ABSTRACT

JEERUPHAN, THANAWIT. Random Walk with Jump Dependent Cookies on $\mathbb{Z}$. (Under the direction of Min Kang.)

Consider a random walk on $\mathbb{Z}$ in a cookie environment. At each site, the process will eat a cookie placed at each site which will act like a stimulator for the process to have a bias toward either left or right jump. Normally, the process is symmetric simple random walk with the equal probability 1/2 to make left or right jump. With the presence of the cookies, the process will jump to the right with probability p and left with probability $1-p$, where p is not necessarily 1/2. Pinsky came up with the process such that the jump transition does not depend on the external parameter such as number of cookies but instead the first time the process going leftwards. We generalizes the model to be that the second left jump will change the environment. We have the result when the second left jump changes the environment in deterministic environment. We also know about the range or how likely the walker will grow at the maximum or minimum in Pinsky's environment. And in the special case, we have a stationary distribution of the growth of the range.

Random Walk with Jump Dependent Cookies on $\mathbb{Z}$

by
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DEDICATION

To my parents and my family.

BIOGRAPHY

Thanawit Jeeruphan was born in Bangkok, Thailand. He finished high school from Triam Udom Suksa school, the well-known high school in Bangkok, Thailand. He has been inspired by the beauty of Mathematics since he was in high school. So he chose to receive a scholarship from the Royal Thai government to pursue a degree in Mathematics. He received bachelor's degree in Mathematics with First-class honours from Chulalongkorn University in Bangkok, Thailand. Three years later, he also received master's degree in Mathematics from the same university. From 2009-2014, he has been pursuing a PhD degree in Applied Mathematics at North Carolina State University with the scholarship received from the Royal Thai government.

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Chapter 1

Various Models of Random Walks

Random walk is a mathematical formalization of a path that consists of succession of random steps. Random walks have been used in many fields such as Physics, Economics, etc. Random walk can be considered on graph, line, plane or higher dimension. Time-wise, random walk can be discrete-time or continuous-time. Most of the time, random walks are assumed to be Markov chain or Markov process. We will introduce several models which are related to this work.

Random walk has a rich structure and been widely studied. Some models have very well known results for low dimension but sometimes the method they used is breakdown when they try to extend to a higher dimension. Some models give you an interesting properties which have not been seen in a simple setting. The model that inspired our work is the Pinsky's one and we also try to look into some other models to apply the idea to Pinsky's model.

1.1 Stochastic Process

In real world's problem, we usually encounter some uncertainties in the development of the processes or experiments such as stock markets or the signal fluctuation of blood pressure and temperature. Suppose that we have known the initial condition, the result may be different each time we have the experiment unlike the ordinary differential equations. So we have to use the stochastic process to study these kind of experiments or problems. Let us review some basic definitions.

Stochastic process or random process is a collection of random variables which represent the evolution of some random values over time. In discrete-time, a stochastic process is a sequence of random variables together with a time series associated with these random variables.

Definition 1.1 (Stochastic Process)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(S, \mathcal{S})$ be a measurable space, a ***stochastic process*** $X : \Omega \times T \to S$ is a collection

$$\{X_t \mid t \in T\},$$

where T is a totally ordered set (or time). The space S is then called the ***state space***.

If T, which is usually represent *time*, is equal to $\mathbb{N}$, we call ***discrete-time stochastic process*** and when $T = [0, \infty)$ we call ***continuous-time stochastic process***.

Let us introduce a filtration. A filtration provides you information about what you have at that specific time. A filtration is an increasing sequence of $\sigma-$algebras on a measurable space. We know that sigma-algebra is a collection of subsets such that being known how to measure them. Next, we will give the formal definition of filtration.

Definition 1.2 (Filtration)**.** Given a probability space $(\Omega, \mathcal{F}, P)$, a ***filtration*** is a weakly increasing collection of sigma algebras on $\Omega, \{\mathcal{F}_t, t \in T\}$, indexed by some totally ordered set T, and bounded above by $\mathcal{F}$. For $s, t \in T$ with $s < t$,

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}.$$

A stochastic process X on the same time set T is said to be ***adapted*** to the filtration if, for every $t \in T, X_t$ is $\mathcal{F}_t$-measurable.

In other words, adapted process is a process that cannot see into the future or X_n is known at time n for every realization and for every n. The natural filtration is a filtration generated by that process and records the past history of the process at each time. Therefore; a stochastic process is always adapted to its natural filtration. A natural filtration is a simplest filtration available for studying the process in a sense that all the information concerning the process, and only that information, is available in the natural filtration.

Definition 1.3 (Natural Filtration)**.** Given a stochastic process $X = \{X_t\}_{t\in T}$, the ***natural filtration*** for this process is the filtration where $\mathcal{F}_t$ is generated by all values of X_s up to time $s = t$, i.e.

$$\mathcal{F}_t = \sigma\left(\left\{X_s^{-1}(A) | s \leq t, A \in \mathcal{S}\right\}\right).$$

Next, we will present the definition of Markov property. A Markov property is referred to a memoryless property of a stochastic process. A stochastic process has the Markov property if given the present, the future does not depend on the past. In reality, history always affects or shapes the current and future of our life. However, a Markov property gives us a good start of how to study the random process by eliminating the need of looking into the past.

Definition 1.4 (Markov Property)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t, t \in T\}$, for some totally ordered index set T, and let $(S, \mathcal{S})$ be a measurable space. An S-valued stochastic process $X = \{X_t\}_{t\in T}$ adapted to the filtration is said to have the ***Markov property*** with respect to the $\{\mathcal{F}_t\}$ if, for each $A \in \mathcal{S}$ and each $s, t \in T$ with $s < t$,

$$P\left(X_t \in A | \mathcal{F}_s\right) = P\left(X_t \in A | X_s\right).$$

A ***Markov process*** is a stochastic process which satisfies the Markov property with respect to its natural filtration.

We will give a definition of a hitting time or first hit time which is important for studying stochastic process. A hitting time is the first time at which a given process hits a given subset of the state space.

Definition 1.5 (Hitting Time)**.** Given $(\Omega, \mathcal{F}, P)$ be a probability space and a measurable state space S, let $X : \Omega \times T \to S$ be the stochastic process, and let A be a measurable subset of the state space S. Then the ***first hit time*** $\tau_A : \Omega \to T$ is a random variable defined by

$$\tau_A(\omega) = \inf \{t \in T | X_t(\omega) \in A\}.$$

And the k^{th} ***hitting time*** by

$$\tau_A^{(k)}(\omega) = \inf \left\{t > \tau_A^{(k-1)} | X_t(\omega) \in A\right\},$$

where $\tau_A^{(1)} = \tau_A$

If the set A is a singleton, we often write τ_a in stead of $\tau_{\{a\}}$. Now we are ready to present the idea of transience and recurrence which is the main aspect of this dissertation. A recurrent state of a process is a state such that the process will visit that state infinitely many time. A transient state is a state such that the process will visit finitely many time.

Definition 1.6 (Transience/Recurrence)**.** Let $X : \Omega \times T \to S$ be the stochastic process. State $i \in S$ is said to be ***transient*** if

$$P(X_n = i \text{ i.o.} | X_0 = i) = 0.$$

And called ***recurrent*** if not transient or

$$P(X_n = i \text{ i.o.} | X_0 = i) = 1.$$

The process is said to be transient (recurrent) if every site in state space is transient (recurrent).

Because of the tail event, the probability has to be either 0 or 1 by Kolmogorov's zero-one law. There is another definition of transient/recurrent as follows. Roughly speaking, the transient state means that starting from state i, there is a non-zero probability that it will never return to state i in finite time.

Definition 1.7 (Transience/Recurrence)**.** Let $X : \Omega \times T \to S$ be the stochastic process. State $i \in S$ is said to be ***transient*** if

$$P(\tau_i < \infty | X_0 = i) < 1.$$

And called ***recurrent*** if not transient or

$$P(\tau_i < \infty | X_0 = i) = 1.$$

The process is said to be transient (recurrent) if every site in state space is transient (recurrent).

Normally, these two definitions are equivalent whenever the process has Markov property. Definition 1.6 agrees with our common sense. And it is easy to see that definition 1.6 implies definition 1.7.

1.2 Lattice Random Walk

Random walks play an important roles in various fields such as economics by using as a "random walk hypothesis" to model the shares prices. Random Walks have been used as a simplified models of physical Brownian motion and diffusion such as a random movement of molecules in fluid. A popular and elementary example of random walk is a lattice random walk. At each step, the walker jumps to another location on the lattice according to some probability distribution. But the interesting class of random walk which is related to our work is the nearest neighbor random walk. When the walker want to move, the only possible sites he can visit are the sites closest to the current position. And each probability of jumping to those sites can be different. The nearest neighbor random walk can be rigorously defined using the following definition.

Definition 1.8. Given a discrete-time stochastic process $\{X_n\}_{n=0}^{\infty}$ with Markov property on state space $\mathbb{Z}^d$, let $\{e_i\}_{i=1}^{d}$ be the unit vector in $i-$direction, a ***nearest neighbor random walk or simple random walk on*** $\mathbb{Z}^{\boldsymbol{d}}$ is a process equipped with transition probability satisfying

1. Either $X_{n+1} - X_n = e_i$ or $X_{n+1} - X_n = -e_i$ for some i.

2. $\sum_{i=1}^{d} (P(X_{n+1} = X_n + e_i) + P(X_{n+1} = X_n - e_i)) = 1$.

The first line tell you that each increment will be in the basis directions only. Whenever the probability of jumping in each direction are equal, we will call a ***symmetric*** simple random walk.

Definition 1.9. A ***simple symmetric random walk on*** $\mathbb{Z}^{\boldsymbol{d}}$ is the simple random walk with the transition probability

$$P(X_{n+1} = X_n \pm e_i) = \frac{1}{2d}.$$

We will only focus ourselves on one-dimensional case. In one-dimensional walk, we will have 2 distinct directions, backward-forward or left-right. So the transition probability will describe that 2 possibilities. And we will often let p be the probability of jumping to the right. There are various aspects to study about random walk. We will only focus on the transient and recurrent property. The following theorem will give the criterion whether the process is recurrent or transient.

Theorem 1.10. *Let $\{X_n\}_{n=0}^{\infty}$ be a simple random walk with transition probability $P(X_{n+1} = x+1 \mid X_n = x) = p$ and $P(X_{n+1} = x-1 \mid X_n = x) = 1 - p = q$.*

(i) If $p \neq q$ then X_n is transient ; moreover, if $q < p$ then $\lim_{n\to\infty} X_n = +\infty$, while if $q > p$ then $\lim_{n\to\infty} X_n = -\infty$. a.s.

(ii) If $p = q$ then X_n is recurrent ; moreover,

$$\overline{\lim_{n\to\infty}} X_n = +\infty, \underline{\lim_{n\to\infty}} X_n = -\infty. \text{ a.s.}$$

Sketch of proof of theorem 1.10 By the strong law of large number, we know that

$$\lim_{n\to\infty} \frac{X_n}{n} = EZ_1 = p - q \qquad \text{P-}a.s.$$

Thus it is easy to see that the process will move with the asymptotic average velocity close to $p - q$. When p is not equal to q, the random walk, with probability 1, will eventually drift to infinity. $\lim_{n\to\infty} X_n = +\infty$ if $p > q$ and $\lim_{n\to\infty} X_n = +\infty$ if $p < q$. This means that the process is transient. Whenever $p = q$, the process is recurrent. One can look for more results in [10, Chapter 7, p.123].

1.3 Random Walk in Random Environment, RWRE

From purely mathematical point of view, the most natural question arises to ourselves after knowing the simple random walk: Is it possible, at each jump, to have different probability distribution? In other areas, random walk provide a basic model to describe the transport processes. And some of the case, the medium where the system evolves may have some defects, impurities which we can call them as irregularity. So one may think about treating those irregularity as a random environment.

In this section we will give the definition and transience/recurrence result in Random walk in random environment(RWRE). RWRE is a process in which related to randomness in not only the transition mechanism or the way the walker jumps but also the environment which will affect the randomness of the walk. For example, we can think of the movement of the water molecules in ice water. In normal water the molecules will move like a Brownian motion. When the ice is presented, we have to take into account the change of the temperature and when the molecules of water hit the ice. One can look at lecture note by Zeituni [15] for more properties and results of RWRE. Basically, the definition of a RWRE involves two components: first, the *environment,* which is fixed throughout the time while the process evolved. Second, the random walk whose transition probabilities depend on the given but fixed environment.

In order to define the random walk, we have to introduce the environment first and then we can talk about the random walk in that specific environment. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space of the environment. Let $\omega \in \Omega$, a random walk in the environment ω is the Markov chain $\{X_n\}_{n=0}^{\infty}$ on $\mathbb{Z}^d$, using the probability space $(\Omega', \mathcal{F}', P_x^{\omega})$, with the transition probability similar to the one in definition 1.8 but with P_x^ω is stead of regular P. P_x^ω means the probability measure under the environment ω conditioned on the staring point x.

$P_x^\omega(\cdot)$ can be referred as the *quenched* law of the random walk $\{X_n\}_{n=0}^\infty$. We define the semi-direct product $\mathbf{P}_x = \mathbb{P} \times P_x^\omega$ on $\Omega \times \Omega'$ by $\mathbf{P}_x[\cdot] := \mathbb{E}\left[P_x^\omega\left[\cdot\right]\right]$, which will be called *annealed* measure. So we can understand that the quenched law is one realization of the environment which is fixed. The annealed law is the average over all possible environments. It is easy to see that this random walk is not Markovian. We will introduce the simplest model in RWRE and compare it to the simple random walk. Now we consider the simplest RWRE.

Example 1.1 (Simple RWRE in 1-d)**.** Let $\{X_n\}_{n=0}^\infty$ be a RWRE in one dimensional $\mathbb{Z}$ with the transition probability

$$P_x^\omega(X_{n+1} = X_n + 1|X_n = y) = p_y^\omega$$
$$P_x^\omega(X_{n+1} = X_n - 1|X_n = y) = q_y^\omega = 1 - p_y^\omega.$$

where p_y is i.i.d. random variable on $(\Omega, \mathcal{F}, \mathbb{P})$

We can see that when you want to make a jump, there are 2 parameters which you have to consider, the current position and in which specific environment you are. In this simple RWRE, we had a theorem about the transient and recurrent dated back in 1975 by Solomon [14].

Theorem 1.11 (Transient/Recurrent in RWRE)**.** *Let $\{X_n\}_{n=0}^\infty$ with the transition probability as in example 1.1. Set $\rho_y := \frac{q_y}{p_y}, y \in \mathbb{Z}$ and $\eta := \mathbb{E}\ln\rho_0$.*

(i) If $\eta \neq 0$ then X_n is transient ($\mathbf{P}_0 - a.s.$); moreover, if $\eta < 0$ then $\lim_{n\to\infty} X_n = +\infty$, while if $\eta > 0$ then $\lim_{n\to\infty} X_n = -\infty(\mathbf{P}_0 - a.s.)$.

(ii) If $\eta = 0$ then X_n is recurrent ($\mathbf{P}_0 - a.s.$); moreover,

$$\overline{\lim_{n\to\infty}}\, X_n = +\infty, \underline{\lim_{n\to\infty}}\, X_n = -\infty, \mathbf{P}_0 - a.s.$$

We can think about the η as a ratio of the probability of jumping to the right vs jumping to the left. If $\eta < 0$, then balance is shifted to the right making the process transient and eventually go to infinity. Similarly, when $\eta > 0$, the process is also transient but will eventually go to the left instead. Whenever $\eta = 0$, the process is in a balanced situation meaning that the process will be recurrent. We can also relate this parameter back to the simple random walk in deterministic environment. In deterministic environment, $\eta = \ln \rho_0 = \ln(q - p)$ which we also have similar result depend on the equality of p and q. Next we will discuss the speed of RWRE.

Theorem 1.12 (Asymptotic Velocity in RWRE)**.** *The limit $v := \lim_{t \to \infty} \frac{X_t}{t}$ exists ($\mathbf{P}_0 - a.s.$) and is given by*

$$v = \begin{cases} \frac{1-\mathbb{E}\rho_0}{1+\mathbb{E}\rho_0} & \text{if } \mathbb{E}\rho_0 < 1, \\ -\frac{1-\mathbb{E}\rho_0^{-1}}{1+\mathbb{E}\rho_0^{-1}} & \text{if } \mathbb{E}\rho_0^{-1} < 1, \\ 0 & \text{otherwise }. \end{cases}$$

With this theorem we can see that if $\mathbb{E}\rho_0 < 1$ or the force for leftward jump is less than the force for rightward jump in average, the process will have non trivial velocity similar to the case of simple random walk.

1.4 Cookie Random Walk or Multi-Excited Random Walk

This model was introduced by Zerner [16] in order to generalize the idea of excited random walk. The excited random walk means the walker will be excited or having bias only for the first time of visiting each site. After that first visit, the walker will jump with equal probability. We consider the cookie random walk as follows, placing a number of cookies at each site. Whenever the process arrives at that specific site, the walker eats one cookie and jumps with the probability encoded by that cookie. If there is no cookie at that site, the walker jumps with the symmetric probability or equal probabilities. We will give the definition of special environment in which the random walk will be.

Definition 1.13 (Cookie Environment)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where $\mathcal{F}$ is a canonical $\sigma-$algebra. A ***cookie environment*** is an element

$$\omega = (\omega(z))_{z\in\mathbb{Z}} = (\omega(z,i)_{i\geq 1})_{z\in\mathbb{Z}} \in \Omega := \left([1/2,1]^{\mathbb{N}}\right)^{\mathbb{Z}},$$

where $\mathbb{N}$ is the set of strictly positive integers. We will refer to $\omega(z,i)$ as to the strength of the i-th cookie at site z.

For example, suppose that $\omega(0) = (\frac{7}{10}, \frac{7}{10}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$. This can be viewed as follows. There are 2 cookies at site 0 with $\frac{7}{10}$ being the probability of jumping to the right and $\frac{3}{10}$ for leftward jumping probability. Now we formally define the random walk in cookies environment.

Definition 1.14 (Cookie Random Walk)**.** Given a starting point $x \in \mathbb{Z}$ and a cookie environment $\omega \in \Omega$, consider stochastic process $\{X_n\}_{n=0}^{\infty}$ where the state space is $\mathbb{Z}$, together with a probability space $(\Omega', \mathcal{F}', P_{x,\omega})$, which satisfies $P_{x,\omega}-$a.s.

$$P_{x,\omega}\left[X_0 = x\right] = 1,$$
$$P_{x,\omega}\left[X_{n+1} = X_n + 1|\mathcal{F}_n\right] = \omega\left(X_n, V_n(X_n)\right),$$
$$P_{x,\omega}\left[X_{n+1} = X_n - 1|\mathcal{F}_n\right] = 1 - \omega\left(X_n, V_n(X_n)\right).$$

Where $V_n(x) = \sum_{k=0}^{n} \chi_x(X_k)$ is a number of visits to site x up to time n and $\mathcal{F}_n = \sigma\left(X_0, X_1, \dots, X_n\right)$ is a filtration of its past.

We can see that $\{X_n\}_{n=0}^{\infty}$ itself is in general not a Markov chain since the transition probabilities depend on the history of the process. Next we will present the theorem about the transient and recurrent in cookie environment from [16, Theorem 12].

Theorem 1.15 (Transience/Recurrence)**.** *Assume that* $(\omega(x))_{x\geq 0}$ *is stationary and ergodic under* $\mathbb{P}$*. If*

$$\mathbb{P}[\omega(0) = (1, 1/2, 1/2, 1/2, \dots)] < 1$$

then

$$(X_n)_{n\geq 0} \textit{ is recurrent if and only if } \mathbb{E}[\sum_{i\geq 1}(2\omega(0,i)-1)] \leq 1.$$

This theorem tells us that when we exclude the trivial environment, the first jump of each site will always be the right jump, the transient and recurrent can be determined by the total drifted stored at site 0. In deterministic version when every site has the same number of cookies and each cookie encodes the same probability, we have the following result.

Corollary 1.16. *Let $\{X_n\}_{n=0}^{\infty}$ be random walk in deterministic cookies environment with k cookies at each site and probability p to jump to the right, the process is recurrent if*

$$p \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{2k}\right]. \tag{1.2}$$

This corollary easily follows from theorem 1.15. Consider the theorem 1.15,

$$\mathbb{E}[\sum_{i\geq 1}(2\omega(0,i)-1)] \leq 1$$

replace $\omega(0,i)$ by p, we obtain,

$$\sum_{i}^{k}(2p-1) \leq 1$$

$$p \leq \frac{1}{2} + \frac{1}{2k}.$$

Notice that normally simple asymmetric random walk is always transient. This cookie random walk is in between the symmetric and asymmetric random walk. The more we place the cookies at each site, the smaller value of p, probability to jumping to the right, have to be. Next, Basdevant and Singh [2, Theorem 1.1] give the criteria whether the process will have non trivial speed.

Theorem 1.17 (Speed of Deterministic Cookie Random Walk)**.** *Let $\{X_n\}_{n=0}^{\infty}$ be random walk in deterministic cookie environment with k cookies at each site and probability p to jump to the* *right, the process is recurrent if*

$$\lim_{n\to\infty} \frac{X_n}{n} = 0 \text{ } \textit{if } p \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{k}\right]$$

Unlike the simple random walk and RWRE, there is possibility of having trivial speed in the transient process. Or we can say that 3 cookies with suitable strength is minimum number of cookie for the process to have non-trivial speed.

1.5 Other Random Walks

There are several random walks which is not directly related to our model but some of the idea from that model will be used. All of the following models are non-markovian or the history will directly or indirectly determine the way the process evolves.

Self-Avoiding Random Walk

Self-avoiding random walk is a lattice random walk where the visiting site is always distinct. Revisit the old site is prohibited. So most of the time we will consider this model with the dimension greater than one since the process in one dimension is always transient either to the left or to the right based on the first move. One can look for more details in [13].

Edge/Vertex Reinforce Random Walk, ERRW/VRRW

Edge Reinforced Random Walk(ERRW) is the process which the transition probabilities of the walk will be determined by how frequent of using that edge in the past. Similarly for the Vertex Reinforce Random Walk(VRRW) where the transition mechanism will based on the past visit to the sites or vertices.

Example 1.3 (Edge/Vertex Reinforce Random Walk, ERRW/VRRW)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and $G = (V, E, \sim)$ be a non-oriented connected locally finite graph without loops. Let $(a_e)_{e\in E}$ be a sequence of initial weights associated to each edge $e \in E$.

Let $\{X_n\}_{n=0}^{\infty}$ be a random process which takes values on V, and set $\mathcal{F}_n$ be a filtration of its past, $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$. Let $\{X_n\}_{n=0}^{\infty}$ with starting point $x_0 \in V$ and weights $(a_e)_{e \in E}$, if $X_0 = x_0$, and for all $n \in \mathbb{N}$

$$P(X_{n+1} = j | \mathcal{F}_n) = \chi_{\{j \sim X_n\}} \times \frac{F(X_n, j)}{\sum_{k \sim X_n} F(X_n, k)}.$$

where $F(X_n, j)$ takes values in [0,1]. This random walk will be called ***Edge Reinforced Random Walk, ERRW*** if the function $F(X_n, j)$ depends on the edge and ***Vertex Reinforced Random Walk, VRRW*** if the function $F(X_n, j)$ depends on the vertex only.

One example of $F(X_n, j)$ is $F(X_n, j) = a_{\{X_n, j\}} + \sum_{k=1}^{n} \chi_{\{\{X_{k-1}, X_k\} = \{X_n, j\}\}}$. This means that the more you using edge $\{X_n, j\}$, the more you will be using that edge connecting X_n and j again in the future. Similarly, the VRRW can be defined in similar fashion using the different $F(X_n, j)$ such as $F(X_n, j) = a_{\{X_n, j\}} + \sum_{k=1}^{n} \chi_{\{\{X_k = j\}\}}$, which will only depend on the vertex. For more details see [11].

Random Walk Perturbed at Extrema

Another non-Markovian random walk is called ***Random Walk Perturbed at Extrema***, firstly introduced by Davis [6]. This model shares a lot of similarity with the ordinary random walk. But when you are at the maximum or at the minimum you can jump with different probability and not necessary the same for the max and min. Suppose we have a simple random walk in 1-D, at each time n, there will be a range of random walk or the set of the previous points the walker has visited up to time n. Then we define the transition probability based on what kind of point it is, being the internal point or the extrema point i.e. maximum and minimum.

Example 1.4 (Random Walk Perturbed at Extrema)**.** Let $\{X_n\}_{n=0}^{\infty}$ be a simple random walk in $\mathbb{Z}$ with the transition probability

$$P\left(X_{n+1}=x+1|X_n=x\right)=1-P\left(X_{n+1}=x-1|X_n=x\right)$$
$$=\begin{cases} p_M & \text{if } x=\max_{0\le k\le n} X_k \\ p_m & \text{if } x=\min_{0\le k\le n} X_k \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

This is called ***random walk perturbed at extrema***. When the size of the range is 0, i.e. x is both maximum and minimum, choose $\frac{1}{2}$.

This process can be viewed a special case of cookie environment random walk. The idea was introduced by [3]. On the positive axis at site x, place a geometric number of cookies M_x at that site. $\{M_x\}_{x\in\mathbb{Z}^+}$ are i.i.d. random variables which have a distribution $P(M_x=i)=p_M(1-p_M)^{i-1}$ where $i\in\mathbb{N}\setminus\{0\}$. When the walker visit the site with more than one cookie left, eats one cookie and always jumps to the left. If the visit is the time that the walker will finished the last cookie, the walker has to make a right jump. If there is no cookie at that site, the walker will jump with equal probability.

On the negative axis we also place a geometric number of cookie m_x similar to the positive axis but with the different distribution. $\{m_x\}_{x\in\mathbb{Z}^-}$ are i.i.d. random variables with $P(m_x=i)=p_m^{i-1}(1-p_m), i\in\mathbb{N}\setminus\{0\}$. And swap the left and right jump when there is some cookies left at the site.

1.6 Pinsky's Random Walk

Let us introduce another kind of environment called "have your cookie and eat it" environment introduced by Pinsky [9]. As we can see from the cookie environment, the number of cookie is an external parameter in the sense that it relates only to how frequent the walker visits that specific site. This environment, we will later refer it as "Pinsky's model".

Pinsky's model can be viewed as follows, place an infinite number of cookies at each site. The walker must have a cookie before each jump. And the walker jumps according to that probability encoded in that cookie. Whenever the walker makes a first left jump at that site, the cookies, at the site the walker jumped from, will have symmetric probability afterward. When the walker visit the specific site again, the walker still has to eat that cookie but the cookie will contribute nothing to the probability to jump. Or we can viewed the rest of the cookies as a placebo.

We can think about using the same environment in the cookie random walk but this time the number or cookie is not known and can not be viewed separately from the process. And the most important thing is that this process is again not Markovian. Pinsky has a theorem about the transient/recurrent with this "have your own cookie and eat it" environment.

Theorem 1.18. *Let $\{X_n\}_{n=0}^{\infty}$ be a random walk in a deterministic "have your own cookie and eat it" environment $\{\omega(x)\}_{x\in\mathbb{Z}}$*

(i) Let $\omega(x) = p$ for all x in $\mathbb{Z}$. Then

$$\begin{cases} P_1(\tau_0 = \infty) = 0, & \text{if } p \leq \frac{2}{3}; \\ \frac{3p-2}{p} \leq P_1(\tau_0 = \infty) \leq \frac{3p-2}{p(2p-1)}, & \text{if } p \in (\frac{2}{3}, 1). \end{cases}$$

In particular the process is recurrent if $p \leq \frac{2}{3}$ and transient if $p > \frac{2}{3}$.

(ii) Let ω be periodic with period $N > 1$. Then the process is recurrent if $\frac{1}{N}\sum_{x=1}^{N} \frac{\omega(x)}{1-\omega(x)} \leq 2$ and transient if $\frac{1}{N}\sum_{x=1}^{N} \frac{\omega(x)}{1-\omega(x)} > 2$.

Again, this is not clear at first whether the random walk is recurrent or transient, but there is a connection to the result in corollary 1.16. When the process is recurrent, the walker will visit to that site infinity many time. So the cookie will running out with the probability 1. The cookie will change when the first left jump happens like the environment waits for the first success. Then the expected number of jump from that site when the cookie is present is

$\sum_{m=1}^{\infty} mp^{m-1}(1-p) = \frac{1}{1-p}$. Thus, it is similar to the corollary 1.16 with $\frac{1}{1-p}$ number of cookie. Notice that this is not an integer for most values of p. Substitute back in (1.2), we get that $p \leq \frac{2}{3}$ which agrees with Pinksy's result. We will finished this chapter with the theorem about the random environment assuming stationary and ergodic.

Theorem 1.19. *Let $\{X_n\}_{n=0}^{\infty}$ be a random walk in a stationary and ergodic "have your own cookie and eat it" environment $\{\omega(x)\}_{x\in\mathbb{Z}}$, then the process is P_x^{ω}-recurrent if $\mathbb{E}\frac{\omega(0)}{1-\omega(0)} \leq 2$ and P_x^{ω}-transient if $\mathbb{E}\frac{\omega(0)}{1-\omega(0)} > 2$ for $\mathbb{P}$-almost every environment ω*

Chapter 2

Main Results

In survey paper by Kosygina and Zerner [8], Pinsky's result has been noted. Eventhough the Pinksy's environment, "have your cookie and eat it", does not meet the requirements of the theorem [8, Theorem 3.10, p. 13], the conclusion of the theorem is still valid. So it is interesting to see and pursue if we can generalize or have a better assumption for unified theorem.

In this dissertation we will create the new kind of environment inspired by the Pinsky's environment, "have your own cookie and eat it" environment, called ***"k^{th} Left Jump Breaks Cookies(kLJBC)."*** The random walk with "k^{th} Left Jump Breaks Cookies(kLJBC)" can be thought as follows. We place an infinite number of cookies at each site on $\mathbb{Z}$, when the walker visits the certain site, the walker has one cookie and jump accordingly to the probability defined by the cookie. If the walker makes a right jump, the environment is still the same. Also the first $k-1$ times the walker makes a left jump, nothing has change at these steps. When the walker makes a k^{th} jump, the environment at the site the walker jumps from will change to symmetric environment, i.e. with equal probability of jumping left and right. This kind of environment extends the Pinksy's work and adds more level of self-interaction to the environment.

2.1 Preliminary

The first proposition will confirm our intuition when we have 2 different environments, one being dominated by the others. The random walk in dominating environment will be likely to go to the right than the dominated one. In other word, we have monotonicity with respect to starting points and environments.

Proposition 2.1. *Let ω_1 and ω_2 be environments with $\omega_1(x) \leq \omega_2(x)$, for all $x \in \mathbb{Z}$. Denote probabilities for the random walk in the "kLJBC" environment ω_i by $P_{\cdot}^{\omega_i}, i = 1, 2$. Then*

$$P_y^{\omega_1}(\tau_z \leq \tau_x \wedge N) \leq P_y^{\omega_2}(\tau_z \leq \tau_x \wedge N), \text{ for } x < y < z \text{ and } N > 0 \tag{2.1}$$

In particular then,

$$P_y^{\omega_1}(\tau_z \leq \tau_x) \leq P_y^{\omega_2}(\tau_z \leq \tau_x),$$

$$P_y^{\omega_1}(\tau_x = \infty) \leq P_y^{\omega_2}(\tau_x = \infty) \text{ and}$$

$$P_y^{\omega_1}(\tau_z = \infty) \geq P_y^{\omega_2}(\tau_z = \infty).$$

Proof. We will prove this by using the coupling only on the "space" meaning that the time parameter can be different. On a probability space$(\Omega, \mathcal{F}, \mathcal{P})$, define a sequence of i.i.d. random variables $\{U_k\}_{k=1}^{\infty}$ uniformly distributed on $[0, 1]$. Create $\{X_n^1\}$ and $\{X_n^2\}$, stochastic processes on $(\Omega, \mathcal{F}, \mathcal{P})$ such that the distribution of processes match the distributions of $P_y^{\omega_1}$ and $P_y^{\omega_2}$, respectively. Now we coupling the two processes $\{X_n^1\}_{n=1}^{\infty}$ and $\{X_n^2\}_{n=1}^{\infty}$.

First of all we set $X_0^1 = y = X_0^2$ to start from the same point. Then we define the mechanism whether each process will jump right or left depends on U_k. If $U_1 \leq \omega_i(y)$ the i-process will jump to the right i.e. $X_1^i = y + 1$ for $i = 1, 2$. When $U_1 > \omega_i(y)$, the i-process will take a left jump,$X_1^i = y + 1$. If $U_1 \leq \omega_1(y)$ or $U_1 > \omega_2(y)$, the processes will move together. If this case happens, use U_2 to couple again.

Then there will be the time that the first process will take left jump while the second process will jump to the right, mark the first time they separate as $n_1 \geq 1$. At this time, we have $X^1_{n^1} = X^2_{n_1} - 2$. Now we stop the second process and use U_{n_1+1}, the next unused random variable, to move $\{X^1_n\}$. Since this kind of process has more tendency to jump to the right and even the simple symmetric random walk is recurrent and irreducible, the first process will come back to same site as $X^2_{n_1}$, where the second process stopped and waiting, in finite time with probability 1. We mark the first time when the first process catch up with the second process as $m_1 > n_1$. Then continue coupling both processes again using U_{m_1+1}, the next available random variable.

There will be the second time the process will part again, mark the second time the processes differ as n_2. We have $X^1_{m_1+n_2} = X^2_{n_1+n_2} - 2$. Freeze the second process again and move only the first process using $U_{m_1+n_2+1}$. And again at time $m_2 > n_2$ both processes will be at the same level $X^2_{n_2} = X^1_{m_2}$. Continue this and we will have the sequence $\{(n_i, m_i)\}_{i=1}^{\infty}$. Let $N_j = \sum_{k=1}^{j} n_j$ and $M_j = \sum_{k=1}^{j} m_j$

Let τ^i_w denote the hitting time of w for the process $\{X^i_n\}$. From the construction above, one has, for each $j \geq 1$

$$\{\tau^1_z \leq \tau^1_x \wedge N\} \cap \{\tau^1_z \leq M_j\} \subset \{\tau^2_z \leq \tau^2_x \wedge N\} \cap \{\tau^2_z \leq N_j\}.$$

When they part, the process 1 can move around but not exceed the position of the second process. The only case they are different is when the first process moves to the left and the second process moves to the right. Then the statement is clear. Let $j \to \infty$, we have $\{\tau^1_z \leq \tau^1_x \wedge N\} \subset \{\tau^2_z \leq \tau^2_x \wedge N\}$. Hence, $\mathcal{P}(\tau^1_z \leq \tau^1_x \wedge N) \leq \mathcal{P}(\tau^2_z \leq \tau^2_x \wedge N)$. Since $\tau_z \geq z - y$, $\lim_{z\to\infty} \tau_z \geq \lim_{z\to\infty} z - y = \infty$. We have

$$\mathcal{P}(\infty \leq \tau^1_x \wedge N) \leq \mathcal{P}(\infty \leq \tau^2_x \wedge N)$$

or

$$\mathcal{P}(\tau^1_x = \infty) \leq \mathcal{P}(\tau^2_x = \infty).$$

Similarly, since $\tau_x \geq y - x$, $\lim_{x \to -\infty} \tau_x \geq \lim_{x \to -\infty} y - x = \infty$. We get

$$\mathcal{P}(\tau_z^1 \leq \infty \wedge N) \leq \mathcal{P}(\tau_z^2 \leq \infty \wedge N)$$

or

$$\mathcal{P}(\tau_z^1 \leq N) \leq \mathcal{P}(\tau_z^2 \leq N)$$

and

$$\mathcal{P}(\tau_z^1 \geq N) \geq \mathcal{P}(\tau_z^2 \geq N).$$

If we let $N \to \infty$ in above equation, we have

$$\mathcal{P}(\tau_z^1 = \infty) \geq \mathcal{P}(\tau_z^2 = \infty).$$

$\square$

Next we will categorize transience and recurrence based on $\phi \equiv P_1(\tau_0 = \infty)$, whether the process which started at 1 will reach 0 in finite time. Since this random walk is not Markovian, we must use the suitable definition which will reflect the intuitive meaning of transient and recurrent. Definition 1.6 is more suitable since the first visit back to the starting point will not guarantee the successive coming back. So we will use the definition of transient and recurrent as the following

Definition 2.2. Let $\{X_n\}_{n=0}^{\infty}$ be a random walk in "kLJBC" environment the process is called

(i) ***Transient*** if $P_y(X_n = x \text{ i.o.}) = 0$,

(ii) ***Recurrent*** if $P_y(X_n = x \text{ i.o.}) = 1$.

Lemma 2.3. *Let $\{X_n\}_{n=0}^{\infty}$ be a random walk in a deterministic "kLJBC" environment $\{\omega(x)\}_{x\in\mathbb{Z}}$, where $\omega(x) \in [\frac{1}{2}, 1)$ for all $x \in \mathbb{Z}$. Assume that the environment ω is deterministic and periodic: for some $N \geq 1$, $\omega(x + N) = \omega(x)$, for all $x \in \mathbb{Z}$.*

(i) If $P_1(\tau_0 = \infty) = 0$, *then the process is recurrent.*

(ii) If $P_1(\tau_0 = \infty) > 0$, *then the process is transient and* $\lim_{n\to\infty} X_n = \infty$ *a.s..*

Proof. (i) In the first case, $P_1(\tau_0 = \infty) = 0$ means the process started from site 1 will visit site 0 in finite time with probability 1. In order to reach 0, the process has to make a left jump at some sites, obviously at 1. So the environment will change into ω' such that $\omega'(x) \leq \omega(x), \forall x \in \mathbb{Z}$ if the process makes k^{th} left jump at some sites in order to reach 0. It is possible that the left jump may not be the k^{th} yet. In that case the environment at that site will still be the same. Anyway, we denote the new environment by ω'. Let $\omega^s(x) = \frac{1}{2}, \forall x \in \mathbb{Z}$ be the environment for the simple symmetric random walk.

By the proposition 2.1, $\omega^s(x) \leq \omega'(x)$ implies $P_0^{\omega^s}(\tau_1 = \infty) \geq P_0^{\omega'}(\tau_1 = \infty)$. But since the simple symmetric random walk is recurrent, $0 = P_0^{\omega^s}(\tau_1 = \infty) \geq P_0^{\omega'}(\tau_1 = \infty)$. Therefore the process will hit 1 again in finite time with the probability 1. At the time process reaches 1, the current environment, ω^2 will satisfy $\omega^2(x) \leq \omega'(x), \forall x \in \mathbb{Z}$. By the proposition 2.1 again, site 0 will be visited again in finite time with the probability 1. And this process will continue indefinitely; hence, we get $P_1(X_n = 0 \text{ i.o.}) = 1$.

Let $x \in \mathbb{Z}$. There is positive probability that the process at 0 will jump to the site x directly in $|x|$ steps regardless of the environment. $P_1(X_n = 0 \text{ i.o.}) = 1$. will lead to $P_1(X_n = x \text{ i.o.}) = 1$.

Now let $y > 1$. There is positive probability that the walker started from site 1 will jump directly to y in $y-1$ steps. The environment in this case is still intact. Thus $P_1(X_n = x \text{ i.o.}) = 1$ will lead to $P_y(X_n = x \text{ i.o.}) = 1$. Now we will consider the case when $y \leq 0$. By comparing to simple symmetric random walk, the walker starts from y will definitely hit 1 with probability 1. Call the current environment $\omega' \leq \omega$. Now we have 2 cases, $x > 1$ or $x < 0$. For $x < 0$, by Using the proposition 2.1, $P_1(X_n = x \text{ i.o.}) = 1$ implies $P_1^{\omega'}(X_n = x \text{ i.o.}) = 1$. Hence, we will have $P_y(X_n = x \text{ i.o.}) = 1$. For $x > 0$, we use the original assumption and proposition 2.1 to get $P_1^{\omega'}(\tau_0 = \infty) = 0$ and therefore $P_1^{\omega'}(X_n = x \text{ i.o.}) = 1$. Hence we will have $P_y(X_n = x \text{ i.o.}) = 1$.

(ii) Let $P_1(\tau_0 = \infty) > 0$. First we will show that $P_y(\tau_x = \infty) > 0$, for all $x < y$. Assume that there is $x_0 < y_0$ such that $P_{y_0}(\tau_{x_0} = \infty) = 0$ and we will have a contradiction. By argument in proof of part (i.) (The process started at y_0 will reach x_0 with probability 1 in finite time, then the process will hit y_0 with probability 1 by comparing to simple symmetric random walk. After that the process will jump to x_0 again by the assumption with different environment and come back to y_0 again....), $P_{y_0}(\tau_{x_0} = \infty) = 0$ will lead to $P_{y_0}(X_n = x \text{ i.o.}) = 1$ for all $x \in \mathbb{Z}$. In particular, $P_{y_0}(X_n = 0 \text{ i.o.}) = 1$. If $y_0 = 1$, we arrive at a contradiction.

Now consider the case $y_0 < 1$. There is a positive probability that the process starts from y_0 will hit site 1 in $1 - y_0$ steps and the environment will still be the same. Therefore, $P_{y_0}(X_n = 0 \text{ i.o.}) = 1$ will lead to $P_1(X_n = 0 \text{ i.o.}) = 1$ which is a contradiction. Now consider the case $y_0 > 1$. From $P_1(\tau_0 = \infty) > 0$, there is a positive probability that the process started at 1 will hit site y_0 before site 0. After the walker hits site y_0, the walker continues without hitting 0 at any time. But when the process hit y_0, on that event, let the ω' be the current environment which will satisfy $\omega' \leq \omega$. Since $P_{y_0}(X_n = 0 \text{ i.o.}) = 1$ implies $P_{y_0}^{\omega'}(X_n = 0 \text{ i.o.}) = 1$ using the proposition 2.1, the process will indeed visit 0 with the probability 1, which is a contradiction.

From the fact that $P_y(\tau_x = \infty) > 0$, for all $x < y$, and from the periodicity of the environment, there is a strictly positive probability called $\phi_1 \equiv \inf_{x \in \mathbb{Z}} P_x(\tau_{x-1} = \infty) = \inf_{x \in \{0,1,\ldots,N-1\}} P_x(\tau_{x-1} = \infty) > 0$. We will prove transience using this fact. Fix x and y. Since the environment of simple symmetric random walk, called ω^s, will be less than or equal to current environment, by proposition 2.1, $P_y^{\omega}(\tau_z < \infty) \geq P_y^{\omega^s}(\tau_z < \infty) = 1$, for all $z > y$. So the process will have a new maximum infinitely often. The current environment at any new maximum is still intact or equal to ω. The environment to the right of that maximum is also ω. Hence, with probability at least ϕ_1 the process will never ever go below that maximum again. Therefore; we have transience i.e. $P_y(X_n = x \text{ i.o.}) = 0$ and $\lim_{n \to \infty} X_n = \infty$ almost surely. □

2.2 2-left-jump-break-cookie Random Walk

Next theorem will be the special case where k is equal to 2 or the second left jump break cookie environment.

Theorem 2.4. *Let* $\{X_n\}_{n=o}^{\infty}$ *be a random walk in a deterministic "2LJBC" environment* $\{\omega(x)\}_{x\in Z}$*, where* $\omega(x) \in [\frac{1}{2}, 1)$ *for all* $x \in \mathbb{Z}$*.*

1. *Let* $\omega(x) = p$ *for all* x *in* $\mathbb{Z}$*. Then*

$$\begin{cases} P_1(\tau_0 = \infty) = 0, & \text{if } p \leq \frac{3}{5}; \\ \frac{-3+4p+\sqrt{-3+8p-4p^2}}{2p} \leq P_1(\tau_0 = \infty) \leq \frac{3-5p}{2-5p+3p^2-2p^3}, & \text{if } p \in (\frac{3}{5}, 1). \end{cases}$$

   In particular, the process is recurrent if $p \leq \frac{3}{5}$ *and transient if* $p > \frac{3}{5}$*.*

2. *Let* ω *be periodic with period* $N > 1$*. Then the process is recurrent if* $\frac{1}{N}\sum_{x=1}^{N} \frac{\omega(x)}{1-\omega(x)} \leq \frac{3}{2}$ *and transient if* $\frac{1}{N}\sum_{x=1}^{N} \frac{\omega(x)}{1-\omega(x)} > \frac{3}{2}$*.*

Remark 1. Comparing the result to the Pinksky's result, this result is perfectly understandable since the more you delay changing environment, the smaller p should be in order to have the recurrent process.

Remark 2. As we have described in Pinsky's model, the number of cookies is like a geometric number of cookie represent waiting time for the first success. This 2LJBC is like a negative binomial represent waiting time for the second success or second left jump. Similar situation happens, when the process is recurrent the process will visit certain site infinitely many times. The expected number of jump from that site when the effective cookies is present is $\sum_{m=2}^{\infty}\binom{m-1}{m-2}p^{m-2}(1-p)^2 = \frac{2}{1-p}$. Thus, it is again similar to the corollary 1.16 with $\frac{2}{1-p}$ number of cookie. Substitute back in (1.2), we get that $p \leq \frac{3}{5}$ for recurrent process.

Proof. For $x \in \mathbb{Z}$, define $\nu_x^1 = \inf\{n \geq 1 : X_{n-1} = x, X_n = x-1\}$, the first time the process jumps to the left from site x. Similarly, let $\nu_x^k = \inf\{n > \nu_x^{k-1} : X_{n-1} = x, X_n = x-1\}$, the k^{th} time the process jumps to the left from site x. Let $D_0^x = 0$ and for $n \geq 1$ let

$$D_n^x = \sum_{m=0}^{n} \chi_{\{X_m = x, m < \nu_x^2\}}.$$

D_n^x is the number of visits at site x before the 2^{nd} left jump occurs and before time n. Define, the total drift,

$$D_n = \sum_{x \in \mathbb{Z}} (2\omega(x) - 1) D_n^x.$$

Notice that $2\omega(x) - 1$ comes from the expectation of increment, multiplied by the number of visits at that site. So D_n is an increasing sequence because $\omega(x) \in [\frac{1}{2}, 1)$. Let $M_n = X_n - D_n$. It is clear that D_n is predictable process. From Doob decomposition theorem, $\{M_n\}_{n=0}^{\infty}$ is a martingale with respect to its natural filtration. By Doob's optional stopping theorem applying to $\{M_n\}_{n=0}^{\infty}$, we have

$$\begin{aligned}
E_1[M_0] &= E_1[M_{\tau_0 \wedge \tau_n}] \\
E_1[X_0 - D_0] &= E_1[X_{\tau_0 \wedge \tau_n} - D_{\tau_0 \wedge \tau_n}] \\
E_1[X_0] - E_1[D_0] &= E_1[X_{\tau_0 \wedge \tau_n}] - E_1[D_{\tau_0 \wedge \tau_n}] \\
1 - 0 &= (0)P_1(\tau_0 \leq \tau_n) + nP_1(\tau_0 > \tau_n) - E_1[D_{\tau_0 \wedge \tau_n}.]
\end{aligned}$$

After rearranging the term, we get,

$$\begin{aligned}
nP_1(\tau_0 > \tau_n) &= 1 + E_1[D_{\tau_0 \wedge \tau_n}] \\
nP_1(\tau_0 > \tau_n) &= 1 + \sum_{x=1}^{n-1} (2\omega(x) - 1) E_1[D_{\tau_0 \wedge \tau_n}^x]. \qquad (2.2)
\end{aligned}$$

Notice that

$$\lim_{n \to \infty} P_1(\tau_0 > \tau_n) = P_1(\tau_0 = \infty),$$

which is the term we are looking for. Let $\phi_x = P_{x+1}(\tau_x = \infty)$. Since $\{\tau_0 > \tau_n\} \supseteq \{\tau_0 > \tau_{n+1}\}$, $P_1(\tau_0 > \tau_n)$ is a sequence of number which is decreasing and bounded below. Therefore the limit exists and

$$\tau_0 > \tau_n \geq n - 1$$
$$\lim_{n\to\infty} \tau_0 > \lim_{n\to\infty} \tau_n \geq \lim_{n\to\infty} n - 1$$
$$\tau_0 \geq \infty.$$

We will assume that the process is transient i.e. $\phi > 0$. Then we will create the random variable that stochastically dominates $D^x_{\tau_0}$ under P_1. First of all, let $\delta^{(l)}_x = P^{\omega^{(l)}}_{x-1}(\tau_0 < \tau_x)$ be the probability of not returning to x before reaching 0, conditioned on the process returning to x from $x-1$ at least $l-1$ times. Notice that the environment may not be the same at each time we consider this term. We can suppress the l whenever $l = 1$. We also define $\phi^{(l)}_x = P^{\omega^{(l)}}_{x+1}(\tau_x = \infty)$, the probability of not visiting site x starting from site $x + 1$ and conditioned on the process returning to x from $x + 1$ for $l - 1$ times.

Let $R_l = \omega(x)(1 - \phi^{(l)}_x)$ be a probability of jumping to the right for the l^{th} time and come back to the starting point x with probability one. Let $R^e_l = \omega(x)\phi^{(l)}_x$ be a probability of jumping to the right for the l^{th} time and will not come back to the starting point x, so that will end the counting.

Let $L_l = (1 - \omega(x))(1 - \delta^{(l)}_x)$ be a probability of jumping to the left for the l^{th} time and come back to the staring point x with probability one. Let $L^e_l = (1 - \omega(x))\delta^{(l)}_x$ be a probability of jumping to the left for the l^{th} time and hit 0 before returning back to x, so the counting will be ended.

Let $L^e_k = (1 - \omega(x))$ be a probability of jumping to the left for the k^{th} time which will be the last left jump before the environment change to symmetric random walk, so the counting will be ended also in this situation.

All notation above can be summarized here:

$$\begin{aligned}
\delta_x^{(l)} &= P_{x-1}^{\omega^{(l)}}(\tau_0 < \tau_x) \\
R_l &= \omega(x)(1-\phi_x^{(l)}) && (l^{th} \text{ Right jump}) \\
R_l^e &= \omega(x)\phi_x^{(l)} && (l^{th} \text{ Right jump end counting}) \\
L_l &= (1-\omega(x))(1-\delta_x^{(l)}) && (l^{th} \text{ Left jump}) \\
L_l^e &= (1-\omega(x))\delta_x^{(l)} && (l^{th} \text{ Left jump end counting}) \\
L_k^e &= (1-\omega(x)) && (k^{th} \text{ Left jump end counting}).
\end{aligned} \tag{2.3}$$

Notice that there is a relationship as follows

$$\begin{aligned}
L_2^e + R_{m-1}^e &= 1 - R_{m-1} \\
L_l^e + R_m^e &= (1-L_l) - R_m.
\end{aligned} \tag{2.4}$$

When $k = 1$ the environment is similar to geometric distribution i.e. waiting for the first success. In this case, the success means left jump. When $k > 1$ it will be similar to negative binomial i.e. waiting for the k^{th} success or k^{th} left jump in this situation. Now we will restrict our case to $k = 2$.

For $x \geq 1$, we will have $D_{\tau_0}^x = 0$ when $\tau_0 < \tau_x$ or the number of visit to site 0 is 0. This means the process started at 1 must reach site 0 before site x. $D_{\tau_0}^x > 0$ when $\tau_0 > \tau_x$, or visits site x before site 0 provided that the process started from site 1.

When $D_{\tau_0}^x = 1$ it means that at time τ_x which the process is at site x for the first time, the process will jump to the left, i.e. to $x-1$ and eventually hit 0 with the probability $(1-\omega(x))\delta_x$ or jump to the right and never come back to x again with the probability $\omega(x)\phi(x)$. By using the notation above, the probability is equal to $L_1^e + R_1^e$.

In case $D_{\tau_0}^x = 2$, there are 2 cases whether the process has already made a left jump at the first time. So we have either 2 left jumps, right jump then left jump, left jump then right jump or 2 right jumps. We get $L_1L_2^e + R_1L_1^e + L_1R_1^e + R_1R_2^e = L_1(L_2^e + R_1^e) + R_1(L_1^e + R_2^e)$.

If $D^x_{\tau_0} = 3$, we have $R_1L_1L^e_2 + L_1R_1L^e_2 + R_1R_2L^e_1 + R_1L_1R^e_2 + L_1R_1R^e_2 + R_1R_2R^e_3 = 2R_1L_1(L^e_2 + R^e_2) + R_1R_2(L^e_1 + R^e_3)$. Notice that the order of L and R doesn't play any important role.
If $D^x_{\tau_0} = 4$, we have $3R_1R_2L_1(L^e_2 + R^e_3) + R_1R_2R_3(L^e_1 + R^e_4)$. So we have

$$P(D^x_{\tau_0} = m) = \begin{cases} L^e_1 + R^e_1 & \text{if } m = 1 \\ L_1(L^e_2 + R^e_1) + R_1(L^e_1 + R^e_2) & \text{if } m = 2 \\ (m-1)(\prod_{i=1}^{k-2} R_i)L_1(L^e_2 + R^e_{k-1}) + (\prod_{i=1}^{k-1} R_i)(L^e_1 + R^e_k) & \text{if } m > 2. \end{cases}$$

First of all, we will try to consider the sequences $\left\{\phi^{(l)}_x\right\}$ and $\left\{\delta^{(l)}_x\right\}$. Both of the sequences will be monotonic. The former one will be non-increasing and non-decreasing for the latter sequence. Therefore; $\{R_i\}$ and $\{L_i\}$ are non-decreasing and non-increasing, respectively. Now we will try to find the random variable which will be stochastically dominated by $D^x_{\tau_0}$. For this domination, we can come up with anything that serve our purpose. Secondly, we take a look at $P(2 \leq D^x_{\tau_0} \leq M)$, Consider $P(2 \leq D^x_{\tau_0} \leq 3)$, we get

$$\begin{aligned} &P(2 \leq D^x_{\tau_0} \leq 3) \\ &\quad = L(L^e_2 + R^e_1) + R_1(L^e + R^e_2) + 2LR_1(L^e + R^e_2) + R_1R_2(L^e + R^e_3) \end{aligned}$$

by (2.4), we get,

$$\begin{aligned} &= L(1 - R_1) + R_1((1 - L) - R_2) + 2LR_1(1 - R_2) + R_1R_2((1 - L) - R_3) \\ &= L + R_1 - 3LR_1R_2 - R_1R_2R_3. \end{aligned} \tag{2.5}$$

Consider again that $P(2 \leq D^x_{\tau_0} \leq 4)$, by adding more term to (2.5), we get

$$\begin{aligned} &P(2 \leq D^x_{\tau_0} \leq 4) \\ &\quad = L + R_1 - 3LR_1R_2 - R_1R_2R_3 + 3LR_1R_2(L^e_2 + R^e_3) + R_1R_2R_3(L^e + R^e_4) \end{aligned}$$

using (2.5), we obtain,

$$
\begin{aligned}
&= L + R_1 - 3LR_1R_2 - R_1R_2R_3 + 3LR_1R_2(1-R_3) + R_1R_2R_3((1-L)-R_4) \\
&= L + R_1 - 4LR_1R_2R_3 - R_1R_2R_3R_4.
\end{aligned}
$$

Therefore; we can prove by math induction that

$$
\begin{aligned}
P(2 \le D^x_{\tau_0} \le M) &= L + R_1 - ML\prod_{i=1}^{M-1} R_i - \prod_{i=1}^{M} R_i \qquad (2.6)\\
&= R\left(\frac{L}{R} + 1 - \frac{ML}{R}\prod_{i=1}^{M-1} R_i - \frac{1}{R}\prod_{i=1}^{M} R_i\right).
\end{aligned}
$$

After carefully consider $P_1(1 < D^x_{\tau_0} < M)$, we can have the smaller term by modifying R_i. This can be done because of transient assumption, $\phi_x > 0$. Even $\phi_x^{(i)}$ maybe 0, we will have a bigger quantity. Therefore; we set $\phi_x = 0$, or change $R_i := \omega(x)(1-\phi_x^{(i)})$ to $\omega(x)$ in the parenthesis to have the desired domination. So we choose 2 independent random variables I_x and V_x satisfying

$$
P(I_x = 1) = 1 - P(I_x = 0) = P_1(\tau_x < \tau_0), \qquad (2.7)
$$

$$
P(V_x = l) = \begin{cases} L^e + R^e & \text{if } l = 1 \\ R(L(L_2^e)\frac{1}{\omega(x)} + (L^e)) & \text{if } k = 2 \\ R\left((k-1)(\omega(x))^{k-3}L(L_2^e) + (\omega(x))^{k-2}(L^e)\right) & \text{if } k > 2 \end{cases}
$$

Notice that when $\phi_x = 0$, R^e will vanish. Furthermore, $P_1(D^x_{\tau_0} < m) \ge P(I_xV_x < m)$ by knowing that the the first term of both distributions are the same and the way we defined distribution of the I_x. Then $D^x_{\tau_0}$ is indeed being stochastically dominated by I_xV_x. Therefore; $E_1D^x_{\tau_0} \le EI_xV_x = P_1(\tau_x < \tau_0)EV_x$. Since

$$
EV_x = \frac{2 - \phi_x(2 - \omega(x) + (\omega(x))^2)}{1 - \omega(x)} + \delta_x\left(\phi_x - \frac{1-\phi_x}{1-\omega(x)} - \phi_x\omega(x)\right),
$$

by stochastic domination, we get

$$E_1 D_{\tau_0}^x \leq \left(\frac{2 - \phi_x(2 - \omega(x) + (\omega(x))^2)}{1 - \omega(x)} + \delta_x \left(\phi_x - \frac{1 - \phi_x}{1 - \omega(x)} - \phi_x \omega(x) \right) \right) P_1(\tau_x < \tau_0). \quad (2.8)$$

Substituting (2.8) into (2.2) and using monotonicity of D_n gives

$$\begin{aligned} P_1(\tau_0 > \tau_n) \leq \frac{1}{n} + \frac{1}{n} \sum_{x=1}^{n-1} (2\omega(x) - 1) \left(\frac{2 - \phi_x(2 - \omega(x) + (\omega(x))^2)}{1 - \omega(x)} \right) P_1(\tau_x < \tau_0) + \\ + \frac{1}{n} \sum_{x=1}^{n-1} (2\omega(x) - 1) \left(\delta_x \left(\phi_x - \frac{1 - \phi_x}{1 - \omega(x)} - \phi_x \omega(x) \right) \right) P_1(\tau_x < \tau_0) \quad (2.9) \end{aligned}$$

Consider part (i) of the theorem which we have $\omega(x) = p$ for all x and $\phi_x \equiv \phi$ independent of x. By letting $n \to \infty$, we have

$$\begin{aligned} \lim_{n\to\infty} P_1(\tau_0 > \tau_n) \leq \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n-1} (2p - 1) \left(\frac{2 - \phi(2 - p + p^2)}{1 - p} \right) P_1(\tau_x < \tau_0) + \\ + \lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n-1} (2p - 1) \left(\delta_x \left(\phi - \frac{1 - \phi}{1 - p} - \phi p \right) \right) P_1(\tau_x < \tau_0) \\ P_1(\tau_0 = \infty) \leq (2p - 1) \left(\frac{2 - \phi(2 - p + p^2)}{1 - p} \right) P_1(\tau_0 = \infty), \end{aligned}$$

provided that $\phi - \frac{1-\phi}{1-p} - \phi p \leq 0$ or $\phi \leq \frac{1}{2-2p+p^2}$. Thus

$$(2p - 1) \left(\frac{2 - \phi(2 - p + p^2)}{1 - p} \right) \geq 1,$$

or

$$\phi = P_1(\tau_0 = \infty) \leq \frac{3 - 5p}{2 - 5p + 3p^2 - 2p^3}.$$

As we can see from figure 2.1, this result is valid since $\frac{1}{2-2p+p^2} \geq \frac{3-5p}{2-5p+3p^2-2p^3}$. Because of transient assumption $\phi > 0$, it follows that $p > \frac{3}{5}$ is a necessary condition for transience. And this gives the upper bound on $P_1(\tau_0 = \infty)$ in part (i) of the theorem.

Now consider part (ii) of the theorem. In this case, both $\omega(x)$ and ϕ_x are periodic with

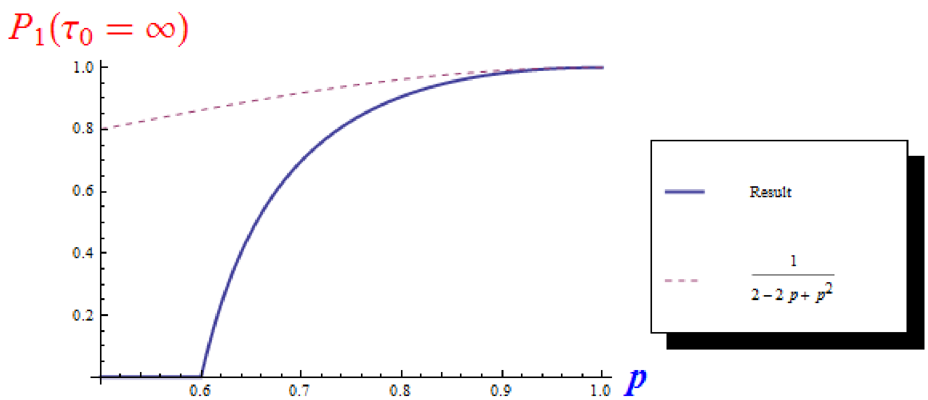

Figure 2.1: Plot showing the result and the constraint

period $N > 1$. By transience assumption and Lemma 2.3, $\phi_x > 0$ for all x. Therefore; letting $n \to \infty$ in (2.9), we get

$$\frac{1}{N}\sum_{x=1}^{N}(2\omega(x)-1)\left(\frac{2-\phi_x\left(2-\omega(x)+(\omega(x))^2\right)}{1-\omega(x)}\right) \geq 1. \tag{2.10}$$

Since (2.10) was derived under the assumption that $\phi_x > 0$, for all x, we get $\frac{1}{N}\sum_{x=1}^{N}\frac{2(2\omega(x)-1)}{1-\omega(x)} > 1$, which is a necessary condition for transience. This is equivalent to

$$\begin{aligned}\frac{1}{N}\sum_{x=1}^{N}\frac{2\omega(x)-1}{1-\omega(x)} &> \frac{1}{2}\\ \frac{1}{N}\sum_{x=1}^{N}\left(\frac{\omega(x)}{1-\omega(x)}+\frac{\omega(x)-1}{1-\omega(x)}\right) &> \frac{1}{2}\\ \frac{1}{N}\sum_{x=1}^{N}\frac{\omega(x)}{1-\omega(x)} &> \frac{3}{2}.\end{aligned}$$

Now we consider the necessary condition for the recurrence in part (i) and (ii), and to the lower bound on $P_1(\tau_0 = \infty)$. We make no assumption of transience or recurrence for now. Under P_1, consider $D^x_{\tau_0\wedge\tau_n}$ which appeared on the right hand side of (2.2). If $\tau_0 < \tau_n$, then $D^x_{\tau_0\wedge\tau_n} = 0$. Let $\epsilon_{x,n}$ be the probability of not returning to x before reaching n and after the first jump to the right from x to $x+1$.

Similarly, let $\epsilon_{x,n}^{(l)}$ be the probability, conditioned on the process returning to x from $x+1$ at least $l-1$ times and not return to x before hitting n, that after l^{th} jumps rightward from x to $x+1$. Each time the process jumps from x to $x+1$, denote the current environment ω'', will satisfy $\omega'' \leq \omega'$ where ω' was the environment in effect the previous time the process jumped from x to $x+1$. Thus by the proposition 2.1, it follows that

$$P_{x+1}^{\omega''}(\tau_n < \tau_x) \leq P_{x+1}^{\omega'}(\tau_n < \tau_x),$$

or $\epsilon_{x,n}^{(l)} \leq \epsilon_{x,n}$ for $l > 1$. Notice that the difference between $\phi_x^{(l)}$ and $\epsilon_{x,n}^{(l)}$ is just that the latter have a wall at n. Thus everything should follows as in the case of $D_{\tau_0}^x$. So we also introduce the similar notation for this part

$$\begin{aligned}
\delta_x^{(l)} &= P_{x-1}^{\omega^{(l)}}(\tau_0 < \tau_x) & & \\
R_l &= \omega(x)(1-\epsilon_{x,n}^{(l)}) & & (l^{th} \text{ Right jump}) \\
R_l^e &= \omega(x)\epsilon_{x,n}^{(l)} & & (l^{th} \text{ Right jump end counting}) \\
L_l &= (1-\omega(x))(1-\delta_x^{(l)}) & & (l^{th} \text{ Left jump}) \\
L_l^e &= (1-\omega(x))\delta_x^{(l)} & & (l^{th} \text{ Left jump end counting}) \\
L_k^e &= (1-\omega(x)) & & (k^{th} \text{ Left jump end counting}).
\end{aligned}$$

Again, we have the same relationships

$$\begin{aligned}
L_2^e + R_{m-1}^e &= 1 - R_{m-1} \\
L_l^e + R_m^e &= (1-L_l) - R_m.
\end{aligned}$$

We also have the distribution of $D^x_{\tau_0\wedge\tau_n}$ as follows,

$$\begin{cases} P(D^x_{\tau_0\wedge\tau_n}=0)=P_1(\tau_0<\tau_x); \\ P(D^x_{\tau_0\wedge\tau_n}=1)=L^e+R^e_1; \\ P(D^x_{\tau_0\wedge\tau_n}=m)=(m-1)L\left(L^e_2+R^e_{m-1}\right)\prod_{i=1}^{m-2}R_i+\left(L^e+R^e_m\right)\prod_{i=1}^{m-1}R_i, m\geq 2. \end{cases}$$

Now we have to find a random variable that will be dominated by $D^x_{\tau_0\wedge\tau_n}$. From (2.6), we need the bigger quantity. Therefore; we replace R_i by R_1 in (2.6). We will get bigger quantity since $\epsilon^{(i)}_{x,n}\leq\epsilon_{x,n}$ and $R_1\leq R_i$. Now we create a random variable Z_x which is independent to I_x as in (2.7) such that

$$P(Z_x=m)=\begin{cases} L^e+R^e & \text{if } k=1 \\ (m-1)L\left(R_1\right)^{m-2}(1-R_1)+\left(R_1\right)^{m-1}\left((1-L)-R_1\right) & \text{if } k\geq 2. \end{cases}$$

which will serve our purpose. Then we have

$$EZ_x=\frac{2-\delta_x(1-\omega(x))-(2-\epsilon_{x,n})\omega(x)}{(1-(1-\epsilon_{x,n})\omega(x))^2}.$$

Since $D^x_{\tau_0\wedge\tau_n}$ stochastically dominates I_xZ_x, we obtain

$$E_1D_{\tau_n\wedge\tau_0}\geq EI_xZ_x=\frac{2-\delta_x(1-\omega(x))-(2-\epsilon_{x,n})\omega(x)}{(1-(1-\epsilon_{x,n})\omega(x))^2}P_1(\tau_x<\tau_0).$$

From $\{\tau_x<\tau_0\}\supseteq\{\tau_n<\tau_0\}$, for all $x=1,2,\dots,n-1$, we get $P_1(\tau_x<\tau_0)\geq P_1(\tau_n<\tau_0)$. Using this fact we will have

$$\begin{aligned} E_1D_{\tau_n\wedge\tau_0}\geq EI_xZ_x&=\frac{2-\delta_x(1-\omega(x))-(2-\epsilon_{x,n})\omega(x)}{(1-(1-\epsilon_{x,n})\omega(x))^2}P_1(\tau_x<\tau_0) \\ &\geq\frac{2-\delta_x(1-\omega(x))-(2-\epsilon_{x,n})\omega(x)}{(1-(1-\epsilon_{x,n})\omega(x))^2}P_1(\tau_n<\tau_0). \end{aligned} \tag{2.11}$$

From (2.2) and (2.11) it follows that for any n,

$$\frac{1}{n}\sum_{x=1}^{n-1}(2\omega(x)-1)\frac{2-\delta_x(1-\omega(x))-(2-\epsilon_{x,n})\omega(x)}{(1-(1-\epsilon_{x,n})\omega(x))^2}<1. \tag{2.12}$$

Now consider part (i) of the theorem. In this case $\omega(x)=p$ for all x, then $\epsilon_{x,n}$ will depend only on $n-x$ and $\lim_{(n-x)\to\infty}\epsilon_{x,n}=\phi=P_1(\tau_0=\infty)$. If the process is recurrent, then we can conclude that $\phi=0$. Since δ_x, probability of visiting site 0 starting from site $x-1$ before visiting site x, depends only on x but not n, we know that when x is very large $\delta_x\to 0$. All the terms related to the value of δ_x when x is small will be annihilated by $\frac{1}{n}$.

By letting $n\to\infty$ in (2.12), we have $\frac{2(2p-1)}{1-p}\leq 1$ which is a necessary condition for recurrence. This is equivalence to $p\leq\frac{3}{5}$. In transient case, $\phi>0$ will present. Therefore; we have $\frac{(2p-1)(2-(2-\phi)p)}{(1-(1-\phi)p)^2}\leq 1$ or

$$\phi=P_1(\tau_0=\infty)\geq\frac{-3+4p+\sqrt{-3+8p-4p^2}}{2p}.$$

which gives the lower bound on $P_1(\tau_0=\infty)$ in part (i).

Now assume recurrence and periodic in part (ii), $\lim_{n\to\infty}\epsilon_{x,n}=0$. Let $n\to\infty$ in (2.12). Hence the necessary condition for recurrence is $\frac{1}{N}\sum_{x=1}^{N}\frac{2\omega(x)-1}{1-\omega(x)}\leq\frac{1}{2}$ or $\frac{1}{N}\sum_{x=1}^{N}\frac{\omega(x)}{1-\omega(x)}\leq\frac{3}{2}$ $\square$

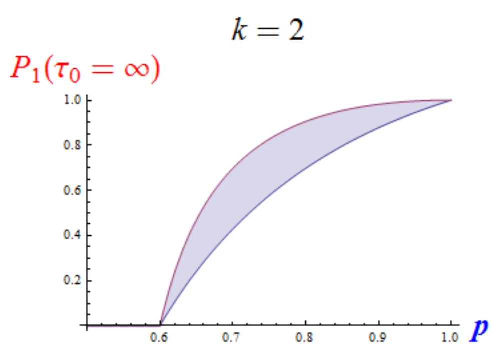

Figure 2.2: Plot showing possible value of $\phi=P_1(\tau_0=\infty)$ in second left jump break cookie environment.

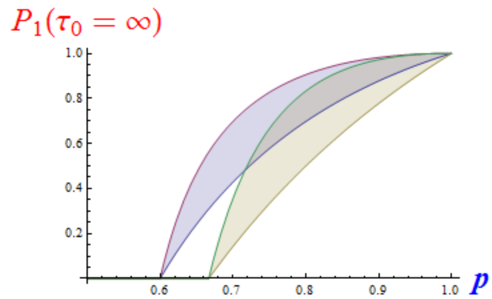

Figure 2.3: Plot comparing the possible value of $\phi = P_1(\tau_0 = \infty)$ between second left jump break cookie environment and "have your own cookie and eat it environment".

2.3 k-left-jump-break-cookie Random Walk

In this section we will prove the theorem about the general k^{th}-left jump break cookie random walk.

Theorem 2.5. *Let* $\{X_n\}_{n=0}^{\infty}$ *be a random walk in a deterministic "kLJBC" environment* $\{\omega(x)\}_{x\in Z}$*, where* $\omega(x) \in [\frac{1}{2}, 1)$ *for all* $x \in \mathbb{Z}$*.*

1. *Let* $\omega(x) = p$ *for all* x *in* $\mathbb{Z}$*. Then*

$$\begin{cases} P_1(\tau_0 = \infty) = 0, & \text{if } p \le \frac{k+1}{2k+1}; \\ \phi = P_1(\tau_0 = \infty) \le \frac{(k+1)-(2k+1)p}{k-(2k+1)p+3p^2-2p^3}, & \text{if } p \in (\frac{k+1}{2k+1}, 1). \end{cases}$$

   In particular the process is recurrent if $p \le \frac{k+1}{2k+1}$

2. *Let* ω *be periodic with period* $N > 1$*. Then the process is recurrent if* $\frac{1}{N}\sum_{x=1}^{N} \frac{\omega(x)}{1-\omega(x)} \le 1 + \frac{1}{k}$*.*

Remark 3. The idea of the proof of this theorem is similar to the one that we have used in proving second left jump case. And we get only one side of the argument working.

Proof. For general k^{th}, we will use the same notation as in (2.3). We can think as the follow. When $D^x_{\tau_0} = l$, this means that there is l positions to be filled by L or R in those spots and there are only k number of L's to fill where R is unlimited. We focus on the rightmost spot first. Then we separate into 2 groups after trimming the end or the last step.

Case 1. Using all available L. Start from the rightmost spot, we can fill either L^e_k or R^e without the problem. Then fill out the rest, $k-1$ spots out of $l-1$ spots will be filled by L. So we have $\binom{l-1}{k-1} R^{l-k} L^{k-1} \left(L^e_k + R^e\right)$. But since there is the change of environment between each jump so we have to distinguish the jump. We obtain

$$\binom{l-1}{k-1}\left(\prod_{j=1}^{l-k} R_j\right)\left(\prod_{s=1}^{k-1} L_s\right)\left(L^e_k + R^e_{l-k+1}\right).$$

Case 2. Using only some L. Start from the rightmost spot again since we have to denoted it by superscript e, we can fill either L^e or R^e. Then we can choose how many L we will be used after filling out the rightmost spot, ranging from 0 to $k-2$. For each i number of L we used, we have $\binom{l-1}{i} R^{l-i-1} L^i \left(L^e + R^e\right)$. Similar to the previous case, for all possible values of choosing L, we have

$$\sum_{i=0}^{k-2} \binom{l-1}{i} \left(\prod_{j=1}^{l-i-1} R_j\right) \left(\prod_{s=1}^{i} L_s\right) \left(L_{i+1}^e + R_{l-i}^e\right).$$

Therefore; we have the distribution of $D_{\tau_0}^x$ as follows,

$$P(D_{\tau_0}^x = l) = \sum_{i=0}^{k-2} \binom{l-1}{i} \left(\prod_{j=1}^{l-i-1} R_j\right) \left(\prod_{s=1}^{i} L_s\right) \left(L_{i+1}^e + R_{l-i}^e\right) + \tag{2.13}$$

$$+ \binom{l-1}{k-1} \left(\prod_{j=1}^{l-k} R_j\right) \left(\prod_{s=1}^{k-1} L_s\right) \left(L_k^e + R_{l-k+1}^e\right) \tag{2.14}$$

Similar to the case when $k = 2$, we assume transient and consider $D_{\tau_0}^x$ instead of $D_{\tau_0 \wedge \tau_n}^x$. Since the possibility of stop counting $D_{\tau_0}^x$ is either hit 0 or getting lost to infinity. We will define the distribution of dominating random variable as follow.

When $D_{\tau_0}^x = 1$, there are 2 possibility either make a left jump and hit 0 or getting lost to infinity.

When $D_{\tau_0}^x \geq 2$, we assume that there is no more getting lost to infinity. The only way to stop counting is either hit 0 or make the last possible left jump allowed. This is clear that this is indeed domination we looking for since we get rid of some possibility of not coming back in $D_{\tau_0}^x$. Therefore, we have the distribution as follow.

Let I_x and V_x be independent random variables satisfying

$$P(I_x = 1) = 1 - P(I_x = 0) = P_1(\tau_x < \tau_0), \tag{2.15}$$

$$P(V_x = l) = \begin{cases} L^e + R^e & \text{if } l = 1 \\ R\left(\sum_{i=1}^{k-1} \binom{l-1}{i-1}(\omega(x))^{l-i-1} L_{k-1}^{i-1}\left(L_{k-1}^e\right) + \binom{l-1}{k-1}(\omega(x))^{l-k-1} L_{k-1}^{k-1}\left(L_k^e\right)\right) & \text{if } l > 1 \end{cases}$$

By computing the expectation of V_x, we obtain

$$\begin{aligned} EV_x = & \frac{k - \phi_x(k - \omega(x) + \omega(x)^2)}{1 - \omega(x)} \\ & + \delta_x^{(k-1)} \left(\frac{\frac{k(k-1)}{2} - \phi_x\left(\frac{k(k-1)}{2} + 1 - 2\omega(x) + \omega(x)^2\right)}{1 - p} \right) + C_x, \end{aligned} \tag{2.16}$$

where $C_x = O\left(\left(\delta_x^{(k-1)}\right)^2\right)$ and will vanish when x is very large. Substitute back into (2.2) and using monotonicity of D_n, we get

$$\begin{aligned} E_1 D_{\tau_0}^x \leq & \frac{k - \phi_x(k - \omega(x) + \omega(x)^2)}{1 - \omega(x)} P_1(\tau_x < \tau_0) + \\ & + \left(\delta_x^{(k-1)} \left(\frac{\frac{k(k-1)}{2} - \phi_x\left(\frac{k(k-1)}{2} + 1 - 2\omega(x) + \omega(x)^2\right)}{1 - \omega(x)} \right) + C_x \right) P_1(\tau_x < \tau_0). \end{aligned} \tag{2.17}$$

Substituting (2.17) into (2.2) and using monotonicity of D_n gives

$$\begin{aligned} P_1(\tau_0 > \tau_n) \leq & \frac{1}{n} + \frac{1}{n} \sum_{x=1}^{n-1} (2\omega(x) - 1) \left(\frac{k - \phi_x(k - \omega(x) + \omega(x)^2)}{1 - \omega(x)} + \right. \\ & \left. + \delta_x^{(k-1)} \left(\frac{\frac{k(k-1)}{2} - \phi_x\left(\frac{k(k-1)}{2} + 1 - 2\omega(x) + \omega(x)^2\right)}{1 - \omega(x)} \right) + C_x \right) P_1(\tau_x < \tau_0). \end{aligned} \tag{2.18}$$

Consider part (i) of the theorem which we have $\omega(x) = p$ for all x and $\phi \equiv \phi_x$ independent of

x. By letting $n \to \infty$, we have

$$\lim_{n\to\infty} P_1(\tau_0 > \tau_n) \leq \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n-1} (2p-1) \left(\frac{k - \phi(k - p + p^2)}{1-p} \right) P_1(\tau_x < \tau_0) +$$
$$+ \lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n-1} (2p-1) \delta_x^{(k-1)} \left(\frac{\frac{k(k-1)}{2} - \phi \left(\frac{k(k-1)}{2} + 1 - 2p + p^2 \right)}{1-p} \right) P_1(\tau_x < \tau_0) +$$
$$+ \lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n-1} (2p-1) C_x P_1(\tau_x < \tau_0)$$

computing the limit, we obtain,

$$P_1(\tau_0 = \infty) \leq (2p-1) \left(\frac{k - k\phi + p\phi - \phi p^2}{1-p} \right) P_1(\tau_0 = \infty),$$

provided that $\frac{\frac{k(k-1)}{2} - \phi \left(\frac{k(k-1)}{2} + 1 - 2p + p^2 \right)}{1-p} \leq 0$. Thus

$$(2p-1) \left(\frac{k - \phi(k - p + p^2)}{1-p} \right) \geq 1,$$

or

$$\phi = P_1(\tau_0 = \infty) \leq \frac{(k+1) - (2k+1)p}{k - (2k+1)p + 3p^2 - 2p^3}. \tag{2.19}$$

with the constraint $\phi \leq \frac{\frac{k(k-1)}{2}}{\frac{k(k-1)}{2} + 1 - 2p + p^2}$. But this constraint is always greater than the result we have when $k \geq 2$ and $\frac{1}{2} \leq p \leq 1$. So we have the desired result. Because of transient assumption $\phi > 0$, it follows that $p > \frac{k+1}{2k+1}$ is a necessary condition for transience. And this gives the desired upper bound on $P_1(\tau_0 = \infty)$ in part (i) of the theorem.

Now consider part (ii) of the theorem. In this case $\omega(x)$ and ϕ_x are periodic with period $N > 1$. By transience assumption and Lemma 2.3, $\phi_x > 0$ for all x. Therefore; letting $n \to \infty$ in (2.18), we get

$$\frac{1}{N} \sum_{x=1}^{N} (2\omega(x) - 1) \left(\frac{k - \phi_x(k - \omega(x) + (\omega(x))^2)}{1 - \omega(x)} \right) \geq 1. \tag{2.20}$$

Since (2.20) was derived under the assumption that $\phi_x > 0$, for all x, we get $\frac{1}{N} \sum_{x=1}^{N} \frac{k(2\omega(x)-1)}{1-\omega(x)} >$

1 which is a necessary condition for transience. And this is equivalent to

$$\frac{1}{N}\sum_{x=1}^{N}\frac{\omega(x)}{1-\omega(x)} > 1+\frac{1}{k}. \tag{2.21}$$

□

Chapter 3

Range

In this chapter we will try to take a look about the speed of the range or how fast the maximum/minimum point moves. Let us introduce the notation we need, let

$$\mathcal{M}(k) = \max_{i=1,2,\ldots,k} X_i,$$

$$\mathfrak{m}(k) = \min_{i=1,2,\ldots,k} X_i,$$

be the maximum and minimum of the site visited in previous k steps, respectively. Let $\mathcal{R}(k) = \mathcal{M}(k) - \mathfrak{m}(k)$ be the size of the range of the random walk for the previous k steps. Let $\mathfrak{t}_n = \min\{k : \mathcal{R}(k) = n\}$ be the stopping time when this range reaches n. Also, $\Delta_{n,n+1} = \mathfrak{t}_{n+1} - \mathfrak{t}_n$, the time required for size of the range n to increase by 1.

Let $D_{\mathcal{M}}(n) = \mathbb{E}(\Delta_{n,n+1}|X_{\mathfrak{t}_n} = \mathcal{M}(\mathfrak{t}_n))$ and $D_{\mathfrak{m}}(n) = \mathbb{E}(\Delta_{n,n+1}|X_{\mathfrak{t}_n} = \mathfrak{m}(\mathfrak{t}_n))$, the average of the time needed to increase the size of range one step given that the previous one have occurs at the maximum(resp. minimum). Next, let $p_{\mathcal{M}}(n) = \mathbb{P}(X_{\mathfrak{t}_{n+1}} = \mathcal{M}(\mathfrak{t}_{n+1})|X_{\mathfrak{t}_n} = \mathcal{M}(\mathfrak{t}_n))$ and $p_{\mathfrak{m}}(n) = \mathbb{P}(X_{\mathfrak{t}_{n+1}} = \mathfrak{m}(\mathfrak{t}_{n+1})|X_{\mathfrak{t}_n} = \mathfrak{m}(\mathfrak{t}_n))$ be the probabilities that once the range increased at the maximum (resp. minimum) the next increase will take place again at the maximum(resp. minimum). Without lost of generality we can assume that the range is now $\{0, 1, \ldots, n\}$.

In Pinsky's random walk, if you are at the minimum, you have made at least one left jump at each site from the maximum all the way through the minimum. So at the minimum, we can precisely calculate the probability as follows.

$$p_{\mathfrak{m}} = (1-p) + p\left(\frac{n-1}{n}p_{\mathfrak{m}} + \frac{1}{n}(1-p_{\mathcal{M}})\right),$$
$$p_{\mathcal{M}} = \frac{1}{2} + \frac{1}{2}\left(\frac{n-1}{n}p_{\mathcal{M}} + \frac{1}{n}(1-p_{\mathfrak{m}})\right).$$

Solving for $p_{\mathfrak{m}}$ and $p_{\mathcal{M}}$, we get

$$p_{\mathfrak{m}} = p_{\mathfrak{m}}(n) = \frac{1-p+n(1-p)}{1+n(1-p)}.$$
$$p_{\mathcal{M}} = p_{\mathcal{M}}(n) = \frac{p+n(1-p)}{1+n(1-p)}.$$

The $p_{\mathcal{M}}$ in this case appear just to help us calculate $p_{\mathfrak{m}}$. For large n, both probability is closed to 1, i.e.

$$p_{\mathfrak{m}}(n) = 1 - \frac{p}{1-p} \times \frac{1}{n} + O(n^{-2}) \text{ and } p_{\mathcal{M}}(n) = 1 - \frac{1}{n} + O(n^{-2}). \tag{3.1}$$

Now consider the case of being at the maximum. If you are at the maximum, the environment will be mixed between $\frac{1}{2}$ and p. And if you have made any left jump (not increasing the range) whenever you come back to the maximum site again, the probability will not be the same. So we will try to find the bound instead.

One trivial upper bound is to assume that the change will not happen at all i.e. we will have a asymmetric random walk. We will see that the probability of growing at the maximum, will be $1 - O(x)$ where x decays exponentially.

Suppose that the bias will not go away even you have made a left jump at that site. This will be served as our upper bound. Let $\beta(x)$ be the probability of reaching the rightmost point in the interval before hitting the leftmost point. When the size of range equals n and $0 \leq x \leq n$, we have the exit probability from the starting point x to the right is

$$\beta(x) = \frac{\left(\frac{1-p}{p}\right)^x - 1}{\left(\frac{1-p}{p}\right)^n - 1}.$$

Recalled that $p_{\mathfrak{m}}$ is the probability of growing at the minimum, provided that last time the range also grows at the minimum. Similarly for $p_{\mathcal{M}}$, we add the superscript in order to tell that this is just the upper bound of the quantity we are interested in. Thus, we obtain,

$$\begin{aligned} p^u_{\mathfrak{m}}(n) &= (1-p) + p\left((1-\beta(1))\, p^u_{\mathfrak{m}} + \beta(1)(1-p^u_{\mathcal{M}})\right) \\ p^u_{\mathcal{M}}(n) &= p + (1-p)\left((1-\beta(n-1))\,(1-p^u_{\mathfrak{m}}) + \beta(n-1)p^u_{\mathcal{M}}\right). \end{aligned}$$

Solve two equations above we have

$$\begin{aligned} p^u_{\mathfrak{m}}(n) &= \frac{-b(n)p^2 + 2b(n)p - b(n) - p + 1}{a(n)p^2 - b(n)p^2 + 2b(n)p - b(n) - p + 1} \\ p^u_{\mathcal{M}}(n) &= \frac{p(a(n)p - p + 1)}{a(n)p^2 - b(n)p^2 + 2b(n)p - b(n) - p + 1}, \end{aligned}$$

where $b(n) = \beta(n-1)$ and $a(n) = \beta(1)$. Again, we will only focus on the growth at the maximum. Since $\lim_{n\to\infty} b(n) = 1$, we get

$$\lim_{n\to\infty} p^u_{\mathcal{M}}(n) = \frac{p(a(n)p - p + 1)}{a(n)p^2 - p^2 + p} = 1.$$

Let $x(n) := (\frac{1-p}{p})^n$. Then we get

$$\begin{aligned} a(n) &= \beta(1) = \frac{\frac{p}{1-p} - 1}{x(n) - 1} \\ b(n) &= \beta(n-1) = \frac{\frac{p}{1-p}x(n) - 1}{x(n) - 1}. \end{aligned} \tag{3.2}$$

From,

$$p^u_{\mathcal{M}}(n) = \frac{p(a(n)p - p + 1)}{a(n)p^2 - b(n)p^2 + 2b(n)p - b(n) - p + 1}$$

Using (3.2), change a, b into x

$$\begin{aligned}
&= -\frac{p(p - x + px)}{-p^2 + x - 2px + p^2x} \\
&= \frac{p^2 - p(1-p)x}{p^2 - (1-p)^2x} \\
&= 1 - \left(\frac{3}{p} - 2 - \frac{1}{p^2}\right) x + O(x^2) \\
&= 1 - \left(\frac{3}{p} - 2 - \frac{1}{p^2}\right) (\frac{1-p}{p})^n + O((\frac{1-p}{p})^{2n}).
\end{aligned}$$

Thus we have the upper bound which decays exponentially which will not match the growth at the minimum which decays polynomially. Thus this kind of upper bound is too rough. So we try to utilize the fact that the environment will change to symmetric one after you have made the first left jump.

Recall that $\nu_x = \inf\{n \geq 1 : X_{n-1} = x, X_n = x - 1\}$, the first left jump at site x. We define

$$s_x^{(j)} = \begin{cases} p, & \text{if } \nu_x > \tau_j, \\ \frac{1}{2}, & \text{if } \nu_x < \tau_j. \end{cases} \tag{3.3}$$

Let $\gamma_{x,y} = \inf\{n > \tau_y | X_n = x\}$ or the hitting time of x after visited site y. Also let $P^{\{s_x^{(n)}\}}(\cdot) = P(\cdot \mid \{s_x^{(n)}\}_{x=1}^{\mathcal{M}-1})$, or the probability distribution knowing all the information of any left jump made at site 0 through site n.

For the walker started at the maximum, it can jump directly to the right to make a range grows by 1 or makes k step(s) back to the left and then comes back at the maximum and makes a right jump.

To jump back k step(s), it is the same as you jump back one step at a time but not necessary consecutive. So the probability can be written as

$$P_{\mathcal{M}}^{\{s_x^{(n)}\}}\left(\tau_{\mathcal{M}-k} < \tau_{\mathcal{M}+1}\right) =$$
$$P_{\mathcal{M}}^{\{s_x^{(n)}\}}\left(\tau_{\mathcal{M}-1} < \tau_{\mathcal{M}+1}\right) \prod_{i=1}^{k-1} P_{\mathcal{M}}^{\{s_x^{(n)}\}}\left(\tau_{\mathcal{M}-i-1} < \tau_{\mathcal{M}+1} \mid \tau_{\mathcal{M}-i} < \tau_{\mathcal{M}+1}\right). \tag{3.4}$$

It is clear that the first step jumping left at the maximum is

$$P_{\mathcal{M}}^{\{s_x^{(n)}\}}\left(\tau_{\mathcal{M}-1} < \tau_{\mathcal{M}+1}\right) = 1 - p = 1 - s_{\mathcal{M}}^n. \tag{3.5}$$

Now we try to calculate $P_{\mathcal{M}}^{\{s_x^{(n)}\}}\left(\tau_{\mathcal{M}-i-1} < \tau_{\mathcal{M}+1} \mid \tau_{\mathcal{M}-i} < \tau_{\mathcal{M}+1}\right)$. We know that the environment on the right hand side of $\mathcal{M}-i$ is $\frac{1}{2}$ and $s_{\mathcal{M}-i}^n$ at the site $\mathcal{M}-i$.
Let $p_i = P_{\mathcal{M}}^{\{s_x^{(n)}\}}\left(\tau_{\mathcal{M}-i-1} > \tau_{\mathcal{M}+1} \mid \tau_{\mathcal{M}-i} < \tau_{\mathcal{M}+1}\right)$, the probability starting from site $\mathcal{M}$, visiting $\mathcal{M}-i$ then reaching site $\mathcal{M}+1$ before visiting site $\mathcal{M}-i-1$. Since the environment on the right hand side of $\mathcal{M}-i$ is $\frac{1}{2}$, we can use the martingale to calculate this quantity. When the walker makes a right jump, we can think about the walker is in the interval of length $i+1$ starting from $\mathcal{M}-i+1$ to $\mathcal{M}+1$ and all the environments in between are $1/2$ or being symmetric. So the exit probability to the left, back to the starting point is $\frac{i}{i+1}$.

Since all the environment in the interval have been changed, we can treat the environment as we restarting the calculation again. So we multiply this with the starting probability, p_1. On the other hand, the exit probability to the right is $\frac{1}{i+1}$ when the walker exits at this side it means that we are done. So we have,

$$\begin{aligned} p_i &= s_{\mathcal{M}-i}^n\left(\frac{i}{i+1}p_i + \frac{1}{i+1}\right) \\ (i+1)p_i - i s_{\mathcal{M}-i}^n p_i &= s_{\mathcal{M}-i}^n \\ p_i &= \frac{s_{\mathcal{M}-i}^n}{i+1-i s_{\mathcal{M}-i}^n}. \end{aligned}$$

Therefore; we have

$$P_{\mathcal{M}}^{\{s_x^{(n)}\}}\left(\tau_{\mathcal{M}-i-1} < \tau_{\mathcal{M}+1} \mid \tau_{\mathcal{M}-i} < \tau_{\mathcal{M}+1}\right) = 1 - P_{\mathcal{M}}^{\{s_x^{(n)}\}}\left(\tau_{\mathcal{M}-i-1} > \tau_{\mathcal{M}+1} \mid \tau_{\mathcal{M}-i} < \tau_{\mathcal{M}+1}\right)$$
$$= 1 - \frac{s_{\mathcal{M}-i}^n}{i+1-is_{\mathcal{M}-i}^n}. \tag{3.6}$$

Substitute (3.5) and (3.6) into (3.4) we get

$$\begin{aligned} P_{\mathcal{M}}^{\{s_x^{(n)}\}}\left(\tau_{\mathcal{M}-k} < \tau_{\mathcal{M}+1}\right) &= (1 - s_{\mathcal{M}}^n)\prod_{i=1}^{k-1}\left(1 - \frac{s_{\mathcal{M}-i}^n}{i+1-is_{\mathcal{M}-i}^n}\right) \\ &= \prod_{i=0}^{k-1}\left(1 - \frac{s_{\mathcal{M}-i}^n}{i+1-is_{\mathcal{M}-i}^n}\right). \end{aligned}$$

Then we have

$$\begin{aligned} &P_{\mathcal{M}}^{\{s_x^{(n)}\}}\left(\gamma_{\mathcal{M}+1,\mathcal{M}-k} < \tau_{\mathcal{M}-k-1}\right) \\ &\quad = P_{\mathcal{M}}^{\{s_x^{(n)}\}}\left(\tau_{\mathcal{M}-k} < \tau_{\mathcal{M}+1}\right) P_{\mathcal{M}-k}^{\{s_x^{(n)}\}}\left(\tau_{\mathcal{M}+1} < \tau_{\mathcal{M}-k-1} \mid \tau_{\mathcal{M}-k} < \tau_{\mathcal{M}+1}\right) \\ &\quad = (1 - s_{\mathcal{M}}^n)\left(\frac{s_{\mathcal{M}-k}^n}{k+1-ks_{\mathcal{M}-k}^n}\right)\prod_{i=1}^{k-1}\left(1 - \frac{s_{\mathcal{M}-i}^n}{i+1-is_{\mathcal{M}-i}^n}\right). \end{aligned}$$

If we collect all possible k steps, we get

$$\begin{aligned} \sum_{k=0}^{n} P_{\mathcal{M}}^{\{s_x^{(n)}\}}\left(\gamma_{\mathcal{M}+1,\mathcal{M}-k} < \tau_{\mathcal{M}-k-1}\right) &= \sum_{k=0}^{n}(1-p)\left(\frac{s_{\mathcal{M}-k}^n}{k+1-ks_{\mathcal{M}-k}^n}\right)\prod_{i=1}^{k-1}\left(1 - \frac{s_{\mathcal{M}-i}^n}{i+1-is_{\mathcal{M}-i}^n}\right) \\ &= 1 - \prod_{k=0}^{n}\left(1 - \frac{s_{\mathcal{M}-k}^n}{k+1-ks_{\mathcal{M}-k}^n}\right). \end{aligned}$$

This is the probability to grow at the maximum given that you are at the maximum. From the definition of s_x^j in (3.3), we cannot draw any conclusion that it will be $\frac{1}{2}$ or p but we can find upper bound and lower bound by substitute $s^n_{x\mathcal{M}-k}$ with p and $\frac{1}{2}$, respectively. So we calculate

$$\prod_{k=0}^{n}\left(1-\frac{s^n_{\mathcal{M}-k}}{k+1-ks^n_{\mathcal{M}-k}}\right)=\prod_{k=0}^{n}\left(1-\frac{p}{k+1-kp}\right)$$
$$=\frac{(1-p)\Gamma\left(n+2\right)\Gamma\left(\frac{1}{1-p}\right)}{\Gamma\left(n+1+\frac{1}{1-p}\right)}$$

Since $\frac{1}{2}\leq p\leq 1$, $\frac{1}{1-p}\geq 2$. Let $\frac{1}{1-p}=2+x$, where $x\geq 0$. From the limit for asymptotic approximations of gamma function,

$$\lim_{n\to\infty}\frac{\Gamma(n+\alpha)}{\Gamma(n)n^{\alpha}}=1,\qquad \alpha\in\mathbb{R},$$

we can see that

$$\frac{(1-p)\Gamma\left(n+2\right)\Gamma\left(\frac{1}{1-p}\right)}{\Gamma\left(n+1+\frac{1}{1-p}\right)}=\frac{(1-p)\Gamma\left(n+2\right)\Gamma\left(2+x\right)}{\Gamma\left(n+3+x\right)}\sim C\frac{1}{n^{1+x}}=C\frac{1}{n^{\frac{p}{1-p}}}.$$

So the growth at the maximum in Pinsky's model will depend on p, strength of the cookie, i.e. $1-O(\frac{1}{n^{\frac{p}{1-p}}})$. But the growth at the minimum is always $1-O(\frac{1}{n})$. So we look at the special case or the lower bound of the probability of growing at the maximum where p is $\frac{1}{2}$, this is just a simple symmetric random walk which will exhibit a stationary distribution of growing at the maximum and minimum as in [2, Proposition 1]

Theorem 3.1. *Let Z_n be an induced chain on $\{0,1\}$. Let $Z_n=0$ if $X_{\mathfrak{t}_n}=\mathfrak{m}(\mathfrak{t}_n)$ or if you are at the minimum with the probability $p_{\mathfrak{m}}(n)=1-\frac{p}{1-p}\times\frac{1}{n}+O(n^{-2})$. Let $Z_n=1$ if $X_{\mathfrak{t}_n}=\mathcal{M}(\mathfrak{t}_n)$ or if you are at the maximum with the probability $p^l_{\mathcal{M}}(n)=1-\frac{1}{n}+O(n^{-2})$. Thus, there is a limiting distribution*

$$\lim_{n\to\infty}\mathbb{P}(Z_n=0)=1-p=:\pi_{\mathcal{M}}$$

.

After we have the stationary distribution, by [2, Theorem 1], we get that when the size of the range is very large, the average of the time required to grow one unit is two times the probability of jumping to the right times the size of the range at that time. And also the average of time to grow to the the current size is equal to the probability times the square of current size.

Theorem 3.2. *(a) For large n*

$$\mathbb{E}\Delta_{n,n+1} = 2pn + o(n).$$

(b) Asymptotically,

$$\mathbb{E}\mathfrak{t}_n = pn^2 + o(n^2).$$

So we know that the growth will not be less than this quantity above in Pinsky's model or when the first left jump change the environment. In summary, we have

$$p_{\mathfrak{m}}(n) = 1 - \frac{p}{1-p} \times \frac{1}{n} + O(n^{-2})$$

and

$$1 - \frac{1}{n} + O(n^{-2}) \leq p_{\mathcal{M}}(n) \leq 1 - O(\frac{1}{n^{\frac{p}{1-p}}}) \leq 1 - \left(\frac{3}{p} - 2 - \frac{1}{p^2}\right)(\frac{1-p}{p})^n + O((\frac{1-p}{p})^{2n}).$$

Chapter 4

Summary and Future Research

4.1 Summary

The research done in this dissertation extended the Pinky's model [9] to include the second left jump break cookie, only in deterministic case. We had to deal with some quantity which will not arise in the Pinsky's model. We also saw some similarities in cookie random walk and Pinsky's model. The crucial thing we had to look was the average of the total drift. In stationary and ergodic cookie random walk, it was the average of the drift stored at site zero. In Pinsky's and our model we look at the total drift throughout the entire range. We also has some results on the general k^{th} case. Even though, we had only one side of the theorem similar to the case of second left jump. For Pinsky's model when the process was recurrence, it was like having a geometric number of cookies waiting for the first left jump to change the behavior at that site. In our model it was a negative binomial number of cookies waiting for the k^{th} left jump.

Moreover, We had investigated the behavior of the growth of the length [2]. They had a stationary distribution telling when the growth would take place at the same extrema site. And the average speed of the growth followed. In our case, we tried to look for the same thing. But we wanted to use the simplest model for better understanding. So, we used the Pinsky's model as our base model for this part.

At the minimum, we had the precise probability due the the nature of underlying distribution which is symmetric. We knew the probability of the growth at the maximum site was based on the history. We can find the lower and upper bound to bound this quantity. For the lower bound, the trivial lower bound was the walker had made alt least one left jump throughout the interval except the minimum one. In this special case, we can reduce the model to random walk perturbed at the extrema and can use the result from [2]. For upper bound, we also use the trivial one which resulted in the probability decreasing exponentially. After utilizing the nature of the environment, we had a better upper bound which decreasing in the order of polynomial to the power of the ratio between the probability of jumping to the right and jumping to the left. This makes sense when the probability of jumping to the right is increasing. This will imply that the growth will more likely to take place at the maximum. But this upper bound cannot be matched with the lower bound to had a stationary distribution.

4.2 Application and Simulation

Cookie random walk is one form of the biased random walk wildly used to model biological systems such as a biased random walk model for the trajectories of swimming micro-organisms[7] and biased random walk models for chemotaxis and related diffusion approximations[1]. For example, in ecology to model animal movement, phathophysiology such as movements of cancer cell in blood stream for invasion or movement of a white blood cell to eliminate the treat to the organ. All of these models can be modeled using biased random walk. One advantage of random walk model is that it can distinguish the underlying mechanism from the observed data. So we have much better understanding of various movement mechanism in nature around us.

We also found that there are several models which are non-markovian. We will explain a few examples. Elephant walk [12], for example, which is a discrete-time random walk with memory of full history can be described as follows. For the first step, there is probability q to make 1 right jump and $1-q$ for making left jump. For the time $t+1$, choose the previous time t' from $\{1, 2, 3, \ldots, t\}$ uniformly and move exactly like that t' step with probability p. If at time t' the walker move to the right, at time $t+1$ the walker will also move to the right with probability p. Similarly, left-jump at time t' will lead to the left-jump at time $t+1$ also. With probability $1-p$ the walker will move in the opposite direction from time t'. If $p = \frac{1}{2}$, then there is no memory in the model. The walker move independent from its past. If $p < \frac{1}{2}$, we will call a reformer or one who try to move differently from their past. If $p > \frac{1}{2}$, it will be more like a traditional type, prefer to move similar to their past. Schütz and Trimper found the critical value of p which is the memory parameter. This value will separate the walker in to 2 cases, a weakly localized regime and escape regime which is similar to transience and recurrence in our context. Moreover, in escape regime, they found another critical value which will make the process superdiffusive.They also know that the probability distribution is governed by non-Markovian Fokker-Planck equation.

As we have already discussed some models in the group of Reinforced random walk (RRW). We may consider our model is also one kind of RRW. In this type of process, the walker will modify the chemical environment of themselves or the chemical of other individuals within the system. Therefore this type of models has been widely used to describe cell locomotion. It allows the modelers to modify the transition probabilities using the information about the site it visited or the path the walker took. In one dimensional RRW there is a "master" equation as follows [5, equation (3.17)]

$$\frac{\partial}{\partial t}p(x,t) = T^{+}(x-\delta,t)p(x-\delta,t) + T^{-}(x+\delta,t)p(x+\delta,t) - (T^{+}(x,t) + T^{-}(x,t))p(x,t).$$

Where $T^{+}(x,t)$ is called the transition rate from x to $x+\delta$ and $T^{-}(x,t)$ is from x to $x-\delta$. There is a discrete time version of the above equation as follows,

$$p(x,t+\tau) = p(x,t)(1-l-r) + p(x-\delta,t)r + p(x+\delta,t)l.$$

Where τ is time step, l and r are probability of making a left jump and right jump with distant δ, respectively. Let $w(x,t)$ be a concentration of a control substance. Using the continuum limit, we derived,

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(d(w)\frac{\partial p}{\partial x} - \chi(w)p\frac{\partial w}{\partial x}\right).$$

Also notice that is is a one-dimensional version of the Fokker-Planck equation. Several possible models can be derived from this equation such as 'local model', 'barrier model', 'normalized barrier model' and 'variable mean waiting time(VMWT)'. From modeler point of view this is convenient since the functional form of $d(w)$(which describes the effects of a control substance on random motility) and $\chi(w)$(which describes directional effect) can tell a relationship between microscopic rules about the transition rates which govern individual cell behavior and macroscopic scale of behavior at a population level.

There are two main types of mechanisms for movement in response to stimulus, kinesis and taxis. Kinesis refers to a situation that the organism samples only at a single point and adjust its speed accordingly. Taxis refers to a situation that the organism is able to detect the preferential direction and bias its turn without necessarily altering total speed or turning rate. Most of the time the random walk use the transition rates which we have to assume that we have full control over migrating organism. But we can see that in barrier model, the organism need at least 2 sensors to calculate the transition rate without moving. We call this 'tropotaxis'. But in the case of having only one sensors, the same taxis can be produced by moving itself around in various directions to sample the local various stimulus. This kind of taxis call 'klinotaxis'. Biased random walk (BRW) can be used for the simplest model of taxis where there is no correlation between successive step direction. More realistic taxis model can be derived in the form of Biased and Correlated random walk. The taxis will be determined by weighting among the global directional bias(goal attractiveness, which controls the advection), the local directional bias(persistence, which controls motility) together with the level of random noise in the system.

We also have a simulation using Monte Carlo method in APL2. There is a programming code in Appendix 2. Basically, we let the walker moves and when the walker visits the site 0, we count as a successful event and end that experiment. Then we start the experiment again

Table 4.1: Table represent the probability of successful coming back to site 0 within 10,000 steps started from site 1

	.65	.7	.75	.8	.85	.9	.95
$k=1$	-	.6860	.4878	.3365	.2163	.1212	.0538
$k=2$	.6524	.4890	.3565	.2627	.1782	.1142	.0524

and again independent of previous results. In other words, we have a sequence of iid random variable for which each trial is Bernoulli random variable with the success p_0. By the law of large number, the average will converge to $p_0 = P_1(\tau_0 \leq N)$. And if we let $N \to \infty$ we will have $P_1(\tau_0 \leq \infty) = 1 - P_1(\tau_0 \leq \infty) = 1 - \phi$. We simulate for 10,000 times. We use the model k^{th} left jump break cookie random walk when $k = 1$ and 2. Next will be the plot between the simulation of **unsuccessful** event or subtract our data from 1.

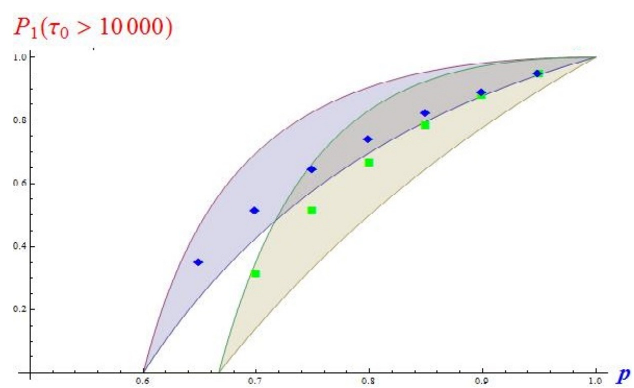

Figure 4.1: Plot comparing the theoretical and simulation results

We can see from the graph that both results are fitted perfectly within their respective bounds.

4.3 Future research

We saw that both first left jump and k^{th} left jump cookies model acted like a geometric and negative binomial, respectively. We can arise to change the underlining distribution such as uniform distribution, the effect of cookie will fade away gradually, which we think it will be similar to one form of Vertex-Reinforced Random Walk. The transition probability will depend on how frequent the number of visit is. The important part but not yet been answer is the speed of the walk in k^{th} left jump model or at least second left jump. We can see also that even in the Pinsky's model, the speed criterion is not yet completed. Or we can try to unify the assumption in which all the cookies-related model will work. Because right now the Pinsky' model have been treated separately but having the similar result. The most promising maybe the second part of the general k^{th} left jump theorem which we suspect it will true but not yet completed. We think that the other part of the proof in general k^{th} left jump model can be done with similar idea we used in proving second left jump.

In Pinsky's random walk [9], stationary and ergodic results followed from the deterministic case with some treatment of the environment. So it is possible to try to extend our results to include the stationary and ergodic environment case also.

About the range problem, lots of questions have been unanswered. We know precisely about the speed of Pinsky's model in transient case when the probability of jumping to the right is higher than $\frac{10}{11}$. Pinsky himself also believes that it should be true for all transient case. We can think about the speed of the maximum point or the growth of the length. It should show some connections to the speed in transient case. We can think about this in other cookie models too since there is a possibility of being transient but having zero speed. Or may be the scaling factor is not precise in those models.

We can also try to consider Pinksy's model when there is only limited amount of memory similar to [2] which consider the limited memory of random walk perturbed at extrema. Since our memory is limited, it is natural to ask the questions like that. Using the idea and knowledge for generalizing Pinsky's model, we think about a learning process in which the past will influence the decision at the current time like in ERRW or VRRW. Or using this idea that the first or second or k^{th} of something will change the environment or the decision making at that current time.

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APPENDICES

Appendix A

Doob's Optional Stopping Theorem

Definition A.1 (Martingale)**.** Let $M = \{M_n\}_{n=0}^{\infty}$ with $M_n \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ for n=0,1,.... Let $\{\mathcal{F}_n\}_{n=0}^{\infty}$ be a filtration on $(\Omega, \mathcal{F}, P)$.

(i) The process M is called ***martingale*** if M is $\{\mathcal{F}_n\}_{n=0}^{\infty}$-adapted and

$$E(M_{n+1} \mid \mathcal{F}_n) = M_n \text{ a.s. for } n = 0, 1, \ldots.$$

(ii) The process M is called ***sub-martingale*** or ***super-martingale***, if M is $\{\mathcal{F}_n\}_{n=0}^{\infty}$-adapted and

$$E(M_{n+1} \mid \mathcal{F}_n) \geq M_n \text{ a.s.} \quad \text{or} \quad E(M_{n+1} \mid \mathcal{F}_n) \leq M_n \text{ a.s.},$$

for $n = 0, 1, \ldots$, respectively.

Definition A.2 (Martingale)**.** A discrete-time stochastic process $\{X_n\}_{n=0}^{\infty}$ is ***martingale*** if

(a) $E(|X_n|) < \infty$ for all n, and

(b) $E(X_{n+1} \mid X_0, X_1, \ldots, X_n) = X_n$.

Definition A.3 (Stopping Time)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$. A random variable (time) $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ is called ***stopping time*** if

$$\{\omega \in \Omega \mid \tau(\omega) = n\} \in \mathcal{F}_n \text{ for } n = 0, 1, 2, \ldots.$$

Theorem A.4 (Doob Decomposition Theorem)**.** *Any submartingale* $\{X_n\}_{n=0}^{\infty}$ *, can be written in a unique way as* $X_n = M_n + A_n$*, where* M_n *is a martingale and* A_n *is a predictable increasing sequence with* $A_0 = 0$*.*

Theorem A.5 (Doob's Optional Stopping Theorem)**.** *Let* $X = \{X_n\}_{n=0}^{\infty}$ *be a martingale with respect to* $\{\mathcal{F}_n\}_{n=0}^{\infty}$ *and* $\tau_1, \tau_2 : \Omega \to \mathcal{N}$ *be stopping times such that*

$$\tau_1(\omega) \leq \tau_2(\omega) \leq T < \infty$$

for all $\omega \in \Omega$ *and for some* $T > 0$*. Then*

$$E(X_{\tau_2} \mid \mathcal{F}_{\tau_1}) = X_{\tau_1} \;\; a.s.$$

Appendix B

APL2 simulation Code

The first code will be the code that generate uniform distribution on $[0, 1]$.

```
[0]   Y←UNI X
[1]   Y←(?Xρ¯2+2⋆31)÷¯2+2⋆31
```

The next code will give the result of the position of the walker after a certain steps.

```
 Y←SIZE THESIS1 P;J1;JF;COMPARE;COUNT;APPEND;TEMP;Y1;BCOUNT
⍝'Y' FOR SEQUENCE OF POSITION OF THE WALK
⍝'Y1' FOR POSITION OF THE WALK
⍝'JI' FOR THE COLLECTION OF THE I TH JUMP
⍝'JF' FOR THE COLLECTION OF THE FINAL K TH JUMP
⍝'COMPARE' FOR PROBABILITY OF THE JUMP TO THE RIGHT DEPEND ON WHETHER
⍝   K JUMP HAPPENS BEFORE
⍝'APPEND' FOR 1 OR -1 DEPEND ON LEFT OR RIGHT JUMP HAPPEND
⍝'COUNT' FOR EACH EXPERIMENT
⍝'BCOUNT' FOR WHOLE EXPERIMENT
⍝'TEMP' FOR A SEQUENCE OF RANDOM VARIABLE IID UNIFORM ON [0,1]
 Y←⍳0
 BCOUNT←1
BLOOP:Y1←1
 COUNT←1
 JF←⍳0
 TEMP←⍳0
LOOP:COMPARE←0.5
 TEMP←TEMP,UNI 1
 APPEND←1
⍝CONDITION TO CHANGE FROM.5 TO P IF IT IS NOT AFTER THE K LEFT JUMP
 →((+/Y1)∊JF)/AFTERFINALJUMP
 COMPARE←(1-P[2])
⍝MECHANISM TO MOVE THE WALKER
AFTERFINALJUMP:→((TEMP[COUNT])<COMPARE)/RJ
 APPEND←¯1
 JF←JF,(+/Y1)
RJ:Y1←Y1,APPEND
 COUNT←COUNT+1
 →(COUNT≤P[1])/LOOP
 Y←Y,(+/Y1)
 BCOUNT←BCOUNT+1
 →(BCOUNT≤SIZE)/BLOOP
 ⎕←Y
```

The following code will check whether the walker come back to 0 from 1 within 10000 steps in Pinsky's model.

```
 Y←SIZE THESIS2 P;J1;JF;COMPARE;COUNT;APPEND;TEMP;Y1;TRY
⍝'Y' FOR SEQUENCE OF POSITION OF THE WALK
⍝'Y1' FOR SEQUENCE OF THE JUMP(LEFT OR RIGHT)
⍝'JI' FOR THE COLLECTION OF THE I TH JUMP
⍝'JF' FOR THE COLLECTION OF THE FINAL K TH JUMP
⍝'COMPARE' FOR PROBABILITY OF THE JUMP TO THE RIGHT DEPEND ON WHETHER
⍝      K JUMP HAPPENS BEFORE
⍝'APPEND' FOR 1 OR -1 DEPEND ON LEFT OR RIGHT JUMP HAPPEND
⍝'COUNT' FOR EACH EXPERIMENT
⍝'TRY' FOR NUMBER OF EXPERIMENT
⍝'TEMP' FOR A SEQUENCE OF RANDOM VARIABLE IID UNIFORM ON [0,1]
 Y←0
 TRY←1
TRYLOOP:Y1←1
 COUNT←1
 JF←⍳0
 TEMP←⍳0
LOOP:COMPARE←0.5
 TEMP←TEMP,UNI 1
 APPEND←1
⍝CONDITION TO CHANGE FROM.5 TO P IF IT IS NOT AFTER THE K LEFT JUMP
 →((+/Y1)∊JF)/AFTERFINALJUMP
 COMPARE←(1-P)
⍝MECHANISM TO MOVE THE WALKER
AFTERFINALJUMP:→((TEMP[COUNT])>COMPARE)/RJ
 APPEND←¯1
 JF←JF,(+/Y1)
RJ:Y1←Y1,APPEND
 COUNT←COUNT+1
⍝CONDITION HITTING ZERO
 →(0≠(+/Y1))/CONT
 Y←Y+1
 →(0=(+/Y1))/END
⍝CONDITION HOW MANY STEPS
CONT:→(COUNT≤10000)/LOOP
END:TRY←TRY+1
 →(TRY≤SIZE)/TRYLOOP
 ⎕←Y
```

The following code will check whether the walker come back to 0 from 1 within 10000 steps in the second left jump break cookies model.

```
 Y←SIZE THESIS3 P;J1;JF;COMPARE;COUNT;APPEND;TEMP;Y1;TRY
⍝'Y' FOR NUMBER OF SUCCESSFUL VISIT TO 0
⍝'Y1' FOR SEQUENCE OF THE JUMP(LEFT OR RIGHT)
⍝'JI' FOR THE COLLECTION OF THE I TH JUMP
⍝'JF' FOR THE COLLECTION OF THE FINAL K TH JUMP
⍝'COMPARE' FOR PROBABILITY OF THE JUMP TO THE RIGHT DEPEND ON WHETHER
⍝      K JUMP HAPPENS BEFORE
⍝'APPEND' FOR 1 OR -1 DEPEND ON LEFT OR RIGHT JUMP HAPPEND
⍝'COUNT' FOR EACH EXPERIMENT
⍝'TRY' FOR NUMBER OF EXPERIMENT
⍝'TEMP' FOR A SEQUENCE OF RANDOM VARIABLE IID UNIFORM ON [0,1]
 Y←0
 TRY←1
TRYLOOP:Y1←1
 COUNT←1
 J1←⍳0
 JF←⍳0
 TEMP←⍳0
LOOP:COMPARE←0.5
 TEMP←TEMP,UNI 1
 APPEND←1
⍝CONDITION TO CHANGE FROM.5 TO P IF IT IS NOT AFTER THE K LEFT JUMP
 →((+/Y1)∊JF)/AFTERFINALJUMP
 COMPARE←(1-P)
⍝MECHANISM TO MOVE THE WALKER
AFTERFINALJUMP:→((TEMP[COUNT])>COMPARE)/RJ
 APPEND←¯1
⍝RECORDING LEFT JUMP CHECK WHETHER LEFT JUMP EVER HAPPENS BEFORE
 →(1-((+/Y1)∊J1))/ADDJ1
 JF←JF,(+/Y1)
ADDJ1:J1←J1,(+/Y1)
RJ:Y1←Y1,APPEND
 COUNT←COUNT+1
⍝CONDITION HITTING ZERO
 →(0≠(+/Y1))/CONT
 Y←Y+1
 →(0=(+/Y1))/END
⍝CONDITION HOW MANY STEPS
CONT:→(COUNT≤10000)/LOOP
END:TRY←TRY+1
 →(TRY≤SIZE)/TRYLOOP
 ⎕←Y
```