t^E_z(z)$, $U^0(z)$, $W^0(z)$ can be obtained by substituting evanescent wavenumber $k$ which is either complex or purely imaginary into Eq. (2.34) and Eq. (2.31).

Similar to the evanescent displacement excitations, the wave field $1$ is now excited by edge evanescent traction excitations, after substituting Eq. (4.16) and Eq. (4.17) into Eq. (3.30), the complex reciprocity relation for waveguide excited by prescribed edge evanescent tractions can be simplified as

$$\sum_{m=0}^{\infty} i\omega \alpha m \int_{-h/2}^{h/2} \left[ U^m(z) t^m_x(z) + W^m(z) t^m_z(z) - U^m(z) t^m_x(z) - W^m(z) t^m_z(z) \right] e^{i(k_0-k_0)^z} dz$$

$$= i\omega A^E \int_{-h/2}^{h/2} \left[ U^E(z) t^E_x(z) + W^E(z) t^E_z(z) - U^E(z) t^E_x(z) - W^E(z) t^E_z(z) \right] dz$$  \hfill (4.18)

After applying orthogonality conditions on Eq. (4.18), it follows that
and thus

\[ A_m = 0 \]  \hspace{1cm} (4.20)

which means there are no propagating Lamb waves survived in the mode expansion of the excited field at \( x = x \) if time-harmonic traction excitations are used to excite the plate. Therefore, applying evanescent traction distributions at the edge of the plate can be used to generate purely evanescent Lamb modes.

### 4.2.2.2 Finite Element Analysis Verification

Similar to evanescent displacement excitations, a finite element analysis is performed to study the generation of evanescent Lamb modes by prescribing time-harmonic traction excitations at \( x = 0 \). The governing equation has been given in Eq. (4.13). The prescribed non-dimensional evanescent traction distribution at \( x = 0 \) can be obtained from Eq. (2.42) and (2.44) as \( \overline{T}_x(z) = -\overline{\sigma}_x(z) \) and \( \overline{T}_z(z) = -\overline{\tau}_z(z) \). The amplitude coefficient \( A \) is chosen to be one for convenience. Figures 4.4(a), 4.4(b), and 4.4(c) show the normalized traction distributions prescribed at \( x = 0 \) for the generation of evanescent \( A_1, S_1/S_2 \) and \( A_2/A_3 \) modes, respectively. Similar to the evanescent displacement excitations, both the tractions \( \overline{T}_x \) and \( \overline{T}_z \) \((z) \) applied at \( x = 0 \) consist of the summation of two time-harmonic excitations which are out of phase with each other and can be written as

\[ \overline{T}_x = \text{Re}(\overline{T}_x)e^{-i\omega x} + \text{Im}(\overline{T}_x)e^{i(\pi/2-\omega x)} \]  \hspace{1cm} (4.21)

\[ \overline{T}_z = \text{Re}(\overline{T}_z)e^{-i\omega x} + \text{Im}(\overline{T}_z)e^{i(\pi/2-\omega x)} \]  \hspace{1cm} (4.22)
Figure 4.4 Time-harmonic normalized traction distribution at $x = 0$ for the generation of 
(a) $A_1$ evanescent mode; (b) $S_1/S_2$ evanescent mode; (c) $A_2/A_3$ evanescent mode at $\omega h/c_T = 0.0946$ ($f h = 0.3 \text{ MHz:mm}$).
Figures 4.5(a), (b) and (c) show the normalized evanescent displacement field for $A_1$, $S_1/S_2$, and $A_2/A_3$ evanescent modes under evanescent edge traction excitations, respectively. Similar to the evanescent displacement excitations, the FEA results shown in Figure 8 confirm that the time-harmonic evanescent edge traction excitations can be used to generate Lamb evanescent field which is decayed exponentially away from the boundary. This result validates the theoretical formulations discussed in Section 4.2.2.1 that propagating Lamb waves vanished in the mode expansion of the excited field by time-harmonic traction excitations. Consequently, applying time-harmonic evanescent traction excitations at the edge is capable generating a purely Lamb evanescent field.
Figure 4.5 Evanescent Lamb wave displacement field generated by edge traction excitations at $\omega h/c_T = 0.0946$ ($fh = 0.3 \text{ MHz} \cdot \text{mm}$), (a) $A_1$ evanescent mode; (b) $S_1/S_2$ evanescent mode; (c) $A_2/A_3$ evanescent mode.
(c)
4.3 Conversion of Evanescent into Propagating Lamb Waves through Narrow Apertures

The generation of purely Lamb evanescent field has been demonstrated by prescribing evanescent displacement/traction distributions at the edge of the plate. For purely evanescent Lamb modes, since they only have reactive power flow which cannot be consumed and decays exponentially as they propagate out, there will be no power that can be radiated to the far-field. In order to convert evanescent waves into propagating so that far-field sensors can receive the real power flow, part of the reactive power that confined in the evanescent waves has to be released and becomes radiative. In this Chapter, the conversion of evanescent Lamb waves into propagating is investigated by prescribing time-harmonic evanescent displacement or traction excitations through a narrow aperture with thickness $h_1 < h$ at the edge of the semi-infinite plate (shown in Figure 4.6(a) and 4.6(b)). As evanescent excitations are imposed in the narrow aperture, in a special case where $h_1 = h$, only evanescent Lamb modes can be generated. However, the sudden thickness increase, say from $h_1$ to $h_2$, enables part of the reactive power to be released and becomes radiative and thus propagating Lamb waves can be converted from the evanescent excitations. The possible propagating Lamb modes that can be converted from evanescent modes can be described by the use of the complex dispersion curve and group velocity dispersive curve shown in Figure 4.7(a) and 4.7(b). Figure 4.8 shows a description of evanescent Lamb wave generation and the conversion of evanescent into propagating Lamb waves with fixed $h_1$ and $h$ by varying the excitation angular frequency $\omega$. Using Figure 4.7 and Figure 4.8, at a given evanescent
excitation frequency, the value of $\omega h / c_r$ and the aperture symmetric or asymmetric with respect to neutral $x$ axis determine the permissible propagating Lamb modes. Below $A_1$ cut-off frequency $\pi$, only propagating $A_0$ and $S_0$ mode can be converted and more propagating Lamb modes can be converted if $\omega h / c_r$ is greater than $\pi$. The conversion process is first investigated by using the FEA. An analytical model based on the complex reciprocity and orthogonality relations with the aid of FEA are then utilized to quantify the amplitude coefficient of the converted propagating modes. The amplitude coefficient is verified through the validation of a complex power conservation relation. Power analysis is conducted to study the power conversion efficiency from evanescent into propagating Lamb waves.
Figure 4.6 Cross section of a semi-infinite plate for studying conversion of evanescent waves into propagating. The plate (thickness = $h$) is subject to evanescent excitations at the left boundary through a small aperture (height = $h_1$). (a) Symmetric aperture with respect to $x$ axis, (b) asymmetric aperture with respect to $x$ axis.
Figure 4.7 (a) Complex dispersion curves and (b) group velocity dispersive curves used for studying converting evanescent into propagating Lamb waves. (The green solid dot denotes the $A_1$ evanescent mode excited at $\omega h_1/c_T$ and the red solid dots represent possible propagating Lamb modes that can be converted depending on the frequency $\omega h/c_T$).
4.3.1 Conversion of Evanescent into Propagating Lamb Waves- Finite Element Modeling

A frequency domain finite element analysis (Structural Mechanics Module, COMSOL Multiphysics) is first adopted to study the conversion of evanescent into propagating Lamb waves as evanescent excitations are incident through the narrow aperture at the plate edge. From Figure 4.7(a), for any given frequency below \( A_1 \) cut-off, \( A_1 \) evanescent mode has the smallest imaginary wavenumber and thus decays the slowest among all evanescent modes. Therefore, for illustration, an \( A_1 \) evanescent mode is chosen to study the conversion of evanescent into propagating Lamb waves. The time-harmonic \( A_1 \) evanescent displacement/traction distribution was applied through an aperture at the left boundary.
For convenience, the aperture height is chosen to be half of the plate thickness ($h_1 = 3\text{mm}$ and $h_1/h = 1/2$) and a perfect matched layer is adopted to avoid any reflection. The excitation frequency is 100 kHz and $A$ is chosen to be $1 \times 10^{-6}$. Figures 4.9(a) and 4.9(b) show the conversion of $A_1$ evanescent Lamb wave into $A_0$ propagating mode under edge evanescent displacement and traction excitations through a narrow aperture, respectively. From the figure, in the region close to the edge of evanescent excitation, the waves still decay very rapidly, and this is because there is an infinite number of non-propagating modes (evanescent waves) generated at $fh = 0.6 \text{MHz:mm}$ and these waves decay exponentially. After the waves propagate for a certain distance, a steady state propagating $A_0$ mode starts to form and is clearly shown in Figures 4.9(a) and (b). Since the aperture is symmetric with respect to $x$ axis and the frequency $\omega h/c_r$ is below $A_1$ cut-off frequency, only $A_0$ propagating mode can be converted and propagate to the far-field. The wavelength of the converted propagating Lamb mode is found to be 20.1 mm which agrees with the wavelength of the converted propagating $A_0$ Lamb mode.
Figure 4.9 Conversion of $A_1$ evanescent Lamb wave into $A_0$ propagating mode through a narrow aperture ($h_1 = 3\text{mm}, h = 6\text{mm}$). Excitation frequency $f = 100 \text{kHz}$. (a) Total displacement field- under displacement excitations, (b) total displacement field- under traction excitations.
4.3.2 Determination of Amplitude Coefficient of the Converted Propagating Lamb Waves-Theoretical Formulations

To quantify how efficient the conversion of evanescent to propagating Lamb waves is, the amplitude coefficient of the propagating modes that are converted from evanescent waves needs to be determined. To this end, wave field 2 is, therefore, chosen as a specific virtual propagating mode to determine the amplitude coefficient of the converted propagating mode.

The known virtual field 2 is assumed as

\[
\mathbf{u}_2^v(x, z) = \left[ u_{x}^{\text{ev}}(z) \mathbf{i} + u_{z}^{\text{ev}}(z) \mathbf{k} \right] e^{-ik_x x}
\]

(4.23)

\[
\mathbf{\sigma}_2^v(x, z) = \begin{pmatrix} \sigma_{x}^{\text{ev}}(z) & 0 & \tau_{xz}^{\text{ev}}(z) \\ 0 & 0 & 0 \\ \tau_{xz}^{\text{ev}}(z) & 0 & \sigma_{z}^{\text{ev}}(z) \end{pmatrix} e^{-ik_z z}
\]

(4.24)

where \( k_n \) is chosen as the wavenumber of the possible converted propagating Lamb wave, and \( u_{x}^{\text{ev}}(z), u_{z}^{\text{ev}}(z), \sigma_{x}^{\text{ev}}(z), \tau_{xz}^{\text{ev}}(z) \) and \( \sigma_{z}^{\text{ev}}(z) \) can be obtained by substituting real \( k_n \) into Eq. (2.30) and Eq. (2.32).

4.3.2.1 Evanescent Displacement Excitations

For evanescent displacement excitations correspond to an evanescent mode prescribed through the narrow aperture which is symmetric with respect to x axis (Figure 4.6(a)), the wave field 1 at \( x=0 \) are

\[
u_{x}^{\text{ev}}(z) = A^{\text{ev}} U^{\text{ev}}(z) \quad (-h_1 / 2 \leq z \leq h_1 / 2)
\]

(4.25)

\[
u_{z}^{\text{ev}}(z) = A^{\text{ev}} W^{\text{ev}}(z) \quad (-h_1 / 2 \leq z \leq h_1 / 2)
\]

(4.26)
The corresponding stresses \( \sigma^0_x(z) \) and \( \tau^0_{xz}(z) \) from \(-h_1/2 \leq z \leq h_1/2\) at \( x = 0 \), and the displacements \( u^0_x(z) \), \( u^0_z(z) \) from \(-h_1/2 \leq z \leq h/2\) and \( h_1/2 \leq z \leq h/2\) at \( x=0 \) can be obtained from the FEA which is presented in Section 4.3.1. Since the region \(-h_1/2 \leq z \leq h/2\) and \( h_1/2 \leq z \leq h/2\) at \( x = 0 \) are traction free, \( \sigma^0_x(z) = 0 \), \( \tau^0_{xz}(z) = 0 \). The wave field \( \mathbf{1} \) at any arbitrary position \( x = x \) can be written as

\[
\mathbf{u}_1(x, z) = \sum_{m=0}^{\infty} A_m \left[ U^m(z) i + W^m(z) k \right] e^{ikx} \tag{4.27}
\]

\[
\mathbf{\sigma}_1(x, z) = \sum_{m=0}^{\infty} A_m \begin{pmatrix} t^m_x(z) & 0 & t^m_z(z) \\ 0 & 0 & 0 \\ t^m_{xz}(z) & 0 & t^m_z(z) \end{pmatrix} e^{ikx} \tag{4.28}
\]

Therefore, after applying the complex reciprocity relation (Eq. (3.30)) on the two wave fields, the complex reciprocity relation for this problem can be simplified as the following relation between the contribution from the integral at \( x=0 \) from evanescent displacement excitation and the integral at \( x=x \)
For evanescent displacements applied through the non-symmetric aperture (Figure 4.6(b)), the complex reciprocity relation can be easily derived as

\[
\sum_{m=0}^{\infty} i \omega A_m \int_{-h/2}^{h/2} \left[ U^{n^*}(z) t^{m^*}_{x}(z) + W^{n^*}(z) t^{m^*}_{z}(z) \right] e^{i(k_m-k'_m)x} \bigg|_{x=x} \, dz \\
= \omega \int_{-h/2}^{h/2} \left[ U^{n^*}(z) \sigma^{0}_{x}(z) + W^{n^*}(z) \tau^{0}_{z}(z) \right] \bigg|_{x=0} \, dz \\
- \omega A^{EW} \int_{-h/2}^{h/2} \left[ U^{EW}(z) t^{n^*}_{x}(z) + W^{EW}(z) t^{n^*}_{z}(z) \right] \bigg|_{x=0} \, dz \\
- 2\omega \int_{h/2}^{h/2} \left[ u^{0}_{x}(z) t^{n^*}_{x}(z) + u^{0}_{z}(z) t^{n^*}_{z}(z) \right] \bigg|_{x=0} \, dz
\]

(4.29)

The actual number of summation on the left side of Eq. (4.29) and Eq. (4.30) is finite since \(x (x > 20h)\) is chosen large enough and only propagating modes (real-valued wavenumbers) prevail at \(x = x\). The complex reciprocity relations of Eq. (4.29) and (4.30) can be further reduced as
\[ 4 \sum_{m=0}^{\infty} A_m P_m e^{i(k_m - k_n)x} = \text{Integral } I + \text{Integral } II + \text{Integral } III \quad (4.31) \]

From the orthogonality condition, the only term that can survive in the summation of Eq. (4.31) is for \( k_m = k_n^* \). Since \( n \) is a propagating mode, the condition for \( k_m = k_n^* \) is \( m = n \). It then follows that

\[ A_m = \frac{\text{Integral } I + \text{Integral } II + \text{Integral } III}{4P_{nn}} \quad (4.32) \]

where \text{Integral } I, \text{Integral } II and \text{Integral } III are defined in Eq.(4.29) and Eq.(4.30) for symmetric and asymmetric aperture, respectively. Eq. (4.32) defines the amplitude coefficient of any converted propagating Lamb wave modes due to evanescent displacement excitations. For a given evanescent mode, the propagating mode that can be converted is either symmetric or antisymmetric if the aperture is symmetric while non-symmetric aperture can be used to convert both symmetric and antisymmetric propagating Lamb modes. It is found from Eq. (4.30) - Eq. (4.32) that the propagating amplitude coefficient \( A_m \) is independent of location, since only the displacements and stresses at \( x = 0 \) contribute to the calculation. Therefore, after knowing the stresses and displacements distributions under evanescent excitations at the plate edge, the amplitude of the converted propagating modes can be easily obtained.

4.3.2.2 Evanescent Traction Excitations

If the evanescent traction excitations are prescribed through the symmetric narrow aperture (Figure 4.6(a)), wave field 1 at \( x = 0 \) are
The corresponding displacements $u_x^0(z), u_z^0(z)$ at $x=0$ under edge traction excitations needs to be obtained from FEA and $\sigma_x^0(z)=0, \tau_x^0(z)=0$ for $-h_1/2 \leq z \leq h/2$ and $h_1/2 \leq z \leq h/2$ at $x=0$. The wave field $\mathbf{1}$ at any arbitrary position $x_0 = x$ is given in Eq. (4.27) and Eq. (4.28). Similar to the evanescent edge displacement excitations and applying the complex reciprocity relations (Eq. (3.30)) on the two wave fields, the complex reciprocity relation for this problem can be simplified as the following relation between the contribution from the integral at $x=0$ from evanescent traction excitations and the integral at $x=x_0$:

$$\sum_{m=0}^{n} \int_{-h/2}^{h/2} \left[ U^n(z) t_x^m(z) + W^n(z) t_{xz}^m(z) \right] e^{i(k_x-k_{x_0})x} \Big|_{x=x_0} dz$$

$$= i\omega A_{ew}^{x} \int_{-h/2}^{h/2} \left[ U^n(z) t_x^{x_0} + W^n(z) t_{xz}^{x_0} \right] \Big|_{x=x_0} dz$$

Similarly, for evanescent tractions applied through a non-symmetric aperture (Figure 4.6(b)), applying the complex reciprocity relation on the waveguide yields
The complex reciprocity relation can be further simplified by using the orthogonality condition as

\[ 4A_m P_{mm} e^{i(k_m - k_n)x} = \text{Integral I + Integral II} \]  \hspace{1cm} (4.37)

Therefore, from Eq. (4.37), the amplitude coefficient of any converted propagating Lamb wave modes due to evanescent edge traction excitations can be written as

\[ A_m = \frac{\text{Integral I + Integral II}}{4P_{mm}} \]  \hspace{1cm} (4.38)

where \text{Integral I} and \text{Integral II} are defined in Eq.(4.35) and Eq.(4.36) for symmetric and non-symmetric aperture, respectively.

### 4.3.3 Power Conversion Efficiency for the Propagating Lamb Waves

The conversion of evanescent into propagating Lamb waves is presented by using the FEA model to investigate the conversion process and an analytical model based on the complex reciprocity theorem via FEA example is proposed to calculate the amplitude of the converted propagating Lamb modes. As propagating Lamb waves can be converted from evanescent excitations, real power can be radiated to the far-field and carried by propagating
Lamb waves. It is crucial to know the amount of propagating power that can be converted and the propagating power conversion efficiency from the evanescent excitations at the plate edge. Figure 4.10 shows the representation of stresses and displacements in the aperture due to evanescent Lamb mode excitation and at any arbitrary far-field section of the plate for the calculation of power delivered to the plate and power flow outward by propagating Lamb modes.

4.3.3.1 Verification on the Amplitude Coefficient \( (A_m) \) of the Converted Propagating Lamb Waves

Before the power conversion efficiency of the converted propagating Lamb waves can be defined, the amplitude coefficient \( (A_m) \) of the converted propagating Lamb waves obtained from Section 4.3.2 has to be validated first. The amplitude coefficient is verified through the
validation of a complex power conservation relation. As the plate is excited either by evanescent displacement or traction excitation through a narrow aperture at the edge (shown in Figure 4.10), the complex power flow that is delivered into the plate through the aperture is

\[
P_{in} = P_{c0} = -\frac{1}{2} i \omega \int_{-h/2}^{h/2} \left[ u_0^e(z) \sigma_x^0(z) + u_0^t(z) \tau_z^0(z) \right] dz = P_R^0 + i P_I^0 \quad (4.40a)
\]

for symmetric aperture, and

\[
P_{in} = P_{c0} = -\frac{1}{2} i \omega \int_{-h/2}^{h/2} \left[ u_0^e(z) \sigma_x^0(z) + u_0^t(z) \tau_z^0(z) \right] dz = P_R^0 + i P_I^0 \quad (4.40b)
\]

for non-symmetric aperture. Where \( P_{c0} \) denotes complex power delivered to the plate due to source excitation, \( P_R^0 \) represents average-power flow into the plate edge, and \( P_I^0 \) denotes the peak reactive power that is provided to the plate. The complex power flow into the plate needs to be evaluated from the FEA, since the analytical solutions for the stresses or displacements in the aperture corresponding to the evanescent displacements or tractions excitation are difficult to obtain. As already discussed in Section 3.4.1, the propagating Lamb waves contribute to the real part of the complex power flow input while the imaginary part of the complex input is attributed to the evanescent waves. Therefore, after the plate is excited, an infinite number of evanescent Lamb waves are generated in the region close to the source with exponentially decaying reactive power away from the source, as well as the converted propagating Lamb waves that can carry radiative energy out. The converted propagating real power flow is the part which can be received by the far-field sensors. The amount of this
position independent real power flow for the converted propagating Lamb waves are given from Eq. (3.70) as

\[ P^{PW} = \sum_{m} -\frac{1}{2} i\omega \int_{-h/2}^{h/2} \left[ u_x^{m*}(z) \sigma_x^{m}(z) + u_z^{m*}(z) \tau_{xz}^{m}(z) \right] dz \]

\[ = \sum_{m} -\frac{1}{2} i\omega |A_m|^2 \int_{-h/2}^{h/2} \left[ U^{m*}(z) t_x^{m}(z) + W^{m*}(z) t_{xz}^{m}(z) \right] dz \]  

(4.41)

where \( A_m \) is the amplitude coefficient for the converted propagating Lamb modes and it has been defined in Eq. (4.32) and Eq. (4.38). As the real part of complex power flow is totally contributed from propagating Lamb waves, a complex power conservation relation must be satisfied, i.e.,

\[ P^{PW} = P_{R}^{0} \]  

(4.42)

The real power flow for propagating Lamb modes must equal to the real part of complex power flow into the plate due to the source excitation. Figure 4.1 shows the comparison of the real part of the complex power flow into the plate edge where it is excited by \( A_1 \) evanescent mode through a symmetric aperture and the real power flow for the converted propagating Lamb modes as a function of normalized frequency. It is seen from both figures, the power flow carried out by propagating Lamb modes match well with the real part of the complex power flow into the plate for both displacement and traction excitations. The validation of this complex power conservation relation verifies the accuracy of the amplitude coefficient (\( A_m \)) of the converted propagating Lamb mode obtained from the theoretical model based on the complex reciprocity theorem via FEA.
Figure 4.11 Comparison of the real part of the complex power into the plate through a symmetric aperture ($h_1 = 3$mm and $h = 6$mm) and the real power flow for converted propagating Lamb waves, (a) $A_1$ evanescent displacement excitations and (b) $A_1$ evanescent traction excitations. (Log power is used in the plots).
\[ \log(P) \]

(a) \( A_1 \) evanescent displacement excitations

(b) \( A_1 \) evanescent traction excitations
4.3.3.2 Propagating Power Conversion Efficiency ($\xi$)

In order to quantify how efficient that propagating power can be converted due to evanescent excitations through the narrow aperture at the plate edge, the propagating power conversion efficiency needs to be defined. Using Figure 4.11, the propagating power conversion efficiency can be defined as the ratio between real power flow outward through any far-filed cross section and the absolute value of the complex power flow into the plate

$$\xi = \frac{P_{PW}}{|P_C|} = \frac{P_R^0}{|P_C|}$$

Eq. (4.43) describes how much propagating power takes portion in the complex power flow input and the higher this value is, the more efficient of the conversion. The propagating power conversion efficiency for each converted propagating mode can be readily obtained as

$$\xi_m = \frac{P_{PW,m}}{|P_C^0|}$$

where $P_{PW,m}$ denotes the real power flow for each converted propagating mode and can be readily obtained from Eq. (4.41). Figure 4.12(a) and (b) show the complex power flow into the plate edge through a symmetric aperture ($h_1 = 3mm$ and $h = 6mm$) under $A_1$ evanescent displacement and traction excitations, respectively, as a function of normalized frequency.

Figure 4.13(a) and (b) show the propagating power conversion efficiency upon $A_1$ evanescent excitations as a function of normalized frequency under displacement and traction excitations through a symmetric aperture at the edge, respectively. For evanescent displacement excitation, the conversion efficiency for $A_0$ mode can be as high as over 95% in
a wide frequency range below $A_1$ cut-off frequency and starts to decrease as the frequency increases. Above the $A_1$ cut-off, $A_1$ propagating wave appears and the conversion efficiency has a peak value about 85% in the frequency region under consideration. The conversion efficiency of $A_1$ mode is greater than $A_0$ mode in the frequency range above $A_1$ cut-off. For evanescent traction excitation, the $A_0$ mode exhibits high conversion efficiency at low frequency and the conversion efficiency decreases as the frequency increases in the frequency range below $A_1$ cut-off. As the excitation frequency exceeds $A_1$ cut-off, $A_0$ mode conversion efficiency begins to increase and the conversion is more efficient than $A_1$ mode. It is noticed that the conversion efficiency can be extremely low, i.e. almost no conversion occurs for certain excitation frequency. The $A_0$ mode conversion efficiency shows opposite trends as frequency increases for displacement and traction excitation for the frequency range under consideration.
Figure 4.12 Complex power flow input through a symmetric aperture \((h_1 = 3mm \text{ and } h = 6mm)\) as a function of normalized frequency under (a) \(A_1\) evanescent displacement excitation, (b) \(A_1\) evanescent traction excitation. (The complex power is in log scale).
A1 evanescent displacement excitations

(a)

A1 evanescent traction excitations

(b)
Figure 4.13 Propagating power efficiency of converting evanescent $A_1$ Lamb wave into propagating Lamb waves as a function of frequency. The plate is excited by the (a) evanescent $A_1$ displacement distribution, (b) evanescent $A_1$ traction distribution at the left edge through a symmetric aperture.
$h_1 = 3\text{mm}$

$A_1$ evanescent mode

or

$A_0 + A_1$

$h = 6\text{mm}$

Displacement excitation

$\xi(\%)$

$\omega h / c_T$
Traction excitation

$\xi (%)$

$\omega h/c_T$

A_0

A_1

(b)
Figure 4.14(a) and (b) show the power conversion efficiency of $A_1$ evanescent waves into propagating as a function of normalized frequency under displacement and traction excitations through a non-symmetric aperture at the edge, respectively. The frequency range for the current study is chosen from 20 kHz to 471.2 kHz (or $\bar{\omega}$ from 0.24 to 5.6), which is below the $S_1$ cut-off frequency for a 6 mm plate. Different from evanescent excitations applied through symmetric aperture, propagating $S_0$ mode can be converted in addition to the $A_0$ and $A_1$ modes since excitations are imposed through a non-symmetric aperture. However, the conversion efficiency for $S_0$ mode under evanescent traction excitations is almost zero across the entire frequency range below the $A_1$ cut-off frequency. Above the $A_1$ cut-off, the $S_0$ mode conversion begins, but the efficiency is still lower than $A_0$ and $A_1$ modes. A large amount of $S_0$ mode can be converted if the plate is excited by evanescent displacement excitations. At frequencies below the $A_1$ cut-off, both the $S_0$ and $A_0$ modes contribute in the converted propagating modes. The conversion efficiency for $S_0$ mode can reach as high as 62% and is larger than $A_0$ and $A_1$ modes in a wide frequency range. It is seen from both Figure 4.14 and Figure 4.15 that the power conversion efficiency for different propagating Lamb modes depends strongly on frequency, and the choice of excitation methods, either displacement or traction excitations, can affect the conversion efficiency for different Lamb modes.
4.4 Summary

In this Chapter, it is demonstrated it is feasible to convert evanescent Lamb waves into propagating waves by applying evanescent displacement or traction excitations through a narrow aperture from the edge of a semi-infinite plate. It is shown both theoretically and numerically that applying evanescent displacement and traction excitation obtained from Lamb wave displacement and stress responses at the edge of a semi-infinite plate can be used to generate a purely evanescent field. The conversion phenomenon was investigated through the theoretical formulation of the amplitude coefficient for the converted propagating Lamb mode and the amplitude coefficient is then proved accurate through the validation of a complex power conservation relation. Finally, a power conversion analysis quantitatively provided the power conversion efficiency for different converted propagating modes under different excitation frequencies. It was found that the conversion efficiency for different propagating modes varies dramatically as excitation frequency changes. It is worth noting that the propagating power converted from evanescent waves excited by displacement and traction excitations exhibit marked difference which deserves future investigation.
Figure 4.14 Propagating power efficiency of converting evanescent $A_1$ Lamb wave into propagating as a function of frequency. The plate is excited by the (a) evanescent $A_1$ displacement distribution, (b) evanescent $A_1$ traction distribution at the left boundary through a non-symmetric aperture with thickness $h_1/h = 0.5$. 
$h_1 = 3 \text{ mm}$

$A_1$ evanescent mode

or

$h = 6 \text{ mm}$

$A_0 + S_0 + A_1$

Displacement excitation

(a)
Chapter 5

Conversion of Evanescent into Propagating SH Waves in Plates
5.1 Introduction

In this Chapter, generation of a purely SH evanescent field is first studied by imposing time-harmonic SH evanescent displacements at the edge of a 2-D semi-infinite plate. The evanescent SH wave generation is theoretically studied by using the complex reciprocity and orthogonality relations and a 2-D finite element model is proposed to model SH evanescent wave propagation. The conversion of evanescent into SH propagating waves is then demonstrated by prescribing the evanescent displacements distributions through a narrow aperture at the edge of a 2-D semi-infinite plate. A FEA model is first utilized to model the conversion process and an analytical model with the aid of FEA based on the complex reciprocity and orthogonality relations are then proposed to quantify the amplitude coefficient of the converted SH propagating waves from evanescent excitations. Power analysis is performed finally to calculate the propagating power conversion efficiency from the evanescent excitations.

5.2 Generation of Evanescent SH Waves

From the steady-state time-harmonic complex integral form reciprocity relation given in Eq. (3.17b), the SH waves may be excited by the body forces $\mathbf{f}$ at the right hand side of Eq. (3.17b), by traction force $\mathbf{\sigma} \cdot \mathbf{n}$ and displacement sources $\mathbf{u}$ at the plate boundaries. Therefore, a time-harmonic SH evanescent displacement excitation prescribed at the left boundary $x = 0$ of a semi-infinite plate, as shown in Figure 5.1, is proposed to generate a purely SH evanescent field. For the semi-infinite plate under time-harmonic motion represented by $e^{-iot}$,
Im(k) > 0 represents a forward decaying SH evanescent wave away from the left boundary in the +x direction. In this Section, the theoretical formulations based on the reciprocity relation and the orthogonality relation for the excitation of SH evanescent wave field is presented first. A finite element method is then used to solve the two-dimensional partial differential equation to model the SH evanescent wave.

5.2.1 Through Edge Displacement Excitations

5.2.1.1 Theoretical Formulation

The complex reciprocity relation in Eq. (3.30) requires two wave fields 1 and 2 to be judiciously chosen. The wave field 1 (real field) is chosen as the field generated from an
actual excitation and the excited field in the waveguide can be represented by the mode expansion (the time-harmonic term $e^{i\omega t}$ is omitted in the context herein) as

$$\mathbf{u}_i(x, z) = \sum_{m=0}^{\infty} u_i^m(z) e^{ik_x x}$$  \hspace{1cm} (5.1)

$$\sigma_i(x, z) = \sum_{m=0}^{\infty} \begin{pmatrix} 0 & \tau_{xy}^m(z) & 0 \\ \tau_{xy}^m(z) & 0 & \tau_{yz}^m(z) \\ 0 & \tau_{yz}^m(z) & 0 \end{pmatrix} e^{ik_x x}$$  \hspace{1cm} (5.2)

The wave field $2$ is taken to be a specific virtual mode

$$\mathbf{u}_2(x, z) = u_2^\nu(z) e^{-ik_x x}$$  \hspace{1cm} (5.3)

$$\sigma_2(x, z) = \begin{pmatrix} 0 & \tau_{xy}^\nu(z) & 0 \\ \tau_{xy}^\nu(z) & 0 & \tau_{yz}^\nu(z) \\ 0 & \tau_{yz}^\nu(z) & 0 \end{pmatrix} e^{-ik_x x}$$  \hspace{1cm} (5.4)

Substituting Eq. (5.1)- Eq. (5.3) into Eq. (3.30), the complex reciprocity relation at $x=0$ gives

$$\int_{-h/2}^{h/2} F_{mn} \bigg|_{x=0} dz = \sum_{m=0}^{\infty} i\omega \int_{-h/2}^{h/2} \left[ u_i^m(z)\tau_{xy}^m(z) - u_2^\nu(z)\tau_{xy}^\nu(z) \right] dz$$  \hspace{1cm} (5.5)

For the edge displacement excitations, the evanescent displacement of an evanescent SH mode applied at $x = 0$ can be obtained from Eq. (2.56) as

$$u_y^{EW} = A^{EW} V^{EW}(z)$$  \hspace{1cm} (5.6)

The corresponding stress at $x = 0$ can be expressed via Eq. (2.58a) as

$$\tau_{xy}^0 = A^{EW} t_{xy}^0(z)$$  \hspace{1cm} (5.7)
where $A^{EW}$ is the amplitude coefficient for the applied SH evanescent displacement, $V^{EW}(z)$, $t_{xy}^{0}(z)$ can be obtained by substituting imaginary wavenumber $k$ into Eq. (2.57) and Eq. (2.59a). After substituting Eq. (5.6) and Eq. (5.7) into Eq. (5.5), the complex reciprocity at $x = 0$ can be rewritten as

$$
\left. \int_{-h/2}^{h/2} F_{mn} \right|_{x=0} dz = i \omega A^{EW} \left. \int_{-h/2}^{h/2} \left( u_{xy}^{m} t_{xy}^{0} - V^{EW} t_{xy}^{n} \right) \right|_{x=0} dz
$$

(5.8)

The complex reciprocity relation at $x=x$ becomes

$$
\left. \int_{-h/2}^{h/2} F_{mn} \right|_{x=x} dz = \sum_{m=0}^{\infty} i \omega A_{m} \left. \int_{-h/2}^{h/2} \left( u_{xy}^{m} t_{xy}^{m} - V^{m} t_{xy}^{n} \right) \right|_{x=x} dz
$$

(5.9)

Substituting Eq. (5.8) and Eq. (5.9) into Eq. (3.30), the complex reciprocity relation for the waveguide excited by edge evanescent displacement can be simplified as

$$
4 \sum_{m=0}^{\infty} A_{m} P_{mn} e^{i(k_{w} - k_{n})x} = i \omega A^{EW} \left. \int_{-h/2}^{h/2} \left[ u_{xy}^{n}(z) t_{xy}^{EW}(z) - V^{EW}(z) t_{xy}^{n}(z) \right] \right|_{x=0} dz
$$

(5.10)

where $P_{mn}$ is defined in Eq. (3.46b). If the wave field 2 is taken to be a propagating SH mode, since the orthogonality relation is automatically satisfied between evanescent modes and propagating modes, the right-hand side of Eq. (5.10) is zero. On the left-hand side of Eq. (5.10), the orthogonality relation requires that the only nonzero value for $P_{mn}$ is for $m=n$, and it follows that

$$
4 A_{m} P_{mm} e^{i(k_{w} - k_{n})x} = 0
$$

(5.11)

and thus

$$
A_{m} = 0
$$

(5.12)
which means there are no propagating SH waves survived at $x = x$ in the mode expansion of the real wave field 1 if the plate is excited by the time-harmonic edge evanescent displacement distributions.

5.2.1.2 Finite Element Verification

The finite element method (FEM) has been extensively and successfully used in the modeling of guided wave propagation in plates. The three-dimensional (3-D) solid model is generally required to perform the numerical analysis on the modeling of guided waves interacting with defects. However, the 3-D solid models are usually computational expensive, so that the used of the simplified two-dimensional (2-D) models is preferred when possible. With the assumption of plane strain, 2-D models are adopted in many studies when the modeling of Lamb waves is needed. The 2-D plane strain elements model requires that the displacements have to be in the assumed plane of strain. For Lamb waves, both the displacements $u_x$ and $u_z$ are in the $x$-$z$ plane and thus Lamb waves modeling can be easily implemented in most commercially available software. However, in the case of SH waves, the particle deformation $u_y$ is normal to the plane of wave propagation, no such 2-D approximation can be made for the SH waves. Demma (2003) proposed an approximate approach to model the SH wave propagation in plates by using a 2-D axisymmetric model of a pipe with large diameter (infinite diameter to approximate a plate) so that the axisymmetric analysis allows displacements in the direction normal to the element. However, this is still an approximation method to model SH waves using a 2-D model. In order to model the SH waves propagation in two-dimensional, in this dissertation, the commercially available
software COMSOL is utilized to solve the governing equation, i.e., partial differential equation (PDE) for SH waves in plates by a finite element method.

The governing equation for SH wave is given in Eq. (2.45). By assuming a steady-state time-harmonic SH wave proportional to $e^{-i\omega t}$, Eq. (2.45) is thereby becomes

$$\mu \left( \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2} \right) = -\rho \omega^2 u_y$$

(5.13)

where $u_y$ is the displacement in the Fourier domain. This PDE needs to be written in a specific COMSOL formalism in frequency domain in the following form

$$\nabla \cdot (c \nabla u) + au = 0 \quad \text{in } \Omega$$

$$\mathbf{n} \cdot (c \nabla u) = g \quad \text{on } \partial \Omega$$

$$hu = r \quad \text{on } \partial \Omega$$

(5.14)

where $u$ is a frequency-dependent variable to be determined in the domain $\Omega$ of boundary $\partial \Omega$. The outward unit vector on $\partial \Omega$ is denoted by $\mathbf{n}$. The first PDE equation must be satisfied in the domain $\Omega$. The second and the third equations define the traction boundary condition and the displacement boundary condition, respectively at the boundary $\partial \Omega$, where $g$ is the traction vector at the boundary and $r$ is the amplitude of the prescribed displacements. $h$ is a $2 \times 2$ matrix, the elements of which are either 0 or 1 depending on the direction of the prescribed displacements and on the orientation of the boundary. The coefficients $c$ and $a$ which related to the stiffness of the material and the inertial effect, respectively, depend on the number of space dimensions of the problem under considerations.
For the 2-D SH waves modeling, by comparing Eq. (5.13) and Eq. (5.14), the coefficients \( c \) and \( a \) for the SH wave PDE are determined to be

\[
c = \mu, \quad a = \rho \omega^2
\]  

(5.15)

The stress boundary condition and prescribed displacement at the boundary are

\[
\mathbf{n} \cdot (\mu \nabla \mathbf{u}) = 0 \quad \text{at} \quad z = \pm h / 2
\]

\[
u = u_y \quad \text{at} \quad x = 0
\]

(5.16)

where the prescribed displacement \( u_y \) is the displacement for SH waves given in Eq. (2.56).

For simulating guided wave propagation in a semi-infinite plate, the ‘perfectly matched layers (PMLs)’ are commonly used to eliminate the reflection from the right-hand side of the plate. However, the PML element is unavailable in the PDE module of COMSOL. Another solution is to add several absorbing regions continuously to the end of the plate to eliminate reflections of guided waves from the edge of the plate (Liu, 2003). The stiffness matrix of each absorbing region is made complex by artificially adding a gradually increasing imaginary part to the initial real part of the original plate as the absorbing regions get far away from the end of the plate. In this dissertation by taking advantage of the PDE module in COMSOL for wave propagation modeling (Castaings, 2004), the imaginary part of the shear modulus which is the required input data, can be defined as an exponentially increasing function along the \( x \) direction of the absorber which is solely defined as a single region. The shear modulus for the absorber is defined using the following closed form solution

\[
\mu_{\text{absorber}} = \mu_{\text{original plate}} + i \mu (x - L)e^{(x-L)/a}
\]

(5.17)
where $L$ is the length of the original plate and $a$ is a coefficient which is chosen to ensure a large imaginary part of $\mu$ for adequate absorbing at the end of the absorber, and $a = 0.005$ is used in this model. In this model, the original plate is assumed to be a lossless isotropic plate and thus its imaginary part of the shear modulus is zero. It is noted from Eq. (5.17) that at the boundary of the original plate and the absorbing region, $\mu_{\text{absorber}} = \mu_{\text{original plate}}$, and this is to guarantee there is no abrupt change of acoustic impedance between them. The exponential increasing of the imaginary part of shear modulus away from the boundary ensures a smooth impedance change in the absorbing region to minimize the reflection. The element size is chosen to be at least $1/20^{th}$ of the wavelength. Convergence test showed that the solutions converge towards the same consistent results when the element order was greater than or equal to 3. Therefore, free triangular elements with cubic Lagrange polynomials behavior was selected for the simulations in this dissertation. The mesh size can be customized to fit the requirements for different regions of complex structures, and the mesh can be easily locally refined (e.g., corner refinement, distribution) if small detailed mesh is needed. Figure 5.2 shows the schematic of a semi-infinite plate with extra absorbing region for the generation of a SH pure evanescent field by prescribing evanescent displacement across the entire thickness of the plate.
To model the evanescent SH wave, the original plate is chosen to be an aluminum plate with the length \( L = 30 \text{ mm} \) and thickness \( h \), density \( \rho = 2713 \text{ kg/m}^3 \), Young’s modulus \( E = 72.6 \text{ GPa} \) and Poisson’s ratio \( \nu = 0.33 \). The absorbing region that is 10 mm long has the same thickness as the original plate is added to the end of the original plate. Generation of \( SH_1 \) and \( SH_2 \) evanescent modes at \( \omega h/c_T = 0.5944 \) (\( f h = 0.3 \text{ MHz:mm} \)) is investigated. Following Eq. (5.16), the prescribed evanescent displacement \( u_y \) needs to be applied at the left edge of the plate. The displacement distribution applied at \( x = 0 \) for the generation of \( SH_1 \) and \( SH_2 \) evanescent modes can be obtained from Eq. (2.56) and are plotted in Figure 5.3(a) and (b), respectively, with unit amplitude coefficient \( A_m = 1 \).
Figure 5.3 Time-harmonic displacement distribution applied at \( x = 0 \) for the generation of
(a) \( SH_1 \) evanescent mode; (b) \( SH_2 \) evanescent mode at \( \omega h/c_T = 0.5944 \) (\( fh = 0.3 \, MHz\cdot mm \)).

Figure 5.4(a) and (b) show the antiplane SH evanescent displacement field for \( SH_1 \) mode and \( SH_2 \) mode, respectively. From Figure 10(a), the displacement \( u_y \) is antisymmetric with respect to the neural \( x \) axis, and this is exactly the property of antisymmetric SH mode. The displacement shown in Figure 5.4(b) is symmetric about the \( x \) axis and this represents the symmetric SH mode. The exponentially decaying feature of both the \( SH_1 \) and \( SH_2 \) field provides evidence that the generated displacement fields are indeed the SH evanescent wave fields. The FE modeling results confirm that the time-harmonic edge displacement excitations prescribed at the left edge of the plate can be used to generate evanescent SH field which is exponentially decay away from the boundary. The FEA results agree with the theoretical proof demonstrated in Section 5.2.1.1 that the propagating SH wave is absent in the mode expansion of the excited field from time-harmonic displacement excitation. As a
result, both the theoretical prediction and FEA results verify that imposing time-harmonic evanescent displacement at the edge can be used to generate a purely SH evanescent field.
Figure 5.4  SH evanescent displacement field in the antiplane direction generated by edge
displacement excitations at $\omega h/c_T = 0.5944$ ($fh = 0.3 \text{ MHz} \cdot \text{mm}$) (a) $SH_1$ evanescent mode;
(b) $SH_2$ evanescent mode.
5.3 Conversion of Evanescent into Propagating SH Waves through Narrow Apertures

Generation of evanescent SH waves has been demonstrated by prescribing time-harmonic evanescent displacement distributions at the edge of the plate. The complex SH dispersion curves and group velocity dispersive curve shown in Figure 5.5(a) and Figure 5.5(b) are used to study the SH wave conversion, if a $SH_1$ evanescent mode is generated at normalized frequency $\omega h_1/c_T$ by prescribing evanescent displacement partially through a narrow aperture $h_1$ at the edge of a plate with thickness $h$, where $h > h_1$, in the plate at normalized frequency $\omega h/c_T$, propagating SH modes will appear and the value of $\omega h/c_T$ and the aperture symmetric or asymmetric manner with respect to neutral $x$ axis determine the possible propagating SH modes that can be converted. Below the $SH_1$ cut-off frequency $\pi$, only propagating $SH_0$ mode may be converted and more propagating mode could be converted if $\omega h/c_T$ is greater than $\pi$. Figure 5.6 shows the description of evanescent SH wave generation and the conversion of evanescent into propagating SH modes with fixed $h_1$ and $h$ by varying the excitation angular frequency $\omega$. In this Section, the conversion of evanescent SH waves into propagating is investigated by imposing time-harmonic evanescent displacement distributions through a narrow aperture (thickness = $h_1 < h$) at the edge of a semi-infinite plate, as shown in Figure 5.7. In Section 5.3.1, a finite element analysis is first utilized to model the conversion of SH evanescent waves into propagating. An analytical model with the aid of FEA based on the complex reciprocity theorem is then proposed in Section 5.3.2 to quantify the amplitude coefficient of the converted propagating SH modes.
from evanescent excitation. Power analysis on the conversion is performed in Section 5.3.3 where the propagating power conversion efficiency from the evanescent excitations is quantified.
Figure. 5.5 (a) Complex dispersion curves and (b) group velocity dispersive curves used for the study of converting evanescent into propagating SH waves. (The green solid dot denotes the $SH_1$ evanescent mode excited at $\omega h_1/c_T$ and the pink solid dots represent possible propagating SH modes that can be converted depends on the frequency $\omega h/c_T$).
Symmetric propagating waves (real wavenumbers)
Antisymmetric propagating waves (real wavenumbers)
Symmetric evanescent waves (imaginary wavenumbers)
Antisymmetric evanescent waves (imaginary wavenumbers)

\( \omega h/c_T \)

\( \text{Im}(kh) \)

\( \text{Re}(kh) \)

0 < \( \omega h / c_T \) < \( \pi \)

\( SH_1 \) evanescent mode excitation

\( \omega h/c_T \)

\( c_g/c_T \)

0 < \( \omega h / c_T \) < \( \pi \)

\( SH_1 \) evanescent mode excitation

10 \( Thc\omega \pi / \ll 

10 \( Thc\omega \pi / \gg 

\( SPH \)
Figure 5.6 Description of evanescent SH wave generation and evanescent into propagating conversion.

Figure 5.7 Cross section of a semi-infinite plate for studying conversion of evanescent SH waves into propagating. The plate (thickness = $h$) is subject to evanescent displacement.
5.3.1 Conversion of Evanescent SH Waves into Propagating-Finite Element Modeling

As already discussed in Section 5.2.1.2, by solving the PDE for SH guided waves using the finite element method, a 2-D FE model can be used to model the SH wave propagation in plates. In order to study the conversion of evanescent into propagating SH waves, a similar finite element model is utilized to investigate the conversion phenomenon of evanescent into propagating and to quantify the amplitude coefficient of the converted propagating SH mode. Since $SH_1$ evanescent mode decays the slowest among all the antisymmetric SH modes, $SH_1$ evanescent mode is selected to study the conversion of evanescent into propagating SH waves. Figure 5.8 shows schematic of a semi-infinite plate and an absorbing region is added adjacent to the main plate. In the figure, evanescent displacement excitations are prescribed partially through a symmetric aperture for the study of conversion of evanescent into propagating SH modes. In this FE model, the main plate is 200 $mm$ long and the absorbing layer is 30 $mm$ in length. The plate thickness is 6 $mm$ and the aperture size is 3 $mm$. Using the complex dispersive curves shown in Figure 5.5(a), the frequency range that can be used to generate a $SH_1$ evanescent mode through the 3 $mm$ aperture is from 0 to 528.51 $kHz$. However, in the 6$mm$ plate, the excitation frequency has to be greater than the $SH_1$ cut-off frequency 264.25 $kHz$ so that propagating $SH_1$ mode can be converted. Therefore, the frequency range that can be used to study the conversion of evanescent $SH_1$ mode into $SH_1$ propagating mode is from 264.25 $kHz$ to 528.51 $kHz$. Figure 5.9(a), (b) and (c) show the displacement field $u_y$ of the plate under the edge evanescent $SH_1$ displacement excitations at 250 $kHz$ ($\omega h/c_T = 2.972$), 300 $kHz$ ($\omega h/c_T = 3.566$) and at 400 $kHz$ ($\omega h/c_T = 4.7547$),
respectively. As expected and shown in Figure 5.9(a), since the plate is excited at 250 kHz ($\omega h/c_T = 2.972$) which is below the $SH_1$ cut-off frequency although the excitations are partially applied at the plate edge, no propagating SH mode can be converted at this frequency and waves still exhibit exponentially decay. It is shown from Figure 5.9(b) and (c) that as evanescent $SH_1$ mode is excited through the symmetric aperture at the edge with frequencies greater than $SH_1$ cut-off, the wave decays dramatically in the region close the to the excitation source, but the propagating SH mode appears as the wave propagates out. The wavelength of the propagating SH mode is 24.5 mm at 300 kHz and is 10.6 mm at 400 kHz which confirm the converted propagating mode is $SH_1$ mode.

Figure 5.8 Schematic of a semi-infinite plate using an absorbing region adjacent to the main plate to study the conversion of evanescent into propagating SH waves by prescribing evanescent displacement distributions through a symmetric aperture ($h_1$) at the edge of the plate ($h$).
Figure 5.9 Displacement field $u_y$ of the plate under the edge evanescent displacement excitations, (a) $SH_1$ mode conversion at 250 kHz ($\omega h/c_T = 2.972 < \pi$); (b) $SH_1$ mode conversion at 300 kHz ($\omega h/c_T = 3.566$); (c) $SH_1$ mode conversion at 400 kHz ($\omega h/c_T = 4.7547$).
5.3.2 Determination of Amplitude Coefficient of the Converted Propagating SH Waves- Theoretical Formulations

As it has been discussed above that propagating SH mode can be converted from evanescent excitations, it is important to know how efficient the conversion of evanescent into propagating SH waves is. To quantify the propagating power conversion efficiency, we first need to quantify the amplitude coefficient of the propagating modes that are converted from evanescent SH waves. Therefore, as required by the complex reciprocity theorem, wave field $2$, is chosen as a specific virtual propagating mode to determine the amplitude coefficient of the converted propagating SH mode from evanescent excitations by applying the complex reciprocity theorem. The known virtual wave field $2$ can be written as

$$u_2^*(x, z) = u_y^{n*}(z) je^{-ik_n x}$$  \hspace{1cm} (5.18a)

$$\sigma_2^*(x, z) = \begin{bmatrix} 0 & \tau_x^{n*}(z) & 0 \\ \tau_x^{n*}(z) & 0 & \tau_y^{n*}(z) \\ 0 & \tau_y^{n*}(z) & 0 \end{bmatrix} e^{-ik_n x}$$  \hspace{1cm} (5.18b)

where $k_n$ is chosen as the wavenumber of the possible converted propagating SH wave, and $u_y^{n*}(z)$, $\tau_x^{n*}(z)$, $\tau_y^{n*}(z)$ can be obtained by substituting $k_n$ into Eq. (3.56) and Eq. (2.58).

For SH evanescent displacement distributions prescribed through a narrow aperture which is symmetric with respect to the $x$ axis, as shown in Figure 5.7(a), the real wave field 1 at $x = 0$ is

$$u_y^{EW}(z) = A^{EW} V^{EW}(z) \quad (-h_1 / 2 \leq z \leq h_1 / 2)$$  \hspace{1cm} (5.19)
where \( u_{y}^{EW}(z) \) is the applied evanescent SH mode displacement at a given frequency, \( A^{EW} \) is the amplitude coefficient for the applied displacement and \( V^{EW}(z) \) is obtained by substituting imaginary wavenumber \( k \) into Eq. (2.59). The corresponding stress \( \tau_{xy}^{0}(z) \) from \(-h_{1}/2 \leq z \leq h_{1}/2 \) at \( x=0 \) and the displacement \( u_{y}^{0}(z) \) from \(-h_{1}/2 \leq z \leq h/2 \) and \( h_{1}/2 \leq z \leq h/2 \) at \( x=0 \) can be obtained from the FEA. The stress \( \tau_{xy}^{0}(z) = 0 \) in the region \(-h/2 \leq z \leq h/2 \) and \( h_{1}/2 \leq z \leq h/2 \) where the traction is free. The wave field \( \mathbf{1} \) at any arbitrary position \( x = x \) can be written as

\[
\mathbf{u}_{1}(x,z) = \sum_{m=0}^{\infty} u_{y}^{m}(z) e^{ik_{x}x} 
\]

(5.20a)

\[
\sigma_{1}(x,z) = \sum_{m=0}^{\infty} \begin{pmatrix} 0 & \tau_{xy}^{m}(z) & 0 \\ \tau_{xy}^{m}(z) & 0 & \tau_{yz}^{m}(z) \\ 0 & \tau_{yz}^{m}(z) & 0 \end{pmatrix} e^{ik_{x}x} 
\]

(5.20b)

After applying the complex reciprocity relation for SH waves on the two wave fields, the SH wave complex reciprocity relation can be simplified as the following relation between the contribution from the integral at \( x = 0 \) due to evanescent displacement excitation and the integral at \( x = x \)

\[
\sum_{m=0}^{\infty} i\omega A_{m} \int_{-h/2}^{h/2} [V_{\ast}^{m}(z)t_{xy}^{m}(z) - V^{m}(z)t_{xy}^{\ast}(z)] e^{i(k_{x}x-k_{y}z)} dx \bigg|_{x=x} dz
\]

\[
= i\omega \left[ \int_{-h/2}^{h/2} V_{\ast}^{m}(z)t_{xy}^{0}(z) \bigg|_{x=0} dz - A^{EW} \int_{-h/2}^{h/2} V^{EW}(z)t_{xy}^{0}(z) \bigg|_{x=0} dz - 2 \int_{h/2}^{-h/2} u_{y}^{0}(z)t_{xy}^{0}(z) \bigg|_{x=0} dz \right] 
\]

(5.21)
If SH evanescent displacement distributions are imposed through an asymmetric aperture, as shown in Figure 5.7(b), the real wave field \( u_y^{EW} \) at \( x = 0 \) is

\[
u_y^{EW} = A^{EW} V^{EW}(z) \quad (-h/2 \leq z \leq h_1 - h/2)
\]

(5.22)

The corresponding stress \( \tau_{xy}^0(z) \) from \( h/2 \leq z \leq h_1 - h/2 \) at \( x=0 \) and the displacement \( u_y^0(z) \) from \( h_1 - h/2 < z < h/2 \) at \( x=0 \) can be obtained from the FEA. The stress \( \tau_{xy}^0(z) = 0 \) in the region \( h_1 - h/2 < z < h/2 \). The wave field \( 1 \) at \( x = x \) and the virtual wave field \( 2 \) are given in Eq. (5.20) and Eq. (5.18). The complex reciprocity relation for the excitations through asymmetric aperture is then easily derived as

\[
\sum_{m=0}^{\infty} i \omega A_m \int_{-h/2}^{h/2} [V^n(z)r_{xy}^m(z) - V^m(z)r_{xy}^n(z)] e^{i(k_m - k_n)x} |_{x=x} \, dz
\]

\[
= i \omega \left[ \int_{-h/2}^{h_1 - h/2} V^n(z)r_{xy}^0(z) |_{x=0} \, dz - A^{EW} \int_{-h/2}^{h_1 - h/2} V^{EW}(z)r_{xy}^n(z) |_{x=0} \, dz - \int_{h_1 - h/2}^{h/2} u_y^0(z)t_{xy}^n(z) |_{x=0} \, dz \right]
\]

(5.23)

Since only the converted propagating SH modes are of interest for this study, the arbitrary position \( x \) is chosen large enough \( (x > 20h) \) so that the actual number of summation in the left-hand side of Eq. (5.23) is finite and only propagating modes dominate at \( x = x \). After using the orthogonality relation on the left-hand side of Eq. (5.23), the only non-zero term exists in the summation is for \( k_m = k_n^* \), i.e., \( m = n \), and the complex reciprocity relation can be further simplified as

\[
4P_{mn} A_m = i \omega \left[ \int_{-h/2}^{h/2} V^n(z)r_{xy}^0(z) \, dz - A^{EW} \int_{-h/2}^{h_1 - h/2} V^{EW}(z)r_{xy}^n(z) \, dz - \int_{h_1 - h/2}^{h/2} u_y^0(z)t_{xy}^n(z) \, dz \right]
\]

(5.24)
for symmetric aperture, and

\[ 4P_{mn}A_m = i\omega \left[ \int_{-h/2}^{h/2} V^n(z)\tau_{xy}^0(z)dz - \int_{-h/2}^{h/2} V^{*n}(z)\tau_{xy}^0(z)dz - \int_{h/2}^{h/2} u_{ij}^{0}(z)\tau_{xy}(z)dz \right] \]

(5.25)

for excitations through asymmetric aperture.

Therefore, from Eq. (5.24) and Eq. (5.25), the amplitude coefficient for the propagating SH mode converted from evanescent displacement excitation is

\[ A_m = \frac{i\omega \left[ \int_{-h/2}^{h/2} V^n(z)\tau_{xy}^0(z)dz - \int_{-h/2}^{h/2} V^{*n}(z)\tau_{xy}^0(z)dz - \int_{h/2}^{h/2} u_{ij}^{0}(z)\tau_{xy}(z)dz \right]}{4P_{mn}} \]

(5.26)

for excitations through symmetric aperture, and

\[ A_m = \frac{i\omega \left[ \int_{-h/2}^{h/2} V^n(z)\tau_{xy}^0(z)dz - \int_{-h/2}^{h/2} V^{*n}(z)\tau_{xy}^0(z)dz - \int_{h/2}^{h/2} u_{ij}^{0}(z)\tau_{xy}(z)dz \right]}{4P_{mn}} \]

(5.27)

for excitations through asymmetric aperture.

For evanescent SH mode excitations prescribed through a symmetric aperture, the converted propagating SH mode can be either symmetric or antisymmetric depending on the excitation mode that is used, while both the symmetric and antisymmetric modes can be converted if the evanescent excitation is imposed through a asymmetric aperture in regardless of the excitation mode. It is seen from Eq. (5.26) and Eq. (5.27), the amplitude coefficient \( A_m \) for the converted propagating SH mode is \( x \) location independent. In other words, the amplitude of the converted propagating SH modes can be easily determined by knowing the
prescribed evanescent displacement and the corresponding stresses and displacements at \( x = 0 \).

5.3.3 Power Conversion Efficiency for Propagating SH Waves

The conversion process of evanescent into propagating SH mode is demonstrated through a FE model and an analytical model based on the complex reciprocity theorem with the aid of FEA is presented to quantify the amplitude coefficient of the converted propagating SH mode. The amount of propagating power that can be transmitted to the far-field by the converted propagating waves and the propagating power conversion efficiency upon the evanescent excitation imposed at the edge is another crucial issue needs to be addressed. Figure 5.10 shows the representation of stresses and displacements in the aperture due to evanescent SH mode excitation and at any arbitrary far-field section of the plate for the calculation of power delivered into the plate and the power flow outward by propagating SH mode.

Figure 5.10 Representation of stresses and displacements in the aperture due to evanescent SH mode excitation and at any arbitrary far-field section of the plate for the calculation of power delivered to the plate and power flow outward by propagating SH modes.
5.3.3.1 Verification on the Amplitude Coefficient \((A_m)\) of the Converted Propagating SH Waves

Before the power conversion efficiency of the converted propagating SH waves can be defined, the amplitude coefficient \((A_m)\) of the converted propagating Lamb waves obtained from Section 5.3.2 has to be validated first. The amplitude coefficient is verified through the validation of a complex power conservation relation which has been discussed in Section 3.3.

When SH evanescent displacement is prescribed through the narrow aperture at the edge of the plate, the power input into the plate through the aperture can be written as

\[
P_{in} = P_c^0 = -\frac{1}{2} i\omega \int_{-h/2}^{h/2} u^{*}_{zy}(z) \tau_{xy}^0(z) \bigg|_{x=0} \, dz = P_R^0 + iP_I^0
\]  

(5.28)

for symmetric aperture, and

\[
P_{in} = P_c^0 = -\frac{1}{2} i\omega \int_{-h/2}^{h-h/2} u^{*}_{zy}(z) \tau_{xy}^0(z) \bigg|_{x=0} \, dz = P_R^0 + iP_I^0
\]  

(5.29)

for non-symmetric aperture, where \(P_c^0\) denotes complex power delivered to the plate due to source excitation, \(u^{EW}_{zy}\) is the applied SH evanescent displacement, \(\tau_{xy}^0\) is the corresponding shear stress that can be obtained from FEA. \(P_R^0\) represents time-averaged power flow into aperture, and \(P_I^0\) denotes the peak reactive power that is provided to the plate edge. As discussed in Section 5.2.1, if the SH evanescent displacement is applied through the entire thickness of the plate, the corresponding stress \(\tau_{xy}^0\) which is pure imaginary is out-of-phase with the applied displacement and thus the complex power flow \(P_c^0\) into the plate is pure
imaginary. It is actually the phase relation between the applied evanescent displacement $u_{y}^{\text{EW}}$ and the corresponding stress $\tau_{x y}^{0}$ that can determine the value of $P_{c}^{0}$. For the current problem, as both evanescent and propagating waves are generated when the plate edge is partially excited by the evanescent displacement excitations, the corresponding stress $\tau_{x y}^{0}$ at $x = 0$ will therefore become complex which make the power flow input to the plate complex valued. Figure 5.1 shows the complex power flow into the plate edge through a symmetric aperture ($h_{1} = 3\text{mm}$ and $h = 6\text{mm}$) under $SH_{1}$ evanescent displacement excitation as a function of normalized frequency obtained from numerical study through FEA. The converted propagating SH waves possess real power and contribute entirely to the real part of complex power flow input and therefore can be received by the far-field sensors. The real power flow for the converted propagating SH waves can be obtained from Eq. (3.81) as

$$P_{\text{PW}} = \sum_{m=0}^{\infty} \frac{1}{4} G \omega |A_m|^2 h k_{m}^R$$

(5.30)

where $A_m$ is the amplitude coefficient for the converted propagating SH modes and it has been defined in Eq. (5.26) and Eq. (5.27), $k_m^R$ is the wavenumber for the converted SH modes. A complex power conservation relation exists between the power flow for converted propagating SH modes and the real part of complex power input from the source, i.e.,

$$P_{\text{PW}} = P_{R}^{0}$$

(5.31)

Figure 5.12 shows the comparison of the real part of the complex power flow into the plate edge where it is excited by $SH_{1}$ evanescent mode through a symmetric aperture and the
real power flow for the converted propagating SH modes as a function of normalized frequency. It is noticed from the figure, a very good agreement exists between the real part of complex power input and the real power flow for converted propagating SH mode, which validates the accuracy of the complex power conservation and the amplitude coefficient of the converted propagating mode determined from the theoretical model based on the complex reciprocity theorem via FEA.

Figure 5.11 Complex power flow delivered into a symmetric aperture \((h_1 = 3\, mm\) and \(h = 6\, mm\)) under \(SH_1\) evanescent displacement excitation as a function of normalized frequency. (The complex power is in log scale).
Figure 5.12 Comparison of the real part of the complex power input to the plate through a symmetric aperture under $SH_1$ evanescent displacement excitation and the real power flow for the converted propagating SH modes. (Log power is used in the plots).

5.3.3.2 Propagating Power Conversion Efficiency ($\xi$)

As SH evanescent displacement prescribed through the aperture is used to excite the plate, the complex power is delivered into the plate and only the propagating SH mode can carry energy out and transfer it to the far-field which can be captured by the sensors. Therefore, the amount of propagating power that can be transferred to the far-field is crucial when processing the signals received by the sensors and the ratio of converted propagating power passing through any far-field cross section to the absolute value of complex power
input due to evanescent excitation is defined to measure the efficiency of the conversion. The propagating power conversion efficiency for converting evanescent into propagating SH wave is thereby

$$\xi = \frac{P_{PW}^m}{|P_C^0|} = \frac{P_R^0}{|P_C^0|}$$ (5.32)

The conversion efficiency for each single converted propagating SH mode can be easily obtained as

$$\xi_m = \frac{P_{PW,m}^m}{|P_C^0|}$$ (5.33)

where $P_{PW,m}^m$ denotes the real power flow for each converted propagating mode and can be readily obtained from Eq. (5.30).

Figure 5.13 shows the power conversion efficiency of $SH_1$ evanescent waves into propagating $SH_1$ mode as a function of normalized frequency. The $SH_1$ evanescent displacement is imposed through a symmetric aperture at the edge and only antisymmetric SH mode can converted. In the Figure, below the $SH_1$ cut-off frequency of the plate, no antisymmetric propagating SH mode falls into this range and the wave in the plate is pure evanescent due to the $SH_1$ evanescent mode generated in the aperture. As the excitation frequency exceeds the $SH_1$ cut-off frequency of the plate, the conversion of propagating $SH_1$ mode starts which means propagating power can be radiated out. The conversion efficiency can be as high as 99% and remains significant across the entire frequency range under consideration. Since the aperture size is half of the plate thickness ($h_1/h=1/2$), the upper limit
of the normalized frequency is chosen to be $2\pi$ so that the normalized frequency $\omega h_1/c_T$ of the aperture still below the $SH_1$ cut-off frequency $\pi$ to make $SH_1$ evanescent mode can be generated through the aperture.

Figure 5.14 shows the power conversion efficiency of $SH_1$ evanescent mode into propagating $SH_0$ and propagating $SH_1$ modes as a function of normalized frequency. The $SH_1$ evanescent displacement excitation is now prescribed through an asymmetric aperture and both symmetric and antisymmetric modes are thereby possible to be converted. From the SH wave dispersion curves shown in Figure 5.5(a), $SH_0$ mode is the only mode without cut-off frequency, which means there will be always $SH_0$ propagating mode that can be converted from $SH_1$ evanescent excitation. As already discussed above, the $SH_1$ propagating mode appears after the frequency is greater than the $SH_1$ cut-off frequency of the plate. It should be noticed that, the conversion efficiency for both $SH_0$ and $SH_1$ mode is relatively low (less than 10%) in a large frequency range in comparison with the symmetric aperture case. Although $SH_0$ mode can always be converted, the conversion efficiency is almost zero in the low frequency range and the conversion efficiency starts to increase as the frequency increases. The largest conversion efficiency is found to be 42.3% and 22.2% for $SH_0$ and $SH_1$ mode, respectively.

5.4 Summary

In this Chapter, the conversion of evanescent into propagating SH waves is successfully demonstrated by prescribing evanescent SH displacement through a narrow aperture at the
edge of a semi-infinite plate. Generation of a purely evanescent SH field is first proved theoretically by imposing evanescent SH displacement across the entire thickness of the plate edge. A 2-D model is then proposed to model the generation of purely evanescent SH wave field by solving the governing equation (PDE) of SH waves in plates using a finite element method. The conversion of evanescent into propagating SH waves is then investigated using the proposed finite element model and a theoretical model based on the reciprocity and orthogonality relations with the aid of FEA is proposed to determine the amplitude coefficient of the converted propagating SH wave. Finally, the power conversion analysis quantitatively provides the power conversion efficiency for the propagating SH mode and it is found that the conversion frequency can be significant and varies dramatically on the excitation frequencies.
Figure 5.13 Power conversion efficiency of converting evanescent $SH_1$ mode into propagating $SH_1$ mode as a function of normalized frequency. The plate is excited by evanescent $SH_1$ displacement at the left edge through a symmetric aperture.
Figure 5.14 Power conversion efficiency of converting evanescent \( SH_1 \) mode into propagating \( SH \) modes as a function of normalized frequency. The plate is excited by evanescent \( SH_1 \) displacement at the left edge through a nonsymmetric aperture.
Chapter 6

Conclusions and Future Work
6.1 Conclusions and Contributions beyond Previous Work

The following conclusions can be drawn from this dissertation:

- A comprehensive study on the propagating and evanescent guided waves in plates (Lamb waves and SH waves) is conducted, including displacements and stresses, and power flow for propagating and evanescent guided waves in plates.

1. The phase information of the displacements and stresses components plays a key role in differentiating the propagating and evanescent guided waves in plates.

2. The power flow delivered into an elastic plate is separated into radiative and reactive parts. This type of separation of power flow is proved to be related to the usual separation made on the basis of propagating and evanescent guided waves. From the theoretical formulations of power flow for propagating and evanescent guided waves in plates, it is proved that propagating guided waves in plates have pure real (average) power flow that is radiative and can propagate energy to the far-field, and evanescent guided waves in plates have pure imaginary power flow, and this part of power which is reactive oscillates between the source and the near field and is stored in the plate.

- The successful demonstration of converting evanescent into propagating guided waves in the plate may hold promise for subwavelength damage detection and imaging for structural health monitoring. The analysis methods proposed in this dissertation can be used as a theoretical foundation for the future numerical and experimental investigation on the conversion of evanescent into propagating Lamb waves in a damaged plate.
1. The proposed method for the generation of a purely evanescent guided wave field by simply prescribing evanescent displacements or stresses distributions at the edge of the plate is key to understanding the behavior of evanescent guided waves in plates. The proposed method is proved both theoretically and numerically. It lays a foundation for the study of evanescent guided waves in plates scattered from geometric features such as cracks, holes, stiffeners, and etc.

2. A 2-D model by solving the governing equation of SH waves using a finite element method is proposed to simplify the modeling of SH guided waves. This 2-D model is important for the study of conversion of evanescent into propagating SH waves.

3. The conversion of evanescent into propagating guided waves in plates is demonstrated by imposing evanescent displacements or tractions excitations through a narrow aperture at the edge boundary of the plate. An analytical model based the complex reciprocity theorem with the aid of finite element analysis has been proposed and validated for the determination of the amplitude coefficient of the converted propagating guided waves upon evanescent excitations.

4. The presented power conversion analysis method provides a powerful quantitative approach to describe the amount of propagating power that can be converted upon evanescent excitations. The conservation relation, i.e., the real part of the complex power input to the plate is proved to be equal to the propagating power flow outward any far-field cross section of the plate. The power conversion efficiency for the converted propagating
guided waves is defined and the conversion efficiency can be significant and is strongly frequency dependent.

6.2 Future Work

To further explore the conversion of evanescent into propagating guided waves as evanescent waves scattered from geometrical discontinuities in plates, based on the present dissertation work, some recommendations are proposed for future investigation:

1. The power conversion efficiency for the converted propagating Lamb waves excited by evanescent displacement and traction excitations exhibit remarkable difference, the underlying mechanism that causes this difference deserves future investigation.

2. Experimental method needs to be proposed to generate a purely evanescent guided wave field in plates and experimental study is necessary to verify the amplitude coefficient of the converted propagating waves determined from the theoretical model with the aid of finite element analysis presented in this dissertation.

3. After knowing the properties of evanescent guided waves, e.g., phase information for displacements and stresses distributions and the power flow for evanescent waves, evanescent waves that are produced and mixed with propagating waves at the close proximity of structural discontinuities can be identified and separated from the propagating waves using numerical method (e.g., finite element analysis). Since laser vibrometer is being widely used in guided wave based damage detection, evanescent wave information can be collected directly using laser vibrometer in the region very close to the discontinuities.
Obtaining evanescent waves directly in the near field is promising for NDI, but may be unsuitable for SHM.

4. The proposed analytical model with the finite element analysis for the determination of amplitude coefficient of the converted propagating guided waves from evanescent excitations can be extended to quantify the mode conversion process of propagating guided waves into evanescent.

5. Evanescent guided wave is proved in the dissertation only carry reactive power flow in plates, and this part of power which is stored in the plate oscillates between the source and the near field and cannot be consumed. A rigorous theoretical description and physical explanation on how the reactive power is stored by evanescent waves deserves future investigation and may have great scientific meaning for the study of evanescent waves in the elastic medium.
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